




Median geometry for spaces with measured walls and for groups

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Abstract

We show that uniform lattices of isometries of products of real hyperbolic spaces act properly discontinuously and cocompactly on a median space. For lattices in products of at least two factors, this is the strongest degree of compatibility possible with median geometry. The result follows from an analysis of a quasification of median geometry that provides a geometric characterization of spaces at a finite Hausdorff distance from a median space. The case of complex hyperbolic metric spaces is different; we show that these spaces cannot be at finite Hausdorff distance from a median space.

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1 Introduction

1.a. Median spaces and spaces with walls. A *median space* is a metric space (X, d) such that every triple of points $x_1, x_2, x_3 \in X$ admits a unique *median point*, i.e. a point $m \in X$ satisfying

$$d(x_i, m) + d(m, x_j) = d(x_i, x_j), \quad \text{for all } i, j \in \{1, 2, 3\}, i \neq j.$$

The map $X \times X \times X \rightarrow X$, $(x_1, x_2, x_3) \mapsto m$, endows X with a *ternary algebra structure*, called *median algebra*. These were studied in [2, 25, 44, 57, 58]. The geometric study of median spaces was started in [52, 55], and more recently by Bowditch in [11–13].

Examples of median metric spaces are trees, \mathbb{R}^n with the ℓ^1 metric for any $n \geq 1$, and CAT(0) cube complexes on which the Euclidean metric on cubes is replaced by the ℓ^1 metric. According to Chepoi and Gerasimov [20, 34, 35] the class of 1-skeleta of CAT(0) cube complexes coincides with the class of *median graphs*, which are simplicial graphs whose 0-skeleton with the combinatorial distance is median. General median spaces can be thought of as non-discrete versions of 0-skeleta of CAT(0) cube complexes, in the same way in which real trees are non-discrete generalizations of simplicial trees. Indeed, according to Bowditch [11, 12], the metric of a complete connected finite rank median metric space has a bi-Lipschitz equivariant deformation that is CAT(0).¹

The interest of median geometry comes, among other things, from the relevance of median graphs in graph theory and computer science [21], and in optimization theory (see [50, 61] and references therein). Moreover, two important properties of infinite groups, Kazhdan's Property (T) and a-T-menability, can be reformulated in terms of actions on median spaces [18]. The δ -hyperbolicity of metric spaces, characterized by the fact that all geodesic triangles are δ -thin, is connected with both median and Banach space geometry (e.g. Yu [62] showed that every hyperbolic group acts properly on an ℓ^p -space, for p large enough). In this paper, we investigate the following common generalization.

Definition 1.1 [Roughly median (pseudo-)metric spaces and groups] We say that a (pseudo-)metric space (X, pdist) is *roughly median* (or δ -median if we want to emphasize the constant) if there is $\delta, D \geq 0$ such that, given any three points x, y, z , there is a point $m \in X$

$$m \in I_{2\delta}(x, y) \cap I_{2\delta}(y, z) \cap I_{2\delta}(z, x) \subseteq B(m, D)$$

where a *rough interval* or δ -interval is the set

$$I_\delta(a, b) = \{t \in X \mid \text{pdist}(a, t) + \text{pdist}(t, b) \leq \text{pdist}(a, b) + \delta\}.$$

¹ Finite rank seems to be the optimal condition for the existence of such a deformation (see Sect. 2 for the notion of rank).

Namely, we require any triple intersection of δ -intervals to be non-empty and of diameter at most D . A group is called *roughly median* if it admits a geometric action (i.e. properly discontinuous and cocompact) on a roughly median metric space.

This natural notion encompasses both median geometry and hyperbolicity, and is an instance of coarse median geometry, but is more restrictive as, for instance, the Euclidean space \mathbb{R}^2 is not roughly median, but is coarsely median. Whether any coarse median space can be given a rough median metric is unclear. In [51] the authors have a notion of intervals that allows to express a rough median, but their relation with metric intervals is unclear. The class of roughly median groups is contained in the a priori larger class of coarse median groups, and includes mapping class groups [53], see Proposition 2.12. It is stable under direct products, relative hyperbolicity (see Proposition 2.11) and rough isometries, that is quasi-isometries with multiplicative constant one.

Another structure that appears naturally in the study of actions on Banach spaces is that of *space with measured walls* (see Definition 3.1). We proved in [18] that, in some sense, the category of spaces with measured walls is equivalent to the one of subspaces of pseudo-metric median spaces (see Sect. 2), by constructing, for any given wall space X , a median space $\mathcal{M}(X)$ containing it.

In the present work, we show that a δ -median structure on a wall space X is in fact equivalent to being at finite distance from the associated median space $\mathcal{M}(X)$.

Theorem 1.2 *Let (X, \mathcal{W}, μ) be a space with measured walls, μ -locally finite (Definition 4.9), and such that all its half-spaces are quasi-convex (Definition 4.5). The following are equivalent:*

- (1) *the metric space (X, dist) associated to the wall structure is δ -median;*
- (2) *the medianization $\mathcal{M}(X)$ is at finite Hausdorff distance from X ;*
- (3) *X admits an isometric embedding φ into a median space \mathcal{M} such that \mathcal{M} is within finite Hausdorff distance from $\varphi(X)$.*

A more precise and expanded formulation of the above can be found in Theorem 4.14, and combined with Proposition 5.1 allows us to deduce the following.

Corollary 1.3 *The real hyperbolic space \mathbb{H}^n embeds isometrically and $\text{Isom}(\mathbb{H}^n)$ -equivariantly into a locally compact median space at finite Hausdorff distance from the embedded \mathbb{H}^n .*

In particular, the full isometry group of the real hyperbolic space \mathbb{H}^n , as well as all its uniform lattices, act properly and cocompactly on the locally compact median space associated to the usual measured walls structure on \mathbb{H}^n .

1.b. Cubulable and medianizable groups. The previous results bring to light the existence, in the class of finitely generated groups, of several degrees of compatibility with median geometry, starting with cubulable, and proceeding in a decreasing order of strength, as follows.

Definition 1.4 A group is said to be

- *cubulable* if it acts geometrically (i.e. properly discontinuously and cocompactly) on a CAT(0) cube complex;
- *strongly medianizable* if it acts geometrically on a median space of finite rank (Definition 2.1);
- *medianizable* if it acts geometrically on a median space.

Note that the cocompactness assumption is crucial here, and the study of proper actions on median spaces, according to [18], is the one of *a-T-menability* (also sometimes called *Haagerup property*, or *antiT*). If a finitely generated group is cubulable, the CAT(0) cube complex on which it acts has finite dimension, so finite rank as a median space and if it is medianizable then the median space on which it acts is locally compact, without necessarily having finite rank in the sense of median spaces.

The difference between strongly medianizable versus cubulable is unclear, at this point it even seems possible that the two properties are equivalent. Some evidence comes from the fact that key properties known for cubulable groups (Tits alternative, superrigidity) have been proven for strongly medianizable groups as well [28, 30, 31]. Recent work by Messaci [47], showing that a connected locally compact median space of finite rank which admits a transitive action is isometric to \mathbb{R}^n endowed with the ℓ^1 -metric, provides further evidence for the equivalence of the two properties. Note that a Rips-type theorem for median spaces of finite rank as discussed in Sect. 1.d would give a weaker result than the equivalence between cubulable and strongly medianizable, if the connection between stabilizers is similar to the one established for actions on trees, as in [8, 9, 32, 39, 56].

On the other hand, the distinction between medianizable and strongly medianizable groups is clear: there are interesting examples of groups that are in the former class but not in the latter. Indeed, Corollary 1.3 implies the following.

Corollary 1.5 *Uniform lattices in a product $SO(n_1, 1) \times \cdots \times SO(n_k, 1)$, with $k \geq 1$, are medianizable.*

For $k \geq 2$, irreducible uniform lattices in products $SO(n_1, 1) \times \cdots \times SO(n_k, 1)$ are not cubulable, due to results of Chatterji–Fernós–Iozzi [19]. Moreover, by work of Fioravanti [28, 31], they are not strongly medianizable either and, in fact, any isometric action on a finite rank median space has a fix point. Thus, Corollary 1.5 is the best one can hope for, in terms of median geometry, for these lattices. It is unknown if these same lattices can act properly on an infinite dimensional CAT(0) cube complex.

Even in the case of one factor ($k = 1$), Corollary 1.5 may turn out to be significant. It is not known if all arithmetic uniform lattices in $SO(n, 1)$, with n odd and larger than 3, are cubulable, cocompactly or even just properly. It is for instance the case for the uniform lattices described in [48, 59] and [46, §6]. The general consensus seems to be that these lattices are cubulable, except for the construction in dimension 7 [3], especially for the lattices constructed using octonions instead of quaternions [46, Theorem 6.7]. For the congruence subgroups of these latter lattices, it is proved in [4] that the first Betti number is always zero. Thus, to cubulate these lattices one

would have to find finite index subgroups other than congruence subgroups, and it is not known if such subgroups exist.

1.c. Other rank one symmetric spaces. Quaternionic hyperbolic spaces and the Cayley hyperbolic plane over the field of octonions cannot be equipped with measured walls structures, due to the fact that their groups of isometries have Property (T). As far as the complex hyperbolic spaces are concerned, we explained in [18] how, using results of Faraut and Harzallah [27], they can also be equipped with structures of spaces with measured walls (the wall metric being in this case $\text{dist}^{\frac{1}{2}}$, where dist is the hyperbolic metric), hence their groups of isometries also act properly by isometries on a median space. However, the action is no longer cobounded.

Corollary 1.6 (1) *The space $(\mathbb{H}_{\mathbb{C}}^n, \text{dist})$ cannot be isometrically embedded into a median space. In particular, the hyperbolic distance dist cannot be a wall metric.*
 (2) *For any $\alpha \in [1/2, 1)$, whenever dist^α is a wall metric, $(\mathbb{H}_{\mathbb{C}}^n, \text{dist}^\alpha)$ cannot be at bounded Hausdorff distance from a median space.*

1.d. Possible further applications and open questions.

Infinite dimensional real hyperbolic spaces. These spaces can be defined either as quotients of stabilizers of quadratic forms of signature 1 on infinite dimensional real vector spaces [37, §6.A.III], or by an infinite dimensional hyperboloid model [14, 49]. They have a natural structure of measured walls, inducing a metric that equals the hyperbolic metric (see Remark 4.18 and references therein for details).

