Geometric and topological rigidity of manifolds with non-negative Ricci curvature



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First of all, I would like to thank my supervisor, Andrea Mondino, for his guidance throughout the PhD. His expertise across many branches of geometry and analysis allowed me to work on a variety of different topics, and to greatly improve as a mathematician.

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Abstract

This thesis focuses on new geometric and topological rigidity results for manifolds with non-negative Ricci curvature. We obtain alternative proofs and generalizations of the results from [139], where non-compact 3-manifolds with non-negative Ricci curvature are classified. These are obtained by using methods and tools from metric geometry and optimal transport. In particular, we rely on the theory of Ricci limit spaces and of metric measure spaces with Ricci curvature bounded from below in a synthetic sense, i.e. RCD spaces.

Chapter 1 contains a general overview of Ricci curvature and of the methods used in the thesis, as well as a thorough discussion of the original contributions of the author.

Chapter 2 contains preliminaries.

Chapter 3 is based on joint work with Andrea Mondino [74]. Here, we show that a 3-manifold with non-negative Ricci curvature and disconnected mean-convex boundary is necessarily a product, where one of the factors is a real interval. This result is then used to give an alternative proof of the rigidity of 3-manifolds with non-negative Ricci curvature and non-zero first Betti number. The original proof of this first Betti number rigidity result is due to [139].

Chapter 4 is based on joint work with Mattia Magnabosco and Daniele Semola [73]. Here, we show that an n-manifold with non-negative Ricci curvature and first Betti number equal to $n-2$ has universal covering that splits $\mathbb{R}^{n-2}$ isometrically. This generalizes the aforementioned rigidity of 3-manifolds with non-negative Ricci curvature and non-zero first Betti number to higher dimensions.

Chapter 5 is also based on joint work with Mattia Magnabosco and Daniele Semola [73]. Here, we show that manifolds with non-negative Ricci curvature have vanishing simplicial volume. This provides a higher dimensional

analogue of the fact that a contractible 3-manifold with non-negative Ricci curvature is homeomorphic to $\mathbb{R}^3$. The original proof of this result in the 3-dimensional case is due to [139].

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Chapter 1

Introduction

We obtain new topological and geometric rigidity results for manifolds with non-negative Ricci curvature. We do so by employing techniques from the theory of Ricci limit spaces, and RCD spaces. The main novel contributions of the thesis are taken from the papers [74], written in collaboration with my supervisor Andrea Mondino, and [73], written in collaboration with Mattia Magnabosco and Daniele Semola.

The main references that were used for this Introduction are [175, 2, 186].

1.1 Ricci curvature

Curvature is a way to quantify how much a geometric object deviates from being 'Euclidean'. On a Riemannian manifold, curvature is described by a four-tensor (the Riemann curvature tensor) which measures the failure of the second covariant derivatives to commute. This complex mathematical object has been the subject of intense study, but many of its properties remain not fully understood.

A related central notion of curvature is that of *sectional curvature*. Given a Riemannian manifold (M, g) and a point $p \in M$, sectional curvature is a function K defined on the Grassmannian of 2-planes in the tangent space T_pM. Given such a plane Π, the sectional curvature $K(\Pi)$ measures the Gaussian curvature in p of the surface given by the image of Π under the exponential map. Notably, given two orthonormal vectors $u, v \in T_pM$ spanning $\Pi \subset T_pM$, the sectional curvature $K(\Pi)$ appears in the Taylor expansion of the distance between the two corresponding geodesics starting at p:

$$\mathsf{d}(\exp_p(tu), \exp_p(tv)) = \sqrt{2}t\Big(1 - \frac{K(\Pi)}{12}t^2 + O(t^3)\Big) \quad \text{as } t \to 0. \tag{1.1}$$

Equation (1.1) reflects the general idea that sectional curvature encodes how much *distances* deviate from being 'Euclidean'. In particular, it follows from (1.1) that,

if a manifold M has non-negative sectional curvature, then distances between short geodesics starting from p are smaller than distances between the corresponding geodesics in the tangent space T_pM. An opposite relationship holds if one assumes non-positive sectional curvature instead. These facts are encoded in the celebrated Toponogov Theorem, which shows that lower and upper sectional curvature bounds on a Riemannian manifold can be characterized solely by comparing the behavior of the Riemannian distance with the Euclidean one. This characterization of lower and upper sectional curvature bounds by means of distance comparison (and without taking derivatives) allowed to define sectional curvature upper and lower bounds on metric spaces, giving rise to the theory of Alexandrov spaces (see [43, 44]).

Another important notion of curvature is that of *Ricci curvature*. Ricci curvature is described by a 2-tensor which is obtained as a contraction of the full Riemann curvature tensor. Given a manifold (M, g) and an orthonormal basis $\{e_i\}_{i=1}^n \subset T_pM$, the evaluation of the Ricci curvature tensor $\mathsf{Ric}(e_i, e_i)$ can be expressed as a sum of sectional curvatures:

$$\mathsf{Ric}(e_i, e_i) = \sum_{j \neq i} K(\Pi_{i,j}), \tag{1.2}$$

where $\Pi_{i,j} \subset T_pM$ is the plane spanned by $\{e_i, e_j\}$. Even though it is not immediate from equation (1.2), Ricci curvature is associated to infinitesimal control over volumes. Let us give an instance of this phenomenon (taken from [186, Chapter 14]). Let $\varphi : M \to \mathbb{R}$ be a smooth function and, for every $t > 0$, consider the map $T_t : M \to M$ given by $T_t(x) := \exp_x(t\nabla\varphi(x))$. Consider the differential $DT_t : TM \to TM$ and the infinitesimal volume distortion of the map T_t given by $\mathcal{J}(t) : M \to \mathbb{R}$, where $\mathcal{J}(t)(x) := \det(D_xT_t)$. Finally, let $X \in TM$ be the vector field given by $X := \frac{\mathrm{d}}{\mathrm{d}t}T_t(x)$ and let n be the dimension of M. After dealing with the regularity issues that arise due to the possible presence of a cut locus, one obtains that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{J}(t)^{1/n} + \frac{\mathsf{Ric}(X, X)}{n}\mathcal{J}(t)^{1/n} \leq 0. \tag{1.3}$$

The previous equation shows the existence of a link between Ricci curvature and volume distortion. Moreover, we observe an interesting phenomenon: the dimensional parameter n appears explicitly in the formula (1.3). This is a feature of many other formulas that link Ricci curvature lower bounds to geometric control of the underlying manifold (as for instance the bounds on the Laplacians of distance functions [45], and Bishop-Gromov inequality and Lévy–Gromov isoperimetric inequality [108], among others).

Ricci curvature plays an important role in different branches of geometry. Notably, in Einstein's theory of general relativity (see, for instance, [157]), the interaction between matter and curvature of a Lorentzian spacetime (M, g) is described by the equation

$$\mathsf{Ric} - \frac{1}{2}\mathsf{R}g + \Lambda g = kT,$$

where R denotes the trace of the Ricci tensor (i.e. scalar curvature), T denotes the stress-energy tensor (which is associated to the presence of matter), and k, Λ are constants.

Another branch of geometry in which Ricci curvature plays a central role is the theory of geometric flows. In particular, the Ricci flow (see, for instance, [185]), defined by the evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}g_t = -2\mathsf{Ric}_{g_t},$$

has led to a number of profound applications, including Perelman's proof of the Poincaré Conjecture.

In this thesis, we focus on manifolds with non-negative Ricci curvature. As we shall see, manifolds satisfying this curvature assumption have a lot of restrictions on their geometric and topological properties. In sharp contrast, any manifold of dimension at least 3 admits a complete metric with Ricci curvature ≤ -1 (see [142]).

Among the many classical results restricting the geometry of manifolds with non-negative Ricci curvature we find: Kapovitch-Wilking's Theorem [126], stating that the fundamental group is virtually nilpotent; Bishop-Gromov's inequality ([108]), controlling the volume growth of balls; the Splitting Theorem by Cheeger-Gromoll [58], providing a sufficient condition for these manifolds to be products; and the Lévy–Gromov isoperimetric inequality ([108]), which gives estimates on the isoperimetric ratio.

At the same time, manifolds with non-negative Ricci curvature still enjoy a lot of topological and geometric flexibility. For example: they can have infinite Betti numbers by Sha-Yang's work [176] (see also [19]), and they can have fundamental group which is not finitely generated by Brué-Naber-Semola's conterexample to the Milnor conjecture [38]. Both these phonomena are forbidden by non-negative sectional curvature, thanks to work of Gromov (see [105] and [165, Theorem 12.5.1]).

To conclude the section, we mention two celebrated techniques that have been used to study manifolds with Ricci curvature bounded from below. These are also the techniques used in this thesis.

A fruitful approach was developed by Schoen-Yau in the Eighties, through the use of minimal hypersurfaces. Roughly speaking, they showed that stable minimal hypersurfaces in manifolds with non-negative Ricci curvature had to be very rigid ([174]). Then, linking topological assumptions on the ambient manifold to existence results for such minimal hypersurfaces, they obtained restrictions on the topology of the base manifold. This approach later led to the complete topological classification of 3-manifolds with non-negative Ricci curvature in [139].

Another approach is based on work of Gromov in the Eighties ([108]), and Cheeger-Colding in the late Nineties ([53, 54, 55, 56]). In particular, Gromov observed that the class of manifolds with Ricci curvature bounded below by K, and dimension and diameter bounded above by n and D respectively, is precompact in the Gromov Hausdorff topology. A similar result holds removing the diameter bound and considering the *pointed* Gromov Hausdorff topology. The metric (measure) spaces arising as limits of such manifolds, i.e. the so-called Ricci limit spaces, were then systematically studied by Cheeger-Colding. This approach has had many applications. At the same time, several conjectures on the geometry of Ricci limit spaces remain open (see [155]).

1.2 Ricci limit spaces and RCD spaces

As mentioned in the previous section, a powerful precompactness result due to Gromov implies that the class $\mathcal{M}_{K,n}$ of manifolds with Ricci curvature bounded from below by K, and dimension bounded from above by n, is precompact in the pointed Gromov Hausdorff topology. Metric spaces arising as limits of such manifolds are called Ricci limit spaces. Due to Fukaya ([91]), it was soon realized that, in order not to lose geometric information when passing from a sequence of Riemannian manifolds to a Ricci limit space, the latter object had to be equipped with a limiting measure, which arises from the volume measures of the approximating manifolds. Moreover, Gromov (motivated in part by the work of Fukaya) observed in [109, p. 85] that, since Ricci curvature manifests itself with control over volumes (rather than dstances), a theory of $\mathsf{Ric} \geq K$ on metric spaces should include a measure as a given object alongside the distance. For these reasons, it became clear that the right setting to study Ricci limit spaces is that of *metric measure spaces*, i.e. metric spaces equipped with a reference measure. We also mention that the theory of Bakry-Émery [29], which preceded the theory of Ricci limit spaces, offered a kind of non-smooth Ricci curvature

lower bound (together with a dimensional upper bound) in the setting of measure spaces equipped with a Dirichlet form.

Since its birth, the theory of Ricci limit spaces has had several applications. Three ways to deduce properties of manifolds in $\mathcal{M}_{K,n}$ through the study of Ricci limit spaces are the following:

- To prove that manifolds in $\mathcal{M}_{K,n}$ cannot exhibit arbitrarily bad behavior, one argues by contradiction. This gives rise to a sequence of manifolds with increasingly bad behavior. By passing to a subsequence, these manifolds converge to a Ricci limit space which will inherit the increasingly pathological behavior of the approximating manifolds. If this was ruled out a priori, one obtains a contradiction.

- To prove that a qualitative property fails on $\mathcal{M}_{K,n}$, one can show that it is stable under pointed Gromov Hausdorff convergence, and that it fails for a Ricci limit space.

- To prove properties of non-compact manifolds in $\mathcal{M}_{0,n}$, it is sometimes useful to look at their 'tangent cones at infinity'. These are Ricci limit spaces which encode some of the geometric features of the initial manifold at large scales.

The first strategy was used, for instance, in [60, 125], to show that manifolds with bounded Ricci curvature, and a lower volume bound, enjoy a priori estimates on the L^2-norm of the full Riemann curvature tensor. An example of the second strategy is provided in [81], where it is shown that, on Riemannian manifolds, one cannot hope to obtain Calderon-Zygmund type estimates for harmonic functions depending only on a lower bound of the Ricci curvature and an upper bound on the dimension. An example where tangent cones at infinity are used to study the geometry of manifolds with non-negative Ricci curvature is [159], which gives an alternative proof of the fact that the fundamental group of 3-manifolds with non-negative Ricci curvature is finitely generated (which also followed from the earlier [139]).

A natural question that arose in the context of Ricci limit spaces was whether it were possible to describe Ricci curvature lower bounds synthetically, i.e. without relying on the differential structure of the underlying manifold. This question was motivated by the synthetic notions of lower sectional curvature bounds that had been previously studied in the context of Alexandrov spaces (see, for instance, [44]).

The key technical tools to obtain the aforementioned synthetic notions of Ricci curvature bounds are provided by the theory of optimal transport. We recall here

some basic definitions. Given a metric measure space $(X, \mathsf{d}, \mathfrak{m})$, one denotes by $\mathcal{P}_2(X)$ the set of probability measures with finite second moment, i.e. those probability measures μ such that $\int_X \mathsf{d}^2(x,y)\, d\mu(y) < +\infty$ for some (and thus for all) $x \in X$. The squared Wasserstein distance between two measures $\mu, \nu \in \mathcal{P}_2(X)$ is defined as

$$\inf \left\{ \int_{X\times X} \mathsf{d}(x,y)^2 \, \mathrm{d}\pi(x,y) \right\},$$

where the infimum is taken among probability measures π on $X \times X$ with $(p_1)_{\#}\pi = \mu$, and $(p_2)_{\#}\pi = \nu$, where $p_1, p_2 : X \times X \to X$ are the projections on the first and second factor respectively. The Shannon Entropy $\mathrm{Ent}_{\mathfrak{m}} : \mathcal{P}_2(X) \to (-\infty, +\infty]$ is defined as

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) = \int_X \rho \log \rho \, \mathrm{d}\mathfrak{m},$$

whenever $\mu \ll \mathfrak{m}$ has density ρ and $\int_X \rho \log \rho \, \mathrm{d}\mathfrak{m}$ is well defined, and it is set to be $+\infty$ otherwise.

It was first conjectured in [158] that a lower Ricci curvature bound on a Riemannian manifold (equipped with its volume measure) (M, g, Vol) should be equivalent to a convexity property of the Shannon entropy on the space $(\mathcal{P}_2(M), W_2)$. This fact was then proved in [66, 169]. In 2004, Sturm and Lott-Villani ([182, 183, 144]) independently defined the Ricci curvature lower bound $\mathsf{Ric} \geq K$ on a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ by requiring the K-convexity of the Shannon entropy on the space $(\mathcal{P}_2(X), W_2)$. This gave birth to the theory of $\mathsf{CD}(K, \infty)$ spaces (where CD stands for curvature dimension), i.e. metric measure spaces with Ricci curvature bounded below by K in a synthetic sense and no upper dimensional bound. Spaces having both a synthetic notion of lower Ricci curvature bound and an upper dimensional bound, i.e. $\mathsf{CD}(K, N)$ spaces, were then introduced by considering different entropies. The theory of CD spaces then grew rapidly, and it was soon realized that this class was far larger than that of Ricci limit spaces. In particular, the Finsler structures $(\mathbb{R}^n, \|\cdot\|, \lambda^n)$, where $\|\cdot\|$ is any norm, satisfy the $\mathsf{CD}(0, n)$ condition, but are not Ricci limit spaces. To rule out such structures, Ambrosio-Gigli-Savaré ([11]) introduced the class of $\mathsf{RCD}(K, \infty)$ spaces (i.e. spaces with a 'Riemannian' curvature dimension condition), by requiring the additional property that the Sobolev space $\mathsf{W}^{1,2}(X)$ is a Hilbert space. The corresponding class of $\mathsf{RCD}(K, N)$ spaces was then introduced by Gigli ([95]), combining the $\mathsf{CD}(K, N)$ condition with the requirement that the Sobolev space $\mathsf{W}^{1,2}(X)$ is a Hilbert space. A notable property of the $\mathsf{RCD}(K, N)$ condition is that, unlike its smooth counterpart, it is stable with respect to pointed measured Gromov Hausdorff convergence ([99]) and metric quotients ([94]). This fact is at

the heart of many applications of the $\mathsf{RCD}(K,N)$ theory in Riemannian geometry, including the ones presented in the thesis.

To this day, the problem of characterizing Ricci limit spaces within the RCD class remains open. A gap between the two classes is known to exist, if one restricts to non-collapsed $\mathsf{RCD}(K,N)$ spaces (i.e. spaces where $\mathfrak{m} = \mathsf{H}^N$, introduced in [80]): the cone $C(\mathbb{RP}^2)$ is a non-collapsed $\mathsf{RCD}(0,3)$ space due to [130], but it cannot be approximated with manifolds in $\mathcal{M}_{0,3}$ due to [178, 179].

Nevertheless, the interest in the RCD theory has gone beyond its initial tie to Ricci limit spaces, as shown by the many families of spaces that were shown to fit into the RCD framework, including: finite dimensional Alexandrov spaces ([166]); stratified spaces ([32]); singular Kähler–Einstein spaces ([184]); and conical spaces ([130]). Moreover, the connection between Ricci curvature lower bounds and optimal transport has been investigated also in non-smooth spaces of Lorentzian signature ([48]).

1.3 Contributions

There is a strong link between the novel contributions presented in the thesis and the celebrated classification result of non-compact 3-manifolds with non-negative Ricci curvature by Gang Liu [139] (after Schoen-Yau [174]). Let us first recall Liu's result, and briefly comment on his proof.

Theorem 1.3.1 (Liu '11)**.** *Let (M^3, g) be a non-compact manifold with non-negative Ricci curvature. Then, either the universal covering splits a line, or M is homeomorphic to $\mathbb{R}^3$.*

The previous theorem provides a full topological classification of the non-compact manifolds admitting a complete metric of non-negative Ricci curvature. Indeed, if the universal covering of M splits a line, then M has non-negative sectional curvature, so that all the possible topological types are determined (by [57]). We also recall that the classification problem for compact 3-manifolds with non-negative Ricci curvature had been previously solved by Hamilton in [117] using Ricci flow. Among the novel results of the thesis, we obtain a generalization of this theorem to higher dimensions, and we provide alternative proofs in dimension 3.

We now briefly comment on the proof of Liu's Theorem. Using observations from [174], one can reduce the proof of Theorem 1.3.1 to the following two theorems.

Theorem 1.3.2 (Liu '11). *If (M^3, g) has non-negative Ricci curvature and $\pi_1(M) \cong \mathbb{Z}$, then the universal covering splits a line*

Theorem 1.3.3 (Liu '11). *If (M^3, g) has non-negative Ricci curvature and it is contractible, then it is homeomorphic to $\mathbb{R}^3$.*

Surprisingly, even though Theorems 1.3.2 and 1.3.3 seem to be quite different, they have very similar proofs. Let us first comment on some aspects of the proof of Theorem 1.3.2. The starting point is an idea from [174]: The assumption on the fundamental group implies the existence of a properly embedded two-sided stable minimal hypersurface $\Sigma \subset M$. In this way, one translates a topological assumption (i.e. $\pi_1(M) = \mathbb{Z}$) into a geometric one (i.e. existence of the aforementioned surface, whose minimality depends on the underlying Riemannian metric). Moreover, again by [174], $\Sigma \subset M$ is totally geodesic and the Ricci curvature of M vanishes in the normal direction to Σ, so that, imprecisely, one obtains an infinitesimal splitting. To obtain the full splitting of the universal covering, Liu then refines these concepts, and shows that these techniques can be used to construct not only one minimal hypersurface, but an entire foliation. The principle underlying Theorem 1.3.3 is very similar: If the statement were not true, then M would contain a properly immersed (and not necessarily embedded) two-sided stable minimal hypersurface $\Sigma \subset M$. Again by [174], Σ is totally geodesic, so that one obtains an infinitesimal splitting, and then a similar foliation argument as in Theorem 1.3.2 allows to conclude. We remark that in Theorem 1.3.2, the minimal surface Σ is properly embedded as it is constructed from a minimization problem in a chosen homology class. On the other hand, in Theorem 1.3.3, $\Sigma \subset M$ is only properly immersed, as it is constructed from solving a Plateau problem.

We now examine the original contributions of the thesis chapter by chapter. Chapter 3 is taken from [74], written in collaboration with Andrea Mondino. Chapters 4 and 5 are taken from [73], written in collaboration with Mattia Magnabosco and Daniele Semola.

1.3.1 Chapter 3

The main result of Chapter 3 is Theorem 1.3.4 below, which is taken from [74], written in collaboration with Andrea Mondino. For the precise definition of mean-convexity, we refer to Chapter 3.

Theorem 1.3.4. *Let (M^3, g) be a manifold with non-negative Ricci curvature and disconnected mean-convex boundary. Then, $M \cong \Sigma \times [0, l]$ isometrically, for some $l > 0$, and some surface Σ.*

As we outlined in the previous section, a key result to understand the topology of 3-manifolds with non-negative Ricci curvature is the fact that stable minimal hypersurfaces in such ambient manifolds are totally geodesic ([174]). Theorem 1.3.4 provides a related rigidity result, showing that if a 3-manifold with non-negative Ricci curvature is bounded by multiple minimal surfaces (or, more generally, it has mean-convex boundary), then not only such surfaces are totally geodesic and isometric, but the manifold is a product. We stress that no stability is assumed in Theorem 1.3.4, and that the dimension constraint is sharp, as there exists a flat 4-manifold with disconnected minimal boundary which is not a product (i.e. the portion of $\mathbb{R}^4$ between a hyperplane and a catenoid disjoint from it). We also remark that, according to the assumptions of Theorem 1.3.4, the boundary of M could have more than 2 connected components, and that part of the conclusion is that there are exactly two such components.

Theorem 1.3.4 is part of a series of classical and modern results dealing with the following question (posed, for example, in [170]): If two minimal hypersurfaces $\Sigma_1, \Sigma_2 \subset M^n$ do not intersect, under what conditions on the ambient space M do they determine each other's geometry? Among the classical results on this problem in Euclidean space, we find Hoffman-Meeks' Half Space Theorem [120], stating that a connected, proper, non-planar minimal surface in $\mathbb{R}^3$ cannot be contained in a half space; and Bombieri-De Giorgi-Miranda's proof that the only entire positive solutions of the minimal surface equation in Euclidean space are the constant functions [34]. Other related results in the Riemannian setting are given, for instance, in [170, 65, 86]. These results all assume that the ambient space is a product with $\mathbb{R}$, and they show that minimal hypersurfaces that are disjoint from a chosen slice and satisfy some additional assumptions are themselves horizontal slices. Notable results where the existence of disjoint minimal hypersurfaces forces both the minimal hypersurfaces to be isometric and the ambient space to be a product are [128] (see also [67, 131]) and [20]. These results are more closely related to Theorem 1.3.4. In [128], Kasue shows that a manifold of arbitrary dimension with non-negative Ricci curvature and mean-convex disconnected boundary is a product, if at least one boundary component is compact. As noted above, the theorem fails in dimension ≥ 4 if one removes the compactness assumption. In [20], a variant of Theorem 1.3.4 is proved by Anderson-Rodriguez assuming an additional uniform sectional curvature bound, and relying on

foliation techniques by Schoen-Yau [174]. Our proof of Theorem 1.3.4 is completely different from Anderson-Rodriguez's one, and it is closer in spirit to Kasue's result. We also mention that our proof relies on a gluing procedure in which a non-smooth metric space arises, and we refer to the next section for more details. Finally, Theorem 1.3.4 allows to obtain an alternative proof of Theorem 1.3.2 (see Section 3.3).

1.3.1.1 Strategy of the proof

We first explicitly state the result by Kasue [128] that was described in the previous section.

Theorem 1.3.5 (Kasue)**.** *Let (M^n, g) be a manifold with non-negative Ricci curvature and disconnected mean-convex boundary. If one boundary component is compact, then $M \cong \Sigma \times [0, l]$ isometrically, for some $l > 0$ and some compact $(n-1)$-manifold Σ^{n-1}.*

Theorem 1.3.4 can be reduced to a variant of Kasue's result, i.e. Theorem 1.3.6, where a boundary component is not required to be compact, but it is instead required to be totally geodesic, parabolic, and with non-negative Ricci curvature. We recall that a manifold is said to be parabolic if it admits no positive Green's function for the Laplacian, and we refer to [104] for more background on this notion.

Theorem 1.3.6. *Let (M^n, g) be a Riemannian manifold with non-negative Ricci curvature and disconnected mean-convex boundary. If one boundary component is totally geodesic, parabolic, and has non-negative Ricci curvature, then $M \cong \Sigma \times [0, l]$ isometrically for some $l > 0$ and some $(n-1)$-manifold Σ.*

Theorem 1.3.4 can be reduced to Theorem 1.3.6 as follows: By using techniques from [147], under the assumptions of Theorem 1.3.4, one can construct a stable properly embedded two-sided minimal surface $N^2 \subset M^3$, which either coincides with a boundary component of M, or it is disjoint from ∂M. This is obtained by a geometric variant of Perron's method, where one solves a suitable area-minimization problem with the mean-convex components acting as barriers. The stable minimal surface $N \subset M$ is then totally geodesic by [174] and, by Gauss-Codazzi equations, it has non-negative sectional curvature. In particular, it is also parabolic. Hence, by replacing M with each of its regions having N as a boundary component, we reduce to a subcase of Theorem 1.3.6.

To prove Theorem 1.3.6, we argue as follows. Let $S \subset \partial M$ be the boundary component which is totally geodesic, parabolic, and has non-negative Ricci curvature. Consider the metric space (X, d) obtained by gluing M and $S \times [0, +\infty)$ along their

isometric boundaries. Let us emphasize that (X, d) is a non-smooth space: the glued Riemannian metric is only of class $C^{0,1}$ along S. This lack of smoothness motivates the use of synthetic tools developed for non-smooth spaces, particularly those based on optimal transport. Consider the distance function $\mathsf{d}_{S'}$ from $S' := \partial M \setminus S$ in the glued space and its restriction to $S \times [0, +\infty)$. We denote by d_S the distance from S in $S \times [0, +\infty)$. If we can show that $\mathsf{d}_{S'}$ is constant on $S \times \{0\}$, then the statement follows by arguments analogous to Kasue's ones.

Thanks to the assumption that S is totally geodesic and has non-negative Ricci curvature (and previous results on the Laplacian of distance functions in metric measure spaces with synthetic Ricci curvature lower bounds, see [95, 49, 131, 152, 100]), it holds $\Delta \mathsf{d}_{S'} \leq 0$ on $S \times (0, +\infty)$ in distributional sense. In particular, it holds $\Delta(\mathsf{d}_{S'} - \mathsf{d}_S) \leq 0$ and, calling ν the exterior normal of $S \times \{0\}$ in $S \times [0, +\infty)$, it also (formally) holds $\nabla(\mathsf{d}_{S'} - \mathsf{d}_S) \cdot \nu \geq 0$ on $S \times \{0\}$.

If $S \times [0, +\infty)$ has sufficiently small volume growth at infinity, these facts allow to deduce that $\mathsf{d}_{S'} - \mathsf{d}_S$ is constant by an integration by parts argument, cf. [124]. Although S is assumed to be parabolic (which for manifolds with non-negative Ricci curvature amounts to a control on the volume growth at infinity), in general $S \times [0, +\infty)$ does not satisfy the required volume growth bounds to apply the aforementioned integration by parts strategy directly. In this case, we multiply the volume measure of $S \times [0, +\infty)$ by a suitable weight, cf. [75]. Making an appropriate choice of such a weight, we obtain that $\mathsf{d}_{S'} - \mathsf{d}_S$ satisfies the previous Laplacian bound also in the weighted space, which, in addition, has the desired volume growth. We then carry out a weighted integration by parts argument to obtain that $\mathsf{d}_{S'} - \mathsf{d}_S$ is constant, which forces $\mathsf{d}_{S'}$ to be constant on $S \times \{0\}$.

1.3.2 Chapter 4

The main result of Chapter 4 is Theorem 1.3.7 below, which is taken from [73], written in collaboration with Mattia Magnabosco and Daniele Semola. This introductory part, as well as the introduction of the next chapter, are also largely taken from [73].

Theorem 1.3.7. *Let (M, g) be a complete Riemannian n-manifold with $\mathsf{Ric} \geq 0$ for $n \geq 3$. If $\pi_1(M)$ contains a subgroup isomorphic to $\mathbb{Z}^{n-2}$, then the universal cover $(\overline{M}, \overline{g})$ of (M, g) splits a factor $\mathbb{R}^{n-2}$ isometrically.*

Theorem 1.3.7 is a higher dimensional version of Theorem 1.3.2. A variant of this result had been obtained by Anderson in [17], under the additional assumption that the sectional curvature is uniformly bounded. The validity of Theorem 1.3.7 in all

dimensions without further assumptions had remained an open question since then. Very recently, it was stated explicitly as a conjecture by Ye in [191, Conjecture 1] (see also the more recent [162, Question 1.6]).

We also mention that the assumptions are sharp. Indeed, in [156], Nabonnand constructed a complete (M^4, g) with $\mathsf{Ric} > 0$ and $\pi_1(M^4) \simeq \mathbb{Z}$. Bergery generalized such construction later in [31] by building examples of (M^n, g) with $\mathsf{Ric} > 0$ and $\pi_1(M^n) \simeq \mathbb{Z}^{n-3}$. These examples show that Theorem 1.3.7 is sharp, in the sense that its conclusion might fail if we just assume that $\pi_1(M^n)$ has a subgroup isomorphic to $\mathbb{Z}^{n-3}$. Finally, we mention that an immediate consequence of Theorem 1.3.7 is that an n-manifold with non-negative Ricci curvature and $\pi_1 \cong \mathbb{Z}^{n-2}$ has non-negative sectional curvature. This allows to obtain new topological restrictions for manifolds with non-negative Ricci curvature: n-manifolds with $\pi_1 \cong \mathbb{Z}^{n-2}$ and infinite topological type do not admit complete Riemannian metrics with non-negative Ricci curvature, as this would violate the soul Theorem [57]. On the other hand, as already mentioned, manifolds with non-negative Ricci curvature can have infinite topological type by [176].

In the next section, we discuss the main steps to prove Theorem 1.3.7. The tools that we use come from the theory of Ricci limits and RCD spaces.

1.3.2.1 Strategy of the proof

The proof of Theorem 1.3.7 hinges on two main steps.

The first step is a blow-down argument. It is not difficult to check that every equivariant Gromov Hausdorff tangent cone at infinity (see Sections 2.1.3 and 2.2.2) of $(\overline{M}, \overline{g}, \overline{p}, \mathbb{Z}^{n-2})$ is an $\mathsf{RCD}(0, n)$ space whose isomorphisms group contains a closed $\mathbb{R}^{n-2}$ subgroup. Here $(\overline{M}, \overline{g}, \overline{p}, \mathbb{Z}^{n-2})$ is the universal cover of (M, g) endowed with the action by deck transformations of $\mathbb{Z}^{n-2} < \pi_1(M)$ and $\overline{p} \in \overline{M}$ is any reference point. See Lemma 2.2.23 below for the precise statement, which is essentially [162, Lemma 3.1]. Our first key ingredient is the following splitting result:

Proposition 1.3.8. *Let $(\widetilde{X}, \widetilde{\mathsf{d}}, \widetilde{\mathfrak{m}}, \widetilde{q}, G)$ denote any equivariant blow-down of $(\overline{M}, \overline{g}, \overline{p}, \mathbb{Z}^{n-2})$. Then $(\widetilde{X}, \widetilde{\mathsf{d}}, \widetilde{\mathfrak{m}}, \widetilde{q}, G)$ splits a factor $\mathbb{R}^{n-2}$ isometrically and $\mathbb{R}^{n-2} < G$ acts by translations on this Euclidean factor.*

The proof of Proposition 1.3.8 relies on some recently developed RCD technology. The most delicate case is when the blow-down is homeomorphic to $\mathbb{R}^{n-2} \times \mathbb{R}_+$ and has rectifiable dimension $n-1$. We are going to argue that $(\widetilde{X}, \widetilde{\mathsf{d}}, \widetilde{\mathfrak{m}})$ is a measured warped product, i.e., ignoring the possible nonsmoothness, the metric has the form

$\mathrm{d}r^2 + f^2(r) g_{\mathbb{R}^{n-2}}$ and $\overline{\mathfrak{m}} = h(r)\,\mathrm{d}r\,\mathrm{d}\lambda^{n-2}$, for some functions $f, h : \mathbb{R}_+ \to \mathbb{R}_+$. The $\mathsf{RCD}(0,n)$ condition in this setting is equivalent to a lower bound on the weighted Ricci tensor interpreted in the sense of distributions, thanks to [151]. Such distributional lower Ricci bound yields some restrictions on f and h which, eventually, imply that f must be constant. Hence $\widetilde{X}$ splits a factor $\mathbb{R}^{n-2}$.

The second main step in the proof of Theorem 1.3.7 is to pass from the splitting of every tangent cone at infinity of $\overline{M}$, to the splitting of $\overline{M}$ itself.

Remark 1.3.9. In general, the splitting of all blow-downs of a complete manifold with $\mathsf{Ric} \geq 0$ does not imply the splitting of the original manifold. Kasue and Washio's construction from [129, p. 913-914] was the first counterexample. Later Anderson observed that the n-dimensional Schwarzschild metrics on $\mathbb{R}^2 \times S^{n-2}$ for $n \geq 4$ are Ricci-flat counterexamples, cf. with [15, p. 71].

That we can obtain the splitting of the universal cover starting from the splitting of the blow-downs is very much special to our situation. Thanks to the Cheeger-Colding theory, see in particular [53, 56], the following holds. If a blow-down of a complete (M^n, g) with $\mathsf{Ric} \geq 0$ splits a factor $\mathbb{R}^k$ isometrically for some $0 \leq k \leq n-1$, then there exist two sequences $\varepsilon_i \to 0$, $R_i \to \infty$ and points $p_i \in M$ such that

$$\mathsf{d}_{\mathrm{GH}}\left(B_r(p_i), B_r^{\mathbb{R}^k \times Y_{r,i}}((0, q_{r,i}))\right) < \varepsilon_i r\,, \text{ for every } i \in \mathbb{N} \text{ and } \ 0 < r < R_i\,, \tag{1.4}$$

where $Y_{r,i}$ denotes some complete metric space, $q_{r,i} \in Y_{r,i}$, and d_{GH} denotes the Gromov Hausdorff distance. We will argue that the points p_i can be chosen to remain at a uniformly bounded distance from some fixed point $p \in M$ as $i \to \infty$ and this is clearly enough to complete the proof of Theorem 1.3.7. Ruling out the possibility that the points p_i escape to infinity requires two new ideas. The first one is the use of Cheeger and Naber's slicing theorem, see [60, Theorem 1.23]. This tool crucially refines the general picture that we sketched above under the assumption that $k = n-2$. At an heuristic level, it guarantees that if the points p_i escape to infinity, they must do so along the directions of the (approximate) $(n-2)$-splitting. To deal with this scenario, the second key tool is a careful study of the behavior of $(n-2)$ linearly independent harmonic functions with almost linear growth under compositions with the isometric action by $\mathbb{Z}^{n-2}$. The existence of such functions follows from Proposition 1.3.8 together with [122, Theorem 1.10], obtained by Huang and Huang by developing the methods previously introduced by Cheeger, Jiang, and Naber in [59].

Remark 1.3.10. The proof of Theorem 1.3.7 goes in the *opposite direction* with respect to the proof of Theorem 1.3.2. Indeed, in Theorem 1.3.2 an infinitesimal splitting originally obtained along a stable minimal hypersurface is propagated through the whole manifold. In our case we first obtain the splitting at infinity, i.e., at the level of blow-downs, and then upgrade such information to a splitting at finite scales.

Nevertheless, the proofs of Theorems 1.3.2, 1.3.4, and Proposition 1.3.8 all share a common feature: at different steps, the sought splitting reduces to showing that certain functions satisfying an elliptic partial differential relation are constant, and this is achieved using that the underlying space (which is a smooth manifold in Theorem 1.3.2, and it is a metric measure space in Theorem 1.3.4, and Proposition 1.3.8) is parabolic.

Remark 1.3.11. While we were writing up [73], the preprint [123] appeared on arXiv. There, Huang and Huang obtain Theorem 1.3.7 assuming in addition that the universal covering has Euclidean volume growth. In their setting, the splitting of the blow-downs of the universal covering is an immediate consequence of the so-called cone-splitting principle. Nevertheless, to pass from the splitting of the blow-downs of the universal covering to the splitting of the universal covering itself, they employ a strategy analogous to ours based on the slicing theorem.

1.3.3 Chapter 5

The main result of Chapter 5 is Theorem 1.3.12 below (which is again one of the main results from [73]). We recall that the simplicial volume of a connected oriented (possibly non-compact) manifold M is a proper homotopy invariant, denoted by $||M|| \in [0, \infty]$, which measures the geometric complexity of the real fundamental class of M (see Section 2.3 for the precise definition and some background). It was introduced by Gromov in [110]. Here we obtain the following:

Theorem 1.3.12. *Let $n \geq 2$. If (M^n, g) is a smooth, complete, oriented Riemannian manifold with $\mathsf{Ric} \geq 0$, then $||M|| = 0$.*

Despite the simplicial volume being a topological invariant, its interplay with (Ricci) curvature became clear since the earliest developments of the theory. In particular, Gromov in [110, p. 14] obtained the following *Isolation Theorem* for the simplicial volume under lower Ricci curvature bounds:

Theorem 1.3.13 (Gromov '82)**.** *For every $n \in \mathbb{N}$ there exists $\varepsilon(n) > 0$ such that the following holds. Let (M^n, g) be a complete Riemannian manifold with $\mathsf{Ric} \geq -(n-1)$. If $\mathrm{vol}(B_1(p)) \leq \varepsilon(n)$ for every $p \in M$, then $||M|| = 0$.*

An immediate corollary of Theorem 1.3.13 is that the simplicial volume of any closed manifold with $\mathsf{Ric} \geq 0$ vanishes. Thanks to Theorem 1.3.12, the closedness assumption can be dropped. We note that Theorem 1.3.12 does not follow from Gromov's isolation theorem by scaling even in the case of non-compact manifolds with non-negative Ricci curvature and sub-Euclidean volume growth. Indeed, the sub-Euclidean volume growth does not guarantee the existence of a fixed radius such that *all* balls of that radius have scale-invariantly small volume.

Theorem 1.3.12 yields new topological restrictions to the existence of complete metrics with $\mathsf{Ric} \geq 0$ in all dimensions, see [73, Section 6]. Moreover, it provides an alternative proof of Theorem 1.3.3, using that a contractible 3-manifold with vanishing simplicial volume must be homeomorphic to $\mathbb{R}^3$ thanks to a result by Bargagnati-Frigerio [30]. Combining this fact with Theorem 1.3.7, one obtains a new proof of the topological classification of open 3-manifolds with $\mathsf{Ric} \geq 0$, i.e. Theorem 1.3.1. Such proof is independent of minimal surfaces methods, and it extends to $\mathsf{RCD}(0,3)$ spaces whose underlying space is a topological 3-manifold up to homeomorphism (see [73, Section 5]).

1.3.3.1 Strategy of the proof

The proof of Theorem 1.3.12 hinges on a general vanishing theorem for the simplicial volume of non-compact manifolds originally stated by Gromov in [110, p. 58] with a sketch of proof, and proved in all details more recently by Frigerio and Moraschini in [90, Theorem 9, Corollary 11]. Their result, stated below as Theorem 2.3.12, guarantees that the simplicial volume of an open triangulable n-manifold M vanishes if one can construct a countable open cover of M with relatively compact domains which is amenable at infinity (see Definition 2.3.11 below for the relevant terminology) and has multiplicity less than or equal to n. To construct such cover in our setting we distinguish two cases. If the volume growth of (M^n, g) is Euclidean, then the cover basically consists of metric annuli. To obtain amenability at infinity we show that the relative fundamental group of a large metric annulus inside a slightly larger one is always finite in this setup, see Theorem 5.1.2 below for the precise statement. The proof relies on a contradiction argument based on the Cheeger-Colding theory.

When the volume growth is not maximal the construction of the sought cover is more delicate. We are going to combine a conformal deformation argument together with a general result due to Papasoglu in [163] (motivated by some earlier work of Guth [112, 113] and Gromov's [110]) to show that a conformal rescaling of (M^n, g) has small $(n-1)$-Urysohn width. This helps us to build a geometrically well-behaved

cover of M with multiplicity $\leq n$. The amenability at infinity of such cover will follow from Kapovitch and Wilking's Margulis lemma for manifolds with non-negative Ricci curvature from [126].

The extension of Theorem 1.3.12 to triangulable $\mathsf{RCD}(0,n)$ manifolds follows from the same strategy. The only point one has to be careful about is the application of the Margulis lemma. While a Margulis lemma for RCD spaces has been recently obtained in [84], the local version corresponding to [126, Theorem 1] is not yet available in the literature. We overcome this by obtaining a local Margulis lemma for $\mathsf{RCD}(K,N)$ spaces (and, more generally, for essentially nonbranching and semi-locally simply connected $\mathsf{CD}(K,N)$ spaces) where the bound on the index of the nilpotent subgroup is not effective. Despite being weaker than the statement obtained in the smooth case in [126], our result, whose proof relies on [36], is sufficient for our purposes.

1.4 Contributions not included in the thesis

In the context of my doctoral studies, I also worked on the papers [69, 70, 71, 72, 75, 76], which are not included in the thesis due to page constraints, and whose content is briefly described below. We first describe [76], since it is the most closely related to the content of the thesis, as it deals with topological restrictions of manifolds satisfying appropriate curvature bounds.

- [76] was written in collaboration with Andrea Mondino. We consider manifolds with almost non-negative Ricci curvature and strictly positive integral lower bounds on the sum of the lowest k eigenvalues of the Ricci tensor. The geometry of manifolds with joint lower bounds on the Ricci and the scalar curvature has been investigated in many recent papers (see, for instance, [154, 62, 195, 162]), while the question of understanding the geometry of manifolds with a lower bound on the first k eigenvalues of the Ricci tensor (but no uniform lower bound on the Ricci tensor itself) was stated in [107] (and studied, for instance, in [68, 134, 187, 89, 103]).

  We show that, if (M^n, g) is a Riemannian manifold satisfying the aforementioned curvature bounds for $k = 2$, then M is contained in a neighbourhood of controlled width of an isometrically embedded 1-dimensional sub-manifold. From this, we deduce several metric and topological consequences: M has at most linear volume growth and at most two ends, it has bounded 1-Urysohn

width, the first Betti number of M is bounded above by 1, and there is precise information on elements of infinite order in $\pi_1(M)$.

If (M^n, g) is a Riemannian manifold satisfying such bounds for $k \geq 2$ and additionally the Ricci curvature is non-negative, then we show that M has at most $(k-1)$-dimensional behavior at large scales.

If $k = n = \dim(M)$, so that the integral lower bound is on the scalar curvature, assuming in addition that the $(n-2)$-Ricci curvature (see [76, Section 3.2]) is non-negative, then we prove that the dimension drop at large scales improves to $n-2$.

From the above results we deduce topological restrictions, such as upper bounds on the first Betti number.

The remaining papers deal with properties of sets minimizing the perimeter in RCD spaces. We introduce the subject very briefly, and then list the main contributions of the author.

Sets of finite perimeter in metric measure spaces were first studied by Ambrosio and Miranda in [4, 149, 3] as part of the broader development of Sobolev theory in the setting of metric measure spaces ([114, 52, 177]). In the context of RCD spaces, these objects were then systematically studied by Ambrosio, Brué, Semola, and others (see, for instance, [6, 41, 39]).

This led to the study of sets minimizing the perimeter and isoperimetric sets in RCD spaces. Thanks to the stability and compactness properties of both RCD spaces and extremal sets of the area functional on RCD spaces, this theory has allowed to obtain many new properties of such sets in the smooth category as well (see [152, 22, 23, 24, 25, 26, 27], among others).

We now list the main contributions of the author. These are mostly rigidity results of two types:

1. In [71, 72] the existence of a perimeter minimizer in a Riemannian manifold with suitable curvature conditions forces the ambient space to be a product (through techniques relying on RCD spaces).

2. In [75, 70] (which rely on [69]) we study Liouville-type properties of perimeter minimizers in RCD spaces. Here we assume that the ambient space splits a line, and we show that under suitable conditions the only boundaries of perimeter minimizing sets are the horizontal slices.

Let us briefly explain the content of each one of the aforementioned papers.

- [71] was written in collaboration with Francesco Fiorani. We show that the Hausdorff dimension of the singular set of perimeter minimizers in non-collapsed limits of manifolds with two-sided bounds on the Ricci curvature is at most $n-5$, where n is the dimension of the ambient space. The estimate is sharp.

  As a byproduct, combining with the results of [60], one obtains that if a Ricci flat 4-manifold with Euclidean volume growth at infinity contains a perimeter minimizer, then it is isometric to $\mathbb{R}^4$.

- [72] was written in collaboration with Mattia Magnabosco. We study whether the existence of a perimeter minimizing set in a Riemannian manifold (M, g) forces an isometric splitting. We show that this is the case when M has non-negative sectional curvature and quadratic volume growth at infinity. Moreover, we obtain that the boundary of the perimeter minimizing set is identified with a slice in the product structure of M.

  Previous related results had been obtained in [17, 20, 16, 46]. These results treat the analogous problem under various curvature assumptions, but always assuming a lower as well as an upper curvature bound. As a byproduct of our result, 4-manifolds with non-negative sectional curvature and scalar curvature ≥ 1, which contain a perimeter minimizing set, split off a line isometrically. This fact provides a global variant of a recent result from [61] regarding stable minimal hypersurfaces in 4-manifolds which satisfy the aforementioned curvature assumptions.

  Finally, we mention that the main result of this paper is obtained by applying the RCD theory to a suitable tangent cone at infinity of the manifold in question.

- In [69], we consider the area functional on proper $\mathsf{RCD}(K, \infty)$ spaces. In particular, we study the relationship between the area functional and the heat flow, extending a series of results that were previously obtained for sets of finite perimeter in this context.

  As a first consequence, we obtain that the area formula holds on proper $\mathsf{RCD}(K, \infty)$ spaces, i.e. the area of a function of bounded variation coincides with the perimeter of its hypograph. We remark that different versions of this result on finite dimensional $\mathsf{RCD}(K, N)$ spaces were known due to [14, 21]. As a second consequence, we deduce that, in our setting, the area functional coincides with

the relaxed area functional considered in [115] (a fact that is known to fail on general metric measure spaces due to [116]). As a byproduct, we obtain that minimizers of the area functional satisfy the mild regularity properties obtained in [116] for the relaxed area functional.

- In [70], we show that if $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}(K, N)$ space and $u \in \mathsf{W}^{1,1}_{loc}(X)$ is a solution of the minimal surface equation, then u is harmonic on its graph (which has a natural metric measure space structure). If $K = 0$, this allows to obtain an Harnack inequality for u, which in turn implies that $\mathsf{RCD}(0, N)$ spaces satisfy the Bernstein Property, meaning that any positive solution to the minimal surface equation must be constant. This fact had been proved in the setting of Riemannian manifolds in increasing generality in the works [170, 65, 86]. We remark that, once the harmonicity of u on its graph is obtained, the Bernstein Property for $\mathsf{RCD}(0, N)$ spaces essentially follows by repeating the argument of [86]. On the other hand, to prove that u is indeed harmonic on its graph, we develop a new blow-up argument which relies on Cheeger's Differentiation Theorem from [52].

- [75] was written in collaboration with my supervisor Andrea Mondino. This paper is similar in spirit to the previous one, as it deals with a similar (but independent) problem. We show that any parabolic $\mathsf{RCD}(0, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ satisfies the Half Space Property, meaning that, if $E \subset X \times \mathbb{R}$ is locally perimeter minimizing set and it is contained in a half space $X \times \mathbb{R}_+$, then ∂E is a locally finite union of horizontal slices. A variant of the result is then proved for $\mathsf{RCD}(K, N)$ spaces as well. The validity of the Half Space Property had been previously investigated in the setting of Riemannian manifolds in increasing generality in the works [170, 65, 85] (in particular, [85] shows its validity on parabolic manifolds without assuming any lower curvature bound). Nevertheless, our proof is completely different, and it relies on the tools developed in [152].

Chapter 2

Preliminaries

This chapter contains preliminaries. In Section 2.1, we recall some general results about analysis on metric measure spaces. In Section 2.2, we recall the main definitions and properties concerning RCD spaces. Finally, in Section 2.3, we give a brief overview of the results on the simplicial volume that will be used later in the thesis.

2.1 Analysis on metric measure spaces

In this section, we recall some facts about analysis on metric measure spaces. The purpose is dual: We fix the notational conventions that will be used throughout the thesis, and we introduce some of the tools that will be central to our results.

2.1.1 Notation and standard tools

For our purposes, a *metric measure space* is a triple $(X, \mathsf{d}, \mathfrak{m})$, where (X, d) is a complete separable metric space and $\mathfrak{m}$ is a non-negative Borel measure on X. We mainly consider *proper* metric measure spaces, i.e. spaces where balls are compact, whose measure $\mathfrak{m}$ is finite on bounded sets and has full support.

Given an open set $\Omega \subset X$, we denote by $\mathsf{Lip}(\Omega)$ the set of Lipschitz functions on Ω. The families of locally Lipschitz functions, bounded Lipschitz functions, and compactly supported Lipschitz functions are denoted by $\mathsf{Lip}_{loc}(\Omega)$, $\mathsf{Lip}_b(\Omega)$, and $\mathsf{Lip}_c(\Omega)$ respectively. If $f \in \mathsf{Lip}(\Omega)$, we denote its Lipschitz constant by $\mathsf{L}(f)$. If $f \in \mathsf{Lip}_{loc}(\Omega)$, and $x \in \Omega$ is an accumulation point of Ω, the *local Lipschitz constant* of f at x is defined as

$$\mathsf{lip}(f)(x) := \limsup_{y \to x} \frac{|f(x) - f(y)|}{\mathsf{d}(x, y)}.$$

We denote by $C(\Omega)$ the set of continuous functions on Ω, and by $C_b(\Omega)$ and $C_c(\Omega)$, respectively, the subsets of bounded, and compactly supported, continuous functions.

Given a closed interval $[a,b] \subset \mathbb{R}$ and a (continuous) curve $\gamma : [a,b] \to X$, the length of γ is defined as

$$\sup\{\sum_{i=1}^{N} \mathsf{d}(\gamma(a_i), \gamma(a_{i-1})) : a \leq a_0 \leq \cdots \leq a_N \leq b,\ N \in \mathbb{N}\}. \tag{2.1}$$

We say that $\gamma : [a,b] \to X$ is an absolutely continuous curve (denoted $\gamma \in AC([a,b],(X,\mathsf{d}))$) if there exists $g \in \mathsf{L}^1((a,b))$ such that for every $a \leq t \leq s \leq b$ it holds

$$\mathsf{d}(\gamma(t),\gamma(s)) \leq \int_t^s g(z)\,dz. \tag{2.2}$$

If $\gamma \in AC([a,b],(X,\mathsf{d}))$, by [5, Theorem 9.2], for almost every $t \in (a,b)$ the following limit exists:

$$|\dot{\gamma}|(t) := \lim_{h\to 0} \frac{\mathsf{d}(\gamma(t+h),\gamma(t))}{h}. \tag{2.3}$$

Moreover, $|\dot{\gamma}|$ belongs to $\mathsf{L}^1((a,b))$ and is called the *metric derivative* of γ. We say that a curve $\gamma : [a,b] \to X$ is a *geodesic* if its length coincides with the distance between its endpoints. Unless otherwise specified, geodesics have constant unit metric derivative. The metric space (X,d) is said to be *geodesic* if for every $x, y \in X$, there exists a geodesic connecting them.

For every $p \in [0,+\infty]$, we denote by $\mathsf{L}^p(X,\mathfrak{m})$ the space of p-summable functions. We often use the abbreviated notation $\mathsf{L}^p(X)$, when there is no ambiguity regarding the underlying measure. As for the other functions spaces, $\mathsf{L}^p_{loc}(X,\mathfrak{m})$ denotes the localized version of $\mathsf{L}^p(X,\mathfrak{m})$. If $p \in [1,+\infty]$, and $f \in \mathsf{L}^p(X,\mathfrak{m})$, we denote its norm by $\|f\|_{\mathsf{L}^p(X,\mathfrak{m})}$ (or simply $\|f\|_{\mathsf{L}^p(X)}$ or $\|f\|_p$). Moreover, given $x \in X$ and $r > 0$, we set

$$f_{x,r} := \fint_{B_r(x)} f\,d\mathfrak{m}.$$

We now recall the main definitions concerning Hausdorff measures, covering theorems, and differentiation of functions in metric measure spaces.

Definition 2.1.1 (Hausdorff measure)**.** Let (X,d) be a metric space. Let $k \geq 0$, and let $\omega_k := \pi^{k/2}/\Gamma(1+k/2)$, where Γ is the Euler Gamma function. If $\delta \in (0,+\infty]$, and $A \subset X$, we set

$$\mathsf{H}^k_\delta(A) := \frac{\omega_k}{2^k} \inf\Big\{ \sum_{i\in\mathbb{N}} \operatorname{diam}(A_i)^k : \bigcup_{i\in\mathbb{N}} A_i = A, \quad \operatorname{diam}(A_i) < \delta \Big\}.$$

The k^{th}-dimensional Hausdorff measure of A is defined as

$$\mathsf{H}^k(A) := \lim_{\delta\to 0} \mathsf{H}^k_\delta(A).$$

Definition 2.1.2 (Hausdorff dimension)**.** Let (X, d) be a metric space and let $A \subset X$. The *Hausdorff dimension* of A is defined as

$$\dim_H(A) := \inf\Big\{d \geq 0 : \mathsf{H}^d(A) = 0\Big\}.$$

We recall a general Covering Theorem for metric spaces.

Theorem 2.1.3. *(Vitali Covering Theorem) Let (X, d) be a metric space, and let $\mathcal{V}$ be a family of balls with uniformly bounded radii. Then, there exists a disjoint subfamily $\mathcal{V}' \subset \mathcal{V}$ such that*

$$\bigcup_{B_r(x)\in\mathcal{V}} B_r(x) \subset \bigcup_{B_r(x)\in\mathcal{V}'} B_{5r}(x).$$

We now recall the definition and a few key properties of doubling metric measure spaces.

Definition 2.1.4 (Doubling metric measure spaces)**.** A metric measure space is *locally uniformly doubling* if there exists a non-decreasing function $C : (0, +\infty) \to (0, +\infty)$ such that, for every $x \in X$ and $r \in (0, R)$, it holds

$$\mathfrak{m}(B_{2r}(x)) \leq C(R)\mathfrak{m}(B_r(x)).$$

A metric measure space is *doubling* if C in the above definition can be chosen to be constant.

