

# Fixed point ratios in actions of finite classical groups, IV

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## Abstract

This is the final paper in a series of four on fixed point ratios in non-subspace actions of finite classical groups. Our main result states that if  $G$  is a finite almost simple classical group and  $\Omega$  is a faithful transitive non-subspace  $G$ -set then either  $\text{fpr}(x) \lesssim |x^G|^{-1/2}$  for all elements  $x \in G$  of prime order, or  $(G, \Omega)$  is one of a small number of known exceptions. In this paper we assume  $G_\omega$  is either an almost simple irreducible subgroup in Aschbacher's  $\mathcal{S}$  collection, or a subgroup in a small additional set  $\mathcal{N}$  which arises when  $G$  has socle  $\text{Sp}_4(q)'$  ( $q$  even) or  $\text{P}\Omega_8^+(q)$ . This completes the proof of the main theorem.

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## 1. Introduction

If  $G$  is a permutation group on a finite set  $\Omega$  then the *fixed point ratio* of an element  $x \in G$ , which we denote by  $\text{fpr}(x)$ , is defined to be the proportion of points in  $\Omega$  fixed by  $x$ . Our main result on fixed point ratios, which we refer to as Theorem 1, states that if  $\Omega$  is a faithful, transitive, non-subspace  $G$ -set, where  $G$  is a finite almost simple classical group with socle  $G_0$ , then

$$\text{fpr}(x) < |x^G|^{-\frac{1}{2} + \frac{1}{n} + \iota}$$

for all elements  $x \in G$  of prime order, where either  $\iota = 0$  or  $(G_0, \Omega, \iota)$  belongs to a short list of known exceptions (see [4, Table 1] for the list of exceptional cases). Here a transitive  $G$ -set is

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said to be *non-subspace* if the point stabilizer  $G_\omega$  is a non-subspace subgroup of  $G$ , i.e.  $G_\omega \cap G_0$  is contained in a maximal subgroup of  $G_0$  which acts irreducibly on the natural  $G_0$ -module (see [4, Definition 1]). In general,  $n = \dim V$  (see Remark 1.2).

The proof of Theorem 1 is based on Aschbacher’s main theorem on the subgroup structure of finite classical groups. Recall that in [1], eight collections of subgroups of  $G$  are defined, labeled  $\mathcal{C}_i$  for  $1 \leq i \leq 8$ , and in general it is shown that if  $H$  is a maximal subgroup of  $G$  not containing  $G_0$  then either  $H$  is contained in one of the  $\mathcal{C}_i$  collections, or it belongs to a family  $\mathcal{S}$  of almost simple groups which satisfy various irreducibility conditions (see [19] for a detailed description of these subgroup collections). Due to the existence of certain outer automorphisms, a small additional collection  $\mathcal{N}$  arises when  $G_0$  is  $\text{Sp}_4(q)'$  ( $q$  even) or  $\text{P}\Omega_8^+(q)$  (see Table 3.1 and [5, 3.1]).

This is the final paper in a series of four. In [4] we provided some background and motivation, stated our main results and described applications to base sizes and monodromy groups. In [5,6] we established Theorem 1 in the case where  $G_\omega$  is a non-subspace subgroup contained in a member of one of the  $\mathcal{C}_i$  collections. Therefore, it just remains to consider the collections  $\mathcal{S}$  and  $\mathcal{N}$ . In this paper we complete the proof of Theorem 1.

**Theorem 1.1.** *Let  $G$  be a finite almost simple classical group acting transitively and faithfully on a set  $\Omega$  with point stabilizer  $G_\omega \leq H$ , where  $H \leq G$  is a maximal non-subspace subgroup in one of the collections  $\mathcal{S}$  or  $\mathcal{N}$ . Then*

$$\text{fpr}(x) < |x^G|^{-\frac{1}{2} + \frac{1}{n} + \iota}$$

for all elements  $x \in G$  of prime order, where  $\iota = 0$  or  $(G_0, H, \iota)$  is listed in Table 1.1, where  $G_0$  denotes the socle of  $G$ .

**Remark 1.2.** The integer  $n$  in the statement of Theorem 1.1 is defined as follows: if  $G_0 \in \{\text{Sp}_4(2)', \text{SL}_3(2)\}$  then  $n = 2$ , otherwise  $n$  is the minimal degree of a non-trivial irreducible  $K\widehat{G}_0$ -module, where  $\widehat{G}_0$  is a covering group of  $G_0$  and  $K$  is the algebraic closure of  $\mathbb{F}_q$ . We also note that each subgroup appearing in Table 1.1 is a member of the collection  $\mathcal{S}$ . Further, the *type of  $H$*  refers to the socle of the almost simple group  $H \cap G_0$ .

Table 1.1  
The exceptional cases with  $\iota > 0$

| $G_0$                   | Type of $H$       | $\iota$ |
|-------------------------|-------------------|---------|
| $\text{P}\Omega_8^+(q)$ | $\Omega_7(q)$     | 0.219   |
| $\Omega_7(q)$           | $G_2(q)$          | 0.108   |
| $\Omega_{10}^-(2)$      | $A_{12}$          | 0.087   |
| $\text{Sp}_8(2)$        | $A_{10}$          | 0.062   |
| $\Omega_8^+(2)$         | $A_9$             | 0.124   |
| $\text{P}\Omega_8^+(3)$ | $\Omega_8^+(2)$   | 0.081   |
| $\Omega_7(3)$           | $\text{Sp}_6(2)$  | 0.065   |
| $\text{PSU}_6(2)$       | $\text{PSU}_4(3)$ | 0.076   |
| $\text{Sp}_6(2)$        | $\text{SU}_3(3)$  | 0.054   |
| $\text{PSU}_4(3)$       | $\text{PSL}_3(4)$ | 0.011   |
| $\text{SL}_4(2)$        | $A_7$             | 0.164   |

**Notation.** Our notation for classical groups is standard (see [19], for example) and we continue with the notation and terminology introduced in the previous papers [4–6] in this series. In particular, if  $H \leq G$  is a non-subspace subgroup and  $x \in H$  has prime order then

$$f(x, H) := \frac{\log|x^G \cap H|}{\log|x^G|}$$

and thus Theorem 1 states that  $f(x, H) < 1/2 + 1/n + \epsilon$  (see [5, (1)]). In addition, we adopt the standard Aschbacher–Seitz [2] notation for unipotent involutions in symplectic and orthogonal groups and we define the *associated partition* of a unipotent element  $x \in \text{PGL}(V)$  to be the partition of  $\dim V$  which encodes the Jordan normal form of  $x$  on  $V$  (see [5, §3.3]). Also, for each  $x \in \text{PGL}(V)$  we define  $\nu(x)$  to be the codimension of the largest eigenspace of  $x$  on  $V$  (see [5, 3.16]). We write  $i_r(S)$  for the number of elements of order  $r$  in  $S$ , where  $S$  is a subset of a finite group.

**2. Proof of Theorem 1.1:  $H \in \mathcal{S}$**

Let  $G$  be a finite almost simple classical group over  $\mathbb{F}_q$ , with socle  $G_0$  and natural module  $V$  of dimension  $n$ . We write  $q = p^f$ , where  $p$  is prime. If  $H$  is a maximal subgroup of  $G$  in Aschbacher’s  $\mathcal{S}$  collection then  $H \cap G_0$  is almost simple, with socle  $H_0$ . Let  $\hat{H}_0$  be the full covering group of  $H_0$ . Then  $\hat{H}_0$  acts absolutely irreducibly on  $V$  and is defined over no proper subfield of  $\mathbb{F}_{q^u}$ , where  $u = 2$  if  $G_0$  is unitary, otherwise  $u = 1$ . In addition,  $\hat{H}_0$  fixes a non-degenerate form on  $V$  only if  $G_0$  fixes a form of the same type (see [19, §1.2], for example).

Our strategy for dealing with these subgroups is as follows. First we define three sets of irreducible inclusions  $H < G$ , denoted by the letters  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  (see Tables 2.1–2.3). In  $\mathcal{A}$ ,  $H_0$  is an alternating group,  $q = p$  is prime and  $V$  is the fully deleted permutation module for  $H_0$  over  $\mathbb{F}_p$ . We establish Theorem 1.1 for these inclusions in Proposition 2.5, while the collections  $\mathcal{B}$  and  $\mathcal{C}$  are considered in Propositions 2.6 and 2.10, respectively. If an irreducible embedding  $H < G$  is not one of the inclusions in  $\mathcal{A}$ ,  $\mathcal{B}$  or  $\mathcal{C}$  then Theorems 2.2 and 2.4 imply that the following hold:

- (i) If  $n \geq 6$  then  $\nu(x) > \max(2, \frac{1}{2}\sqrt{n})$  for all  $1 \neq x \in H \cap \text{PGL}(V)$ .
- (ii)  $|H| < |\mathbb{F}|^{2n+4}$ , where  $V$  is defined over the field  $\mathbb{F}$ .

(Here  $\nu(x)$  denotes the codimension of the largest eigenspace of  $x$  on  $V$ —see [5, 3.16].) In particular, if  $x \in H \cap \text{PGL}(V)$  has prime order then the bound on  $\nu(x)$  in (i) yields a lower bound for  $|x^G|$  via [5, 3.38]. Then an upper bound for  $f(x, H)$  follows since (ii) gives  $|x^G \cap H| < |\mathbb{F}|^{2n+4}$ . In this way, we reduce to a small collection of inclusions which we can deal with on a case-by-case basis (see Tables 2.10 and 2.11). Finally we consider the remaining cases with  $n < 6$  in Proposition 2.22.

**Definition 2.1.** Let  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  be the set of irreducible inclusions  $H < G$  recorded in Tables 2.1–2.3, respectively. In Table 2.1, we have  $H_0 = A_d$  with  $d \geq 5$ . We write  $M(\lambda)$  for the unique irreducible  $\mathbb{F}_q H_0$ -module of highest weight  $\lambda$  (up to quasi-equivalence) and we follow [3] in labeling the fundamental dominant weights  $\lambda_i$ .

**Theorem 2.2.** [22, 4.2] *If  $H \in \mathcal{S}$  then one of the following holds:*

- (i)  $H_0$  is alternating, embedded in  $G_0$  as in  $\mathcal{A}$ ;

Table 2.1  
The collection  $\mathcal{A}$ ,  $H_0 = A_d$

|                    | $d$                   | $p$ | $G_0$  |
|--------------------|-----------------------|-----|--|
| ( $\mathcal{A}1$ ) | arbitrary             | odd | $\begin{cases} \text{P}\Omega_{d-1}^\epsilon(p) & \text{if } (d, p) = 1 \\ \text{P}\Omega_{d-2}^\epsilon(p) & \text{otherwise} \end{cases}$  |
| ( $\mathcal{A}2$ ) | $d \equiv 2 \pmod{4}$ | 2   | $\text{Sp}_{d-2}(2)$   |
| ( $\mathcal{A}3$ ) | $d \equiv 0 \pmod{4}$ | 2   | $\begin{cases} \Omega_{d-2}^+(2) & \text{if } d \equiv 0 \pmod{8} \\ \Omega_{d-2}^-(2) & \text{if } d \equiv 4 \pmod{8} \end{cases}$         |
| ( $\mathcal{A}4$ ) | odd                   | 2   | $\begin{cases} \Omega_{d-1}^+(2) & \text{if } d \equiv \pm 1 \pmod{8} \\ \Omega_{d-1}^-(2) & \text{if } d \equiv \pm 3 \pmod{8} \end{cases}$ |

Table 2.2  
The collection  $\mathcal{B}$

|                    | $H_0$   | $G_0$  | Representation of $H_0$             |
|--------------------|---|--|-------------------------------------|
| ( $\mathcal{B}1$ ) | $\text{PSL}_d(q)$ , $d \geq 5$  | $\text{PSL}_{\frac{1}{2}d(d-1)}(q)$  | $\bigwedge^2 V_d$                   |
| ( $\mathcal{B}2$ ) | $\begin{cases} \Omega_7(q), & p > 2 \\ \text{Sp}_6(q), & p = 2 \end{cases}$ | $\text{P}\Omega_8^+(q)$  | spin representation                 |
| ( $\mathcal{B}3$ ) | $\begin{cases} \Omega_9(q), & p > 2 \\ \text{Sp}_8(q), & p = 2 \end{cases}$ | $\text{P}\Omega_{16}^+(q)$   | spin representation                 |
| ( $\mathcal{B}4$ ) | $\text{P}\Omega_{10}^+(q)$  | $\text{PSL}_{16}(q)$   | one of the two spin representations |
| ( $\mathcal{B}5$ ) | $E_6(q)$  | $\text{PSL}_{27}(q)$   | $M(\lambda_1)$ or $M(\lambda_6)$    |
| ( $\mathcal{B}6$ ) | $E_7(q)$  | $\begin{cases} \text{PSp}_{56}(q), & p > 2 \\ \Omega_{56}^+(q), & p = 2 \end{cases}$ | $M(\lambda_7)$                      |
| ( $\mathcal{B}7$ ) | $M_{24}$  | $\text{SL}_{11}(2)$  |                                     |
| ( $\mathcal{B}8$ ) | $\text{Co}_1$   | $\Omega_{24}^+(2)$   |                                     |

- (ii)  $H_0$  is embedded in  $G_0$  as in  $\mathcal{B}$ ;
- (iii)  $|H| < |\mathbb{F}|^{2n+4}$ , where  $V$  is defined over  $\mathbb{F}$  and  $n = \dim V$ .

**Remark 2.3.** If  $G_0 = \text{PSU}_n(q)$  then part (iii) of Theorem 2.2 reads  $|H| < q^{4n+8}$  because the natural  $G_0$ -module is a vector space over the field  $\mathbb{F}_{q^2}$  and not  $\mathbb{F}_q$ .

**Theorem 2.4.** [12, 7.1] *If  $H \in \mathcal{S}$  and  $n = \dim V \geq 6$  then one of the following holds:*

- (i)  $H_0$  is alternating, embedded in  $G_0$  as in  $\mathcal{A}$ ;
- (ii)  $H_0$  is embedded in  $G_0$  as in  $\mathcal{C}$ ;
- (iii)  $\nu(x) > \max(2, \frac{1}{2}\sqrt{n})$  for all non-trivial elements  $x \in H \cap \text{PGL}(V)$ .

2.1. The  $\mathcal{A}$  collection

**Proposition 2.5.** *The conclusion to Theorem 1.1 holds for the collection  $\mathcal{A}$ .*

**Proof.** Let  $H < G$  be an inclusion in the collection  $\mathcal{A}$ , where  $H$  has socle  $H_0 = A_d$ . Let  $V$  denote the fully deleted permutation module for  $H_0$  over  $\mathbb{F}_p$ , i.e.  $V = U/(U \cap W)$ , where  $U$  and

Table 2.3  
The collection  $\mathcal{C}$

|                     | $H_0$   | $G_0$   |                                  |
|---------------------|---|---|----------------------------------|
| ( $\mathcal{C}1$ )  | $\mathrm{PSL}_3^\epsilon(q), p > 2$   | $\mathrm{PSL}_6^\epsilon(q)$  | $S^2V_3$                         |
| ( $\mathcal{C}2$ )  | $\begin{cases} \Omega_7(q), & p > 2 \\ \mathrm{Sp}_6(q), & p = 2 \end{cases}$ | $\mathrm{P}\Omega_8^+(q)$   | spin representation              |
| ( $\mathcal{C}3$ )  | ${}^3D_4(q_0), q = q_0^3$   | $\mathrm{P}\Omega_8^+(q)$   | minimal module                   |
| ( $\mathcal{C}4$ )  | $G_2(q)'$   | $\begin{cases} \Omega_7(q), & p > 2 \\ \mathrm{Sp}_6(q), & p = 2 \end{cases}$ | $M(\lambda_1)$                   |
| ( $\mathcal{C}5$ )  | $G_2(q), p = 3$   | $\Omega_7(q)$   | $M(\lambda_2)$                   |
| ( $\mathcal{C}6$ )  | $A_6$   | $\mathrm{PSL}_6^\epsilon(p)$  | $p \equiv \epsilon(3), p \geq 5$ |
| ( $\mathcal{C}7$ )  | $A_7$   | $\mathrm{PSL}_6^\epsilon(p)$  | $p \equiv \epsilon(3), p \geq 5$ |
| ( $\mathcal{C}8$ )  | $\mathrm{PSL}_2(7)$   | $\mathrm{P}\Omega_6^\epsilon(p)$  | $p \neq 2, 7$                    |
| ( $\mathcal{C}9$ )  | $\mathrm{PSL}_3(4)$   | $\mathrm{PSL}_6^\epsilon(p)$  | $p \equiv \epsilon(3), p \geq 5$ |
| ( $\mathcal{C}10$ ) | $\mathrm{PSL}_3(4)$   | $\mathrm{P}\Omega_6^-(3)$   |                                  |
| ( $\mathcal{C}11$ ) | $\mathrm{SU}_3(3)$  | $\mathrm{PSp}_6(p)$   | $p \neq 3$                       |
| ( $\mathcal{C}12$ ) | $\mathrm{SU}_3(3)$  | $\mathrm{PSL}_7^\epsilon(p)$  | $p \equiv \epsilon(3), p \geq 5$ |
| ( $\mathcal{C}13$ ) | $\mathrm{SU}_3(3)$  | $\Omega_7(p)$   | $p \geq 5$                       |
| ( $\mathcal{C}14$ ) | $\mathrm{SU}_4(2)$  | $\mathrm{P}\Omega_6^\epsilon(p)$  | $p \equiv \epsilon(3), p \geq 5$ |
| ( $\mathcal{C}15$ ) | $\mathrm{PSU}_4(3)$   | $\mathrm{PSL}_6^\epsilon(p)$  | $p \equiv \epsilon(3), p \geq 5$ |
| ( $\mathcal{C}16$ ) | $\mathrm{PSU}_4(3)$   | $\mathrm{PSU}_6(2)$   |                                  |
| ( $\mathcal{C}17$ ) | $\mathrm{SU}_5(2)$  | $\mathrm{PSp}_{10}(p)$  | $p \geq 3$                       |
| ( $\mathcal{C}18$ ) | $\mathrm{Sp}_6(2)$  | $\Omega_7(p)$   | $p \geq 3$                       |
| ( $\mathcal{C}19$ ) | $\Omega_8^+(2)$   | $\mathrm{P}\Omega_8^+(p)$   | $p \geq 3$                       |
| ( $\mathcal{C}20$ ) | $M_{12}$  | $\mathrm{PSL}_6(3)$   |                                  |
| ( $\mathcal{C}21$ ) | $M_{22}$  | $\mathrm{PSU}_6(2)$   |                                  |
| ( $\mathcal{C}22$ ) | $J_2$   | $\mathrm{PSp}_6(q)$   | $p \geq 3$                       |

$W$  are the submodules of  $\mathbb{F}_p^d$  defined by

$$U = \left\{ (a_1, \dots, a_d) : \sum_{i=1}^d a_i = 0 \right\}, \quad W = \{ (a, \dots, a) : a \in \mathbb{F}_p \}$$

with respect to the natural action of the symmetric group  $S_d$  on the coordinates of  $\mathbb{F}_p^d$ . Observe that  $H \leq S_d \leq \mathrm{PGL}(V)$ . Let  $x \in H$  be an element of prime order  $r$  and let  $h$  denote the number of  $r$ -cycles in the cycle-shape of  $x$ . Write  $K$  for the algebraic closure of  $\mathbb{F}_p$  and let  $\bar{G}$  be a simple algebraic group of adjoint type over  $K$  such that  $\bar{G}_\sigma$  has socle  $G_0$ , where  $\sigma$  is a suitable Frobenius morphism of  $\bar{G}$ . According to [5, 3.3] we may assume  $G$  is without triality if  $G_0 = \mathrm{P}\Omega_8^+(p)$ . In order to establish Theorem 1.1 for the inclusions listed in Table 2.1 we will show that  $f(x, H) < 1/2 + 1/n$ , with the exception of the following cases.

| $H_0$       | $A_{12}$           | $A_{10}$           | $A_9$           | $A_7$           |
|-------------|--------------------|--------------------|-----------------|-----------------|
| $G_0$       | $\Omega_{10}^-(2)$ | $\mathrm{Sp}_8(2)$ | $\Omega_8^+(2)$ | $\Omega_6^+(2)$ |
| $f(x, H) <$ | 0.687*             | 0.687*             | 0.749*          | 0.914*          |

(Here  $n = n(G)$  is the integer defined in Remark 1.2.) These bounds are obtained through direct calculation and agree with the relevant entries in Table 1.1. (Note that  $\Omega_6^+(2) \cong \text{SL}_4(2)$  and we list the case  $(\Omega_6^+(2), A_7)$  in Table 1.1 under  $G_0 = \text{SL}_4(2)$ .) The asterisks indicate that these cases are exceptions to the main statement of Theorem 1.1.