Some important groups have interesting actions on these spaces, such as the groups of automorphisms of (products of) regular trees and their lattices (including the Burger–Mozes examples) [14] and the groups of birational transformations of complex Kähler surfaces [15]. It seems natural to ask the following.

Question 1.7 Is the infinite dimensional hyperbolic space at finite Hausdorff distance from the median space associated to it?

Rips-type theorems for median spaces. Our results are also relevant in the setting of potential extensions of Rips-type theorems to actions on median spaces [18, Question 1.11]. The existing Rips-type theorems provide conditions under which a non-trivial (i.e. without global fixed point), minimal action of a group G on a real tree $T \neq \mathbb{R}$, yields a non-trivial cocompact action on a simplicial tree, with stabilizers of edges virtually cyclic extensions of stabilizers of arcs in T (see [8, 9, 32, 39, 56] and references therein).

For instance, Bestvina and Feighn prove in [8, Theorem 9.5] that if G is finitely presented, and the action $G \curvearrowright T$ is stable (a condition satisfied by free actions) then the required action on a simplicial tree can always be produced (with virtually cyclic stabilizers of edges, if the initial action was free).

The interest of a Rips-type theorem for median spaces is that it would relate the negation of property (T) to actions on CAT(0) cube complexes, and it would provide conditions under which a-T-menability implies weak amenability with Cowling-Haagerup constant 1 [18, §1.3].

The results in this paper show that *one cannot expect, for actions on median spaces, a theorem similar to the one of Bestvina–Feighn*: uniform irreducible lattices in products

$SO(n_1, 1) \times \cdots \times SO(n_k, 1)$ are finitely presented, they act properly discontinuously, minimally and with compact quotient on median spaces, but cannot act non-trivially on a CAT(0) cube complex [19] when $k \geq 2$. Still, the following question may still have a positive answer

Question 1.8 It is possible to obtain Rips-type theorems for stable (e.g. proper) actions of finitely presented groups on median spaces of finite rank?

This is consistent with the case of real trees, which are median spaces of rank one. For such Rips theorems, the most appropriate condition corresponding to the condition $T \neq \mathbb{R}$ for trees seems to be “median space with no global fixed point at infinity under the full isometry group, and which is not within bounded Hausdorff distance from a space \mathbb{R}^n with the ℓ^1 norm”. This follows from the fact that a finite rank median metric space has a bi-Lipschitz equivariant metric that is CAT(0), and from the results in [16].

1.e. Plan of the paper. In Sect. 2 we recall the definition of (pseudo-)median space, and define and discuss other relevant notions, such as that of δ -tripodic and of geodesically δ -tripodic spaces. Section 3 recalls the construction of a median space associated to a space with walls (this is done in more detail in [18, 28, 29]). Section 4 is devoted to the proof of Theorem 4.14. In Sect. 5 we prove that the medianization of the real hyperbolic space is locally compact. Section 6 discusses the complex hyperbolic case and Corollary 1.6.

2 Tripodic and median spaces

Recall that a *pseudo-metric space* (X, pdist) is a space such that pdist satisfies all the properties of a distance, except for “ $\text{pdist}(x, y) = 0 \Rightarrow x = y$ ”. Its *separation* is the metric space \widehat{X} obtained as a quotient by the equivalence relation $\text{pdist}(x, y) = 0$.

Definition 2.1 A point a in the pseudo-metric space X is said to be *between* two other points x, y in X if

$$\text{pdist}(x, a) + \text{pdist}(a, y) = \text{pdist}(x, y).$$

The *interval* $I(x, y)$ of endpoints x and y is the set of points that are between x and y . A point a is said to be δ -*between* two other points x, y if

$$\text{pdist}(x, a) + \text{pdist}(a, y) \leq \text{pdist}(x, y) + \delta.$$

When $\delta = 0$, a is between x and y and it belongs to the interval $I(x, y)$. The δ -interval $I_\delta(x, y)$ of Definition 1.1 is the set of points that are δ -between x and y .

A subset A in X is called *convex* if every point a that is between two points x, y in A lies in A . When X is a metric space, this is equivalent to the fact that $I(x, y) \subseteq A$ for every two points x, y in A .

A *median pseudo-metric space* is a space for which, given any triple of points x, y, z , the set $I(x, y) \cap I(y, z) \cap I(x, z)$ is non-empty and has diameter zero. Any

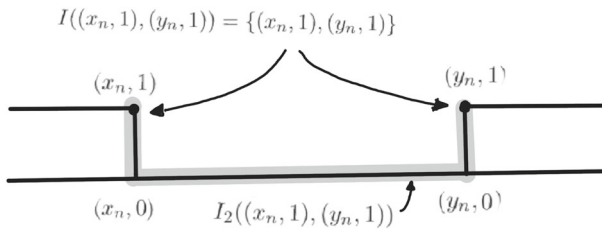


Fig. 1 A line $T \times [0, 1]$ with pieces of a rectangle removed

point in the latter set is called *median point for the triple* x, y, z . When pdist is a metric, this coincides with the notion of *median metric space* recalled in the introduction.

The *rank* of a median metric space X is the supremum over the set of integers k such that X contains an isometric copy of the set of vertices $\{-a, a\}^k$ of the cube of edge length $2a$ endowed with the induced ℓ_1 -metric, for some $a > 0$. Equivalently, it is the supremum over the set of integers k so that there exist k pairwise transverse convex walls. The *rank* of a median pseudo-metric space (X, pdist) equals the rank of its separation.

Given $R > 0$, the *open R -neighbourhood* of a subset A , i.e. $\{x \in X : \text{pdist}(x, A) < R\}$, is denoted by $\mathcal{N}_R(A)$. In particular, if $A = \{a\}$ then $\mathcal{N}_R(A) = B(a, R)$ is the *open R -ball centered at a* . We use the notation $\overline{\mathcal{N}}_R(A)$ and $\overline{B}(a, R)$ to designate the corresponding *closed neighborhood* and *closed ball*, defined by non-strict inequalities.

Remark 2.2 In any metric space, since for $a \in \overline{\mathcal{N}}_R(I(x, y))$ there is a point in $I(x, y)$ at distance less than R , one can see that

$$\overline{\mathcal{N}}_R(I(x, y)) \subseteq I_{2R}(x, y). \tag{1}$$

The reverse inclusion is true in median metric spaces, but false in general. It is false in a Euclidean space, even with $\overline{\mathcal{N}}_R(I(x, y))$ replaced by $\overline{\mathcal{N}}_D(I(x, y))$, for some uniform $D = D(R)$. The following examples show spaces that are at bounded distance from a median space where the reverse inequality is still false.

Example 2.3 Consider a simplicial tree T , the median space $M = T \times [0, 1]$, and the subspace X of M obtained by removing relative interiors of rectangles $[x_n, y_n] \times [0, 1]$ and, in the copy $T \times \{1\}$, the relative interiors of geodesics $[x_n, y_n] \times \{1\}$, with $[x_n, y_n]$ pairwise disjoint geodesics in T of respective length n . In the space X with the metric induced from M , $I((x_n, 1), (y_n, 1)) = \{(x_n, 1), (y_n, 1)\}$, while $I_2((x_n, 1), (y_n, 1))$ contains the entire geodesic $[x_n, y_n] \times \{0\}$ (see Fig. 1).

Another example would be to consider \mathbb{R}^2 with the L^1 norm, remove the interior of pairwise disjoint translates of squares $[0, n] \times [0, n]$, $n \in \mathbb{N}$, and replace them with the other faces of a height one parallelepiped over that square, with the interior of the bottom square removed.

Remark 2.4 Let (X_1, d_1) and (X_2, d_2) be metric space, and $\delta_1, \delta_2 \geq 0$. Then, for any $x = (x_1, x_2), y = (y_1, y_2) \in (X_1 \times X_2, d_1 + d_2)$, a direct computation shows that

$$I_{\delta_1}(x_1, y_1) \times I_{\delta_2}(x_2, y_2) \subseteq I_{\delta_1+\delta_2}(x, y) \subseteq I_{\delta_1+\delta_2}(x_1, y_1) \times I_{\delta_1+\delta_2}(x_2, y_2).$$

The second inclusion is false in general for the ℓ^2 combination of the metrics, even with different constants. For instance, in the Euclidean plane \mathbb{R}^2 , taking $x = (0, 0)$ and $y = (2R, 0)$, one checks that $t = (R, \sqrt{R\delta}) \in I_\delta(x, y)$ is not contained in the rectangle $I_\delta(0, 2R) \times I_\delta(0, 0)$, when R is large enough.

The following generalizations of both median spaces and Gromov hyperbolic spaces requires, loosely speaking, that the set of ‘quasi-centers’ of any triple of points is non-empty, or that they come from a thin geodesic triangle.

Definition 2.5 [(Geodesically) tripod (pseudo-)metric spaces and groups] A (pseudo-)metric space (X, pdist) is called *tripodic* (or δ -*tripodic* when we want to specify the constant) if, given any three points x, y, z ,

$$I_{2\delta}(x, y) \cap I_{2\delta}(y, z) \cap I_{2\delta}(z, x) \neq \emptyset.$$

We say that X is *geodesically tripod (or δ -tripodic)* if given any three points x, y, z , there exists a δ -thin geodesic triangle with vertices x, y, z , namely a geodesic triangle such that each of its sides is contained in the δ -neighborhood of the union of the other two sides.

A finitely generated group G is *(geodesically) tripod (or δ -tripodic)* if it admits a geometric action on a (geodesically) tripod metric space.

Remark 2.6 Roughly median spaces and groups (Definition 1.1) are in particular tripod. A tripod space is then roughly median if the diameter of those non-empty triple intersections is uniformly bounded. Similarly, we can talk about geodesically roughly median spaces, when one can actually see those thin triangles in the space, which is not always the case, as Example 2.3 shows.

Example 2.7 (1) In metric graph theory, 0-tripodic graphs are called modular graphs.

- Thus, all the examples of modular graphs are also examples of 0-tripodic spaces.
- (2) Pseudo-modular graphs (in particular Helly graphs) are 1-tripodic [17, §2.2.2.2].
 - (3) Quasi-median graphs of finite cubical dimension d are d -tripodic by work of Genevois [33, Proposition 2.84].

