Expectations and Equilibrium in Macroeconomics

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Abstract

Expectations play a central role in macroeconomics. How they are modelled not only shapes predictions about the decisions of economic agents, but also influences the equilibrium properties of the model. This thesis consists of three self-contained chapters, each contributing to the broad theme of expectations and equilibrium in macroeconomics.

The first chapter studies the heterogeneous attention choices of households and firms. Using the rational inattention framework, I show that households find it optimal to pay more attention to supply shocks, because these shocks most affect their real income, while firms optimally pay more attention to demand shocks due to their larger impact on profits. The model reconciles survey evidence on heterogeneous beliefs across households and firms, generates state-dependent Phillips curve slopes, and shows that central bank communication that fails to consider the heterogeneous attention choices can backfire.

The second chapter, co-authored with Guido Ascari, moves to the financial markets. Investors are assumed to possess limited memory and to form expectations in a time-varying way. The former guarantees a tendency to revert to fundamentals. The latter induces momentum in asset prices and is motivated by a novel empirical observation about a time-varying mapping from price-dividend ratio to return expectations in survey data. Using the simulated method of moments, we show that the model matches a host of asset-pricing features, and generates survey-consistent subjective investor beliefs.

The third chapter, co-authored with David Murakami and Ivan Shchapov, shows that the presence of an occasionally binding constraint from the effective lower bound (ELB) in New Keynesian models often leads to multiple or no equilibria. The problem stems from a strong feedback loop between expectations (of inflation and output) and current outcomes at the ELB. We show that simple fiscal policy rules can introduce additional stabilising forces that dampen this loop, thereby ensuring the existence and uniqueness of an equilibrium.

Chapter 1

Rational Inattention Choices in Firms and Households

Abstract

Survey data shows that households associate higher expected inflation with lower expected output growth, while firms and professionals associate higher expected inflation with higher expected growth. Standard macroeconomic models struggle to explain the asymmetry. This paper develops a dynamic general equilibrium model with rationally inattentive households and firms, and shows that the asymmetry in agents' beliefs can be explained by their asymmetric attention to different shocks, driven by their respective objectives. Households find it optimal to pay more attention to supply shocks because these shocks most affect their real income, while firms optimally pay more attention to demand shocks because of their larger impact on profits. Attention choices can influence the shocks' propagation, offering a joint explanation for the flattening of the Phillips curve in recent decades, and its steepening in the post-pandemic period. Finally, I show that policy communication that fails to consider the heterogeneous attention choices may backfire.

1.1 Introduction

Expectations are a key driver of economic decisions.[1] Households' and firms' expectations about future macroeconomic variables are central to their consumption and pricing decisions (Born et al., 2022; Coibion et al., 2023), thereby influencing aggregate output and prices. However, survey evidence shows that expectations of different agents differ widely (Carroll, 2003; Mankiw et al., 2003; Candia et al., 2020). These discrepancies between households and firms are a challenge for theories of expectations and their macroeconomic effects.

One particular point of difference is the way in which different agents perceive the relationship between output growth and inflation. Figure 1.1, based on Candia et al. (2020), plots joint expectations over inflation and output growth for different economic agents in the United States. Households associate higher expected inflation with lower expected output growth. In contrast, firms and professional forecasters associate higher expected inflation with higher expected growth, although the correlation for firms is weak.[2] The negative association by households is labelled as a supply-side view, as supply shocks are expansionary for output and reduce inflation, leading to the negative comovement between output and inflation. Similarly, the positive association by firms and professional forecasters is labelled as a demand-side view, as demand shocks are expansionary for output and inflation (Candia et al., 2020).

What drives these asymmetric views, and how does belief asymmetry affect aggregate outcomes? Standard macroeconomics models assume full-information rational expectations (FIRE) cannot address these questions, as they imply that all agents have the same beliefs, and thus rule out any role for belief heterogeneity. Recent advances in the theory of expectations that depart from FIRE also struggle to account for the systematic heterogeneity observed across agents (e.g., Evans and Honkapohja (2001a); Woodford (2003); Gabaix and Laibson (2017); Bordalo et al. (2018)). In principle, existing expectation models could potentially explain these contrasting views by imposing different partial information or subjective models for different agents (Han, 2024; Andre et al., 2022). For example, one might assume that households observe mostly supply shocks, while firms observe more demand shocks. However, such assumptions lack a theoretical ba-

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[1] Throughout the paper, I use the words "beliefs," "expectations," and "views" as synonyms.

[2] The cross-sectional patterns are consistently observed across various countries (Candia et al., 2020) and in randomised controlled trials (Coibion et al., 2021, 2023). Moreover, all these patterns also hold when controlling for individual-level fixed effects (see Appendix A.1).

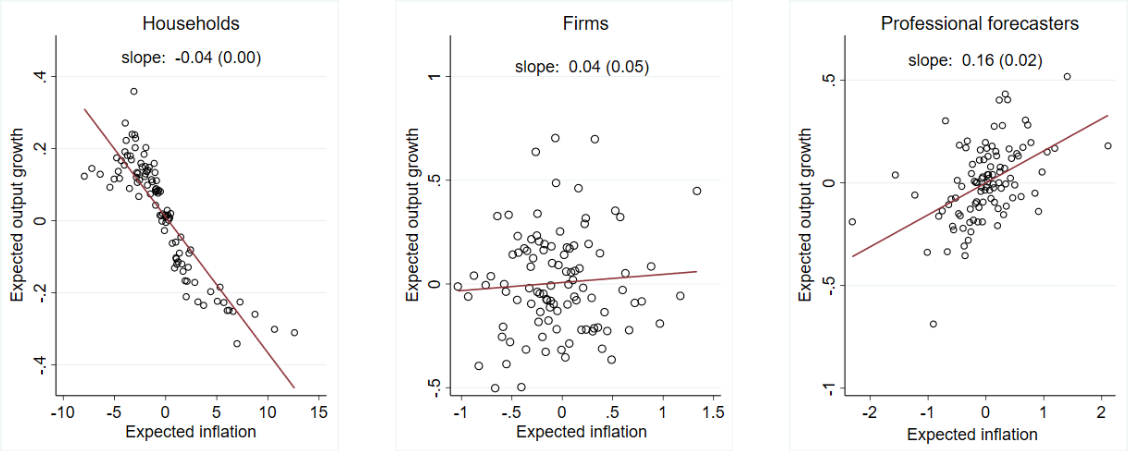

Figure 1.1: Correlation between expected inflation and expected output

Notes: Each panel plots the cross-section of forecasts of output growth and inflation after removing time-fixed effects. I present the resulting correlations in binscatter form for each agent. Table A.3 provides a summary of the associated regression statistics. Data sources: Michigan Survey of Consumers, The Livingston Survey, The Survey of Professional Forecasters.

sis for why these information differences arise. The contribution of this paper is to allow agents to endogenously choose their (partial) information sets and to show that they optimally acquire information in a way that generates such differing partial information, leading to the observed heterogeneity in their views. Using this framework, I examine the implications for business cycle fluctuations, monetary policy, and policy communication.

I develop a dynamic general equilibrium model with rationally inattentive households and firms. The economy is close to a simple New Keynesian model. Households and firms are rationally inattentive, and they can choose *what* and *how much* information to acquire, subject to an attention cost which increases with the informativeness of the signal.[3] The assumption of rational inattention generates *endogenous* and *asymmetric* attention choices for households and firms. The endogeneity of attention choices stems from optimising agents who pay more attention to the economic shocks that matter most for their objectives. The attention choices of firms and households are asymmetric as they have different objectives. With standard utility and profit functions, I show that it is optimal for households to pay more attention to supply shocks than to demand shocks, as supply shocks (which lead to negative comovement in output and inflation) most affect their real income and optimal consumption.[4] Firms, on the other

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[3] Formally, I model the cost of attention as the Shannon mutual information times a scaling parameter, following Sims (2003). More precise (less noisy) signals are therefore more costly.

[4] Although Kamdar (2018) also features rationally inattentive households, the mechanism is dif-

hand, optimally allocate more attention to demand shocks than to supply shocks, as demand shocks (which lead to positive comovement in output and inflation) have a greater impact on their input costs and pricing decisions.

The asymmetric attention choices can jointly explain the contrasting views by households and firms in the data, as their expectations reflect their respective partial information sets. Since professional forecasters' expectations do not affect economic outcomes, I do not introduce them explicitly in the model but instead assume they have full information and thus their expectations depend on the equilibrium correlation between output and inflation. The calibrated model *quantitatively* matches the survey expectations of households, firms, and professional forecasters (see Figure 1.4).

Furthermore, rich interactions between attention allocations arise in the general equilibrium model where both agents are rationally inattentive. In particular, attention allocation choices of firms and households are substitutes for demand shocks (households pay less attention if firms pay more attention), while complements for supply shocks (households pay less attention if firms pay less). The common thread behind these interactions is the externality that emerges in attention when the variables that agents try to learn are endogenous to others' behaviour.[5] These interactions have important implications for the propagation of shocks. For example, the strategic complementarity in the case of supply shocks can trigger a downward spiral of inattention, dampening the economy's overall response to supply shocks.

An important feature of the model is that attention choices are endogenous to monetary policy conduct and the stochastic economic environment. As conditions change, households and firms reallocate attention in ways that significantly influence macroeconomic outcomes. Specifically, a shift to more hawkish monetary policy at the beginning of the Great Moderation stabilises prices, reducing firms' attention. This further dampens their price adjustment behaviour, making prices even less sensitive to output fluctuations. It also shifts households' at-

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ferent. Kamdar (2018) explains the negatively correlated posterior beliefs on labour market slackness and price by households as a direct result of information compression, whereas in this paper, households' supply-side view arises from the optimal responses of firms and thus the results are robust across different information structures.

[5] The strategic interactions in information acquisition have been studied in several studies, Maćkowiak and Wiederholt (2009) and Hellwig and Veldkamp (2009) among others, which argue complementarity (substitutability) in information choices arises from the complementarity (substitutability) in actions. Here I demonstrate that complementarity (substitutability) can also arise through the value of information in a general equilibrium model with multiple inattentive agents.

tention from supply shocks to demand shocks, amplifying output gap volatility. Both forces contributed to the flattening of the Phillips curve observed over recent decades.[6] In the post-pandemic period, heightened volatility of supply and demand shocks reversed this mechanism: firms increased their attention, households adjusted in tandem, and thus the Phillips curve steepened, consistent with the post-pandemic U.S. aggregate and regional evidence.[7]

The model has broader implications for communication. When agents only pay attention to partial and biased information, communication policies may not work as intended. First, standard theory predicts that news about higher future inflation should raise households' spending today, a key mechanism of forward guidance. However, households with a supply-side biased information set may misinterpret the higher inflation as originating from a contractionary supply shock, leading them to lower output growth expectations and reduce spending. Second, central bank may commit to a lower interest rate path during periods of economic slack to stimulate demand. However, firms with a demand-side biased partial information set may misinterpret the systematic response in interest rate as an expansionary monetary policy shock and raise prices, which dampens the demand further. These findings highlight that policymakers need to carefully craft their communication strategies, taking into consideration how different agents perceive the information.

Related Literature. This study contributes to the research agenda that seeks to develop a data-consistent model of expectation formation. Three closely related studies are Kamdar (2018), Bhandari et al. (2024), and Han (2024). Kamdar (2018) and Bhandari et al. (2024) both look at the same facet of consumer surveys, attributing the observation to pessimism. Han (2024) explains observed heterogeneity by exogenously assuming different partial information for different agents. In contrast to these papers, I argue that agents' partial information is optimally chosen based on their respective objectives, and show that households' supply-side view arises from the optimal responses of firms.[8]

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[6] The flattening of the Phillips curve in recent decades has been widely documented; see, for example, Coibion and Gorodnichenko (2015), Blanchard (2016), Bullard (2018), and Hooper et al. (2020); Kishaba and Okuda (2023).

[7] See Hobijn et al. (2023), Furlanetto and Lepetit (2024), Gelain and Lopez (2024) and Cerrato and Gitti (2022) for evidence on the post-pandemic steepening of the Phillips curve.

[8] This study also differs from Kamdar (2018) on the household side in several critical aspects. As mentioned in footnote 3, Kamdar (2018)'s results rely on information compression, as a result, agents' belief does not approach the true data-generating process as information costs decrease. In contrast, this model converges to the full-information equilibrium as information costs ap-

This paper broadly relates to the rational inattention literature following Sims (2003). The core premise of this literature is that incentives drive attention, implying that agents pay more attention to more volatile and more important variables (e.g., Maćkowiak and Wiederholt (2009); Kohlhas and Walther (2021); Flynn and Sastry (2024)). Here I show that agents' attention to particular shocks can be higher than their attention to others. Another contribution of this paper is that it solves a dynamic general equilibrium model in which both firms and households are rationally inattentive. While Maćkowiak and Wiederholt (2015) also features two-sided rational inattention, this paper extends the analysis by studying further expectation-related moments.

This paper also connects to a vast literature in macroeconomics on the role of imperfect information in business cycle dynamics (Lucas (1972); Woodford (2001); Eusepi and Preston (2010); Blanchard et al. (2013); Angeletos and La'o (2013); Chahrour and Ulbricht (2023) among others), and in the effect of policy (for e.g, Amador and Weill (2010); Paciello (2012); Angeletos and Lian (2018)). The contribution of this paper is to highlight the macroeconomic consequences when agents endogenously choose different partial information, and offer new insights on communication when different agents in the economy have heterogeneous attention choices and views.

Layout. The paper is organised as follows. In Section 1.2, I provide a closed-form characterisation of households' and firms' attention choices under rational inattention in the illustrative model. In Section 1.3, I study the full dynamic general equilibrium model, where I calibrate the model and analyse the impact on macroeconomic dynamics. In Section 1.4, I show that the model can jointly explain the flattening of the Phillips curve over recent decades and its steepening in the post-pandemic. In Section 1.5, I discuss the implications for communication. Section 1.6 concludes.

1.2 Attention Choices in Firms and Households

In this section, I present a simple model with rational inattention to illustrate the heterogeneous attention choices of households and firms. The full model is presented and solved quantitatively in Section 1.3.

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proach zero. Moreover, this paper focuses on the correlation of expectations rather than posterior beliefs, making the results directly relevant to survey evidence, where questions pertain to agents' expectations rather than their posterior beliefs.

1.2.1 Environment

Households. There is a continuum of hand-to-mouth households indexed by $i \in [0,1]$. Household i in each period chooses consumption $C_{i,t}$ to maximise its expected utility and supplies labour $L_{i,t}$ such that the budget constraint binds. Household i's period utility at time t is

$$U\left(C_{i,t}, L_{i,t}\right) = \left[\frac{C_{i,t}^{1-\gamma}}{1-\gamma} - \frac{L_{i,t}^{1+\eta}}{1+\eta}\right] \tag{1.1}$$

$$s.t.\ P_t C_{i,t} = W_t L_{i,t}, \quad C_{i,t} = \left[\int_0^1 C_{i,j,t}^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}} \tag{1.2}$$

where $C_{i,j,t}$ is household i's demand for variety j given its price $P_{j,t}$ and $C_{i,t}$ is the final consumption good aggregated with a constant elasticity of substitution $\theta > 1$ across varieties. W_t is the nominal wage, and $P_t = [\int_0^1 P_{j,t}^{1/(\theta^{-1}})dj]^{\theta-1}$ is the aggregate price index. The parameter $\gamma > 1$ is the risk aversion coefficient and the parameter η is the inverse of Frisch elasticity of labour supply.

Firms. There is a continuum of firms producing differentiated goods, each indexed by $j \in [0,1]$. Each firm j is a monopoly producer of its own variety and faces a demand curve $Y_{j,t} = (P_{j,t}/P_t)^{-\theta} Y_t$, where $Y_t = \int_0^1 Y_{j,t} dj$ is the aggregate output. Firm j hires labour $L_{j,t}$, pays wages W_t per worker, and produces with a linear technology

$$Y_{j,t} = A_t L_{j,t} \tag{1.3}$$

where A_t is the aggregate productivity.

In each period, firm j sets the price $P_{j,t}$ for its own product to maximise its expected profit and produces a sufficient quantity of goods to meet the demand $Y_{j,t}$. Prices are fully flexible (nonetheless, as discussed later in Section 1.2.5, price stickiness arises endogenously from firms' inattention). The profit of firm j at time t, discounted by the household's marginal utility of consumption, is expressed as

$$\Pi_{j,t}(P_{j,t}, L_{j,t}, Y_{j,t}) = \frac{1}{P_t C_t^{\gamma}} \left[P_{j,t} Y_{j,t} - (1-\theta^{-1}) W_t L_{j,t}\right] \tag{1.4}$$

where $(1-\theta^{-1})W_t$ denotes the subsidised wage rate, with the subsidy θ^{-1} paid to eliminate steady-state distortions introduced by monopolistic competition.

Central Bank. For analytical tractability, I assume that central bank directly controls the nominal aggregate demand $Q_t \equiv P_t Y_t$. This assumption allows for a closed-form characterisation of the solution.[9] I consider a more standard Taylor rule in the quantitative model in Section 1.3. I further assume that the central bank has full information and interpret it as the model counterpart of the professional forecasters in the survey.

Shocks. The economy is subject to both demand and supply shocks. I model the demand shock as a shock to the nominal aggregate demand ($q_t \equiv \log Q_t$), and the supply shock as a shock to all firms' productivity levels ($a_t \equiv \log A_t$). The two exogenous processes follow Gaussian white noise distributions with variances $\sigma_q^2 > 0$ and $\sigma_a^2 > 0$, and are mutually independent.

1.2.2 Attention Costs and Information Structure

Costly Attention. In this environment, agents must pay attention in order to be aware of the economic conditions. While the cost of attention can, in principle, take many different forms (see e.g., Hébert and Woodford (2018)), I follow Sims (2003) and model the attention costs as linear in Shannon's mutual information $\mu\mathcal{I}\left(X; S^t|S^{t-1}\right)$, where μ is the marginal cost of attention. Specifically, $s_t \in \mathcal{S}^t$ denotes the signals at time t, and $\mathcal{S}^t$ is the set of available signals. The history of signals up to time t is denoted by $S^t = S^{t-1} \cup s_t$. Mutual information is defined as

$$\mathcal{I}\left(X; S^t|S^{t-1}\right) \equiv h(X|S^{t-1}) - \mathbb{E}\left[h(X|S^t)|S^{t-1}\right]$$

This measures the reduction in entropy of the object X due to information gained from signal S^t conditional on the history of signals S^{t-1}.

This formulation assumes that agents do not forget information over time and thus the information chosen today can have a continuation value. In the simple model presented in this section, this condition does not matter as shocks are i.i.d., so the knowledge about the shocks today does not affect future priors. However, in the full model presented in Section 1.3, where shock processes are more complex and intertemporal decisions are involved, past information becomes useful to agents.

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[9] Assuming that the monetary authority directly controls the nominal aggregate demand is a popular framework in the rational inattention literature to study the effects of monetary policy on pricing. See for example Mankiw et al. (2003); Woodford (2003); Maćkowiak and Wiederholt (2009); Paciello (2012); Afrouzi and Yang (2021) among others.

Information Structure. It is necessary to specify the information structure, i.e., the available signal set $\mathcal{S}^t$. I consider two popular approaches in the literature. One approach, optimal signal design, explored by Sims (2003) and Maćkowiak et al. (2018), allows agents full flexibility when designing the conditional distribution of their signals given the state of the economy. An alternative approach, constrained information structure, restricts agents to acquiring N separate, conditionally independent signals about N different components in their optimal action.[10] In the current context, I partition the signal into one subvector that contains only information on nominal aggregate demand shock q_t and another subvector that contains only information on the productivity shock a_t.

The choice of information structure typically depends on the problem at hand. In this context, optimal signal design is more realistic than restricting agents to separate signals for different shocks.[11] However, for analytical tractability and interpretability, in Section 1.2.4 and 1.2.5, I solve the attention problem under a constrained information structure. In Section 1.2.5, I compare the predictions of each approach and find that the choice of information structure does not significantly affect the results. In other sections, including the quantitative model in Section 1.3, I adopt optimal signal design to better capture how households and firms acquire information.

Timing. In the initial period $t = 0$, households and firms make their ex ante attention choices, which we can think of as determining the form and precision of the associated signals. In each subsequent period $t > 0$, shocks (q_t, a_t) realise. The economy proceeds through three stages: (i) depending on their respective attention choices, households and firms receive different forms of signals with different precision levels; (ii) based on their respective signals, households choose their consumption and firms set their prices for their own varieties. (iii) after their choices are committed, households supply labour to cover their consumption and firms produce sufficient goods to meet the demand. Finally, the real wage adjusts to clear the labour market.

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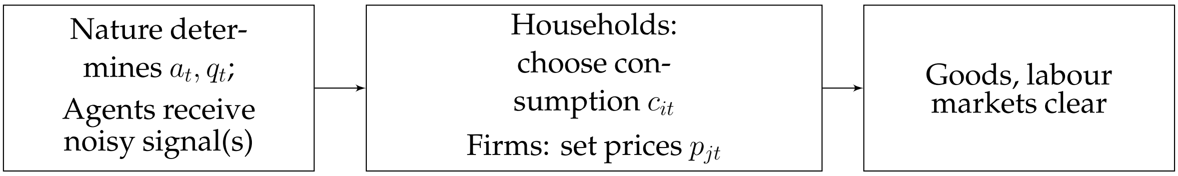
[10] Maćkowiak and Wiederholt (2009) partitions firms' signals into two subvectors, with one subvector containing information about idiosyncratic conditions and another about aggregate conditions.

[11] The model implied optimal signals align with the survey evidence on agents' attention choices. See Appendix A.1.2 for details on households' and firms' attention choices in the survey.



1.2.3 Attention Problems of Households and Firms

Households. For tractability, I simplify the households' utility function (1.1) using quadratic approximations (derivation see Appendix A.2.1). After the approximation, household i's objective (1.1) at time t can be expressed as the utility loss from deviating from the optimal consumption level $c^*_{i,t}$ – the consumption level that households would choose under full information[12]

$$\left[-\frac{(\gamma+\eta)}{2}\left(c_{i,t}-c^*_{i,t}\right)^2\right] + \text{terms independent of } \{c_{i,t}\} \tag{1.5}$$

Here, lowercase letters denote the logs of the corresponding variables. $c_{i,t}$ is the actual consumption choice made by the household i. When the household deviates from its optimal choice, the utility loss is proportional to the risk aversion coefficient γ and the inverse of Frisch elasticity of labour supply η. Households that are more risk-averse and less elastic in labour supply lose more utility by choosing a suboptimal consumption level.

The optimal consumption under full information is obtained by equating the marginal rate of substitution between consumption and leisure to the real wage[13]

$$c^*_{i,t} = \frac{1+\eta}{(\gamma+\eta)}\left(w_t - p_t\right) \tag{1.6}$$

The equation states that optimal consumption is a function of real wage. If households know the real wage, they can achieve the optimal consumption level. This also implies that households want to learn about real wages to guide their consumption decisions. This aligns with the survey evidence from the Michigan Survey of Consumers, which shows that households pay more attention to developments related to the real labour market than to prices (see Appendix A.1.2 for details).

Substituting the optimal consumption from Equation (1.6) into the utility

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[12] The first-order term of this approximation drops out due to the envelope theorem: there are no first-order costs of deviating from c^*_{it}. Full derivation see A.2.1.

[13] The optimal consumption is derived by substituting $l_{i,t}$ using the budget constraint $p_t + c_{i,t} = w_t + l_{i,t}$ into the intra-temporal Euler equation $\gamma c_{i,t} + \eta l_{i,t} = w_t - p_t$.

function (1.5), and adding the attention cost term, household $i \in [0,1]$'s attention problem is formally defined as

$$\max_{\{s_{i,t}\in\mathcal{S}_h^t\}} \mathbb{E}_t^h \left[-\frac{(\gamma+\eta)}{2} \left(c_{i,t} - \frac{1+\eta}{\gamma+\eta} (w_t - p_t) \right)^2 - \mu^h \mathcal{I}\left(a_t, q_t; s_{i,t}\right) \right] \tag{1.7}$$

The first term in Equation (1.7) captures the benefits of attention, as $c_{i,t}$ gets closer to the optimal level, which is a function of the real wage. The second term reflects the cost of attention, measured by the marginal cost of attention $\mu^h > 0$ times the expected reduction in entropy after observing the signal $s_{i,t} \in \mathcal{S}_h^t$, where $\mathcal{S}_h^t$ is the set of all available signals for households at time t.

Firms. I simplify the firms' profit function (1.4) using quadratic approximations (derivation see Appendix A.2.2), which yields

$$\left[-\frac{\theta-1}{2} \left(p_{j,t} - p_{j,t}^* \right)^2 \right] + \text{terms independent of } \{p_{j,t}\} \tag{1.8}$$

where lowercase letters denote the logs of the corresponding variables. Equation (1.8) states that firm j experiences a profit loss from setting a price $p_{j,t}$ that deviates from its optimal price level under full information $p_{j,t}^*$. Moreover, the magnitude of profit losses is proportional to firm's demand elasticity $(\theta - 1)$. In other words, firms with more elastic demand experience larger profit losses when charging a suboptimal price. In this simple setup, firm's optimal price under full information is its nominal marginal cost

$$p_{j,t}^* = w_t - a_t \tag{1.9}$$

This implies that firms seek information on nominal marginal cost to guide their pricing decisions. This aligns with the survey evidence from the Business Inflation Expectation survey, which suggests that firms have strong incentives to pay attention to their unit costs when setting prices (see Appendix A.1.2 for details).

Substituting the optimal price (1.9) into the profit function (1.8), and adding the attention cost, firm j's attention problem is formally defined as

$$\max_{\{s_{j,t}\in\mathcal{S}_f^t\}} \mathbb{E}_t^f \left[-\frac{\theta-1}{2} \left(p_{j,t} - (w_t - a_t) \right)^2 - \mu^f \mathcal{I}\left(q_t, a_t; s_{j,t}\right) \right] \tag{1.10}$$

The first term captures the benefit of paying attention, that the firm's price $p_{j,t}$

gets closer to the optimal level, i.e., firm j's nominal marginal cost. The second term is the cost of attention, measured by firm's marginal cost of attention $\mu^f > 0$ times the expected entropy reduction about the optimal price $p^*_{j,t}$ after observing $s_{j,t} \in \mathcal{S}^t_f$.

The equilibrium of the model is defined as in Definition 1.

Definition 1 (Equilibrium). *Given the processes for the monetary policy and productivity shocks* $\{q_t, a_t\}_{t\geq 0}$, *a general equilibrium of this economy is an allocation for every household* $i \in [0,1]$, $\Omega_i \equiv \{s_{i,t} \in \mathcal{S}_{i,t}, C_{i,t}, L_{i,t}\}_{t=0}^{\infty}$, *given their initial set of signals; an allocation for every firm* $j \in [0,1]$, $\Omega_j \equiv \{s_{j,t} \in \mathcal{S}_{j,t}, P_{j,t}, L_{j,t}, Y_{j,t}\}_{t=0}^{\infty}$ *given their initial set of signals; a set of prices* $\{P_t, W_t\}_{t=0}^{\infty}$, *such that*

1. *Given the processes for* $\{P_t, W_t\}_{t=0}^{\infty}$ *and all firms' decisions* $\{\Omega_j\}_{j\in[0,1]}$, *every household* i*'s allocation solves the attention problem (1.7);*
2. *Given the processes for* $\{P_t, W_t\}_{t=0}^{\infty}$ *and all households' allocations* $\{\Omega_i\}_{i\in[0,1]}$, *every firm* j*'s allocation solves the attention problem (1.10);*
3. *The equilibrium processes* $\{P_t, W_t\}_{t=0}^{\infty}$ *are consistent with the allocations of households and firms,* $\{\Omega_i\}_{i\in[0,1]}$ *and* $\{\Omega_j\}_{j\in[0,1]}$.

Solving for equilibrium with two-sided rational inattention is complex, as agents' attention choices and decisions depend on endogenous variables as well as each other's attention choices and decisions. To provide intuition for the attention choices of households and firms, I simplify the model by first considering the case where only households are subject to rational inattention, while firms have full information (Section 1.2.4). Next, I examine the case where only firms are rationally inattentive while households have full information (Section 1.2.5). Finally, in Section 1.2.7, I solve the general equilibrium model with two-sided rational inattention analytically, and explain the rich interactions in attention allocation between households and firms.

1.2.4 Households' Attention Choices

I begin by analysing the case where households are subject to rational inattention while firms have full information. In this case, firms set prices at their optimal level according to Equation (1.9), which implies that the real wage is fully determined by productivity

$$w_t - p_t = a_t \tag{1.11}$$

From Equation (1.11), the real wage is not affected by demand shocks q_t, this is due to firms' optimising behaviour – following a demand shock, nominal wages rise, firms with full information increase prices one-to-one with nominal wage, and the real wage is thus unaffected. This follows the classical dichotomy.

To develop the intuition for households' attention choices, imagine that a measure of zero of households have no information, while all others have full information. Since all other households have full information, the optimal consumption remains $c^*_{i,t} = \frac{1+\eta}{\gamma+\eta}(w_t - p_t) = \frac{1+\eta}{\gamma+\eta}a_t$. However, households with no information fail to adjust their consumption (i.e., $c_{i,t} = 0$), resulting in an expected utility loss proportional to

$$\mathbb{E}_{i,t}\left[-\left(c_{i,t} - c^*_{i,t}\right)^2\right] = \mathbb{E}_{i,t}\left[-\left(0 - \frac{1+\eta}{\gamma+\eta}a_t\right)^2\right] = -\left(\frac{1+\eta}{\gamma+\eta}\right)^2\sigma_a^2$$

This indicates that, as long as firms have full information and adjust their prices to fully track changes in the nominal marginal cost, there is no utility loss for households from misinformation about demand shocks, even if they pay no attention to those shocks. The expected utility loss arises solely from misinformation about supply shocks. Furthermore, the expected loss is higher when (i) optimal consumption is more responsive to productivity shocks (i.e., high $(1+\eta)/(\gamma+\eta)$) (ii) shocks are more volatile (i.e., high σ_a^2). Figure 1.2a illustrates this with a contour plot showing the utility loss when a_t and q_t are misperceived. The plot consists of horizontal lines, indicating no loss from not attending to q_t.

Under constrained information structure, households can obtain N separate, conditionally independent signals. In this context, households can obtain one signal about the nominal aggregate demand shock and another signal about the productivity shock[14], i.e.,

$$s_{i,t} = \{s_{i,q,t}, s_{i,a,t}\} \tag{1.12}$$

where

$$s_{i,q,t} = q_t + e_{i,q,t} \quad \text{and} \quad s_{i,a,t} = a_t + e_{i,a,t} \tag{1.13}$$

and $\{s_{i,q,t}, q_t\}$ and $\{s_{i,a,t}, a_t\}$ are independent, follow stationary Gaussian processes, and all noise terms are mean-zero and independently distributed across households.

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[14] In the households' attention problem, both the constrained and flexible information structures yield the same signal form since optimal consumption depends solely on productivity shocks.

Upon receiving these signals, consumption $c_{i,t} = \mathbb{E}[c^*_{i,t}|s_{i,t}] = \frac{1+\eta}{\gamma+\eta}\mathbb{E}[a_t|s_{i,a,t}]$ maximises the expected utility for any given posterior belief. For ease of notation, define $\lambda_{h,a} \equiv \frac{1+\eta}{\gamma+\eta}$. And further define $\sigma^2_{a|s}$ as the posterior uncertainty about a_t. Substituting $c_{i,t}$ and real wage (1.11) into Equation (1.7) yields

$$\begin{aligned} &\max_{\{s_{i,t}\in\mathcal{S}_i^t\}} \mathbb{E}_t^i \left[-\frac{\gamma+\eta}{2} \left(\lambda_{h,a}\mathbb{E}[a_t|s_{i,a,t}] - \lambda_{h,a}a_t\right)^2 - \mu^h \mathcal{I}\left(a_t, q_t; s_{i,t}\right) \right] \\ &= \frac{1}{2} \max_{\sigma^2_{a|s}\le\sigma^2_a} \left[-(\gamma+\eta){\lambda_{h,a}}^2\sigma^2_{a|s} - \mu^h \ln \frac{\sigma^2_a}{\sigma^2_{a|s}} \right] \end{aligned} \tag{1.14}$$

Solving this problem characterises households' attention choices, as summarised in Proposition 1.

Proposition 1. *When firms have full information, and households can obtain a signal vector of the form* $s_{i,t} = \{s_{i,q,t}, s_{i,a,t}\}$

1. *Households only attend to signal about supply shocks* $s_{i,a,t}$. *The attention weight on supply shocks (the Kalman-gain) is*

$$\xi_{h,a} = \max\left(0, 1 - \frac{\mu^h}{(\gamma+\eta)\lambda^2_{h,a}\sigma^2_a}\right)$$

   and the attention weight on demand shock is $\xi_{h,q} = 0$.

2. *Households' consumption evolves according to*

$$c_{i,t} = \lambda_{h,a}\mathbb{E}[a_t|s_{i,a,t}] = \xi_{h,a}\lambda_{h,a}\left(a_t + e_{i,a,t}\right).$$

Proof. See Appendix A.2.3.

The Proposition 1 shows that households never pay attention to demand shocks, as such information has no value for them. This is because, as long as firms have full information and set prices to offset changes in q_t, optimal consumption is unaffected by the demand shocks. Therefore, when attention is costly, households would not choose to acquire such information. Moreover, households pay more attention to supply shocks if (i) the information generates a higher payoff (i.e., higher $\lambda_{h,a}$), and (ii) households are sufficiently uncertain about it (i.e., higher prior uncertainty σ^2_a), and (iii) attention costs are relatively low (i.e., low μ^h).

The Proposition 1 shows that information on demand shocks q_t has no value for households when firms have full information. However, if firms are inattentive, they under-react due to incomplete information, and prices adjust only partially to demand shocks. As a result, demand shocks have a real impact. Then, information about demand shocks becomes valuable for households – but only secondarily. The intuition is summarised in Corollary 1, while derivation and solution are presented in Section 1.2.7.

Corollary 1. *When firms are inattentive and price adjustments are sub-optimal, households have incentives to pay attention to demand shocks.*

1.2.5 Firms' Attention Choices

I analyse the case where firms are subject to rational inattention while households have full information.[15] When households have full information, all households equate the marginal rate of substitution between consumption and labour to the real wage, i.e., $\gamma c_{i,t} + \eta l_{i,t} = w_t - p_t$ and the budget constraint holds as $p_t + c_{i,t} = w_t + l_{i,t}$, $\forall i$. This implies that the nominal marginal cost takes the following form

$$w_t - a_t = q_t - \frac{1+\eta}{\gamma+\eta} a_t \tag{1.15}$$

To develop the intuition for firms' attention choices, imagine that a measure of zero of firms have no information while all other firms have full information. Since all other firms have full information, the optimal price remains $p^*_{j,t} = q_t - \frac{1+\eta}{\gamma+\eta} a_t$. However, firms without information fail to adjust their prices (i.e., $p_{j,t} = 0$), resulting in expected profit losses proportional to

$$\mathbb{E}_{j,t}\left[-\left(p_{j,t} - p^*_{j,t}\right)^2\right] = \mathbb{E}_{j,t}\left[-\left(0 - \left(q_t - \frac{1+\eta}{\gamma+\eta} a_t\right)\right)^2\right] = -\left[\sigma_q^2 + \left(\frac{1+\eta}{\gamma+\eta}\right)^2 \sigma_a^2\right] \tag{1.16}$$

As shown in Equation (1.16), misinformation about both shocks are expected to cause profit loss. The magnitude of expected profit loss due to misinformation about a particular shock depends on (i) the volatility of each shock, with more volatile shocks leading to a greater expected loss from misinformation; (ii) the

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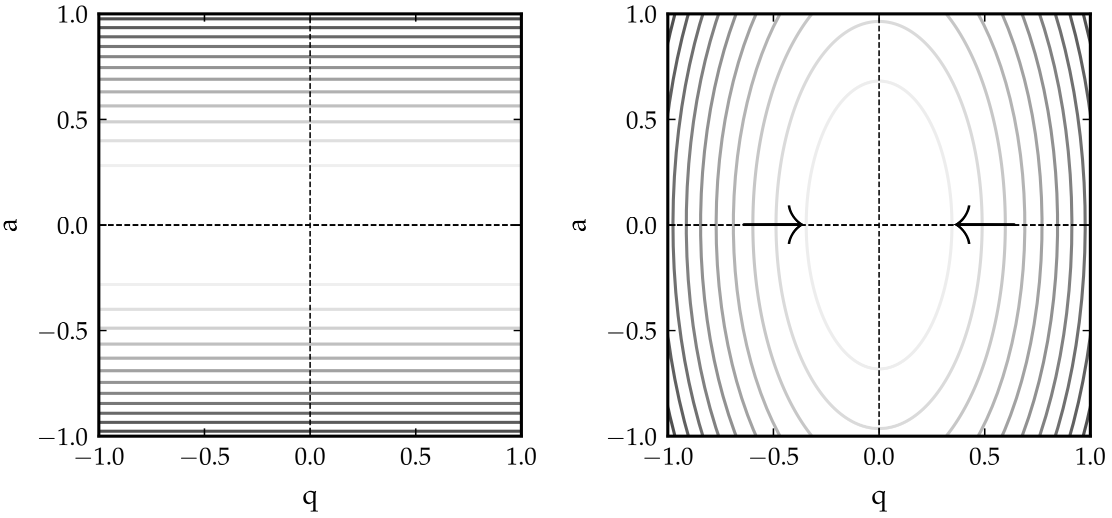
[15] For tractability, I assume that there is no general equilibrium feedback through strategic complementarity in price setting. However, this feedback effect is included in the quantitative model (see Section 1.3).

responsiveness of optimal price to each shock. In particular, for relatively high values of risk aversion coefficient $\gamma > 1$, misinformation about demand shocks can result in greater profit loss than misinformation about supply shocks. Under standard parameter values, I show misinformation about demand shocks may incur larger profit loss for firms (see Section 1.3 for detailed parameterisation). This is illustrated in Figure 1.2b.



(a) Households' utility losses

(b) Firms' profit losses

Figure 1.2: Losses from misinformation about (q, a)

Notes: Figure 1.2a shows a contour plot of households' utility losses when q and a are misperceived. It shows that the losses occur only along a varying a, which is therefore the only component that households would pay attention to. Figure 1.2b shows a contour plot of firms' profit losses when unit shocks q and a are misperceived. It shows that the descent of losses is steeper in the case of demand shocks q, which is therefore the more important component that firms need to pay attention to.

Suppose firms can obtain separate, conditionally independent signals about q_t and a_t, as defined in Equation (1.12) and (1.13). For ease of notation, let $\lambda_{f,q} \equiv 1$ and $\lambda_{f,a} \equiv -\frac{1+\eta}{\gamma+\eta}$. Under this notation, the nominal marginal cost is given by $w_t - a_t = \lambda_{f,q} q_t + \lambda_{f,a} a_t$.

Firms' attention choices are characterised by the following Proposition 2.

Proposition 2. *When households have full information, and firms can obtain a signal vector of the form* $s_{j,t} = (s_{j,q,t}, s_{j,a,t})$

1. *Firms optimally allocate attention towards both signals,* $s_{j,q,t}$ *and* $s_{j,a,t}$*. The atten-*

tion weights (Kalman gain) on each signal are given by

$$\xi_{f,q} = \max\left(0, 1 - \frac{\mu^f}{(\theta-1)\lambda_{f,q}^2\sigma_q^2}\right), \tag{1.17a}$$

$$\xi_{f,a} = \max\left(0, 1 - \frac{\mu^f}{(\theta-1)\lambda_{f,a}^2\sigma_a^2}\right). \tag{1.17b}$$

2. *Firms' prices evolve according to*

$$p_{j,t} = \lambda_{f,q}\xi_{f,q}\left[q_t + e_{j,q,t}\right] + \lambda_{f,a}\xi_{f,a}\left[a_t + e_{j,a,t}\right]. \tag{1.18}$$

Proof. See Appendix A.2.4.

The Proposition 2 shows that allocation of attention to q_t and a_t is independent, and firms have incentives to pay attention to both shocks. They choose to pay more attention to a particular shock, if (i) the shock is particularly volatile (i.e., σ_q^2 or σ_a^2 is large), (ii) the optimal price is particularly responsive to that shock (i.e., $\lambda_{f,q}$ or $\lambda_{f,a}$ large).

For relatively high values of γ, the attention weight can be higher for demand shocks, i.e., $\xi_{f,q} \gtrapprox \xi_{f,a}$, in which cases firms find it optimal to pay more attention to demand shocks. The intuition is that, following a positive productivity shock, the optimal price should decrease on impact as $p_{j,t}^* = w_t - a_t$. This reduction in prices leads to a surge in demand c_t. For $\gamma > 1$, the income effect dominates, and labour supply decreases, which in turn causes wages to rise. This offsets the initial downward pressure on prices, so the optimal price $p_{j,t}^*$ is less affected by productivity shocks when γ is large.

Comparison to Optimal Signal Design. Proposition 2 characterises the solution to the attention problem under constrained information structure. Alternatively, firms can freely design their optimal signal. Following the characterisations of optimal signal design in Maćkowiak et al. (2018), the optimal signal is a single signal about their optimal action, i.e., the nominal marginal cost. The prior uncertainty about the optimal price is $\sigma_p^2 \equiv \lambda_{f,q}^2\sigma_q^2 + \lambda_{f,a}^2\sigma_a^2$, the solution to the firms' attention problem is characterised in Proposition 3 below.

Proposition 3. *When households have full information and firms can freely design their optimal signal, firms will pay more attention to demand shocks. Formally,*

1. *Firms optimally obtain a single signal about their optimal price* $p^*_{j,t}$

$$s_{j,t} = p^*_{j,t} + e_{j,t} = \lambda_{f,q} q_t + \lambda_{f,a} a_t + e_{j,t}$$

   where $e_{j,t}$ *is the idiosyncratic noise in the signal.*

2. *The optimal signal for firms skews towards nominal aggregate shocks as* $|\lambda_{f,q}| > |\lambda_{f,a}|$.

3. *Firm's price evolves according to*

$$p_{j,t} = \xi_f \left(p^*_{j,t} + e_{j.t} \right) = \xi_f \lambda_{f,q} q_t + \xi_f \lambda_{f,a} a_t + \epsilon_{j,t}$$

   where the Kalman-gain of the firm's signal under optimal information structure is

$$\xi_f = \max \left(0, 1 - \frac{\mu^f}{(\theta - 1)\left(\lambda^2_{f,q}\sigma^2_q + \lambda^2_{f,a}\sigma^2_a\right)} \right).$$

Proof: See Appendix A.2.5.

From the first part of Proposition 3, firms optimally obtain a single signal about their optimal price $p^*_{j,t}$. The optimal signal is skewed towards q_t as optimal price is more responsive to q_t, i.e., $|\lambda_{f,q}| = 1$ is greater than $|\lambda_{f,a}| = \frac{1+\eta}{\gamma+\eta}$ when $\gamma > 1$. As a result, more attention is allocated to nominal aggregate demand shocks. The results relate to Kohlhas and Walther (2021), which shows that the asymmetry of attention under optimal signal design depends on the weights $\lambda_{f,q}$ and $\lambda_{f,a}$ in agents' optimal action through their influences on $p^*_{j,t}$. The last part of Proposition 3 shows that firms attention is higher if (i) either shock is more volatile (i.e., large σ^2_q or σ^2_a); (ii) the loss from misinformation is high (i.e., high θ or $\lambda_{f,q}$ or $\lambda_{f,a}$); and (iii) the marginal cost of firms μ^f is relatively low.

The key difference between optimal signal design and the constrained information structure is evident from Proposition 2 and Proposition 3. With optimal signal design, a higher attention weight allocated to one shock over another is driven by the optimal signal being skewed towards that shock. As a result, the volatility of the shocks does not affect the relative attention, instead, relative attention depends solely on the relative responsiveness of price to the shock, i.e., $\lambda_{f,q}/\lambda_{f,a}$. In contrast, under the constrained information structure, higher attention weight is given to a shock either because optimal price is more responsive to that shock or because that shock is particularly volatile. Therefore, in this case, relative attention depends on both $\lambda_{f,q}/\lambda_{f,a}$ and σ_q^2/σ_a^2.

In summary, attention choices of households and firms differ significantly. Households tend to allocate substantially more attention to supply shocks than to demand shocks, while firms pay attention to both shocks, with slightly more attention to demand shocks.

So far, I have solved the attention problem assuming one side is rationally inattentive and the other side is fully informed. Before addressing the case where both households and firms are rationally inattentive, I first explain how attention choices lead to the supply-side view by households and demand-side view by firms.

1.2.6 Implications of Attention Choices on Beliefs

Suppose the true data-generating processes are characterised by

$$y_t = \Psi_{y,q} q_t + \Psi_{y,a} a_t, \tag{1.19}$$

$$p_t = \Psi_{p,q} q_t - \Psi_{p,a} a_t. \tag{1.20}$$

Here, Ψs denote the responses of aggregate output y_t and aggregate price p_t to demand and supply shocks. The specific values are determined endogenously in equilibrium, which depend on the equilibrium attention choices and decisions made by firms and households. The specific values are not central to the discussion in this section. Nonetheless, a positive demand shock is typically expansionary and inflationary (i.e., $\Psi_{y,q} > 0$ and $\Psi_{p,q} > 0$), while a positive supply shock tends to increase output but decrease prices (i.e., $\Psi_{y,a} > 0$ and $\Psi_{p,a} < 0$).

Define the expected output growth of agent k as $\mathbb{E}^k(y_{t+1} - y_t)$ and expected inflation as $\mathbb{E}^k(\pi_{t+1}) = \mathbb{E}^k(p_{t+1} - p_t)$, where $k = \{h, f, cb\}$ represents households, firms and professional forecasters. With these definitions in place, I can derive the unconditional covariance between expected output growth and expected inflation

$$Cov\left(\mathbb{E}^k(y_{t+1} - y_t), \mathbb{E}^k(\pi_{t+1})\right) = \Psi_{y,q}\Psi_{p,q}\xi_{k,q}^2\sigma_q^2 - \Psi_{y,a}\Psi_{p,a}\xi_{k,a}^2\sigma_a^2 \tag{1.21}$$

Equation (1.21) characterises agents' perceived correlation between expected output growth and expected inflation. Here, $\xi_{k,q}$ is the attention weight that agent k assigns to demand shocks, while $\xi_{k,a}$ is the attention weight on supply shocks.

Both $\xi_{k,q}$ and $\xi_{k,a}$ range between 0 and 1, where a value of 1 corresponds to the full information case, and 0 indicates that agents receive no information. The covariance is the sum of two components: the first component is positive, indicating that conditional on demand shocks, the covariance is positive; the second component is negative, indicating that conditional on supply shocks, the covariance is negative. The unconditional covariance is the sum of these two components.

Full Information Benchmark. If all the agents have full information, then the attention weights for all agents k on both shocks equal 1. The covariance is thus the same across all agents, and it equals to

$$Cov\left(\mathbb{E}(y_{t+1} - y_t), \mathbb{E}(\pi_{t+1})\right) = \Psi_{y,q}\Psi_{p,q}\sigma_q^2 - \Psi_{y,a}\Psi_{p,a}\sigma_a^2 \tag{1.22}$$

The covariance (1.22) is the same across all agents and can be either positive or negative, depending on the parameterisation, which contradicts the survey evidence showing that agents hold different views.

Rational Inattention Framework. In the current model, rationally inattentive households have little incentive to pay attention to demand shocks, i.e., $\xi_{h,q} \ll \xi_{h,a}$. As a result, the second component in Equation (1.21) dominates, leading to a negative unconditional covariance between expected output growth and expected inflation, i.e., a supply-side view. Firms allocate attention to both shocks, with slightly more attention towards demand shocks $\xi_{f,q} \gtrapprox \xi_{f,a}$, resulting in a weak positive unconditional covariance. Professional forecasters are assumed to have full information (i.e., $\xi_{cb,q} = \xi_{cb,a} = 1$), thus their view is determined by Equation (1.22), which depends on the equilibrium output and price responses. Formally, the findings are summarised in Proposition 4.

Proposition 4. *The asymmetric attention choices are sufficient on their own to explain the contrasting views held by different agents. In particular*

1. *Households optimally pay more attention to supply shocks, and thereby form a negative correlation between output growth and inflation in their expectations;*

2. *Firms find it optimal to pay attention to both shocks, with slightly more attention towards demand shocks, and thus form a weak-positive correlation between output growth and inflation in their expectations;*

3. *Professional forecasters have full information and their view reflects the correlation between output and inflation in equilibrium (Equation 1.22).*

Using the simple model, I analytically show that the proposed mechanism can potentially match survey expectations. To quantitatively evaluate the model and determine the numerical values of the covariance, I extend the simple model into a more plausible setting and solve it numerically in Section 1.3.

Moreover, from Proposition 4, the model generates over-identifying restrictions that I can use for calibrating the marginal cost of attention parameters (μ^h and μ^f). Importantly, as the attention parameters change, they affect both (i) the attention weights that agents put on different shocks ($\xi_{k,a}$ and $\xi_{k,q}$), which then affect households' and firms' perceived correlation between expected output and inflation by Equation (1.21), and (ii) the equilibrium responses of aggregate output and prices ($\Psi_{y,q}$, $\Psi_{y,a}$, $\Psi_{p,q}$, $\Psi_{p,a}$), and thus determine the professional forecasters' perceived correlation by Equation (1.22).

1.2.7 Strategic Interactions in Attention Allocation

This section solves for the equilibrium where both households and firms are subject to rational inattention, and discusses the strategic interactions in attention allocation between households and firms. As described in Section 1.2.3, when both agents are subject to rational inattention, their optimal actions depend on the exogenous shocks, endogenous variables, as well as each other's attention choices (Equation (1.6) and (1.9)). And the equilibrium is characterised by a fixed-point problem (see Definition 1).

For illustrative purposes, I solve the model separately for demand shocks and supply shocks, and discuss the strategic interactions in attention allocation between households and firms in each case.

Substitutability in Attention Allocation in Demand Shocks. I begin by guessing that in equilibrium, the nominal wage is a linear function of demand shock, i.e., $w_t = H_{w,q} q_t$ (this guess will be verified). Given this, the rational inattention problem of firm j (1.10) becomes[16]

$$\begin{aligned}
&\max_{\{s_{j,t}\in\mathcal{S}_f^t\}} \mathbb{E}_t^f \left[-\frac{\theta-1}{2}\left(p_{j,t}-\left(w_t-a_t\right)\right)^2-\mu^f\mathcal{I}\left(q_t,a_t;s_{j,t}\right)\right] \\
&= -\frac{1}{2}\max_{\sigma^2_{f,q|s}\geq\sigma^2_q}\left[\left(\theta-1\right)H^2_{w,q}\sigma^2_{f,q|s}+\mu^f\ln\frac{\sigma^2_q}{\sigma^2_{f,q|s}}\right]
\end{aligned}$$

where $\sigma^2_{f,q|s}$ denotes the posterior uncertainty about q_t by firms. Solving the first-order condition gives

$$p_{j,t}=\xi_{f,q}\left(w_t+e_{j,t}\right), \qquad \xi_{f,q}\equiv\max\left(0,1-\frac{\mu^f}{\left(\theta-1\right)H^2_{w,q}\sigma^2_q}\right)$$

where $e_{j,t}$ is firm j's rational inattention error, assumed to be mean-zero and independently distributed across firms. Note that firms' attention $\xi_{f,q}$ increases if the equilibrium nominal wage is very responsive to demand shocks q_t, as indicated by a higher value of $H_{w,q}$.

As firms have the same prior and attention choices, and their rational inattention errors are independently distributed, I can aggregate the price decisions $p_{j,t}$ over firms, which gives the aggregate price level

$$p_t\equiv\int_0^1 p_{j,t}dj=\xi_{f,q}w_t=\xi_{f,q}H_{w,q}q_t \tag{1.23}$$

The attention weight $\xi_{f,q}$ governs how responsive the aggregate price level is to changes in the nominal wage. In particular, if $\xi_{f,q}=1$, all firms are fully attentive, and the prices move one-to-one with equilibrium nominal wage $p_t=w_t$, in which case the real wage is unaffected. If $\xi_{f,q}=0$, firms pay no attention and do not respond to q_t. When $\xi_{f,q}\in(0,1)$, the price level rises less than optimal, that is, firms make "pricing mistakes" due to incomplete information and set the price too low, i.e., $p_t<w_t$.[17]

Substituting the aggregate price level (1.23) and the guess $w_t=H_{w,q}q_t$ into

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[16] The derivation follows the same steps as in Section 1.2.5.

[17] Here, by "pricing mistakes" I mean deviations from the full information perspective. Under rational inattention, however, firms' decisions are optimal ex ante.

households' attention problem (1.7) yields

$$\max_{\{s_{i,t}\in\mathcal{S}_h^t\}_{t\geq 0}} \mathbb{E}\left[-\frac{(\gamma+\eta)}{2}\left(c_{i,t}-\frac{1+\eta}{\gamma+\eta}(w_t-p_t)\right)^2-\mu^h\mathcal{I}\left(q_t;s_{i,t}\right)\right]$$
$$=-\frac{1}{2}\max_{\sigma^2_{h,q|s}\geq\sigma^2_q}\left[(\gamma+\eta)\left[\frac{1+\eta}{\gamma+\eta}(1-\xi_{f,q})H_{w,q}\right]^2\sigma^2_{h,q|s}+\mu^h\ln\frac{\sigma^2_q}{\sigma^2_{h,q|s}}\right]$$

The first term in the equation represents the benefit of paying attention, and it decreases with firms' attention $\xi_{f,q}$. When firms pay full attention, $\xi_{f,q}=1$ and $p_t=w_t$, households receive no benefit from paying attention (as discussed in Section 1.2.4). This is because any fluctuation in the nominal wage is exactly offset by an equivalent change in the price level, leaving the real wage and optimal consumption level unchanged ($w_t-p_t=0, c^*_{i,t}=0$). In this case, as attention is costly, households do not pay attention. However, as firms pay less attention and set the prices below the optimal level, i.e., $p_t=\xi_{f,q}w_t$ with $\xi_{f,q}<1$, it becomes beneficial for households to pay attention. The benefit increases as firms make larger "pricing mistakes". Therefore, in the case of demand shocks, the attention levels of households and firms are substitutes, that is, if firms pay less attention to demand shocks, households will pay more attention.

Solving the first order condition in steady state, the consumption choice by household i is given by

$$c_{i,t}=\xi_{h,q}\left[\frac{1+\eta}{\gamma+\eta}\left(1-\xi_{f,q}\right)w_t+e_{i,t}\right]$$

with

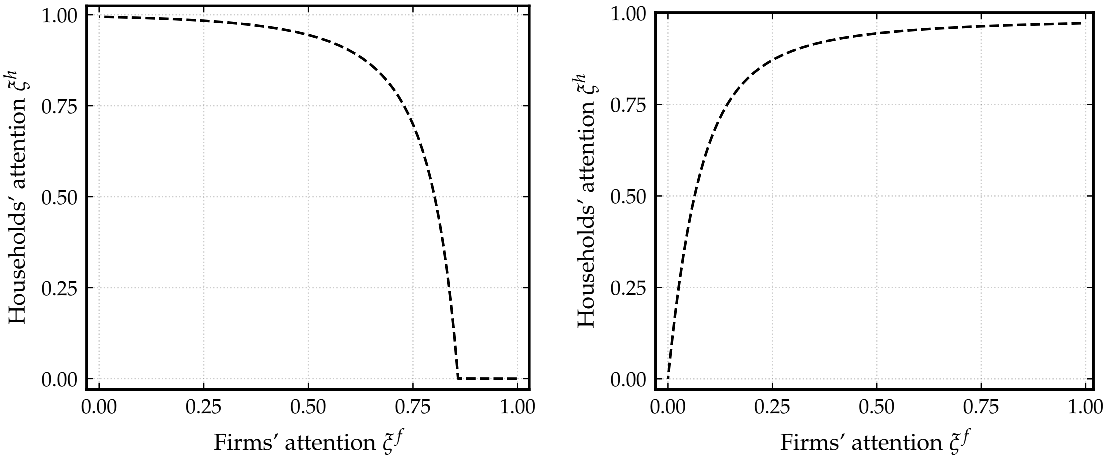
$$\xi_{h,q}\equiv\max\left(0,1-\frac{\mu^h}{(\gamma+\eta)\left[\frac{1+\eta}{\gamma+\eta}(1-\xi_{f,q})H_{w,q}\right]^2\sigma^2_q}\right)$$

where $e_{i,t}$ is the idiosyncratic noise in the signal, which is assumed to be mean-zero and independently distributed across households. Note that households have incentives to pay attention to demand shocks only when firms are sufficiently inattentive, indicated by sufficiently low $\xi_{f,q}$. Formally, the attention level of households is inversely related to the attention level of firms, i.e., $\partial\xi_{h,q}/\partial\xi_{f,q}<0$, as illustrated in Figure 1.3a.