Theorem 2.1.5 (Characterization of doubling spaces)**.** *A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is doubling if and only if there exist constants $C, s > 0$, such that, for every $x, y \in X$ and every $R \geq r > 0$ such that $x \in B_R(y)$, it holds*

$$\frac{\mathfrak{m}(B_r(x))}{\mathfrak{m}(B_R(y))} \geq C\Big(\frac{r}{R}\Big)^s.$$

Theorem 2.1.6 (Lebesgue Theorem)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a doubling metric measure space, and let $f \in \mathsf{L}^1_{loc}(X)$. Then, for $\mathfrak{m}$-a.e. $x \in X$, it holds*

$$\lim_{r\to 0} \fint_{B_r(x)} |f - f(x)|\, d\mathfrak{m} = 0.$$

2.1.2 Sobolev spaces

We now recall the main definitions concerning Sobolev Spaces in the setting of metric measure spaces. Several approaches are possible. Here, we consider the relaxation approach due to Cheeger (see [52]). Under mild assumptions, these different approaches give rise to the same Sobolev Spaces and to the same notion of weak gradient for $1 < p < +\infty$ (see [12]). In the case $p = 1$, this will still be the case in the class of proper RCD spaces (due to [97]).

Definition 2.1.7. (Sobolev Spaces) Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space, let $\Omega \subset X$ be an open set and let $p > 1$. A function $f \in \mathsf{L}^p(\Omega)$ belongs to the Sobolev space $\mathsf{W}^{1,p}(\Omega)$, if there exists a sequence of locally Lipschitz functions $\{f_i\}_{i\in\mathbb{N}} \subset \mathsf{Lip}_{loc}(\Omega)$ converging to f in $\mathsf{L}^p(\Omega)$ such that

$$\limsup_{i\to+\infty} \int_\Omega \mathsf{lip}(f_i)^p \, \mathrm{d}\mathfrak{m} < +\infty.$$

A function $f \in \mathsf{L}^p_{loc}(\Omega)$ is said to be in the Sobolev space $\mathsf{W}^{1,p}_{loc}(\Omega)$ if, for every $\eta \in \mathsf{Lip}_c(\Omega)$, it holds $f\eta \in \mathsf{W}^{1,p}(\Omega)$. Moreover, we denote by $\mathsf{W}^{1,p}_0(\Omega)$ the closure in $\mathsf{W}^{1,p}(\Omega)$ of $\mathsf{Lip}_c(\Omega)$.

For every $f \in \mathsf{W}^{1,p}(\Omega)$, through a minimization procedure, one can define the *relaxed gradient* $|\nabla f|$ (a priori depending on p, but independent of the exponent in the spaces that we consider later in the thesis). This object is such that

$$\begin{aligned}&\int_\Omega |\nabla f|^p \, \mathrm{d}\mathfrak{m}\\ &= \inf\left\{ \liminf_{n\to+\infty} \int \mathsf{lip}(f_n)^p \, \mathrm{d}\mathfrak{m} \,\middle|\, (f_n)_n \subset \mathsf{L}^p(\Omega) \cap \mathsf{Lip}_{loc}(\Omega), \|f_n - f\|_{\mathsf{L}^p(\Omega)} \to 0 \right\}. \quad (2.4)\end{aligned}$$

The quantity in (2.4) is called the p-Cheeger energy and it is denoted by $\mathsf{Ch}_p(f)$. In the case $p = 2$, we write Ch instead of Ch_2. The Sobolev norm of f is defined as $\|f\|_{\mathsf{W}^{1,p}(\Omega)} := \|f\|_{\mathsf{L}^p(\Omega)} + \mathsf{Ch}_p(f)^{1/p}$. One can check that with this norm the space $\mathsf{W}^{1,p}(\Omega)$ is Banach.

Remark 2.1.8. The original definition of Sobolev Space given in [52] is slightly different from the one given above. Indeed, Cheeger considers generalized upper gradients in place of local Lipschitz constants. Nevertheless, the two definitions give rise to the same spaces due to [7].

We now consider functions of bounded variation following [149] (see also [8]).

Definition 2.1.9. (Functions of bounded variation) Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space and let $\Omega \subset X$ be an open set. A function $f \in \mathsf{L}^1(\Omega)$ is said to be of bounded variation if there exists a sequence of locally Lipschitz functions $\{f_i\}_{i\in\mathbb{N}} \subset \mathsf{Lip}_{loc}(\Omega)$ converging to f in $\mathsf{L}^1(\Omega)$ such that

$$\limsup_{i\to+\infty} \int_\Omega \mathsf{lip}(f_i)\, \mathrm{d}\mathfrak{m} < +\infty.$$

The space of such functions is denoted $\mathsf{BV}(\Omega)$. A function $f \in \mathsf{L}^1_{loc}(\Omega)$ is said to be in $\mathsf{BV}_{loc}(\Omega)$ if, for every $\eta \in \mathsf{Lip}_c(\Omega)$, it holds $f\eta \in \mathsf{BV}(\Omega)$.

For every $f \in \mathsf{BV}(\Omega)$ and every open set $A \subset \Omega$ one defines

$$|Df|(A) =$$

$$\inf\Big\{ \liminf_{n\to+\infty} \int_A \mathsf{lip}(f_n)\, \mathrm{d}\mathfrak{m} \Big| (f_n)_n \subset \mathsf{L}^1(A) \cap \mathsf{Lip}_{loc}(A), \|f_n - f\|_{\mathsf{L}^1(A)} \to 0 \Big\}.$$

One can check that the quantity in the previous expression is the restriction to the open subsets of Ω of a finite measure. The corresponding BV norm is defined as $\|f\|_{\mathsf{BV}(\Omega)} := \|f\|_{\mathsf{L}^1(\Omega)} + |Df|(\Omega)$. One can check that with this norm the space $\mathsf{BV}(\Omega)$ is Banach. The function f belongs to $\mathsf{W}^{1,1}(\Omega)$ if $f \in \mathsf{BV}(\Omega)$ and $|Df| << \mathfrak{m}$. In this case, we denote by $|\nabla f|$ the density of $|Df|$ with respect to $\mathfrak{m}$.

Relaxed gradients are local and satisfy the chain rule:

- For every $f, g \in \mathsf{W}^{1,p}(X)$, one has $|\nabla f| = |\nabla g|$ on $\mathfrak{m}$-a.e. point of $\{f = g\}$.
- For every $f \in \mathsf{W}^{1,p}(X)$, and $\varphi \in \mathsf{Lip}(\mathbb{R})$ with $\varphi(0) = 0$, it holds $|\nabla \varphi \circ f| = |\varphi'(f)||\nabla f|$ on $\mathfrak{m}$-a.e. point of X.

Definition 2.1.10 (Poincaré inequality)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space and let $1 \le p < +\infty$. The space X supports a $(1, p)$-Poincaré inequality if there exist $C, \lambda > 0$ such that, for every $f \in \mathsf{W}^{1,p}(X)$, $x \in X$ and $r > 0$, it holds

$$⨍_{B_r(x)} |f - f_{x,r}|\, d\mathfrak{m} \le Cr\Big(⨍_{B_{\lambda r}(x)} |\nabla f|^p\, d\mathfrak{m}\Big)^{1/p}.$$

Doubling metric measure spaces supporting a Poincaré inequality (called PI spaces) enjoy a lot of regularity properties. The next theorem can be found in [119, Theorem 13.5.1].

Theorem 2.1.11 (Gradient of Lipschitz functions)**.** *Let $1 < p < +\infty$. Let $(X, \mathsf{d}, \mathfrak{m})$ be a doubling space supporting a $(1, p)$-Poincaré inequality. Then, for every $f \in \mathsf{Lip}(X)$, it holds $\mathsf{lip}(f) = |\nabla f|$ on $\mathfrak{m}$-a.e. $x \in X$.*

Notably, the previous result fails for $p = 1$ (see [14] and [116]).

Definition 2.1.12 (Infinitesimal Hilbertianity)**.** A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian if the space $\mathsf{W}^{1,2}(X)$ is a Hilbert space.

For every $f, g \in \mathsf{W}^{1,2}(X)$, one can define $\nabla f \cdot \nabla g \in \mathsf{L}^1(X)$ as

$$\nabla f \cdot \nabla g := \frac{|\nabla(f+g)|^2 - |\nabla(f-g)|^2}{4}.$$

An important consequence of infinitesimal Hilbertianity is the following.

Theorem 2.1.13 (Products of gradients)**.** *If $(X, \mathsf{d}, \mathfrak{m})$ is Infinitesimally Hilbertian, the function $\cdot : \mathsf{W}^{1,2}(X) \times \mathsf{W}^{1,2}(X) \to \mathsf{L}^1(X)$ defined as $(f, g) \mapsto \nabla f \cdot \nabla g$ is symmetric, bilinear, it satisfies the chain rule and the Leibniz rule in both entries, and for every $f, g \in \mathsf{W}^{1,2}(X)$, it holds $|\nabla f \cdot \nabla g| \leq |\nabla f||\nabla g|$.*

We then consider the Laplacian in the metric setting.

Definition 2.1.14 (Laplacian)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be infinitesimally Hilbertian and let $\Omega \subset X$ be an open set. Let $f \in \mathsf{W}^{1,2}(\Omega)$. We say that $f \in D(\Delta, \Omega)$ if there exists a function $h \in \mathsf{L}^2(\Omega)$ such that

$$\int_\Omega gh \, \mathrm{d}\mathfrak{m} = -\int_\Omega \nabla g \cdot \nabla f \, \mathrm{d}\mathfrak{m} \quad \text{for any } g \in \mathsf{W}^{1,2}_0(\Omega).$$

In this case we say that $\Delta f = h$ in Ω.

We also have the following more general definition.

Definition 2.1.15 (Distributional Laplacian)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be infinitesimally Hilbertian and let $\Omega \subset X$ be an open set. Let $f \in \mathsf{W}^{1,1}_{loc}(\Omega)$ and let μ be a Radon measure on Ω. We say that $\Delta f = \mu$ in Ω in distributional sense if

$$\int_\Omega g \, d\mu = -\int_\Omega \nabla g \cdot \nabla f \, \mathrm{d}\mathfrak{m} \quad \text{for any } g \in \mathsf{Lip}_c(\Omega).$$

2.1.3 Convergence of metric spaces

We now recall the main definitions concerning Gromov Hausdorff convergence, referring to [44], [186], and [99] for an overview on the subject.

Definition 2.1.16 (Gromov Hausdorff approximations)**.** Let (X, d_x, x) and (Y, d_y, y) be pointed metric spaces and let $\delta > 0$. A map $f : X \to Y$ is a δ-GH map if

- $f(x) = y$.
- $\sup_{a,b\in X} \left|\mathsf{d}_x(a,b) - \mathsf{d}_y(f(a),f(b))\right| \leq \delta$.
- The image of f is a δ-net in Y.

Definition 2.1.17 (Pointed Gromov Hausdorff convergence)**.** A sequence of pointed metric spaces (X_n, d_n, x_n) converges to (X, d, x) in pointed Gromov Hausdorff convergence if, for every $\varepsilon, R > 0$, there exists N such that, for every $n \geq N$, there exists a ε-GH map $f_n^{\varepsilon,R} : \overline{B}_R(x_n) \to \overline{B}_R(x)$.

The previous definition of pointed Gromov Hausdorff convergence differs slightly from the one given in [44, p. 272]. Nevertheless, when working on length spaces, the two definitions coincide.

Definition 2.1.18 (Pointed measured Gromov Hausdorff convergence)**.** We say that a sequence of pointed metric measure spaces $(X_n, \mathsf{d}_n, \mathsf{m}_n, x_n)$ converges to $(X, \mathsf{d}, \mathsf{m}, x)$ in pointed measured Gromov Hausdorff convergence, if it converges in pointed Gromov Hausdorff sense, and the maps $f_n^{\varepsilon,R} : \overline{B}_R(x_n) \to \overline{B}_R(x)$ given by Definition 2.1.17 satisfy $(f_n^{\varepsilon,R})_\#(\mathsf{m}_n \llcorner \overline{B}_R(x_n)) \to \mathsf{m} \llcorner \overline{B}_R(x)$ weakly in duality with continuous bounded supported functions on X.

We remark that pointed measured Gromov Hausdorff convergence is metrizable, and the corresponding distance will be denoted d_{pmGH}.

Definition 2.1.19 (Tangent space)**.** Let $(X, \mathsf{d}, \mathsf{m})$ be a metric measure space and let $x \in X$. We denote by $\mathrm{Tan}_x(X)$ the (possibly empty) collection of (isometry classes of) metric measure spaces that are pointed measured Gromov Hausdorff limit points as $r \downarrow 0$ of the family $(X, r^{-1}\mathsf{d}, \mathsf{m}(B_r(x))^{-1}\mathsf{m}, x)$.

In the case of a sequence of uniformly locally doubling metric measure spaces $(X_i, \mathsf{d}_i, \mathsf{m}_i, x_i)$ (as in the case of $\mathsf{RCD}(K,N)$ spaces), pointed measured Gromov Hausdorff convergence to $(X, \mathsf{d}, \mathsf{m}, x)$ can be equivalently characterized by asking for the existence of a proper metric space (Z, d_z) such that all the metric spaces (X_i, d_i) are isometrically embedded into (Z, d_z), $x_i \to x$, $X_i \to X$ in Hausdorff sense in compact sets, and $\mathsf{m}_i \to \mathsf{m}$ weakly in duality with continuous bounded supported functions in Z.

The fundamental result concerning Gromov Hausdorff convergence is the following Precompactness Theorem due to Gromov.

Theorem 2.1.20 (Gromov's Compactness Theorem)**.** *Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ be a sequence of pointed locally uniformly doubling metric measure spaces such that $\mathfrak{m}_n(B_1(x_n)) = 1$ for every $n \in \mathbb{N}$. Then, modulo passing to a subsequence, they converge in pointed measured Gromov Hausdorff sense to a limiting space $(X, \mathsf{d}, \mathfrak{m}, x)$.*

We now recall some basic facts about equivariant pointed Gromov Hausdorff convergence, the main references being [92, 93]. In the next definition, a quadruple (Y, d, G, p) consists of a pointed metric space (Y, d, p) (which is assumed to be a complete length space), and a closed subgroup $G \subset \mathrm{Iso}(Y)$. We denote $G(R) := \{g \in G : \mathsf{d}(p, gp) \leq R\}$.

Definition 2.1.21 (Equivariant Gromov Hausdorff convergence)**.** Two spaces $(Y_1, \mathsf{d}_1, y_1, G_1)$ and $(Y_2, \mathsf{d}_2, y_2, G_2)$ are ε-close in the *equivariant pointed Gromov Hausdorff* sense, if there exists a triple (f, ψ, φ) of maps

$$f : B_{1/\varepsilon}(y_1) \to B_{1/\varepsilon}(y_2)\,,\ \varphi : G_1(1/\varepsilon) \to G_2(1/\varepsilon)\,,\ \psi : G_2(1/\varepsilon) \to G_1(1/\varepsilon)\,,$$

with the following properties:

1. $f(y_1) = y_2$;
2. $f(B_{1/\varepsilon}(y_1))$ is an ε-net in $B_{1/\varepsilon}(y_2)$;
3. $|\mathsf{d}_2(f(a), f(b)) - \mathsf{d}_1(a, b)| < \varepsilon$ for every $a, b \in B_{1/\varepsilon}(y_1)$;
4. If $g \in G_1(1/\varepsilon)$, and $x, gx \in B_{1/\varepsilon}(y_1)$, then $\mathsf{d}_2(f(gx), \varphi(g)f(x)) < \varepsilon$;
5. If $h \in G_2(1/\varepsilon)$, and $x, \psi(h)x \in B_{1/\varepsilon}(y_1)$, then $\mathsf{d}_2(f(\psi(h)x), hf(x)) < \varepsilon$.

Lemma 2.1.22. *Let $(X_i, \mathsf{d}_i, G_i, x_i)$ be a sequence converging in equivariant pGH sense to (X, d, G, x). Let $\gamma_i \in G_i$ be such that $\gamma_i \cdot x_i$ converges to $\overline{x} \in X$ in the space realizing the pGH convergence. Then there exists $\gamma \in G$ such that for every $y \in X$ and every sequence $y_i \to y$, it holds $\gamma_i \cdot y_i \to \gamma \cdot y$. In this case we say that $\gamma_i \to \gamma$.*

Theorem 2.1.23 (Key Theorem of equivariant convergence)**.** *Let (Y_i, d_i, y_i) be a sequence of metric spaces converging in pGH sense to (X, d, x). Let G_i be closed subgroups of $\mathrm{Iso}(Y_i)$. Then, up to a not relabelled subsequence, there exists a closed subgroup $G < \mathrm{Iso}(X)$ such that*

$$(Y_i, \mathsf{d}_i, G_i, y_i) \to (X, \mathsf{d}, G, x)$$

in equivariant pGH sense. Let Y_i/G_i be the orbit space of the action of G_i on Y_i, let $\mathsf{d}_{i/\sim}$ be the induced quotient distance on Y_i/G_i, and let $[y_i] \in Y_i/G_i$ be the orbit of y_i. Using an analogous notation for X/G, $\mathsf{d}_{/\sim}$, and $[x] \in X/G$, it holds

$$(Y_i/G_i, \mathsf{d}_{i/\sim}, [y_i]) \to (X/G, \mathsf{d}_{/\sim}, [x])$$

in pGH sense.

In the next definition, a quintuple $(X, \mathsf{d}, \mathfrak{m}, G, p)$ consists of a pointed metric measure space $(X, \mathsf{d}, \mathfrak{m}, p)$ and a closed subgroup $G \subset \mathrm{Iso}(X)$.

Definition 2.1.24 (Equivariant measured Gromov Hausdorff convergence)**.** Two metric measure spaces $(Y_1, \mathsf{d}_1, \mathfrak{m}_1, G_1, y_1)$ and $(Y_2, \mathsf{d}_2, \mathfrak{m}_2, G_2, y_2)$ are ε-close in the *equivariant pointed measured Gromov Hausdorff* sense, if there exists a triple (f, ψ, φ) of maps

$$f : B_{1/\varepsilon}(y_1) \to B_{1/\varepsilon}(y_2),\ \varphi : G_1(1/\varepsilon) \to G_2(1/\varepsilon),\ \psi : G_2(1/\varepsilon) \to G_1(1/\varepsilon)\,,$$

satisfying the properties of Definition 2.1.21 and such that $f_\#(\mathfrak{m}_1 \llcorner B_{1/\varepsilon}(y_1))$ is ε-close to $\mathfrak{m}_2 \llcorner B_{1/\varepsilon}(y_2)$ w.r.t. weak convergence in Y_2. We stress that, in this case, the groups G_1 and G_2 are required to act by measure-preserving isometries.

2.2 RCD spaces

In this section, we recall some properties of RCD spaces that will be used later on in the thesis. We refer to [2] for a more thorough introduction to the subject. In this section all metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$ considered are assumed to satisfy $\mathfrak{m}(B_r(x)) \leq c_1 e^{c_2 r^2}$ for some $c_1, c_2 > 0$, $x \in X$ and $r > 0$ large enough. This is needed so that the entropies considered are well defined in their domains.

2.2.1 RCD(K, ∞) spaces

In this subsection we recall the main properties of RCD(K, ∞) spaces. We will denote by $\mathcal{P}_2(X)$ the set of probability measures μ such that for every $x \in X$ we have $\int_X \mathsf{d}^2(x, y)\, d\mu(y) < +\infty$ and on this space we will consider the Wasserstein metric W_2. When we refer to a Wasserstein geodesic between two measures $\mu_1, \mu_2 \in \mathcal{P}_2(X)$, we mean an absolutely continuous curve taking values in $(\mathcal{P}_2, W_2)$ whose length coincides with the distance $W_2(\mu_1, \mu_2)$. On a space $(X, \mathsf{d}, \mathfrak{m})$, we consider the relative entropy functional $\mathrm{Ent}_\mathfrak{m} : \mathcal{P}_2(X) \to (-\infty, +\infty]$ defined as

$$\mathrm{Ent}_\mathfrak{m}(\mu) = \begin{cases} \int_X \rho \log \rho \, d\mathfrak{m} & \text{if } \mu = \rho\mathfrak{m} \\ +\infty & \text{otherwise.} \end{cases}$$

Definition 2.2.1 ($\mathsf{CD}(K,\infty)$ spaces)**.** The metric measure space $(X,\mathsf{d},\mathfrak{m})$ satisfies the $\mathsf{CD}(K,\infty)$ condition if for every two absolutely continuous measures $\mu_0=\rho_0\mathfrak{m}$ and $\mu_1=\rho_1\mathfrak{m}$ in $\mathcal{P}_2(X)$, there exists a Wasserstein geodesic $(\mu_s)_{s\in[0,1]}\in\mathsf{Geo}(\mathcal{P}_2(X))$ joining μ_0 and μ_1 such that

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_s)\leq(1-s)\mathrm{Ent}_{\mathfrak{m}}(\mu_0)+s\mathrm{Ent}_{\mathfrak{m}}(\mu_1)-\frac{s(1-s)}{2}KW_2(\mu_0,\mu_1)^2.$$

Definition 2.2.2 ($\mathsf{RCD}(K,\infty)$ spaces)**.** A metric measure space $(X,\mathsf{d},\mathfrak{m})$ is said to be an $\mathsf{RCD}(K,\infty)$ space if it satisfies the $\mathsf{CD}(K,\infty)$ condition and the Sobolev space $\mathsf{W}^{1,2}(X)$ is a Hilbert space.

There is a characterization of $\mathsf{RCD}(K,\infty)$ which does not rely on optimal transport. This result was found in [11] (see also [10]), and links the $\mathsf{RCD}(K,\infty)$ condition to a weak Bochner-type inequality.

Theorem 2.2.3 (Bochner inequality on $\mathsf{RCD}(K,\infty)$)**.** *A metric measure space $(X,\mathsf{d},\mathfrak{m})$ is an $\mathsf{RCD}(K,\infty)$ space if and only if:*

1. *It is infinitesimally Hilbertian.*
2. *Every $f\in\mathsf{W}^{1,2}(X)$ such that $|\nabla f|\leq 1$ $\mathfrak{m}$-a.e. admits a 1-Lipschitz representative.*
3. *For every $f\in D(\Delta)$ and for every non-negative $g\in D(\Delta)\cap\mathsf{L}^\infty(X)$ such that $\Delta f\in\mathsf{W}^{1,2}(X)$ and $\Delta g\in\mathsf{L}^\infty(X)$, it holds*

$$\frac{1}{2}\int_X|\nabla f|^2\Delta g\,d\mathfrak{m}\geq\int_X\Big[\nabla f\cdot\nabla\Delta f+K|\nabla f|^2\Big]g\,d\mathfrak{m}.$$

We now recall some further key properties of $\mathsf{RCD}(K,\infty)$ spaces. First of all, $\mathsf{RCD}(K,\infty)$ spaces are essentially non-branching length spaces (due to a combination of [9, 78, 168]). Moreover, these spaces satisfy a Poincaré inequality (see [167]).

Proposition 2.2.4. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(K,\infty)$ space with $K\leq 0$, then for every $f\in\mathsf{W}^{1,2}(X)$, $x\in X$ and $r>0$ the following Poincaré inequality holds:*

$$\int_{B_r(x)}|f-f_{x,r}|\,d\mathfrak{m}\leq 4re^{|K|r^2}\int_{B_{2r}(x)}|\nabla f|\,d\mathfrak{m}.$$

Another important tool when working with RCD spaces is the heat flow. We recall here some of the main properties (referring to [9, 11] for more details).

Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(K,\infty)$ space. The heat flow $P_t:\mathsf{L}^2(X)\to\mathsf{L}^2(X)$ is the $\mathsf{L}^2(X)$-gradient flow of the Cheeger energy Ch. Under the RCD assumption P_t is

linear, continuous, self adjoint, and contractive in $\mathsf{L}^2(X)$. Moreover the definition can be extended to all $\mathsf{L}^p(X)$ spaces for $p \in [1, +\infty)$.

A key result in the RCD theory is that the dual heat semigroup $\overline{P}_t : \mathcal{P}_2(X) \to \mathcal{P}_2(X)$ defined by

$$\int_X f \, \mathrm{d}\overline{P}_t\mu := \int_X P_t f \, d\mu \quad \forall \mu \in \mathcal{P}_2(X), \quad \forall f \in \mathsf{Lip}_c(X),$$

is K-contractive with respect to the Wasserstein distance W_2 and for $t > 0$ maps probability measures into absolutely continuous probability measures. So we can introduce the heat kernel $p_t : X \times X \to [0, +\infty)$ by

$$p_t(x, \cdot)\mathfrak{m} := \overline{P}_t \delta_x.$$

The heat kernel is symmetric, continuous and stochastically complete i.e,

$$\int_X p_t(x, y) \, d\mathfrak{m}(y) = 1.$$

This allows us to define $P_t(f)$ when $f \in \mathsf{L}^\infty(X)$ by setting

$$P_t(f)(x) := \int_X f(y) p_t(x, y) \, d\mathfrak{m}(y).$$

Moreover, if $f \in \mathsf{L}^\infty(X)$, for every $t > 0$, we have $P_t f \in \mathsf{Lip}(X)$ and

$$\sqrt{2I_{2K}(t)} \, \mathsf{lip}(P_t(f)) \leq \|f\|_{\mathsf{L}^\infty(X)},$$

where $I_L(t) := \int_0^t e^{Lr} \, dr$. The heat flow has good approximation properties, in particular if $f \in \mathsf{W}^{1,2}(X)$, then $P_t(f) \to f$ in $\mathsf{W}^{1,2}(X)$; while if $f \in \mathsf{L}^\infty(X)$, then $P_t f \in \mathsf{Lip}(X)$ for every $t > 0$.

A key result in the theory of $\mathsf{RCD}(K, \infty)$ spaces is the so called Bakry-Émery contraction estimate of the heat flow. This was first obtained in [11] for functions in $\mathsf{W}^{1,2}$, while the generalization to functions in $\mathsf{W}^{1,p}$ for $p \in (1, +\infty)$ followed from [172]. Yet another generalization was then obtained in [97], where it was proved that, for a *proper* $\mathsf{RCD}(K, \infty)$ space $(X, \mathsf{d}, \mathfrak{m})$, if $f \in \mathsf{BV}(X)$, then

$$|DP_t(f)| \leq e^{-Kt} \overline{P}_t(|Df|). \tag{2.5}$$

Combining the previous contraction estimate with a lower semicontinuity argument we obtain that, if X is proper and $\Omega \subset X$ is open with $|Df|(\partial\Omega) = 0$, then

$$\lim_{t \to 0} |DP_t(f)|(\Omega) = |Df|(\Omega). \tag{2.6}$$

2.2.2 $\mathsf{RCD}(K,N)$ spaces

In this subsection, we recall the definition and the main properties of $\mathsf{RCD}(K,N)$ spaces. Given a metric measure space $(X,\mathsf{d},\mathfrak{m})$ and a measure $\mu=\rho\mathfrak{m}+\mu^{\perp}$, where $\mu^{\perp}$ is the singular part in its decomposition, we define for every $N>1$

$$\mathcal{E}_N(\mu):=-\int_X \rho^{1-1/N}\,d\mathfrak{m}.$$

We denote by $\mathcal{P}(\mathsf{Geo}(X))$ the space of probability measures on geodesics in X and, given a curve $\gamma: I\subset\mathbb{R}\to X$ and $s\in I$, we define $e_s(\gamma):=\gamma(s)$. We recall that given $\mu_1,\mu_2\in\mathcal{P}_2(X)$ there exists $\boldsymbol{\eta}\in P(\mathsf{Geo}(X))$ such that $s\mapsto(e_s)_{\#}\boldsymbol{\eta}$ is a Wasserstein geodesic, and $(e_0,e_1)_{\#}\boldsymbol{\eta}$ is an optimal transport plan between μ_1 and μ_2. Such $\boldsymbol{\eta}\in P(\mathsf{Geo}(X))$ will be called an optimal dynamic transport plan between μ_1 and μ_2.

We then set

$$s_k(r):=\begin{cases}\frac{\sin\sqrt{k}r}{\sqrt{k}} & k>0\\ r & k=0\\ \frac{\sinh\sqrt{-k}r}{\sqrt{-k}} & k<0\end{cases},\qquad \sigma_k^{(s)}(\theta):=\begin{cases}\frac{s_k(s\theta)}{s_k(\theta)} & k\theta^2\neq 0 \text{ and } k\theta^2<\pi^2\\ s & k\theta^2=0\\ +\infty & k\theta^2\geq\pi^2\end{cases}$$

and $\tau_{K,N}^{(s)}(\theta):=s^{1/N}\sigma^{(s)}_{K/(N-1)}(\theta)^{1-1/N}$.

With this in mind we can state the finite dimensional curvature dimension condition.

Definition 2.2.5 ($\mathsf{CD}(K,N)$ spaces)**.** A metric measure space $(X,\mathsf{d},\mathfrak{m})$ satisfies the $\mathsf{CD}(K,N)$ condition for $K\in\mathbb{R}$ and $N\in(1,+\infty)$ if the following happens. For every two absolutely continuous measures with bounded support $\mu_0=\rho_0\mathfrak{m},\mu_1=\rho_1\mathfrak{m}\in\mathcal{P}_2(X)$ there exists an optimal dynamic transport plan $\boldsymbol{\eta}\in P(\mathsf{Geo}(X))$ between μ_0 and μ_1 such that for every $N'\geq N$ and $s\in[0,1]$ we have

$$\begin{aligned}&\mathcal{E}_{N'}((e_s)_{\#}\boldsymbol{\eta})\\ &\quad\leq-\int_{\mathsf{Geo}(X)}\tau_{K,N'}^{(1-s)}(\mathsf{d}(\gamma_0,\gamma_1))\rho_0^{-1/N'}(\gamma_0)+\tau_{K,N'}^{(s)}(\mathsf{d}(\gamma_0,\gamma_1))\rho_1^{-1/N'}(\gamma_1)\,\mathrm{d}\boldsymbol{\eta}(\gamma).\end{aligned}$$

Definition 2.2.6 ($\mathsf{RCD}(K,N)$ spaces)**.** A metric measure space $(X,\mathsf{d},\mathfrak{m})$ is said to be an $\mathsf{RCD}(K,N)$ space if it satisfies the $\mathsf{CD}(K,N)$ condition and the Sobolev space $\mathsf{W}^{1,2}(X)$ is a Hilbert space.

As for $\mathsf{RCD}(K,\infty)$ spaces, also $\mathsf{RCD}(K,N)$ spaces can be characterized in terms of a Bochner formula. This was done independently in [13] and [87].

Theorem 2.2.7 (Bochner Formula on $\mathsf{RCD}(K,N)$ spaces)**.** *A metric measure space $(X,\mathsf{d},\mathfrak{m})$ is an $\mathsf{RCD}(K,N)$ space if and only if:*

1. *It is infinitesimally Hilbertian.*
2. *Every $f \in \mathsf{W}^{1,2}(X)$ such that $|\nabla f| \leq 1$ $\mathfrak{m}$-a.e. admits a 1-Lipschitz representative.*
3. *For every $f \in D(\Delta)$ and for every non-negative $g \in D(\Delta) \cap \mathsf{L}^\infty(X)$ such that $\Delta f \in \mathsf{W}^{1,2}(X)$ and $\Delta g \in \mathsf{L}^\infty(X)$, it holds*

$$\frac{1}{2}\int_X |\nabla f|^2 \Delta g \, d\mathfrak{m} \geq \int_X \Big[\nabla f \cdot \nabla \Delta f + K|\nabla f|^2 + \frac{(\Delta f)^2}{N}\Big] g \, d\mathfrak{m}.$$

In the next proposition we will denote by $V_{K,N}(r)$ the volume of the ball of radius r in the model space for the $\mathsf{CD}(K,N)$ condition (see [183, Example 1.8]). The next results show that RCD spaces are doubling, so that, since they also satisfy a Poincaré inequality, they fit into the setting of PI spaces. This result is a consequence of [183].

Proposition 2.2.8. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(K,N)$ space. Then for every $x \in X$ and $R > r > 0$ we have*

$$\frac{\mathfrak{m}(B_R(x))}{\mathfrak{m}(B_r(x))} \leq \frac{V_{K,N}(R)}{V_{K,N}(r)}.$$

Corollary 2.2.9. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(K,N)$ space. Then $\mathfrak{m}$ is locally uniformly doubling i.e., there exists a non-decreasing function $C : (0,+\infty) \to (0,+\infty)$ such that for every $x \in X$ and $r \in (0,R)$*

$$\mathfrak{m}(B_{2r}(x)) \leq C(R)\mathfrak{m}(B_r(x)).$$

If $K \geq 0$ then C can be chosen to be the constant function taking value 2^N.

As a consequence of Corollary 2.2.9, $\mathsf{RCD}(K,N)$ spaces are proper. Being proper length spaces, $\mathsf{RCD}(K,N)$ spaces are geodesic spaces.

Combining Gromov's Precompactness Theorem (Theorem 2.1.20), and the stability of the $\mathsf{RCD}(K,N)$ condition [99] (after [182, 183, 144, 11]), one obtains that $\mathsf{RCD}(K,N)$ spaces form a compact family w.r.t. pointed measured Gromov Hausdorff convergence.

Theorem 2.2.10 (Compactness of the $\mathsf{RCD}(K,N)$ class)**.** *Let $(X_n,\mathsf{d}_n,\mathfrak{m}_n,x_n)$ be a sequence of pointed $\mathsf{RCD}(K,N)$ spaces such that $\mathfrak{m}_n(B_1(x_n)) = 1$ for every $n \in \mathbb{N}$. Then modulo passing to a subsequence they converge in pointed measured Gromov Hausdorff sense to an $\mathsf{RCD}(K,N)$ space.*

Definition 2.2.11 (Tangent cones at infinity)**.** Let $(X, \mathsf{d}, \mathfrak{m}, x)$ be a pointed $\mathsf{RCD}(0, N)$ space, and consider a sequence $r_i \uparrow +\infty$. By Theorem 2.2.10, up to a subsequence, the spaces $(X, \mathsf{d}/r_i, \mathfrak{m}(B_{r_i}(x))^{-1}\mathfrak{m}, x)$ converge in pmGH sense to a limiting $\mathsf{RCD}(0, N)$ space $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$. Such X_∞ is called a *tangent cone at infinity* (or *blow-down*) of X.

The proof of the next lemma is the same as [145, Lemma 2.1].

Lemma 2.2.12. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(0, n)$ space. Then, the set of its tangent cones at infinity is connected in the pointed Gromov Hausdorff topology.*

Similarly, given a pointed $\mathsf{RCD}(K, N)$ space $(X, \mathsf{d}, \mathfrak{m}, x)$, the set $\mathrm{Tan}_x(X)$ is connected in the pointed Gromov Hausdorff topology.

The next lemma follows repeating the proof of [192, Proposition 3.3].

Lemma 2.2.13. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(0, n)$ space such that all of its tangent cones at infinity are 1-dimensional, and one tangent cone at infinity is $\mathbb{R}$. Then, $X \cong \mathbb{R} \times Y$ isometrically for some compact metric space Y.*

We now give a few conditions on how to obtain RCD spaces.

Remark 2.2.14. Let $\alpha, \beta \in (0, +\infty)$. If $(X, \mathsf{d}, \mathfrak{m})$ is $\mathsf{RCD}(K, N)$ then $(X, \alpha\mathsf{d}, \beta\mathfrak{m})$ is $\mathsf{RCD}(K/\alpha^2, N)$.

Given a smooth manifold (M, g) we denote by d_g and $\mathfrak{m}_g$ respectively the distance induced by g and the volume measure. If the boundary ∂M is non-empty, we denote by $\mathrm{I\!I}^M_{\partial M}$ its second fundamental form, with the convention that $\mathrm{I\!I}^M_{\partial M}(X, Y) := g(\nabla_X \nu, Y)$ and ν is the exterior unit normal. We say that the boundary is convex if the second fundamental form is non-negative. The next proposition can be obtained repeating an argument in [118, Theorem 2.4].

Proposition 2.2.15. *Let (M, g) be a smooth geodesically complete n-dimensional manifold with convex boundary and let $V \in C^\infty(M)$. Then $(M, \mathsf{d}_g, e^{-V}\mathfrak{m}_g)$ is an $\mathsf{RCD}(K, N)$ space for $N > n$ if and only if*

$$\mathsf{Ric} + \mathrm{Hess}_V - \frac{\nabla V \otimes \nabla V}{N - n} \geq K. \tag{2.7}$$

Proposition 2.2.16. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, N)$ space and $(X', \mathsf{d}', \mathfrak{m}')$ be an $\mathsf{RCD}(K', N')$ space for $K, K' \leq 0$ and let $\mathsf{d} \times \mathsf{d}'$ be their product distance. Then the product space*

$$(X \times X', \mathsf{d} \times \mathsf{d}', \mathfrak{m} \otimes \mathfrak{m}') = (X, \mathsf{d}, \mathfrak{m}) \otimes (X', \mathsf{d}', \mathfrak{m}')$$

is an $\mathsf{RCD}(K + K', N + N')$ space.

A key result in the theory of finite dimensional RCD spaces is the Splitting Theorem due to [96] (after [58] and [53]). We say that a metric space (X, d) contains a line if there exists an isometric map $\varphi : \mathbb{R} \to X$. We denote by d_e the Euclidean distance and by λ^k the k-dimensional Lebesgue measure.

Theorem 2.2.17 (Splitting Theorem)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(0, N)$ space which contains a line. Then, there exists an $\mathsf{RCD}(0, N-1)$ space $(Y, \mathsf{d}_y, \mathfrak{m}_y)$ such that*

$$(X, \mathsf{d}, \mathfrak{m}) = (Y, \mathsf{d}_y, \mathfrak{m}_y) \otimes (\mathbb{R}, \mathsf{d}_e, \lambda^1).$$

We then collect some results concerning the essential dimension of RCD spaces. The next result follows from [98, 150, 42]. Given an $\mathsf{RCD}(K, N)$ space $(X, \mathsf{d}, \mathfrak{m})$, for every $k \in \mathbb{N}$, we set $\mathrm{R}_k := \{x \in X : \mathrm{Tan}_x(X) = \{(\mathbb{R}^k, \mathsf{d}_e, \lambda^k)\}\}$.

Theorem 2.2.18 (Structure theory)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, N)$ space. There exists $k \in \mathbb{N} \cap [1, N]$, called essential dimension of X, such that for $\mathfrak{m}$-a.e. $x \in X$, we have $\mathrm{Tan}_x(X) = \{(\mathbb{R}^k, \mathsf{d}_e, c\lambda^k)\}$ for some constant $c > 0$. Any such point will be called a regular point for X. Moreover, it holds $\mathfrak{m} \ll \mathsf{H}^k \llcorner \mathrm{R}_k$.*

We now recall some facts about the localization technique for $\mathsf{RCD}(K, N)$ spaces, the main reference being [50]. We remark that these results hold for essentially non-branching $\mathsf{CD}(K, N)$ spaces, so that they hold, in particular, for $\mathsf{RCD}(K, N)$ spaces.

Definition 2.2.19 (Disintegration of a measure)**.** Let $(R, \mathcal{R}, \mathfrak{m})$ be a measure space and let Q be a set. Let $P : R \to Q$ be a map, and let $\mathcal{Q}$ and $\mathfrak{q}$ be respectively the push-forward σ-algebra, and the measure, induced on Q by P.

A disintegration of $\mathfrak{m}$ consistent with P is a map $f : \mathcal{R} \times Q \to \mathbb{R}_+$ such that

- $f(\cdot, q)$ is a measure on $(R, \mathcal{R})$ concentrated in $P^{-1}(q)$ for every $q \in Q$.
- $f(B, \cdot)$ is $\mathfrak{q}$-measurable for every $B \in \mathcal{R}$.
- For every $(B, C) \in \mathcal{R} \times \mathcal{Q}$, it holds $\mathfrak{m}(B \cap P^{-1}(C)) = \int_C f(B, q)\, d\mathfrak{q}(q)$.

In the next theorem, we use the notation from Definition 2.2.19. This formulation of the localization technique follows combining [50, 49], with the fact that $\mathsf{RCD}(K, N)$ spaces are non-branching due to [83].

Theorem 2.2.20 (Localization technique)**.** *Let $N > 1$. Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, N)$ space, let $C \subset X$ be a closed set, and let d_C be the distance function from C. Let $\{r_q\}_{q \in Q}$ be the collection of maximal geodesics realizing d_C in $X \setminus C$ minus their*

endpoints Σ. *Let* $P : X \setminus (C \cup \Sigma) \to Q$ *be the map associating to every point the corresponding geodesic in* Q. *Then,* $\mathfrak{m}(\Sigma) = 0$, *and* $\mathfrak{m} \llcorner (X \setminus C)$ *admits a disintegration* f *consistent with* P. *Moreover, for* q*-a.e.* $q \in Q$, *the measure* $f(\cdot, q)$ *is absolutely continuous w.r.t.* $\mathsf{H}^1 \llcorner r_q$, *and its density* h_q *satisfies in weak sense*

$$(h_q^{\frac{1}{N-1}})'' + \frac{K}{N-1} h_q^{\frac{1}{N-1}} \leq 0 \quad \text{on } r_q.$$

We now recall some facts about equivariant Gromov Hausdorff convergence and group actions on RCD spaces. Combining Gromov's Precompactness Theorem, the stability of the $\mathsf{RCD}(K, N)$ condition under pGH convergence [99], and a simple argument which relies on Theorem 2.1.23, one obtains the following.

Proposition 2.2.21. *Let* $(Y_i, \mathsf{d}_i, \mathfrak{m}_i, y_i)$ *be a sequence of normalized* $\mathsf{RCD}(K, N)$ *spaces, and let* G_i *be closed subgroups of* $\mathrm{Iso}(Y_i)$. *Then, up to passing to a subsequence, there exists an* $\mathsf{RCD}(K, N)$ *space* $(X, \mathsf{d}, \mathfrak{m}, x)$, *and a closed subgroup* $G < \mathrm{Iso}(X)$, *such that*

$$(Y_i, \mathsf{d}_i, \mathfrak{m}_i, G_i, y_i) \to (X, \mathsf{d}, \mathfrak{m}, G, x)$$

in equivariant pmGH sense.

In the next results, when we say that a subgroup $G < \mathrm{Iso}(X)$ is isomorphic to $\mathbb{Z}$ (or $\mathbb{R}$), we implicitly mean that G is equipped with the topology induced by $\mathrm{Iso}(X)$, that $\mathbb{Z}$ (or $\mathbb{R}$) has its natural topology, and that there exists an isomorphism of topological groups $\varphi : G \to \mathbb{Z}$ (or $\varphi : G \to \mathbb{R}$). We denote this by writing $G \cong \mathbb{Z}$ (or $G \cong \mathbb{R}$).

Lemma 2.2.22. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$. *Let* $\mathbb{R}^k \cong G < \mathrm{Iso}(X)$ *be a closed subgroup. Then, the orbit* $Gp \subset X$ *is homeomorphic to* $\mathbb{R}^k$.

Proof. Let $\varphi : \mathbb{R}^k \to G$ be the isomorphism of topological groups given by the statement. Since G acts freely, the map $\psi : G \to Gp \subset X$ is a homeomorphism, so that $\psi \circ \varphi : \mathbb{R}^k \to Gp$ is a homeomorphism as well. □

Lemma 2.2.23. *Let* $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ *be a sequence of normalized* $\mathsf{RCD}(K, N)$ *spaces. Let* $\mathbb{Z}^k \cong G_i < \mathrm{Iso}(X_i)$ *be closed subgroups with generators* $\{\gamma_{i,j}\}_{j=1}^k$ *such that* $\mathsf{d}_i(x_i, \gamma_{i,j} \cdot x_i) \to 0$ *as* $i \uparrow \infty$, *for every* $j \in \{1, \cdots, k\}$. *Then, up to passing to a subsequence, one has equivariant pmGH convergence*

$$(X_i, \mathsf{d}_i, \mathfrak{m}_i, G_i, x_i) \to (X, \mathsf{d}, \mathfrak{m}, G, x)$$

for a closed subgroup $G < \mathrm{Iso}(X)$ *such that* G *contains a closed subgroup isomorphic to* $\mathbb{R}^k$.

Proof. The proof is the same as [162, Lemma 3.1] using that $\mathrm{Iso}(X)$ is a Lie Group by [111] or [180]. □

The following theorem follows from the proof of [153, Theorem 1.3].

Theorem 2.2.24. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{RCD}(0,n)$ space such that $\mathrm{Iso}(X)$ contains a closed subgroup isomorphic to $\mathbb{Z}^k$ acting cocompactly. Then, X splits off a factor $\mathbb{R}^k$.*

The following lemma is well-known, and its proof is an adaptation of [165, Theorem 7.3.11 point 2].

Lemma 2.2.25. *Let (X, d) be a metric space. If $X \cong \mathbb{R}^k \times Y$ and Y contains no lines, then $\mathrm{Iso}(X) \cong \mathrm{Iso}(\mathbb{R}^k) \times \mathrm{Iso}(Y)$.*

Proof. Let $h \in \mathrm{Iso}(X)$ and $y \in Y$. Since Y contains no lines, it holds $h(\mathbb{R}^k \times \{y\}) = \mathbb{R}^k \times \{y_h\}$ for some $y_h \in Y$. Since $h \in \mathrm{Iso}(X)$, then the map $f_h : Y \to Y$ such that $f_h(y) = y_h$ belongs to $\mathrm{Iso}(Y)$.

Given $a \in \mathbb{R}^k$, $y_1, y_2 \in Y$, let $a_1, a_2 \in \mathbb{R}^k$ be such that $h((a, y_1)) = (a_1, f_h(y_1))$ and $h((a, y_2)) = (a_2, f_h(y_2))$. It holds that

$$\begin{aligned} |a_1 - a_2|^2 + |f_h(y_1) - f_h(y_2)|^2 &= \mathsf{d}^2\Big((a_1, f_h(y_1)), (a_2, f_h(y_2))\Big) \\ &= \mathsf{d}^2\Big((a, y_1), (a, y_2)\Big) = |y_1 - y_2|^2 = |f_h(y_1) - f_h(y_2)|^2. \end{aligned}$$

It follows that $a_1 = a_2$, proving that $h = (g_h, f_h)$, for some map $g_h : \mathbb{R}^k \to \mathbb{R}^k$. Since $h \in \mathrm{Iso}(X)$ and $f_h \in \mathrm{Iso}(Y)$, then $g_h \in \mathrm{Iso}(\mathbb{R}^k)$, proving that $\mathrm{Iso}(X) < \mathrm{Iso}(\mathbb{R}^k) \times \mathrm{Iso}(Y)$. The reverse inclusion is obvious. □

We conclude the section with Cheeger's Differentiation Theorem. The next definition is given in [52] for PI spaces.

Definition 2.2.26 (Differentiability)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K,N)$ space, and let $x \in X$ be a point with $\mathrm{Tan}_x(X) = \{\mathbb{R}^d\}$. A function $f \in \mathsf{Lip}(X)$ is differentiable in x if the following happens. Let $r_i \downarrow 0$, $\delta_i \downarrow 0$, and let

$$\psi_i : (\overline{B}^X_{r_i}(x), \mathsf{d}/r_i) \to (\overline{B}^{\mathbb{R}^d}_1(0), \mathsf{d}_{eu}), \qquad \varphi_i : (\overline{B}^{\mathbb{R}^d}_1(0), \mathsf{d}_{eu}) \to (\overline{B}^X_{r_i}(x), \mathsf{d}/r_i)$$

be δ_i-GH approximations. Then, up to passing to a subsequence, there exists a linear function $f_\infty : \mathbb{R}^d \to \mathbb{R}$ with $\mathsf{lip}(f_\infty) = \mathsf{lip}(f)(x)$, such that

$$\Big\| \frac{f - f(x)}{r_i} - f_\infty \circ \psi_i \Big\|_{\infty, \overline{B}^X_{r_i}(x)} + \Big\| \frac{f \circ \varphi_i - f(x)}{r_i} - f_\infty \Big\|_{\infty, \overline{B}^{\mathbb{R}^d}_1(0)} \to 0\,.$$

The next theorem follows from [52, Theorem 10.2].

Theorem 2.2.27 (Cheeger's Differentiation Theorem)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K,N)$ space, and let $f \in \mathsf{Lip}(X)$. Then, f is differentiable at $\mathfrak{m}$-a.e. $x \in X$.*

2.2.3 Non-collapsed $\mathsf{RCD}(K,N)$ spaces

We now recall some basic facts about non-collapsed RCD spaces, the main references being [80, 79]. This is only a partial account of this very rich theory, and mostly aims at recalling Proposition 2.2.31, which will be used later on in the thesis.

Definition 2.2.28 (Non-collapsed RCD spaces)**.** Let $K \in \mathbb{R}$ and $N \geq 1$. An $\mathsf{RCD}(K,N)$ space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is called *non-collapsed*, if $\mathfrak{m} = \mathsf{H}^N$.

It is a consequence of the structure theory of $\mathsf{RCD}(K,N)$ spaces that, if $(X,\mathsf{d},\mathsf{H}^N)$ is non-collapsed, then $N \in \mathbb{N}$. In particular, from now on in all statements about non-collapsed $\mathsf{RCD}(K,N)$ spaces we tacitly assume that N is an integer.

Theorem 2.2.29 (Non-collapsed and collapsed convergence)**.** *Let $K \in \mathbb{R}$, $n \in \mathbb{N}$, and let $(X_i,\mathsf{d}_i,\mathsf{H}^n_i,x_i)$ be pointed $\mathsf{RCD}(K,n)$ spaces converging to (X,d,x) in pGH convergence. Then exactly one of the following alternatives holds.*

1. *Either: $\limsup_{i\to+\infty}\mathsf{H}^n_i(B_1(x_i)) > 0$. In this case the $\limsup$ is a limit and the pGH convergence can be improved to pmGH convergence to $(X,\mathsf{d},\mathsf{H}^n,x)$.*
2. *Or: $\lim_{i\to+\infty}\mathsf{H}^n_i(B_1(x_i)) = 0$. In this case the Hausdorff dimension of X is bounded above by $n-1$.*

Theorem 2.2.30 (Tangent cones are metric cones)**.** *Let $(X,\mathsf{d},\mathsf{H}^N)$ be an $\mathsf{RCD}(K,N)$ space and let $x \in X$. Then every element of $\mathrm{Tan}_x(X)$ is a metric cone with vertex at the base point.*

Proposition 2.2.31. *Let $(X,\mathsf{d},\mathsf{H}^N,x)$ be a non-compact $\mathsf{RCD}(0,N)$ space with maximal volume growth, i.e.*

$$\lim_{r\to+\infty}\frac{\mathsf{H}^N(B_r(x))}{r^N} > 0. \tag{2.8}$$

Then, any tangent cone at infinity of X is a metric cone.

2.3 Simplicial volume

We collect some basic definitions and results on the simplicial volume of non-compact manifolds without boundary, mostly aiming at stating Theorem 2.3.12 below, which we use later in the thesis. In particular, we do not mention many important tools and facts related to the simplicial volume, like for example bounded cohomology. The key references for this section are [110, 140, 90]. All manifolds are assumed to be connected.

Definition 2.3.1 (Locally finite real fundamental class)**.** Let X be an oriented topological manifold without boundary. A locally finite singular real n-chain is a formal sum

$$c = \sum_{i \in I} a_i \sigma_i$$

such that

1. $a_i \in \mathbb{R}$,
2. $\sigma_i : \Delta^n \to X$ is a continuous map from the n-simplex Δ^n into X.
3. Every compact subset of X intersects the image of only finitely many simplices σ_j.

Locally finite singular real n-chains form a vector space, which we denote $C_n^{\mathrm{lf}}(X, \mathbb{R})$. Moreover, the usual boundary operator is well-defined on such chains and the resulting homology is denoted by $H_*^{\mathrm{lf}}(X, \mathbb{R})$. Then, if X is n-dimensional, it holds $H_n^{\mathrm{lf}}(X, \mathbb{R}) \cong \mathbb{R}$, and there exists a preferred generator (in the sense of [140, Theorem 5.4]), called the fundamental class, which we denote by $[X] \in H_n^{\mathrm{lf}}(X, \mathbb{R})$.

Definition 2.3.2 (Simplicial volume)**.** Let X be an oriented topological n-manifold without boundary, and let $c = \sum_{i \in I} a_i \sigma_i$ be a locally finite singular real n-chain. The ℓ^1-norm (strictly speaking, this is not a norm as it can be infinite) of c is defined as

$$|c|_1 = \sum_i |a_i| \in [0, \infty].$$

The simplicial volume of X, denoted $||X||$, is the infimum of the ℓ^1-norms of the n-chains representing the fundamental class of X, i.e.

$$||X|| := \inf\{|c|_1 : c \in C_n^{\mathrm{lf}}(X, \mathbb{R}) \text{ is a cycle and } [c] = [X] \text{ in } H_n^{\mathrm{lf}}(X, \mathbb{R})\}. \tag{2.9}$$

Next, we turn our attention to Gromov's vanishing criterion for the simplicial volume of compact manifolds, i.e. Theorem 2.3.7, taken from [110, p.41]. The criterion uses the notion of amenable group.

Definition 2.3.3 (Amenable group)**.** Let G be a group and let $\ell^\infty(G, \mathbb{R})$ be the vector space of bounded $\mathbb{R}$-valued functions on G. We say that G is *amenable* if there exists a linear map $\mathrm{m} : \ell^\infty(G, \mathbb{R}) \to \mathbb{R}$ with the following properties:

1. $\mathrm{m}(1) = 1$.

2. If $f \in \ell^\infty(G, \mathbb{R})$ satisfies $f \geq 0$ pointwise, then $\mathrm{m}(f) \geq 0$.

3. For all $g \in G$ and $f \in \ell^\infty(G, \mathbb{R})$, it holds $\mathrm{m}(f) = \mathrm{m}(g \cdot f)$, where $(g \cdot f)(h) := f(g^{-1} \cdot h)$ for every $h \in G$.

We refer to [141, Section 9] for some key properties of amenable groups. For our purposes, it will be sufficient to know that virtually nilpotent groups are amenable (which follows from [141, Definition 9.1.8] and the fact that nilpotent groups are solvable). We also mention that in the literature there is an alternative definition of amenability for locally compact topological groups [171, Definition 1.1.4]. This definition is equivalent to Definition 2.3.3 for discrete groups, but it is not equivalent for general topological groups, as shown in [171, Example 1.2.9]. In the rest of the thesis, whenever we say that a group is amenable, we mean that it satisfies the properties described in Definition 2.3.3.

Remark 2.3.4. It is now known that the fundamental group of a manifold with non-negative Ricci curvature is virtually nilpotent (of index depending only on the dimension) by [126]. Before [126], this fact was only known for manifolds with non-negative Ricci curvature and finitely generated fundamental group, using for example that such group must have polynomial growth by [148] and then applying the famous [106]. This was still enough to obtain that manifolds with non-negative Ricci curvature have amenable fundamental group (see, for instance, [141, Corollary 9.1.7]).

Definition 2.3.5 (Amenable subsets of a topological space)**.** Let X be a path-connected topological space, Y be a subset of X and $i : Y \to X$ be the inclusion map. We say that Y is amenable (in X) if for every path-connected component Y' of Y the image of $i_* : \pi_1(Y') \to \pi_1(X)$ is an amenable subgroup of $\pi_1(X)$.

Definition 2.3.6 (Covers and multiplicity)**.** Let X be a path-connected topological space, and let $\{U_i\}_{i \in I}$ be a cover of X, i.e. $U_i \subset X$ for every $i \in I$ and $X = \bigcup_{i \in I} U_i$. We say that the cover is open if U_i is open in X for every $i \in I$, and amenable if each U_i is amenable in X for every $i \in I$. The multiplicity of $\{U_i\}_{i \in I}$ is defined as

$$\begin{aligned}\mathrm{mult}\big(\{U_i\}_{i \in I}\big) := \sup\{n \in \mathbb{N} \,:\, \exists\, i_1, \dots, i_n \in I\,,\, i_h \neq i_k \\ \text{for } h \neq k,\, U_{i_1} \cap \dots \cap U_{i_n} \neq \emptyset\} \in \mathbb{N} \cup \{\infty\}\,.\end{aligned}$$

Theorem 2.3.7 (Vanishing criterion for compact manifolds)**.** *Let X be an oriented closed topological n-manifold. Suppose that X admits an open cover $\{U_i\}_i$ such that:*

1. *$U_i \subset X$ is amenable in X for every i;*

2. *the multiplicity of* $\{U_i\}$ *is at most* n.