Remark 2.8 Geodesically δ -tripodic implies tripod, but the converse implication is not true in general. Example 2.3 depict spaces that are 1-tripodic, but not geodesically 1-tripodic. These examples display the key difference between the two notions: in a δ -tripodic space in general we may not have that $I_\delta(x, y)$ is a subset of $\mathcal{N}_D(I(x, y))$ for all $x, y \in X$, for some uniform constant D ; in a geodesically δ -tripodic space, on the other hand, for any $\alpha \geq 0$ there exists $D = D(\delta, \alpha) \geq 0$ such that $I_\alpha(x, y) \subseteq \mathcal{N}_D(I(x, y))$. Indeed, take $t \in I_\alpha(x, y)$ and consider a δ -thin triangle with vertices x, y, t . Take $p \in I(x, y)$, at distance less than δ from both $p_x \in I(x, t)$ and $p_y \in I(y, t)$.

Then $\text{pdist}(x, p) \leq \text{pdist}(x, p_x) + \delta$ and $\text{pdist}(y, p) \leq \text{pdist}(y, p_y) + \delta$ and from $t \in I_\alpha(x, y)$ we get that

$$\begin{aligned} &\text{pdist}(x, p_x) + \text{pdist}(p_x, t) + \text{pdist}(t, p_y) + \text{pdist}(p_y, y) \\ &\leq \text{pdist}(x, p) + \text{pdist}(p, y) + \alpha \end{aligned}$$

hence $\text{pdist}(p_x, t) + \text{pdist}(t, p_y) \leq 2\delta + \alpha$ and so $2\text{pdist}(t, p_y) - 2\delta \leq 2\delta + \alpha$ since $\text{pdist}(p_x, t) \leq \text{pdist}(p_y, t) + 2\delta$. Then $\text{pdist}(t, p) \leq \text{pdist}(t, p_y) + \delta \leq D(\alpha, \delta)$, so that $t \in \mathcal{N}_D(I(x, y))$.

- Example 2.9** (1) Geodesic Gromov hyperbolic spaces are (geodesically) δ -tripodic;
 (2) (geodesic) median spaces are (geodesically) 0-tripodic;
 (3) if (X_1, d_1) and (X_2, d_2) are δ_1 and δ_2 -tripodic respectively, then, from the inclusions in Remark 2.4, one deduces that the product $(X_1 \times X_2, d_1 + d_2)$ is $(\delta_1 + \delta_2)$ -tripodic. In particular, products of hyperbolic spaces are δ -tripodic.
 (4) Euclidean (and Hilbert) spaces cannot be δ -tripodic for any $\delta < \infty$ unless if they are of real dimension one.
 (5) According to Huang and Osajda [43], certain Artin groups and Garside groups (e.g. braid groups) are tripodic.

With the above definition and $D = 0$, a 0-median space is a median space. It is immediate from the definitions that a roughly median space is coarse median in the sense of Bowditch [10].

Remark 2.10 The property of being roughly median is strictly stronger than that of tripodic: the building associated to $Sp(4, \mathbb{Q}_p)$ admits a metric that is 0-tripodic. Indeed, this metric corresponds to a consistent choice of a pair of orthogonal roots in each flat. This is possible because it has two pairs of roots of different lengths. For such a metric, intervals grow exponentially, medians are not unique and can have arbitrarily large diameters. The metric cannot be median because $Sp(4, \mathbb{Q}_p)$ has property (T). A similar argument works for $Sp(2n, \mathbb{Q}_p)$ and $SL_4(\mathbb{Q}_p)$. They cannot be coarse median either [40], hence in particular not roughly median.

Being tripodic or roughly median behaves well under rough isometries, which are quasi-isometries with multiplicative constant 1, see in Arnt’s PhD thesis [1]. Those notions also behave well under relative hyperbolicity, as the following stability result shows.

Proposition 2.11 *Let G be a finitely generated group hyperbolic relative to a finite family of finitely generated subgroups H_1, \dots, H_n . Assume that for every $i \in \{1, \dots, n\}$ the subgroup H_i is δ -tripodic (resp δ -median). Then the group G is α -tripodic (resp roughly median) for some $\alpha \geq 0$.*

In particular, any lattice in $SO(n, 1)$ is δ -tripodic. However, non-uniform lattices in $SU(3, 1)$ are not δ -tripodic: a group that is δ -tripodic must have sub-cubic Dehn function [26], and these lattices have cubic Dehn functions. Note that Elder’s proof in [26] is purely metric, thus even though the result is formulated for word metrics and Dehn functions, it is also true for metrics quasi-isometric to word metrics and for metric generalisations of Dehn functions, as defined in [37, 5.F].

Proof of Proposition 2.11 Assume that for every $i \in \{1, \dots, n\}$ the subgroup H_i admits a geometric action on a (geodesically) δ -tripodic metric space X_i (resp. roughly median). According to Section 4 of [38] or [45], there is a metric space Y on which G acts geometrically and such that the X_i 's are isometrically embedded. Let us show that this metric space Y is (geodesically) α -tripodic (resp. roughly median) if the subspaces X_i are. According to [23, 24], relative hyperbolicity implies that there is a constant δ such that for any triangle Δ , consisting of a triple of points x, y, z in G and (discrete) geodesics between those points, there exists a coset gH_i such that if we denote by $x_1, x_2, y_1, y_2, z_1, z_2 \in gH_i$ the entrance and exit points of the geodesics of Δ from this coset, we have that $d(x_1, x_2), d(y_1, y_2), d(z_1, z_2) \leq \delta$. Now, since X_i is (geodesically) δ_i -tripodic (resp. roughly median) and isometrically embedded in Y , there is a δ_i -centre for the triple x_1, y_1, z_1 , which will work for the triple x, y, z as well. \square

The geodesical versions of δ -tripodic (resp. δ -median) are a little more subtle as one needs to control where the actual geodesics travel, and are left to an interested reader. It is straightforward to check that products of geodesic hyperbolic spaces are roughly geodesically median. There has been a lot of lively research recently around the question of the existence of an (equivariant) embedding of a group into a finite product of hyperbolic spaces. This started with the work of Bestvina, Bromberg, Fujiwara [5], and later Bestvina, Bromberg, Fujiwara and Sisto [7], who introduced a set of axioms allowing to construct such an embedding, and applied it, for instance, to mapping class groups. Hagen and Petyt later proved that, at least in the case of mapping class groups, this embedding endows the group with a structure of δ -median space [41]. After that, Bestvina, Bromberg and Fujiwara showed that in certain cases (e.g. for mapping class groups) the construction can be adapted to yield an equivariant quasi-isometric embedding into a finite product of quasi-trees. This construction has been proven to induce a roughly median structure on mapping class groups by Petyt [53] (but it is unclear if it can be obtained using word metrics).

Proposition 2.12 (Petyt) *Mapping class groups are roughly median.*

Proof Let G be a mapping class group, that acts on a product of quasi-trees as provided by [6]. According to Proposition 3.2 of [53], under the orbit map, any hierarchy path in the finite index, colour-preserving subgroup of G , turns into a quasi-geodesic in each of the quasi-trees. It remains to see that the orbit is coarsely median-preserving. This is the content of the first two paragraphs of the proof of Theorem A, just below the statement of Proposition 3.2 on p.12, which shows that the median of 3 points in G lies on a triangle of hierarchy paths, which turns into triangles of quasi-geodesics in the quasi-trees. \square

Alternatively, one can also use Proposition 3.9 of [41], which embeds the mapping class group in a product of hyperbolic spaces with coarsely median-preserving orbits (with the same argument).

3 The median space associated to a space with measured walls

From [42], we recall that a *wall* of a set X is a partition $X = h \sqcup h^c$ (where h is possibly empty or the whole X). A collection \mathcal{H} of subsets of X is called a *collection of half-spaces* if for every $h \in \mathcal{H}$ the complementary subset h^c is also in \mathcal{H} . We call *collection of walls* on X the collection $\mathcal{W}_{\mathcal{H}}$ of pairs $w = \{h, h^c\}$ with $h \in \mathcal{H}$. For a wall $w = \{h, h^c\}$ we call h and h^c *the two half-spaces bounding w* . Another wall $w' = \{h', h'^c\}$ is said to *intersect* $w = \{h, h^c\}$ if each half-space bounding w intersects non-trivially each half-space bounding w' , and the walls w, w' are *parallel* otherwise. We say that a wall $w = \{h, h^c\}$ *separates* two disjoint subsets A, B in X if $A \subseteq h$ and $B \subseteq h^c$ or vice-versa and denote by $\mathcal{W}(A|B)$ the set of walls separating A and B . In particular $\mathcal{W}(A|\emptyset)$ is the set of walls $w = \{h, h^c\}$ such that $A \subseteq h$ or $A \subseteq h^c$; hence $\mathcal{W}(\emptyset|\emptyset) = \mathcal{W}$. We use the notation $\mathcal{W}(x|y)$ to designate $\mathcal{W}(\{x\}|\{y\})$.

Definition 3.1 (*Space with measured walls* [22]) A *space with measured walls* is a set X , with \mathcal{W} a collection of walls, \mathcal{B} a σ -algebra of subsets of \mathcal{W} and μ a measure on \mathcal{B} , such that for every two points $x, y \in X$ the set of separating walls $\mathcal{W}(x|y)$ is in \mathcal{B} and has finite measure. We denote by pdist_{μ} the pseudo-metric on X defined by

$$\text{pdist}_{\mu}(x, y) = \mu(\mathcal{W}(x|y)),$$

and we call it the *wall pseudo-metric*.

Remark 3.2 Consider the set \mathcal{H} of half-spaces determined by \mathcal{W} , and the natural projection map $\mathfrak{p} : \mathcal{H} \rightarrow \mathcal{W}, h \mapsto \{h, h^c\}$. The pre-images of the sets in \mathcal{B} define a σ -algebra on \mathcal{H} , which we still (abusing notations) denote by \mathcal{B} ; hence on \mathcal{H} can be defined a pull-back measure that we also denote by μ . This allows us to work either in \mathcal{H} or in \mathcal{W} .

For the definition of homomorphisms of spaces with measured walls we use a slightly modified terminology, in accordance with the one in [28, 29].

