Parabolic systems of forward-backward type exhibiting (p, q)-type growth

Miles Oliver Caddick

The Queen's College

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy

Trinity Term 2018

Abstract

This thesis is concerned with systems of nonlinear equations exhibiting both forward-backward type behaviour, and non-standard growth conditions. A motivating problem in one spatial dimension with application to the Met Office is discussed before proceeding to consider higher dimensional problems.

In the higher dimensional setting, in the absence of a monotonicity condition we work within the framework of Young measure solutions. We prove existence of large-data global-in-time Young measure solutions to initial-boundary-value problems for multidimensional nonlinear systems of forward-backward type of the form $\partial_t u - \operatorname{div}(a(Du)) + Bu = F$, where $B \in \mathbb{R}^{m\times m}$, $Bv\cdot v \leq 0$ for all $v \in \mathbb{R}^m$, F is an m-component vector-function defined on a bounded open Lipschitz domain $\Omega \subset \mathbb{R}^n$, and a is a locally Lipschitz mapping of the form $a(A) = K(A)A$, where $K : \mathbb{R}^{m\times n} \to \mathbb{R}$. The long-time behaviour of these Young measure solutions is then studied, and under suitable assumptions on the source term we show convergence to Young measure solutions of the corresponding time-independent equations. We also discuss how the results proven can be adapted to cover mappings a which have different structure.

We develop a numerical algorithm for the approximate solution of problems in this class, and we prove the convergence of the algorithm to a Young measure solution of the system under consideration.

Acknowledgements

First and foremost, to my two supervisors. To Professor Endre Süli, as without his patience, generosity, and guidance this work could not have been completed. And to Professor Mike Cullen, for introducing to me the motivating problem behind my studies, and for inviting me to visit him in the Met Office.

Professors Melanie Ruplin, Jan Kristensen, Luc Nguyen and Yves Capdeboscq agreed to be assessors for Transfer and Confirmation of Status. During these assessments they all gave valuable comments and suggestions, and I would like to thank them for taking their time to do this. Furthermore I'd like to thank Professor Jan Kristensen and Dr Filip Rindler for agreeing to be my examiners for my viva.

To Jonathan Whyman for encouraging me to get involved in the Consultative Committee for Graduates, and Sandhya Patel for helping me to find me feet when I did so.

To Luca Alasio, Matthew Schrecker, Francesco Della Porta, Aleksander Klimek, Bogdan Raita, Ellya Kawecki, Kevin Schlegel and Matteo Croci for the many discussions and for being so generous with their time. Also, despite his numerous, and frankly completely unforgivable, pranks, to Dr. Franz Gmeineder for his sympathetic ear and frequent pep-talks.

To Tabea Tscherpel, for being an extremely conscientious and considerate friend and housemate, and to Tom Cuthbertson, for asking truly deep mathematical questions.

To the Lunch Party, the MCR Committee of 2016 – 2017, and the Board Games @ Queen's group, for giving me multiple opportunities to escape from the office and switch off from my DPhil.

To Sarah Gouldesbrough for teaching me about Queen's, and Tim Westwood for his good sense of humour. Their constant source of friendship and lunchtime companionship will remain a fond memory.

To my parents, for their unwavering support.

Finally, to Abigail Linton, for allowing me to escape when it all got a bit too much.

Statement of Originality

In Section 5.1, plots of numerical experiments have been included. Figures 5.1−5.4 were generated by Matteo Croci.

The contents of Sections 3.1−3.3, and Chapter 4, have been arranged into a draft paper with Professor Endre Süli and Professor Mike Cullen [15], which is awaiting submission to a journal.

Contents

Chapter 1

Introduction and Literature Review

1.1 Introduction

1.1.1 Systems under consideration

This work is concerned with particular nonlinear systems of partial differential equations which exhibit a *forward-backward*, or *non-monotone* behaviour, and whose nonlinearity has some non-standard growth conditions similar to those studied in [11], [12], [50], [58] and [59], among others. More explicitly, we study systems of equations of the form

$$\partial_t u - \operatorname{div}(a(Du)) - Bu = F, \tag{1.1.1.1}$$

with $u : [0,T] \times \Omega \to \mathbb{R}^m$ and $\Omega \subset \mathbb{R}^n$. Throughout we will take real numbers p_i (for $i = 1, \ldots, m$) to satisfy $p_i > \max\{1, \frac{2n}{n+2}\}$, and shall denote the Hölder conjugate of p_i by p_i', meaning that

$$\frac{1}{p_i} + \frac{1}{p_i'} = 1.$$

We shall also define $p := \min_{i=1,\ldots,m}\{p_i\}$ and $q := \max_{i=1,\ldots,m}\{p_i\}$, and impose

$$q - p < 1.$$

Further, we shall denote the Hölder conjugates of p and q by p' and q' respectively. We also assume that

$$F \in \bigtimes_{i=1}^{m} L^{p_i'}(Q_T), \tag{1.1.1.2}$$

where, here and throughout the thesis, we define $Q_T := [0,T] \times \Omega$. Further, we assume

$$Bu \cdot u \leq 0 \tag{1.1.1.3}$$

with $B \in \mathbb{R}^{m\times m}$ a constant matrix, and that the systems we shall be considering are subject to some initial data

$$u(0,\cdot) = u_0(\cdot) \in L^2(\Omega;\mathbb{R}^m), \tag{1.1.1.4}$$

and boundary data of Dirichlet type

$$u|_{(0,T]\times\partial\Omega} = g, \tag{1.1.1.5}$$

where for the most part we shall consider the case $g \equiv 0$. Here Du denotes the spatial derivative of $u\colon (0,T)\times\Omega \to \mathbb{R}^m$, $\Omega \subset \mathbb{R}^n$ is a bounded open subset of $\mathbb{R}^n$ with a sufficiently smooth boundary, and $a : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ is a locally Lipschitz mapping with the following properties:

$$a(A) = K(A)A, \tag{1.1.1.6}$$

where $K : \mathbb{R}^{m\times n} \to \mathbb{R}$ is continuous, and

$$c_0 \sum_{i=1}^{m} (\mu_i^2 + |A_i|^2)^{\frac{p_i-2}{2}} \leq K(A) \leq c_1 \sum_{i=1}^{m} (\mu_i^2 + |A_i|^2)^{\frac{p_i-2}{2}} \tag{1.1.1.7}$$

with $0 < c_0 < c_1$, $p_i > 1$ for all $i = 1,\ldots,m$, and μ_i is a constant defined by

$$\mu_i \begin{cases} \in \mathbb{R} \text{ if } p_i \geq 2 \\ \neq 0 \text{ if } 1 < p_i < 2, \end{cases} \tag{1.1.1.8}$$

and where $|A_i|$ denotes the Euclidean norm of the i-th row of the matrix A. Note that we make no assumptions here about the monotonicity of the map a.

1.1.2 Physical relevance

The motivation for studying equations of this form comes from the following system of equations, which is of interest to the Met Office:

$$\partial_t w - \partial_z(K(\partial_z w)\partial_z w) - Aw = F \quad \text{for } (t,z) \in (0,T] \times (0,1), \tag{1.1.2.1}$$

$$w(t,1) = f(t) \quad \text{for } t \in (0,T], \tag{1.1.2.2}$$

$$w_1(t,0) = g_1(t) \quad \text{for } t \in (0,T], \tag{1.1.2.3}$$

$$w_2(t,0) = g_2(t) \quad \text{for } t \in (0,T], \tag{1.1.2.4}$$

$$\partial_z w_3(t,0) = h(t) \quad \text{for } t \in (0,T], \tag{1.1.2.5}$$

subject to the initial condition

$$w(0,z) = w_0(z) \quad \text{for } z \in (0,1). \tag{1.1.2.6}$$

Here, $w : (0,T) \times (0,1) \to \mathbb{R}^3$ is a vector with components $w = (w_1, w_2, w_3)$, A is the skew-symmetric matrix

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and F is a forcing term given by the constant vector

$$F(t,x) = \begin{pmatrix} -V \\ U \\ 0 \end{pmatrix}. \tag{1.1.2.7}$$

In the Met Office model, the first two components of w represent wind velocity, whereas the third represents temperature. The function K is given by

$$K(\partial_z w) = \lambda^2(|\partial_z w_1|^2 + |\partial_z w_2|^2)^{\frac{1}{2}} k(\partial_z w), \tag{1.1.2.8}$$

where k is referred to as a stability function and λ^2 is some mixing length which is a strictly positive, continuous and bounded function of the spatial variable z. We give the precise definition of K below:

$$(|\partial_z w_1|^2 + |\partial_z w_2|^2)^{\frac{1}{2}} k(\partial_z w) = \begin{cases} \frac{(|\partial_z w_1|^2+|\partial_z w_2|^2)^{\frac{3}{2}}}{|\partial_z w_1|^2+|\partial_z w_2|^2+\partial_z w_3} & \text{if } \partial_z w_3 > 0, \\ \sqrt{|\partial_z w_1|^2 + |\partial_z w_2|^2 - \partial_z w_3} & \text{if } \partial_z w_3 \leq 0. \end{cases} \tag{1.1.2.9}$$