Then, $||X|| = 0$.

Corollary 2.3.8. *If* X *is an oriented closed topological manifold with amenable fundamental group, then* $||X|| = 0$. *In particular, if* X *is a closed oriented smooth Riemannian manifold with non-negative Ricci curvature, then* $||X|| = 0$.

We now turn our attention to vanishing criteria for the simplicial volume of non-compact manifolds. First we recall the analog of Corollary 2.3.8 for non-compact manifolds (from [140, p.75]).

Theorem 2.3.9 (Simplicial volume of non-compact manifolds with amenable fundamental group)**.** *If* X *is an oriented topological manifold without boundary with amenable fundamental group, then* $||X|| \in \{0, \infty\}$. *In particular, if* X *is an oriented smooth Riemannian manifold with non-negative Ricci curvature, then* $||X|| \in \{0, +\infty\}$.

The first result relating the simplicial volume of non-compact manifolds and lower Ricci curvature bounds is the so-called Isolation Theorem from [110], which we stated in the Introduction as Theorem 1.3.13, and which we recall below.

Theorem 2.3.10 (Isolation Theorem)**.** *For every* $n \in \mathbb{N}$ *there exists* $\varepsilon(n) > 0$ *such that the following holds. Let* (M^n, g) *be a complete Riemannian manifold with* $\mathsf{Ric} \geq -(n-1)$. *If* $\mathrm{vol}(B_1(p)) \leq \varepsilon(n)$ *for every* $p \in M$, *then* $||M|| = 0$.

Finally, we turn our attention to the vanishing criterion that we rely on in Chapter 5, i.e. Theorem 2.3.12. This result was stated in [110, p.58] with a sketched proof, and proved in full detail in [90, Corollary 11]. We first recall a relevant definition.

Definition 2.3.11. A sequence of subsets $\{U_i\}_{i\in\mathbb{N}}$ of a path-connected non-compact topological space X is said to be amenable at infinity if there exists a sequence $\{W_i\}_{i\in\mathbb{N}}$ of open subsets of X such that the following conditions hold:

(i) each W_i is large, i.e. its complement in X is relatively compact;

(ii) the family $\{W_i\}_{i\in\mathbb{N}}$ is locally finite;

(iii) $U_i \subset W_i$ for every $i \in \mathbb{N}$;

(iv) there exists $\bar{i} \in \mathbb{N}$ such that U_i is an amenable subset of W_i for every $i \geq \bar{i}$.

Theorem 2.3.12 (Vanishing criterion for the simplicial volume of open manifolds)**.** *Let M^n be an oriented, open, and triangulable manifold of dimension n. Suppose that $\{U_i\}_{i\in\mathbb{N}}$ is an amenable open cover of M such that each U_i is relatively compact in M, the sequence $\{U_i\}_{i\in\mathbb{N}}$ is amenable at infinity and* $\operatorname{mult}\big(\{U_i\}_{i\in\mathbb{N}}\big) \leq n$. *Then* $\|M\| = 0$.

Chapter 3

Rigidity of disjoint minimal surfaces

The main result of this section is Theorem 1.3.4, which we restate below.

Theorem 1.3.4. *Let (M^3, g) be a manifold with non-negative Ricci curvature and disconnected mean-convex boundary. Then, $M \cong \Sigma \times [0, l]$ isometrically, for some $l > 0$, and some surface Σ.*

As explained in Section 1.3.1.1, a key step to prove this result is to prove Theorem 1.3.6, which we also restate below.

Theorem 1.3.6. *Let (M^n, g) be a Riemannian manifold with non-negative Ricci curvature and disconnected mean-convex boundary. If one boundary component is totally geodesic, parabolic, and has non-negative Ricci curvature, then $M \cong \Sigma \times [0, l]$ isometrically for some $l > 0$ and some $(n-1)$-manifold Σ.*

We prove Theorem 1.3.4 assuming Theorem 1.3.6 in section 3.1. Then, in Section 3.2, we prove Theorem 1.3.6.

We now recall some notational conventions. In a Riemannian manifold (M^n, g) with boundary ∂M and external unit normal ν, we denote by $\mathrm{II}^M_{\partial M}$ the second fundamental form of the boundary, with the convention that $\mathrm{II}^M_{\partial M}(X, Y) := g(\nabla_X \nu, Y)$. We set $H^M_{\partial M} := (\mathrm{tr}\mathrm{II}^M_{\partial M})/(n-1)$ to be the associated mean curvature (the superscript M will be omitted if there is no ambiguity). We say that a manifold with boundary (M, g) is convex (resp. mean-convex) if its boundary satisfies $\mathrm{II}_{\partial M} \geq 0$ (resp. $H_{\partial M} \geq 0$). Similarly, a connected component S of ∂M is said to be convex (resp. mean-convex) if it satisfies $\mathrm{II}_S \geq 0$ (resp. $H_S \geq 0$).

3.1 Proof of Theorem 1.3.4 assuming Theorem 1.3.6

The proof of Theorem 1.3.4 is very simple, if, in addition to Theorem 1.3.6, one assumes the validity of the following two Lemmas. To different degrees, these are well known to experts, and their proofs are given later in the section. Before stating the lemmas, we recall the definition of perimeter of a set. Let (M, g) be a Riemannian manifold, and let $E \subset M$ be a Borel set. Given an open set $A \subset M$, the perimeter of E relative to A is defined as

$$P(E, A) := \inf \left\{ \liminf_{k\to\infty} \int_A |\nabla f_k| \, \mathrm{dVol} \ : \ f_k \in C^\infty(A), \ f_k \to \chi_E \ \text{ in } \mathsf{L}^1_{\mathrm{loc}}(A) \right\}.$$

Lemma 3.1.1. *Let (M^n, g) be a smooth Riemannian manifold, let $E \subset M$ be a smooth open set with mean-convex boundary, and let $x \in \partial E$. Then there exists a ball B centered in x such that, for every $E \subset E' \subset M$ with $E' \Delta E \subset\subset B$, it holds that $P(E, B) \leq P(E', B)$.*

Lemma 3.1.2. *Let (M^n, g) be a Riemannian manifold with $n \leq 7$ and mean-convex disconnected boundary ∂M, and let $\Sigma_1 \subset \partial M$ be a connected component. Then, at least one of the following two assertions holds.*

1. *One connected component $\Sigma \subset \partial M$ of ∂M is minimal and stable.*
2. *There exists a smooth open set $E \subset M$ such that ∂E is a non-empty union of stable minimal hypersurfaces, $\partial E \cap \partial M = \emptyset$, and $E \supset \Sigma_1$.*

We now prove Theorem 1.3.4 assuming the Lemmas above and Theorem 1.3.6.

Proof of Theorem 1.3.4. If one boundary component Σ of ∂M is minimal and stable, then it is totally geodesic and has non-negative sectional curvature by [174]. Hence, Σ is parabolic by Bishop-Gromov's inequality, so that the statement follows by Theorem 1.3.6. If no connected component of ∂M is minimal and stable, we consider the set $E \subset M$ given by Lemma 3.1.2 relative to a connected component Σ of ∂M. As before, each connected component of ∂E is parabolic and it has non-negative sectional curvature by [174]. By Theorem 1.3.6, it follows that $\overline{E} = \Sigma \times [0, l]$ isometrically for some $l > 0$. Applying the same argument to $M \setminus E$, we conclude. □

We now turn our attention to Lemma 3.1.1. It is well known that minimal hypersurfaces locally minimize the area (see [136]). Lemma 3.1.1 can be viewed as an analogue of this fact for sets with mean-convex boundary, and its proof follows by adapting the argument in [136]. We include the proof for completeness.

Proof of Lemma 3.1.1. Let B be a small ball of M centered in x. Let $C \subset \partial E$ be an $(n-2)$-dimensional submanifold of ∂E containing x, and let $P_0 : \partial E \cap B \to C$ be the closest point projection inside M restricted to ∂E. Observe that if B is small enough, this map is well defined and smooth. Using the smooth coarea formula (see [51, Exercise III.12 point c]) w.r.t. $P_0 : \partial E \cap B \to C$, we can rewrite the area of $\partial E \cap B$ as

$$\mathsf{H}^{n-1}(\partial E \cap B) = \int_C \int_{B \cap \partial E_c} 1/J_{n-2}(P_0)\, \mathrm{d}\mathsf{H}^1\, \mathrm{d}\mathsf{H}^{n-2}(c), \tag{3.1}$$

where $\partial E_c := P_0^{-1}(c) \subset \partial E \cap B$ and $J_{n-2}(P_0)$ is the appropriate coarea factor, i.e. for every $x \in \partial E \cap B$

$$J_{n-2}(P_0)(x) := |\det(d_x P_{0|\mathrm{Ker}(d_x P_0)^{\perp, T_x \partial E}})|,$$

where $\mathrm{Ker}(d_x P_0)^{\perp, T_x \partial E}$ denotes the orthogonal complement of $\mathrm{Ker}(d_x P_0)$ in $T_x \partial E$. Let now $P_1 : B \to \partial E$ be the natural nearest point projection in M. Modulo shrinking B, also P_1 is well defined and smooth. Set now $P := P_0 \circ P_1 : B \to C$ and, given a point $y \in \partial E \cap B$, consider the coarea factor $J_{n-2}(P)(y)$. We next relate $J_{n-2}(P)(y)$ to $J_{n-2}(P_0)(y)$. Observe that, using the identification $T_y \partial E \subset T_y M$, it holds

$$\mathrm{Ker}(d_y P) = \mathrm{Ker}(d_y P_0) \oplus T_y \partial E^{\perp, T_y M},$$

so that $\mathrm{Ker}(d_y P)^{\perp, T_y M} = \mathrm{Ker}(d_y P_0)^{\perp, T_y \partial E}$. Moreover, by the chain rule and Binet's Formula, it holds

$$\begin{aligned} J_{n-2}(P)(y) &:= |\det(d_y P_{|\mathrm{Ker}(d_y P)^{\perp, T_y M}})| \\ &= |\det(d_y P_{1|\mathrm{Ker}(d_y P_0)^{\perp, T_y \partial E}})||\det(d_y P_{0|\mathrm{Ker}(d_y P_0)^{\perp, T_y \partial E}})| \\ &= J_{n-2}(P_0)(y), \end{aligned}$$

where the last identity follows from the fact that $\mathrm{Ker}(d_y P_0)^{\perp, T_y \partial E} \subset T_y \partial E$. Hence, we can rewrite (3.1) as

$$\begin{aligned} \mathsf{H}^{n-1}(\partial E \cap B) &= \int_C \int_{B \cap \partial E_c} 1/J_{n-2}(P)\, \mathrm{d}\mathsf{H}^1\, \mathrm{d}\mathsf{H}^{n-2}(c) \\ &= \int_C \mathrm{length}_c(\partial E_c \cap B)\, \mathrm{d}\mathsf{H}^{n-2}(c), \end{aligned} \tag{3.2}$$

where length_c denotes the length of a curve in the smooth surface $\{P = c\} \subset B$ with conformally rescaled metric $J_{n-2}(P)^{-2} g$.

Observe now that since every point in the image of P is a regular value, then ∂E_c is a smooth curve, and it is equal to the boundary of

$$E_c := E \cap \{P = c\},$$

inside the smooth surface $\{P = c\}$.

Key Step: E_c is convex in $\{P = c\}$, endowed with the metric $J_{n-2}(P)^{-2}g$. Assume by contradiction that this is not the case, so that there exists $c_0 \in C$ such that ∂E_{c_0} has a point of strictly negative (mean) curvature in $\{P = c_0\}$ with the metric $J_{n-2}(P)^{-2}g$. By continuity, there is a neighbourhood $U \subset B$ of x_0 in M, such that every point of $\partial E_c \cap U$ for some $c \in C$ has strictly negative mean curvature in the corresponding rescaled metric of $\{P = c\}$. Let now $X \in TU \subset TM$ be a smooth vector field such that:

1. $X_{|\partial E \cap U}$ is non-vanishing and outer pointing;
2. if $p \in \partial E_c \cap U$ for some $c \in C$, then $X(p) \in T_p\{P = c\} \subset T_pM$.

Since $\{P = c\}$ intersects ∂E orthogonally in M, a vector field with these properties can be obtained, modulo restricting U, by considering an extension of the outer unit normal of E in M. Consider now a non-negative function $\varphi \in C^\infty_c(\partial E \cap U)$ which is equal to 1 in a neighbourhood of x_0, and consider for every $t > 0$ small enough the flow $\varphi_t(\cdot) : \partial E \to M$ w.r.t. φX. Set $\Sigma^t := \varphi_t(\partial E)$, and observe that these only differ from ∂E in a compact set of U, and converge smoothly to ∂E as $t \to 0$. Moreover, setting $\Sigma^t_c := \Sigma^t \cap \{P = c\}$, one has that $\varphi_t(\partial E_c) = \Sigma^t_c$ by the properties of X.

By the first variation of the area w.r.t. compactly supported variations (see [137, Chapter 1]), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}^{n-1}(\Sigma^t \cap U)_{|t=0} = \int_{\partial E} \varphi H_{\partial E} \cdot X \, \mathrm{d}\mathsf{H}^{n-1} \geq 0, \tag{3.3}$$

where $H_{\partial E}$ is the mean curvature vector, i.e. a normal vector to ∂E whose modulus is the modulus of the mean curvature of ∂E in E (with the understanding that this vector points outwards if the mean curvature is non-negative, and inwards otherwise).

At the same time, using the coarea formula as in the first part of the proof, and denoting by $\mathrm{length}^t_c(\Sigma^t_c \cap U)$ the length of the curve $\Sigma^t_c \cap U$ with respect to the conformally rescaled metric $J^{-2}_{n-2}(P_{|\Sigma^t})g$ on Σ^t, one has

$$\begin{aligned}\mathsf{H}^{n-1}(\Sigma^t \cap U) &= \int_C \mathrm{length}^t_c(\Sigma^t_c \cap U) \, \mathrm{d}\mathsf{H}^{n-2}(c) \\ &= \int_C \mathrm{length}^t_c(\varphi_t(\partial E_c \cap U)) \, \mathrm{d}\mathsf{H}^{n-2}(c).\end{aligned}$$

We now claim that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{length}^t_c(\varphi_t(\partial E_c \cap U))_{|t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{length}_c(\varphi_t(\partial E_c \cap U))_{|t=0}. \tag{3.4}$$

Assume for the moment that the claim is true. In this case, passing the derivatives under the integral sign, using the first variation of the area in each slice $P = c$, and denoting by $H^c_{\partial E_c}$ the mean curvature of ∂E_c in $\{P = c\}$ with the conformally rescaled metric, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}^{n-1}(\Sigma^t \cap U)_{|t=0} &= \int_C \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{length}_c(\varphi_t(\partial E_c \cap U))_{|t=0}\, \mathrm{d}\mathsf{H}^{n-2}(c)\\ &= \int_C \int_{\partial E_c} \varphi(H^c_{\partial E_c} \cdot X)\, J^{-1}_{n-2}(P)\mathrm{d}\mathsf{H}^1\mathrm{d}\mathsf{H}^{n-2}(c) < 0.\end{aligned}$$

The final strict inequality comes from our mean curvature assumptions on $\partial E_c \cap U$ inside $\{P = c\}$ with the rescaled metric, giving the desired contradiction.

To conclude the proof of the key step, we need to prove (3.4). Observe that as $t \to 0$, we have that $J_{n-2}(P_{|\Sigma^t}) \to J_{n-2}(P_{|\partial E})$ smoothly (in the natural sense, pulling back everything to ∂E via φ_t), and that $J_{n-2}(P_{|\partial E})$ is strictly positive. Hence, it suffices to show that

$$\sup_{U\cap\Sigma^t} \frac{|J_{n-2}(P_{|\Sigma^t}) - J_{n-2}(P)|}{t} \to 0 \quad \text{as } t \to 0. \tag{3.5}$$

For every $x \in \Sigma^t_c$, since $\mathrm{Ker}(d_x P) = T_x\{P = c\}$, we have

$$\mathrm{Ker}(d_x P) \cap T_x\Sigma^t = T_x\Sigma^t_c, \tag{3.6}$$

so that

$$(\mathrm{Ker}(d_x P) \cap T_x\Sigma^t)^{\perp, T_x\Sigma^t} = T_x\Sigma^{t\,\perp, T\Sigma^t}_c. \tag{3.7}$$

Denote by $\Pi_{t,x} : T_x\Sigma^{t\,\perp, T\Sigma^t}_c \to \mathrm{Ker}(d_x P)^{\perp, T_x M}$ the restriction of the orthogonal projection inside $T_x M$. It holds,

$$\begin{aligned}J_{n-2}(P_{|\Sigma^t})(x) &= |\det(d_x P_{|(\mathrm{Ker}(d_x P)\cap T_x\Sigma^t)^{\perp, T_x\Sigma^t}})|\\ &= |\det(d_x P_{|\mathrm{Ker}(d_x P)^{\perp, T_x M}} \circ \Pi_{t,x})|\\ &= J_{n-2}(P)(x)|\det(\Pi_{t,x})|.\end{aligned}$$

Hence, it suffices to show that

$$\sup_{x\in\Sigma^t\cap U} \frac{\big||\det(\Pi_{t,x})| - 1\big|}{t} \to 0 \quad \text{as } t \to 0. \tag{3.8}$$

This follows because for every $y \in \partial E \cap U$, setting $y_t := \varphi_t(y)$, the function $t \to \det(\Pi_{t,y_t})$ is smooth in $[0, \varepsilon]$ for some $\varepsilon > 0$, and it is a product of the cosines of the angles formed by (an orthonormal basis of) $T_{y_t}\Sigma^{t\,\perp, T\Sigma^t}_c$ with $\mathrm{Ker}(d_{y_t} P)^{\perp, T_{y_t} M}$. These

angles vary smoothly, and they are all equal to 0 at $t = 0$. Expanding the cosine, we obtain that $\det(\Pi_{t,y_t})$ has vanishing first derivative in 0, which then implies

$$\frac{\big||\det(\Pi_{t,y_t})| - 1\big|}{t} \to 0 \quad \text{for every } y \in \partial E \cap U \text{ as } t \to 0. \tag{3.9}$$

Moreover, there is a compact set $K \subset \partial E \cap U$ such that $y_t = \varphi_t(y) = y$ for every $y \in \partial E \cap U \setminus K$, so that (3.8) amounts to showing that

$$\sup_{y \in K} \frac{\big||\det(\Pi_{t,y_t})| - 1\big|}{t} \to 0 \quad \text{as } t \to 0. \tag{3.10}$$

Observe now that the function $(y,t) \mapsto (\det(\Pi_{t,y_t}) - 1)/t$ defined on $(\partial E \cap U) \times [0, \varepsilon]$ for some $\varepsilon > 0$ is also smooth (because $(y,t) \mapsto \det(\Pi_{t,y_t})$ is smooth, and it is equal to 1 in $t = 0$). Then, (3.10) follows from (3.9) via a compactness argument. This concludes the proof of the key step.

We now complete the proof of the lemma. By the Key Step, each set $E_c \subset \{P = c\}$ has convex boundary. By a standard property of convex sets (which follows by combining [33] and [1, Lemma 1 Section 3]), for every $c \in C$ and $x \in \partial E_c$, there exists a neighbourhood U_x^c of x in $\{P = c\}$, such that any curve of $\{P = c\}$ which lies outside of E_c and connects two points of $\partial E_c \cap U_x^c$ is longer than the corresponding portion of ∂E_c with the same endpoints. The neighbourhoods $U_x^c \subset \{P = c\}$ vary continuously as c and x vary. Hence, having fixed $x \in \partial E$, there exists a neighbourhood U_x in M such that for every $c \in C$ and $y \in \partial E_c \cap U_x$ one can choose $U_y^c := U_x \cap \{P = c\}$.

Fix now a ball $B \subset U_x$ containing x and let $E' \subset M$ be a smooth set such that $E \subset E'$ and $E' \Delta E \subset\subset B$. Denoting $\partial E'_c := E' \cap \{P = c\}$, it holds

$$\begin{aligned} \mathsf{H}^{n-1}(\partial E' \cap B) &= \int_C \int_{\partial E'_c \cap B} J^{-1}_{n-2}(P_{|\partial E'}) \, \mathrm{d}\mathsf{H}^1 \mathrm{d}\mathsf{H}^{n-2}(c) \\ &\geq \int_C \mathrm{length}_c(\partial E'_c \cap B) \, \mathrm{d}\mathsf{H}^{n-2}(c), \end{aligned}$$

where the last inequality is justified by the fact that $J_{n-2}(P_{|\partial E'}) \leq J_{n-2}(P)$ immediately from the definition. Using that $\mathrm{length}_c(\partial E'_c \cap B) \geq \mathrm{length}_c(\partial E_c \cap B)$ and (3.2), we then deduce

$$\begin{aligned} \mathsf{H}^{n-1}(\partial E' \cap B) &\geq \int_C \mathrm{length}_c(\partial E'_c \cap B) \, \mathrm{d}\mathsf{H}^{n-2}(c) \\ &\geq \int_C \mathrm{length}_c(\partial E_c \cap B) \, \mathrm{d}\mathsf{H}^{n-2}(c) = \mathsf{H}^{n-1}(\partial E \cap B). \end{aligned}$$

In the case when $E' \subset M$ is not smooth, the statement follows by an approximation argument combined with the previous part of the proof. □

We now turn our attention to Lemma 3.1.2. The classical [147, lemma 4, section 11] states that a manifold with mean-convex disconnected boundary and dimension ≤ 7 contains a properly embedded stable minimal hypersurface. Lemma 3.1.2 is a reformulation of this result. The logic behind the duality in Lemma 3.1.2 is that if the minimal hypersurface given by Meeks-Schoen-Yau touches the boundary, then it coincides with a boundary component by the maximum principle. We give a proof using sets minimizing the perimeter, which does not formally invoke Meeks-Schoen-Yau's result, although the underlying idea is the same.

We recall a few facts about sets minimizing the perimeter in manifolds. In Euclidean space, the subject is by now classical, see for instance the monographs [102, 146]. When the ambient space is not $\mathbb{R}^n$, many properties have been obtained in the more general setting of metric measure spaces. An account of this theory can be found in [152].

A set $E \subset M$ is said to have locally finite perimeter if $P(E, B_r(x)) < +\infty$ for all $x \in M$ and $r > 0$. If E has locally finite perimeter, there exists a unique Radon measure μ such that $\mu(A) = P(E, A)$ if $A \subset M$ is open. This measure is denoted $P(E, \cdot)$.

A set of locally finite perimeter $E \subset M$ is *perimeter minimizing* in an open set $A \subset M$ if, for every $F \subset M$ such that $F \Delta E \subset\subset A$, it holds that $P(E, A) \leq P(F, A)$. The set $E \subset M$ is *locally perimeter minimizing* in A if for every $x \in A$ there exists $r_x > 0$ such that E is perimeter minimizing in $B_{r_x}(x)$.

If $E \subset M^n$ locally minimizes the perimeter in M, then it admits both an open and a closed representative, and these have the same topological boundary, which is a smooth hypersurface outside of a set $\Sigma \subset \partial E$ of Hausdorff dimension at most $n-8$. Whenever we refer to the boundary of a locally perimeter minimizing set, we mean the topological boundary of its open (or closed) representative.

Proof of Lemma 3.1.2. Set $\Sigma_2 := \partial M \setminus \Sigma_1$. Let $(\widetilde{M}, \widetilde{g})$ be an enlargement of M beyond its boundary i.e., an open manifold containing open sets $E_1, E_2 \subset \widetilde{M}$ with $\partial E_1 = \Sigma_1$, $\partial E_2 = \Sigma_2$, and $\widetilde{M} \setminus (E_1 \cup E_2) = M$ isometrically. Unless otherwise specified, in the rest of the proof, all balls and boundaries are taken in $\widetilde{M}$. Let $p \in M$, and for $i \in \mathbb{N}$ large enough consider the minimization problem

$$\min\{P(E, B_i(p)) : E \supset E_1,\ E \cap E_2 = \emptyset,\ E = E_1 \text{ in } B_i(p) \setminus B_{i-1}(p)\}. \tag{3.11}$$

For every $i \in \mathbb{N}$ sufficiently large, we denote by $E^i \subset \widetilde{M}$ the set realizing the minimum in the previous minimization problem. Such set exists by the direct method of calculus

of variations. We construct the set $E \subset M$ in several steps. We then show that either E satisfies item 2, or item 1 holds.

Step 1: The sequence E^i can be taken to be increasing.

Assume that the partial sequence $\{E^i\}_{i=1}^l$ is increasing. We need to show that E^{l+1} can be chosen so that $E^{l+1} \supset E^l$. Indeed, let $\widetilde{E}^{l+1}$ be a minimum of the variational problem (3.11), and set $E^{l+1} := \widetilde{E}^{l+1} \cup E^l$. The set E^{l+1} is trivially a competitor for the variational problem (3.11). Moreover,

$$\begin{aligned} P(E^{l+1}, B_{l+1}(p)) &= P(\widetilde{E}^{l+1} \cup E^l, B_{l+1}(p)) \\ &\leq P(\widetilde{E}^{l+1}, B_{l+1}(p)) + P(E^l, B_{l+1}(p)) - P(\widetilde{E}^{l+1} \cap E^l, B_{l+1}(p)). \end{aligned} \tag{3.12}$$

In addition, $P(E^l, B_l(p)) \leq P(\widetilde{E}^{l+1} \cap E^l, B_l(p))$, since $\widetilde{E}^{l+1} \cap E^l$ is a competitor for E^l in (3.11). Since $E^l = \widetilde{E}^{l+1} \cap E^l$ in $B_{l+1} \setminus B_{l-1}(p)$, we deduce $P(E^l, B_{l+1}(p)) \leq P(\widetilde{E}^{l+1} \cap E^l, B_{l+1}(p))$. Combining with (3.12), we obtain

$$P(E^{l+1}, B_{l+1}(p)) \leq P(\widetilde{E}^{l+1}, B_{l+1}(p)).$$

This concludes the proof of Step 1.

Step 2: There exists a locally finite covering with balls $\{B_\alpha\}_\alpha$ of $\overline{M} \subset \widetilde{M}$ such that, for every α, if i is large enough (depending on α), the sets $E^i \subset \widetilde{M}$ minimize the perimeter in the ball B_α.

It is sufficient to show that for every $x \in \overline{M} \subset \widetilde{M}$ there exists a ball B of $\widetilde{M}$ containing x such that the sets E^i are perimeter minimizing in B. If $x \in M \setminus \partial M$, one can take any ball centered in x which does not intersect ∂M. So, we consider the case $x \in \partial M$. We assume that $x \in \Sigma_2$, the case $x \in \Sigma_1$ being analogous. Since ∂M is mean-convex, by Lemma 3.1.1, there exists a ball $B \subset \widetilde{M}$ centered in x such that for every $M \subset C \subset \widetilde{M}$ with $C \Delta M \subset\subset B$, it holds $P(M, B) \leq P(C, B)$. This ball can be taken small so that it is disjoint from Σ_1. We show that each E^i is perimeter minimizing in the ball $B \subset \widetilde{M}$. Let $C \subset \widetilde{M}$ be such that $C \Delta E^i \subset\subset B$. Since E^i minimizes the variational problem (3.11), it holds

$$P(E^i, B) \leq P(C \cap M, B). \tag{3.13}$$

By our choice of B, it holds $P(M, B) \leq P(C \cup M, B)$. Combining with (3.13), we deduce

$$P(E^i, B) \leq P(C \cap M, B) \leq P(C, B) + P(M, B) - P(C \cup M, B) \leq P(C, B),$$

proving our claim.

Hence, up to passing to a subsequence, there exists $E \subset \widetilde{M}$ locally minimizing the perimeter such that $E^i \to E$ in $\mathsf{L}^1_{loc}(\widetilde{M})$ (see [102, Theorem 1.19 and Lemma 9.1]). Since each E^i minimizes the perimeter in $B_{i-1}(p) \cap (M \setminus \partial M)$ by construction, E also minimizes the perimeter in $M \setminus \partial M$. By construction, it also holds $E \supset E_1$ and $E \cap E_2 = \emptyset$.

Step 3: Let $U \subset \widetilde{M}$ be open and bounded. If $C \subset \widetilde{M}$ is such that $E \supset C \supset E_1$, and $C \Delta E \subset\subset U$, then $P(E, U) \leq P(C, U)$.

The claim follows combining Step 1 and a standard argument (see, for instance, [82, Proposition 1.3]). We report the argument for the sake of completeness. Consider $C \subset \widetilde{M}$ as in the statement of step 3. The sets $C \cap E^i$ satisfy $P(E^i, U) \leq P(C \cap E^i, U)$ if i is large enough. Hence,

$$P(E^i \cup C, U) \leq P(E^i, U) + P(C, U) - P(C \cap E^i, U) \leq P(C, U).$$

The sets $C \cup E^i$ converge in $\mathsf{L}^1(U)$ to E by Step 1. Hence, by lower semicontinuity, it holds $P(E, U) \leq P(C, U)$, concluding the proof of Step 3.

We now conclude the proof of the lemma. First of all, E is a smooth set by Step 2 and the assumption on the dimension. Assume that $x \in \partial E \cap \partial M$. We wish to show that in this case item 1 is satisfied. Since ∂M is mean-convex and ∂E locally lies on the interior side of ∂M, by the maximum principle, we infer that $\partial E = \partial M$ in a neighbourhood of x. By a standard argument, it follows that ∂E contains a connected component $\Sigma \subset \partial M$ of ∂M. Since E is locally perimeter minimizing in $\widetilde{M}$, such connected component $\Sigma \subset \partial E$ has vanishing mean curvature. By Step 3, it follows that Σ is a stable minimal hypersurface.

Assume now that $\partial E \cap \partial M = \emptyset$. We wish to show that E satisfies the conditions in item 2. Indeed, E is locally perimeter minimizing, so that it has an open representative. By construction, E minimizes the perimeter in $M \setminus \partial M$. Since $E \supset E_1$ and $E \cap E_2 = \emptyset$, the boundary ∂E in $\widetilde{M}$ is nonempty, and since $\partial E \cap \partial M = \emptyset$ by assumption, we have that the boundary of E in M is non-empty as well. □

3.2 Proof of Theorem 1.3.6

We prove a slightly more general version of Theorem 1.3.6, where instead of assuming that one boundary component is totally geodesic, we only assume that it is convex. This is Theorem 3.2.1 below.

Theorem 3.2.1. *Let (M^n, g) be a Riemannian manifold with $\mathsf{Ric}_M \geq 0$ and disconnected mean-convex boundary. Assume that one connected component $S \subset \partial M$ is parabolic, convex, and satisfies $\mathsf{Ric}_S \geq 0$.*

Then, there exists $l > 0$ such that M is isometric to $S \times [0, l]$ with the product metric.

The proof of Theorem 3.2.1 uses the following two Lemmas, which we will prove at the end of the section. Lemma 3.2.2 is a standard result, while Lemma 3.2.3 is a variant of the usual Laplacian bounds for the distance function from a mean-convex set in a manifold with non-negative Ricci curvature [127].

Lemma 3.2.2. *Let (M^n, g) be a Riemannian manifold with $\mathsf{Ric}_M \geq 0$ and mean-convex boundary. Let S_1 be a union of connected components of ∂M so that $\partial M \setminus S_1 \neq \emptyset$ and let $S_2 := \partial M \setminus S_1$. Assume that $\mathsf{d}_{S_1} + \mathsf{d}_{S_2}$ is constant. Then, there exists $l > 0$ such that M is isometric to $S_1 \times [0, l]$ with the product metric.*

Lemma 3.2.3. *Let (M^n, g) be a Riemannian manifold with $\mathsf{Ric}_M \geq 0$. Let S_1 be a union of connected components of ∂M. Assume that $\partial M \setminus S_1 \neq \emptyset$ and set $S_2 := \partial M \setminus S_1$. Let $(N, \widetilde{g})$ be a second manifold with $\mathsf{Ric}_N \geq 0$, and whose boundary ∂N is isometric to S_1. Let (X, d) be the metric space obtained by gluing M and N along the two isometric copies of S_1, and let d_{S_2} be the distance from S_2 in X. Assume that $\mathrm{I\!I}^M_{S_1} + \mathrm{I\!I}^N_{\partial N} \geq 0$. If $H_{S_2} \geq 0$, then $\Delta \mathsf{d}_{S_2} \leq 0$ in $X \setminus S_2$ in distributional sense.*

We now prove Theorem 3.2.1 assuming Lemmas 3.2.2 and 3.2.3.

Proof of Theorem 3.2.1 assuming Lemmas 3.2.2 and 3.2.3. Consider the manifold $S \times (0, +\infty)$ equipped with the product metric $\widetilde{g}$. We denote by Vol the Riemannian volume in $S \times (0, +\infty)$ w.r.t. the metric $\widetilde{g}$ and by Vol_S the Riemannian volume on S w.r.t. the metric g_S, which is the restriction of g to S.

Let $\mathsf{d}_{S\times\{0\}}$ be the distance from $S \times \{0\}$ in $S \times (0, +\infty)$ and let d_S be the distance from S in M. By standard computations (see [165]), it holds

$$\Delta \mathsf{d}_{S\times\{0\}} = 0, \quad \text{on } S \times (0, +\infty).$$

Similarly, the second fundamental form of $S \times \{0\}$ in $S \times (0, +\infty)$ satisfies $\mathrm{I\!I}_{S\times\{0\}} = 0$. Moreover, $\mathsf{Ric}_S \geq 0$ together with Gauss' equation implies that $\mathsf{Ric}_{S\times(0,+\infty)} \geq 0$.

Consider the metric space obtained by gluing M and $S \times (0, +\infty)$ along the two boundaries isometric to S. On this metric space, we consider the distance function from $S' := \partial M \setminus S$ and we denote it by $\mathsf{d}_{S'}$. In particular, this allows us to define $\mathsf{d}_{S'}$

on $S \times (0, +\infty)$ (seen as a subset of the previously defined metric space). We claim that the function $\overline{\mathsf{d}} := \mathsf{d}_{S'} - \mathsf{d}_{S\times\{0\}}$ is constant on $S \times (0, +\infty)$.

Let us first show how this is enough to conclude the proof. If $\overline{\mathsf{d}} := \mathsf{d}_{S'} - \mathsf{d}_{S\times\{0\}}$ is constant on $S \times (0, +\infty)$, then $\mathsf{d}_{S'}$ is constant when restricted to S, which in turn implies that there exists a geodesic $\gamma \subset M$ joining S and S' whose length is equal to the distance between S and S'. By triangle inequality, $\mathsf{d}_S + \mathsf{d}_{S'} : M \to \mathbb{R}$ attains its minimum on any point of γ, so that, being superharmonic by Lemma 3.2.3, it is constant by the Maximum Principle. The conclusion then follows by Lemma 3.2.2.

Hence, we only need to prove that $\overline{\mathsf{d}} := \mathsf{d}_{S'} - \mathsf{d}_{S\times\{0\}}$ is constant on $S \times (0, +\infty)$. By Lemma 3.2.3, it holds $\Delta \mathsf{d}_{S'} \leq 0$ in distributional sense on $S \times (0, +\infty)$. Hence,

$$\Delta(\mathsf{d}_{S'} - \mathsf{d}_{S\times\{0\}}) \leq 0 \quad \text{on } S \times (0, +\infty) \text{ in distributional sense.} \tag{3.14}$$

Consider the weight function $V \in \mathsf{Lip}(S \times (0, +\infty))$ given by $V = \mathsf{d}_{S'}$. We denote by $\Delta_w \overline{\mathsf{d}}$ the weighted Laplacian of $\overline{\mathsf{d}}$ w.r.t. V, i.e. the measure $\Delta_w \overline{\mathsf{d}} := \Delta \overline{\mathsf{d}} - \nabla V \cdot \nabla \overline{\mathsf{d}}$. Thanks to (3.14) and the definition of V, it holds

$$\Delta_w \overline{\mathsf{d}} \leq 0 \quad \text{on } S \times (0, +\infty) \text{ in distributional sense.} \tag{3.15}$$

Set $v := (1 + \overline{\mathsf{d}})^{-1}$ and note that, by the chain rule, it holds in distributional sense on $S \times (0, +\infty)$

$$\Delta_w v = -(1 + \overline{\mathsf{d}})^{-2} \Delta_w \overline{\mathsf{d}} + 2(1 + \overline{\mathsf{d}})^{-3} |\nabla \overline{\mathsf{d}}|^2 \geq 0. \tag{3.16}$$

Let $\varphi \in \mathsf{Lip}_c(S \times [0, +\infty))$ be a positive function. Consider the functions $\varphi_\varepsilon \in \mathsf{Lip}(S \times [0, +\infty))$ defined by $\varphi_\varepsilon(x, t) := (1 - \varepsilon^{-1} t) \vee 0$ (where $\vee$ denotes the maximum). Integrating by parts, it holds

$$\begin{aligned} &\int_{S\times(0,+\infty)} (\nabla v \cdot \nabla \varphi)\, e^{-V} \mathsf{dVol} \\ &= \int_{S\times(0,+\infty)} [\nabla v \cdot \nabla(\varphi\varphi_\varepsilon)]\, e^{-V} \mathsf{dVol} + \int_{S\times(0,+\infty)} [\nabla v \cdot \nabla(\varphi(1 - \varphi_\varepsilon))]\, e^{-V} \mathsf{dVol} \\ &= \int_{S\times(0,+\infty)} [\nabla v \cdot (\varphi \nabla \varphi_\varepsilon + \varphi_\varepsilon \nabla \varphi)]\, e^{-V} \mathsf{dVol} - \int_{S\times(0,+\infty)} \varphi(1 - \varphi_\varepsilon)\, e^{-V} \mathrm{d}\Delta_w v. \end{aligned} \tag{3.17}$$

Observe now that $\nabla \varphi_\varepsilon = -\varepsilon^{-1} \nabla \mathsf{d}_{S\times\{0\}} 1_{[0,\varepsilon]}$, so that

$$\nabla v \cdot \nabla \varphi_\varepsilon = \varepsilon^{-1} 1_{[0,\varepsilon]} (1 + \overline{\mathsf{d}})^{-2} \nabla \overline{\mathsf{d}} \cdot \nabla \mathsf{d}_{S\times\{0\}} \leq 0. \tag{3.18}$$

Hence, passing to the limit in (3.17) as $\varepsilon \to 0$ and taking into account (3.16), it holds

$$\int_{S\times(0,+\infty)} (\nabla v \cdot \nabla \varphi)\, e^{-V} \mathsf{dVol} \leq 0. \tag{3.19}$$

Let $\psi \in \mathsf{Lip}_c(S \times [0,+\infty))$ be a positive function. We repeat a computation from [190] (see also [137, Lemma 7.1]). By (3.19), it holds

$$\begin{aligned} 0 &\geq \int_{S\times(0,+\infty)} [\nabla(\psi^2 v)\cdot\nabla v]\, e^{-V}\mathrm{d}\mathsf{Vol} \\ &= 2\int_{S\times(0,+\infty)} \psi v(\nabla\psi\cdot\nabla v)\, e^{-V}\mathrm{d}\mathsf{Vol} + \int_{S\times(0,+\infty)} \psi^2|\nabla v|^2\, e^{-V}\mathrm{d}\mathsf{Vol}, \end{aligned}$$

so that

$$\int_{S\times(0,+\infty)} \psi^2|\nabla v|^2\, e^{-V}\mathrm{d}\mathsf{Vol} \leq 2\int_{S\times(0,+\infty)} |\psi v\nabla\psi\cdot\nabla v|\, e^{-V}\mathrm{d}\mathsf{Vol}. \tag{3.20}$$

By Young's inequality, we then obtain

$$\begin{aligned} 2\int_{S\times(0,+\infty)} |\psi v\nabla\psi\cdot\nabla v|\, e^{-V}\mathrm{d}\mathsf{Vol} \leq \frac{1}{2}\int_{S\times(0,+\infty)} \psi^2|\nabla v|^2\, e^{-V}\mathrm{d}\mathsf{Vol} \\ + 8\int_{S\times(0,+\infty)} |\nabla\psi|^2 v^2\, e^{-V}\mathrm{d}\mathsf{Vol}, \end{aligned}$$

which, combining with (3.20), gives

$$\int_{S\times(0,+\infty)} \psi^2|\nabla v|^2\, e^{-V}\mathrm{d}\mathsf{Vol} \leq 16\int_{S\times(0,+\infty)} |\nabla\psi|^2 v^2\, e^{-V}\mathrm{d}\mathsf{Vol}.$$

Recalling the definition of v, this implies

$$\int_{S\times(0,+\infty)} \psi^2 \frac{|\nabla\bar{\mathsf{d}}|^2}{(1+\bar{\mathsf{d}})^2}\, e^{-V}\mathrm{d}\mathsf{Vol} \leq 16\int_{S\times(0,+\infty)} |\nabla\psi|^2\, e^{-V}\mathrm{d}\mathsf{Vol}. \tag{3.21}$$

Let $x \in S$ and $r > 0$. Let $\varphi_1 \in \mathsf{Lip}_c(S)$ be such that $\varphi_1 \equiv 1$ in $B_r(x) \subset S$ and $\varphi_2 \in \mathsf{Lip}_c([0,+\infty))$ be such that $\varphi_2 \equiv 1$ in $[0,r)$, both taking values in $[0,1]$. Let $\psi \in \mathsf{Lip}_c(S\times[0,+\infty))$ be defined by $\psi(y,t) := \varphi_1(y)\varphi_2(t)$. We denote by $\nabla^S\varphi_1$ the gradient of φ_1 w.r.t. the metric g_S on S. It holds

$$|\nabla\psi|^2(y,t) \leq |\nabla^S\varphi_1|^2(y) + \varphi_1^2(y)(\varphi_2'(t))^2.$$

Hence, with this choice of ψ, the r.h.s. in (3.21) is bounded from above by

$$\begin{aligned} 16\int_0^{+\infty} e^{-t}\,\mathrm{d}t \int_S |\nabla^S\varphi_1|^2\,\mathrm{d}\mathsf{Vol}_S \\ + 16\int_S \varphi_1^2\,\mathrm{d}\mathsf{Vol}_S \int_0^{+\infty} |\varphi_2'|^2 e^{-t}\,\mathrm{d}t. \end{aligned} \tag{3.22}$$

Taking an infimum over all the admissible $\varphi_2 \in \mathsf{Lip}_c([0,+\infty))$ such that $\varphi_2 \equiv 1$ on $[0,r)$, we deduce

$$\int_{B_r(x)\times[0,r]} \frac{|\nabla\bar{\mathsf{d}}|^2}{(1+\bar{\mathsf{d}})^2}\, e^{-V}\mathrm{d}\mathsf{Vol} \leq 16\int_0^{+\infty} e^{-t}\,\mathrm{d}t \int_S |\nabla^S\varphi_1|^2\,\mathrm{d}\mathsf{Vol}_S. \tag{3.23}$$

Since S is parabolic, taking the infimum of the r.h.s. over all the functions $\varphi_1 \in \mathsf{Lip}_c(S)$ such that $\varphi_1 \equiv 1$ on $B_r(x) \subset S$, we deduce $|\nabla \overline{\mathsf{d}}| \equiv 0$ on $B_r(x) \times [0, r]$. The constancy of $\overline{\mathsf{d}}$ follows from the arbitrariness of $r > 0$, concluding the proof. □

Lemma 3.2.2 is standard, and it follows from [67] and [128]. We sketch a proof for the sake of completeness.

Proof of Lemma 3.2.2. Since $\mathsf{d}_{S_1} + \mathsf{d}_{S_2} \equiv D$ is constant, every point of M lies on a unique geodesic realizing the distance from S_1 to S_2. In particular, d_{S_1} and d_{S_2} are both smooth and, being distances from mean-convex boundaries in a manifold with non-negative Ricci curvature, they satisfy

$$\Delta \mathsf{d}_{S_1} \leq 0 \quad \text{and} \quad \Delta \mathsf{d}_{S_2} \leq 0.$$

Hence, $\Delta \mathsf{d}_{S_1} = 0$. Using $t = \mathsf{d}_{S_1}$ as a coordinate on M, we can write $g = \mathrm{d}t^2 + g^{\perp}(t)$, where $g^{\perp}(t)$ is a metric on the t enlargement of S_1 in M.

Using Cauchy-Schwarz inequality first, and then plugging d_{S_1} into the Bochner Formula, we obtain

$$0 = \frac{(\Delta \mathsf{d}_{S_1})^2}{n-1} \leq |\operatorname{Hess}(\mathsf{d}_{S_1})|^2 = -\mathsf{Ric}_M(\nabla \mathsf{d}_{S_1}, \nabla \mathsf{d}_{S_1}) \leq 0.$$

In particular, all inequalities in the previous expression are equalities, forcing $\operatorname{Hess}(\mathsf{d}_{S_1}) = 0$ everywhere on M. Denoting by L_{∂_t} the Lie derivative in the direction of $\nabla \mathsf{d}_{S_1}$, the previous equality implies

$$L_{\partial_t} g^{\perp} = 0. \tag{3.24}$$

Consider now the map $\varphi : S_1 \times (0, D) \to M \setminus (S_1 \cup S_2)$ which sends (x, t) to the point at distance t from x along the flow line of $\nabla \mathsf{d}_{S_1}$. This map is bijective since every point of M lies on a unique geodesic realizing the distance from S_1 to S_2. Equipping $S_1 \times (0, D)$ with the product metric, we obtain that φ is an isometry by (3.24) and the definition of Lie derivative, concluding the proof. □

We now prove Lemma 3.2.3. Let us spend a few words on the proof. The glued space X is a manifold with a C^0 Riemannian metric, which induces a metric measure space structure $(X, \mathsf{d}, \mathsf{H}^n)$. Observe that $\Delta \mathsf{d}_{S_2} \leq 0$ near S_2 in X by a standard result on the Laplacian of distance functions from mean-convex sets in manifolds with non-negative Ricci curvature. Then, such Laplacian bound can be globalized to the whole X using a by now classical procedure from [49, 152, 131] (see also section 2.2.2). The idea is that, thanks to the assumption on the second fundamental forms of $S_1 \subset M$ and $\partial N \subset N$, the glued space X behaves locally like a non-branching CD space away

from S_2. Hence, its volume measure can be disintegrated along the transport rays of the distance function d_{S_2}, and the problem of globalizing the bound $\Delta \mathsf{d}_{S_2} \leq 0$ reduces to an ODE problem on such transport rays. To carry out this procedure, let us specify what we mean when we say that X behaves locally like a CD space away from S_2.

Recall that, given $N \in [1, +\infty)$ and $\mu = \rho \mathsf{H}^n \in \mathcal{P}_2(X)$, the Rényi entropy of μ w.r.t. H^n is defined as

$$\mathcal{E}_N(\mu|\mathsf{H}^n) := -\int_X \rho^{1-1/N} \, \mathrm{d}\mathsf{H}^n.$$

Comparing with the definition of Rényi entropy given in Section 2.2.2, here we also highlight the base measure H^n, as this will be convenient for the proof of Lemma 3.2.3. Given $K \in \mathbb{R}$, two measures $\mu_1 = \rho_1 \mathsf{H}^n, \mu_2 = \rho_2 \mathsf{H}^n \in \mathcal{P}_2(X)$, and an optimal plan Π between them, we set

$$T^{(t)}_{K,N}(\Pi|\mathsf{H}^n) := \int_{X\times X} [\tau^{1-t}_{K,N}(\mathsf{d}(x_1,x_2))\rho_1^{-1/N}(x_1) + \tau^{t}_{K,N}(\mathsf{d}(x_1,x_2))\rho_2^{-1/N}(x_2)] \, d\Pi(x_1,x_2),$$

where, for every $s \in [0,1]$, $\tau^s_{\cdot,\cdot}(\cdot)$ is the appropriate distortion coefficient (see Section 2.2.2). We say that X *satisfies the* $\mathrm{CD}(K,N)$ *condition in* $U \subset X$, if the following holds. For every pair $\mu_1 = \rho_1 \mathsf{H}^n, \mu_2 = \rho_2 \mathsf{H}^n \in \mathcal{P}_2(X)$ of measures supported in U, there exists a Wasserstein geodesic $\{\xi_t\}_{t\in[0,1]} \subset \mathcal{P}_2(X)$ from μ_1 to μ_2 and an optimal coupling Π of μ_1 and μ_2 such that, for every $t \in [0,1]$ and every $N' \geq N$, it holds

$$\mathcal{E}_{N'}(\xi_t|\mathsf{H}^n) \leq -T^{(t)}_{K,N'}(\Pi|\mathsf{H}^n).$$

Observe that this is just a repackaging of Definition 2.2.5, where we require that the convexity inequality holds for measures supported in a fixed set $U \subset X$. This condition differs from the $\mathsf{CD}_{loc}(K,N)$ condition that appears in the literature (see for instance [28, 47]), since we are not requiring that the convexity of the entropy holds in a neighbourhood of every point. We now prove Lemma 3.2.3

Proof of Lemma 3.2.3. From now on, we refer to S_1, S_2, M and N as subsets of X. Balls $B_r(x) \subset X$ are always defined w.r.t. the distance d on X. The space X admits a smooth structure such that the metric g^X on X defined by $g^X_{|M} = g$ and $g^X_{|N} = \widetilde{g}$ is locally Lipschitz continuous and it induces the distance d.

Step 1. We show that, for every $x \in X \setminus S_2$, there exists $r > 0$ such that X satisfies $\mathsf{CD}(0,n)$ in $B_r(x)$.

Suppose first that the point $x \in X \setminus S_2$ does not belong to S_1. In this case, there exists a convex ball $B_r(x)$ not intersecting $S_1 \cup S_2$. Consider two probability measures $\mu_1 = \rho_1 \mathsf{H}^n, \mu_2 = \rho_2 \mathsf{H}^n \in \mathcal{P}_2(B_r(x))$. Any Wasserstein geodesic $\{\xi_t\}_{t\in[0,1]} \subset \mathcal{P}_2(X)$ between μ_1 and μ_2 is concentrated on geodesics connecting points in $B_r(x)$. These geodesics are themselves contained in $B_r(x)$ by convexity, so that they lie in the region where X is a smooth manifold with non-negative Ricci. Hence, $\{\xi_t\}_{t\in[0,1]}$ satisfies the required convexity condition of the entropy by [181].

Suppose now that $x \in S_1$. If $R > 0$ is small enough, using the same arguments of [173] (here we use that $\mathrm{I\!I}^M_{S_1} + \mathrm{I\!I}^N_{\partial N} \geq 0$, see also [164]), we obtain the following. There exists a sequence of smooth Riemannian metrics g_k on $\overline{B}_R(x) \subset X$, converging uniformly to g^X on $\overline{B}_R(x)$, such that $\mathsf{Ric}_{(B_R(x),g_k)} \geq -1/k$. We set $r := R/8$. If k is large enough, geodesics in $(\overline{B}_R(x), g_k)$ connecting points in $B_r(x)$ are contained in $B_{R/2}(x)$. In particular, $(\overline{B}_R(x), g_k)$ satisfies $\mathsf{CD}(-1/k, n)$ in $B_r(x)$ by the argument used in the previous case. We now consider $(\overline{B}_R(x), g^X)$.

We denote by V and V_k the volume measures relative to g^X and g_k in $\overline{B}_R(x)$. Fix two probability measures $\mu_1, \mu_2 \in \mathcal{P}(B_r(x))$ that are absolutely continuous w.r.t. V (or, equivalently, V_k) and have compactly supported continuous densities μ_1/V and μ_2/V. Thanks to [186, Corollary 29.22], it is enough to verify the convexity property of the $\mathsf{CD}(0, n)$ condition under this extra continuity assumption.

For every k, let $\{\xi^k_t\}_{t\in[0,1]}$ be a Wasserstein geodesic between μ_1 and μ_2 in $(\overline{B}_R(x), g_k)$, and let Π_k be an optimal plan between the same measures such that, for every $t \in [0, 1]$ and every $N \geq n$, it holds

$$\mathcal{E}_N(\xi^k_t | V_k) \leq -T^{(t)}_{-1/k,N}(\Pi_k | V_k).$$

By [186, Theorem 28.9], the following holds. There exists a Wasserstein geodesic $\{\xi_t\}_{t\in[0,1]}$ between μ_1 and μ_2 in $(\overline{B}_R(x), g^X)$ which arises as limit of $\{\xi^k_t\}_{t\in[0,1]}$. The measures Π_k converge weakly to an optimal plan Π between μ_1 and μ_2 in $(\overline{B}_R(x), g^X)$.

Since $g_k \to g_X$ uniformly on $\overline{B}_R(x)$, we also have the following. The Riemannian distances d_k induced by g_k on $\overline{B}_R(x) \times \overline{B}_R(x)$ converge uniformly to the distance d' induced by g^X on $\overline{B}_R(x) \times \overline{B}_R(x)$, which coincides with d on $B_r(x) \times B_r(x)$ by the choice of r. The densities of V_k w.r.t. V converge uniformly to 1 on $\overline{B}_R(x)$.

Hence, for every $t \in [0, 1]$ and every $N \geq n$, we have uniform convergence

$$\begin{aligned}
&\tau^t_{-1/k,N} \circ \mathsf{d}_k \to \tau^t_{0,N} \circ \mathsf{d} \quad \text{in } B_r(x) \times B_r(x),\\
&(\mu_1/V_k) \to (\mu_1/V) \quad \text{in } B_r(x),\\
&(\mu_2/V_k) \to (\mu_2/V) \quad \text{in } B_r(x).
\end{aligned}$$

Combining this with the weak convergence of Π_k to Π, we deduce

$$T^{(t)}_{-1/k,N}(\Pi_k|V_k) \to T^{(t)}_{0,N}(\Pi|V) \quad \text{for all } t \in [0,1] \text{ and } N \geq n.$$

Hence, by [186, Theorem 29.20], it holds

$$\mathcal{E}_N(\xi_t|V) \leq -T^{(t)}_{0,N}(\Pi|V) \quad \text{for every } t \in [0,1] \text{ and every } N \geq n.$$

Finally, since geodesics in $(\overline{B}_R(x), g^X)$ connecting points in $B_r(x)$ are contained in $B_{R/2}(x)$, the Wasserstein geodesic $\{\xi_t\}_{t\in[0,1]}$ in $(\overline{B}_R(x), g^X)$ is also a Wasserstein geodesic in X. Hence, for every $x \in X \setminus S_2$, there exists $r > 0$ such that X satisfies $\mathsf{CD}(0,n)$ in $B_r(x)$.

Step 2. We show that, for every $x \in X \setminus S_2$, there exists $r > 0$ such that geodesics of X contained in $B_r(x)$ do not branch.

If $x \in X \setminus (S_1 \cup S_2)$, there exists a smooth neighbourhood of x, implying the claim. Hence, suppose that $x \in S_1$. Since M and N are smooth manifolds, there exists $k' \in \mathbb{R}$ such that both their sectional curvatures in a neighbourhood of x are bounded below by k'. Hence, using the same arguments of [135] (and using again the condition on the second fundamental forms), we obtain the following. There exist $R > 0$ and a sequence of Riemannian metrics g_k on $\overline{B}_R(x) \subset X$, converging uniformly to g^X on $\overline{B}_R(x)$, such that $\mathsf{Sec}_{(B_R(x),g_k)} \geq k' - 1$.