First consider  $(\mathcal{A}2)$ . Here  $d \equiv 2 \pmod{4}$  and we may assume  $d \geq 10$  since  $H_0 \cong G_0$  if  $d = 6$ . Referring to a general  $d$ -tuple  $(a_1, \dots, a_d)$ , one can check that the elements defined by

$$\begin{aligned} e_i: & \quad a_{2i-1} = a_{2i} = 1, \quad a_j = 0 \text{ for all other } j; \\ f_i: & \quad a_j = a_d = 1 \text{ for all } j \leq 2i - 1, \text{ otherwise } a_k = 0; \\ g: & \quad a_j = 1 \text{ for all } j, \end{aligned}$$

where  $1 \leq i \leq \frac{1}{2}(d - 2)$ , form a basis for  $U$ . Since  $p$  divides  $d$  we have  $\dim V = d - 2$  and it is easy to see that

$$\{e_i + (U \cap W), f_i + (U \cap W): 1 \leq i \leq (d - 2)/2\} = \{\bar{e}_i, \bar{f}_i: 1 \leq i \leq (d - 2)/2\}$$

is a standard symplectic basis for  $V$  with respect to the form on  $V$  induced from the symmetric bilinear form  $f$  on  $U$  defined by

$$f((a_1, \dots, a_d), (b_1, \dots, b_d)) = \sum_{i=1}^d a_i b_i.$$

If  $r$  is odd then  $x$  is  $\bar{G}$ -conjugate to  $[I_{d-2-h(r-1)}, \omega I_h, \dots, \omega^{r-1} I_h]$ , where  $\omega \in K$  is a primitive  $r$ th root of unity. If  $r = 2$  and  $h < d/2$  then by replacing  $x$  with a suitable conjugate we may assume that  $x$  interchanges the first two coordinates, while fixing the last two. This implies that  $f(\bar{f}_1, \bar{f}_1 x) = f(\bar{f}_1, \bar{e}_1 + \bar{f}_1) = 1$  and thus  $x$  is  $\bar{G}$ -conjugate to either  $b_h$  or  $c_h$ , the precise type depending on the parity of  $h$ . (As remarked in the Introduction, in this paper we adopt the standard Aschbacher–Seitz [2] notation for unipotent involutions.) If  $h = d/2$  then the action of  $x$  on the above basis for  $V$  is given by

$$x: \bar{e}_i \mapsto \bar{e}_i, \quad \bar{f}_i \mapsto \bar{f}_i + \sum_{j \neq i} \bar{e}_j$$

and therefore  $x$  is  $G$ -conjugate to  $a_{d/2-1}$ . It follows that  $x^G \cap H \subseteq x^{S_d}$  for all  $x \in H$  of prime order and thus

$$|x^G \cap H| \leq |x^{S_d}| = \frac{d!}{h!(d - hr)!r^h}. \tag{1}$$

If  $r = 2$  and  $h < d/2$  then  $|x^G| > 2^{h(d-h-1)-1}$  (see [5, 3.22]) and (1) implies that  $f(x, H) < 1/2 + 1/(d - 2)$  with the exception of the following cases.

|             |         |         |         |         |
|-------------|---------|---------|---------|---------|
| $(h, d)$    | (4, 10) | (3, 10) | (2, 10) | (1, 10) |
| $f(x, H) <$ | 0.615   | 0.636*  | 0.666*  | 0.687*  |

Here the upper bounds for  $f(x, H)$  are obtained through direct calculation. For instance, if  $(h, d) = (1, 10)$  then  $f(x, H) < 0.687^*$  since  $|x^G \cap H| = 45$  and  $|x^G| = 255$ . As before, the asterisks indicate that the case  $(G_0, H_0) = (\text{Sp}_8(2), A_{10})$  appears in Table 1.1. Now, if  $r = 2$  and  $h = d/2$  then  $|x^G| > 2^{d^2/4-d}$  and (1) is sufficient unless  $d = 10$ , where direct calculation yields  $f(x, H) < 0.619$ . If  $r$  is odd then

$$|x^G| > \frac{1}{2} \left(\frac{2}{3}\right)^{\frac{1}{2}(r-1)} 2^{\frac{1}{2}h(r-1)(2d-hr-3)}$$

and we are left to deal with the following cases.

|             |            |            |            |            |
|-------------|------------|------------|------------|------------|
| $(r, h, d)$ | (3, 1, 10) | (3, 2, 10) | (3, 3, 10) | (5, 2, 10) |
| $f(x, H) <$ | 0.590      | 0.598      | 0.614      | 0.593      |

The cases (A3) and (A4) are similar. For example, in (A4)  $d$  is odd and  $x$  is given as follows up to  $\bar{G}$ -conjugacy:

$$x = \begin{cases} [I_{d-1-h(r-1)}, \omega I_h, \dots, \omega^{r-1} I_h] & \text{if } r > 2, \\ b_h \text{ or } c_h & \text{if } r = 2. \end{cases}$$

Note that we may assume  $d \geq 7$  since  $G_0 = \Omega_4^-(2)$  if  $d = 5$  and thus  $n = 2$  (see Remark 1.2). If  $d = 7$  then  $n = 4$  since  $G_0 = \Omega_6^+(2)$  and we obtain the following results through direct calculation. Here  $\zeta = 1$  if  $G = O_6^+(2)$ , otherwise  $\zeta = 0$ .

|                |        |        |        |        |        |        |                      |
|----------------|--------|--------|--------|--------|--------|--------|----------------------|
| $(h, r)$       | (1, 2) | (2, 2) | (3, 2) | (1, 3) | (2, 3) | (1, 5) | (1, 7)               |
| $ x^G \cap H $ | 21     | 105    | 105    | 70     | 280    | 504    | $2^\zeta \cdot 360$  |
| $ x^G $        | 28     | 210    | 420    | 112    | 1120   | 1344   | $2^\zeta \cdot 2880$ |
| $f(x, H) <$    | 0.914* | 0.871* | 0.771* | 0.901* | 0.803* | 0.864* | 0.760*               |

Similarly, if  $d = 9$  then  $f(x, H) \leq (\log 36)/(\log 120) < 0.749^*$ , with equality if and only if  $x \in S_9$  has cycle-shape  $(2, 1^7)$ . In case (A3) we may assume  $d \geq 12$  since  $A_8 \cong \Omega_6^+(2)$ . In particular, if  $d = 12$  then  $f(x, H) \leq (\log 10395)/(\log 706860) < 0.687^*$ , with equality if and only if  $x \in A_{12}$  has cycle-shape  $(2^6)$ .

Now assume  $p$  is odd. If  $r = p$  and  $(d, p) = 1$  then  $x$  is  $\bar{G}$ -conjugate to  $[J_p^h, I_{d-1-hp}]$ , otherwise  $p$  divides  $d$  and  $x$  is given as follows (up to  $\bar{G}$ -conjugacy)

$$x = \begin{cases} [J_p^h, I_{d-2-hp}] & \text{if } h < d/p, \\ [J_p^{h-2}, J_{p-1}^2] & \text{if } h = d/p > 1, \\ [J_{p-2}] & \text{if } p = d \end{cases}$$

where  $J_i$  denotes a standard Jordan block of size  $i$ . In both cases, the desired result quickly follows. For example, if  $p$  divides  $d$  and  $h = d/p > 1$  then

$$|x^G| > \frac{1}{4} \left(\frac{p}{p+1}\right) p^{\frac{1}{2}(p-1)(2dh-5h-ph^2)}$$

(see [5, 3.21]) and (1) is sufficient unless  $(h, p) = (3, 3)$ , where direct calculation yields  $f(x, H) < 0.619$ . If  $x$  is semisimple and  $r$  is odd then  $x$  is conjugate to  $[I_{d-e-h(r-1)}, \omega I_h, \dots, \omega^{r-1} I_h]$ , where  $e = 1$  if  $(d, p) = 1$ , otherwise  $e = 2$ . Therefore (1) holds and the result quickly follows. Finally, let us assume  $x$  is a semisimple involution. If  $d$  is coprime to  $p$  then  $x$  is  $\text{PO}_{d-1}(K)$ -conjugate to  $[-I_h, I_{d-1-h}]$ . In particular, if  $d$  is even and  $h \geq d/2 - 1$  then

$$|x^G \cap H| \leq \frac{d!}{(d/2 - 1)!2^{d/2}} + \frac{d!}{(d/2)!2^{d/2}} = \frac{(d/2 + 1)d!}{(d/2)!2^{d/2}}$$

and the bound  $|x^G| > \frac{1}{4}(p + 1)^{-1} p^{d^2/4 - d/2 + 1}$  is good enough. If not, then (1) holds and the result quickly follows. Similar reasoning applies when  $p$  divides  $d$ .  $\square$

2.2. The  $\mathcal{B}$  collection

**Proposition 2.6.** *The conclusion to Theorem 1.1 holds for the collection  $\mathcal{B}$ .*

**Proof.** Let  $H < G$  be an inclusion in the collection  $\mathcal{B}$ . For easy reference, we partition the proof into a number of separate lemmas, beginning with the embedding labeled  $(\mathcal{B}2)$ .

**Lemma 2.7.** *The conclusion to Theorem 1.1 holds for  $(\mathcal{B}2)$ .*

**Proof.** This embedding is obtained by restricting a spin representation  $\psi$  of  $G_0 = \text{P}\Omega_8^+(q)$  to the stabilizer of a 1-dimensional non-singular subspace of  $V$ . Set  $\bar{G} = \text{PSO}_8(K)$ , where  $K$  is the algebraic closure of  $\mathbb{F}_q$ . We may assume that  $G$  does not contain a triality automorphism (see [5, 3.3]). We claim that

$$f(x, H) \leq \frac{\log 672}{\log 2240} \approx 0.844^* \tag{2}$$

for all elements  $x \in H$  of prime order, so this case is included in Table 1.1.

**Case 1.**  $p = 2$ .

Here  $H_0 = \text{Sp}_6(q)$  and we define  $\bar{H} = \text{Sp}_6(K)$ . If  $x \in H - \text{PGL}(V)$  is a field automorphism of prime order  $r$  then  $q = q_0^r$  and (2) holds since [5, 3.43, 3.48] imply that

$$|x^G \cap H| \leq |\text{Sp}_6(q) : \text{Sp}_6(q^{1/r})| < 2q^{21(1-\frac{1}{r})}, \quad |x^G| \geq |\Omega_8^+(q) : \Omega_8^+(q^{1/r})| > \frac{1}{2}q^{28(1-\frac{1}{r})}.$$

If  $x$  is an involutory graph-field automorphism then similar bounds hold (with  $r = 2$ ). For the remainder of Case 1 we will assume  $x \in H \cap \text{PGL}(V)$  has prime order  $r$ . If  $r = 2$  then using [7, Table 6] and the proof of [5, 3.22] we obtain the results recorded in the next table.

Now assume  $r$  is odd. Let  $i \geq 1$  be minimal such that  $r$  divides  $q^i - 1$  and write  $\theta$  for the natural embedding of  $\text{Sp}_6(q)$  in  $\Omega_8^+(q)$  as the stabilizer of a 1-dimensional non-singular subspace of the natural module (see [19, 4.1.7]). If  $x \in H$  is  $\bar{H}$ -conjugate to the diagonal matrix  $\text{diag}[\mu_1, \mu_2, \mu_3] \in \text{GL}_3 < \bar{H}$  then  $\theta(x)$  is  $\bar{G}$ -conjugate to  $\text{diag}[1, \mu_1, \mu_2, \mu_3] \in$



| $\mathrm{Sp}_6(q)$ -class | $\mathrm{O}_8^+(q)$ -class | $ x^G \cap H  \leq$     | $ x^G  \geq$              | $f(x, H) <$ |
|---------------------------|----------------------------|-------------------------|---------------------------|-------------|
| $a_2$                     | $a_2$                      | $(q^2 + 1)(q^6 - 1)$    | $(q^2 + 1)^2(q^6 - 1)$    | 0.800*      |
| $b_1, c_2$                | $a_4$                      | $q^4(q^6 - 1)$          | $q^2(q^4 - 1)(q^6 - 1)$   | 0.840*      |
| $b_3$                     | $c_4$                      | $q^2(q^4 - 1)(q^6 - 1)$ | $q^2(q^4 - 1)^2(q^6 - 1)$ | 0.753*      |

$\mathrm{GL}_4 < \bar{G}$  and thus [5, 3.55(iv)] implies that the possibilities for  $C_{\bar{H}}(x)$  and  $C_{\bar{G}}(x)$  are as follows.

| $i$  | $C_{\bar{H}}(x)$                       | $C_{\bar{G}}(x)$  |
|------|--|---|
| 6, 3 | $\mathrm{GL}_1^3$                      | $\mathrm{SO}_2 \times \mathrm{GL}_1^3$  |
| 4    | $\mathrm{Sp}_2 \times \mathrm{GL}_1^2$ | $\mathrm{GL}_2^2$   |
| 2, 1 | $\mathrm{Sp}_4 \times \mathrm{GL}_1$   | $\mathrm{GL}_4$   |
|      | $\mathrm{Sp}_2 \times \mathrm{GL}_2$   | $\mathrm{SO}_4 \times \mathrm{GL}_2$  |
|      | $\mathrm{Sp}_2 \times \mathrm{GL}_1^2$ | $\mathrm{GL}_2^2$   |
|      | $\mathrm{GL}_3$                        | $\mathrm{SO}_2 \times \mathrm{GL}_3$ or $\mathrm{GL}_3 \times \mathrm{GL}_1$                        |
|      | $\mathrm{GL}_2 \times \mathrm{GL}_1$   | $\mathrm{SO}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_1$ or $\mathrm{GL}_2 \times \mathrm{GL}_1^2$ |
|      | $\mathrm{GL}_1^3$                      | $\mathrm{SO}_2 \times \mathrm{GL}_1^3$ or $\mathrm{GL}_1^4$   |

It is now straightforward to check that (2) holds. For instance, when  $q = 2$  we obtain the following results. (Here we adopt the notation of [8] for labeling  $\mathrm{Sp}_6(2)$ -classes.)

| $\mathrm{Sp}_6(2)$ -class | $i$ | $C_{\bar{H}}(x)$                       | $C_{\bar{G}}(x)$                       | $ x^G \cap H $ | $ x^G $    | $f(x, H) <$ |
|---------------------------|-----|--|--|----------------|------------|-------------|
| 3A                        | 2   | $\mathrm{Sp}_4 \times \mathrm{GL}_1$   | $\mathrm{GL}_4$                        | 672            | 2240       | 0.844*      |
| 3B                        | 2   | $\mathrm{GL}_3$                        | $\mathrm{SO}_2 \times \mathrm{GL}_3$   | 2240           | 89 600     | 0.677*      |
| 3C                        | 2   | $\mathrm{Sp}_2 \times \mathrm{GL}_2$   | $\mathrm{SO}_4 \times \mathrm{GL}_2$   | 13 440         | 268 800    | 0.761*      |
| 5A                        | 4   | $\mathrm{Sp}_2 \times \mathrm{GL}_1^2$ | $\mathrm{GL}_2^2$                      | 48 384         | 580 608    | 0.813*      |
| 7A                        | 3   | $\mathrm{GL}_1^3$                      | $\mathrm{SO}_2 \times \mathrm{GL}_1^3$ | 207 360        | 24 883 200 | 0.719*      |

We conclude that (2) holds, with equality if  $x$  belongs to the  $\mathrm{Sp}_6(2)$ -class 3A.

**Case 2.**  $p \neq 2$ .

Here  $H_0 = \Omega_7(q)$  and it is easy to see that (2) holds if  $x \in H - \mathrm{PGL}(V)$ . Let us assume  $x \in H \cap \mathrm{PGL}(V)$  has prime order  $r$ . Recall that if  $r = p$  then the associated partition of  $x$  (with respect to  $V$ ) is the partition of  $n = \dim V$  which encodes the Jordan normal form of  $x$  on  $V$  (see [5, §3.3]). If  $\lambda'$  (respectively  $\lambda$ ) denotes the associated partition of  $x \in H$  (respectively  $\psi(x) \in G$ ) then [7, Table 7] indicates that the possibilities for  $\lambda'$  and  $\lambda$  are as follows. Here the symbol  $\ddagger$  (respectively  $\ddot{\ddagger}$ ) signifies the condition  $p \geq 7$  (respectively  $p \geq 5$ ).

|             |                |                            |              |               |            |              |
|-------------|----------------|----------------------------|--------------|---------------|------------|--------------|
| $\lambda'$  | $(7)^\ddagger$ | $(5, 1^2)^\ddot{\ddagger}$ | $(3^2, 1)$   | $(3, 2^2)$    | $(3, 1^4)$ | $(2^2, 1^3)$ |
| $\lambda$   | $(7, 1)$       | $(4^2)$                    | $(3^2, 1^2)$ | $(3, 2^2, 1)$ | $(2^4)$    | $(2^2, 1^4)$ |
| $f(x, H) <$ | 0.773*         | 0.818*                     | 0.778*       | 0.782*        | 0.835*     | 0.795*       |

We now explain how these bounds are derived. If  $p = 7$  and  $\lambda = (7, 1)$  then [5, 3.18] implies that

$$|x^G \cap H| \leq \frac{|O_7(q)|}{|O_1(q)|q^3}, \quad |x^G| \geq \frac{1}{2} \frac{|O_8^+(q)|}{|O_1(q)|^2q^4}$$

and thus  $f(x, H) < 0.773^*$  as claimed. The case  $\lambda = (4^2)$  is similar. For  $p = 3$  we require precise values for  $|x^G \cap H|$  and  $|x^G|$ . For example, suppose  $\lambda = (2^4)$ . First observe that the partition  $\lambda' = (3, 1^4)$  corresponds to precisely two distinct  $H_0$ -classes, represented by  $x_+$  and  $x_-$ , where

$$|x_\epsilon^{H_0}| = \frac{|O_7(q)|}{|O_4^\epsilon(q)||O_1(q)|q^5} = \frac{1}{2}q^2(q^2 + \epsilon)(q^6 - 1).$$

In the natural embedding  $H_0 \hookrightarrow G_0$ , the elements  $x_+$  and  $x_-$  represent the two distinct  $G_0$ -classes with associated partition  $(3, 1^5)$ . These  $G_0$ -classes fuse in  $\text{Inndiag}(G_0)$  and

$$|\psi(x_\epsilon)^{G_0}| = |x_\epsilon^{G_0}| = \frac{|O_8^+(q)|}{|O_5(q)||O_1(q)|q^6} = \frac{1}{2}q^2(q^4 - 1)(q^6 - 1).$$

(Here  $\text{Inndiag}(G_0)$  is the group of *inner-diagonal* automorphisms of  $G_0$ , see [11, 2.5.10].) Hence

$$f(x, H) \leq \max_{\epsilon = \pm} \left\{ \frac{\log|x_\epsilon^{H_0}|}{\log|x_\epsilon^{G_0}|}, \frac{\log(|x_+^{H_0}| + |x_-^{H_0}|)}{\log 2|x_\epsilon^{G_0}|} \right\} < 0.835^* \tag{3}$$

for all  $q \geq 3$ . The other bounds are derived in a similar fashion.

Now assume  $x$  is a semisimple involution. Here [5, 3.55(iii)] implies that  $\psi(x)$  is  $\bar{G}$ -conjugate to either  $[-iI_4, iI_4]$  or  $[-I_4, I_4]$ , where  $i \in K$  satisfies  $i^2 = -1$ , and the following results hold.

| $x$                        | $\psi(x)$       | $f(x, H) <$ |
|----------------------------|-----------------|-------------|
| $[-I_2, I_5], [-I_6, I_1]$ | $[-iI_4, iI_4]$ | 0.829*      |
| $[-I_4, I_3]$              | $[-I_4, I_4]$   | 0.778*      |

For example, if  $\psi(x)$  is conjugate to  $[-iI_4, iI_4]$  and  $C_G(x)$  is of type  $\text{GL}_4^\epsilon(q)$  then

$$|x^G \cap H| \leq |O_7(q) : O_6^\epsilon(q)O_1(q)| + |O_7(q) : O_5(q)O_2^\epsilon(q)|, \quad |x^G| \geq \frac{1}{2} |SO_8^+(q) : \text{GL}_4^\epsilon(q)|$$

and thus  $f(x, H) < 0.829^*$  if  $\epsilon = +$ , while  $f(x, H) < 0.826^*$  if  $\epsilon = -$ . For semisimple elements of odd prime order we argue as in Case 1.  $\square$

**Lemma 2.8.** *The conclusion to Theorem 1.1 holds for  $(\mathcal{B}3)$  and  $(\mathcal{B}4)$ .*

**Proof.** First consider  $(\mathcal{B}4)$ . Fix a spin representation which embeds  $H_0 = \text{P}\Omega_{10}^+(q)$  in  $G$ , where  $G$  has socle  $G_0 = \text{PSL}_{16}(q)$ , and suppose  $x \in H$  has prime order. If  $x \in H \cap \text{PGL}(V)$  then  $|x^G| > \frac{1}{2}q^{96}$  since  $v(x) \geq 4$  (see the proof of [12, 7.5]) and [5, 3.49] implies that the same bound holds if  $x \in H - \text{PGL}(V)$ . The desired result now follows since  $|x^G \cap H| < |H| < 2 \log_2 q \cdot q^{45}$ .