When firms are sufficiently inattentive to nominal aggregate demand shocks, the shocks can have a real impact. Aggregating over the consumption decisions over all households yields

$$c_t \equiv \int_0^1 c_{i,t} di = \xi_{h,q} \left[\frac{1+\eta}{\gamma+\eta} \left(1-\xi_{f,q}\right) H_{w,q} \right] q_t$$

So far, I have shown that, given the guess for the nominal wage, I can solve for the attention and decisions of households and firms. However, the nominal wage is also endogenous to the equilibrium decisions of households and firms, and an equilibrium requires these two processes to be consistent.



(a) Substitutability in demand shocks (b) Complementarity in supply shocks

Figure 1.3: Strategic interactions in attention allocation

Notes: The figure plots the attention levels (Kalman gain) of households and firms. I assume the marginal cost of attention of households (μ^h) is fixed and vary marginal cost of attention of firms (μ_f). As the cost of firms' information decreases (μ_f declines), the firms' attention level increases. Households' attention varies with firms' attention.

Complementarity in Attention Allocation in Productivity Shocks. In the case of productivity shocks, the optimal price $p^*_{j,t} = w_t - a_t$ is a function of both endogenous and exogenous variables.[18] While solving for the equilibrium follows the same guess-and-verify method as before, the intuition in the case of productivity shocks is less straightforward. To gain insight into the interaction between households' and firms' attention, imagine for a moment that the labour supply is *perfectly* elastic ($\eta \to \infty$), and thus the wage does not move much following a productivity shock ($w_t = 0$), and the optimal price decision simplifies to $p_{j,t} = -a_t$. Intuitively, when firms pay full attention, the price drop is the most

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[18] This contrasts with the case of demand shocks, where the optimal price is solely a function of endogenous variables, i.e., $p^*_{j,t} = w_t$.

significant. This, in turn, suggests that optimal consumption will experience the most substantial increase, incentivising households to pay more attention. Thus, in the case of a productivity shock, attention levels of households and firms are complements.

Generalising to the case where labour supply is not perfectly elastic, I first guess that in equilibrium nominal wage is a linear function of the productivity shock, i.e., $w_t = H_{w,a}a_t$. Given this guess, the rational inattention problem of firm j (1.10) becomes

$$\max_{\sigma^2_{a|f,a|s} \geq \sigma^2_a} -\frac{1}{2}\left[(\theta-1)\left(H_{w,a}-1\right)^2\sigma^2_{f,a|s} + \mu^f \ln\frac{\sigma^2_a}{\sigma^2_{f,a|s}}\right]$$

where $\sigma^2_{f,a|s}$ denotes the posterior uncertainty about a_t of firms. Solving the attention problem gives

$$p_{j,t} = \xi_{f,a}\left(w_t - a_t + e_{j,t}\right), \qquad \xi_{f,a} \equiv \max\left(0, 1 - \frac{\mu^f}{(\theta-1)\left(H_{w,a}-1\right)^2\sigma^2_a}\right)$$

where $e_{j,t}$ is firm j's idiosyncratic noise, with zero mean and independently distributed across firms. Aggregating over j gives

$$p_t \equiv \int_0^1 p_{j,t}dj = \xi_{f,a}\left(w_t - a_t\right) = \xi_{f,a}\left(H_{w,a}-1\right)a_t \tag{1.24}$$

The aggregate price depends on the equilibrium wage, productivity shock, and firms' attention choices.

Substituting the aggregate price (1.24) and the guess $w_t = H_{w,a}a_t$ into household i's rational inattention problem (1.7) yields

$$\max_{\sigma^2_{h,a|s} \geq \sigma^2_a} -\frac{1}{2}\left[(\gamma+\eta)\left[\frac{1+\eta}{\gamma+\eta}\left(H_{w,a} - \xi_{f,a}\left(H_{w,a}-1\right)\right)\right]^2\sigma^2_{h,a|s} + \mu^h \ln\frac{\sigma^2_a}{\sigma^2_{h,a|s}}\right]$$

where $\sigma^2_{h,a|s}$ denotes the posterior uncertainty of households about a_t. The solution is characterised by

$$c_{i,t} = \xi_{h,a}\left[\frac{1+\eta}{\gamma+\eta}\left(w_t - \xi_{f,a}\left(w_t - a_t\right)\right) + e_{i,t}\right],$$

$$\text{with}\quad \xi_{h,a} \equiv \max\left(0, 1 - \frac{\mu^h}{(\gamma+\eta)\left[\frac{1+\eta}{\gamma+\eta}\left(H_{w,a} - \xi_{f,a}\left(H_{w,a}-1\right)\right)\right]^2 \sigma_a^2}\right)$$

The solution implies that $\partial\xi_{h,a}/\partial\xi_{f,a} > 0$, meaning that as firms allocate more attention to supply shocks (high $\xi_{f,a}$), households tend to allocate more attention as well (high $\xi_{h,a}$), and vice versa. Consequently, in the case of a productivity shock, attention choices made by households and firms are complements, as illustrated in the right panel of Figure 1.3b.

1.3 Quantitative Model

In this section, I extend the simple model from Section 1.2 to a dynamic setting. The objective is to (i) assess whether the proposed mechanism can quantitatively match the survey evidence (i.e., Figure 1.1); and (ii) quantify the consequences of asymmetric attention by households and firms on business cycles.

1.3.1 Extended Model

In this section, I extend the simple model from Section 1.2 in three dimensions. First, I relax the assumption of hand-to-mouth behaviour and allow households to engage in intertemporal substitution through trading nominal bonds. Second, I allow for strategic complementarities in pricing by assuming segmented labour markets, which matters quantitatively for the inflation dynamics. Third, I assume the central bank sets the interest rate following a standard Taylor rule, which reflects a more plausible monetary policy framework.

Households. There is a continuum of households, indexed by $i \in [0,1]$. Each period, household i chooses the consumption level $C_{i,t}$ and bond holdings $B_{i,t}$ based on its information set $S_i^t = \{s_{i,\tau}\}_{\tau=0}^{\tau=t}$. After deciding on consumption and bond holdings, household i supplies labour $L_{i,t}$ at given wage W_t such that the budget constraint holds. Formally, household i's expected present value of utility is given by

$$\mathbb{E}^i\left[\sum_{t=0}^{\infty}\beta^t\left(\frac{C_{i,t}^{1-\gamma}}{1-\gamma} - \frac{L_{i,t}^{1+\eta}}{1+\eta}\right)\right], \quad C_{i,t} = \left[\int_0^1 C_{i,j,t}^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}} \tag{1.25}$$

less the cost of attention. Here B_t is the nominal bond holdings at t that yield a nominal return of R_t at $t+1$, D_t is the aggregate profits of firms, and T_t is the net lump-sum transfers (or taxes, if negative). Household i takes $\{P_t, R_t, W_t, D_t, T_t\}$ as given. The budget constraint is

$$s.t.\ P_t C_{i,t} + B_{i,t} = W_t L_{i,t} + R_{t-1} B_{i,t-1} + D_t + T_t \tag{1.26}$$

Firms. There is a continuum of firms producing differentiated goods, indexed by $j \in [0,1]$. Firm j faces a demand curve given by $Y_{j,t} = (P_{j,t}/P_t)^{-\theta} Y_t$. Firm j takes the wage $W_{j,t}$ and demand for its goods as given. In each period, firm j sets the price for its own variety $P_{j,t}$, based on its information, and then hires sufficient labour $L_{j,t}$ to produce to meet its demand according to production function $Y_{j,t} = A_t L_{j,t}$. Formally, firm j's expected present value of profit discounted by households' marginal utility of consumption is given by

$$\mathbb{E}^j \left[\sum_{t=0}^{\infty} C_t^{-\gamma} \left[P_{j,t} Y_{j,t} - (1-\theta^{-1}) \frac{W_{j,t}}{A_t} \left(\frac{P_{j,t}}{P_t} \right)^{-\theta} Y_t \right] \right] \tag{1.27}$$

less the cost of attention. Here A_t is the aggregate productivity, with $a_t \equiv \log(A_t)$ follows a AR(1) process: $a_t = \rho_a a_{t-1} + \sigma_a \varepsilon_t$, with $\varepsilon_t \sim N(0,1)$. Other variables are defined similarly as in Section 1.2.

Central Bank. I assume the central bank has full information – it knows the shocks, households' and firms' actions, and the equilibrium outcomes. Monetary policy is specified as the following standard Taylor rule with interest rate smoothing

$$\frac{R_t}{\bar{R}} = \left(\frac{R_{t-1}}{\bar{R}} \right)^{\rho} \left[\left(\frac{P_t}{P_{t-1}} \right)^{\phi_\pi} \left(\frac{Y_t}{Y_t^n} \right)^{\phi_y} \right]^{1-\rho} e^{-\sigma_u u_t} \tag{1.28}$$

where R_t is the nominal interest rate, $\bar{R}$ is the steady state nominal rate, $Y_t = C_t$ is aggregate output, Y_t^n is natural level of output in the economy with no frictions, and $u_t \sim N(0,1)$ is a monetary policy shock. I specify the rule such that a positive u_t shock corresponds to an expansionary monetary policy shock. Denote $i_t \equiv \log(R_t)$, the log-linearised Taylor rule is

$$i_t = \rho i_{t-1} + (1-\rho)(\phi_\pi \pi_t + \phi_x x_t) - u_t \tag{1.29}$$

I interpret the central bank in the model as the counterpart of professional forecasters in the survey.

Fiscal Authority. The government has to finance maturing nominal government bonds and the wage subsidy, by collecting lump-sum taxes or issuing new bonds. The government's budget constraint is

$$\frac{B_t}{P_t} = \frac{R_{t-1}}{\Pi_t}\frac{B_{t-1}}{P_{t-1}} + \theta^{-1}\frac{W_t L_t}{P_t} + \frac{T_t}{P_t}$$

How the fiscal authority finances its expenditures matters a great deal for the macroeconomic outcomes. Here I consider two assumptions about how the government satisfies its intertemporal budget constraint: (i) government debt is held constant, and transfers fully adjust; (ii) let government debt absorb the majority of the fiscal imbalance in the short run, and adjust the path of lump-sum tax to satisfy long-run solvency.[19] In particular, the government raises taxes to repay all the interest payments and repay a portion $\bar{\tau}$ of existing debt

$$-\frac{T_t}{P_t} = \frac{R_{t-1}}{\Pi_t}\frac{B_{t-1}}{P_{t-1}} + \bar{\tau}\left(\frac{B_{t-1}}{P_{t-1}} - \frac{\bar{B}}{\bar{P}}\right)$$

Following common practice in the New Keynesian literature, I restrict the value of τ such that monetary policy is active and fiscal policy is passive in the sense of Leeper (1991).

Timing. The timing is specified similarly to Section 1.2, with two key differences. First, households now choose not only $c_{i,t}$ but also $b_{i.t}$, which affects their attention choices. Second, because there is persistence (from both the shock process and the intertemporal relationship) in the model, the history of the past signals is relevant for current decisions. Therefore, the information sets for households and firms are $S_i^t \equiv \{s_{i,t} \cup S_i^{t-1}\}$ and $S_j^t \equiv \{s_{j,t} \cup S_j^{t-1}\}$, respectively.

1.3.2 Households' Attention Problem

Analogous to Section 1.2.3, I derive an expression for the expected discounted sum of utility losses when actions of household i deviate from the optimal actions. Household chooses real bond holdings, $\tilde{b}_{i,t}$, and consumption level, $c_{i,t}$, in each period t. This is equivalent to directly choosing the vector x_t in Equation

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[19] For the model solution under the second assumption see Online Appendix.

(1.31) if the household knows its own past actions (for detailed derivation see Appendix A.3.1). Formally, household i's rational inattention problem is

$$\max_{s_{i,t}\in\mathcal{S}_h^t}\sum_{t=0}^{\infty}\beta^t\mathbb{E}\left[\frac{1}{2}\left(x_{i,t}-x_{i,t}^*\right)'\Theta\left(x_{i,t}-x_{i,t}^*\right)-\mu^h\mathcal{I}\left(\left\{x_{i,t-j}^*\right\}_{j=0}^{\infty};s_{i,t}|S_i^{t-1}\right)|s_i^{-1}\right] \tag{1.30}$$

Here S_i^{t-1} denotes the history of signals up to time $t-1$. The choice vector is

$$x_{i,t}=\begin{pmatrix}\omega_B\left(\tilde{b}_{i,t}-\tilde{b}_{i,t-1}\right)\\ -\omega_B\left(\frac{1}{\beta}\tilde{b}_{i,t-1}-\tilde{b}_{i,t}\right)+\left(\gamma\frac{\omega_W}{\eta}+1\right)c_{i,t}\end{pmatrix} \tag{1.31}$$

and

$$\Theta=-\bar{C}^{1-\gamma}\begin{bmatrix}\left(\gamma-\frac{\gamma^2\omega_W}{\gamma\omega_W+\eta}\right)\frac{1}{\beta} & 0\\ 0 & \frac{\omega_W}{\gamma\omega_W+\eta}\end{bmatrix} \tag{1.32}$$

Moreover, $x_{i,t}^*$ is the optimal choice vector for household i, which is given by

$$x_{i,t}^*=\begin{pmatrix}z_t-(1-\beta)\sum_{s=t}^{\infty}\beta^{s-t}\mathbb{E}_t\left[z_s\right]+\frac{\beta}{\gamma}\left(1+\omega_W\frac{\gamma}{\eta}\right)\sum_{s=t}^{\infty}\beta^{s-t}\mathbb{E}_t\left(i_s-\pi_{s+1}\right)\\ \omega_W\left(\frac{1}{\eta}+1\right)\tilde{w}_t+\left[\frac{1}{\beta}\omega_B\left(i_{t-1}-\pi_t\right)+\omega_D\tilde{d}_t+\omega_T\tilde{\tau}_t\right]\end{pmatrix} \tag{1.33}$$

The lowercase variables denote the log deviations of the corresponding variables, and variables with a tilde indicate that they are real variables. Moreover, $z_t\equiv\omega_W\left(1+1/\eta\right)\tilde{w}_t+\frac{1}{\beta}\omega_B\left(i_{t-1}-\pi_t\right)+\omega_D\tilde{d}_t+\omega_T\tilde{\tau}_t$. The coefficients $(\omega_B,\omega_W,\omega_D,\omega_T)$ denote the steady-state ratios of $\left(\frac{\bar{B}}{\bar{C}\bar{P}},\frac{\bar{W}\bar{L}}{\bar{C}\bar{P}},\frac{\bar{D}}{\bar{C}\bar{P}},\frac{\bar{T}}{\bar{C}\bar{P}}\right)$.

The first element of the choice vector $x_{i,t}$ is the change in bond holdings, and the second element of $x_{i,t}$ is the component of the marginal rate of substitution between consumption and leisure. These two elements are directly chosen by household through their choice of real bond holdings $\tilde{b}_{i,t}$ and $c_{i,t}$. The formulation of the optimal choice vector (1.33) implies that: (i) it is optimal to increase bond holdings when income is high relative to permanent income or when the real return on bond is high; (ii) it is optimal to equate the marginal rate of substitution between consumption and leisure to the real wage.[20] When the household deviates from these optimal choices, the utility loss is determined by the matrix Θ. This matrix is diagonal, because a suboptimal marginal rate of substitution between consumption and leisure does not affect the optimal change in bond

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[20] In the formulation, I replaced the labour supply using the budget constraint.

holdings, and a suboptimal change in bond holdings does not affect the optimal marginal rate of substitution between consumption and leisure.

1.3.3 Firms' Attention Problem

After a log-quadratic approximation, I derive firm j's present value of expected profit loss

$$\max_{s_{j,t} \in \mathcal{S}_f^t} \sum_{t=0}^{\infty} \beta^t \mathbb{E}\left[-\frac{\theta-1}{2}\left(p_{j,t}-p_{j,t}^*\right)^2 - \mu^f \mathcal{I}\left(p_{j,t}^*; s_{j,t} | s_j^{t-1}\right) | s_j^{-1}\right] \tag{1.34}$$

where

$$p_{j,t}^* = w_{j,t} - a_t = p_t + \alpha\left[y_t - \frac{1+\eta}{\eta+\gamma} a_t\right] \tag{1.35}$$

where $\alpha \equiv \frac{(\eta+\gamma)}{(1+\theta\eta)}$ is the pricing complementarity. Equation (1.35) implies it is optimal for firm j to increase its price if its nominal marginal cost increases, and vice versa.

1.3.4 Equilibrium

In the quantitative model, I extend the definition of equilibrium 1 in Section 1.2.3.

Definition 2 (Equilibrium). *Given exogenous processes for productivity and monetary policy shocks $\{a_t, u_t\}$ and initial sets of signals for households and firms, a general equilibrium for this economy is an allocation for every household $i \in [0,1]$, $\Omega_i^h \equiv \{s_{i,t} \in \mathcal{S}_{i,t}^h, C_{i,t}, B_{i,t}, L_{i,t}\}_{t=0}^{\infty}$, an allocation for every firm $j \in [0,1]$, $\Omega_j^f \equiv \{s_{j,t} \in \mathcal{S}_{j,t}^f, P_{j,t}, L_{j,t}, Y_{j,t}\}_{t=0}^{\infty}$, a set of prices $\{P_t, R_t, W_t\}$. Aggregate variables are obtained by aggregating individual actions, such that*

1. *Given the set of prices and $\{\Omega_j^f\}_{j\in[0,1]}$, the households' allocation solves the problem in Equation (1.30)*
2. *Given the set of prices and $\{\Omega_i^h\}_{i\in[0,1]}$, the firms' allocation solves the problem in Equation (1.34)*
3. *Central bank sets the nominal interest rate according to the rule in Equation (1.29)*
4. *Good market clears, labour market clears, and bond market clears.*

Computing the Equilibrium. I solve a dynamic general equilibrium model in which both agents are rationally inattentive. As defined in Section 1.3.4, the equilibrium is characterised by a fixed-point problem. Specifically, given the processes for the optimal actions of households and firms, $(x_{i,t}^*, p_{j,t}^*)$, I can solve their respective attention problems. In the meanwhile, the processes $(x^*_{i,t}, p^*_{j,t})$ are endogenous to the decisions of households and firms. In equilibrium, these two processes must be consistent with each other.

I start by guessing the MA representation of the optimal actions $(x^*_{i,t}, p^*_{j,t})$ as functions of the productivity (ε_t) and monetary policy (u_t) shocks. I then approximate the processes with truncated $\text{MA}(200)$ processes.[21] I then solve the problem numerically using the algorithm for dynamic rational inattention problems (DRIPs) developed in Afrouzi and Yang (2021). Next, I solve the implied state-space representations of other variables in the model, based on which I update the guess for the MA representation of the optimal actions $(x^*_{i,t}, p^*_{j,t})$, until the model converges. Appendix A.3.2 provides a detailed description of the implementation.

1.3.5 Calibration

The model is calibrated at a quarterly frequency. Table 1.1 summarises the assigned values for the non-rational inattention parameters, which are estimated outside the model, as well as the calibrated values for the marginal attention costs of households and firms.

Table 1.1: Parameter values

Parameter	**Value**	**Source / Moment matched**
Panel A. Assigned parameters		
Time discount factor (β)	0.99	Quarterly frequency
Elasticity of substitution across firms (θ)	10	Firms' average markup
Risk aversion coefficient (γ)	3.5	Households' risk aversion level
Inverse of Frisch elasticity (η)	2.5	Aruoba et al. (2017)
Taylor rule: smoothing (ρ)	0.936	Estimates 1985-2017
Taylor rule: response to inflation (ϕ_π)	1.62	Estimates 1985-2017
Taylor rule: response to output gap (ϕ_x)	0.225	Estimates 1985-2017
Persistence of productivity shocks (ρ_a)	0.93	Fernald (2014), 1981-2022
S.D of productivity shocks (σ_a)	0.86×10^{-2}	Fernald (2014), 1981-2022
S.D of monetary shocks (σ_u)	0.41×10^{-2}	Estimates 1985-2017
Panel B. Calibrated parameters		
Attention cost of households (μ^h)	0.0106	Slope coefficients in Figure 1.1
Attention cost of firms (μ^f)	0.0095	Slope coefficients in Figure 1.1

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[21] With a length of 200, I can get arbitrarily close to the true $\text{MA}(\infty)$ processes. Increasing the length does not significantly change the results.

Non-Rational Inattention Parameters. I assign values for the non-rational inattention parameters following the literature. I assume the inverse of the Frisch elasticity (η) to be 2.5 and the risk aversion coefficient (γ) to be 3.5, which are standard values in business cycle models. I set the elasticity of substitution across firms (θ) to 10, corresponding to a markup of 11 percent.

I estimate the Taylor rule using real-time U.S. data. Specifically, I use the federal funds rate as a measure of the nominal interest rate, and the Tealbook forecast of inflation and output gap. I employ quarterly data from 1985:1 to 2017:4. The point estimates suggest a smoothing factor of approximately 0.936, with responses to inflation and the output gap of 1.62 and 0.225, respectively.[22] I then compute the model-consistent measure of the monetary policy shock u_t from the data, rewriting the monetary policy rule (1.29) as $u_t = i_t - \rho i_{t-1} - (1-\rho)[\phi_\pi \pi_t + \phi_x(y_t - y_t^n)]$. The standard deviation of u_t is estimated to be 0.41×10^{-2}.

To calibrate the parameters of the stochastic process for aggregate productivity, I use data on total factor productivity (TFP) reported by Fernald (2014), from 1981:1 to 2022:4. I regress the log of TFP on a constant and a time trend. I then regress the residual on its own lag. Based on the point estimates from this regression, I set the autocorrelation of aggregate technology to 0.93 and the standard deviation of the aggregate technology shock ε_t equal to 0.86×10^{-2}.

Rational Inattention Parameters. As described in Section 1.2.6, the model generates over-identifying restrictions on the attention cost parameters (μ^h and μ^f) as these parameters determine jointly agents' attention choices as well as the equilibrium responses of output and inflation to shocks, which affect the perceived correlation between output and inflation of households and firms. It also affects the perceived correlation of professional forecasters, which depends on the equilibrium correlation between expected inflation and expected output growth.

I calibrate the values for μ^h and μ^f to match the slope coefficients for the households, firms and professional forecasters in the Figure 1.1. Holding the non-rational inattention parameters constant at the selected values, and solving over a grid of attention cost values, I find that $\mu^h = 0.0106$ and $\mu^f = 0.0095$ could generate data-consistent slope coefficients. The calibrated attention parameters also suggest that households face higher information frictions than firms, con-

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[22] Because empirical Taylor rule is estimated using annualised rates while the Taylor rule in the model is expressed in quarterly rates, I rescale the coefficient on the output gap in the model, yielding $\phi_x = 0.9/4 = 0.225$.

sistent with findings from other survey-based studies (see for e.g., Link et al. (2023)).

1.3.6 Results

I simulate the model using the parameter values from Table 1.1. Table 1.2 reports the moments for expected inflation and output growth regressions – including the slope coefficients, their associated p-values, and the R-squared values for all agents. Column 2 reports the data moments. Note that the magnitude of the slope coefficient for households does not have a meaningful quantitative interpretation; only the sign matters. This is because in the Michigan Survey of Consumers, households do not provide quantitative forecasts for growth, I assign numerical values to their growth expectations following Candia et al. (2020).[23] However, the magnitudes of slope coefficients for firms and professional forecasters are quantitatively meaningful.

I simulate the model 1,000 times and report the median of the results in Column 3, and the 90 percent confidence interval in Column 4. In each simulation, the time horizon and the numbers of households and firms align with the survey data.

Table 1.2: Moments in the data and the model

Moment	**Data**	**Model**	**90% interval**
Slope coef. of HHs' expectations	-0.038	-0.047	$[-0.061, -0.034]$
Slope coef. of Firms' expectations	0.039	0.005	$[-0.008, 0.024]$
Slope coef. of CB's expectations	0.156	0.155	$[0.103, 0.208]$
R-squared value of HHs' expectations	0.022	0.045	$[0.028, 0.063]$
R-squared value of Firms' expectations	0.002	0.001	$[0.000, 0.004]$
R-squared value of CB's expectations	0.016	0.194	$[0.089, 0.319]$
P-value of HHs' expectations	0.000	0.000	$[0.000, 0.000]$
P-value of Firms' expectations	0.428	0.406	$[0.002, 0.865]$
P-value of CB's expectations	0.000	0.000	$[0.000, 0.000]$

Notes: The table presents the data moments and model moments under calibration in Table 1.1. The time horizon in each simulation is consistent with the survey data. The numbers of households and firms in the simulation align with the survey sample size. I simulate 1,000 times and report the median of the results in Column 3, and the 90 percent interval in Column 4.

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[23] In the Michigan Survey of Consumers, respondents are asked about whether they expect business conditions in the next year to improve, stay the same or deteriorate. Following Candia et al. (2020), I assign point values to each answer ranging from 1 (improve) to -1 (deteriorate).

The model matches the slope of the professional forecasters. In the model, I assume the central bank (the model counterpart of professional forecasters) has full information. Consequently, their beliefs are the correct beliefs about the dynamics of future inflation and output growth.

The model matches the moments for households and firms. First, by emphasising the attention mechanism, the model successfully replicates the negative slope seen in households' expectations as well as the weakly positive slope observed in firms' expectations. In particular, as households pay more attention to supply shocks, their information sets contain mostly supply shocks, and they would base their expectation on their partial information. Firms, on the other hand, pay attention to both shocks, with slightly more attention to demand shocks. Therefore, their information sets contain more demand shocks, which affects their expectations. In accordance with this dual attention to both shocks by firms, the p-value of the slope coefficient for firms is not statistically significant, in line with the survey evidence.

By simulating the model, I generate the model counterpart of Figure 1.1, as illustrated in Figure 1.4. These two figures exhibit striking similarities, providing support for the model's validity. It is worth noting that the survey data displays a wider dispersion than the model, potentially stemming from inherent noise in the beliefs of households and firms. Nevertheless, this specific aspect falls beyond the scope of the current model, which primarily focuses on the correlation between expected inflation and expected growth. For comprehensive investigations into belief noise, I recommend referring to the literature on this topic, for example Juodis and Kučinskas (2023).

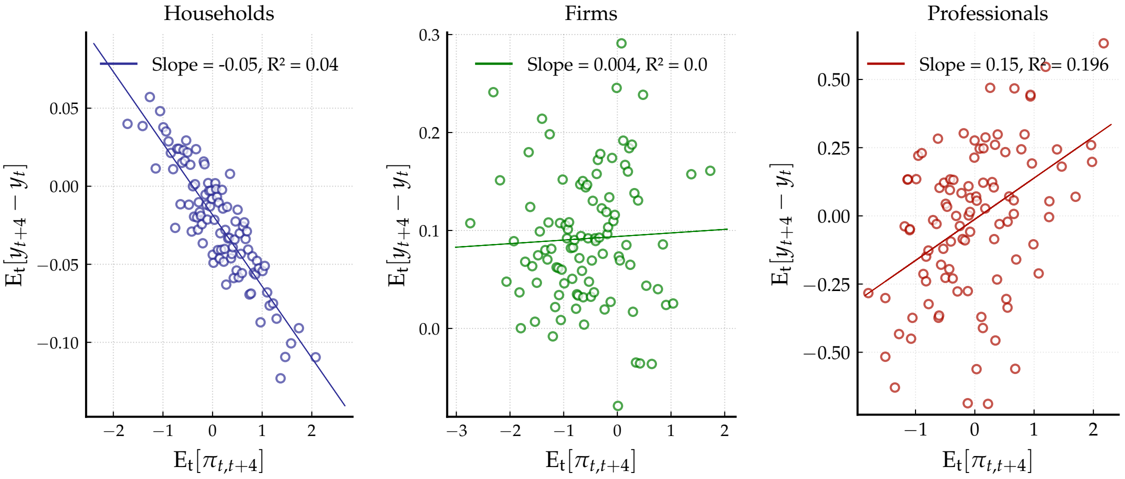

Figure 1.4: Simulated expected inflation and expected output

Notes: The figure plots the simulated expected inflation and expected output growth for households, firms, and professional forecasters. The parameterisation values are from Table 1.1.

1.3.7 Quantifying the Real Consequences of Inattention

Attention choices of households and firms significantly affect the response of aggregate output to shocks compared to the full information case. In the case of supply shocks, the overall response of aggregate output under rational inattention is lower than the full information benchmark. In particular, firms' inattention and pricing complementarity dampen the aggregate output response by around 70 percent, households' inattention dampens it by 24 percent, and the strategic complementarity between households' and firms' attention allocation further dampens the response by 7 percent.

In the case of demand shocks, when firms have full information, demand shocks do not have a real impact on output, this follows the classical dichotomy. When firms are inattentive, demand shocks have real impacts. This implies that firms' inattention amplifies the real effects of demand shocks, by increasing money non-neutrality. Introducing inattentive households (without strategic interactions) lowers the output response by around 43 percent, as households are inattentive and under-react. Strategic substitutability between households and firms further reduces the output response by 24 percent. This is because as households pay less attention to demand shocks, firms pay more attention, and money is more neutral.

1.3.8 Empirical Validation

Attention Matters for Beliefs. The model predicts that households pay more attention to the real side of the economy, such as real wages, and their attention choice matters for their perceived relationship between inflation and output growth. To test this prediction, I utilise additional data from the Michigan Survey of Consumers. I find that households pay significantly more attention to employment-related developments than to price-related developments, and by running a simple regression, I show that households who pay attention to employment-related developments hold an even stronger supply-side view compared to those who do not. This provides further support to the model which predicts that attention choices matter for agents' perceived relationship between expected inflation and output growth. For detailed data construction and empirical specification, see Appendix A.1.3.

The finding persists when I divide the employment-related developments into positive and negative ones (see Columns 2 and 3 in Table A.2), which would allay any concern that the results are biased by employment-related developments more likely being negative than positive. These results also rule out pessimism as the sole explanation for the negative correlation between inflation and output. Bhandari et al. (2022) find that increased pessimism generates an upward bias in unemployment and inflation forecasts, contributing to the negative correlation between inflation and real activity. However, the results in this paper suggest that attention choices are a key driver of households' supply-side view.

Forecast Errors. The model predicts that households pay much less attention to demand shocks. If this is true, they are more likely to make larger forecast errors during periods dominated by demand shocks. To test this, I use the supply shocks and demand shocks identified by Eickmeier and Hofmann (2022)[24] The forecast error for one-year-ahead inflation is measured as the absolute difference between the median forecast from the Michigan Survey of Consumers and the realised inflation for the corresponding period. I find that forecast errors during periods dominated by demand shocks are about 1.6 times larger than during periods dominated by supply shocks.

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[24] I use their identified shocks because their empirical analysis adopts the same definition of supply and demand shocks as in this paper: supply shocks move inflation and output in opposite directions, while demand shocks move both variables in the same direction. They estimated structural demand and supply factors for the period 1970Q1–2022Q2.

1.4 The Phillips Curve Dynamics

An important feature of the model is that attention choices of households and firms are endogenous to economic conditions. When the economic conditions change – due to shifts in policy or changes in stochastic environment – households and firms reallocate their attention. The reallocation of attention in turn has a significant impact on the outcomes. In this section, I show that the attention model can jointly explain the flattening of the Phillips curve in recent decades before the pandemic and its steepening in the post-pandemic period.

1.4.1 Flattening of the Phillips Curve in Recent Decades

The decades preceding the start of the COVID-19 pandemic saw a decline in the correlation between inflation and real activity and many raised the possibility that the Phillips curve had flattened or even disappeared (Coibion and Gorodnichenko, 2015; Blanchard, 2016; Bullard, 2018; Hooper et al., 2020; Stock and Watson, 2020; Smith et al., 2025). Many explanations have been proposed in the literature to rationalise the flattening of the Phillips curve. In this section, I show that the shift to a more hawkish monetary policy and the resulting reallocation of attention by households and firms are consistent with, and quantitatively relevant, for the observed flattening of Phillips curve.

To capture the shift in monetary policy, I simulate the model under two regimes: $\phi_\pi = 1.3$ (dovish) and $\phi_\pi = 1.62$ (the baseline calibration). The $\phi_\pi = 1.3$ is calibrated to match the monetary policy before the Great Moderation. This exercise follows the spirit of Maćkowiak and Wiederholt (2015) and Afrouzi and Yang (2021).[25] The model predicts a flatter Phillips curve under more hawkish policy, along with lower volatility in inflation and output (see Figure 1.5).

The intuition is straightforward. First, a more hawkish monetary policy stabilises the prices, reducing firms' attention. This, in turn, further dampens their price adjustment behaviour, making prices even less sensitive to output fluctuations, and the Phillips curve is flatter.

Second, households reallocate their attention as firms pay less attention. In the case of productivity shocks, households pay less attention as firms pay less

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[25] Afrouzi and Yang (2021) develop a model with rationally inattentive firms and show that more hawkish monetary policy reduces firms' attention to input costs, flattening the Phillips curve. In contrast, the model here features rational inattention on both sides, which is crucial for output dynamics. Maćkowiak and Wiederholt (2015) model inattention on both sides but absent from the strategic interaction in attention allocation between households and firms.

attention, due to the complementarity in their attention allocations. As a result, consumption (output) responds less to the productivity shock, which increases deviations of output from the efficient output level and raises output gap volatility. In the case of monetary policy shocks, when firms pay less attention, monetary policy shocks have a larger real impact, which incentivises households to pay more attention to those shocks. This also leads to a more volatile output gap. These two attention channels are absent in standard New Keynesian models with full or exogenous dispersed information, and both contribute to the flattening of the Phillips curve.

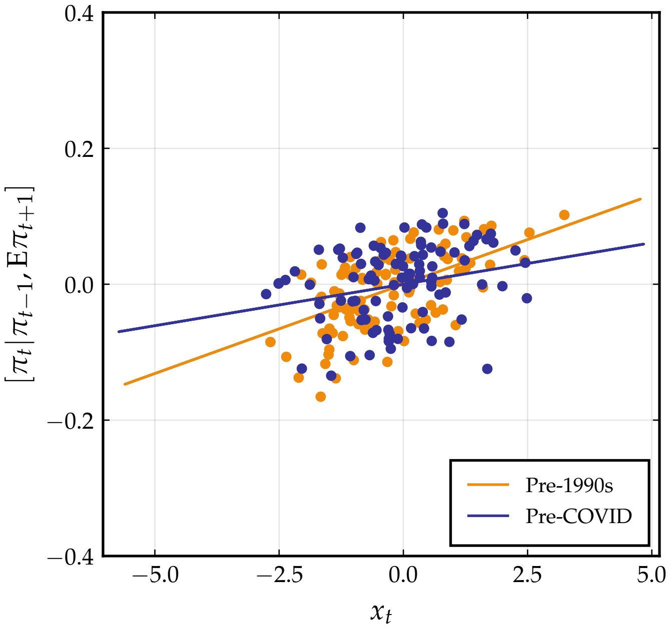


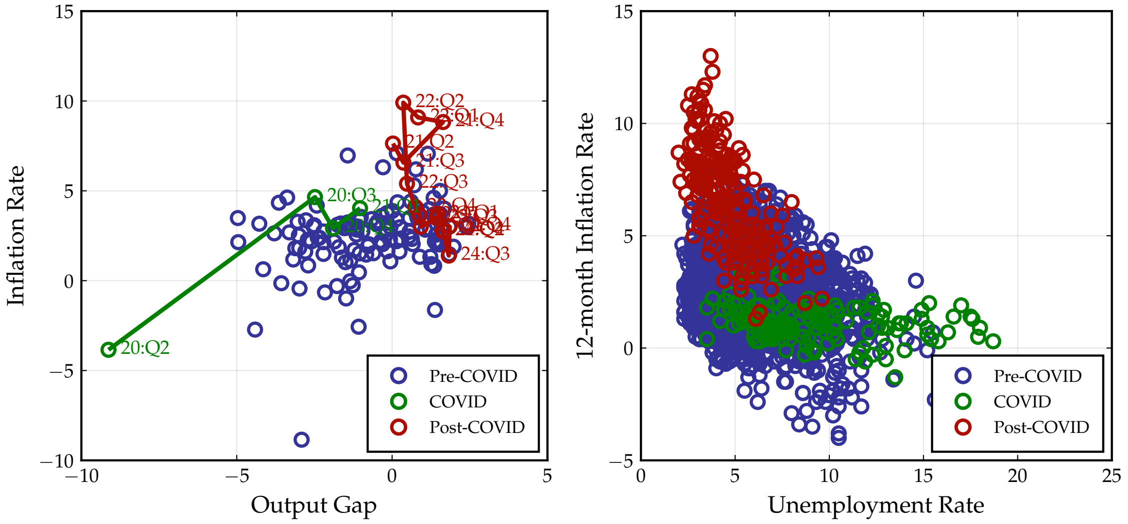
Figure 1.5: The flattening of the Phillips curve over recent decades

Notes: This figure compares the model-implied Phillips curve under dovish and hawkish monetary policies. In the former case, I assume $\phi_\pi = 1.3$, and in the latter case $\phi_\pi = 1.62$. The rest of the parameters are kept the same across the two regimes, as specified in Table 1.1. I simulate the model for 50,000 periods.

1.4.2 Steepening of the Phillips Curve in the Post-pandemic Era

The post-COVID data signal that the Phillips curve is back. Figure 1.6a plots the relationship between the inflation rate and the output gap in the U.S. before, during, and after COVID. The raw data suggest a steepening of the Phillips curve in the post-COVID period. Similar findings are reported by Hobijn et al. (2023), Furlanetto and Lepetit (2024), and Gelain and Lopez (2024) among others.

The MSA-level data show a similar pattern. Figure 1.6b shows the relationship between inflation and unemployment rates across 21 U.S. metropolitan statistical areas (MSAs) over the same period.[26] The panel variation in inflation and unemployment provides a larger sample. The regional raw data reveal a notable steepening in the post-COVID period relative to the pre-COVID period. Using MSA-level data, Cerrato and Gitti (2022) estimate that the slope of the Phillips curve has tripled between the pre-COVID and the post-COVID periods.



(a) Aggregate level

(b) Regional level

Figure 1.6: The Phillips correlation before, during, and after the COVID-19

Notes: Figure 1.6a shows the relationship between CPI and the output gap. Figure 1.6b shows the relationship between the 12-month, all-items inflation rate and the unemployment rate at the US metropolitan area level. Blue dots represent observations from the pre-COVID period (Jan 1990–Feb 2020), green dots correspond to the COVID period (Mar 2020–Feb 2021), and red dots denote the post-COVID period (Mar 2021–Sep 2022). Data sources: FRED, Federal Reserve Bank of St. Louis; the U.S. Bureau of Labor Statistics.

As the post-pandemic period was characterised by large supply and demand shocks (Di Giovanni et al., 2023; Shapiro, 2024), I assume a 20 percent increase in the volatility of both shocks in the post-COVID calibration. The rest of the parameters follows the baseline model (see Table 1.1). I then simulate the model under these calibrations and compare the results with the pre-COVID baseline. The simulated results are shown in Figure 1.8, which shows a steeper Phillips curve in the post-COVID period relative to the baseline. Although the model relies on stylised assumptions and is not calibrated to match the data exactly, it captures the qualitative shift in slope across periods. By contrast, a standard

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[26] Output gap data are not available at the MSA-level, but unemployment serves as a widely used proxy for economic slack.

New Keynesian model does not replicate this pattern: the slope of the Phillips curve remains roughly constant across regimes.

The intuition is straightforward. When shocks become more volatile, rationally inattentive firms increase their attention (see Proposition 2). Focusing on demand shocks, as volatility rises, firms pay more attention to them. This implies that they would raise prices more aggressively following a positive demand shock. Therefore, such shock only has moderate real impact. This led to a steepening of Phillips curve.[27]

Figure 1.7 illustrates this mechanism by showing simulated inflation dynamics following a demand shock under three scenarios: the pre-COVID period, the post-COVID period with heightened attention, and the post-COVID period assuming attention remains unchanged. Without the increase in attention, the rise in inflation is driven purely by larger shocks in the post-pandemic period, and the effect is relatively modest (green circles versus blue triangles). With endogenous attention, however, firms respond more aggressively to the same shocks, which amplifies the inflation response significantly (red squares). This mechanism is supported by empirical evidence on firms' price adjustment during and after the COVID period: Montag and Villar (2023) find that both the frequency and the size of price changes in the U.S. increased sharply during those periods.

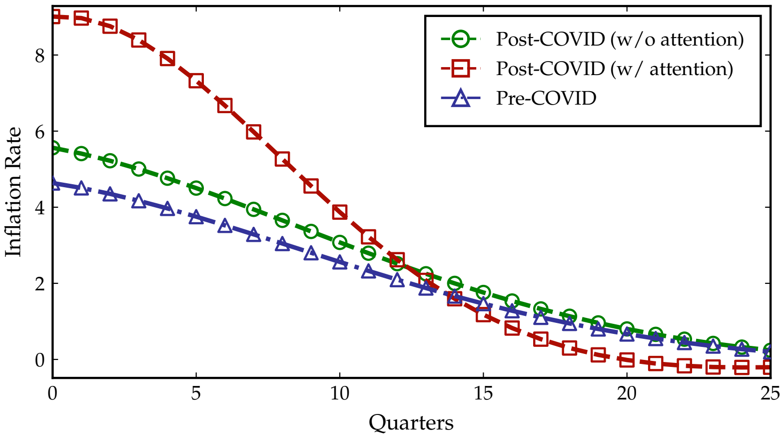

Figure 1.7: Inflation dynamics

Notes: This figure plots inflation dynamics following a demand shock under three scenarios: Pre-COVID (blue triangles), Post-COVID with reallocated attention (red squares), and Post-COVID with attention held constant (green circles).

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[27] The adjustment in household attention would depend on the calibration. On one hand, increased volatility tends to raise households' attention. On the other hand, as firms pay more attention to demand shocks, the real effects of those shocks become more muted, which reduces the incentive for households to pay attention. Nonetheless, these adjustments play a less central role in explaining the steepening.

Figure 1.6, as well as the findings of Cerrato and Gitti (2022), also suggest a brief flattening of the Phillips correlation during the COVID period. The model is also able to catch this dynamic. In particular, I calibrate the volatility of supply shocks during the COVID period to be 20 percent higher than in the pre-COVID – consistent with evidence that supply shocks were the dominant force at the time – while keeping the volatility of demand shocks at the baseline level. The result is shown in Figure 1.8. In response to the increased volatility of supply shocks, households rationally allocate more attention to these shocks, leading to a stronger output response and smaller deviations from the efficient level. Firms also increase their attention to supply shocks, causing prices to adjust more sharply. Together, these adjustments generate a more negative slope in the regression of inflation on the output gap following supply shocks, contributing to the temporary flattening observed during the COVID period.

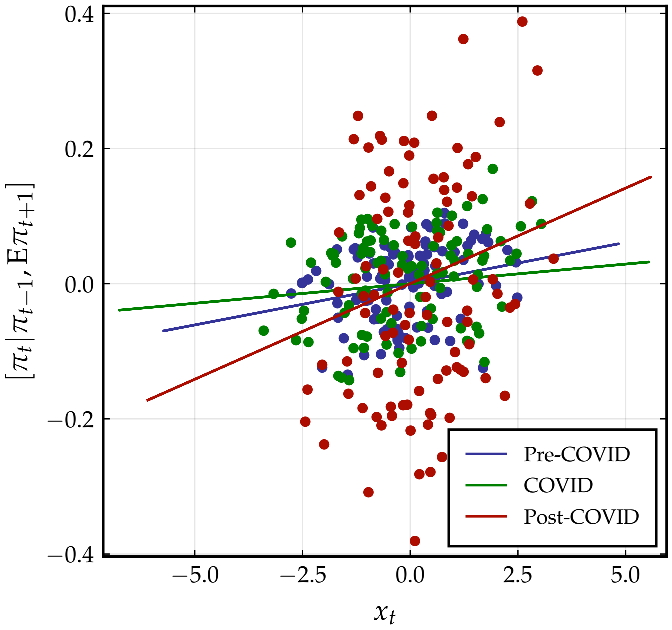

Figure 1.8: Model Phillips curve before, during, and after the COVID-19

Notes: This figure plots model implied Phillips curve before, during, and after the COVID-19. Most parameters are calibrated as in the baseline model (see Table 1.1). I assume that the volatility of supply shocks increases by 20 percent during the COVID period, and that the volatility of demand shocks also rises by 20 percent in the post-COVID period. I simulate the model for 50,000 periods.

Understanding the dynamics of the slope of the Phillips curve is crucial, as it determines the trade-off between inflation and real activity faced by the monetary authorities. In particular, the steepening of the Phillips curve during the post-COVID period might suggest that contractionary monetary policy can reduce inflation with smaller output losses, and therefore encouraging a stronger policy response to inflation. However, this argument is only partially correct. If households' and firms' attention were fixed, a stronger monetary policy response to inflation can indeed reduce inflation at a limited cost to output. But as discussed in Section 1.4.1, monetary policy itself can influence agents' attention, and a more hawkish stance flattens the Phillips curve, ultimately worsening the inflation-output trade-off.

1.5 Communication with Agents with Heterogeneous Attention Choices

Rational inattention has several implications for policy communication. In Section 1.5.1, I formally demonstrate how the misalignment of interests limits the effectiveness of the policy communication. In Section 1.5.2 and 1.5.3, I examine two experiments where information provision could potentially have adverse effects on the economy.

1.5.1 The Veil of Inattention

For communication to be effective, the receiver must be *able* and *willing* to absorb, process and use the information. This section provides a rationale for why households and firms are inattentive to policy communication. To fix ideas, consider a central bank communicating its monetary policy actions to the public

$$S_{p,t} = i_t + \nu_t, \nu_t \sim N(0, \sigma_\nu^2)$$

However, whether households and firms choose to absorb this information depends on how relevant they believe the signal is for their decisions. Formally, the benefit of processing the central bank signal is proportional to

$$\mathbb{E}[x_t^*|S_{k,t}, S_{p,t}] - \mathbb{E}[x_t^*|S_{k,t}] \propto \underbrace{\frac{\Sigma_0}{\Sigma_0 + \Delta_{k,p}\sigma_\nu^2}}_{\text{signal-to-noise ratio}} \times \underbrace{\Delta_{k,p}}_{\text{relevance of signal}} \times \underbrace{\left(S_p - \mathbb{E}[S_{p,t}|S_k^t]\right)}_{\text{marginal new info from } S_{p,t}} \tag{1.36}$$

where Σ_0 is the prior uncertainty about the objective x_t^*, S_k^t is agent k's information set. $\Delta_{p,k}$, $k = \{h, f\}$ captures how relevant the central bank's signal is to agents' objective, i.e., how much $S_{p,t}$ matters for households' optimal consumption and bond decisions, or for firms' pricing decision. The benefit from processing the signal about i_t is discounted by the term $\Delta_{k,p}$ because the i_t may not be of direct relevance to households' and firms' interest. Therefore, if processing central bank information requires cognitive effort or attention costs, households and firms may rationally ignore it.[28] However, if the signal better aligns with the audience's interests, they are more likely to pay attention to the signal.

This analysis relates to Angeletos and Sastry (2021), which study whether communications should aim to anchor expectations of the policy instrument (such as interest rate path) or the targeted outcome (aggregate output or prices), while the focus of this paper is on the incentives of rationally inattentive agents. Agents are more likely to pay attention when the content is directly relevant to their decisions (in the extreme case, when the central bank provides signals about their optimal actions). In this sense, communicating targeted outcomes is better than communicating policy instruments.

1.5.2 Communication about Future Inflation

Even assuming that the policy communication can reach the general public, does it always improve the economic outcomes? Here, I consider the effects of releasing information about future inflation to households in response to a demand shock. As shown in the left panel of Figure 1.9, following a demand shock, households raise their inflation expectations. However, due to inattention, the increase in expectations (grey square) is smaller than the actual rise in inflation (grey circle). Suppose the central bank communicates the correct expected future inflation to households (i.e., a one-time signal about $\mathbb{E}_t\pi_{t+1}$), households' inflation expectations align more closely with accurate one (black triangle line).

However, releasing this information has unintended consequences. As households revise their inflation expectations upward, they also adjust their output growth expectations. The right panel of Figure 1.9 shows that households revise their expected output growth downward (moving from the grey square line to the black triangle line), deviating further from the full-information benchmark (grey circle).

This downward revision occurs because rationally inattentive households misinterpret the higher inflation as originating from a contractionary supply

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[28] These cognitive costs help explain why households are generally inattentive to policy announcements but do update expectations when presented with information in randomized controlled trials (RCTs).

shock. Since households' information sets primarily consist of supply shocks, they are inclined to interpret the inflation increase within this context. As a result, they expect lower growth. This is consistent with findings from RCTs: for example, Coibion et al. (2023) show that an exogenous increase in households' inflation expectations lowers their growth expectations.

Lower growth expectations, in turn, lead households to anticipate lower future income and to reduce current spending. This behaviour contrasts with the predictions of standard New Keynesian models with full information, in which an increase in inflation expectations stimulate current spending -— a key mechanism of forward guidance. These results suggest that communication aimed at stimulating the economy by raising inflation expectations may backfire.

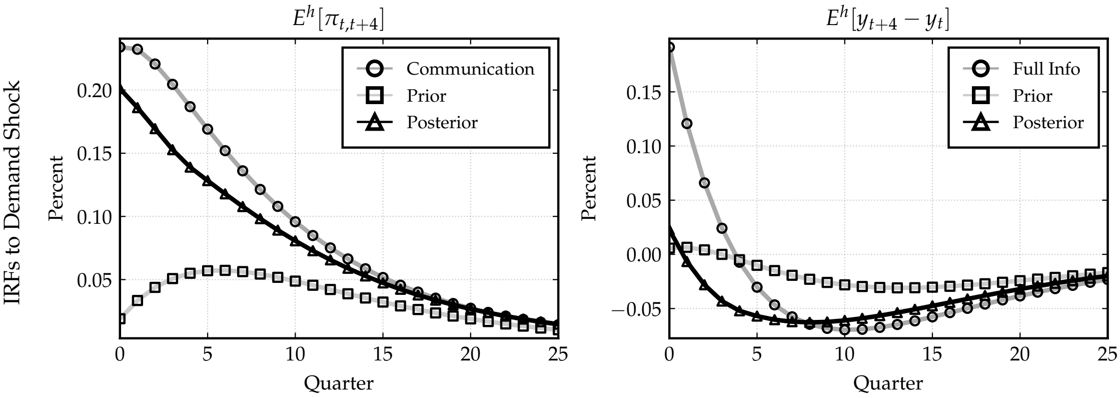

Figure 1.9: Communication of higher future inflation

Notes: The figure plots the impulse responses of households' expected inflation and expected growth following the communication of a higher future inflation trajectory. Households revise their inflation expectations upward after the communication, but at the same time revise their growth expectations downward.

If the same information were provided to firms, they would raise both their inflation expectations and output growth expectations (see Figure A.8 in Appendix A.4). This implies that while providing the information to households and firms might align their expectations on one dimension, it could cause even greater divergence on another. Such divergence may result in inefficient fluctuations in the economy.

1.5.3 Communication about Lower Interest Rate Path

In this section, I compare the impulse response functions to a positive supply shock under the baseline model and a scenario in which the central bank also provides a one-time perfectly informative signal about the future interest rate path.

In response to a positive supply shock, since agents are inattentive, the output response is lower than the potential level of output, creating a temporary negative output gap. The central bank then systematically responds to the negative output gap by lowering the interest rates. The response in interest rates is shown in the left panel of Figure 1.10. Suppose the central bank communicates the lower interest rate path to firms. In that case, firms might misinterpret the systematic response in interest rates as an expansionary demand shock due to their skewed information set, and thus raise their inflation expectations (middle panel) and prices (right panel). As firms raise prices, aggregate demand falls further, worsening the economic slack.

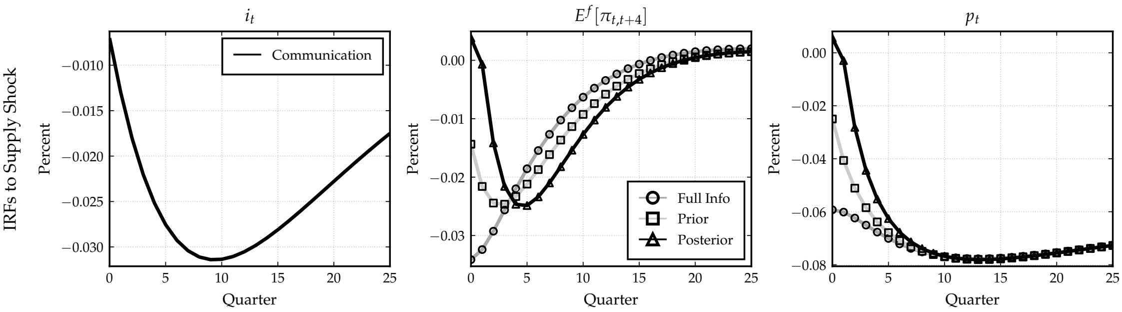

Figure 1.10: Communication of lower interest rate

Notes: The figure plots the impulse responses of firms' expected inflation and expected growth following the communication of a lower trajectory of future interest rates. Firms revise their inflation expectations in the wrong direction following the communication and as a result the price adjustments are even more sub-optimal.

1.6 Conclusion

This paper proposes a model with two-sided rational inattention and demonstrates its implications for belief formation, economic fluctuations, and policy communication. I show that households and firms face distinct incentives when choosing what economic information to pay attention to. Households focus more on supply shocks, while firms pay attention to both supply and demand shocks, with a slight emphasis on demand.

Using a calibrated dynamic model, I show that the asymmetric attention choices of households and firms generate divergent views about inflation and output growth, matching the patterns observed in survey data. Furthermore, I show the model can jointly explain for the flattening of the Phillips curve in the decades preceding the pandemic and its steepening in the post-COVID period, highlighting the mechanism of attention reallocation. I further study the role of communication when agents are rationally inattentive. I show that providing more accurate information does not always improve outcomes. When communication is misaligned with agents' incentives or incomplete information skews their interpretation, even well-intentioned announcements can backfire.

These findings suggest that the effectiveness of policy and communication depends critically on how different agents in the economy allocate attention. Rational inattention is not merely a friction, but a force that shapes macroeconomic outcomes and policy transmission.

This paper takes a preliminary step towards understanding the heterogeneous information choices among different groups of agents, and their consequences for business cycles and policy. While this paper compares households and firms, there is significant variation within these groups, driven by differences in characteristics such as income levels, education, or firm size. Future research can delve deeper into the heterogeneity within these groups. Moreover, this paper focuses on how households and firms allocate attention to aggregate economic conditions. However, there are also household- or firm-level factors that capture their attention. A valuable extension would be to explore how agents allocate attention between aggregate economic shocks and idiosyncratic, individual-level factors. Understanding how they allocate attention between these two dimensions could offer new insights into the decision-making processes of households and firms and their responses to broader economic policies.

Chapter 2

Sunspots, Expectations and Asset Pricing[*]

Abstract

We propose a theory of asset pricing in which agents, constrained by limited memory, form expectations based on past observations in ways pinned by sunspot realisations. Limited memory guarantees a tendency to revert to fundamentals. Sunspot shocks induce 'momentum' in asset prices and is motivated by a novel empirical observation about a time-varying mapping from the price-dividend ratio to return expectations in survey data. The simulated method of moments estimates show that the model quantitatively replicates a host of asset-pricing features, including equity premium, excess volatility, persistence of price-dividend ratio, predictability of excess returns, and the consumption correlation puzzle. The model also generates empirically plausible subjective expectations.

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[*] This chapter is co-authored with Guido Ascari. The views expressed are those of the authors and do not necessarily reflect official positions of De Nederlandsche Bank.

2.1 Introduction

Expectations play a central role in asset pricing. Asset prices are essentially forward-looking, implying that expectations about the asset's future payouts and future prices would determine the equilibrium today. The predominant modelling approach assumes rational expectations (RE). Under RE, investors' expectations are tied down by the true underlying law of motion that generates future asset payouts (Sargent, 2008).

However, two observations motivate the need for a different approach.

First, recent empirical evidence suggests that investor expectations systematically deviate from RE. Among others, investor expectations of returns tend to be extrapolative, in the sense that they are positively correlated with current or past realised prices. In this respect, the paper uncovers a novel empirical fact: the strength of this extrapolative behaviour varies over time. Figure 2.1 plots the coefficient from a time-varying parameter regression of expected returns on the price-dividend (PD) ratio. A visual inspection suggests a time-varying mapping from the realised PD ratio to return expectations, that is, the way investors map observations to expectations appears to be time-varying.[1]

Second, RE assumes that agents retain full memory of all past data. This assumption is both conceptually implausible and inconsistent with empirical and experimental findings (see e.g., Jonides et al., 2008). In practice, agents possess limited memory, and this feature has been increasingly recognised in the recent finance and macroeconomics literature (see Section 2.1 for a brief review).

In response to these considerations, we propose a new asset-pricing framework that incorporates both time-varying expectation formation and limited memory. The model builds on the long-run risk (henceforth LRR) framework of Bansal and Yaron (2004), with a representative agent featuring Epstein and Zin's (1989) recursive preferences, and a small predictable component in the consumption and dividend growth process. We explore two forms of memory constraint: (i) decay memory, where the influence of historical data on expectations gradually fades as it recedes into the past; and (ii) finite memory, where a past data point no longer influences agents' expectations after a given period of time.