Hence, for k fixed, every point $y \in B_R(x)$ has a neighbourhood $U_y \subset B_R(x)$, depending on k, where triplets of points satisfy the comparison property of Alexandrov spaces with curvature bounded below by $k' - 1$ (w.r.t. the distance induced by g_k in $\overline{B}_R(x)$). This implies, by the proof of Toponogov's Theorem (see [43, Theorem 10.3.1]), the existence of $0 < r \ll R$ (this time independent of k) such that triplets of points in $B_r(x)$ satisfy the comparison property of Alexandrov spaces with curvature bounded below by $k' - 1$ (w.r.t. the distance induced by g_k in $\overline{B}_R(x)$).

One can check that since $g_k \to g^X$ uniformly, then also $(\overline{B}_R(x), g^X)$ satisfies the same triangle comparison property for points contained in $B_r(x)$. This implies that geodesics in $B_r(x)$ do not branch (see [44]).

Step 3. We now prove the statement.

Since $\mathsf{Ric}_M \geq 0$ and $H_{S_2} \geq 0$, it holds $\Delta \mathsf{d}_{S_2} \leq 0$ in a neighbourhood of S_2 in classical sense (see, for instance, [67]). We now use a globalization technique that was previously used in [152] and [131].

Consider the partition of the transport set of d_{S_2} in transport rays $\{r_\alpha\}_{\alpha\in Q}$, Q being a set of indices with a measure q induced by the partition. By [49, Theorem

3.4], there exists a disintegration of the measure H^n into measures $\{h_\alpha\}_{\alpha\in Q}$, each concentrated on the corresponding transport ray. By [50, Theorem 4.2] (the proof works in our setting since the space X is non-branching and $\mathsf{CD}(0,n)$ around every point $x \in X \setminus S_2$), q-almost every measure h_α is absolutely continuous w.r.t. the Lebesgue measure of the corresponding transport ray, it admits a locally Lipschitz continuous density (that we still denote by h_α), and it satisfies in weak sense

$$(h_\alpha^{\frac{1}{n-1}})'' \leq 0. \tag{3.25}$$

By [49, Corollary 4.16] (as before, the proof works in our setting thanks to the previous steps), it holds

$$\Delta \mathsf{d}_{S_2} = (\log h_\alpha)' + [\Delta \mathsf{d}_{S_2}]^{sing} \quad \text{on } X \setminus S_2$$

in distributional sense, where $[\Delta \mathsf{d}_{S_2}]^{sing}$ is a negative singular measure and the transport rays of d_{S_2} are now oriented from S_2 to $X \setminus S_2$.

Hence, q-almost every density h_α satisfies $(\log h_\alpha)' \leq 0$ near S_2. Setting

$$f_\alpha := h_\alpha^{\frac{1}{n-1}},$$

we obtain that on the corresponding transport ray r_α, thanks to (3.25), it holds in weak sense

$$\begin{cases} f_\alpha'' \leq 0 & \text{on } r_\alpha \\ f_\alpha' \leq 0 & \text{in a neighbourhood of } S_2 \cap r_\alpha \text{ in } r_\alpha. \end{cases}$$

By Riccati Comparison (see, for instance, [131, Lemma 3.9]), f_α then satisfies $(\log f_\alpha)' \leq 0$. In particular,

$$\Delta \mathsf{d}_{S_2} \leq (\log h_\alpha)' = (n-1)(\log f_\alpha)' \leq 0,$$

concluding the proof. □

3.3 An alternative proof of Theorem 1.3.2 based on Theorem 1.3.4

Here, we show how Theorem 1.3.4 can be used to obtain a new proof of Theorem 1.3.2. The main difference with the original proof by Gang Liu, is that we skip the construction of a foliation of minimal surfaces, and rely instead on Theorem 1.3.4. We stress that this alternative proof is possible because the assumptions of Theorem 1.3.2 allow to construct properly embedded stable minimal hypersurfaces in the ambient manifold. The embeddedness of such hypersurfaces is crucial for this alternative approach to work. This is the reason why no alternative proof of Theorem

1.3.3 follows by these tools. Indeed, as explained in the introduction, arguing by contradiction in Theorem 1.3.3, one can still construct stable minimal surfaces in the ambient manifold, but these will only be properly immersed.

Proof of Theorem 1.3.2 based on Theorem 1.3.4. Modulo passing to the orientable double covering, we assume that M is orientable. Let $\gamma \subset M$ be a smooth loop generating $\pi_1(M)$. By the argument of [174, p.217, proof of Lemma 3], there exists a properly embedded orientable stable minimal hypersurface $\Sigma \subset M$ intersecting (a perturbation of) γ transversally with algebraic intersection number 1.

Claim The image of the map $i_\# : \pi_1(\Sigma) \to \pi_1(M)$ induced by the inclusion is trivial.

Indeed, if this were not the case, then $i_\#(\pi_1(\Sigma)) \subset \pi_1(M)$, being a subgroup of $\pi_1(M) \cong \mathbb{Z}$, would contain a copy of $\mathrm{span}(\gamma^k)$ for some $k \in \mathbb{N}$. Hence, γ^k can be homotopically deformed into a loop fully contained in Σ, which can be then deformed into a loop which is disjoint from Σ (moving away from Σ in its tubular neighbourhood, which exists since M and Σ are both orientable). Hence, the intersection number of γ^k with Σ is zero. At the same time, since the intersection number of γ and Σ is one, it follows that the intersection number of Σ with γ^k is k, a contradiction.

We now conclude the proof. Let $p : \widetilde{M} \to M$ be the universal covering of M, and let $q \in \Sigma \cap \gamma$. Let Σ_1, Σ_2 be lifts of Σ passing through distinct points $q_1, q_2 \in \widetilde{M} \cap p^{-1}(q)$. Observe that these exist by the previous claim and the standard lifting criterion. These two hypersurfaces are disjoint, since otherwise we would lose uniqueness of the lift at the points that they share. Consider a 3-manifold with boundary $V \subset \widetilde{M}$ such that $\partial V = \Sigma_1 \cup \Sigma_2$ topologically (such V exists since $\widetilde{M}$ is simply connected, and Σ_1, Σ_2 are properly embedded codimension-1 submanifolds without boundary). By Theorem 1.3.4, it holds $V \cong \Sigma \times [0, l]$ isometrically for some $l > 0$. Iterating this procedure for every pair of points $q_1, q_2 \in \widetilde{M} \cap p^{-1}(q)$, we conclude. □

Chapter 4

First Betti number rigidity

The goal of this section is to prove the following splitting-type theorem for $\mathsf{RCD}(0,n)$ spaces, which implies Theorem 1.3.7 from Section 1.3.2:

Theorem 4.0.1. *Let $(X,\mathsf{d},\mathsf{H}^n)$ be an $\mathsf{RCD}(0,n)$ space for some $n\in\mathbb{N}$, $n\geq 2$. If $\pi_1(X)$ has a subgroup isomorphic to $\mathbb{Z}^{n-2}$ then the universal cover of X splits a factor $\mathbb{R}^{n-2}$ as a metric measure space.*

The first key ingredient towards the proof of Theorem 4.0.1 is the following splitting theorem for the blow-downs of X:

Proposition 4.0.2. *Let $(\widetilde{X},\widetilde{\mathsf{d}},\widetilde{\mathfrak{m}},\widetilde{q},G)$ denote any equivariant blow-down of the universal cover $(\overline{X},\overline{\mathsf{d}},\mathsf{H}^n,\overline{p},\mathbb{Z}^{n-2})$ of $(X,\mathsf{d},\mathsf{H}^n)$, endowed with the action of $\mathbb{Z}^{n-2}<\pi_1(X)$ by deck transformations. Then $(\widetilde{X},\widetilde{\mathsf{d}},\widetilde{\mathfrak{m}},\widetilde{q},G)$ splits a factor $\mathbb{R}^{n-2}$ as a metric measure space and there is a closed subgroup $\mathbb{R}^{n-2}<G$ acting by translations on such Euclidean factor.*

A corollary of Proposition 4.0.2 is that the blow-down of $(X,\mathsf{d},\mathfrak{m})$ is unique as a metric space. Such information is crucial for the proof of the following:

Proposition 4.0.3. *Under the assumptions of Theorem 4.0.1, there exist linearly independent nonconstant harmonic functions $u_1,\dots,u_{n-2}:\overline{X}\to\mathbb{R}$ such that the following hold:*

i) for every $\varepsilon>0$ there exists $C_\varepsilon>0$ such that $|u_i(x)|\leq C_\varepsilon\mathsf{d}(x,\overline{p})^{1+\varepsilon}+C_\varepsilon$ for every $x\in\overline{X}$ and $i=1,\dots,n-2$, i.e., the harmonic functions have almost linear growth;

ii) for every $\varepsilon>0$ there exists $R_\varepsilon>0$ such that for every $r>R_\varepsilon$ there is a lower triangular $(n-2)\times(n-2)$ matrix $T_{r,\varepsilon}$ such that $T_{r,\varepsilon}\circ U:B_r(\overline{p})\to\mathbb{R}^{n-2}$ is an ε-splitting map. Here we set $U:=(u_1,\dots,u_{n-2}):\overline{X}\to\mathbb{R}^{n-2}$;

iii) the map $\mathbb{Z}^{n-2} \ni \gamma \mapsto U(\gamma(\overline{p})) \in \mathbb{R}^{n-2}$ *is an injective group homomorphism;*

iv) if $0 < \varepsilon < \varepsilon_0$*, then*

$$\liminf_{r\to\infty} \frac{\lambda^{n-2}\left(T_{r,\varepsilon} \circ U(B_r(\overline{p}))\right)}{r^{n-2}} > 0\,. \tag{4.1}$$

Items i) and ii) of Proposition 4.0.3 follow from [122, Theorem 1.10], taking into account the metric uniqueness of the blow-down which follows from Proposition 4.0.2 (see Corollary 4.1.2). On the other hand, the proofs of items iii) and iv) require some new ideas. The *non-degeneracy* condition (4.1), together with the almost-splitting property (with arbitrary precision) up to linear transformation in ii) and the equivariance from iii) play a key role in the proof of Theorem 4.0.1.

Remark 4.0.4. For the sake of clarity we note that a posteriori the functions u_i could just be taken to be the coordinates of the $\mathbb{R}^{n-2}$ factor of $\overline{X}$. It is straightforward to check that with such choice items i)–iv) hold and one can choose all the matrices $T_{r,\varepsilon} = \mathrm{Id}$. However, at this stage of the proof we do not know that $\overline{X}$ splits a factor $\mathbb{R}^{n-2}$ yet.

We will prove Proposition 4.0.2 in Section 4.1 below. Proposition 4.0.3 is a slightly imprecise combination of Proposition 4.2.1 and Proposition 4.2.3, which are the bulk of Section 4.2. In Section 4.3 we will combine these tools to complete the proof of Theorem 4.0.1.

4.1 Splitting of the blow-downs

The goal of this section is to prove Proposition 4.0.2. The proof is much simpler if one restricts to the case where the universal covering of the space in question has Euclidean volume growth. In such case, which is the one dealt with in the recent [123], the sought splitting follows from the so-called cone-splitting principle. To treat the remaining cases, the key step is to prove the following splitting theorem via $\mathbb{R}^k$-action.

Theorem 4.1.1. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(0, N)$ *space satisfying the following conditions for some integer* $1 \le k \le N$*:*

1. $\dim_{\mathrm{e}}(X) \le k + 1$*, where* $\dim_e(X)$ *denotes the essential dimension.*
2. $\int_1^{+\infty} \frac{t^{k+1}}{\mathfrak{m}(B_t(p))}\,\mathrm{d}t = +\infty$ *for some* $p \in X$*.*
3. *There exists a closed subgroup* $H < \mathrm{Iso}(X)$ *which is isomorphic to* $\mathbb{R}^k$*.*

Then, $X \cong Y \times \mathbb{R}^k$ *as metric measure spaces for some* $\mathsf{RCD}(0, N-k)$ *space* $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$.

As we shall see, Theorem 4.1.1 easily implies the following uniqueness and structure theorem for the blow-downs of the universal cover in the setting of Proposition 4.0.2.

Corollary 4.1.2. *Let* $(X, \mathsf{d}, \mathsf{H}^n)$ *be an* $\mathsf{RCD}(0, n)$ *space for some* $n \in \mathbb{N}$, $n \geq 2$. *If* $\pi_1(X)$ *has a subgroup isomorphic to* $\mathbb{Z}^{n-2}$, *then the blow-down of the universal cover* $(\overline{X}, \overline{\mathsf{d}}, \mathsf{H}^n)$ *of* $(X, \mathsf{d}, \mathsf{H}^n)$ *is unique as a (pointed) metric space and (pointed) isometric to a metric cone (pointed at a tip).*

Assuming the validity of Theorem 4.1.1 and Corollary 4.1.2, whose proofs will be discussed later in this section, we complete the proof of Proposition 4.0.2.

Proof of Proposition 4.0.2. We assume first that $\overline{X}$ has Euclidean volume growth. Then every blow-down $(\widetilde{X}, \widetilde{\mathsf{d}}, \widetilde{q})$ is an n-dimensional metric cone with tip $\widetilde{q}$ by [79, Theorem 1.1] and [80, Theorem 1.3] after [53, Theorem 7.6]. By Lemma 2.2.23, there exists a closed subgroup of G isomorphic to $\mathbb{R}^{n-2}$ which acts freely on $\widetilde{X}$ since the only compact subgroup of $\mathbb{R}^{n-2}$ is the trivial group. Hence, the set of tips of $\widetilde{X}$ contains a subset homeomorphic to $\mathbb{R}^{n-2}$, so that $\widetilde{X} \cong \mathbb{R}^{n-2} \times C(Z)$ isometrically, by cone-splitting, where Z is either isometric to a circle S^1_r for $0 < r < 1$ or to an interval $[0, a]$ for $0 < a \leq \pi$.

Assume now that $\overline{X}$ does not have Euclidean volume growth. In this case, we have that $\dim_{\mathrm{e}}(\widetilde{X}) \leq \dim_{\mathsf{H}}(\widetilde{X}) \leq n-1$, by [80, Theorem 1.4] after [54, Theorem 3.1]. By Lemma 2.2.23 there exists a closed subgroup of G isomorphic to $\mathbb{R}^{n-2}$. Moreover, by the Bishop-Gromov inequality, there exists $c > 0$ such that

$$\widetilde{\mathfrak{m}}(B_r(\widetilde{q})) \leq cr^n \quad \text{for every } r > 1\,. \tag{4.2}$$

Hence, Theorem 4.1.1 applies with $k = n-2$ thus yielding that $\widetilde{X} \cong \mathbb{R}^{n-2} \times Z$ as metric measure space.

By Corollary 4.1.2, the blow-down of $\overline{X}$ is unique as a metric space and isometric to a metric cone. The second part of the statement follows immediately from [160, Theorem 3.2]. We note that such result is stated for smooth complete Riemannian manifolds therein but the proof works verbatim in the RCD case. □

We are going to prove Theorem 4.1.1 by induction on the dimension k of the group $\mathbb{R}^k \cong H < \mathrm{Iso}(X)$. The most challenging part of the proof is the base step of the induction, corresponding to $k = 1$. As shown by Proposition 4.1.10, we can easily

reduce to the case where $X/H \cong \mathbb{R}_+$ isometrically. The proof of the splitting theorem in this case consists of two main steps. The first step of the proof is to show that $(X, \mathsf{d}, \mathfrak{m})$ is isomorphic to a measured warped product $(\mathbb{R}_+ \times_\varphi \mathbb{R}, \mathsf{d}_\varphi, h(s)\lambda^1(\mathrm{d}s)\lambda^1(\mathrm{d}t))$ for some $\varphi \in C(\mathbb{R}_+, (0, +\infty])$ and $h \in \mathsf{Lip}_{\mathrm{loc}}((0, +\infty)) \cap \mathsf{L}^1_{\mathrm{loc}}([0, +\infty))$. Establishing such structural result is the goal of Section 4.1.1 (see in particular Proposition 4.1.3). In Section 4.1.2, using the assumption that X has non-negative Ricci curvature in synthetic sense, we deduce that the warping factor φ is superharmonic on a suitably weighted half-line. Such weighted half-line turns out to have slow volume growth, thanks to assumption (2), thus forcing the warping factor φ to be constant and yielding the sought splitting (see in particular Proposition 4.1.29). In Section 4.1.3 we deal with the inductive step. In Section 4.1.4 we exploit Theorem 4.1.1 to prove Corollary 4.1.2.

4.1.1 Proof of Theorem 4.1.1, base step: measured warped product structure

Our setup in this section is that we consider an $\mathsf{RCD}(0, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ such that $\dim_{\mathrm{e}}(X) \leq 2$,

$$\int_1^{+\infty} \frac{t^2}{\mathfrak{m}(B_t(p))}\,\mathrm{d}t = +\infty\,, \tag{4.3}$$

for some $p \in X$ and there is a closed subgroup $H < \mathrm{Iso}(X)$ isomorphic to $\mathbb{R}$. To complete the proof of the base step of Theorem 4.1.1 we are going to argue that X splits a line isometrically. To do so, as mentioned above, our first goal is to show that X is a measured warped product. Such result holds independently of the volume growth assumption (4.3). More precisely, we are going to prove the following.

Proposition 4.1.3. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(0, N)$ space with $\dim_{\mathrm{e}}(X) \leq 2$ such that there is a closed subgroup $H < \mathrm{Iso}(X)$ isomorphic to $\mathbb{R}$. Then either $(X, \mathsf{d}, \mathfrak{m})$ splits a line or there exist an increasing function $h \in \mathsf{Lip}_{\mathrm{loc}}((0, +\infty)) \cap \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$ and a continuous and strictly positive function $\varphi : (0, +\infty) \to (0, +\infty)$ such that $(X, \mathsf{d}, \mathfrak{m}, H)$ is equivariantly isomorphic to a measured warped product on $\mathbb{R}_+ \times \mathbb{R}$ with metric and measure given respectively by*

$$\mathrm{d}r^2 + \varphi(r)^2\,\mathrm{d}x^2\,, \qquad h(r)\,\mathrm{d}r\,\mathrm{d}x\,, \tag{4.4}$$

and with the $\mathbb{R}$-action being by translation on the x component.

Note that the conclusion of Proposition 4.1.3 would be pretty standard if $(X, \mathsf{d}, \mathfrak{m})$ is assumed to be a smooth weighted Riemannian manifold. Such assumption is inadmissible in our setup, and this is a source of several headaches. Indeed, we will be forced to prove the measured warped product structure simultaneously with some regularity of the warping function φ and weight function h.

Remark 4.1.4. On any (sufficiently regular) measured warped product with the structure in (4.4), the projection onto the Euclidean component $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a harmonic function. Such observation is at the root of all the subsequent arguments in this section.

Remark 4.1.5. Proposition 4.1.3 generalizes in particular the structural results obtained in the recent [77]. In their case, $(X, \mathsf{d}, \mathfrak{m})$ is the blow-down of a complete Riemannian manifold (M, g) with $\mathsf{Ric} \geq 0$ and the explicit form of the metric g was exploited to obtain the measured warped product structure of the blow-down.

We start by observing that there is a natural RCD structure on the quotient space X/H.

Remark 4.1.6 (Quotients of RCD spaces by a group isomorphic to $\mathbb{R}$)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, N)$ space and let $\mathbb{R} \cong H < \mathrm{Iso}(X)$ be a closed subgroup. Consider the discrete subgroup $\mathbb{Z} < H \cong \mathbb{R}$ generated by $1 \in \mathbb{R}$. We consider the quotient space $X/\mathbb{Z}$ equipped with the quotient distance d' and the unique measure $\mathfrak{m}'$ such that the projection $\pi : X \to X/\mathbb{Z}$ is locally measure preserving. Since π is locally an isomorphism of metric measure spaces, the quotient space $(X/\mathbb{Z}, \mathsf{d}', \mathfrak{m}')$ is locally $\mathsf{RCD}(K, N)$. Thus, by the local to global property [47, 138], $(X/\mathbb{Z}, \mathsf{d}', \mathfrak{m}')$ is $\mathsf{RCD}(K, N)$ as well.

Consider the action of $\mathbb{R}/\mathbb{Z}$ on $X/\mathbb{Z}$. Since $\mathbb{R}/\mathbb{Z}$ is a compact subgroup of $\mathrm{Iso}(X/\mathbb{Z})$, thanks to [94], the quotient space $(X/\mathbb{Z})/(\mathbb{R}/\mathbb{Z})$ is an $\mathsf{RCD}(K, N)$ space when equipped with the quotient distance d'' and the quotient measure $\mathfrak{m}''$. Moreover the metric space $((X/\mathbb{Z})/(\mathbb{R}/\mathbb{Z}), \mathsf{d}'')$ is isomorphic to $X/\mathbb{R}$ with its natural quotient distance.

We continue with the following observation regarding the dimension of the quotient of an RCD space with respect to an action of $\mathbb{R}$ by measure preserving isometries.

Lemma 4.1.7. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, N)$ space such that $H < \mathrm{Iso}(X)$ is a closed subgroup isomorphic to $\mathbb{R}$. Consider the $\mathsf{RCD}(K, N)$ structure $(X/H, \widetilde{\mathsf{d}}, \widetilde{\mathfrak{m}})$ on the quotient space X/H given by Remark 4.1.6. Then, we have*

$$\dim_{\mathrm{e}}(X/H) \leq \dim_{\mathrm{e}}(X) - 1\,.$$

Proof. Let $d \in \mathbb{N}$ be the essential dimension of $(X/H, \widetilde{\mathsf{d}}, \widetilde{\mathfrak{m}})$ and let d' be the essential dimension of $(X, \mathsf{d}, \mathfrak{m})$. Call $\pi : X \to X/H$ the projection map. By [150], there exists $x \in X$ such that $\mathrm{Tan}_x(X) = \{\mathbb{R}^{d'}\}$ and $\mathrm{Tan}_{\pi(x)}(X/H) = \{\mathbb{R}^d\}$. By Lemma 2.2.23, for a sequence $r_i \uparrow +\infty$, we have the pointed equivariant Gromov Hausdorff convergence

$$(X, r_i\mathsf{d}, H, x) \to (\mathbb{R}^{d'}, \mathsf{d}_{\mathrm{eu}}, G, 0)\,,$$

where $G < \mathrm{Iso}(\mathbb{R}^{d'})$, and G contains a subgroup isomorphic to $\mathbb{R}$. By Theorem 2.1.23, $\mathbb{R}^d \cong \mathbb{R}^{d'}/G$, so that $d \leq d' - 1$, as claimed. □

We record two well-known consequences of the splitting theorem for $\mathsf{RCD}(0, N)$ spaces from [96]. Since their proofs are standard we omit them.

Lemma 4.1.8. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(0, N)$ space with a closed subgroup $H < \mathrm{Iso}(X)$ such that X/H is homeomorphic (and hence isometric) to $\mathbb{R}$. Then, there exists a splitting $X \cong Y \times \mathbb{R}$ such that the H-action is trivial on the $\mathbb{R}$ factor.*

Lemma 4.1.9. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a non-compact $\mathsf{RCD}(0, N)$ space such that $\mathrm{Iso}(X)$ contains a closed subgroup H, with X/H compact. Then, $X \cong Y \times \mathbb{R}$ as metric measure spaces.*

Combining Lemma 4.1.7, Lemma 4.1.8, Lemma 4.1.9 and the classification of RCD spaces with essential dimension less than or equal to one obtained in [133], we immediately obtain the following.

Lemma 4.1.10. *Under the same assumptions and with the same notation as in Proposition 4.1.3, either $X \cong Y \times \mathbb{R}$ as a metric measure space and the H action is by translation on the second factor, or $X/H \cong \mathbb{R}_+$ as a metric space.*

Thanks to Lemma 4.1.10, to complete the proof of Proposition 4.1.3 we can assume without loss of generality that X/H is isometric to $\mathbb{R}_+$. This is the most delicate case in the proof, and it is our focus for the rest of the section. The next two lemmas show that, in this case, X admits an explicit and natural parametrization (as a set) with a map $\Phi : \mathbb{R} \times \mathbb{R}_+ \to X$.

Lemma 4.1.11. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(0, N)$ space with a closed subgroup $H < \mathrm{Iso}(X)$ such that $X/H \cong \mathbb{R}_+$ as metric spaces. Then there exists a ray $r : [0, +\infty) \subset X$ such that*

$$X = \bigcup_{t \geq 0} (H \cdot r(t))\,, \tag{4.5}$$

and

$$\mathsf{d}(H \cdot r(t), H \cdot r(s)) = |t - s| \quad \text{for every } t, s \geq 0\,. \tag{4.6}$$

In particular, the canonical projection to the quotient space $\pi : X \to X/H \cong \mathbb{R}_+$ *coincides with the distance* $\mathsf{d}(\cdot, H \cdot r(0)) : X \to \mathbb{R}_+$ *from the orbit of* $r(0)$*.*

Proof. Consider the canonical projection map $\pi : X \to X/H \cong \mathbb{R}_+$. Fix $s > 0$, and let $H \cdot x_0$ and $H \cdot x_s$ be the preimages of 0 and s via π. By definition of quotient distance, we have that

$$s = \mathsf{d}(H \cdot x_0, H \cdot x_s)\,.$$

It is elementary to check that there exists a geodesic $\gamma_s : [0, s] \to X$ realizing the distance between $H \cdot x_0$ and $H \cdot x_s$. Modulo translating γ_s with an element of H, we may assume without loss of generality that $\gamma_s(0) = x_0$.

We claim that with such choice $\pi(\gamma_s(t)) = t$ for every $t \in [0, s]$. To this aim observe that $\pi : X \to X/H$ is a 1-Lipschitz map, so that $\pi \circ \gamma_s$ is a 1-Lipschitz curve. At the same time $\pi(\gamma_s(0)) = 0$ and $\pi(\gamma_s(s)) = s$, so that $\pi(\gamma_s(t)) = t$ for every $t \in [0, s]$. It follows that

$$\mathsf{d}(H \cdot \gamma_s(t_1), H \cdot \gamma_s(t_2)) = |t_1 - t_2| \quad \text{for every } t_1, t_2 \in [0, s]\,.$$

Choose a sequence $s_i \uparrow +\infty$ and consider the corresponding geodesics $\gamma_{s_i} \subset X$. Modulo passing to a subsequence, they converge to a limiting ray $r : [0, +\infty) \to X$, with the property that $\pi(r(t)) = t$ for every $t \geq 0$ and

$$\mathsf{d}(H \cdot r(t_1), H \cdot r(t_2)) = |t_1 - t_2| \quad \text{for every } t_1, t_2 \in \mathbb{R}_+\,.$$

Since the image of $\pi \circ r$ is the whole $\mathbb{R}_+$, also (4.5) follows. □

Lemma 4.1.12. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(0, N)$ *space with a closed subgroup* $H < \mathrm{Iso}(X)$ *isomorphic to* $\mathbb{R}$ *such that* $X/H \cong \mathbb{R}_+$*. Let* $r \subset X$ *be the ray given by Lemma 4.1.11. Then the map*

$$\Phi : \mathbb{R} \times \mathbb{R}_+ \to X\,, \quad \Phi(t, s) := t \cdot r(s)$$

is bijective. Moreover, denoting by $\mathsf{d}(\cdot, H \cdot r(0))$ *the distance from the orbit of* $r(0)$ *in* X*, the inverse* $\Psi := \Phi^{-1} : X \to \mathbb{R} \times \mathbb{R}_+$ *can be written as* $\Psi(x) = (f(x), \mathsf{d}(x, H \cdot r(0)))$ *for some function* $f : X \to \mathbb{R}$*. Furthermore,* $f : X \to \mathbb{R}$ *is equivariant with respect to the* $H \cong \mathbb{R}$*-action on* X *and the standard* $\mathbb{R}$*-action on* $\mathbb{R}$ *by translations.*

Proof. The map Φ is surjective by (4.5). Thus we only need to check the injectivity. Let $(t_1, s_1), (t_2, s_2) \in H \times \mathbb{R}_+$ be such that

$$t_1 \cdot r(s_1) = t_2 \cdot r(s_2)\,.$$

Then $r(s_1)$ and $r(s_2)$ belong to the same orbit of H in X. Thus $s_1 = s_2$ by (4.6). Since the action of H is free (because the only compact subgroup of H is the trivial group), it follows, $t_1 = t_2$, thus concluding the proof that Φ is bijective. To prove the representation formula for the inverse $\Psi := \Phi^{-1}$ we observe that, given any $x = \Phi(t, s) = t \cdot r(s)$,

$$\mathsf{d}(x, H \cdot r(0)) = \mathsf{d}(t \cdot r(s), H \cdot r(0)) = \mathsf{d}(H \cdot r(s), H \cdot r(0)) = s\,, \tag{4.7}$$

where we applied (4.6) for the last equality. □

Our next key goal is to show that the function $f : X \to \mathbb{R}$ introduced in Lemma 4.1.12 above is harmonic. Such conclusion, and its proof, are of course motivated by the observation we made in Remark 4.1.4. Nevertheless, the full argument is quite technical and it requires a few intermediate steps. We start from a mild regularity result for f. It will be our starting point for the forthcoming computations.

Lemma 4.1.13. *Under the same assumptions and with the same notation as in Lemma 4.1.11, let $f : X \to \mathbb{R}$ be the map given by Lemma 4.1.12. Then, $f \in \mathsf{Lip}_{\mathrm{loc}}(X)$.*

Proof. Let $K \subset\subset X$ and assume by contradiction that there exist distinct points $x_i, y_i \in K$ such that

$$\frac{|f(x_i) - f(y_i)|}{\mathsf{d}(x_i, y_i)} \to +\infty \quad \text{as } i \uparrow +\infty\,. \tag{4.8}$$

Using the same notation as in Lemma 4.1.12, we can write $x_i = t_i^x \cdot r(s_i^x)$ and $y_i = t_i^y \cdot r(s_i^y)$ for some $t_i^x, t_i^y \in \mathbb{R}$ and $s_i^x, s_i^y \in \mathbb{R}_+$, so that (4.8) reads as

$$\frac{\mathsf{d}(x_i, y_i)}{|t_i^x - t_i^y|} \to 0 \quad \text{as } i \uparrow +\infty\,. \tag{4.9}$$

By the equivariance of f we can assume without loss of generality that $t_i^y = 0$ for every $i \in \mathbb{N}$. Moreover, we can also assume that $t_i^x > 0$ for every $i \in \mathbb{N}$. By the triangle inequality, we get that

$$\mathsf{d}(r(s_i^y), 1 \cdot r(s_i^y)) \le \left\lfloor \frac{1}{t_i^x} \right\rfloor \mathsf{d}\Big(r(s_i^y), (t_i^x) \cdot r(s_i^y)\Big) \tag{4.10}$$

$$+ \mathsf{d}\left(\left\lfloor \frac{1}{t_i^x} \right\rfloor t_i^x \cdot r(s_i^y), 1 \cdot r(s_i^y) \right). \tag{4.11}$$

We remark that, since $\mathbb{R}$ acts additively, the element $1 \in \mathbb{R}$ does not induce the identity on X, so that $1 \cdot r(s_i^y) \neq r(s_i^y)$ in the previous inequality. We claim that the right-hand side converges to zero as $i \uparrow +\infty$. If the claim holds, we readily get a contradiction using that $H \cong \mathbb{R}$ acts freely on X. To prove the claim, using (4.6) we can control the first summand as

$$\left\lfloor \frac{1}{t_i^x} \right\rfloor \mathsf{d}\Big(r(s_i^y), t_i^x \cdot r(s_i^y)\Big) \leq \frac{1}{t_i^x}\Big(\mathsf{d}(x_i, y_i) + \mathsf{d}(r(s_i^x), r(s_i^y))\Big) \tag{4.12}$$

$$\leq 2\frac{\mathsf{d}(x_i, y_i)}{t_i^x}\,, \tag{4.13}$$

and note that the last term converges to zero by (4.9). Hence, to conclude, we need to show that

$$\mathsf{d}\left(\left\lfloor \frac{1}{t_i^x} \right\rfloor t_i^x \cdot r(s_i^y), 1 \cdot r(s_i^y)\right) \to 0\,.$$

This follows combining that

$$\left\lfloor \frac{1}{t_i^x} \right\rfloor t_i^x \to 1\,,$$

and that the sequence $r(s_i^y)$ is precompact in X. □

Corollary 4.1.14. *The map $\Phi : \mathbb{R} \times \mathbb{R}_+ \to X$ introduced in Lemma 4.1.12 is a homeomorphism. Moreover, $\dim_{\mathrm{e}}(X) = 2$.*

Proof. The first conclusion follows from Lemma 4.1.12 and Lemma 4.1.13. In particular, X is homeomorphic to $\mathbb{R}_+ \times \mathbb{R}$ and thus it has topological dimension equal to 2. Combining with [150, Theorem 1.1], it follows that the essential dimension of X is at most 2. At the same time, Lemma 4.1.7 implies that $\dim_{\mathrm{e}}(X) \geq 2$. □

We continue by studying the structure of the measure $\mathfrak{m}$.

Lemma 4.1.15. *Under the same assumptions and with the same notation as in Lemma 4.1.12, we have the following. There exists a non-decreasing function $h \in \mathsf{Lip}_{\mathrm{loc}}((0, +\infty)) \cap \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that, for every $A \subset\subset X$,*

$$\mathfrak{m}(A) = \int_{\mathbb{R}_+} h(s)\, \lambda^1(f(A \cap \{\mathsf{d}_{H\cdot r(0)} = s\}))\, ds\,. \tag{4.14}$$

Moreover, h is strictly positive on $(0, +\infty)$.

Proof. Let $A \subset\subset X$. By the coarea formula applied to the function $\mathsf{d}_{H\cdot r(0)}$,

$$\mathfrak{m}(A) = \int_{\mathbb{R}_+} P(\{\mathsf{d}_{H\cdot r(0)} > s\}, A)\, ds\,,$$

where $P(\{\mathsf{d}_{H\cdot r(0)} > s\}, A)$ denotes the perimeter of the set $\{\mathsf{d}_{H\cdot r(0)} > s\}$ evaluated on A. We claim that there exists a non-negative function $h \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$ independent of A such that for λ^1-a.e. $s \in \mathbb{R}_+$ we have

$$P(\{\mathsf{d}_{H\cdot r(0)} > s\}, A) = h(s)\, \lambda^1(f(A \cap \{\mathsf{d}_{H\cdot r(0)} = s\}))\,. \tag{4.15}$$

Since the measure $P(\{\mathsf{d}_{H\cdot r(0)} > s\}, \cdot)$ is concentrated on the set $\{\mathsf{d}_{H\cdot r(0)} = s\}$, and the restriction $f_{|\{\mathsf{d}_{H\cdot r(0)}=s\}} : \{\mathsf{d}_{H\cdot r(0)} = s\} \to \mathbb{R}$ is bijective by Lemma 4.1.12, (4.15) follows if we can prove the identity of measures

$$f_{\#}P(\{\mathsf{d}_{H\cdot r(0)} > s\}, \cdot) = h(s)\lambda^1 \quad \text{for } \lambda^1\text{-a.e. } s \in \mathbb{R}_+\,.$$

Fix $s \geq 0$. For every $a < b$ and $t \in \mathbb{R}$, by equivariance of f and invariance of the perimeter we have

$$f_{\#}P(\{\mathsf{d}_{H\cdot r(0)} > s\}, \cdot)(a, b) = f_{\#}P(\{\mathsf{d}_{H\cdot r(0)} > s\}, \cdot)(a + t, b + t)\,.$$

Since the only translation-invariant measures on $\mathbb{R}$ are multiples of λ^1, $f_{\#}P(\{\mathsf{d}_{H\cdot r(0)} > s\}, \cdot) = h(s)\lambda^1$ for some $h(s) \geq 0$. Since this holds for every $s \geq 0$, (4.14) holds for some function $h \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$.

We are left to show that $h \in \mathsf{Lip}_{\mathrm{loc}}((0, +\infty))$ and that h is increasing and strictly positive on $(0, +\infty)$. Let $\mathbb{Z} < H \cong \mathbb{R}$ be the copy of $\mathbb{Z}$ in H generated by $1 \in H \cong \mathbb{R}$. Consider the quotient space $X/\mathbb{Z}$ with its canonically induced $\mathsf{RCD}(0, N)$ structure $(X/\mathbb{Z}, \mathsf{d}', \mathfrak{m}')$ (cf. Remark 4.1.6). Let $f_{\mathbb{Z}} : X/\mathbb{Z} \to H/\mathbb{Z}$ be defined such that $f_{\mathbb{Z}}(\mathbb{Z} \cdot x) := \mathbb{Z} \cdot f(x)$, and let $\widetilde{r} \subset X/\mathbb{Z}$ be the projection of r under the canonical quotient map. Observe that

$$\bigcup_{t\in(0,1)\,,\ s\in\mathbb{R}_+} t \cdot r(s) \tag{4.16}$$

is a fundamental domain of the covering $\pi : X \to X/\mathbb{Z}$. Therefore, applying (4.14), we obtain that for every $A \subset\subset X/\mathbb{Z}$

$$\mathfrak{m}'(A) = \int_{\mathbb{R}_+} h(s)\, \lambda^1(f_{\mathbb{Z}}(A \cap \{\widetilde{\mathsf{d}}_{(H/\mathbb{Z})\cdot\widetilde{r}(0)} = s\}))\, ds\,. \tag{4.17}$$

Consider now the quotient $X/H \cong (X/\mathbb{Z})/(H/\mathbb{Z})$, with its $\mathsf{RCD}(0, N)$ structure $(X/H, \mathsf{d}'', \mathfrak{m}'')$ (cf. Remark 4.1.6). Since $X/H \cong \mathbb{R}_+$, [47, Theorem A.2] implies

that $\mathfrak{m}'' = \overline{h}\lambda^1$ for a non-decreasing, locally Lipschitz, and strictly positive function $\overline{h} : (0, +\infty) \to \mathbb{R}$. On the other hand, for every $s \geq 0$, we have that

$$f_{\mathbb{Z}}(\{\widetilde{\mathsf{d}}_{(H/\mathbb{Z})\cdot\tilde{r}(0)} = s\}) = H/\mathbb{Z}$$

and therefore

$$\lambda^1(f_{\mathbb{Z}}(\{\widetilde{\mathsf{d}}_{(H/\mathbb{Z})\cdot\tilde{r}(0)} = s\})) = 1\,.$$

Thanks to (4.17), we infer that $\overline{h} = h$, thus completing the proof. □

The following corollary is an immediate consequence of Lemma 4.1.15.

Corollary 4.1.16. *Under the same assumptions and with the same notation as in Lemma 4.1.12, if $A \subset\subset X$ is a Borel set, then $\mathfrak{m}(A) = 0$ if and only if $\lambda^2(\Psi(A)) = 0$.*

For our next purposes it is helpful to study the behavior of the tangent spaces of $(X, \mathsf{d}, \mathfrak{m})$.

Lemma 4.1.17. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(0, N)$ space with a closed subgroup $H < \mathrm{Iso}(X)$ isomorphic to $\mathbb{R}$ and such that $X/H \cong \mathbb{R}_+$ as metric spaces. Let $\pi : X \to [0, +\infty)$ be the canonical projection to the quotient. Then, we have the following:*

i) every $x \in \pi^{-1}((0, +\infty))$ is a regular point of $(X, \mathsf{d}, \mathfrak{m})$ and the (unique) equivariant blow-up of $(X, \mathsf{d}, \mathfrak{m}, x, H)$ is $(\mathbb{R}^2, \mathsf{d}_{\mathrm{eucl}}, \lambda^2)$ endowed with an $\mathbb{R}$-action by translations;

ii) for every $x \in \pi^{-1}(\{0\})$ the blow-up of $(X, \mathsf{d}, \mathfrak{m})$ at x is homeomorphic to $\mathbb{R} \times \mathbb{R}_+$; in particular x is not a regular point.

Proof. If $x \in \pi^{-1}((0, +\infty))$, then the blow-up of $X/H \cong \mathbb{R}_+$ at $\pi(x)$ is isomorphic to $\mathbb{R}$. By Theorem 2.1.23 and Lemma 4.1.8, every equivariant blow-up of $(X, \mathsf{d}, \mathfrak{m}, x, H)$ admits a splitting $X = Y \times \mathbb{R}$ with the blow-up action being trivial on the $\mathbb{R}$-factor. By lower semicontinuity of the essential dimension under pmGH convergence, see [132, Theorem 1.5], and Corollary 4.1.14, $\dim_{\mathrm{e}}(Y) \leq 1$. By Lemma 2.2.23 the limit group contains a closed subgroup isomorphic to $\mathbb{R}$. Item (i) thus follows from the classification of 1-dimensional RCD spaces, see [133].

If $\pi(x) = 0$, then every blow-up of $X/H \cong \mathbb{R}_+$ at $\pi(x)$ is isometric to $\mathbb{R}_+$. By Lemma 2.2.23 the limit group H_∞ of every equivariant blow-up (X_∞, H_∞) of $(X, \mathsf{d}, \mathfrak{m}, x, H)$ contains a closed subgroup $H'_\infty < H_\infty$ isomorphic to $\mathbb{R}$. By Corollary 4.1.14, lower semicontinuity of the essential dimension again, and Lemma 4.1.7, the quotient X_∞/H'_∞ has dimension at most 1. At the same time, $X_\infty/H_\infty \cong \mathbb{R}_+$ by

Theorem 2.1.23, so that X_∞/H'_∞ is non-compact. In particular, X_∞/H'_∞ is either a ray or a line. If $X_\infty/H'_\infty \cong \mathbb{R}_+$, then X_∞ is homeomorphic to $\mathbb{R}\times\mathbb{R}_+$ by Corollary 4.1.14. If $X_\infty/H'_\infty \cong \mathbb{R}$, then X_∞ is isometric to $\mathbb{R}^2$ by Lemma 4.1.8. We show that this is not possible. Let $r\subset X$ be the ray given by Lemma 4.1.11 and assume without loss of generality that $x = r(0)$. As $(X,\mathsf{d}/r_i)\to X_\infty\cong\mathbb{R}^2$ for some sequence $r_i\downarrow 0$, up to a subsequence, the ray $r\subset X$ converges to a ray $r_\infty\subset\mathbb{R}^2$, and the sets $\{y\in X : \mathsf{d}(y,r)\geq r_i\}$ converge to $\{y\in\mathbb{R}^2 : \mathsf{d}_{\mathrm{eucl}}(y,r_\infty)\geq 1\}$ in Hausdorff sense in the space realizing the pGH convergence. At the same time, one can write $\{y\in X : \mathsf{d}(y,r)\geq r_i\}$ as the union of two sets at distance greater than 1 from each other and distance smaller than 2 from $r(0)$ in $(X,\mathsf{d}/r_i)$ (by Corollary 4.1.14, $X\setminus r$ consists of 2 path connected components, and the sought sets can be taken to be the complements of the 1-neighbourhood of r in each connected component of $(X\setminus r,\mathsf{d}/r_i)$). Hence, also $\{y\in\mathbb{R}^2 : \mathsf{d}_{\mathrm{eucl}}(y,r_\infty)\geq 1\}$ must admit such partition with $r(0)$ replaced by $r_\infty(0)$, which is a contradiction. □

Lemma 4.1.18. *Under the same assumptions and with the same notation as in Lemma 4.1.11, let $\gamma:[0,T]\to X$ be a segment. If the image of γ has three distinct points lying in $\{\mathsf{d}_{\mathbb{R}\cdot r(0)}=0\}$, then it is fully contained in $\{\mathsf{d}_{\mathbb{R}\cdot r(0)}=0\}$.*

Proof. The conclusion follows from Lemma 4.1.17 thanks to the Hölder continuity of tangent cones in the interior of geodesics, see [83] after [64]. □

Remark 4.1.19. Under the assumptions of Lemma 4.1.11 it is possible that a segment $\gamma:[0,T]\to X$ has $\gamma(0)$ and $\gamma(T)$ lying in the singular orbit $\mathbb{R}\cdot r(0)$ with all the interior points lying in the regular part. Examples where this actually happens appear in [161, 77].

Motivated by Remark 4.1.4, our next goal is to show that the function f we introduced before is harmonic. Such statement corresponds to Lemma 4.1.23 below. The next two lemmas are instrumental to its proof. Oversimplifying, we are going to rely on a blow-up argument to compute the partial derivatives of any sufficiently regular test function by using $\nabla\mathsf{d}_{H\cdot r(0)}$ and ∇f as an orthogonal basis for the tangent bundle of X.

More precisely, in Lemma 4.1.21 we show that gradients in (X,d) can be computed as on a manifold with a warped product metric of the form $dr^2+|\nabla f(r)|^{-2}dx^2$.

Lemma 4.1.20. *Under the same assumptions and with the same notation as in Lemma 4.1.11, for every $s > 0$ and every sequence $r_i \downarrow 0$, up to passing to a subsequence, there exist $\delta_i \downarrow 0$ and δ_i-GH approximations*

$$\psi_i : (\overline{B}^X_{r_i}(r(s)), \mathsf{d}/r_i) \to (\overline{B}^{\mathbb{R}^2}_1(0), \mathsf{d}_{\mathrm{eucl}})$$

satisfying the following conditions:

1. *$\psi_i(r(s + r_i)) = (0, 1)$ and $\psi_i(r(s - r_i)) = (0, -1)$.*
2. *$\psi_i(H \cdot r(s)) \to \{y = 0\}$ in Hausdorff sense in $\mathbb{R}^2$.*

Assume in addition that the function f given by Lemma 4.1.12 is differentiable in $r(s)$ according to Theorem 2.2.26, and $\mathsf{lip}(f)(r(s)) > 0$. Modulo passing to a subsequence, the linear functions $f_\infty, \mathsf{d}_\infty : \mathbb{R}^2 \to \mathbb{R}$ such that $\mathsf{lip}(\mathsf{d}_\infty) = 1$ and $\mathsf{lip}(f_\infty) = \mathsf{lip}(f)(r(s))$ obtained by blowing up $\mathsf{d}_{H\cdot r(0)}$ and f at $r(s)$ have orthogonal gradients and they span the space of linear functions on $\mathbb{R}^2$.

Proof. Since $\mathrm{Tan}_{r(s)}(X) = \{\mathbb{R}^2\}$ by Lemma 4.1.17, up to passing to a (non-relabeled) subsequence, there exist $\delta_i \downarrow 0$, and δ_i-GH approximations

$$\psi_i : (\overline{B}^X_{ir_i}(r(s)), \mathsf{d}/r_i) \to (\overline{B}^{\mathbb{R}^2}_i(0), \mathsf{d}_{\mathrm{eucl}}),$$

with $\psi_i(r(s)) = 0$. It is a standard observation that one can take the second components of the GH-maps ψ_i to be just $(\mathsf{d}_{H\cdot r(0)} - s)/r_i$, since $r(s)$ is an intermediate point of a ray for $\mathsf{d}_{H\cdot r(0)}(\cdot)$ by (4.6). With such choice, condition 1 is trivially satisfied, up to a slight perturbation of the first component of the GH-maps. Condition 2 follows immediately from Lemma 4.1.17 (i).

Consider now the linear maps $\mathsf{d}_\infty, f_\infty : \mathbb{R}^2 \to \mathbb{R}$ obtained by blowing up $\mathsf{d}_{H\cdot r(0)}$ and f at $r(s)$ via Theorem 2.2.26. Since $\mathsf{d}_{H\cdot r(0)}(r(s+r_i)) = s+r_i$ and $\psi_i(r(s+r_i)) = (0, 1)$ for every i, it follows that $\mathsf{d}_\infty((0, 1)) = 1$. Since $|\nabla \mathsf{d}_\infty| \leq 1$, $\nabla \mathsf{d}_\infty$ is vertical. Since $f(r(s + r_i)) = 0$, $f_\infty((0, 1)) = 0$, so that ∇f_∞ is horizontal. Since $\nabla f_\infty \neq 0$ by assumption, the set $\{\mathsf{d}_\infty, f_\infty\}$ spans the space of linear functions on $\mathbb{R}^2$. □

Lemma 4.1.21. *Under the same assumptions and with the same notation as in Lemma 4.1.15, let $\varphi \in \mathsf{Lip}_c(X)$, and let $\widetilde{\varphi} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be defined as $\varphi \circ \Psi^{-1}$, so that $\varphi(x) = \widetilde{\varphi}(f(x), \mathsf{d}_{\mathbb{R}\cdot r(0)}(x))$. Then we have the following:*

1. *For $h\lambda^1$-a.e. $s \in \mathbb{R}_+$ with $\mathsf{lip}(f) \neq 0$ on $(\mathsf{d}_{\mathbb{R}\cdot r(0)})^{-1}(s)$,*

$$\widetilde{\varphi}(\cdot, s) \in \mathsf{Lip}_c(\mathbb{R}). \tag{4.18}$$

2. *For* $\mathfrak{m}$*-a.e.* $x \in X$ *with* $\mathsf{lip}(f)(x) \neq 0$, $\widetilde{\varphi}(\cdot, \mathsf{d}_{\mathbb{R}\cdot r(0)}(x))$ *is differentiable in* $f(x)$, *with*

$$\nabla\varphi \cdot \nabla f(x) = \widetilde{\varphi}_{x_1}(f(x), \mathsf{d}_{\mathbb{R}\cdot r(0)}(x))|\nabla f|^2(x)\,. \tag{4.19}$$

3. *For* $\mathfrak{m}$*-a.e.* $x \in X$, $\widetilde{\varphi}(f(x), \cdot)$ *is differentiable in* $\mathsf{d}_{\mathbb{R}\cdot r(0)}(x)$, *with*

$$\nabla\varphi \cdot \nabla \mathsf{d}_{\mathbb{R}\cdot r(0)}(x) = \widetilde{\varphi}_{x_2}(f(x), \mathsf{d}_{\mathbb{R}\cdot r(0)}(x))\,. \tag{4.20}$$

In particular, for $\mathfrak{m}$*-a.e.* $x \in X$ *with* $\mathsf{lip}(f)(x) \neq 0$, $\widetilde{\varphi}(\cdot,\cdot)$ *has partial derivatives in* $(f(x), \mathsf{d}_{\mathbb{R}\cdot r(0)}(x))$, *and*

$$|\nabla\varphi|^2(x) = \widetilde{\varphi}^2_{x_2}(f(x), \mathsf{d}_{\mathbb{R}\cdot r(0)}(x)) + \widetilde{\varphi}^2_{x_1}(f(x), \mathsf{d}_{\mathbb{R}\cdot r(0)}(x))|\nabla f|^2(x)\,.$$

Proof. **Proof of 1:** Fix $\varepsilon > 0$. We show that, for $h\lambda^1$-a.e. $s \in \mathbb{R}_+$ such that $\mathsf{lip}(f) \geq \varepsilon$ on $(\mathsf{d}_{\mathbb{R}\cdot r(0)})^{-1}(s)$, (4.18) holds. Fix such an $s \in (0, +\infty)$, and assume that for every $x \in (\mathsf{d}_{\mathbb{R}\cdot r(0)})^{-1}(s)$ the function f is differentiable in x. By Theorem 2.2.27, $h\lambda^1$-a.e. $s \in \mathbb{R}_+$ such that $\mathsf{lip}(f) \geq \varepsilon$ on $(\mathsf{d}_{\mathbb{R}\cdot r(0)})^{-1}(s)$ has this property. Let $t \in \mathbb{R}$ and consider a sequence $t_i \to t$. We show that there exists $C > 0$ independent of t_i and t such that

$$\limsup_{i\to+\infty} \frac{|\widetilde{\varphi}(t_i, s) - \widetilde{\varphi}(t, s)|}{|t_i - t|} < C\,, \tag{4.21}$$

which would then imply item 1. Let $x_s := r(s) \in (\mathsf{d}_{\mathbb{R}\cdot r(0)})^{-1}(s)$. We have that

$$\frac{|\varphi(t_i \cdot x_s) - \varphi(t \cdot x_s)|}{\mathsf{d}(t \cdot x_s, t_i \cdot x_s)} < \mathsf{L}(\varphi)\,,$$

where $\mathsf{L}(\varphi)$ denotes the Lipschitz constant of φ, so that

$$\frac{|\widetilde{\varphi}(t_i, s) - \widetilde{\varphi}(t, s)|}{|t_i - t|} \frac{|t_i - t|}{\mathsf{d}(t \cdot x_s, t_i \cdot x_s)} < \mathsf{L}(\varphi)\,.$$

Hence, to conclude the proof of (4.21) it is enough to show that for i large enough

$$\frac{|t_i - t|}{\mathsf{d}(t \cdot x_s, t_i \cdot x_s)} \geq \frac{\varepsilon}{2} > 0\,. \tag{4.22}$$

Observe that $r_i := \mathsf{d}(t \cdot x_s, t_i \cdot x_s) \to 0$ as $i \uparrow +\infty$ and, consider the GH maps

$$\psi_i : (\overline{B}^X_{r_i}(x_s), \mathsf{d}/r_i) \to (\overline{B}^{\mathbb{R}^2}_1(0), \mathsf{d}_{\mathrm{eucl}})\,,$$

introduced in Lemma 4.1.20. Up to subsequences, since f is differentiable in x_s with $\mathsf{lip}(f)(x) \geq \varepsilon$, there exists a linear function $f_\infty : \mathbb{R}^2 \to \mathbb{R}$ with $\mathsf{lip}(f_\infty) \geq \varepsilon$, such that

$$\left\| \frac{f - f(x_s)}{r_i} - f_\infty \circ \psi_i \right\|_{\infty, , \overline{B}^X_{r_i}(x_s)} \to 0 \tag{4.23}$$

as $i \to \infty$. Thanks to Lemma 4.1.20,

$$|f_\infty(\pm 1, 0)| \geq \varepsilon\,. \tag{4.24}$$

The same lemma gives that $\psi_i((t_i - t)\cdot x_s)$ converges to a point in $\{(\pm 1, 0)\}$. Evaluating (4.23) in $(t_i - t)\cdot x_s$, we deduce

$$\Big|\frac{t_i - t}{\mathsf{d}(t\cdot x_s, t_i \cdot x_s)} - f_\infty(\pm 1, 0)\Big| \to 0\,,$$

which, combined with (4.24), gives (4.22). This concludes the proof of item 1.

Proof of 2:. Thanks to Lemma 4.1.15 and item 1, for $\mathfrak{m}$-a.e. $x \in X$ with $\mathsf{lip}(f)(x) \neq 0$, $\widetilde{\varphi}(\cdot, \mathsf{d}_{\mathbb{R}\cdot r(0)}(x))$ is differentiable in $f(x)$. Hence, we only need to check (4.19).