For the remainder, let us consider the embedding  $(\mathcal{B}3)$ . This is obtained by restricting a spin representation  $\psi$  of  $P\Omega_{10}^+(q)$  to the stabilizer of a 1-dimensional non-singular subspace of the natural  $P\Omega_{10}^+(q)$ -module. If  $x \in H - PGL(V)$  has prime order  $r$  then  $x$  induces a field automorphism on both  $H_0$  and  $G_0$ , so by [5, 3.43, 3.48] we have

$$|x^G \cap H| \leq |Sp_8(q) : Sp_8(q^{1/r})| < 2q^{36(1-\frac{1}{r})}, \quad |x^G| > \frac{1}{4}q^{120(1-\frac{1}{r})}.$$

These bounds are always sufficient. Now assume  $x \in H \cap PGL(V)$  has prime order  $r$ . If  $r = p = 2$  then we easily derive the following bounds.

| $Sp_8(q)$ -class     | $a_2$ | $b_1, c_2$ | $b_3, a_4, c_4$ |
|----------------------|-------|------------|-----------------|
| $O_{16}^+(q)$ -class | $a_4$ | $a_8$      | $c_8$           |
| $f(x, H) <$          | 0.278 | 0.250      | 0.319           |

If  $r = p > 2$  and  $\dim x^{SO_9} > 22$  then the  $PSO_{16}(K)$ -class of  $\psi(x)$  is given in [7, Table 5] and it is very easy to check that  $f(x, H) < 9/16$ . To deal with the remaining classes, we simply extend [7, Table 5] to all unipotent elements of prime order: each remaining class has a representative in a Levi  $D_4$  subgroup of  $B_4$ ; the spin module restricts to a direct sum of two non-isomorphic spin modules for  $D_4$  and [7, Table 7] applies. In this way we obtain the following results (up to conjugacy)

| $x$                 | $\psi(x)$             | $\dim x^{SO_9}$ | $\dim \psi(x)^{SO_{16}}$ |
|---------------------|-----------------------|-----------------|--------------------------|
| $[J_3^2, I_3]$      | $[J_3^4, I_4]$        | 22              | 76                       |
| $[J_3, J_2^2, I_2]$ | $[J_3^2, J_2^4, I_2]$ | 20              | 70                       |
| $[J_3, I_6]$        | $[J_2^8]$             | 14              | 56                       |
| $[J_2^4, I_1]$      | $[J_3, J_2^4, I_5]$   | 16              | 60                       |
| $[J_2^2, I_5]$      | $[J_2^4, I_8]$        | 12              | 44                       |

and the reader can readily check that the subsequent bounds on  $|x^G \cap H|$  and  $|x^G|$  from [5, 3.18] are always sufficient. The argument for semisimple elements is straightforward.  $\square$

**Lemma 2.9.** *The conclusion to Theorem 1.1 holds for the remaining embeddings in  $\mathcal{B}$ .*

**Proof.** We begin with  $(\mathcal{B}1)$ . Note that we may assume  $d \geq 5$  since  $\rho$  is an isomorphism when  $d = 3$ , while  $H_0 \cong P\Omega_6^+(q)$  is a  $\mathcal{C}_3$ -subgroup if  $d = 4$  (see [5, 8.2]). Let  $x \in H$  be an element of prime order  $r$  and observe that  $H \cap PGL(V) \leq PGL_d(q)$ . If  $x \in H - PGL(V)$  then [5, 3.49] implies that the trivial bound  $|x^G \cap H| < |H| < 2 \log_2 q \cdot q^{d^2-1}$  is always sufficient so let us assume  $x \in H \cap PGL(V)$ . If  $y \in H$  is a long root element (with respect to  $H_0$ ) then  $\nu(y) = d - 2$  (with respect to  $V$ ) and thus [7, 2.8] implies that  $\nu(x) \geq d - 2$  if  $r = p$ . Moreover, [5, 3.22] gives  $|x^G| > \frac{1}{2}q^{d^3-5d^2+10d-8}$  and the result follows since  $|x^G \cap H| < q^{d^2-1}$ . Similarly, if  $x$  is semisimple then  $\nu(x) \geq d - 1$  (minimal if  $\nu(x) = 1$  with respect to the natural  $H_0$ -module) and this time the result follows via [5, 3.36].

The other cases are just as easy. For example, consider  $(\mathcal{B}7)$ , so  $(G, H) = (SL_{11}(2), M_{24})$ . If  $x \in H$  has odd prime order then the values of the corresponding Brauer character (see [16, p. 267]) reveal that  $\nu(x) \geq 6$ , whence  $|x^G| > 2^{65}$  and the bound  $|x^G \cap H| < |M_{24}|$  is

sufficient. On the other hand, if  $x$  is an involution then Theorem 2.4 gives  $v(x) \geq 3$ , hence  $|x^G| > 2^{47}$  and the result follows since  $i_2(M_{24}) = 43\,263$ . Similarly, for  $(\mathcal{B}8)$  we have  $(G, H) = (\Omega_{24}^+(2), Co_1)$  and  $v(x) \geq 5$  for all non-trivial elements  $x \in H$  (see [12, Table 1]). In particular, if  $x$  has odd prime order  $r$  then  $v(x) \geq 6$ , hence  $|x^G| > \frac{1}{3}2^{102}$  and the bound  $|x^G \cap H| \leq i_r(Co_1)$  is always sufficient. The remaining cases are left to the reader.  $\square$

This completes the proof of Proposition 2.6.  $\square$

2.3. The  $\mathcal{C}$  collection

**Proposition 2.10.** *The conclusion to Theorem 1.1 holds for the collection  $\mathcal{C}$ .*

**Proof.** Let  $H < G$  be an inclusion in the collection  $\mathcal{C}$ . As before, we partition the proof into a number of separate lemmas. We begin by assuming  $H_0$  is a simple group of Lie type in defining characteristic; these are the inclusions labeled  $(\mathcal{C}1)$ – $(\mathcal{C}5)$  in Table 2.3. Note that we have already considered  $(\mathcal{C}2)$  in Lemma 2.7.

**Lemma 2.11.** *The conclusion to Theorem 1.1 holds for  $(\mathcal{C}1)$ .*

**Proof.** Let  $\rho: \text{SL}_3^\epsilon(q) \rightarrow \text{SL}_6^\epsilon(q)$  be the corresponding irreducible representation and note that  $q$  is odd (see Table 2.3). First assume  $x \in H \cap \text{PGL}(V) \leq \text{PGL}_3^\epsilon(q)$  has prime order  $r$ . Now, if  $v(x) \geq 3$  then [5, 3.38] implies that  $|x^G| > \frac{1}{2}(q + 1)^{-2}q^{19}$  and in this case the trivial bound  $|x^G \cap H| < q^8$  is always sufficient. A straightforward calculation reveals that  $\rho$  sends a long root element to the class containing  $[J_3, J_2, I_1]$  and thus [7, 2.8] implies that  $v(x) \geq 3$  if  $r = p$ . In fact, it is easy to see that  $v(x) \geq 3$  if  $r$  is odd. On the other hand, if  $r = 2$  then  $\rho(x)$  is conjugate to  $[-I_2, I_4]$  and the result follows since  $|x^G \cap H| < 2q^4$  and  $|x^G| > \frac{3}{8}q^{16}$ .

Now assume  $x \in H - \text{PGL}(V)$ . If  $x$  is a field automorphism of prime order  $r$  then  $q = q_0^r$  and the desired result follows since  $|x^G \cap H| < |H| < 2 \log_3 q \cdot q^8$  and  $|x^G| > \frac{1}{12}q^{35/2}$ . The same bounds apply if  $\epsilon = +$ ,  $q = q_0^2$  and  $x$  is an involutory graph-field automorphism. Finally, if  $x$  is an involutory graph automorphism then

$$|x^G \cap H| \leq \frac{|\text{PGL}_3^\epsilon(q)|}{|\text{SO}_3(q)|} < 2q^5, \quad |x^G| \geq \frac{|\text{PSL}_6^\epsilon(q)|}{|\text{Sp}_6(q)|} > \frac{1}{2}(q + 1)^{-1}q^{14}$$

(see [5, Table 3.10]) and the result follows.  $\square$

**Lemma 2.12.** *The conclusion to Theorem 1.1 holds for  $(\mathcal{C}3)$ .*

**Proof.** Here  $q = q_0^3$  and  $H_0 = C_{G_0}(\psi)$ , where  $\psi \in \text{Aut}(G_0)$  is a triality graph-field automorphism (see the proof of [19, 2.3.4]). According to [18, Table 1], the maximality of  $H$  in  $G$  implies that  $G \cap \text{PGL}(V) = G_0$ . First assume  $x \in H - \text{PGL}(V)$  has prime order  $r$ . If  $r > 3$  then  $x$  is a field automorphism of  $G_0$ ,  $q_0 = q_1^r$  and the result follows since

$$|x^G \cap H| \leq |{}^3D_4(q_0) : {}^3D_4(q_0^{1/r})| < 2q_0^{28(1-\frac{1}{r})}, \quad |x^G| > \frac{1}{4}q_0^{84(1-\frac{1}{r})}.$$

If  $r = 2$  or  $3$  then [21, 1.3] gives

$$|x^G \cap H| \leq i_r(\text{Aut}({}^3D_4(q_0))) < 2(q_0 + 1)q_0^{15+4\delta_{3,r}} \tag{4}$$

and the desired result follows since [5, 3.49] states that  $|x^G| > \frac{1}{8}q_0^{42}$ .

Now suppose  $x \in H \cap \text{PGL}(V)$  has prime order  $r$ . If  $r = p > 2$  and  $x$  has associated partition  $\lambda = (2^2, 1^4)$  then  $x$  lies in the  $H_0$ -class labeled  $A_1$  in [26]. Moreover, from [26, p. 677] we deduce that

$$|x^G \cap H| = (q_0^2 - 1)(q_0^8 + q_0^4 + 1), \quad |x^G| = (q_0^6 + 1)^2(q_0^{18} - 1)$$

and thus  $f(x, H) < 1/3$  for all  $q_0 \geq 3$ . On the other hand, if  $\lambda \neq (2^2, 1^4)$  then [5, 3.55(i)] implies that  $|x^G| > \frac{1}{8}q_0^{48}$  (minimal if  $\lambda = (3, 2^2, 1)$ ) and the desired result follows since  $|x^G \cap H| < q_0^{28}$ . Next suppose  $r = p = 2$ . Since  $x$  is centralized by  $\psi$ , it follows that  $x$  is  $G_0$ -conjugate to  $a_2$  or  $c_4$  (see [5, 3.55(ii)]), whence  $|x^G| > \frac{1}{2}q_0^{30}$  and (4) implies that  $f(x, H) < 0.607$ . Similarly, if  $x$  is a semisimple involution then  $C_G(x)$  is of type  $O_4^+(q)^2$ , so  $|x^G| > \frac{1}{8}q_0^{48}$  and (4) is always sufficient. Finally, if  $x$  is a semisimple element of odd prime order then  $v(x) \geq 4$  since no element with  $v(x) = 2$  is fixed by  $\psi$  (see [5, 3.55(iv)]). For the same reason,  $C_{\bar{G}}(x)$  is not of type  $GL_4$ , whence  $\dim x^{\bar{G}} \geq 18$  and the subsequent bounds

$$|x^G \cap H| < |{}^3D_4(q_0)| < q_0^{28}, \quad |x^G| > \frac{1}{2}(q_0^3 + 1)^{-1}q_0^{57}$$

are always sufficient.  $\square$

**Lemma 2.13.** *The conclusion to Theorem 1.1 holds for (C4) and (C5).*

**Proof.** First observe that if  $p = 3$  then  $H_0 = G_2(q)'$  admits an involutory graph automorphism  $\tau$  which interchanges the two 7-dimensional  $G_2(q)$ -modules  $M(\lambda_1)$  and  $M(\lambda_2)$ . Therefore we need only consider the embedding (C4) and thus  $n = 7 - \delta_{2,p}$ . We claim that

$$f(x, H) < 0.750^* \tag{5}$$

for all elements  $x \in H$  of prime order, so this case is included in Table 1.1.

If  $x \in H - \text{PGL}(V)$  has prime order  $r$  then  $x$  is a field automorphism of  $G_0$  and  $q = q_0^r$ . Applying [5, 3.43] we deduce that

$$|x^G \cap H| \leq |G_2(q) : G_2(q_0)| < 2q^{14(1-\frac{1}{r})},$$

$$|x^G| \geq (2, q - 1)^{-1} |\text{Sp}_6(q) : \text{Sp}_6(q_0)| > \frac{1}{4}q^{21(1-\frac{1}{r})}$$

and one can check that these bounds are sufficient unless  $(r, q) = (2, 4)$ , where direct calculation yields  $f(x, H) < 0.670^*$ . For the remainder we may assume that  $x \in H \cap \text{PGL}(V) = G_2(q)$  has prime order  $r$ .

Table 2.4  
( $\mathcal{C}4$ ),  $r = p > 3$

| $G_2$ -class  | $\lambda$    | $ x^G \cap H  \leq$     | $ x^G  \geq$                                | $f(x, H) <$ |
|---------------|--------------|-------------------------|---|-------------|
| $A_1$         | $(2^2, 1^3)$ | $q^6 - 1$               | $(q^2 + 1)(q^6 - 1)$                        | 0.750*      |
| $\tilde{A}_1$ | $(3, 2^2)$   | $q^2(q^6 - 1)$          | $\frac{1}{2}q^2(q^4 - 1)(q^6 - 1)$          | 0.692*      |
| $G_2(a_1)$    | $(3^2, 1)$   | $q^2(q^2 - 1)(q^6 - 1)$ | $\frac{1}{2}q^3(q - 1)(q^4 - 1)(q^6 - 1)$   | 0.743*      |
| $G_2^\dagger$ | $(7)$        | $q^4(q^2 - 1)(q^6 - 1)$ | $\frac{1}{2}q^6(q^2 - 1)(q^4 - 1)(q^6 - 1)$ | 0.680*      |

Table 2.5  
( $\mathcal{C}4$ ),  $r = p \leq 3$

| $p$ | $G_2$ -class                     | $G$ -class   | $ x^G \cap H  \leq$                | $ x^G  \geq$                              | $f(x, H) <$ |
|-----|----------------------------------|--------------|------------------------------------|---|-------------|
| 3   | $A_1$                            | $(2^2, 1^3)$ | $q^6 - 1$                          | $(q^2 + 1)(q^6 - 1)$                      | 0.750*      |
|     | $\tilde{A}_1, \tilde{A}_1^{(3)}$ | $(3, 2^2)$   | $q^2(q^6 - 1)$                     | $\frac{1}{2}q^2(q^4 - 1)(q^6 - 1)$        | 0.705*      |
|     | $G_2(a_1)$                       | $(3^2, 1)$   | $\frac{1}{2}q^2(q^2 - 1)(q^6 - 1)$ | $\frac{1}{2}q^3(q - 1)(q^4 - 1)(q^6 - 1)$ | 0.714*      |
| 2   | $A_1$                            | $a_2$        | $q^6 - 1$                          | $(q^2 + 1)(q^6 - 1)$                      | 0.750*      |
|     | $\tilde{A}_1$                    | $b_3$        | $q^2(q^6 - 1)$                     | $q^2(q^4 - 1)(q^6 - 1)$                   | 0.672*      |

**Case 1.**  $r = p$ .

In [20, Table 1], Lawther gives the Jordan normal form on  $\bar{V} = V \otimes K$  of a representative of each unipotent class in  $G_2$  (the algebraic group). For  $p > 3$ , the size of each unipotent class in  $G_2(q)$  is given in [15, p. 158] and we derive the results in Table 2.4. Here  $\lambda$  denotes the associated partition of  $x$  with respect to  $\bar{V}$  and we adopt Lawther’s notation for labeling the unipotent classes in the algebraic group  $G_2$ . The symbol  $\dagger$  appearing in the final row denotes the additional condition  $p \geq 7$  which ensures  $x$  has prime order.

Now assume  $p \leq 3$ . Here detailed information on the unipotent classes in  $G_2(q)$  is given by Enomoto in [9]. In particular, centralizer orders for unipotent elements are listed in [9, Tables 1–2] and using this data, together with [20, Table 1], we derive the results listed in Table 2.5. The fourth row in Table 2.5 is worth noting. Here  $p = 3$  and there are precisely two distinct  $G_2(q)$ -classes which correspond to the  $G_2$ -class  $G_2(a_1)$ , with representatives  $x_+$  and  $x_-$ , where  $|C_{G_2(q)}(x_\epsilon)| = 2q^4$  and

$$|x_\epsilon^{\Omega_7(q)}| = \frac{|\mathcal{O}_7(q)|}{|\mathcal{O}_2^\epsilon(q)||\mathcal{O}_1(q)|q^6} = \frac{1}{2}q^3(q - \epsilon)(q^4 - 1)(q^6 - 1),$$

i.e. the elements  $x_+$  and  $x_-$  represent the two distinct  $\Omega_7(q)$ -classes with associated partition  $\lambda = (3^2, 1)$ . The entries in the fourth row follow immediately. As a final remark, we note that (5) is best possible since

$$\lim_{q \rightarrow \infty} \frac{\log[q^6 - 1]}{\log[(q^2 + 1)(q^6 - 1)]} = \frac{3}{4}.$$

Table 2.6  
( $\mathcal{C}4$ ),  $r \neq p$

| $q$ | $G_2(q)$ -class of $x$ | $f(x, H) <$ |
|-----|------------------------|-------------|
| 4   | 3B                     | 0.717*      |
|     | 5A, 5B                 | 0.719*      |
|     | 13A, 13B               | 0.684*      |
| 3   | 2A                     | 0.701*      |
|     | 7A                     | 0.680*      |
|     | 13A, 13B               | 0.646*      |
| 2   | 3B                     | 0.685*      |

**Case 2.**  $r \neq p$ .

If  $q < 5$  then the values of the associated Brauer character are listed in [16] and we can compute  $f(x, H)$  precisely. Indeed, the reader can check that  $f(x, H) < 1/2 + 1/n$  with the exception of the cases listed in Table 2.6.

Now assume  $q \geq 5$ . Here we follow the proof of [7, 7.7]. By replacing  $x$  with a suitable conjugate we may assume that  $x \in A_{2.2} < G_2$  (as algebraic groups over  $K = \overline{\mathbb{F}}_q$ ) where  $A_2$  is generated by long root subgroups and

$$V \downarrow A_2 = \begin{cases} V_3 \oplus V_3^* & \text{if } p = 2, \\ V_3 \oplus V_3^* \oplus 0 & \text{if } p \neq 2 \end{cases} \tag{6}$$

where  $V_3$  and  $0$  denote the natural and trivial  $A_2$ -modules, respectively. First assume  $r$  is odd. If  $x \in G_0$  is a regular semisimple element then (6) implies that  $x$  is also regular as an element of  $G_2(q)$  and thus  $f(x, H) \leq \log_b a$ , where

$$a = f \cdot \frac{|G_2(q)|}{(q-1)^2}, \quad b = f \cdot \frac{|\mathrm{Sp}_6(q)|}{(q+1)^3}$$

and  $f = \log_p q$ . This yields  $f(x, H) < 0.704^*$  for all  $q \geq 5$ . On the other hand, if  $x$  is not regular then it is easy to see that  $f(x, H)$  is maximal if  $x = [\mu, \mu^{-1}, 1] \in A_2$  for some  $1 \neq \mu \in K$ . Here  $C_{G_2}(x) = A_1 T_1$  and we calculate that  $f(x, H) < 0.732^*$  for all  $q \geq 5$  since  $f(x, H) \leq \log_\beta \alpha$ , where

$$\alpha = f \cdot \frac{|G_2(q)|}{|\mathrm{GL}_2(q)|}, \quad \beta = f \cdot \frac{|\mathrm{Sp}_6(q)|}{|\mathrm{Sp}_2(q)||\mathrm{GU}_2(q)|}.$$

Finally, suppose  $x$  is a semisimple involution. Now, there is a unique class of involutions in  $G_2(q)$  and (6) implies that  $x$  acts on  $V$  as  $[-I_4, I_3]$ . Therefore

$$|x^G \cap H| = q^4(q^4 + q^2 + 1), \quad |x^G| = \frac{|\mathrm{O}_7(q)|}{|\mathrm{O}_4^+(q)||\mathrm{O}_3(q)|} = \frac{1}{2}q^6(q^2 + 1)(q^4 + q^2 + 1)$$

(since  $x \in G_0$ ) and we conclude that  $f(x, H) < 0.691^*$  for all  $q \geq 5$ .  $\square$

To complete the proof of Proposition 2.10, let us assume  $H_0$  is not a simple group of Lie type in defining characteristic. These are the cases labeled ( $\mathcal{C}6$ )–( $\mathcal{C}22$ ) in Table 2.3.

**Lemma 2.14.** *The conclusion to Theorem 1.1 holds for (C19).*

**Proof.** Here  $H_0 = \Omega_8^+(2)$ ,  $G_0 = P\Omega_8^+(p)$  and the embedding  $\rho : H \rightarrow G$  corresponds to the reduction modulo  $p$  of the complex 8-dimensional representation of  $H_0$  which arises from the natural action of the Weyl group  $W(E_8) = 2.\Omega_8^+(2).2$  on a maximal torus  $T_8 < E_8$ , where the algebraic group  $E_8$  is defined over the complex numbers. Now  $H \cap PGL(V) \leq O_8^+(2)$  and we claim that

$$f(x, H) < \begin{cases} 0.706^* & \text{if } p = 3, \\ 0.500 & \text{if } p \geq 5 \end{cases} \tag{7}$$

for all elements  $x \in H$  of prime order. In particular, the case  $p = 3$  is an exception to the main statement of Theorem 1.1 and is therefore listed in Table 1.1. To justify (7), we first assume  $x \in H \cap PGL(V)$  has prime order  $r$ . Throughout,  $K$  denotes the algebraic closure of  $\mathbb{F}_p$  and  $\bar{G} = \text{PSO}_8(K)$ .

**Case 1.**  $x \in H \cap PGL(V)$ ,  $r \neq p$ .

For semisimple elements, we use the values of the associated Brauer character  $\chi$  to compute  $f(x, H)$  precisely. These values are available in [16] when  $p \leq 7$  and in [8] for  $p > 7$ . When  $p = 3$  we obtain the results displayed in Table 2.7. Here we adopt the standard Atlas notation for labeling class representatives (see [8]) and we set  $\zeta = 1$  if  $G$  contains a triality automorphism, otherwise  $\zeta = 0$ .