Definition 3.3 Let (X, \mathcal{W}, μ) and (X', \mathcal{W}', μ') be two spaces with measured walls. A map $\phi : X \rightarrow X'$ is a *homomorphism of spaces with measured walls* if:

- for any $w' = \{h', h'^c\} \in \mathcal{W}'$ we have $\{\phi^{-1}(h'), \phi^{-1}(h'^c)\} \in \mathcal{W}$; this latter wall we denote by $\phi^*(w')$;
- the map $\phi^* : \mathcal{W}' \rightarrow \mathcal{W}$ is measurable and $(\phi^*)_*\mu' = \mu$.

We say that ϕ is a *monomorphism of spaces with measured walls* if ϕ^* is surjective, and that ϕ is a *coarsely surjective monomorphism* if moreover there exists $D > 0$ such that X' is contained in the closed D -neighbourhood of $\phi(X)$, and for every half-space h of X there exists a half-space h' of X' at Hausdorff distance less than D of $\phi(h)$. Both neighbourhoods are considered with respect to the wall pseudo-metric $\text{pdist}_{\mu'}$.

Note that being a monomorphism does not necessarily imply injectivity. On the other hand, being a homomorphism already does imply that the map is an isometric embedding with respect to the wall pseudo-distances.

Example 3.4 The plane \mathbb{R}^2 with the Euclidean distance can be isometrically embedded in a median space, *via* the map

$$\begin{aligned} \mathbb{R}^2 &\rightarrow L^1([0, 2\pi]) \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \frac{1}{4} (x \sin t + y \cos t). \end{aligned}$$

Example 3.5 As mentioned in the introduction, a median metric space is endowed with a structure of convex measured walls, such that the wall pseudo-metric pdist_μ coincides with the median metric, and isometries are automorphisms of the space with measured walls [18, Section 5]. Thus, every L^1 -space has a measured walls space structure, and every L^p -space, with $p \in (1, 2]$ (in particular, every Hilbert space), has a measured walls space structure, since they all embed isometrically in an L^1 -space [60].

Example 3.6 (*Finite dimensional real hyperbolic space*) Define the half-spaces of the real hyperbolic space \mathbb{H}^n to be closed or open geometric half-spaces, so that the boundary of half-spaces is an isometric copy of \mathbb{H}^{n-1} (a geometric hyperplane of \mathbb{H}^n). More precisely, if we use the model of one sheet of a hyperboloid,

$$\mathcal{H}^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1}^2 - \sum_{i=1}^n x_i^2 = 1, x_{n+1} > 0\}$$

for \mathbb{H}^n , then a closed half-space h is defined by $x_1 \geq 0$, with boundary hyperplane ∂h defined by the equation $x_1 = 0$. The connected component of the identity, $SO_I(n, 1)$, of the group stabilizing the form $x_{n+1}^2 - \sum_{i=1}^n x_i^2$, acts transitively on the set of unit tangent vectors of \mathbb{H}^n identified with \mathcal{H}^n , hence all closed half-spaces are in the same orbit. The stabilizer of the half-space h can be identified with the stabilizer of the vector $(1, 0, \dots, 0)^t$ in $SO_I(n, 1)$, thus with $SO_I(n - 1, 1)$ (identified with the subgroup of $SO_I(n, 1)$ with 1 in the upper left corner and all the other entries of the first row and the first column zero). Thus, we can identify the set of closed half-spaces with $SO(n, 1)/SO(n - 1, 1)$.

An element in the stabilizer of the hyperplane $x_1 = 0$ can either fix h or swap it with the opposite closed half-space, as it contains for instance the symmetry represented by the diagonal matrix σ with diagonal $(-1, -1, 1, \dots, 1)$. Thus, the stabilizer of the hyperplane ∂h is the semidirect product of $SO_I(n - 1, 1)$ with $\mathbb{Z}_2 = \{id, \sigma\}$. The set of hyperplanes can therefore be identified with $SO(n, 1)/[SO(n - 1, 1) \rtimes \mathbb{Z}_2]$.

The set of walls is of the form $\{h, h^c\}$, with h a closed half-space and h^c the complementary open half-space, and can therefore be identified with the set of closed half-spaces, hence with $SO(n, 1)/SO(n - 1, 1)$. Since the stabilizer of a wall/a closed half-space is unimodular, there is a $SO(n, 1)$ -invariant borelian measure $\mu_{\mathbb{H}^n}$ on the set of walls. The set of walls separating two points has a compact closure, therefore it has finite measure and thus $(\mathbb{H}^n, \mathcal{W}_{\mathbb{H}^n}, \mu_{\mathbb{H}^n})$ is a space with measured walls. By Crofton’s formula [22, Proposition 3], up to multiplying the measure $\mu_{\mathbb{H}^n}$ by some positive constant, the wall pseudo-metric on \mathbb{H}^n coincides with the usual hyperbolic distance. The case of the infinite dimensional hyperbolic space seems more complicated, see Remark 4.18.

We now recall how a space with measured walls naturally embeds in a median space. This is done in detail in [18, 28, 29] and we just give the outline here.

Definition 3.7 Given a space with measured walls (X, \mathcal{W}, μ) , a section \mathfrak{s} for the projection $\mathfrak{p} : \mathcal{H} \rightarrow \mathcal{W}$ is called *admissible* if its image contains, together with a half-space h , all the half-spaces h' containing h . We denote by $\overline{\mathcal{M}}(X)$ the set of admissible sections.

Remark 3.8 We identify an admissible section \mathfrak{s} with its image $\sigma = \mathfrak{s}(\mathcal{W})$; with this identification, an admissible section becomes a collection of half-spaces, σ , such that:

- for every wall $w = \{h, h^c\}$ either h or h^c is in σ , but never both;
- if $h \subset h'$ and $h \in \sigma$ then $h' \in \sigma$.

Such a collection of half-spaces is commonly called *ultrafilter* in the literature on median spaces.

For any $x \in X$ we denote by σ_x the image of the section of \mathfrak{p} associating to each wall $\{h, h^c\}$ the half-space containing x . That is, σ_x is the set of half-spaces $h \in \mathcal{H}$ such that $x \in h$. This is an admissible section and

$$\mathfrak{p}(\sigma_x \triangle \sigma_y) = \mathcal{W}(x|y).$$

Let now x_0 denote some base point in X . We define

$$\mathcal{B}_0 := \{A \subseteq \mathcal{H} \text{ such that } A \triangle \sigma_{x_0} \in \mathcal{B} \text{ and } \mu(A \triangle \sigma_{x_0}) < +\infty\},$$

and endowed with the following map

$$\text{pdist}_\mu(A, B) = \mu(A \triangle B),$$

the set \mathcal{B}_0 becomes a median pseudo-metric space. Indeed, the identity $A \triangle \sigma_{x_1} = (A \triangle \sigma_{x_0}) \triangle (\sigma_{x_0} \triangle \sigma_{x_1})$ proves the triangle inequality, and the fact that $\sigma_{x_0} \triangle \sigma_{x_1}$ is measurable with finite measure shows that the median pseudo-metric space \mathcal{B}_0 is independent of the chosen base point x_0 . In particular, $\sigma_x \in \mathcal{B}_0$ for any $x \in X$ and the elements in $\overline{\mathcal{M}}(X) \setminus \mathcal{B}_0$ are the elements at infinite distance from the sections coming from the elements of X , and form the Roller boundary (that we will not discuss here). The map

$$\chi^{x_0} : \mathcal{B}_0 \rightarrow \mathcal{S}^1(\mathcal{H}, \mu), \quad \chi^{x_0}(A) = \chi_{A \triangle \sigma_{x_0}} \tag{2}$$

is an isometric embedding of \mathcal{B}_0 into the median subspace $\mathcal{S}^1(\mathcal{H}, \mu) \subseteq \mathcal{L}^1(\mathcal{H}, \mu)$, where $\mathcal{S}^1(\mathcal{H}, \mu) = \{\chi_B \mid B \text{ is measurable and } \mu(B) < +\infty\}$. Notice that for $x, y \in X$ we have $\text{pdist}_\mu(x, y) = \mu(\sigma_x \triangle \sigma_y)$, thus $x \mapsto \sigma_x$ is an isometric embedding of X into $(\mathcal{B}_0, \text{pdist}_\mu)$.

Definition 3.9 The median space associated to X is the metric space $\mathcal{M}(X)$ obtained as the separation of the pseudo-metric space

$$\overline{\mathcal{M}}(X) \cap \mathcal{B}_0.$$

Since each admissible section σ_x belongs to $\mathcal{M}(X)$, it follows that X isometrically embeds in $\mathcal{M}(X)$. We will denote by $\iota : X \rightarrow \mathcal{M}(X)$ this isometric embedding. Given two elements τ, τ' in $\mathcal{M}(X)$, we denote

$$\mathcal{W}(\tau|\tau') = \{w = \{h, h^c\} \in \mathcal{W} \mid h \in \tau, h^c \in \tau'\} = \mathfrak{p}(\tau \Delta \tau').$$

The following result emphasizes that the construction of $\mathcal{M}(X)$ is the right one.

Proposition 3.10 (Proposition 3.14, [18]) *The space $\mathcal{M}(X)$ is a median subspace of $\widehat{\mathcal{B}}_0$. Let (X', \mathcal{W}') be another space with measured walls. Any homomorphism of spaces with measured walls $\phi : X \rightarrow X'$ induces an isometric embedding $\mathcal{M}(X) \rightarrow \mathcal{M}(X')$. In particular the group of automorphisms of (X, \mathcal{W}) acts by isometries on $\mathcal{M}(X)$.*

Remark 3.11 To all intents and purposes, the space $\mathcal{M}(X)$ can be replaced by $\mathcal{M}_0(X)$, the metric completion of the median closure of X in $\mathcal{M}(X)$. The space $\mathcal{M}_0(X)$ may in general be different from $\mathcal{M}(X)$, but is equal to it when the space $\mathcal{M}_0(X)$ is locally convex [29].

All the results in [18] and in this paper that are formulated for $\mathcal{M}(X)$ also hold for $\mathcal{M}_0(X)$. Moreover $\mathcal{M}_0(X)$ has the advantage of being a complete geodesic metric median space when X is connected. Indeed, according to Bowditch [12], a complete median space is geodesic if and only if it is connected. The median map is 1-Lipschitz, therefore the image of X by it in $\mathcal{M}(X)$ is connected, and as the median completion of X in $\mathcal{M}(X)$ equals the increasing union of (connected) sets obtained by iterative applications of the median map to X , it is itself connected. The space $\mathcal{M}_0(X)$ is the metric completion of this latter median completion, hence it is itself connected.