We show that the limited memory assumption, despite being an arguably minor deviation from RE, opens up a very different world from the point of

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[1] This is confirmed by the Hansen (1992) stability test. Section 2.6.4 presents a detailed discussion of this time-varying parameter estimation.

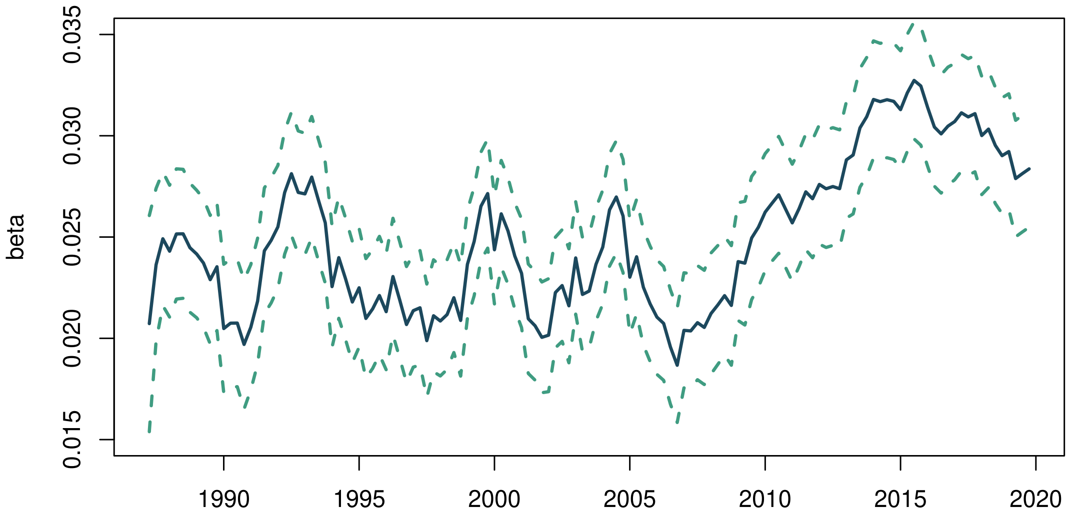

Figure 2.1: Time-varying parameter estimations

Notes: This figure plots the coefficient from a time-varying parameter regression of expected excess return on the log PD ratio based on survey data constructed by Nagel and Xu (2021). See Section 2.6.4 for details. Dashed lines represent the 95% confidence intervals based on standard errors for the coefficient.

view of admissible dynamics. As a consequence of limited memory, backward-looking solutions of an explosive system are no longer explosive; hence the typical saddle-point dynamic system admits infinite possible solutions. In other words, once we perturb the original Muth (1961) RE solution with the limited memory assumption, there is an infinite number of bounded solutions. Hence, the Blanchard and Kahn's (1980a) stability criterion to select the unique stable solution of a saddle-point dynamic system is not applicable, and Muth's (1961) original problem of the multiplicity of solutions remains. This raises the question of how to select the equilibrium path among all bounded solutions.

Building on the extensive macroeconomics literature on the indeterminacy of RE equilibria, we appeal to the existence of a sunspot shock to select one among these stable paths, modelling our sunspot shock as in Ascari et al. (2019).[2] This sunspot shock has an appealing economic interpretation, as it generates variations in how agents weight past information when forming expectations, as sug-

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[2] Ascari et al. (2019) generalise RE solutions to accommodate temporarily unstable paths. They build solutions that randomly jump between all the admissible RE paths, both stable and unstable. There is an important distinction between their framework and ours. In their framework, time variation in the solution allows temporary walks on unstable RE trajectories, and they thus need to impose an exogenous force for the system eventually to converge to the unique stable solution. In our framework, instead, the limited memory assumption provides such a stabilising force, and expectations never explode.

gested by the survey evidence (see Figure 2.1). As a result, equilibrium asset prices switch randomly across the infinite limited memory equilibria as agents change how they combine past data to calculate their expectations. Moreover, this approach has two other attractive properties. First, as explained by Ascari et al. (2019), the sunspot is multiplicative rather than additive, thereby generating both a time-varying parameter solution and endogenous stochastic volatility. Second, this approach encompasses the usual RE equilibrium as a special case.

The model gives rise to a time-varying equity premium and endogenous stochastic volatility. The proposed mechanism affects the risk premium through two channels. First, expectation shocks introduce an additional source of risk, generating a higher average equity premium. Second, because expectation shocks enter the asset-pricing solution non-linearly, it also affects the risk premium agent demands for the LRR in a time-varying way. This second channel generates time variation in the equity premium and thus gives rise to endogenous stochastic volatility. Moreover, the model predicts that endogenous stochastic volatility is procyclical, increasing when asset prices are high. As such, the model naturally generates boom-and-bust dynamics.

Based on a simulated method of moments (SMM) approach, the model replicates a host of key asset-pricing features, including equity premium, excessive volatility, and persistence of PD ratio, without yielding counterfactual high-level correlation between returns and fundamentals. It also replicates two key empirical patterns: a negative correlation between *realised* excess returns and lagged PD ratios in actual return data, and a positive correlation between *expected* excess returns and lagged PD ratios in survey data.

We compare our model's performance with that of the Bansal and Yaron (2004) model. Notice that, unlike Bansal and Yaron (2004), we do not assume an exogenous process for stochastic volatility. Instead, stochastic volatility arises endogenously from the time-varying expectation formation process and is thus disciplined by the model's theoretical structure. Despite this constraint imposed by the theoretical structure, our model quantitatively outperforms the Bansal and Yaron (2004) model, suggesting that the assumed expectation formation process is corroborated by the data.

Related Literature Although the variant of consumption-based RE models has been the benchmark for asset pricing models, they have difficulties in answering questions such as what drives the periodic booms and busts in the financial markets, or why subjective excess returns expectations (from survey data) are virtually unrelated to dividend growth but strongly positively correlate with price levels. These add to other well-documented puzzles, includes the equity premium puzzle (see, e.g., Mehra and Prescott, 1985) and the asset price volatility puzzles (see, e.g., Shiller, 1981), that pose significant challenge on the standard RE models.

Our contribution relates to several strands of literature.

By introducing sunspots into an asset pricing model, our contribution leverages on the literatures on indeterminacy of equilibria in macroeconomics (e.g., Benhabib and Farmer, 1999), on rational bubbles (e.g., Blanchard and Watson, 1982) and on learning (e.g., Evans and Honkapohja, 2001b). Challe (2004) theoretically shows that models with sunspots shocks may also account for the predictability of asset returns, a feature of the data that our model is able to reproduce quantitatively. Farmer (2015) presents a model where asset price fluctuations are caused by sunspots shocks. However, both papers use an overlapping generation framework to induce indeterminacy, while we assume infinitely-lived agents with limited memory.

A large literature proposes alternative expectation formation processes that deviate from full rationality to explain these phenomena, such as learning. Branch and Evans (2010) propose a learning model where agent's switches between alternative underparameterised forecasting rules, depending on which performs best. As a results, multiple equilibria arise depending on the deep parameters of the model. As theirs, our framework also features multiple equilibria, although they arise through a completely different mechanism due to the finite memory assumption. Branch and Evans (2011) show that periods of excess volatility, bubbles and crashes could arise when agents learn about both the expected return of an asset and its conditional variance (i.e., return risk). The learning process yields stochastic volatility in their model, because shocks to fundamentals lead agents to adjust their perceived risk. Similarly in our framework, stochastic volatility, bubbles and crashes arises due to the interaction between the multiplicative sunspots a là Ascari et al. (2019) and shocks to fundamentals. Lansing (2010) is another paper that finds that near-rational bubbles can emerge through learning dynamics.

Moreover, learning models in which agents have memory constraints are closely related to our modelling assumption. Evans and Honkapohja (2001b) and Honkapohja and Mitra (2003) present early analysis of learning with limited memory. Nagel and Xu (2021) study asset price behaviour in an economy where agents learn about the asset dividend growth with fading memory. The fading memory in the Bayesian framework produces perpetual learning, which induces substantial long-run uncertainty. This paper does not feature learning, instead, we show that the rational expectation with limited memory, by itself, gives history dependent expectations, which implies a specific updating rule for expectations (see Section 2.2.1 for a comparison between the updating of expectations in our framework with respect to the learning approach). On top of the behavioural finance literature, limited memory and memory constraints are also been embedded in macroeconomics (see, e.g., Woodford, 2018; Angeletos and Lian, 2021).

Our paper also relates to studies on explaining the extrapolative beliefs of investors. Adam et al. (2016) argue that past price increases generate optimism about future capital gains and thus a further rise in asset prices. Greenwood and Shleifer (2014) analyse time series data of investor expectations for future stock market returns and find strong evidence of extrapolation; that is, investors' expected returns are positively correlated with the price-to-dividend (PD) ratio. Furthermore, Adam et al. (2017) establish that this finding is consistent across a variety of surveys. Though large behavioural finance literature attribute such extrapolating beliefs to some behavioural biases, we contribute to this discussion by providing a theory that could generate such extrapolative behaviour with minimal deviation from the standard RE.

Finally, our model also shares some similarities with asset pricing models with time preference shocks. For example, Albuquerque et al. (2016) stress the importance of demand shocks coming from stochastic changes in agents' rate of time preference in resolving the correlation puzzle (see Cochrane, 2009, for the discussion on correlation puzzle). As explained later, despite being very different in terms of assumptions and modelling, our time-varying expectation shock has some similarity regarding economic intuition with the valuation shock in Albuquerque et al. (2016). In particular, both models yields stochastic changes in agents' valuation of assets, which in turn determines the equilibrium distribution of prices. However, Albuquerque et al. (2016)'s valuation risk comes from changes in agents' rate of time preference and it alters the valuation of assets in the absence of any shocks to the fundamentals. In our model, instead, the existence and the magnitude of the change in valuations would depend crucially on the history of fundamental shocks.

The remainder of the paper is structured as follows. Section 2.2 presents a simple example to explain our approach to modelling limited memory and time-varying expectations. Section 2.3 describes how we incorporate the time-varying expectation formation process into the Bansal and Yaron (2004) asset-pricing model. Section 2.4 derives analytical results that explain the model's ability to replicate the dynamics of the PD ratio and the equity premium. Section 2.5 describes the data sources and the estimation procedure. Section 2.6 discusses the quantitative results and compares them with the model of Bansal and Yaron (2004). Section 2.7 presents robustness checks. Finally, Section 2.8 concludes.

2.2 Time-Varying Expectations

This section presents our approach to modelling expectations. It modifies the standard RE assumption by introducing limited memory and time variation in expectation formation, as in Ascari et al. (2019).

We use a simple forward-looking equation to illustrate the basic intuition

$$y_t = \theta \mathbb{E}_t y_{t+1} + \varepsilon_t, \tag{2.1}$$

where ε_t is an i.i.d. shock $\sim N(0, \sigma_\varepsilon^2)$, and $\mathbb{E}_t y_{t+1} = \mathbb{E}(y_{t+1}|\mathcal{I}_t)$ is the expected value of y_{t+1} conditional on the information set at time t. Here ε_t can be interpreted as the asset's dividend and y_t as its price.

Any forward-looking equation such as equation (2.1) implies a fundamental degree of freedom, and an infinite number of solutions, because one can find an infinite number of pairs $(y_t, \mathbb{E}_t y_{t+1})$ that satisfy it. To see this, recall that Muth's (1961) seminal RE idea assumes agents form their expectations so that the expected forecast error cannot be systematic or predictable, i.e., $\mathbb{E}_{t-1}(\eta_t) = 0,$ where $\eta_t = y_t - \mathbb{E}_{t-1} y_t$ is the forecast error. The RE requirement, however, is generally not enough to pin down a unique solution, as is evident by rewriting equation (2.1) using conditional expectations $\xi_t = \mathbb{E}_t(y_{t+1})$ as

$$\xi_t = \theta^{-1} \left(\xi_{t-1} - \varepsilon_t + \eta_t \right). \tag{2.2}$$

Any process η_t such that $\mathbb{E}_{t-1}(\eta_t) = 0$ defines a different solution to (2.2).[3] Any

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[3] One could interpret the error of expectations η_t as a martingale difference process, and the requirement of a zero expected error simply implies that the solution is characterised up to an arbitrary martingale.

forecast error of the form

$$\eta_t = b\varepsilon_t + \zeta_t, \tag{2.3}$$

then yields a RE solution, where ζ_t is a mean-zero non-fundamental/sunspot disturbance, uncorrelated with the fundamental one. Equation (2.3) shows that there are two main degrees of freedom in the admissible solutions: the parameter b and the disturbance ζ_t.

The point was evident in Muth's (1961) original formulation that looks for solutions to equation (2.1) expressed as a weighted sum of past, current, and expected future values of the structural shocks (hence, abstracting from sunspot disturbances)

$$y_t = \sum_{j=1}^{\infty} u_j \varepsilon_{t-j} + b\varepsilon_t + \sum_{j=1}^{\infty} c_j \mathbb{E}_t \varepsilon_{t+j}, \tag{2.4}$$

where u_j, b and c_j are coefficients to be determined. Plug equation (2.4) back into (2.1), and use the undetermined coefficient method to derive the set of admissible solutions as

$$y_t = (b-1)\sum_{j=1}^{\infty} \frac{1}{\theta^j}\varepsilon_{t-j} + b\varepsilon_t + b\sum_{j=1}^{\infty}\theta^j \mathbb{E}_t\varepsilon_{t+j} = (b-1)\sum_{j=1}^{\infty}\frac{1}{\theta^j}\varepsilon_{t-j} + b\varepsilon_t, \tag{2.5}$$

given that $\mathbb{E}_t\varepsilon_{t+j} = 0, \forall j > 0$.[4] Equation (2.5) shows that all the infinite solutions of equation (2.1) that are functions only of the history of the structural shocks can be parameterised by a free parameter $b \in (-\infty, +\infty)$. A particular value of b defines a particular solution. Following the terminology used by Blanchard (1979), two important solutions often considered in the literature are: (i) the pure forward-looking solution corresponding, to $b = 1 : y_t^F = \varepsilon_t$; (ii) the pure backward-looking solution, corresponding to $b = 0 : y_t^B = -\sum_{j=1}^{\infty}\theta^{-j}\varepsilon_{t-j} = \theta^{-1}(y_{t-1}^B - \varepsilon_{t-1})$. Moreover, Muth (1961) stresses that b has a natural interpretation: it defines the way agents form their expectations. This is easy to see by writing the implied expected value of y_{t+1} at t as (assuming $b \neq 0$)

$$\mathbb{E}_t y_{t+1} = (b-1)\sum_{i=1}^{\infty}\left(\frac{1}{b\theta}\right)^i y_{t+1-i}, \tag{2.6}$$

which shows that agents combine past values of the observable variable, $\{y_{t-i}\}_{i=0}^{\infty}$,

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[4] Without loss of generality, we assume that expected future shocks are zero. Appendix B.1 contains all the derivations for the equations in this section.

to form their expectation about its future value, $\mathbb{E}_t y_{t+1}$.[5] The weights on past values are determined by b. First, b measures the extent to which past observations matter for expectations in absolute terms. If $b = 1$, for example, past values have no effect; this is the forward-looking solution. Second, b determines the relative weight ($\frac{1}{b\theta}$) placed on the past data when agents form their expectations. Hence, any given value of b pins down a particular way in which agents combine past data to form their expectations, thus leading to one particular RE solution.

In other words, there is a multiplicity of solutions satisfying the rationality condition, meaning that additional conditions must be imposed in order to pin down a unique equilibrium. Blanchard and Kahn (1980a) famously proposed stability (i.e., boundedness) of the solution as such a condition. In the case where $\theta < 1$ and the agent has full information and retains full memory of the past history of the shocks (i.e., the agent knows $h^t = \{\varepsilon_t, \varepsilon_{t-1}, \ldots\}$), the backward-looking component in the solution (2.5) is explosive. Therefore, the stability condition pins down the pure forward-looking solution (corresponding to $b = 1$, $y_t = \varepsilon_t$, $\mathbb{E}_t y_{t+1} = 0$), which is indeed the unique bounded one.

1. Limited Memory. We twist this framework and deviate from the usual RE in one fundamental way. As in Muth's (1961) original formulation, we look for solutions to Equation (2.1) expressed as a weighted sum of past, current and expected future values of the structural shocks, but we assume limited memory. We investigate two different specifications of limited memory: (i) finite memory; (ii) decay memory. In both cases, we assume that the agent does not internalize the memory constraints, that is, the agent at t does not anticipate that at time $t+1$ she will experience some loss of memory of the data known at time t.

(i) Finite Memory. We assume the agent remembers the past structural shocks only up to T periods ago. In other words, a past data point does not influence the agent's expectations after an extended period of time. Under this specification, the expectation can only condition on a subset of structural shocks in the past: $\mathbb{I}_t = \{\varepsilon_t, \varepsilon_{t-1}, \ldots, \varepsilon_{t-T+1}, \varepsilon_{t-T}\}$. Then, denoting the expectations under limited

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[5] One of the purposes of Muth (1961) original paper is to write the expectation at time t as an exponentially weighted average of past observations – as in the adaptive expectations or constant gain learning framework – because he showed in a previous paper, Muth (1960), that, under some assumptions, this is the optimal estimator.

memory as $\bar{\mathbb{E}}$, we guess the solution has the following formulation

$$y_t = \sum_{j=1}^{T} u_j \varepsilon_{t-j} + b\varepsilon_t + \sum_{j=1}^{\infty} c_j \bar{\mathbb{E}}_t \varepsilon_{t+j}. \tag{2.7}$$

Equation (2.7) is similar to equation (2.4) but with an additional memory constraint. Using as before the method of undetermined coefficients to find the parameters u_j, b and c_j consistent with a solution as equation (2.7) yields

$$y_t = (b-1)\sum_{j=1}^{T} \frac{1}{\theta^j}\varepsilon_{t-j} + b\varepsilon_t + b\sum_{j=1}^{\infty} \theta^j \bar{\mathbb{E}}_t \varepsilon_{t+j} = (b-1)\sum_{j=1}^{T} \frac{1}{\theta^j}\varepsilon_{t-j} + b\varepsilon_t, \tag{2.8}$$

this solution mirrors Muth's Equation (2.5), and again admits infinite solutions parameterised by b.

(ii) Decay Memory. We assume the agent progressively loses memory of past structural shocks at a rate λ, that is, historical data's influence on expectations gradually fades over time as it recedes into the past. The solution conditions on the decayed memory of structural shocks in the past: $\mathbb{I}_t = \{\varepsilon_t, \lambda\varepsilon_{t-1}, \lambda^2\varepsilon_{t-2}, \ldots\}$. Following Muth (1961)'s formulation, we guess the solution has the following formulation

$$y_t = \sum_{j=1}^{\infty} u_j \lambda^j \varepsilon_{t-j} + b\varepsilon_t + \sum_{j=1}^{\infty} c_j \bar{\mathbb{E}}_t \varepsilon_{t+j}, \tag{2.9}$$

Using the undetermined coefficients method, the solution has a similar form as equation (2.5) but with a constant decay rate on past shocks.

$$y_t = (b-1)\sum_{j=1}^{\infty} \left(\frac{\lambda}{\theta}\right)^j \varepsilon_{t-j} + b\varepsilon_t + b\sum_{j=1}^{\infty} \theta^j \bar{\mathbb{E}}_t \varepsilon_{t+j} = (b-1)\sum_{j=1}^{\infty} \left(\frac{\lambda}{\theta}\right)^j \varepsilon_{t-j} + b\varepsilon_t, \tag{2.10}$$

Despite these apparently minor differences with respect to the original RE formulation, the assumption of limited memory has a major implication for the stability property of the differential equation (2.1). Under limited memory, the backward-looking solution might not be explosive anymore, even in the case where $\theta < 1$. Hence, any linear combination (i.e., any given b) of the backward- and forward-looking solutions is an admissible solution according to the stability criterion of Blanchard and Kahn (1980a). The original pure backward looking solution, corresponding to $b = 0$, is now equal to $y_t^B = -\sum_{j=1}^{T} \theta^{-j} \varepsilon_{t-j}$ from (2.8) in the finite memory case or to $y_t^B = -\sum_{j=1}^{\infty} \left(\frac{\lambda}{\theta}\right)^j \varepsilon_{t-j}$ from Equation (2.10) in the decay memory case. In the former case, y_t^B is always bounded. In the latter case, where $\lambda > \theta$, y_t^B is explosive so that the stability criterion pins down the unique stable solution, which is the forward-looking one, i.e., $b = 1$. In the case where $\lambda < \theta$, however, the Blanchard and Kahn's (1980a) stability condition can no longer select a unique solution, because solution (2.10) is always bounded for any value of b. It follows that, in the finite memory case or whenever the degree of memory decay is sufficiently high in the decay memory case, the original problem of the multiplicity of solutions remains, even when the RE solution would be saddle point stable, i.e., $\theta < 1$. As a result, b is no longer constrained by the stability condition, and thus can take any value.

In the limited memory case, b has the same interpretation as in RE: it pins down a particular way in which agents combine past data to form their expectations. In the decay memory case, for example, future expectations follow

$$\bar{\mathbb{E}}_t y_{t+1} = (b-1)\sum_{i=1}^{\infty}\left(\frac{1}{\theta b}\right)^i \lambda^{i-1} y_{t+1-i}, \tag{2.11}$$

which carries the same intuitive interpretation as equation (2.6), though now agents form their expectations conditional on a fraction of past observations, i.e., $\mathbb{I}_t = \{y_t, \lambda y_{t-1},\ \lambda^2 y_{t-2}, \ldots\}$. As b is not constrained to be equal to one by the stability condition, past observations matter for expectations even in the case of a pure forward-looking equation such as (2.1). The value of b would determine how agent maps discounted past observations to form their expectations. Given that b is now a free parameter, this raises the following question: how can we pin down a unique solution (i.e., a unique b) from all the admissible bounded solutions?

2. Time-varying Expectation Formation Process. As in the macroeconomic literature on indeterminacy, we appeal to the existence of a sunspot shock to choose one among all these stable paths. Following the approach in Ascari et al. (2019), we assume that b_t is time-varying, so that b_t follows a random walk, $b_t = b_{t-1} + \sigma_b \xi_t$, where $\xi \sim$ i.i.d $N(0,1)$ is the sunspot shock. This assumption is convenient and has a natural interpretation. As discussed above, there is an infinite number of possible ways agents could combine (the memory of) past observations mimicking the RE. All of them are admissible and they are parameterised by b_t. The sunspot shock hence captures the fact that our agent can change over time how to combine past data to form her expectations. Any given value of b_t picks a particular solution. Two brief comments follow. First, as explained by Ascari et al. (2019), this sunspot is a multiplicative sunspot, rather than an additive one. Hence, it generates both a time-varying parameter solution and endogenous stochastic volatility. Second, this setup encompasses the standard RE equilibrium as a special case, corresponding to $b_t = 1, \forall t$.

Therefore, the solution is randomised among different admissible stable equilibria. Under these assumptions, the solution in the finite memory case in the finite memory case with time-varying expectations is

$$y_t = \sum_{j=1}^{T} \left(\frac{1}{\theta}\right)^j (b_t - 1)\varepsilon_{t-j} + b_t \varepsilon_t \tag{2.12}$$

and in the decay memory case is

$$y_t = \sum_{j=1}^{\infty} \left(\frac{\lambda}{\theta}\right)^j (b_t - 1)\varepsilon_{t-j} + b_t \varepsilon_t. \tag{2.13}$$

For detailed derivation see Appendix B.1.

2.2.1 How Price Expectations are Updated

Our approach is different from learning. Our agents observe what a fully informed RE agent would observe, but the RE formation process is perturbed by limited memory. This implies a multiplicity of ways agents can combine the memory of past observables, which is then assumed to be time-varying and pinned down by an exogenous sunspot shock. As such, our agents are not learning.

However, our assumptions do impose a theoretical structure on expectation formation which implies a specific updating rule for expectations. This section derives the updating of expectations in our approach – for brevity, we focus only on the decay memory case – highlighting similarities and differences with respect to the learning approach.

Corresponding to (2.6) or (2.11), the expectation in the decay memory case with time-varying b_t is

$$\bar{\mathbb{E}}_t y_{t+1} = (b_t - 1) \sum_{i=1}^{\infty} \left(\frac{\lambda^{i-1}}{\theta^i \prod_{j=0}^{i-1} b_{t-j}} \right) y_{t+1-i}, \tag{2.14}$$

which can be written recursively as

$$\bar{\mathbb{E}}_t y_{t+1} = \frac{1}{\theta}\left[\frac{\nu_t}{\nu_{t-1}}\lambda\bar{\mathbb{E}}_{t-1}y_t + \nu_t\left(y_t - \lambda\bar{\mathbb{E}}_{t-1}y_t\right)\right], \tag{2.15}$$

where $\nu_t = \frac{b_t - 1}{b_t}$. This expression is similar to the updating implied by constant gain learning, employed by, e.g., Adam et al. (2016) and Nagel and Xu (2021), which is

$$\bar{\mathbb{E}}_t y_{t+1} = \bar{\mathbb{E}}_{t-1}y_t + \nu\left(y_t - \bar{\mathbb{E}}_{t-1}y_t\right), \tag{2.16}$$

where ν is the gain parameter. The gain parameter measures how much agents update their expectation based on new information.

Equations (2.15) and (2.16) highlight three differences between our approach and constant gain learning. The first difference arises from the RE assumption. Notice that the updating rule (2.15) is multiplied by $1/\theta$, this is because under RE agents take the model into account when formulating their forecasts. In other words, the agent knows the objective underlying law of motion. To see this, combining equation (2.2) and equation (2.3) yields the updating rule under RE as [6]

$$\mathbb{E}_t y_{t+1} = \frac{1}{\theta}\left[\mathbb{E}_{t-1}y_t + \nu\left(y_t - \mathbb{E}_{t-1}y_t\right)\right] \quad \text{where} \quad \nu = \frac{b-1}{b}. \tag{2.17}$$

The updating rule under RE is similar to constant gain learning, but the updating is multiplied by $1/\theta$ and the constant gain is given by $\nu = \frac{b-1}{b}$.

Second, the term λ appears in front of the past expectation in equation (2.15) as a result of the decay memory assumption. Agents discount the previous period's expectation by a factor λ, because they partially lose past memory.

Third, the time-varying expectation assumption makes not only the gain parameter to be time-varying, ν_t, but also changes the way past expectation affects current expectation, i.e., the term $\frac{\nu_t}{\nu_{t-1}}$. As clear from (2.14), a change in b_t changes the weights agents use for past data in forming their expectations – thus agents also change the weight of the last observation, i.e., the gain parameter.[7]

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[6] From (2.3) $y_t - \mathbb{E}_{t-1}y_t = \eta_t = b\varepsilon_t$, abstracting for simplicity from the additive sunspot shock ξ_t.

[7] Note that with a constant b and no decay memory, $\lambda = 1$, then (2.15) becomes (2.17). Moreover, when $\theta < 1$ and $b = 1$ we get the usual RE forward-looking solution $\mathbb{E}_t y_{t+1} = 0, \forall t$.

2.3 An Asset Pricing Model with Time-varying Expectations

This section presents an asset-pricing model that embeds the assumptions on time-varying expectation formation and limited memory described above in a structure based on the seminal model of Bansal and Yaron (2004).[8]

The representative agent has Epstein and Zin (1989) recursive preferences, so the utility function satisfies

$$V_t = [(1-\delta)C_t^{1-1/\psi} + \delta(\mathbb{E}_t V_{t+1}^{1-\gamma})^{\frac{1-1/\psi}{1-\gamma}}]^{\frac{1}{1-1/\psi}},$$

where C_t is consumption, γ is the coefficient of risk aversion, ψ is the intertemporal elasticity of substitution (IES), and δ is the discount factor. From the Euler equation, the asset pricing equation for the gross return from any asset i ($R_{i,t+1}$) is

$$\mathbb{E}_t[\delta^\theta G_{c,t+1}^{-\frac{\theta}{\psi}} R_{a,t+1}^{-(1-\theta)} R_{i,t+1}] = 1, \tag{2.18}$$

where $\theta = \frac{1-\gamma}{1-1/\psi}$, $G_{c,t+1} = (C_{t+1}/C_t)$, and $R_{a,t+1}$ is the unobservable return on an asset that delivers aggregate consumption as its dividends each period, or the so-called 'return on the wealth portfolio'. This is the usual asset pricing equation $\mathbb{E}_t[M_{t+1}R_{i,t+1}] = 1$, where $M_{t+1} = \delta^\theta G_{c,t+1}^{-\frac{\theta}{\psi}} R_{a,t+1}^{-(1-\theta)}$ is the stochastic discount factor (SDF) for Epstein and Zin (1989) preferences. In logs[9]

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} g_{c,t+1} + (\theta - 1) r_{a,t+1}. \tag{2.19}$$

Process for consumption and dividends. As in Bansal and Yaron (2004), the consumption growth, $g_{c,t}$, and the dividend growth, $g_{d,t}$, processes contain a small predictable component x_t – the so-called LRR in Bansal and Yaron (2004) – with x_t evolving according to a AR(1) process, so that

$$x_{t+1} = \rho x_t + \varphi_e \sigma e_{t+1}, \tag{2.20}$$

$$g_{c,t+1} = \mu + x_t + \sigma \eta_{t+1}, \tag{2.21}$$

$$g_{d,t+1} = \mu_d + \phi x_t + \varphi_d \sigma u_{t+1}. \tag{2.22}$$

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[8] Appendices B.2.1 and B.3.1 contain all the derivations for the equations in this section.

[9] Lower-case variables indicate logs. So for example, $g_{c,t+1} = \log G_{c,t+1} = \log(C_{t+1}/C_t)$.

The exogenous shocks e_{t+1},u_{t+1} and η_{t+1} are white noise and mutually independent, ρ is the persistence of the expected growth rate process. $\varphi_d > 1$ captures the fact that the evolution of dividend is much more volatile than that of consumption, while ϕ implies that the persistent component x_t induces correlation between consumption and dividend growth. In contrast to Bansal and Yaron's (2004) model, we do not assume stochastic volatility a priori,[10] but it arises endogenously through our assumptions on expectations, as we will show later. Therefore, the stochastic volatility in our model is disciplined by the model's structure.

The solution method involves two steps. First, as Bansal and Yaron (2004) and Albuquerque et al. (2016) among others, we solve the model using the approximation proposed by Campbell and Shiller (1988), which involves linearising the expressions for the returns and exploiting the properties of the log-normal distribution. Second, we assume the relevant state variables for deriving the solution and then apply the undetermined coefficient method to derive the solution.

Solution for the return on the wealth portfolio. We first solve for the log return on the wealth portfolio $r_{a,t+1}$, as it determines the SDF and therefore the market portfolio return $r_{i,t+1}$ given (2.18). Applying the log-linear approximations in Campbell and Shiller (1988), the (approximated) log return on the wealth portfolio can be written as

$$r_{a,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{c,t+1} \tag{2.23}$$

where $z_t = \log(P_{C,t}/C_t)$ is the log of the price-consumption (PC) ratio, as this asset delivers aggregate consumption as its dividends each period, $P_{C,t} =$ is the price of this asset, and the κ's are the following approximation coefficients

$$\kappa_0 = \log(1 + \exp(\bar{z})) - \kappa_1 \bar{z}; \qquad \kappa_1 = \frac{\exp(\bar{z})}{1 + \exp(\bar{z})}. \tag{2.24}$$

To find the solution for z_t, we use the method of undetermined coefficients, as described in Section 2.2. We guess that the solution for z is a linear function of the past, present, and expected future values of the endogenous state variable x, but subject to memory constraints and a time-varying parameter, indicated by b_t.

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[10] In Bansal and Yaron's (2004) model, the variance follows an exogenous AR(1) process.

In the case of finite memory

$$z_t = A_{0,t} + \left(1 - \frac{1}{\psi}\right)\left[\sum_{j=1}^{T} u_{j,t} x_{t-j} + b_t x_t + \sum_{j=1}^{\infty} c_{j,t}\mathbb{E}_t x_{t+j}\right], \tag{2.25}$$

while in the case of decay memory

$$z_t = A_{0,t} + \left(1 - \frac{1}{\psi}\right)\left[\sum_{j=1}^{\infty} u_{j,t}\lambda^j x_{t-j} + b_t x_t + \sum_{j=1}^{\infty} c_{j,t}\mathbb{E}_t x_{t+j}\right]. \tag{2.26}$$

The time-varying expectation parameter follows the random walk process $b_t = b_{t-1} + \sigma_b \xi_t$, where $\xi_t \sim$ i.i.d $N(0,1)$ and is assumed to be uncorrelated with all other fundamental shocks in the model. The parameters $A_{0,t}$, $u_{j,t}$ and $c_{j,t}$ are coefficients to be determined. Substituting equation (2.25) and equation (2.26) into the Euler equation (2.18) yields the solution for z_t under finite memory

$$z_t = A_{0,t} + \left(1 - \frac{1}{\psi}\right)\left[\sum_{j=1}^{T}\left(\frac{1}{\kappa_1}\right)^j (b_t - 1)x_{t-j} + b_t x_t + b_t\sum_{j=1}^{\infty}(\kappa_1\rho)^j x_t\right], \tag{2.27}$$

and under decay memory

$$z_t = A_{0,t} + \left(1 - \frac{1}{\psi}\right)\left[\sum_{j=1}^{\infty}\left(\frac{\lambda}{\kappa_1}\right)^j (b_t - 1)x_{t-j} + \frac{b_t}{1-\kappa_1\rho} x_t\right]. \tag{2.28}$$

Given the solution for z_t, equation (2.23) yields the log wealth return $r_{a,t+1}$, which in turn yields the log of the SDF from (2.19).

To avoid repetition, we present only the decay memory case in what follows. Similar expressions hold for the finite memory case, see Appendix B.3.1 and Appendix B.3.2 for this and Section 2.4, respectively.

Solution for the market return. The same procedure solves for the market return, which has an analogous expression to equation (2.23), that is

$$r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m} z_{m,t+1} - z_{m,t} + g_{d,t+1}, \tag{2.29}$$

where κ_m's are coefficients as in equation (2.24) consistent with the average PD ratio $\bar{z}_m$. Similarly to equation (2.26), the guess for $z_{m,t}$ is

$$z_{m,t} = A_{0,m,t} + (\phi - \frac{1}{\psi})\left[\sum_{j=1}^{\infty} u_{j,t}\lambda^j x_{t-j} + b_t x_t + \sum_{j=1}^{\infty} c_{j,t}\mathbb{E}_t x_{t+j}\right], \tag{2.30}$$

where the parameters $A_{0,m,t}$, $u_{j,t}$, and $c_{j,t}$ are coefficients to be determined by substituting the conjectured equation (2.30) into the Euler equation (2.18). The solution for $z_{m,t}$ is

$$z_{m,t} = A_{0,m,t} + (\phi - \frac{1}{\psi})\left[\sum_{j=1}^{\infty}\left(\frac{\lambda}{\kappa_{1,m}}\right)^j (b_t - 1)x_{t-j} + \frac{b_t}{1-\kappa_{1,m}\rho}x_t\right]. \tag{2.31}$$

As the expectation parameter b_t enters into the solution in a multiplicative way, the PD ratios (2.31) are constant in the absence of fundamental shocks (assume $x_{t-j} = 0, \forall j \in [0, \infty)$). Appendix B.2.1 provides the expressions for $A_{0,t}$, $A_{0,m,t}$ and the solution for the risk-free rate $r_{f,t+1}$.

From equation (2.28) and equation (2.31), we can see that, for $\lambda < \kappa_1$ and $\lambda < \kappa_{1,m}$, both z_t and $z_{m,t}$ are bounded for any given bounded level of b_t. Hence, as explained in the previous section, the decay memory assumption implies a multiplicity of bounded solutions, and a given value of b_t pins down a particular solution among the infinite admissible ones.[11]

2.4 Analytical Results

This section derives analytical results that provide the intuition for why the asset-pricing model with time-varying expectation formation process and limited memory can potentially explain many asset pricing puzzles. In particular, subsection 2.4.1 shows that the PD ratio can persistently deviate from its fundamental values, and these deviations are often associated with high price volatility. Subsection 2.4.2 shows that the model could generate a higher and time-varying equity premium as a result of the expectation formation process. Again, we focus mainly on the decay memory case to avoid repetition, but similar arguments apply to the finite memory case.[12]

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[11] The same applies to the finite memory case because the backward-looking summations are finite.

[12] See Appendix B.2.2 for the derivations in this section and Appendix B.3.2 for the corresponding derivations in the finite memory case.

2.4.1 Price-Dividend Ratio

To highlight the difference between our model solution for the PD ratio and the standard RE solution, we rewrite (2.31) as a combination of usual RE solution and a bounded backward-looking component[13]

$$z_{m,t} = \underbrace{b_t \left(A_{0,m} + \frac{\phi - \frac{1}{\psi}}{1 - \kappa_{1,m}\rho} x_t \right)}_{\text{fundamental eq., } z_{m,t}^{RE}} + (1 - b_t) \underbrace{\left(A_{0,m} - \sum_{j=1}^{\infty} \left(\frac{\lambda}{\kappa_{1,m}} \right)^j \left(\phi - \frac{1}{\psi} \right) x_{t-j} \right)}_{\text{bounded backward-looking eq.}} \tag{2.32}$$

The solution encompasses the usual RE result as a special case, corresponding to $b_t = 1$

$$z_{m,t}^{RE} = A_{0,m} + \frac{\phi - \frac{1}{\psi}}{1 - \kappa_1 \rho} x_t. \tag{2.33}$$

From equation (2.33), it is evident that, under the standard RE, the volatility of the PD ratio is roughly equal to the volatility of dividend growth, inherited from the long-run risk process. As the dividend process is very stable, the fundamental RE solution in standard asset-pricing models has difficulty matching the high price volatility observed in the data. Moreover, from equation (2.33), the volatility of the PD ratio is constant without additional assumptions on x_t, which is inconsistent with the stochastic price volatility observed in the data.

When $b_t \neq 1$, the PD ratios deviate from the usual RE values and we can distinguish the 'fundamental regime' and the 'trend-following regime', following the Boswijk et al.'s (2007) terminology. As the value of the expectation parameter b_t varies, the model yields different ways in which agent maps past observations to form their expectations, and thus different equilibrium prices. When the value of $b \to 1$, the model solution is close to the usual RE solution and asset prices are close to the fundamental values, so we are near the 'fundamental regime'. When b differs substantially from 1, the model enters the 'trend-following regime', where persistent under- and over-valuations of asset prices appear, even though economic fundamentals remain stable. To see this, define $\hat{z}_{m,t}$ as the deviation from the usual RE solution, i.e., $\hat{z}_{m,t} = z_{m,t} - z_{m,t}^{RE}$, then, for

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[13] The time-varying components in the A_0 and $A_{0,m}$ were abbreviated when deriving the analytical solutions of this Section as they do not affect the main argument, but their effects were obviously considered when doing the quantitative analysis.

$b_t \neq 1$,[14]

$$\hat{z}_{m,t+1} = \frac{\lambda}{\kappa_{1,m}} \frac{b_{t+1}-1}{b_t-1} \hat{z}_{m,t} + (b_{t+1}-1)\left(\phi - \frac{1}{\psi}\right) \frac{1}{1-\kappa_{1,m}\rho} \varphi_e \sigma e_{t+1} + \underbrace{(b_{t+1}-1)\left(\phi - \frac{1}{\psi}\right) \frac{1}{1-\kappa_{1,m}\rho} (1-\lambda)\rho x_t}_{\text{Due to memory loss, approach to zero as } \lambda \to 1} . \tag{2.34}$$

This implies that $\hat{z}_{m,t+1}$ positively depends on the deviation in the last period when b_t and b_{t+1} are on the same side relative to 1. Given that b_t is a very persistent process, it follows that persistent under- and over-valuations of asset prices arise. Moreover, even if expectations about stock prices are very high at a given point in time, the PD ratio has the tendency to return to fundamentals in the absence of fundamental shocks as $\lambda/\kappa_{1,m} < 1$. In other words, the expectation formation process affects the response to the fundamental shocks, through b_t, and its persistence induces "momentum" in stock prices, while the decay memory assumption, through λ, entails "mean-reversion" over long horizons to stable fundamentals. Together, the model is able to explain periodic booms and busts in the financial market.

Equation (2.34) also shows that our model provides a micro-structure for stochastic volatility, which arises from the time-variation in b_t. Stochastic volatility is a desirable feature in the asset pricing literature. However, in many models, such stochastic volatility is disentangled from the fundamentals. For example, Bansal and Yaron (2004) model stochastic volatility by adding fluctuations in economic uncertainty, which are completely free in the sense that they follow an independent process unrelated to any of the fundamentals. In contrast, in our model, stochastic volatility arises as a by-product of time-variation in b_t, and is thus tied to the particular assumptions about the expectation formation process. This implies certain restrictions on the stochastic volatility due to the structure of the assumed expectation process – one expectation formation parameter b_t governs both the level and volatility of the PD ratio at once.

Formally, the conditional variance of $z_{m,t+1}$ is equal to (here we assume $\rho = 0$

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[14] In the finite memory case (2.34) is given by

$$\hat{z}_{m,t+1} = \frac{(b_{t+1}-1)}{\kappa_{1,m}(b_t-1)} \hat{z}_{m,t} + (b_{t+1}-1)\left(\phi - \frac{1}{\psi}\right)\left(\frac{1}{1-\kappa_{1,m}\rho}\varphi_e \sigma e_{t+1} - \frac{1}{\kappa_{1,m}^T} x_{t-T}\right).$$

for simplicity, the full derivation for $\rho \neq 0$ can be found in the Appendix B.2.1)

$$\mathbb{V}ar_t(z_{m,t+1}) = \left(\frac{z_{m,t} - z_{m,t}^{RE}}{b_t - 1}\right)^2 \sigma_b^2 + \left(\phi - \frac{1}{\psi}\right)^2 (b_t \varphi_e \sigma)^2, \qquad \text{for } b_t \neq 1, \quad (2.35\text{a})$$

$$\mathbb{V}ar_t(z_{m,t+1}) = \left(\phi - \frac{1}{\psi}\right)^2 \varphi_e^2 \sigma^2, \qquad \text{for } b_t = 1. \quad (2.35\text{b})$$

Our model implies that the time-varying expectations affect not only the level of the conditional variance of the PD ratio through σ_b^2 as a new source of risk (captured by the first term in equation (2.35a)), but also induce time-variation in how the variance of structural shocks feed into PD volatility, because the coefficients that multiply σ^2 depend on b_t (captured by the second term in equation (2.35a)). This follows from equation (2.32). Think equation (2.32) as expressing $z_{m,t}$ as a function of x_t; thus, a change in b_t affects both intercept of the PD ratio (the second term), and the slope of the PD ratio (the response of PD ratio to x_t).

Moreover, from equation (2.35), the volatility of the PD ratio increases when the PD ratio deviates further from the fundamental solution; in other words, the price volatility increases in bubbly markets, a phenomenon often observed in the data.

2.4.2 Equity Premium

In our framework, the model can generate a higher and time-varying equity premium, as there are now two sources of systemic risk: the first relates to fluctuations in expected consumption growth; the second relates to fluctuations in the expectation formation process.

The innovation to pricing kernel, i.e., the SDF is given by (derivation see Appendix B.2.2)

$$m_{t+1} - \mathbb{E}_t m_{t+1} = -\lambda_{m,\eta}\sigma\eta_{t+1} - \lambda_{m,e,t+1}\sigma e_{t+1} - \lambda_{m,\xi,t+1}\sigma_b\xi_{t+1}, \qquad (2.36)$$

where $\lambda_{m,\eta}$, $\lambda_{m,e,t+1}$ and $\lambda_{m,\xi,t+1}$ capture the pricing kernel's exposure to the consumption growth shocks, η_{t+1}, to the LRR shocks, e_{t+1}, and to the time-varying expectation shocks, ξ_{t+1}, respectively, and are given by

$$\lambda_{m,\eta} = -\left(-\frac{\theta}{\psi} + \theta - 1\right) = \gamma, \tag{2.37a}$$

$$\lambda_{m,e,t+1} = (1-\theta)\kappa_1 \left(1 - \frac{1}{\psi}\right) b_{t+1} \frac{1}{1-\kappa_1\rho}\varphi_e, \tag{2.37b}$$

$$\lambda_{m,\xi,t+1} = (1-\theta)\kappa_1 \left(1 - \frac{1}{\psi}\right) \left[\sum_{j=1}^{\infty} (\frac{\lambda}{\kappa_1})^j x_{t+1-j} + \frac{1}{1-\kappa_1\rho}\rho x_t\right]. \tag{2.37c}$$

The pricing kernel's exposure to LRR $\lambda_{m,e,t+1}$ rises with the persistence parameter ρ, or as agents become more forward-looking (higher b_{t+1}). The pricing kernel's exposure to the expectation risk $\lambda_{m,\xi,t+1}$ rises with the difference between the expected pure forward-looking solution and the pure backward-looking solution, as can be shown by rewriting Equation (2.37c) as

$$\lambda_{m,\xi,t+1} = (1-\theta)\kappa_1(1 - \frac{1}{\psi})\left(\underbrace{\mathbb{E}_t\left(z_{t+1}|b_{t+1}=1\right)}_{\mathbb{E}_t z^f_{t+1}} - \underbrace{\mathbb{E}_t\left(z_{t+1}|b_{t+1}=0\right)}_{\mathbb{E}_t z^b_{t+1}}\right). \tag{2.38}$$

It is instructive to compare this with the standard RE case (i.e., $b_{t+1} = 1$ and $\sigma_b = 0$), where equation (2.36) corresponds to the Bansal and Yaron's (2004) model without stochastic volatility (see equation (6) on p. 1486 therein). The Bansal and Yaron's (2004) model with exogenously assumed stochastic volatility also delivers an expression with three terms, but in their case, the last term arises from their assumption on stochastic volatility (see equation (10) on p. 1487), rather than the shock to expectation formation as in our model.

Similarly, the innovation in the market return is given by

$$r_{m,t+1} - \mathbb{E}_t r_{m,t+1} = \beta_{m,u}\sigma u_{t+1} + \beta_{m,e,t+1}\sigma e_{t+1} + \beta_{m,\xi,t+1}\sigma_b \xi_{t+1}, \tag{2.39}$$

where $\beta_{m,u}$, $\beta_{m,e,t+1}$ and $\beta_{m,\xi,t+1}$ are time-dependent convolutions, as given in Appendix B.2.2. The conditional equity premium for the market portfolio $r_{m,t+1}$ is equal to $\mathbb{E}_t(r_{m,t+1} - r_{f,t}) = -Cov(m_{t+1} - \mathbb{E}_t m_{t+1}, r_{m,t+1} - \mathbb{E}_t r_{m,t+1}) - 0.5\mathbb{V}ar_t(r_{m,t+1})$, which yields

$$\mathbb{E}_t(r_{m,t+1} - r_{f,t}) = \vartheta_{e,t}\sigma^2 + \vartheta_{\xi,t}\sigma_b^2 - 0.5\mathbb{V}ar_t(r_{m,t+1}), \tag{2.40}$$

with

$$\vartheta_{e,t} = (1-\theta)\left(1-\frac{1}{\psi}\right)\frac{\kappa_1 b_t}{1-\kappa_1\rho}\left(\phi-\frac{1}{\psi}\right)\frac{\kappa_{1,m} b_t}{1-\kappa_{m,1}\rho}\varphi_e^2\sigma^2, \tag{2.41a}$$

$$\vartheta_{\xi,t} = \frac{\vartheta_{e,t}}{b_t^2\varphi_e^2\sigma^2}\left[(1-\kappa_1\rho)\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_1})^j x_{t-j}+x_t\right]\left[(1-\kappa_{m,1})\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_{1,m}})^j x_{t-j}+x_t\right] \tag{2.41b}$$

$$\mathbb{V}ar_t(r_{m,t+1}) = (\beta_{m,u}+\beta_{m,e,t+1})^2\sigma^2+\beta_{m,\xi,t+1}^2\sigma_b^2 \tag{2.41c}$$

The equity premium is determined by two sources of risk: the fluctuations in consumption growth and the fluctuations in expectations. The second term in equation (2.40) shows that the equity premium must compensate for the risk due to time-varying expectation formation process (σ_b^2). Moreover, $\vartheta_{e,t}$ illustrates that the realisation of b_t also determines how the equity premium compensates for consumption growth volatility (σ^2). Therefore, as with the PD ratio, the expectation formation process affects both the level of the equity premium – i.e., the intercept of the equation for the equity premium (2.40) – and its slope – i.e., how it reacts to consumption growth volatility.

2.5 Data and Methodology

We estimate our asset-pricing model with a time-varying expectation formation process using the simulated method of moments (SMM). This section first describes the data sources (subsection 2.5.1) and then the methodology employed (subsection 2.5.2). Subsection 2.5.3 formally examines which moments should be included in the SMM, as including all moments of interest may violate certain regularity conditions.

2.5.1 Data Sources

The seasonally adjusted per-capita annual consumption growth used in this paper is from Barro and Ursúa (2012) and was extended by the authors to 2019 using data from the Bureau of Economic Analysis (BEA) (National Income and Product Accounts: Table 7.1 Selected Per Capita Product and Income Series). The real S&P 500 return, PD ratio and dividend growth are from Robert Shiller's website. The data on the annualised nominal return to one-month Treasury bills, deflated by the CPI, are taken from Robert Shiller's website. We use the data from 1929 to 2018 which includes several crisis episodes in financial markets.

2.5.2 Simulated Method of Moments

This paper applies the simulated method of moments (SMM) estimation to evaluate the ability of the model to match salient features of data. The SMM approach aims to find model parameter values that make model simulated moments match the data moments as closely as possible.

In this application, the moments of interest include: the mean, standard deviation and the first-order autocorrelation of consumption growth; the mean, standard deviation and the first-order autocorrelation of dividend growth; the mean, standard deviation and the first-order autocorrelation of PD ratio; the mean and standard deviation of real stock return; the contemporaneous correlation between consumption growth and dividend growth; the contemporaneous correlation between stock return and consumption growth; the correlation between stock returns and one-period lagged consumption growth; the mean and standard deviation of the risk-free rate. Moreover, we include also excess return predictability, that is, the coefficient c^2 and the R^2 in the following regression:

$$r_{s,t+n} - r_{f,t} = c_n^1 + c_n^2 \log(PD_t) + u_{t,n} \tag{2.42}$$

where the dependent variable ($r_{s,t+n} - r_{f,t}$) is the observed real excess return of stocks over bonds from t to $t+n$ years (here we consider a five-year horizon, $n=5$), and $u_{n,t}$ is the regression residual.

There are 11 parameters we are trying to estimate, namely: the coefficient of relative risk aversion γ; the elasticity of intertemporal substitution ψ; the rate of time preference δ; the drift in the log consumption and in the dividend growth processes μ; the persistence of the expected growth rate process ρ; the volatility of innovation σ; the volatility of the persistent component of the growth process (i.e., the LRR) φ_e; the elasticity of dividend growth to the persistent component of the growth process ϕ; the volatility of dividend growth process φ_d; the volatility of the innovations in the expectation formation process σ_b; the decay rate of memory λ (or the limit period of memory T in the case of finite memory). We summarise these parameters in the vector $\theta = \{\gamma, \psi, \delta, \mu, \rho, \sigma, \varphi_e, \phi, \varphi_d, \sigma_b, \lambda \text{ (or } T)\}$.

Formally, the SMM entails the following. Let $(y_1, ..., y_N)$ denote the observed data sample with size N. The sample moments are defined as $\hat{M}_N \equiv \frac{1}{N}\sum_{t=1}^{N} h(y_t)$ for a given moment function h. Some of the statistics of interest we considered here are functions of moments so that $\hat{S}_N \equiv S(\hat{M}_N)$. We base our SMM estimates and tests on matching the statistics $\hat{S}_N$. Let $\hat{S}_N \in R^s$ denote a vector of statistics that will be matched in the estimation given the N observations in the data. $\tilde{S}(\theta)$ is a vector of moments implied by the model for some parameter value θ. The SMM parameter estimate θ_N is formally defined as

$$\hat{\theta}_N \equiv \arg\min_{\theta} [\hat{S}_N - \tilde{S}(\theta)]' \hat{\Sigma}_{S,N}^{-1} [\hat{S}_N - \tilde{S}(\theta)]. \tag{2.43}$$

The SMM estimates $\hat{\theta}_N$ then selects the model parameter values such that the model moments $\tilde{S}(\theta)$ fit the observed moments $\hat{S}_N$ as closely as possible in terms of a quadratic form with weighting matrix $\hat{\Sigma}_{S,N}^{-1}$. In our paper, we create 1,000 synthetic time series using the Monte Carlo procedure, each length equal to our sample size.[15] In all synthetic time series, we initialise the value of b_t to be 1 – corresponding to the fundamental solution (usual RE solution). $\tilde{S}(\theta)$ that enters the criterion function is the mean value of the sample moments across the synthetic time series for a given parameter vector of $\theta \in \Theta$.[16] Let ν^i denote a realisation of shocks drawn randomly from their known distributions, and let $(y_1(\theta, \nu^i), \ldots, y_N(\theta, \nu^i))$ denote the random variables corresponding to a history of length N generated by the model for shock realisation ν^i and parameter value θ. Then, the model statistics $\tilde{S}(\theta)$ are computed as

$$\tilde{S}(\theta) \equiv \frac{1}{K} \sum_{i=1}^{K} S(M_N(\theta, \nu^i)) = \frac{1}{K} \sum_{i=1}^{K} S\left(N \sum_{t=1}^{N} h(y_t(\theta, \nu^i)) \right) \tag{2.44}$$

where we use $K = 1,000$. In other words, $\tilde{S}(\theta)$ is the average across a large number of simulations of length N of the statistics $S(M_N(\theta, \nu^i))$ implied by each simulation.

The weighting matrix in our estimation, $\hat{\Sigma}_{S,N}^{-1}$, is the inverse of the variance-covariance matrix of the empirical moment conditions $\hat{S}_N$, as required for efficient SMM estimation. The weighting matrix is estimated using the Newey-West estimator – which is heteroscedasticity and autocorrelation consistent – with a

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[15] We assume the agent makes decisions on a monthly basis and we compute model moments at an annual frequency.

[16] An unconstrained minimisation of the objective function over the parameter space Θ can be numerically unstable and computationally costly. Therefore, additional restrictions have been imposed on Θ. These restrictions can only deteriorate the model's empirical performance, so that the goodness of fit results presented in the next section represent a lower bound on what the model can achieve.

lag of 3.[17]

The SMM approach also allows a formal econometric test to evaluate individual and overall goodness of fit based on asymptotic distribution. Under the null hypothesis of the test that the model is correct, we have

$$\hat{W}_N \equiv N[\hat{S}_N - \tilde{S}(\hat{\theta}_N)]'\hat{\Sigma}_{S,N}^{-1}[\hat{S}_N - \tilde{S}(\hat{\theta}_N)] \rightarrow \chi^2(s-n) \text{ as } N \rightarrow \infty, \tag{2.45}$$

where convergence is in distribution. In addition, t-statistics can be conducted based on the asymptotic distribution for each element of the deviations $\hat{S}_N - \tilde{S}(\hat{\theta}_N)$ to evaluate how close each individual moment is to the corresponding data moment.

2.5.3 Which Moments to Match?

The validity of the SMM requires certain regularity conditions, as documented by Adda and Cooper (2003) and Davidson and MacKinnon (2004). As some components of the moment functions listed above are not sample moments but non-linear functions of sample moments, this paper is concerned with violating one of the regularity conditions of standard SMM, that is, the non-singularity of the covariance matrix of the moment vector. The violation of the singularity requirement would result in the test $\hat{W}_N$ varying greatly with small model changes or testing procedures. Moreover, if the non-singularity condition is violated, the maximisation algorithm would become nearly unstable, as evident from equation (2.45): the formula for $\hat{W}_N$ nearly divides zero by zero, the objective function is nearly undefined, and the asymptotic distribution may not be a good approximation to the true distribution of the test statistic. Therefore, to ensure that $\hat{\Sigma}_{S,N}$ is invertible, we need to carefully select statistics so that they do not give rise to multicollinearity. To decide which statistics to use, following Adam et al. (2016) and Adam et al. (2017), we exclude some moments from the estimation that are nearly redundant. The idea is to compute the variability of each statistic that cannot be explained by a linear combination of the remaining statistics, similarly to the R^2 coefficient of a regression of each statistic on all the other statistics. The regression coefficients and the ensuing R^2 are computed from $\hat{\Sigma}_{S,N}$ in a standard manner. We exclude those statistics with $R^2 < 0.04$ as they are nearly redundant. Specifically, they are the R^2 of the excess return predictability regression (2.42)

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[17] We follow the common practice of specifying the lag as the smallest integer greater than or equal to $(T^{1/4})$.

