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Generating topological order from a two-dimensional cluster state using a duality mapping

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Abstract. In this paper, we prove, extend and review possible mappings between the two-dimensional (2D) cluster state, Wen's model, the 2D Ising chain and Kitaev's toric code model. We introduce a 2D duality transformation to map the 2D lattice cluster state into the topologically ordered Wen model. Then, we investigate how this mapping could be achieved physically, which allows us to discuss the rate at which a topologically ordered system can be achieved. Next, using a lattice fermionization method, Wen's model is mapped into a series of 1D Ising interactions. Considering the boundary terms with this mapping then reveals how the Ising chains interact with one another. The duality of these models can be taken as a starting point to address questions as to how their gate operations in different quantum computational models can be related to each other.

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Contents

1. Introduction

The study of the physical properties of lattice spin systems is one of the most challenging, yet most fascinating, areas of condensed matter physics. These systems exhibit a plethora of exotic quantum phases, some of which are even believed to be useful in quantum computation. Those proposals have in common that they exploit the special ground state properties of a spin system, e.g. its entanglement or topological structure, to overcome the notorious problems of the standard circuit model of quantum computation, such as the individual control of qubits or decoherence. Two prominent examples of such proposals are measurement-based quantum computation (MBQC) and topological quantum computation (TQC).

In the case of MBQC, the computational task is achieved via single-qubit measurements on a highly entangled state [1]. Not only are these measurements simpler to realize experimentally but also this approach reduces the problem of quantum control because the interaction between qubits after the state preparation is not necessary. If the entangled state is appropriately chosen, MBQC is equivalent to the standard circuit model, i.e. it provides us with a universal quantum computer. One example of such a state in two dimensions is the so-called cluster state [2].

TQC, on the other hand, uses a 2D topologically ordered system with non-Abelian braiding statistics and braids the quasi-particle excitations [3] in such a way that logical qubits are processed to perform computational tasks [4]. This model of quantum computation is robust against errors because the long-range entanglement of the system prevents perturbations of small areas of physical qubits from causing errors to the logical qubit. A simple model with topological order and Abelian braiding statistics that may help us to better understand TQC is Kitaev's toric code model [5, 6], which was originally designed for obtaining a topologically protected quantum memory.

While these two proposals make use of two very different properties of the systems to achieve quantum computation, they share the rare property that they are the ground states of some nontrivially interacting exactly solvable spin system. The present work shows a method of directly mapping between the 2D cluster state on a square lattice and Wen's model, which in turn relates the cluster state with the toric code [7] and the Ising model [8]. In this case, the boundary terms transform nontrivially and the ground state degeneracy is changed. These

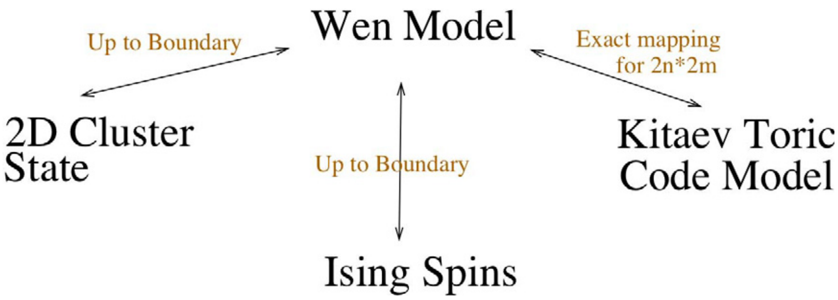

Figure 1. Summary of the mapping between the 2D cluster state, the Wen model, the Ising model and Kitaev's toric code model. The 2D cluster state is mapped, up to the boundary terms, to Wen's model by using a duality transformation. Wen's model can be diagonialized using a fermionization approach leading, up to the boundary terms, to a Hamiltonian of anti-ferromagnetic Ising spins. On the other hand, the Wen model can be mapped exactly to Kitaev's toric code model when the lattice is of even by even dimensions.

mappings are summarized in figure 1. This work can be considered a generalization of the 1D work done in [9], which transforms the 1D cluster state into the 1D Ising model. This particular work considers the boundary terms and topological properties of these models.

Motivated by the importance of 2D spin models for the realization of a quantum computing device, and fascinated by the possibility of mapping these models to each other even though at first glance they seem unrelated, the goal of this work is to prove new methods of mapping between the 2D cluster state and Wen's model, and to extend known results of existing mappings among Kitaev's toric code model, Wen's model and the Ising model.