4 Distances to the associated median space

Notation 4.1 Let (X, \mathcal{W}, μ) be a space with measured walls. For any subset $Y \subseteq X$, we denote by $\mathcal{W}(Y)$ the set of walls separating two points of Y (we also say that these walls *cut* Y). We denote by $\mathcal{H}(Y) = \mathfrak{p}^{-1}(\mathcal{W}(Y))$ the corresponding set of half-spaces. The sets $\mathcal{W}(Y)$ and $\mathcal{H}(Y)$ are not *a priori* measurable, unless Y is countable for instance, in which case $\mathcal{W}(Y) = \bigcup_{y, y' \in Y} \mathcal{W}(y|y')$. We write $\overline{\mu}(\mathcal{W}(Y)) \leq K$ if for any measurable subset $\mathcal{E} \subseteq \mathcal{W}(Y)$ we have that $\mu(\mathcal{W}(\mathcal{E})) \leq K$, hence we may talk about *a priori* non-measurable sets having bounded measure.

In this section we investigate under what circumstances a space with measured walls is within bounded Hausdorff distance of its associated median space, or a complete median space in general. Such a measured wall space would have to satisfy metric properties similar to those of a median space, up to bounded perturbation. We briefly recall here a few more properties of median spaces, as discussed in [18], ending with the property that will be central to our approach.

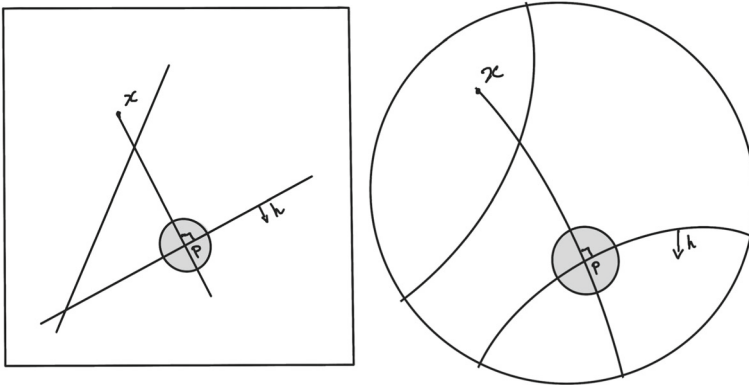


Fig. 2 The Euclidean plane with most walls close to x and crossing h , versus the hyperbolic plane where only hyperplanes near p can cross h

Definition 4.2 Given a metric space (X, dist) , a subset A and a point x in X , an ϵ -projection p of x on A is a point $p \in A$ such that $\text{pdist}(x, p) < \text{pdist}(x, A) + \epsilon$. A gate between a point x and a subset Y of X is a point $p \in Y$ that is between x and any point $y \in Y$. A subset Y is gate convex if every point $x \in X \setminus Y$ has a gate.

Every gate convex set is closed and convex. The converse is true if X is a complete median space. In particular, in a complete median space the closure of every half-space is gate convex. This is the property that we will quasify and use to discuss the finiteness of the Hausdorff distance of a measured wall space to its associated median space.

Definition 4.3 A measured wall space (X, \mathcal{W}, μ) is said to have quasi-gated half-spaces if there are constants $\epsilon, K \geq 0$ such that every half-space h of X is (ϵ, K) -gated, namely, for every point x outside h , and every ϵ -projection p of x on h (for the wall pseudo-distance), the set of walls separating x from p and intersecting h has measure bounded by K .

Remark 4.4 The Euclidean plane (\mathbb{R}^2, ℓ^2) does not have quasi-gated half spaces. Indeed, any straight line (or hyperplane) corresponds to a wall, and half-spaces are given by a side h of that hyperplane. For a point $x \in h^c$, any ϵ -projection p has to be in the ϵ -ball centered around the orthogonal projection of x on the hyperplane \bar{h} . But the measure of the hyperplanes separating x from p and intersecting h is the same as the one of all hyperplanes separating x from p , since the ones not intersecting \bar{h} are the ones parallel to \bar{h} , which is a measure 0 set of hyperplanes. Hence that measure is almost equal to the distance between x and p , so cannot be bounded by any constant independent of the distance between x and p . This phenomena doesn't happen in the hyperbolic plane: there walls correspond to bi-infinite geodesics, and given a point x and a wall \bar{h} , again any ϵ -projection is in the ϵ -ball around the nearest point projection of x onto \bar{h} , but now if x is far enough from \bar{h} , most geodesic passing near x will fail to intersect \bar{h} , only the ones close enough to that ϵ -projection will (see Fig. 2).

A consequence of the property that half-spaces are quasi-gated is a property of quasi-convexity of walls. The latter quasi-convexity property is slightly different from the one introduced in [18], where we defined *quasi-convex sets* to be sets Y such that for every $a, b \in Y$, $I(a, b) \subseteq \overline{N}_M(Y)$ for some uniform $M \geq 0$. Here, we extend the notion of quasi-convexity as follows.

Definition 4.5 Given a metric space (X, dist) , and constants $\delta, M \geq 0$, a subset $Y \subseteq X$ is called (δ, M) -*quasi-convex* if, for every $a, b \in Y$, then

$$I_\delta(a, b) \subseteq \overline{N}_M(Y).$$

With this terminology, we obtain the following.

Lemma 4.6 *Let (X, \mathcal{W}, μ) be a measured wall space with quasi-gated half-spaces. Let $\epsilon, K \geq 0$ be constants such that all its half-spaces are (ϵ, K) -gated.*

Then, for every $\delta \geq 0$, all the half-spaces of X are $(\delta, K + \delta)$ -quasi-convex.

Proof Let $\delta \geq 0$, x, y two distinct points in a half-space h and $a \in I_\delta(x, y)$. Let p be an ϵ -projection of a onto h . Then $\mathcal{W}(a|p) \setminus \mathcal{W}(a|h)$ has measure at most K .

On the other hand

$$\mathcal{W}(a|h) \sqcup \mathcal{W}(x|y) \subseteq \mathcal{W}(a|x) \cup \mathcal{W}(a|y),$$

whence

$$\mu(\mathcal{W}(a|h)) + \text{pdist}_\mu(x, y) \leq \text{pdist}_\mu(x, a) + \text{pdist}_\mu(a, y) \Rightarrow \mu(\mathcal{W}(a|h)) \leq \delta.$$

It follows that $\text{pdist}_\mu(a, p) \leq K + \delta$. □

Remark 4.7 If we don't assume gated or quasi-gated, but we know instead that for any point $x \in X$ and any half-space $h \in \mathcal{H}$ not containing the point x , the set $\mathcal{W}(x|h)$ is measurable and its measure is equal to the pseudo-distance $\text{pdist}_\mu(x, h)$ between x and h , then the half-spaces are convex with respect to pdist_μ , in the sense that if a and b are two points in a half-space h and $x \in I(a, b)$, then $\text{pdist}_\mu(x, h) = \mu(\mathcal{W}(x|h)) = 0$. Indeed, $x \in I(a, b)$ satisfies by definition that $\text{pdist}_\mu(a, x) + \text{pdist}_\mu(x, b) = \text{pdist}_\mu(a, b)$. Hence, since in general $\text{pdist}_\mu(a, b) = \mu(\mathcal{W}(a, x | b)) + \mu(\mathcal{W}(a | x, b))$, we have

$$\begin{aligned} 2\mu(\mathcal{W}(x|h)) + \text{pdist}_\mu(a, b) &= \underbrace{\mu(\mathcal{W}(a, x | b)) + \mu(\mathcal{W}(x | h))}_{\mu(\mathcal{W}(x|b))} \\ &\quad + \underbrace{\mu(\mathcal{W}(a | x, b)) + \mu(\mathcal{W}(x | h))}_{\mu(\mathcal{W}(x|a))} \end{aligned}$$

since $\mathcal{W}(a, x | b) \sqcup \mathcal{W}(x | h) \subseteq \mathcal{W}(x | b)$, and similarly $\mathcal{W}(a | x, b) \sqcup \mathcal{W}(x | h) \subseteq \mathcal{W}(x | a)$. Combining the above, we get $0 = \text{pdist}_\mu(a, x) + \text{pdist}_\mu(x, b) - \text{pdist}_\mu(a, b) \geq 2\text{pdist}_\mu(x, h) \geq 0$.

The condition that $\text{pdist}_\mu(x, h) = \mu(\mathcal{W}(x|h))$ is satisfied for \mathbb{R}^2 when endowed with the ℓ^1 metric, but not with the Euclidean metric, as in the latter case the hyperplanes are all the straight lines; we can have points arbitrarily far from a hyperplane, but the measure of the set of hyperplanes separating a point from another hyperplane is always 0, since this set can only contain hyperplanes of a fixed slope.

Having quasi-gated half-spaces also suffices to ensure that walls separating a point x from a wall actually cut a small ball around x , which again is not true in the Euclidean plane.

Lemma 4.8 *Let (X, \mathcal{W}, μ) be a measured wall space with the property that there are constants $\epsilon, K \geq 0$ such that all its half-spaces are (ϵ, K) -gated.*

Then for every $\tau \in \mathcal{M}(X)$, and $x \in X$, ϵ -projection of τ onto $X \subseteq \mathcal{M}(X)$, each wall in $\mathcal{W}(x|\tau)$ cuts the ball $\bar{B}(x, 2K + \epsilon) \subseteq X$.

Proof Take $w = \{h, h^c\} \in \mathcal{W}(x|\tau)$, so that $x \in h$ and $h^c \in \tau$. Let $p \in h^c$ be a point such that $\text{pdist}_\mu(p, x) \leq \text{pdist}_\mu(h^c, x) + \epsilon$. By assumption, we have that

$$\mu(\mathcal{W}(x|p) \setminus \mathcal{W}(x|h^c)) \leq K.$$

We can write

$$\begin{aligned} \text{pdist}_\mu(\tau, \sigma_p) &= \mu((\tau \triangle \sigma_x) \triangle (\sigma_p \triangle \sigma_x)) \\ &= \mu(\mathcal{W}(x|\tau) \setminus \mathcal{W}(x|p)) + \mu(\mathcal{W}(x|p) \setminus \mathcal{W}(x|\tau)). \end{aligned}$$

By the admissibility of τ , $\mathcal{W}(x|h^c) \subseteq \mathcal{W}(x|\tau)$. It follows that $\mathcal{W}(x|p) \setminus \mathcal{W}(x|\tau) \subseteq \mathcal{W}(x|p) \setminus \mathcal{W}(x|h^c)$, so that, using the assumption once again, we have $\mu(\mathcal{W}(x|p) \setminus \mathcal{W}(x|\tau)) \leq K$.