Table 2.1: Parameter estimates

Parameter	Finite memory	Decay memory
γ	3.6721	3.9015
	(0.0026)	(0.1551)
δ	0.9965	0.9961
	(0.0003)	(0.0003)
ψ	1.1150	1.1148
	(0.001)	(0.0085)
ρ	0.9941	0.9915
	(0.0013)	(0.0020)
φ_e	0.1097	0.0788
	(0.0012)	(0.0121)
σ	0.003	0.004
	(0.0003)	(0.0002)
μ	0.0017	0.0016
	(0.0001)	(0.0001)
ϕ	2.5317	2.5344
	(0.1212)	(0.1851)
φ_d	9.7109	6.2188
	(1.4423)	(0.5413)
σ_b	0.0123	0.0245
	(0.0014)	(0.0051)
	$T = 6.8750$	$\lambda = 0.9419$
	(2.0583)	(0.0016)

Notes: The table presents estimated parameter values for the finite memory and decay memory models (with standard errors reported in parentheses). The model simulates on a monthly basis and appropriately compounds to the annual frequency. The Simulated Method of Moment is used to obtain the estimates.

and the first-order autocorrelation of consumption growth, $\rho_{\Delta c/c,-1}$ (for which $R^2 = 0.0283$ and $R^2 = 0.0140$, respectively). After we drop these two statistics, the weighting matrix becomes non-singular. Moreover, even though we drop these two statistics, the model is able to match them.[18]

2.6 Quantitative Results

This section presents the results for both the finite memory model and the decay memory model. Table 2.1 reports our parameter estimates along with standard errors. The estimated parameters are very close between the two different specifications of our models. First, the coefficient of risk aversion, γ, is estimated to be 3.67 for the finite memory model and 3.9 for the decay memory model. These val-

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[18] This applies to both the decay memory and finite memory cases.

ues are smaller than the value of 10 used in Bansal and Yaron (2004) to generate the observed equity premium but larger than the value of 1.5 estimated by Albuquerque et al. (2016) in their benchmark model. Second, the IES, ψ, is precisely estimated to be 1.115 in both models. This value is larger than one, and close to the value of 1.5 calibrated by Bansal and Yaron (2004) and the estimate of 1.46 reported by Albuquerque et al. (2016). Third, the parameter ρ, which governs the persistence of the LRR that affects both consumption and dividend growth, is very high at 0.99, but consistent with the value of 0.979 used by Bansal and Yaron (2004). Fourth, the memory limit is similar between the two specifications. The finite memory is estimated to be 6.87 years (equivalent to 83 months) and the memory decay parameter is 0.94 per month, implying that the memory weight after 83 months is around 0.006. Finally, the parameter ϕ that measures the effect of the LRR on dividend growth is close to the calibrated value of 3 in Bansal and Yaron (2004). The main difference between the finite and decay specifications is the estimated variance of the sunspot shock, which is larger in the latter case.

The estimated model closely matches the data moments, and the model's performance is robust across the two specifications for limited memory. Table 2.2 reports the data moments (with standard errors reported in parentheses) and the model moments (with t-statistics for each moment reported in parentheses) under the two specifications. Taking sampling uncertainty into account, the model moments are all statistically indistinguishable from the US data moments at the 5% significance level.[19] In particular, both models closely match the observed main asset pricing moments: level and volatility of the stock returns, the PD ratio, the risk-free rate, and the dividend yield, without yielding unrealistically strong correlations between stock returns and measurable fundamentals. The model also succeeds in replicating the return predictability and the autocorrelation of the consumption process, even though these two statistics are not included in the set of moments to be matched. The p-value of the Wald test, a measure of the model's overall performance, indicates that both models cannot be rejected at 5% significance level, even though the finite memory model only marginally so.

The model matches quantitatively many asset pricing features, particularly for the decay memory case. This is notably impressive given the result in Section 2.6.5, where we estimate the standard LRR model by Bansal and Yaron

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[19] The t-statistics based on formal asymptotic distribution are all at or below two in absolute value, with the only exceptions being the mean of the PD ratio and the standard deviation of bond return for the finite memory model.

Table 2.2: Quantitative model performance

	U.S. data	Finite memory	Decay memory
Mean stock return E_{r^s}	7.79	6.00	6.27
	(1.83)	(0.99)	(0.83)
Mean bond return E_{r^b}	0.45	1.05	1.05
	(0.49)	(-1.20)	(-1.23)
Mean PD ratio E_{PD}	32.05	35.84	34.80
	(1.43)	(-2.47)	(-1.91)
Mean dividend growth $E_{\triangle D/D}$	1.74	2.77	2.56
	(1.12)	(-0.97)	(-0.73)
Std. dev. stock return σ_{r^s}	18.71	19.05	19.26
	(0.94)	(-0.36)	(-0.56)
Std. dev. PD ratio σ_{PD}	16.40	17.91	17.68
	(2.05)	(-0.69)	(-0.62)
Std. dev. dividend growth $\sigma_{\triangle D/D}$	10.67	11.29	11.08
	(1.60)	(-0.40)	(-0.25)
Std. dev. bond return σ_{r^b}	3.91	2.70	3.28
	(0.43)	(2.48)	(1.46)
Autocorr. PD ratio $\rho_{PD,-1}$	0.90	0.85	0.80
	(0.12)	(0.41)	(0.74)
Mean consumption growth $E_{\triangle C/C}$	2.01	2.15	2.00
	(0.32)	(-0.46)	(0.03)
Std. dev. consumption growth $\sigma_{\triangle C/C}$	2.96	2.79	2.93
	(0.32)	(0.56)	(0.08)
Autocorr. consumption growth $\rho_{\triangle C/C,-1}$	0.61	0.80	0.62
	(0.12)	(-0.07)	(-0.00)
Autocorr. dividend growth $\rho_{\triangle D/D,-1}$	0.24	0.31	0.79
	(0.37)	(-0.09)	(-0.08)
Corr. $corr_{\triangle C/C,\triangle D/D}$	0.47	0.55	0.50
	(0.13)	(-0.58)	(-0.25)
Predictability β_{PD}	-0.0110	-0.0098	-0.0090
	(0.003)	(-0.38)	(-0.70)
Predictability R^2	0.1327	0.0804	0.0756
	(0.086)	(0.60)	(0.66)
Contemporaneous corr. between	0.03	0.07	0.24
stock return and consumption growth	(0.11)	(-0.31)	(-1.88)
Corr. between stock return and	-0.13	0.52	0.10
one-period lag consumption growth	(0.27)	(-0.53)	(-0.45)
Test statistic $\hat{W}_N$		10.5926	7.7713
p-value of $\hat{W}_N$		6.01%	16.93%

Note: The table compares the asset-pricing moments from the US data (column 2, with standard errors reported in parentheses) with the one implied by the finite memory and decay memory models (column 3 and 4, respectively, the t-ratios for each moment are reported in parentheses). The t-ratios are calculated as (data moment - model moment)/(estimated standard deviation of the model moment). The measure of the overall goodness of fit $\hat{W}_N$, defined as (2.45), and the corresponding p-value are reported in the last two rows of the table.

(2004), with exogenous stochastic volatility. The latter model is overall rejected by the data, as the p-value of the Wald test is zero (see Table 2.6). The process for stochastic volatility in the Bansal and Yaron (2004) model follows an independent process, which is free and unrelated to any of the fundamentals. Our benchmark model is the same, but, in contrast, stochastic volatility is constrained because it arises endogenously. Our modelling assumptions regarding the perturbation of the standard RE Muth's (1961) type of solution imposes certain restrictions on the way stochastic volatility can affect the model variables. The superior quantitative performance of our benchmark model provides an implicit empirical corroboration of this theoretical microstructure implied by the particular assumptions about the expectation formation process. As we argue below, our assumptions enable the model to match all these asset pricing facts, because the expectation shock amplifies the fundamental uncertainty (Section 2.6.1) and generates persistent disconnection of asset prices from the dynamics of the fundamentals (Sections 2.6.2 and 2.6.3).

Moreover, the model generates empirically plausible subjective expectations when compared to survey evidence on expectations, as it matches statistically the evidence about the expected excess market return being positively correlated with lagged price-dividend ratios (Section 2.6.4). This result, together with the evidence of time-variation in this correlation, therefore, provides also an explicit empirical corroboration of our mechanism.

2.6.1 The Equity Premium

Both the finite and decay memory models are capable of producing a large equity premium (around 5%) with a relatively moderate estimated degree of risk aversion ($\gamma = 3.67$ or 3.90). The intuition behind this is that the model requires additional compensation for expectation risks on top of fundamental risks. This compensation for expectation risks increases the equity premium by 40% in the decay memory case – and only by 10% in the finite memory case – relative to a model estimated by setting the variance of the time preference shock to zero and $b_t = 1$ for $\forall t$. Note that the model requires fundamental shocks, because the expectation shock simply amplifies the risks connected to fundamentals, but would not create *per se* an equity premium.

Notably, in our model, the compensation for expectation risks increases as the gap between the expected forward-looking solution and the expected backward-looking solution widens. As the gap increases, the change in the relative weights (the expectation shock) on those two solutions can induce large price fluctuations, which increases the agent's desire to hedge the risk. Moreover, the estimated values of risk aversion and of the IES imply $\theta < 1$, so that $\gamma > 1/\phi$ which Epstein et al. (2014) show to be the condition for a preference for early resolution of uncertainty. Hence, as in standard LRR models, LRRs are penalised more heavily than current risks in our model, because they are resolved in the distant future. In the CRRA case ($\theta = 1$), the equity premium is not affected by the two sources of risks, both the consumption and expectation risks, i.e., $\vartheta_{e,t}, \vartheta_{\xi,t} = 0$ in equation (2.40).

From an analytical point of view, equations (2.28) and (2.31) help to explain why the model embodies a compensation for expectation risk and that compensation is particularly pronounced for stocks relative to bonds. Equations (2.28) and (2.31) show the solution for the price-consumption ratio and the price-dividend ratio, respectively. First, from Table 2.2, the parameter ϕ is larger than one, which implies that dividends are more sensitive both to changes in the LRR, $x's$, and to changes in the expectation formation process, b_t, (i.e., see (2.28) and (2.31)). Second, under our parameter estimates, $\kappa_1 < \kappa_{1,m} \approx 1$, which implies that the price-dividend ratio is more sensitive to changes in expected future price-dividend ratio, and thus any changes in the expectation formation process would have a leveraged impact on the current price-dividend ratio.

Our expectation risk shares a certain similarity with the 'valuation risk' in Albuquerque et al. (2016) in that both risks are due to stochastic changes in agents' valuation of assets in the absence of changes in fundamentals. Albuquerque et al. (2016) introduce a time preference shock that changes agents' relative valuation of present consumption against future consumption. A shock that increases agents' valuation of the present relative to the future would drive down the asset price, as they want to sell stocks and consume more. In our model, expectation sunspot shocks also change agents' valuation of assets. In particular, assume an agent that buys an asset at a certain date and then at a later date changes her expectations such that she expects the future price to be lower than she initially expected. Since the shock is common to all agents (who are identical), they sell their assets (both stocks and bonds), driving down the prices.

However, there is a fundamental difference between our expectation shock and the valuation shock. Our sunspot process creates uncertainty over the future valuations, because it changes the way agents form expectations, that is, the weight between the backward- and forward-looking solution, and bonds and stocks are exposed differently to this risk. As such, this expectation shock interacts with the fundamental shocks, increasing the risk and inducing stochastic volatility. Here, we insert it into the Bansal and Yaron (2004) model of LRR without stochastic volatility. Albuquerque et al.'s (2016) valuation shock is a state variable of the model and it is a fundamental demand shock to time preference, that exhibits exogenous stochastic volatility.[20] In principle, one could embed the expectation shock in a model with a (demand) valuation shock rather than a (supply) LRR shock. The expectation shock would then amplify a different fundamental shock. In other words, our shock interacts with fundamental shocks amplifying the risks connected to fundamentals. As such it creates stochastic volatility, but, contrary to a valuation shock, our model would not imply an equity premium in the absence of uncertainty on the consumption process, i.e., $\sigma = 0$.

Finally, a problem with some explanations of the equity premium is that they imply counterfactually high levels of volatility for the risk-free rate (for example, Boldrin et al. 2001). Table 2.2 shows that the volatilities of the risk-free rate and the stock market returns implied by our model are similar to those in the data.

2.6.2 The Correlation Puzzle

As well documented by Albuquerque et al. (2016), the correlation and covariance between stock returns and measurable fundamentals, especially consumption growth, are weak. Simultaneously accounting for the equity premium and the correlation puzzle is challenging for models with all uncertainty loaded to the supply side, such as Bansal and Yaron's (2004) LRR models. Albuquerque et al. (2016) consider this problem one of their main motivation for introducing a valuation risk from the demand side. Our LRR model, augmented by expectations risk, does relatively well at matching the correlation between stock returns and consumption growth in the data (both contemporaneous and with one-period lagged consumption). The intuition for this result is related to the ability of our model to generate boom-and-bust bubbly behaviour, as explained in Section 2.4 and shown in the next section.

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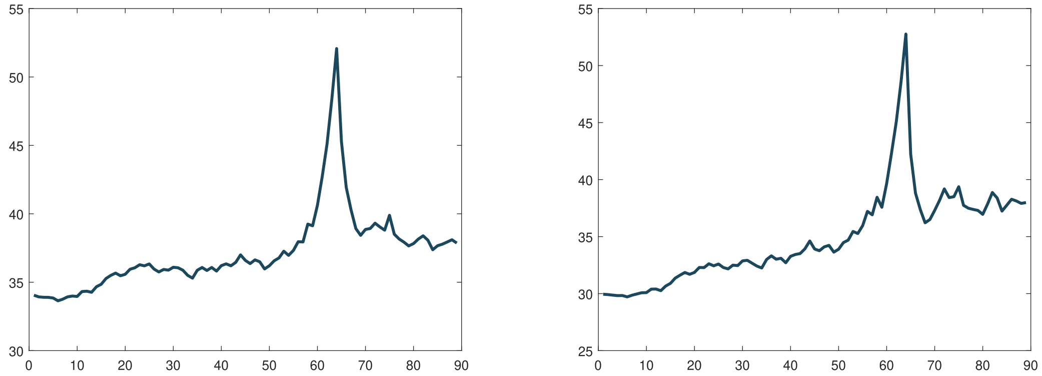
[20] Moreover, both the growth rates of consumption and dividends are affected by the innovation to the persistent component of the time-preference shock.



(a) Finite memory

(b) Decay memory

Figure 2.2: Simulated price-dividend ratio

Notes: The figure shows the simulated PD ratio using the estimated model from Table 2.1. The simulated time series is able to generate booms and busts, as observed in the data.

2.6.3 Price Dividend Ratio

In Table 2.2, we show that the model matches the mean, volatility, and persistence of the PD ratio very well. Indeed, the model is capable of generating ample boom-and-bust type movements, one of the main characteristics of the behaviour of the PD ratio in the data. Figure 2.2 shows a particular realisation of the PD behaviour from simulating both model specifications. Both simulations exhibit a clear boom-bust cycle. This kind of behaviour arises as the PD ratio persistently deviates from the fundamental one, as evident in equation (2.34). The fact that our sunspot shock is very persistent is key for generating “momentum" in stock prices, while the limited memory assumption entails “mean reversion" over long horizons to stable fundamentals. The fact that the PD ratio persistent drifts away from its fundamental RE value – temporarily delinking stock prices from fundamentals – also explains why the model is able to match the low correlation between the fundamentals, i.e., consumption growth, and stock returns, as we discussed in the previous paragraph.

Moreover, another important feature observed in the financial markets is that the PD ratio volatility increases in a bubbly market. Figure 2.2 suggests that the model is able to replicate this feature. Equation (2.35a) provides the analytical explanation behind this, because it shows that the conditional variance of $z_{m,t+1}$ increases in the deviation of the PD ratio, $z_{m,t}$, from its fundamental value, $z_{m,t}^{RE}$.

Finally, another well-known feature of the data is the predictability of excess

Table 2.3: Predictability of excess returns

	Slope coefficient				R^2		
	Data	Finite memory	Decay memory		Data	Finite memory	Decay memory
c_1^2	-0.0022	-0.0019	4	R_1^2	0.0391	0.0208	0.0235
	(0.0010)	(-0.27)	(-0.36)		(0.0375)	(0.49)	(0.42)
c_3^2	-0.0062	-0.0059	-0.0054	R_3^2	0.0890	0.0544	0.0553
	(0.0025)	(-0.14)	(-0.34)		(0.0872)	(0.53)	(0.52)
c_5^2	-0.0110	-0.0098	-0.0090	R_5^2	0.1327	0.0804	0.0756
	(0.0034)	(-0.33)	(-0.58)		(0.0872)	(0.60)	(0.66)

Notes: The table reports the results of regression excess return over holding periods of one, three, and five years on the lagged price-dividend ratio based on the parameter estimates in Table 2.1. $c^2(n)$ and the $R^2(n)$ in the table represents the coefficients and the R^2, respectively, in the following regression: $X_{t,n} = c_n^1 + c_n^2 PD_t + u_{t,n}$, where $X_{t,n}$ is the observed real excess return of stocks over bonds from t to $t+n$ years. The standard error (for estimated data moments) and t-ratios (for model implied moments) are reported in parentheses.

returns from lagged PD ratios. Following Albuquerque et al. (2016), Table 2.3 reports the results of regressing real excess returns on equity over holding periods of one, three, and five years on the lagged price-dividend ratio, that is

$$r_{s,t+n} - r_{f,t} = c_n^1 + c_n^2 \log(PD_t) + u_{t,n} \tag{2.46}$$

where the dependent variable $r_{s,t+n} - r_{f,t}$ is the observed real excess return of stocks over bonds from t to $t+n$ years, and $u_{n,t}$ is the regression residual. The second column in Table 2.3 reports the slope coefficients c_1^2, c_3^2 and c_5^2, while the sixth column reports the R^2's. The slope coefficients are all negative, signalling that high PD ratios are associated with lower future excess returns. The other columns in Table 2.3 show the corresponding results from our simulated data. Our model matches both the slope coefficients and the R^2 values of the regression, and the $t-$statistics are all well within the significance level.

2.6.4 Implication for Expectations

Traditional rational expectations models give rise to an important counterfactual prediction for the behaviour of investors' return expectations. Reflecting the data feature just discussed in Table 2.3, the rationally expected return should correlate negatively with the PD ratio. However, the available survey data on investors' return expectations suggest the opposite. Based on the UBS Survey, the CFO survey and the Shiller individual investor survey, Adam et al. (2017) concludes a positive correlation between the PD ratio and survey expected returns, despite the fact that actual returns are negatively correlated to the PD ratio. Nagel and Xu (2021) shows that this conclusion holds true for both individual investors and professional forecasters. Although the previous literature documented the positive correlation between price-dividend and expected returns, there is much less formal evidence on the stability of this correlation. This is important for us because our model implies time variation in expectation formation, and hence time variation in this correlation, as stressed in the Introduction. Therefore, in this section, we first confirm the well-documented facts that the PD ratio plays a role in investors' future return expectations. Then, we use econometric tests to show that the way agents map observed PD ratios to calculate their return expectation is time-varying.

We use two primary datasets to inform our study. Firstly, the UBS/Gallup Survey data, which is based on a representative sample of approximately 1,000 US investors with at least US$10,000 in financial wealth. We use the data from February 1999 onwards when the survey was conducted on a regular monthly basis until April 2007. After rigorous data cleaning procedures, we derive around 600 observations for each month. While the UBS survey data exhibits commendable consistency, it covers a relatively short time frame.

Complementing the UBS data, we also draw upon the survey data constructed by Nagel and Xu (2021). This dataset, which spans from 1987:Q2 to 2022:Q4, provides quarterly survey-implied 1-year-ahead stock excess return expectations. For the purposes of this study, we have chosen to exclude data post-2019 to maintain temporal consistency with other datasets and ensure relevance to our research objectives.

Figure 2.3 visualises the survey-implied one-year ahead expected excess return on the market portfolio in relation to the prevailing Price-Dividend (PD) ratio using the two datasets. Both figures indicate a clear positive correlation between the PD ratio and the expected excess returns 1-year-ahead.[21]

To test our two hypotheses: (i) PD ratio plays a role in investors' future return expectations; (ii) the mapping from the PD ratio to expectations is time-varying, we consider the following regression as in Nagel and Xu (2021) and Adam and

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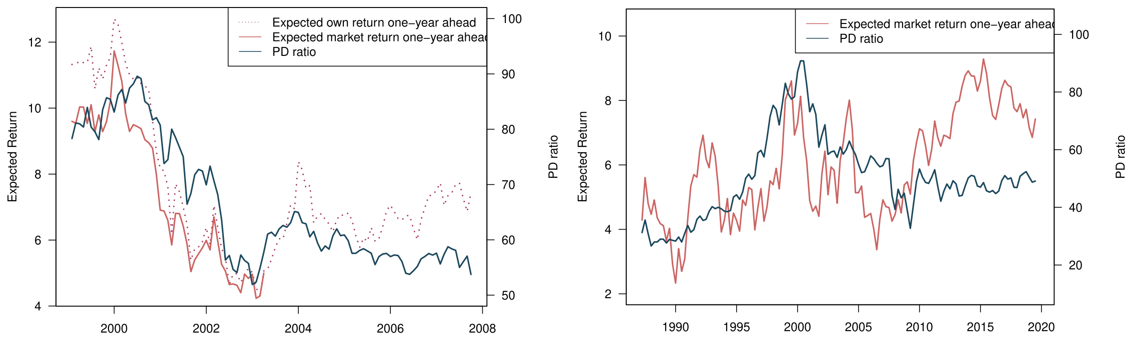
[21] Regarding the UBS survey, the expected return on an individual's own portfolio over the upcoming year is closely aligned with the expected market return for the same period. Moreover, as evident from the figure, the UBS survey changed its survey questions over time. Before 2003 the respondents reported both the return expectations on their own portfolio and expectations on market return one year ahead, while from 2003 onwards, respondents report only the return they expect on their own portfolio one year ahead.



(a) UBS survey

(b) Nagel and Xu (2021) data

Figure 2.3: PD ratio and expected return one year ahead

Nagel (2022)

$$\hat{\mathbb{E}}_t r_{s,t+1} - r_{f,t} = \beta_0 + \beta_t \log(PD_t) + \varepsilon_t, \qquad t = 1, \dots, T \tag{2.47}$$

where $\hat{\mathbb{E}}_t r_{s,t+1}$ is the survey expected market return of one-year ahead at time t. Following the Nagel and Xu's (2021) approach, we impute the market return expectations by regressing expected market returns on own portfolio expectations using the part of the sample where both are available and using the fitted value from this regression when the market return expectation is not reported.

The first hypothesis can be tested by assuming $\beta_t = \beta$ for all t. Table 2.4 indicates that the estimated coefficients of the log PD ratio (i.e., β) are around 0.0269 for the UBS survey and 0.0229 for the Nagel and Xu (2021) data. We estimate the same regression on model-generated data and the result is remarkably close. As Table 2.4 shows, the model-implied regression coefficient is 0.0240.[22]

Table 2.4: Survey return expectations and PD ratio

	UBS survey		Nagel and Xu's (2021) data		Model		
	Estimate	(SE)	Estimate	(SE)	Mean	5%	95%
$\log(PD)$	0.0269	(0.009)	0.0229	(0.005)	0.0240	0.0112	0.0368
R^2	0.08		0.14		0.22		

To test the parameter stability, we follow the methodology proposed by Nyblom (1989) and Hansen (1992). We extend the standard regression model and

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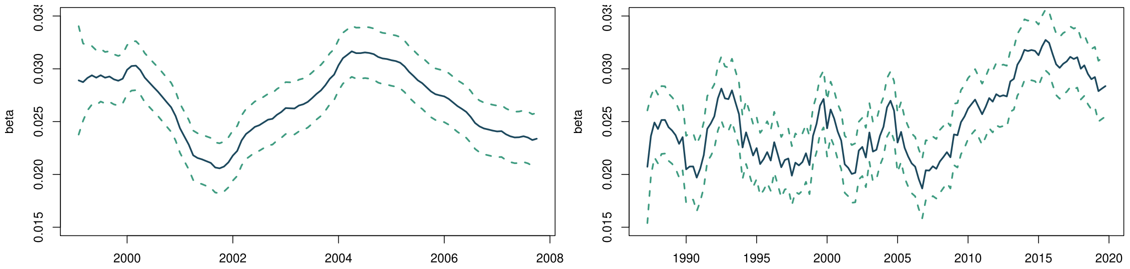
[22] The regression coefficient for the finite memory model is 0.0148. Both are calculated as the mean of 1,000 simulations.



(a) UBS survey

(b) Nagel and Xu (2021) data

Figure 2.4: Time-varying parameter estimations

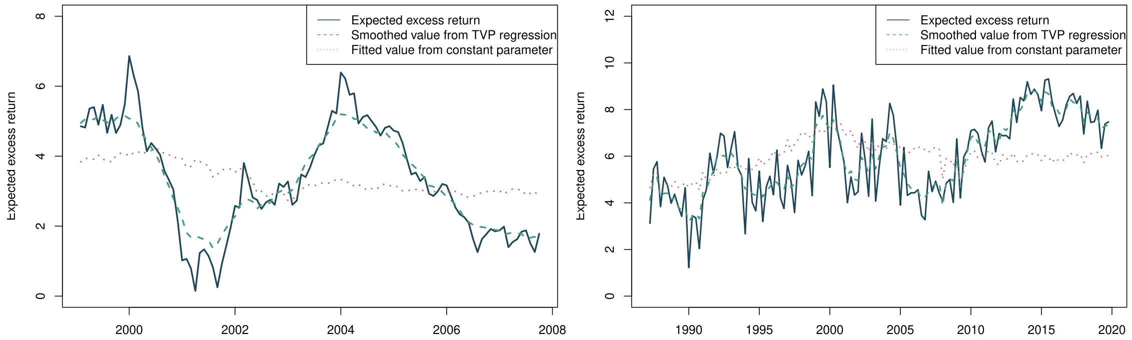
allow the regression coefficients to evolve randomly over time, specifically,

$$\beta_t = \beta_{t-1} + \nu_t \tag{2.48}$$

where ε_t and ν_t are uncorrelated. ν_t is i.i.d $N(0, \tau^2 G)$ (where G is assumed to be known), so that the coefficient β_t follows a random walk and thus evolves smoothly but randomly over the sample period. When ε_t is i.i.d $N(0, \sigma_\varepsilon^2)$, this is referred as the "time-varying parameter" model (see e.g., Cooley and Prescott, 1976). Under the null hypothesis $\beta_t = \beta$ for all t. A rejection of the null hypothesis implies the parameter is unstable, and thus investors' way of forming expectations based on past price-dividend ratio is time-varying. The Kalman filter is then applied, where β_t is the unobserved state vector, (2.48) the state equation, and (2.47) the measurement equation. Figure 2.4 plots the estimation of β_t for both datasets. Figure 2.5 plots the actual value of expected excess return versus the fitted value from constant parameter regression and the smoothed value from TVP regression. It is evident that the smoothed expected excess returns from the TVP regression fit much better to the actual data, which further suggests the rejection of a constant β. When we apply a formal test for time-varying coefficient $\beta_t = \beta$ following Hansen's (1992) approach, the Lagrange multipliers for both datasets are estimated to be greater than the critical value of 0.47, therefore, we reject the null hypothesis that β_t is constant at 5 percent confidence level.

2.6.5 Comparison with Bansal and Yaron (2004)

This subsection discusses the relation between our model and the LRR model pioneered by Bansal and Yaron (2004). Both models feature low-frequency fluctuations in consumption growth and stochastic volatility, which induce changes in the agent's SDF. In contrast to our model, however, Bansal and Yaron (2004)



(a) UBS survey (b) Nagel and Xu (2021) data

Figure 2.5: Actual and fitted expected excess return

assume RE and an exogenous AR(1) process for stochastic volatility, $\sigma_{t+1}^2 = \sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}$, where σ^2 is the unconditional variance of consumption and ν_1 is the persistence of the volatility process. Consequently, the pricing kernel is affected by the volatility shock w_{t+1}

$$m_{t+1} - \mathbb{E}_t m_{t+1} = \lambda'_{m,\eta}\sigma_t\eta_{t+1} - \lambda'_{m,e}\sigma_t e_{t+1} - \lambda'_{m,w}\sigma_w w_{t+1} \tag{2.49}$$

and the equity premium in the presence of time-varying volatility becomes

$$\mathbb{E}_t(r_{m,t+1} - r_{f,t}) = \beta'_{m,e}\lambda'_{m,e}\sigma_t^2 + \beta'_{m,w}\lambda'_{m,w}\sigma_w^2 - 0.5\mathbb{V}ar_t(r_{m,t+1}) \tag{2.50}$$

A few observations follow. While Bansal and Yaron (2004) introduces time-varying consumption volatility, we introduce a time-varying expectation formation process. Comparing the equity premium equations (2.50) and (2.40), it is clear that both specifications can induce a time-varying compensation for the LRR. Despite this similarity, the two models are not observationally equivalent. First, they have different implications for the correlation between observed consumption growth and asset returns. In our case, the changes in the compensation to expected consumption growth are driven by the expectation formation parameter b_t, so that the correlation between consumption growth and asset returns is relatively weak, as observed in the data. Second, in Bansal and Yaron's (2004) setting, the compensation for the volatility shock is constant. In contrast, the compensation for the expectation formation shock varies over time and positively correlates with the deviation of current prices from the fundamental one. Moreover, our time-varying component arises endogenously from the time-varying expectation formation process, whereas the time-varying volatility in Bansal and

Yaron (2004) is exogenous. Despite this, our model has better quantitatively performance than the Bansal and Yaron's (2004) one, as we show next.

Estimation Result. In evaluating the quantitative performance of the LRR model, we first focus on Bansal and Yaron's (2004) original calibration.[23] Then, for a fair comparison, we also utilise the SMM method described in the previous section to estimate the model. The parameter values in both cases are reported in Table 2.5. In the estimation, we assume that the agent's decision interval is monthly and then appropriately compounded to match the annual data. Notice, however, that the structural parameters in this case are estimated less precisely, especially the risk-aversion factor and the IES. The larger standard errors point out difficulties in the estimation, as well documented in the literature (Bansal et al., 2016).

The model moments are reported in Table 2.6. Columns 3 and 4 display the model moments under calibration and the ones obtained by estimating the model using the SMM, respectively. It turns out that our model outperforms the LRR in terms of both individual moments and overall fit. In particular, despite the LRR model matching relatively well the levels of equity premium and price-dividend ratio, the model falls short of the data on some dimensions – the high volatility of price-dividend ratio and market returns as well as the high persistence in the price-dividend ratio (of which t-ratios greater than 2). This might be explained by the fact that the LRR model loads all the uncertainty onto the supply side and thus the price fluctuations only come from fluctuations in the fundamentals, which is considerably small in the data.

Appendix B.4 shows that our key insights hold also in a very different framework, as a simple version of the Lucas (1978) model. Adam et al. (2016) use this model to show that a departure from RE – in the form of a learning model – enables even such a simple asset pricing model to reproduce a variety of stylised asset pricing facts quantitatively. We then use the same Lucas (1978) model, but we embed in it our deviation from RE. We show that the expectation formation mechanism proposed in our paper improves the replication of both individual moments and the overall goodness of fit with respect to the learning mechanism in Adam et al. (2016), for both the finite memory and the decay memory model. Therefore, the improvement in the quantitative performance of our approach is quite robust, because most of the results continue to hold even when applying it

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[23] In Bansal and Yaron (2004)'s original calibration, their parameters were originally calibrated based on the period 1929 to 1998; therefore, we slightly adjusted their original calibration to better match the data moments in this paper.

Table 2.5: LRR parameter estimates

Parameter	Bansal and Yaron (2004)'s original calibration	Estimated value
γ	10	12.6277
		(6.0513)
δ	0.9989	0.9999
		(1.0119)
ψ	1.5	1.0890
		(0.9666)
ρ	0.975	0.8763
		(0.7481)
φ_e	0.0373	0.1085
		(0.8499)
σ	0.0072	0.0065
		(0.9980)
μ	0.0015	0.0020
		(1.0034)
μ_d	0.0008	0.0020
		$(-)$
ϕ	2.5	3.0579
		(1.1576)
φ_d	5.96	5.1702
		(1.2144)
ν_1	0.999	0.9236
		(0.9616)
σ_w	2.4×10^{-6}	2.4569×10^{-6}
		(1.0042)

Notes: The table presents Bansal and Yaron's (2004) original parameter values in column 2 (however, as their parameters were calibrated based on period 1929 to 1998, we slightly adjusted their original calibration to better match the data moments in this paper) and the estimated parameter values in column 3 (with standard errors reported in parentheses). The SMM is used to obtain the estimates. The model simulates on a monthly basis and appropriately compounds to the annual frequency.

Table 2.6: LRR model performance

	U.S. Data	Calibrated model	Estimated model
Mean stock return E_{r^s}	7.79	5.26	5.31
	(1.83)		(1.43)
Mean bond return E_{r^b}	0.45	0.45	-1.16
	(0.49)		(2.11)
Mean PD ratio E_{PD}	32.05	33.13	35.60
	(1.43)		(-0.71)
Mean dividend growth $E_{\triangle D/D}$	1.74	2.16	3.20
	(1.12)		(-1.82)
Std. dev. stock return σ_{r^s}	18.71	18.26	14.64
	(0.94)		(2.23)
Std. dev. PD ratio σ_{PD}	16.40	5.47	3.24
	(2.05)		(3.42)
Std. dev. dividend growth $\sigma_{\triangle D/D}$	10.67	11.29	12.76
	(1.60)		(-0.79)
Std. dev. bond return σ_{r^b}	3.91	0.94	1.25
	(0.43)		(3.59)
Autocorr. PD ratio $\rho_{PD,-1}$	0.90	0.56	0.17
	(0.12)		(7.66)
Mean consumption growth $E_{\triangle C/C}$	2.01	1.85	2.50
	(0.32)		(-1.44)
Std. dev. consumption growth $\sigma_{\triangle C/C}$	2.96	2.79	2.70
	(0.32)		(0.42)
Autocorr. consump. growth $\rho_{\triangle C/C,-1}$	0.61	0.80	0.14
	(0.12)		(0.31)
Autocorr. dividend growth $\rho_{\triangle D/D,-1}$	0.24	0.31	0.03
	(0.37)		(0.45)
Corr. $corr_{\triangle C/C,\triangle D/D}$	0.47	0.55	0.18
	(0.13)		(1.96)
Predictability β_{PD}	-0.0110	-0.0098	-0.0145
	(0.003)		(0.9524)
Predictability R^2	0.1327	0.0804	0.0348
	(0.086)		(1.0622)
Contemporaneous correlation between stock return and consumption growth	0.03 (0.11)	0.07	0.14 (-0.95)
Correlation between stock return and one-period lag consumption growth	-0.13 (0.27)	0.52	0.03 (-0.59)
Test statistic $\hat{W}_N$			558
p-value of $\hat{W}_N$			0

Notes: The table compares the asset-pricing moments from the US data (Column 2, with standard errors reported in parentheses) with the one implied by the calibrated and estimated Bansal and Yaron's (2004) model (Column 3 and 4, respectively, the t-ratios for each moment are reported in parentheses). The t-ratios are calculated as (data moment - model moment)/(estimated standard deviation of the model moment). The measure of the overall goodness of fit $\hat{W}_N$, defined as (2.45), and the corresponding p-value are reported in the last two rows of the table.

to the simplest version of the Lucas (1978) model with time separable preferences and standard stochastic driving processes as in Adam et al. (2016).

2.7 Robustness Checks

This section presents robustness checks for the quantitative results in the previous section, to deal with two possible concerns: (i) the random walk assumption for the stochastic process governing b_t; (ii) the speed of the fading memory. We show that our results survive to changes in both (i) and (ii).

First, the random walk assumption is not crucial for our results. We use this assumption as a benchmark because it is very often used in the time-varying parameters empirical literature and it is very convenient to convey intuition with the analytical expressions in Section 2.4. However, one might be concerned about the fact that it would imply possible unbounded solutions or undefined unconditional moments, which might pose challenges to our econometric approach. We then estimate the model assuming that $b_t \sim AR(1)$ with an unconditional mean equal to 1, the RE benchmark, such that: $(b_t - 1) = \rho_b(b_{t-1} - 1) + \sigma_b \xi_t$, with $\xi_t \sim$ i.i.d $N(0, 1)$. The first two columns of Table 2.7 and columns two and three in Table 2.8 show the estimated parameter values and the quantitative model performance, respectively. The estimated value of ρ_b is around 0.997 for both the decay and the finite memory cases, signalling that indeed the model calls for a very persistent process for the time-varying expectations. Other parameters are almost unaffected, with the exception of a higher degree of risk aversion, γ, for the decay memory case. The overall empirical performance of the model is also only marginally affected. Again both models cannot be rejected at 5% significance level according to the p-value of the Wald test, with the decay memory model performing relatively better with a p-value higher than 10%.

Second, our estimated degree of memory decay implies a memory span of roughly seven years and a half-life of just one year. This is faster than other estimates from the literature. For example, Nagel and Xu (2021) find that the estimated impact of events on memory has a half-life of ten years, corresponding to a value of λ of 0.9942. Moreover, this latter value is more in line with the estimated persistence of the fundamental LRR, x_t, that generates a similar half-life of ten years. Hence, one might argue that it seems odd that investors use only seven years of lagged data when the half-life of the fundamental shock is ten years. We then estimate the model imposing λ the same value as in Nagel and Xu (2021), still assuming that b_t follows the same $AR(1)$ as above. The last columns in Table 2.7 and Table 2.8 show the estimated parameter values and the quantitative model performance. The results are very similar to our benchmark case, both in terms of estimated parameters and in terms of model moments. The overall empirical performance of the model is also similar to our benchmark model with a p-value of the Wald test higher than 15%.

Table 2.7: Parameter estimates, $b_t \sim AR(1)$

Parameter	Finite	Decay	Decay (fixed λ)
γ	3.8857	5.7056	4.1241
	(0.1225)	(0.1306)	(0.1808)
δ	0.9956	0.9946	0.9947
	(0.0005)	(0.0005)	(0.0002)
ψ	1.1624	1.3697	1.5550
	(0.0462)	(0.0229)	(0.0924)
ρ	0.9925	0.9750	0.9752
	(0.0012)	(0.0093)	(0.0019)
φ_e	0.1050	0.0683	0.1764
	(0.0181)	(0.0152)	(0.0077)
σ	0.003	0.0061	0.0033
	(0.0003)	(0.0004)	(0.0002)
μ	0.0017	0.0016	0.0016
	(0.0002)	(0.0001)	(0.0001)
ϕ	2.1430	2.4629	1.9047
	(0.2451)	(0.1684)	(0.0474)
φ_d	8.5846	3.8976	8.0736
	(1.0564)	(0.4815)	(0.4968)
σ_b	0.0302	0.0457	0.0412
	(0.0075)	(0.0111)	(0.0046)
ρ_b	0.9968	0.9971	0.9969
	(0.0013)	(0.0004)	(0.0006)
	$T = 8.7072$	$\lambda = 0.9419$	0.9942
	(1.6249)	(0.0006)	$(-)$

Notes: The table presents estimated parameter values for the finite memory and decay memory models (with standard errors reported in parentheses). The model simulates on a monthly basis and appropriately compounds to the annual frequency. The Simulated Method of Moment method is used to obtain the estimates.

Table 2.8: Quantitative model performance, $b_t \sim AR(1)$

	U.S. Data	Finite	Decay	Decay (fixed λ)
Mean stock return E_{r^s}	7.79	5.82	6.03	6.32
	(1.83)	(1.05)	(0.96)	(0.76)
Mean bond return E_{r^b}	0.45	0.81	0.83	1.10
	(0.49)	(-0.83)	(-1.00)	(-1.12)
Mean PD ratio E_{PD}	32.05	34.67	34.91	35.08
	(1.43)	(-1.94)	(-1.85)	(-0.95)
Mean dividend growth $E_{\Delta D/D}$	1.74	2.71	2.42	2.42
	(1.12)	(-0.88)	(-0.61)	(-0.59)
Std. dev. stock return σ_{r^s}	18.71	19.25	18.56	19.25
	(0.94)	(-0.59)	(0.28)	(-0.35)
Std. dev. PD ratio σ_{PD}	16.40	18.09	18.68	18.50
	(2.05)	(-0.83)	(-1.04)	(-0.78)
Std. dev. dividend growth $\sigma_{\Delta D/D}$	10.67	11.29	9.83	10.87
	(1.60)	(-0.39)	(0.60)	(-0.10)
Std. dev. bond return σ_{r^b}	3.91	3.59	3.86	3.84
	(0.43)	(0.74)	(0.12)	(0.11)
Autocorr. PD ratio $\rho_{PD,-1}$	0.90	0.81	0.79	0.78
	(0.12)	(0.70)	(0.92)	(0.92)
Mean consumption growth $E_{\Delta C/C}$	2.01	2.15	2.03	1.94
	(0.32)	(-0.49)	(-0.08)	(0.18)
Std. dev. consumption growth $\sigma_{\Delta C/C}$	2.96	2.98	2.98	3.11
	(0.32)	(-0.08)	(-0.10)	(-0.32)
Autocorr. consumption growth $\rho_{\Delta C/C,-1}$	0.61	0.76	0.36	0.67
	(0.12)	(-0.05)	(0.09)	(-0.02)
Autocorr. dividend growth $\rho_{\Delta D/D,-1}$	0.24	0.24	0.20	0.19
	(0.37)	(-0.00)	(0.05)	(0.06)
Corr. $corr_{\Delta C/C,\Delta D/D}$	0.47	0.48	0.36	0.47
	(0.13)	(-0.08)	(0.98)	(0.03)
Predictability β_{PD}	-0.0110	-0.0129	-0.0122	-0.0119
	(0.003)	(0.69)	(0.45)	(0.35)
Predictability R^2	0.1327	0.0909	0.0847	0.0804
	(0.086)	(0.49)	(0.58)	(0.60)
Contemporaneous correlation between stock return and consumption growth	0.03 (0.11)	0.48 (-1.76)	0.17 (-1.48)	0.24 (-1.75)
Correlation between stock return and one-period lag consumption growth	-0.13 (0.27)	0.10 (-0.44)	0.05 (-0.35)	0.07 (-0.39)
Test statistic $\hat{W}_N$		9.1351	9.49	7.358
p-value of $\hat{W}_N$		5.78%	11.8%	15.8%

2.8 Conclusion

We propose a novel mechanism for asset pricing models based on two features: (i) limited memory; (ii) time-varying expectations. The first assumption guarantees that the model is stable over long horizons, but opens up the possibility of many different temporary equilibria. The second assumption borrows from Ascari et al. (2019) the idea of modelling a multiplicative sunspot shock to select one equilibrium among all admissible ones. This sunspot shock has an appealing economic interpretation as a time-varying expectation formation process, because it entails a change in the way agents combine past data to calculate their expectations.

These two assumptions allow for the possibility of temporary explosive trajectories and boom-and-bust behaviour, typical of asset price dynamics. Intuitively, our persistent sunspot shock can induce “momentum" in stock prices, while the limited memory assumption implies “mean reversion" over long horizons to stable fundamentals. We thus embed this mechanism into the standard Bansal and Yaron (2004) model of LRR, featuring Epstein-Zin preferences and a persistent predictable component in LRR. However, contrary to Bansal and Yaron (2004), we do not assume stochastic volatility, because our mechanism generates it endogenously.

The resulting model is able to quantitatively reproduce a variety of stylised asset pricing facts, such as the equity premium, excessive volatility and persistence of price-dividend ratio, the relatively weak correlation between returns and fundamentals and the observed predictability of excess returns by lagged price-dividend ratios. Moreover, despite the assumed time-varying expectation formation process imposes some theoretical structure on the stochastic volatility, the quantitative performance of our model outperforms that of the Bansal and Yaron (2004) model, implying that the expectation process seems corroborated by the data.

Furthermore, the model also generates empirically plausible subjective expectations. We use both the survey data in Nagel and Xu (2021) and the UBS/-Gallup Survey data to show that there is a positive correlation between the PD ratio and survey expected returns. Our model is able to replicate very closely the regression coefficient of expected one-year-ahead survey market return on the PD ratio in both surveys. Although the previous literature studied the positive correlation between price-dividend and expected returns, we uncover a novel fact showing that this relationship varies over time, as implied by the time-varying expectation formation process in our model. This empirical fact was one of the main motivations to introduce this process in our model in the first place.

Chapter 3

Restoring Existence and Uniqueness at the Effective Lower Bound with Simple Fiscal Policy[†]

Abstract

The presence of an occasionally binding constraint from the effective lower bound (ELB) in New Keynesian models often leads to multiple or no equilibria. The problem stems from a strong feedback loop between expectations (of inflation and output) and current outcomes at the ELB. We show that simple fiscal policy rules can introduce additional stabilising forces that dampen this loop, thereby ensuring the existence and uniqueness of an equilibrium.

[†] This chapter is co-authored with David Murakami and Ivan Shchapov.

3.1 Introduction

The canonical New Keynesian (NK) model with an occasionally binding effective lower bound (ELB) constraint can admit either no solutions (non-existence) or multiple solutions (non-uniqueness).[1] These findings have been demonstrated by Ascari and Mavroeidis (2022) (henceforce, AM) in a stochastic environment with rational expectations and Holden (2023) under perfect foresight. With non-existence, the economy cannot settle into a consistent, stable outcome; while with multiplicity, non-fundamental shifts in beliefs might trigger belief-driven recessions. To prevent such outcomes, policymakers aim to ensure their actions will produce uniqueness.

This paper demonstrates that simple Ricardian fiscal policy (FP) ensures a unique minimum state variable (MSV) solution. Our key finding is if FP is persistent and reactive to inflation and output fluctuations, it guarantees a unique MSV solution, while also satisfying Blanchard-Kahn (BK) local determinacy conditions. This paper identifies two critical properties for achieving uniqueness of an MSV solution. First, at the ELB, FP stabilises the economy when monetary policy is constrained, establishing an equilibrium path. Second, a countercyclical rule-based FP eliminates belief-driven equilibria when it is sufficiently persistent.

Building on the work of Gourieroux et al. (1980) (GLM), AM derive two main results using a system of linearised equations and endogenous regime switching. First, they demonstrate that achieving solution existence in ELB-constrained NK models poses a non-trivial challenge when the Taylor principle is satisfied. Additionally, AM identify conditions that restrict the support of stochastic shocks, necessary to ensure model existence. However, these support restrictions prove cumbersome, and depend on model structural parameters and past realisations of state variables in backward-looking models. Second, even with support restrictions to ensure existence, the model may still exhibit multiple MSV solutions, potentially up to 2^k solutions, where k represents the number of discrete shock states.

This concern extends beyond the conventional scope of the ELB literature, which mainly examined sunspot shocks or belief-driven fluctuations between steady states.[2] However, general conditions to ensure existence and unique-

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[1] Referred to as “incoherent” and “incomplete”, respectively, in Ascari and Mavroeidis (2022) although the meanings are equivalent.

[2] See, for example, Eggertsson and Woodford (2003); Guerrieri and Iacoviello (2015); Kulish et al. (2017); Aruoba et al. (2018, 2021); Angeletos and Lian (2023).

ness of MSV solutions in the macroeconomics DSGE literature remain limited, although recent papers have provided sufficiency conditions for MSV equilibrium existence in NK models (Eggertsson, 2011; Christiano et al., 2018; Nakata, 2018; Nakata and Schmidt, 2019). Compared to this strand of literature, this paper studies both existence and uniqueness.

As highlighted in follow-up work, Ascari et al. (2023) showed that multiplicity of MSV solutions emerges from the interplay between rational expectations and the inherently non-linear nature of the ELB constraint. In addition, Holden (2024) presents an alternative approach to ensuring existence and uniqueness at the ELB via adjusting the monetary authority's inflation target to be consistent with the Fisher equation with a positive nominal interest rate. Compared to these papers, this paper maintains the FIRE framework and proposes mechanisms specifically emphasising the role of FP to address issues identified by AM and Holden.

Thus, our paper adds to the studies that explore FP, the ELB, and multiple equilibria interactions. Seminal work by Benhabib et al. (2001) examined how Ricardian FP with active monetary policy leads to unique convergence to a steady state equilibrium. However, convergence was not always to a unique steady state and could include an unintended liquidity trap steady state. Benhabib et al. (2002) extended this to establish convergence to a non-liquidity trap steady state. Both studies assumed perfect foresight environments, while this paper maintains FIRE.[3] Critically, we argue that the problem of non-existence and non-uniqueness stems from the strong feedback loop between expectations (of inflation and output in a canonical NK model) and current realisations at the ELB. The core aim of our paper is to show that certain FP rules can introduce stabilising forces that dampen this feedback loop. This in turn can help ensure not only existence but also uniqueness of equilibrium.

Our paper is closely related to the contributions of Schmidt (2016), Tamanyu (2021), and Nakata and Schmidt (2022), which addressed the aforementioned classical concerns of the literature on the ELB. These studies show how expectations-driven liquidity traps could be avoided with appropriate FP, emphasising fiscal rule variations. Meanwhile, examples of a more policy-focused contribution that our work is related to are Correia et al. (2013); Seidl and Seyrich (2023) which show that distortionary tax policy can perfectly replicate the unique rational expectations equilibrium without the ELB constraint. While these results were

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[3] See Definition 3 and Propositions 5 and 6 of Benhabib et al. (2001).

quantitatively demonstrated in a perfect foresight environment with agents making expectation errors, our work – using a textbook NK setup – encompasses the mechanisms of their basic model as a special case.

It is notable that the aforementioned literature on the ELB and FP primarily focused on the elimination of a liquidity trap steady state, often assuming restrictions on the shock process or stochastic environment. Our primary contribution is to simultaneously consider existence and uniqueness of MSV solutions, as well as local determinacy (BK conditions) concerning the ELB and FP instruments. Additionally, despite the paper looking into fiscal and monetary policy interactions,[4] it refrains from examining fiscal policy potency or fiscal multipliers at the ELB.

The paper is structured as follows: Section 3.2 provides an overview of existence and uniqueness (E&U) conditions for an MSV solution within the context of an ELB-bound NK model and describes the methodology used to verify these conditions. Section 3.3 demonstrates how Ricardian FP restores E&U of an MSV solution in a purely forward-looking reference NK model constrained by the ELB. Section 3.4 assesses E&U conditions for an NK model with FP featuring policy inertia; that is, a model with an endogenous state. Finally, Section 3.5 concludes the paper.

3.2 Verifying a Unique MSV Solution of the New Keynesian Model with the ELB

In this section, we provide a sketch of AM's methodology to verify E&U of systems of piecewise-linear equations, applying the methodology to a textbook NK model subject to the ELB. Further explanation and derivation can be found in Appendix C.1 or in AM. For brevity, in this Section and Section 3.3, we abstract from models that feature endogenous states. We revisit E&U conditions for models with endogenous states in Section 3.4.

General Verification for Linear Models. Let Y_t be a $n \times 1$ vector of endogenous variables, X_t be a $n_x \times 1$ vector of exogenous state variables, and $s_t \in \{0, 1\}$ be an indicator variable that is equal to 1 when some inequality constraint is slack and 0 otherwise. Additionally, let Ω_t denote the information set, thus allowing us to

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[4] This literature is vast – see, for example, Galí et al. (2007); Davig and Leeper (2011); Eggertsson and Krugman (2012); Billi and Walsh (2022); Hills and Nakata (2018).

write: $Y_{t+1|t} = \mathbb{E}_t[Y_{t+1}|\Omega_t]$ and $X_{t+1|t} = \mathbb{E}_t[X_{t+1}|\Omega_t]$. The system can be written in the canonical form

$$\begin{aligned} \mathbf{0} &= \left(\boldsymbol{A}_{s_{t,i}} Y_t + \boldsymbol{B}_{s_{t,i}} Y_{t+1|t} + \boldsymbol{C}_{s_{t,i}} X_t + \boldsymbol{D}_{s_{t,i}} X_{t+1|t}\right), \\ s_{t,i} &= \mathbf{1}\left\{\left[\boldsymbol{a}' Y_t + \boldsymbol{b}' Y_{t+1|t} + \boldsymbol{c}' X_t + \boldsymbol{d}' X_{t+1|t}\right] > 0\right\}, \end{aligned} \tag{3.1}$$

where $\boldsymbol{A}_{s_i}$, $\boldsymbol{B}_{s_i}$, $\boldsymbol{C}_{s_i}$, and $\boldsymbol{D}_{s_i}$ are coefficient matrices, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ are coefficient vectors, and $\mathbf{1}\cdot$ is an indicator function that is equal to unity if the inequality constraint is slack and zero otherwise. Without loss of generality, we assume that shocks X_t are k-state stationary first-order Markov processes with transition kernel $\boldsymbol{K}$. Collecting all possible states of X_t for states $i = 1, .., k$ into a $n_x \times k$ matrix $\mathbf{X}$. Let $\boldsymbol{e}_i$ denote the i-th column of the $k \times k$ identity matrix $\boldsymbol{I}_k$, such that $\mathbf{X}\boldsymbol{e}_i$, the i-th column of $\mathbf{X}$, is the i-th state of X_t. Moreover, the elements of the transition kernel $\boldsymbol{K}$ are $\boldsymbol{K}_{ij} = \Pr(X_{t+1} = \mathbf{X}\boldsymbol{e}_j | X_t = \mathbf{X}\boldsymbol{e}_i)$.

The existence of MSV solutions of the system (3.1) cannot be analysed directly using the GLM theorem due to the existence of expectations of the endogenous variables $Y_{t+1|t}$. However, we can convert Equation (3.1) into a model without expectations of endogenous variables by imposing certain rules for expectations. While there are multiple possible rules, here we focus on FIRE, as in AM.

Under FIRE, the agents' expectation of X_{t+1}, based on Ω_t, should be consistent with the actual outcomes in equilibrium, i.e., $\mathbb{E}_t\left[X_{t+1}|\Omega_t = X_{t+1}|X_t = \mathbf{X}\boldsymbol{e}_i\right] = \mathbf{X}\boldsymbol{K}'\boldsymbol{e}_i$.[5] Moreover, define $\mathbf{Y}$ as an $n \times k$ matrix whose i-th column, $\mathbf{Y}\boldsymbol{e}_i$ is the corresponding i-th state value of Y_t, i.e., $Y_t = f(X_t = \mathbf{X}\boldsymbol{e}_i)$ along an MSV solution. Therefore, under rational expectations, the expectation of Y_{t+1} is pinned down by $\mathbb{E}[Y_{t+1}|\Omega_t = Y_{t+1}|Y_t = \mathbf{Y}\boldsymbol{e}_i] = \mathbb{E}[Y_{t+1}|X_t = \mathbf{X}\boldsymbol{e}_i] = \mathbf{Y}\boldsymbol{K}'\boldsymbol{e}_i$. Substituting the expectations into Equation (3.1), we rewrite the problem as

$$\begin{aligned} \mathbf{0} &= \left(\boldsymbol{A}_{s_i}\mathbf{Y} + \boldsymbol{B}_{s_i}\mathbf{Y}\boldsymbol{K}' + \boldsymbol{C}_{s_i}\mathbf{X} + \boldsymbol{D}_{s_i}\mathbf{X}\boldsymbol{K}'\right)\boldsymbol{e}_i, \\ s_i &= \mathbf{1}\left\{\left[\boldsymbol{a}'\mathbf{Y} + \boldsymbol{b}'\mathbf{Y}\boldsymbol{K}' + \boldsymbol{c}'\mathbf{X} + \boldsymbol{d}'\mathbf{X}\boldsymbol{K}'\right]\boldsymbol{e}_i > 0\right\}, \quad i = 1, ..., k. \end{aligned} \tag{3.2}$$

The system (3.2) relates $\mathbf{Y}$ to $\mathbf{X}$, and can be expressed as $F(\mathbf{Y}) = \lambda(\mathbf{X})$, where $\lambda(\cdot)$ is some function of $\mathbf{X}$,[6] and $F(\cdot)$ is a piecewise linear function of $\mathbf{Y}$. To formalise this, consider a subset $J \subseteq \{1, ..., k\}$. For each subset J, define the corresponding

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[5] For example, suppose $k = 2$. At time t, X_t is in state h, i.e., $X_t = X(k = h)$. With probability p it will stay in state h, and with probability $1 - p$ it will switch to state l. Then, the expectation of X_{t+1} under FIRE should be $\mathbb{E}[X_{t+1}|X_t = X(k = h)] = pX(k = h) + (1 - p)X(k = l)$.