Let $x \in X \setminus (\mathbb{R}\cdot r(0))$ be any point such that $\widetilde{\varphi}(\cdot, \mathsf{d}_{\mathbb{R}\cdot r(0)}(x))$ is differentiable in $f(x)$, $\mathsf{lip}(f)(x) \neq 0$, and the functions f, φ, $\mathsf{d}_{\mathbb{R}\cdot r(0)}$, $\varphi - f$, and $\varphi + f$ are differentiable at x. Let $r_i \downarrow 0$ and, up to passing to a subsequence, consider the corresponding linear functions $(\varphi + f)_\infty, (\varphi - f)_\infty, \varphi_\infty, \mathsf{d}_\infty, f_\infty : \mathbb{R}^2 \to \mathbb{R}$ obtained by blowing-up as in Definition 2.2.26. By Lemma 4.1.20, there exist $a, b \in \mathbb{R}$ such that $\varphi_\infty = a\mathsf{d}_\infty + bf_\infty$. Furthermore, $(\varphi \pm f)_\infty = \varphi_\infty \pm f_\infty$. Hence, using that $\nabla d_\infty \cdot \nabla f_\infty = 0$ by Lemma 4.1.20,

$$\begin{aligned}
\nabla\varphi \cdot \nabla f(x) =& \frac{\mathsf{lip}(\varphi + f)^2(x) - \mathsf{lip}(\varphi - f)^2(x)}{4} && (4.25)\\
=& \frac{|\nabla(\varphi + f)_\infty|^2 - |\nabla(\varphi - f)_\infty|^2}{4} && (4.26)\\
=& \nabla\varphi_\infty \cdot \nabla f_\infty = b|\nabla f_\infty|^2 = b|\nabla f|^2(x)\,. && (4.27)
\end{aligned}$$

To conclude, we only need to show that $b = \widetilde{\varphi}_{x_1}(f(x), \mathsf{d}_{\mathbb{R}\cdot r(0)}(x))$. To this aim, let $t_i \downarrow 0$ be such that $\mathsf{d}(x, t_i \cdot x) = r_i$. Then, as $i \to \infty$, we have that

$$\Big|\frac{\varphi(t_i \cdot x) - \varphi(x)}{\mathsf{d}(x, t_i \cdot x)} - a\mathsf{d}_\infty(\psi_i(t_i \cdot x)) - bf_\infty(\psi_i(t_i \cdot x))\Big| \to 0\,. \tag{4.28}$$

By item 2 of Lemma 4.1.20, using also that $\nabla\mathsf{d}_\infty \subset \mathbb{R}^2$ is vertical, we infer that $\mathsf{d}_\infty(\psi_i(t_i \cdot x)) \to 0$. Rewriting (4.28) in terms of $\widetilde{\varphi}$, we obtain

$$\Big|\frac{\widetilde{\varphi}(f(x) + t_i, \mathsf{d}_{\mathbb{R}\cdot r(0)}(x)) - \widetilde{\varphi}(f(x), \mathsf{d}_{\mathbb{R}\cdot r(0)}(x))}{t_i}\frac{t_i}{\mathsf{d}(x, t_i \cdot x)} - bf_\infty(\psi_i(t_i \cdot x))\Big| \to 0\,. \tag{4.29}$$

Arguing as in the proof of item 1, we obtain that

$$\Big|\frac{t_i}{\mathsf{d}(x, t_i \cdot x)} - f_\infty(\psi_i(t_i \cdot x))\Big| \to 0, \quad \text{and} \quad f_\infty(\psi_i(t_i \cdot x)) \to f_\infty(\pm 1, 0) \neq 0\,.$$

Since $\widetilde{\varphi}(\cdot, \mathsf{d}_{\mathbb{R}\cdot r(0)}(x))$ is differentiable in $f(x)$ by assumption, (4.29) implies that $b = \widetilde{\varphi}_{x_1}(f(x), \mathsf{d}_{\mathbb{R}\cdot r(0)}(x))$, as claimed.

Proof of 3: For every $t \in \mathbb{R}$, the map $\widetilde{\varphi}(t, \cdot)$ is Lipschitz and hence differentiable λ^1-a.e. in $\mathbb{R}_+$. By Lemma 4.1.16, for $\mathfrak{m}$-a.e. $x \in X$, the map $\widetilde{\varphi}(f(x), \cdot)$ is differentiable in $\mathsf{d}_{\mathbb{R}\cdot r(0)}(x)$. The identity (4.20) follows repeating the argument used for (4.19) and taking into account that $|\nabla \mathsf{d}_{\mathbb{R}\cdot r(0)}| \equiv 1$.

Finally, for $\mathfrak{m}$-a.e. point $x \in X$ where (4.19) and (4.20) hold, we have

$$|\nabla\varphi|^2 = |\nabla\varphi_\infty|^2 = |\nabla\varphi_\infty \cdot \nabla \mathsf{d}_\infty|^2 + \left|\nabla\varphi_\infty \cdot \frac{\nabla f_\infty}{|\nabla f_\infty|}\right|^2 \tag{4.30}$$

$$= |\nabla\varphi \cdot \nabla \mathsf{d}_{\mathbb{R}\cdot r(0)}|^2(x) + |\nabla f|^{-2}|\nabla\varphi \cdot \nabla f|^2(x) \tag{4.31}$$

$$= \widetilde{\varphi}^2_{x_2}(f(x), \mathsf{d}_{\mathbb{R}\cdot r(0)}(x)) + \widetilde{\varphi}^2_{x_1}(f(x), \mathsf{d}_{\mathbb{R}\cdot r(0)}(x))|\nabla f|^2(x)\,. \tag{4.32}$$

□

Remark 4.1.22. Item 3 in Lemma 4.1.21 is well known, as it follows for instance from [49, Theorem 4.5]. The computation of the directional derivative in the other direction is more subtle instead.

The next lemma and the subsequent corollary contain the main regularity result concerning the map $f : X \to \mathbb{R}$ introduced in Lemma 4.1.12.

Lemma 4.1.23. *Under the same assumptions and with the same notation as in Lemma 4.1.12, the function $f : X \to \mathbb{R}$ is harmonic.*

Proof. Let $\Psi : X \to \mathbb{R} \times \mathbb{R}_+$ be the map given by Lemma 4.1.12 (with the usual identification $H \cong \mathbb{R}$), and let $h \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$ be the density from Lemma 4.1.15. Let $\varphi \in \mathsf{Lip}_c(X)$, and define $\widetilde{\varphi} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ by $\widetilde{\varphi} := \varphi \circ \Psi^{-1}$, so that $\varphi(x) = \widetilde{\varphi}(f(x), \mathsf{d}_{\mathbb{R}\cdot r(0)}(x))$. To prove that f is harmonic we need to check that $\int_X \nabla\varphi\cdot\nabla f \,\mathrm{d}\mathfrak{m} = 0$. By item 2 of Lemma 4.1.21, we have

$$\int_X \nabla\varphi \cdot \nabla f \,\mathrm{d}\mathfrak{m} = \int_X \widetilde{\varphi}_{x_1}(f(x), \mathsf{d}_{\mathbb{R}\cdot r(0)}(x))|\nabla f|^2(x)\,\mathrm{d}\mathfrak{m}(x)\,.$$

By Lemma 4.1.15 and the fact that $|\nabla f|$ is invariant by translations by $H \cong \mathbb{R}$, which follows from the equivariance of f, we deduce

$$\int_X \widetilde{\varphi}_{x_1}(f(x), \mathsf{d}_{\mathbb{R}\cdot r(0)}(x))|\nabla f|^2(x)\,\mathrm{d}\mathfrak{m}(x)$$

$$= \int_{\mathbb{R}_+} |\nabla f|^2(\Psi^{-1}(0,s))h(s) \int_{\mathbb{R}} \widetilde{\varphi}_{x_1}(t,s)\,\mathrm{d}t\,ds\,. \tag{4.33}$$

By item 1 of Lemma 4.1.21, for $h\lambda^1$-a.e. $s \in \mathbb{R}_+$, we have that $\widetilde{\varphi}(\cdot, s) \in \mathsf{Lip}_c(\mathbb{R})$. Thus $\int_{\mathbb{R}} \widetilde{\varphi}_{x_1}(t,s)\,\mathrm{d}t = 0$. Combining with (4.33), the statement follows. □

Corollary 4.1.24. *Under the same assumptions and with the same notation as in Lemma 4.1.12, the following hold:*

1. $|\nabla f|^2 \in \mathsf{W}^{1,2}_{\mathrm{loc}}(X) \cap \mathsf{L}^\infty_{\mathrm{loc}}(X)$, *and* $|\nabla f|^2$ *has non-negative measure-valued Laplacian.*

2. $|\nabla f|^2$ *admits a continuous representative, which is invariant under the action of* H *and increases along the ray* $r : (0, \infty) \to X$. *Equivalently, the function canonically induced by* $|\nabla f|^2$ *on the quotient space* X/H *is increasing.*

3. *The function induced by* $|\nabla f|^2$ *on the quotient* $X/H \cong \mathbb{R}_+$ *belongs to* $\mathsf{W}^{1,2}_{\mathrm{loc}}((0,+\infty)) \cap \mathsf{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}_+, h\mathrm{d}\lambda^1)$, *where* $(\mathbb{R}_+, \mathsf{d}_{\mathrm{eucl}}, h\mathrm{d}\lambda^1)$ *is the quotient* $\mathsf{RCD}(0,N)$ *structure on* X/H *(see Remark 4.1.6).*

4. *The continuous representative of* $|\nabla f|^2$ *is strictly positive on* $X \setminus (\mathbb{R} \cdot r(0))$.

Proof. **Proof of 1:** By Lemma 4.1.13, $|\nabla f|^2 \in \mathsf{L}^\infty_{\mathrm{loc}}(X)$. By Lemma 4.1.23 and [172, Lemma 3.2], $|\nabla f|^2 \in \mathsf{W}^{1,2}_{\mathrm{loc}}(X) \cap \mathsf{L}^\infty_{\mathrm{loc}}(X)$ and has non-negative measure-valued Laplacian.

Proof of 2 and 3: We first prove simultaneously that item 3 holds, and that $|\nabla f|^2$ has a continuous representative in $X \setminus (\mathbb{R} \cdot r(0))$. As we already observed, by the equivariance of f with respect to the $\mathbb{R}$-action, for every $a \in \mathbb{R}$ we have that

$$|\nabla f|^2(x) = |\nabla f|^2(a \cdot x) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X\,. \tag{4.34}$$

Fix $R > 0$, and let $\varphi \in \mathsf{Lip}(X)$ be a cut-off function such that $\varphi(a \cdot x) = \varphi(x)$ for every $x \in X$ and every $a \in \mathbb{R}$, $\varphi \equiv 1$ on $\{\mathsf{d}_{\mathbb{R}\cdot r(0)} < R\}$, and $\varphi \equiv 0$ on $\{\mathsf{d}_{\mathbb{R}\cdot r(0)} > R+1\}$. Such a function can be obtained as a composition of the form $k \circ \mathsf{d}_{\mathbb{R}\cdot r(0)}$, for some $k \in \mathsf{Lip}_c(\mathbb{R}_+)$. Since $R > 0$ is arbitrary, it is enough to show that $\varphi|\nabla f|^2$ has a continuous representative. For every $t > 0$, let $f_t := P_t(\varphi|\nabla f|^2) \in \mathsf{Lip}(X)$. Given any $a \in \mathbb{R}$, by (4.34), the invariance of the cut-off function φ and of the heat flow, f_t is invariant under the $H \cong \mathbb{R}$-action as well. Moreover, for every open set $A \subset\subset X$, we have

$$\lim_{t\to 0} \mathsf{Ch}(f_t, A) = \mathsf{Ch}(\varphi|\nabla f|^2, A) < +\infty\,. \tag{4.35}$$

Consider now the quotient space $X/\mathbb{Z}$ with its natural $\mathsf{RCD}(0,N)$ structure, and observe that by invariance the functions f_t naturally induce functions $f_t^{\mathbb{Z}} : X/\mathbb{Z} \to \mathbb{R}$. From (4.35), for every $A \subset\subset X/\mathbb{Z}$, we have that

$$\limsup_{t\to 0} \mathsf{Ch}(f_t^{\mathbb{Z}}, A) < +\infty\,. \tag{4.36}$$

Consider now the space $X/\mathbb{R} \cong \mathbb{R}_+$ obtained by quotienting $X/\mathbb{Z}$ by $\mathbb{R}/\mathbb{Z}$, endowed with its $\mathsf{RCD}(0,N)$ structure $(\mathbb{R}_+, \mathsf{d}_{\mathrm{eucl}}, \widetilde{\mathfrak{m}})$. Consider the functions $f_t^{\mathbb{R}} : X/\mathbb{R} \to \mathbb{R}$ naturally induced by $f_t^{\mathbb{Z}}$ on $X/\mathbb{R}$. By [94, Corollary 5.5] and (4.36), for every $A \subset\subset X/\mathbb{R}$, we have

$$\limsup_{t\to 0} \mathsf{Ch}(f_t^{\mathbb{R}}, A) < +\infty\,.$$

Hence, the functions $f_t^{\mathbb{R}}$ converge in $\mathsf{L}^2_{\mathrm{loc}}$ in $(\mathbb{R}_+, \mathsf{d}_{\mathrm{eucl}}, \widetilde{\mathfrak{m}})$ to a limiting function $f_0^{\mathbb{R}} \in \mathsf{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}_+, \widetilde{\mathfrak{m}})$. By [47, Theorem A.2], $\widetilde{\mathfrak{m}} = h\,\mathrm{d}\lambda^1$, where $h^{1/(N-1)} : (0,+\infty) \to (0,+\infty)$ is a positive and concave function. Hence, for every $\varepsilon > 0$, the functions $f_t^{\mathbb{R}}$ have uniformly bounded Cheeger energies in $((\varepsilon, \varepsilon^{-1}), \mathsf{d}_{\mathrm{eucl}}, \lambda^1)$, where now we are using the standard Euclidean structure. Thus the functions $f_t^{\mathbb{R}}$ converge to $f_0^{\mathbb{R}}$ in $\mathsf{L}^2_{\mathrm{loc}}$ on $((0,+\infty), \mathsf{d}_{\mathrm{eucl}}, \lambda^1)$ as well. Therefore $f_0^{\mathbb{R}} \in \mathsf{W}^{1,2}_{\mathrm{loc}}(0,+\infty)$ w.r.t. the standard Euclidean structure. In particular, $f_0^{\mathbb{R}}$ is continuous in $(0,+\infty)$. At the same time, $f_0^{\mathbb{R}}$ induces a continuous *pointwise defined* function $F : \{\mathsf{d}_{\mathbb{R}\cdot r(0)} > 0\} \subset X \to \mathbb{R}$, which is the $\mathsf{L}^2_{\mathrm{loc}}$ limit of the functions f_t in X. It follows that F is a continuous representative of $\varphi|\nabla f|^2$ in $\{\mathsf{d}_{\mathbb{R}\cdot r(0)} > 0\}$. This proves item 3, and shows that $|\nabla f|^2$ has a continuous representative in $\{\mathsf{d}_{\mathbb{R}\cdot r(0)} > 0\} \subset X$, as we claimed.

We now claim that $f_0^{\mathbb{R}}$ is monotone in an interval $(0,\eta)$ for $\eta > 0$ small enough. Since $\Delta(\varphi|\nabla f|^2) \geq 0$ in $\{\mathsf{d}_{\mathbb{R}\cdot r(0)} < R\}$, by the strong maximum principle [101, Theorem 2.8], the function $f_0^{\mathbb{R}}$ is constant in a neighbourhood of each local maximum $(0,R)$. Hence, if $f_0^{\mathbb{R}}$ is not monotone in $(0,\eta)$, there exists a triplet

$$0 < a < b < c < \eta\,,$$

such that

$$f_0^{\mathbb{R}}(b) < \min\{f_0^{\mathbb{R}}(a), f_0^{\mathbb{R}}(c)\}. \tag{4.37}$$

Thus, if $f_0^{\mathbb{R}}$ is not monotone in any $(0,\eta)$, there exist two triplets

$$0 < a < b < c < a' < b' < c' < R\,,$$

such that

$$f_0^{\mathbb{R}}(b) < \min\{f_0^{\mathbb{R}}(a), f_0^{\mathbb{R}}(c)\}, \quad f_0^{\mathbb{R}}(b') < \min\{f_0^{\mathbb{R}}(a'), f_0^{\mathbb{R}}(c')\}\,. \tag{4.38}$$

It follows that $f_{0\,|[b,b']}^{\mathbb{R}}$ has a global maximum point $m \in (b,b')$. Thus, by the strong maximum principle again, $f_0^{\mathbb{R}}$ must be constant in $[b,b']$, a contradiction. This shows that there exists $\eta > 0$ such that $f_0^{\mathbb{R}}$ is monotone in $(0,\eta)$. Since $\varphi|\nabla f|^2$ is essentially bounded, $f_0^{\mathbb{R}}$ is bounded. Thus $f_0^{\mathbb{R}}$ extends to a continuous monotone function on $[0,R)$. Hence, $\varphi|\nabla f|^2$ has a continuous representative on X. Using again that

$\Delta(\varphi|\nabla f|^2) \geq 0$ on $\{\mathsf{d}_{\mathbb{R}\cdot r(0)} < R\} \subset X$, it follows that $f_0^{\mathbb{R}}$ is increasing in $[0, R)$. Since by construction $f_0^{\mathbb{R}}$ is the function induced on $X/\mathbb{R}$ by the continuous representative of $\varphi|\nabla f|^2$, $|\nabla f|^2$ is increasing on the quotient $X/\mathbb{R}$ or, equivalently, on r.

Proof of 4: If $|\nabla f|^2$ vanishes on $r(s)$ for some $s > 0$, then it vanishes on the whole $\{\mathsf{d}_{\mathbb{R}\cdot r(0)} < s\}$ by invariance and monotonicity. If this is the case, then f must be constant in $\{\mathsf{d}_{\mathbb{R}\cdot r(0)} < s\}$, a contradiction. □

As noted earlier, we proved in Lemma 4.1.21 that gradients in (X, d) can be computed as on a manifold with a warped product metric of the form $dr^2 + |\nabla f(r)|^{-2} dx^2$. By Lemma 4.1.15, the measure $\mathfrak{m}$ on (X, d) has a similar warped product structure. By Corollary 4.1.24, the warping function has a reasonable amount of regularity. Making rigorous the isomorphic identification of $(X, \mathsf{d}, \mathfrak{m})$ with a measured warped product is still quite technical at this stage, but the proof below follows a by now well established strategy essentially introduced in the proof of the splitting theorem for $\mathsf{RCD}(0, N)$ spaces in [96].

Lemma 4.1.25. *Under the same assumptions and with the same notation as in Lemma 4.1.12, the restriction*

$$\Psi^{-1}_{|\mathbb{R}\times(0,+\infty)} : \big(\mathbb{R} \times (0, +\infty), \mathsf{d}_{\mathrm{eucl}}\big) \to \big(\{\mathsf{d}_{\mathbb{R}\cdot r(0)} > 0\}, \mathsf{d}\big) \subset (X, \mathsf{d})$$

is locally Lipschitz.

Proof. We denote by $f : X \to \mathbb{R}$ the map introduced in Lemma 4.1.12 under the usual identification $H \cong \mathbb{R}$. We recall that $\Psi^{-1}(a, b) = a \cdot r(b)$. Let $K \subset\subset \mathbb{R} \times (0, +\infty)$ be open. We claim that there exists $C > 1$ such that

$$|f(a \cdot y) - f(y)| = |a| \geq C\mathsf{d}(y, a \cdot y) \quad \text{for every } y, a \cdot y \in \Psi^{-1}(K)\,. \tag{4.39}$$

If the claim holds, given pairs $(a, b), (\widetilde{a}, \widetilde{b}) \in K$, we deduce

$$\begin{aligned}\mathsf{d}(\Psi^{-1}(a, b), \Psi^{-1}(\widetilde{a}, \widetilde{b})) &\leq |b - \widetilde{b}| + \mathsf{d}(a \cdot r(b), [(\widetilde{a} - a) + a] \cdot r(b)) \\ &\leq |b - \widetilde{b}| + C^{-1}|a - \widetilde{a}| \leq \sqrt{2}C^{-1}\mathsf{d}_{\mathrm{eucl}}((a, b), (\widetilde{a}, \widetilde{b}))\,, \end{aligned} \tag{4.40}$$

which implies the statement. To finish the proof we are left to prove (4.39). By item 4 of Corollary 4.1.24, there exists $0 < \widetilde{C} < 1$ such that $|\nabla f| \geq \widetilde{C}$ on $\Psi^{-1}(K)$. Let $y \in \Psi^{-1}(K)$ and let $a > 0$ be such that $a \cdot y \in \Psi^{-1}(K)$. Let $y_i \to y$ be differentiability points of f in $\Psi^{-1}(K)$. Since $|\nabla f| \geq \widetilde{C} > 0$ on $\Psi^{-1}(K)$, using Lemma 4.1.20, for every $i \in \mathbb{N}$ we have that

$$\frac{a/j}{\mathsf{d}(a/j \cdot y_i, y_i)} = \frac{f(a/j \cdot y_i) - f(y_i)}{\mathsf{d}(a/j \cdot y_i, y_i)} \geq \widetilde{C}/2 \quad \text{for every } j \in \mathbb{N} \text{ large enough.}$$

Hence

$$a = j(a/j) \geq \frac{\widetilde{C}}{2}\Big(\mathsf{d}(y_i, a/j \cdot y_i) + \cdots + \mathsf{d}\big([(j-1)a/j] \cdot y_i, a \cdot y_i\big)\Big) \tag{4.41}$$

$$\geq \frac{\widetilde{C}}{2}\mathsf{d}(y_i, a \cdot y_i)\,. \tag{4.42}$$

Passing to the limit as $y_i \to y$, we obtain (4.39) when $a > 0$. The case $a < 0$ is analogous. □

Remark 4.1.26. The examples discussed in [161, 77] show that, in general, the map $\Psi : (X, \mathsf{d}) \to \mathbb{R} \times \mathbb{R}_+$ need not be globally bi-Lipschitz.

By the invariance of $|\nabla f|$ with respect to the $H \cong \mathbb{R}$-action we can identify it with a function on $X/\mathbb{R} \cong \mathbb{R}_+$. We will tacitly make such identification from now on. On $\mathbb{R} \times \mathbb{R}_+$, we consider the C^0 Riemannian metric (with extended values) given by $g(x_1, x_2) := |\nabla f|^{-2}(x_2)\, dx_1^2 + dx_2^2$. Since the continuous representative of $|\nabla f|$ is strictly positive on $(0, +\infty)$ by Corollary 4.1.24, every two points in $\mathbb{R} \times \mathbb{R}_+$ can be joined by a curve of finite length w.r.t. g. Recall that the length of an absolutely continuous curve $\gamma \in AC([0,1], \mathbb{R} \times \mathbb{R}_+)$ with respect to g is defined as

$$\mathrm{length}_g(\gamma) := \int_0^1 g(\gamma', \gamma')^{1/2}\, \mathrm{d}t\,. \tag{4.43}$$

Thus, the metric g induces a real-valued distance d_g on $\mathbb{R} \times \mathbb{R}_+$ by setting

$$\mathsf{d}_g(a, b) := \inf\big\{\mathrm{length}_g(\gamma) : \gamma \in AC([0,1], \mathbb{R} \times \mathbb{R}_+) \text{ and } \gamma(0) = a, \gamma(1) = b\big\}\,. \tag{4.44}$$

The next two lemmas establish some technical variants of the Sobolev-to-Lipschitz property on (X, d) and $(\mathbb{R} \times \mathbb{R}_+, \mathsf{d}_g)$ which are going to be helpful later to complete the proof of Proposition 4.1.3. We recall that we denote by λ^k the k-dimensional Lebesgue measure.

Lemma 4.1.27. *Consider the C^0 Riemannian metric (with extended values) $g(x_1, x_2) := |\nabla f|^{-2}(x_2)\, dx_1^2 + dx_2^2$ and the induced real-valued distance d_g on $\mathbb{R} \times \mathbb{R}_+$. If $\varphi \in \mathsf{Lip}_{\mathrm{loc}}(\mathbb{R} \times (0, +\infty), \mathsf{d}_g)$ has $\mathsf{lip}(\varphi) \leq 1$ at λ^2-a.e. point, then φ admits a 1-Lipschitz extension to $(\mathbb{R} \times \mathbb{R}_+, \mathsf{d}_g)$.*

Proof. Let $a, b \in \mathbb{R} \times (0, +\infty)$. We need to show that $|\varphi(a) - \varphi(b)| \leq \mathsf{d}_g(a, b)$. Without loss of generality we assume that $a, b \in \mathbb{R} \times (1, +\infty)$. By the monotonicity of $|\nabla f|^2$ obtained in Corollary 4.1.24, the set $\mathbb{R} \times (1, +\infty)$ is geodesically convex in $(\mathbb{R} \times \mathbb{R}_+, \mathsf{d}_g)$. Hence, for every $\varepsilon > 0$, there exists a smooth curve $\gamma_\varepsilon : [0,1] \to \mathbb{R} \times (1, +\infty)$

such that $\mathsf{d}_g(a,b) \geq \mathrm{length}(\gamma_\varepsilon) - \varepsilon$. By perturbing γ_ε, one can construct a curve $\widetilde{\gamma}_\varepsilon : [0,1] \to \mathbb{R} \times (0,+\infty)$ such that $\mathsf{d}_g(a,b) \geq \mathrm{length}(\widetilde{\gamma}_\varepsilon) - 2\varepsilon$ and, for λ^1-a.e. $t \in [0,1]$, $\mathsf{lip}(\varphi)(\widetilde{\gamma}_\varepsilon(t)) \leq 1$. Therefore

$$\begin{aligned} |\varphi(a) - \varphi(b)| \leq \int_0^1 (\varphi \circ \widetilde{\gamma}_\varepsilon)' \, \mathrm{d}t &\leq \int_0^1 g(\widetilde{\gamma}_\varepsilon', \widetilde{\gamma}_\varepsilon')^{1/2} \, \mathrm{d}t \\ &= \mathrm{length}(\widetilde{\gamma}_\varepsilon) \leq \mathsf{d}_g(a,b) + 2\varepsilon \,, \end{aligned}$$

where for the second inequality we used [151, Proposition 4.22] which guarantees that

$$\lim_{t \to 0} \frac{\mathsf{d}_g\big(\widetilde{\gamma}_\varepsilon(t+s), \widetilde{\gamma}_\varepsilon(s)\big)}{t} = g(\widetilde{\gamma}_\varepsilon'(s), \widetilde{\gamma}_\varepsilon'(s))^{1/2} \quad \text{for } \lambda^1\text{-a.e. } s \in (0,1) \,. \tag{4.45}$$

Since $\varepsilon > 0$ is arbitrary, the statement follows. □

Lemma 4.1.28. *Under the same assumptions and with the same notation as in Lemma 4.1.11, let $\varphi \in \mathsf{Lip}_{\mathrm{loc}}(X \setminus \{\mathsf{d}_{\mathbb{R} \cdot r(0)} = 0\}) \cap C(X)$ be such that $|\nabla \varphi| \leq 1$ at $\mathfrak{m}$-a.e. point. Then, φ is 1-Lipschitz.*

Proof. Our first claim is that $\mathsf{lip}(\varphi) \leq 1$ everywhere on $X \setminus \{\mathsf{d}_{\mathbb{R} \cdot r(0)} = 0\}$. To this aim, fix $x \in X \setminus \{\mathsf{d}_{\mathbb{R} \cdot r(0)} = 0\}$, and let $\eta \in \mathsf{Lip}_c(X \setminus \{\mathsf{d}_{\mathbb{R} \cdot r(0)} = 0\})$ be a function which is equal to 1 on a neighbourhood of x. By the Bakry-Émery contraction estimate from [11, Theorem 6.2],

$$\mathsf{lip}(P_t(\eta\varphi)) \leq P_t(\mathsf{lip}(\eta\varphi)) \quad \text{pointwise} \,. \tag{4.46}$$

On a sufficiently small neighbourhood of x, $P_t(\mathsf{lip}(\eta\varphi)) \leq 1 + o(1)$ and $P_t(\eta\varphi)$ converges uniformly to φ as $t \to 0$, by the well-known properties of the heat flow on $\mathsf{RCD}(K,N)$ spaces. It follows that, possibly considering a smaller neighbourhood of x, φ is a uniform limit of $(1+o(1))$-Lipschitz functions. The claim follows.

Let $a, b \in X$ and let $\gamma \subset X$ be any geodesic connecting them. By Lemma 4.1.18, either γ has at most two points in $\{\mathsf{d}_{\mathbb{R} \cdot r(0)} = 0\}$, or it is fully contained in $\{\mathsf{d}_{\mathbb{R} \cdot r(0)} = 0\}$. In the former case, $\varphi \circ \gamma$ is absolutely continuous and

$$|\varphi(a) - \varphi(b)| \leq \int_0^{\mathrm{length}(\gamma)} (\varphi \circ \gamma)'(t) \, \mathrm{d}t \leq \mathrm{length}(\gamma) \leq \mathsf{d}(a,b) \,. \tag{4.47}$$

So, assume that $\gamma \subset \{\mathsf{d}_{\mathbb{R} \cdot r(0)} = 0\}$. Fix $\varepsilon > 0$. Let x' be the midpoint of γ. Let x'' be the point at distance ε from x' on the ray starting from x' which realizes the distance from $\{\mathsf{d}_{\mathbb{R} \cdot r(0)} = 0\}$. Let γ_a and γ_b be geodesics from a to x'', and from x'' to b. These geodesics have at most two points each in $\{\mathsf{d}_{\mathbb{R} \cdot r(0)} = 0\}$, and they have length at most

$\mathsf{d}(a,b)/2+\varepsilon$. Let η be the curve (parameterized by arc length) obtained by joining these two geodesics, and observe that

$$|\varphi(a)-\varphi(b)| \le \int_0^{\mathrm{length}(\eta)} (\varphi\circ\eta)'(t)\,\mathrm{d}t \le \mathrm{length}(\eta) \le \mathsf{d}(a,b)+2\varepsilon\,. \tag{4.48}$$

The conclusion follows by the arbitrariness of ε. □

We are ready to complete the proof of the measured warped product structure.

Proof of Proposition 4.1.3. Thanks to Lemma 4.1.10 it suffices to prove the following statement:

Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(0,N)$ space such that $\mathrm{Iso}(X)$ contains a closed subgroup $H<\mathrm{Iso}(X)$ isomorphic to $\mathbb{R}$ and $X/H\cong\mathbb{R}_+$. Let $\Psi: X\to\mathbb{R}\times\mathbb{R}_+$, $f: X\to\mathbb{R}$ and $r:[0,+\infty)\to X$ be as in Lemma 4.1.12, under the usual identification $H\cong\mathbb{R}$. On $\mathbb{R}\times\mathbb{R}_+$, consider the C^0 Riemannian metric (with extended values) given by $g(x_1,x_2):=|\nabla f|^{-2}(x_2)\,dx_1^2+dx_2^2$ and the induced real-valued distance d_g, and let h be the density from Lemma 4.1.15. Then, we claim that

$$\Psi:(X,\mathsf{d},\mathfrak{m})\to(\mathbb{R}\times\mathbb{R}_+,\mathsf{d}_g,h(x_2)dx_1dx_2)$$

is a measure preserving isometry.

The fact that Ψ is measure preserving readily follows from Lemma 4.1.15. In the remaining part of the proof we prove that it is an isometry. We denote by ∇_g gradients w.r.t. g, and by $|\cdot|_g$ the norm of a vector w.r.t. g. Let $\varphi\in\mathsf{Lip}(X)$ be such that $\mathsf{L}(\varphi)\le 1$ and define $\widetilde{\varphi}:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$ as $\widetilde{\varphi}:=\varphi\circ\Psi^{-1}$. We claim that $\widetilde{\varphi}$ is 1-Lipschitz w.r.t. d_g. By Lemma 4.1.25 and Corollary 4.1.24, $\widetilde{\varphi}\in\mathsf{Lip}_{\mathrm{loc}}(\mathbb{R}\times(0,+\infty),\mathsf{d}_g)$. By Lemma 4.1.21, combined with the fact that $|\nabla f|^2\neq 0$ on $\mathfrak{m}$-a.e. $x\in X$ by Corollary 4.1.24,

$$\begin{aligned}|\nabla_g\widetilde{\varphi}|_g^2(\Psi(x)) =&\widetilde{\varphi}_{x_2}^2(f(x),\mathsf{d}_{\mathbb{R}\cdot r(0)}(x))+\widetilde{\varphi}_{x_1}^2(f(x),\mathsf{d}_{\mathbb{R}\cdot r(0)}(x))|\nabla f|^2(x)\\ =&|\nabla\varphi|^2(x)\quad\text{for }\mathfrak{m}\text{-a.e. } x\in X\,.\end{aligned}\tag{4.49}$$

By [151, Proposition 4.25], $\mathsf{lip}_g(\widetilde{\varphi})=|\nabla_g\widetilde{\varphi}|_g$ on λ^2-a.e. point of $\mathbb{R}\times\mathbb{R}_+$, where $\mathsf{lip}_g(\widetilde{\varphi})$ denotes the local Lipschitz constant of $\widetilde{\varphi}$ w.r.t. d_g. By Corollary 4.1.16 and (4.49), we infer that $\mathsf{lip}_g(\widetilde{\varphi})\le 1$ on λ^2-a.e. point of $\mathbb{R}\times(0,+\infty)$. Hence $\widetilde{\varphi}$ is 1-Lipschitz in $(\mathbb{R}\times\mathbb{R}_+,\mathsf{d}_g)$ by Lemma 4.1.27.

Fix any $x\in X$ and let $\mathsf{d}_x: X\to\mathbb{R}$ be defined as $\mathsf{d}_x(y):=\mathsf{d}(x,y)$. The function $\mathsf{d}_x\circ\Psi^{-1}$ is 1-Lipschitz in $(\mathbb{R}\times\mathbb{R}_+,\mathsf{d}_g)$ by the previous part of the proof. Hence

$$\mathsf{d}(x,y)=|\mathsf{d}_x\circ\Psi^{-1}(\Psi(x))-\mathsf{d}_x\circ\Psi^{-1}(\Psi(y))|\le\mathsf{d}_g(\Psi(x),\Psi(y))\,.\tag{4.50}$$

Let $\widetilde{\varphi} \in \mathsf{Lip}(\mathbb{R}\times\mathbb{R}_+, \mathsf{d}_g)$ be a 1-Lipschitz function, and let $\varphi : X \to \mathbb{R}$ be defined as $\varphi := \widetilde{\varphi}\circ\Psi$. We claim that φ is 1-Lipschitz. By Lemma 4.1.13, $\varphi \in \mathsf{Lip}_{\mathrm{loc}}(X\setminus\{\mathsf{d}_{\mathbb{R}\cdot r(0)} = 0\})$. Arguing as before, we deduce that $|\nabla\varphi| \le 1$ on $\mathfrak{m}$-a.e. point of X. Thus φ is 1-Lipschitz on X by Lemma 4.1.28. Consider any point $a \in \mathbb{R}\times\mathbb{R}_+$, and let $\mathsf{d}_{g,a} : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ be defined as $\mathsf{d}_{g,a}(b) := \mathsf{d}_g(a,b)$. By the previous considerations $\mathsf{d}_{g,a}\circ\Psi$ is 1-Lipschitz on X. Hence,

$$\mathsf{d}_g(a,b) = |\mathsf{d}_{g,a}\circ\Psi(\Psi^{-1}(a)) - \mathsf{d}_{g,a}\circ\Psi(\Psi^{-1}(b))| \le \mathsf{d}(\Psi^{-1}(a),\Psi^{-1}(b))\,. \tag{4.51}$$

The combination of (4.50) and (4.51) shows that Ψ is an isometry, thus completing the proof. □

4.1.2 Proof of Theorem 4.1.1, base step: splitting

Theorem 4.1.29 below corresponds to the base case $k = 1$ in Theorem 4.1.1. Proving it is the goal of this section.

Theorem 4.1.29. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(0,N)$ space such that $\mathrm{Iso}(X)$ contains a closed subgroup $H < \mathrm{Iso}(X)$ isomorphic to $\mathbb{R}$, $\dim_{\mathrm{e}}(X) \le 2$, and*

$$\int_1^{+\infty} \frac{t^2}{\mathfrak{m}(B_t(p))}\,\mathrm{d}t = +\infty\,. \tag{4.52}$$

Then $X \cong Y\times\mathbb{R}$ as metric measure spaces for some $\mathsf{RCD}(0,N-1)$ space $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$.

The proof of Theorem 4.1.29 relies on the measured warped product rigidity obtained in Proposition 4.1.3, and on the equivalence between synthetic and distributional lower bounds on the weighted Ricci curvature valid for sufficiently regular weighted Riemannian metrics (Theorem 4.1.30 below).

Given a Riemannian metric g on $\mathbb{R}^n$ such that $g_{ij} \in \mathsf{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)\cap C(\mathbb{R}^n)$, one can compute the distributional Christoffel symbols, defined as

$$\Gamma^k_{ij} := \frac{1}{2}\,g^{k\ell}\left(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\right)\,. \tag{4.53}$$

Thanks to the regularity of g, the Christoffel symbols belong to $\mathsf{L}^2_{\mathrm{loc}}(\mathbb{R}^n)$. Hence, one can define the distributional Ricci curvature as

$$(\mathsf{Ric}_g)_{ij} := \partial_k\Gamma^k_{ij} - \partial_j\Gamma^k_{ik} + \Gamma^k_{ij}\Gamma^\ell_{k\ell} - \Gamma^k_{i\ell}\Gamma^\ell_{jk}\,. \tag{4.54}$$

Given a function $V \in \mathsf{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)\cap C(\mathbb{R}^n)$, its distributional Hessian is defined as

$$(\mathrm{Hess}_g(V))_{ij} := \partial^2_{i,j}V - \Gamma^k_{ij}\,\partial_k V\,. \tag{4.55}$$

The next result is proved in [151].

Theorem 4.1.30. *Let (M^n, g) be a Riemannian manifold with boundary with g a continuous Riemannian metric with Christoffel symbols in $\mathsf{L}^2_{loc}(M)$. Let $V \in C^0(M) \cap \mathsf{W}^{1,2}_{loc}(M)$, and assume that $(M, \mathsf{d}_g, e^{-V}\mathsf{dVol}_g)$ is an $\mathsf{RCD}(0,\infty)$ space. Then,*

$$\mathsf{Ric}_M + \mathrm{Hess}_V \geq 0 \quad \text{on } M \setminus \partial M \text{ in distributional sense.}$$

Consider now a function $\varphi \in \mathsf{W}^{1,2}_{\mathrm{loc}}((0,+\infty)) \cap C((0,+\infty))$, and the manifold $(0,+\infty) \times \mathbb{R}$ equipped with the warped product metric $g(s,t) := \mathrm{d}s^2 + \varphi^2(s)\mathrm{d}t^2$. We will denote as $(0,+\infty) \times_\varphi \mathbb{R}$ the resulting Riemannian manifold. Specializing (4.53) to this simplified setting, we deduce

$$\begin{aligned}
\Gamma^s_{t,t} &= -\varphi\varphi', \\
\Gamma^t_{t,s} &= \Gamma^t_{s,t} = \frac{\varphi'}{\varphi}, \\
\Gamma^s_{s,s} &= \Gamma^t_{t,t} = \Gamma^t_{s,s} = 0\,.
\end{aligned}$$

Hence, using (4.54), we obtain

$$(\mathsf{Ric}_{(0,+\infty)\times_\varphi\mathbb{R}})_{t,t} := -(\varphi\varphi')' + (\varphi')^2\,. \tag{4.56}$$

Similarly, using (4.55), we deduce

$$(\mathrm{Hess}_{(0,+\infty)\times_\varphi\mathbb{R}}(V))_{t,t} := \varphi\varphi' V'\,. \tag{4.57}$$

Proof of Theorem 4.1.29. Thanks to Proposition 4.1.3, we can assume that $(X, \mathsf{d}, \mathfrak{m})$ is equivariantly isomorphic to a measured warped product $\mathbb{R}_+ \times_\varphi \mathbb{R}$, with metric and measure given respectively by

$$g(r,x) := \mathrm{d}r^2 + \varphi(r)^2\,\mathrm{d}x^2\,, \qquad \mathfrak{m} := h(r)\,\mathrm{d}r\,\mathrm{d}x\,, \tag{4.58}$$

for some increasing function $h \in \mathsf{Lip}_{\mathrm{loc}}((0,+\infty)) \cap \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$ and some continuous and strictly positive function $\varphi : (0,+\infty) \to (0,+\infty)$, with the $\mathbb{R}$-action being by translation on the x component and d the length distance induced by g. Under such identification, the equivariant function $f : X \to \mathbb{R}$ introduced in Lemma 4.1.12 admits the clear expression $f(r,x) = x$ and we have $\varphi = |\nabla f|^{-1}$, by direct computation.

The 2-dimensional Hausdorff measure with respect to g on X can be written in coordinates as $\varphi(r)drdx$. By (4.58), there exists a function $F : \mathbb{R}_+ \to \mathbb{R}_+$ such that $F\varphi \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_+, \lambda^1)$ and

$$\mathfrak{m} = F(r)\varphi(r)drdx\,. \tag{4.59}$$

Let $\mathsf{d}_{g,0} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $\mathsf{d}_{g,0}(a) := \mathsf{d}_g(a, \mathbb{R} \times \{0\})$. According to [47, Section 7], consider the disintegration of $\mathfrak{m}$ induced by the 1-Lipschitz function $\mathsf{d}_{g,0}$. By [47, Main Theorem 1.1] and [47, Theorem A.2],

$$(F\varphi)^{\frac{1}{N-1}} : (0, +\infty) \to (0, +\infty) \tag{4.60}$$

is a concave function. Since $\varphi \in C^0(0, +\infty) \cap \mathsf{W}^{1,2}_{\mathrm{loc}}(0, +\infty)$ and it is non-increasing and strictly positive by Corollary 4.1.24, $F \in C^0[0, +\infty) \cap \mathsf{W}^{1,2}_{\mathrm{loc}}(0, +\infty)$. Using again that $\varphi \in C^0(0, +\infty) \cap \mathsf{W}^{1,2}_{\mathrm{loc}}(0, +\infty)$ we deduce that the distributional Christoffel symbols of g are in $\mathsf{L}^2_{\mathrm{loc}}((0, +\infty) \times \mathbb{R})$. Note that this condition is independent of whether we consider as reference measure $\mathfrak{m}$ or λ^2. The remaining part of the proof is divided into three steps. In Step 1 we use the equivalence between synthetic and distributional lower bound on the Ricci curvature to deduce that φ is super-harmonic on the weighted half-lines $([\delta, +\infty), \mathsf{d}_{\mathrm{eucl}}, F\lambda^1)$ for every $\delta > 0$. In Step 2 we reformulate the growth condition (4.52) into a growth condition for F. In Step 3 we combine the results of the previous two steps to show that φ is constant. This is clearly enough to complete the proof of the splitting of $(X, \mathsf{d}, \mathfrak{m})$.

Step 1: We claim that for every $\delta > 0$, and every non-negative function $\psi \in \mathsf{Lip}_c([\delta, +\infty))$, there holds

$$\int_{[\delta, +\infty)} \varphi' \psi' \, F \mathrm{d}\lambda^1 \geq 0 \,. \tag{4.61}$$

Note that the integral in (4.61) is well defined and finite thanks to Corollary 4.1.24. Set $V := -\log(F) : \mathbb{R}_+ \to \mathbb{R}$. By Theorem 4.1.30, since $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}(0, N)$ space, and thus in particular it is an $\mathsf{RCD}(0, \infty)$ space, we have that

$$\mathsf{Ric}_{\mathbb{R}_+ \times_\varphi \mathbb{R}} + \mathrm{Hess}_{\mathbb{R}_+ \times_\varphi \mathbb{R}}(V) \geq 0 \quad \text{distributionally on } (0, +\infty) \times \mathbb{R} \,. \tag{4.62}$$

Using (4.56) and (4.57), we obtain that

$$-(\varphi\varphi')' + (\varphi')^2 + \varphi\varphi' V' \geq 0 \quad \text{in distributional sense in } (\delta, +\infty) \,. \tag{4.63}$$

By a straightforward manipulation, (4.63) is equivalent to

$$\varphi'' - \varphi' V' \leq 0 \quad \text{in distributional sense on } (\delta, +\infty) \,,$$

which in turn is equivalent to

$$\int_\delta^{+\infty} \varphi' \psi' \, F \mathrm{d}\lambda^1 \geq 0 \quad \text{for every non-negative } \psi \in \mathsf{Lip}_c((\delta, +\infty)) \,. \tag{4.64}$$

We continue proving (4.61) for any non-negative function $\psi \in \mathsf{Lip}_c([\delta, +\infty))$. For every $\varepsilon > 0$, consider the function $\eta_\varepsilon \in \mathsf{Lip}_c([\delta, +\infty))$ defined by

$$\eta_\varepsilon(t) := 1 \wedge (1 - \varepsilon^{-1}(t - \delta - \varepsilon)) \vee 0\,. \tag{4.65}$$

Using the Leibniz rule for the second equality, we have that

$$\begin{aligned}
&\int_{[\delta,+\infty)} \varphi'\psi'\, F \mathrm{d}\lambda^1 \\
&= \int_{[\delta,+\infty)} \varphi'(\psi\eta_\varepsilon)'\, F \mathrm{d}\lambda^1 + \int_{[\delta,+\infty)} \varphi'(\psi(1-\eta_\varepsilon))'\, F \mathrm{d}\lambda^1 \\
&= \int_{[\delta,+\infty)} \varphi'(\psi\eta_\varepsilon' + \eta_\varepsilon\psi')\, F \mathrm{d}\lambda^1 + \int_{[\delta,+\infty)} \varphi'(\psi(1-\eta_\varepsilon))'\, F \mathrm{d}\lambda^1.
\end{aligned}$$

Since $\varphi'\eta_\varepsilon' \geq 0$ in $(0, +\infty)$ by Corollary 4.1.24, passing to the limit in the previous equation as $\varepsilon \to 0$ and taking into account (4.64), we deduce (4.61).

Step 2: Let $A : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $A(r) := \int_0^r F(t)\,\mathrm{d}t$. Note that $A(r)$ is the measure of the r-ball centered at 0 in the weighted half-line $([0, +\infty), \mathsf{d}_{\mathrm{eucl}}, F\lambda^1)$. We claim that

$$\int_1^{+\infty} \frac{t}{A(t)}\,\mathrm{d}t = +\infty\,. \tag{4.66}$$

By triangle inequality, there holds

$$B_r^g((0,0)) \supset \Big\{(a,b) \in \mathbb{R} \times \mathbb{R}_+ : b \in (0, r/2),\, a \in \Big(-r\frac{\varphi(b)^{-1}}{2}, r\frac{\varphi(b)^{-1}}{2}\Big)\Big\}\,.$$

Hence,

$$\mathfrak{m}(B_r^g(0,0)) \geq \int_0^{r/2} \int_{-r\frac{\varphi(s)^{-1}}{2}}^{r\frac{\varphi(s)^{-1}}{2}} \varphi(s)F(s)\,dxds = r\int_0^{r/2} F(t)\,\mathrm{d}t = rA(r/2)\,.$$

Therefore, using (4.52), we conclude that

$$\int_1^{+\infty} \frac{t}{A(t)}\,\mathrm{d}t = \int_2^{+\infty} \frac{t}{4A(t/2)}\,\mathrm{d}t \geq \int_2^{+\infty} \frac{t^2}{4\mathfrak{m}\big(B_t^g((0,0))\big)}\,\mathrm{d}t = +\infty\,,$$

thus proving (4.66).

Step 3: We claim that φ is constant. As noted before, this is clearly enough to prove that $(X, \mathsf{d}, \mathfrak{m})$ splits a line.

By Lemma 4.1.31 below and Step 1, for every $\delta > 0$ we have that

$$\int_\delta^{+\infty} \psi^2 \frac{\varphi'^2}{(1+\varphi)^2}\, F \mathrm{d}t \leq 16 \int_\delta^{+\infty} \psi'^2\, F \mathrm{d}t\,, \tag{4.67}$$

for every non-negative function $\psi \in \mathsf{Lip}_c([\delta, +\infty))$. Fix any $R > \delta$. We claim that φ is constant on $[\delta, R]$. Using the notation of Step 2, consider the functions $\psi_i \in \mathsf{Lip}_c([\delta, +\infty))$ defined by

$$\psi_i(t) := \Big(\int_{R \vee t \wedge i}^{i} \frac{s}{A(s)}\, ds \Big) \Big/ \Big(\int_R^i \frac{s}{A(s)}\, ds \Big) .$$

We claim that

$$\int_\delta^{+\infty} \psi_i'^2\, F \mathrm{d}t \to 0 \quad \text{as } i \to +\infty , \tag{4.68}$$

which, combined with (4.67), would imply that φ is constant on $[\delta, R]$. To prove the claim, we can compute

$$\begin{aligned} \int_\delta^{+\infty} \psi_i'^2\, F \mathrm{d}t &= \Big(\int_R^i t^2 F(t)/A^2(t)\, \mathrm{d}t \Big) \Big/ \Big(\int_R^i \frac{s}{A(s)}\, ds \Big)^2 \\ &= \Big(-\frac{i^2}{A(i)} + \frac{R^2}{A(R)} + 2 \int_R^i \frac{s}{A(s)}\, ds \Big) \Big/ \Big(\int_R^i \frac{s}{A(s)}\, ds \Big)^2 . \end{aligned}$$

The right-hand side of the previous equality converges to zero as $i \to +\infty$ thanks to Step 2, thus proving (4.68). It follows from (4.67) that φ is constant on $[\delta, R]$. By the arbitrariness of δ and R, φ is constant on $(0, +\infty)$, as we claimed. □

The next lemma follows by adapting Yau's result from [190] (see also [137, Lemma 7.1]).

Lemma 4.1.31. *Let $f, u \in \mathsf{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ be positive functions such that*

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla \varphi\, f \mathrm{d}\lambda^n \geq 0 \quad \textit{for every non-negative } \varphi \in \mathsf{W}^{1,2}(\mathbb{R}^n) \cap C_c(\mathbb{R}^n) . \tag{4.69}$$

Then,

$$\int_{\mathbb{R}^n} \varphi^2 \frac{|\nabla u|^2}{(1+u)^2}\, f \mathrm{d}\lambda^n \leq 16 \int_{\mathbb{R}^n} |\nabla \varphi|^2\, f \mathrm{d}\lambda^n ,$$

for every non-negative $\varphi \in \mathsf{W}^{1,2}(\mathbb{R}^n) \cap C_c(\mathbb{R}^n)$.

Proof. We define $v \in \mathsf{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ as $v := (1+u)^{-1}$. Using the chain rule and the Leibniz rule, for every $\varphi \in \mathsf{W}^{1,2}(\mathbb{R}^n) \cap C_c(\mathbb{R}^n)$, we get

$$0 \geq \int_{\mathbb{R}^n} \nabla(\varphi^2 v) \cdot \nabla v\, f \mathrm{d}\lambda^n = 2 \int_{\mathbb{R}^n} \varphi v \nabla \varphi \cdot \nabla v\, f \mathrm{d}\lambda^n + \int_{\mathbb{R}^n} \varphi^2 |\nabla v|^2\, f \mathrm{d}\lambda^n ,$$

so that

$$\int_{\mathbb{R}^n} \varphi^2 |\nabla v|^2\, f \mathrm{d}\lambda^n \leq 2 \int_{\mathbb{R}^n} |\varphi v \nabla \varphi \cdot \nabla v|\, f \mathrm{d}\lambda^n . \tag{4.70}$$

Applying Young's inequality we get

$$2\int_{\mathbb{R}^n}\varphi v\nabla\varphi\cdot\nabla v\,f\mathrm{d}\lambda^n\le\frac{1}{2}\int_{\mathbb{R}^n}\varphi^2|\nabla v|^2\,f\mathrm{d}\lambda^n+8\int_{\mathbb{R}^n}|\nabla\varphi|^2v^2\,f\mathrm{d}\lambda^n\,,$$

which combined with (4.70) gives

$$\int_{\mathbb{R}^n}\varphi^2|\nabla v|^2\,f\mathrm{d}\lambda^n\le 16\int_{\mathbb{R}^n}|\nabla\varphi|^2v^2\,f\mathrm{d}\lambda^n\,.$$

Recalling the definition of v, the statement follows. □

4.1.3 Proof of Theorem 4.1.1, inductive step

The goal of this section is to complete the inductive proof of Theorem 4.1.1 by proving the inductive step. The next three lemmas are needed to carry out such inductive step.

Lemma 4.1.32. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(0,N)$ space satisfying the following conditions:*

1. $\dim_{\mathrm{e}}(X)\le k+1$;
2. $\int_1^{+\infty}\frac{t^{k+1}}{\mathfrak{m}(B_t(p))}\,\mathrm{d}t=+\infty$ *for some* $p\in X$;
3. *There exists a closed subgroup* $\mathbb{R}^k\cong H<\mathrm{Iso}(X)$.

Let $H''\le H$ be a closed subgroup isomorphic to $\mathbb{R}$. Consider the quotient space $(X/H'',\mathsf{d}'')$ equipped with the measure $\mathfrak{m}''$ provided by Remark 4.1.6. Then, denoting by $B''_t(p'')$ balls in $(X/H'',\mathsf{d}'')$ centered in $p''=H''\cdot p$, we have that:

1. $\dim_{\mathrm{e}}(X/H'')\le k$.
2. $\int_1^{+\infty}\frac{t^k}{\mathfrak{m}''(B''_t(p''))}\,\mathrm{d}t=+\infty$.
3. $\mathrm{Iso}(X/H'')$ *contains a closed subgroup isomorphic to* $\mathbb{R}^{k-1}$.

Proof. Item 1 immediately follows from Lemma 4.1.7. To prove item 3, it suffices to observe that the quotient $\mathbb{R}^{k-1}\cong H/H''$ is isomorphic to a closed subgroup of $\mathrm{Iso}(X/H'')$. Hence, we only need to show item 2, i.e. that

$$\int_1^{+\infty}\frac{t^k}{\mathfrak{m}''(B''_t(p''))}\,\mathrm{d}t=+\infty\,. \tag{4.71}$$

Let $H'\le H''$ be the subgroup isomorphic to $\mathbb{Z}$ generated by $1\in H''\cong\mathbb{R}$, and consider the natural $\mathsf{RCD}(0,N)$ structure $(X/H',\mathsf{d}',\mathfrak{m}')$. We denote by $B'_t(p')$ balls

in $(X/H', \mathsf{d}')$ centered in $p' := p + H'$. As an intermediate step, we show that there exists $c > 0$ such that

$$\mathfrak{m}''(B''_t(p'')) \leq \mathfrak{m}'(B'_{t+c}(p')) \quad \text{for every } t > 0\,. \tag{4.72}$$

To this aim, we observe that by definition of $\mathfrak{m}''$,

$$\mathfrak{m}''(B''_t(p'')) = \mathfrak{m}'\{x \in X/H' : \mathsf{d}'(x + H''/H', p') \leq t\}\,. \tag{4.73}$$

Since H''/H' is a compact group, there exists $c > 0$, such that $p' + H''/H' \subset B'_c(p')$. Combining with (4.73), (4.72) follows. Our next claim is that there exists $c' > 0$ such that

$$t\mathfrak{m}'(B'_{t/2}(p')) \leq c'\mathfrak{m}(B_t(p)) \quad \text{for every } t \geq 0 \text{ large enough}\,. \tag{4.74}$$

Note that the combination of (4.74) and (4.72) would give (4.71). To prove the claim, consider the projection $\pi : X \to X/H'$, and observe that it is a covering map. Let $F \subset X$ be the fundamental domain containing p provided by Lemma 5.3.6. By construction,

$$\mathfrak{m}'(B'_t(p')) \leq \mathfrak{m}(B_t(p) \cap F)\,, \quad \forall t > 0\,. \tag{4.75}$$

Assume w.l.o.g. that the generator h of $H' \cong \mathbb{Z}$ satisfies $\mathsf{d}(p, p + h) = 1$. By the triangle inequality,

$$B_t(p) \supset \bigcup_{i=1}^{\lfloor t/2 \rfloor} h^i + (B_{t/2}(p) \cap F)\,. \tag{4.76}$$

Since $\mathfrak{m}((h^i + F) \cap F) = 0$ by Lemma 5.3.6 again, we deduce that

$$\mathfrak{m}(B_t(p)) \geq \lfloor t/2 \rfloor \mathfrak{m}'(B'_{t/2}(p'))\,, \tag{4.77}$$

thus yielding (4.74). □

The following lemma is well known, thus its elementary proof will be omitted.