This paper is organized as follows. We start with a review of the cluster state model and study its topological properties. Next, we prove that in the thermodynamic limit the 2D cluster state model can be mapped to Wen's model [10], which is topologically ordered, using a global unitary transformation. In the case of open boundary conditions, Chen and Hu [8] have shown how Wen's model can be diagonalized using a Jordan–Wigner transformation [11, 12]. We first review their technique and then turn our attention to the more involved case of periodic boundary conditions. We not only give a constructive way of obtaining the spectrum of the Wen Hamiltonian in this way but also present a discussion about the ground state degeneracy using stablizers. Finally, we review the local unitary transformation that takes Wen's model into Kitaev's toric code model [7], establishing a link between the 2D cluster state and Kitaev's toric code. This relationship is then used to interpret the boundary terms of the mapping between the cluster state and Wen's model when the cluster state on a square periodic lattice is transformed.

2. The family of cluster states

Consider a D-dimensional square lattice and associate with each site of the lattice a spin-half particle. Then, the cluster state, which is a special case of a graph state on a square lattice, is defined as the common eigenstate of the 'stabilizers' $K_i = X_i \prod_{j\in\mathcal{N}(i)} Z_j$ with eigenvalue $+1$ $\forall i$. Here, X_i and Z_i are the Pauli spin operators acting on the ith site and $\mathcal{N}(i)$ denotes the neighboring sites of the ith spin. Any cluster state is the nondegenerate ground state of the Hamiltonian $H_C = -\sum_i K_i$ with eigenvalue $-N$.

More insights into the structure of the family of cluster states can be obtained using its operational construction: a cluster state can be prepared via the application of controlled phase gates $U_{i,j} = [\mathbb{1} + Z_i + Z_j - Z_i Z_j]/2$ to the product state $|+\rangle^{\otimes N}$ in the following way,

$$|C\rangle = \left(\prod_i U_{i,\mathcal{N}(i)}\right) |+\rangle^{\otimes N}. \tag{1}$$

The operational representation allows us to write down immediately the complete spectrum of H_C: If we define $|C_{\vec{n}}\rangle = \prod_j (Z_j)^{n_j} |C\rangle$, where $\vec{n} \equiv (n_1, n_2, \ldots, n_N)$ and $n_j \in \{0, 1\}$, then one can easily show that $K_i |C_{\vec{n}}\rangle = (-1)^{n_i} |C_{\vec{n}}\rangle$ $\forall i$ and that $\langle C_{\vec{n}} | C_{\vec{m}}\rangle = \delta_{\vec{n},\vec{m}}$ holds. Further, we can identify that the energy eigenvalue of $|C_{\vec{n}}\rangle$ is a function of the number of spin flips, $E_{\vec{n}} = -N + 2\sum_i n_i$.

The cluster state is a highly entangled state in the sense that the state has the largest relative entropy that any deterministic state can reach for a given number of qubits [13]. The relative entropy of entanglement of the cluster state is $N/2$ [14]. Entanglement also scales with N in other multipartite entanglement measures [15].

Next, we discuss the topological properties of the family of cluster states defined in equation (1). As has been proposed in [16] and [17], the von Neuman entropy serves as a measure to characterize the topological order of a many-body system $\mathcal{S}$ in the following way: let A be a subsystem of $\mathcal{S}$, and let $L = |\partial A|$ be the length of the boundary between A and $\mathcal{S} \setminus A$. Then, the von Neuman entropy is given by

$$S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A) = \alpha L - \gamma + \cdots, \tag{2}$$

where ρ_A denotes the reduced density matrix of subsystem A. The ellipsis represents terms that vanish in the limit $L \to \infty$. The scale invariant part γ characterizes the global feature of the entanglement in the ground state, called the *topological entanglement entropy*.

Making use of the operational representation of the cluster state given in equation (1), the topological entanglement entropy of the cluster state can be readily calculated. To this end, we make use of the fact that any cluster state can be obtained via a local application of controlled phase gates U_k^{cp} and that every individual spin of the system is in a completely mixed state. If we further denote by $\mathcal{N}$ those neighboring sites of the boundary ∂A that lie outside of region A, then we have that $\rho_A = \mathrm{Tr}_{\mathcal{N}}[\mathbf{U} \mathbb{1}_{\mathcal{N}} \otimes |C\rangle_A \langle C| \mathbf{U}^\dagger]$. Here, $|C\rangle_A$ denotes the cluster state defined on subsystem A alone, $\mathbb{1}_{\mathcal{N}}$ is the state of the spins in boundary $\mathcal{N}$ and $\mathbf{U} = (\prod_{k\in\partial A} U_k^{\mathrm{cp}})$ is the unitary of the controlled phase gates applied to the spins on ∂A and $\mathcal{N}$. Thus, $\rho_I = \frac{1}{2^L} \sum_{\{n_k \in \partial A\}=0}^{1} \hat{B}_{\vec{n}} |C\rangle_I \langle C| \hat{B}_{\vec{n}}$, where $\hat{B}_{\vec{n}} = \prod_{k\in\partial A} (Z_k)^{n_k}$, $\vec{n} = (n_1, n_2, \ldots)$, and $L = \sum_{k\in\partial A}$ is the number of qubits in the boundary ∂A. In this case, the boundary ∂A is defined. With this result at hand, we use equation (2) to arrive at