[6] Appendix C.1.1.3 shows that the function is unbounded for the class of models considered in this paper.

cone $\boldsymbol{C}_J$ as the set of matrices $\mathbf{Y} \in \mathbb{R}^{n\times k}$ for which the regime indicators s_i satisfy:

$$\boldsymbol{C}_J = \{\mathbf{Y} | \mathbf{Y} \in \mathbb{R}^{n\times k}, s_i = 1 \text{ if } i \in J \text{ and } s_i = 0 \text{ if } i \notin J\}.$$

For example, the cone of the subset $J = 1$, $\boldsymbol{C}_{\{1\}}$ implies that $s_1 = 1$ but all others $s_{i\neq 1} = 0$.[7] There are 2^k such cones. Let us associate an invertible linear mapping $\boldsymbol{\mathcal{A}}_J$ with each cone. Then the piecewise-linear function $F(\mathbf{Y})$ can be expressed as:

$$F(\mathbf{Y}) = \sum_J \boldsymbol{\mathcal{A}}_J \mathbf{1}\left\{\boldsymbol{C}_J\right\} \operatorname{vec}(\mathbf{Y}), \tag{3.3}$$

where $\mathbf{1}_{\boldsymbol{C}_J} = 1$ if $\mathbf{Y} \in \boldsymbol{C}_J$ and $\mathbf{1}_{\boldsymbol{C}_J} = 1$ if $\mathbf{Y} \notin \boldsymbol{C}_J$.[8] $\operatorname{vec}(\cdot)$ is the vector operator function.[9]

If $F(\mathbf{Y})$ in Equation (3.3) is invertible, then the linear system has a unique MSV solution. A necessary and sufficient condition for the invertibility of $F(\cdot)$, as stipulated in GLM, is that all the determinants of $\boldsymbol{\mathcal{A}}_J$, $J \subseteq \{1, ..., k\}$ have the same sign. Failure of this requirement implies that either no MSV solution exists or that there may be multiple MSV solutions:

Theorem 1 (GLM)**.** *Suppose that the mapping $F(\cdot)$ defined in Equation* (3.3) *is continuous. A necessary and sufficient condition for $F(\cdot)$ to be invertible is that all the determinants* $\det \mathcal{A}_J$, $J \subseteq \{1, ..., k\}$ *have the same sign.*

Application to a Canonical New Keynesian Model with the ELB. Consider the canonical NK model as in, for example, Galí (2015). The model in its linearised in terms of log-deviations from steady state can be written in three equations, the dynamic IS equation (DISE), New Keynesian Phillips Curve (NKPC),

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[7] Using the state-space formulation, $\boldsymbol{C}_{\{1\}}$ means

$0 = \left(\boldsymbol{A}_1\mathbf{Y} + \boldsymbol{B}_1\mathbf{Y}\boldsymbol{K}' + \boldsymbol{C}_1\mathbf{X} + \boldsymbol{D}_1\mathbf{X}\boldsymbol{K}'\right)\boldsymbol{e}_i$, if $\left(\boldsymbol{a}'\mathbf{Y} + \boldsymbol{b}'\mathbf{Y}\boldsymbol{K}' + \boldsymbol{c}'\mathbf{X} + \boldsymbol{d}'\mathbf{X}\boldsymbol{K}'\right)\boldsymbol{e}_i > 0$, $i = 1$, and

$0 = \left(\boldsymbol{A}_0\mathbf{Y} + \boldsymbol{B}_0\mathbf{Y}\boldsymbol{K}' + \boldsymbol{C}_0\mathbf{X} + \boldsymbol{D}_0\mathbf{X}\boldsymbol{K}'\right)\boldsymbol{e}_i$, if $\left(\boldsymbol{a}'\mathbf{Y} + \boldsymbol{b}'\mathbf{Y}\boldsymbol{K}' + \boldsymbol{c}'\mathbf{X} + \boldsymbol{d}'\mathbf{X}\boldsymbol{K}'\right)\boldsymbol{e}_i \leq 0$, $i = 2, ..., k$.

[8] This indicator tells you which segment you are in. If you are in the cone $\boldsymbol{C}_J$, then F calls the mapping $\boldsymbol{\mathcal{A}}_J$.

[9] The transformation of (3.2) into (3.3) is generally non-trivial (in which the expressions of $\boldsymbol{\mathcal{A}}_J$ require Kronecker product operations) as it presents a Sylvester equation in $\mathbf{Y}$. See, for example, Kolmogorov and Fomin (1957). However, there are two exceptions that allow straightforward computation of the $\boldsymbol{\mathcal{A}}_J$: $n = 1$ and $n = k > 1$. We make use of this simplifying assumption both in this example and the analytical derivation in Appendix C.1.

and the Taylor rule (TR):

$$\text{DISE: } \hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}) + \varepsilon_t, \tag{3.4a}$$

$$\text{NKPC: } \hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \hat{y}_t + u_t, \tag{3.4b}$$

$$\text{TR: } \hat{i}_t = \max\left\{-\mu, \phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t\right\}, \tag{3.4c}$$

and where ε_t is a demand shock, u_t is a cost-push shock, $\hat{y}_t$ is the output gap, $\hat{\pi}_t$ is inflation, and $\hat{i}_t$ is the nominal interest rate. The parameters of interest in the model are: σ, the coefficient of relative risk aversion; β, the representative household's subjective discount factor; κ, the slope of the NKPC; $\mu = \ln(r\pi^*)$, the ELB of the nominal interest rate in deviation from the steady state, where $r = 1/\beta$ is the steady state gross real interest rate and π^* is the gross inflation target of the monetary authority; ϕ_y, the monetary authority's response parameter to output fluctuations; and ϕ_π, the monetary authority's responsiveness to inflation. In particular, we consider an active monetary policy rule: $\phi_\pi > 1$. We can write the model in canonical form (3.1) with $Y_t = [\hat{\pi}_t \;\; \hat{y}_t]'$ and $X_t = [u_t \;\; \varepsilon_t \;\; \mu]'$, and with coefficient matrices given in Appendix C.1.1.1.

To simplify the analysis, we assume $u_t = 0, \forall t$, $\phi_y = 0$, and that ε_t follows a two-state Markov process ($k = 2$) with states $\varepsilon_t = (\varepsilon^T, 0)$ and a transition kernel:

$$\boldsymbol{K} = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}, \tag{3.5}$$

where p is the probability of remaining in the first state and q is the probability of remaining in the second state.

In this model, the expectation of the endogenous state variable under FIRE is given by $\mathbb{E}_t[Y_{t+1}|X_t = \mathbf{X}\boldsymbol{e}_i] = \mathbf{Y}\mathbf{K}'\boldsymbol{e}_i$. Given this, the E&U of an MSV solution can be checked following the strategy outlined earlier in this section. First, under FIRE, the model can be written in canonical form (3.2). Then write the piecewise linear function $F(\mathbf{Y})$, where the mappings are given by $\boldsymbol{\mathcal{A}}_J$:[10]

$$\begin{aligned}
&\boldsymbol{\mathcal{A}}_{J_1} = \boldsymbol{I}_2 \otimes \boldsymbol{A}_1 + \boldsymbol{K} \otimes \boldsymbol{B}_1 && J_1 = \{1,2\},\\
&\boldsymbol{\mathcal{A}}_{J_2} = \boldsymbol{e}_1\boldsymbol{e}_1' \otimes \boldsymbol{A}_0 + \boldsymbol{e}_1\boldsymbol{e}_1'\boldsymbol{K} \otimes \boldsymbol{B}_0 + \boldsymbol{e}_2\boldsymbol{e}_2' \otimes \boldsymbol{A}_1 + \boldsymbol{e}_2\boldsymbol{e}_2'\boldsymbol{K} \otimes \boldsymbol{B}_1 && J_2 = \{2\},\\
&\boldsymbol{\mathcal{A}}_{J_3} = \boldsymbol{e}_1\boldsymbol{e}_1' \otimes \boldsymbol{A}_1 + \boldsymbol{e}_1\boldsymbol{e}_1'\boldsymbol{K} \otimes \boldsymbol{B}_1 + \boldsymbol{e}_2\boldsymbol{e}_2' \otimes \boldsymbol{A}_0 + \boldsymbol{e}_2\boldsymbol{e}_2'\boldsymbol{K} \otimes \boldsymbol{B}_0 && J_3 = \{1\},\\
&\boldsymbol{\mathcal{A}}_{J_4} = \boldsymbol{I}_2 \otimes \boldsymbol{A}_0 + \boldsymbol{K} \otimes \boldsymbol{B}_0, && J_4 = \emptyset,
\end{aligned} \tag{3.6}$$

where $\boldsymbol{e}_1 = [1 \quad 0]'$ and $\boldsymbol{e}_2 = [0 \quad 1]'$, and matrices $(\boldsymbol{A}_0, \boldsymbol{B}_0)$ and $(\boldsymbol{A}_1, \boldsymbol{B}_1)$ are defined in Appendix C.1.1.1, corresponding to the coefficient matrices of Y_t and $Y_{t+1|t}$ when the inequality constraint is binding and slack, respectively. $\otimes$ is the Kronecker product. J follows the same definition as before. Specifically, $J_1 = \{1,2\}$ implies that the constraints are slack in both states (i.e., $s_i = 1$ for $i = \{1,2\}$). $J_2 = \{2\}$ implies that the constraint is slack in state $i = 2$ but binding in state $i = 1$ (i.e., $s_2 = 1$ and $s_1 = 0$). $J_3 = \{1\}$ implies that the constraint is slack in state $i = 1$ but binding in state $i = 2$ (i.e., $s_1 = 1$ and $s_2 = 0$). $J_4 = \emptyset$ implies that the constraint is binding in both states (i.e., $s_i = 0$ for $i = 1,2$).

We solve the determinants of $\boldsymbol{\mathcal{A}}_J$ and the expressions can be found in the Appendix C.1.1.2. Under an active monetary policy rule, the determinants do not have the same sign for some values of p and q. This implies that the system does not generally have a unique MSV solution. We show the results analytically and explain the intuition through the following special case where $q = 1$.

A Special Case ($q = 1$). We consider the special case where $p < 1$ (transitory state) and $q = 1$ (absorbing state), with the support of ε_t equal to ε^T and 0, respectively. Note that $q = 1$ implies that once the system enters this state where $\varepsilon_t = 0$, the shock vanishes forever. This simplifying assumption allows us to isolate the transitory state equilibrium, without needing to account for expected transitions. Under this assumption, the model admits two absorbing states when $\varepsilon_t = 0$: a positive interest rate (PIR) steady state, where $\{\hat{\pi}, \hat{y}, \hat{i}\} = \{0, 0, 0\}$, and a zero interest rate (ZIR) steady state, $\{\hat{\pi}, \hat{y}, \hat{i}\} = \{-\mu, -\mu(1-\beta)/\kappa, -\mu\}$.[11]

In the transitory states, the equilibrium is characterised by $(\hat{\pi}^T, \hat{y}^T)$ and with probability $1 - p$ the equilibrium transitions to the absorbing state, which can be either a PIR or ZIR equilibrium. Here, we consider the dynamic system around

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[10] The matrix form of the mappings can be found in the Appendix C.1.1.

[11] For a derivation see Appendix C.1.1.4.

the PIR absorbing state.[12] The AS and AD relations can be written as:

$$\hat{\pi}^T = \frac{\kappa}{1-p\beta}\hat{y}^T \quad AS, \tag{3.7a}$$

$$\hat{\pi}^T = \begin{cases} \frac{\sigma(1-p)}{p-\phi_\pi}\hat{y}^T - \frac{\sigma}{(p-\phi_\pi)}\varepsilon^T & AD^{TR} \text{ for } \hat{\pi}^T \geq -\frac{\mu}{\phi_\pi}, \\ \frac{\sigma(1-p)}{p}\hat{y}^T - \frac{\mu}{p} - \frac{\sigma}{p}\varepsilon^T & AD^{ELB} \text{ for } \hat{\pi}^T \leq -\frac{\mu}{\phi_\pi}. \end{cases} \tag{3.7b}$$

AD^{TR} is downward-sloping when $\phi_\pi > 1$, while both AS and AD^{ELB} are upward-sloping. Define θ as the ratio of the slopes of the AD^{ELB} and AS,

$$\theta \equiv \frac{\sigma(1-p)(1-p\beta)}{p\kappa}, \tag{3.8}$$

where it is decreasing in p, the probability of staying in the transitory state. For a small p, θ can be greater than 1, which implies that the slope of AD^{ELB} is steeper than the slope of AS.[13] For large values of p, $\theta < 1$, i.e., the slope of AD^{ELB} is flatter than the slope of AS. These two possibilities are illustrated in Figure 3.1. As discussed by AM, for the case where $\theta > 1$, there is always a solution for any value of ε^T. But when $\theta \leq 1$, there are two solutions when ε^T is small and no solution if the shock ε^T is large. For the existence of the solution, we need to impose a lower bound on the shock ε^T:

$$\varepsilon^T \geq -\frac{\mu p}{\sigma}\left(\frac{\theta}{\phi_\pi} + \frac{\phi_\pi - p}{p\phi_\pi}\right). \tag{3.9}$$

Put simply, these support restrictions ensure that a negative shock to AD does not lead it shifting too far to the left or above of AS such that there is no intersection, as shown in $AD_1^{TR,ELB}$ of Figure 3.1b.

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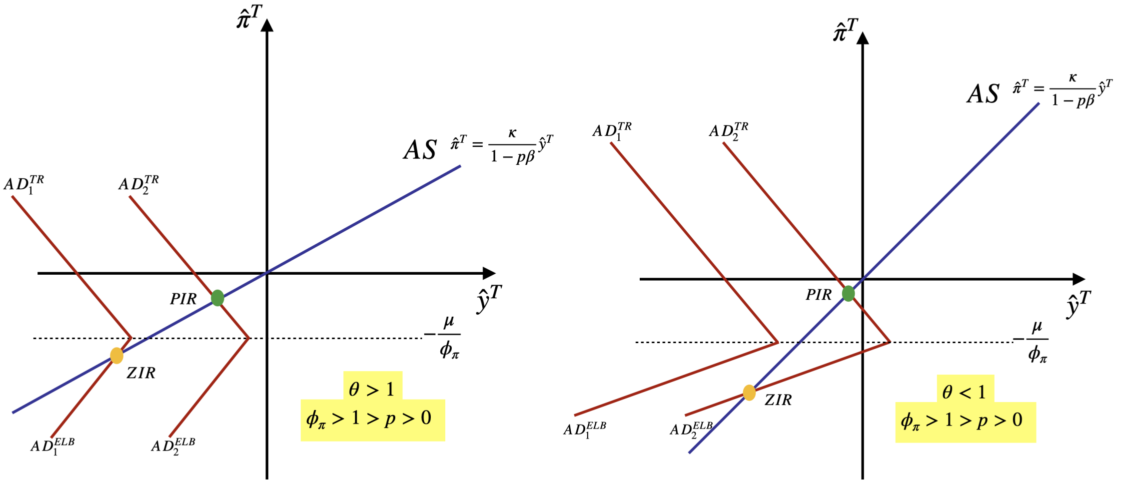
[12] We relegate the discussion of the ZIR absorbing state to Appendix C.1.1.4.

[13] If we instead assume perfect foresight with $\hat{\pi}_{t+1} = p\hat{\pi}_t$ and $\hat{y}_{t+1} = p\hat{y}_t$, then the condition $\theta > 1$ exactly matches the $M_{11} > 0$ condition in Holden (2023). Moreover, if we further assume $p = 0$, i.e., the economy jumps to the PIR steady state in the next period, then both the condition $\theta > 1$ and the condition $M_{11} > 0$ are always satisfied.



(a) PIR absorbing states ($\theta > 1$)

(b) PIR absorbing states ($\theta < 1$)

Figure 3.1: Transitory states of the New Keynesian model ($\phi_\pi > 1$)

Notes: The figure shows how adverse demand shocks of different size, ε^T, shift the demand curve. Left panel shows the case where the equilibrium is unique. Right panel illustrates non-existence and non-uniqueness.

We now explain the intuition behind the non-existence of a solution for a large negative demand shock ε^T. First, note that output gap in the transitory state when $\varepsilon_t = \varepsilon^T$ is given by:

$$\hat{y}_t = \Lambda \mathbb{E}_t \hat{y}_{t+1} - \frac{1}{\sigma} \max\left\{ \phi_\pi \frac{\kappa}{1-\beta p} \hat{y}_t, -\mu \right\} + \varepsilon^T,$$

where

$$\Lambda \equiv \left[1 + \frac{\kappa}{\sigma(1-\beta p)} \right] > 1.$$

For small values of $\varepsilon^T < 0$, the ELB constraint is slack, and the above equation is stable. However, for sufficiently large $\varepsilon^T < 0$ (very negative, to be precise), the ELB will be binding and the output gap will be given by:

$$\hat{y}_t = \Lambda \mathbb{E}_t \hat{y}_{t+1} + \frac{\mu}{\sigma} + \varepsilon^T, \tag{3.10}$$

where $\mathbb{E}_t \hat{y}_{t+1} = p\hat{y}_t$ under rational expectations. We focus on the feedback loop between $\hat{y}_t$ and $\mathbb{E}_t \hat{y}_{t+1}$. When a sufficiently large negative demand shock eventuates, the contemporaneous output gap becomes negative ($\hat{y}_t < 0$). This leads to $\mathbb{E}_t \hat{y}_{t+1} < 0$. If $p\Lambda > 1$, Equation (3.10) implies an even more negative output gap and thus an even more negative $\mathbb{E}_t \hat{y}_{t+1}$. The feedback loop implies that the output gap (and its expectation) fall indefinitely. This leads to no fixed point where expectations settle into an equilibrium, and thus the solution does not exist.[14]

Another way to interpret the non-existence of the solution in this system is that the equilibrium is inconsistent with the ELB constraint. To see this, set $\hat{y}_t = \hat{y}^T$ and substitute $\mathbb{E}_t \hat{y}_{t+1} = p\hat{y}^T$ into Equation (3.10), to write the output gap as:

$$\hat{y}^T = \frac{1}{1 - p\Lambda}\left(\frac{\mu}{\sigma} + \varepsilon^T\right). \tag{3.11}$$

If again $p\Lambda > 1$ and there is a large negative realisation of ε^T, then $\hat{y}^T$ and $\hat{\pi}^T$ can potentially be positive following ε^T, which precludes existence of a solution in which the ELB binds. Therefore, for a solution to exist we require either:

1. p to be small such that $p\Lambda < 1$, which the solution exists for any ε^T, this corresponds to the condition $\theta > 1$ and is illustrated in Figure 3.1a; or

2. For $p\Lambda > 1$ (corresponding to $\theta < 1$), instead we restrict ε^T to be small such that the equilibrium exists, as illustrated in Figure 3.1b.

Either way, both options require restrictions on the support of the demand shock. The interpretation of such restrictions on p and on ε^T is straightforward. The negative demand shock ε^T increases real interest rates when nominal interest rates are at the ELB. As real interest rates rise, intertemporal substitution effects induce households to save more and consume less, which puts downward pressure on inflation and output. Conversely, income effects exert upward pressure on inflation and output. Since $\Lambda > 1$, the income effect is strong at the ELB: current output $\hat{y}_t$ responds by proportionally more than an increase in $\mathbb{E}_t \hat{y}_{t+1}$. For high values of p such that $p\Lambda > 1$, the income effect dominates the substitution effect, leading to a positive inflation and output gap for sufficiently large negative values of ε^T. This implies the potential non-existence of a solution at the ELB.

From the discussion above, the non-existence of a solution in such models arises from the strong income effect at the ELB, as evident from Equation (3.10). One way to resolve this issue is to relax the assumption of rational expectations. In models with non-rational expectations, even when the income effect is strong, i.e., $\Lambda > 1$, the expectation $\mathbb{E}_t \hat{y}_{t+1}$ is dampened. As a result, despite the strong feedback mechanism, the impact on $\hat{y}_t$ is muted. This is investigated in detail in Ascari et al. (2023). Alternatively, we could introduce additional stabilising

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[14] Such feedback loop does not exist under perfect foresight as y_{t+1} is known with certainty, therefore, the standard three-equation NK model with ELB has a unique equilibrium under perfect foresight, see Holden (2023).

forces to reduce the feedback loop between $\mathbb{E}_t\hat{y}_{t+1}$ and $\hat{y}_t$, i.e., a smaller Λ. In what follows, we show that a simple fiscal policy rule can ensure an MSV solution.

3.3 Fiscal Policy and Existence and Uniqueness

In this section, we show how fiscal policy that consists of government spending financed through lump-sum taxes can restore uniqueness of MSV solution in a baseline NK model subject to the ELB.

Model. We augment the baseline NK model with a simple FP setup. The model is otherwise standard, and derivation is given in Appendix C.2. In what follows, we show that under simple fiscal feedback rules, the model can generate a unique MSV solution despite the ELB under certain restrictions on FP. The model is described by the DISE, NKPC, TR, and the natural interest rate given by:

$$\hat{x}_t = \mathbb{E}_t\hat{x}_{t+1} - \frac{s_c}{\sigma}(\hat{i}_t - \mathbb{E}_t\hat{\pi}_{t+1} - \hat{r}_t^n), \tag{3.12a}$$

$$\hat{\pi}_t = \beta\mathbb{E}_t\hat{\pi}_{t+1} + \kappa_y\hat{x}_t, \tag{3.12b}$$

$$\hat{i}_t = \max\left\{-\mu, \phi_\pi\hat{\pi}_t + \phi_y\hat{x}_t\right\}, \tag{3.12c}$$

$$\hat{r}_t^n = -\Gamma\mathbb{E}_t\Delta\hat{g}_{t+1} + \sigma\varepsilon_t, \tag{3.12d}$$

where $s_c \equiv C/Y$ is the steady-state consumption-output ratio, $s_g \equiv G/Y$ is the steady-state government expenditure-output ratio, $\kappa_y = \frac{(1-\gamma)(1-\gamma\beta)}{\gamma}(\varphi + \sigma/s_c)$ is the slope of the NKPC, $\Gamma \equiv (s_g\sigma\varphi)/(s_c\varphi + \sigma)$ and ε_t is a demand shock. Other parameters are defined in the same way as in Section 3.2.

Permanent Fiscal Policy Change. The model in (3.12) is closed with a rule for government expenditure given by

$$\hat{g}_{t+1} - \hat{g}_t = \psi_\pi\hat{\pi}_t + \psi_y\hat{x}_t, \tag{3.13}$$

where ψ_π and ψ_y denote the sensitivity of the government spending growth rate to deviations of inflation and the output gap, respectively. Throughout this section, we assume that future government spending depends on the current realisations of endogenous variables such that the model is purely forward-looking. This assumption allows us to check if the model possesses a unique MSV solution analytically. We relax this assumption in Section 3.4.

Similar to Section 3.2, define $Y_t = [\hat{\pi}_t \quad \hat{x}_t]'$, and $X_t = [\varepsilon_t \quad \mu]'$, so the system can be written as the canonical form (3.2). The coefficient matrices can be found in Appendix C.2.3. For analytical tractability, we assume $\phi_y = \psi_y = 0$. Moreover, ε_t follows a two-state Markov chain process ($k = 2$) with states $\varepsilon_t = (\varepsilon^T, 0)$, with a transition kernel as in Equation (3.5).

We can write the piecewise linear function $F(\mathbf{Y})$, where the mappings follow the same formulation as in Equation (3.6). If all the determinants ($\det \boldsymbol{\mathcal{A}}_J, J \subseteq \{1, ..., k\}$) have the same sign, then the model exists a unique MSV solution. Suppose the BK condition is satisfied, i.e., $\phi_\pi + \Gamma\psi_\pi > 1$, the results are summarised in Proposition 5.

Proposition 5. *When the Blanchard-Kahn condition is satisfied, the New Keynesian model with fiscal policy as defined in* (3.12) *has a unique minimum state variable solution if*

$$\Gamma\psi_\pi > \max 1, \Phi^g_{p,q,\beta,\kappa} + \frac{\phi_\pi (1-q)\left[1 + \frac{\sigma(1-\beta p-\beta q+\beta)}{\kappa_y s_c}\right]}{\Gamma\psi_\pi + \phi_\pi - 1}. \tag{3.14}$$

Proof: *Appendix C.2.4.*

From Proposition 5, if the state $i = 2$ is absorbing, i.e., $q = 1$, the second term in the $\max$ operator simplifies to $\Phi^g_{p,q,\beta,\kappa}$, which is bounded from above by 1. We can characterise the existence of a solution in the special case in Corollary 2.

Corollary 2. *Suppose state $i = 2$ is absorbing ($q = 1$), the baseline New Keynesian model with fiscal policy as defined in* (3.12) *has a unique minimum state variable solution if*

$$\Gamma\psi_\pi > 1, \textit{ where } \Gamma \equiv \frac{s_g \sigma \varphi}{(s_c \varphi + \sigma)}. \tag{3.15}$$

Here the parameter Γ captures how the natural rate responds to the expected growth in government spending (see Equation (3.12)). For large values of Γ, the restriction on ψ_π is relatively relaxed.

Similar to the baseline NK case, below we characterise the conditions for existence of an MSV solution in the special case for this model with FP where $p < 1$ (transitory state) and $q = 1$ (absorbing state), with the support of ε_t given by ε^T and 0, respectively.

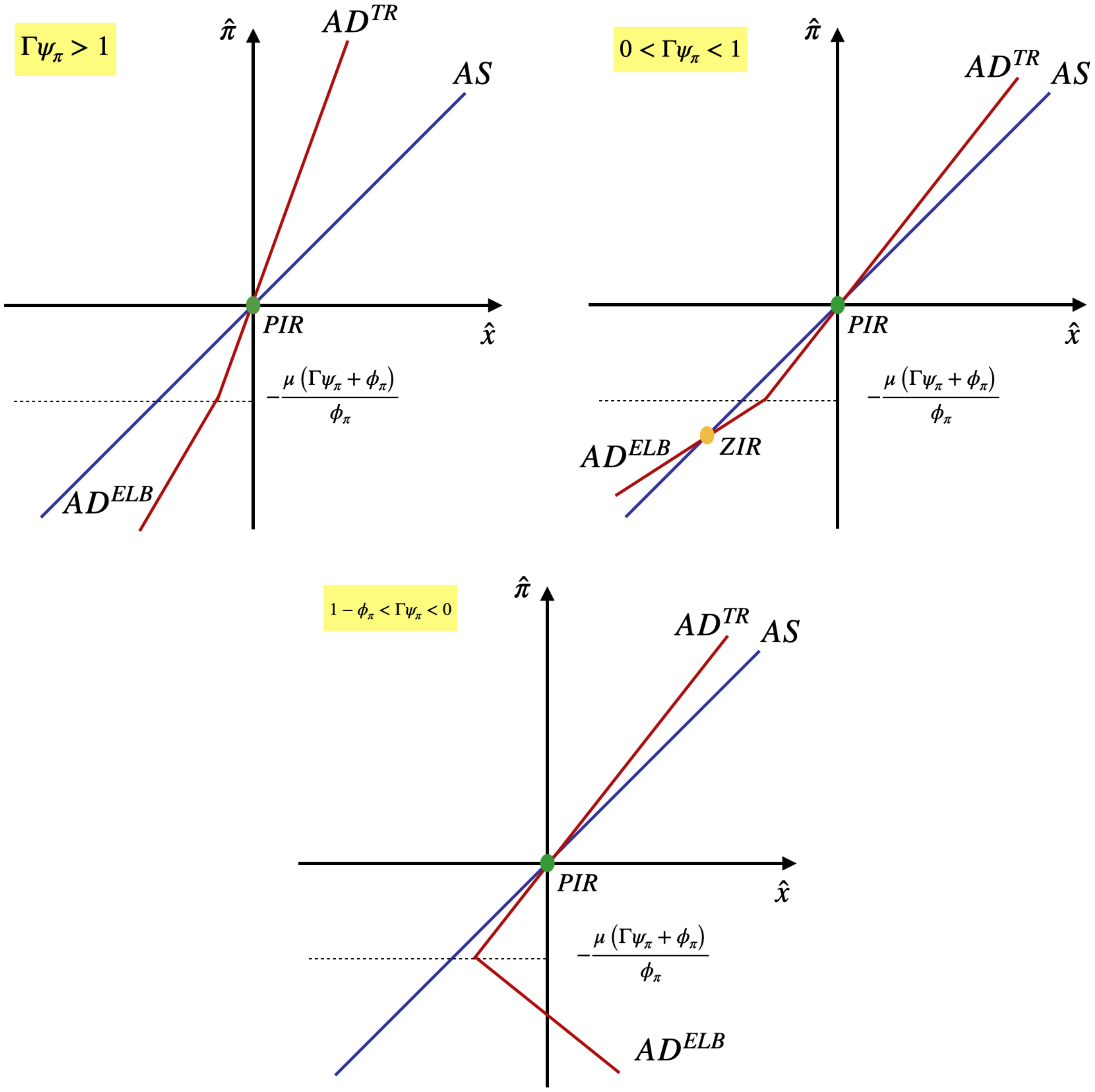

Figure 3.2: Existence and uniqueness with simple fiscal rule: Absorbing state

Notes: Figure shows range for $\Gamma\psi_\pi$ for existence and uniqueness of a unique minimum state variable solution (left and bottom figures). Right panel shows a case of multiplicity.

Absorbing State. First, considering the absorbing state of the model, where $\varepsilon_t = 0$. In the absorbing state we have $\hat{\pi}_t = \hat{\pi}_{t+1} = \hat{\pi}$ and $\hat{x}_t = \hat{x}_{t+1} = \hat{x}$. Hence, we can write the following AS and AD relations:

$$\hat{\pi} = \frac{\kappa_y}{1-\beta}\hat{x} \quad AS, \tag{3.16a}$$

$$\hat{\pi} = \begin{cases} (\phi_\pi + \Gamma\psi_\pi)\frac{\kappa_y}{1-\beta}\hat{x} & AD^{TR}, \text{ for } \hat{\pi} > -\frac{\mu(\Gamma\psi_\pi+\phi_\pi)}{\phi_\pi}, \\ -\mu + \Gamma\psi_\pi\frac{\kappa_y}{1-\beta}\hat{x} & AD^{ELB}, \text{ for } \hat{\pi} \leq -\frac{\mu(\Gamma\psi_\pi+\phi_\pi)}{\phi_\pi}. \end{cases} \tag{3.16b}$$

We plot AS (3.16a) and AD (3.16b) in Figure 3.2. When the BK condition holds (i.e., $\phi_\pi + \Gamma\psi_\pi > 1$), the slope of AD^{TR} must be positive. Then E&U of the solution depends on the FP rule coefficient ψ_π. When ψ_π is positive, to ensure a unique intersection between AD and AS we require the slope of the AD^{ELB} to be steeper than the that of AS, as shown in the left panel of Figure 3.2. From Equations (3.16a) and (3.16b), this condition is satisfied if $\Gamma\psi_\pi > 1$. In this case, we end up with one unique PIR equilibrium, $\{\hat{\pi}, \hat{y}, \hat{i}\} = \{0, 0, 0\}$.[15] However, when ψ_π is positive but less than one, multiple equilibria arise, as illustrated in the right panel of Figure 3.2.

Note that the BK condition requires $\phi_\pi + \Gamma\psi_\pi > 1$ to hold, and consequently if $\phi_\pi > 1$, ψ_π can take on negative values while still satisfying the BK condition. In this case, the slope of AD^{ELB} becomes negative and AD and AS also intersects uniquely at a PIR equilibrium.[16] This is shown in the bottom panel of Figure 3.2.

To conclude, the E&U of an MSV solution in the absorbing state requires either $\Gamma\psi_\pi > 1$ or $1 - \phi_\pi < \Gamma\psi_\pi < 0$.[17] When $\Gamma\psi_\pi > 1$, the slope of AD^{ELB} is steeper than that of AS, ensuring a single intersection and a unique equilibrium. When $1 - \phi_\pi < \Gamma\psi_\pi < 0$, the slope of AD^{ELB} is negative, again leading to a unique equilibrium. In contrast, when $0 < \Gamma\psi_\pi < 1$, multiple equilibria emerge.

Transitory State. We proceed with analysing the transitory equilibria with $\varepsilon_t = \varepsilon^T$. As before, the economy remains in a transitory state with probability p, and with probability $1-p$ jumps to the PIR absorbing state. AS and AD are given by:

$$\hat{\pi}^T = \frac{\kappa_y}{1 - p\beta}\hat{x}^T \quad AS, \tag{3.17a}$$

$$\hat{\pi}^T = \begin{cases} \frac{\sigma(1-p)}{s_c(p-\phi_\pi-\Gamma\psi_\pi)}\hat{x}^T - \frac{\sigma}{(-\phi_\pi+p-\Gamma\psi_\pi)}\varepsilon^T & AD^{TR} \text{ for } \hat{\pi}^T \geq -\frac{\mu}{\phi_\pi}, \\ \frac{\sigma(1-p)}{s_c(p-\Gamma\psi_\pi)}\hat{x}^T - \frac{\mu}{(p-\Gamma\psi_\pi)} - \frac{\sigma}{(p-\Gamma\psi_\pi)}\varepsilon^T & AD^{ELB} \text{ for } \hat{\pi}^T \leq -\frac{\mu}{\phi_\pi}. \end{cases} \tag{3.17b}$$

First, if the BK condition holds, the slope of AD^{TR} is always negative. If the slope of AD^{ELB} is also negative (which implies $\Gamma\psi_\pi > p$), it should be flatter than AD^{TR}. In this case, there is only one intersection between the AS and AD curves, as shown in the left panel of Figure 3.3. If the slope of AD^{ELB} is instead positive,

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[15] For the unique solution to be the PIR equilibrium, we also require $-\mu \leq 0$, which holds if $(r\pi^*)^{-1} \leq 1$.

[16] However, if $\phi_\pi \leq 1$, ψ_π cannot take on negative values and this case is ruled out.

[17] The second condition exists only if $\phi_\pi > 1$.

i.e., $\Gamma\psi_\pi < p$, then there is a unique equilibrium if the slope of AD^{ELB} is steeper than AS. Define $\tilde{\theta}$ as the ratio of the slope of AD^{ELB} to the slope of AS:

$$\tilde{\theta} \equiv \frac{\sigma\left(1-p\right)\left(1-\beta p\right)}{s_c\kappa_y\left(p-\Gamma\psi_\pi\right)}. \tag{3.18}$$

Then E&U of an MSV solution requires $\tilde{\theta} > 1$, which implies

$$\Gamma\psi_\pi > \underbrace{p - \frac{\sigma\left(1-p\right)\left(1-\beta p\right)}{s_c\kappa_y}}_{\equiv\Phi^g_{p,q=1,\beta,\kappa_y}}. \tag{3.19}$$

This condition coincides with the second term in Equation (3.14) when $q = 1$. Note that condition (3.19) is less restrictive than $\Gamma\psi_\pi > p$. The solution is plotted in the right panel of Figure 3.3. For $\tilde{\theta} < 1$, where the slope of AD^{ELB} is flatter than AS, there is either no solution or multiple solutions as illustrated in the bottom panel of Figure 3.3.

To conclude, E&U of an MSV solution with an absorbing state requires either $\Gamma\psi_\pi > 1$ or $1 - \phi_\pi < \Gamma\psi_\pi < 0$, and the E&U condition in a transitory state requires $\Gamma\psi_\pi > \Phi^g_{p,q=1,\beta,\kappa}$. The overlap between the two cases is the requirement of $\Gamma\psi_\pi > 1$, as outlined in Corollary 2.

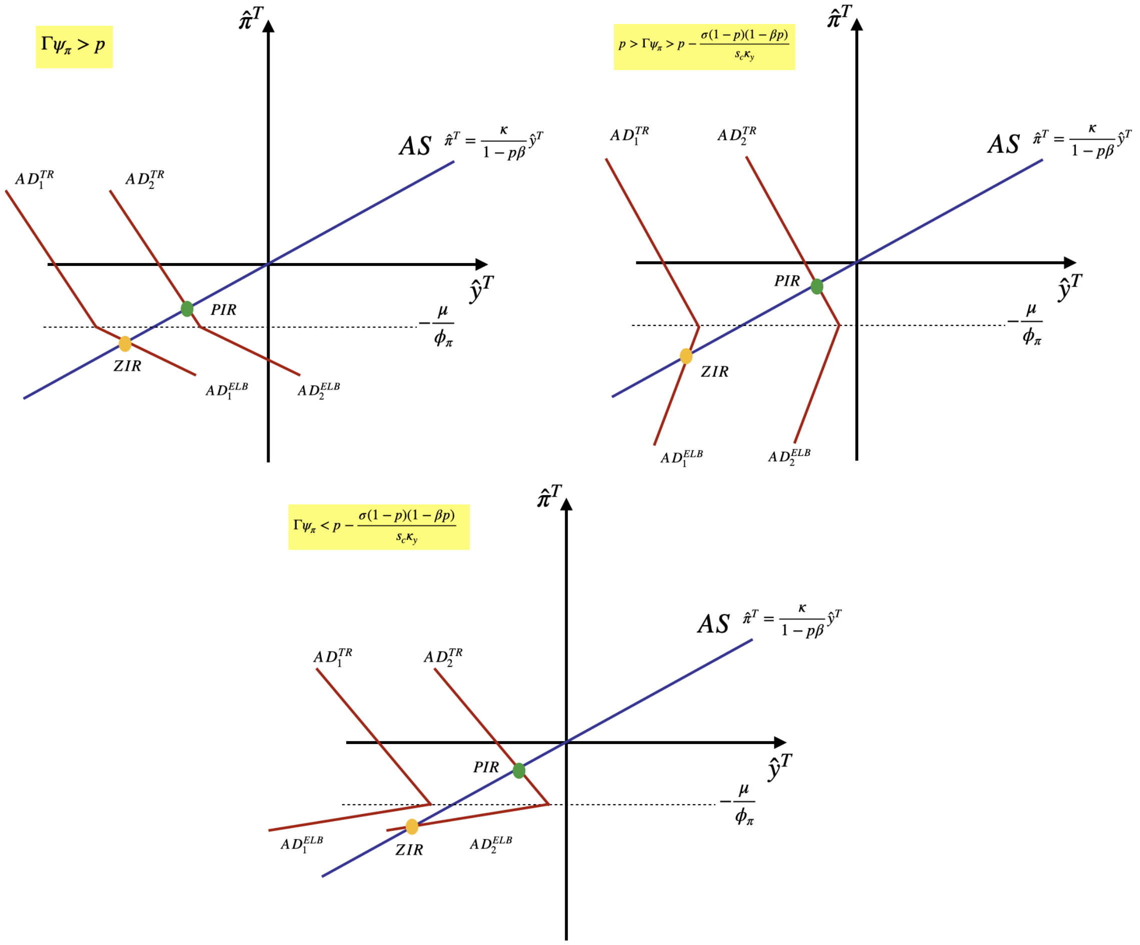

Figure 3.3: Existence and uniqueness with simple fiscal rule: Transitory state

Notes: Top row shows transitory state with a positive interest rate absorbing state with active fiscal policy. Top left panel shows procyclical fiscal policy. Top right shows countercyclical fiscal policy. Bottom panels shows passive procyclical fiscal policy regime. Passive fiscal policy in general implies non-existence of solution or two solutions as a special case. Active fiscal policy implies existence of unique solution.

Comparison to Baseline NK model and the Importance of a Forward-looking Rule. Thus far we have characterised the requirements for FP such that there exists a unique MSV solution when both $\phi_y = \psi_y = 0$. Before we extend to the special case to where ϕ_y and ψ_y are different from zero, we first discuss the intuition why such FP can guarantee a unique solution in NK models with the ELB.

For the baseline NK model considered in Section 3.2, the E&U of an MSV solution requires θ as defined in Equation (3.8) to be greater than unity, i.e.,

$$\theta > 1.$$

This condition simply stated that AD^{ELB} has to be steeper than AS. However, as the slope of AD^{ELB} depends on exogenous uncertainty p, so too does the ratio θ. From the expression, for some values of p, the condition $\theta > 1$ does not hold. Therefore, the baseline NK model does not generally have a unique MSV solution.

Analogously, for the NK model with FP following (3.13), the relative slope between AD^{ELB} and AS was given in Equation (3.18). Then, E&U requires $\tilde{\theta}$ to satisfy the following conditions:

$$\text{either } \tilde{\theta} < 0, \text{ or } \tilde{\theta} > 1. \tag{3.20}$$

That is, AD^{ELB} can have a negative slope or have a positive slope and be steeper than AS. In contrast to the baseline NK model, the slope of AD^{ELB} in the NK model with FP also depends on the fiscal policy rule, and so too does $\tilde{\theta}$. With certain fiscal policy satisfying Proposition 5, the condition (3.20) always holds regardless the value of p. Thus, one can argue that if the effects of uncertainty p can be counteracted by FP, the model will have a unique MSV solution thus satisfying the E&U conditions. Our work is thus complimentary to the results of Nakata and Schmidt (2022), particularly the role of FP in ruling out sunspot equilibria.

The importance of persistence implied by (3.13) cannot be overstated and is a key point of this paper. To highlight this, consider the case where the fiscal targeting rule is given in deviations and not in growth rates, i.e., $\hat{g}_t = \psi'_\pi \hat{\pi}_t$. This will imply the following AD^{ELB} and AS slope ratio:

$$\hat{\theta} \equiv \frac{\sigma (1-p)(1-\beta p)}{s_c \kappa_y \left[p - \Gamma \psi'_\pi (p-1)\right]}.$$

To satisfy the E&U conditions, the value of $\hat{\theta}$ can be either negative or greater than unity. However, this does not hold for sufficiently high values of p. Therefore, the model with FP specified as $\hat{g}_t = \psi'_\pi \hat{\pi}_t$ does not generally satisfy conditions for E&U for an MSV solution.

The intuition for why a contemporaneous FP rule would fail lies in the fact that the strength of the FP would depend on the uncertainty parameter p. In particular, in the transitory state, when $\hat{g}_t = \psi'_\pi \hat{\pi}^T$, we have $\mathbb{E}_t \hat{g}_{t+1} = \psi'_\pi \mathbb{E}_t \hat{\pi}_{t+1} =$

$\psi'_\pi p\hat{\pi}^T$. Therefore, the natural rate would depend on p,

$$r_t^n = \Gamma\psi'_\pi(1-p)\hat{\pi}.$$

Hence, the effect of FP depends on the value of p. And for large value of p, ψ'_π can be unbounded. For example, when $p = 1$, FP has no effect on the system. This highlights the importance of commitment to future changes in policy that depend on contemporaneous deviations of endogenous variables as in, for example, (3.13).

Besides the aforementioned link with Nakata and Schmidt (2022), other approaches in the literature – such as price level targeting (PLT) in Holden (2023), implicit inflation target adjustment in Holden (2024), and unconventional monetary policy in AM and Ikeda et al. (2021) – rely on a similar mechanism to guarantee uniqueness. As argued in Holden (2023), PLT rules can restore uniqueness in the presence of an occasionally binding ELB constraint as such a policy implies a promise about future inflation given inflation today. If monetary policy is committed to a given price level path, the monetary authority promises that a period of low inflation today will be followed by a period of high inflation in the future. Thus, agents expecting high prices in the future increase their consumption in periods of low inflation and, by implication, the system has a unique solution around the PIR absorbing state. The commitment to higher inflation in the future delivers sufficient information about the expected dynamics of the system that alleviates uncertainty that would otherwise engender multiplicity and, by implication, pins down the unique solution similar to persistent fiscal policy.

Output Gap Targeting. Here we relax the assumption $\phi_y = \psi_y = 0$ and find the E&U conditions quantitatively. The model is calibrated according to the values in Table 3.1. These parameter values are standard in the NK DSGE literature.

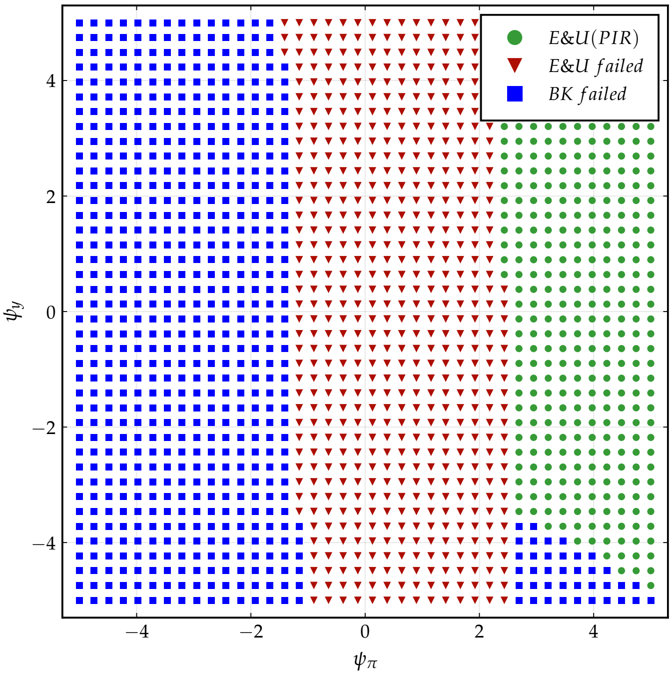

Figure 3.4: Existence and uniqueness region under with simple fiscal Rule

Notes: Green circles denote regions where the solution is unique and locally determinate. Red triangles denote the region where the solution is either non-unique or non-existent. Blue squares denote regions where the solution is not locally determinate.

Table 3.1: Model calibration

Parameter	Value	Description
σ	3	Coefficient of relative risk-aversion
φ	2	Frisch elasticity of labour supply
β	0.99	Discount factor
γ	3/4	Calvo probability
ϵ	10	Elasticity of substitution between goods
s_c	0.7	Fraction of consumption in output
s_g	1-s_c	Fraction of government spending in output
ϕ_π	1.5	Weight on inflation, Taylor rule
ϕ_y	0.2	Weight on output gap, Taylor rule
κ_y	$\frac{(1-\gamma)(1-\gamma\beta)}{\gamma}(\varphi+\sigma/s_c)$	Slope of NKPC

The region for which the model satisfies the E&U conditions as a function of the fiscal authority's reaction parameters, ψ_π and ψ_y, are shown in Figure 3.4. Using our baseline calibration, we see that the model generally satisfies the E&U conditions in the negative orthant of ψ_π and ψ_y space $\mathbb{R}^2_-$, and when ψ_π is sufficiently large. Mechanically, a strong enough reaction on the part of the fiscal authority to inflation and output deviations leads to a unique MSV solution by ensuring an intersection between AD and AS.[18] Furthermore, we note that the degree of reaction on the part of the fiscal authority to the output gap is largely irrelevant as to whether or not the model satisfies the E&U conditions. Moreover, the rule in Equation (3.13) nests the special case where FP can fully replicate monetary policy as considered in Correia et al. (2013); Seidl and Seyrich (2023), who termed this as "unconventional fiscal policy". This is the case if FP activates at the ELB, and its feedback coefficients are set such that they exactly mirror the effects of the counterfactual unconstrained monetary policy. Further details and derivations of this special case are provided in Appendix C.2.5.

Existence and Uniqueness with Distortionary Taxes. As shown above, if FP reacts to exogenous disturbances aggressively enough, the model satisfies the E&U conditions through standard aggregate demand channels. But satisfaction of E&U conditions is not exclusive to the fiscal setup we have discussed.

Consider the case similar to Correia et al. (2013); Seidl and Seyrich (2023) where the fiscal authority levies consumption subsidy and wage taxes, τ^c and τ^w, respectively. The expression for the natural rate in the model in (3.12) becomes

$$\hat{r}_t^n = \sigma \varepsilon_t - \Psi^c \Delta \hat{\tau}_{t+1}^c, \tag{3.21}$$

where $\Delta \hat{\tau}_{t+1}^c$ is the consumption subsidy growth rate. If the fiscal authority sets $\Delta \hat{\tau}_{t+1}^c$ according to a rule in (3.13), a unique MSV solution can be achieved under similar parametric restrictions on feedback coefficients.[19]

3.4 New Keynesian Model with Less Persistent Fiscal Policy

In Section 3.3, we showed that a permanent change in government spending ensures the existence of a unique MSV solution conditional on satisfying a certain set of conditions. For analytical and computational tractability, the model was purely forward-looking. But in this section we augment the fiscal expenditure rule (3.13) so that the model contains an endogenous state variable. More pre-

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[18] This is illustrated for a simple case in Figure 3.2.

[19] See Appendix C.2.2 for model derivation details.

cisely, we replace (3.13) with:

$$\hat{g}_{t+1} - g_t = \rho_g \left(g_t - g_{t-1}\right) + \left(1-\rho_g\right)\left(\psi_\pi^* \hat{\pi}_t + \psi_y^* \hat{x}_t\right), \tag{3.22}$$

where $\rho_g \in (-1, 0]$ is a mean-reversion parameter, and where the rest of the model is as given in (3.12). Notably, if $\rho_g \neq 0$ the system has an endogenous state variable Δg_t, and if $\rho_g = 0$, the government spending nests the fiscal rule in Section 3.3. The persistence of government spending is given by $1 + \rho_g \leq 1$, which decreases as ρ_g becomes more negative.[20] In addition, the parameters $\psi_\pi \equiv \left(1-\rho_g\right)\psi_\pi^*$ and $\psi_y \equiv \left(1-\rho_g\right)\psi_y^*$ in (3.22) capture the overall responsiveness of the growth of government spending $\Delta\hat{g}_{t+1}$ to current inflation $\hat{\pi}_t$ and the current output gap $\hat{x}_t$, respectively.

Define $Y_t = [\hat{\pi}_t \;\; \hat{x}_t \;\; \mathbb{E}_t \Delta \hat{g}_{t+1}]'$ and $X_t = [\varepsilon_t \;\; \mu]'$, then we can write the model in canonical form,

$$\begin{aligned} \mathbf{0} &= \left(\boldsymbol{A}_{s_{t,i}} Y_t + \boldsymbol{B}_{s_{t,i}} Y_{t+1|t} + \boldsymbol{C}_{s_{t,i}} X_t + \boldsymbol{D}_{s_{t,i}} X_{t+1|t} + \boldsymbol{H}_{s_{t,i}} Y_{t-1}\right), \\ s_{t,i} &= \mathbf{1}\left[\boldsymbol{a}' Y_t + \boldsymbol{b}' Y_{t+1|t} + \boldsymbol{c}' X_t + \boldsymbol{d}' X_{t+1|t} + \boldsymbol{h}' Y_{t-1} > 0\right], \end{aligned} \tag{3.23}$$

where the coefficient matrices given in Appendix C.3.1. We also assume as before that ε_t follows a 2-state Markov chain with transition kernel $\boldsymbol{K}$ ($k = 2$).

With an endogenous state variable, the MSV solutions are of the form $Y_t = f\left(Y_{t-1}, X_t\right)$. The key difference compared to Section 3.3 is that the support of Y_t will vary endogenously over time through the evolution of Y_{t-1}, and thus, it can no longer be characterised by a constant matrix $\mathbf{Y}$. Hence, when there are endogenous state variables, we have

$$\mathbb{E}\left[Y_{t+1} | Y_t = \mathbf{Y}_t \boldsymbol{e}_i, X_t = \mathbf{X}\boldsymbol{e}_i\right] = \mathbf{Y}_{t+1}^i \boldsymbol{K}' \boldsymbol{e}_i,$$

where $\mathbf{Y}_{t+1}^i \in \mathbb{R}^{n\times k}$ gives the support of Y_{t+1} when Y_t is in the i-th state. Therefore, from this expression, we can see that the support of Y_t grows at 2^t for any given initial condition Y_0, thence, the MSV solution cannot be represented by any finite-dimensional system of piecewise-linear equations. This implies that we cannot check the E&U condition directly using the GLM Theorem.

Since there is only one endogenous state variable in this model, for ease of notation, define $y_t \equiv g' \boldsymbol{Y}_t$, where $\boldsymbol{g} = [0 \;\; 0 \;\; 1]'$. Moreover, we can write

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[20] Government spending can be rewritten as $\hat{g}_{t+1} = (1+\rho_g) g_t + (1-\rho_g)\left(\psi_\pi^* \hat{\pi}_t + \psi_y^* \hat{x}_t\right)$ when $g_{t-1} = 0$.

$\boldsymbol{H}_{s_{t,i}}\boldsymbol{Y}_{t-1} = \boldsymbol{h}_{s_{t,i}}y_{t-1}$. Suppose that the support at time t can be represented in the form $\mathbf{Y}_t = \boldsymbol{G}y_{t-1} + \boldsymbol{Z}$, where matrix $\boldsymbol{G}$ captures how the support $\mathbf{Y}_t$ depends on the endogenous state y_{t-1}, and matrix $\boldsymbol{Z}$ represents the part of $\mathbf{Y}_t$ that depends on exogenous variable X_t.[21] The matrix $\boldsymbol{G}$ is an $n \times k$ matrix with each column being the coefficient of y_{t-1} for each different state of X_t.

With these definitions, we can rewrite the Equation (3.23) as

$$\begin{aligned} \mathbf{0} = &\left(\boldsymbol{A}_{s_{t,i}}\boldsymbol{G}\boldsymbol{e}_i + \boldsymbol{B}_{s_{t,i}}\boldsymbol{G}\boldsymbol{K}'\boldsymbol{e}_i\boldsymbol{g}'\boldsymbol{G}\boldsymbol{e}_i + \boldsymbol{h}_{s_{t,i}}\right) y_{t-1} \\ &+ \left(\boldsymbol{A}_{s_{t,i}}\boldsymbol{Z} + \boldsymbol{B}_{s_{t,i}}\boldsymbol{G}\boldsymbol{K}'\boldsymbol{e}_i\boldsymbol{g}'\boldsymbol{Z} + \boldsymbol{B}_{s_{t,i}}\boldsymbol{Z}\boldsymbol{K}' + \boldsymbol{C}_{s_{t,i}}\mathbf{X} + \boldsymbol{D}_{s_{t,i}}\mathbf{X}\boldsymbol{K}'\right)\boldsymbol{e}_i \end{aligned} \tag{3.24}$$

for all $i = 1, \ldots, k$. For each subset $J \subseteq \{1, ..., k\}$, we can solve for the corresponding $\boldsymbol{G}$ and $\boldsymbol{Z}$ using the method of undetermined coefficients. For example, for subset $J = \{1, 2\}$, the constraints are slack in both states (i.e., $s_{t,1} = s_{t,2} = 1$), we can solve for the corresponding $\boldsymbol{G}$ and $\boldsymbol{Z}$. However, the system is not piecewise linear in $\boldsymbol{G}$ and $\boldsymbol{Z}$, and so we cannot use the GLM theorem to check for E&U. In the fashion of AM, we have to check all possible 2^k regime configurations J. Moreover, since y_{t-1} is endogenous, we need to solve the system backwards from some terminal condition y_T, and then check E&U for all possible values of y_T.

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[21] In the case where the system does not have endogenous states (such as in Section 3.2), we have $\boldsymbol{G} = 0$, and the support $\mathbf{Y}_t$ is thus time-invariant.

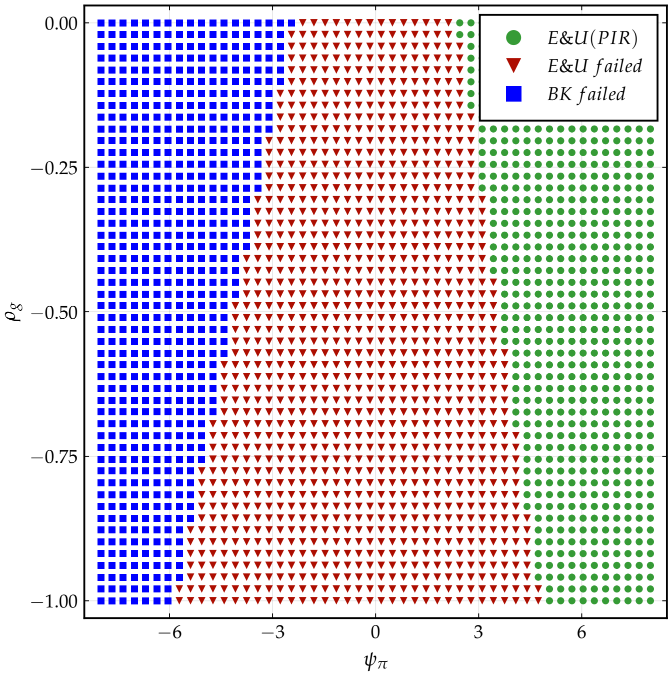

Figure 3.5: Existence and uniqueness region with less persistent fiscal rule

Notes: Green circles denote the parameter space where the solution is unique and locally determinate. Red triangles denote the region where the solution is either non-unique or non-existent. Blue squares denote the region where the solution is locally indeterminate.

We first present computational results for E&U conditions for the parameters ρ_g and ψ_π in (3.22). These are illustrated in Figure 3.5. As government spending becomes less persistent ($\rho_g \to -1$), a stronger contemporaneous fiscal response to inflation is required for E&U. As we decrease inertia in the government spending rule ($\rho_g \to 0$), E&U of a unique MSV solution is easier to satisfy (i.e., requires a lower value of ψ_π).

We now provide some intuition through the derivation of analytical results.[22] Suppose that the economy starts in a ZIR transitory state equilibrium where where $\varepsilon_t = \varepsilon^T$ and then converges to a PIR in the absorbing state where $\varepsilon_t = 0$. In other words, we wish to show the restrictions on FP needed for a ZIR equilibrium to exist in the temporary state, given that the agents expect to move to the stable manifold of the PIR equilibrium system as soon as the shock vanishes.

---

[22] As in Section 3.3, assume that $\phi_y = \psi_y = 0$, and ε_t follows a two-state Markov chain process ($k = 2$) with states $\varepsilon_t = (\varepsilon^T, 0)$ with a transition kernel $\boldsymbol{K}$ specified as Equation (3.5).