It should be noted that this is a one-dimensional version of a higher-dimensional model. The model from which this problem stems can be found in [4], and is summarised below:

$$\frac{D\mathbf{u}}{Dt} - f\mathbf{k} \times \mathbf{u} + \nabla\phi = \frac{\partial}{\partial z}\left(K_m \frac{\partial \mathbf{u}}{\partial z}\right),$$

$$\frac{\partial \phi}{\partial z} = g\frac{\theta - \theta_0}{\theta_0},$$

$$\frac{D\theta}{Dt} = \frac{\partial}{\partial z}\left(K_h \frac{\partial \theta}{\partial z}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$, u, v are components of the horizontal wind velocity vector $\mathbf{u}$, w represents the vertical wind velocity, f is the Coriolis parameter (which is a constant governed by the rotation of the Earth), $\mathbf{k}$ is the unit vertical vector, ϕ is the geopotential (that is, the potential of the Earth's gravitational field), θ is the potential temperature, θ_0 is a surface reference potential temperature, $K_m = 0.7K_h$ (to be defined below) and ∇ denotes the horizontal gradient operator (so $\nabla \cdot \mathbf{u} + \frac{\partial w}{\partial z} = 0$).

To obtain the formulation given in (1.1.2.1) from this we consider a one-dimensional "vertical slice", meaning that we consider the horizontal wind vector $\mathbf{u}$, temperature θ and geopotential ϕ to depend only on the height, z, above the surface of the earth, and we set the Coriolis parameter $f = 1$. In addition to this, we also ignore the fact that

K_m and K_h differ by a multiplicative constant, simply calling them both K. Finally, we assume that λ, which in reality is a smooth function of $z \in [0,1]$ which is bounded in the sense that there are constants c_1 and c_2 such that $0 < c_1 \le \lambda(z) \le c_2$, is identically equal to 1.

In the function K in (1.1.2.8) given above, the function k defined in (1.1.2.9) is really a function of the 'gradient Richardson number', defined by Beare and Cullen in [4] as

$$Ri = \frac{g}{(|\partial_z w_1|^2 + |\partial_z w_2|^2)\theta_0} \partial_z w_3,$$

where θ_0 is some constant reference surface potential temperature and g is the gravitational constant. Then k is defined by

$$k(Ri) = \begin{cases} \frac{1}{1+10Ri} & \text{if } Ri \ge 0, \\ \sqrt{1-16Ri} & \text{if } Ri < 0, \end{cases}$$

which, ignoring constants, is easily seen to agree with (1.1.2.9). As a final simplification we set all constants to 1.

As mentioned above, the model under consideration is a one-dimensional version of a higher dimensional model, which governs the evolution of the Atmospheric Boundary Layer (henceforth ABL). The atmosphere of the Earth is divided into many different layers, the lowest of which is called the troposphere. According to [64], the ABL is the lowest part of the troposphere, and is the part which is most influenced by surface-level forcings, reacting to these over a timescale of roughly one hour. Such forcings include, for example, heat transfer, pollutant emission and flow modification induced by the terrain. In [38] it is instead defined as "the layer of air directly above the Earth's surface in which the effects of the surface (friction, heating and cooling) are felt directly on time scales of less than one day, and in which significant fluxes of momentum, heat or matter are carried by turbulent motions on a scale of the order of the depth of the boundary layer or less".

The effective modelling of the ABL is important. We spend almost our entire lives living within the ABL, and it is here that pollution dissipates and where phenomena such as fog occur. Additionally, weather forecasting of quantities such as dew, frost and temperature are in fact forecasts of the ABL, and the evolution of storms and hurricanes is tied to the inflow of moist air in the ABL (according to [64]). Thus for correct modelling of elements of the weather system it is important to have a well-posed model for the ABL.

1.1.3 Properties (or lack thereof) of the function K

Classical monotone operator theory allows us to deal with well-posedness of PDE problems when our function K satisfies certain coercivity, continuity, monotonicity and boundedness assumptions. This theory is typically applied to the case where $K(\partial_z w) = |\partial_z w|^{p-2}$ for $p \in (1, \infty)$ to show existence and uniqueness of solutions to the parabolic p-Laplacian problem, and to continuous functions of $\partial_z w$ satisfying the same coercivity, monotonicity and boundedness conditions as the p-Laplacian. However, our model fails to fall into this category on multiple levels.

Firstly, the growth rates exhibited by K differ according to which component we are looking at, making this vaguely reminiscent of the (p, q)-growth problems studied in (among others) [12].

Secondly, there is a term of the form $\frac{1}{1+\partial_z w_3}$ appearing in the function k when we have $\partial_z w_3 > 0$, which is reminiscent of so-called linear growth problems. Such problems have been considered in, for example, [2], where existence and uniqueness have been established in a suitable function space for dimensions larger than or equal to 2. However, to this author's knowledge there are no results for (p, q)-growth problems which exhibit some form of linear growth property, which is what would be needed to study the particular systems we shall be considering in this thesis (under the assumption of monotonicity).

The third is that the operator $a(\xi) := K(\xi)\xi$ is simply not a monotone operator. We give the relevant calculations in an appendix, but we see that taking $\xi_1 = (0.035, 0, -0.01)$ and $\xi_2 = (0.05, 0, 0)$, as well as $\xi_1 = (-0.2, -0.1, 0.2)$ and $\xi_2 = (-0.1, 0, 0.5)$, it holds that at the points ξ_1 and ξ_2 the mapping a fails to satisfy the monotonicity condition.

Finally, there exists no function ψ for which $a(\xi) = \nabla\psi(\xi)$. To see this, consider the circle C parameterised by $r(\theta) = (1 + \cos(\theta), 0, \sin(\theta))$. Then it can be shown (we refer to the appendix once again for calculations) that

$$\oint_C a(x)\ \mathrm{d}x \neq 0.$$

This means that the vector field a is not conservative, and thus cannot be written as the gradient of some C^1 function.

1.1.4 Literature Review

Problems with these types of nonstandard growth conditions were first considered in the elliptic setting in the early 1990s by Marcellini in his paper [50]. In this, he investigated the existence and regularity of solutions of elliptic equations of the following type:

$$\begin{cases} \mathrm{div}(a(x,Du)) = b(x) \text{ for } x \in \Omega, \\ u(x) = g(x) \text{ for } x \in \partial\Omega, \end{cases} \tag{1.1.4.1}$$

where $\Omega \subset \mathbb{R}^n$ for $n \geq 2$ and the component functions of a are assumed to be locally Lipschitz. Under some non-standard growth conditions (referred to throughout this as (p,q)-growth conditions) of the following form:

$$\sum_{i,j} \partial_{\xi_j} a^i(x,\xi)\lambda_i\lambda_j \geq m(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2, \tag{1.1.4.2}$$

$$|\partial_{\xi_j} a^i(x,\xi)| \leq M(1+|\xi|^2)^{\frac{q-2}{2}}, \tag{1.1.4.3}$$

$$|\partial_{\xi_j} a^i(x,\xi) - \partial_{\xi_i} a^j(x,\xi)| \leq M(1+|\xi|^2)^{\frac{p+q-4}{4}}, \tag{1.1.4.4}$$

$$|\partial_{x_s} a^i(x,\xi)| \leq (1+|\xi|^2)^{\frac{p+q-2}{4}}, \tag{1.1.4.5}$$

the following range being imposed on p and q:

$$2 \leq p \leq q < \frac{p(n+2)}{n},$$

a bound on the component functions of a of the form

$$|a^i(x,0)| \leq M$$

and the following regularity assumed on b and the boundary data g:

$$b \in L^{\frac{p}{p-1}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega),$$

$$g \in W^{1,\frac{p(q-1)}{p-1}}(\Omega),$$

it was shown that there is a weak solution u to (1.1.4.1) in $W^{1,p}(\Omega) \cap W^{1,q}_{\mathrm{loc}}(\Omega)$ with the following higher interior regularity

$$u \in W^{1,p}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap W^{2,2}_{\mathrm{loc}}(\Omega). \tag{1.1.4.6}$$