On the other hand

$$\begin{aligned} \mu(\mathcal{W}(x|\tau) \setminus \mathcal{W}(x|p)) &= \text{pdist}_\mu(\tau, \sigma_x) - \mu(\mathcal{W}(x|\tau) \cap \mathcal{W}(x|p)) \\ &\leq \text{pdist}_\mu(\tau, \iota(X)) + \epsilon - \mu(\mathcal{W}(x|h^c) \cap \mathcal{W}(x|p)). \end{aligned}$$

Now $\mu(\mathcal{W}(x|h^c) \cap \mathcal{W}(x|p)) = \mu(\mathcal{W}(x|p)) - \mu(\mathcal{W}(x|p) \setminus \mathcal{W}(x|h^c))$, which by assumption is larger than $\text{pdist}_\mu(x, p) - K$. Combining all, we get that

$$\text{pdist}_\mu(\tau, \sigma_p) \leq K + \text{pdist}_\mu(\tau, \iota(X)) + \epsilon - \text{pdist}_\mu(x, p) + K,$$

hence that $\text{pdist}_\mu(x, p) \leq 2K + \epsilon$, as desired. □

This result is a quasification of the property that median spaces have, and that *closed walls are gate convex*. It will suffice to conclude that X is within bounded distance from $\mathcal{M}(X)$, provided that the set of walls intersecting a ball has finite measure. This is the condition that we introduce next.

Definition 4.9 A wall space (X, \mathcal{W}, μ) is called μ -locally finite if for every $R > 0$, there exists $M = M(R) \geq 0$ such that $\bar{\mu}(\mathcal{W}(B(x, R))) \leq M < +\infty$ for every $x \in X$, where the open ball $B(x, R)$ is with respect to the wall metric pdist_μ . When we want to avoid specifying the measure μ and there is no risk of confusion, we simply say that the measured wall space is *measurably locally finite*.

- Example 4.10** (1) The measured wall space structures on \mathbb{R}^n endowed with either the Euclidean norm or the ℓ^1 -norm are measurably locally finite, as is the structure on the real hyperbolic space \mathbb{H}^n .
 (2) Discrete wall spaces whose wall distance is uniformly proper are μ -locally finite.
 (3) More generally, if for each ball we have a covering

$$\mathcal{W}(B(x, R)) \subseteq \mathcal{W}(x|a_1) \cup \dots \cup \mathcal{W}(x|a_n(R)),$$

with $d(x, a_i) \leq g(R)$, then the measured wall space is measurably locally finite.

Lemma 4.8 implies the following.

Lemma 4.11 *Let (X, \mathcal{W}, μ) be a measured wall space with quasi-gated half-spaces and μ -locally finite. Then $\mathcal{M}(X)$ is within Hausdorff distance from $\iota(X)$, and in particular is roughly median.*

Precisely, if there are constants $\epsilon, K \geq 0$ such that all half-spaces are (ϵ, K) -gated, then $\mathcal{M}(X)$ is within Hausdorff distance at most

$$\delta = \delta(K, \epsilon) := \bar{\mu}(\mathcal{W}(\bar{B}(x, 2K + \epsilon)))$$

from $\iota(X)$, and hence X is δ -median (Definition 1.1).

To complete the characterization of measured wall spaces within finite distance of their associated median space, we have to consider a converse of the statement in Lemma 4.11.

Remark 4.12 The property alone that $\mathcal{M}(X)$ is within bounded distance from $\iota(X)$ does not suffice to imply that half-spaces are quasi-gated. Example 2.3 shows this, since the space X is within bounded Hausdorff distance of $\mathcal{M}(X) = M$, but for the midpoints $(a_n, 0)$ of the geodesics $[x_n, y_n] \times \{0\}$ in X , there exist no points p_n in the half-space $h = T \times [1/2, 1] \cap X$ such that $\mathcal{W}(a_n, p_n) \setminus \mathcal{W}(a_n, h)$ has uniformly bounded measure. Indeed, for every $p_n \in h$, the measure of $\mathcal{W}(a_n, p_n) \setminus \mathcal{W}(a_n, h)$ is at least $n/2 - 1$. The reason for this is that in this example half-spaces are not quasi-convex in the sense of Definition 4.5. Quasi-convexity is therefore a condition independent of the “bounded distance from $\mathcal{M}(X)$ ” condition and, as it is also a necessary condition for the half-spaces to be quasi-gated, it has to be added to obtain a converse. This is the subject of the next lemma.

It is to be noted that, when the quasi-convexity of the half-spaces is required, the condition “within bounded distance from $\mathcal{M}(X)$ ” can be weakened to “tripodic”.

Lemma 4.13 *Let (X, \mathcal{W}, μ) be a measured wall space that is δ -tripodic when endowed with the wall pseudo-metric, for some $\delta \geq 0$, and such that there is $M \geq 0$ such that all the half-spaces are $(2\delta, M)$ -quasi-convex.*

For every half-space h of X , every point x outside h , and every ϵ -projection p of x on h , the walls separating x from p and intersecting h must intersect $\bar{B}(p, 2\delta + M + \epsilon)$. Namely, for every $k \geq 2\delta + M + \epsilon$ then

$$\mathcal{W}(x|p) \cap \mathcal{W}(h) \subseteq \mathcal{W}(\bar{B}(p, k)).$$

In particular, if X is μ -locally finite, then all its half-spaces are (ϵ, K) -gated for $K \geq \bar{\mu}(\bar{B}(p, 2\delta + M + \epsilon))$.

Proof Let x be a point outside a half-space h_0 , and let p be a ϵ -projection of x on h_0 . Let \mathcal{E} be a measurable subset of \mathcal{W} contained in $\mathcal{W}(x|p) \setminus \mathcal{W}(x|h_0)$. For any wall $w = \{h, h^c\} \in \mathcal{E}$ assume the notation is such that $x \in h$ (so $p \in h^c$). Since the wall w does not separate x from h_0 , the intersection $h \cap h_0$ is not empty. Take $q \in h \cap h_0$. As X is δ -tripodic, there exists a point m that is 2δ -between any two of the three points x, p, q . In particular, since the walls are $(2\delta, M)$ -quasi-convex, $m \in I_{2\delta}(x, q) \subseteq \bar{N}_M(h)$, and $m \in I_{2\delta}(p, q) \subseteq \bar{N}_M(h_0)$ so

$$\text{pdist}(x, m) \geq \text{pdist}(x, h_0) - M \geq \text{pdist}(x, p) - \epsilon - M$$

hence $\text{pdist}(x, p) - \text{pdist}(x, m) \leq M + \epsilon$. Finally, from $m \in I_{2\delta}(x, p)$ we compute

$$\text{pdist}(p, m) \leq \text{pdist}(p, x) - \text{pdist}(m, x) + 2\delta \leq 2\delta + M + \epsilon.$$

This shows that $w \in \mathcal{W}(\bar{B}(p, \epsilon + M + 2\delta))$. □

Combining all the preceding lemmas, we obtain the following result, that immediately implies Theorem 1.2.

Theorem 4.14 *Let (X, \mathcal{W}, μ) be a space with measured walls that is μ -locally finite (see Definition 4.9). The following are equivalent:*

- (1) *there are constants $\delta, M \geq 0$ such that the Hausdorff distance from $\iota(X)$ to the associated median space $\mathcal{M}(X)$ is at most δ and all the half-spaces of X are $(2\delta, M)$ -quasi-convex;*
- (2) *there are constants $\delta, M \geq 0$ such that there exists a median space \mathcal{M} and a monomorphism $\varphi : X \rightarrow \mathcal{M}$ such that \mathcal{M} is within Hausdorff distance at most δ from $\varphi(X)$, and all the half-spaces of X are $(2\delta, M)$ -quasi-convex;*
- (3) *there exists δ and M non-negative constants such that X with the wall pseudo-metric is δ -tripodic, and all the half-spaces of X are $(2\delta, M)$ -quasi-convex;*
- (4) *the space X has (ϵ, K) -gated half-spaces, for some ϵ and K non-negative constants.*

Proof That (1) implies (2) is immediate, and that (2) implies (3) follows from the fact that roughly median spaces are in particular tripodic. That (3) implies (4) is the content of Lemma 4.13, and that (4) implies (1) follows from the combination of Lemmas 4.11 and 4.6. □

Note that the property of being at finite Hausdorff distance from its median space is not inherited by subsets, not even when they are geodesic. Indeed, every L^2 space can be embedded isometrically into an L^1 space [60]. A particular case of the equivalence in Theorem 4.14 is the following.