$$S(\rho_I) = -\frac{1}{2^L} \log \frac{1}{2^L} \sum_{\{n_k \in \text{ boundary}\}=0}^{1} = L. \tag{3}$$

A comparison with equation (2) reveals that the topological entanglement entropy $\gamma = 0$.

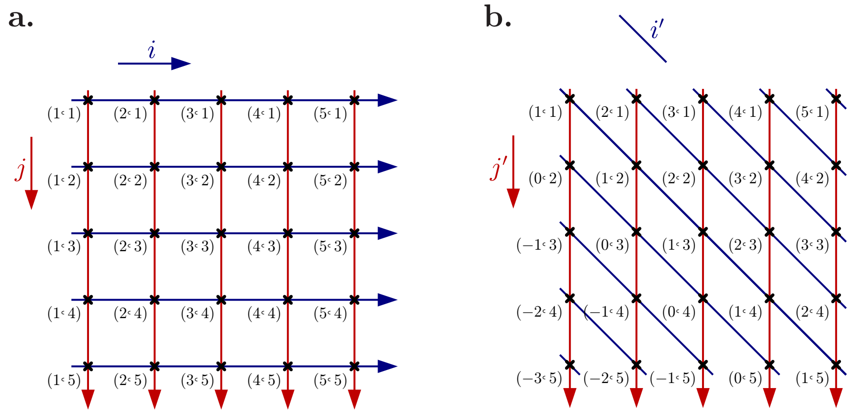

Figure 2. (a) The standard coordinates; the arrowheads show the direction in which the coordinates increase. (b) The new coordinates on a 6×6 lattice. The red arrows show the j' coordinates that still increase, but the diagonal blue lines do not have arrowheads because they show the lines under which the i' coordinates remain constant.

Thus, a cluster state on an arbitrary graph is the unique ground state of a stabilizer Hamiltonian without topological order. However, as we will show in the next section, the 2D cluster state is linked to a topologically ordered system via a simple duality transformation. This seems like a contradiction because the energy spectrum will not be conserved between a cluster-state model with a non-degenerate ground state and a topologically-ordered model. This is resolved later in this paper when we consider boundary terms.

3. Mapping of the two-dimensional (2D) cluster state to a topologically ordered system

In this section, we show how the 2D cluster state Hamiltonian can be transformed into a Hamiltonian with topological order using a simple unitary transformation. Recall that the Hamiltonian of the 2D cluster state on a square lattice is of the form

$$H_C^{2D} = -\sum X_{i,j} Z_{i-1,j} Z_{i+1,j} Z_{i,j-1} Z_{i,j+1}. \tag{4}$$

The duality mapping between the cluster-state model and Wen's model can be written as follows: firstly, the indices are transformed $(i, j) \to (i', j')$ such that $j = j'$ and $i' = i - j + 1$. The effect of this index transformation is shown in figure 2. Then the mapping can be conveniently written as

$$\mu^x_{i',j'} = Z_{i',j'} Z_{i',j'+1}$$

and

$$\mu^z_{i',j'} = \prod_{k=1}^{j'} X_{i',k},$$

where the modes are ordered in a diagonal way on the lattice, as depicted by the operator indices in figure 3.

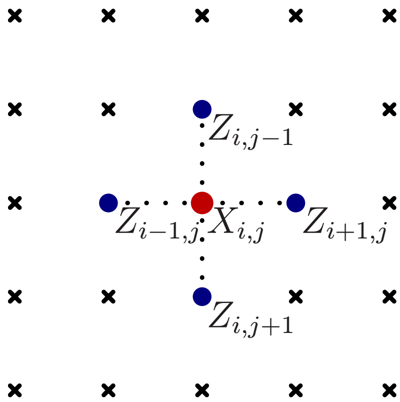

Figure 3. Graphical representation of the term $X_{i,j}Z_{i-1,j}Z_{i+1,j}Z_{i,j-1}Z_{i,j+1}$ of the 2D cluster Hamiltonian.