In addition to this, it was shown that if the coefficients a^i are $C^{1,\alpha}_{loc}$ and b is $C^{0,\alpha}_{loc}$ then in fact there is a classical solution to (1.1.4.1) lying in the space

$$(g + W^{1,p}_0(\Omega)) \cap C^{2,\alpha}_{loc}(\Omega). \tag{1.1.4.7}$$

These results have, under certain assumptions, been extended through the work of I. V. Nezhinskaya, see [53], [54] and [55], under the assumption that Ω is a bounded domain with a C^2 boundary (compared to the assumption that Ω is a bounded open set used in [50]). Across the two papers [53] and [54] the solvability of the problem (1.1.4.1) (as well as its parabolic counterpart) are considered, but with slightly different growth conditions:

$$\sum_{i,j} \partial_{\xi_j} a^i(x,\xi)\lambda_i\lambda_j \geq m(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2, \tag{1.1.4.8}$$

$$\sum_{i,j} \left| \frac{\partial a_i}{\partial \xi_j}(x,\xi) \right| \leq m_1(1+|\xi|^2)^{\frac{q-2}{2}}, \tag{1.1.4.9}$$

$$\sum_{i,j} \left| \frac{\partial a_i}{\partial x_j}(x,\xi) \right| \leq m_2(1+|\xi|^2)^{\frac{q-1}{2}}. \tag{1.1.4.10}$$

We also have a different admissible range of p and q. In the elliptic case it is shown that under this smaller range of

$$2 \leq p \leq q \leq p\left(1 + \frac{2}{n^2+n}\right), \tag{1.1.4.11}$$

that there is a weak solution to the problem which is of the class

$$C^{1+\alpha}(\bar{\Omega}) \cap W^{2,2}(\Omega)$$

(compared to (1.1.4.6)). Further it is shown that one can obtain a classical solution with the regularity

$$C^{2,\alpha}(\bar{\Omega})$$

(compared to (1.1.4.7)). Nezhinskaya also considers the parabolic version of (1.1.4.1) in [54], showing that under suitable growth conditions (but most importantly condition (1.1.4.11)) one obtains a classical solution of the parabolic problem.

The final of Nezhinskaya's papers which we discuss, [55], is of particular interest, as it is the only example found by this author where Neumann data is considered alongside the p, q-growth conditions, albeit only in an elliptic setting. Nezhinskaya considers the following Neumann boundary-value problem:

$$\operatorname{div}(a(x, u_x)) = B(x, u) \tag{1.1.4.12}$$

for $x \in \Omega$ (a bounded domain in $\mathbb{R}^m$ with $n \geq 2$ with a C^2 boundary $\partial\Omega$), with the following boundary condition:

$$A_i(x, u_x) \cos(\mathbf{n}(x), x_i^h) + \Psi(x, u) = 0. \tag{1.1.4.13}$$

Here $\mathbf{n}(x)$ denotes the outward unit normal to $\partial\Omega$ at the point x, and the following growth conditions are satisfied:

$$\sum_{i,j} \partial_{\xi_j} a^i(x, \xi)\lambda_i\lambda_j \geq m(1 + |\xi|^2)^{\frac{p-2}{2}}|\lambda|^2, \tag{1.1.4.14}$$

$$\sum_{i,j} \left| \frac{\partial a_i}{\partial \xi_j}(x, \xi) \right| \leq m_1(1 + |\xi|)^{q-2}, \tag{1.1.4.15}$$

$$\sum_{i,j} \left| \frac{\partial a_i}{\partial x_j}(x, \xi) \right| \leq m_2(1 + |\xi|)^{q-1}, \tag{1.1.4.16}$$

along with (1.1.4.11). The particular form of the boundary data considered aids with smoothing out the boundary in order to obtain the necessary estimates. Under the above assumptions, Nezhinskaya obtains the necessary boundary estimates needed in order to pass to the limit in a sequence of approximating problems in order to obtain a solution.

More recently, Cupini, Marcellini and Mascolo in [22] proved analogous results to those in [50]. In particular, the admissible range of p and q is extended, and the criteria that $p \geq 2$ is removed. In its place, the authors assume that

$$1 < p \leq q \leq p+1 \text{ and, if } p < n, \text{ that } q < p\frac{n-1}{n-p},$$

and show that if $b \in L^{\infty}_{loc}(\Omega)$, $g \in W^{1,p'(q-1)}(\Omega)$ (where Ω is once again a bounded open subset of $\mathbb{R}^m$), and assumptions (1.1.4.2)−(1.1.4.5) hold, then there exists a weak solution u to (1.1.4.1) in the sense that for every open subset Ω' of Ω satisfying $\bar{\Omega}' \subset \Omega$, and for every $\phi \in W^{1,q}_0(\Omega')$, it holds that

$$\int_{\Omega} a(x, Du) \cdot D\phi(x) + b(x)\phi(x) \, \mathrm{d}x = 0.$$

The authors also demonstrate that these weak solutions u satisfy (1.1.4.6).

There are four papers in particular that have been encountered by this author which relate to parabolic problems with these non-standard growth conditions. The first two are the papers by Bögelein, Duzaar and Marcellini, see [11] and [12], both of which deal with parabolic equations or systems of the form

$$\partial_t u - \operatorname{div}(Df(Du)) = 0,$$

with f a C^2 convex function satisfying growth conditions similar to

$$|\xi|^p \leq f(\xi) \leq L(1 + |\xi|^q). \tag{1.1.4.17}$$

In these papers, the nonlinear term is approximated by $f_\epsilon(\xi) = f(\xi) + \epsilon|\xi|^q$ so that standard growth rates are satisfied. Monotone operator theory then applies to give a unique solution u_ϵ, and local estimates on the gradient of the solutions are obtained to justify passing to the limit in the weak formulation. We make this more precise in later chapters, commenting here that the paper [11] deals only with equations (rather than systems), and in both papers the lower growth rate is assumed to be at least 2. The third and fourth papers are by Singer, [58] and [59]. In the second of these he considers systems in which the non-linear term is allowed some space and time dependence (in particular it is assumed that we have some Hölder continuity condition in the spatial variable). The proofs follow along the same lines as those found in [12], however it should be noted that the assumption of Hölder continuity in the space variable allows for an extension of the result in [12] to include the case where the lower growth rate is larger than $\frac{2n}{n+2}$, at the cost of having a shorter admissible range of the upper and lower growth exponents. The first of these mimics the result of [11], but considering the case where the lowest growth exponent is assumed to be larger than $\frac{2n}{n+2}$. The *variational solutions* discussed in section 2.1 of [12] are expanded upon in [13], wherein the authors show the existence of global in time variational solutions to equations of the form

$$\partial_t u - \operatorname{div}((Df)(x, Du)) = h(x)$$

for f a convex function satisfying

$$f(x, u, \xi) \geq \nu|\xi|^p - g(x)(1 + |u|) \text{ and } \int_\Omega f(x, u_0, Du_0)\, \mathrm{d}x < \infty,$$

where $g \in L^{p'}(\Omega)$, $g \geq 0$ and $p > 1$, subject to L^2 initial data. These solutions have the properties that $\partial_t u \in L^2(Q_\infty; \mathbb{R}^m)$ and $u \in C^{0,\frac{1}{2}}([0, T]; L^2(\Omega; \mathbb{R}^m))$ for all $T > 0$.

Continuing on this variational note, there has been a significant amount of research done into the regularity of minimisers of integrals in which the integrand satifies (1.1.4.17). We do not delve into explicit details here; regularity of solutions is not something which we concern ourselves with for the most part, and due to the assumptions we shall make regarding the function a, the variational structure presented here is not something which we shall be able to use. Results regarding the regularity of such minimisers can be found in [9, 10, 17, 21, 30–33, 49] (among others), and references contained within.

Up until now we have only been discussing the case where the operator defined by the function a is monotone and has a particular structure, and as such the authors have been in a position to study *weak solutions* of the equations or systems. On the *non-monotone*, or *forward-backward* side of things, the situation is a little different. For the most part, authors have been confined to studying *Young measure solutions*. Typically Young measure solutions are a pair (u, ν), with u being a function and ν being a Young measure, satisfying

$$\int_0^T \int_\Omega \partial_t u \cdot \psi + \langle \nu_{t,x}, a \rangle : D\psi \, \mathrm{d}x \, \mathrm{d}t = 0$$

for all ψ in some suitable function space. Young measure solutions to parabolic partial differential equations have been studied by a number of authors. Frehse and Specovius-Neugebauer in [37] showed the existence of Hölder continuous Young measure solutions to two-dimensional non-monotone quasilinear problems exhibiting quadratic growth (that is, $p = q = n = 2$), under the assumption that their nonlinearity a was expressible as

$$a = \nabla\Phi \tag{1.1.4.18}$$

for some C^1 function Φ. Young measure solutions to certain forward backward parabolic equations were also studied by Demoulini in [23]: under the assumptions that $m = 1$, $p = q = 2$ and that (1.1.4.18) is satisfied, a sequence of approximating solutions was constructed by means of minimising an integral which involves the convexification of Φ. Using this construction, a family of Young measures was generated displaying an independence property, from which it was possible to deduce a uniqueness result regarding the function u (but not regarding the family of Young measures). From this, a weak-strong uniqueness result follows if the classical solution which is assumed to exist satisfies additional constraints on its gradient.