Lemma 4.1.33. *Let (X, d) be a metric space, and let $H < \mathrm{Iso}(X \times \mathbb{R})$ be a closed subgroup. Consider the resulting quotient space $(X \times \mathbb{R})/H$, and assume that it is isometric to $Y \times \mathbb{R}$ for some metric space (Y, d_y). Let $\pi : X \times \mathbb{R} \to Y \times \mathbb{R}$ be the projection on the quotient, and assume that $\pi((x, t)) = (y, t)$ for some $x \in X$, $y \in Y$ and every $t \in \mathbb{R}$. Then*

$$H < \mathrm{Iso}(X) \times \{\mathrm{Id}\} < \mathrm{Iso}(X \times \mathbb{R})\,. \tag{4.78}$$

Applying iteratively Lemma 4.1.33 and the splitting theorem for $\mathsf{RCD}(0,N)$ spaces from [96], and using that lines can be lifted under metric submetries, we obtain the following:

Lemma 4.1.34. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(0,N)$ space, and let $H<\mathrm{Iso}(X)$ be a closed subgroup. Assume that $X/H\cong\mathbb{R}^k\times Y$ isometrically. Then, $X\cong\mathbb{R}^k\times Y'$ isometrically, and*

$$H<\mathrm{Iso}(Y')\times\{\mathrm{Id}\}<\mathrm{Iso}(Y'\times\mathbb{R}^k)\,. \tag{4.79}$$

Proof of Theorem 4.1.1. We argue by induction. The base step corresponds to Theorem 4.1.29. For the inductive step, we assume that the statement holds for every j such that $1\le j\le k-1$, and we show that it holds for k. Let $H'\cong\mathbb{R}$ be a closed subgroup of H, and consider the quotient space $(X/H',\mathsf{d}'',\mathfrak{m}'')$ with the induced $\mathsf{RCD}(0,N)$ structure, as in Remark 4.1.6. By Lemma 4.1.32 the space X/H' satisfies the hypotheses of the theorem for $k-1$. Thus, by the inductive hypothesis, $X/H'\cong\mathbb{R}^{k-1}\times Y$ as metric measure spaces, for some $\mathsf{RCD}(0,N-k+1)$ space $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$. By Lemma 4.1.34, we also have that $X\cong\mathbb{R}^{k-1}\times Y'$ isometrically, and that

$$H'<\{\mathrm{Id}\}\times\mathrm{Iso}(Y')<\mathrm{Iso}(\mathbb{R}^{k-1})\times\mathrm{Iso}(Y')\,. \tag{4.80}$$

Hence, Y' splits off a line isometrically thanks to the base case $k=1$. □

4.1.4 Proof of Corollary 4.1.2

We distinguish two cases, according to the volume growth rate at infinity. The case of Euclidean volume growth is much easier and does not require Theorem 4.1.1.

Lemma 4.1.35. *Let $(X,\mathsf{d},\mathsf{H}^n)$ be an $\mathsf{RCD}(0,n)$ space with a closed subgroup $\mathbb{Z}^{n-2}\cong H<\mathrm{Iso}(X)$ and such that*

$$\lim_{s\to+\infty}\frac{\mathsf{H}^n(B_s(p))}{s^n}=v>0\quad\text{for some } p\in X\,. \tag{4.81}$$

Then, either $X\cong\mathbb{R}^{n-1}\times Y$ isometrically, or the blow-down X_∞ of X is unique and isometric to $\mathbb{R}^{n-2}\times Z$, for some $Z\in\{C(S^1_r),C([0,a])\}$ with $r\in(0,1)$, $a\in(0,\pi)$. In the latter case, if (X_∞,H_∞) is an equivariant blow-down of (X,H), then $H_\infty<\mathrm{Iso}(\mathbb{R}^{n-2})\times\mathrm{Iso}(Z)$ and $\pi_{\mathrm{Iso}(\mathbb{R}^{n-2})}(H_\infty)\cong\mathbb{R}^{n-2}$ acts by translations on $\mathbb{R}^{n-2}$.

Proof. By (4.81) and [79, 80] after [53, Theorem 7.6], every blow-down $(X_\infty,\mathsf{d}_\infty,p_\infty)$ of X is an n-dimensional metric cone centered at its tip p_∞. Fix one such X_∞. By Lemma 2.2.23, there exists a closed subgroup of $\mathrm{Iso}(X_\infty)$ isomorphic to $\mathbb{R}^{n-2}$.

By cone-splitting, $X_\infty \cong \mathbb{R}^{n-2} \times C(W)$ isometrically, where $W \in \{S^1_r, [0,a]\}$. If $W = S^1_1$, then $X \cong \mathbb{R}^n$ by [80, Theorem 1.6]. If $W = [0,\pi]$, then $X \cong \mathbb{R}^{n-1} \times \mathbb{R}_+$ by [37, Theorem 8.1]. Thus, we reduce to the case when $Z \in \{C(S^1_r), C([0,a])\}$ and $r \in (0,1), a \in (0,\pi)$. By volume convergence, see [80, Theorem 1.3] after [63, 54], $\mathsf{H}^n(B_1(p_\infty)) = v$. Thus, the radius $r \in (0,1)$ or the parameter $a \in (0,\pi)$ such that $X_\infty \cong \mathbb{R}^{n-2} \times C(S^1_r)$ or $X_\infty \cong \mathbb{R}^{n-2} \times C([0,a])$ are determined by the asymptotic volume ratio of X. In particular, there are at most two non-isometric choices for the blow-down of X. By Lemma 2.2.12, the blow-down of X must be unique.

The second part of the statement now readily follows from [160, Theorem 3.2]. □

In the case where the volume growth is non-Euclidean, we begin by studying the blow-down of the quotient space with respect to the $\mathbb{Z}^{n-2}$-action.

Lemma 4.1.36. *Let $(X, \mathsf{d}, \mathsf{H}^n)$ be an $\mathsf{RCD}(0,n)$ space with a closed subgroup $\mathbb{Z}^{n-2} \cong H < \mathrm{Iso}(X)$ and such that*

$$\lim_{s\to+\infty} \frac{\mathsf{H}^n(B_s(p))}{s^n} = 0 \quad \text{for some } p \in X\,. \tag{4.82}$$

Then, either $X \cong \mathbb{R}^{n-2} \times Z$ as metric measure spaces, or the blow-down of X/H is unique up to pointed isometry and pointed isometric to $(\mathbb{R}_+, \mathsf{d}_{\mathrm{eucl}}, 0)$.

Proof. Set $Y := X/H$, and consider its induced $\mathsf{RCD}(0,n)$ structure $(Y, \overline{\mathsf{d}}, \overline{\mathfrak{m}})$. By Theorem 2.2.24, we can assume without loss of generality that Y is non-compact. Consider any blow-down $(Y_\infty, \overline{\mathsf{d}}_\infty, \overline{\mathfrak{m}}_\infty, y_\infty)$ of Y. Up to taking a subsequence, the corresponding rescalings of X converge to an equivariant blow-down $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, H_\infty, x_\infty)$ of $(X, \mathsf{d}, \mathfrak{m}, H)$. By Lemma 2.2.23, H_∞ contains a closed subgroup isomorphic to $\mathbb{R}^{n-2}$. By Theorem 2.1.23, $X_\infty/H_\infty \cong Y_\infty$ isometrically. Thus, by Lemma 4.1.7, $\dim_{\mathrm{e}}(Y_\infty) \leq 1$. Since Y is non-compact, $\dim_{\mathrm{e}}(Y_\infty) = 1$ and Y_∞ is non-compact. Thus, by the classification of 1-dimensional RCD spaces from [133], Y_∞ is isometric to $\mathbb{R}$ or $\mathbb{R}_+$.

If Y has a blow-down isometric to $\mathbb{R}$, then $Y \cong \mathbb{R} \times Y'$, for some compact $\mathsf{RCD}(0, n-1)$ space Y' by Lemma 2.2.13. By Lemma 4.1.34, $X \cong \mathbb{R} \times X'$, $H < \{\mathrm{Id}\} \times \mathrm{Iso}(X') < \mathrm{Iso}(\mathbb{R}) \times \mathrm{Iso}(X')$. Hence, with a slight abuse of notation, $X'/H \cong Y'$. By Theorem 2.2.24 again, X' splits off $n-2$ Euclidean factors.

Assume that every blow-down (Y_∞, y_∞) of Y is isometric to $\mathbb{R}_+$. In this case, we claim that $y_\infty = 0$. If not, by taking a blow-up of Y_∞ at y_∞ and using that the blow-up of a blow-down is still a blow-down, we obtain a blow-down of Y isometric to $\mathbb{R}$, thus reaching a contradiction. □

The combination of Lemma 4.1.37 below with Lemma 4.1.35 is clearly sufficient to prove Corollary 4.1.2.

In the statement of Lemma 4.1.37 below, we obtain an Abelian group $H_\infty < \mathrm{Iso}(\mathbb{R}^{n-2} \times Z)$ with $Z \in \{\mathbb{R}, \mathbb{R}_+\}$ such that $(\mathbb{R}^{n-2} \times Z)/H_\infty \cong \mathbb{R}_+$. Then, we are led to consider $H_\infty^2 := \{h^2 : h \in H_\infty\}$ to obtain a free action. Intuitively, this is necessary in the case where $Z = \mathbb{R}$ to kill the reflections fixing a hyperplane of $\mathbb{R}^{n-1}$.

Lemma 4.1.37. *Let $(X, \mathsf{d}, \mathsf{H}^n)$ be an $\mathsf{RCD}(0,n)$ space with a closed subgroup $\mathbb{Z}^{n-2} \cong H < \mathrm{Iso}(X)$ and such that*

$$\lim_{s \to +\infty} \frac{\mathsf{H}^n(B_s(p))}{s^n} = 0 \quad \text{for some } p \in X\,. \tag{4.83}$$

Then, either $X \cong \mathbb{R}^{n-2} \times Y$ isometrically, or the blow-down X_∞ of X is unique up to pointed isometry and isometric to $\mathbb{R}^{n-2} \times Z$, where $Z \in \{\mathbb{R}, \mathbb{R}_+\}$. Moreover, if (X_∞, H_∞) is an equivariant blow-down of (X, H), then $H_\infty^2 := \{h^2 : h \in H_\infty\}$ acts freely on X_∞.

Proof. Set $Y := X/H$. By Lemma 4.1.36, we can assume without loss of generality that the blow-down of Y is unique (up to pointed isometry) and pointed isometric to $(\mathbb{R}_+, \mathsf{d}_{\mathrm{eucl}}, 0)$. Let $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, H_\infty, x_\infty)$ be an equivariant blow-down of $(X, \mathsf{d}, \mathfrak{m}, H)$. By Theorem 2.1.23, $X_\infty/H_\infty = \mathbb{R}_+$, with the orbit $x_\infty \cdot H_\infty$ corresponding to $0 \in \mathbb{R}_+$. By Lemma 2.2.23, H_∞ contains a closed subgroup isomorphic to $\mathbb{R}^{n-2}$. By (4.83), $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ is an $\mathsf{RCD}(0,n)$ space with $\dim_{\mathrm{e}}(X_\infty) \le n-1$. Thus, by Theorem 4.1.1, $X_\infty := \mathbb{R}^{n-2} \times Z'$, where $\dim_{\mathrm{e}}(Z') \le 1$. Since $X_\infty/H_\infty \cong \mathbb{R}_+$, by Lemma 4.1.7 we infer that $\dim_{\mathrm{e}}(Z') = 1$. If a blow-down X is isometric to $\mathbb{R}^{n-2} \times K$ where K is compact then by taking another blow-down we obtain a contradiction. Therefore, all blow-downs of X are isometric to $\mathbb{R}^{n-2} \times Z$ with $Z \in \{\mathbb{R}, \mathbb{R}_+\}$. Moreover, if $X_\infty \cong \mathbb{R}^{n-2} \times \mathbb{R}_+$, then $x_\infty \in \partial X_\infty$. Hence, the space of *pointed* blow-downs of X contains at most two elements. Since such space is connected by Lemma 2.2.12, it follows that blow-down of X is unique up to pointed isometries.

To prove the second part of the statement, let G_∞ denote a closed subgroup of H_∞ isomorphic to $\mathbb{R}^{n-2}$ obtained from Lemma 2.2.23. Observe that H_∞ is abelian, since all the approximating groups along the blow-down sequence are. The quotient group H_∞/G_∞ acts effectively by measure preserving isometries on X_∞/G_∞. By Lemma 4.1.7, X_∞/G_∞ has essential dimension at most 1, and $(X_\infty/G_\infty)/(H_\infty/G_\infty) \cong \mathbb{R}_+$. It follows that X_∞/G_∞ is isometric to either $\mathbb{R}$ or $\mathbb{R}_+$, and that H_∞/G_∞ is either generated by a reflection of $\mathbb{R}$, or the identity on $\mathbb{R}_+$. In both cases, if $h \in H_\infty$, then $h^2 \in G_\infty$. Since $G_\infty \cong \mathbb{R}^{n-2}$, h^2 acts freely on X_∞. □

Corollary 4.1.38. *Let $(X, \mathsf{d}, \mathsf{H}^n)$ be an $\mathsf{RCD}(0,n)$ space with a closed subgroup $\mathbb{Z}^{n-2} \cong H < \mathrm{Iso}(X)$, and assume that X does not split a factor $\mathbb{R}^{n-2}$. Then there is $k \in \{n-2, n-1\}$ such that the blow-down of X is unique (up to pointed isometry), pointed isometric to $(\mathbb{R}^k \times Y, (0,y))$ for some pointed metric space (Y, d_Y, y), and $\mathrm{Iso}(\mathbb{R}^k \times Y) \cdot (0,y) \subset \mathbb{R}^k \times \{y\}$.*

Proof. The conclusion follows combining Lemmas 4.1.37 and 4.1.35 with Lemma 2.2.25. □

4.2 Equivariant harmonic functions with almost linear growth

We continue to work under the assumption that $(X, \mathsf{d}, \mathsf{H}^n)$ is an $\mathsf{RCD}(0,n)$ space such that $\mathrm{Iso}(X)$ contains a closed subgroup $H \cong \mathbb{Z}^{n-2}$. Our first goal in this section is to construct an $\mathbb{R}^k$-valued harmonic map $u : X \to \mathbb{R}^k$, with $k \in \{n-2, n-1\}$ determined by the dimension of the maximal Euclidean factor split-off by the blow-down of X which is:

i) equivariant with respect to the $H^2 < H$-action (we pass to H^2 to obtain a limiting free action in the equivariant blow-downs; This is a consequence of Lemma 4.1.37 and the comments preceding it);

ii) almost-splitting up to linear transformation at every sufficiently large scale;

iii) *non-degenerate* along the H^2-orbit of the base-point, after the linear corrections appearing in ii).

For the precise meaning of the non-degeneracy condition in iii) we address the reader to the statement of Proposition 4.2.3.

Proposition 4.2.1 below addresses the first two items of the plan.

Proposition 4.2.1. *Let $(X, \mathsf{d}, \mathsf{H}^n)$ be an $\mathsf{RCD}(0,n)$ space with a closed subgroup $\mathbb{Z}^{n-2} \cong H < \mathrm{Iso}(X)$. Let $H^2 := \{h^2 : h \in H\}$. Then there exist $k \in \{n-2, n-1, n\}$ and a harmonic map $u : X \to \mathbb{R}^k$ satisfying the following:*

1. *For every $\gamma \in H$ we have $u \circ \gamma^2(\cdot) = u(\cdot) + u(\gamma^2 \cdot p)$, and the map $\varphi : H^2 \to \mathbb{R}^k$ given by $\varphi(\gamma^2) := u(\gamma^2 \cdot p)$ is an injective group homomorphism.*

2. *For every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that for every $R > R_\varepsilon$ there exists a lower triangular matrix $T_R \in \mathbb{R}^{k\times k}$ with positive diagonal entries such that $T_R u : B_R(p) \to \mathbb{R}^k$ is an ε-splitting map.*

Proof. The statement holds trivially if $X \cong \mathbb{R}^{n-2} \times Z$. Thus we assume without loss of generality that this is not the case. By Lemmas 4.1.35 and 4.1.37, the blow-down X_∞ of X is unique (up to isometry) and isometric to $\mathbb{R}^k \times Y$ for some $k \geq n-2$ and some metric space (Y, d_Y) that does not split a line. By Theorem [122, Theorem 1.10], there exists a harmonic map $u = (u_1, \cdots, u_k) : X \to \mathbb{R}^k$ vanishing in $p \in X$ such that:

1. $\{u_1, \cdots, u_k\}$ is a basis for

$$\Big\{ v \in \mathsf{W}^{1,2}_{\mathrm{loc}}(X) : \Delta v = 0, \text{and for every } \varepsilon > 0 \text{ there exist } R_\varepsilon, C_\varepsilon > 0 : \\ |v(x)| \leq C_\varepsilon \mathsf{d}(x,p)^{1+\varepsilon} + C_\varepsilon \text{ for every } x \in X \Big\}. \quad (4.84)$$

2. For every $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that for every $R > R_\varepsilon$ there exists a lower triangular matrix $T_R \in \mathbb{R}^{k\times k}$ with positive diagonal entries, such that $T_R u : B_R(p) \to \mathbb{R}^k$ is an ε-splitting map.

Let $\gamma \in H$. We claim that $u \circ \gamma^2(\cdot) = u(\cdot) + u(\gamma^2 \cdot p)$. To this aim observe that along any sequence of rescalings converging to the equivariant blow-down $(X_\infty, H_\infty, p_\infty)$ of (X, H), γ converges to a limiting element $\gamma_\infty \in H_\infty$ by Lemma 2.1.22 and γ^2 converges to γ^2_∞. Since γ_∞ fixes p_∞, by Lemmas 4.1.35 and 4.1.37, we get that $\pi_{\mathrm{Iso}(\mathbb{R}^k)}(\gamma^2_\infty) = \mathrm{Id}_{\mathbb{R}^k}$.

Consider a sequence $\varepsilon_i \downarrow 0$, and corresponding $R_i \uparrow +\infty$ and $T_i \in \mathbb{R}^{k\times k}$ such that $T_i u : B_{R_i}(p) \to \mathbb{R}^k$ is an ε_i-splitting map for every $i \in \mathbb{N}$. By 1, there exist matrices $A_i \in \mathbb{R}^{k\times k}$, such that

$$\frac{T_i u \circ \gamma^2(\cdot)}{R_i} = \frac{A_i T_i u(\cdot) + T_i u(\gamma^2 \cdot p)}{R_i}. \quad (4.85)$$

To conclude the proof of the claim, it suffices to show that $A_i = \mathrm{Id}_{\mathbb{R}^k}$ for every $i \in \mathbb{N}$.

Since the sequence of maps $T_i u / R_i$ converges to $\pi_{\mathbb{R}^k}$ in the blow-down and γ^2 converges to an element γ^2_∞ such that $\pi_{\mathrm{Iso}(\mathbb{R}^k)}(\gamma^2_\infty) = \mathrm{Id}_{\mathbb{R}^k}$, passing to the limit in (4.85) we obtain that $A_i \to \mathrm{Id}_{\mathbb{R}^k}$. Note that the matrices A_i are all similar to each other by (4.85). Thus, all their (complex) eigenvalues are equal to 1. To conclude

the proof it suffices to show that all the matrices A_i are diagonalizable. To this aim, let $B \in \mathbb{R}^{k\times k}$ be such that

$$u \circ \gamma^2(\cdot) = Bu(\cdot) + u(\gamma^2 \cdot p)\,. \tag{4.86}$$

The matrix B is easily seen to be similar to the matrices A_i, so that it suffices to show that B is diagonalizable. Iterating (4.86), we deduce

$$u \circ i\gamma^2(\cdot) = B^i u(\cdot) + u(i\gamma^2 \cdot p) \quad \text{for every } i \in \mathbb{N}\,, \tag{4.87}$$

where $i\gamma^2$ denotes the sum in H^2 of i copies of γ^2. Let $M \in \mathbb{C}^{k\times k}$ be the invertible matrix such that $MBM^{-1} = J$ is the Jordan form of B. From (4.87) we deduce that

$$Mu \circ i\gamma^2(\cdot) = J^i Mu(\cdot) + Mu(i\gamma^2 \cdot p) \quad \text{for every } i \in \mathbb{N}\,. \tag{4.88}$$

By the triangle inequality there exists $c > 0$ such that $\mathsf{d}(i\gamma^2 \cdot p, p) \le ci$ for every $i \in \mathbb{N}$. Fix now $R > 0$. By the Cheng-Yau gradient estimate [121, Theorem 1.2] and the almost linear growth assumption in item 1, there exists a constant $c(R,n) > 0$ such that

$$\sup_{B_R(i\gamma^2\cdot p)} |\nabla u| \le c(R,n) i^{1/2} \quad \text{for every } i \in \mathbb{N}\,. \tag{4.89}$$

Hence, from (4.88) and (4.89) we deduce that

$$\begin{aligned} \sup_{x\in B_R(p)} |J^i Mu| &= \sup_{x\in B_R(p)} |Mu(i\gamma^2 \cdot x) - Mu(i\gamma^2 \cdot p)| \\ &\le |M| R \sup_{B_R(i\gamma^2\cdot p)} |\nabla u| \le C i^{1/2}\,, \end{aligned} \tag{4.90}$$

for some constant $C > 0$. Our next claim is that if $R > 0$ is sufficiently large then

$$\lambda^k\big(u(B_R(p))\big) > 0\,. \tag{4.91}$$

Note that the claim is almost straightforward in the smooth Riemannian case. Indeed, if $R > 0$ is sufficiently large then $T_R u$ is an almost splitting map. Thus, by a standard propagation of regularity argument originally due to [54], there exists a point $x \in B_R(p)$ such that $\{\nabla u_1(x), \dots, \nabla u_k(x)\}$ are linearly independent. In the RCD case one can argue in a similar way to find a regular point $x \in B_R(p)$ and $0 < r < R$ such that $T_R u : B_r(x) \to \mathbb{R}^k$ is still an almost splitting map. Then one can notice that $T_R u : B_r(x) \to \mathbb{R}^k$ can be extended to an almost splitting map $\overline{T_R u} : B_{r/2}(x) \to \mathbb{R}^n$ (with the first k components given by those of $T_R u$). From [40] we have that $\lambda^n(\overline{T_R u}(B_r(x))) > 0$. Thus, from Fubini we deduce that $\lambda^k(u(B_r(x))) > 0$ and (4.91)

follows. By (4.91) there exist linearly independent vectors $\{e_i\}_{i=1}^k \subset Mu(B_R(p)) \subset \mathbb{C}^k$. Thus, possibly increasing $C > 0$, (4.90) implies that

$$|J^i| \leq Ci^{1/2}\,. \tag{4.92}$$

If J is not diagonal, using that it has all eigenvalues equal to 1, then at least one entry of J^i has modulus greater than i, a contradiction. This shows that $u \circ \gamma^2(\cdot) = u(\cdot) + u(\gamma^2 \cdot p)$, as we claimed.

Consider now the map $\varphi : H^2 \to \mathbb{R}^k$ defined by $\varphi(\gamma^2) := u(\gamma^2 \cdot p)$. The map is a group homomorphism thanks to the identity $u \circ \gamma^2(\cdot) = u(\cdot) + u(\gamma^2 \cdot p)$. Indeed, given any $\gamma_1, \gamma_2 \in H^2$, we have that

$$\begin{aligned} u \circ \gamma_1 \circ \gamma_2(\cdot) =& [u(\cdot) + u(\gamma_1 \cdot p)] \circ \gamma_2 = u \circ \gamma_2(\cdot) + u(\gamma_1 \cdot p) \\ =& u(\cdot) + u(\gamma_1 \cdot p) + u(\gamma_2 \cdot p)\,. \end{aligned}$$

Therefore $\varphi(\gamma_1 + \gamma_2) = \varphi(\gamma_1) + \varphi(\gamma_2)$.

To complete the proof, assume by contradiction that φ is not injective. Under such assumption there exists a subgroup $H' < H$ isomorphic to $\mathbb{Z}$ such that $u(H' \cdot p) = \{0\}$. Since we assumed at the beginning of the proof that X does not split off $\mathbb{R}^{n-2}$ isometrically, by Corollary 4.1.38 the blow-down of X is pointed isometric to $(\mathbb{R}^k \times Y, (0, y))$, with Y containing no lines, and it satisfies $\mathrm{Iso}(\mathbb{R}^k \times Y) \cdot (0, y) \subset \mathbb{R}^k \times \{y\}$. Moreover, the orbit $H' \cdot p$ pointed Hausdorff converges along a blow down sequence to an orbit $H'_\infty \cdot (0, y)$ for some $H'_\infty < \mathrm{Iso}(\mathbb{R}^k \times Y)$. By Lemma 2.2.23, H_∞ has a closed subgroup isomorphic to $\mathbb{R}$. Hence, $\pi_{\mathbb{R}^k}(H'_\infty \cdot (0, y))$ contains a set homeomorphic to $\mathbb{R}$. At the same time, by item 2, the map $\pi_{\mathbb{R}^k}$ is approximated, along a chosen blow down sequence, by maps of the form $T_i u / R_i$ for some $R_i \uparrow +\infty$ and some matrices $T_i \in \mathbb{R}^{k \times k}$. Since all these maps vanish on $H' \cdot p$, we deduce that $\pi_{\mathbb{R}^k}(H'_\infty \cdot (0, y)) = \{0\}$, a contradiction. □

The next lemma will be needed for the proof of Proposition 4.2.3. The result holds for actions of $\mathbb{Z}^k$, but we only state it for $k = n - 2$, as this is the case that we will use later on.

Lemma 4.2.2. *Let (X, d) be a proper separable metric space with a closed subgroup $H < \mathrm{Iso}(X)$ isomorphic to $\mathbb{Z}^{n-2}$ and generated by $\{\gamma_1, \cdots, \gamma_{n-2}\}$. Let $R > 0$. Then, there are natural numbers $\{k_j\}_{j=1}^{n-2}$ and $C > 0$ such that*

$$R/2n \leq \mathsf{d}\Big(\mathrm{span}(\gamma_1, \cdots, \gamma_{j-1}) \cdot p, k_j \gamma_j \cdot p\Big) \leq R/2n + C \quad \text{for } j = 1, \ldots, n-2\,, \tag{4.93}$$

and for every natural number $t < k_j$ we have

$$\mathsf{d}\Big(\mathrm{span}(\gamma_1, \cdots, \gamma_{j-1}) \cdot p, t\gamma_j \cdot p\Big) \leq R/2n\,. \tag{4.94}$$

Proof. Let k_1 be the smallest natural number such that

$$\mathsf{d}\Big(p, k_1\gamma_1 \cdot p\Big) \geq R/2n\,. \tag{4.95}$$

Observe that such k_1 exists since $\mathrm{span}(\gamma_1)\cdot p$ is non-compact. We claim that for every $j \in \{1, \cdots, n-3\}$ we have

$$\limsup_{s\to+\infty} \mathsf{d}(\mathrm{span}(\gamma_1, \cdots, \gamma_j) \cdot p, s\gamma_{j+1} \cdot p) = +\infty\,. \tag{4.96}$$

Assume by contradiction that this is not the case, i.e.,

$$\limsup_{s\to+\infty} \mathsf{d}(\mathrm{span}(\gamma_1, \cdots, \gamma_j) \cdot p, s\gamma_{j+1} \cdot p) < c < +\infty\,. \tag{4.97}$$

Then for every $s \in \mathbb{N}$ there exists $h_s \in \mathrm{span}(\gamma_1, \cdots, \gamma_j)$ such that $\mathsf{d}(p, (h_s + s\gamma_{j+1})\cdot p) \leq c$. Up to a not relabelled subsequence, $h_s + s\gamma_{j+1} \to h_\infty \in H$, thus contradicting the assumption that $H \cong \mathbb{Z}^{n-2}$ as a topological group. Hence (4.96) holds.

For every $j \in \{1, \cdots, n-3\}$, we let k_{j+1} be the smallest natural number such that

$$\mathsf{d}\Big(\mathrm{span}(\gamma_1, \cdots, \gamma_j) \cdot p, k_{j+1}\gamma_{j+1} \cdot p\Big) \geq R/2n\,. \tag{4.98}$$

These numbers exist by (4.96). At the same time, there exists a constant $C > 0$ such that

$$\mathsf{d}(p, \gamma_j \cdot p) \leq C \quad \text{for every } j \in \{1, \cdots n-2\}\,. \tag{4.99}$$

Therefore

$$\begin{aligned} \mathsf{d}\Big(\mathrm{span}(\gamma_1, \cdots, \gamma_{j-1}) \cdot p, k_j\gamma_j \cdot p\Big) &\leq \mathsf{d}\Big(\mathrm{span}(\gamma_1, \cdots, \gamma_{j-1}) \cdot p, (k_j-1)\gamma_j \cdot p\Big) \\ &\quad + \mathsf{d}\Big((k_j-1)\gamma_j \cdot p, k_j\gamma_j \cdot p\Big) \\ &\leq R/2n + C\,. \end{aligned} \tag{4.100}$$

Combining with (4.98), we obtain

$$R/2n \leq \mathsf{d}\Big(\mathrm{span}(\gamma_1, \cdots, \gamma_{j-1}) \cdot p, k_j\gamma_j \cdot p\Big) \leq R/2n + C\,, \tag{4.101}$$

thus concluding the proof. □

Proposition 4.2.3 formulates and establishes a precise version of the non-degeneracy condition referred to in item iii) of the plan discussed at the beginning of the section.

Proposition 4.2.3. *Let $(X, \mathsf{d}, \mathsf{H}^n)$ be an $\mathsf{RCD}(0,n)$ space with a closed subgroup $\mathbb{Z}^{n-2} \cong H < \mathrm{Iso}(X)$. Assume that X does not split a factor $\mathbb{R}^{n-2}$. Let $k \in \{n-2, n-1\}$ be given by Corollary 4.1.38 such that the blow-down of X is unique and isometric to $\mathbb{R}^k \times Y$, where Y does not split any line. Let $u : X \to \mathbb{R}^k$ be the harmonic map given by Proposition 4.2.1 and let $\varepsilon_i \downarrow 0$, $R_i \uparrow +\infty$, and $T_i \in \mathbb{R}^{k\times k}$ be such that $T_i u : B_{R_i}(p) \to \mathbb{R}^k$ is an ε_i-splitting map for every $i \in \mathbb{N}$.*

If $k = n-1$, for every $i \in \mathbb{N}$ we let $\Sigma_i \subset \mathbb{R}^{n-1}$ be the hyperplane containing $T_i u(H^2 \cdot p)$ and let $\pi_i : \mathbb{R}^{n-1} \to \Sigma_i \cong \mathbb{R}^{n-2}$ be the orthogonal projection on Σ_i. Define $v_i : B_{R_i}(p) \to \mathbb{R}^{n-2}$ by $v_i := T_i u$ if $k = n-2$, and $v_i := \pi_i \circ T_i u$ if $k = n-1$. Then v_i is a ε_i-splitting map for every $i \in \mathbb{N}$ and there exist constants $C_1, C_2 > 0$ such that for every i sufficiently large

$$\lambda^{n-2}\Big(v_i(\{x \in B_{R_i}(p) : \mathsf{d}(x, H^2 \cdot p) \le C_1\})\Big) \ge C_2 R_i^{n-2}. \tag{4.102}$$

Proof. By Lemmas 4.1.35 and 4.1.37, either $X \cong \mathbb{R}^{n-2} \times Y$ isometrically, or the blow-down of X is unique and isomorphic to $\mathbb{R}^k \times Y'$ for some $k \in \{n-2, n-1\}$. Let $\{\gamma_j\}_{j=1}^{n-2}$ be generators of H^2 and, for every i, let $\{k_{i,j}\}_{j=1}^{n-2}$ be the natural numbers given by Lemma 4.2.2 relative to the scale R_i. We divide the rest of the proof in three steps.

Step 1: Let $\Sigma \subset \mathbb{R}^k$ be the $(n-2)$-dimensional vector space containing $u(H \cdot p)$ and consider the map $T_{i|\Sigma} : \Sigma \to \Sigma_i$. We claim that

$$\liminf_{i\to+\infty} \frac{\det(T_{i|\Sigma})\Pi_{j=1}^{n-2} k_{i,j}}{R_i^{n-2}} > 0\,. \tag{4.103}$$

By construction of the $\{k_{i,j}\}_{j=1}^{n-2}$ (see Lemma 4.2.2), for every $j \in \{2, \cdots, n-2\}$ there exists $h_{i,j} \in \mathrm{span}(\gamma_1, \cdots, \gamma_{j-1})$ such that

$$\begin{aligned} \frac{\mathsf{d}\Big(p, (h_{i,j} + k_{i,j}\gamma_j) \cdot p\Big)}{R_i} &= \frac{\mathsf{d}\Big(\mathrm{span}(\gamma_1, \cdots, \gamma_{j-1}) \cdot p, k_{i,j}\gamma_j \cdot p\Big)}{R_i} \\ &\to 1/2n \quad \text{as } i \uparrow +\infty\,. \end{aligned} \tag{4.104}$$

Let $(X_\infty, \mathsf{d}_\infty, H^2_\infty, p_\infty)$ be an equivariant blow-down obtained as limit of the sequence $(X, \mathsf{d}/R_i, H^2)$. Recall that $X_\infty \cong \mathbb{R}^k \times Y$, and set $p_\infty = (0, y_\infty)$. The groups $H^2_j := \mathrm{span}(\gamma_1, \cdots, \gamma_j)$ converge, up to not relabelled subsequences, to groups $H^2_{\infty,j} < H^2_\infty$, and the orbits $H^2_j \cdot p$ converge to $H^2_{\infty,j} \cdot p_\infty$. Thanks to (4.104) and Lemma 4.1.38, the points $q_{i,j} := (h_{i,j} + k_{i,j}\gamma_j) \cdot p$ converge to limit points

$$q_{\infty,j} \in H^2_{\infty,j} \cdot p_\infty \subset \mathbb{R}^k \times \{y_\infty\} \subset X_\infty \tag{4.105}$$

such that

$$\mathsf{d}_\infty(H^2_{\infty,j-1}\cdot p_\infty, q_{\infty,j}) = \mathsf{d}_\infty(p_\infty, q_{\infty,j}) = 1/2n\,. \tag{4.106}$$

Thanks to (4.105) and (4.106) we have

$$\mathsf{d}_{\rm eucl}(\pi_{\mathbb{R}^k}(H^2_{\infty,j-1}\cdot p_\infty), \pi_{\mathbb{R}^k}(q_{\infty,j})) = 1/2n\,. \tag{4.107}$$

The maps T_iu/R_i converge to the projection $\pi_{\mathbb{R}^k}: X_\infty \to \mathbb{R}^k$ along the blow-down sequence, so that

$$T_iu/R_i(q_{i,j}) \to \pi_{\mathbb{R}^k}(q_{\infty,j}) \in \mathbb{R}^k \quad \text{as } i\uparrow +\infty\,. \tag{4.108}$$

The set $\pi_{\mathbb{R}^k}(H^2_{\infty,j-1}\cdot p_\infty)$ is contained in a $(j-1)$-dimensional vector space since it is the limit of the nets $T_iu/R_i(H^2_{j-1}\cdot p)$. Moreover, it contains the set $\{\pi_{\mathbb{R}^k}(q_{\infty,s})\}_{s=1}^{j-1}$. At the same time, by Lemma 2.2.23, $H^2_{\infty,j-1}$ contains a closed group isomorphic to $\mathbb{R}^{j-1}$. Thus, by invariance of the domain, $\pi_{\mathbb{R}^k}(H^2_{\infty,j-1}\cdot p_\infty)$ coincides with the aforementioned $(j-1)$-dimensional vector space. It follows from (4.107) that $\{\pi_{\mathbb{R}^k}(q_{\infty,j})\}_{j=1}^{n-2}$ are linearly independent. Hence we obtain that

$$\frac{\det(T_{i|\Sigma})\Pi_{j=1}^{n-2}k_{i,j}}{R_i^{n-2}}\Big(u(\gamma_1\cdot p)\wedge\cdots\wedge u(\gamma_{n-2}\cdot p)\Big) \tag{4.109}$$

$$= \frac{1}{R_i^{n-2}}\Big([T_iu(q_{i,1})]\wedge\cdots\wedge[T_iu(q_{i,n-2})]\Big) \tag{4.110}$$

$$\to \pi_{\mathbb{R}^k}(q_{\infty,1})\wedge\cdots\wedge\pi_{\mathbb{R}^k}(q_{\infty,n-2})\neq 0\,, \tag{4.111}$$

thus completing the proof of (4.103).

Step 2: Arguing as we did in the proof of (4.91), we can show that there exists $i_0\in\mathbb{N}$ such that the ε_{i_0}-splitting map $T_{i_0}u: B_{R_{i_0}}(p)\to\mathbb{R}^k$ satisfies

$$\lambda^k(T_{i_0}(u(B_{R_{i_0}}(p)))) > 0\,. \tag{4.112}$$

Thus, setting $C_1 := R_{i_0}$, we have

$$\lambda^k(u(B_{C_1}(p))) > 0\,. \tag{4.113}$$

Moreover, modulo passing to a sublattice of H^2, we may suppose that the sets $u(h\cdot B_{C_1}(p))$ are all disjoint as h varies in H^2. We claim that

$$\#\big((H^2\cdot p)\cap B_{R_i-C_1}(p)\big) \geq \Pi_{j=1}^{n-2}k_{i,j}\,. \tag{4.114}$$

To prove (4.114), fix $j\in\{2,\cdots,n-2\}$. For every natural number $m_{i,j}\leq k_{i,j}$ there exists $h_{i,j}^{m_{i,j}}\in H^2_{j-1}$ such that

$$\mathsf{d}(p,(h_{i,j}^{m_{i,j}}+m_{i,j}\gamma_j)\cdot p) \leq R_i/2n + C < \frac{R_i-C_1}{n}\,, \tag{4.115}$$

where $C > 0$ is the constant of Lemma 4.2.2 and the last inequality holds for i large enough. Hence by triangle inequality we have

$$\mathsf{d}\Big(p, \Big[\sum_{j=1}^{n-2} \big(h_{i,j}^{m_{i,j}} + m_{i,j}\gamma_j\big)\Big] \cdot p\Big) < R_i - C_1\,, \tag{4.116}$$

for every

$$(m_{i,1}, \cdots, m_{i,n-2}) \in \mathbb{N}^{n-2} \cap \Pi_{j=1}^{n-2}[1, k_{i,j}]\,. \tag{4.117}$$

Given (4.116) and (4.117), to prove (4.114) it suffices to establish the following claim: if

$$(m_{i,1}, \cdots, m_{i,n-2}) \neq (m'_{i,1}, \cdots, m'_{i,n-2})\,, \tag{4.118}$$

then

$$\sum_{j=1}^{n-2} \big(h_{i,j}^{m_{i,j}} + m_{i,j}\gamma_j\big) \neq \sum_{j=1}^{n-2} \big(h_{i,j}^{m'_{i,j}} + m'_{i,j}\gamma_j\big)\,. \tag{4.119}$$

To prove the claim, assume by contradiction that (4.118) holds but

$$\sum_{j=1}^{n-2} \big(h_{i,j}^{m_{i,j}} + m_{i,j}\gamma_j\big) = \sum_{j=1}^{n-2} \big(h_{i,j}^{m'_{i,j}} + m'_{i,j}\gamma_j\big)\,. \tag{4.120}$$

Then $m_{i,n-2} = m'_{i,n-2}$ since $\{\gamma_1, \ldots, \gamma_{n-2}\}$ is a basis of H^2 and

$$\sum_{j=1}^{n-2} h_{i,j}^{m_{i,j}}\,,\ \sum_{j=1}^{n-2} h_{i,j}^{m'_{i,j}} \in \mathrm{span}(\gamma_1, \cdots, \gamma_{n-3})\,. \tag{4.121}$$

Hence $h_{i,n-2}^{m_{i,n-2}} = h_{i,n-2}^{m'_{i,n-2}}$ and thus

$$\sum_{j=1}^{n-3} \big(h_{i,j}^{m_{i,j}} + m_{i,j}\gamma_j\big) = \sum_{j=1}^{n-3} \big(h_{i,j}^{m'_{i,j}} + m'_{i,j}\gamma_j\big)\,. \tag{4.122}$$

Iterating this procedure, we deduce $(m_{i,1}, \cdots, m_{i,n-2}) = (m'_{i,1}, \cdots, m'_{i,n-2})$, a contradiction.

Step 3: To conclude the proof we treat the case $k = n-2$ and $k = n-1$ separately.

Assume first that $k = n-2$. Using (4.114) and the equivariance of u, we obtain that for every $i \in \mathbb{N}$ there holds

$$\begin{aligned}
&\lambda^{n-2}\Big(\frac{T_i}{R_i}\big(u\big(\{x \in B_{R_i}(p) : \mathsf{d}(x, H^2 \cdot p) \leq C_1\}\big)\big)\Big) \\
&= \frac{\det(T_i)}{R_i^{n-2}}\lambda^{n-2}\Big(u\big(\{x \in B_{R_i}(p) : \mathsf{d}(x, H^2 \cdot p) \leq C_1\}\big)\Big) \\
&\geq \frac{\det(T_i)\Pi_{j=1}^{n-2}k_{i,j}}{R_i^{n-2}}\lambda^{n-2}\Big(u(B_{C_1}(p))\Big)\,.
\end{aligned}$$

Combining with (4.103) and (4.113), we obtain the result for $k = n-2$.

Assume now that $k = n-1$. Let $\mathcal{H}_i$ be the set of affine hyperplanes of $\mathbb{R}^{n-1}$ parallel to Σ_i. Observe that for every $A \subset \mathbb{R}^{n-1}$ we have

$$\begin{aligned}\lambda^{n-2}(\pi_i \circ T_i(A)) &\geq \sup_{H\in\mathcal{H}_i} \lambda^{n-2}(T_i(A)\cap H)\\ &= \sup_{H\in\mathcal{H}_i} \lambda^{n-2}\Big(T_i(A\cap T_i^{-1}(H))\Big)\,. \end{aligned} \tag{4.123}$$

Let $v \in \mathbb{R}^{n-1}$ be a unit vector normal to $\Sigma := T_i^{-1}(\Sigma_i) \subset \mathbb{R}^{n-1}$. We have that

$$\begin{aligned}\sup_{H\in\mathcal{H}_i} \lambda^{n-2}\Big(T_i(A\cap T_i^{-1}(H))\Big) &= \sup_{t\in\mathbb{R}} \lambda^{n-2}\Big(T_i\big((A+tv)\cap T_i^{-1}(\Sigma_i)\big)\Big)\\ &= \det(T_{i|\Sigma}) \sup_{t\in\mathbb{R}} \lambda^{n-2}\big((A+tv)\cap\Sigma\big)\,. \end{aligned} \tag{4.124}$$

Applying (4.123) and (4.124) with

$$A := u(\{x \in B_{R_i}(p) : \mathsf{d}(x, H^2\cdot p) \leq C_1\})\,, \tag{4.125}$$

we deduce that

$$\begin{aligned}&\lambda^{n-2}\Big(\pi_i \circ \frac{T_i}{R_i}\left(u(\{x \in B_{R_i}(p) : \mathsf{d}(x, p+H^2) \leq C_1\})\right)\Big)\\ &\quad\geq \frac{\det(T_{i|\Sigma})}{R_i^{n-2}} \sup_{t\in\mathbb{R}} \lambda^{n-2}\big([u(\{x \in B_{R_i}(p) : \mathsf{d}(x, H^2\cdot p) \leq C_1\}) + tv]\cap\Sigma\big)\,. \end{aligned} \tag{4.126}$$

By (4.114), $[u(\{x \in B_{R_i}(p) : \mathsf{d}(x, H^2\cdot p) \leq C_1\}) + tv]\cap\Sigma$ contains $\Pi_{j=1}^{n-2} k_{i,j}$ disjoint copies of $[u(B_{C_1}(p)) + tv]\cap\Sigma$. Hence

$$\begin{aligned}&\lambda^{n-2}\Big(\pi_i \circ \frac{T_i}{R_i} u(\{x \in B_{R_i}(p) : \mathsf{d}(x, p+H^2) \leq C_1\})\Big)\\ &\quad\geq \frac{\det(T_{i|\Sigma})\Pi_{j=1}^{n-2} k_{i,j}}{R_i^{n-2}} \sup_{t\in\mathbb{R}} \lambda^{n-2}([u(B_{C_1}(p)) + tv]\cap\Sigma)\,. \end{aligned} \tag{4.127}$$

The statement then follows combining (4.127) with (4.103) and (4.113). □

4.3 Proof of Theorem 4.0.1

Let $(\overline{X}, \overline{\mathsf{d}}, \mathsf{H}^n, p)$ denote the universal cover of $(X, \mathsf{d}, \mathsf{H}^n)$. Since $\pi_1(X)$ has a subgroup isomorphic to $\mathbb{Z}^{n-2}$ there is a closed subgroup $\mathbb{Z}^{n-2} \cong H < \mathrm{Iso}(\overline{X})$. By Corollary 4.1.38, either $\overline{X} \cong \mathbb{R}^{n-2} \times Y$ isometrically, or the blow-down of $\overline{X}$ is unique up to isometry and isometric to $\mathbb{R}^k \times Y'$ for some $k \in \{n-2, n-1\}$.

Let $\varepsilon_i \downarrow 0$, $R_i \uparrow +\infty$, and $v_i : B_{R_i}(p) \to \mathbb{R}^{n-2}$ be the numbers and the ε_i-splitting maps given by Proposition 4.2.3. Fix $\widetilde{\varepsilon} > 0$ sufficiently small. By the slicing theorem, see [35] after [60, Theorem 1.23], for every i large enough there exists $A_{i,\widetilde{\varepsilon}} \subset B^{\mathbb{R}^{n-2}}_{R_i/2}(0)$ with $\lambda^{n-2}(B^{\mathbb{R}^{n-2}}_{R_i/2}(0) \setminus A_{i,\widetilde{\varepsilon}}) < \widetilde{\varepsilon} R_i^{n-2}$ such that the following holds: For every $x \in v_i^{-1}(A_{i,\widetilde{\varepsilon}}) \cap B_{R_i}(p)$, and for every $r \in (0, R_i/2]$, there exists a matrix $A_r \in \mathbb{R}^{(n-2)\times(n-2)}$ such that $A_r v_{i|B_r(x)} \to \mathbb{R}^{n-2}$ is an $\widetilde{\varepsilon}$-splitting map.

By (4.102) in Proposition 4.2.3, there exists $C_1 > 0$ such that for i large enough we have

$$A_{i,\widetilde{\varepsilon}} \cap v_i\big(\{x \in B_{R_i}(p) : \mathsf{d}(x, H^2 \cdot p) \leq C_1\}\big) \neq \emptyset\,. \tag{4.128}$$

Fix now a sequence $\widetilde{\varepsilon}_j \downarrow 0$. For every $j \in \mathbb{N}$ let $i(j)$ be such that

$$A_{i(j),\widetilde{\varepsilon}_j} \cap v_{i(j)}\big(\{x \in B_{R_{i(j)}}(p) : \mathsf{d}(x, H^2 \cdot p) \leq C_1\}\big) \neq \emptyset\,. \tag{4.129}$$

Accordingly, for every j we take $x_j \in B_{R_{i(j)}}(p)$ such that

i) $v_{i(j)}(x_j) \in A_{i(j),\widetilde{\varepsilon}_j}$;

ii) $\mathsf{d}(x_j, H^2 \cdot p) \leq C_1$.

Fix any $R \gg 1$. Applying [37, Theorem 3.8] (after [60, Lemma 1.21]) we obtain a sequence $\delta_j \downarrow 0$ such that the balls $B_R(x_j)$ are $\delta_j R$-close in GH sense to balls

$$B^{\mathbb{R}^{n-2}\times Y_j}_R((0, y_i))\,, \tag{4.130}$$

for some pointed metric spaces (Y_j, y_j). Since $\mathsf{d}(x_j, H^2 \cdot p) \leq C_1$, modulo applying an isometry of H^2, we can assume that $x_j \in \overline{B}_{C_1}(p)$ for every j. Up to another subsequence, $x_j \to x_\infty \in \overline{B}_{C_1}(p)$, so that $B_R(x_\infty)$ is isometric to $B^{\mathbb{R}^{n-2}\times Y}_R((0, y))$ for some pointed metric space (Y, y). Hence, the same holds for $B_{R-C_1}(p)$. Since $R \gg 1$ is arbitrary, the statement follows.

Chapter 5

Simplicial volume and non-negative Ricci curvature

The goal of this section is to prove the following vanishing result for the simplicial volume of $\mathsf{RCD}(0,n)$ manifolds, which implies Theorem 1.3.12 (since smooth manifolds are triangulable by [189]):

Theorem 5.0.1. *For every $n \geq 2$, the following holds. If $(X, \mathsf{d}, \mathsf{H}^n)$ is an $\mathsf{RCD}(0,n)$ space and X is an oriented triangulable topological n-manifold, then $||X|| = 0$.*

Remark 5.0.2. The assumption that X is triangulable is related to the presence of the same assumption in [90, Corollary 11] on which our proof depends. As remarked in [90] it seems possible that such assumption is not necessary.

The closest available result in the literature is due to J. Lott and Z. Shen [143, Corollary 1.1]. They show that a complete Riemannian manifold (M^n, g) with quadratic curvature decay and such that

$$\liminf_{r\to\infty} \frac{\mathrm{vol}(B_r(p))}{r^n} = 0 \tag{5.1}$$

for some $p \in M$ satisfies $||M|| = 0$.

For the proof of Theorem 5.0.1 we are going to distinguish two cases: the Euclidean volume growth case, i.e., the case when

$$\limsup_{r\to\infty} \frac{\mathsf{H}^n(B_r(p))}{r^n} > 0\,, \tag{5.2}$$

and the non-Euclidean volume growth case, when

$$\limsup_{r\to\infty} \frac{\mathsf{H}^n(B_r(p))}{r^n} = 0\,, \tag{5.3}$$

discussed in Section 5.1 and Section 5.2 respectively. In both cases, the key technical tool is provided by [90, Corollary 11], which we stated in the preliminaries as Theorem 2.3.12.

5.1 Proof of Theorem 5.0.1: Euclidean volume growth case

The goal of this section is to prove Theorem 5.0.1 under the assumption that the space has maximal volume growth. Thanks to Theorem 2.3.12, it is sufficient to prove Theorem 5.1.1 below. We recall that given a covering $\{U_i\}_i$ of a metric space, we denote by $\mathrm{mult}\big(\{U_i\}_{i\in\mathbb{N}}\big)$ its multiplicity (see Definition 2.3.6).

Theorem 5.1.1. *Let $(X, \mathsf{d}, \mathsf{H}^n, p)$ be an $\mathsf{RCD}(0,n)$ space. If X has maximal volume growth, then it admits a cover $\{U_i\}_{i\in\mathbb{N}}$ which is amenable, open, amenable at infinity and has $\mathrm{mult}\big(\{U_i\}_{i\in\mathbb{N}}\big) \leq 2$.*

Theorem 5.1.1, in turn, is an immediate consequence of Theorem 5.1.2 below, which is the focus of this section.

Theorem 5.1.2. *Let $n \geq 3$, and let $(X, \mathsf{d}, \mathsf{H}^n, p)$ be an $\mathsf{RCD}(0,n)$ space with maximal volume growth*

$$\lim_{r\to+\infty} \frac{\mathsf{H}^n(B_r(p))}{r^n} \geq v > 0. \tag{5.4}$$

Then, for every r sufficiently large, $B_{4r}(p) \setminus B_{3r}(p)$ is contained in a path-connected component C_r of $B_{5r}(p) \setminus B_{2r}(p)$. Moreover, there exists a constant $C(n,v) > 0$ such that the image through the push-forward of the inclusion map

$$i_* : \pi_1(C_r, c) \to \pi_1(B_{6r}(p) \setminus B_r(p), c) \tag{5.5}$$

has cardinality at most $C(n,v)$ for every sufficiently large $r > 0$ and every $c \in C_r$.

The proof of the first part of the statement of Theorem 5.1.2 is straightforward. To prove (5.5) we argue by contradiction. Hence, we find a sequence of annuli $A_i := B_{6r_i}(p) \setminus B_{r_i}(p)$ such that the images of the maps as in (5.5) have large cardinality. Then we consider their universal coverings $\widetilde{A}_i$, and suitable subsets $\widetilde{A}'_i \subset \widetilde{A}_i$ which are coverings of the corresponding path-connected components C_{r_i}. Thanks to our assumption by contradiction, the coverings $\widetilde{A}'_i$ are non-compact. At the same time, scaling down distances, the sets C_{r_i} converge to a metric space Y which is isometric to an annulus in a metric cone over an $\mathsf{RCD}(n-2, n-1)$ space with volume bounded below by a constant depending on v. Such annulus is homotopically equivalent to the cross-section. Thus, in particular, if $n \geq 3$ then $\#\pi_1(Y) < C(n,v)$ as an application of [153, Theorem 1.2]. Similarly, the coverings $\widetilde{A}'_i$ equipped with their deck transformations converge in equivariant Gromov Hausdorff sense to a pair $(\widetilde{Y}, H)$ with $H < \mathrm{Iso}(\widetilde{Y})$ such that $\widetilde{Y}/H \cong Y$ isometrically.

To finish the proof, we are going to combine the statement that $\#\pi_1(Y) < C(n,v)$ with some ideas originally developed in [17, 18, 194] to deduce that H has also cardinality bounded by $C(n,v)$. This then leads to a contradiction.

Making completely rigorous the first part of the plan sketched above involves a number of steps. The proofs of the next two lemmas follow the lines of [194, Lemma 3.3].

Lemma 5.1.3. *Let $(X, \mathsf{d}, \mathsf{H}^n, p)$ be an $\mathsf{RCD}(K,n)$ space with*

$$\mathsf{H}^n(B_1(p)) \geq v > 0. \tag{5.6}$$

Then there exists $k(v,n,K) \in \mathbb{N}$ such that the image through the push-forward of the inclusion map

$$i_* : \pi_1(B_1(p),p) \to \pi_1(B_4(p),p) \tag{5.7}$$

contains at most k elements that can be represented by loops of length less than or equal to 2 centered in p.

Proof. We denote the image of $i_* : \pi_1(B_1(p),p) \to \pi_1(B_4(p),p)$ by G. Let $\pi : Y \to B_4(p)$ be the universal covering of $B_4(p)$. Consider on Y the distance $\widetilde{\mathsf{d}}$ obtained by lifting the length structure of $B_4(p)$. Let $\widetilde{\mathsf{H}}^n$ denote the corresponding Hausdorff measure. Let $\widetilde{p} \in \pi^{-1}(p)$, and let $F \subset Y$ be the set containing $\widetilde{p}$ constructed in Lemma 5.3.6. Call $G_2 \subset G$ the set of elements that can be represented by loops of length at most 2 centered in p. Let

$$B := \bigcup_{g \in G_2} g(F \cap B_1^{\widetilde{\mathsf{d}}}(\widetilde{p})) \subset Y\,. \tag{5.8}$$

We claim that $B \subset B_3^{\widetilde{\mathsf{d}}}(\widetilde{p})$. Indeed, if $g \in G_2$, we can join $\widetilde{p}$ to $g(\widetilde{p})$ with a curve of length at most 2 and we can join $g(\widetilde{p})$ to any element of $g(F \cap B_1^{\widetilde{\mathsf{d}}}(\widetilde{p}))$ by a curve of length at most 1. Hence, $B \subset B_3^{\widetilde{\mathsf{d}}}(\widetilde{p})$ by triangle inequality.