A straightforward calculation shows that under this dual transformation, the cluster-state Hamiltonian is transformed into Wen's model [10],

$$H_W = -\sum \mu^z_{i,j-1}\mu^z_{i-1,j}\mu^x_{i-1,j-1}\mu^x_{i,j}, \tag{5}$$

where we have now reverted to the original indices.

Wen's model is of special interest because it exhibits a new kind of phase transition where a change of sign in the Hamiltonian leads to a change in the quantum order of the system while the symmetry is preserved. The model is different from Kitaev's toric code model [5] but it can be mapped to it through a local unitary transformation. Further, Wen's model corresponds to the low-energy sector of Kitaev's honeycomb lattice model in the limit $J_z \gg J_x, J_y$ [10].

This mapping can be discussed in more depth by considering the physical operation that performs such a transformation. The mapping is performed using the following circuit [18] along each of the diagonals of a periodic lattice of $N \times N$ spins,

$$U_{i'} = \left[\prod_{j=1}^{N-1}\left(\mathbb{1}_{i',N-j}\otimes|0\rangle\langle 0|_{i',N-j+1} + X_{i',N-j}\otimes|1\rangle\langle 1|_{i',N-j+1}\right)\right]^{\mathrm{T}} \times \left[\prod_{j=1}^{N}\frac{1}{\sqrt{2}}(X_{i',j}+Z_{i',j})\right].$$

The transpose acting on the product of the controlled-NOT gates reverses their order (of course the controlled-NOT gate is symmetric, so all that the transpose does is to reverse the order of the gates). The order is important because all of the controlled-NOT gates have to be performed sequentially. They cannot be performed simultaneously because they do not commute with one another on pairs of nearest neighbour sites. With this in mind, and assuming that each of the controlled-NOT gates requires only one unit of time to be performed, and all of the Hadamard gates can be performed simultaneously (as they all occur on different sites), and each of the $U_{i'}$ operators can be performed simultaneously, then the time it takes to transform a cluster state into a topologically ordered system scales linearly with the length of the boundary of the lattice. This agrees with the result obtained by Bravyi *et al* [19], who found that the optimal time to generate a topologically ordered system scales with the boundary of the system and not with the volume of the system. This matches the rate at which topological order is generated in [6], which is also recognized in [19].

4. Wen's model and its explicit solution

Wen's model describes spins on a 2D lattice that are subject to spin–spin interactions on 2×2-plaquettes [10],

$$\hat{H}_{\mathcal{W}} = g \sum_{i,j} W_{ij}, \tag{6}$$

where $W_{ij} = X_{ij} Z_{i+1,j} Z_{i,j+1} X_{i+1,j+1}$ and g is the interaction strength. This model was designed to be exactly solvable using a projective construction, which is the same method adopted by Kitaev to solve the honeycomb model [17]. This paper uses an alternative approach to diagonalize this Hamiltonian introduced by Chen and Hu [8] and generalizes this approach to periodic boundary conditions. This method uses the Jordan–Wigner transformation to write the spin-half operators in terms of fermionic creation and annihilation operators,

$$2c_{ij}^{\dagger} = \left(\prod_{j'<j} \prod_{i'} Y_{i'j'} \right) \left(\prod_{i'<i} Y_{i'j} \right) (X_{ij} + i Z_{ij}), \tag{7}$$

which obey canonical anti-commutation relations (CAR) $\{c_k^{\dagger}, c_l\} = \delta_{kl}$. Equivalently, one can use the Hermitian Majorana operators,

$$\alpha_{ij} = -\mathrm{i}(c_{ij}^{\dagger} - c_{ij}), \quad \beta_{ij} = c_{ij}^{\dagger} + c_{ij}, \tag{8}$$

which also fulfill the CAR. Now we define new fermionic modes as a linear superposition of Majorana fermions on two neighboring sites (i, j)–$(i, j+1)$, $d_{ij} = (\alpha_{ij} + \mathrm{i}\beta_{i,j+1})/2$. With these new modes, it follows that

$$\mathrm{i}\alpha_{ij}\beta_{i,j+1} = 2d_{ij}^{\dagger}d_{ij} - 1 = X_{ij}(Y_{i+1,j}Y_{i+2,j} \cdots Y_{i-1,j+1})X_{i,j+1}, \tag{9}$$

and the interaction operators in Wen's model take the simple form

$$W_{ij} = (2d_{ij}^{\dagger}d_{ij} - 1)(2d_{i+1,j}^{\dagger}d_{i+1,j} - 1). \tag{10}$$