In the PhD thesis of Thanh [65] and some of the references contained therein (in particular [68]), possible regularising terms are discussed for forward-backward parabolic equations, the inclusion of which allows for solutions with higher regularity to be obtained than those considered here; Thanh also obtained results concerning the long-time behaviour of certain Young measure solutions and (in the case of an equation in one space dimension) the support of the Young measure appearing in the Young measure solutions. These results are achieved through studying the so-called *Sobolev regularisation*

$$\partial_t u = \operatorname{div}(a(Du)) + \epsilon\Delta\partial_t u,$$

and making certain structural assumptions on the function a. In particular, in [65] it is assumed that a is globally Lipschitz satisfying (p, p) growth conditions with $p \in (1, 2]$ and that (1.1.4.18) holds. The results deduced in [65] are applied only to equations, as opposed to systems. More recently, Kim and Yan in [45] have expanded on their earlier work in [44] to show the existence of infinitely many Lipschitz solutions (and under further assumptions, the existence of a unique classical solution) for particular classes of forward-backward parabolic equations with Neumann boundary conditions. Although their work allows us to consider better notions of solution than the Young measure solutions which we shall discuss here, many of their structural assumptions regarding their nonlinearities are incompatible with our setting. Furthermore, their focus is solely on equations, whereas we are concering ourselves with systems.

On the numerical side, Carstensen and Roubíček [18] have previously considered the numerical computation of Young measures arising in minimisation problems in the calculus of variations. As we lack this variational structure ourselves (in this work we are not assuming (1.1.4.18)), neither of the methods in [18] and [23] will be suitable in our setting. In [5] the authors consider numerical experiments for equations which have a regularising term added, but only for small values of the regularising parameter and do not consider numerical schemes for the limiting case.

Furthermore, recent work by Fjordholm, Käpelli, Mishra and Tadmor in [35] (and by Fjordholm, Mishra and Tadmor in [36]) has focused on the development of a numerical scheme for the numerical approximation of measure-valued solutions to systems of hyperbolic conservation laws. Such measure measure-valued solutions were discussed by DiPerna in [25] for equations of the form

$$\partial_t u + \partial_x f(u) = 0,$$

with $f : \mathbb{R}^m \to \mathbb{R}^n$, subject to

$$u(0,x) = u_0(x),$$

and were obtained as limits of diffusive and dispersive equations of the form

$$\partial_t u + \partial_x f(u) = \epsilon \partial_{xx} u$$

and

$$\partial_t u + \partial_x f(u) = \epsilon \partial_{xxx} u.$$

He then showed that under certain assumptions in the scalar case one could deduce that the measure-valued solution was in fact a solution in the usual distributional sense. Measure-valued solutions to incompressible fluid equations were introduced in a three-dimensional setting by DiPerna and Majda in [26]. In this paper, measure-valued solutions were defined for the Euler equations, and were obtained as limits of weak Leray-Hopf solutions to the Navier-Stokes equations. We once again do not discuss details in depth, but instead highlight certain publications which demonstrate a key difference between the measure-valued solutions for hyperbolic conservation laws and the forward-backward nonlinear systems of PDEs that we are considering in this thesis. One desirable property to have is that of *weak-strong uniqueness*; the property by which measure-valued solutions agree with classical solutions (under the assumption that the classical solutions exist). Such a result for systems of hyperbolic conservation laws of the form

$$\partial_t u + \mathrm{div}_x f(u) = 0, \tag{1.1.4.19}$$

with $u : [0,\infty) \times \Omega \to \mathbb{R}^m$ with $\Omega \subset \mathbb{R}^m$ an open set in $\mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}^{m\times n}$ a C^2 map, was obtained by Brenier, De Lellis and Székelyhidi in [14] under the assumption that (1.1.4.19) possesses a strictly convex entropy. Similar results have also been obtained for compressible fluid models (see [34] and [42]). Such results, at least to this author's knowledge, are sadly sparse for the nonlinear forward-backward systems which we shall be considering in this thesis. As mentioned previously, Demoulini obtained one such result for equations satisfying particular growth and structural conditions (see the remarks after

Theorem 2.8 in [23]), and Kim and Yan (see Theorem 1.2 in [45]) obtained results in this direction (once again for particular equations with particular assumptions in place) but none of these results hold in our setting.

It is worth noting that there has been a significant amount of research carried out on equations of the form

$$\partial_t u - \Delta(\phi(u)) = 0,$$

where the function ϕ is non-monotone, and usually satisfying some additional structural assumptions. We mention these types of equations only in passing; in one spatial dimension, if we were to formally take an extra spatial derivative of (1.1.2.1) and set $F \equiv 0$, we would obtain the following:

$$\partial_t \partial_z w - \partial_{zz}(K(\partial_z w)\partial_z w) - A\partial_z w = 0.$$

Setting $u = \partial_z w$ gives

$$\partial_t u - \partial_{zz}(K(u)u) - Au = 0. \tag{1.1.4.20}$$

Defining $\phi(u) := K(u)u$, we see that this can then be written as

$$\partial_t u - \partial_{zz}(\phi(u)) - Au = 0.$$

A wealth of information can be found in [7, 8, 16, 41, 51, 60–62, 65–69] and the references contained therein.

1.2 Outline of thesis

The structure of this thesis is as follows.

1.2.1 Chapter 2

In Chapter 2 we shall review some of the key aspects of monotone operator theory. We shall deduce some conditions on K for which we can apply monotone operator theory, before moving on to decouple the system of equations obtained from the Met Office model.

We shall demonstrate that one can apply monotone operator theory to each of these decoupled systems (under certain conditions), before presenting a small (but important) correction to [12].

1.2.2 Chapter 3

In Chapter 3 we shall begin with the definition of a *Young measure solution*, and we shall introduce some necessary results and definitions needed in order to prove the existence of these solutions for the systems under consideration. We shall then prove the existence of a Young measure solution, and relate this back to the system of equations which is of interest to the Met Office. We then move on to discuss the long-time behaviour of the Young measure solutions which have been constructed in the existence proof. We end the chapter by considering extensions of the results to different nonlinearities which possess similar growth conditions, and discussing what one can do if one instead regularises the problem to some fourth-order partial differential equation to overcome the lack of monotonicity.

In this chapter we shall encounter some original content. Specifically, original results shall be obtained which are related to the following

- Existence of Young measure solutions to non-monotone systems of parabolic equations with (p, q)-type growth conditions.
- The study of the long-time behaviour of Young measure solutions of non-monotone parabolic systems and equations with (p, q)-type growth conditions, under suitable assumptions regarding the behaviour of the source term.
- The study of parabolic systems of the form (1.1.1.1) where it is assumed that (1.1.4.18) does not hold.

1.2.3 Chapter 4

In Chapter 4 we shall start by giving an overview of the algorithm presented in [35] and [36] for the approximation of measure-valued solutions of nonlinear hyperbolic systems. After introducing additional definitions and further preliminary results, we shall consider a fully discrete numerical scheme, and show that we can use a backward Euler method to handle the time-discretisation. Following on from this, we shall develop and analyse a numerical scheme (motivated by the one used in [35] and [36]) for the approximation of Young measure solutions of the parabolic systems of equations which we have been considering up until this point.

The analysis, and subsequent proof of convergence to a Young measure solution of a non-linear forward-backward system of parabolic equations, of a numerical scheme forms original content in this chapter.

1.2.4 Chapter 5

The final chapter of this thesis will discuss the attempts made by this author in conjunction with Matteo Croci to implement the numerical scheme developed in Chapter 4. We shall discuss directions in which further research could go.

Chapter 2

Systems of Parabolic Equations with (p, q)-growth conditions

As previously mentioned there are many aspects of the Met Office's model (1.1.2.1)−(1.1.2.9) which make it difficult to fit it into the existing theory of weak solutions for nonlinear parabolic systems. In this chapter we explore conditions under which we can assert the existence of a unique weak solution for systems of the form (1.1.2.1).

We start with a recap of the familiar monotone operator theory, and discuss what we mean by "standard growth conditions". Following this, we make attempts to apply this theory to the Met Office model. To do this, we shall consider the system formed by the velocity components (with prescribed temperature) and the temperature component (with prescribed velocity) separately, showing that each of these systems satisfies a monotonicity condition.

Finally in this chapter we discuss the work of Bögelein, Duzaar and Marcellini, in particular [12], which details the existence of weak solutions to systems of PDEs exhibiting growth conditions to which our growth conditions from the Met Office model can be thought of being analogous.