Theorem 4.15 *Let (X, \mathcal{W}, μ) be a space with measured walls that is μ -locally finite and has quasi-convex half-spaces. The following are equivalent:*

- (1) *the Hausdorff distance from $\iota(X)$ to the associated median space $\mathcal{M}(X)$ is finite;*
- (2) *there exists a median space \mathcal{M} and a monomorphism $\varphi : X \rightarrow \mathcal{M}$ such that \mathcal{M} is within finite Hausdorff distance from $\varphi(X)$;*
- (3) *there exists δ such that the pseudo-metric space (X, pdist_μ) is δ -tripodic;*
- (4) *there are ϵ and K non-negative constants such that the space X has (ϵ, K) -gated half-spaces.*

Corollary 4.16 *A space with measured walls that is μ -locally finite and geodesically δ -tripodic (e.g. geodesic Gromov hyperbolic or finite product of geodesic Gromov hyperbolic spaces) with respect to the distance pdist_μ is at finite Hausdorff distance of its associated median space.*

Corollary 4.17 *A finite product of finite dimensional real hyperbolic spaces is within bounded distance from its associated median space.*

Remark 4.18 The infinite dimensional real hyperbolic space can be described in several ways (see Example 3.6 for the finite dimensional case). For instance, given the Hilbert space ℓ^2 and $O(1, \infty)$ the space of bounded operators preserving the form $-x_1^2 + \sum_{i=2}^\infty x_i^2$, one can define \mathbb{H}^∞ as the infinite dimensional Riemannian symmetric space of constant negative curvature $O(1, \infty)/[O(1) \times O(\infty)]$, where $O(\infty)$ represents the space of orthogonal operators that keep the first coordinate x_1 fixed [37, §6.A.III]. Alternatively, given a Hilbert space H and a 1-dimensional subspace L in it, one can define a bilinear symmetric form determined by the quadratic form

$$B(v, v) = \|v_L\|^2 - \|v_{L^\perp}\|^2,$$

where v_L and v_{L^\perp} are the orthogonal projections of v on L , respectively on its orthogonal L^\perp . For a fixed non-zero vector e in L one can define the hyperboloid model of the infinite dimensional hyperbolic space as

$$\mathbb{H}^\infty = \{v \in H \mid B(v, v) = 1, B(v, e) > 0\},$$

with distance defined by $\cosh(\text{dist}_{\mathbb{H}^\infty}(x, y)) = B(x, y)$, turning it into a complete metric space with constant curvature -1 (in an appropriate sense) [49]. Note that there is a generalization of this construction, \mathbb{H}^α , of Hilbert dimension α , for every cardinal $\alpha \geq 2$ [14]. For simplicity, here we restrict to $\alpha = \aleph_0$.

Finally, \mathbb{H}^∞ can also be defined as the direct limit of the sequence of metric spaces $(\mathbb{H}^n)_{n \in \mathbb{N}}$, where $\mathbb{H}^n \hookrightarrow \mathbb{H}^{n+1}$ is the natural inclusion obtained by adding a 0 direction.

It is unclear how to endow the space $X = \mathbb{H}^\infty$ with a measured walls structure, and even if it were, only a few of the previous arguments would work. In particular, the following are true:

- (1) for every $\lambda > 0$, the coarse interval $I_\lambda(x, y)$ is contained in $\mathcal{N}_{\lambda+\delta}(I(x, y))$, for every x, y , where δ only depends on the hyperbolicity constant. Therefore, all the half-spaces are quasi-convex in the most general sense.
- (2) the half-spaces are quasi-gated only in the following sense: given a half-space h , a point x outside h and an ϵ -projection p of x onto h , every wall in $\mathcal{W}(x|p) \setminus \mathcal{W}(x|h)$ intersects a ball $B(p, \delta)$, with δ depending only on the hyperbolicity constant and on ϵ (Lemma 4.13).

However, as X is not measurably locally finite, the rest of the argument fails. It is nevertheless natural to ask whether the result cannot be obtained by different means. An answer to Question 1.7 cannot be obtained by taking a limit of the finite dimensional case either, as the bound we obtain for the distance to the associated median space in that case depends on volumes of balls, that increase with the dimension.

Remark 4.19 To complete the picture, we note that for a measured wall space X satisfying the equivalent assumptions of Theorem 4.14, a connection can be established between its half-spaces h and corresponding sets $h_{\mathcal{M}}$ of $\mathcal{M}(X)$, that are not half-spaces in general, defined, for each $h \in \mathcal{H}$, by

$$h_{\mathcal{M}} = \{[\sigma] \in \mathcal{M}(X) \text{ such that } h \in \sigma \text{ for some } \sigma \in [\sigma]\}.$$

These sets are convex, but their complements in $\mathcal{M}(X)$ might not be. Note that the complement of a set $h_{\mathcal{M}}$ is in general not $(h^c)_{\mathcal{M}}$, since we can have $h_{\mathcal{M}} \cap (h^c)_{\mathcal{M}} \neq \emptyset$, see Remark 2.18 in [29].

5 Local compactness of the medianization of real hyperbolic spaces

In this section, we will show that the medianization of a real hyperbolic space is locally compact. To do so, we will show that balls in that medianization are *totally bounded*, namely they can be covered by finitely many balls of any given radius. In a metric space, a subset is compact if and only if it is complete and totally bounded. More precisely, the aim of this section is to show the following.

Proposition 5.1 *Let $n \geq 1$. For any $R \geq 0$ and any $\epsilon > 0$, any ball of radius R in $\mathcal{M}(\mathbb{H}^n)$ can be covered by a finite collection of balls of radius ϵ .*

We start by identifying the walls in \mathbb{H}^n .

Notation 5.2 Fix a basepoint x_0 in \mathbb{H}^n . For every point $x \neq x_0$, we denote by H_x the unique hyperplane orthogonal to the unique geodesic $[x, x_0]$ and containing x , and by W_x the unique wall determined by H_x and whose closed component contains x_0 .

Lemma 5.3 *Let $n \geq 2$. For every admissible section $\tau \in \mathcal{M}(\mathbb{H}^n)$ and non-trivial geodesic $[a, x_0]$ in \mathbb{H}^n , one of the following cases occurs:*

- (1) *there exists $\theta \in (a, x_0)$ such that τ coincides with σ_a on every W_x with $x \in (\theta, x_0]$ and τ coincides with σ_{x_0} on every W_x with $x \in [a, \theta)$;*

- (2) τ coincides with σ_a on every W_x with $x \in (a, x_0)$;
- (3) τ coincides with σ_{x_0} on every W_x with $x \in (a, x_0)$.

Proof Suppose that we are neither in case (2) nor in case (3). Thus, there exists a point $u \in (a, x_0)$ such that τ coincides with σ_a on W_u and a point $v \in (a, x_0)$ such that the section τ coincides with σ_{x_0} on W_v . Clearly for every $x \in [u, x_0]$, the section τ coincides with σ_a on W_x , and for every $y \in [a, v]$, it coincides with σ_{x_0} on W_y . In particular u must be between v and x_0 . Let δ be the supremum of the distances to x_0 of points such as u and let θ be the point on $[a, x_0]$ that is at distance δ from x_0 . As θ appears as limit of a sequence of points u_n as above and $(\theta, x_0] = \bigcup [u_n, x_0]$, the first property in (1) follows, and the second property follows according to the choice of θ and the fact that for every $x \in (a, x_0)$, then τ must equal either σ_a or σ_{x_0} . \square

Notation 5.4 We denote by $\theta_\tau(a)$ the point θ in case (1), the endpoint a in case (2) and the endpoint x_0 in case (3). Note that when a varies on the sphere $S(x_0, R)$, the parameter $\theta_\tau(a)$ defines a map

$$\begin{aligned} \delta_\tau : S(x_0, R) &\rightarrow [0, R] \\ a &\mapsto \text{dist}(x_0, \theta_\tau(a)). \end{aligned}$$

Lemma 5.5 For every admissible section $\tau \in \mathcal{M}(\mathbb{H}^n)$, the map δ_τ defined above is continuous.

Proof Let a_0 be a fixed point on $S(x_0, R)$ and let a be a point on $S(x_0, R)$ such that the angle between $[x_0, a]$ and $[x_0, a_0]$ is at most ϵ . For every point y_0 on $[a_0, x_0]$, the point y on $[a, x_0]$ that is nearest to a such that H_y does not intersect H_{y_0} satisfies

$$\text{dist}(x_0, y) \leq \text{dist}(x_0, y_0) + \kappa \epsilon R,$$

for a universal constant κ . This implies that $\text{dist}(x_0, \theta_\tau(a)) \leq \text{dist}(x_0, \theta_\tau(a_0)) + \kappa \epsilon R$. The opposite inequality is obtained by swapping a_0 and a in the previous argument. \square

Remark 5.6 According to Lemmas 4.13 and 4.8, given τ an arbitrary admissible section in $\mathcal{M}(\mathbb{H}^n)$, and $x \in \mathbb{H}^n$ a point such that $\text{pdist}_\mu(\tau, \sigma_x) \leq \text{pdist}_\mu(\tau, \iota(\mathbb{H}^n)) + 1$, any wall that separates x from τ also cuts the ball $B(x, \rho)$, where ρ only depends on the hyperbolicity constant and the dimension of \mathbb{H}^n . This implies that (in the Notation 4.1 where $\mathcal{H}(Y)$ is the set of half-spaces cutting Y)

$$\tau = [\sigma_x \cap \mathcal{H}(B(x, \rho))^c] \sqcup [\tau \cap \mathcal{H}(B(x, \rho))],$$

meaning that τ and σ_x , as admissible sections, coincide on the walls that do not intersect $B(x, \rho)$ but may differ on the walls intersecting that ball, see Fig. 3. In particular,

$$\text{dist}_{\mathcal{M}(X)}(\tau, \sigma_x) \leq \mu(\mathcal{H}(B(x, \rho))) := \Delta_\rho.$$

We now have everything in place for the proof of Proposition 5.1.

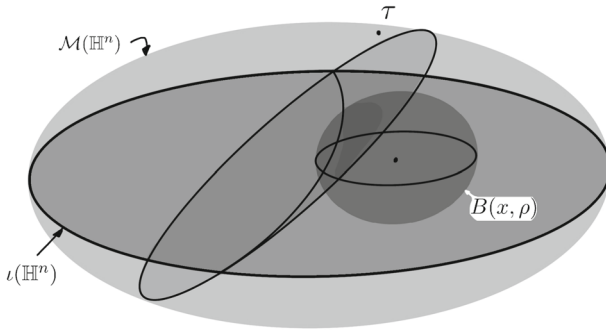


Fig. 3 A wall separating τ from a 1-projection x in $\iota(\mathbb{H}^n)$ also cuts the ball $B(x, \rho)$

Proof of Proposition 5.1 According to Corollary 4.17, the median space $\mathcal{M}(\mathbb{H}^n)$ is at finite Hausdorff distance from \mathbb{H}^n , so it suffices to consider balls centered at some σ_{x_0} , for $x_0 \in \mathbb{H}^n$ fixed. Let τ be an arbitrary admissible section in this ball $B(\sigma_{x_0}, R)$, any 1-projection in $\iota(\mathbb{H}^n) \subseteq \mathcal{M}(\mathbb{H}^n)$ of that given τ is within distance $R + \Delta_\rho$ from x_0 . In particular, for $R' = R + \Delta_\rho + \rho$ we can write (see Remark 5.6 above)

$$\tau = [\sigma_{x_0} \cap \mathcal{H}(B(x_0, R'))^c] \sqcup [\tau \cap \mathcal{H}(B(x_0, R'))],$$

meaning that τ coincides with σ_{x_0} on walls that are far enough from x_0 .