Wen's model transforms differently, depending on the boundary conditions. We consider first an $N \times M$ lattice with open boundary conditions. Here, the Hamiltonian equation (6) takes the form

$$H_{\mathcal{W}}^{\text{open}} = g \sum_{j=1}^{M-1} \sum_{i=1}^{N-1} (2\hat{\mathcal{N}}_{ij} - 1)(2\hat{\mathcal{N}}_{i+1j} - 1), \tag{11}$$

where $\hat{\mathcal{N}}_{ij} \equiv d_{ij}^{\dagger}d_{ij}$. In this case, neighboring fermions in each row are coupled with Ising-type interactions. However, there are no inter-row interactions, so the Hamiltonian takes the form of $M-1$ independent Ising chains. With this in mind, it is obvious that each row has a two-fold degenerate ground state,

$$|\psi^{+}\rangle_j = \prod_{i=1}^{N/2} d_{2ij}^{\dagger}|\Omega\rangle, \quad |\psi^{-}\rangle_j = \prod_{i=1}^{N/2} d_{2i-1j}^{\dagger}|\Omega\rangle, \tag{12}$$

where $|\Omega\rangle$ is the state of vacuum. The states $|\psi^+\rangle_j$ and $|\psi^-\rangle_j$ have anti-ferromagnetic excitations in the fermionic bases. The ground state of the Hamiltonian is then given by

$$|\psi_{\bar{n}}\rangle = \bigotimes_{j=1}^{M} |\psi^{n_j}\rangle_j, \tag{13}$$

where $\bar{n} = (n_1, n_2, \ldots)$ and $n_j \in \{+, -\}$.[9]

In the case of periodic boundary conditions, the diagonalized Hamiltonian takes a different form. The periodicity of the system written in terms of spin-half operators imposes the constraint $W_{i+N,j+M} = W_{i,j}$. Then, using the same fermionic transformation rules, the Hamiltonian takes the form

$$H_{\mathcal{W}}^{\text{periodic}} = g \sum_{j=1}^{M} \sum_{i=1}^{N-1} (2\hat{\mathcal{N}}_{ij} - 1)(2\hat{\mathcal{N}}_{i+1j} - 1) + g \sum_{j=1}^{M} \hat{L}_j, \tag{14}$$

where $\hat{L}_j$ is of the form

$$\hat{L}_j = \prod_{i=2}^{N-1} (2\hat{\mathcal{N}}_{ij} - 1) \prod_{i=1}^{N} \beta_{ij} \prod_{i=1}^{N} \alpha_{ij+1}, \tag{15}$$

introducing an interaction between rows $j-1$ and $j+1$ (because the β_{ij} operator affects the $(i, j-1)$-th site). Since the operators α_{ij} and β_{ij} flip the excitation of the fermion mode d_{ij}, it follows that for even N,

$$\prod_{i=1}^{N} \alpha_{ij} |\psi^{\pm}\rangle_j = |\psi^{\mp}\rangle_j, \quad \prod_{i=1}^{N} \beta_{ij} |\psi^{\pm}\rangle_{j-1} = |\psi^{\mp}\rangle_{j-1}. \tag{16}$$

For the sake of simplicity, we deal with only the case when N is even in this paper. Then, the condition that the boundary operators impose is that the two eigenstates $|\psi^{\pm}\rangle_j$ in each row should be superposed, and one of the ground states of the Hamiltonian (14) is given by

$$|\Psi_0^{(1)}\rangle = \bigotimes_{j=1}^{M} (|\psi^+\rangle_j \pm |\psi^-\rangle_j). \tag{17}$$

The two ground states can be seen as the product of a Greenberger–Horne–Zeilinger-type entangled state [20]. Further, in the case of periodic boundary conditions and an even number of rows (i.e. M even), $|\Psi_0^{(1)}\rangle$ is not the unique ground state of Wen's model, but we have two other solutions. The complete set of ground states is given by

$$|\Psi_0\rangle = \bigotimes_{j=1}^{M/2} (|\psi^+\rangle_{2j} \pm |\psi^-\rangle_{2j})(|\psi^+\rangle_{2j+1} \pm |\psi^-\rangle_{2j+1}). \tag{18}$$

The ground states of Wen's model can be understood using the stabilizer formalism [21]. In the case when $g < 0$, the ground states of the model will be stabilized by all of the terms in the Hamiltonian W_{ij}. In fact, these terms will form a generating set for the entire stabilizer group. However, in the case of infinite or periodic boundary conditions, the operators W_{ij}