2.1 A review of the theory of monotone operators

In this section we start by introducing some notation and preliminary standard results regarding Sobolev spaces, before moving on to present a review of the theory of monotone operators. We end by showing that the parabolic p-Laplacian problem is well-posed. In particular we show that if our function $K(\xi)\xi$ was monotone and exhibited standard growth conditions then we would have a unique weak solution under suitable assumptions on the regularity of the given initial and boundary data.

2.1.1 Notation and Preliminaries

The definitions of Sobolev spaces and their duals are well known for scalar functions. As we shall be mostly concerning ourselves with systems in this work, we here give the corresponding notions of Lebesgue and Sobolev spaces for vector-valued functions within the context of Bochner spaces which will be used in the bulk of our analysis.

Definition 2.1.1.1. We say that a function $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m$, given by $u = (u_1, \ldots, u_m)$, belongs to the Lebesgue space $L^p(Q_T; \mathbb{R}^m)$ (where $Q_T = [0,T] \times \Omega$) if it holds that each component function $u_i : \mathbb{R}^m \to \mathbb{R}$ belongs to the Lebesgue space $L^p(Q_T)$.

Definition 2.1.1.2. We say that a function $u : \mathbb{R}^m \to \mathbb{R}^m$, given by $u = (u_1, \ldots, u_m)$, belongs to the Bochner space $L^p(0,T; W_0^{1,p}(\Omega; \mathbb{R}^m))$ if it holds that each component function $u_i : \mathbb{R}^m \to \mathbb{R}$ belongs to the Bochner space $L^p(0,T; W_0^{1,p}(\Omega))$.

We also take the opportunity here to recall a few useful results regarding Bochner spaces. The first of these is the Aubin–Lions Lemma, found as *Lemma 7.7* in [56].

Lemma 2.1.1.3. *Let X_0, X and X_1 be Banach spaces with X_0 compactly embedded in X, which is in turn continuously embedded in X_1, and let $1 \leq p, q \leq \infty$. Let*

$$W^{p,q}(X_0, X_1) = \{u \in L^p(0,T; X_0) : \partial_t u \in L^q(0,T; X_1)\}.$$

Then, if $p < \infty$ it holds that $W^{p,q}(X_0, X_1)$ is compactly embedded into $L^p(0,T; X)$. In addition, if $p = \infty$ and $q > 1$ it holds that $W^{p,q}(X_0, X_1)$ is compactly embedded into $C([0,T]; X)$.

The second result can be found as *Lemma 7.3* in [56].

Lemma 2.1.1.4. *It holds that* $W^{p,p'}(W_0^{1,p}(\Omega;\mathbb{R}^m), W^{-1,p'}(\Omega;\mathbb{R}^m))$ *is embedded continuously in* $C([0,T];L^2(\Omega;\mathbb{R}^m))$*, and the following integration-by-parts formula holds for any* $t_1, t_2 \in [0,T]$ *and any* $u_1, u_2 \in W^{p,p'}(W_0^{1,p}(\Omega;\mathbb{R}^m), W^{-1,p'}(\Omega;\mathbb{R}^m))$*:*

$$\begin{aligned}&\int_\Omega u_1(t_2,x)u_2(t_2,x) - u_1(t_1,x)u_2(t_1,x)\ \mathrm{d}x \\ &\quad = \int_{t_1}^{t_2} \langle \partial_t u_1(t), u_2(t)\rangle_{W^{-1,p'}(\Omega;\mathbb{R}^m), W_0^{1,p}(\Omega;\mathbb{R}^m)} + \langle u_1(t), \partial_t u_2(t)\rangle_{W^{-1,p'}(\Omega;\mathbb{R}^m), W_0^{1,p}(\Omega;\mathbb{R}^m)}\ \mathrm{d}t.\end{aligned}$$

2.1.2 Monotone operator theory

We begin with some preliminary definitions. Throughout, V will denote a reflexive Banach space and V' will be the dual of V. We denote by $\langle \cdot\ ,\cdot\rangle$ an appropriate duality pairing between V and V'. The content of this subsection comes from [56] and [57].

Definition 2.1.2.1. An operator $S\colon V \to V'$ is hemicontinuous if for each fixed u_1, u_2, $v \in V$ the function $h : \mathbb{R} \to \mathbb{R}$, given by

$$h(\lambda) = \langle S(u_1 + \lambda u_2), v\rangle,$$

is continuous.

Definition 2.1.2.2. An operator $S : V \to V'$ is monotone if for all u_1, $u_2 \in V$ we have

$$\langle S(u_1) - S(u_2), u_1 - u_2\rangle \geq 0,$$

and is called strictly monotone if the above inequality is an equality only when $u_1 = u_2$. If for all u_1, $u_2 \in V$ we have

$$\langle S(u_1) - S(u_2), u_1 - u_2\rangle \geq a(\|u_1 - u_2\|_V)\|u_1 - u_2\|_V,$$

where a is a nonnegative strictly monotonic increasing function on $[0,\infty)$ which is strictly positive on $(0,\infty)$, then S is uniformly monotone.

Definition 2.1.2.3. An operator $S : V \to V'$ is pseudomonotone if it is bounded and

- $u_j \rightharpoonup u$ in V, and
- $\limsup_{j\to\infty} \langle S(u_j), u_j - u\rangle \leq 0$

imply that for all $v \in V$,

$$\langle S(u), u - v\rangle \leq \liminf_{j\to\infty} \langle S(u_j), u_j - v\rangle.$$

Remark 2.1.2.4. *In [57], Proposition 2.5, it is shown that a hemicontinuous, bounded, monotone map $S : V \to V'$ is pseudomonotone.*

Definition 2.1.2.5. An operator $S : V \to V'$ is coercive if

$$\lim_{\|u\|_V\to\infty} \frac{\langle S(u), u\rangle}{\|u\|_V} = +\infty.$$

Theorem 2.1.2.6. *If $S : V \to V'$ is pseudomontone and coercive then it is also surjective, meaning that there is at least one solution to the equation $S(u) = F$ for any $F \in V'$.*

We omit the proof, which can be found in [56], pp. 33−34. We remark that in the case where S is some differential operator, so that $S(u) = F$ represents a PDE, the stronger assumption of uniform montonicity gives the existence of a unique weak solution which depends continuously upon the given data (see, for example, [57], Remark 2.17 and Remark 7.4).

We now discuss the adaption of this theory to the case where we have a parabolic problem which we wish to solve.

Definition 2.1.2.7. Let V be a real separable and reflexive Banach space and H a real separable Hilbert space with scalar product $(\cdot, \cdot)$ such that the imbedding $V \subset H$ is continuous and V is dense in H. Then the formula $\langle \tilde{v}, u\rangle = (v, u)$ for $u \in V$ and $v \in H$ defines a continuous linear function $\tilde{v}$ over V and it generates a bijection between H and a subset of V'. We may then write $V \subset H \subset V'$, which we call an evolution triple, or Gelfand triple.

We then have the following theorem, found as Theorem 7.1 in [57].

Theorem 2.1.2.8. *Let $V \subset H \subset V'$ be an evolution triple, let $1 < p < \infty$ and let $0 < T < \infty$. Assume that for all fixed $t \in [0,T]$ the map $\tilde{S}(t) : V \to V'$ is monotone, coercive and hemicontinuous. Assume also that it is bounded in the sense that*

$$\|\tilde{S}(t)(v)\|_{V'} \leq c_1 \|v\|_V^{p-1} + k_1(t)$$

holds for all $v \in V$ and all $t \in [0,T]$ with a suitable constant c_1 and a function $k_1 \in L^{p'}(0,T)$. Finally for arbitrary fixed $u, v \in V$ assume that the function $t \mapsto \langle \tilde{S}(t)(u), v\rangle$ for $t \in [0,T]$ is measurable. Then for arbitrary $F \in L^{p'}(0,T;V')$ and $u_0 \in H$ there exists a unique solution to $\partial_t u - \operatorname{div}(a(Du)) = F$ lying in the space

$$W^{p,p'}(V,V') = \{u \in L^p(0,T;V), \partial_t u \in L^{p'}(0,T;V')\}$$

with $u(0) = u_0$, and $[S(u)](t) = [\tilde{S}(t)](u(t))$ where

$$\langle [S(u)](t), v\rangle = \langle [\tilde{S}(t)](u(t)), v\rangle = \int_\Omega a(Du(t)) : Dv \; \mathrm{d}x$$

for all $v \in V$.

We remark (see [57], remark 7.4) that if in addition to this we have uniform monotonicity of $\tilde{S}(t)$ in the sense that

$$\langle [\tilde{S}(t)](v_1) - [\tilde{S}(t)](v_2), v_1 - v_2\rangle \geq c\|v_1 - v_2\|_V^p$$

for $v_1, v_2 \in V$ with some constant $c > 0$ for all $t \in [0,T]$, then we have continuous dependence upon our given data.