Absorbing State. We proceed as in Section 3.3 by first considering the absorbing state of the model, when $\varepsilon_t = 0$. We solve the system as in (3.16):

$$\hat{\pi} = \frac{\kappa_y}{1-\beta}\hat{x} \quad AS, \tag{3.25a}$$

$$\hat{\pi} = \begin{cases} \left(\phi_\pi + \frac{\Gamma\psi_\pi}{1-\rho_g}\right)\frac{\kappa_y}{1-\beta}\hat{x} & AD^{TR} \text{ for } \hat{\pi} > -\frac{\mu[\Gamma\psi_\pi+(1-\rho_g)\phi_\pi]}{(1-\rho_g)\phi_\pi}, \\ -\mu + \frac{\Gamma\psi_\pi}{1-\rho_g}\frac{\kappa_y}{1-\beta}\hat{x} & AD^{ELB} \text{ for } \hat{\pi} \leq -\frac{\mu[\Gamma\psi_\pi+(1-\rho_g)\phi_\pi]}{(1-\rho_g)\phi_\pi}. \end{cases} \tag{3.25b}$$

If $\rho_g = 0$, the system (3.25) will have the same expression as in (3.16). Therefore, the graphical representation would be the similar to Figure 3.2, with slight differences in the slopes of the curves. When AD^{ELB} is steeper than AS or when AD^{ELB} is negative sloping, i.e., $\Gamma\psi_\pi > 1-\rho_g$ or $\Gamma\psi_\pi < 0$ we have a unique equilibrium, which corresponds to the PIR equilibrium, given by $\left(\hat{\pi}, \hat{x}, \hat{\imath}, \Delta\hat{g}\right) = (0,0,0,0)$. Whereas for $0 < \Gamma\psi_\pi < 1-\rho_g$, the system admits two equilibria, a PIR equilibrium and a ZIR equilibrium. The ZIR equilibrium is given by:

$$\left(\hat{\pi}, \hat{x}, \hat{\imath}, \Delta\hat{g}\right) = \left(-\frac{(1-\rho_g)\,\mu}{1-\rho_g-\Gamma\psi_\pi}, -\frac{(1-\rho_g)\,(1-\beta)\,\mu}{(1-\rho_g-\Gamma\psi_\pi)\,\kappa_y}, -\mu, -\frac{(1-\rho_g)\,\psi_\pi\mu}{1-\rho_g-\Gamma\psi_\pi}\right).$$

The transitory state admits endogenous dynamics because of the presence of the endogenous variable $\Delta\hat{g}$. Off of the steady states, the economy will travel along a stable trajectory that leads to the steady states. We then explore the conditions under which the following hold:

1. When the shock disappears the economy will converge to the PIR along the stable manifold;
2. the solution is an MSV solution in the sense that it depends just on state variables;
3. in the transitory state where $\varepsilon_t = \varepsilon^T$, the economy will be in a ZIR.

Under these assumptions, once the shock disappears then we must be on the unique stable manifold that leads to the PIR.[23] Assumption (i) is key as it pins down the expectations in the absorbing state, similar to the proof in Section 3.3. However, rather than jump to the PIR steady state as when the model is forward-looking, the system will arrive there inertially along the unique stable manifold.

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[23] Here, we focus solely on convergence to PIR steady state. This is motivated by the fact that the appearance of ZIR steady state necessarily implies the presence of multiple equilibria, which trivially undermines the uniqueness of the solution.

To find the MSV solution of the PIR system, we use the method of undetermined coefficients and assume a solution of this form:

$$\begin{aligned}\hat{\pi}_t &= \gamma_\pi \Delta \hat{g}_t,\\ \hat{x}_t &= \gamma_x \Delta \hat{g}_t,\\ \Delta \hat{g}_{t+1} &= \gamma_g \Delta \hat{g}_t.\end{aligned}$$

Substituting into the system (3.12) gives the following equation in γ_g:

$$\begin{aligned}0 = {} & (1-\beta\gamma_g)(\gamma_g-\rho_g)(\gamma_g-1)\\ & -\frac{s_c\kappa_y}{\sigma}\left[(\gamma_g-\rho_g)(1-\rho_g)\frac{\phi_\pi-\gamma_g}{\psi_\pi}+\Gamma\psi_\pi\Big(\rho_g+\gamma_g-\rho_g\Big)\right].\end{aligned} \tag{3.26}$$

If there exists a unique solution within the unit circle, i.e., $|\gamma_g| < 1$, the dynamics along the stable trajectory are given by the recursions:

$$\begin{aligned}\hat{\pi}_{t+j} &= \gamma_\pi \gamma_g^j \Delta \hat{g}_t,\\ \hat{x}_{t+j} &= \gamma_x \gamma_g^j \Delta \hat{g}_t,\\ \Delta \hat{g}_{t+j+1} &= \gamma_g^{j+1} \Delta \hat{g}_t.\end{aligned}$$

If $\Delta \hat{g}_t = \nu_g$ then simply

$$\begin{aligned}\hat{\pi}_t &= \gamma_\pi \nu_g,\\ \hat{x}_t &= \gamma_x \nu_g,\\ \hat{g}_{t+1} &= \gamma_g \nu_g.\end{aligned}$$

Importantly, if $|\gamma_g| < 1$, the system will never be in a ZIR state when the shock vanishes, since $\hat{g}_{t+1} = \gamma_g \nu_g < \nu_g$. The value of $|\gamma_g|$ depends on the fiscal policy rule, ρ_g, and ψ_π. Holding other parameters constant, a more negative ρ_g requires higher a ψ_π to ensure that $|\gamma_g| < 1$.

Transitory State. Next, consider the ZIR transitory state. Assume that $\Delta \hat{g}_t = \nu_g$, to eliminate the endogenous state, allowing the system to be written as completely forward-looking. If an MSV solution exists, it will be constant (π^T, x^T) and with probability $(1-p)$ we are back on the manifold of the PIR absorbing state. The expectations are thus:

$$\begin{aligned}\mathbb{E}_t\hat{\pi}_{t+1} &= p\hat{\pi}^T + (1-p)\,\gamma_\pi\nu_g,\\ \mathbb{E}_t\hat{x}_{t+1} &= p\hat{x}^T + (1-p)\,\gamma_x\nu_g.\end{aligned}$$

We can then write the ZIR transitory system as:

$$\hat{\pi}^T = \frac{\kappa_y}{(1-\beta p)}\hat{x}^T + \frac{\beta\,(1-p)\,\gamma_\pi\nu_g}{(1-\beta p)} \quad AS, \tag{3.27a}$$

$$\hat{\pi}^T = \begin{cases} \frac{\sigma(1-p)}{s_c(p-\phi_\pi-\Gamma\psi_\pi)}\hat{x}^T - \frac{\sigma(1-p)\gamma_x\nu_g}{s_c(p-\phi_\pi-\Gamma\psi_\pi)} - \frac{(1-p)\gamma_\pi\nu_g-\Gamma\rho_g\nu_g+\sigma\varepsilon^T}{p-\phi_\pi-\Gamma\psi_\pi} & AD^{TR} \text{ for } \hat{\pi}^T \geq -\frac{\mu}{\phi_\pi},\\ \frac{\sigma(1-p)}{s_c(p-\Gamma\psi_\pi)}\hat{x}^T - \frac{\sigma(1-p)\gamma_x\nu_g}{s_c(p-\Gamma\psi_\pi)} - \frac{\mu+(1-p)\gamma_\pi\nu_g-\Gamma\rho_g\nu_g+\sigma\varepsilon^T}{p-\Gamma\psi_\pi} & AD^{ELB} \text{ for } \hat{\pi}^T \leq -\frac{\mu}{\phi_\pi}. \end{cases} \tag{3.27b}$$

We could plot this system and it would be similar to Figure 3.3, since the slopes are identical but different intercepts. The same reasoning therefore applies. For E&U, FP must satisfy the same condition specified in Equation (3.19). The results are summarised by Proposition 6.

Proposition 6. *When the Blanchard-Kahn condition is satisfied, the New Keynesian model with fiscal policy as defined in* (3.22) *has a unique minimum state variable solution such that the economy is in a zero interest rate transitory state when* $\varepsilon_t = \varepsilon^T$ *and converges to a positive interest rate absorbing state* ($q = 1$) *when* $\varepsilon_t = 0$ *if*

$$\Gamma\psi_\pi > \max 1-\rho_g, p - \frac{\sigma\,(1-p)\,(1-\beta p)}{s_c\kappa_y}, \tag{3.28}$$

and where $|\gamma_g| < 1$ *depends on the fiscal policy as specified in Equation* (3.26).

Proposition 6 nests Proposition 5 by setting $\rho_g = 0$. From Proposition 6, inertia in the government spending rule affects the solution in two ways. First, for Equation (3.28) to hold, a larger value of ψ_π is required as $\rho_g \leq 0$ becomes more negative. That is, if government spending is less persistent (i.e., ρ_g is more negative), then to ensure uniqueness we require a larger fiscal response to current inflation (i.e., a higher ψ_π) than we would otherwise. When $\rho_g = 0$, we are back to the solution in Section 3.3, i.e., $\Gamma\psi_\pi > 1$.

Second, to ensure that the system transitions to a PIR equilibrium when the shock vanishes, we require that $|\gamma_g| < 1$. Importantly, the value of $|\gamma_g|$ depends on the fiscal policy rule parameters ρ_g and ψ_π. Fixing other parameter values as in Table 3.1 and setting $\phi_y = \psi_y = 0$, a more negative ρ_g increases the absolute value of γ_g, implying that more inertial government spending reduces the speed at which the economy transitions to the steady state.[24] On the other hand, a more aggressive response to inflation reduces the absolute value of γ_g and pushes the economy back to the steady state more quickly. Therefore, if government spending exhibits greater inertia (a very negative ρ_g), a larger response in $\Delta\mathbb{E}_t g_{t+1}$ to current inflation (a very high ψ_π) is required. For example, the following two combinations both yield $|\gamma_g| = -0.1$: $(\rho_g, \psi_\pi) = (-0.1, 1.4)$ and $(\rho_g, \psi_\pi) = (-0.2, 25.9)$.

Note that by setting $\rho_g = 0$, we obtain $\gamma_g = 0$. Consequently, in the next period we have $\left(\hat{\pi}, \hat{x}, \hat{\imath}, \Delta\hat{g}\right) = (0, 0, 0, 0)$. This is because when $\rho_g = 0$ the system no longer has endogenous state variables, therefore, it immediately jumps to the PIR equilibrium rather than arriving there inertially along the unique stable manifold.[25]

3.5 Conclusion

This paper explores how fiscal policy can restore equilibrium uniqueness in a baseline New Keynesian model subject to an occasionally binding effective lower bound constraint on the interest rate. Our findings suggest that simple fiscal policy can guarantee a unique solution that is also locally determinate. We establish that in order to guarantee a minimum state variable solution and local determinacy, fiscal policy needs to be sufficiently persistent and aggressive in its response to inflation.

First, we analytically verified that if the fiscal authority is able to credibly commit to a sufficiently strong countercyclical permanent policy change in response to an exogenous disturbance, solution uniqueness in the model is restored. This conclusion is rationalised by the fact that fiscal policy is not constrained by the effective lower bound and provides an active policy response when monetary policy is constrained. Moreover, by committing to a permanent policy change, the fiscal authority is able to alleviate the fundamental uncertainty

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[24] If government spending is very persistent (for example, if $\rho_g = -1$), then $|\gamma_g|$ might exceed unity.

[25] These results also hold for an inertial rule for government spending in levels,

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + (1 - \rho_g)(\psi_\pi^* \hat{\pi}_t + \psi_y^* \hat{x}_t), \quad \rho_g \in [0, 1].$$

that engenders multiplicity of equilibria in the baseline New Keynesian model.

Second, we find that the fiscal response need not imply a permanent policy change but rather it has to be sufficiently persistent to guarantee existence and uniqueness of a minimum state variable solution. The persistence property of the policy rule, coupled with it being sufficiently countercyclical, are needed to eliminate belief-driven equilibria and pin down a unique solution. By showing this, we address the main concerns raised by Ascari and Mavroeidis (2022) about New Keynesian models featuring occasionally binding constraints.

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APPENDIX

Appendix to Chapter 1

A.1 Supplementary Evidence on Expectations and Attention Choices

A.1.1 Individual-level Evidence

The pattern plotted in Figure 1.1 also holds when controlling for individual-level fixed effects. I leverage the panel dimension of the surveys. Focusing on respondents who appear at least three times in the Michigan Survey of Consumers (MSC), I conduct individual regressions – mirroring the approach above – for each respondent.[26] The results explicitly characterise how households' beliefs about the inflation and output growth expectations evolve jointly. Of the 4,276 respondents interviewed at least three times, 75.3% demonstrate a negative slope, implying that when households increase their inflation forecasts between subsequent interviews, they also predict more adverse economic conditions going forward.

The firm pool is relatively stable, with firms being asked between 1 and 38 times. I focus on the firms that have at least 5 observations and run the regression for each firm. Around 54.3% of the firms show a positive slope and 45.7% show a negative slope. The Survey of Professional Forecasters is also a panel with relatively longer time span, and I focus on forecasters with at least 10 observations, of which 73.7% have a positive slope.

A.1.2 Stylised Facts on Attention Choices

More evidence on attention choices can be obtained by looking at surveys of what information agents have. The Michigan Survey of Consumers asks respon-

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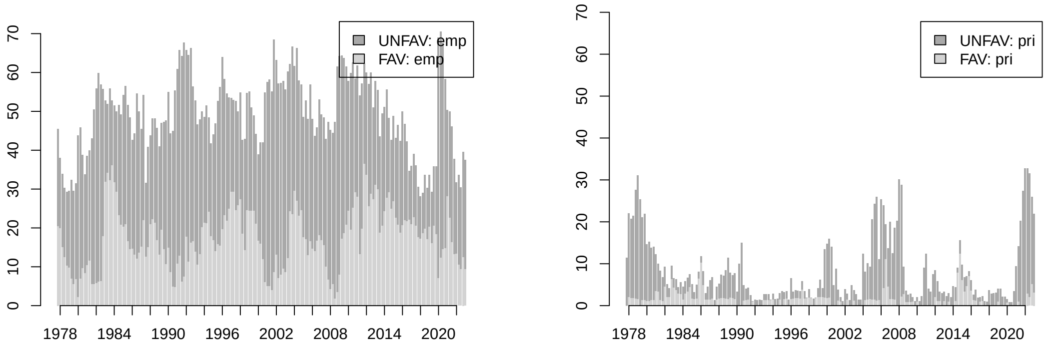
[26] The survey features a rotating panel sample design. Typically, any given survey sample from the MSC comprises two-thirds new respondents and one-third being re-interviewed. This setup creates a short panel where each cross-sectional unit appears in the survey more than once.

dents to report news related to business conditions that they have heard during the last few months while making their predictions about inflation and output growth in the next year.[27] Figure A.6 shows spike plots for news heard that is price-related or employment-related over time. The news households consistently pay attention to is employment-related, while news about prices stands out only in particular periods, indicating a consistently high level of attention to the real side of the economy among households.



(a) News about employment

(b) News about prices

Figure A.6: Spike plots of news heard categories

Note: These plots show the fraction of survey respondents having heard news in each category in the relevant quarter. Each category further distinguishes between favourable (depicted in light grey) and unfavourable news (shown in dark grey).

On the firm side, the Business Inflation Expectations (BIE) survey reveals that firms have strong incentives to pay attention to nominal marginal cost, and these play a significant role in their price-setting strategies. Specifically, from 2011 to 2023, 69% of respondents in the BIE survey indicated that labour costs would affect the prices of their products and/or services in the upcoming 12 months.

A.1.3 Attention Choices of Households Shape Their Beliefs

The Michigan Survey shows that consumers overall pay more attention to news related to the labour market, but does the degree to which individual households pay attention to the different types of news affect how they perceive the relationship between growth and inflation? To test this, I run the following regression:

$$\begin{aligned}\mathbb{E}_t^i[Growth] = \beta_0 + \beta_1 \mathbb{E}_t^i[Inflation] + \gamma_1 \mathbb{E}_t^i[Inflation] \times News_{i,t}^{labour} + \gamma_2 \mathbb{E}_t^i[Inflation] \times News_{i,t}^{price} \\ + \alpha_1 News_{i,t}^{labour} + \alpha_2 News_{i,t}^{price} + \alpha_t + u_{i,t} \qquad \text{(A.29)}\end{aligned}$$

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[27] A detailed description of the question, along with a comprehensive list of categories, is available on the Michigan Survey of Consumers.

Here the labour news $News_{i,t}^{labour}$ is a binary variable, taking a value of 1 if a respondent i reports having heard news about labour market conditions recently, and 0 otherwise. Similarly, the price news variable $News_{i,t}^{price}$ is set to 1 if the respondent i has recently heard news related to prices, and 0 otherwise. A supply-side view corresponds to a negative β_1. If the coefficient of the cross term is negative $\gamma_1 < 0$ or $\gamma_2 < 0$, attention to that news contributes to a supply-side view. Conversely, a positive coefficient $\gamma_1 > 0$ or $\gamma_2 > 0$ suggests that paying attention to this news contributes to a more demand-side view.

Table A.2 reports the results of this regression and finds $\gamma_1 < 0$ and statistically significant – households who pay attention to labour market news hold an even stronger supply-side view compared to those who do not. Conversely, attention to price-related news appears to contribute to a demand-side view $\gamma_2 > 0$, though its impact is relatively muted. The results still hold when I divide the labour news into positive news and negative news (see Column 2 and 3 in Table A.2), which should allay any concern that the results are biased by labour news being more likely to be negative than positive. These results also rule out pessimism as the sole explanation for the negative correlation between inflation and output.[28]

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[28] Bhandari et al. (2022) find that increased pessimism generates an upward bias in unemployment and inflation forecasts, contributing to the negative correlation between inflation and real activity. However, the results in this paper suggest that pessimism is only part of the explanation behind households' supply-side view.

Table A.2: Attention choices matter for beliefs

	Growth forecasts		
	All	labour news (+)	labour news (-)
Inflation forecasts	-0.047^{***}	-0.047^{***}	-0.047^{***}
	(0.001)	(0.001)	(0.001)
Inflation forecasts × labour news	-0.0186^{**}	-0.019^{*}	-0.013^{*}
	(0.007)	(0.010)	(0.008)
Inflation forecasts × price news	0.006	0.006	0.006
	(0.027)	(0.027)	(0.027)
Labour news	-0.091^{***}	0.150^{***}	-0.237^{***}
	(0.025)	(0.025)	(0.022)
Price news	0.061	0.061	0.061
	(0.073)	(0.073)	(0.073)
Intercept	0.019	0.019	0.020
	(0.002)	(0.002)	(0.002)

Note: The table presents the results of regression (A.29). Column 1 presents the results for the full sample. Column 2 and 3 show that the results are robust even when dividing labour news into favourable/unfavourable categories.

A.2 Proofs for Section 1.2

The derivation in this section and Appendix A.3 largely build on the rational inattention literature, particularly Maćkowiak and Wiederholt (2025) and Afrouzi and Yang (2021).

A.2.1 Approximation of Household's Utility Function

Household i's per-period utility at time t is given by:

$$U\left(C_{i,t}, L_{i,t}\right) = \frac{C_{i,t}^{1-\gamma}}{1-\gamma} - \frac{L_{i,t}^{1+\eta}}{1+\eta}$$

As households are hand-to-mouth, labour supply can be substituted using the budget constraint $L_{it} = (P_t C_t)/W_t$. The utility then becomes

$$U\left(C_{i,t}, L_{i,t}\right) = \frac{C_{i,t}^{1-\gamma}}{1-\gamma} - \frac{\left(\frac{P_t C_{i,t}}{W_t}\right)^{1+\eta}}{1+\eta}$$

Households take wages and prices as given, meaning the only choice variable is consumption C_{it}. Expressing the per-period utility function in terms of log-deviations from the non-stochastic steady state yields

$$\hat{u}\left(c_{i,t}, p_t, w_t\right) = \left[\frac{\left(\bar{C}e^{c_{i,t}}\right)^{1-\gamma}}{1-\gamma} - \frac{\left(\frac{\bar{P}e^{p_t}\bar{C}e^{c_{i,t}}}{\bar{W}e^{w_t}}\right)^{1+\eta}}{1+\eta}\right]$$

The per-period utility of household i depends on choice variable $c_{i,t}$ and variables that the household takes as given, namely $\{w_t, p_t\}$. For any given $\{w_t, p_t\}$, the consumption level that maximises utility is

$$c_{i,t}^{*} = \arg\max_{c_{i,t}} \hat{u}\left(c_{i,t}, p_t, w_t\right) \Leftrightarrow \hat{u}_1\left(c_{i,t}^{*}, p_t, w_t\right) = 0$$

Taking a second-order approximation of the utility function $L\left(c_{i,t}, p_t, w_t\right) \equiv \hat{u}\left(c_{i,t}, p_t, w_t\right) - \hat{u}(c_{i,t}^{*}, p_t, w_t)$ around the steady state yields

$$\begin{aligned} L\left(c_{i,t}, p_t, w_t\right) = \frac{1}{2}\hat{u}_{11}\left(c_{i,t}^2 - {c_{i,t}^{*}}^2\right) + \hat{u}_{12}p_t\left(c_{i,t} - c_{i,t}^{*}\right) \\ + \hat{u}_{12}w_t\left(c_{i,t} - c_{i,t}^{*}\right) + \mathcal{O}\left(\|c_{i,t}, p_t, w_t\|^3\right) \end{aligned} \qquad \text{(A.30)}$$

where $\hat{u}_{1,n}, n \in \{1, 2, 3\}$ denotes the second-order derivatives of the utility function with respect to $c_{i,t}$, $c_{i,t}$ and p_t, and $c_{i,t}$ and w_t around the approximation point. Since $c_{i,t}^{*}$ maximises utility for any p_t and w_t,

$$\hat{u}_1(c_{i,t}^{*}, p_t, w_t) = 0 \Rightarrow \hat{u}_{11}c_{i,t}^{*} + \hat{u}_{12}p_t + \hat{u}_{13}w_t + \mathcal{O}\left(\|p_t, w_t\|^2\right) = 0$$

Combining this with Equation (A.30) I obtain

$$\begin{aligned} \hat{u}\left(c_{i,t}, p_t, w_t\right) &= L\left(c_{i,t}, p_t, w_t\right) + \hat{u}\left(c_{i,t}^{*}, p_t, w_t\right) \\ &= \frac{1}{2}\hat{u}_{11}\left(c_{i,t} - c_{i,t}^{*}\right)^2 + \mathcal{O}\left(\|c_{i,t}, p_t, w_t\|^3\right) + \text{terms independent of } c_{i,t} \end{aligned}$$

Given the utility specification, $\hat{u}_{11} = -(\gamma + \eta)$ in the steady state. Moreover, optimal consumption is given by

$$c_{i,t}^{*} = \frac{1+\eta}{\gamma+\eta}\left(w_t - p_t\right)$$

Therefore, household i's objective (1.1) can be approximated as

$$\left[-\frac{(\gamma+\eta)}{2}\left(c_{i,t} - c_{i,t}^{*}\right)^2\right] + \text{terms independent of } \{c_{i,t}\}_{t\geq 0}$$

A.2.2 Approximation of Firm's Profit Function

First, substitute the production function and demand function into firm j's per-period profit function

$$\Pi\left(P_{j,t}, W_t, X_t\right) = \frac{1}{P_t C_t}\left[P_{j,t}\left(\frac{P_{j,t}}{P_t}\right)^{-\theta} Y_t - (1-\theta^{-1})\frac{W_t}{A_t}\left(\frac{P_{j,t}}{P_t}\right)^{-\theta} Y_t\right]$$

The per-period profit function can then be expressed in terms of log-deviations from the non-stochastic steady state

$$\hat{\pi}\left(p_{jt}, w_t, a_t, x_t\right) = \bar{C}e^{c_t}e^{-\theta(p_{jt}-p_t)-p_t}\left[e^{p_{jt}} - \left(1-\theta^{-1}\right)e^{w_t-a_t}\right]$$

where the lowercase letters denote the log-deviations of the corresponding variable. For any given $\{w_t, p_t, y_t, a_t\}$,

$$p_{jt}^{*} = \arg\max_{p_{jt}} \hat{\pi}\left(p_{jt}, p_t, w_t, y_t, a_t\right) \Leftrightarrow \hat{\pi}_1\left(p_{jt}, p_t, w_t, y_t, a_t\right) = 0$$

Define function $L\left(p_{jt}, p_t, w_t, y_t, a_t\right) \equiv \hat{\pi}\left(p_{jt}, p_t, w_t, y_t, a_t\right) - \hat{\pi}\left(p_{jt}^{*}, p_t, w_t, y_t, a_t\right)$, and take a second-order approximation around the steady state

$$\begin{aligned} L\left(p_{jt}, p_t, w_t, y_t, a_t\right) = {} & \frac{1}{2}\hat{\pi}_{11}\left(p_{jt}^2 - p_{jt}^{*\,2}\right) + \hat{\pi}_{12}p_t\left(p_{jt} - p_{jt}^{*}\right) + \hat{\pi}_{13}w_t\left(p_{jt} - p_{jt}^{*}\right) \\ & + \hat{\pi}_{14}y_t\left(p_{jt} - p_{jt}^{*}\right) + \hat{\pi}_{15}a_t\left(p_{jt} - p_{jt}^{*}\right) + \mathcal{O}\left(\left\|p_{jt}, p_t, w_t, y_t, a_t\right\|^3\right) \end{aligned} \tag{A.31}$$

Here $\hat{p}_{1,n}, n \in \{1,2,3,4,5\}$ denotes second-order derivatives of the profit function with respect to p_{jt}, p_{jt} and p_t, p_{jt} and w_t, p_{jt} and y_t, and p_{jt} and a_t around the approximation point. Note that since p_{jt}^{*} maximises the profit function for any given $\{w_t, p_t, y_t, a_t\}$, I have

$$\hat{\pi}\left(p_{jt}^{*}, p_t, w_t, y_t, a_t\right) = 0 \Rightarrow \hat{\pi}_{11}p_{jt}^{*} + \hat{\pi}_{12}p_t + \hat{\pi}_{13}w_t + \hat{\pi}_{13}y_t + \hat{\pi}_{13}a_t + \mathcal{O}\left(\left\|p_t, w_t, a_t, y_t\right\|^2\right) = 0$$

Combining this result with Equation (A.31) I obtain

$$\begin{aligned} \hat{\pi}\left(p_{jt}, p_t, w_t, y_t, a_t\right) &= L\left(p_{jt}, p_t, w_t, y_t, a_t\right) + \hat{\pi}\left(p_{jt}^{*}, p_t, w_t, y_t, a_t\right) \\ &= \frac{1}{2}\hat{\pi}_{11}\left(p_{jt} - p_{jt}^{*}\right)^2 + \mathcal{O}\left(\left\|p_{jt}, p_t, w_t, y_t, a_t\right\|^3\right) + \text{terms independent of } p_{jt} \end{aligned}$$

Given the particular profit function, $\hat{\pi}_{11} = -(\theta - 1)$ in the steady state. And the optimal price

$$p^*_{jt} = w_t - a_t$$

Hence, the firm j's objective (1.4) is approximated by

$$\sum_{t=0}^{\infty} \beta^t \mathbb{E}^j \left[-\frac{\theta - 1}{2} \left(p_{jt} - p^*_{jt} \right)^2 \right] + \text{terms independent of } \{p_{jt}\}_{t \geq 0}$$

A.2.3 Proof of Proposition 1

Upon reception of a signal $s_{i,a,t} = a_t + e_{i,a,t}$, the consumption $c_{i,t} = \lambda_{h,a} \mathbb{E}[a_t | s_{i,a,t}]$ maximises the expected utility (1.14) for any given posterior belief. Bayesian updating with Gaussian prior uncertainty and signals delivers

$$\mathbb{E}\left[a_t | s_{i,a,t}\right] = \xi_{h,a}[a_t + e_{i,a,t}]$$

where $\xi_{h,a} \equiv (1 - \sigma^2_{a|s}/\sigma^2_a) \in [0, 1]$, and $\xi_{h,a}$ is the Kalman-gain on the signal. Now rewrite the problem (1.14) in terms of choice variable $\xi_{h,a}$

$$\max_{\xi_{h,a} \in [0,1]} \left[-\left(\gamma + \eta\right) \lambda^2_{h,a} \left(1 - \xi_{h,a}\right) \sigma^2_a - \mu^h \ln \frac{1}{1 - \xi_{h,a}} \right]$$

Solving the first order condition, the solution is

$$\xi_{h,a} = \max \left(0, 1 - \frac{\mu^h}{(\gamma + \eta)\lambda^2_{h,a}\sigma^2_a} \right)$$

A.2.4 Proof of Proposition 2

By the independence assumption, I can solve the firms attention choices for aggregate demand shock and the productivity shock separately.

In the case of demand shocks, the signals take the form $s_{j,q,t} = q_t + e_{j,q,t}$. To derive firms' attention choices, it is instructive to first express the firms' ex-ante expected utility as a function of their attention choices. Note that firm j's prior uncertainty about q_t is simply σ^2_q, and denote firm j's posterior uncertainty as

$\sigma_{q|s_j} \equiv var(q_t|s_{j,q,t})$. The firm j's attention problem is then

$$\max_{\{s_{j,q,t}\in\mathcal{S}_f^t\}} \mathbb{E}_t^f \left[-\frac{\theta-1}{2}\left(\mathbb{E}[p^*_{j,t}|s_{j,q,t}] - p^*_{j,t}\right)^2 - \mu^f \mathcal{I}\left(q_t; s_{j,q,t}\right)\right]$$

$$= \frac{1}{2}\max_{\sigma^2_{q|s_j}\leq\sigma^2_q} \left[-(\theta-1)\lambda^2_{f,q}\sigma^2_{q|s_j} - \mu^f \ln\frac{\sigma^2_q}{\sigma^2_{q|s_j}}\right] \tag{A.32}$$

For every realisation of the signal at time t, the firm will set price $p_{j,t} = \mathbb{E}[p^*_{j,t}|s_{j,q,t}]$. Hence, the expected profit depends on the expected square deviation of $\mathbb{E}[p^*_{j,t}|s_{j,q,t}]$ from $p^*_{j,t}$, which reduces to the conditional variance in (A.32).

Upon reception of a signal $s_{j,q,t} = q_t + e_{j,q,t}$, the price $p_{j,t} = \mathbb{E}[p^*_{j,t}|s_{j,q,t}]$ maximises the expected profit for any given posterior belief. Bayesian updating with Gaussian prior uncertainty and signals yields

$$\mathbb{E}\left[p^*_{j,t}|s_{j,q,t}\right] = \xi_{f,q}\lambda_{f,q}[q_t + e_{j,q,t}]$$

where $\xi_{f,q} \equiv (1 - \sigma^2_{q|s_j}/\sigma^2_q) \in [0,1]$ is the attention weight on the signal. I can now rewrite the problem (A.32) in terms of the choice variable $\xi_{f,q}$

$$\max_{\xi_{f,q}\in[0,1]} \left[-\left(\theta-1\right)\lambda^2_{f,q}\left(1-\xi_{f,q}\right)\sigma^2_q - \mu^f \ln\frac{1}{1-\xi_{f,q}}\right]$$

Solving gives the expression in Equation (1.17a)

$$\xi_{f,q} = \max\left(0, 1 - \frac{\mu^f}{(\theta-1)\lambda^2_{f,q}\sigma^2_q}\right)$$

By the same procedure, I can solve the attention problem for supply shocks a_t.

In the case of productivity shocks, the firm's attention problem is

$$\max_{\sigma^2_{a|s_j}\leq\sigma^2_a} \left[-\left(\theta-1\right)\lambda^2_{f,a}\sigma^2_{a|s_j} - \mu^f \ln\left(\frac{\sigma^2_a}{\sigma^2_{a|s_j}}\right)\right]$$

where $\lambda_{f,a} = -\frac{1+\eta}{\gamma+\eta}$, σ^2_a is the prior variance of firm j's belief about the productivity shock and $\sigma^2_{a|s_j}$ denotes the posterior variance.

Upon reception of a signal $s_{j,a,t} = a_t + e_{j,a,t}$, the price $p_{j,t} = \mathbb{E}[p^*_{j,t}|s_{j,a,t}]$ maximises the expected profit for any given posterior belief. Bayesian updating with

Gaussian prior uncertainty and signals yields

$$\mathbb{E}\left[p^*_{j,t}|s_{j,a,t}\right] = \xi_{f,a}\lambda_{f,a}[a_t + e_{j,a,t}]$$

where $\xi_{f,a} \equiv (1 - \sigma^2_{a|s_j}/\sigma^2_a) \in [0,1]$, and $\lambda_{f,a}\xi_{f,a}$ reflects the attention weight on the signal. I can now rewrite the firms' attention problem in terms of choice variable $\xi_{f,a}$

$$\max_{\xi_{f,a}\in[0,1]} \left[-(\theta-1)\,\lambda^2_{f,a}\,(1-\xi_{f,a})\,\sigma^2_a - \mu^f \ln\frac{1}{1-\xi_{f,a}}\right]$$

Solving gives the expression in Equation (1.17b)

$$\xi_{f,a} = \max\left(0, 1 - \frac{\mu^f}{(\theta-1)\lambda^2_{f,a}\sigma^2_a}\right)$$

Combining these results together gives the Proposition 2.

A.2.5 Proof of Corollary 3

Under optimal signal design, firms optimally choose to receive a single signal of the optimal price, i.e., $s_{j,t} = p^*_{j,t} + e_{j,t} = \lambda_{f,q}q_t + \lambda_{f,a}a_t + e_{j,t}$ where $e_{j,t}$ is the attention error. Upon receiving this signal, the price $p_{j,t} = \mathbb{E}[p^*_{j,t}|s_{j,t}]$ maximises the expected profit for any given posterior belief. Therefore, the objective can be expressed as

$$\begin{aligned}
&\max_{\{s_{j,t}\in\mathcal{S}^t_f\}} \mathbb{E}^f_t\left[-\frac{\theta-1}{2}\left(\mathbb{E}[p^*_{j,t}|s_{j,t}] - p^*_{j,t}\right)^2 - \mu^f\mathcal{I}\left(q_t, a_t; s_{j,t}\right)\right] \\
&= \frac{1}{2}\max_{\sigma^2_{p|s}\le\sigma^2_p}\left[-(\theta-1)\sigma^2_{p|s} - \mu^f \ln\left(\frac{\sigma^2_p}{\sigma^2_{p|s}}\right)\right]
\end{aligned}$$

where $\sigma^2_p \equiv \lambda^2_{f,q}\sigma^2_q + \lambda^2_{f,a}\sigma^2_a$ denotes the prior uncertainty about $p^*_{j,t}$ and $\sigma^2_{p|s}$ denotes the posterior uncertainty. Solve the model, the firm sets a price according to

$$p_{j,t} = \xi_f\left(p^*_{j,t} + e_{j.t}\right) = \xi_f\left(\lambda_{f,q}q_t + \lambda_{f,a}a_t + e_{j,t}\right) \tag{A.33}$$

with

$$\xi_f = \max\left(0, 1 - \frac{\mu^f}{(\theta-1)\sigma^2_p}\right)$$

From Equation (A.33), the weights on the demand shock (q_t) and the supply shock (a_t) are $\xi_f\lambda_{f,q}$ and $\xi_f\lambda_{f,a}$, respectively.

A.3 Proofs for Quantitative Model

A.3.1 Approximation of Households' Utility Function

First, using the flow budget constraint (1.26) to substitute for labour in the utility function and expressing all variables in terms of log-deviations from the non-stochastic steady state I obtain the following expression for the per-period utility of household i in period t:

$$u = \left(\frac{\bar{C}^{1-\gamma}}{1-\gamma} e^{(1-\gamma)c_{i,t}} - \frac{\left[\frac{\bar{P}\bar{C}e^{p_t + c_{i,t}} + \bar{B}e^{b_{i,t}} - \bar{R}\bar{B}e^{i_{t-1} + b_{i,t-1}} - \bar{D}e^{d_t} - \bar{T}e^{\tau_t}}{\bar{W}e^{w_t}} \right]^{1+\eta}}{1+\eta} \right)$$

Here, the lowercase letters denote the log-deviations of the corresponding variables. $c_{i,t}$ is the consumption by household i, $\tilde{b}_{i,t}$ is the real bond holdings, $\tilde{d}_t$ is the real dividends, and $\tilde{\tau}_t$ is the real transfers (taxes if negative). Define the steady state ratios

$$(\omega_B, \omega_W, \omega_D, \omega_T) = \left(\frac{\bar{B}}{\bar{C}\bar{P}}, \frac{\bar{W}\bar{L}}{\bar{C}\bar{P}}, \frac{\bar{D}}{\bar{C}\bar{P}}, \frac{\bar{T}}{\bar{C}\bar{P}} \right)$$

In period t, household i chooses $v_t \equiv (\tilde{b}_{i,t}, c_{i,t})'$, the choices made in previous period represented by $v_{t-1} = (\tilde{b}_{i,t-1}, 0)'$. Households take following variables as given $\zeta_t \equiv [\tilde{d}_t, i_{t-1}, \tilde{w}_t, \tilde{\tau}_t, \pi_t]'$.

A log-quadratic approximation to the expected discounted sum of per-period utility around the non-stochastic steady state yields

$$\sum_{t=0}^{\infty} \beta^t \mathbb{E}_i^h \left[\frac{1}{2} (v_t - v_t^*)' \Theta_0 (v_t - v_t^*) + (v_t - v_t^*) \Theta_1 \left(v_{t+1} - v_{t+1}^* \right) \right] \tag{A.34}$$

where

$$\Theta_0 = -\bar{C}_i^{1-\gamma} \begin{bmatrix} \frac{\eta}{\omega_W} \left[1 + \frac{1}{\beta} \right] \omega_B^2 & \frac{\eta}{\omega_W} \omega_B \\ \frac{\eta}{\omega_W} \omega_B & \left(\gamma + \frac{\eta}{\omega_W} \right) \end{bmatrix}, \qquad \Theta_1 = \bar{C}^{1-\gamma} \begin{bmatrix} \frac{\eta}{\omega_W} \omega_B^2 & \frac{\eta}{\omega_W} \omega_B \\ 0 & 0 \end{bmatrix}$$

The sequence of optimal bond holdings under full information is given by

$$\omega_B \left(\frac{1}{\beta} \tilde{b}_{i,t-1}^* - \tilde{b}_{i,t}^* \right) + c_{i,t}^* = \mathbb{E}_t \left[\omega_B \left(\frac{1}{\beta} \tilde{b}_{i,t}^* - \tilde{b}_{i,t+1}^* \right) + c_{i,t+1}^* \right] \tag{A.35}$$

and the optimality choice for consumption

$$-\omega_B\left(\frac{1}{\beta}\tilde{b}^*_{i,t-1}-\tilde{b}^*_{i,t}\right)+\left(\gamma\frac{\omega_W}{\eta}+1\right)c^*_{i,t}=\omega_W\left(\frac{1}{\eta}+1\right)\tilde{w}_t+\left[\frac{1}{\beta}\omega_B\left(i_{t-1}-\pi_t\right)+\omega_D\tilde{d}_t+\omega_T\tilde{\tau}_t\right] \tag{A.36}$$

Together with the log-linearised budget constraint

$$c_{i,t}=\omega_W\left(\tilde{w}_t+l_{i,.t}\right)+\frac{1}{\beta}\omega_B\left(i_{t-1}-\pi_t\right)+\omega_B\left(\frac{1}{\beta}\tilde{b}_{i,t-1}-\tilde{b}_{i,t}\right)+\omega_D\tilde{d}_t+\omega_D\tilde{\tau}_t \tag{A.37}$$

Under full information, combining the optimality choice for consumption (A.36) with the optimal bond holdings (A.35) yields the standard inter-temporal Euler equation

$$c^*_{i,t}=\mathbb{E}_t\left[c^*_{i,t+1}-\frac{1}{\gamma}\left(i_t-\pi_{t+1}\right)\right]$$

Similarly, combining the log-linearised budget constraint (A.37) with the optimality condition for consumption (A.36) gives the standard intra-temporal Euler equation

$$\tilde{w}_t=\gamma c^*_{i,t}+\eta l^*_{i,t} \tag{A.38}$$

To solve for the optimal bond holdings under full information, I follow the transformation in Maćkowiak and Wiederholt (2025). Specifically, I substitute Equation (A.38) into the budget constraint (A.37) and rearranging the expression

$$\left(1+\omega_W\frac{\gamma}{\eta}\right)c^*_{i,t}=\omega_W\left(1+\frac{1}{\eta}\right)\tilde{w}_t+\omega_B\left(\frac{1}{\beta}\left(\tilde{b}^*_{i,t-1}+i_{t-1}-\pi_t\right)-\tilde{b}^*_{i,t}\right)+\omega_D\tilde{d}_t+\omega_D\tilde{\tau}_t$$

Sum over time from $t=0$ to ∞, discounting each period by β^t

$$\left(1+\omega_W\frac{\gamma}{\eta}\right)\sum_{s=t}^{t+N}\beta^{s-t}c^*_{i,s}=\omega_B\frac{1}{\beta}\tilde{b}^*_{i,t-1}+\sum_{s=t}^{t+N}\beta^{s-t}\left[z_s\right]-\omega_B\beta^N\tilde{b}^*_{i,t+N} \tag{A.39}$$

Here $z_t\equiv\omega_W\left(1+\frac{1}{\eta}\right)\tilde{w}_t+\frac{1}{\beta}\omega_B\left(i_{t-1}-\pi_t\right)+\omega_D\tilde{d}_t+\omega_D\tilde{\tau}_t$.

Taking expectations on both sides of Equation (A.39), and as $N\to\infty$, I get

$$\left(1+\omega_W\frac{\gamma}{\eta}\right)\sum_{s=t}^{\infty}\beta^{s-t}\mathbb{E}_t\left[c^*_{i,s}\right]=\omega_B\frac{1}{\beta}\tilde{b}^*_{i,t-1}+\sum_{s=t}^{\infty}\beta^{s-t}\mathbb{E}_t\left[z_s\right] \tag{A.40}$$

Next, using the Euler Equation and the law of iterated expectations yields

$$\sum_{s=t}^{\infty} \beta^{s-t} \mathbb{E}_t \left[c_{i,s}^* \right] = \frac{1}{1-\beta} c_{i,t}^* + \frac{1}{\gamma} \frac{1}{1-\beta} \sum_{s=t+1}^{\infty} \beta^{s-t} \mathbb{E}_t \left(r_{s-1} - \pi_s \right) \tag{A.41}$$

Combining Equation (A.41) with the budget constraint (A.37) yields

$$\omega_B \tilde{b}_{i,t}^* = \omega_B \tilde{b}_{i,t-1}^* + z_t - (1-\beta) \sum_{s=t}^{t+N} \beta^{s-t} \mathbb{E}_t \left[z_s \right] + \left(1 + \omega_W \frac{\gamma}{\eta} \right) \frac{1}{\gamma} \sum_{s=t+1}^{\infty} \beta^{s-t} \mathbb{E}_t \left(r_{s-1} - \pi_s \right) \tag{A.42}$$

Note that the off-diagonal element of Θ_0 in Equation (A.34) is non-zero, indicating that a suboptimal bond holding $b_{i,t}^*$ affects the optimal consumption choice $c_{i,t}^*$, and vice versa. Moreover, the second term in Equation (A.34) shows that a suboptimal bond holding today affects tomorrow's bond holding decision. These intra-and inter-temporal relationships complicate the attention problem. Therefore, similar to Maćkowiak and Wiederholt (2025), I do the following transformation such that I could express Equation (A.34) as[29]

$$\begin{aligned} &\sum_{t=0}^{\infty} \beta^t \mathbb{E}_i^h \left[\frac{1}{2} \left(v_t - v_t^* \right)' \Theta_0 \left(v_t - v_t^* \right) + \left(v_t - v_t^* \right) \Theta_1 \left(v_{t+1} - v_{t+1}^* \right) \right] \\ &= \sum_{t=0}^{\infty} \beta^t \mathbb{E}_{i,-1} \left[\frac{1}{2} \left(x_{i,t} - x_{i,t}^* \right)' \Theta \left(x_{i,t} - x_{i,t}^* \right) \right] \end{aligned} \tag{A.43}$$

where, instead of choosing $v_{i,t} = (\tilde{b}_{i,t}, c_{i,t})'$ directly, I assume that household i chooses the a transformation of $v_{i,t}$, which is $x_{i,t}$

$$x_{i,t} = \begin{pmatrix} \omega_B \left(\tilde{b}_{i,t} - \tilde{b}_{i,t-1} \right) \\ -\omega_B \left(\frac{1}{\beta} \tilde{b}_{i,t-1} - \tilde{b}_{i,t} \right) + \left(\gamma \frac{\omega_W}{\eta} + 1 \right) c_{i,t} \end{pmatrix}$$

This transformation diagonalises the loss function and separates the interdependencies between bond holdings and consumption, thereby simplifying the atten-

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[29] The proof of this transformation is lengthy and omitted here for brevity, but I can provide it upon request.

tion allocation problem. And the corresponding Θ in Equation (A.43) is

$$\Theta = -\bar{C}^{1-\gamma} \begin{bmatrix} \frac{\eta}{\omega_W}\left[1 - \frac{1}{\left(1+\omega_W \frac{\gamma}{\eta}\right)}\right]\frac{1}{\beta} & 0 \\ 0 & \frac{\eta}{\omega_W}\frac{1}{\left(1+\omega_W \frac{\gamma}{\eta}\right)} \end{bmatrix}$$

In this transformed space, a suboptimal choice of the first element in $x_{i,t}$ does not affect the optimal choice of the second element, and vice versa.

And the optimal choice of $x^*_{i,t}$ under full information is

$$x^*_{i,t} = \begin{pmatrix} z_t - (1-\beta)\sum_{s=t}^{\infty}\beta^{s-t}\mathbb{E}_t\left[z_s\right] + \frac{\beta}{\gamma}\left(1+\omega_W\frac{\gamma}{\eta}\right)\sum_{s=t}^{\infty}\beta^{s-t}\mathbb{E}_t\left(i_s - \pi_{s+1}\right) \\ \omega_W\left(\frac{1}{\eta}+1\right)\tilde{w}_t + \left[\frac{1}{\beta}\omega_B\left(i_{t-1}-\pi_t\right) + \omega_D\tilde{d}_t + \omega_T\tilde{\tau}_t\right] \end{pmatrix}$$

A.3.2 Solution Algorithm

In this economy, firms want to track their optimal price $p^*_{j,t}$ as given in Equation (1.35), while households want to track their optimal vector $x^*_{i,t}$ as given in Equation (1.33). It is evident from Equation (1.35) and (1.33), the optimal actions are determined in the equilibrium. However, since these variables follow Gaussian processes, Wold's theorem implies that they can be represented by a moving average of infinite order, i.e., an $MA(\infty)$ process. In particular,

$$p^*_{j,t} = \Phi_a(L)\varepsilon^a_t + \Phi_u(L)\varepsilon^u_t$$

$$x^*_{i,t} = \Psi_a(L)\varepsilon^a_t + \Psi_u(L)\varepsilon^u_t$$

where $\Phi_a(.)$, $\Phi_u(.)$, $\Gamma_a(.)$ and $\Gamma_u(.)$ are lag polynomials. However, to bypass the issue of unit root, following Afrouzi and Yang (2021), I define $\tilde{\varepsilon}^u_t \equiv (1-L)^{-1}\varepsilon^u_t = \sum_{k=0}^{\infty}\varepsilon^u_{t-k}$. I then rewrite the state-space representation as

$$p^*_{j,t} = \Phi_a(L)\varepsilon^a_t + \phi_u(L)\tilde{\varepsilon}^u_t$$

$$x^*_{i,t} = \Psi_a(L)\varepsilon^a_t + \psi_u(L)\tilde{\varepsilon}^u_t$$

where $\phi_u(L) = (1-L)\Phi_u(L)$ and $\psi_u(L) = (1-L)\Psi_u(L)$ are in l_2. Therefore, the processes can be approximated arbitrarily well with truncation.

The equilibrium should be uniquely determined by the history of monetary and productivity shocks. Let $\nu_t = (\varepsilon^a_t, \varepsilon^u_t)$ and $\tilde{\nu}_t = (\varepsilon^a_t, \tilde{\varepsilon}^u_t)$. Define the vector of shocks as $\vec{g}_t \equiv (\nu_t, \nu_{t-1}, \ldots, \nu_{t-(L+1)})$ and $\vec{\mathrm{g}}_t \equiv (\tilde{\nu}_t, \tilde{\nu}_{t-1}, \ldots, \tilde{\nu}_{t-(L+1)})$. These vectors are related via the transformation $\vec{g}_t = (I - \Lambda M')\,\vec{\mathrm{g}}_t$, where I is the identity matrix, Λ is a diagonal matrix with $\Lambda_{(2k,2k)} = 1$ and $\Lambda_{(2k-1,2k-1)} = 0$ for all $k = 1, 2, \ldots, L$, and M is a shift matrix. Note that the exogenous processes can be represented by

$$\begin{aligned} a_t &= H_a' \vec{\mathrm{x}}_t, \quad H_a' = (1, 0, \rho_a, 0, \rho_a^2, 0, \ldots, \rho_a^{L-1}, 0) \\ \varepsilon_t^u &= H_u' \vec{\mathrm{x}}_t, \quad H_u' = (0, 1, 0, 0, 0, 0, \ldots, 0, 0) \end{aligned}$$

The optimal price can be represented by $p_{j,t}^* \approx H_{p,(n)}' \vec{\mathrm{g}}_t$, the optimal action for households can be represented by $x_{i,t}^* \approx H_{x,(n)}' \vec{\mathrm{g}}_t$. The objective is to iteratively solve for the coefficients $H_{p,(n)}'$ and $H_{x,(n)}'$. In particular, given the guess $H_{(p,(n-1))}$ and $H_{(x,(n-1))}$, the optimal actions are

$$p_{j,t}^* = H_{(p,(n-1))}' \vec{\mathrm{g}}_t; \qquad x_{1,i,t}^* = H_{(x1,(n-1))}' \vec{\mathrm{g}}_t; \qquad x_{2,i,t}^* = H_{(x2,(n-1))}' \vec{\mathrm{g}}_t;$$

Here $x_{1,i,t}^*$ and $x_{2,i,t}^*$ denote the first and second elements in the optimal action vector $x_{i,t}^*$. If government debt is held constant, then it is optimal for households not to pay attention to $x_{1,i,t}^*$, as the interest rate is determined such that there will be no change in total bond holdings. In this case, $x_{1,i,t}^* = 0$ for any shocks. However, if the government debt absorbs some of the fiscal imbalances, then households will pay attention to $x_{1,i,t}^*$. For simplicity, the derivation here considers the case where government bond holdings remain constant.

Aggregating over firms and households yields the aggregate price level, the aggregate change in bond holdings and aggregate consumption. For example,

$$\begin{aligned} p_t = \int_0^1 p_{j,t} dj = H_{p,(n-1)}' \int_0^1 \mathbb{E}_{j,t}\left[\vec{\mathrm{g}}_t\right] dj &\approx H_{p,(n-1)}' \left[\sum_{k=0}^{\infty} \left[\left(I - K_{(n)} Y_{(n)}' \right) A \right]^k K_{(n)} Y_{(n)}' {M'}^k \right] \vec{\mathrm{g}}_t \\ &= H_{p,(n-1)}' X_{(n)} \vec{\mathrm{g}}_t \equiv H_p' \vec{\mathrm{g}}_t \end{aligned}$$

By same procedure, I get

$$x_{2,t} = \int_0^1 x_{2,i,t} di \approx H_{(x2,(n-1))}' Z_{(n)} \vec{\mathrm{g}}_t = H_{(x2)}' \vec{\mathrm{g}}_t$$

Follow directly, I get expressions for inflation and total consumption

$$\begin{aligned} \pi_t &= H'_\pi \vec{g}_t = \left[H'_p(I - \Lambda M')^{-1}(I - M'))\right] \vec{g}_t \\ c_t &= H'_c \vec{g}_t = \frac{1}{\left(\gamma \frac{\omega_W}{\eta} + 1\right)} H'_{x2}(I - \Lambda M')^{-1} \vec{g}_t \end{aligned}$$

By the production function $y_t = a_t + l_t$ and goods market clears, the aggregate labour demand is $l_t = H'_l \vec{g}_t = (H_c - H_a)' \vec{g}_t$. And the interest rate is determined by the Taylor rule 1.29

$$i_t = H'_i \vec{g}_t = \left((1 - \rho)\left(\phi_\pi H'_\pi + \phi_x \left(H'_c - \frac{1 + \eta}{\gamma + \eta} H'_a\right)\right) + H'_u\right)(I - \rho M')^{-1} \vec{g}_t$$

I then solve for the implied representations of the other variables in the model.

$$\begin{aligned} \omega_t &= H'_\omega \vec{g}_t = \frac{\eta}{\omega_W}\left(-\frac{1}{\gamma}\left(1 + \omega_W \frac{\gamma}{\eta}\right)(H'_i - H'_\pi M)(I - M')^{-1} - (H'_c - \omega_W H'_l)\right) \vec{g}_t \\ d_t &= H'_d \vec{g}_t = \frac{1}{\omega_D}\left(H'_c - (1 - \frac{1}{\theta})\omega_W (H'_\omega + H'_l)\right) \vec{g}_t \\ \tau_t &= H'_\tau \vec{g}_t = \frac{1}{\omega_T}\left(-\frac{1}{\beta}\omega_B (H'_i M' - H'_\pi) - \frac{1}{\theta}\omega_W (H'_\omega + H'_l)\right) \vec{g}_t \end{aligned}$$

Given these variables, I use Equation (1.35) and Equation (1.33) to update my guess for the MA representations of $H_{p,(n)}$ and $H_{x2,(n)}$

$$\begin{aligned} H_{p,(n)} &= \left((H'_\omega + H'_p(I - \Lambda M')^{-1} - H'_a)(I - \Lambda M')\right)' \\ H_{x2,(n)} &= \left(\left(\frac{1}{\beta}\omega_B (H'_i M' - H'_\pi) + \omega_D H'_d + \omega_T H'_\tau\right) + \omega_W (1 + \frac{1}{\eta}) H'_\omega\right)\left(I - \Lambda M')^{-1}\right)' \end{aligned}$$

I repeat the above procedures until both $H_{p,(n)}$ and $H_{x2,(n)}$ converge.

A.4 Appendix Table and Figures

Table A.3: Perceived relationship between inflation and growth

	Growth forecasts			
	Households		Firms	Professional forecasters
	Full sample	Great Moderation		
Inflation forecasts	-0.038^{***}	-0.034^{***}	0.039	0.156^{***}
	(0.001)	(0.001)	(0.020)	(0.023)
Observations	232,848	143,680	337	2,886
R^2	0.022	0.017	0.002	0.016

Note: The table provides statistics corresponding to Figure 1.1. In the Michigan Survey of Consumers, respondents are not asked to provide a quantitative forecast for output growth, they are asked about whether they expect business conditions over the next year to improve, stay the same or deteriorate. Following Candia et al. (2020), I assign point values to each answer ranging from 1 (improve) to -1 (deteriorate).

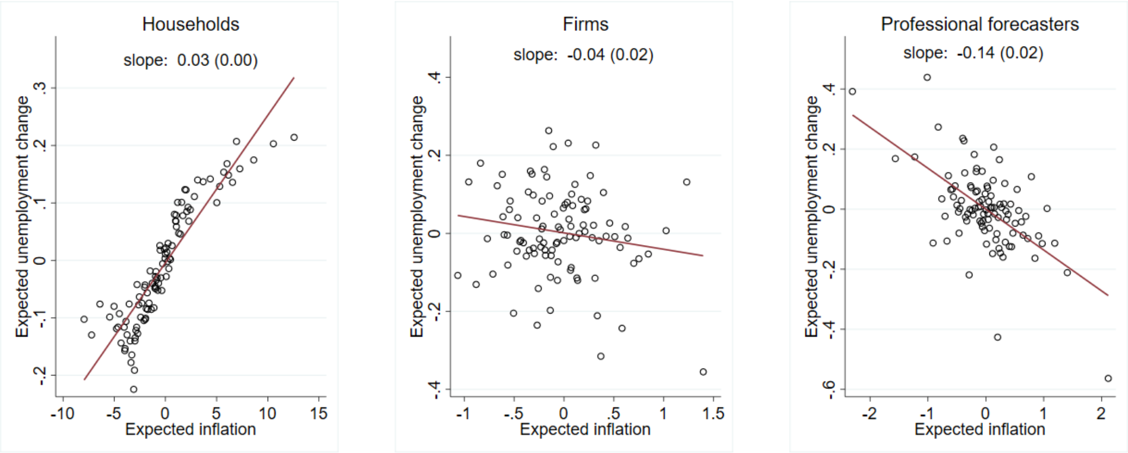

Figure A.7: Correlation between expected inflation and unemployment change

Notes: Each panel plots a bin-scatter for the joint distribution of expectations for change in unemployment rate and inflation in the next year across different economic agents in the United States. For each variable, I take out the time fixed effect so that all variables are mean zero. Data Sources: Michigan Survey of Consumers; The Livingston Survey; The Survey of Professional Forecasters.

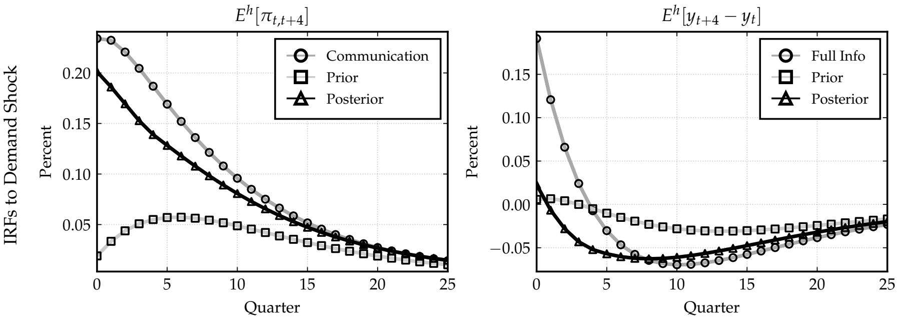

Figure A.8: Communication of higher future inflation to firms

Notes: The figure plots the impulse responses of firms' expected inflation and expected growth following the communication of a higher trajectory for future inflation. Firms revise their inflation expectations upward after the communication, at the same time revise their growth expectations upward.