[9] Note that if we defined the Hamiltonian equation (11) for periodic boundary conditions, i.e. on a torus, the ground state would be 2^M-fold degenerate. However, if we interchanged rows and columns, the degeneracy would change to 2^N, as we now have a torus of a different size.

do not form an independent set because of the condition that $\prod_{i,j} W_{ij} = \mathbb{1}$. This implies that $\prod_{i\neq k, j\neq l} W_{ij} = W_{kl}$ and if one of the interaction operators is removed, then the generating set will become independent. Following the stabilizer formalism, if a stabilizer group describing N qubits contains k generators in the generating set, then the stabilizer group will describe 2^{N-k} states [21]. As there are N operators W_{ij}, of which $N-1$ are independent, it is clear that in the general case, periodic lattices will be twofold degenerate. In the special case when either the lattice is infinite or the lattice has dimensions of an even by even number of spins, the interaction operators also follow the condition that $\prod_{i+j=\text{even}} W_{ij} = \mathbb{1}$. This means that there are only $N-2$ independent operators in the generating set, so it follows that the ground state will be fourfold degenerate.

5. The connection to Kitaev's toric code model

In this section, we demonstrate that Wen's model can be transformed into Kitaev's toric code model using only local unitary transformations. Since local transformations have no influence on topological effects, this allows us to evaluate the entanglement of Wen's model by inspecting the ground state of the toric code model. This equivalence, coupled with the duality mapping, also provides the connection between the 2D cluster state and Kitaev's toric code model. The connection between Wen's model and Kitaev's toric code model has been studied by Nussinov and Ortiz [7]. We investigate this transformation for periodic boundary conditions, and look for the necessary conditions for the transformation to be faithful and when the mapping is mismatched. Moreover, we show how to construct the exact ground states of Kitaev's toric code model in a spin basis.

The toric code describes a family of simple spin systems with local interactions in which the existence of anyons can be demonstrated. Due to their braiding statistics, the state can be used for topological quantum computation [22]. Kitaev's toric code model on a square lattice,

$$H_K = -\sum_v A_v - \sum_p B_p, \tag{19}$$

is a sum of constraint operators associated with vertices v and plaquettes p, namely

$$A_v = \prod_{j\in v} X_j, \quad B_p = \prod_{j\in p} Z_j, \tag{20}$$

where $v = \{(i, j), (i+1, j), (i, j+1), (i+1, j+1)\}, \forall i, j$, where $i+j = \text{even}$, and $p = \{(i, j), (i+1, j), (i, j+1), (i+1, j+1)\}, \forall i, j$, where $i+j = \text{odd}$.

Wen's model can be mapped into the toric code using a completely local unitary transformation, $UH_{\mathcal{W}}U^\dagger = H_K$, where

$$U = \prod_{i+j=\text{even}} \frac{1}{\sqrt{2}} \left(X_{i,j} + Z_{i,j}\right). \tag{21}$$

Note that for periodic boundary conditions, this relation is only valid for lattices with an even number of sites in each direction. Let us briefly explain the action of this unitary. If $i + j$ is even, $UW_{ij}U^\dagger = A_{ij}$, whereas if $i + j$ is odd, we find that $UW_{ij}U^\dagger = B_{ij}$, which is a plaquette on p. The conventional method of graphically representing the toric code represents the spins as edges on a graph, instead of using the vertices [5, 6]. One can see that this is equivalent to replacing the vertices with diagonal lines. If we then replace the spins on even sites with diagonal lines

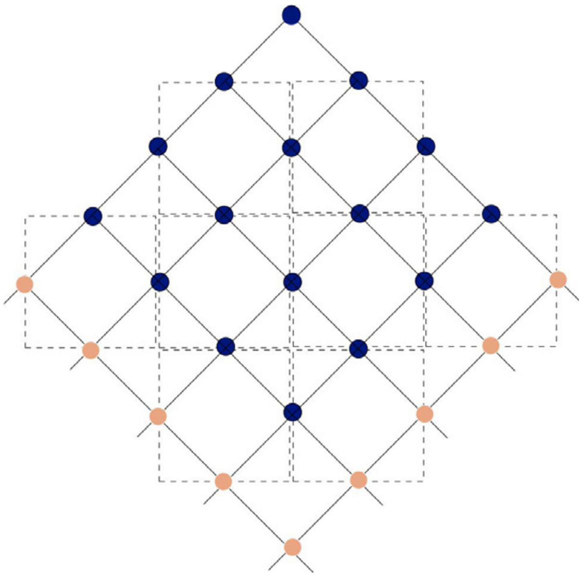

Figure 4. Transformation of a four-by-four square lattice from Wen's spin model into Kitaev's toric code model. The blue dots are the actual spins in the four-by-four lattice and the yellow dots are the spins that are translated from the first row and the first column. The solid line represents the lattice for the Wen model and the dashed lines denote the transformed lattice in the toric code model. They are both periodic under translation.