2.1.3 Standard growth conditions

We start by defining what we mean by *standard growth conditions.*

Definition 2.1.3.1. Let $1 < p < \infty$. We say that a function $a : \mathbb{R}^m \to \mathbb{R}^m$ satisfies *standard p-growth conditions*, or *(p,p)-growth conditions*, if there exist constants c_1, c_2

and c_3 such that

$$|a(\xi)| \leq c_1(1 + |\xi|^{p-1}), \text{ and}$$
$$a(\xi) \cdot \xi \geq c_2|\xi|^p - c_3,$$

where $c_1, c_2 > 0$ and $c_3 \geq 0$.

Remark 2.1.3.2. *In this subsection we shall be considering $p \geq 2$.*

It is well-known that if the mapping a appearing in (1.1.2.1) is a continuous and monotone operator satisfying standard growth conditions, then (1.1.2.1)$-$(1.1.2.7) has a unique weak solution (for a sufficiently regular right-hand side). In this section we ask the question: how could we alter the formulation of K appearing in (1.1.2.8)$-$(1.1.2.9) in order to deduce the existence of a unique weak solution?

We show that if $a(\xi) = K(\xi)\xi$ has standard growth conditions, is continuous and is monotone, then we can apply Theorem 2.1.2.8 in order to obtain a unique weak solution to the system (1.1.2.1)$-$(1.1.2.6). We begin with the weak formulation to this problem. As we have a mixture of Dirichlet and Neumann boundary conditions we need to choose our space of test functions in a way which reflects this. We choose our test functions

$$\phi = (\phi_1, \phi_2, \phi_3) \in L^p(0, T; W_0^{1,p}(0, 1)) \times L^p(0, T; W_0^{1,p}(0, 1)) \times L^p(0, T; W_*^{1,p}(0, 1))$$
$$=: L^p(0, T; W_{*,0}^{1,p}(0, 1)), \tag{2.1.3.1}$$

where by $L^p(0, T; W_*^{1,p}(0, 1))$ we denote the space of $L^p(0, T; W^{1,p}(0, 1))$ functions v which satisfy $v(t, 1) = 0$ for all $t \in (0, T)$. We now consider the smooth function $u = (u_1, u_2, u_3)$ defined on Q_T which satisfies:

$$u(t, 1) = f(t),$$
$$u_1(t, 0) = g_1(t),$$
$$u_2(t, 0) = g_2(t),$$
$$\partial_x u_3(t, 0) = 0,$$

where f, g_1, g_2 and h are the smooth boundary data found in the system (1.1.2.1) − (1.1.2.7). In this system we now make the substitution $w = v + u$, and search for a solution $v \in L^p(0,T; W^{1,p}_{*,0}(0,1))$. We remark that although we do not satisfy Dirichlet data everywhere, we are still able to use the Friedrichs-Poincaré inequality as we have Dirichlet boundary conditions in all components on at least one part of the boundary. Taking the inner product of (1.1.2.1) with a test function chosen as in (2.1.3.1), integrating over $Q_T = (0,T) \times (0,1)$ and formally integrating by parts in time we arrive at the following weak formulation:

$$\begin{aligned} \int_{Q_T} \partial_t(v+u) \cdot \phi + K(\partial_x(v+u))\partial_x(v+u) \cdot \partial_x \phi - A(v+u) \cdot \phi \,\mathrm{d}x\,\mathrm{d}t \\ = \left(\int_{Q_T} F \cdot \phi \,\mathrm{d}x\,\mathrm{d}t \right) + \int_0^T K(0,0,h(t))h(t)\phi_3(t,0)\,\mathrm{d}t. \end{aligned} \tag{2.1.3.2}$$

From this we see that we need $F \in L^{p'}(0,T; (W^{1,p}_{*,0}(0,1))')$, and that if

$$K(\xi) \le c_1(1 + |\xi|^{p-2}),$$

then we can take $h \in L^p(0,T)$ in order to satisfy the conditions of the right-hand side needed in Theorem 2.1.2.8. We now deduce conditions under which we have monotonicity, coercivity, hemicontinuity and boundedness properties needed to apply Theorem 2.1.2.8. We first define the operator $[S(v)](t) = [\tilde{S}(t)](v(t))$ by the inner product:

$$\begin{aligned} \langle [\tilde{S}(t)]v(t), \phi(t) \rangle = \int_0^1 K(\partial_x(v(t,x) + u(t,x)))\partial_x(v(t,x) + u(t,x)) \cdot \partial_x \phi(t,x) \\ - A(v(t,x) + u(t,x)) \cdot \phi(t,x)\,\mathrm{d}x \end{aligned} \tag{2.1.3.3}$$

for all $\phi \in L^p(0,T; W^{1,p}_{*,0}(0,1))$. We begin by checking coercivity, and impose that

$$K(\xi) \ge c_2|\xi|^{p-2} - c_3.$$

Under this assumption we compute

$$\langle [\tilde{S}(t)](v(t)), v(t) \rangle = \int_0^1 K(\partial_x(v+u))(\partial_x(v+u)) \cdot \partial_x v - A(v+u) \cdot v \; dz$$

$$
\begin{aligned}
&\geq c_2\|v+u\|^p_{W^{1,p}(0,1;\mathbb{R}^3)} - c_3\int_0^1 |\partial_x(v+u)|^2 \,\mathrm{d}x \\
&\quad - \int_0^1 c_2|\partial_x(v+u)|^{p-2}\partial_x(v+u)\cdot\partial_x u \,\mathrm{d}x \\
&\quad + \int_0^1 c_3(\partial_x(v+u)\cdot\partial_x u) - c(|u|^2+|v|^2)\, dz \\
&\geq c_2\|v+u\|^p_{W^{1,p}(0,1;\mathbb{R}^3)} - c_3\|v+u\|^2_{W^{1,2}(0,1;\mathbb{R}^3)} - \frac{c_2}{p'}\|v+u\|^p_{W^{1,p}(0,1;\mathbb{R}^3)} \\
&\quad - \frac{c_2}{p}\|u\|^p_{W^{1,p}(0,1;\mathbb{R}^3)} - c_3\|v+u\|^2_{W^{1,2}(0,1;\mathbb{R}^3)} - c_3\|u\|^2_{W^{1,2}(0,1;\mathbb{R}^3)} \\
&\quad - c(\|u\|^2_{L^2(0,1;\mathbb{R}^3)} + \|v\|^2_{L^2(0,1;\mathbb{R}^3)}),
\end{aligned}
$$

from which we obtain coercivity by noting that by the reverse triangle inequality:

$$
\|v+u\|^p_{W^{1,p}(0,1;\mathbb{R}^3)} \geq c(\|v\|^p_{W^{1,p}_{*,0}(0,1;\mathbb{R}^3)} + P(\|u\|_{W^{1,p}(0,1;\mathbb{R}^3)}, \|v\|_{W^{1,p}_{*,0}(0,1;\mathbb{R}^3)})),
$$

where P is an expression of growth at most $p-1$ in its second argument. The highest power of $\|v\|_{W^{1,p}_{*,0}(0,1;\mathbb{R}^3)}$ appearing in the above inequality therefore has a positive coefficient and the power is larger than 1, giving us the desired coercivity.

For the sake of ensuring uniform monotonicity we make the assumption that

$$
[K(x)x - K(y)y]\cdot(x-y) \geq c|x-y|^p
$$

and compute:

$$
\begin{aligned}
&\langle [\tilde{S}(t)](v^1(t)) - [\tilde{S}(t)](v^2(t)), v^1(t)-v^2(t)\rangle \\
&= \int_0^1 \Big[K(\partial_x(v^1+u))(\partial_x(v^1+u)) - K(\partial_x(v^1+u))(\partial_x(v^1+u))\Big]\cdot(\partial_x(v^1-v^2))\,\mathrm{d}x \\
&\geq c\|\partial_x(v^1-v^2)\|^p_{L^p(0,1)},
\end{aligned}
$$

noting that the lower order term involving A disappears as A is a rotation matrix by 90 degrees in the first two components.