Combining such inclusion with Lemma 5.3.6 we get

$$\begin{aligned}(\#G_2)\cdot \mathsf{H}^n(B_1(p)) &= \sum_{g\in G_2} \widetilde{\mathsf{H}}^n\big(g(F\cap B_1^{\widetilde{\mathsf{d}}}(\widetilde{p}))\big) &(5.9)\\ &\leq \widetilde{\mathsf{H}}^n(B) \leq \widetilde{\mathsf{H}}^n(B_3^{\widetilde{\mathsf{d}}}(\widetilde{p}))\,. &(5.10)\end{aligned}$$

Thus, by Lemma 5.3.5, it holds

$$(\#G_2)\cdot \mathsf{H}^n(B_1(p)) \leq v_{K,n}(3), \tag{5.11}$$

where $v_{K,N}(r)$ denotes the volume of the ball of radius r in the model space for the $\mathsf{CD}(K,N)$ condition. Hence, combining with (5.6), we obtain that

$$\#G_2 \leq v_{K,n}(3)/v\,, \tag{5.12}$$

thus concluding the proof. □

Lemma 5.1.4. *Let $(X,\mathsf{d},\mathsf{H}^n,p)$ be an $\mathsf{RCD}(0,n)$ space with*

$$\lim_{r\to+\infty}\frac{\mathsf{H}^n(B_r(p))}{r^n}\geq v>0\,. \tag{5.13}$$

Then there exist $\varepsilon(n,v)>0$, $C(n,v)\in\mathbb{N}$ such that the image through the push-forward of the inclusion map

$$i_*:\pi_1(B_\varepsilon(p),p)\to\pi_1(B_1(p),p) \tag{5.14}$$

has cardinality at most C.

Proof. Let $m_0:=\lceil 1+\omega_n 3^n/v\rceil$, and choose $\varepsilon(n,v):=(10m_0)^{-1}$. We denote the image of $i_*:\pi_1(B_\varepsilon(p),p)\to\pi_1(B_1(p),p)$ by G. It is a standard observation that G can be generated by loops of length at most 2ε. Thus, by Lemma 5.1.3, G is generated by at most $k(n,v)$ elements $\{\gamma_1,\cdots,\gamma_k\}$ of length at most 2ε centered in p. For every $m\in\mathbb{N}$, let $U(m)$ be the ball of radius m in G centered in the identity w.r.t. the word metric and the set of generators considered above. Let $\pi:Y\to B_1(p)$ be the universal covering of $B_1(p)$ and let $\widetilde{p}$ be a preimage of p. Endow Y with the distance $\widetilde{\mathsf{d}}$ induced by the lift of the length structure of $B_1(p)$ (cf. with Remark 5.3.4) and the corresponding Hausdorff measure $\widetilde{\mathsf{H}}^n$. Let $F\subset Y$ be the set containing $\widetilde{p}$ as in Lemma 5.3.6.

We define

$$B:=\bigcup_{g\in U(m_0)}g(F\cap B^{\widetilde{\mathsf{d}}}_{m_0\varepsilon}(\widetilde{p}))\subset Y\,. \tag{5.15}$$

We claim that if $x\in B$ then x is joined to $\widetilde{p}$ by a curve of length at most $3m_0\varepsilon$. Indeed, if

$$x\in g(F\cap B^{\widetilde{\mathsf{d}}}_{m_0\varepsilon}(\widetilde{p}))\quad\text{for some}\quad g\in U(m_0)\,, \tag{5.16}$$

then there is a curve from $\widetilde{p}$ to $g(\widetilde{p})$ of length at most $2m_0\varepsilon$, and another curve of length at most $m_0\varepsilon$ joining $g(\widetilde{p})$ and x.

Since $B\subset B^{\widetilde{\mathsf{d}}}_{3m_0\varepsilon}(\widetilde{p})$, Lemma 5.3.5 implies that

$$\widetilde{\mathsf{H}}^n(B)\leq\omega_n(3m_0\varepsilon)^n\,. \tag{5.17}$$

We claim that $\#G \leq \#U(m_0)$. Note that if the claim holds then the statement immediately follows.

Assume by contradiction that

$$\#G > \#U(m_0)\,. \tag{5.18}$$

Since $\#U(m+1) > \#U(m)$ unless $U(m) = G$ for every $m \in \mathbb{N}$, it follows that $m_0 \leq \#U(m_0)$. Hence, we obtain that

$$m_0 \cdot \mathsf{H}^n(B_{m_0\varepsilon}(p)) \leq \#U(m_0) \cdot \widetilde{\mathsf{H}}^n(F \cap B^{\widetilde{\mathsf{d}}}_{m_0\varepsilon}(\widetilde{p})) = \widetilde{\mathsf{H}}^n(B)\,. \tag{5.19}$$

Combining with (5.17), we deduce that

$$m_0 \mathsf{H}^n(B_{m_0\varepsilon}(p)) \leq \omega_n (3m_0\varepsilon)^n\,. \tag{5.20}$$

Exploiting (5.13), we obtain that

$$m_0 \leq \omega_n (3m_0\varepsilon)^n / \mathsf{H}^n(B_{m_0\varepsilon}(p)) \leq \omega_n 3^n / v\,, \tag{5.21}$$

thus contradicting our choice of m_0. □

The following lemma is well-known (cf. with [193]), and it can be obtained arguing as for the proof of the analogous result in the smooth setting, see [18]. We discuss the proof for the sake of completeness.

Lemma 5.1.5. *Let $K > 0$ and let $(X, \mathsf{d}, \mathsf{H}^n)$ be an $\mathsf{RCD}(K,n)$ space with*

$$\mathsf{H}^n(X) \geq v > 0\,. \tag{5.22}$$

Then, $\pi_1(X)$ has cardinality at most $C(n, v, K)$.

Proof. Let $(\widetilde{X}, \widetilde{\mathsf{d}}, \widetilde{\mathsf{H}}^n)$ be the universal covering of X, which is simply connected and is an $\mathsf{RCD}(K,n)$ space by [188] and [153]. By Bonnet-Myers [183, Corollary 2.6], it holds $\mathrm{diam}(\widetilde{X}) \leq C(n,K)$, so that Bishop-Gromov's inequality [183, Theorem 2.3] implies that $\mathsf{H}^n(\widetilde{X}) \leq C(n,K)$. Consider now a fundamental domain $F \subset \widetilde{X}$ as given by Lemma 5.3.6. For every $g \in \pi_1(X)$, it holds $g \cdot F \cap F = \emptyset$ and $\widetilde{\mathsf{H}}^n(g \cdot F) = \mathsf{H}^n(X)$. Hence,

$$\#\pi_1(X)\, v \leq \#\pi_1(X)\, \mathsf{H}^n(X) \leq \widetilde{\mathsf{H}}^n(\widetilde{X}) \leq C(n,K)\,, \tag{5.23}$$

concluding the proof. □

The next two lemmas contain technical results concerning sequences of metric spaces converging in the pGH sense to metric cones.

Lemma 5.1.6. *Let $Z = C(Y)$ be the metric cone defined over some metric space (Y, d_Y) with $\operatorname{diam}(Y) \leq \pi$ and let d be the associated cone distance. Calling $\overline{p}$ its tip, for every $\varepsilon \in (0, 1/2)$, we define the annuli*

$$A := B_6(\overline{p}) \setminus \overline{B}_1(\overline{p}), \quad A_\varepsilon := \overline{B}_{6-\varepsilon}(\overline{p}) \setminus B_{1+\varepsilon}(\overline{p}), \quad A^\varepsilon := \overline{B}_{6+\varepsilon}(\overline{p}) \setminus B_{1-\varepsilon}(\overline{p}). \tag{5.24}$$

Denote by $\widehat{\mathsf{d}}$, $\widehat{\mathsf{d}}_\varepsilon$ and $\widehat{\mathsf{d}}^\varepsilon$ the length distances induced by d on the annuli A, A_ε and A^ε, respectively. Then, for every $\varepsilon \in (0, 1/2)$ we have that

$$\frac{1}{1+\varepsilon}\widehat{\mathsf{d}}_\varepsilon(p,q) \leq \widehat{\mathsf{d}}(p,q) \leq \frac{1}{1-\varepsilon}\widehat{\mathsf{d}}^\varepsilon(p,q), \tag{5.25}$$

for every $p, q \in \widetilde{A} := \overline{B}_5(\overline{p}) \setminus B_2(\overline{p})$.

Proof. Recall that $Z = C(Y)$ is defined as

$$Z = (Y \times [0,\infty))/\sim, \tag{5.26}$$

where the equivalence relation $\sim$ identifies all points of the form $(y, 0)$ for all $y \in Y$, and the cone distance d is defined by

$$\mathsf{d}\big((x,r),(y,s)\big) = \sqrt{r^2 + s^2 - 2rs\,\cos(d_Y(x,y))}, \quad \text{where } x, y \in Y,\ r, s \geq 0. \tag{5.27}$$

We are only going to prove the first inequality in (5.25), as the proof of the second is analogous. Given any two points $p, q \in \widetilde{A}$, for every sufficiently small $\delta > 0$ there exists a curve $\gamma : [0,1] \to A$ such that $\gamma(0) = p$, $\gamma(1) = q$ and $\operatorname{length}^{\mathsf{d}}(\gamma) < \widehat{\mathsf{d}}(p,q) + \delta$. If $\gamma(t) = (y(t), r(t))$, with $y : [0,1] \to Y$ and $r : [0,1] \to (1,6)$, recalling that $\vee$ denotes the maximum and $\wedge$ the minimum, we define $\widetilde{\gamma} : [0,1] \to A_\varepsilon$ as

$$\widetilde{\gamma}(t) := (y(t), \widetilde{r}(t)) := \big(y(t), (1+\varepsilon) \vee r(t) \wedge (6-\varepsilon)\big). \tag{5.28}$$

Since $\varepsilon \in (0, 1/2)$, $\widetilde{\gamma}(0) = p$ and $\widetilde{\gamma}(1) = q$. Then, we observe that, for every $s, t \in [0,1]$, there holds

$$\begin{aligned}
\mathsf{d}(\widetilde{\gamma}(s), \widetilde{\gamma}(t))^2 =& (\widetilde{r}(s) - \widetilde{r}(t))^2 + 2\widetilde{r}(s)\widetilde{r}(t)\big(1 - \cos(d_Y(y(s), y(t)))\big) && (5.29)\\
\leq& (r(s) - r(t))^2 + 2(1+\varepsilon)^2 r(s) r(t)\big(1 - \cos(d_Y(y(s), y(t)))\big) && (5.30)\\
\leq& (1+\varepsilon)^2 \mathsf{d}(\gamma(s), \gamma(t))^2. && (5.31)
\end{aligned}$$

Thus, we conclude that

$$\widehat{\mathsf{d}}_\varepsilon(p,q) \leq \operatorname{length}^{\mathsf{d}}(\widetilde{\gamma}) \leq (1+\varepsilon)\operatorname{length}^{\mathsf{d}}(\gamma) < (1+\varepsilon)(\widehat{\mathsf{d}}(p,q) + \delta). \tag{5.32}$$

Since (5.32) holds for every $\delta > 0$, the first inequality in (5.25) follows. □

Lemma 5.1.7. *Let $(X, \mathsf{d}, \mathsf{H}^n, \overline{p})$ be an $\mathsf{RCD}(0,n)$ space with*

$$\lim_{r\to+\infty} \frac{\mathsf{H}^n(B_r(\overline{p}))}{r^n} \geq v > 0\,. \tag{5.33}$$

Let $r_i \uparrow +\infty$ be a sequence such that $(X, \mathsf{d}_i := \mathsf{d}/r_i, \overline{p})$ converges in the pGH sense to a metric cone $(C(Y), \mathsf{d}_\infty, p_\infty)$. For every $i \in \mathbb{N}$, set $A_i := B_{6r_i}(\overline{p}) \setminus \overline{B}_{r_i}(\overline{p})$ and $\widetilde{A}_i := \overline{B}_{5r_i}(\overline{p}) \setminus B_{2r_i}(\overline{p})$. Moreover, we define $A := B_6^{\mathsf{d}_\infty}(p_\infty) \setminus \overline{B}_1^{\mathsf{d}_\infty}(p_\infty)$ and $\widetilde{A} := \overline{B}_5^{\mathsf{d}_\infty}(p_\infty) \setminus B_2^{\mathsf{d}_\infty}(p_\infty)$. For every $i \in \mathbb{N}$, we call $\widehat{\mathsf{d}}_i$ the length distance induced by d_i on A_i and, similarly, we call $\widehat{\mathsf{d}}_\infty$ the length distance induced by d_∞ on A. Then, $(\widetilde{A}_i, \widehat{\mathsf{d}}_i)$ converges in the GH sense to $(\widetilde{A}, \widehat{\mathsf{d}}_\infty)$.

Proof. Since $(X, \mathsf{d}_i, \overline{p})$ converges in the pGH sense to $(C(Y), \mathsf{d}_\infty, p_\infty)$, there exists a sequence $(\varepsilon_i)_{i\in\mathbb{N}} \to 0$ such that for every $i \in \mathbb{N}$ there are ε_i-GH maps $f_i : B^{\mathsf{d}_\infty}_{1/\varepsilon_i}(p_\infty) \to B^{\mathsf{d}_i}_{1/\varepsilon_i}(\overline{p})$ and $g_i : B^{\mathsf{d}_i}_{1/\varepsilon_i}(\overline{p}) \to B^{\mathsf{d}_\infty}_{1/\varepsilon_i}(p_\infty)$ with $f_i(p_\infty) = \overline{p}$ and $g_i(\overline{p}) = p_\infty$. In particular, $f_i\big(B^{\mathsf{d}_\infty}_{1/\varepsilon_i}(p_\infty)\big)$ is an ε_i-net in $B^{\mathsf{d}_i}_{1/\varepsilon_i}(\overline{p})$, $g_i\big(B^{\mathsf{d}_i}_{1/\varepsilon_i}(\overline{p})\big)$ is an ε_i-net in $B^{\mathsf{d}_\infty}_{1/\varepsilon_i}(p_\infty)$ and

$$\begin{aligned} |\mathsf{d}_i(f_i(x_\infty), f_i(y_\infty)) - \mathsf{d}_\infty(x_\infty, y_\infty)| < \varepsilon_i\,, \quad \forall\, x_\infty, y_\infty \in B^{\mathsf{d}_\infty}_{1/\varepsilon_i}(p_\infty)\,, \\ |\mathsf{d}_\infty(g_i(x_i), g_i(y_i)) - \mathsf{d}_i(x_i, y_i)| < \varepsilon_i\,, \quad \forall\, x_i, y_i \in B^{\mathsf{d}_i}_{1/\varepsilon_i}(\overline{p})\,. \end{aligned} \tag{5.34}$$

Moreover, we can construct such f_i and g_i so that

$$\mathsf{d}_\infty(g_i(f_i(x_\infty)), x_\infty) < 2\varepsilon_i, \quad \forall x_\infty \in B^{\mathsf{d}_\infty}_{1/\varepsilon_i}(p_\infty)\,. \tag{5.35}$$

In the following we are implicitly going to consider only indices i sufficiently large so that $\varepsilon_i < 1/7$ and thus f_i and g_i are defined on $B_7^{\mathsf{d}_\infty}(p_\infty)$ and $B_7^{\mathsf{d}_i}(\overline{p})$, respectively.

Fix any two points $p, q \in \widetilde{A}$ and $\varepsilon > 0$ sufficiently small. Consider the annulus

$$A_\varepsilon := \overline{B}^{\mathsf{d}_\infty}_{6-\varepsilon}(p_\infty) \setminus B^{\mathsf{d}_\infty}_{1+\varepsilon}(p_\infty) \subset C(Y)\,. \tag{5.36}$$

According to Lemma 5.1.6, take a constant-speed curve $\gamma_\varepsilon : [0,1] \to A_\varepsilon$ such that $\gamma_\varepsilon(0) = p$, $\gamma_\varepsilon(1) = q$ and $\mathrm{length}^{\mathsf{d}_\infty}(\gamma_\varepsilon) \leq (1+\varepsilon)\widehat{\mathsf{d}}_\infty(p,q)$. Partition the interval $[0,1]$ with times $t_0 = 0 < t_1 < \cdots < t_N = 1$ such that $|t_{j+1} - t_j| < \varepsilon(8\widehat{\mathsf{d}}_\infty(p,q))^{-1}$ for every $j = 0, \dots, N-1$. Setting $x_j = \gamma_\varepsilon(t_j)$ for $j = 0, \dots, N$, we have that

$$\mathsf{d}_\infty(x_j, x_{j+1}) < \varepsilon/4\,, \quad \text{for } j = 0, \dots, N-1\,. \tag{5.37}$$

Moreover,

$$\sum_{j=0}^{N-1} \mathsf{d}_\infty(x_j, x_{j+1}) \leq \mathrm{length}^{\mathsf{d}_\infty}(\gamma_\varepsilon) \leq (1+\varepsilon)\widehat{\mathsf{d}}_\infty(p,q)\,. \tag{5.38}$$

Take any $i \in \mathbb{N}$ sufficiently large such that $\varepsilon_i < \varepsilon/8$. Setting $x_j^i = f_i(x_j)$ and recalling that f_i is a ε_i-GH map, on the one hand we have that

$$\mathsf{d}_i(x_j^i, x_{j+1}^i) < \varepsilon_i + \mathsf{d}_\infty(x_j, x_{j+1}) < \varepsilon/2\,, \quad \text{for } j = 0, \dots, N-1\,. \tag{5.39}$$

On the other hand, recalling that $f_i(p_\infty) = \overline{p}$, we get that

$$x_j^i \in \overline{B}^{\mathsf{d}_i}_{6-3\varepsilon/4}(\overline{p}) \setminus B^{\mathsf{d}_i}_{1+3\varepsilon/4}(\overline{p}) \subset X_i\,, \quad \text{for } j = 0, \dots, N\,. \tag{5.40}$$

Combining (5.39) and (5.40), we deduce that $\widehat{\mathsf{d}}_i(x_j^i, x_{j+1}^i) = \mathsf{d}_i(x_j^i, x_{j+1}^i)$ for $j = 0, \dots, N-1$. Thus, introducing the simplified notations $p_i = f_i(p) = f_i(x_0) = x_0^i$ and $q_i = f_i(q) = f_i(x_N) = x_N^i$, we obtain that

$$\begin{aligned}
\widehat{\mathsf{d}}_i(p_i, q_i) =& \widehat{\mathsf{d}}_i(x_0^i, x_N^i) \leq \sum_{j=0}^{N-1} \widehat{\mathsf{d}}_i(x_j^i, x_{j+1}^i) && (5.41)\\
=& \sum_{j=0}^{N-1} \mathsf{d}_i(x_j^i, x_{j+1}^i) \leq \sum_{j=0}^{N-1} \mathsf{d}_\infty(x_j, x_{j+1}) + \varepsilon_i N && (5.42)\\
\leq& (1+\varepsilon)\widehat{\mathsf{d}}_\infty(p, q) + \varepsilon_i N\,. && (5.43)
\end{aligned}$$

Consider a constant-speed curve $\eta_i : [0,1] \to A_i$ such that $\eta_i(0) = p_i$, $\eta_i(1) = q_i$ and $\mathrm{length}^{\mathsf{d}_i}(\eta_i) = \widehat{\mathsf{d}}_i(p_i, q_i)$. Partition the interval $[0,1]$ with times $s_0 = 0 < s_1 < \cdots < s_M = 1$ such that $|s_{j+1} - s_j| < \varepsilon(16\widehat{\mathsf{d}}_\infty(p,q) + \varepsilon N)^{-1}$. Then, calling $y_j^i = \eta_i(s_j)$ for $j = 0, \dots, M$, the inequality (5.41) guarantees that

$$\mathsf{d}_i(y_j^i, y_{j+1}^i) < \varepsilon/8\,. \tag{5.44}$$

Define $z_j^i := g_i(y_j^i) \in C(Y)$ for $j = 0, \dots, M$, and observe that by construction (see (5.35)) we have that

$$\mathsf{d}_\infty(z_0^i, p) < 2\varepsilon_i \quad \text{and} \quad \mathsf{d}_\infty(z_M^i, q) < 2\varepsilon_i\,. \tag{5.45}$$

On the other hand, since g_i is a ε_i-GH map,

$$\mathsf{d}_\infty(z_j^i, z_{j+1}^i) < \varepsilon_i + \mathsf{d}_\infty(y_j^i, y_{j+1}^i) < \varepsilon/2\,, \quad \text{for } j = 0, \dots, M-1\,. \tag{5.46}$$

Moreover, recalling that $g_i(\overline{p}) = p_\infty$, we get that

$$z_j^i \in \overline{B}^{\mathsf{d}_\infty}_{6+\varepsilon/4}(p_\infty) \setminus B^{\mathsf{d}_\infty}_{1-\varepsilon/4}(p_\infty) \subset C(Y)\,, \quad \text{for } j = 0, \dots, M\,. \tag{5.47}$$

Therefore

$$\widehat{\mathsf{d}}_i(p_i,q_i) \geq \sum_{j=0}^{M-1} \mathsf{d}_i(y_j^i, y_{j+1}^i) \geq \sum_{j=0}^{M-1} \mathsf{d}_\infty(z_j^i, z_{j+1}^i) - \varepsilon_i M$$
$$\geq \mathsf{d}_\infty(z_0^i, p) + \sum_{j=0}^{M-1} \mathsf{d}_\infty(z_j^i, z_{j+1}^i) + \mathsf{d}_\infty(z_M^i, q) - \varepsilon_i(M+4)\,. \quad (5.48)$$

Let $\widehat{\mathsf{d}}^\varepsilon$ denote the length distance on $\overline{B}^{\mathsf{d}_\infty}_{6+\varepsilon}(p_\infty) \setminus B^{\mathsf{d}_\infty}_{1-\varepsilon}(p_\infty)$. Combining (5.46) and (5.47), we know that $\widehat{\mathsf{d}}^\varepsilon(z_j^i, z_{j+1}^i) = \mathsf{d}_\infty(z_j^i, z_{j+1}^i)$ for $j = 0, \dots, M-1$, and similarly $\widehat{\mathsf{d}}^\varepsilon(z_0^i, p) = \mathsf{d}_\infty(z_0^i, p)$ and $\widehat{\mathsf{d}}^\varepsilon(z_M^i, q) = \mathsf{d}_\infty(z_M^i, q)$. Then, the right-hand side of (5.48) is equal to

$$\widehat{\mathsf{d}}^\varepsilon(z_0^i, p) + \sum_{j=0}^{M-1} \widehat{\mathsf{d}}^\varepsilon(z_j^i, z_{j+1}^i) + \widehat{\mathsf{d}}^\varepsilon(z_M^i, q) - \varepsilon_i(M+4)$$
$$\geq \widehat{\mathsf{d}}^\varepsilon(p,q) - \varepsilon_i(M+4) \geq (1-\varepsilon)\widehat{\mathsf{d}}_\infty(p,q) - \varepsilon_i(M+4)\,,$$

where we applied Lemma 5.1.6 for the last inequality. In particular,

$$\widehat{\mathsf{d}}_i(p_i,q_i) \geq (1-\varepsilon)\widehat{\mathsf{d}}_\infty(p,q) - \varepsilon_i(M+4)\,. \quad (5.49)$$

Since p, q and ε were chosen arbitrarily and $\widehat{\mathsf{d}}_\infty$ is clearly bounded on $\widetilde{A} \times \widetilde{A}$, the combination of (5.41) and (5.49) implies that given any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for every $i \geq N(\varepsilon)$ we have

$$\left|\widehat{\mathsf{d}}_i(f_i(p), f_i(q)) - \widehat{\mathsf{d}}_\infty(p,q)\right| \leq \varepsilon/3\,, \quad \forall p, q \in \widetilde{A}\,. \quad (5.50)$$

Consider the closest point projection $\pi : C(Y) \to \overline{B}^{\mathsf{d}_\infty}_{5-\varepsilon/3}(p_\infty) \setminus B^{\mathsf{d}_\infty}_{2+\varepsilon/3}(p_\infty)$ and observe that

$$\widehat{\mathsf{d}}_\infty(\pi(p), p) \leq \frac{\varepsilon}{3}\,, \quad \forall p \in \widetilde{A}\,. \quad (5.51)$$

Thus for every $p, q \in \widetilde{A}$ and every $i \geq N(\varepsilon)$ we have that

$$\left|\widehat{\mathsf{d}}_i(f_i(\pi(p)), f_i(\pi(q))) - \widehat{\mathsf{d}}_\infty(p,q)\right|$$
$$\leq \left|\widehat{\mathsf{d}}_i(f_i(\pi(p)), f_i(\pi(q))) - \widehat{\mathsf{d}}_\infty(\pi(p), \pi(q))\right| + \left|\widehat{\mathsf{d}}_\infty(\pi(p), \pi(q)) - \widehat{\mathsf{d}}_\infty(p,q)\right|$$
$$< \frac{\varepsilon}{3} + \left|\widehat{\mathsf{d}}_\infty(\pi(p), \pi(q)) - \widehat{\mathsf{d}}_\infty(\pi(p), q)\right| + \left|\widehat{\mathsf{d}}_\infty(\pi(p), q) - \widehat{\mathsf{d}}_\infty(p,q)\right| < \varepsilon\,. \quad (5.52)$$

On the other hand, since f_i is an ε_i-GH map,

$$\mathsf{d}_i\big(x_i, f_i\big(\overline{B}^{\mathsf{d}_\infty}_{5+2\varepsilon_i}(p_\infty) \setminus B^{\mathsf{d}_\infty}_{2-2\varepsilon_i}(p_\infty)\big)\big) \leq \varepsilon_i\,, \quad \text{for every } x_i \in \widetilde{A}_i\,. \quad (5.53)$$

Therefore, noting that

$$f_i\big(\overline{B}^{\mathsf{d}_\infty}_{5-\varepsilon/3}(p_\infty) \setminus B^{\mathsf{d}_\infty}_{2+\varepsilon/3}(p_\infty)\big) \subset \widetilde{A}_i \tag{5.54}$$

for every i sufficiently large, we have that

$$f_i(\pi(\widetilde{A})) = f_i\big(\overline{B}^{\mathsf{d}_\infty}_{5-\varepsilon/3}(p_\infty) \setminus B^{\mathsf{d}_\infty}_{2+\varepsilon/3}(p_\infty)\big) \tag{5.55}$$

is a $(4\varepsilon_i + \varepsilon/3)$-net in $\widetilde{A}_i$, with respect to d_i. Thus, up to taking a bigger $N(\varepsilon)$, for every $i \geq N(\varepsilon)$ we have that $f_i(\pi(\widetilde{A}))$ is a ε-net in $\widetilde{A}_i$, with respect to d_i. Since the distances d_i and $\widehat{\mathsf{d}}_i$ coincide when restricted to balls of radius $\frac{1}{2}$ centered in $\widetilde{A}_i$, for ε sufficiently small we conclude that $f_i(\pi(\widetilde{A}))$ is an ε-net in $\widetilde{A}_i$ with respect to $\widehat{\mathsf{d}}_i$. This argument proves that for every $i \geq N(\varepsilon)$ the map $f_i \circ \pi : (\widetilde{A}, \widehat{\mathsf{d}}) \to (\widetilde{A}_i, \widehat{\mathsf{d}}_i)$ is an ε-GH map. The conclusion follows from the arbitrariness of ε. □

Proof of Theorem 5.1.2. We first show that, for every r sufficiently large, $B_{4r}(p) \setminus B_{3r}(p)$ is contained in a path-connected component C_r of $B_{5r}(p) \setminus B_{2r}(p)$.

Assume that this is not the case, so that there exists a sequence $r_i \uparrow +\infty$ violating the statement. Hence, for every $i \in \mathbb{N}$, there are two points $x_i, y_i \in B_{4r_i}(p) \setminus B_{3r_i}(p)$ such that no path in $B_{5r_i}(p) \setminus B_{2r_i}(p)$ connects them. By [79, 80] after [53, Theorem 7.6], up to passing to a subsequence, the spaces $(X, \mathsf{d}/r_i)$ converge in pGH sense to a metric cone $C(Z)$ centered in its tip, where (Z, d_Z) is a path-connected metric space. Hence, there are limit points $x_i \to x_\infty$, $y_i \to y_\infty$ in $\overline{B}_4^{C(Z)}(p) \setminus B_3^{C(Z)}(p)$. Consider a collection $\{z_j\}_{j=1}^k \subset \overline{B}_4^{C(Z)}(p) \setminus B_3^{C(Z)}(p)$ for some $k \in \mathbb{N}$ such that $z_1 = x_\infty$, $z_k = y_\infty$ and $\mathsf{d}^{C(Z)}(z_j, z_{j+1}) \leq 1/2$, for every $j \in \{1, \cdots, k-1\}$. For every $i \in \mathbb{N}$ and every $j \in \{2, \cdots, k-1\}$ consider points $x_j^i \in B_{5r_i}(p) \setminus B_{2r_i}(p)$ such that $x_j^i \to z_j$, and set $x_1^i := x_i$ and $x_k^i := y_i$. Consider a curve obtained by concatenating geodesics connecting z_j^i and z_{j+1}^i as j varies between 1 and $k-1$. For i large enough, this curve is contained in $B_{5r_i}(p) \setminus B_{2r_i}(p)$, a contradiction.

Let $L > 0$ be large to be chosen later, and assume by contradiction that the images of the maps in (5.5) have cardinality greater than L for some sequence $r_i \uparrow +\infty$. Let $A_i := B_{6r_i}(p) \setminus \overline{B}_{r_i}(p)$, and let $\overline{\mathsf{d}}_i$ be the induced length distance on A_i. Since A_i is an open set of an $\mathsf{RCD}(0,n)$ space, it is semi-locally simply connected by [188]. Hence, there exists a simply connected universal covering $\widetilde{A}_i$ of A_i. On $\widetilde{A}_i$, we consider the distance $\widetilde{\mathsf{d}}_i$ obtained by lifting the length structure of A_i as in Remark 5.3.4. The deck transformations of this covering form a group of isometries of $(\widetilde{A}_i, \widetilde{\mathsf{d}}_i)$ isomorphic to $\pi_1(A_i)$, and the corresponding quotient space is isometric to $(A_i, \overline{\mathsf{d}}_i)$.

Let $p_i : \widetilde{A}_i \to A_i$ be the covering map, and let $\widetilde{A}'_i$ be a path-connected component of $p_i^{-1}(C_{r_i})$. Let $G_i < \pi_1(A_i)$ be the subgroup of deck transformations of $p_i : \widetilde{A}_i \to A_i$ sending $\widetilde{A}'_i$ into itself. Then, since G_i acts freely on $\widetilde{A}_i$, it can be naturally identified with a group of isometries of $(\widetilde{A}'_i, \widetilde{\mathsf{d}}_i)$. Moreover, since $\pi_1(A_i)$ acts transitively on the fibers of $p_i : \widetilde{A}_i \to A_i$, it follows that G_i acts transitively on the fibers of the restriction

$$p_i|_{\widetilde{A}'_i} : \widetilde{A}'_i \to C_{r_i} \,. \tag{5.56}$$

Thus, $(\widetilde{A}'_i, \widetilde{\mathsf{d}}_i)/G_i \cong (C_{r_i}, \overline{\mathsf{d}}_i)$ isometrically. In addition, since loops contained in C_{r_i} can be lifted to paths contained in $\widetilde{A}'_i$, each group G_i contains an isomorphic copy of the image of $i_* : \pi_1(C_{r_i}, \bar{c}_i) \to \pi_1(A_i, \bar{c}_i)$ where $\bar{c}_i \in C_{r_i}$.

The spaces $(\widetilde{A}'_i, \widetilde{\mathsf{d}}_i/r_i)$ are locally uniformly doubling by Lemma 5.3.5. Hence, modulo passing to a subsequence, one has the equivariant GH convergence

$$(\widetilde{A}'_i, \widetilde{\mathsf{d}}_i/r_i, G_i) \to (Z, \mathsf{d}_z, G) \,. \tag{5.57}$$

Observe now that by an argument similar to the one used at the beginning of the proof, for every $\varepsilon > 0$, each $(C_{r_i}, \overline{\mathsf{d}}_i/r_i)$ is ε-close to $(B_{5r_i}(p) \setminus B_{2r_i}(p), \overline{\mathsf{d}}_i/r_i)$ in GH sense for i large enough. Combining with Lemma 5.1.7, we deduce that the spaces $(C_{r_i}, \overline{\mathsf{d}}_i/r_i)$ converge to an annulus $(C_\infty, \overline{\mathsf{d}}_\infty)$ of a tangent cone at infinity $C(Y)$ of X, where Y is a non-collapsed $\mathsf{RCD}(n-2, n-1)$ space, and $\overline{\mathsf{d}}_\infty$ is the length distance induced by $C(Y)$ on C_∞. By [80, Theorem 1.2] (after [63]), $\mathsf{H}^{n-1}(Y)$ is bounded from below by a constant depending on n and v, so that $\#\pi_1(C_\infty) < C(n, v)$ by Lemma 5.1.5. Note that here we are using the assumption that $n \geq 3$. In the remainder of the proof, we denote by $C(n, v)$ a generic constant depending on n and v only, without renaming it at every step. Since $\widetilde{A}'_i/G_i \cong C_{r_i}$, by Theorem 2.1.23, we have that

$$(Z, \mathsf{d}_z)/G \cong (C_\infty, \overline{\mathsf{d}}_\infty) \,. \tag{5.58}$$

We now claim that $\#G \leq C(n, v)$. We divide the proof of this fact in steps.

Step 1: We claim that $\Gamma_\varepsilon := \{g \in G : g \cdot x \in B_\varepsilon(x) \text{ for some } x \in Z\}$ has cardinality at most $C(n, v)$, where $\varepsilon := \varepsilon'/10$ and $\varepsilon'(n, v)$ is given by Lemma 5.1.4.

Assume that there are $N > 0$ distinct elements $\{g_k\}_{k=1}^N \subset \Gamma_\varepsilon$ and (possibly repeated) points $x_k \in Z$ such that $g_k \cdot x_k \in B_\varepsilon(x_k)$. For every $k \in \{1, \dots, N\}$, choose points $x_{i,k} \in \widetilde{A}'_i$ converging to x_k, and isometries $g_{i,k} \in G_i$ converging to g_k. If $k \neq k'$, since $g_k \neq g_{k'}$, we have that $g_{i,k} \neq g_{i,k'}$ for i large enough.

Consider the correspondence between deck transformations of $\widetilde{A}_i$ and $\pi_1(A_i)$. Each $g_{i,k}$ corresponds to a loop $\gamma_{i,k}$ of $\pi_1(A_i)$ centered in $p_i(x_{i,k}) \in A_i$ obtained by projecting a curve of $\widetilde{A}_i$ from $x_{i,k}$ to $g_{i,k} \cdot x_{i,k}$. In particular, such curve can be chosen to satisfy

$$\text{length}^{\overline{\mathsf{d}}_i}(\gamma_{i,k}) \leq 2\widetilde{\mathsf{d}}_i(x_{i,k}, g_{i,k} \cdot x_{i,k})\,. \tag{5.59}$$

Since $x_{i,k} \to x_k$ and $g_{i,k} \cdot x_{i,k} \to g_k \cdot x_k \in B_\varepsilon^{\mathsf{d}_z}(x_k)$ as $(\widetilde{A}_i', \widetilde{\mathsf{d}}_i/r_i) \to (Z, \mathsf{d}_z)$, it follows that

$$\limsup_{i\to+\infty} \text{length}^{\overline{\mathsf{d}}_i/r_i}(\gamma_{i,k}) \leq 2 \limsup_{i\to+\infty} \frac{\widetilde{\mathsf{d}}_i(x_{i,k}, g_{i,k} \cdot x_{i,k})}{r_i} \leq 2\varepsilon\,. \tag{5.60}$$

Observe now that $C_\infty \cong Z/G$ is $C(n)$-doubling and has diameter bounded above by a possibly different $C(n)$. As a consequence, we deduce that there exists $T(N,n) > 0$ (with the property that $T(N,n) \uparrow +\infty$ as $N \uparrow +\infty$) such that, if C_∞ contains a collection of points $\{c_l\}_{l=1}^N$, then there are at least T points of such collection contained in a ball $B_{\varepsilon/2}^{\overline{\mathsf{d}}_\infty}(c)$. Using this principle with the collection $\{G \cdot x_k\}_{k=1}^N \subset C_\infty$, we find T different indexes $k_1, \dots, k_T$ such that $\overline{\mathsf{d}}_\infty(G \cdot x_{k_j}, G \cdot x_\infty) < \varepsilon/2$ for all $j = 1, \dots, T$ and some $G \cdot x_\infty \in Z/G$. By Theorem 2.1.23, the projections $G_i \cdot x_{i,k_j} \in (C_{r_i}, \overline{\mathsf{d}}_i/r_i)$ converge to $G \cdot x_{k_j}$ in the space realizing the convergence $(C_{r_i}, \overline{\mathsf{d}}_i/r_i) \to (C_\infty, \overline{\mathsf{d}}_\infty)$, for all $j = 1, \dots, T$. Moreover, in the same space, there are points $a_i \in C_{r_i}$ such that $a_i \to G \cdot x_\infty$. Hence, for i large enough

$$G_i \cdot x_{i,k_j} \in B_\varepsilon^{\overline{\mathsf{d}}_i/r_i}(a_i) \subset C_{r_i} \tag{5.61}$$

for all $j = 1, \dots, T$.

Observe that for ε small enough we have $B_\varepsilon^{\overline{\mathsf{d}}_i/r_i}(a_i) = B_\varepsilon^{\mathsf{d}/r_i}(a_i)$. Moreover, for every k and every i, it holds that $G_i \cdot x_{i,k} \in C_{r_i}$ is identified with $p_i(x_{i,k}) \in A_i$ via the inclusion $C_{r_i} \to A_i$. Hence, combining with (5.60), the balls $B_{3r_i\varepsilon}^{\mathsf{d}}(a_i) \subset X$ contain T loops $\gamma_{i,k}$ corresponding to different elements of $\pi_1(A_i)$. By Lemma 5.1.4, it follows that $T \leq C(n,v)$, so that $N \leq C(n,v)$ as well.

We record two important consequences of Step 1:

i) the set $\text{Fix}(G) := \{g \in G : g \cdot x = x \text{ for some } x \in Z\}$ has cardinality at most $C(n,v)$;

ii) for every $z \in Z$, there is $\varepsilon > 0$ such that the almost stabilizer $\Gamma_\varepsilon(z) := \{g \in G : g \cdot z \in B_\varepsilon(z)\}$ has cardinality at most $C(n,v)$.

Step 2: We claim that $H := \langle \text{Fix}(G) \rangle$ is normal and has cardinality at most $C(n,v)$.

Let $h \in \mathrm{Fix}(G)$ be such that $h \cdot z = z$ for some $z \in Z$, and let $g \in G$. Then $g^{-1}hg$ fixes $g^{-1}(z)$, so that $g^{-1}hg \in \mathrm{Fix}(G)$. Let now $h \in H$, so that $h = g_1 \cdots g_k$ for some $\{g_j\}_{j=1}^k \subset \mathrm{Fix}(G)$. For every $g \in G$, we have that

$$g^{-1}hg = g^{-1}g_1 g\, g^{-1}g_2 \cdots g_{k-1} g\, g^{-1} g_k g\,. \tag{5.62}$$

Since $g^{-1}g_j g$ belongs to $\mathrm{Fix}(G)$ for every j, it follows that $g^{-1}hg \in H$, proving that H is normal.

To prove that H has cardinality at most $C(n,v)$, we first claim that $H/Z(H)$ has cardinality at most $C(n,v)$, where $Z(H) < H$ is the center of H. Let $T : H \to \mathrm{Sym}(\mathrm{Fix}(G))$ be the homomorphism defined so that $T(h)$ is the permutation of $\mathrm{Fix}(G)$ given by $T(h)(g) := h^{-1}gh$. By Step 1, $H/\mathrm{Ker}(T)$ has cardinality at most $C(n,v)$. Since $\mathrm{Ker}(T)$ is in the center of H it follows that $H/Z(H)$ has cardinality at most $C(n,v)$, as claimed.

By Schur's Theorem, the derived group $H' \leq H$ has cardinality at most $C(n,v)$. Therefore it suffices to show that H/H' has cardinality at most $C(n,v)$ as well. The group H/H' is abelian, and it is generated by at most $C(n,v)$ elements of order at most $C(n,v)$ (i.e. the classes of the elements in $\mathrm{Fix}(G)$). Hence, H/H' has cardinality at most $C(n,v)$, concluding the proof of Step 2.

Step 3: We claim that G/H acts properly discontinuously on Z/H.

We first show that G/H acts freely. Indeed if $(g+H)\cdot x = H\cdot x$ for some $H\cdot x \in Z/H$ and some $g + H \in G/H$ then there exists $h \in H$ such that $(g+h)\cdot x = x$, so that $g + h \in H$ and $g + H = H$. Hence, the action is free.

To conclude, it suffices to show that if $H \cdot x \in Z/H$ and $g_i + H \in G/H$ are such that $(g_i + H)\cdot x \to H \cdot x$, then $g_i + H = H$ for i large enough. To this aim, observe that there are elements $h_i \in H$ such that $(g_i + h_i)\cdot x \to x$. By Step 1, it follows that $g_i + h_i$ fixes x for i large enough, so that $g_i \in H$ because G/H acts freely.

To complete the proof that $\#G \leq C(n,v)$, note that since G/H acts properly discontinuously on Z/H by Step 3, G/H is a subgroup of $\pi_1(C_\infty)$. Thus it has cardinality at most $C(n,v)$. Combining with Step 2, we deduce that G has cardinality at most $C(n,v)$ as well.

We now conclude the proof of the theorem. Since Z/G is compact, we deduce that Z is compact. Let $a_i \in \widetilde{A}'_i$ be elements such that $a_i \to z \in Z$ as $(\widetilde{A}'_i, \widetilde{\mathsf{d}}_i/r_i) \to (Z, \mathsf{d}_z)$. Let $\varepsilon(n,v) > 0$ be given by Lemma 5.1.4, so that the almost stabilizers $\Gamma_{\varepsilon,a_i} := \langle \{g \in G_i : \widetilde{\mathsf{d}}_i(a_i, g\cdot a_i) < \varepsilon r_i\}\rangle$ all have cardinality at most $C(n,v)$. Recall that we are assuming by contradiction that $\#G_i \geq L$, where L is a large number, and

that we aim to show that if L is chosen too large in a way that depends only on n and v, then we have a contradiction. In particular, the index of Γ_{ε,a_i} in G_i is at least $S := L/C(n,v)$. Consider elements $\{g_i^j\}_{j=1}^S \subset G_i$ belonging to different left cosets of Γ_{ε,a_i} in G_i. Since Z is compact, for each $j \in \{1, \cdots S\}$, there exists $g^j \in G$ such that $g_i^j \to g^j$.

We claim that the elements $\{g^j\}_{j=1}^S \subset G$ are all distinct. If this is the case, we conclude that

$$L/C(n,v) = S \leq \#G \leq C(n,v)\,, \tag{5.63}$$

contradicting a suitably large choice of L. We show that $g^1 \neq g^2$, the other cases being analogous. If $g^1 = g^2$, then $\mathsf{d}_z(g^1 \cdot z, g^2 \cdot z) = 0$, so that

$$\widetilde{\mathsf{d}}_i(g_i^1 \cdot a_i, g_i^2 \cdot a_i) < \varepsilon r_i \tag{5.64}$$

if i is large enough. This implies that $g_i^1 \in g_i^2 + \Gamma_{\varepsilon,a_i}$, contradicting that the elements g_i^j belong to different left cosets of Γ_{ε,a_i} in G_i. □

5.2 Proof of Theorem 5.0.1: non-Euclidean volume growth case

The goal of this section is to prove Theorem 5.0.1 under the assumption that the space X does not have maximal volume growth. We observe that if X is compact, then $\pi_1(X)$ is virtually abelian and thus amenable by [153, Theorem 1.3], so that $\|X\| = 0$ by [110, Corollary C, page 248]. In the following, we will assume that X is non-compact. To prove the vanishing theorem in this case, thanks to Theorem 2.3.12, it is sufficient to prove the following:

Theorem 5.2.1. *Let $(X, \mathsf{d}, \mathsf{H}^n)$ be a non-compact $\mathsf{RCD}(0,n)$ space such that*

$$\lim_{R\to\infty} \frac{\mathsf{H}^n(B_R(p))}{R^n} = 0\,, \tag{5.65}$$

for some $p \in X$. Then X admits a cover with precompact open sets $\{U_i\}_{i\in\mathbb{N}}$ which is amenable, amenable at infinity and has $\mathrm{mult}\big(\{U_i\}_{i\in\mathbb{N}}\big) \leq n$.

As said before, Theorem 5.2.1 generalizes some results due to Gromov in [110], where he assumed a uniform small upper bound on the volume of unit balls. The rough idea behind the proof of Theorem 5.2.1 is the following. In view of (5.65), [80, Theorem 1.2] implies that all the blow-downs of X have Hausdorff dimension $\leq n-1$. In particular, they have topological dimension at most $n-1$ and thus

they admit arbitrarily small open covers with multiplicity at most n. Ideally, one would like to combine the existence of such covers with small multiplicity with the Margulis lemma from [126] (and the subsequent partial generalization to RCD spaces in [84]) to construct the sought cover of X. There are several difficulties when trying to implement this strategy. The most severe ones are that:

i) in general, it is not clear whether the pointed GH-approximations from the (rescaled)-space to its blow-downs can be chosen to be continuous, due to the possibly poor topological behavior of blow-downs in the collapsed case;

ii) the blow-down of X need not be unique and need not be invariant under rescaling. Thus it is hard to turn the local information obtained via a blow-down argument into a global one;

iii) a local version of the Margulis lemma analogous to the one obtained in [126] in the smooth case is not available yet for RCD spaces, although a more global Margulis lemma was recently obtained in such context in [84].

The key tool to circumvent the difficulties in i) and ii) will be a recent result by Papasoglu stated below as Theorem 5.2.3. Using Theorem 5.2.3 we show in Proposition 5.2.6 that a metric space with volume growth of polynomial degree less than n, after a suitable conformal rescaling of the metric, has small $(n-1)$-Urysohn width (see Definition 5.2.2) outside of sufficiently large compact sets. To complete the proof of Theorem 5.2.1 we are going to combine such control on the $(n-1)$-width with Theorem 5.3.1, where we obtain a weaker version of the Margulis lemma from [126] with no quantitative bound on the index of the nilpotent subgroup.

Definition 5.2.2. Let $q \in \mathbb{N}$. We say that a metric space (X, d) has q-Urysohn width less than or equal to W, and write $UW_q(X, \mathsf{d}) \leq W$ if for every $\varepsilon > 0$ there exist a q-dimensional simplicial complex Y and a continuous map $\pi : X \to Y$ such that every fiber $\pi^{-1}(y)$, $y \in Y$, has diameter $\leq W + \varepsilon$.

In the following, we will denote by H^n_∞ the n-dimensional Hausdorff content for $n \geq 0$. The next statement corresponds to [163, Theorem 3.3].

Theorem 5.2.3. *For every $n \in \mathbb{N}$ there exists $\varepsilon_n > 0$ with the following property. Let (X, d) be a proper metric space and $R > 0$ be such that for every $x \in X$ there is an open set U_x with the properties:*

(i) $B_R(x) \subset U_x \subset B_{10R}(x)$

(ii) $\mathsf{H}^{n-1}_{\infty}(\partial U_x) \leq \varepsilon_n R^{n-1}$.

Then for every $\varepsilon > 0$ *there exist a* locally finite $(n-1)$*-dimensional simplicial complex* Y *and a* proper *continuous map* $\pi : X \to Y$ *such that every fiber* $\pi^{-1}(y)$, $y \in Y$, *has diameter* $\leq R + \varepsilon$. *In particular,* $UW_{n-1}(X) \leq R$.

The statement of [163, Theorem 3.3] does not mention that, in the previous theorem, the simplicial complex Y and the map π can be taken to be locally finite and proper, respectively. However, this is a consequence of the proof in [163].

The following Coarea Inequality follows from [88, Theorem 1.1].

Proposition 5.2.4 (Coarea inequality)**.** *Let* (X, d) *be a proper metric space and* $n > 1$. *Then for every* $x \in X$ *and* $0 < \alpha < \beta < \infty$ *we have that*

$$\int_{\alpha}^{\beta} \mathsf{H}^{n-1}(\partial B_s(x))ds \leq C_n \left(\mathsf{H}^n(B_\beta(x)) - \mathsf{H}^n(B_\alpha(x))\right), \tag{5.66}$$

where C_n *is a dimensional constant.*

To prove Theorem 5.2.1 the key idea is to apply Theorem 5.2.3 to a conformal rescaling of (X, d). Lemma 5.2.5 below introduces the relevant conformal rescaling and establishes its basic properties. The forthcoming Proposition 5.2.6 establishes the sought estimate for the Urysohn width of the conformal rescaling away from a bounded set.

We recall that the set of absolutely continuous curves in a metric space (X, d), denoted by $AC([a, b], (X, \mathsf{d}))$, is defined in (2.2).

Lemma 5.2.5. *Let* (X, d, p) *be a complete, separable and geodesic pointed metric space. Given any curve* $\gamma \in AC([0, 1], (X, \mathsf{d}))$ *we set*

$$\widetilde{L}(\gamma) := \int_0^1 \frac{|\dot{\gamma}|(t)}{\mathsf{d}(p, \gamma(t)) \vee 1} \mathrm{d}t\,, \tag{5.67}$$

where $|\dot{\gamma}|$ *denotes the metric derivative. Accordingly, we define the function* $\widetilde{\mathsf{d}} : X \times X \to \mathbb{R}_+$ *as*

$$\widetilde{\mathsf{d}}(x, y) := \inf\{\widetilde{L}(\gamma) \,:\, \gamma \in AC([0, 1], (X, \mathsf{d})), \gamma(0) = x,\, \gamma(1) = y\}\,. \tag{5.68}$$

Then $\widetilde{\mathsf{d}}$ *is a distance on* X. *Moreover, there exists* $0 < \delta_0 \leq \frac{1}{6}$ *such that for all* $\delta \in (0, \delta_0]$ *and every* $q \in X$ *with* $\mathsf{d}(p, q) > 10$, *we have that*

$$\widetilde{B}_\delta(q) \subset B_{2\delta \mathsf{d}(p,q)}(q) \subset B_{3\delta \mathsf{d}(p,q)}(q) \subset \widetilde{B}_{6\delta}(q)\,, \tag{5.69}$$

where we denote by $\widetilde{B}$ *balls with respect to the distance* $\widetilde{\mathsf{d}}$.

Proof. It is straightforward to check that $\widetilde{\mathsf{d}}$ is symmetric, finite, and satisfies the triangle inequality. Thus, to prove that $\widetilde{\mathsf{d}}$ is a distance we need only to check positivity. To this aim, consider $a, b \in X$ with $\widetilde{\mathsf{d}}(a,b) = 0$ and assume by contradiction that $\delta := \mathsf{d}(a,b) > 0$. Let $\gamma \in AC([0,1],(X,\mathsf{d}))$ be any curve connecting a and b and call $t_0 \in [0,1]$ the first intersection time of γ with $\partial B_{\delta/2}(a)$. For every $t \in [0,t_0]$, it holds

$$\mathsf{d}(p,\gamma(t)) \le \mathsf{d}(p,a) + \mathsf{d}(a,\gamma(t)) \le \mathsf{d}(p,a) + \delta/2. \tag{5.70}$$

Then, we can estimate

$$\begin{aligned}
\int_0^1 \frac{|\dot\gamma|(t)}{\mathsf{d}(p,\gamma(t)) \vee 1}\mathrm{d}t &\ge \int_0^{t_0} \frac{|\dot\gamma|(t)}{\mathsf{d}(p,\gamma(t)) \vee 1}\mathrm{d}t && (5.71)\\
&\ge \int_0^{t_0} \frac{|\dot\gamma|(t)}{\mathsf{d}(p,a) + \delta/2 + 1}\mathrm{d}t && (5.72)\\
&\ge \frac{\delta/2}{\mathsf{d}(p,a) + \delta/2 + 1} > 0\,. && (5.73)
\end{aligned}$$

Since γ was arbitrary, we obtain the desired contradiction.

Until the end of the proof we fix $\delta_0 \le 1/6$ with the following property: for every $\delta \in (0,\delta_0]$, if $x > 0$ is such that $\log(1+x) \le \frac{3}{2}\delta$, then $x \le 2\delta$.

We start by proving the third inclusion in (5.69). Take $q \in X$ with $\mathsf{d}(p,q) > 10$, any $\delta \in (0,\delta_0]$ and consider any $a \in B_{3\delta\mathsf{d}(p,q)}(q)$. Let $\gamma \in AC([0,1],(X,\mathsf{d}))$ be a constant speed geodesic (with respect to d), joining a to q. In particular, we have that

$$|\dot\gamma| \equiv \mathsf{d}(a,q) \le 3\delta\mathsf{d}(p,q)\,, \quad \gamma([0,1]) \subset B_{\mathsf{d}(p,q)/2}(q) \subset X \setminus B_{\mathsf{d}(p,q)/2}(p)\,. \tag{5.74}$$

Thus, for every $t \in [0,1]$ we have that $\mathsf{d}(p,\gamma(t)) \vee 1 \ge \frac{1}{2}\mathsf{d}(p,q)$ and therefore

$$\widetilde{\mathsf{d}}(a,q) \le \widetilde{L}(\gamma) = \int_0^1 \frac{|\dot\gamma|(t)}{\mathsf{d}(p,\gamma(t)) \vee 1}\mathrm{d}t \le \int_0^1 \frac{3\delta\mathsf{d}(p,q)}{\frac{1}{2}\mathsf{d}(p,q)}\mathrm{d}t = 6\delta\,. \tag{5.75}$$

Since a was arbitrary, we deduce that $B_{3\delta\mathsf{d}(p,q)}(q) \subset \widetilde{B}_{6\delta}(q)$.