The third row of Table 2.7 merits some explanation. The notation here indicates that if  $x \in H_0$  is  $H_0$ -conjugate to  $2B$  (respectively  $2C$ ) then  $\rho(x)$  is  $G_0$ -conjugate to  $2A$  (respectively  $2B$ ) and so on. In particular, if  $x \in \{2B, 2C, 2D\}$  then  $f(x, H) < 0.678^*$  if  $\zeta = 0$ , otherwise  $f(x, H) < 0.705^*$ . Similar notation applies if  $x \in \{5A, 5B, 5C\}$ . We conclude that (7) holds when  $p = 3$ . For  $p \geq 5$ , we can do entirely similar calculations and the reader can check that  $f(x, H) < 1/2$ . In particular, we note that if  $p \geq 5$  and  $h$  is in the  $H_0$ -class  $3E$  then  $\chi(\rho(h)) = 2$  and thus  $\rho(h)$  is  $\bar{G}$ -conjugate to  $[I_4, \omega I_2, \omega^2 I_2]$ , where  $\omega \in K$  is a primitive cube root of unity.

**Case 2.**  $x \in H \cap PGL(V)$ ,  $r = p$ .

Here  $p \in \{3, 5, 7\}$  and we derive the results presented in Table 2.8. In the table,  $\lambda$  denotes the associated partition of  $x \in G$  and  $\zeta$  is defined as before. Note that we only list those partitions  $\lambda$  for which  $x^G \cap H$  is non-empty. We now explain how we obtain the results in Table 2.8. First observe that a triality graph automorphism  $\tau$  of  $G_0$  induces a triality automorphism on  $H_0$ . If  $x$

Table 2.7  
(C19),  $r \neq p$ ,  $p = 3$

| $H_0$ -class of $x$ | $G_0$ -class of $x$ | $ x^G \cap H $           | $ x^G $                          | $f(x, H) <$ |          |
|---------------------|---------------------|--------------------------|----------------------------------|-------------|----------|
| $2A, 2E$            | $2D$                | 58 275                   | 14 926 275                       | 0.665*      |          |
| $(2B, 2C, 2D)$      | $(2A, 2B, 2C)$      | $3^\zeta \cdot 3780$     | $3^\zeta \cdot 189\,540$         | 0.678*      | (0.705*) |
| $2F$                | $2E$                | 120                      | 1080                             | 0.686*      |          |
| $2G$                | $2F$                | 37 800                   | 7 960 680                        | 0.664*      |          |
| $(5A, 5B, 5C)$      | $(5A, 5B, 5C)$      | $3^\zeta \cdot 580\,608$ | $3^\zeta \cdot 2\,751\,211\,008$ | 0.611       | (0.630*) |
| $7A$                | $7A$                | 24 883 200               | 176 863 564 800                  | 0.658*      |          |



Table 2.8  
(C19),  $r = p$

| $p$ | $\lambda$         | $ x^G \cap H $      | $ x^G  \geq$                | $f(x, H) <$ |          |
|-----|-------------------|---------------------|-----------------------------|-------------|----------|
| 3   | $(2^4), (3, 1^5)$ | $3^{\zeta} .2240$   | $3^{\zeta} .262080$         | 0.619       | (0.650*) |
|     | $(3, 2^2, 1)$     | 89 600              | 10 483 200                  | 0.706*      |          |
|     | $(3^2, 1^2)$      | 268 800             | 377 395 200                 | 0.634*      |          |
| 5   | $(4^2), (5, 1^3)$ | $3^{\zeta} .580608$ | $3^{\zeta} .47528208000000$ | 0.422       | (0.441)  |
| 7   | $(7, 1)$          | 24 883 200          | 93 756 760 664 555 520 000  | 0.371       |          |

is  $H_0$ -conjugate to  $3A$  then  $x \in A_9 < \Omega_8^+(2)$  has cycle-shape  $(3, 1^6)$  (see (A4) in Table 2.1). Without loss, we may assume that the restriction of  $\rho$  to  $A_9$  factors through  $\Omega_7(3)$  as follows

$$\rho : A_9 \xrightarrow{\rho_1} \Omega_7(3) \xrightarrow{\rho_2} \text{P}\Omega_8^+(3), \tag{8}$$

where  $\rho_1$  is the irreducible representation afforded by the fully deleted permutation module for  $A_9$  over  $\mathbb{F}_3$  (see (A1) in Table 2.1) and  $\rho_2$  is the restriction of a spin representation of  $\text{P}\Omega_8^+(3)$  (see (B2) in Table 2.2). From the proofs of Proposition 2.5 and Lemma 2.7 we deduce that  $\rho(x)$  acts on  $V$  with associated partition  $\lambda = (2^4)$  and thus [5, 3.55(i)] implies that  $\rho(x)$  and  $\rho(x)^\tau$  belong to distinct  $G_0$ -classes. The three  $H_0$ -classes  $\{3A, 3B, 3C\}$  are permuted by a triality automorphism of  $H_0$  (see [5, 3.55(iv)]) and therefore they are fused in  $G$  if and only if  $G$  contains a triality automorphism. This explains the entries in the second row and the case  $p = 5$  is entirely similar. The entries in rows 3 and 4 are also derived via (8). Finally, if  $p = 7$  then  $x \in A_9 < \Omega_8^+(2)$  has cycle-shape  $(7, 1^2)$  and we may assume that the restriction of  $\rho$  to  $A_9$  is the irreducible representation afforded by the fully deleted permutation module for  $A_9$  over  $\mathbb{F}_7$ . Then the proof of Proposition 2.5 implies that  $\lambda = (7, 1)$  and the entries in the final row follow at once.

**Case 3.**  $x \in H - \text{PGL}(V)$ .

Here  $x$  is a triality graph automorphism since  $G_0$  is defined over the prime field. As previously remarked,  $x$  induces a triality graph automorphism on  $H_0$ . Now, if we assume  $p > 7$  then the obvious bounds (see [5, Table 3.10])

$$|x^G \cap H| \leq 2|\Omega_8^+(2) : G_2(2)| + 2|\Omega_8^+(2) : \text{PGU}_3(2)| = 1\,641\,600, \quad |x^G| > \frac{1}{8}p^{14}$$

are always sufficient. Next assume  $p \in \{5, 7\}$ . Let  $\tau$  be a  $G_2$ -type triality automorphism of  $H_0$  (see [5, 3.47]) and observe that  $C_{G_0}(\tau) = G_2(p)$  by Lagrange’s Theorem. According to [11, p. 215], the four distinct  $H_0$ -classes of triality automorphisms in  $\text{Aut}(H_0)$  are represented by the elements  $\tau^\pm$  and  $(h\tau)^\pm$ , where  $h \in H_0$  lies in the  $H_0$ -class  $3E$  and  $[h, \tau] = 1$ . As remarked in Case 1,  $\rho(h)$  is  $\tilde{G}$ -conjugate to  $[I_4, \omega I_2, \omega^2 I_2]$ , so  $\rho(h)$  lies in the  $G_0$ -class  $3E$  and thus  $h\tau$  is a non- $G_2$  triality of  $G_0$ . We conclude that the centralizers  $C_{H_0}(x)$  and  $C_{G_0}(x)$  are of the same type. In particular, if  $x$  is a  $G_2$ -type triality then  $f(x, H) < 0.486$  since

$$|x^G \cap H| = 2^\xi |\Omega_8^+(2) : G_2(2)| = 2^\xi .14\,400, \quad |x^G| > 2^{\xi-3} p^{14}$$

(see [5, 3.48]) where  $\xi = 1$  if  $G$  contains an involutory graph automorphism, otherwise  $\xi = 0$ . Similarly, one can check that  $f(x, H) < 1/2$  if  $x$  is a non- $G_2$  triality. Finally, let us assume  $p = 3$ .

Now, if  $y \in x^G \cap H$  and  $C_{H_0}(y) = G_2(2)$  then  $C_{G_0}(y) = G_2(3)$  since  $|G_2(2)| > 3^5 |\text{SL}_2(3)|$ . Furthermore, if  $C_{H_0}(y) = \text{PGU}_3(2)$  and  $C_{G_0}(y) = G_2(3)$  then  $|x^{G_0} \cap H_0| \geq |\Omega_8^+(2) : \text{PGU}_3(2)| = 806\,400$  and thus  $\text{fpr}(x) > 0.691$  since  $|x^{G_0}| = |\text{P}\Omega_8^+(3) : G_2(3)| = 1\,166\,400$ . This contradicts [23, Theorem 1] and thus  $C_{H_0}(x)$  and  $C_{G_0}(x)$  are of the same type. In particular, if  $x$  is a  $G_2$ -type triality then  $f(x, H) < 0.701^*$  since  $|x^G \cap H| = 2^\xi \cdot 14\,400$  and  $|x^G| = 2^\xi \cdot 1\,166\,400$ . Similarly, if  $x$  is a non- $G_2$  triality then  $f(x, H) < 0.673^*$ .  $\square$

**Lemma 2.15.** *The conclusion to Theorem 1.1 holds for (C18).*

**Proof.** Here  $H = \text{Sp}_6(2) \leq \text{PGL}(V)$  and  $\rho : \text{Sp}_6(2) \rightarrow \Omega_7(p)$  is the restriction of the map in (C19). More precisely,  $\rho$  factors through  $\Omega_8^+(2)$  as follows:

$$\rho : \text{Sp}_6(2) \xrightarrow{\rho_1} \Omega_8^+(2) \xrightarrow{\rho_2} \text{P}\Omega_8^+(p), \tag{9}$$

where  $\rho_1$  is the restriction of a spin representation (see (B2) in Table 2.2) and  $\rho_2$  is the embedding (C19). Let  $x \in H$  be an element of prime order  $r$ . We claim that

$$f(x, H) < \begin{cases} 0.707^* & \text{if } p = 3, \\ 0.500 & \text{if } p \geq 5 \end{cases}$$

and thus the case  $p = 3$  appears in Table 1.1.

Let  $x \in H$  be an element of prime order  $r$  and let  $\chi$  be the associated Brauer character. Since  $\chi$  is given in [8,16], we can compute  $f(x, H)$  precisely when  $r \neq p$ . For example, when  $p = 3$  we derive the following results.

| $\text{Sp}_6(2)$ -class of $x$ | $\Omega_7(3)$ -class of $x$ | $ x^G \cap H $ | $ x^G $     | $f(x, H) <$ |
|--------------------------------|-----------------------------|----------------|-------------|-------------|
| 2A                             | 2A                          | 63             | 351         | 0.707*      |
| 2B, 2D                         | 2C                          | 4095           | 331 695     | 0.655*      |
| 2C                             | 2B                          | 945            | 22 113      | 0.685*      |
| 5A                             | 5A                          | 48 384         | 38 211 264  | 0.618       |
| 7A                             | 7A                          | 207 360        | 327 525 120 | 0.625       |

We can do entirely similar calculations when  $p \geq 5$  and the bound  $f(x, H) < 1/2$  quickly follows. Now assume  $r = p$ . Here  $r \in \{3, 5, 7\}$  and in view of (9) and our earlier work we derive the following results, where  $\lambda$  denotes the associated partition of  $\rho(x) \in G$ .

| $p$ | $\text{Sp}_6(2)$ -class of $x$ | $\lambda$  | $ x^G \cap H $ | $ x^G  \geq$        | $f(x, H) <$ |
|-----|--------------------------------|------------|----------------|---------------------|-------------|
| 3   | 3A                             | $(3, 1^4)$ | 672            | 26 208              | 0.640       |
|     | 3B                             | $(3, 2^2)$ | 2240           | 262 080             | 0.619       |
|     | 3C                             | $(3^2, 1)$ | 13 440         | 1 572 480           | 0.667*      |
| 5   | 5A                             | $(5, 1^2)$ | 48 384         | 30 466 800 000      | 0.447       |
| 7   | 7A                             | $(7)$      | 207 360        | 797 251 366 195 200 | 0.357       |

For example, suppose  $x$  is  $\text{Sp}_6(2)$ -conjugate to 3A, i.e.  $x = [I_4, \omega, \omega^2]$ , where  $\omega \in K$  is a primitive cube root of unity. Then the proof of Lemma 2.7 implies that  $y = \rho_1(x)$  is  $\text{O}_8^+(2)$ -conjugate to  $[\omega I_4, \omega^2 I_4]$ , the proof of Lemma 2.14 gives  $\rho_2(y) = [J_3, I_5]$  and thus  $\lambda = (3, 1^4)$ . The other results are obtained in a similar fashion.  $\square$

**Lemma 2.16.** *The conclusion to Theorem 1.1 holds for (C16) and (C21).*

**Proof.** Both of these cases can be analysed using GAP [10]. For the embedding (C16) we find that  $f(x, H) \leq (\log 666)/(\log 6336) < 0.743^*$ , with equality possible if  $x$  lies in the  $G_0$ -class  $2D$ . Therefore this case is recorded in Table 1.1. For (C21) we calculate that  $f(x, H) < 0.663$  for all elements  $x \in H$  of prime order.  $\square$

**Lemma 2.17.** *The conclusion to Theorem 1.1 holds for (C11).*

**Proof.** Here  $H_0 = \text{SU}_3(3)$  and  $G_0 = \text{PSP}_6(p)$  with  $p \neq 3$ . If  $p = 2$  then using GAP [10] we deduce that  $f(x, H) \leq (\log 63)/(\log 315) < 0.721^*$  for all elements  $x$  of prime order, with equality if  $x$  is  $G$ -conjugate to  $a_2$ . In particular, the case  $p = 2$  is listed in Table 1.1. Now assume  $p \geq 5$  and let  $x \in H$  be an element of prime order  $r$ . We claim that  $f(x, H) < 1/2$ . If  $x \in H_0$  is an involution then  $x$  is  $G$ -conjugate to  $[-I_4, I_2]$ , whence  $|x^G \cap H| = 63$ ,  $|x^G| = p^4(p^4 + p^2 + 1)$  and thus  $f(x, H) < 0.321$ . The values of the corresponding Brauer character imply that  $\nu(x) \geq 3$  if  $r \neq p$  is odd, so [5, 3.36] gives  $|x^G| > \frac{1}{4}(p + 1)^{-1}p^{13}$  and the bound  $|x^G \cap H| \leq i_r(H)$  is sufficient. Finally, if  $r = p$  then  $p = 7$  and we deduce that  $f(x, H) < 0.401$  since  $|x^G \cap H| \leq i_7(H) = 1728$  and

$$|x^G| \geq \frac{|\text{Sp}_6(7)|}{|\text{Sp}_2(7)||\text{O}_2^-(7)|7^7} = 123\,530\,400$$

since  $x$  is not a long root element by [17, Theorem II].  $\square$

**Lemma 2.18.** *The conclusion to Theorem 1.1 holds for the remaining embeddings in C.*

**Proof.** In each of the remaining cases we claim that

$$f(x, H) < \frac{1}{2} + \frac{1}{n} \tag{10}$$

for all elements  $x \in G$  of prime order, where  $n$  is defined as in Remark 1.2. Let  $V$  denote the natural  $G_0$ -module, write  $\chi$  for the corresponding Brauer character and let  $x \in H$  be an element of prime order  $r$ . If  $x \in H - \text{PGL}(V)$  then the reader can check that the bound  $|x^G \cap H| \leq i_r(\text{Aut}(H_0) - H_0)$ , with the lower bound on  $|x^G|$  from [5, 3.49], is always sufficient.

Now suppose  $x \in H \cap \text{PGL}(V)$ . If  $r \neq p$  then  $\chi(x)$  is listed in [8,16] and (10) is easily checked. Now assume  $r = p$  and let  $\lambda$  denote the associated partition of  $x$  with respect to  $V$ . By [17],  $x$  is not a long root element, i.e.  $\lambda \neq (2^2, 1^{n-4})$  if  $G_0$  is orthogonal, otherwise  $\lambda \neq (2, 1^{n-2})$ . From this observation we derive a lower bound for  $|x^G|$  and we find that the upper bound  $|x^G \cap H| \leq i_r(H \cap \text{PGL}(V))$  is always sufficient. For instance, for (C14) we have  $H_0 = \text{SU}_4(2)$ ,  $n = 4$  and the hypothesis  $r = p$  implies that  $p = 5$ . Moreover, since  $x$  is not a long root element, we have

$$|x^G| \geq \frac{|\text{O}_6^-(5)|}{|\text{O}_3(5)||\text{O}_1(5)|5^4} = 196\,560$$

and thus  $f(x, H) < 0.702$  since  $i_5(H) = 5184$ .  $\square$

This completes the proof of Proposition 2.10.  $\square$

2.4. The remaining cases

Now assume that the irreducible embedding  $H < G$  is not in  $\mathcal{A}$ ,  $\mathcal{B}$  or  $\mathcal{C}$ . Then Theorems 2.2 and 2.4 apply and we use work of Lübeck [25] and Hiss–Malle [14] to quickly reduce to a small collection of irreducible embeddings which we can deal with on a case-by-case basis. This is the collection  $\mathcal{D}$  (see Tables 2.10 and 2.11). Of course, Theorem 2.4 only applies if  $\dim V \geq 6$  and the remaining cases are considered in Proposition 2.22.

**Proposition 2.19.** *The conclusion of Theorem 1.1 holds if  $\dim V \geq 6$  and the inclusion  $H < G$  is not in one of the collections  $\mathcal{A}$ ,  $\mathcal{B}$  or  $\mathcal{C}$ .*

**Proof.** First observe that Theorem 2.2 implies that

$$|x^G \cap H| < |H| < \begin{cases} q^{4n+8} & \text{if } G_0 = \text{PSU}_n(q), \\ q^{2n+4} & \text{otherwise.} \end{cases} \tag{11}$$

If  $x \in H \cap \text{PGL}(V)$  has prime order then Theorem 2.4 and [5, 3.38] imply that  $|x^G| > g(n, q)$  for some function  $g$ . For example, if  $G_0 = \text{PSp}_n(q)$  then

$$g(n, q) = \frac{1}{8}(q + 1)^{-1} \max(q^{\alpha(n-\alpha)+1}, q^{3n-8}),$$

where  $\alpha = \lceil \frac{1}{2}\sqrt{n} \rceil + \beta$  and  $\beta = 1$  if  $n = 4m^2$  for some  $m \in \mathbb{N}$ , otherwise  $\beta = 0$ . Let  $N = N(G_0)$  be the smallest integer such that the bounds (11) and  $|x^G| > g(n, q)$  imply that  $f(x, H) < 1/2 + 1/n$  for all  $n \geq N$  and all values of  $q$ . Then  $N$  is given in Table 2.9. If  $x \in H - \text{PGL}(V)$  then lower bounds for  $|x^G|$  are derived in [5, 3.49] and it is easy to check that (11) is always sufficient if  $n \geq N(G_0)$ . For the remainder we may assume  $6 \leq n < N$ .

**Case 1.**  $H_0$  is a simple group of Lie type in defining characteristic.

Let  $\rho : \hat{H}_0 \rightarrow \text{GL}(V)$  be the absolutely irreducible representation corresponding to the embedding  $H < G$ , where  $\hat{H}_0$  is the full covering group of  $H_0$ . First assume  $H_0 = \text{PSL}_2(q')$ , where  $q'$  is a power of  $p$ . Then  $G_0$  is either symplectic or orthogonal, and [19, 5.4.6(i)] implies that  $q' = q^i$  and  $n = l^i \geq 2^i$  for some  $i \geq 1$ , where  $l$  is the dimension of an irreducible  $K\hat{H}_0$ -module and  $K$  is the algebraic closure of  $\mathbb{F}_q$ . Of course

$$|x^G \cap H| < |\text{Aut}(H_0)| \leq i \log_2 q \cdot q^i (q^{2^i} - 1) \tag{12}$$

Table 2.9  
The values  $N = N(G_0)$

| $G_0$ | $\text{PSL}_n(q)$ | $\text{PSU}_n(q)$ | $\text{PSp}_n(q)$ | $\text{P}\Omega_n^\epsilon(q)$ |
|-------|-------------------|-------------------|-------------------|--------------------------------|
| $N$   | 10                | 64                | 64                | 64                             |

and applying Theorem 2.4 and [5, 3.38, 3.49] we deduce that

$$|x^G| > \frac{1}{8}(q + 1)^{-1}q^{3n-11} \tag{13}$$

for all elements  $x \in H$  of prime order. One can check that these bounds are sufficient unless  $(i, n) \in \{(3, 8), (1, 6)\}$ . Suppose  $(i, n) = (3, 8)$ . If  $p = 2$  then  $G_0 = \Omega_8^+(q)$  and  $H$  is not maximal (see [18]) so we may assume  $p$  is odd and thus  $G_0 = \text{PSp}_8(q)$ . If  $x$  has odd prime order then the desired result follows by applying (12) since Theorem 2.4 gives  $\nu(x) \geq 3$  and thus  $|x^G| > \frac{1}{4}q^{18}$  (minimal if  $x$  is unipotent with associated partition  $\lambda = (2^3, 1^2)$ ). On the other hand, if  $x$  is an involution then [5, 3.37] gives  $|x^G| > \frac{1}{4}q^{16}$  and one can easily check that the bound

$$|x^G \cap H| \leq i_2(\text{Aut}(\text{PSL}_2(q^3))) < 2(1 + q^{-3})q^6$$

(see [5, 3.14]) is always sufficient. Now assume  $(i, n) = (1, 6)$ . Here  $p$  must be odd since  $n$  is divisible by 3. Now, if  $G_0 = \text{PSp}_6(q)$  then  $|x^G| > \frac{1}{4}(q + 1)^{-1}q^{13}$  and (12) is sufficient. If  $G_0$  is orthogonal then we require  $f(x, H) < 3/4$  (see Remark 1.2) and thus the above bounds (12) and (13) are sufficient for all  $q \geq 7$ . If  $q < 7$  then  $H_0 \cong A_5$  since  $\text{PSL}_2(3)$  is not simple. However, we can rule out this case since neither  $A_5$  nor  $2.A_5$  admit an irreducible representation of degree 6 in characteristic 2 or 5.