To show that the ball of radius R is totally bounded, it suffices to find, for every $\epsilon > 0$, a finite number of admissible sections τ_1, \dots, τ_m in the closed ball of radius R around σ_{x_0} such that for every admissible section τ at distance at most R from σ_{x_0} , there exists some τ_i such that the symmetric difference

$$\tau \Delta \tau_i = [\tau \cap \mathcal{H}(B(x_0, R'))] \Delta [\tau_i \cap \mathcal{H}(B(x_0, R'))] (= \text{pdist}(\tau, \tau_i))$$

has measure at most $C\epsilon$, where C is a constant depending on the hyperbolicity constant and the dimension of \mathbb{H}^n . To construct those admissible sections, we consider a_1, \dots, a_k points on the sphere $S(x_0, R')$ such that for every $a \in S(x_0, R')$ there exists an a_i such that the angle between $[x_0, a]$ and $[x_0, a_i]$ is at most $\epsilon/2R'$. For each $i \in \{1, 2, \dots, k\}$ we consider $b_1(i), \dots, b_n(i)$ on $[x_0, a_i]$ ordered from x_0 to a_i , dividing it into subgeodesics of equal length ϵ . This process subdivides the ball of radius R' into a grid of mesh ϵ .

On the finite set of walls

$$W_F = \bigcup_{i=1}^k \bigcup_{j=1}^n \{W_{b_j(i)}\}$$

the number of admissible sections is finite. Let τ_1, \dots, τ_m be admissible sections at distance at most R from σ_{x_0} such that all the possible restrictions $\tau|_{W_F}$ of an admissible section at distance at most R from σ_{x_0} are realized by some τ_ℓ . In particular, for any

τ at distance at most R from σ_{x_0} there exists τ_ℓ such that τ restricted to W_F coincides with τ_ℓ .

We are left to show that the distance from τ to τ_ℓ in $\mathcal{M}(X)$ is at most $C\epsilon$. To this end, assume that $\theta_\tau(a_r)$ is inbetween $b_j(i)$ and $b_{j+1}(i)$, possibly equal to $b_j(i)$. Then, on the set $W_{b_{j+1}(i)}$, the section τ coincides with σ_{x_0} and on the set $W_{b_j(i)}$, the section τ coincides with σ_a . Whence the same is true for τ_ℓ and $\theta_{\tau_\ell}(a_r)$ is inbetween $b_j(i)$ and $b_{j+1}(i)$. It follows that the distance between $\theta_\tau(a_r)$ and $\theta_{\tau_\ell}(a_r)$ is at most ϵ .

For an arbitrary $a \in S(x_0, R')$, the above and Lemma 5.5 imply that the distance between $\theta_\tau(a)$ and $\theta_{\tau_\ell}(a)$ is at most $(1 + \kappa)\epsilon$, for the constant κ of Lemma 5.5 above.

Thus, for every $a \in S(x_0, R')$, the sections τ and τ_ℓ coincide on every W_x with $x \in [a, x_0]$, except maybe on a small sub-interval of length at most $(1 + \kappa)\epsilon$. The distance between τ and τ_ℓ , given by the measure of the walls on which they differ, is hence bounded by $C\epsilon$, where C depends on the dimension n . □

Proposition 5.1 combined with Corollary 4.17 imply the following.

Corollary 5.7 *Let X be a finite product of finite dimensional real hyperbolic spaces. Every uniform lattice in $\text{Isom}(X)$ acts on the metric completion $\overline{\mathcal{M}(X)}$ of the median space associated to X properly discontinuously and with compact quotient.*

Remark 5.8 It would be interesting to know if $\mathcal{M}(X)$ is already complete, but this is not immediate in our opinion. It follows from results of [28, 31] that irreducible lattices as in Corollary 5.7 cannot act properly discontinuously cocompactly on a median space of finite rank. This implies that:

- the median space $\mathcal{M}(X)$ has infinite rank;
- the action described in Corollary 5.7 is the best type of action on a median space that can be found for such lattices.

6 The complex hyperbolic space

In the case of complex hyperbolic spaces $\mathbb{H}_{\mathbb{C}}^n$, their hyperbolic metric $\text{dist}_{\mathbb{H}_{\mathbb{C}}}$ cannot be induced by a wall structure (see the proof of Corollary 1.6, at the end of this section). The square root of this metric however is known to come from a wall structure (possibly also larger powers $\text{dist}_{\mathbb{H}_{\mathbb{C}}}^\alpha$ with $\alpha \in (1/2, 1)$ might, but nothing is proven in this respect). Here we explain that no metric $\text{dist}_{\mathbb{H}_{\mathbb{C}}}^\alpha$ can be δ -tripodic, hence the wall space it would be induced by cannot be within bounded Hausdorff distance from a median space. This is explained in Proposition 6.2 below, which follows from simple considerations about snowflaked metrics. For the sake of completeness, we provide the entire (easy computational) argument here.

Lemma 6.1 *Let $a \geq b \geq 0$, let $0 < \alpha < 1$ and $\beta > 1$. The following inequalities hold:*

- (1) $(a + b)^\beta \geq a^\beta + b^\beta$;
- (2) $a^\alpha + b^\alpha - (a + b)^\alpha \geq b^\alpha(2 - 2^\alpha) \geq 0$.

Proof The inequalities are trivial when $b = 0$. Thus we may assume that $b > 0$, in fact without loss of generality we may assume that $b = 1$ and $a \geq 1$.

(1) follows from the fact that the function $f(x) = (x + 1)^\beta - x^\beta - 1$ is increasing for $x \geq 1$ and $f(1) = 2^\beta - 2 > 0$.

(2) follows from the fact that the function $g(x) = x^\alpha + 1 - (x + 1)^\alpha$ is increasing for $x \geq 1$ and $g(1) = 2 - 2^\alpha$. □

Proposition 6.2 (Snowflaked metric spaces) *Let (X, dist) be a metric space and let $0 < \alpha < 1$. This implies that dist^α is a metric, and the following holds.*

- (1) *For every two points x, y in X , the interval with respect to the metric dist^α reduces to $\{x, y\}$.*
- (2) *If (X, dist^α) is δ -tripodic, for some constant $\delta > 0$, then X has bounded diameter.*

Proof That dist^α is a metric follows from Lemma 6.1.

(1) We denote the distance $\mathbf{d} = \text{dist}^\alpha$, and set $\beta = \frac{1}{\alpha}$. We thus have that $\text{dist} = \mathbf{d}^\beta$.

Let z be a point between two points x, y in X , with respect to the metric \mathbf{d} . Thus $\mathbf{d}(x, y) = \mathbf{d}(x, z) + \mathbf{d}(z, y)$, whence

$$\text{dist}(x, y) = [\mathbf{d}(x, z) + \mathbf{d}(z, y)]^\beta \geq \text{dist}(x, z) + \text{dist}(z, y) \geq \text{dist}(x, y). \tag{3}$$

The first inequality in (3) follows from Lemma 6.1(1). Note that equality holds if and only if one of the two numbers is zero. Since the first and the last terms in (3) are equal, all inequalities become equalities, in particular from the above we can deduce that either $\mathbf{d}(x, z) = 0$ or $\mathbf{d}(z, y) = 0$.

(2) The proof is a coarse version of the proof of (1). Take three points $x, y, z \in X$ and consider a δ -median point m with respect to the metric \mathbf{d} . Without loss of generality we can assume that $\text{dist}(m, x) \leq \text{dist}(m, y) \leq \text{dist}(m, z)$. We can write

$$\begin{aligned} \mathbf{d}(x, y) + \delta &\geq \mathbf{d}(x, m) + \mathbf{d}(m, y) \geq (\text{dist}(x, m) + \text{dist}(m, y))^\alpha \geq \text{dist}(x, y)^\alpha \\ &= \mathbf{d}(x, y). \end{aligned} \tag{4}$$

The second inequality is Lemma 6.1 (2). Using (4) and again Lemma 6.1(2), we can write that

$$\delta \geq \text{dist}(x, m)^\alpha + \text{dist}(m, y)^\alpha - [\text{dist}(x, m) + \text{dist}(m, y)]^\alpha \geq (2 - 2^\alpha)\text{dist}(m, x). \tag{5}$$

If we repeat the arguments in (4) and (5), with x, y replaced by y, z , we obtain that

$$(2 - 2^\alpha)\text{dist}(m, y) \leq \delta,$$

whence

$$\text{dist}(x, y) \leq \frac{2\delta}{2 - 2^\alpha}.$$

This shows that only close enough triples of points admit a δ -median point, and hence (X, \mathbf{d}) is δ -tripodic only if it has bounded diameter, that is only when (X, dist) has bounded diameter. \square

Corollary 6.3 *Let (X, dist) be an unbounded geodesic Gromov hyperbolic metric space, and let $0 < \alpha < 1$. Then (X, dist^α) cannot be δ -tripodic for any $\delta > 0$.*

In particular if the snowflaked metric dist^α is induced by a measured walls structure, then no isometric embedding of (X, dist^α) into a median metric space \mathcal{M} has image within bounded distance of \mathcal{M} .

Consider now the hyperbolic space $\mathbb{H}_{\mathbb{C}}^n$ with $n \geq 2$, endowed with the hyperbolic distance dist . Recall that $\mathbb{H}_{\mathbb{C}}^n$ admits a structure of space with measured walls such that the induced distance is $\text{dist}^{1/2}$. Possibly the exponent $1/2$ can be increased to some other value $\alpha \in (1/2, 1)$. In either case, the space with measured walls thus obtained cannot be within finite distance of a median space due to Proposition 6.2. Thus we can finish the proof of Corollary 1.6 stated in the introduction.

Proof of Corollary 1.6 (1) An isometric embedding of any metric space Y in a space with walls X gives a wall structure on Y (by restricting the walls of X to the isometrically embedded subspace Y). But a wall on a manifold gives a codimension one totally geodesic submanifold as the intersection of the topological closure of each side. However, according to [36, §3.1.11], every totally geodesic submanifold of $\mathbb{H}_{\mathbb{C}}^n$ is either totally real or complex-linear, and in particular has real-codimension at least 2 when $n \geq 2$. See also Remark 2.11 of [54].

(2) is an immediate consequence of Corollary 6.3. \square

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