For the first inclusion in (5.69), take any $b \in \widetilde{B}_\delta(q)$ and consider a curve $\gamma \in AC([0,1],(X,\mathsf{d}))$ joining q to b such that $\widetilde{L}(\gamma) \le \frac{3}{2}\delta$. Since $\widetilde{L}$ is invariant by reparametrisation, we can assume that γ has constant speed with respect to the distance d, i.e. $|\dot\gamma| \equiv L$ for some $L > 0$. We claim that

$$\gamma([0,1]) \subset X \setminus B_2(p)\,. \tag{5.76}$$

Arguing by contradiction, let $t_1 \in [0,1]$ be the first intersection time of γ with $\partial B_2(p)$ and let $t_0 \in [0,1]$ be defined as

$$t_0 := \sup\{t < t_1 \,:\, \gamma(t) \notin B_4(p)\}\,. \tag{5.77}$$

With such choices we can estimate

$$\frac{3}{2}\delta \geq \int_{t_0}^{t_1} \frac{|\dot\gamma|(t)}{\mathsf{d}(p,\gamma(t)) \vee 1}\mathrm{d}t = \int_{t_0}^{t_1} \frac{|\dot\gamma|(t)}{\mathsf{d}(p,\gamma(t))}\mathrm{d}t \geq \int_{t_0}^{t_1} \frac{|\dot\gamma|(t)}{4}\mathrm{d}t \geq \frac{1}{2}\,, \tag{5.78}$$

thus yielding the desired contradiction, since $\delta \leq \delta_0 \leq 1/6$. On the other hand, we have that

$$\mathsf{d}(p,\gamma(t)) \leq \mathsf{d}(p,q) + \mathsf{d}(q,\gamma(t)) \leq \mathsf{d}(p,q) + Lt\,. \tag{5.79}$$

Thus, taking into account (5.76), we deduce that

$$\frac{3}{2}\delta \geq \int_0^1 \frac{|\dot\gamma|(t)}{\mathsf{d}(p,\gamma(t))}\mathrm{d}t \geq \int_0^1 \frac{L}{\mathsf{d}(p,q)+Lt}\mathrm{d}t = \log\left(1 + \frac{L}{\mathsf{d}(p,q)}\right). \tag{5.80}$$

Then, our choice of δ_0 implies that $L/\mathsf{d}(p,q) \leq 2\delta$. We can conclude by observing that $\mathsf{d}(q,b) \leq L \leq 2\delta\mathsf{d}(p,q)$. Since $b \in \widetilde{B}_\delta(q)$ was arbitrary, we proved that $\widetilde{B}_\delta(q) \subset B_{2\delta\mathsf{d}(p,q)}(q)$. □

Proposition 5.2.6. *Fix $n \in \mathbb{N}$. Let (X,d,p) be a complete, separable, proper, non-compact and geodesic pointed metric space. Let $\widetilde{\mathsf{d}}$ and δ_0 be the distance and the parameter given by Lemma 5.2.5. Assume that*

$$\lim_{R\to\infty} \frac{\mathsf{H}^n(B_R(p))}{R^n} = 0\,. \tag{5.81}$$

Then, for every $\delta \in (0,\delta_0]$ there exists $T(\delta) > 20$ such that the following holds. There exist a locally finite $(n-1)$-dimensional simplicial complex Y and a proper continuous map $\pi : (X \setminus B_{T(\delta)}(p), \widetilde{\mathsf{d}}) \to Y$ with fibers of diameter less than δ. In particular,

$$UW_{n-1}(X \setminus B_{T(\delta)}(p), \widetilde{\mathsf{d}}) \leq \delta\,. \tag{5.82}$$

Proof. Our goal is to show that $(X \setminus B_{T(\delta)}(p), \widetilde{\mathsf{d}})$ satisfies the assumptions of Theorem 5.2.3.

Fix any $\delta \in (0,\delta_0]$ and take any $q \in X$ such that $\mathsf{d}(p,q) > 20$. Given any $\eta \in [0,\delta]$, observe that for every $q' \in \partial B_{(2\delta+\eta)\mathsf{d}(p,q)}(q)$ we have

$$\mathsf{d}(p,q') \geq \mathsf{d}(p,q) - (2\delta+\eta)\mathsf{d}(p,q) \geq \frac{1}{2}\mathsf{d}(p,q)\,. \tag{5.83}$$

Suppose now that $A \subset \partial B_{(2\delta+\eta)\mathsf{d}(p,q)}(q)$ has $\mathrm{diam}_{\widetilde{\mathsf{d}}}(A) = \rho \ll \delta_0$, where $\mathrm{diam}_{\widetilde{\mathsf{d}}}$ denotes the diameter computed with respect to the distance $\widetilde{\mathsf{d}}$. In particular, there exist

$a, b \in A$ such that $\widetilde{\mathsf{d}}(a,b) > \frac{\rho}{2}$, thus $b \notin \widetilde{B}_{\rho/2}(a)$. Taking into account (5.83) (which also ensures that $\mathsf{d}(p,a) > 10$) and applying Lemma 5.2.5, we obtain

$$B_{\rho \mathsf{d}(p,q)/8}(a) \subset B_{\rho \mathsf{d}(p,a)/4}(a) \subset \widetilde{B}_{\rho/2}(a)\,. \tag{5.84}$$

Thus, $b \notin B_{\rho \mathsf{d}(p,q)/8}(a)$ and, denoting by $\mathrm{diam}_{\mathsf{d}}$ the diameter computed with respect to the distance d, we conclude that

$$\mathrm{diam}_{\mathsf{d}}(A) \geq \frac{\mathsf{d}(p,q)}{8}\rho = \frac{\mathsf{d}(p,q)}{8}\,\mathrm{diam}_{\widetilde{\mathsf{d}}}(A)\,. \tag{5.85}$$

In the following, for every $d, \zeta > 0$, we denote by H^d_ζ and $\widetilde{\mathsf{H}}^d_\zeta$ the ζ-approximate d-dimensional Hausdorff measures with respect to d and $\widetilde{\mathsf{d}}$, respectively. Accordingly, we denote by H^d and $\widetilde{\mathsf{H}}^d$ the respective d-dimensional Hausdorff measures. Thanks to (5.85), for every $\rho \ll \delta_0$ we obtain that

$$\begin{aligned}
&\widetilde{\mathsf{H}}^{n-1}_\rho\big(\partial B_{(2\delta+\eta)\mathsf{d}(p,q)}(q)\big)\\
&= \inf\left\{\sum_{i\in\mathbb{N}} \mathrm{diam}_{\widetilde{\mathsf{d}}}(A_i)^{n-1} \;:\; \bigcup_{i\in\mathbb{N}} A_i = \partial B_{(2\delta+\eta)\mathsf{d}(p,q)}(q),\ \mathrm{diam}_{\widetilde{\mathsf{d}}}(A_i) \leq \rho\right\}\\
&\leq \inf\left\{\frac{8^{n-1}}{\mathsf{d}(p,q)^{n-1}}\sum_{i\in\mathbb{N}} \mathrm{diam}_{\mathsf{d}}(A_i)^{n-1} \;:\; \bigcup_{i\in\mathbb{N}} A_i = \partial B_{(2\delta+\eta)\mathsf{d}(p,q)}(q)\,,\right.\\
&\qquad\left.\mathrm{diam}_{\mathsf{d}}(A_i) \leq \rho\,\frac{\mathsf{d}(p,q)}{8}\right\} = \frac{8^{n-1}}{\mathsf{d}(p,q)^{n-1}}\;\mathsf{H}^{n-1}_{\rho\frac{\mathsf{d}(p,q)}{8}}\big(\partial B_{(2\delta+\eta)\mathsf{d}(p,q)}(q)\big)\,.
\end{aligned} \tag{5.86}$$

Taking the limit as $\rho \to 0$, we conclude that for every $q \in X$ with $\mathsf{d}(p,q) > 20$ and every $\eta \in [0,\delta]$ there holds

$$\widetilde{\mathsf{H}}^{n-1}\big(\partial B_{(2\delta+\eta)\mathsf{d}(p,q)}(q)\big) \leq \frac{c}{\mathsf{d}(p,q)^{n-1}}\;\mathsf{H}^{n-1}\big(\partial B_{(2\delta+\eta)\mathsf{d}(p,q)}(q)\big)\,, \tag{5.87}$$

where, for notational simplicity, we put $c = 8^{n-1}$.

Applying Proposition 5.2.4 with $x = q$, $\alpha = 2\delta\mathsf{d}(p,q)$ and $\beta = 3\delta\mathsf{d}(p,q)$ we obtain

$$⨍_{2\delta\mathsf{d}(p,q)}^{3\delta\mathsf{d}(p,q)} \mathsf{H}^{n-1}(\partial B_s(q))ds \leq C_n\,\frac{\mathsf{H}^n(B_{3\delta\mathsf{d}(p,q)}(q)) - \mathsf{H}^n(B_{2\delta\mathsf{d}(p,q)}(q))}{\delta\mathsf{d}(p,q)}\,. \tag{5.88}$$

Thus there exists $\eta(q) \in (0,\delta)$ such that

$$\mathsf{H}^{n-1}(\partial B_{(2\delta+\eta(q))\mathsf{d}(p,q)}(q)) \leq C_n\,\frac{\mathsf{H}^n(B_{3\delta\mathsf{d}(p,q)}(q)) - \mathsf{H}^n(B_{2\delta\mathsf{d}(p,q)}(q))}{\delta\mathsf{d}(p,q)}\,. \tag{5.89}$$

The combination of (5.87) and (5.89) yields

$$\widetilde{\mathsf{H}}^{n-1}\big(\partial B_{(2\delta+\eta(q))\mathsf{d}(p,q)}(q)\big) \le c\,C_n \frac{\mathsf{H}^n(B_{3\delta\mathsf{d}(p,q)}(q))}{\delta \mathsf{d}(p,q)^n} \tag{5.90}$$

$$\le c\,C_n \frac{\mathsf{H}^n(B_{2\mathsf{d}(p,q)}(p))}{\delta \mathsf{d}(p,q)^n}\,. \tag{5.91}$$

Thanks to (5.81) there exists $T(\delta) > 20$ such that

$$\widetilde{\mathsf{H}}^{n-1}\big(\partial B_{(2\delta+\eta(q))\mathsf{d}(p,q)}(q)\big) \le \varepsilon_n \delta^{n-1}\,, \qquad \forall q \in X \setminus B_{T(\delta)}(p)\,, \tag{5.92}$$

where ε_n is the constant given by Theorem 5.2.3. Finally, we deduce that

$$\widetilde{\mathsf{H}}^{n-1}_\infty\big(\partial B_{(2\delta+\eta(q))\mathsf{d}(p,q)}(q) \cap (X \setminus B_{T(\delta)}(p))\big) \le \widetilde{\mathsf{H}}^{n-1}_\infty\big(\partial B_{(2\delta+\eta(q))\mathsf{d}(p,q)}(q)\big) \tag{5.93}$$

$$\le \widetilde{\mathsf{H}}^{n-1}\big(\partial B_{(2\delta+\eta(q))\mathsf{d}(p,q)}(q)\big) \tag{5.94}$$

$$\le \varepsilon_n \delta^{n-1}\,. \tag{5.95}$$

Thanks to Lemma 5.2.5 we can apply Theorem 5.2.3 to the metric space $(X \setminus B_{T(\delta)}(p), \widetilde{\mathsf{d}})$ and conclude the proof. □

Proof of Theorem 5.2.1. We fix $\delta < \min\{\delta_0/2, \varepsilon/8\}$, where δ_0 is the parameter identified in Lemma 5.2.5 and $\varepsilon = \varepsilon(n)$ is the constant appearing in Corollary 5.3.2.

By Proposition 5.2.6 there exist a locally finite $(n-1)$-dimensional simplicial complex Y and a proper continuous map $\pi : X \setminus B_{T(\delta)}(p) \to Y$ such that every fiber $\pi^{-1}(y)$, $y \in Y$, has diameter $\le \delta$. Thus, every $y \in Y$ has a precompact open neighborhood V_y such that $\mathrm{diam}_{\widetilde{\mathsf{d}}}(\pi^{-1}(V_y)) \le 2\delta$. Indeed, if this is not the case, then we can find $y \in Y$ and a sequence of decreasing open neighborhoods $\{V_y^i\}_{i\in\mathbb{N}}$ of y such that

$$\mathrm{diam}_{\widetilde{\mathsf{d}}}(\pi^{-1}(V_y^i)) > 2\delta\,, \quad \cap_{i\in\mathbb{N}} \overline{V}_y^i = \{y\}\,. \tag{5.96}$$

Thus, for every i we can find $x_i \in \pi^{-1}(V_y^i)$ such that $\widetilde{\mathsf{d}}(x_i, \pi^{-1}(y)) > \frac{\delta}{2}$. Since π is proper, the sequence $\{x_i\}_{i\in\mathbb{N}}$ is contained in a compact set and thus (up to subsequences) it converges to a limit $\overline{x} \in X$. By passing to the limit in the previous conditions, on the one hand we have $\widetilde{\mathsf{d}}(\overline{x}, \pi^{-1}(y)) \ge \frac{\delta}{2}$. On the other hand, we have that

$$\overline{x} \in \cap_{i\in\mathbb{N}} \pi^{-1}(\overline{V}_y^i) = \pi^{-1}(y)\,, \tag{5.97}$$

resulting in a contradiction.

Consider the covering $\{V_y\}_{y\in Y}$ of Y. Since Y is proper, because it is a locally finite simplicial complex, and has topological dimension $n-1$, there exists a *locally*

finite refinement $\{\widehat{V}_\alpha\}_{\alpha\in\mathbb{N}}$ of $\{V_y\}_{y\in Y}$ with multiplicity at most n. After discarding all sets $\widehat{V}_\alpha$ such that $\widehat{V}_\alpha \cap \pi(X \setminus B_{T(\delta)}(p)) = \emptyset$, we obtain a locally finite covering of $\pi(X \setminus B_{T(\delta)}(p))$, still denoted as $\{\widehat{V}_\alpha\}_{\alpha\in\mathbb{N}}$, with multiplicity at most n. For every $\alpha \in \mathbb{N}$ we define $\widehat{U}_\alpha := \pi^{-1}(\widehat{V}_\alpha)$. Observe that $\{\widehat{U}_\alpha\}_{\alpha\in\mathbb{N}}$ is a locally finite open cover of $X \setminus B_{T(\delta)}(p)$ with $\mathrm{mult}\big(\{\widehat{U}_\alpha\}_{\alpha\in\mathbb{N}}\big) \leq n$. Set

$$A := \{\alpha \in \mathbb{N} \,:\, \widehat{U}_\alpha \cap B_{2T(\delta)}(p) = \emptyset\}\,. \tag{5.98}$$

Consider an enumeration $m : \mathbb{N}\setminus\{0\} \to A$ and define the open cover $\{U_i\}_{i\in\mathbb{N}}$ of X as

$$U_0 := B_{2T(\delta)}(p) \cup \bigcup_{\alpha\in\mathbb{N}\setminus A} \widehat{U}_\alpha\,, \qquad U_i = \widehat{U}_{m(i)}, \quad i \in \mathbb{N}\,. \tag{5.99}$$

By construction, $\{U_i\}_{i\in\mathbb{N}}$ is a locally finite open cover of X such that $\mathrm{mult}\big(\{U_i\}_{i\in\mathbb{N}}\big) \leq n$. Moreover, Corollary 5.3.2 guarantees that for every bounded (open) set $U \subset X$, and $u \in U$, the image of the natural map induced by inclusion $i_* : \pi_1(U,u) \to \pi_1(X,u)$ is a virtually nilpotent and thus amenable subgroup. Therefore, the cover $\{U_i\}_{i\in\mathbb{N}}$ is amenable. For the sake of clarity we note that the main result of [84] would be sufficient to check the last claim.

For the application of Theorem 2.3.12, we are left to prove that $\{U_i\}_{i\in\mathbb{N}}$ is amenable at infinity. For every $\alpha \in \mathbb{N}$ the set $\widehat{V}_\alpha$ is contained in $V_{y(\alpha)}$ for some $y(\alpha) \in Y$. Thus $\mathrm{diam}_{\widetilde{\mathsf{d}}}(U_i) \leq 2\delta$ for every $i \in \mathbb{N}\setminus\{0\}$. In particular, for every $i \in \mathbb{N}\setminus\{0\}$, taking any $p_i \in U_i \subset X \setminus B_{T(\delta)}(p)$, Lemma 5.2.5 yields that

$$U_i \subset \widetilde{B}_{2\delta}(p_i) \subset B_{4\delta\mathsf{d}(p,p_i)}(p_i)\,, \tag{5.100}$$

where we used that $\delta \leq \delta_0/2$. We introduce the sequence $\{W_i\}_{i\in\mathbb{N}}$ by setting

$$W_0 := X\,, \qquad W_i := X \setminus \overline{B}_{\frac{1}{2}\mathsf{d}(p,p_i)}(p)\,, \quad i \in \mathbb{N}\setminus\{0\}\,. \tag{5.101}$$

We claim that conditions (i)-(iv) in Definition 2.3.11 hold. Condition (i) is trivial since X is proper, and condition (iii) follows from (5.100). Moreover, since $\{U_i\}_{i\in\mathbb{N}}$ is locally finite, we deduce that $\mathsf{d}(p,p_i) \to \infty$ as $i \to \infty$ and therefore (ii) holds. To check (iv), we apply Corollary 5.3.2 to obtain that for every $i \in \mathbb{N}\setminus\{0\}$ the image of the inclusion map

$$\pi_1\big(B_{4\delta\mathsf{d}(p,p_i)}(p_i), p_i\big) \to \pi_1\big(B_{4\delta\mathsf{d}(p,p_i)/\varepsilon}(p_i), p_i\big) \tag{5.102}$$

is virtually nilpotent. Our choice of δ guarantees that $B_{4\delta\mathsf{d}(p,p_i)/\varepsilon}(p_i) \subset W_i$. Thus U_i is amenable in W_i for every $i \in \mathbb{N}\setminus\{0\}$. By Theorem 2.3.12 we conclude that $||X|| = 0$. □

5.3 A Margulis lemma for $\mathsf{CD}(K,N)$ spaces

In the proof of the vanishing theorem for the simplicial volume (Theorem 5.0.1) we relied on the following Margulis lemma for RCD spaces:

Theorem 5.3.1. *For every $K \leq 0$ and $N > 1$ there exists a constant $\varepsilon := \varepsilon(K,N) > 0$ such that for every $\mathsf{RCD}(K,N)$ space $(X,\mathsf{d},\mathfrak{m})$ and $p \in X$, the image through the push-forward of the inclusion map*

$$i_* : \pi_1(B_\varepsilon(p),p) \to \pi_1(B_{10}(p),p) \tag{5.103}$$

is virtually nilpotent.

By scaling, Theorem 5.3.1 yields the following:

Corollary 5.3.2. *For every $N > 1$ there exists a constant $\varepsilon := \varepsilon(N) > 0$ such that for every $\mathsf{RCD}(0,N)$ space $(X,\mathsf{d},\mathfrak{m})$, $p \in X$ and $R > 0$ the image of the inclusion map*

$$\pi_1(B_{\varepsilon R}(p),p) \to \pi_1(B_R(p),p) \tag{5.104}$$

is virtually nilpotent.

Remark 5.3.3. With the very same strategy as in the proof of Theorem 5.3.1 one can prove a version of Margulis lemma for non-branching and semi-locally simply connected $\mathsf{CD}(K,N)$ spaces.

For smooth Riemannian manifolds, Theorem 5.3.1 follows from a stronger result due to V. Kapovitch and B. Wilking in [126]. They proved that there exist constants $0 < \varepsilon(K,N) < 1$ and $C(K,N) > 0$ such that if (M^N, g) is a complete Riemannian manifold with $\mathrm{Ric} \geq K$ then the image of

$$i_* : \pi_1(B_\varepsilon(p),p) \to \pi_1(B_{10}(p),p) \tag{5.105}$$

has a nilpotent subgroup whose index is bounded above by $C(K,N)$. In the recent [84] the same effective virtual nilpotency statement was obtained for $\mathsf{RCD}(K,N)$ spaces $(X,\mathsf{d},\mathfrak{m},p)$ for the image of

$$i_* : \pi_1(B_\varepsilon(p),p) \to \pi_1(X,p)\,. \tag{5.106}$$

It is expected that fueling the strategy of [126] with the ideas introduced to deal with the synthetic framework in [84] should lead to a complete generalization of the Margulis lemma from [126] to $\mathsf{RCD}(K,N)$ spaces. Here we obtain a weaker but

local version of the Margulis lemma where the index of the nilpotent subgroup is not controlled effectively. While such statement seems non-optimal, it is sufficient for the purposes of the thesis. The proof follows a different strategy with respect to [126, 84], and it is based on [36].

Besides [36], the main technical tool will be a local Bishop-Gromov inequality valid for universal covers of open subsets of $\mathsf{RCD}(K,N)$ spaces endowed with their natural length structure.

Remark 5.3.4. Let (X,d) be a length metric space (not necessarily complete). Let $\widetilde{X}$ be a path connected topological space, and let $\pi:\widetilde{X}\to X$ be a covering map. We define a distance $\widetilde{\mathsf{d}}$ on $\widetilde{X}$ as

$$\widetilde{\mathsf{d}}(a,b):=\inf\{\mathrm{length}^{\mathsf{d}}(\pi\circ\gamma):\gamma\in C([0,1],\widetilde{X})\ \ \gamma(0)=a,\gamma(1)=b\}\,. \tag{5.107}$$

With this definition, it is clear that $\pi:(\widetilde{X},\widetilde{\mathsf{d}})\to(X,\mathsf{d})$ is a local isometry. Precisely, for every $p\in\widetilde{X}$ there is a neighborhood $N_p\ni p$ such that $\pi|_{N_p}$ is an isometry. It follows that for every $\gamma\in C([0,1],\widetilde{X})$ we have

$$\mathrm{length}^{\widetilde{\mathsf{d}}}(\gamma)=\mathrm{length}^{\mathsf{d}}(\pi\circ\gamma)\,, \tag{5.108}$$

so that $\gamma\in AC([0,1],\widetilde{X})$ if and only if $\pi\circ\gamma\in AC([0,1],X)$. Moreover, $(\widetilde{X},\widetilde{\mathsf{d}})$ is a length metric space.

Finally, if $\mathfrak{m}$ is a measure on X, there is a unique measure $\widetilde{\mathfrak{m}}$ on $\widetilde{X}$ such that $\pi_\#(\widetilde{\mathfrak{m}}\llcorner N_p)=\mathfrak{m}\llcorner\pi(N_p)$ for every N_p as above.

We denote by $v_{K,N}(r)$ the volume of the ball of radius r in the model space for the $\mathsf{CD}(K,N)$ condition.

Lemma 5.3.5. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(K,N)$ space with $K\le 0$ and $N>1$. Let $O\subset X$ be an open, bounded, and path-connected set and call $\overline{\mathsf{d}}$ the length distance induced by d on O. Given a path-connected covering $\pi:\widetilde{O}\to O$, endow $\widetilde{O}$ with the length distance $\widetilde{\mathsf{d}}$ and the measure $\widetilde{\mathfrak{m}}$ obtained by lifting $\overline{\mathsf{d}}$ and $\mathfrak{m}$, cf. Remark 5.3.4. For every $\widetilde{p}\in\widetilde{O}$, the function*

$$(0,\mathsf{d}(\pi(\widetilde{p}),X\setminus O))\ni r\mapsto\frac{\widetilde{\mathfrak{m}}\big(B_r^{\widetilde{\mathsf{d}}}(\widetilde{p})\big)}{v_{K,N}(r)}\,, \tag{5.109}$$

is non-increasing. Moreover, if $\mathfrak{m}=\mathsf{H}^N$ then the function in (5.109) is bounded from above by 1.

Proof. Observe that (O, d) and $(O, \overline{\mathsf{d}})$ are locally isometric, meaning that for every $p \in O$ there is a neighborhood $N_p \ni p$ such that d and $\overline{\mathsf{d}}$ agree on $N_p \times N_p$. In particular, since (X, d) is non-branching by [83], the length space $(O, \overline{\mathsf{d}})$ is non-branching as well. Moreover, since $\pi : (\widetilde{O}, \widetilde{\mathsf{d}}) \to (O, \overline{\mathsf{d}})$ is a local isometry, $(\widetilde{O}, \widetilde{\mathsf{d}})$ is non-branching as well.

Claim 1: every pair $\widetilde{p}, \widetilde{q} \in \widetilde{O}$ such that $\widetilde{\mathsf{d}}(\widetilde{p}, \widetilde{q}) < \mathsf{d}(\pi(\widetilde{p}), X \setminus O)$ is connected by a geodesic in $(\widetilde{O}, \widetilde{\mathsf{d}})$.

To prove Claim 1 observe that, since $(\widetilde{O}, \widetilde{\mathsf{d}})$ is a length space, there is a sequence of curves $(\widetilde{\gamma}_n) \subset AC([0,1], \widetilde{O})$ such that

$$\mathrm{length}^{\widetilde{\mathsf{d}}}(\widetilde{\gamma}_n) \leq \widetilde{\mathsf{d}}(\widetilde{p}, \widetilde{q}) + \frac{1}{n} \tag{5.110}$$

for every $n \in \mathbb{N}$. Consider the sequence $(\gamma_n) \subset AC([0,1], O)$ defined as $\gamma_n := \pi(\widetilde{\gamma}_n)$ for every n. Eventually, these curves will be contained in a compact subset of O. Thus, up to not relabelled subsequences, $\gamma_n \to \gamma_\infty \in AC([0,1], O)$ uniformly. Moreover, by what we observed before,

$$\mathrm{length}^{\overline{\mathsf{d}}}(\gamma_n) = \mathrm{length}^{\widetilde{\mathsf{d}}}(\widetilde{\gamma}_n) \leq \widetilde{\mathsf{d}}(\widetilde{p}, \widetilde{q}) + \frac{1}{n} \tag{5.111}$$

for every n and therefore $\mathrm{length}^{\overline{\mathsf{d}}}(\gamma_\infty) \leq \mathsf{d}(\widetilde{p}, \widetilde{q})$ by lower semicontinuity of the length. Let $\widetilde{\gamma}_\infty$ be the lifting of γ_∞ starting from $\widetilde{p}$. On the one hand, we have that

$$\mathrm{length}^{\widetilde{\mathsf{d}}}(\widetilde{\gamma}_\infty) = \mathrm{length}^{\overline{\mathsf{d}}}(\gamma_\infty) \leq \widetilde{\mathsf{d}}(\widetilde{p}, \widetilde{q})\,. \tag{5.112}$$

On the other hand, since O is semi-locally simply connected, the curves γ_n are eventually homotopically equivalent to γ_∞. Therefore $\widetilde{\gamma}_\infty(1) = \widetilde{q}$. This concludes the proof of the claim.

Take any $\widetilde{p} \in \widetilde{O}$ and any $r_0 < \mathsf{d}(\pi(\widetilde{p}), X \setminus O)$. Consider the 1-Lipschitz function $\widetilde{u} : \widetilde{O} \to \mathbb{R}$, defined as $\widetilde{u}(q) := \widetilde{\mathsf{d}}(\widetilde{p}, q) \wedge r_0$. We are going to consider the disintegration of the measure $\widetilde{\mathfrak{m}}$ with respect to $\widetilde{u}$, following [47, Section 7]. In particular, we define the transport relation

$$\widetilde{R}_{\widetilde{u}} := \{(\widetilde{x}, \widetilde{y}) \in \widetilde{O} \times \widetilde{O} \,:\, \widetilde{x} \neq \widetilde{y} \text{ and } |\widetilde{u}(\widetilde{x}) - \widetilde{u}(\widetilde{y})| = \mathsf{d}(\widetilde{x}, \widetilde{y})\} \tag{5.113}$$

and the associated transport set $\widetilde{T}_{\widetilde{u}} := \mathsf{p}_1(\widetilde{R}_{\widetilde{u}})$, where p_1 denotes the projection on the first factor. By what we observed before,

$$\widetilde{T}_{\widetilde{u}} = \overline{B}^{\widetilde{\mathsf{d}}}_{r_0}(\widetilde{p})\,. \tag{5.114}$$

We introduce also the set of branching points

$$\widetilde{A} := \{x \in \widetilde{T}_{\widetilde{u}} \,:\, \exists z, w \in \widetilde{X} \text{ with } (x,z),(x,w) \in \widetilde{R}_{\widetilde{u}}, (z,w) \notin \widetilde{R}_{\widetilde{u}}\}\,. \tag{5.115}$$

Accordingly, we consider the non-branched transport set $\widetilde{T}^b_{\widetilde{u}} := \widetilde{T}_{\widetilde{u}} \setminus \widetilde{A}$ and the non-branched transport relation

$$\widetilde{R}^b_{\widetilde{u}} := \widetilde{R}_{\widetilde{u}} \cap (\widetilde{T}^b_{\widetilde{u}} \times \widetilde{T}^b_{\widetilde{u}})\,. \tag{5.116}$$

Since $(\widetilde{O}, \widetilde{\mathsf{d}})$ is non-branching, we have that

$$\widetilde{T}^b_{\widetilde{u}} = \overline{B}^{\widetilde{\mathsf{d}}}_{r_0}(\widetilde{p}) \setminus (\mathrm{Cut}(\widetilde{p}) \cup \{\widetilde{p}\})\,. \tag{5.117}$$

With our choice of $\widetilde{u}$, all the defined sets are easily seen to be Borel.

Note that $\widetilde{R}^b_{\widetilde{u}}$ is an equivalence relation on $\widetilde{T}^b_{\widetilde{u}}$. Denoting by $\widetilde{Q} \subset X$ the set of its equivalence classes, every $\alpha \in \widetilde{Q}$ corresponds to a set $\widetilde{X}_\alpha \subset \widetilde{O}$ which is a geodesic and thus isometric to an interval $\widetilde{I}_\alpha \subset \mathbb{R}$. We equip $\widetilde{Q}$ with the σ-algebra induced by the equivalence projection $\mathfrak{Q} : \widetilde{T}^b_{\widetilde{u}} \to \widetilde{Q}$ and with the measure $\widetilde{\mathfrak{q}} := \mathfrak{Q}_\#\big(\widetilde{\mathfrak{m}} \llcorner \widetilde{T}^b_{\widetilde{u}}\big)$. The Disintegration Theorem (see for instance [47, Theorem 6.19]) guarantees the existence of a (essentially unique) *consistent* (see [47, Definition 6.18]) disintegration, i.e. a measurable map $\widetilde{Q} \ni \alpha \mapsto \widetilde{\mathfrak{m}}_\alpha \in \mathcal{P}(\widetilde{O})$ such that

$$\widetilde{\mathfrak{m}} \llcorner \widetilde{T}^b_{\widetilde{u}} = \int_{\widetilde{Q}} \widetilde{\mathfrak{m}}_\alpha \, \widetilde{\mathfrak{q}}(\mathrm{d}\alpha)\,. \tag{5.118}$$

Consider now any point $\widetilde{q} \in B^{\widetilde{\mathsf{d}}}_{r_0}(\widetilde{p})$. By construction, there is an open neighborhood $U_{\widetilde{q}}$ of $\widetilde{q}$ such that the map

$$\pi_{\widetilde{q}} := \pi|_{U_{\widetilde{q}}} : (U_{\widetilde{q}}, \widetilde{\mathsf{d}}) \to (O, \overline{\mathsf{d}}) \tag{5.119}$$

is an isometry. Up to taking a smaller $U_{\widetilde{q}}$, we can assume that $\pi_{\widetilde{q}} : (U_{\widetilde{q}}, \widetilde{\mathsf{d}}) \to (\pi(U_{\widetilde{q}}), \mathsf{d})$ is a bijective isometry. Then, consider a 1-Lipschitz function $u : X \to \mathbb{R}$ such that $u(x) := \widetilde{\mathsf{d}}(\widetilde{p}, \pi^{-1}_{\widetilde{q}}(x))$ for every $x \in \pi(U_{\widetilde{q}})$. Similarly as before, we can get a disintegration of the measure $\mathfrak{m}$ on (X, d), with respect to u (see [49, Section 3] for the adaptation of the discussion above to σ-finite measures). In particular, similarly as before we can define the objects R_u, T_u, R^b_u, T^b_u, Q, $\{X_\beta\}_{\beta \in Q}$ and $\mathfrak{q}$, and according to [49, Theorem 3.4] we find a disintegration

$$\mathfrak{m} \llcorner T^b_u = \int_Q \mathfrak{m}_\beta \, \mathfrak{q}(\mathrm{d}\beta)\,. \tag{5.120}$$

Such disintegration is *strongly consistent* (see [47, Definition 6.18]), meaning that for $\mathfrak{q}$-a.e. $\beta \in Q$ the measure $\mathfrak{m}_\beta$ is concentrated on X_β. Moreover, since $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}(K, N)$ space, [49, Theorem 3.6] ensures that for $\mathfrak{q}$-a.e. $\beta \in Q$ the space $(X_\beta, \mathsf{d}, \mathfrak{m}_\beta)$ is isomorphic (as metric measure space) to $(I_\beta, |\cdot|, h_\beta \lambda^1)$, for an interval $I_\beta \subset \mathbb{R}$ and a measurable function h_β which is a $\mathsf{CD}(K, N)$ density. Thus, by [47, Appendix A], for $\mathfrak{q}$-a.e. $\beta \in Q$ the differential inequality

$$\left(h_\beta^{1/(N-1)}\right)'' + \frac{K}{N-1} h_\beta^{1/(N-1)} \leq 0 \tag{5.121}$$

holds in a weak sense on I_β. Finally, [49, Lemma 3.5] guarantees $\mathfrak{m}(T_u \setminus T_u^b) = 0$.

Define the sets

$$\widetilde{Q}^{\widetilde{q}} := \{\alpha \in \widetilde{Q} \,:\, \widetilde{X}_\alpha \cap U_{\widetilde{q}} \neq \emptyset\}, \qquad Q^{\widetilde{q}} := \{\beta \in Q \,:\, X_\beta \cap \pi_{\widetilde{q}}(U_{\widetilde{q}}) \neq \emptyset\}\,. \tag{5.122}$$

Both sets are easily seen to be measurable.

Claim 2: there exists a measurable bijective map $\Pi : \widetilde{Q}^{\widetilde{q}} \to Q^{\widetilde{q}}$ such that

$$\pi_{\widetilde{q}}(\widetilde{X}_\alpha \cap U_{\widetilde{q}}) = X_{\Pi(\alpha)} \cap \pi_{\widetilde{q}}(U_{\widetilde{q}})\,. \tag{5.123}$$

Such map is defined as follows. Given $\alpha \in \widetilde{Q}^{\widetilde{q}}$, take any $x \in X_\alpha \cap U_{\widetilde{q}}$ and define $\Pi(\alpha)$ as the equivalence class of $\pi_{\widetilde{q}}(x)$. To show that such definition is well posed, given any $\widetilde{q}_1, \widetilde{q}_2 \in U_{\widetilde{q}}$ such that $(\widetilde{q}_1, \widetilde{q}_2) \in \widetilde{R}_{\widetilde{u}}^b$, we observe that by definition

$$|u(\pi_{\widetilde{q}}(\widetilde{q}_1)) - u(\pi_{\widetilde{q}}(\widetilde{q}_2))| = \mathsf{d}(\pi_{\widetilde{q}}(\widetilde{q}_1), \pi_{\widetilde{q}}(\widetilde{q}_2))\,, \tag{5.124}$$

thus $\big(\pi_{\widetilde{q}}(\widetilde{q}_1), \pi_{\widetilde{q}}(\widetilde{q}_2)\big) \in R_u$. Suppose that $\big(\pi_{\widetilde{q}}(\widetilde{q}_1), \pi_{\widetilde{q}}(\widetilde{q}_2)\big) \in R_u \setminus R_u^b$. Then w.l.o.g. there exists $x \in X$ such that $(\pi_{\widetilde{q}}(\widetilde{q}_1), x) \in R_u$ and $(\pi_{\widetilde{q}}(\widetilde{q}_2), x) \notin R_u$. Up to replacing x with a suitable point on the geodesic connecting $\pi_{\widetilde{q}}(\widetilde{q}_1)$ and x inside $\pi_{\widetilde{q}}(U_{\widetilde{q}})$, we can assume that $x \in \pi_{\widetilde{q}}(U_{\widetilde{q}})$. Setting $\widetilde{x} := \pi_{\widetilde{q}}^{-1}(x)$, we obtain that $(\widetilde{q}_1, \widetilde{x}) \in \widetilde{R}_{\widetilde{u}}$ and $(\widetilde{q}_2, \widetilde{x}) \notin \widetilde{R}_{\widetilde{u}}$, a contradiction. This proves that the map Π is well defined. Reverting the same strategy, one shows that Π is bijective and that (5.123) holds. The measurability of Π simply follows from (5.123).

Recalling that $\mathfrak{m}(T_u \setminus T_u^b) = 0$, an argument similar to the one discussed above shows that $\widetilde{\mathfrak{m}}(\widetilde{T}_{\widetilde{u}} \setminus \widetilde{T}_{\widetilde{u}}^b) = 0$. In particular,

$$\widetilde{\mathfrak{m}} \llcorner U_{\widetilde{q}} = \int_{\widetilde{Q}^{\widetilde{q}}} \widetilde{\mathfrak{m}}_\alpha \llcorner U_{\widetilde{q}}\, \widetilde{\mathfrak{q}}(\mathrm{d}\alpha) \quad \text{and} \quad \mathfrak{m} \llcorner \pi_{\widetilde{q}}(U_{\widetilde{q}}) = \int_{Q^{\widetilde{q}}} \mathfrak{m}_\beta \llcorner \pi_{\widetilde{q}}(U_{\widetilde{q}})\, \mathfrak{q}(\mathrm{d}\beta)\,. \tag{5.125}$$

By definition of $\widetilde{\mathfrak{m}}$, we know that $(\pi_{\widetilde{q}})_\#(\widetilde{\mathfrak{m}} \llcorner U_{\widetilde{q}}) = \mathfrak{m} \llcorner \pi_{\widetilde{q}}(U_{\widetilde{q}})$. Thus

$$\widetilde{\mathfrak{m}} \llcorner U_{\widetilde{q}} = \big(\pi_{\widetilde{q}}^{-1}\big)_\# [\mathfrak{m} \llcorner \pi(U_{\widetilde{q}})] = \int_{Q^q} \big(\pi_{\widetilde{q}}^{-1}\big)_\# [\mathfrak{m}_\beta \llcorner \pi_{\widetilde{q}}(U_{\widetilde{q}})]\, \mathfrak{q}(\mathrm{d}\beta) \tag{5.126}$$

$$= \int_{\widetilde{Q}^{\widetilde{q}}} \big(\pi_{\widetilde{q}}^{-1}\big)_\# [\mathfrak{m}_{(\Pi(\alpha))} \llcorner \pi_{\widetilde{q}}(U_{\widetilde{q}})]\, [(\Pi^{-1})_\# \mathfrak{q}](\mathrm{d}\alpha)\,. \tag{5.127}$$

This gives a disintegration of $\widetilde{\mathfrak{m}} \llcorner U_{\widetilde{q}}$ which is (strongly) consistent because (5.120) is strongly consistent. By essential uniqueness of consistent disintegrations, see [47, Theorem 6.19], we deduce that

$$\widetilde{\mathfrak{m}}_\alpha \llcorner U_{\widetilde{q}} = \big(\pi_{\widetilde{q}}^{-1}\big)_\# (\mathfrak{m}_{\Pi(\alpha)} \llcorner \pi(U_{\widetilde{q}}))\,, \qquad \text{for } \widetilde{\mathfrak{q}}\text{-a.e. } \alpha \in \widetilde{Q}^{\widetilde{q}}\,. \tag{5.128}$$

In particular, $\widetilde{\mathfrak{m}}_\alpha|_{U_{\widetilde{q}}}$ is concentrated on $\widetilde{X}_\alpha$, which is isometric to a closed interval $\widetilde{I}_\alpha$. Moreover, by (5.121), for $\widetilde{\mathfrak{q}}$-a.e. $\alpha \in \widetilde{Q}^{\widetilde{q}}$ the metric measure space $(\widetilde{X}_\alpha \cap U_{\widetilde{q}}, \widetilde{\mathsf{d}}, \widetilde{\mathfrak{m}}_\alpha)$ is isomorphic to $(\widetilde{I}_\alpha \cap U_{\widetilde{q}}, |\cdot|, \widetilde{h}_\alpha \lambda^1)$, where $\widetilde{h}_\alpha$ is such that

$$\big(\widetilde{h}_\alpha^{1/(N-1)}\big)'' + \frac{K}{N-1}\, \widetilde{h}_\alpha^{1/(N-1)} \le 0 \quad \text{ weakly on } \quad \widetilde{I}_\alpha \cap U_{\widetilde{q}}\,. \tag{5.129}$$

Here and in the following, with a slight abuse of notation, given any open set $E \subset \widetilde{O}$ we denote by $\widetilde{I}_\alpha \cap E$ the subset of $\widetilde{I}_\alpha$ that corresponds via isometry to $\widetilde{X}_\alpha \cap E$. This last conclusion holds for every $\widetilde{q} \in B^{\widetilde{\mathsf{d}}}_{r_0}(\widetilde{p})$ and for the respective neighborhood $U_{\widetilde{q}}$.

Take any $r_1 < r_0$. Since $\overline{B}^{\widetilde{\mathsf{d}}}_{r_1}(\widetilde{p})$ is compact we can find $\widetilde{q}_1, \dots, \widetilde{q}_M \in B^{\widetilde{\mathsf{d}}}_{r_0}(\widetilde{p})$ such that

$$B^{\widetilde{\mathsf{d}}}_{r_1}(\widetilde{p}) \subset U_{\widetilde{q}_1} \cup \cdots \cup U_{\widetilde{q}_M}\,. \tag{5.130}$$

We let $\widetilde{B}$ be the $\widetilde{\mathfrak{q}}$-null set defined as the collection of elements $\alpha \in \widetilde{Q}$ such that (5.129) does not hold for some $\widetilde{q}_i$ and $U_{\widetilde{q}_i}$, with $i = 1, \dots, M$. Thus, for every $\alpha \in \widetilde{Q} \setminus \widetilde{B}$ the differential inequality (5.129) holds on $\widetilde{I}_\alpha \cap U_{\widetilde{q}_i}$ for every i such that $\widetilde{I}_\alpha \cap U_{\widetilde{q}_i} \neq \emptyset$. Thanks to (5.130) we conclude that for $\widetilde{\mathfrak{q}}$-a.e. $\alpha \in \widetilde{Q}$ the inequality (5.129) holds on $\widetilde{I}_\alpha \cap B^{\widetilde{\mathsf{d}}}_{r_1}(\widetilde{p})$. Thus, for every such α the function $\widetilde{h}_\alpha$ is a $\mathsf{CD}(K,N)$ density on $\widetilde{I}_\alpha \cap B^{\widetilde{\mathsf{d}}}_{r_1}(\widetilde{p})$. Therefore by Bishop-Gromov we have that

$$\frac{\widetilde{\mathfrak{m}}_\alpha\big(B^{\widetilde{\mathsf{d}}}_s(\widetilde{p})\big)}{v_{K,N}(s)} \ge \frac{\widetilde{\mathfrak{m}}_\alpha\big(B^{\widetilde{\mathsf{d}}}_t(\widetilde{p})\big)}{v_{K,N}(t)}\,, \quad \text{ for every } s < t < r_1\,. \tag{5.131}$$

Thus, for every $s < t < r_1$ we get

$$\widetilde{\mathfrak{m}}\big(B^{\widetilde{\mathsf{d}}}_s(\widetilde{p})\big) = \int_{\widetilde{Q}} \widetilde{\mathfrak{m}}_\alpha\big(B^{\widetilde{\mathsf{d}}}_s(\widetilde{p})\big)\, \widetilde{\mathfrak{q}}(d\alpha) \tag{5.132}$$

$$\ge \int_{\widetilde{Q}} \frac{v_{K,N}(s)}{v_{K,N}(t)}\, \widetilde{\mathfrak{m}}_\alpha\big(B^{\widetilde{\mathsf{d}}}_t(\widetilde{p})\big)\, \widetilde{\mathfrak{q}}(d\alpha) = \frac{v_{K,N}(s)}{v_{K,N}(t)}\, \widetilde{\mathfrak{m}}\big(B^{\widetilde{\mathsf{d}}}_t(\widetilde{p})\big)\,. \tag{5.133}$$

The first part of the statement follows since $\widetilde{p} \in \widetilde{O}$ and $r_1 < r_0 < \mathsf{d}(\pi(\widetilde{p}), X \setminus O)$ were arbitrary.

Assume now that $\mathfrak{m} = \mathsf{H}^N$, so that $\widetilde{\mathfrak{m}} = \widetilde{\mathsf{H}}^N$ is the Hausdorff measure induced by $\widetilde{\mathsf{d}}$. To prove the final part of the statement, it is sufficient to show that

$$\lim_{r \to 0} \frac{\widetilde{\mathsf{H}}^N\big(B_r^{\widetilde{\mathsf{d}}}(\widetilde{p})\big)}{v_{K,N}(r)} \leq 1\,. \tag{5.134}$$

To this aim we observe that for $r > 0$ sufficiently small the restriction

$$\pi|_{B_r^{\widetilde{\mathsf{d}}}(\widetilde{p})} : B_r^{\widetilde{\mathsf{d}}}(\widetilde{p}) \to B_r(p) \tag{5.135}$$

is an isometry. Thus the claim follows from [80, Corollary 2.14]. □

We record below another useful technical result about coverings of balls in RCD spaces.

Lemma 5.3.6. *Let $(X, \mathsf{d}, \mathfrak{m}, p)$ be an $\mathsf{RCD}(K, N)$ space for some $K \in \mathbb{R}$ and $1 \leq N < \infty$. Let $r \in (0, +\infty]$, and let $\pi : Y \to B_r(p)$ be a covering, with the convention that $B_\infty(p) := X$. Let $\widetilde{\mathsf{d}}$ and $\widetilde{\mathfrak{m}}$ be respectively the distance and the measure induced on Y by lifting the length structure of $B_r(p)$ and the measure $\mathfrak{m}$, cf. Remark 5.3.4. Let $\widetilde{p} \in \pi^{-1}(p)$ be fixed, and let $F \subset Y$ be the union of the interiors of the lifts to $\widetilde{p}$ of the segments of $B_r(p)$ having p as an endpoint. Then the following hold:*

1. *$\pi|_F : F \to B_r(p)$ is injective;*
2. *For every non-trivial deck transformation $g : Y \to Y$, $F \cap g(F) = \emptyset$;*
3. *For every $s \in (0, r)$, we have that $F \cap \pi^{-1}(B_s(p)) \subset B_s^{\widetilde{\mathsf{d}}}(\widetilde{p})$;*
4. *For every $s \in (0, r)$, we have that $\mathfrak{m}(B_s(p)) = \widetilde{\mathfrak{m}}(F \cap B_s^{\widetilde{\mathsf{d}}}(\widetilde{p}))$.*

Proof. To prove item 1, let $a, b \in F$ such that $\pi(a) = \pi(b)$. Let $\gamma_a, \gamma_b \subset X$ be segments having p as an endpoint whose interiors contain $\pi(a) = \pi(b)$, and such that their lifts to $\widetilde{p}$ contain a and b, respectively. Since $\mathsf{RCD}(K, N)$ spaces are non-branching due to [83], either $\gamma_a \subset \gamma_b$ or $\gamma_b \subset \gamma_a$, which implies $a = b$.

To prove item 2, assume by contradiction that $x \in F \cap g(F)$. Then there are segments $\gamma_{\widetilde{p}}, \gamma_{g(\widetilde{p})} \subset X$ having p as an endpoint, and such that the interiors of the lifts to Y from $\widetilde{p}$ and $g(\widetilde{p})$ respectively both contain x. Again using that $\mathsf{RCD}(K, N)$ spaces are non-branching, we deduce that $\gamma_{\widetilde{p}} \subset \gamma_{g(\widetilde{p})}$ or $\gamma_{g(\widetilde{p})} \subset \gamma_{\widetilde{p}}$. Thus their common part has two different lifts starting from $x \in Y$, a contradiction.

To prove item 3, let $x \in F \cap \pi^{-1}(B_s(p))$, and let $\gamma_x \subset B_s(p)$ be the segment with endpoint p whose lift to $\widetilde{p}$ contains x. The length of γ_x is the same as the length of its lift. Thus $x \in B_s^{\widetilde{\mathsf{d}}}(\widetilde{p})$, concluding the proof.

For item 4, by Vitali's Covering Theorem there exists a countable family of disjoint closed balls

$$\{C_i\}_{i\in\mathbb{N}} \subset \overline{B}_s^{\widetilde{\mathsf{d}}}(\widetilde{p}) \subset Y \tag{5.136}$$

centered in points of $F \cap \overline{B}_s^{\widetilde{\mathsf{d}}}(\widetilde{p})$ such that $\pi|_{C_i} : C_i \to \pi(C_i) \subset \overline{B}_s(p)$ is a measure preserving isometry and

$$\widetilde{\mathfrak{m}}\Big(F \cap \overline{B}_s^{\widetilde{\mathsf{d}}}(\widetilde{p}) \setminus \bigcup_{i\in\mathbb{N}} C_i\Big) = 0\,. \tag{5.137}$$

Vitali's theorem can be applied since $(\overline{B}_s^{\widetilde{\mathsf{d}}}(\widetilde{p}), \widetilde{\mathsf{d}}, \widetilde{\mathfrak{m}})$ is uniformly locally doubling by Lemma 5.3.5. Hence, using item 1 in the last inequality and the fact that $\widetilde{\mathfrak{m}}(\partial B_s^{\widetilde{\mathsf{d}}}(\widetilde{p})) = 0$ by Lemma 5.3.5, we deduce that

$$\begin{aligned}
\widetilde{\mathfrak{m}}\Big(F \cap B_s^{\widetilde{\mathsf{d}}}(\widetilde{p})\Big) &= \sum_{i\in\mathbb{N}} \widetilde{\mathfrak{m}}\Big(F \cap B_s^{\widetilde{\mathsf{d}}}(\widetilde{p}) \cap C_i\Big) && (5.138)\\
&= \sum_{i\in\mathbb{N}} \mathfrak{m}\Big(\pi(F \cap B_s^{\widetilde{\mathsf{d}}}(\widetilde{p}) \cap C_i)\Big) \le \mathfrak{m}(B_s(p))\,. && (5.139)
\end{aligned}$$

To prove the converse inequality, let $\Sigma \subset X$ be the cut-locus of p. We have that $\mathfrak{m}(\Sigma) = 0$ (see, for instance, [47, Remark 7.5]). By the very definition of F,

$$B_s(p) \setminus \Sigma \subset \pi(F \cap B_s^{\widetilde{\mathsf{d}}}(\widetilde{p}))\,. \tag{5.140}$$

Hence, using the collection $\{C_i\}_{i\in\mathbb{N}}$ constructed above and the fact that $\pi : Y \to B_r(p)$ sends $\widetilde{\mathfrak{m}}$-negligible sets to $\mathfrak{m}$-negligible sets, we deduce that

$$\begin{aligned}
\mathfrak{m}(B_s(p)) &= \mathfrak{m}(B_s(p) \setminus \Sigma) \le \mathfrak{m}\Big(\pi(F \cap B_s^{\widetilde{\mathsf{d}}}(\widetilde{p}))\Big) && (5.141)\\
&\le \sum_{i\in\mathbb{N}} \mathfrak{m}\Big(\pi(F \cap B_s^{\widetilde{\mathsf{d}}}(\widetilde{p}) \cap C_i)\Big) && (5.142)\\
&= \sum_{i\in\mathbb{N}} \widetilde{\mathfrak{m}}\Big(F \cap B_s^{\widetilde{\mathsf{d}}}(\widetilde{p}) \cap C_i\Big) = \widetilde{\mathfrak{m}}\Big(F \cap B_s^{\widetilde{\mathsf{d}}}(\widetilde{p})\Big)\,, && (5.143)
\end{aligned}$$

thus concluding the proof. □

Proof of Theorem 5.3.1. Following the notation of Lemma 5.3.5, let $\overline{\mathsf{d}}$ denote the length distance induced by d on $B_{10}(p)$. Endow the universal cover $\pi : \widetilde{X} \to B_{10}(p)$ with the length distance $\widetilde{\mathsf{d}}$ and the measure $\widetilde{\mathfrak{m}}$ obtained by lifting $\overline{\mathsf{d}}$ and $\mathfrak{m}$. Consider the metric space (Y, ρ), where $Y := \pi^{-1}(p)$ and ρ is the restriction of $\widetilde{\mathsf{d}}$ to Y.

Claim: There exists a constant $C(K,N)$ such that (Y,ρ) has *packing constant* $C(K,N)$, meaning that every ball of radius 4 in (Y,ρ) can be covered by at most $C(K,N)$ balls of radius 1.

To prove the claim observe that by Lemma 5.3.5 there exists a constant $C(K,N)$ such that for every $\widetilde{p} \in \pi^{-1}(p)$ we have

$$\widetilde{\mathfrak{m}}\big(B^{\widetilde{\mathsf{d}}}_{9}(\widetilde{p})\big) \leq C(K,N)\, \widetilde{\mathfrak{m}}\big(B^{\widetilde{\mathsf{d}}}_{1/2}(\widetilde{p})\big)\,. \tag{5.144}$$

Fix any $\widetilde{p} \in \pi^{-1}(p)$ and suppose that there are M disjoint balls of radius $1/2$ with centers in $\pi^{-1}(p) \cap B^{\widetilde{\mathsf{d}}}_{4}(\widetilde{p})$, i.e. we find $p_1,\dots,p_M \in \pi^{-1}(p) \cap B^{\widetilde{\mathsf{d}}}_{4}(\widetilde{p})$ such that $B^{\widetilde{\mathsf{d}}}_{1/2}(p_i) \cap B^{\widetilde{\mathsf{d}}}_{1/2}(p_j) = \emptyset$ whenever $i \neq j$. Using (5.144) we deduce that

$$\begin{aligned}\widetilde{\mathfrak{m}}\big(B^{\widetilde{\mathsf{d}}}_{9/2}(\widetilde{p})\big) &\geq \sum_{i=1}^{M} \widetilde{\mathfrak{m}}\big(B^{\widetilde{\mathsf{d}}}_{1/2}(p_i)\big) \\ &\geq \frac{1}{C(K,N)} \sum_{i=1}^{M} \widetilde{\mathfrak{m}}\big(B^{\widetilde{\mathsf{d}}}_{9}(p_i)\big) \geq \frac{M}{C(K,N)} \widetilde{\mathfrak{m}}\big(B^{\widetilde{\mathsf{d}}}_{9/2}(\widetilde{p})\big)\,,\end{aligned}$$

thus $M \leq C(K,N)$. In particular for any $\widetilde{p} \in Y$ there are at most $C(K,N)$ disjoint balls in (Y,ρ) of radius $1/2$ with centers in $B^{\rho}_{4}(\widetilde{p})$.

Consider any $p_0 \in Y$. If $B^{\rho}_{4}(p_0) \setminus B^{\rho}_{1}(p_0) \neq \emptyset$ take $p_1 \in B^{\rho}_{4}(p_0) \setminus B^{\rho}_{1}(p_0)$. Iterate this process by selecting at step i a point $p_i \in B^{\rho}_{4}(p_0) \setminus \big(B^{\rho}_{1}(p_0) \cup \dots \cup B^{\rho}_{1}(p_{i-1})\big)$ if $B^{\rho}_{4}(p_0) \setminus \big(B^{\rho}_{1}(p_0) \cup \dots \cup B^{\rho}_{1}(p_{i-1})\big) \neq \emptyset$. Observe that at each step i the balls $B^{\rho}_{1/2}(p_0),\dots,B^{\rho}_{1/2}(p_i)$ are pairwise disjoint by construction. Therefore this process has to stop within step $C(K,N)-1$, thus producing a cover of $B^{\rho}_{4}(p_0)$ with at most $C(K,N)$ balls of radius 1.

Since X is semi-locally simply connected by [188], $B_{10}(p)$ is semi-locally simply connected as well. Thus we can identify $\pi_1(B_{10}(p),p)$ with the deck transformations of the universal cover $\pi : \widetilde{X} \to B_{10}(p)$. By restricting any such deck transformation to $Y = \pi^{-1}(p)$ we get that $\pi_1(B_{10}(p),p) \leq \mathrm{Iso}(Y,\rho)$, with a slight abuse of notation. By [36, Corollary 11.17] there exists a constant $\varepsilon = \varepsilon(K,N) > 0$ such that for every $\widetilde{p} \in Y$ the *almost stabiliser* $\Gamma_{3\varepsilon}(\widetilde{p}) := \langle S_{3\varepsilon}(\widetilde{p})\rangle$, where $S_{3\varepsilon}(\widetilde{p}) := \{\gamma \in \pi_1(B_{10}(p),p) : \rho(\gamma\cdot\widetilde{p},\widetilde{p}) < 3\varepsilon\}$, is virtually nilpotent. On the other hand, it is a standard observation that every loop in $\pi_1(B_{\varepsilon}(p),p)$ is homotopically equivalent to a concatenation of loops of length less than 3ε. In particular the image of the map (5.103) is isomorphic to a subgroup of the almost stabiliser $\Gamma_{3\varepsilon}(\widetilde{p})$. Therefore it is virtually nilpotent. □

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