that go from the top left corner to the bottom right corner, and spins on odd sites with diagonals that go from the bottom left corner to the top right corner then we have a new graph where edges represent spins. This is represented in figure 4 on a 4×4 lattice. One can see that to obtain a faithful mapping from Wen's model to the toric code model, Wen's model has to be on a lattice of an even number of sites by an even number of sites. If this is not the case, then the toric code picture will not have alternating diagonal lines across the boundary terms, and there will not be perfect star and plaquette operators that consist of only one Pauli operator across the boundaries.

Having these results at hand, we turn our attention now to the ground state of the toric code. We start with an explicit construction of the ground states of the toric code in the case of periodic boundary conditions. Let N_S denote the number of spins on the lattice. One of the four ground states of the toric code can be constructed using the set of vertex operators A_v ($v = 1, \ldots, \frac{N_S}{2} - 1$) as a generating set[10] in the following way,

$$|\psi_0\rangle = \frac{1}{\sqrt{2^{\frac{N_S}{2}-1}}} \sum_{\bar{n}} \prod_{k=1}^{\frac{N_S}{2}-1} A_k^{n_k} |\Omega\rangle \,, \tag{22}$$

where $\bar{n} = (n_1, n_2, \ldots n_{\frac{N_S}{2}-1})$, $n_i \in \{0, 1\}$ and $|\Omega\rangle$ denotes the vacuum in the Z-basis.

To check that this is indeed the ground state of the toric code, we now have to show that $|\psi_0\rangle$ is a stabilizer state of the sets $\{A_v\}_v$ and $\{B_p\}_p$. We start with the plaquette operators. Using

[10] Note that there are in fact $\frac{N_S}{2}$ star operators because there are two qubits per vertex. However, due to the boundary condition $\prod_v A_v = \mathbb{1}$, only $\frac{N_S}{2} - 1$ of them are independent.

the fact that all vertex and all plaquette operators commute, meaning that all of the B_p operators stabilize the vacuum state, we immediately find that

$$B_p\,|\psi_0\rangle = \frac{1}{\sqrt{2^{\frac{N_S}{2}-1}}}\sum_{\bar{n}}\prod_{k=1}^{\frac{N_S}{2}-1} A_k^{n_k} B_p\,|\Omega\rangle = \frac{1}{\sqrt{2^{\frac{N_S}{2}-1}}}\sum_{\bar{n}}\prod_{k=1}^{\frac{N_S}{2}-1} A_k^{n_k}\,|\Omega\rangle = |\psi_0\rangle \quad \forall p.$$

It is slightly more complicated to show that $|\psi_0\rangle$ is also a stabilizer state for the vertex operator. To this end, we rewrite the ground state for some fixed v in the following way,

$$|\psi_0\rangle = (\mathbb{1} + A_v)\,\frac{1}{\sqrt{2^{\frac{N_S}{2}-1}}}\sum_{\bar{n}_v}\prod_{k=1,k\neq v}^{\frac{N_S}{2}-1} A_k^{n_k}\,|\Omega\rangle,$$

where $\bar{n}_v = (n_1, n_1, \ldots, n_{v-1}, n_{v+1}, \ldots, n_{\frac{N_S}{2}-1})$. Since $A_v^2 = \mathbb{1}$, we see immediately that $A_v|\psi_0\rangle = |\psi_0\rangle$ holds.

The other three ground states, $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$, can be found by applying non-contractable loops, w_1 and w_2, composed of Pauli X operators around the torus in the horizontal and vertical directions such that the other ground states are $|\psi_1\rangle = w_1|\psi_0\rangle$, $|\psi_2\rangle = w_2|\psi_0\rangle$ and $|\psi_3\rangle = w_1 w_2|\psi_0\rangle$. The non-contractible loop operators commute with all of the vertex and plaquette operators; thus it is trivial to show that these are also ground states of the toric code Hamiltonian.

6. Interpreting the mapping from the cluster state to Wen's model on a square periodic lattice

In this section, we investigate the duality mapping between the cluster-state model and Wen's model for periodic boundary conditions. Here, special care has to be taken since the spins are ordered on the lattice in a diagonal way. For simplicity, we only consider the case of an $N \times N$ lattice, where N is even. When considering periodic boundary conditions, we need to be more precise in how we define our indices. We define the site $(N+k, j') = (k, j')$ and the site $(1-k, j') = (N+1-k, j')$. Then, to ensure that the new spin operators μ^x, μ^y and μ^z obey the $SU(2)$ commutation relations, the mapping has to be changed as follows: $\mu^x_{i',N} = Z_{i',N}$ when $j' = N$ for all i'. Otherwise, all of the mappings are the same.