Appendix to Chapter 2

B.1 The Simple Example: Derivations

Muth's (1961) Original Formulation. Muth (1961) states that the RE solution should be a function of all the past, present and expected future structural shocks as described in (2.4). Plug this back to the expectation difference equation (2.1) we have

$$\begin{aligned}\sum_{j=1}^{\infty} u_j \varepsilon_{t-j} + b\varepsilon_t + \sum_{j=1}^{\infty} c_j \mathbb{E}_t \varepsilon_{t+j} = \\ \theta \mathbb{E}_t \Big(\sum_{j=1}^{\infty} u_j \varepsilon_{t+1-j} + b\varepsilon_{t+1} + \sum_{j=1}^{\infty} c_j \mathbb{E}_{t+1} \varepsilon_{t+1+j}\Big) + \varepsilon_t\end{aligned} \tag{B1}$$

where u_j, c_j and b are coefficients to be determined. Equal coefficients of ε_{t-j} gives the expression for u's:

$$\begin{aligned}\varepsilon_t :&\quad b = \theta u_1 + 1 \Rightarrow u_1 = \frac{1}{\theta}(b-1);\\ \varepsilon_{t-1} :&\quad u_1 = \theta u_2 \Rightarrow u_2 = \frac{1}{\theta} u_1;\\ &\vdots\\ \varepsilon_{t-T} :&\quad u_T = \theta u_{T+1} \Rightarrow u_{T+1} = \frac{1}{\theta} u_T;\\ &\vdots\end{aligned}$$

and solve for c's:

$$\begin{aligned}\varepsilon_{t+1}: &\quad c_1 = \theta b\\ \varepsilon_{t+2}: &\quad c_2 = \theta c_1\\ &\vdots\\ \varepsilon_{t+T}: &\quad c_T = \theta c_{T-1}\\ &\vdots\end{aligned}$$

The coefficients u's and c's can be represented by the parameter b, and thus the set of solution can be parameterized by b

$$y_t = \sum_{j=1}^{\infty}(\frac{1}{\theta})^j(b-1)\varepsilon_{t-j} + b\varepsilon_t + \sum_{j=1}^{\infty} b\theta^j \mathbb{E}_t\varepsilon_{t+j} \tag{B2}$$

For white noise shocks, the solution becomes

$$y_t = \sum_{j=1}^{\infty}(\frac{1}{\theta})^j(b-1)\varepsilon_{t-j} + b\varepsilon_t \tag{B3}$$

Model (2.1) has an infinite number of solutions (each one corresponds to a particular value of b) due to the presence of the forward-looking component. In fact, the expectation term in that model, under the rational expectations hypothesis, is a conditional mean that can be written as a weighted average of the past observations (see Muth (1961))

$$\mathbb{E}_t y_{t+1} = \sum_{i=1}^{\infty} V_i y_{t+1-i} = \sum_{i=1}^{\infty}(b-1)\left(\frac{1}{\theta b}\right)^i y_{t+1-i} \tag{B4}$$

Proof. Without loss of generality we assume white noise shocks. From Equation (B3), the expectation of y_{t+1} can be expressed as follows

$$\mathbb{E}_t y_{t+1} = \mathbb{E}_t\left[\sum_{j=1}^{\infty}\left(\frac{1}{\theta}\right)^j(b-1)\varepsilon_{t+1-j} + b\varepsilon_{t+1}\right]$$

Since shocks are assumed to be white noise, $\mathbb{E}_t\varepsilon_{t+1} = 0$. Expanding the the summation gives

$$\begin{aligned}\mathbb{E}_t y_{t+1} &= \sum_{j=1}^{\infty} \left(\frac{1}{\theta}\right)^j (b-1)\varepsilon_{t+1-j} \\ &= \left(\frac{1}{\theta}\right)(b-1)\,\varepsilon_t + \left(\frac{1}{\theta}\right)^2 (b-1)\,\varepsilon_{t-1} + \left(\frac{1}{\theta}\right)^3 (b-1)\,\varepsilon_{t-2} + \dots\end{aligned}$$

To facilitate the derivation, we multiply and divide the entire expression by b and factor out the common term $\left(\frac{1}{\theta b}\right)(b-1)$, which yields

$$\mathbb{E}_t y_{t+1} = \left(\frac{1}{\theta b}\right)(b-1)\left[b\varepsilon_t + \left(\frac{1}{\theta}\right) b\varepsilon_{t-1} + \left(\frac{1}{\theta}\right)^2 b\varepsilon_{t-2} + \dots\right]$$

We now decompose the bracketed expression into two parts in order to isolate y_t. Specifically, we write

$$\begin{aligned}&b\varepsilon_t + \left(\frac{1}{\theta}\right) b\varepsilon_{t-1} + \left(\frac{1}{\theta}\right)^2 b\varepsilon_{t-2} + \dots = \\ &\underbrace{\left[b\varepsilon_t + \left(\frac{1}{\theta}\right)(b-1)\,\varepsilon_{t-1} + \left(\frac{1}{\theta}\right)^2 (b-1)\,\varepsilon_{t-2} + \dots\right]}_{=y_t} + \left[\left(\frac{1}{\theta}\right)\varepsilon_{t-1} + \left(\frac{1}{\theta}\right)^2 \varepsilon_{t-2} + \dots\right]\end{aligned}$$

The first group of terms in square brackets corresponds exactly to y_t, while the second group contains the remaining terms. Substituting this decomposition back into the expression for $\mathbb{E}_t y_{t+1}$

$$\mathbb{E}_t y_{t+1} = \left(\frac{1}{\theta b}\right)(b-1)\,y_t + \left(\frac{1}{\theta b}\right)(b-1)\left[\left(\frac{1}{\theta}\right)\varepsilon_{t-1} + \left(\frac{1}{\theta}\right)^2 \varepsilon_{t-2} + \dots\right]$$

Applying the same transformation to the second term recursively, we obtain a representation of $\mathbb{E}_t y_{t+1}$ as an infinite sum of past ys, which yields the final Equation (B4).

Finite Memory. Under finite memory, the information set of the agent at time t is given by $\mathcal{I}_t = \{\varepsilon_t, \varepsilon_{t-1}, \dots, \varepsilon_{t-T}\}$. Assuming time-varying expectations, we guess the solution taking the following form

$$y_t = \sum_{j=1}^{T} u_{j,t}\varepsilon_{t-j} + b_t\varepsilon_t + \sum_{j=1}^{\infty} c_{j,t}\mathbb{E}_t\varepsilon_{t+j} \tag{B5}$$

At time t, Based on this information set, the expectation of y_{t+1} is given by

$$\begin{aligned}\mathbb{E}_t y_{t+1} =&\mathbb{E}\left(y_{t+1}|\mathcal{I}_t\right) = \mathbb{E}\left(y_{t+1}|\varepsilon_t, \varepsilon_{t-1}, \dots, \varepsilon_{t-T}\right) \\ =&\mathbb{E}_t\left(\sum_{j=1}^{T+1} u_{j,t+1}\varepsilon_{t+1-j} + b_{t+1}\varepsilon_{t+1} + \sum_{j=1}^{\infty} c_{j,t+1}\mathbb{E}_{t+1}\varepsilon_{t+1+j}\right) \end{aligned} \tag{B6}$$

This particular form of expectation implies that the agent does *not* internalize the limited memory.

Now use (B5) and (B6) to substitute for y_t and $\mathbb{E}_t y_{t+1}$ in the expectational difference (2.1),

$$\begin{aligned}&\sum_{j=1}^{T} u_{j,t}\varepsilon_{t-j} + b_t\varepsilon_t + \sum_{j=1}^{\infty} c_{j,t}\mathbb{E}_t\varepsilon_{t+j} = \\ &\qquad \theta\mathbb{E}_t\left(\sum_{j=1}^{T+1} u_{j,t+1}\varepsilon_{t+1-j} + b_{t+1}\varepsilon_{t+1} + \sum_{j=1}^{\infty} c_{j,t+1}\mathbb{E}_{t+1}\varepsilon_{t+1+j}\right) + \varepsilon_t\end{aligned}$$

Again, equal coefficients to find an expression for the u's:

$$\begin{aligned}\varepsilon_t :&\quad b_t = \theta\mathbb{E}_t u_{1,t+1} + 1 \Rightarrow \mathbb{E}_t u_{1,t+1} = \frac{1}{\theta}(b_t - 1); \\ \varepsilon_{t-1} :&\quad u_{1,t} = \theta\mathbb{E}_t u_{2,t+1} \Rightarrow \mathbb{E}_t u_{2,t+1} = \frac{1}{\theta}u_{1,t}; \\ \vdots& \\ \varepsilon_{t-T+1} :&\quad u_{T-1,t} = \theta\mathbb{E}_t u_{T,t+1} \Rightarrow \mathbb{E}_t u_{T,t+1} = \frac{1}{\theta}u_{T-1,t}; \\ \varepsilon_{t-T} :&\quad u_{T,t} = \theta\mathbb{E}_t u_{T+1,t+1} \Rightarrow \mathbb{E}_t u_{T+1,t+1} = \frac{1}{\theta}u_{T,t};\end{aligned}$$

and for the c's:

$$\begin{aligned}\varepsilon_{t+1} :&\quad c_{1,t} = \theta\mathbb{E}_t b_{t+1} \\ \varepsilon_{t+2} :&\quad c_{2,t} = \theta\mathbb{E}_t c_{1,t+1} \\ \vdots& \\ \varepsilon_{t+T} :&\quad c_{T,t} = \theta\mathbb{E}_t c_{T-1,t+1} \\ \vdots&\end{aligned}$$

Therefore, parameters $u_{j,t}$ and $c_{j,t}$ are defined by the path of b_t. If b is constant, we have $u_j = \frac{1}{\theta^j} b$ and $c_j = \theta^j b$ are constants as well. And the solution is

$$y_t = \sum_{j=1}^{T} (\frac{1}{\theta})^j (b-1)\varepsilon_{t-j} + b\varepsilon_t + \sum_{j=1}^{\infty} b\theta^j \mathbb{E}_t \varepsilon_{t+j} \tag{B7}$$

If b_t follows a random walk, i.e., $b_t = b_{t-1} + \sigma_b \xi_t$, with ξ_t i.i.d $N(0,1)$, and therefore $\mathbb{E}_t b_{t+1} = b_t$. The solution for the expectational difference equation can be parameterized by b_t. To derive the solution, assume that $u_{1,t+1} = F(b_{t+1})$, we need to find the formulation of F such that $b_t = \frac{1}{\theta}\mathbb{E}_t u_{1,t+1} + 1$ is satisfied, given the stochastic process for b_t. Guess that F is linear,

$$u_{1,t+1} = a_0 + a_1 b_{t+1}$$

then

$$\begin{aligned} b_t &= \theta \mathbb{E}_t[a_0 + a_1 b_{t+1}] + 1 \\ &= \theta[a_0 + a_1 b_t] + 1 \end{aligned}$$

gives

$$a_0 = -\frac{1}{\theta} \qquad a_1 = \frac{1}{\theta}$$

which gives $u_{i,t+1} = -\frac{1}{\theta} + \frac{1}{\theta} b_{t+1}$, bring one period backward we get $u_{i,t} = -\frac{1}{\theta} + \frac{1}{\theta} b_t$. Analogously, assuming $u_{2,t+1} = F(b_{t+1})$ is linear, i.e. $u_{2,t+1} = d_0 + d_1 b_{t+1}$ then

$$\begin{aligned} u_{1,t} &= \theta \mathbb{E}_t[d_0 + d_1 b_{t+1}] \\ &= \theta[d_0 + d_1 b_t] \end{aligned}$$

gives

$$d_0 = -(\frac{1}{\theta})^2 \qquad d_1 = (\frac{1}{\theta})^2$$

By the same reasoning, we can rewrite c_t's as functions of b_t. Then the set of solutions can be represented by b_t.

$$y_t = \sum_{j=1}^{T} (\frac{1}{\theta})^j (b_t - 1)\varepsilon_{t-j} + b_t\varepsilon_t + \sum_{j=1}^{\infty} b_t\theta^j \mathbb{E}_t \varepsilon_{t+j}. \tag{B8}$$

Decay Memory. Under decay memory, the information set of the agent is given by $\mathcal{I}_t = \{\varepsilon_t, \lambda\varepsilon_{t-1}, \lambda^2\varepsilon_{t-2}, \ldots\}$. We guess that the solution takes the following form

$$y_t = \sum_{j=1}^{\infty} u_{j,t}\lambda^j\varepsilon_{t-j} + b_t\varepsilon_t + \sum_{j=1}^{\infty} c_{j,t}\mathbb{E}_t\varepsilon_{t+j} \tag{B9}$$

At time t, Based on this information set, the expectation of y_{t+1} is then given by

$$\begin{aligned}\mathbb{E}_t y_{t+1} =& \mathbb{E}(y_{t+1}|\mathcal{I}_t) = \mathbb{E}(y_{t+1}|\varepsilon_t, \lambda\varepsilon_{t-1}, \lambda^2\varepsilon_{t-2}, \ldots) \\ =& \mathbb{E}_t\big(\sum_{j=1}^{\infty} u_{j,t+1}\lambda^{j-1}\varepsilon_{t+1-j} + b_{t+1}\varepsilon_{t+1} + \sum_{j=1}^{\infty} c_{j,t+1}\mathbb{E}_{t+1}\varepsilon_{t+1+j}\big)\end{aligned} \tag{B10}$$

Now use (B9) and (B10) to substitute for y_t and $\mathbb{E}_t y_{t+1}$ in the expectational difference equation (2.1),

$$\begin{aligned}&\sum_{j=1}^{\infty} u_{j,t}\lambda^j\varepsilon_{t-j} + b_t\varepsilon_t + \sum_{j=1}^{\infty} c_{j,t}\mathbb{E}_t\varepsilon_{t+j} = \\ &\qquad \theta\mathbb{E}_t(\sum_{j=1}^{\infty} u_{j,t+1}\lambda^{j-1}\varepsilon_{t+1-j} + b_{t+1}\varepsilon_{t+1} + \sum_{j=1}^{\infty} c_{j,t+1}\mathbb{E}_{t+1}\varepsilon_{t+1+j}) + \varepsilon_t\end{aligned}$$

Again, equal coefficients to find an expression for the u's:

$$\begin{aligned}\varepsilon_t:&\quad b_t = \theta\mathbb{E}_t u_{1,t+1} + 1 \Rightarrow \mathbb{E}_t u_{1,t+1} = \frac{1}{\theta}(b_t - 1); \\ \varepsilon_{t-1}:&\quad \lambda u_{1,t} = \theta\lambda\mathbb{E}_t u_{2,t+1} \Rightarrow \mathbb{E}_t u_{2,t+1} = \frac{1}{\theta}u_{1,t}; \\ \vdots&\end{aligned}$$

and for the c's:

$$\begin{aligned}\varepsilon_{t+1}:&\quad c_{1,t} = \theta\mathbb{E}_t b_{t+1} \\ \varepsilon_{t+2}:&\quad c_{2,t} = \theta\mathbb{E}_t c_{1,t+1} \\ \vdots&\end{aligned}$$

For constant $b_t = b$, the coefficient for ε_{t-j}, $\forall j$ is $u_j = (b-1)\left(\frac{\lambda}{\theta}\right)^j$, and the coefficient for $\mathbb{E}_t\varepsilon_{t+j}$, $\forall j$ is $c_j = b\theta^j$.

$$y_t = (b-1)\sum_{j=1}^{\infty}(\frac{\lambda}{\theta})^j\varepsilon_{t-j} + b\varepsilon_t + b\sum_{j=1}^{\infty}\theta^j\mathbb{E}_t\varepsilon_{t+j} \tag{B11}$$

For $b_t = b_{t-1} + \sigma_b \xi_t$ follows a random walk process, the solution is

$$y_t = (b_t - 1) \sum_{j=1}^{\infty} (\frac{\lambda}{\theta})^j \varepsilon_{t-j} + b_t \varepsilon_t + b_t \sum_{j=1}^{\infty} \theta^j \mathbb{E}_t \varepsilon_{t+j} \tag{B12}$$

As $\lambda < 1$, the past shocks have dampened impacts on current equilibrium just because the memory loss. In cases where $\lambda > \theta$, transversality condition gives the unique solution, coinciding with $b = 1$. In the case where $\lambda < \theta$, transversality condition cannot help us pick a unique solution.

Under the time-varying expectation formation, the expectation is written as (with white noises)

$$\bar{\mathbb{E}}_t y_{t+1} = (b_t - 1) \sum_{i=1}^{\infty} \left(\frac{1}{\theta}\right)^i \left(\prod_{j=1}^{i} b_{t+1-j}\right)^{-1} \lambda^{i-1} y_{t+1-i} \tag{B13}$$

Proof. Start with the solution (B12), the expectation of y_{t+1} is as follows

$$\mathbb{E}_t y_{t+1} = \mathbb{E}_t \left[\sum_{j=1}^{\infty} \left(\frac{1}{\theta}\right)^j \lambda^{j-1} (b_{t+1} - 1) \varepsilon_{t+1-j} + b_{t+1} \varepsilon_{t+1} \right]$$

Since shocks are assumed to be white noise, $\mathbb{E}_t \varepsilon_{t+1} = 0$. Expanding the the summation gives

$$\begin{aligned} \mathbb{E}_t y_{t+1} &= \left[\sum_{j=1}^{\infty} \left(\frac{1}{\theta}\right)^j \lambda^{j-1} (b_{t+1} - 1) \varepsilon_{t+1-j} \right] \\ &= \left(\frac{1}{\theta}\right) (b_t - 1) \varepsilon_t + \left(\frac{1}{\theta}\right)^2 \lambda (b_t - 1) \varepsilon_{t-1} + \left(\frac{1}{\theta}\right)^3 \lambda^2 (b_t - 1) \varepsilon_{t-2} + \dots \end{aligned}$$

To facilitate the derivation, we multiply and divide the entire expression by b_t and factor out the common term $\left(\frac{1}{\theta b_t}\right)(b_t - 1)$, which yields

$$\mathbb{E}_t y_{t+1} = \left(\frac{1}{\theta b_t}\right) (b_t - 1) \left[b_t \varepsilon_t + \left(\frac{\lambda}{\theta}\right) b_t \varepsilon_{t-1} + \left(\frac{\lambda}{\theta}\right)^2 b_t \varepsilon_{t-2} + \dots \right]$$

We now decompose the bracketed expression into two parts in order to isolate

y_t. Specifically, we write

$$b_t\varepsilon_t + \left(\frac{\lambda}{\theta}\right) b_t\varepsilon_{t-1} + \left(\frac{\lambda}{\theta}\right)^2 b_t\varepsilon_{t-2} + \dots =$$
$$\underbrace{\left[b_t\varepsilon_t + \left(\frac{\lambda}{\theta}\right)(b_t-1)\,\varepsilon_{t-1} + \left(\frac{\lambda}{\theta}\right)^2 (b_t-1)\,\varepsilon_{t-2} + \dots\right]}_{=y_t} + \left[\left(\frac{\lambda}{\theta}\right)\varepsilon_{t-1} + \left(\frac{\lambda}{\theta}\right)^2 \varepsilon_{t-2} + \dots\right]$$

The first group of terms in square brackets corresponds exactly to y_t, while the second group contains the remaining terms. Substituting this decomposition back into the expression for $\mathbb{E}_t y_{t+1}$

$$\mathbb{E}_t y_{t+1} = \left(\frac{1}{\theta b_t}\right)(b_t-1)\,y_t + \left(\frac{1}{\theta b_t}\right)(b_t-1)\left[\left(\frac{\lambda}{\theta}\right)\varepsilon_{t-1} + \left(\frac{\lambda}{\theta}\right)^2 \varepsilon_{t-2} + \dots\right]$$

Focusing on the second term, we multiply and divide the entire expression by b_t and factor out the common term $\left(\frac{\lambda}{\theta b_{t-1}}\right)$, which yields

$$\left(\frac{\lambda}{\theta}\right)\varepsilon_{t-1} + \left(\frac{\lambda}{\theta}\right)^2 \varepsilon_{t-2} + \dots = \left(\frac{\lambda}{\theta b_{t-1}}\right)\left[b_{t-1}\varepsilon_{t-1} + \left(\frac{\lambda}{\theta}\right) b_{t-1}\varepsilon_{t-2} + \left(\frac{\lambda}{\theta}\right)^2 b_{t-1}\varepsilon_{t-3} + \dots\right]$$

Similarly, we can decompose the expression into two parts and isolate y_{t-1}

$$b_{t-1}\varepsilon_{t-1} + \left(\frac{\lambda}{\theta}\right) b_{t-1}\varepsilon_{t-2} + \left(\frac{\lambda}{\theta}\right)^2 b_{t-1}\varepsilon_{t-3} + \dots =$$
$$\underbrace{\left[b_{t-1}\varepsilon_{t-1} + \left(\frac{\lambda}{\theta}\right)(b_{t-1}-1)\,\varepsilon_{t-2} + \left(\frac{\lambda}{\theta}\right)^2 (b_{t-1}-1)\,\varepsilon_{t-3} + \dots\right]}_{=y_{t-1}} + \left[\left(\frac{\lambda}{\theta}\right)\varepsilon_{t-2} + \left(\frac{\lambda}{\theta}\right)^2 \varepsilon_{t-3} + \dots\right]$$

Substituting this decomposition back into the expression for $\mathbb{E}_t y_{t+1}$

$$\begin{aligned}\mathbb{E}_t y_{t+1} = {} & \left(\frac{1}{\theta b_t}\right)(b_t-1)\,y_t + \left(\frac{1}{\theta b_t}\right)\left(\frac{\lambda}{\theta b_{t-1}}\right)(b_t-1)\,y_{t-1} \\ & + \left(\frac{1}{\theta b_t}\right)\left(\frac{\lambda}{\theta b_{t-1}}\right)(b_t-1)\left[\left(\frac{\lambda}{\theta}\right)\varepsilon_{t-2} + \left(\frac{\lambda}{\theta}\right)^2 \varepsilon_{t-3} + \dots\right]\end{aligned}$$

Repeat the same procedure we can arrive at Equation (B13).

Note that for constant b, the expectation under decay memory corresponds to Equation (2.11).

B.2 A Decay Memory Asset Pricing Model

B.2.1 Model Derivations

Recall that the Euler condition in equation (2.18) implies that any asset i should satisfy the following pricing restriction, (this is just the log form of the Euler equation (2.18))

$$\mathbb{E}_t\left[\exp\left(\theta\log(\delta)-\frac{\theta}{\psi}g_{c,t+1}+\theta r_{a,t+1}\right)\right]=1 \tag{B14}$$

where the lowercase letters refer logs.

Solution for Wealth Return $r_{a,t+1}$. Note that when substituting $r_{i,t+1}=r_{a,t+1}$ then (B14) becomes

$$\mathbb{E}_t\left[\exp\left(\theta\log(\delta)-\frac{\theta}{\psi}g_{c,t+1}+\theta r_{a,t+1}\right)\right]=1 \tag{B15}$$

We start by conjecturing that the solutions for the endogenous variables z_t are a linear function of the *discounted* past, present and expected future values of x's but subject to decay memory constraint, i.e. the time t information set is $\mathbb{I}_t=\{x_t,\lambda x_{t-1},\lambda^2 x_{t-2},\ldots\}$. Guess the solution of log price consumption ratio $z_t=\log(P_t/C_t)$ has the following form:

$$z_t=A_{0,t}+\left(1-\frac{1}{\psi}\right)\left(\sum_{j=1}^{\infty}u_{j,t}\lambda^j x_{t-j}+b_t x_t+\sum_{j=1}^{\infty}c_{j,t}\mathbb{E}_t x_{t+j}\right)$$

where the parameters $A_{0,t}$, $u_{j,t}$, b_t and $c_{j,t}$ are coefficients to be determined. Followed by that the expectation of z_{t+1} at time t given information set $\mathbb{I}_t$ has the form

$$\begin{aligned}\bar{\mathbb{E}}_t z_{t+1}&=\mathbb{E}(z_{t+1}|x_t,\lambda x_{t-1},\ldots)\\&=\mathbb{E}_t\left[A_{0,t+1}+\left(1-\frac{1}{\psi}\right)\left(\sum_{j=1}^{\infty}u_{j,t+1}\lambda^{j-1}x_{t+1-j}+b_{t+1}x_{t+1}+\sum_{j=1}^{\infty}c_{j,t+1}\mathbb{E}_{t+1}x_{t+1+j}\right)\right]\end{aligned}$$

Armed with the endogenous variable z_t and its expectation, we plug the approximation $r_{a,t+1}=\kappa_0+\kappa_1 z_{t+1}-z_t+g_{c,t+1}$ into the Euler equation (B15). The solution coefficients can be derived by collecting the terms on the corresponding state variables.

Using the undetermined coefficient methods as in Appendix B.1, the log price-consumption ratio is given by

$$z_t = A_{0,t} + (1-\frac{1}{\psi})\left[\sum_{j=1}^{\infty}(\frac{1}{\kappa_1})^j\lambda^j(b_t-1)x_{t-j} + b_t x_t + b_t\sum_{j=1}^{\infty}(\kappa_1\rho)^j x_t\right]$$

with

$$A_{0,t} = \frac{1}{1-\kappa_1}\left(\log(\delta) - (\frac{1}{\psi}-1)\mu + \kappa_0 + \frac{\theta}{2}(\frac{1}{\psi}-1)^2\left[\sigma^2 - \sigma_b^2\kappa_1 X_t^2 + \kappa_1^2\left(\frac{\varphi_e\sigma}{1-\kappa_1\rho}\right)^2 b_t^2\right]\right)$$

where

$$X_t = \sum_{j=1}^{\infty}(\frac{1}{\kappa_1})^j\lambda^j x_{t-j} + \frac{1}{1-\kappa_1\rho}x_t$$

Solution for Market Return $r_{m,t+1}$**.** When $r_{i,t+1} = r_{m,t+1}$ the log-linearized Euler equation becomes

$$\mathbb{E}_t\left[\exp\left(\theta\log\delta - \frac{\theta}{\psi}g_{c,t+1} + (\theta-1)\,r_{a,t+1} + r_{m,t+1}\right)\right] = 1 \tag{B16}$$

Plug in the linearized expression for $r_{i,t+1}$ and $r_{m,t+1}$

$$1 = \mathbb{E}_t\left[\exp\left(\theta\log(\delta) - \frac{\theta}{\psi}g_{c,t+1} + (\theta-1)(\kappa_0+\kappa_1 z_{t+1} - z_t + g_{c,t+1}) + \kappa_{0,m} + \kappa_{1,m}z_{m,t+1} - z_{m,t} + g_{d,t+1}\right)\right] \tag{B17}$$

Guess that the solution for the endogenous variables $z_{m,t}$ are a linear function of the discounted past, present and expected future values of x's, i.e. the time t information set is $\mathbb{I}_t = \{x_t, \lambda x_{t-1}, \lambda^2 x_{t-2}, ...\}$. Analogously, plugging the guess into the pricing equation (B17) and equating the coefficients of the state variables and constant. Replacing the consumption and dividend growth processes and of the price-consumption and price-dividend ratios, and solving for the expectations, we obtain the solution for $z_{m,t}$:

$$z_{m,t} = A_{0,m,t} + (\phi - \frac{1}{\psi})\left[\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_{1,m}})^j(b_t - 1)x_{t-j} + b_t x_t + b_t\sum_{j=1}^{\infty}(\kappa_{1,m}\rho)^j x_t\right]$$

Solution for the Risk-free Rate $r_{f,t+1}$. To solve for the risk-free rate, we substitute $R_{i,t+1} = R_{f,t+1}$ then (B14) becomes

$$\mathbb{E}_t\left[\exp\left(\theta\log\delta - \frac{\theta}{\psi}g_{c,t+1} + (\theta - 1)\,r_{a,t+1} + r_{f,t+1}\right)\right] = 1$$

In logarithms, the Euler equation is:

$$\begin{aligned}r_{f,t+1} &= -\log(\mathbb{E}_t(\exp(m_{t+1})))\\ &= -\log\left(\mathbb{E}_t(\exp(\theta\log\delta - \frac{\theta}{\psi}g_{c,t+1} + (\theta - 1)\,r_{a,t+1}\right)\end{aligned}$$

Further solve the above expression gives

$$\begin{aligned}r_{f,t+1} =& -\log(\delta) + \frac{1}{\psi}\mu + \frac{1}{\psi}x_t - (1-\theta)\frac{\theta}{2}(\frac{1}{\psi} - 1)^2\sigma^2 - \frac{1}{2}\left[(\frac{\theta}{\psi} + 1 - \theta)\sigma\right]^2\\ &- \frac{1}{2}(1-\theta)(1-\frac{1}{\psi})^2\left[\kappa_1^2(\frac{1}{1-\kappa_1\rho})^2\varphi_e^2\sigma^2 b_t^2 - \sigma_b^2\,[X_t]^2 - \kappa_1^2\sigma_b^2(\frac{1}{1-\kappa_1\rho})^2\varphi_e^2\sigma^2\right]\end{aligned}$$

B.2.2 Analytical Results: Derivations

This section provides the derivation of the analytical results in the main paper.[30]

Price-Dividend Ratio. To derive the persistence of price-dividend ratio (i.e., equation (2.34)), we define $\hat{z}_m$ as the deviation from the usual RE solution,

$$\begin{aligned}\hat{z}_{m,t} \equiv& z_{m,t} - z_{m,t}^{RE}\\ =& (b_t - 1)\left(\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_{1,m}})^j(\phi - \frac{1}{\psi})x_{t-j} + \frac{1}{1-\kappa_{1,m}\rho}(\phi - \frac{1}{\psi})x_t\right)\end{aligned} \tag{B18}$$

Moving the above Equation (B18) one-period forward we get the deviation at

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[30] The time-varying component in the A_0 and $A_{0,m}$ were abbreviated when deriving the analytical solutions as it does not affect the main results, but this impact was considered when doing the quantitative analysis.

time $t+1$, which is given by

$$\hat{z}_{m,t+1} = (b_{t+1}-1)\left(\sum_{j=1}^{\infty}\left(\frac{\lambda}{\kappa_{1,m}}\right)^{j}\left(\phi-\frac{1}{\psi}\right)x_{t+1-j}+\frac{1}{1-\kappa_{1,m}\rho}\left(\phi-\frac{1}{\psi}\right)x_{t+1}\right)$$

Rewriting the summation in the following way

$$\begin{aligned}\hat{z}_{m,t+1} &= (b_{t+1}-1)\frac{\lambda}{\kappa_{1,m}}\left[\sum_{j=1}^{\infty}\left(\frac{\lambda}{\kappa_{1,m}}\right)^{j}\left(\phi-\frac{1}{\psi}\right)x_{t-j}+\left(\phi-\frac{1}{\psi}\right)x_{t}\right]\\ &+ (b_{t+1}-1)\frac{1}{1-\kappa_{1,m}\rho}\left(\phi-\frac{1}{\psi}\right)x_{t+1}\end{aligned}$$

Rearrange Equation (B18) we have

$$\sum_{j=1}^{\infty}\left(\frac{\lambda}{\kappa_{1,m}}\right)^{j}\left(\phi-\frac{1}{\psi}\right)x_{t-j}=\frac{\hat{z}_{m,t}}{(b_t-1)}-\frac{1}{1-\kappa_{1,m}\rho}\left(\phi-\frac{1}{\psi}\right)x_t$$

Substituting this back into the earlier expression for $\hat{z}_{m,t+1}$ and simplify, we obtain

$$\hat{z}_{m,t+1}=\frac{\lambda}{\kappa_{1,m}}\frac{b_{t+1}-1}{b_t-1}\hat{z}_{m,t}+(b_{t+1}-1)\left(\phi-\frac{1}{\psi}\right)\left(\frac{\varphi_e\sigma}{1-\kappa_{1,m}\rho}e_{t+1}+\frac{1}{1-\kappa_{1,m}\rho}(1-\lambda)\rho x_t\right)$$

where we have replaced x_{t+1} using Equation 2.20. As $\lambda \to 1$, the last term approaches to 0. Therefore, for large λ, the deviation is approximated by

$$\hat{z}_{m,t+1}=\frac{\lambda}{\kappa_{1,m}}\frac{b_{t+1}-1}{b_t-1}\hat{z}_{m,t}+(b_{t+1}-1)\left(\phi-\frac{1}{\psi}\right)\left(\frac{\varphi_e\sigma}{1-\kappa_{1,m}\rho}e_{t+1}\right)$$

Volatility of Price-Dividend Ratio. The volatility (2.35) in the text is derived as followed: for $b_t \neq 1$, the variance of $z_{m,t+1}$ is defined as

$$\begin{aligned}\mathbb{V}ar_t(z_{m,t+1}) &= \mathbb{V}ar_t\left(\left(\phi-\frac{1}{\psi}\right)\left[\sum_{j=1}^{\infty}\left(\frac{1}{\kappa_{1,m}}\right)^{j}\lambda^{j-1}(b_{t+1}-1)x_{t+1-j}+b_{t+1}\frac{1}{1-\kappa_{1,m}\rho}x_{t+1}\right]\right)\\ &= \mathbb{V}ar_t\left(\left(\phi-\frac{1}{\psi}\right)\left[b_{t+1}\left(\sum_{j=1}^{\infty}\left(\frac{1}{\kappa_{1,m}}\right)^{j}\lambda^{j-1}x_{t+1-j}+\frac{1}{1-\kappa_{1,m}\rho}\rho x_t\right)+b_{t+1}\frac{\varphi_e\sigma}{1-\kappa_{1,m}\rho}e_{t+1}\right]\right)\end{aligned}$$

where the second line is obtained by substitute in x_{t+1} using Equation (2.20).

Noticing that

$$z_{m,t}^{b} = -\sum_{j=1}^{\infty}\left(\frac{\lambda}{\kappa_{1,m}}\right)^{j} x_{t-j} \qquad \text{and} \qquad z_{m,t}^{RE} = \frac{1}{1-\kappa_{1,m}\rho}x_t$$

We can then simplify $\mathbb{V}ar_t\left(z_{m,t+1}\right)$ using the above expression, yields

$$\mathbb{V}ar_t\left(z_{m,t+1}\right) = \mathbb{V}ar_t\left(\left[b_{t+1}\frac{1}{\kappa_{1,m}}\left(z_{m,t}^{RE} - z_{m,t}^{b}\right) + b_{t+1}\left(\phi - \frac{1}{\psi}\right)\frac{\varphi_e\sigma}{1-\kappa_{1,m}\rho}e_{t+1}\right]\right)$$

Expanding the equation and substituting $z_{m,t}^{RE} - z_{m,t}^{b}$ using $z_{m,t} = (1-b_t)z_{m,t}^{b} + b_t z_{m,t}^{RE}$, we obtain

$$\mathbb{V}ar_t\left(z_{m,t+1}\right) = \frac{\sigma_b^2}{\kappa_{1,m}}\left(\frac{z_{m,t} - z_{m,t}^{RE}}{1-b_t}\right)^2 + b_t^2\left(\phi - \frac{1}{\psi}\right)^2\left(\frac{\varphi_e\sigma}{1-\kappa_{1,m}\rho}\right)^2 + \sigma_b^2\left(\phi - \frac{1}{\psi}\right)^2\left(\frac{\varphi_e\sigma}{1-\kappa_{1,m}\rho}\right)^2$$

Equity Premium: Derivations. First, stochastic discount factor is a function of $r_{a,t+1}$ and $g_{c,t+1}$.

$$\begin{aligned} m_{t+1} &= \theta\log\delta - \frac{\theta}{\psi}g_{c,t+1} + (\theta-1)r_{a,t+1} \\ &= \theta\log\delta - \frac{\theta}{\psi}g_{c,t+1} + (\theta-1)(\kappa_0 + \kappa_1 z_{t+1} - z_t + g_{c,t+1}) \end{aligned}$$

Substituting the equilibrium return for $r_{a,t+1}$ into the equation, it is straightforward to show that the innovation to the pricing kernel is

$$\begin{aligned} m_{t+1} - E_t m_{t+1} &= -\frac{\theta}{\psi}g_{c,t+1} + (\theta-1)(\kappa_1 z_{t+1} + g_{c,t+1}) - \mathbb{E}_t\left[-\frac{\theta}{\psi}g_{c,t+1} + (\theta-1)(\kappa_1 z_{t+1} + g_{c,t+1})\right] \\ &= \left(-\frac{\theta}{\psi} + \theta - 1\right)\left(g_{c,t+1} - \mathbb{E}_t g_{c,t+1}\right) + (\theta-1)\kappa_1\left(z_{t+1} - \mathbb{E}_t z_{t+1}\right) \\ &= -\left(1-\theta+\frac{\theta}{\psi}\right)\sigma\eta_{t+1} - (1-\theta)\kappa_1(1-\frac{1}{\psi})\left[\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_1})^j x_{t+1-j} + \frac{1}{1-\kappa_1\rho}\rho x_t\right]\sigma_b\xi_{t+1} \\ &\quad -(1-\theta)\kappa_1(1-\frac{1}{\psi})b_{t+1}\frac{\varphi_e}{1-\kappa_1\rho}\sigma e_{t+1} \\ &= -\lambda_{m,\eta}\sigma\eta_{t+1} - \lambda_{m,\xi,t+1}\sigma_b\xi_{t+1} - \lambda_{m,e,t+1}\sigma e_{t+1} \end{aligned} \tag{B19}$$

where $\lambda_{m,\eta}$, $\lambda_{m,e,t+1}$ and $\lambda_{m,\xi,t+1}$ captures the pricing kernel's exposure to the independent consumption shocks, η_{t+1}, to the expected growth rate shock, e_{t+1}, and to the time-varying expectation shock, ξ_{t+1}.

Equation (B19) already provides the innovation in m_{t+1}. We now proceed to derive the innovation in the market return. Recall that the return

$$r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m} z_{m,t+1} - z_{m,t} + g_{d,t+1}$$

Then

$$\begin{aligned}
r_{m,t+1} - E_t r_{m,t+1} =& \kappa_{1,m} \left(z_{m,t+1} - E_t z_{m,t+1} \right) + \left(g_{d,t+1} - E_t g_{d,t+1} \right) \\
=& \kappa_{1,m} (\phi - \frac{1}{\psi}) \left[\sum_{j=1}^{\infty} (\frac{\lambda}{\kappa_{1,m}})^j x_{t+1-j} + \frac{1}{1 - \kappa_{m,1}\rho} \rho x_t \right] \sigma_b \xi_{t+1} \\
&+ \kappa_{1,m} (\phi - \frac{1}{\psi}) \frac{1}{1 - \kappa_{m,1}\rho} b_{t+1} \varphi_e \sigma e_{t+1} + \varphi_d \sigma u_{t+1} \\
=& \beta_{m,\xi,t+1} \sigma_b \xi_{t+1} + \beta_{m,e,t+1} \sigma e_{t+1} + \beta_{m,u} \sigma u_{t+1}
\end{aligned} \tag{B20}$$

where

$$\beta_{m,u} = \varphi_d \tag{B21a}$$

$$\beta_{m,\xi,t+1} = \kappa_{1,m} (\phi - \frac{1}{\psi}) \left[\sum_{j=1}^{\infty} (\frac{\lambda}{\kappa_{1,m}})^j x_{t+1-j} + \frac{1}{1 - \kappa_{m,1}\rho} \rho x_t \right] \tag{B21b}$$

$$\beta_{m,e,t+1} = \kappa_{1,m} (\phi - \frac{1}{\psi}) \frac{1}{1 - \kappa_{m,1}\rho} b_{t+1} \tag{B21c}$$

Moreover, it follows that

$$\mathbb{V}ar_t(r_{m,t+1}) = (\beta_{m,u} + \beta_{m,e,t+1})^2 \sigma^2 + \beta_{m,\xi,t+1}^2 \sigma_b^2 \tag{B22}$$

The risk premium for any asset is determined by the conditional variance between the return and m_{t+1}. Thus the risk premium for the market portfolio $r_{m,t+1}$ is equal to $\mathbb{E}_t(r_{m,t+1} - r_{f,t}) = -cov(m_{t+1} - \mathbb{E}_t m_{t+1}, r_{m,t+1} - \mathbb{E}_t r_{m,t+1}) - 0.5\mathbb{V}ar_t(r_{m,t+1})$. Using the innovations in the market return and the pricing kernel, the expression for the equity premium is

$$\mathbb{E}_t(r_{m,t+1} - r_{f,t}) = \beta_{m,e,t} \lambda_{m,e,t} \sigma^2 + \beta_{m,\xi,t} \lambda_{m,\xi,t} \sigma_b^2 - 0.5\mathbb{V}ar_t(r_{m,t+1}) \tag{B23}$$

B.3 A Finite Memory Asset Pricing Model

B.3.1 Model Derivations

In this section, we follow the same steps as in Section B.2.1. Recognize that, in both decay memory and finite memory, the agent has the same preferences and utility function, and thus both model have the same log Euler equation (B14).

Solution for Wealth Return $r_{a,t+1}$. We start by conjecturing that the solutions for the endogenous variables z_i are a linear function of the *finite* past, present and expected future values of x's but subject to finite memory constraint, i.e. the time t information set is $\mathbb{I}_t = \{x_t, x_{t-1}, x_{t-2}, ...x_{t-T}\}$.

$$z_t = A_{0,t} + (1 - \frac{1}{\psi})\left[\sum_{j=1}^{T} u_{j,t}x_{t-j} + b_t x_t + \sum_{j=1}^{\infty} c_{j,t}\mathbb{E}_t x_{t+j}\right] \tag{B24}$$

where the parameters $A_{0,t}$, $u_{j,t}$, b_t and $c_{j,t}$. Under the finite memory assumption, the expectation at time t given information set $\mathbb{I}_t$ would have the following form

$$\begin{aligned}\bar{\mathbb{E}}_t z_{t+1} =& \mathbb{E}(z_{t+1}|x_t, x_{t-1}, x_{t-2}, ...x_{t-T})\\ =& \mathbb{E}_t\left[A_{0,t+1} + (1 - \frac{1}{\psi})\sum_{j=1}^{T+1} u_{t+1,j}x_{t+1-j} + b_{t+1}x_{t+1} + \sum_{j=1}^{\infty}\mathbb{E}_{t+1}c_{t+1,j}x_{t+1+j}\right]\end{aligned} \tag{B25}$$

Approximate that $r_{a,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{c,t+1}$, and by the same procedure as described in Section B.2.1, we get

$$\begin{aligned} z_t &= A_{0,t} + (1 - \frac{1}{\psi})\left[\sum_{j=1}^{T}(\frac{1}{\kappa_1})^j(b_t - 1)x_{t-j} + b_t x_t + b_t\sum_{j=1}^{\infty}(\kappa\rho)^j x_t\right]\\ &= A_{0,t} + (1 - \frac{1}{\psi})\left[\sum_{j=1}^{T}(\frac{1}{\kappa_1})^j(b_t - 1)x_{t-j} + b_t\frac{1}{1-\kappa_1\rho}x_t\right]\end{aligned} \tag{B26}$$

with

$$A_{0,t} = \frac{1}{1-\kappa_1}\left(\log(\delta) + (1 - \frac{1}{\psi})\mu + \kappa_0 + \frac{\theta}{2}(1 - \frac{1}{\psi})^2\left(\sigma^2 + \kappa_1^2(\frac{\varphi_e\sigma}{1-\kappa_1\rho})^2 b_t^2 - \sigma_b^2\Sigma^2\right)\right)$$

where $\Sigma = \sum_{j=1}^{T+1}(\frac{1}{\kappa_1})^j x_{t+1-j} + \frac{\rho x_t}{1-\kappa_1\rho}$.

Solution for Market Return $r_{m,t+1}$. When $r_{i,t+1} = r_{m,t+1}$ the log-linearized Euler equation has the same form as equation (B16). We conjecture that the solutions for the endogenous variables $z_{m,t}$ are a function of the *finite* past, present and expected future values of x's but subject to finite memory constraint, i.e. the time t information set is $\mathbb{I}_t = \{x_t, x_{t-1}, x_{t-2}, ...x_{t-T}\}$.

$$z_{m,t} = A_{0,m,t} + (\phi - \frac{1}{\psi})\left[\sum_{j=1}^{T} u_{j,t}x_{t-j} + b_t x_t + \sum_{j=1}^{\infty} c_{j,t}\mathbb{E}_t x_{t+j}\right] \tag{B27}$$

where the parameters $A_{0,m,t}$, $u_{j,t}$, b_t and $c_{j,t}$ are coefficients to be determined. To use the method of undetermined coefficients, we start by plugging the expressions for the two returns (i.e., $r_{t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{c,t+1}$ and $r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m}z_{m,t+1} - z_{m,t} + g_{d,t+1}$) into the above Euler equation

$$1 = \mathbb{E}_t\left[\exp\left(\theta\log(\delta) - \frac{\theta}{\psi}g_{t+1} + (\theta-1)(\kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1}) + \kappa_{0,m} + \kappa_{1,m}z_{m,t+1} - z_{m,t} + g_{d,t+1}\right)\right]$$

Plugging the conjectured solution of logarithm of price-dividend ratio (B27) into the above equation and collect the terms on the corresponding state variables (i.e., the same procedure as described in Section B.2.1)

$$\begin{aligned} z_{m,t} &= A_{0,m,t} + (\phi - \frac{1}{\psi})\left[\sum_{j=1}^{T}(\frac{1}{\kappa_{1,m}})^j(b_t - 1)x_{t-j} + b_t x_t + b_t\sum_{j=1}^{\infty}(\kappa_{1,m}\rho)^j x_t\right] \\ &= A_{0,m,t} + (\phi - \frac{1}{\psi})\left[\sum_{j=1}^{T}(\frac{1}{\kappa_{1,m}})^j(b_t - 1)x_{t-j} + b_t\frac{1}{1-\kappa_{1,m}\rho}x_t\right] \end{aligned} \tag{B28}$$

B.3.2 Analytical Results: Derivations

This section provides the derivation of the analytical results for the finite memory model.[31] In general it has similar procedure as the decay memory case.

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[31] The time-varying component in the A_0 and $A_{0,m}$ were abbreviated when deriving the analytical solutions as it does not affect the main results, but this impact was considered when doing the quantitative analysis.

Price-Dividend Ratio. Notice that we can rewrite the solution for $z_{m,t}$ (i.e., equation (B28)) as the sum of the usual RE result and a backward-looking component

$$z_{m,t} = \underbrace{b_t \left[A_{0,m} + \frac{\phi - \frac{1}{\psi}}{1-\kappa_{1,m}\rho} x_t \right]}_{\text{usual RE model results, } z_{m,t}^{RE}} + \underbrace{(b_t - 1) \left[A_{0,m} + \sum_{j=1}^{T} (\frac{1}{\kappa_{1,m}})^j \left(\phi - \frac{1}{\psi} \right) x_{t-j} \right]}_{\text{backward-looking eq., } z_{m,t}^{b}} \tag{B29}$$

When $b_t = 1$, the result is back to the usual RE solution, while for $b_t \neq 1$, the asset price can deviate from their fundamental values. Moreover, the deviation from the stable solution is very persistent. To see this, define $\hat{z}$ as the deviation from the usual RE solution, follows the same procedure as in Appendix B.2.2, we obtain

$$\hat{z}_{m,t+1} = \frac{1}{\kappa_{1,m}} \frac{b_{t+1} - 1}{b_t - 1} \hat{z}_{m,t} + (b_{t+1} - 1) \left(\phi - \frac{1}{\psi} \right) \frac{1}{1 - \kappa_{1,m}\rho} \varphi_e \sigma e_{t+1} - \Gamma_{t-T}$$

where Γ_{t-T} denotes deviation in the PD ratio due to memory loss, and it is

$$\Gamma_{t-T} = \frac{1}{\kappa_{1,m}^T} (b_{t+1} - 1) \left(\phi - \frac{1}{\psi} \right) x_{t-T}$$

Moreover, the conditional variance of $z_{m,t+1}$ under finite memory is given by

$$\mathbb{V}ar_t(z_{m,t+1}) = \left[\frac{z_{m,t} - z_{m,t}^{RE}}{\kappa_{1,m}(b_t - 1)} \right]^2 \sigma_b^2 + \left[b_t(\phi - \frac{1}{\psi}) \left(\frac{\varphi_e \sigma}{1 - \kappa_{1,m}\rho} \right) \right]^2 \sigma^2, \quad \text{for } b_t \neq 1$$

$$\mathbb{V}ar_t(z_{m,t+1}) = (\phi - \frac{1}{\psi})^2 \left(\frac{\varphi_e \sigma}{1 - \kappa_{1,m}\rho} \right)^2 \sigma^2, \quad \text{for } b_t = 1$$

Equity Premium. The derivation of the equity premium under finite memory follows the same steps as Appendix B.2.2. First, stochastic discount factor under finite memory can be derived as

$$\begin{aligned} m_{t+1} &= \theta \log \delta - \frac{\theta}{\psi} g_{c,t+1} + (\theta - 1) r_{a,t+1} \\ &= \theta \log \delta - \frac{\theta}{\psi} g_{c,t+1} + (\theta - 1)(\kappa_0 + \kappa_1 z_{t+1} - z_t + g_{c,t+1}) \end{aligned} \tag{B30}$$

Substituting the equilibrium return for $r_{a,t+1}$ into the equation, it is straightforward to show that the innovation to the pricing kernel is

$$\begin{aligned}
m_{t+1}-\bar{\mathbb{E}}_t m_{t+1} &= -\frac{\theta}{\psi}g_{c,t+1}+(\theta-1)(\kappa_1 z_{t+1}+g_{c,t+1})-\bar{\mathbb{E}}_t\left[-\frac{\theta}{\psi}g_{c,t+1}+(\theta-1)(\kappa_1 z_{t+1}+g_{c,t+1})\right]\\
&=\left(-\frac{\theta}{\psi}+\theta-1\right)(g_{c,t+1}-\mathbb{E}_t g_{c,t+1})+(\theta-1)\kappa_1\left(z_{t+1}-\mathbb{E}_t z_{t+1}\right)\\
&=-\left(1-\theta+\frac{\theta}{\psi}\right)\sigma\eta_{t+1}-(1-\theta)\kappa_1(1-\frac{1}{\psi})\left[\sum_{j=1}^{T}(\frac{1}{\kappa_1})^j x_{t+1-j}+\frac{1}{1-\kappa_1\rho}\rho x_t\right]\sigma_b\xi_{t+1}\\
&\quad-(1-\theta)\kappa_1(1-\frac{1}{\psi})b_{t+1}\frac{\varphi_e}{1-\kappa_1\rho}\sigma e_{t+1}\\
&=-\lambda_{m,\eta}\sigma\eta_{t+1}-\lambda_{m,\xi,t+1}\sigma_b\xi_{t+1}-\lambda_{m,e,t+1}\sigma e_{t+1}
\end{aligned} \tag{B31}$$

The expressions $\lambda_{m,\eta}$, $\lambda_{m,e,t+1}$ and $\lambda_{m,\xi,t+1}$ captures the pricing kernel's exposure to the independent consumption shocks, η_{t+1}, to the expected growth rate shock, e_{t+1}, and to the time-varying expectation shock, ξ_{t+1}.

The risk premium for any asset is determined by the conditional variance between the innovations in return and the innovations in the stochastic discount factor. Thus the risk premium for the market portfolio $r_{m,t+1}$ is equal to $\mathbb{E}_t(r_{m,t+1}-r_{f,t})=-cov(m_{t+1}-\mathbb{E}_t m_{t+1}, r_{m,t+1}-\mathbb{E}_t r_{m,t+1})-0.5\mathbb{V}ar_t(r_{m,t+1})$.

Equation (B31) already provides the innovation in m_{t+1}. We now proceed to derive the innovation in the market return.

$$\begin{aligned}
r_{m,t+1}-\bar{\mathbb{E}}_t(r_{m,t+1}) =&\kappa_{1,m}(z_{m,t+1}-\bar{\mathbb{E}}_t z_{m,t+1})+\left(g_{d,t+1}-\bar{\mathbb{E}}_t g_{d,t+1}\right)\\
=&\varphi_d\sigma u_{t+1}+\kappa_{1,m}(\phi-\frac{1}{\psi})\left[\sum_{j=1}^{T}(\frac{1}{\kappa_{1,m}})^j x_{t+1-j}+\frac{1}{1-\kappa_{1,m}\rho}x_{t+1}\right]\sigma_b\xi_{t+1}\\
&+\kappa_{1,m}b_{t+1}\frac{1}{1-\kappa_{1,m}\rho}(\phi-\frac{1}{\psi})\varphi_e\sigma e_{t+1}\\
=&\beta_{m,u}\sigma u_{t+1}+\beta_{m,e,t+1}\sigma e_{t+1}+\beta_{m,\xi,t+1}\sigma_b\xi_{t+1}
\end{aligned} \tag{B32}$$

Moreover, it follows that

$$\mathbb{V}ar_t(r_{m,t+1})=(\beta_{m,u}+\beta_{m,e,t+1})^2\sigma^2+\beta_{m,\xi,t+1}^2\sigma_b^2 \tag{B33}$$

Using the innovations in the market return and the pricing kernel, the expression for the equity premium is time-varying

$$\mathbb{E}_t(r_{m,t+1}-r_{f,t})=\lambda_{m,e,t}\beta_{m,e,t}\sigma^2+\lambda_{m,\xi,t}\beta_{m,\xi,t}\sigma_b^2-0.5\mathbb{V}ar_t(r_{m,t+1}) \tag{B34}$$

The market compensation for expectation variation risks is determined by $\lambda_{m,\xi,t}\beta_{m,\xi,t}$.

B.4 Comparison with the Learning Model

This section shows that our key results hold even in a simple version of the Lucas (1978) model. Adam et al. (2016) use this model to show that a departure from RE - in the form of a learning model - enables even such a simple asset pricing model to reproduce a variety of stylized asset pricing facts quantitatively. We then embed our deviation from RE in such a model. We show that our expectation shock in this paper improves the replication of both individual moments and the overall goodness of fit with respect to the learning mechanism in Adam et al. (2016).

Model Description. The representative agent has standard time-separable CRRA preference. Hence, the problem of the agent is to choose (C_t, S_t, B_t) to maximize the intertemporal utility function

$$\max_{\{C_t \geq 0, S_t, B_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \delta^t u(C_t) = \mathbb{E}_0 \sum_{t=0}^{\infty} \delta^t \frac{(C_t)^{1-\gamma}}{1-\gamma} \tag{B35}$$

subject to the budget constraint

$$C_t + P_t S_t + B_t \leq (P_t + D_t) S_{t-1} + (1 + r_{t-1}) B_{t-1} + Y_t, \tag{B36}$$

where C_t denotes the agent's consumption, P_t the competitive price of stock, S_t the stock hold by the representative agent, B_t the bond holding and Y_t the endowment of income that the agent receives each period. Utility maximisation yields

$$C_t^{-\gamma} P_t = \delta \mathbb{E}_t[C_{t+1}^{-\gamma} P_{t+1}] + \delta \mathbb{E}_t[C_{t+1}^{-\gamma} D_{t+1}] \tag{B37}$$

$$C_t^{-\gamma} = \delta(1 + r_t)\mathbb{E}_t[C_{t+1}^{-\gamma}] \tag{B38}$$

As in Adam et al. (2016), the dividend is assumed to evolve according to $\frac{D_t}{D_{t-1}} = \alpha \varepsilon_t^d$, where $\log \varepsilon_t^d \sim$ i.i.d $N(0, \sigma_d^2)$ and $\alpha \geq 1$. This implies $\mathbb{E}_t(\varepsilon_t^d) = 1$, $\mathbb{E}_{\frac{\Delta D}{D}} \equiv \mathbb{E}\left(\frac{D_t - D_{t-1}}{D_{t-1}}\right) = \alpha - 1$, and $\sigma^2_{\frac{\Delta D}{D}} \equiv \text{var}\left(\frac{D_t - D_{t-1}}{D_{t-1}}\right) = \alpha^2(e^{s_d^2} - 1)$. The consumption growth process is modelled as $\frac{C_t}{C_{t-1}} = \alpha \varepsilon_t^c$, where $\log \varepsilon_t^c \sim$ i.i.d $N(0, \sigma_c^2)$ and $(\log \varepsilon_t^c, \log \varepsilon_t^d)$ are jointly normal. Moreover, the standard deviation of consumption growth is set to be $s^c = \frac{1}{7} s^d$ to capture the lower volatility observed in consumption growth than in dividend growth; the correlation between $\log \varepsilon_t^c$ and $\log \varepsilon_t^d$ is set $\rho_{c,d} = 0.2$ to capture the correlation between dividend and consumption growth.