The final things that we need are some continuity and boundedness properties. It can be shown (see, for example, [57]) that if the function

$$\xi \mapsto K(\xi)\xi$$

is continuous and we have

$$|K(\xi)\xi| \leq c_1(1 + |\xi|^{p-1}),$$

then we satisfy the required boundedness and hemicontinuity properties. Thus by applying Theorem 2.1.2.8 we have the following result:

Theorem 2.1.3.3. *Let $w_0 \in L^2(0,1;\mathbb{R}^3)$, let $h \in L^p(0,1;\mathbb{R}^3)$ and let $u \in W^{1,p}(0,1;\mathbb{R}^3)$ satisfy $u(t,1) = f(t)$, $u_1(t,0) = g_1(t)$,$u_2(t,0) = g_2(t)$ and $\partial_x u_3(t,0) = 0$. If K is continuous and satisfies:*

1. $|K(\xi)\xi| \leq c_1(1 + |\xi|^{p-1})$,
2. $K(\xi) \geq c_2|\xi|^{p-2} - c_3$,
3. $[K(\xi)\xi - K(\eta)\eta] \cdot (\xi - \eta) \geq c|\xi - \eta|^p$,

then there is a unique $v \in L^p(0,T;W^{1,p}_{*,0}(0,1))$ solving the weak formulation (2.1.3.2) *and which depends continuously upon the boundary and initial data.*

From this it follows that there is a weak solution $w \in L^p(0,T;W^{1,p}((0,1),\mathbb{R}^3))$ to (1.1.2.1) $-$ (1.1.2.6). To show uniqueness we let w^1 and w^2 be two weak solutions and consider $w = w^1 - w^2$, noting that as $w \in W^{1,p}_{*,0}(0,1;\mathbb{R}^3)$ it is a valid choice of test function. It holds that w satisfies:

$$\int_0^T \int_0^1 \partial_t w \cdot w + (K(\partial_x w^1)\partial_x w^1 - K(\partial_x w^2)\partial_x w^2) \cdot (\partial_x w^1 - \partial_x w^2) \, \mathrm{d}x \, \mathrm{d}t = 0.$$

By the assumed monotonicity condition it then follows that w satisfies:

$$\|w\|^2_{L^\infty(0,T;L^2(0,1;\mathbb{R}^3))} + \|w\|^p_{L^p(0,T;W^{1,p}_{*,0}(0,1;\mathbb{R}^3))} \leq 0,$$

from which we deduce that $w = 0$ and hence that $w^1 = w^2$.

In the parabolic setting, merely having a monotone operator is enough to guarantee existence and uniqueness (compared to the elliptic setting where one needs the extra condition of strict monotonicity in order to have uniqueness). We give conditions which imply the monotonicity of a map, and following that conditions which give the boundedness, hemicontinuity and coercivity conditions needed.

Lemma 2.1.3.4. *Let $F : \mathbb{R}^m \to \mathbb{R}^n$ be a differentiable function. If Df is positive semidefinite then F is monotone.*

Proof. The proof is taken from [57], Proposition 3.3. Given $x, y \in \mathbb{R}^n$, define a function

$$g(t) = F(tx + (1-t)y) \cdot (x - y).$$

Then,

$$\begin{aligned} \langle F(x) - F(y), x - y \rangle &= g(1) - g(0) \\ &= \int_0^1 g'(t)\ \mathrm{d}t \\ &= \int_0^1 \langle DF(tx + (1-t)y)(x-y), x - y \rangle\ \mathrm{d}t \\ &\geq 0. \end{aligned}$$

□

The following proposition comes from combining various results from sections 3 and 8 of [57]:

Proposition 2.1.3.5. *Let $S : L^p(0, T; V) \to L^{p'}(0, T; V')$ be an operator between a separable reflexive Banach space V and its dual V' defined by*

$$\langle [S(u)](t), v \rangle = \sum_{j=1}^{n} \int_\Omega a_j(t, x, Du(x)) D_j v(x)\ \mathrm{d}x.$$

If the functions a_j are Carathéodory and if there is a constant $c_1 > 0$ and a function

$k_1 \in L^{p'}(Q_T)$ such that for almost every $(t,x) \in Q_T$ and for all ξ,

$$|a_j(t,x,\xi)| \leq c_1|\xi|^{p-1} + k_1(t,x),$$

then the operator $[S(u)](t)$ is bounded and hemicontinuous. Further, if there is a constant $c_2 > 0$ and a function $k_2 \in L^1(Q_T)$ such that for almost every $(t,x) \in \mathbb{R}^{n+1}$ and all ξ,

$$\sum_{j=1}^{n} a_j(t,x,\xi)\xi_j \geq c_2|\xi|^p - k_2(t,x),$$

then the operator $[S(u)](t)$ is coercive.

2.2 Working with the model

In this section we decouple (1.1.2.1) into one system and one equation. We begin by considering the system which governs the velocity, and derive suitable conditions on the temperature for the system to posses a solution, before considering the equation governing the temperature, and deriving suitable conditions on the velocity for the equation to have a solution.

2.2.1 The system formed by the velocity components

We now consider the system formed by the first two components of (1.1.2.1), and show that under suitable conditions on the temperature it possesses a unique solution. That is, we consider the system

$$\partial_t u - \partial_x(K(\partial_x u, \partial_x \theta)\partial_x u) = 0, \tag{2.2.1.1}$$

with the boundary conditions

$$u(t,1) = u(t,0) = 0 \tag{2.2.1.2}$$

and initial data

$$u(0,x) = u_0(x). \tag{2.2.1.3}$$

Here K is given by

$$K(\partial_x u, \partial_x \theta) = \left(\frac{|\partial_x u|^3}{|\partial_x u|^2 + \partial_x \theta}\right)\left(\frac{1+\operatorname{sign}(\partial_x\theta)}{2}\right) + \sqrt{|\partial_x u|^2 - \partial_x\theta}\left(\frac{1-\operatorname{sign}(\partial_x\theta)}{2}\right), \tag{2.2.1.4}$$

and $u : [0,T] \times \Omega \to \mathbb{R}^2$ is a vector-valued function. We have the following proposition:

Proposition 2.2.1.1. *The function K specified above satisfies the following properties:*

1. *K is continuous in its first argument,*
2. *The operator $S(\xi, c) = K(\xi, c)\xi$ is monotone in its first argument, in the sense that for any $c \in \mathbb{R}^2$:*

$$\langle K(\xi, c)\xi - K(\eta, c)\eta, \xi - \eta\rangle \geq 0 \text{ for every } \xi, \eta \in \mathbb{R}^2,$$

3. *There is a constant $c_0 > 0$ such that $K(\partial_x u, \partial_x\theta)\partial_x u \leq c_0(|\partial_x u|^2 + |\partial_x \theta|)$,*
4. *There is a constant $c_1 > 0$ such that $K(\partial_x u, \partial_x\theta)|\partial_x u|^2 \geq c_1(|\partial_x u|^3 - |\partial_x\theta|^{\frac{3}{2}})$.*

Proof. The first statement is obvious from the definition of K. For the second statement we compute:

$$\begin{aligned}
&\langle K(\partial_x u, \partial_x\theta)\partial_x u - K(\partial_x v, \partial_x\theta)\partial_x v, \partial_x u - \partial_x v\rangle \\
&= \left(\frac{1+\operatorname{sign}(\partial_x\theta)}{2}\right)\left\langle\left(\frac{|\partial_x u|^3}{|\partial_x u|^2 + \partial_x\theta}\right)\partial_x u - \left(\frac{|\partial_x v|^3}{|\partial_x v|^2 + \partial_x\theta}\right)\partial_x v, \partial_x u - \partial_x v\right\rangle \\
&\quad + \left(\frac{1-\operatorname{sign}(\partial_x\theta)}{2}\right)\left\langle\sqrt{|\partial_x u|^2 - \partial_x\theta}\,\partial_x u - \sqrt{|\partial_x v|^2 - \partial_x\theta}\,\partial_x v, \partial_x u - \partial_x v\right\rangle \\
&=: \left(\frac{1+\operatorname{sign}(\partial_x\theta)}{2}\right) J_1 + \left(\frac{1-\operatorname{sign}(\partial_x\theta)}{2}\right) J_2
\end{aligned}$$

As the terms involving $\operatorname{sign}(\partial_x\theta)$ are both non-negative, all that remains to be shown is that J_1 and J_2 are both non-negative.