Now assume  $H_0 \neq \text{PSL}_2(q')$ . Here we apply Lübeck’s work [25, Tables A.6-48] on the small degree irreducible representations of simple groups of Lie type. To illustrate the method, suppose  $H_0 = \text{PSL}_m(q')$ , where  $m \geq 3$ . Then [19, 5.4.6(i)] implies that  $q' = q^i$  and  $n = l^{i/u} \geq m^{i/u}$  for some integer  $i \geq 1$ , where  $l$  is the dimension of an irreducible  $K\hat{H}_0$ -module and  $u = 2$  if  $G_0$  is unitary, otherwise  $u = 1$ . In particular,  $i$  is even if  $G_0$  is unitary and we have

$$|x^G \cap H| < |\text{Aut}(H_0)| < 2i \log_2 q \cdot q^{i(m^2-1)}. \tag{14}$$

Now, if  $\rho$  is self-dual then  $G_0$  is symplectic or orthogonal, so (13) holds and we may assume  $n < N(G_0) = 64$ . Inspecting [25] we deduce that (14) is sufficient unless  $\rho$  is one of the embeddings labeled  $(\mathcal{D}1)$ ,  $(\mathcal{D}2)$  or  $(\mathcal{D}4)$  in Table 2.10. If  $\rho$  is not self-dual and  $G_0 = \text{PSL}_n(q)$  then we may assume  $n < 10$  and we find that (14) is always sufficient since Theorem 2.4 and [5, 3.38] together imply that  $|x^G| > \frac{1}{2}q^{6n-19}$ . Finally, if  $\rho$  is not self-dual and  $G_0 = \text{PSU}_n(q)$  then  $i$  is even,  $|x^G| > \frac{1}{2}(q + 1)^{-1}q^{6n-18}$  and close inspection of [25] reveals that (14) is always sufficient.

Proceeding in this way we reduce to the cases listed in Table 2.10. Here  $M(\lambda)^{(l)}$  denotes the twisted module  $M(\lambda)^{\phi^l}$ , where  $\phi$  is a field automorphism of  $\hat{H}_0$  induced by the map  $\mu \mapsto \mu^p$  on field elements. We write  $V_{\text{adj}}$  for the non-trivial composition factor of the adjoint module for  $H_0$ .

**Lemma 2.20.** *The conclusion to Theorem 1.1 holds for the embeddings in Table 2.10.*

**Proof.** Let  $K$  denote the algebraic closure of  $\mathbb{F}_q$  and let  $\bar{G}$  be a simple classical algebraic group over  $K$  of adjoint type with the property that there exists a Frobenius morphism  $\sigma$  of  $\bar{G}$  such that  $\bar{G}_\sigma$  has socle  $G_0$ . Let  $\rho$  be the absolutely irreducible representation corresponding to the embedding  $H < G$ . We claim that (10) holds for all elements  $x \in G$  of prime order.

Table 2.10  
The collection  $\mathcal{D}$ , Part I

|                     | $H_0$   | $G_0$  | Representation of $H_0$                   |
|---------------------|---|--|---|
| ( $\mathcal{D}1$ )  | $\text{PSL}_3^\epsilon(q)$  | $\begin{cases} \Omega_7(q), & p = 3 \\ \text{P}\Omega_8^+(q), & p \neq 3 \end{cases}$              | $V_{\text{adj}}$                          |
| ( $\mathcal{D}2$ )  | $\text{PSL}_4^\epsilon(q), p = 2$   | $\Omega_{14}^{\epsilon'}(q)$   | $V_{\text{adj}}$                          |
| ( $\mathcal{D}3$ )  | $\text{PSU}_5(q)$   | $\text{PSU}_{10}(q)$   | $\wedge^2 V_5$                            |
| ( $\mathcal{D}4$ )  | $\text{PSL}_6^\epsilon(q)$  | $\begin{cases} \text{PSp}_{20}(q), & p > 2 \\ \Omega_{20}^{\epsilon'}(q), & p = 2 \end{cases}$     | $\wedge^3 V_6$                            |
| ( $\mathcal{D}5$ )  | $\text{PSp}_4(q)$   | $\text{P}\Omega_{10}^\epsilon(q)$  | $M(2\lambda_1)$                           |
| ( $\mathcal{D}6$ )  | $\text{PSp}_4(q^2)$   | $\text{P}\Omega_{16}^+(q)$   | $M(\lambda_1) \otimes M(\lambda_1)^{(1)}$ |
| ( $\mathcal{D}7$ )  | $\text{PSp}_6(q)$   | $\begin{cases} \text{P}\Omega_{14}^\epsilon(q), & p \neq 3 \\ \Omega_{13}(q), & p = 3 \end{cases}$ | $M(\lambda_2)$                            |
| ( $\mathcal{D}8$ )  | $\text{PSp}_6(q), p > 2$  | $\text{PSp}_{14}(q)$   | $M(\lambda_3)$                            |
| ( $\mathcal{D}9$ )  | $\text{PSp}_8(q)$   | $\begin{cases} \Omega_{27}(q), & p > 2 \\ \Omega_{26}^\epsilon(q), & p = 2 \end{cases}$            | $M(\lambda_2)$                            |
| ( $\mathcal{D}10$ ) | $\text{P}\Omega_8^-(q_0), q = q_0^2$  | $\text{P}\Omega_8^+(q)$  | one of the two spin representations       |
| ( $\mathcal{D}11$ ) | $\text{P}\Omega_{10}^-(q)$  | $\text{PSU}_{16}(q)$   | spin representation                       |
| ( $\mathcal{D}12$ ) | $\begin{cases} \Omega_{11}(q), & p > 2 \\ \text{Sp}_{10}(q), & p = 2 \end{cases}$ | $\begin{cases} \text{PSp}_{32}(q), & p > 2 \\ \Omega_{32}^+(q), & p = 2 \end{cases}$               | spin representation                       |
| ( $\mathcal{D}13$ ) | $\text{P}\Omega_{12}^+(q)$  | $\begin{cases} \text{PSp}_{32}(q), & p > 2 \\ \Omega_{32}^+(q), & p = 2 \end{cases}$               | one of the two spin representations       |
| ( $\mathcal{D}14$ ) | $F_4(q)$  | $\begin{cases} \text{P}\Omega_{26}^+(q), & p \neq 3 \\ \Omega_{25}(q), & p = 3 \end{cases}$        | $M(\lambda_1)$                            |
| ( $\mathcal{D}15$ ) | $F_4(q), p = 2$   | $\Omega_{26}^+(q)$   | $M(\lambda_4)$                            |
| ( $\mathcal{D}16$ ) | ${}^2E_6(q)$  | $\text{PSU}_{27}(q)$   | $M(\lambda_1)$ or $M(\lambda_6)$          |

**Case i.** The irreducible embedding ( $\mathcal{D}1$ ).

Here  $\rho$  is the representation afforded by the non-trivial composition factor of the adjoint module for  $\text{SL}_3^\epsilon(q)$ . If  $p \neq 3$  then  $q \equiv \epsilon(3)$  (see [18, 2.3.3]) and therefore we may assume  $q > 2$ . Let  $\gamma \in \text{Aut}(H_0)$  be the inverse-transpose graph automorphism and observe that  $\gamma$  acts on  $V$  since  $\rho$  has highest weight  $\lambda = \lambda_1 + \lambda_2$  and this is fixed with respect to the induced action of  $\gamma$  on the set of weights. Therefore  $H \cap \text{PGL}(V) \leq \text{PGL}_3^\epsilon(q) \cdot \langle \gamma \rangle = \tilde{H}$ . We also note that if  $p \neq 3$  then  $\tilde{H}$  is the centralizer in  $\text{PGO}_8^+(q)$  of a non- $G_2$  triality graph automorphism (see [5, 3.47]).

Let  $x \in H \cap \text{PGL}(V)$  be an element of prime order  $r$ , write  $\tilde{H} = \text{PSL}_3(K)$  and choose unipotent  $\tilde{H}$ -class representatives  $u_1 = [J_2, I_1]$  and  $u_2 = [J_3]$ . If  $r = p$  then a straightforward calculation with the adjoint representation reveals that the  $\text{PO}_n(K)$ -class of  $\rho(x)$  is as follows

|           | $p \geq 5$          | $p = 3$        | $p = 2$ |
|-----------|---------------------|----------------|---------|
| $x = u_1$ | $[J_3, J_2^2, I_1]$ | $[J_3, J_2^2]$ | $c_4$   |
| $u_2$     | $[J_5, J_3]$        | $[J_3^2, I_1]$ | –       |

and the desired result follows via [5, 3.18, 3.22]. Similarly, if  $r = 2$  we have

|                   | $p \geq 5$    | $p = 3$       | $p = 2$ |
|-------------------|---------------|---------------|---------|
| $x = [-I_2, I_1]$ | $[-I_4, I_4]$ | $[-I_4, I_3]$ | -       |
| $\gamma$          | $[-I_5, I_3]$ | $[-I_4, I_3]$ | $b_3$   |

and it is easy to check that (10) holds. For example, if  $p = 3$  and  $x$  is conjugate to  $[-I_4, I_3]$  then

$$|x^G \cap H| \leq \frac{|\text{PGL}_3^\epsilon(q)|}{|\text{SO}_3(q)|} + \frac{|\text{GL}_3^\epsilon(q)|}{|\text{GL}_2^\epsilon(q)||\text{GL}_1^\epsilon(q)|} < 2(q + 1)q^4,$$

$$|x^G| \geq \frac{|\text{O}_7(q)|}{|\text{O}_4^-(q)||\text{O}_3(q)|} > \frac{1}{4}q^{12}$$

and we deduce that  $f(x, H) < 0.549$ . Now assume  $x$  is semisimple and  $r$  is odd. We claim that  $|x^G| > \frac{1}{2}(q + 1)^{-1}q^{19-4\delta_{3,p}}$ . For example, if  $p \neq 3$  then  $C_{\bar{G}}(x)$  is not of type  $\text{GL}_4$  (since  $x$  is centralized by a triality graph automorphism, see [5, 3.55(iv)]) nor  $\text{SO}_2 \times \text{GL}_3$  (a calculation with the adjoint representation reveals that  $\nu(x) \neq 5$ ). Therefore

$$|x^G| \geq |\text{O}_8^+(q) : \text{O}_4^+(q)\text{GU}_2(q)| > \frac{1}{2}(q + 1)^{-1}q^{19}$$

as claimed. The desired result now follows since  $|x^G \cap H| \leq |\text{PGL}_3^\epsilon(q)| < q^8$ .

To complete the proof, let us suppose  $x \in H - \text{PGL}(V)$  is an element of prime order  $r$ . First assume  $x$  is a field automorphism, so  $q = q_0^r$ . If  $r$  is odd then  $x^G \cap H \subseteq \text{PGL}_3^\epsilon(q)x$  and  $x$  induces a field automorphism on  $\text{PGL}_3^\epsilon(q)$ . Therefore [5, 3.43, 3.48] imply that

$$|x^G \cap H| \leq |\text{PGL}_3^\epsilon(q) : \text{PGL}_3^\epsilon(q^{1/r})| < 2q^{8(1-\frac{1}{r})}, \quad |x^G| > \frac{1}{4}q^{\frac{1}{2}n(n-1)(1-\frac{1}{r})}$$

and the desired result follows. Similarly, if  $r = 2$  and  $p \neq 3$  then  $\epsilon = +$  (since  $q_0^2 \equiv 1(3)$ ) and the bounds

$$|x^G \cap H| \leq |\text{PGL}_3(q) : \text{PGL}_3(q^{1/2})| + |\text{PGL}_3(q) : \text{PGU}_3(q^{1/2})| < 4q^4 \tag{15}$$

and  $|x^G| > \frac{1}{4}q^{14}$  are always sufficient. On the other hand, if  $(r, p) = (2, 3)$  then  $n = 7$  and again we get  $|x^G \cap H| < 4q^4$  if  $\epsilon = +$ , while  $|x^G \cap H| \leq |\text{PGU}_3(q) : \text{PGO}_3(q)| < 2q^5$  if  $\epsilon = -$ . In both cases the bound  $|x^G| > \frac{1}{4}q^{21/2}$  is good enough. If  $n = 8$  and  $x$  is an involutory graph-field automorphism then  $\epsilon = +$  (since  $q = q_0^2$  and  $p \neq 3$ ) and the bounds (15) and  $|x^G| > \frac{1}{4}q^{14}$  are always sufficient.

Finally, let us assume  $n = 8$  and  $x$  is a triality automorphism. If  $x$  is a graph-field automorphism then  $q = q_0^3$ ,  $|x^G| > \frac{1}{4}q^{56/3}$  and the trivial bound  $|x^G \cap H| < |H| < 6 \log_2 q \cdot q^8$  suffices. If  $x$  is a triality graph automorphism then  $x^G \cap H \subseteq \text{PGL}_3^\epsilon(q) \times \langle \tau \rangle$ , where  $\tau$  is a non- $G_2$  triality graph automorphism which centralizes  $\text{PGL}_3^\epsilon(q)$  (see [5, 3.47]). Applying [5, 3.14, 3.48] we deduce that

$$|x^G \cap H| \leq 2i_3(\text{PGL}_3^\epsilon(q)) + 2 \leq 4(q + 1)q^5, \quad |x^G| > \frac{1}{8}q^{14}$$

and the desired result follows.

**Case ii.** The irreducible embedding ( $\mathcal{D}10$ ).

Here  $\rho$  is the restriction of a spin representation of an orthogonal group  $P\Omega_8^+(q_0^2)$  which contains  $P\Omega_8^-(q_0)$ . Observe that  $H \cap PGL(V) \leq PGO_8^-(q_0) = \hat{H}$  and note that we may assume  $G$  is without triality (see [5, 3.3]). Now, if  $x \in H - PGL(V)$  then [5, 3.50] implies that  $x^G \cap H \subseteq \hat{H}x$ , where  $\hat{H} = \text{Inndiag}(P\Omega_8^-(q_0))$ . In particular, if  $x$  is a field automorphism of odd prime order  $r$  then  $q_0 = q_1^r$  and applying [5, 3.43] we deduce that

$$|x^G \cap H| < 2q_0^{28(1-\frac{1}{r})}, \quad |x^G| > \frac{1}{4}q_0^{56(1-\frac{1}{r})}$$

and the result follows. If  $x$  is an involutory field automorphism then  $x$  induces an involutory graph automorphism on  $\hat{H}$  and therefore [5, 3.14] implies that

$$|x^G \cap H| \leq i_2(\text{Aut}(P\Omega_8^-(q_0))) < 2(q_0 + 1)q_0^{15}.$$

In this case it is easy to check that the bound

$$|x^G| \geq 2^{2(\delta_{2,p}-1)} |O_8^+(q_0^2) : O_8^+(q_0)| = 2^{2(\delta_{2,p}-1)} q_0^{12} (q_0^2 + 1)(q_0^4 + 1)^2 (q_0^6 + 1)$$

is sufficient unless  $q_0 = 3$ . Here  $i_2(\text{Aut}(P\Omega_8^-(3))) = 60\,504\,111$  and the previous bound is in fact sufficient. The argument for an involutory graph-field automorphism is entirely similar.

Assume for the remainder that  $x \in H \cap PGL(V)$  is an element of prime order  $r$ . If  $r = p = 2$  then applying [5, 3.22, 3.55(ii)] we obtain the following results.

| $O_8^-(q_0)$ -class of $x$         | $b_1$ | $a_2$ | $c_2$ | $b_3$ | $c_4$ |
|------------------------------------|-------|-------|-------|-------|-------|
| $O_8^+(q_0^2)$ -class of $\rho(x)$ | $b_1$ | $a_2$ | $a_4$ | $b_3$ | $c_4$ |
| $f(x, H) <$                        | 0.507 | 0.504 | 0.503 | 0.479 | 0.500 |

For example, if  $\rho(x)$  is  $O_8^+(q_0^2)$ -conjugate to  $b_1$  then the proof of [5, 3.22] gives  $|x^G \cap H| = q_0^3(q_0^4 + 1)$ ,  $|x^G| = q_0^6(q_0^8 - 1)$  and thus  $f(x, H) < 0.507$  as claimed. Similarly, when  $r = p > 2$  we derive the bounds listed in the next table. Here the symbol  $\dagger$  (respectively  $\ddagger$ ) indicates the additional condition  $p \geq 5$  (respectively  $p \geq 7$ ).

| $PGO_8^-(q_0)$ -class of $x$         | $[J_2^2, I_4]$ | $[J_3, I_5]$ | $[J_3, J_2^2, I_1]$ | $[J_3^2, I_2]$ | $[J_5, I_3]^\dagger$ | $[J_7, I_1]^\ddagger$ |
|--------------------------------------|----------------|--------------|---------------------|----------------|----------------------|-----------------------|
| $PGO_8^+(q_0^2)$ -class of $\rho(x)$ | $[J_2^2, I_4]$ | $[J_2^4]$    | $[J_3, J_2^2, I_1]$ | $[J_3^2, I_2]$ | $[J_4^2]$            | $[J_7, I_1]$          |
| $f(x, H) <$                          | 0.517          | 0.514        | 0.521               | 0.522          | 0.506                | 0.508                 |

If  $r = 2$  and  $p$  is odd then we may apply [5, 3.37, 3.55(iii)]. For instance, if  $\rho(x)$  is  $\bar{G}$ -conjugate to  $[-iI_4, iI_4]$  then

$$|x^G \cap H| \leq \frac{|O_8^-(q_0)|}{|O_6^+(q_0)||O_2^-(q_0)|} + \frac{|O_8^-(q_0)|}{|O_6^-(q_0)||O_2^+(q_0)|} < 4q_0^{12}, \quad |x^G| > \frac{1}{4}(q_0^2 + 1)^{-1}q_0^{26}$$

and we conclude that  $f(x, H) < 0.586$  for all  $q_0 \geq 3$ .



Finally, let us assume  $r \neq p$  and  $r$  is odd. Let  $i_0 \geq 1$  be minimal such that  $r$  divides  $q_0^{i_0} - 1$  and let  $\hat{x} \in O_8^-(q_0)$  be the lift of  $x$  to an element of order  $r$ . Let  $l_0$  denote the dimension of the 1-eigenspace of  $\hat{x}$  on the natural  $O_8^-(q_0)$ -module and note that  $l_0 \geq 2$  if  $i_0 \neq 8$  (see [5, 3.29]). Further, using [5, 3.55(iv)] we can easily identify the possibilities for  $C_{\bar{G}}(x)$  and  $C_{\bar{G}}(\rho(x))$ , where  $\bar{G} = \text{SO}_8(K)$ , and the desired result quickly follows.

**Case iii.** The inclusions (D7), (D8) and (D16).

In (D7),  $V$  is a section of  $\bigwedge^2 V_6$  and  $H \cap \text{PGL}(V) \leq \text{PGSp}_6(q)$ . If  $x \in H - \text{PGL}(V)$  has prime order then [5, 3.49] gives  $|x^G| > \frac{1}{4}q^{39}$  and the trivial bound

$$|x^G \cap H| < |\text{Aut}(H_0)| < \log_2 q \cdot q^{21} \tag{16}$$

is sufficient. Now assume  $x \in H \cap \text{PGL}(V)$  has prime order  $r$ . If  $r = p$  then we claim that  $\nu(x) \geq 4$  if  $\lambda' = (2, 1^4)$ , otherwise  $\nu(x) \geq 6$ , where  $\lambda' \vdash 6$  denotes the associated partition of  $x$  with respect to the natural  $H_0$ -module. Now, if  $\lambda' = (2, 1^4)$  then an easy calculation with the module  $\bigwedge^2 V_6$  reveals that  $\rho(x)$  acts on  $V$  with Jordan form  $[J_2^4, I_{n-8}]$ . If  $\lambda' \neq (2, 1^4)$  and  $p = 2$  then it is easy to see that  $\nu(x) = 6$ . Similarly, if  $p$  is odd and  $y \in H_0$  is unipotent with associated partition  $\lambda' = (2^2, 1^2)$  then  $\rho(y)$  is conjugate to  $[J_3, J_2^4, I_{n-11}]$  and the claim follows. In particular, if  $\nu(x) = 4$  then  $|x^G \cap H| < q^6$ ,  $|x^G| > \frac{1}{4}q^{32}$  and the result follows, while if  $\nu(x) \neq 4$  our calculation with  $y$  implies that

$$|x^G| \geq \frac{1}{2} \frac{|\text{O}_{13}(q)|}{|\text{Sp}_4(q)||\text{O}_2^-(q)||\text{O}_1(q)|q^{25}} > \frac{1}{8}(q+1)^{-1}q^{43}$$

and (16) is sufficient. Finally, let us assume  $x \in H \cap \text{PGL}(V)$  has order  $r \neq p$ . Then  $\nu(x) \geq 6 - \delta_{3,p}$  and the desired result follows via [5, 3.36] and (16).