Given this model setup, Adam et al. (2016) then assume agents to think that the process for risk-adjusted stock price growth contains both a transitory and a persistent time-varying component

$$\left(\frac{C_t}{C_{t-1}}\right)^{-\gamma} \frac{P_t}{P_{t-1}} = b_t + \epsilon_t \qquad \text{and} \qquad b_t = b_{t-1} + \xi_t, \tag{B39}$$

where ϵ_t and ξ_t are i.i.d white noise also jointly i.i.d. with ε_t^c and ε_t^d. The agents then face a learning problem that consists in optimal filtering out the two unobserved components from the realizations of risk-adjusted stock price growth.

On the contrary, we apply our expectation shock and limited memory assumptions to the model setup, instead of assuming the Adam et al. (2016) mechanism based on the two time-varying components in equation (B39) and the corresponding filtering problem. First, rewrite the Euler Equation (B37) as an expectational difference equation linking the current and expected future value of the PD ratio as

$$\frac{P_t}{D_t} = \delta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \frac{D_{t+1}}{D_t} \frac{P_{t+1}}{D_{t+1}} \right] + \delta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \frac{D_{t+1}}{D_t} \right]. \tag{B40}$$

Then, applying the approach described in Section 2.2, we can rewrite the solution for the PD ratio as (analytical solution and model details can be found in Online Appendix)

$$\frac{P_t}{D_t} = (b_t - 1) \left(\sum_{j=0}^{\infty} \left(\frac{\lambda}{\delta}\right)^j \left(\prod_{i=0}^{j} \eta_{t-i} \right)^{-1} \eta_{t-j} \right) + b_t \frac{\delta \alpha^{1-\gamma} \rho_\varepsilon}{1 - \delta \alpha^{1-\gamma} \rho_\varepsilon}, \tag{B41}$$

where

$$\eta_{t+1} \equiv \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \left(\frac{D_{t+1}}{D_t}\right) = \alpha^{1-\gamma} \left(\varepsilon_{t+1}^c\right)^{-\gamma} \varepsilon_{t+1}^d, \tag{B42}$$

and

$$\rho_\varepsilon = \mathbb{E}_t[(\varepsilon_{t+1}^c)^{-\gamma} \varepsilon_{t+1}^d] = e^{\gamma(1+\gamma)\frac{s_c^2}{2}} e^{-\gamma \rho_{c,d} s_c s_d}. \tag{B43}$$

The standard stable RE solution (corresponding to $b_t = 1$) is $PD_t^F = \frac{\delta \alpha^{1-\gamma} \rho_\varepsilon}{1 - \delta \alpha^{1-\gamma} \rho_\varepsilon}$.

Table B.4: Comparison with Adam et al.'s (2016) learning model

	U.S. Data	Adam et al.'s (2016) results	Finite memory	Decay memory
Quarterly mean stock return E_{r^s}	2.25	1.32	2.24	2.23
	(0.37)	(2.50)	(0.04)	(0.07)
Quarterly mean bond return E_{r^b}	0.15	1.09	1.98	1.98
	(0.19)	(-4.90)	(-9.34)	(-9.52)
Mean PD ratio E_{PD}	123.91	109.66	116.92	119.96
	(21.67)	(0.69)	(0.32)	(0.18)
Mean dividend growth $E_{\Delta D/D}$	0.41	0.22	0.36	0.37
	(0.17)	(1.14)	(0.23)	(0.23)
Std. dev. stock return σ_{r^s}	11.44	5.34	9.01	8.73
	(2.71)	(2.25)	(0.90)	(0.99)
Std. dev. PD ratio σ_{PD}	62.43	40.09	54.78	54.80
	(17.27)	(1.33)	(0.50)	(0.43)
Std.dev. dividend growth $\sigma_{\Delta D/D}$	2.88	1.28	2.29	2.17
	(0.82)	(1.95)	(0.72)	(0.86)
Autocorrel. PD ratio $\rho_{PD,-1}$	0.97	0.96	0.97	0.97
	(0.02)	(0.30)	(-0.03)	(-0.17)
Excess return reg. coefficient $\hat{c}_5$	-0.0041	-0.0050	-0.0047	-0.0044
	(0.0014)	(0.64)	(0.40)	(0.19)
R^2 of excess return regression R_5^2	0.2101	0.2282	0.2951	0.2890
	(0.0824)	(-0.22)	(-1.03)	(-0.95)
Risk aversion coefficient γ		5	5	5
Std. dev. of expect. param σ_b			0.084	0.085
T			6	
λ				0.82
Test statistic $\hat{W}_N$		12.87	9.50	7.25
p-value of $\hat{W}_N$		2.5%	16.7%	20.1%

Notes: The table reports the US asset-pricing moments (column 2, with standard errors reported in parentheses), the model moments from Table III in Adam et al. (2016) (column 3, the t-ratios for each moment are reported in parentheses), as well as the moments implied by the finite memory and decay memory models (column 4 and 5, respectively, the t-ratios for each moment are reported in parentheses). Growth rates and returns are expressed in terms of real quarterly rates of increase. The PD ratio is the price over the quarterly dividend. A t-ratio less than 2 indicates that moments are closely matched with the data. The measure of the overall goodness of fit $\hat{W}_N$ and the corresponding p-value are reported in the last two rows of the table. Moreover, as in Adam et al.'s (2016), we also exclude the mean risk free rate $\hat{E}_{r^b}$ and c_5^2 from the estimation and set $\gamma = 5$ in both models for consistency.

Estimation Results. To facilitate the comparison of the quantitative performance of the two models, we use the data are from Adam et al.'s (2016) database. The data is quarterly US stock market data from 1925Q4 to 2012Q2. The model moments are reported on a quarterly frequency, as in Adam et al. (2016).[32] For comparability, we set the risk aversion coefficient to $\gamma = 5$ as in Adam et al. (2016), then there are 4 free parameters to estimate, namely the growth rate of dividend α, the standard deviation of dividend innovations σ, the standard deviation of the expectation parameter σ_b, and the memory constraint parameter (T in the finite memory case or λ in the decay memory case). Again for comparability, we do not include all the above moments in the estimation for this application, but only the one used in Adam et al. (2016), that is

$$(\hat{E}_{r^s}, \hat{E}_{PD}, \hat{\sigma}_{r^s}, \hat{\sigma}_{PD}, \hat{\rho}_{PD,-1}, \hat{R}_5^2, \hat{E}_{\triangle D/D}, \hat{\sigma}_{\triangle D/D}). \tag{B44}$$

The Adam et al.'s (2016) model does not match the mean bond return $\hat{E}_{r^b}$, so they drop it from the estimation in their favourite specification. Moreover, they also drop $\hat{c}_5^2$ to avoid the near-singularity issue of the $\hat{\Sigma}_{S,N}$ matrix. Nonetheless, they report the model implied mean bond return $\hat{E}_{r^b}$ and $\hat{c}_5^2$ based on the estimated parameters. Hence, we proceed in the same way, again for comparability.

Table B.4 reports the results. Column 1 displays the data moments (standard errors in parenthesis), column 2 the corresponding estimated moments in Adam et al. (2016) (see Table III therein). Columns 3 and 4 display the moments from the estimated finite memory model and decay memory model, respectively (t-statistics in parenthesis). Again, there is very little difference between the finite memory and the decay memory model. Both specifications closely match both the mean and the standard deviation of stock return, whereas the learning mechanism in Adam et al. (2016) is not able to match. None of the models is able to match the mean bond return pointing to the limitation of the CRRA setup regarding the risk-free rate puzzle. Leaving aside the mean bond return, all the t-statistics for the estimated moments in our limited memory setup have an absolute value less than one, suggesting that the individual model moments can match the data moments pretty well, with the only marginal exception of R_5^2 for the finite memory case equal to -1.03. Note that R_5^2 is the only statistic

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[32] The data are available at https://www.klaus-adam.com/published-und-forthcoming/. The original data was downloaded from the Global Financial Database. As they consider the return predictability at the five-year horizon, the effective sample is up to 2007Q1, and due to the seasonal adjustment, the effective starting date is 1927Q2. All data are in real terms. Details can be found in Appendix A of Adam et al. (2016).

for which the t-statistics is lower for the Adam et al.'s (2016) model, while our model matches quite well the standard deviation of dividend growth in contrast to Adam et al. (2016). Indeed, the overall goodness of fit test strongly favours our model that displays p-values of 16.7% and 20.1% for the finite memory and decay memory specifications, respectively, so that both are not rejected well above the 10% significance level.

Appendix to Chapter 3

C.1 Verifying the Existence and Uniqueness of MSV Solution

C.1.1 Canonical New Keynesian Model

C.1.1.1 Coefficient Matrices in Baseline NK Model

When the constraint on $\hat{i}_t$ is binding, the system can be rewritten as follows

$$\underbrace{\begin{bmatrix} 1 & -\kappa \\ 0 & 1 \end{bmatrix}}_{\equiv \boldsymbol{A}_0} \begin{bmatrix} \pi_t \\ y_t \end{bmatrix} + \underbrace{\begin{bmatrix} -\beta & 0 \\ -\sigma^{-1} & -1 \end{bmatrix}}_{\equiv \boldsymbol{B}_0} \begin{bmatrix} \mathbb{E}_t \pi_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -\sigma^{-1} \end{bmatrix}}_{\equiv \boldsymbol{C}_0} \begin{bmatrix} u_t \\ \varepsilon_t \\ \mu \end{bmatrix} = 0 \tag{C1}$$

Whilst when the constraint is slack the system is given by

$$\underbrace{\begin{bmatrix} 1 & -\kappa \\ \sigma^{-1}\phi_\pi & 1 + \sigma^{-1}\phi_y \end{bmatrix}}_{\equiv \boldsymbol{A}_1} \begin{bmatrix} \pi_t \\ y_t \end{bmatrix} + \underbrace{\begin{bmatrix} -\beta & 0 \\ -\sigma^{-1} & -1 \end{bmatrix}}_{\equiv \boldsymbol{B}_1} \begin{bmatrix} \mathbb{E}_t \pi_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{\equiv \boldsymbol{C}_1} \begin{bmatrix} u_t \\ \varepsilon_t \\ \mu \end{bmatrix} = 0 \tag{C2}$$

C.1.1.2 Analytical Derivation of E&U Conditions

In this section, we use the GLM theorem to test the coherence and completeness in the canonical NK model. In the calculation we assume $\phi_y = 0$ and $u_t = 0$. ε_t follows a two-state Markov Chain process as defined in Equation (3.5). Given that we have two endogenous variables and two states. The $\mathbf{Y}$ is a 4×1 vector containing the values of $(\hat{\pi}_t, \hat{y}_t)$ in each of the two states. And $F(\cdot)$ is the piecewise linear function, with the mapping defined in the Section 3.2. The matrix-form of these mappings are given by

$$\boldsymbol{\mathcal{A}}_{J_1} = \begin{bmatrix} A_1 + B_1 p & B_1 (1-p) \\ B_1 (1-q) & A_1 + B_1 q \end{bmatrix}, \quad J_1 = \{1,2\}$$

$$\boldsymbol{\mathcal{A}}_{J_2} = \begin{bmatrix} A_0 + B_0 p & B_0 (1-p) \\ B_1 (1-q) & A_1 + B_1 q \end{bmatrix}, \quad J_2 = \{2\}$$

$$\boldsymbol{\mathcal{A}}_{J_3} = \begin{bmatrix} A_1 + B_1 p & B_1 (1-p) \\ B_0 (1-q) & A_0 + B_0 q \end{bmatrix}, \quad J_3 = \{1\}$$

$$\boldsymbol{\mathcal{A}}_{J_4} = \begin{bmatrix} A_0 + B_0 p & B_0 (1-p) \\ B_0 (1-q) & A_0 + B_0 q \end{bmatrix}, \quad J_4 = \emptyset$$

We can solve for the determinant of each $\boldsymbol{\mathcal{A}}_{J_1}$ to $\boldsymbol{\mathcal{A}}_{J_4}$ and check if they have the same sign. The derivation of the determinants is quite complex; therefore, we do not provide the detailed derivation in the Appendix but can share it upon request. Solve gives

$$\begin{aligned}
\det \boldsymbol{\mathcal{A}}_{J_1} &= \kappa^2 \sigma^{-2} [\phi_\pi - 1] [\phi_\pi - \Phi] \\
\det \boldsymbol{\mathcal{A}}_{J_2} &= \kappa \sigma^{-2} \left(- (\phi_\pi - 1) \kappa \Phi - \phi_\pi (1-q) \kappa \left[1 + \frac{1 - (p+q-1)\beta}{\kappa} \sigma \right] \right) \\
\det \boldsymbol{\mathcal{A}}_{J_3} &= -\sigma^{-2} \kappa^2 (\phi_\pi - \Phi) - \sigma^{-1} (1-q) \left[-\kappa \phi_\pi \left(1 + \sigma^{-1} \kappa + \beta (1-p-q) \right) \right] \\
\det \boldsymbol{\mathcal{A}}_{J_4} &= \kappa^2 \sigma^{-2} \Phi
\end{aligned}$$

where

$$\Phi_{p,q,\beta,\sigma,\kappa} \equiv p + q - 1 - \frac{[1 - p\beta - q\beta + \beta] (2 - p - q) \sigma}{\kappa}$$

note that $\Phi_{p,q,\beta,\sigma,\kappa} < 1$ as the fraction is always positive and the highest value $p + q - 1$ can achieve is 1. Therefore, under an active monetary policy rule $\phi_\pi > 1$, $\det \boldsymbol{\mathcal{A}}_{J_1} > 0$. The GLM theorem requires that all the other determinants to have the same sign. $\det \boldsymbol{\mathcal{A}}_{J_4} > 0$ requires $\Phi_{p,q,\beta,\sigma,\kappa} > 0$. However, in that case, $\det \boldsymbol{\mathcal{A}}_{J_2}$ would be negative, which violates the E&U condition in the GLM theorem. Therefore, a unique MSV solution solution may not exist under an active Taylor rule ($\phi_\pi > 1$) without restrictions on p and q.[33]

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[33] See Appendix C.1.1 for a derivation. The assumption $\phi_y = 0$ is imposed for simplification, but the results can be generalised to the case $\phi_y \neq 0$: see the discussion in Ascari and Mavroeidis (2022).

C.1.1.3 Proof of unboundedness of $\lambda(\mathbf{X})$

The canonical form is given by

$$\begin{aligned}
\mathbf{0} &= \left(\boldsymbol{A}_{s_i}\mathbf{Y} + \boldsymbol{B}_{s_i}\mathbf{Y}\boldsymbol{K}' + \boldsymbol{C}_{s_i}\mathbf{X} + \boldsymbol{D}_{s_i}\mathbf{X}\boldsymbol{K}'\right)\boldsymbol{e}_i,\\
s_i &= \mathbf{1}\left\{\left[\boldsymbol{a}'\mathbf{Y} + \boldsymbol{b}'\mathbf{Y}\boldsymbol{K}' + \boldsymbol{c}'\mathbf{X} + \boldsymbol{d}'\mathbf{X}\boldsymbol{K}'\right]\boldsymbol{e}_i > 0\right\}, \quad i = 1, ..., k.
\end{aligned}$$

The canonical form can be represented as $F(\mathbf{Y}) = \lambda(\mathbf{X})$. Below, we show that $\lambda(\mathbf{X}) = -\left(\boldsymbol{C}_{s_i}\mathbf{X} + \boldsymbol{D}_{s_i}\mathbf{X}\boldsymbol{K}'\right)\boldsymbol{e}_i$ is unbounded.

Assume without loss of generality that $\mathbf{X}$ is a matrix the columns of which pertain to states $1, 2, ..., k$ and rows pertaining to m shocks and l constants

$$\mathbf{X} = \begin{bmatrix}
\varepsilon_{1,1} & \cdots & \varepsilon_{1,k}\\
\varepsilon_{2,1} & \cdots & \varepsilon_{2,k}\\
 & \ddots & \\
\varepsilon_{m,1} & \cdots & \varepsilon_{m,k}\\
\varrho_{1,1} & \cdots & \varrho_{1,k}\\
 & \ddots & \\
\varrho_{l,1} & \cdots & \varrho_{l,k}
\end{bmatrix}.$$

Trivially, $\mathbf{X}$ can be partitioned into two matrices $\mathbf{X}^s$, that includes the first m rows of $\mathbf{X}$, and $\mathbf{X}^c$, that includes the last l rows of $\mathbf{X}$:$\mathbf{X} = \mathbf{X}^s + \mathbf{X}^c$ with

$$\mathbf{X}^s = \begin{bmatrix}\mathbf{X}_{1:m}\\ \mathbf{0}_{l\times k}\end{bmatrix} \quad , \quad \mathbf{X}^c = \begin{bmatrix}\mathbf{0}_{l\times k}\\ \mathbf{X}_{m+1:m+l}\end{bmatrix}.$$

The function $\lambda(\mathbf{X})$ is generally piecewise-linear. In the special case, where the first m columns of $\boldsymbol{C}_{s_i}$ and $\boldsymbol{D}_{s_i}$ pertaining to $\mathbf{X}^s$ are invariant with respect to s_i, the function $\lambda(\mathbf{X}^s + \mathbf{X}^c)$ is linear in $\mathbf{X}^s$, i.e.

$$\boldsymbol{C}_{s_i} = \begin{bmatrix}\boldsymbol{C}^s & \boldsymbol{C}^c_{s_i}\end{bmatrix} \quad , \quad \boldsymbol{D}_{s_i} = \begin{bmatrix}\boldsymbol{D}^s & \boldsymbol{D}^c_{s_i}\end{bmatrix}.$$

This is the case for all the models considered in this paper.

This is an intuitive result as we do not assume that the coefficient on shocks depend upon whether a given constraint is binding.

$\lambda(\mathbf{X})$ is then unbounded as long as $\boldsymbol{C}^s$ and $\boldsymbol{D}^s$ are not null.

Application to baseline NK model. We illustrate the unboundedness of $\lambda(\mathbf{X})$ in the case of the model described in C.1.1.1. The function is given by

$$\lambda(\mathbf{X}) = -\boldsymbol{C}_{s_i}\mathbf{X}\boldsymbol{e}_i = \begin{cases} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \\ \varepsilon_1 & \varepsilon_2 \\ \mu & \mu \end{bmatrix} \boldsymbol{e}_i, & \text{if} \quad s_i = 1 \\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -\sigma^{-1} \end{bmatrix} \begin{bmatrix} u_1 & u_2 \\ \varepsilon_1 & \varepsilon_2 \\ \mu & \mu \end{bmatrix} \boldsymbol{e}_i, & \text{if} \quad s_i = 0 \end{cases} \tag{C3}$$

Observe that the coefficients on shocks u_i and ε_i do not depend on i, i.e.

$$\boldsymbol{C}_{s_i} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{\boldsymbol{C}^s} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sigma^{-1}s_i \end{bmatrix}. \tag{C4}$$

Thus, in the case of a three-equation NK model, $\lambda(\boldsymbol{X})$ is linear and trivially unbounded.

C.1.1.4 Derivation for Graphical Representation

First, consider the absorbing state of the model, where $\varepsilon_t = 0$. In the absorbing state we have $\hat{\pi}_t = \hat{\pi}_{t+1} = \hat{\pi}$ and $\hat{y}_t = \hat{y}_{t+1} = \hat{y}$. Hence, the NKPC can be written as the following aggregate supply (AS) relation:

$$\hat{\pi} = \frac{\kappa}{1-\beta}\hat{y} \quad AS. \tag{C5}$$

Meanwhile, the DISE can be written and rearranged to give a piecewise aggregate demand (AD) relation:

$$\hat{\pi} = \max \begin{cases} \frac{\kappa\phi_\pi}{1-\beta}\hat{y} & AD^{TR}, \\ -\mu & AD^{ELB}. \end{cases} \tag{C6}$$

We plot AS (C5) and AD (C6) in Figure C.9. The figure shows the non-uniqueness problem when the NK model features an active TR. It is clear that the necessary support restriction for existence of a solution is $\mu \geq 0$, i.e., $(r\pi^*)^{-1} < 1$.[34] And when $\mu \geq 0$, the model admits two absorbing states: a positive interest rate (PIR) equilibrium, $\{\hat{\pi}, \hat{y}, \hat{i}\} = \{0, 0, 0\}$, and a zero interest rate (ZIR) equilibrium, $\{\hat{\pi}, \hat{y}, \hat{i}\} = \{-\mu, -\frac{\mu(1-\beta)}{\kappa}, -\mu\}$.

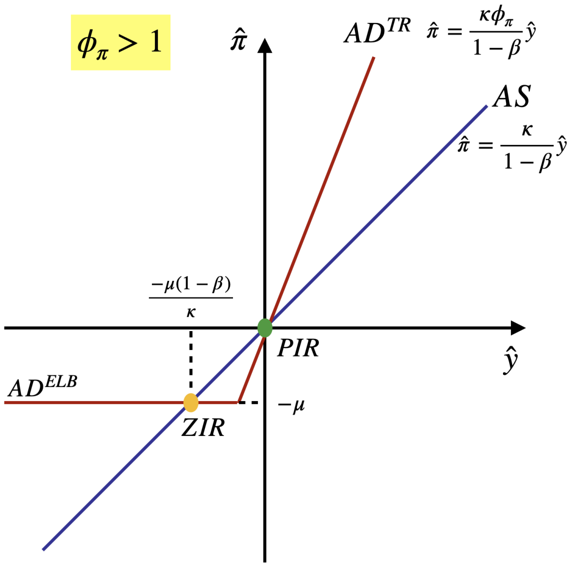

Figure C.9: Absorbing state of the New Keynesian model ($\varepsilon_t = 0$)

Notes: Existence of two steady states in a standard NK model when the Taylor principle is satisfied.

PIR absorbing state. At time t the economy is in a transitory state. With probability p the economy remains in the transitory state, and with $(1-p)$ the economy moves to the PIR absorbing state where $\{\hat{\pi}, \hat{y}, \hat{i}\} = \{0, 0, 0\}$. From (3.4b), we can write:

$$\hat{\pi}^T = \kappa\hat{y}^T + p\beta\hat{\pi}^T,$$

where the second term on the RHS comes from the fact that in period $t+1$ you may remain in a transitory state where $\hat{\pi} \neq 0$, and with probability $1-p$ you jump to the PIR absorbing state where $\hat{\pi} = 0$ Thus, AS is:

$$\hat{\pi}^T = \frac{\kappa}{1-p\beta}\hat{y}^T. \tag{C7}$$

---

[34] As in the figure, when $\mu < 0$, there is no intersection between the AD and AS curve.

For AD, begin by writing the DISE as:

$$\hat{y}^T = p\hat{y}^T - \frac{1}{\sigma}(\hat{i} - p\hat{\pi}^T) + \varepsilon^T.$$

Rearrange and substitute in (3.4c) and $\varepsilon = \varepsilon^T$ to get AD:

$$\hat{\pi}^T = \begin{cases} \frac{\sigma(1-p)}{(p-\phi_\pi)}\hat{y}^T - \frac{\sigma}{(p-\phi_\pi)}\varepsilon^T & AD^{TR} \text{ for } \hat{\pi}^T \geq -\frac{\mu}{\phi_\pi}, \\ \frac{\sigma(1-p)}{p}\hat{y}^T - \frac{\mu}{p} - \frac{\sigma}{p}\varepsilon^T & AD^{ELB} \text{ for } \hat{\pi}^T \leq -\frac{\mu}{\phi_\pi}. \end{cases} \tag{C8}$$

ZIR absorbing state. Here in period t the economy is in a transitory state. With probably p the economy can remain in a transitory state, and with $(1-p)$ it can move to a ZIR absorbing state where $\{\hat{\pi}, \hat{y}, \hat{i}\} = \{-\mu, -\frac{\mu(1-\beta)}{\kappa}, -\mu\}$. Therefore, from (3.4b), AS can be written as:

$$\begin{aligned} \hat{\pi}^T &= \beta[p\hat{\pi}^T + (1-p)(-\mu)] + \kappa\hat{y}^T \\ &= \frac{\kappa}{1-p\beta}\hat{y}^T - \frac{\beta(1-p)}{1-p\beta}\mu. \end{aligned} \tag{C9}$$

To find AD, first begin by writing the DISE as:

$$\hat{y}^T = \left[p\hat{y}^T + (1-p)\left(\frac{-\mu(1-\beta)}{\kappa}\right)\right] - \frac{1}{\sigma}\left[\hat{i} - \left(p\hat{\pi}^T + (1-p)(-\mu)\right)\right] + \varepsilon^T,$$

then substitute in (3.4c) and the ε to get AD:

$$\hat{\pi}^T = \begin{cases} \frac{\sigma(1-p)}{(p-\phi_\pi)}\hat{y}^T + \frac{(1-p)}{(p-\phi_\pi)}\left[\frac{(1-\beta)\sigma}{\kappa} + 1\right]\mu - \frac{\sigma}{(p-\phi_\pi)}\varepsilon^T & AD^{TR} \text{ for } \pi^T \geq -\frac{\mu}{\phi_\pi}, \\ \frac{\sigma(1-p)}{p}\hat{y}^T + \frac{(1-p)}{p}\left[\frac{(1-\beta)\sigma}{\kappa} + 1\right]\mu - \frac{\mu}{p} - \frac{\sigma}{p}\varepsilon^T & AD^{ELB} \text{ for } \hat{\pi}^T \leq -\frac{\mu}{\phi_\pi}. \end{cases} \tag{C10}$$

To find θ simply divide the slop of AD^{TR} by the slope of AS:

$$\theta \equiv \frac{slope_{AD-ELB}}{slope_{AS}} = \frac{\sigma\,(1-p)\,(1-\beta p)}{p\kappa}$$

We plot AD and AS when the economy is in the transitory state and with PIR and ZIR absorbing states for $\theta > 1$ and $\theta < 1$ in Figure C.10. Under ZIR and similar to the PIR case in the main text, if $\theta < 1$ there are either multiple solutions if shocks are sufficiently small or no solutions for shocks that are sufficiently large.

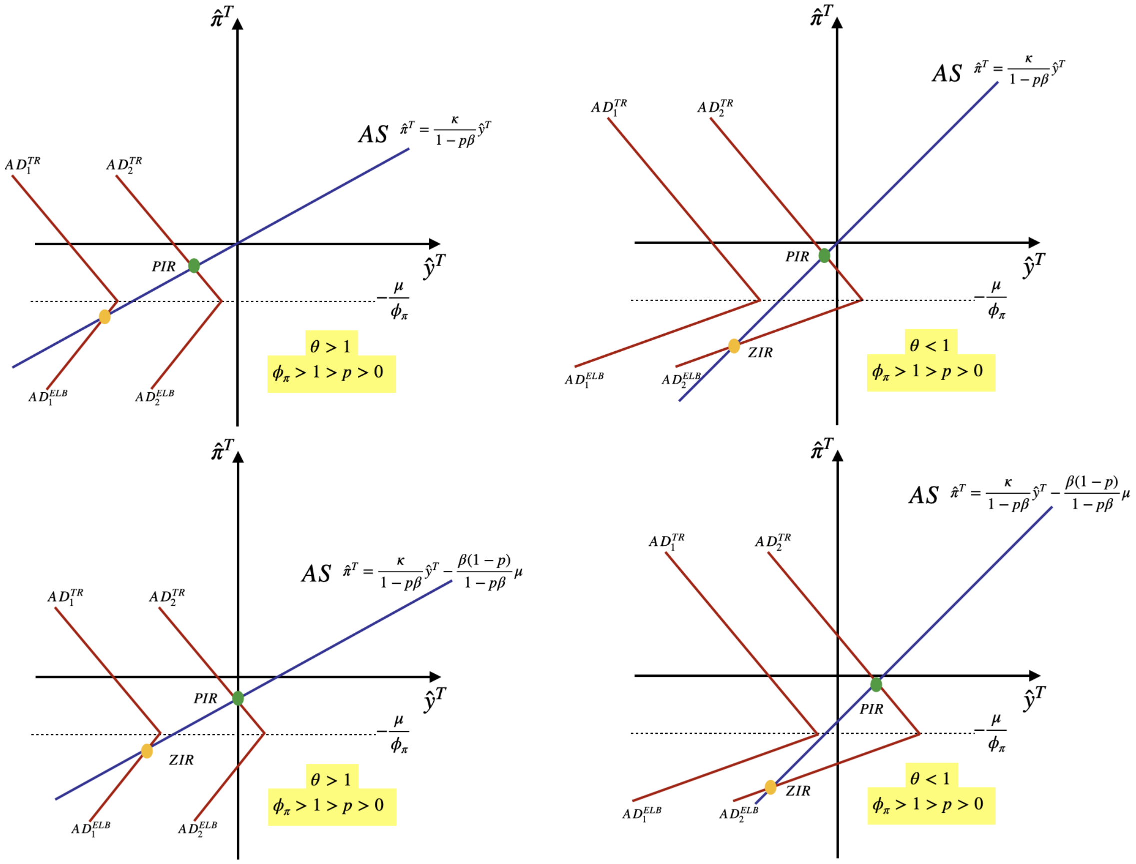

Figure C.10: Transitory state of the New Keynesian model ($\phi_\pi > 1$)

C.1.2 Models with Endogenous States

We no longer assume that $\boldsymbol{H}_{s_t} = \boldsymbol{O}$ and $\boldsymbol{h} = \boldsymbol{0}$ in the canonical form (3.23), but maintain the assumption that $\boldsymbol{X}_t$ follows a k-state stationary Markov process. This implies that, as before, the i-th column of $\mathbf{X}$ gives the value of $\boldsymbol{X}_t$ for a given state i. However, as stipulated by AM, with endogenous states the support of $\boldsymbol{Y}_t$ will vary endogenously over time along the MSV solution given by $\boldsymbol{Y}_t = f(\boldsymbol{Y}_{t-1}, \boldsymbol{X}_t)$. This implies that the solution can no longer be characterised by a time invariant matrix $\mathbf{Y}$. In other words, despite the variables $\boldsymbol{X}_t$ being time invariant (by definition as they are purely forward looking), the support of $\boldsymbol{Y}_t$ must now be a function of $\boldsymbol{Y}_{t-1}$, too. With endogenous states, along an MSV solution we have:

$$\mathbb{E}_t\left[\boldsymbol{Y}_{t+1} | \boldsymbol{Y}_t = \mathbf{Y}_t \boldsymbol{e}_i, \boldsymbol{X}_t = \mathbf{X}\boldsymbol{e}_i\right] = \mathbf{Y}^i_{t+1}\boldsymbol{K}'\boldsymbol{e}_i,$$

Starting from terminal date, T, the model solution is:

$$\mathbf{Y}_T = \boldsymbol{G}_{J_0} y_{T-1} + \boldsymbol{Z}_{J_0}, \tag{C11}$$

where $\boldsymbol{G}_{J_0}$ and $\boldsymbol{Z}_{J_0}$ can be solved from (3.24):

$$\mathbf{0} = \boldsymbol{A}_{s_{t,i}} \boldsymbol{G} \boldsymbol{e}_i + \boldsymbol{h}_{s_{t,i}} + \boldsymbol{B}_{s_{t,i}} \boldsymbol{G} \boldsymbol{K}' \boldsymbol{e}_i \boldsymbol{g}' \boldsymbol{G} \boldsymbol{e}_i, \tag{C12}$$

$$\mathbf{0} = \left(\boldsymbol{A}_{s_{t,1}} \boldsymbol{Z} + \boldsymbol{B}_{s_{t,i}} \boldsymbol{G} \boldsymbol{K}' \boldsymbol{e}_i \boldsymbol{g}' \boldsymbol{Z} + \boldsymbol{B}_{s_{t,i}} \boldsymbol{Z} \boldsymbol{K}' + \boldsymbol{C}_{s_{t,i}} \mathrm{X} + \boldsymbol{D}_{s_{t,i}} \mathrm{X} \boldsymbol{K}' \right) \boldsymbol{e}_i, \tag{C13}$$

$\forall i = 1, ..., k$.

$\mathbf{Y}_T$ is a function of $\boldsymbol{G}_{J_0}$ and $\boldsymbol{Z}_{J_0}$, which are both treated as known.[35] Thus, $\mathbf{Y}_T$ is known and we can solve for $\mathbf{Y}_{T-1}$ from

$$\begin{aligned} \mathbf{0} = &\left(\boldsymbol{A}_{s_{T-1},i} + \boldsymbol{B}_{s_{T-1},i} \boldsymbol{G}_{J_0} \boldsymbol{K}' \boldsymbol{e}_i \boldsymbol{g}' \mathbf{Y}_{T-1} \boldsymbol{e}_i \right) \\ &+ \left(\boldsymbol{B}_{s_{T-1},i} \boldsymbol{Z}_{J_0} \boldsymbol{K}' + \boldsymbol{C}_{s_{T-1},i} \mathrm{X} + \boldsymbol{D}_{s_{T-1},i} \mathrm{X} \boldsymbol{K}' \right) \boldsymbol{e}_i + \boldsymbol{h}_{s_{T-1},i} y_{T-2}. \end{aligned}$$

For every $t \leq T$ the determinants relevant for E&U conditions are given by

$$|\boldsymbol{\mathcal{A}}_{J_0 J_1}| = \prod_i^k \det \left(\boldsymbol{A}_{s_{T-1},i} + \boldsymbol{B}_{s_{T-1},i} \boldsymbol{G}_{J_0} \boldsymbol{K}' \boldsymbol{e}_i \boldsymbol{g}' \right).$$

If $k = 2$, the determinants can be rewritten as

$$\begin{aligned} |\boldsymbol{\mathcal{A}}_{J_0 J_1}| &= \det \left(\boldsymbol{A}_1 + \boldsymbol{B}_1 \boldsymbol{G}_{J_0} \boldsymbol{K}' \boldsymbol{e}_1 \boldsymbol{g}' \right) \det \left(\boldsymbol{A}_1 + \boldsymbol{B}_1 \boldsymbol{G}_{J_0} \boldsymbol{K}' \boldsymbol{e}_2 \boldsymbol{g}' \right), \quad J_1 = \{1, 2\} \text{ (PIR,PIR)}, \\ |\boldsymbol{\mathcal{A}}_{J_0 J_1}| &= \det \left(\boldsymbol{A}_0 + \boldsymbol{B}_0 \boldsymbol{G}_{J_0} \boldsymbol{K}' \boldsymbol{e}_1 \boldsymbol{g}' \right) \det \left(\boldsymbol{A}_1 + \boldsymbol{B}_1 \boldsymbol{G}_{J_0} \boldsymbol{K}' \boldsymbol{e}_2 \boldsymbol{g}' \right), \quad J_1 = \{2\} \text{ (ZIR,PIR)}, \\ |\boldsymbol{\mathcal{A}}_{J_0 J_1}| &= \det \left(\boldsymbol{A}_1 + \boldsymbol{B}_1 \boldsymbol{G}_{J_0} \boldsymbol{K}' \boldsymbol{e}_1 \boldsymbol{g}' \right) \det \left(\boldsymbol{A}_0 + \boldsymbol{B}_0 \boldsymbol{G}_{J_0} \boldsymbol{K}' \boldsymbol{e}_2 \boldsymbol{g}' \right), \quad J_1 = \{1\} \text{ (PIR,ZIR)}, \\ |\boldsymbol{\mathcal{A}}_{J_0 J_1}| &= \det \left(\boldsymbol{A}_0 + \boldsymbol{B}_0 \boldsymbol{G}_{J_0} \boldsymbol{K}' \boldsymbol{e}_1 \boldsymbol{g}' \right) \det \left(\boldsymbol{A}_0 + \boldsymbol{B}_0 \boldsymbol{G}_{J_0} \boldsymbol{K}' \boldsymbol{e}_2 \boldsymbol{g}' \right), \quad J_1 = \{\emptyset\} \text{ (ZIR,ZIR)}. \end{aligned} \tag{C14}$$

If the model has a unique MSV solution, use (C11) with (C12) and (C13), to solve for $\mathbf{Y}_{T-1}$ as a function of y_{T-2}:

$$\begin{aligned} \mathbf{Y}_{T-1} \boldsymbol{e}_i = &- \left(\boldsymbol{A}_{s_{T-1},i} + \boldsymbol{B}_{s_{T-1},i} \boldsymbol{G}_{J_0} \boldsymbol{K}' \boldsymbol{e}_i \boldsymbol{g}' \right)^{-1} \\ &\left[\left(\boldsymbol{B}_{s_{T-1},i} \boldsymbol{Z}_{J_0} \boldsymbol{K}' + \boldsymbol{C}_{s_{T-1},i} \mathrm{X} + \boldsymbol{D}_{s_{T-1},i} \mathrm{X} \boldsymbol{K}' \right) \boldsymbol{e}_i + \boldsymbol{h}_{s_{T-1},i} y_{T-2} \right], \end{aligned}$$

$\forall i = 1, ..., k$.

---

[35] In practice $\boldsymbol{G}_{J_0}$ and $\boldsymbol{Z}_{J_0}$ are precalculated as they are not time-varying per-se but are state dependent. For example, if J_0 always corresponds to the PIR case, then the ELB is never binding and $\boldsymbol{G}_{J_0}$ and $\boldsymbol{Z}_{J_0}$ can easily be obtained from the model policy function (Blanchard and Kahn, 1980b).

C.2 A New Keynesian Model with Fiscal Policy

C.2.1 Model setup

Households. The economy is populated with households indexed with i on a continuum of measure one. The households gain utility from consumption, dislike labour, and have access to one-period risk free bonds. The optimisation problem of the households is thus:

$$\max_{\{C_t, L_t, B_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi} \right) Z_t,$$

subject to the [nominal] period budget constraint given by

$$(1-\tau_t^c) P_t C_t + B_t = (1-\tau_t^w) W_t L_t + R_{t-1} B_{t-1} + P_t T_t,$$

where C_t is consumption, L_t is labour supply, B_t denotes bonds, R_t is nominal interest rate, P_t is the price level, τ_t^c is the consumption subsidy, τ_t^w is the wage tax rate, and T_t are lump-sum taxes.

The consumption bundle C_t consists of a continuum of differentiated goods, and is bundled by a CES aggregator of the form:

$$C_t = \left[\int_0^1 C_t(j)^{\frac{\epsilon-1}{\epsilon}} \, dj \right]^{\frac{\epsilon}{\epsilon-1}}.$$

The utility maximisation problem of the household results in the following intertemporal Euler equation:

$$\beta \mathbb{E}_t \frac{R_t}{\pi_{t+1}} \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{Z_{t+1}}{Z_t} = \mathbb{E}_t \frac{1-\tau_{t+1}^c}{1-\tau_t^c}.$$

The labour supply condition gives the following intratemporal Euler equation:

$$\frac{1-\tau_t^w}{1-\tau_t^c} w_t C_t^{-\sigma} = L_t^{\varphi}.$$

The intratemporal household problem of choosing a consumption bundle results in the following demand for good j:

$$C_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} C_t.$$

Production. Producers use labour as an input to produce differentiated consumption goods according to the following production technology:

$$Y_t(j) = L_t(j).$$

The price-setting problem of an individual firm j follows Rotemberg (1982) where firm j maximises the discounted value of profits,

$$\max_{\{P_t(i)\}} \mathbb{E}_t \sum_{T=t}^{\infty} Q_{t,T} \left[P_t(j) Y_{t,T}(j) - w_T L_T(j) - \frac{\Phi}{2} \left(\frac{P_{t,T}(j)}{P_{t-1,T}(j)} - 1 \right)^2 Y_{t,T} \right],$$

subject to:

$$Y_{t,T}(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t,$$

where Φ denotes a price adjustment cost parameter for the firms.[36] $Y_{t,T}(j)$ denotes demand at time T conditional on the price unchanged since period t. The firm maximises infinite discounted stream of profits, with revenues given by the first term and costs given by the second term. Households own firms, thus their revenues are discounted with the households' discount factor, $Q_{t,T}$:

$$Q_{t,T} = \beta \frac{P_t}{P_T} \left(\frac{C_T}{C_t} \right)^{-\sigma} \frac{Z_T}{Z_t}.$$

The solution to the firm problem results in the following equation for inflation:[37]

$$\pi_t(\pi_t - 1) = \frac{1}{\kappa} \left[\epsilon mc_t + 1 - \epsilon \right] + \mathbb{E}_t \left[Q_{t,t+1}(\pi_{t+1} - 1) \pi_{t+1} \frac{Y_{t+1}}{Y_t} \right].$$

Monetary authority. The monetary authority uses the [gross] nominal interest rate, R_t, as its policy instrument and sets it according to a TR of the form:

$$\frac{R_t}{\bar{R}} = \max \left\{ 1, \left(\frac{\pi_t}{\pi^*} \right)^{\phi_\pi} \left(\frac{Y_t}{Y_t^n} \right)^{\phi_y} \right\},$$

---

[36] We calibrate Φ to the following:

$$\Phi = \frac{\epsilon \gamma}{(1-\gamma)(1-\beta\gamma)},$$

where γ is the probability of firm j being unable to optimally adjust its price in any given period as in a model with Calvo (1983) pricing.

[37] Gross inflation is defined as $\pi_t = P_t / P_{t-1}$

where ϕ_π and ϕ_y is the degree of reaction to contemporaneous inflation and the output deviations from natural level, respectively.

Fiscal authority. The real flow budget constraint for the government is

$$\tau_t^w w_t L_t + T_t = G_t + \tau_t^c C_t. \tag{C15}$$

Market clearing. Markets clear, hence all output is consumed or used for government expenditure,

$$Y_t = C_t + G_t + \frac{\Phi}{2} (\pi_t - 1)^2 Y_t.$$

C.2.2 Log Linearised Equilibrium Conditions

Log linearising the non-linear model equations about a non-inflation deterministic steady state yields the following:

Intertemporal Euler equation:[38]

$$\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - \frac{1}{\sigma} \left(\hat{i}_t + \mathbb{E}_t \left[\varepsilon_t + \Psi^c \Delta \hat{\tau}_{t+1}^c - \hat{\pi}_{t+1} \right] \right), \tag{C16}$$

where $\varepsilon_t \equiv \Delta \hat{z}_{t+1}$ is preference shock.

Labour supply condition:[39]

$$\hat{w}_t = \sigma \hat{c}_t + \varphi \hat{l}_t + \Psi^w \hat{\tau}_t^w - \Psi^c \hat{\tau}_t^c. \tag{C17}$$

Output:

$$\hat{y}_t = \hat{l}_t. \tag{C18}$$

Inflation:

$$\hat{\pi}_t = \frac{1}{\Phi} \epsilon \hat{mc}_t + \beta \mathbb{E}_t \hat{\pi}_{t+1}. \tag{C19}$$

Marginal cost:

$$\hat{mc}_t = \hat{w}_t. \tag{C20}$$

Taylor rule:

$$i_t = \max \left\{ -\mu, \phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t \right\}. \tag{C21}$$

---

[38] We define $\Psi^c = \frac{\bar{\tau}^c}{1-\bar{\tau}^c}$ and $\Delta \hat{\tau}_{t+1}^c = \hat{\tau}_{t+1}^c - \hat{\tau}_t^c$.

[39] We define $\Psi^w = \frac{\bar{\tau}^w}{1-\bar{\tau}^w}$.

Government budget constraint

$$s_g\hat{g}_t + \tau^c s_c(\hat{\tau}^c_t + \hat{c}_t) = \frac{T}{Y}\hat{t}_t + \tau^w \frac{wL}{Y}(\hat{\tau}^w_t + \hat{w}_t + \hat{l}_t).$$

Aggregate resource constraint:

$$\hat{y}_t = s_c\hat{c}_t + s_g\hat{g}_t. \tag{C22}$$

Natural level of output. The natural level of output is attained when $\hat{mc}_t = 0$. Thence, combining (C22), (C20), and (C17) yields

$$\hat{y}^n_t = \frac{s_c}{\sigma + \varphi s_c}\left(\sigma\frac{s_g}{s_c}\hat{g}_t - \Psi^w\hat{\tau}^w_t + \Psi^c\hat{\tau}^c_t\right)$$

Aggregate supply. Combining (C19), (C17), (C18), and (C22) yields

$$\hat{\pi}_t = \underbrace{\frac{\epsilon(\sigma + \varphi s_c)}{\Phi s_c}}_{\kappa_y}\hat{x}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1} \tag{C23}$$

Aggregate demand. Combining (C16) and (C22) yields

$$\hat{x}_t = \mathbb{E}_t\hat{x}_{t+1} - \frac{s_c}{\sigma}(\hat{i}_t - \mathbb{E}_t\hat{\pi}_{t+1} - \hat{r}^n_t), \tag{C24}$$

where the natural rate is given by

$$\hat{r}^n_t = -\Gamma\Delta\hat{g}_{t+1} + \sigma\varepsilon_t - \Psi^c\Delta\hat{\tau}^c_{t+1} + \frac{\sigma}{\sigma + \varphi s_c}\left(\Psi^c\Delta\hat{\tau}^c_{t+1} - \Psi^w\Delta\hat{\tau}^w_{t+1}\right) \tag{C25}$$

Absent of distortionary taxes, i.e. $\hat{\tau}^c_t = \hat{\tau}^w_t = 0$, the expression for $\hat{r}^n_t$ collapses to Equation (3.12d).

Under $\hat{\tau}^c_t = \hat{\tau}^w_t$ and $\hat{g}_t = 0$, the expression collapses to Equation (3.21).

C.2.3 Coefficient Matrices

The coefficients in the canonical representation of the model are:

$$\boldsymbol{A}_0 = \begin{bmatrix} 1 & -\kappa \\ s_c\sigma^{-1}\Gamma\psi_\pi & 1 + s_c\sigma^{-1}\Gamma\psi_y \end{bmatrix}, \quad \boldsymbol{A}_1 = \begin{bmatrix} 1 & -\kappa \\ s_c\sigma^{-1}\left(\phi_\pi + \Gamma\psi_\pi\right) & 1 + s_c\sigma^{-1}\left(\phi_y + \Gamma\psi_y\right) \end{bmatrix},$$

$$\boldsymbol{B}_0 = \boldsymbol{B}_1 = \begin{bmatrix} -\beta & 0 \\ -s_c\sigma^{-1} & -1 \end{bmatrix}, \quad \boldsymbol{C}_0 = \begin{bmatrix} 0 & 0 \\ -s_c & -s_c\sigma^{-1} \end{bmatrix}, \quad \boldsymbol{C}_1 = \begin{bmatrix} 0 & 0 \\ -s_c & 0 \end{bmatrix}, \quad \boldsymbol{D}_0 = \boldsymbol{D}_1 = 0_{2\times 2}$$

$$\boldsymbol{a} = \begin{bmatrix} \phi_\pi, \phi_y \end{bmatrix}', \boldsymbol{c} = \begin{bmatrix} 0, 1 \end{bmatrix}', \quad \boldsymbol{b} = 0_{2\times 1}, \quad \boldsymbol{d} = 0_{2\times 1}.$$

C.2.4 Proof of Proposition 5

We can write the piecewise linear function $F(\mathbf{Y})$, where the mappings follow the same formulation as in Equation (3.6). If all the determinants ($\det \boldsymbol{\mathcal{A}}_J, J \subset 1, \ldots, k$) have the same sign, then the model exists a unique MSV solution. Solving the determinants gives

$$\begin{aligned}
\det A_{J_1} &= \left(\frac{\kappa s_c}{\sigma}\right)^2 (\Gamma\psi_\pi + \phi_\pi - 1)\left(\Gamma\psi_\pi + \phi_\pi - \Phi^g_{p,q,\beta,\kappa}\right), \\
\det A_{J_2} &= \left(\frac{\kappa s_c}{\sigma}\right)^2 (\Gamma\psi_\pi + \phi_\pi - 1)\left(\Gamma\psi_\pi - \Phi^g_{p,q,\beta,\kappa}\right) \\
&\quad - \frac{\kappa s_c}{\sigma^2}\phi_\pi (1-q)\left[\kappa s_c + \sigma(1 - \beta p - \beta q + \beta)\right], \\
\det A_{J_3} &= \left(\frac{\kappa s_c}{\sigma}\right)^2 (\Gamma\psi_\pi - 1)\left(\Gamma\psi_\pi + \phi_\pi - \Phi^g_{p,q,\beta,\kappa}\right) \\
&\quad + \left(\frac{\kappa s_c}{\sigma}\right)^2 (1-q)\left[\frac{\phi_\pi\sigma(1 - \beta p - \beta q + \beta)}{\kappa s_c} + \phi_\pi\right], \\
\det A_{J_4} &= \left(\frac{\kappa s_c}{\sigma}\right)^2 (\Gamma\psi_\pi - 1)\left(\Gamma\psi_\pi - \Phi^g_{p,q,\beta,\kappa}\right),
\end{aligned}$$

with

$$\Phi^g_{p,q,\beta,\kappa} \equiv p + q - 1 - \frac{(2-p-q)\left[1 - \beta p - \beta q + \beta\right]\sigma}{\kappa s_c}.$$

The coefficient $\Phi^g_{p,q,\beta,\kappa}$ cannot be greater than 1 as $0 \leq p, q \leq 1$.

Suppose the local determinacy is satisfied (BK condition), i.e.,

$$\phi_\pi + \Gamma\psi_\pi > 1 \tag{C26}$$

When Equation (C26) is satisfied, the determinant of $\boldsymbol{\mathcal{A}}_{J_1}$ is always positive as $\Phi^g_{p,q,\beta,\kappa} \leq 1$. Therefore, equilibrium uniqueness requires determinant of $\boldsymbol{\mathcal{A}}_{J_2}$, $\boldsymbol{\mathcal{A}}_{J_3}$ and $\boldsymbol{\mathcal{A}}_{J_4}$ are positive.

$\det \boldsymbol{\mathcal{A}}_{J_2} > 0$ requires that

$$\Gamma\psi_\pi - \Phi^g_{p,q,\beta,\kappa} > \frac{\phi_\pi(1-q)\left[1 + \frac{\sigma(1-\beta p - \beta q + \beta)}{\kappa s_c}\right]}{\Gamma\psi_\pi + \phi_\pi - 1}, \tag{C27}$$

where the term on the RHS is always positive, from which we can conclude that $\Gamma\psi_\pi - \Phi^g_{p,q,\beta,\kappa} > 0$.

$\det \boldsymbol{\mathcal{A}}_{J_4} > 0$ requires $\Gamma\psi_\pi - 1$ and $\Gamma\psi_\pi - \Phi^g_{p,q,\beta,\kappa}$ to have the same sign. From Equation (C27), $\Gamma\psi_\pi - \Phi^g_{p,q,\beta,\kappa} > 0$, therefore, E&U requires

$$(\Gamma\psi_\pi - 1) > 0. \tag{C28}$$

For $\det \boldsymbol{\mathcal{A}}_{J_3} > 0$, we require

$$(\Gamma\psi_\pi - 1)\left(\Gamma\psi_\pi + \phi_\pi - \Phi^g_{p,q,\beta,\kappa}\right) > -(1-q)\left[\frac{\phi_\pi \sigma (1 - \beta p - \beta q + \beta)}{\kappa s_c} + \phi_\pi\right] \tag{C29}$$

This is always positive when Equation (C28) holds.

Therefore, for all determinants to be positive, we require both Equation (C27) and (C28) to hold, which is

$$\Gamma\psi_\pi > \max 1, \Phi^g_{p,q,\beta,\kappa} + \frac{\phi_\pi (1-q)\left[1 + \frac{\sigma(1-\beta p-\beta q+\beta)}{\kappa s_c}\right]}{\Gamma\psi_\pi + \phi_\pi - 1}$$

C.2.5 The Unconventional Fiscal Policy Case

Consider the special case of the government spending rule in Equation (3.13): Fiscal policy activates only when monetary policy is constrained following

$$\mathbb{E}_t \Delta \hat{g}_{t+1} = \mathbf{1}\left\{\hat{i}_t = -\mu\right\}(\psi^u_\pi \hat{\pi}_t + \psi^u_y \hat{x}_t), \tag{C30}$$

where ψ^u_π and ψ^u_y denote the coefficients of reaction to inflation and the output gap, respectively.

The presence of the FP instrument in the DISE allows the piecewise linear system to satisfy the E&U conditions, despite the presence of the ELB constraint on $\hat{i}_t$ and an active TR. The instrument $\mathbb{E}_t \Delta \hat{g}_{t+1}$ has the same effect in the NK model as the monetary policy instrument and, hence, it can be set to render the DISE in (3.12a) linear. The E&U conditions are satisfied so long as:

$$\psi^u_\pi = \Gamma^{-1}\phi_\pi, \quad \psi^u_y = \Gamma^{-1}\phi_y, \tag{C31}$$

which also allows (3.12c) to follow an active TR ($\phi_\pi > 1$). It is straightforward to see that since the model is now linear, it implies a unique MSV solution. The rule embeds the mechanism of the simple model in Correia et al. (2013); Seidl and

Seyrich (2023), which showed that a set of tax instruments can replicate monetary policy when the interest rate subject to the ELB constraint.

Uniqueness of an MSV solution in this case is illustrated in Figure C.11 for the special case where $\phi_y = \psi_y^u = 0$. We plot AD and AS for both the absorbing (steady state) case where $\varepsilon_t = 0$ (Figure C.11, Left) and the transitory state with a PIR absorbing state (Figure C.11, Right).

In the absence of active FP, the AD curve is illustrated, as before, with a piecewise red line, which may not intersect AS as shown with $AD^{ELB,TR}$ in the left panel of Figure C.11 and AD_1 in the right panel. Once FP is activate at the ELB, as in the fiscal rule (C30), it fully mimics monetary policy as if the latter were unconstrained. Thus, AD is a linear relation composed of the red AD^{TR} line and the purple AD^u line. In other words, in the presence of active FP stemming from the rule, the model always has a unique solution.

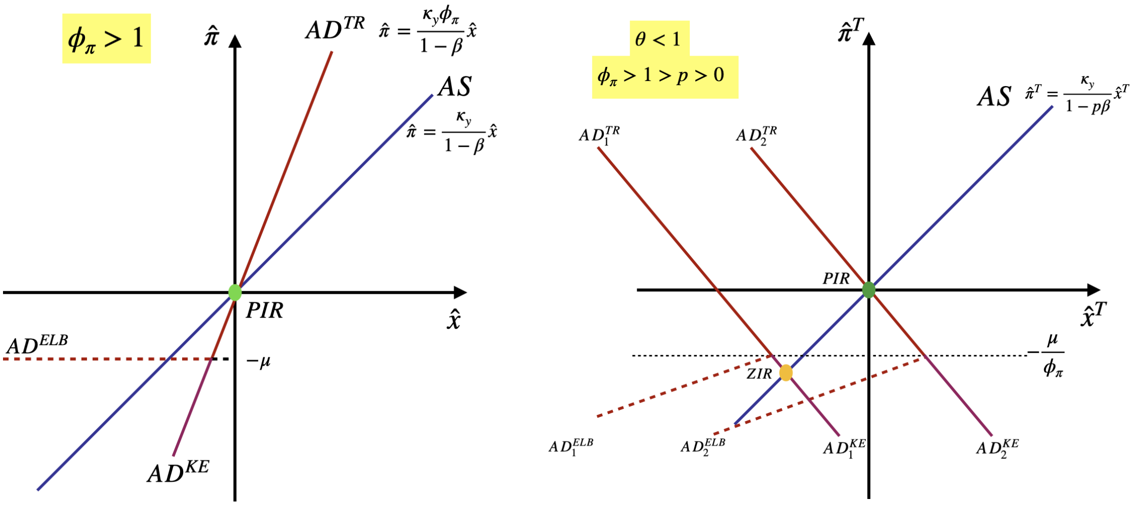

Figure C.11: Existence and uniqueness with unconventional fiscal policy rule

Notes: Left panel illustrates the steady-state equilibrium. Right panel illustrates the transitory state equilibrium with a PIR absorbing state.

C.3 NK Model with Less Persistent Fiscal Policy

C.3.1 Coefficient Matrices

The coefficients in the canonical representation of the model are:

$$\boldsymbol{A}_0 = \begin{bmatrix} 1 & -\kappa_y & 0 \\ 0 & 1 & s_c\sigma^{-1}\Gamma \\ -\left(1-\rho_g\right)\psi_\pi^* & -\left(1-\rho_g\right)\psi_y^* & 1 \end{bmatrix}, \quad \boldsymbol{A}_1 = \begin{bmatrix} 1 & -\kappa_y & 0 \\ s_c\sigma^{-1}\phi_\pi & 1+s_c\sigma^{-1}\phi_y & s_c\sigma^{-1}\Gamma \\ -\left(1-\rho_g\right)\psi_\pi^* & -\left(1-\rho_g\right)\psi_y^* & 1 \end{bmatrix},$$

$$\boldsymbol{B}_0 = \boldsymbol{B}_1 = \begin{bmatrix} -\beta & 0 & 0 \\ -s_c\sigma^{-1} & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{C}_0 = \begin{bmatrix} 0 & 0 \\ -s_c & -s_c\sigma^{-1} \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{C}_1 = \begin{bmatrix} 0 & 0 \\ -s_c & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{D}_0 = \boldsymbol{D}_1 = 0_{3\times 2},$$

$$\boldsymbol{H}_0 = \boldsymbol{H}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\rho_g \end{bmatrix}, \quad \boldsymbol{a} = \begin{bmatrix} \phi_\pi \\ \phi_y \\ 0 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{b} = 0_{3\times 1}, \quad \boldsymbol{d} = 0_{2\times 1}, \quad \boldsymbol{h} = 0_{3\times 1}.$$