Having modified the mapping for square periodic lattices, it is then possible to map the boundary terms for the cluster state. First, it is easy to show that when $2 \leqslant j \leqslant N-1$, the terms $C_{1,j}$ and $C_{N,j}$ map exactly to Wen plaquettes $C_{1,j} = \mu^z_{N,j-1}\mu^x_{1,j-1}\mu^x_{N,j}\mu^z_{1,j}$, and $C_{N,j} = \mu^z_{N-1,j-1}\mu^x_{N,j-1}\mu^x_{N-1,j}\mu^z_{N,j}$ using the standard indices.

The other boundary terms are not as easy to interpret. We have that

$$C_{i,1} = \mu^x_{i-1,1}\mu^z_{i,1}\left[\prod_{k=1}^{n-i}\mu^x_{i+k,k}\right]\left[\prod_{k=1}^{i-1}\mu^x_{k,n-i+k}\right]$$

when $j = 1$ and

$$C_{i,N} = \mu^z_{i-1,N-1}\mu^x_{i,N-1}\mu^z_{i,N}\left[\prod_{k=0}^{N-i}\mu^x_{i+k,1+k}\right]\left[\prod_{k=1}^{i-2}\mu^x_{k,N-i+1+k}\right]$$

when $j = N$.

Whereas mapping these boundary terms leads to quite unintuitive terms in the Hamiltonian, the ground state of this Hamiltonian can be interpreted a little more easily by manipulating the stablizers. Firstly, one can replace the generators of the stablizer group $C_{i,N}$ with the stablizers $C_{i-1,1}C_{i,N}$ $\forall i$. Then in the Wen picture we have that $C_{i-1,1}C_{i,N} = \mu^x_{i-2,1}\mu^z_{i-1,1}\mu^z_{i-1,N-1}\mu^x_{i,N-1}\mu^z_{i,N}$. These terms correspond to the Wen plaquettes interacting between the $(N-1)$th row and the first row. These plaquettes, however, are skewed, as it is the $(i-2)$th and $(i-1)$th spins on the first row that interact with the $(i-1)$th and ith spins on the $(N-1)$th row. They all also interact with one spin on the Nth row. With these terms, we see that we have approximately mapped a cluster state on an $N \times N$ lattice into a periodic Wen model on an $N \times (N-1)$ lattice, which interacts with the Nth row. It is important to stress, however, that manipulating the generators of the stablizer group only gives us a description of the ground state because if we form a Hamiltonian by summing the new generators, there will be a different energy spectrum in the excited states. This exchange only allows us to better interpret the ground state.

Finally, one has to consider the stablizers mapped from the $C_{i,1}$ terms. In the Wen picture, these terms all map into non-contractible loops across the $N \times (N-1)$ periodic lattice in the Wen picture. Then, locally mapping the system into the toric code picture reveals that the $C_{i,1}$ terms map into non-contractible loops of μ^x operators when i is odd and into non-contractible loops of μ^z operators when i is even.

It is important that under unitary transformation the energy spectrum is conserved. In [6], these non-contractible loops are used to encode information on a topological memory. It is these loops that conserve the degeneracy of the ground state, as we are mapping from a cluster-state Hamiltonian that has a non-degenerate ground state.

7. Conclusion

In this work, we have reviewed and proven the relations between four 1D and 2D lattice spin models, namely the 2D cluster state model, Wen's model, Kitaev's toric code and the 1D Ising model. These mappings can be utilized in a plethora of ways. Perhaps the most interesting way is the comparison between the 2D cluster state and the Kitaev model. Both models can be used in the possible realization of quantum computation by making use of two very different ground state properties of the system, i.e. the entanglement and the topological structure. In this respect, we believe that our work can contribute to new insights into questions relating to the fields of quantum information science and condensed matter physics. For example, can we find a simple way to characterize condensed matter systems that can be used in quantum computation [23]?

The other important mapping is that between Wen's model and the Ising model. Since the features of topologically ordered models, such as the entanglement structure, can be difficult to understand intuitively, the mapping of a topologically ordered model to something simpler, such as the Ising model, could improve our understanding of the behavior of such a system. We believe that an extension of the mapping presented in this work to lattices with arbitrary dimension will be a further step toward achieving a better knowledge of the behavior of topologically ordered systems.

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