Now consider the embedding labeled (D8). If  $x \in H - \text{PGL}(V)$  then [5, 3.49] gives  $|x^G| > \frac{1}{4}q^{105/2}$  and the trivial bound

$$|x^G \cap H| < |\text{Aut}(H_0)| < \log_3 q \cdot q^{21} \tag{17}$$

is always sufficient. Now assume  $x \in H \cap \text{PGL}(V) \leq \text{PGSp}_6(q)$  has prime order  $r$ . If  $r \neq p$  then Theorem 2.4 implies that  $\nu(x) \geq 4$  since there is no semisimple element  $y \in G$  with  $\nu(y) = 3$ . Therefore  $|x^G| > \frac{1}{2}q^{40}$  (minimal if  $x = [-I_4, I_{10}]$ ) and the result follows via (17). Finally, suppose  $r = p$ . A well-known theorem of Steinberg states that there are precisely  $q^{2|\Phi^+(\bar{H})|}$  unipotent elements in a finite group of Lie type over  $\mathbb{F}_q$  of the form  $\bar{H}_\sigma$ , where  $\Phi^+(\bar{H})$  is the set of positive roots of  $\bar{H}$ . Therefore  $|x^G \cap H| < q^{18}$  and the desired result follows since  $|x^G| > \frac{1}{4}q^{36}$  (minimal if  $x$  has associated partition  $\lambda = (2^3, 1^8)$ ).

Finally, let us consider (D16). If  $\tau$  is an involutory graph automorphism of the algebraic group  $E_6$  then  $M(\lambda_1)^\tau = M(\lambda_6)$  and so we need only consider  $M(\lambda_1)$ . If  $x \in H - \text{PGL}(V)$  has prime order then [5, 3.49] gives  $|x^G| > \frac{1}{2}q^{348}$  and the result follows since

$$|x^G \cap H| < |\text{Aut}(H_0)| < 2 \log_2 q \cdot q^{78}. \tag{18}$$

Now assume  $x \in H \cap \text{PGL}(V)$  has prime order  $r$ . We claim that  $\nu(x) \geq 7 - \delta_{r,p}$ . This is immediate from [20, Table 5] if  $r = p$  so assume  $x$  is semisimple. Then  $x$  lies in a maximal torus

$T_6 < E_6$  (as algebraic groups defined over the algebraic closure of  $\mathbb{F}_q$ ) and therefore some conjugate of  $x$  lies in a maximal rank subgroup  $A_1 A_5 < E_6$  (algebraic groups). The claim now follows since [24, 2.3] gives

$$V \downarrow A_1 A_5 = (V_2 \otimes V_6) \oplus \left( 0 \otimes \left( \bigwedge^2 V_6 \right)^* \right),$$

where  $V_2$  (respectively  $V_6$ ) denotes the natural module for  $A_1$  (respectively  $A_5$ ) and  $0$  is the trivial 1-dimensional module for  $A_1$ . Therefore [5, 3.38] implies that  $|x^G| > \frac{1}{2}q^{250}$  and (18) is always sufficient.

**Case iv.** The remaining cases in Table 2.10.

These pose few problems. In a number of cases we can calculate directly with the corresponding representation and improve the lower bound on  $\nu(x)$  given in Theorem 2.4. Indeed, there are several such calculations in [7, §7] and many of those results are useful here. For example, consider (S14) and (S15). If  $x \in H \cap \text{PGL}(V)$  has prime order then the proof of [7, 7.4] gives  $\nu(x) \geq 6$  and thus [5, 3.38] implies that  $|x^G| > \frac{1}{2}q^{107}$ . Clearly, the same bound holds if  $x \in H - \text{PGL}(V)$  (see [5, 3.49]) and the desired result follows since  $|x^G \cap H| < |\text{Aut}(H_0)| < 2 \log_2 q \cdot q^{52}$ . The other cases are just as easy. (Note that (S11) is almost identical to the case (S4) in Table 2.2.)

This completes the proof of Lemma 2.20.  $\square$

**Case 2.**  $H_0$  is not a simple group of Lie type in defining characteristic.

Recall that we may assume  $n < N$ , where  $N = N(G_0)$  is given in Table 2.9. We begin with the case  $H_0 = \text{PSL}_2(l)$ , where  $l$  is coprime to  $q$ . The various possibilities for  $G_0$  are listed in [13, Table 2] and for brevity we will only consider the particular case  $G_0 = \text{PSp}_{(l-1)/2}(q)$ , where  $\mathbb{F}_q = \mathbb{F}_p[\sqrt{l}]$  and  $l \equiv 1 \pmod{4}$ ; the other cases are very similar. Here the hypothesis  $6 \leq n \leq 62$  implies that  $13 \leq l \leq 125$  and applying Theorem 2.4 and [5, 3.38] we deduce that

$$|x^G \cap H| < \log_3 l \cdot l(l^2 - 1), \quad |x^G| > \frac{1}{8}(q + 1)^{-1} q^{\frac{1}{2}\alpha(l-1) - \alpha^2 + 1},$$

where  $\alpha = 4$  if  $l \geq 73$ , otherwise  $\alpha = 3$ . These bounds are sufficient with the exception of the cases  $(l, q) \in \{(25, 2), (17, 2), (13, 4), (13, 3)\}$ . Here the desired conclusion quickly follows. For instance, in each case the corresponding Brauer character is available in [16] and we can compute  $f(x, H)$  precisely when  $x$  is semisimple.

For the remainder let us assume  $H_0 \neq \text{PSL}_2(l)$ . In [14], Hiss and Malle list all the absolutely irreducible representations of degree at most 250 of quasisimple finite groups, excluding groups of Lie type in their defining characteristic. Frobenius–Schur indicators are also recorded and further information is given which allows one to calculate the smallest field over which each representation can be written. We make extensive use of these results. To illustrate our approach, let us assume  $G_0 = \text{PSp}_n(q)$  and observe that

$$|x^G \cap H| < |\text{Aut}(H_0)|, \quad |x^G| > \frac{1}{8}(q + 1)^{-1} q^{\alpha(n-\alpha)+1}, \tag{19}$$

where  $\alpha = 4$  if  $n \geq 36$  and  $\alpha = 3$  otherwise. Since we are free to assume  $n \leq 62$ , close inspection of [14, Table 2] reveals that  $|\text{Aut}(H_0)| \leq |\text{Sp}_6(5)|$  and thus (19) is sufficient for all  $n \geq 36$ . For  $n \leq 34$  we consider in turn each pair  $(H_0, n)$  listed in [14, Table 2] with Frobenius–Schur indicator  $-1$  and apply the above bounds (19). Excluding any inclusions which belong to one of the collections  $\mathcal{A}$ ,  $\mathcal{B}$  or  $\mathcal{C}$ , we find that we are left to deal with the following cases:

$$(H_0, G_0) \in \{(G_2(4), \text{PSp}_{12}(3)), (HS, \text{Sp}_{20}(2)), (Suz, \text{PSp}_{12}(3)), (Co_3, \text{Sp}_{22}(2))\}.$$

Similarly, if  $G_0 = \text{PSL}_n^\epsilon(q)$  and  $n \leq 63$  then [14, Table 2] yields  $|\text{Aut}(H_0)| \leq |\text{Sp}_8(3)|$  and applying Theorem 2.4 and [5, 3.38] we reduce to the case  $n \leq 20$ . Further scrutiny of [14, Table 2] reveals that we are left to deal with the single case  $(H_0, G_0) = (J_3, \text{PSU}_9(2))$ . We proceed in the same way when  $G_0$  is orthogonal, noting that  $|x^G \cap H| < |Co_1|$  since  $n \leq 63$ . The cases which we are left to deal with are listed in Table 2.11 (for  $G_0 = \text{P}\Omega_8^+(q)$  we have excluded any subgroups which are not maximal—see [18]). In the table, the † symbol in the row for ( $\mathcal{D}23$ ) denotes the additional condition  $(p, \epsilon) \in \{(2, -), (3, +)\}$ .

**Lemma 2.21.** *The conclusion to Theorem 1.1 holds for the embeddings in Table 2.11.*

**Proof.** Let  $x \in H$  be an element of prime order  $r$ . We claim that (10) holds. If  $x \in H - \text{PGL}(V)$ , which is only possible in cases ( $\mathcal{D}21$ ) and ( $\mathcal{D}24$ ), then the claim follows via [5, 3.49] since  $|x^G \cap H| \leq i_r(\text{Aut}(H_0) - H_0)$ . Now assume  $x \in H \cap \text{PGL}(V)$ .

If  $r = p$  then we bound  $|x^G|$  by applying Theorem 2.4 and [5, 3.22] (note that  $v(x) \geq 4$  if  $G_0$  is orthogonal and  $p$  is odd). Since  $|x^G \cap H| \leq i_r(H \cap \text{PGL}(V))$ , one can check that the subsequent upper bound on  $f(x, H)$  is sufficient except for ( $\mathcal{D}27$ ). Here a more accurate lower bound for  $|x^G|$  suffices. Indeed, [8] gives  $|x^G \cap H| \leq i_3(Suz) = 151\,236\,800$  and we conclude that  $f(x, H) < 0.582 < 7/12$  since

$$|x^G| \geq \frac{|\text{Sp}_{12}(3)|}{|\text{Sp}_6(3)||O_3(3)|3^{24}} = \frac{1}{2}3^2(3^2 + 1)(3^4 + 1)(3^{10} - 1)(3^{12} - 1)$$

(minimal if  $x$  has associated partition  $\lambda = (2^3, 1^6)$ ).

Now suppose  $r \neq p$ . Let  $\chi$  be the corresponding Brauer character and note that  $\chi$  is available in the GAP Character Table Library for each of the embeddings ( $\mathcal{D}17$ )–( $\mathcal{D}27$ ). Therefore, in these cases we can compute  $f(x, H)$  precisely and (10) follows. In the three remaining cases we have  $H = Co_i$ , for  $i \in \{1, 2, 3\}$ . If  $x \in H$  is non-trivial then [12, Table 1] indicates that  $H$  can be generated by five conjugates of  $x$ , whence  $v(x) \geq n/5$  and we conclude that  $v(x) \geq 5$  for all non-trivial elements  $x \in H$ . The desired result now follows as before, using [5, 3.38] and the upper bound  $|x^G \cap H| \leq i_r(H)$ .  $\square$

This completes the proof of Proposition 2.19.  $\square$

**Proposition 2.22.** *The conclusion of Theorem 1.1 holds if  $H \in \mathcal{S}$  and  $\dim V < 6$ .*

**Proof.** First assume  $H_0$  is a simple group of Lie type in defining characteristic. Suppose  $H_0 = \text{PSL}_2(q')$ , where  $q' = p^e$ . Here we may assume  $G_0 \in \{\Omega_5(q), \text{PSp}_4(q)'\}$  since  $n = 2$  if  $G_0 = \Omega_4^-(q)$  (see Remark 1.2). Also recall that there exists an integer  $i \geq 1$  such that  $q' = q^i$  and  $\dim V = l^i$ , where  $l \geq 2$  is the dimension of an irreducible  $K\hat{H}_0$ -module (see the proof of

Table 2.11  
The collection  $\mathcal{D}$ , Part II

|                     | $H_0$     | $G_0$                              |
|---------------------|-----------|------------------------------------|
| ( $\mathcal{D}17$ ) | $A_{10}$  | $\Omega_{16}^+(2)$                 |
| ( $\mathcal{D}18$ ) | $SL_3(3)$ | $\Omega_{12}^-(2)$                 |
| ( $\mathcal{D}19$ ) | $G_2(3)$  | $\Omega_{14}^\epsilon(2)$          |
| ( $\mathcal{D}20$ ) | $G_2(4)$  | $PSp_{12}(3)$                      |
| ( $\mathcal{D}21$ ) | $Sz(8)$   | $P\Omega_8^+(5)$                   |
| ( $\mathcal{D}22$ ) | $M_{11}$  | $\Omega_{10}^-(2)$                 |
| ( $\mathcal{D}23$ ) | $M_{12}$  | $P\Omega_{10}^\epsilon(p)^\dagger$ |
| ( $\mathcal{D}24$ ) | $J_3$     | $PSU_9(2)$                         |
| ( $\mathcal{D}25$ ) | $HS$      | $Sp_{20}(2)$                       |
| ( $\mathcal{D}26$ ) | $M^cL$    | $\Omega_{22}^\epsilon(2)$          |
| ( $\mathcal{D}27$ ) | $Suz$     | $PSp_{12}(3)$                      |
| ( $\mathcal{D}28$ ) | $Co_1$    | $P\Omega_{24}^\epsilon(3)$         |
| ( $\mathcal{D}29$ ) | $Co_2$    | $\Omega_{22}^+(2)$                 |
| ( $\mathcal{D}30$ ) | $Co_3$    | $Sp_{22}(2)$                       |

Proposition 2.19). Clearly, we may assume  $i = 1$  and  $p \geq 5$ . Now, if  $x \in H - PGL(V)$  has prime order  $r$  then  $q = q_0^r$  and  $x$  is a field automorphism. Applying [5, 3.43, 3.48] we get

$$|x^G \cap H| \leq \frac{|PGL_2(q)|}{|PGL_2(q^{1/r})|} < 2q^{3(1-\frac{1}{r})}, \quad |x^G| > \frac{1}{4}q^{10(1-\frac{1}{r})}$$

and the result follows. Now assume  $x \in H \cap PGL(V)$  has prime order  $r$ . If  $r = p$  then

$$|x^G \cap H| \leq \frac{|GL_2(q)|}{|GL_1(q)|q} = q^2 - 1, \quad |x^G| \geq \frac{|Sp_4(q)|}{2|Sp_2(q)|q^3} = \frac{1}{2}(q^4 - 1)$$

and we conclude that  $f(x, H) < 0.554$  for all  $q \geq 5$ . If  $r = 2 < p$  then  $f(x, H) < 0.565$  since  $|x^G \cap H| \leq q^2$  and  $|x^G| \geq \frac{1}{2}q^2(q^2 - 1)$ . Finally, if  $r \neq p$  and  $r$  is odd then

$$|x^G \cap H| \leq 2 \log_5 q \cdot \frac{|GL_2(q)|}{|GL_1(q)|^2} = 2 \log_5 q \cdot q(q + 1),$$

$$|x^G| \geq \frac{|Sp_4(q)|}{|GU_2(q)|} = q^3(q - 1)(q^2 + 1)$$

and again the desired conclusion follows.

Now assume  $H_0 \neq PSL_2(q')$ . By inspecting the tables in [25] (or [19, 5.4.13]) we see that we need only consider the irreducible inclusion  $H_0 = Sz(q) < Sp_4(q) = G_0$ , where  $q = 2^f$  and  $f = 2m + 1 \geq 3$ . Here  $H_0 = C_{G_0}(\psi)$ , where  $\psi$  is an involutory graph-field automorphism of  $G_0$

(see [5, 3.44]). First assume  $x \in H - \text{PGL}(V)$  has prime order  $r$ . If  $x$  is a field automorphism then  $r$  must divide  $f$  and the result follows since

$$|x^G \cap H| \leq |Sz(q) : Sz(q^{1/r})| < 2q^{5(1-\frac{1}{r})}, \quad |x^G| \geq |\text{Sp}_4(q) : \text{Sp}_4(q^{1/r})| > \frac{1}{2}q^{10(1-\frac{1}{r})}.$$

On the other hand, if  $x$  is an involutory graph-field automorphism then we may assume  $x$  centralizes  $H_0$  and we deduce that  $f(x, H) < 0.600$  since

$$|x^G \cap H| \leq i_2(H_0) + 1 = (q - 1)(q^2 + 1) + 1, \\ |x^G| \geq |\text{Sp}_4(q) : Sz(q)| = q^2(q + 1)(q^2 - 1).$$

Now assume  $x \in H \cap \text{PGL}(V)$  has prime order  $r$ . If  $r = 2$  then [5, 3.52] implies that  $x$  is  $\text{Sp}_4(q)$ -conjugate to  $c_2$  and the bounds  $|x^G \cap H| = (q - 1)(q^2 + 1)$  and  $|x^G| = (q^2 - 1)(q^4 - 1)$  are always sufficient. If  $r$  is odd then  $r \geq 5$  and  $v(x) = 3$  (see [5, 3.52]), whence

$$|x^G| \geq \frac{|\text{Sp}_4(q)|}{|\text{GU}_1(q)|^2} = q^4(q - 1)^2(q^2 + 1)$$

and  $f(x, H) < 0.695$  since  $|x^G \cap H| < |\text{Aut}(Sz(q))| = \log_2 q \cdot q^2(q - 1)(q^2 + 1)$ .

For the remainder let us assume  $H_0$  is not a group of Lie type in defining characteristic. In view of Proposition 2.5, we may also assume that the embedding  $H < G$  is not in the collection  $\mathcal{A}$ . Then close inspection of [13, Table 2], [14, Table 2] and [8,16] reveals that we are left to deal with the cases listed in Table 2.12. Here the symbol † (respectively ‡) signifies that  $\epsilon = +$  if and only if  $p \equiv 1, 2, 4(7)$  (respectively  $p \equiv 1, 3, 4, 5, 9(11)$ ).

**Lemma 2.23.** *The conclusion to Theorem 1.1 holds for the embeddings in Table 2.12.*

**Proof.** As usual, let  $V$  denote the natural  $G_0$ -module and let  $\chi$  be the corresponding Brauer character. Observe that the values of  $\chi$  at semisimple elements are available in [8,16]. Let  $x \in H$  be an element of prime order  $r$ .

**Case 1.**  $x \in H \cap \text{PGL}(V)$ ,  $r \neq p$ .

Here we can compute  $f(x, H)$  precisely. For example, consider ( $\mathcal{E}6$ ). From the 2-modular Brauer character table for  $A_7$  (see [16, p. 13]) we derive the following results, where  $\zeta = 1$  if  $G$  contains an involutory graph automorphism of  $G_0$ , otherwise  $\zeta = 0$ .

| $A_7$ -class of $x$ | $\text{SL}_4(2)$ -class of $x$ | $ x^G \cap H $      | $ x^G $              | $f(x, H) <$ |          |
|---------------------|--------------------------------|---------------------|----------------------|-------------|----------|
| 3A                  | 3A                             | 70                  | 112                  | 0.901*      |          |
| 3B                  | 3B                             | 280                 | 1120                 | 0.803*      |          |
| 5A                  | 5A                             | 504                 | 1344                 | 0.864*      |          |
| (7A, 7B)            | (7A, 7B)                       | $2^\zeta \cdot 360$ | $2^\zeta \cdot 2880$ | 0.739       | (0.760*) |

Note that the elements 7A and 7B in  $A_7$  are conjugate in  $G$  if and only if  $\zeta = 1$ ; if they are conjugate then  $f(x, H) < 0.760^*$ , otherwise  $f(x, H) < 0.739$ . As usual, the asterisks in the final column indicate that this case is an exception to the main statement of Theorem 1.1 and it is therefore included in Table 1.1. Similarly, for ( $\mathcal{E}12$ ) we calculate that either  $f(x, H) < 3/4$ , or

Table 2.12  
The collection  $\mathcal{E}$

|                     | $H_0$                | $G_0$                        |                                  |
|---------------------|----------------------|------------------------------|----------------------------------|
| ( $\mathcal{E}1$ )  | $A_5$                | $\mathrm{PSp}_4(p)$          | $p \geq 5$                       |
| ( $\mathcal{E}2$ )  | $A_6$                | $\mathrm{PSL}_3(4)$          |                                  |
| ( $\mathcal{E}3$ )  | $A_6$                | $\mathrm{PSU}_3(5)$          |                                  |
| ( $\mathcal{E}4$ )  | $A_6$                | $\mathrm{PSp}_4(p)$          | $p \geq 5$                       |
| ( $\mathcal{E}5$ )  | $A_7$                | $\mathrm{PSU}_3(5)$          |                                  |
| ( $\mathcal{E}6$ )  | $A_7$                | $\mathrm{SL}_4(2)$           |                                  |
| ( $\mathcal{E}7$ )  | $A_7$                | $\mathrm{PSp}_4(7)$          |                                  |
| ( $\mathcal{E}8$ )  | $A_7$                | $\mathrm{PSL}_4^\epsilon(p)$ | $p \neq 2, 7^\dagger$            |
| ( $\mathcal{E}9$ )  | $\mathrm{PSL}_2(11)$ | $\mathrm{PSL}_5^\epsilon(p)$ | $p \neq 11^{\ddagger}$           |
| ( $\mathcal{E}10$ ) | $\mathrm{SL}_3(2)$   | $\mathrm{PSL}_3^\xi(p)$      | $p \neq 2, 7^\dagger$            |
| ( $\mathcal{E}11$ ) | $\mathrm{SL}_3(2)$   | $\mathrm{PSL}_4^\epsilon(p)$ | $p \neq 2, 7^\dagger$            |
| ( $\mathcal{E}12$ ) | $\mathrm{PSL}_3(4)$  | $\mathrm{PSU}_4(3)$          |                                  |
| ( $\mathcal{E}13$ ) | $\mathrm{SU}_4(2)$   | $\mathrm{PSL}_4^\epsilon(p)$ | $p \equiv \epsilon(3), p \geq 5$ |
| ( $\mathcal{E}14$ ) | $\mathrm{SU}_4(2)$   | $\mathrm{SU}_5(5)$           |                                  |
| ( $\mathcal{E}15$ ) | $M_{11}$             | $\mathrm{SL}_5(3)$           |                                  |

$x$  is  $G_0$ -conjugate to  $2B$  and  $f(x, H) = (\log 120)/(\log 540) < 0.761^*$ . Again, this exceptional case appears in Table 1.1. In each of the remaining cases, the reader can check that  $f(x, H) < 1/2 + 1/n$ , where  $n$  is defined as in Remark 1.2.

**Case 2.**  $x \in H \cap \mathrm{PGL}(V)$ ,  $r = p$ .

First consider ( $\mathcal{E}2$ ). Now  $A_6$  is a maximal subgroup of  $\mathrm{PSL}_3(4)$  (see [8], for example) and therefore  $H \cap \mathrm{PGL}(V) = A_6$ . Now both  $A_6$  and  $\mathrm{PSL}_3(4)$  contain a unique class of involutions, hence  $|x^G \cap H| = 45$ ,  $|x^G| = 315$  and thus  $f(x, H) < 0.662$ . In each of the remaining cases, [17] implies that  $x \in G$  is not a long root element and we use this fact to estimate  $|x^G|$ . Now  $|x^G \cap H| \leq i_r(H \cap \mathrm{PGL}(V))$  and the bound  $f(x, H) < 1/2 + 1/n$  quickly follows, with the exception of the cases ( $\mathcal{E}6$ ) and ( $\mathcal{E}12$ ). For example, consider ( $\mathcal{E}8$ ). Here  $H \cap \mathrm{PGL}(V) = A_7$ , so  $p \in \{3, 5\}$  and  $G_0 = \mathrm{PSU}_4(p)$  (see Table 2.12). Now  $i_3(A_7) = 350$ ,  $i_5(A_7) = 504$  and

$$|x^G| \geq \frac{1}{2} \frac{|\mathrm{GU}_4(p)|}{|\mathrm{GU}_2(p)|p^4} = \frac{1}{2} p(p^3 + 1)(p^4 - 1)$$

since  $x$  is not a long root element. We conclude that  $f(x, H) < 0.722$  if  $p = 3$  and  $f(x, H) < 0.511$  if  $p = 5$ . The other cases are similar. The embeddings ( $\mathcal{E}6$ ) and ( $\mathcal{E}12$ ) can be analysed using GAP [10]: for ( $\mathcal{E}6$ ) we deduce that  $f(x, H) = (\log 105)/(\log 210) < 0.871^*$ , while we get  $f(x, H) = (\log 2240)/(\log 40320) < 0.728$  for ( $\mathcal{E}12$ ).

**Case 3.**  $x \in H - \mathrm{PGL}(V)$ .

Here  $r = 2$  and therefore  $|x^G \cap H| \leq i_2(\mathrm{Aut}(H_0) - H_0)$ . It is easy to derive an accurate lower bound for  $|x^{G_0}|$  (see [5, 3.48]) and thus  $f(x, H) < 1/2 + 1/n$  with the exception of the following cases

- ( $\mathcal{E}2$ ), ( $\mathcal{E}6$ ), ( $\mathcal{E}8$ ), ( $\mathcal{E}12$ ), ( $\mathcal{E}13$ ).

For instance, in ( $\mathcal{E}9$ ) we have  $(H_0, G_0) = (\text{PSL}_2(11), \text{PSL}_5^\epsilon(p))$  and  $f(x, H) < 0.440$  for all  $p \geq 2$  since

$$|x^G \cap H| \leq i_2(\text{PGL}_2(11) - \text{PSL}_2(11)) = 66, \quad |x^G| \geq \frac{|\text{PSL}_5(p)|}{|\text{Sp}_4(p)|} = \frac{p^6(p^3 - 1)(p^5 - 1)}{(5, p - 1)}.$$

For the five exceptional cases we claim that the following bounds hold.

|             | ( $\mathcal{E}2$ ) | ( $\mathcal{E}6$ ) | ( $\mathcal{E}8$ ) | ( $\mathcal{E}12$ ) | ( $\mathcal{E}13$ ) |
|-------------|--------------------|--------------------|--------------------|---------------------|---------------------|
| $f(x, H) <$ | 0.711              | 0.914*             | 0.630              | 0.761*              | 0.445               |

The bounds in cases ( $\mathcal{E}2$ ), ( $\mathcal{E}6$ ) and ( $\mathcal{E}12$ ) are easily checked using GAP [10]. (For ( $\mathcal{E}12$ ) we have  $f(x, H) < 3/4$  unless  $x$  lies in the  $G_0$ -class  $2B$  in which case  $f(x, H) = (\log 120)/(\log 540) < 0.761^*$ .) Now consider ( $\mathcal{E}8$ ). Here  $i_2(S_7 - A_7) = 126$  and we may as well assume  $p = 3$  since

$$|x^G| \geq \frac{|\text{PSL}_4^\epsilon(p)|}{|\text{Sp}_4(p)|} = (4, p - \epsilon)^{-1} p^2(p^3 + 1)$$

(note that Table 2.12 states that  $\epsilon = -$  if  $p = 5$ ). The case  $p = 3$  can be analysed using GAP and the desired result quickly follows.

Finally, let us consider ( $\mathcal{E}13$ ). Here  $H_0 = \text{SU}_4(2)$  and  $G_0 = \text{PSL}_4^\epsilon(p)$ , where  $p \equiv \epsilon(3)$  and  $p \geq 5$ . Now  $x$  induces an involutory graph automorphism on both  $H_0$  and  $G_0$  and we claim that the centralizers  $C_{H_0}(x)$  and  $C_{G_0}(x)$  are of the same type (see [5, 3.47] for a description of the possible types). To see this, let  $\tau$  be a symplectic-type graph automorphism of  $H_0$  and note that  $C_{G_0}(\tau)$  is symplectic since  $\text{Sp}_4(2) \not\leq \text{PSO}_4^\epsilon(p)$ . In the  $2B$ -class of  $H_0$  there is an involution  $h$  such that  $[h, \tau] = 1$  and  $C_{H_0}(h\tau)$  is non-symplectic; moreover  $h$  is  $\tilde{G}$ -conjugate to  $[-iI_2, iI_2]$  (see [16, p. 62]) and therefore  $C_{G_0}(h\tau)$  is orthogonal and the claim follows. We conclude that  $f(x, H) < 0.445$  if  $x$  is symplectic since  $|x^G \cap H| = 36$  and  $|x^G| \geq 3150$ , otherwise  $|x^G \cap H| = 540$ ,  $|x^G| \geq 1\,890\,000$  and  $f(x, H) < 0.436$ .  $\square$

This completes the proof of Proposition 2.22.  $\square$

### 3. Proof of Theorem 1.1: $H \in \mathcal{N}$

To complete the proof of Theorem 1.1, let us assume  $H$  is a subgroup in the collection  $\mathcal{N}$  (see [5, §3.1]). Recall that  $\mathcal{N}$  is empty unless one of the following holds:

- (a)  $G_0 = \text{Sp}_4(q)'$ ,  $p = 2$  and  $G$  contains graph-field automorphisms;
- (b)  $G_0 = \text{P}\Omega_8^+(q)$  and  $G$  contains triality automorphisms.

The subgroups contained in the collection  $\mathcal{N}$  are listed in Table 3.1 (see [5, 3.3]). Here the type of  $H$  gives an approximate group-theoretic structure for  $H \cap \text{PGL}(V)$ .

Table 3.1  
The  $\mathcal{N}$  collection

|       | $G_0$                     | Type of $H$  | Conditions                   |
|-------|---------------------------|--|------------------------------|
| (i)   | $\mathrm{Sp}_4(q)'$       | $\mathrm{O}_2^\epsilon(q) \wr S_2$                           | $p = 2$                      |
| (ii)  | $\mathrm{Sp}_4(q)'$       | $\mathrm{O}_2^-(q^2).2$                                      | $p = 2$                      |
| (iii) | $\mathrm{P}\Omega_8^+(q)$ | $\mathrm{GL}_3^\epsilon(q) \times \mathrm{GL}_1^\epsilon(q)$ | $q \geq 3$ if $\epsilon = +$ |
| (iv)  | $\mathrm{P}\Omega_8^+(q)$ | $\mathrm{O}_2^-(q^2) \times \mathrm{O}_2^-(q^2)$             |                              |
| (v)   | $\mathrm{P}\Omega_8^+(q)$ | $\mathrm{O}_1(q) \wr S_8$                                    | $q = p > 2$                  |
| (vi)  | $\mathrm{P}\Omega_8^+(q)$ | $G_2(q)$   |                              |

3.1. Symplectic groups in dimension four

**Proposition 3.1.** *The conclusion to Theorem 1.1 holds in case (i) of Table 3.1.*

**Proof.** Let  $V$  denote the natural  $\mathrm{Sp}_4(q)$ -module and let  $x \in H$  be an element of prime order  $r$ . Note that we can take  $q > 2$  since  $n = 2$  if  $G_0 = \mathrm{Sp}_4(2)'$  (see Remark 1.2). First assume  $x \in H \cap \mathrm{PGL}(V) = \tilde{H} = \mathrm{O}_2^\epsilon(q) \wr S_2$ . If  $r = 2$  then the following results are easily derived via [5, 3.52].

| $\mathrm{Sp}_4(q)$ -class of $x$ | $ x^G \cap H $     | $ x^G $              | $f(x, H) <$ |
|----------------------------------|--------------------|----------------------|-------------|
| $b_1, a_2$                       | $4(q - \epsilon)$  | $2(q^4 - 1)$         | 0.481       |
| $c_2$                            | $(q - \epsilon)^2$ | $(q^2 - 1)(q^4 - 1)$ | 0.391       |

Similarly, if  $r$  is odd then  $r$  divides  $q - \epsilon$  and the bounds

$$|x^G \cap H| \leq 16 \log_2 q, \quad |x^G| \geq 2|\mathrm{Sp}_4(q) : \mathrm{GU}_2(q)| = 2q^3(q - 1)(q^2 + 1)$$

are sufficient. Now let us assume  $x \in H - \mathrm{PGL}(V)$ , so [5, 3.50] implies that  $x^G \cap H \subseteq \tilde{H}x$ . If  $x$  is a field automorphism of prime order  $r$  then  $q = q_0^r$  and the bounds

$$|x^G \cap H| \leq |\tilde{H}x| \leq 8(q + 1)^2, \quad |x^G| = |\mathrm{Sp}_4(q) : \mathrm{Sp}_4(q^{1/r})| > \frac{1}{2}q^{10(1-\frac{1}{r})}$$

are sufficient unless  $(r, q) = (2, 4)$ ; here we calculate that  $f(x, H) < 0.735$  since  $|\tilde{H}x| \leq 200$  and  $|x^G| = 1360$ . Finally, if  $x$  is an involutory graph-field automorphism then  $\log_2 q$  is odd and the result follows since

$$|x^G \cap H| \leq |\tilde{H}x| \leq 8(q + 1)^2, \quad |x^G| = |\mathrm{Sp}_4(q) : S\mathbb{Z}(q)| = q^2(q + 1)(q^2 - 1). \quad \square$$

**Proposition 3.2.** *The conclusion to Theorem 1.1 holds in case (ii) of Table 3.1.*

**Proof.** Again, we may assume  $q > 2$ . Suppose  $x \in H \cap \mathrm{PGL}(V) = \tilde{H} = \mathrm{O}_2^-(q^2).2$  has prime order  $r$ . If  $r$  is odd then  $r$  divides  $q^2 + 1$  and thus  $|x^G| > \frac{1}{2}q^8$  and the trivial bound  $|x^G \cap H| \leq |\tilde{H}| = 4(q^2 + 1)$  is always sufficient. On the other hand, if  $r = 2$  then  $x$  is  $G$ -conjugate to  $c_2$



and the desired result follows since  $|x^G \cap H| = q^2 + 1$  and  $|x^G| = (q^2 - 1)(q^4 - 1)$ . Finally, if  $x \in H - \text{PGL}(V)$  then  $x^G \cap H \subseteq \tilde{H}x$  and the bounds

$$|x^G \cap H| \leq |\tilde{H}x| = 4(q^2 + 1), \quad |x^G| \geq |\text{Sp}_4(q) : \text{Sz}(q)| = q^2(q + 1)(q^2 - 1)$$

are always sufficient.  $\square$

### 3.2. Orthogonal groups in dimension eight

For the remainder we will adopt the following notation.

**Notation.** Let  $G_0 = \text{P}\Omega_8^+(q)$ , where  $q = p^f$  and  $p$  is prime. Let  $\bar{G} = \text{PSO}_8(K)$ , where  $K$  denotes the algebraic closure of  $\mathbb{F}_q$ , and let  $\sigma$  be a Frobenius morphism of  $\bar{G}$  such that  $\bar{G}_\sigma$  is almost simple with socle  $G_0$  and natural module  $V$  over  $\mathbb{F}_q$ . Then  $G$  denotes an almost simple group which has socle  $G_0$  and contains triality automorphisms.

**Proposition 3.3.** *The conclusion to Theorem 1.1 holds in case (iii) of Table 3.1.*

**Proof.** Let  $H \leq G$  be a subgroup of type  $\text{GL}_3^\epsilon(q) \times \text{GL}_1^\epsilon(q)$  and define

$$B = \frac{\text{GL}_3^\epsilon(q) \times \text{GL}_1^\epsilon(q)}{(2, q - 1)}.$$

If  $q$  is even then  $H \cap \text{PGL}(V) \leq B \cdot \langle \psi_1, \psi_2 \rangle = B \cdot 2^2$ , where  $\psi_1$  acts on  $B$  by sending  $(x_1, x_2)$  to  $(x_1^\gamma, x_2)$  and  $\gamma$  is the familiar inverse-transpose graph automorphism of  $\text{GL}_3^\epsilon(q)$ , while  $\psi_2$  sends  $(x_1, x_2)$  to  $(x_1, x_2^{-1})$ . Similarly, if  $q$  is odd then

$$H \cap \text{PGL}(V) \leq (B \cdot \langle \delta \rangle) \cdot \langle \psi_1, \psi_2 \rangle = (B \cdot 2) \cdot 2^2,$$

where  $\delta \in \bar{G}_\sigma - \text{PSO}_8^+(q)$  is an involution. We claim that  $f(x, H) < 5/8$  for all elements  $x \in G$  of prime order.

**Case 1.**  $x \in H \cap \text{PGL}(V)$ .

Let  $x \in H \cap \text{PGL}(V)$  be an element of prime order  $r$  and note that each  $y \in x^G \cap B$  lifts to an element  $\hat{y} = (\hat{y}_1, \hat{y}_2) \in \hat{B}$ , where  $\hat{B} = \text{GL}_3^\epsilon(q) \times \text{GL}_1^\epsilon(q)$  and

$$|y^B| = |\hat{y}^{\hat{B}}| = |\hat{y}_1^{\text{GL}_3^\epsilon(q)}| |\hat{y}_2^{\text{GL}_1^\epsilon(q)}|$$

(see [5, 3.11]). First assume  $r = p > 2$ . Then  $x^G \cap H \subseteq B$  and  $\lambda \in \{(2^2, 1^4), (3^2, 1^2)\}$ , where  $\lambda$  denotes the associated partition of  $x$ . If  $\lambda = (2^2, 1^4)$  then

$$|x^G \cap H| \leq \frac{|\text{GL}_3^\epsilon(q)|}{|\text{GL}_1^\epsilon(q)|^2 q^3} < 2q^4, \quad |x^G| \geq \frac{|\text{O}_8^+(q)|}{|\text{O}_4^+(q)| |\text{Sp}_2(q)| q^9} > \frac{1}{2} q^{10} \tag{20}$$

and the desired result follows. The case  $\lambda = (3^2, 1^2)$  is very similar. Next assume  $r = p = 2$ . If  $x^G \cap H \subseteq B$  then  $x$  is  $G$ -conjugate to  $a_2$  and [5, 3.22, 3.55(ii)] imply that the bounds in (20) are

valid and so the result follows as before. On the other hand, if  $x^G \cap (H - B)$  is non-empty then  $x^G \cap B = \emptyset$  and there are at most three possibilities for  $x$  up to  $G$ -conjugacy. Firstly, if  $x$  is a  $c_4$ -involution then the bounds

$$|x^G \cap H| \leq |\mathrm{GL}_3^\epsilon(q) : \Omega_3(q)| |\mathrm{GL}_1^\epsilon(q)| = q^2(q - \epsilon)^2(q^3 - \epsilon), \quad |x^G| > \frac{1}{2}q^{16}$$

are always sufficient. Similarly, if  $x$  is conjugate to  $b_3$  then  $|x^G| > \frac{3}{2}q^{15}$  and the desired result follows since

$$|x^G \cap H| \leq 3 \left( \frac{|\mathrm{GL}_3^\epsilon(q)|}{|\Omega_3(q)|} + \frac{|\mathrm{GL}_3^\epsilon(q)|}{|\mathrm{GL}_1^\epsilon(q)|^2 q^3} |\mathrm{GL}_1^\epsilon(q)| \right) = 3(q^3 - \epsilon)(q^2(q - \epsilon) + q^2 - 1).$$

The argument for a  $b_1$ -involution is very similar.

Now assume  $r \neq p$ . First suppose  $r = 2$ . If  $x^G \cap H \subseteq B$  then  $x$  is conjugate to  $[-I_2, I_6]$  and applying [5, 3.55(iii)] we deduce that

$$|x^G \cap H| \leq (2\alpha + 1)(|\mathrm{GL}_3^\epsilon(q) : \mathrm{GL}_2^\epsilon(q)\mathrm{GL}_1^\epsilon(q)| + 1) < (2\alpha + 1)(2q^4 + 1)$$

and  $|x^G| > \frac{3}{4}(q + 1)^{-1}q^{13}$ , where  $\alpha = 1$  if  $q \equiv \epsilon \pmod{4}$ , otherwise  $\alpha = 0$ . The reader can check that these bounds are always sufficient. A similar argument applies if  $x^G \cap (H - B)$  is non-empty. For example, if  $x$  is conjugate to  $[-I_4, I_4]$  then  $|x^G| > \frac{1}{8}q^{16}$  and the desired result follows since

$$|x^G \cap H| \leq 2 \frac{|\mathrm{GL}_3^\epsilon(q)|}{|\mathrm{GL}_2^\epsilon(q)\mathrm{GL}_1^\epsilon(q)|} + \frac{|\mathrm{GL}_3^\epsilon(q)|}{|\mathrm{SO}_3(q)|} \cdot |\mathrm{GL}_1^\epsilon(q)| < 4q^4 + q^2(q + 1)^2(q^3 + 1).$$

Now assume  $r > 2$ . Then  $x^G \cap H \subseteq B$  and  $x$  lifts to a unique element  $\hat{x} \in \Omega_8^+(q)$  of order  $r$ . Write  $\mathcal{E}_x$  for the multiset of eigenvalues of  $\hat{x}$  on the natural  $\mathrm{SO}_8(K)$ -module and let  $i \geq 1$  be minimal such that  $r$  divides  $q^i - 1$ . Also, define the integer  $c = c(i, \epsilon)$  as in the statement of [5, 3.33] and observe that  $c \in \{1, 2, 3\}$  (by definition,  $c = i$  if  $\epsilon = +$ ). If  $c = 2$  then  $x$  is  $\tilde{G}$ -conjugate to  $[I_4, \omega I_2, \omega^{-1} I_2]$ , where  $\omega \in K$  is a primitive  $r$ th root of unity, and [5, 3.55(iv)] implies that  $x$  and  $x^\tau$  are  $\tilde{G}_\sigma$ -conjugate for any triality graph automorphism  $\tau$ . Therefore

$$|x^G \cap H| \leq \log_2 q \cdot \left( \frac{|\mathrm{GL}_3^\epsilon(q)|}{|\mathrm{GL}_1(q^2)\mathrm{GL}_1^\epsilon(q)|} \right) \leq \log_2 q \cdot q^3(q^3 + 1),$$

$$|x^G| \geq \frac{|\mathrm{O}_8^+(q)|}{|\mathrm{O}_4^+(q)\mathrm{GU}_2(q)|} > \frac{1}{2}(q + 1)^{-1}q^{19}$$

and the result follows. The case  $c = 3$  is similar so let us assume  $c = 1$ . We claim that

$$|x^G \cap H| < 3 \log_2 q \cdot 2^6 \cdot 2^{1+\epsilon} q^6, \quad |x^G| > \frac{1}{2} \left( \frac{q}{q + 1} \right)^{\frac{3}{2}(1-\epsilon)+1} q^{\dim x^{\tilde{G}}}. \tag{21}$$

The bound on  $|x^G|$  follows immediately from [5, 3.30] and it is clear that  $|y^B| < 2^{1+\epsilon} q^6$  for all  $y \in x^G \cap H$ , so it remains to show that there are at most  $3 \log_2 q \cdot 2^6$  distinct  $B$ -classes in  $x^G \cap H$ . Here the term  $3 \log_2 q$  accounts for the effect of field and triality graph automorphisms

on  $\mathcal{E}_x$ . Let  $l$  be the dimension of the 1-eigenspace of  $\hat{x}$  on the natural  $\Omega_8^+(q)$ -module and suppose  $y = (y_1, y_2)$  is an element of  $x^G \cap B$  such that  $\mathcal{E}_y \cup \mathcal{E}_y^{-1} = \mathcal{E}_x$ . Evidently, there are at most  $2^{4-l/2}$  distinct possibilities for  $\mathcal{E}_y = \mathcal{E}_{y_1} \cup \mathcal{E}_{y_2}$ , and for each choice there are at most four possibilities for  $\mathcal{E}_{y_2}$ . Therefore there are at most  $2^{6-l/2} \leq 2^6$  choices for  $y$  (up to  $B$ -conjugacy) and (21) follows.

Let us now apply the bounds in (21), beginning with the case  $\epsilon = +$ . If  $\dim x^{\bar{G}} \geq 18$  then it remains to deal with the case  $(r, q) = (3, 4)$ , where  $x$  is  $\bar{G}$ -conjugate to  $[I_4, \omega I_2, \omega^2 I_2]$  or  $[I_2, \omega I_3, \omega^2 I_3]$  and  $\omega \in K$  is a primitive cube root of unity. In the latter case we have

$$|x^G \cap H| \leq 6|\text{GL}_3(4) : \text{GL}_2(4)\text{GL}_1(4)| + 2|\text{GL}_3(4) : \text{GL}_1(4)^3| + 2 = 15458$$

and thus  $f(x, H) < 0.378$  since  $|x^G| \geq |\text{O}_8^+(4) : \text{O}_2^+(4)\text{GL}_3(4)|$ . Similarly, we get  $f(x, H) < 0.358$  if  $x = [I_4, \omega I_2, \omega^2 I_2]$ . Now, if  $\dim x^{\bar{G}} < 18$  then we may assume  $x = [I_6, \mu, \mu^{-1}]$  for some  $1 \neq \mu \in K$ , and thus  $\dim x^{\bar{G}} = 12$ . Furthermore, if  $\tau$  is a triality graph automorphism then  $x^\tau$  is conjugate to  $[\mu I_4, \mu^{-1} I_4]$ , whence  $|x^G| > \frac{3}{2}q^{12}$  and the desired result follows since

$$|x^G \cap H| \leq \log_2 q \cdot (6|\text{GL}_3(q) : \text{GL}_2(q)\text{GL}_1(q)| + 6) < \log_2 q \cdot (12q^4 + 6).$$

The case  $\epsilon = -$  is very similar.

**Case 2.**  $x \in H - \text{PGL}(V)$ .

First assume  $x$  is a field automorphism of prime order  $r$ , so  $q = q_0^r$  and [5, 3.50] implies that  $x^G \cap H \subseteq \tilde{H}x$ , where  $\tilde{H} = H \cap \text{PGL}(V)$ . If we assume  $r > 2$  then the bounds

$$|x^G \cap H| \leq |\tilde{H}x| < 4(q + 1)^2 q^8, \quad |x^G| > \frac{1}{4}q^{28(1-\frac{1}{r})}$$

(see [5, 3.48]) are always sufficient. If  $r = 2$  then applying [5, 3.14] we deduce that

$$|x^G \cap H| \leq (q - \epsilon)^2 i_2(\text{Aut}(\text{PSL}_3^\epsilon(q))) < 2(q + 1)^3 q^4, \quad |x^G| > \frac{1}{4}q^{14}$$

and again the result follows. The same bounds are also valid if  $x$  is an involutory graph-field automorphism. Now, if  $x$  is a triality graph-field automorphism then  $q = q_0^3$ ,  $|x^G| > \frac{1}{4}q^{56/3}$  (see [5, 3.48]) and we find that the trivial bound  $|x^G \cap H| < |H| < 3 \log_2 q \cdot 4(q + 1)^2 q^8$  is sufficient unless  $q = 8$ . In this case we calculate that  $f(x, H) < 0.620$  since

$$|x^G \cap H| < |H| \leq 2^{\zeta+1} 9|\text{GU}_3(8)||\text{GU}_1(8)|, \quad |x^G| = 2^\zeta |\Omega_8^+(8) : {}^3D_4(2)|$$

where  $\zeta = 1$  if  $G$  contains an involutory graph automorphism, otherwise  $\zeta = 0$ .

Finally, let us assume  $x$  is a triality graph automorphism. We claim that

$$|x^G \cap H| \leq 3(q - \epsilon)^2 i_3(\text{PGL}_3^\epsilon(q)). \tag{22}$$

To see this, first observe that there exists an element  $b \in B$  of order three such that  $C_{G_0}(bx) = G_2(q)$  and  $\text{SL}_3^\epsilon(q) \leq C_B(bx)$ . Now, if  $Z = Z(B)$  and  $\tilde{B} := B/Z \cong \text{PGL}_3^\epsilon(q)$  then

$$|x^G \cap H| \leq i_3(B.\langle x \rangle) \leq |Z|.i_3(\tilde{B}.\langle x \rangle) \leq (q - \epsilon)^2 .i_3(\tilde{B}.\langle x \rangle)$$

and thus (22) holds since  $\tilde{B} \cdot \langle x \rangle \cong (\text{PSL}_3^\epsilon(q) \times \langle \tilde{b}x \rangle) \cdot (3, q - \epsilon)$ , where  $\tilde{b}$  is the image of  $b$  in  $\tilde{B}$ . Now, if  $x$  is a non- $G_2$  triality then

$$|x^G| \geq \frac{|\text{P}\Omega_8^+(q)|}{|\text{SL}_2(q)|q^5} = 2^{2(\delta_2, p-1)} q^6 (q^4 - 1)^2 (q^6 - 1)$$

and we find that (22) is always sufficient since  $i_3(\text{PGL}_3^\epsilon(q))$  is given as follows:

|                                 |                 |                                     |                         |
|---------------------------------|-----------------|-------------------------------------|-------------------------|
|                                 | $q \equiv 0(3)$ | $q \equiv \epsilon(3)$              | $q \equiv -\epsilon(3)$ |
| $i_3(\text{PGL}_3^\epsilon(q))$ | $q^6 - 1$       | $q^6 + 2q^4 + 3\epsilon q^3 + 2q^2$ | $q^6 - \epsilon q^3$    |

Now assume  $x$  is a  $G_2$ -type triality. Then

$$|x^G| \geq |\text{P}\Omega_8^+(q) : G_2(q)| \geq 2^{2(\delta_2, p-1)} q^6 (q^4 - 1)^2 \tag{23}$$

and (22) is only sufficient if  $q > 13$ . To deal with the remaining cases we need a more accurate upper bound for  $|x^G \cap H|$ . We claim that

$$|x^G \cap H| \leq 2^\zeta (q + 1)^2 \tag{24}$$

for all values of  $q$ , where  $\zeta = 1$  if  $G$  contains an involutory graph automorphism, otherwise  $\zeta = 0$ . To see this, let  $\{x_\alpha(t) : \alpha \in \Phi, t \in K\}$  be a set of Chevalley generators for the algebraic group  $\text{SO}_8(K)$ , where  $\Phi$  is a root system of type  $D_4$ . Let  $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \subset \Phi$  be a set of simple roots, where  $\alpha_2$  corresponds to the middle node of the associated Dynkin diagram  $D_4$ . Write  $\alpha_0 = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$  for the highest root and consider the subgroup

$$\bar{J} := \langle U_{\pm\alpha_2}, U_{\pm\alpha_0}, h_{\alpha_1}(t), h_{\alpha_3}(u) : t, u \in K^* \rangle = \text{GL}_3(K) \times \text{GL}_1(K) \leq \text{SO}_8(K),$$

where  $U_{\pm\alpha} = \langle x_\alpha(t), x_{-\alpha}(u) : t, u \in K \rangle$  and  $h_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t-1)x_{-\alpha}(1)x_\alpha(-1)$ . Let  $\tau$  be a  $G_2$ -type triality graph automorphism of  $\text{SO}_8(K)$  which centralizes  $\langle U_{\pm\alpha_2}, U_{\pm\alpha_0} \rangle = \text{SL}_3(K) \leq \bar{J}$  and sends  $h_{\alpha_1}(t)$  to  $h_{\alpha_3}(t)$  and  $h_{\alpha_3}(t)$  to  $h_{\alpha_4}(t) = h_{\alpha_3}(t^{-1})h_{\alpha_2}(t^{-2})h_{\alpha_1}(t^{-1}) \times h_{\alpha_0}(t) \in \bar{J}$ . From the well-known Chevalley relations (see [11, 1.12.1], for example) we can determine the elements of order three in the coset  $\bar{J}\tau$ . Furthermore, if  $j\tau \in \bar{J}\tau$  has order three then we can identify  $C_{\text{SO}_8(K)}(j\tau)$  by calculating  $\dim C_{\mathcal{L}(\text{SO}_8(K))}(j\tau)$ , where  $\mathcal{L}(\text{SO}_8(K))$  is the Lie algebra of  $\text{SO}_8(K)$ . In this way we deduce that  $j\tau \in \bar{J}\tau$  is a  $G_2$ -type triality automorphism if and only if  $j \in Z(\bar{J})$ , i.e.

$$j = h_{\alpha_0}(c)h_{\alpha_2}(c^{-2})h_{\alpha_1}(a)h_{\alpha_3}(b)$$

where  $a$  and  $b$  are arbitrary non-zero elements of  $K$  and  $c^3 = (ab)^{-1}$ . This establishes the claim and it is easy to check that the bounds (23) and (24) are always sufficient.  $\square$

**Proposition 3.4.** *The conclusion to Theorem 1.1 holds in case (iv) of Table 3.1.*

**Proof.** Here  $H = N_G(H_0)$ , where  $H_0 = N_{G_0}(L)$  and  $L$  is a Sylow  $l$ -subgroup of  $G_0$  for an odd prime  $l$  which divides  $q^2 + 1$ . According to [18, 3.3.1] we have

$$H_0 \cong (D_{\frac{2}{d}(q^2+1)} \times D_{\frac{2}{d}(q^2+1)}) \cdot 2^2 \leq (\Omega_4^-(q) \times \Omega_4^-(q)) \cdot 2^2,$$

where  $d = (2, q - 1)$ . We claim that every involution in  $H \cap \text{PGL}(V)$  lies in  $\text{Inndiag}(G_0)$ . To see this, suppose  $z \in H \cap \text{PGL}(V)$  is an involution which does not lie in  $\text{Inndiag}(G_0)$ . Then  $z$  must centralize the direct product  $K_0 = \frac{1}{d}(q^2 + 1) \times \frac{1}{d}(q^2 + 1) \leq H_0$ , but this is not possible since each of the direct factors in  $K_0$  acts irreducibly on a 4-space and therefore  $K_0$  is self-centralizing.

Let  $x \in H \cap \text{PGL}(V)$  be an element of prime order  $r$ . If  $r$  is odd then  $r$  divides  $q^2 + 1$ , so  $|x^G| > \frac{1}{2}(q + 1)^{-1}q^{21}$  and the trivial bound

$$|x^G \cap H| \leq |H \cap \text{PGL}(V)| \leq 32(q^2 + 1)^2$$

is always sufficient. If  $r = 2$  then  $|x^G| > \frac{3}{4}(q + 1)^{-1}q^{13}$  (since  $x \in \text{Inndiag}(G_0)$ ) and the previous bound is sufficient unless  $q = 2$ . Here the desired result is easily checked using GAP [10].

Now assume  $x \in H - \text{PGL}(V)$  has prime order. If  $x$  is not a triality graph automorphism then  $q \geq 4$ ,  $|x^G| > \frac{1}{4}q^{14}$  and the bound

$$|x^G \cap H| < |H| \leq 96(q^2 + 1)^2 \tag{25}$$

is always sufficient. Finally, let  $x$  be a triality graph automorphism. If  $q \in \{2, 3\}$  then using GAP [10] we deduce that  $x^G \cap H$  is empty if  $x$  is a  $G_2$ -type triality, while  $|x^G \cap H| \leq 4^{\delta_{3,q}}200$  if  $x$  is a non- $G_2$  triality. If  $q \geq 5$  then the bounds  $|x^G| > \frac{1}{8}q^{14}$  and (25) are always sufficient.  $\square$

**Proposition 3.5.** *The conclusion to Theorem 1.1 holds in case (v) of Table 3.1.*

**Proof.** Here  $q = p$  is odd and  $H = N_G(P)$ , where  $P \leq G_0$  is a group of order 8 which centralizes a non-degenerate 1-decomposition  $\mathcal{D}$  of the natural  $G_0$ -module  $V$  (see [18, 3.4.2]). Now [18, 3.4.2(ii)] gives  $H_0 \cong [2^9].\text{SL}_3(2)$ , where  $[2^9]$  denotes a group of order  $2^9$ , and

$$H \cap \text{PGL}(V) \leq N_{\text{PGO}_8^+(q)}(\mathcal{D}) = 2^7 \cdot S_8 = \tilde{H},$$

where  $\tilde{H}$  is a  $\mathcal{C}_2$ -subgroup of type  $\text{O}_1(q) \wr S_8$ . According to [18, Table I], the maximality of  $H$  in  $G$  implies that  $G \cap \tilde{G}_\sigma = G_0$ , whence  $|G : G_0| \leq 6$  and  $|H| \leq 6 \cdot 2^9 |\text{SL}_3(2)|$ .

First assume  $x \in H \cap \text{PGL}(V)$  is an element of odd prime order  $r$ . Then Lagrange’s Theorem implies that  $r \in \{3, 7\}$  and in view of [5, 3.55] we see that the possibilities for  $x$  (up to  $\tilde{G}_\sigma$ -conjugacy) are as follows, where  $\omega \in K$  is a primitive  $r$ th root of unity.

|         | $p \neq r$                        | $p = r$        |
|---------|-----------------------------------|----------------|
| $r = 3$ | $[I_4, \omega I_2, \omega^2 I_2]$ | $[J_3^2, I_2]$ |
| $r = 7$ | $[I_2, \omega, \dots, \omega^6]$  | $[J_7, I_1]$   |

The result now follows from [6, 2.10] since  $|x^G \cap H| \leq |x^{\tilde{G}} \cap \tilde{H}|$ , where  $\tilde{G} = \text{PGO}_8^+(q)$ . Now assume  $r = 2$ . If  $x$  is conjugate to  $[-I_2, I_6]$  then applying [5, 3.55(iii)] and the proof of [6, 2.10] we calculate that  $f(x, H) < 0.602$  for all  $q \geq 3$  since

$$|x^G \cap H| \leq \binom{8}{2} + \frac{8!}{4!} + \frac{8!}{2!4!} + \frac{8!}{6!} = 2884,$$

$$|x^G| \geq 3 \frac{|\text{SO}_8^+(q)|}{|\text{GU}_4(q)|2} = \frac{3}{2} q^6 (q-1)(q^2+1)(q^3-1).$$

Similarly, if  $x = [-I_3, I_5]$  then  $f(x, H) < 0.591$  since

$$|x^G \cap H| \leq 3 \left[ \binom{8}{3} + \frac{8!}{6!} \binom{6}{2} + \frac{8!}{4!2!} 4 + \frac{8!}{3!2!} \right] = 22\,848,$$

$$|x^G| \geq \frac{3}{2} q^7 (q^4-1)(q^4+q^2+1)$$

(see [6, (48)]). The cases  $x = [-I_1, I_7]$  and  $[-I_4, I_4]$  are very similar.

**Case 2.**  $x \in H - \text{PGL}(V)$ .

Here  $x$  is a triality graph automorphism and the bounds

$$|x^G \cap H| < |H| = 2^\zeta 3 \cdot 2^9 |\text{SL}_3(2)| = 2^\zeta \cdot 258\,048,$$

$$|x^G| \geq 2^\zeta |\text{P}\Omega_8^+(q) : G_2(q)| = 2^{\zeta-2} q^6 (q^4-1)^2$$

are sufficient for all  $q \geq 5$ , where  $\zeta = 1$  if  $G$  contains an involutory graph automorphism, otherwise  $\zeta = 0$ . Finally, if  $q = 3$  then a GAP [10] calculation yields  $|x^G \cap H| \leq 128$  if  $x$  is a  $G_2$ -type triality, otherwise  $|x^G \cap H| \leq 7168$ . The desired result quickly follows.  $\square$

**Proposition 3.6.** *The conclusion to Theorem 1.1 holds in case (vi) of Table 3.1.*

**Proof.** Following [18], we say that a subgroup  $H_0 \leq G_0$  is a  $G_2$ -group if it is isomorphic to  $G_2(q)$ . By [18, 3.1.1(i)], such a subgroup fixes a 1-dimensional non-singular subspace  $U$  of the natural  $G_0$ -module  $V$  and we may identify  $H_0$  with the image of the composition

$$G_2(q) \xrightarrow{\rho} \text{Stab}_{G_0}(U) \hookrightarrow G_0,$$

where  $\rho$  is the irreducible embedding labeled ( $\mathcal{C}4$ ) in Table 2.3. In particular, our earlier work in Lemma 2.13 applies. We also note that  $H_0 = C_{G_0}(\tau)$  for a suitably chosen triality graph automorphism  $\tau$ .

Let  $H$  be a subgroup of  $G$  such that  $H \cap G_0$  is a  $G_2$ -group. Then

$$H \cap \text{PGL}(V) = \begin{cases} G_2(q) \times \langle \gamma \rangle & \text{if } G \text{ contains an involutory graph automorphism,} \\ G_2(q) & \text{otherwise,} \end{cases}$$

where  $\gamma$  is an involution such that  $v(\gamma) = 1$  with respect to  $V$ . We claim that  $f(x, H) < 5/8$  for all prime order elements  $x \in G$ . If  $x \in H \cap \text{PGL}(V)$  then the claim quickly follows from the proof of Lemma 2.13 and we leave the reader to make the necessary minor adjustments. For the remainder, we will assume  $x \in H - \text{PGL}(V)$ .

If  $x$  is a field automorphism of prime order  $r$  then  $q = q_0^r$  and the bounds

$$|x^G \cap H| \leq 2|G_2(q) : G_2(q^{1/r})| < 4q^{14(1-\frac{1}{r})}, \quad |x^G| > \frac{1}{4}q^{28(1-\frac{1}{r})}$$

are always sufficient. The same bounds are valid (with  $r = 2$ ) if  $x$  is an involutory graph-field automorphism. Next fix a triality graph automorphism  $\tau$  such that  $C_{G_0}(\tau) = G_2(q)$ . If  $x$  is a triality graph-field automorphism then  $q = q_0^3$  and without loss we may assume  $x = \tau\phi$ , where  $\phi$  is a field automorphism of order 3 and  $[\tau, \phi] = 1$ . Then  $x^G \cap H \subseteq G_2(q)\phi \times \langle \tau \rangle$  and the result follows via [5, 3.43, 3.48] since  $|x^G \cap H| < 4q^{28/3}$  and  $|x^G| > \frac{1}{4}q^{56/3}$ .

Finally, let us assume  $x$  is a triality graph automorphism, in which case  $x^G \cap H \subseteq G_2(q) \times \langle \tau \rangle$ . If  $x$  is a non- $G_2$  triality then using [21, 1.3(ii)] we deduce that

$$|x^G \cap H| \leq 2^\zeta i_3(G_2(q)) < 2^{1+\zeta} (q + 1)q^9, \\ |x^G| \geq 2^\zeta \frac{|\text{P}\Omega_8^+(q)|}{|\text{PGU}_3(q)|} = 2^\zeta \frac{q^9(q^3 - 1)(q^4 - 1)^2}{(2, q - 1)^2},$$

where  $\zeta = 1$  if  $G$  contains an involutory graph automorphism, otherwise  $\zeta = 0$ . These bounds are always sufficient. On the other hand, if  $x$  is a  $G_2$ -type triality then

$$|x^G \cap H| = 2^\zeta |\{h\tau : h \in G_2(q), h^3 = 1, C_{G_0}(h\tau) = G_2(q)\}|$$

and

$$|x^G| \geq 2^\zeta \frac{|\text{P}\Omega_8^+(q)|}{|G_2(q)|} = 2^\zeta \frac{q^6(q^4 - 1)^2}{(2, q - 1)^2}. \tag{26}$$

If  $p \equiv \epsilon \pmod{3}$  then there are exactly two distinct classes of elements of order three in  $G_2(q)$ , with representatives  $x_1$  and  $x_2$  where

$$|x_1^{G_2(q)}| = \frac{|G_2(q)|}{|\text{SL}_3^\epsilon(q)|} = q^3(q^3 + \epsilon), \quad |x_2^{G_2(q)}| = \frac{|G_2(q)|}{|\text{GL}_2^\epsilon(q)|} = q^5(q + \epsilon)(q^4 + q^2 + 1)$$

and it is easy to see that  $x_1$  (respectively  $x_2$ ) is  $\bar{G}$ -conjugate to  $[I_2, \omega I_3, \omega^2 I_3]$  (respectively  $[I_4, \omega I_2, \omega^2 I_2]$ ) since

$$V \downarrow A_2 = V_3 \oplus V_3^* \oplus 0 \oplus 0,$$

where  $A_2 < G_2$  (algebraic groups) is generated by long root subgroups and  $V_3$  and  $0$  denote the natural and trivial irreducible  $A_2$ -modules, respectively. From [11, p. 215] it follows that

$$|\{h\tau : h \in G_2(q), h^3 = 1, C_{G_0}(h\tau) = G_2(q)\}| = |x_1^{G_2(q)}| + 1 = q^3(q^3 + \epsilon) + 1$$

and thus  $f(x, H) < 5/8$  as required. Finally, assume  $p = 3$  and suppose  $C_{G_0}(h\tau) = G_2(q)$ , where  $h \in G_2(q)$  has order three. In the notation of [20],  $h$  lies in one of the unipotent classes  $A_1, \tilde{A}_1, \tilde{A}_1^{(3)}$  and  $G_2(a_1)$  of the algebraic group  $G_2$ . In fact, by arguing as in the proof of [21, 6.3] we deduce that  $h$  must lie in the class  $\tilde{A}_1$  and thus  $|x^G \cap H| = 2^\zeta q^6$ . As before, the desired result follows via (26).  $\square$

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