For the function $F_1 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F_1(\xi) = \frac{|\xi|^3\xi}{|\xi|^2+c}$ with $c > 0$ one can compute that it has the derivative

$$DF_1(\xi) = \frac{|\xi|^3}{|\xi|^2 + c}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{|\xi|(|\xi|^2 + 3c)}{(|\xi|^2 + c)^2}\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_2^2 \end{pmatrix},$$

which is clearly positive semi-definite. Similarly for the function $F_2(\xi) = \xi\sqrt{|\xi|^2 - c}$ with $c < 0$, one computes

$$DF_2(\xi) = \sqrt{|\xi|^2 - c}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{|\xi|^2 - c}}\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_2^2 \end{pmatrix},$$

which is once again clearly positive semi-definite. Then by Lemma 2.1.3.4 we have the desired monotonicity. For the third point one bounds:

$$\begin{aligned} K(\partial_x u, \partial_x \theta)\partial_x u &\leq |\partial_x u|^2 + |\partial_x u|\sqrt{|\partial_x u|^2 + |\partial_x \theta|} \\ &\leq c_0(|\partial_x u|^2 + |\partial_x \theta|). \end{aligned}$$

Finally, for the fourth point:

$$\begin{aligned} K(\partial_x u, \partial_x \theta)|\partial_x u|^2 &\geq \frac{|\partial_x u|^5}{|\partial_x u|^2 + \partial_x \theta}\left(\frac{1 + \operatorname{sign}(\partial_x \theta)}{2}\right) + |\partial_x u|^3\left(\frac{1 - \operatorname{sign}(\partial_x \theta)}{2}\right) \\ &= \left(|\partial_x u|^3 - \frac{|\partial_x u|^3\partial_x \theta}{|\partial_x u|^2 + \partial_x \theta}\right)\left(\frac{1 + \operatorname{sign}(\partial_x \theta)}{2}\right) + |\partial_x u|^3\left(\frac{1 - \operatorname{sign}(\partial_x \theta)}{2}\right) \\ &\geq |\partial_x u|^3 - |\partial_x u||\partial_x \theta| \\ &\geq c_1(|\partial_x u|^3 - |\partial_x \theta|^{\frac{3}{2}}), \end{aligned}$$

which completes the proof of the proposition. $\square$

Proposition 2.2.1.1 gives us the following proposition which allows us to obtain the existence and uniqueness of solutions to (2.2.1.1).

Proposition 2.2.1.2. *Suppose that $\theta \in L^{\frac{3}{2}}(0, T; W^{1,\frac{3}{2}}(0, 1))$. Then, there exists a unique solution u to* (2.2.1.1) *satisfying*

$$u \in L^3(0, T; W_*^{1,3}(0, 1)).$$

Proof. The proof follows by combining the results of Proposition 2.2.1.1, Proposition 2.1.3.5 and Theorem 2.1.2.8. $\square$

2.2.2 Preliminary dealings with the linear growth term

As a starting step towards handling the linear growth term we prove some a-priori estimates, which will allow us to construct a solution via the Galerkin method to a simplified problem. We consider the equation:

$$\partial_t u - \partial_x \left(\frac{\partial_x u \sqrt{1 - (\partial_x u)_-}}{1 + (\partial_x u)_+} \right) = 0, \qquad (2.2.2.1)$$

subject to the boundary conditions

$$u(t,1) = 0, \quad \partial_x u(t,0) = 0, \qquad (2.2.2.2)$$

and initial data

$$u(0,x) = u_0(x), \qquad (2.2.2.3)$$

with $u_0 \in W^{1,\infty}(\Omega)$. Here, we use the following notation:

$$f_+ = \max(f,0), \quad f_- = \min(f,0).$$

Remark 2.2.2.1. *This is not the usual definition of the negative part of a function, rather it is the negative of the usual definition. We use this non-standard way of defining the negative part in order to closely mimic the formula for K given by* (1.1.2.9).

In the case where $w_1^2 + w_2^2 = 1$ we see that this equation then corresponds to the third component of (1.1.2.1)$-$(1.1.2.7).

Proposition 2.2.2.2. *Let $u_0 \in W^{1,\infty}(0,1)$. Then there is a unique solution u to the problem* (2.2.2.1) $-$ (2.2.2.3) *which depends continuously upon the specified initial datum u_0 with the following regularity:*

1. $u \in L^2(0,T;W^{2,1}(0,1))$,

2. $u \in L^\infty(0,T;W^{1,2}(0,1))$,

3. $\partial_t u \in L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(0,1))$.

Proof. We obtain a solution via the Galerkin method. By considering the following eigenvalue problem:

$$\begin{cases} -\partial_{xx}\phi = \lambda\phi, \\ \partial_x\phi(0) = 0, \\ \phi(1) = 0, \end{cases}$$

one obtains the existence of a complete orthonormal basis of eigenfunctions $\{\phi_i\}_{i=1}^{\infty}$ in $L^2(0,1)$ which are also in $C^\infty(0,1) \cap W_*^{1,p}(0,1)$ for all $p \in [1,\infty]$. We define

$$u_n(t,x) = \sum_{i=1}^{n} \alpha_i(t)\phi_i(x),$$

where the functions α_i satisfy

$$\int_0^1 \partial_t u_n(t,x)\phi_i(x)\ \mathrm{d}x + \int_0^1 \frac{\partial_x u_n(t,x)\sqrt{1-(\partial_x u_n)_-}}{1+(\partial_x u_n(t,x))_+}\partial_x\phi_i(x)\ \mathrm{d}x = 0.$$

From this one obtains the existence of local in time solutions through Peano's Theorem.

In what follows we drop the index n for ease of notation, but we keep in mind that the following computations are carried out on the constructed functions u_n, meaning that we can differentiate in the classical sense in the spatial variable. Testing (2.2.2.1) with u we see that

$$\int_0^1 \partial_t u(t,x)\cdot u(t,x) + \frac{|\partial_x u(t,x)|^2\sqrt{1-(\partial_z u(t,x))_-}}{1+(\partial_x u(t,x))_+}\ \mathrm{d}x = 0.$$

The second term in the above equation is non-negative, and therefore it holds that

$$\int_0^1 \frac{1}{2}\partial_t(u(t,x)^2)\ \mathrm{d}x \le 0.$$

By integrating this in time we see that

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{L^2(0,1)} \le c\|u_0\|_{L^2(0,1)}. \tag{2.2.2.4}$$

Suppose that $m, M \in \mathbb{R}$ are such that our initial datum u_0 satisfies $m \le u_0(x) \le M$ for almost every $x \in [0,1]$. Then one can show that

$$\partial_t(u-M) - \partial_x \left(\frac{\partial_x(u-M)\sqrt{1-(\partial_x(u-M))_-}}{1+(\partial_x u)_+} \right) = 0.$$

Testing against $(u-M)_+$ gives

$$u(t,x) \leq M,$$

for almost every $x \in [0,1]$ and $t \in [0,T]$. Similarly one can show that

$$u(t,x) \geq m$$

for almost every $x \in [0,1]$ and $t \in [0,T]$. Thus if $u_0 \in L^\infty(0,1)$ we have that

$$u \in L^\infty(0,T;L^\infty(0,1)). \tag{2.2.2.5}$$

In order to obtain higher regularity estimates we expand the x-derivative in (2.2.2.1), computing:

$$\begin{aligned}
\partial_x \left(\frac{\partial_x u\sqrt{1-(\partial_x u)_-}}{1+(\partial_x u)_+} \right) &= \partial_{xx} u \left(\frac{\sqrt{1-(\partial_x u)_-}}{1+(\partial_x u)_+} \right) + (\partial_x u)\partial_x \left(\frac{\sqrt{1-(\partial_x u)_-}}{1+(\partial_x u)_+} \right) \\
&= \frac{\partial_{xx} u\sqrt{1-(\partial_x u)_-}}{(1+(\partial_x u)_+)^2} - \frac{\partial_{xx} u(\partial_x u)_-}{2\sqrt{1-(\partial_x u)_-}},
\end{aligned}$$

which transforms (2.2.2.1) into

$$\partial_t u - \frac{\partial_{xx} u\sqrt{1-(\partial_x u)_-}}{(1+(\partial_x u)_+)^2} + \frac{\partial_{xx} u(\partial_x u)_-}{2\sqrt{1-(\partial_x u)_-}} = 0. \tag{2.2.2.6}$$

Testing (2.2.2.6) with $-\partial_{xx} u$ one obtains

$$\frac{1}{2}\partial_t \|\partial_x u(t,\cdot)\|^2_{L^2(0,1)} + \int_0^1 \frac{|\partial_{xx} u(t,x)|^2}{(1+(\partial_x u(t,x))_+)^2} - \frac{|\partial_{xx} u|^2(\partial_x u)_-}{2\sqrt{1-(\partial_x u)_-}} \,\mathrm{d}x = 0, \tag{2.2.2.7}$$

and after using that $\frac{-(\partial_x u)_-}{2\sqrt{1-(\partial_x u)_-}} \geq 0$, one can deduce the inequality:

$$\frac{1}{2}\partial_t \|\partial_x u\|^2_{L^2(0,1)} + \frac{\|\partial_{xx} u\|^2_{L^2(0,1)}}{(1 + \|\partial_x u\|_{L^\infty(0,1)})^2} \leq 0. \tag{2.2.2.8}$$

By the Fundamental Theorem of Calculus, we have

$$\begin{aligned}(\partial_x u(t,x))^2 &= (\partial_x u(t,0))^2 + \int_0^x \partial_s((\partial_s u(t,s))^2)\ \mathrm{d}s \\ &= 2\int_0^x \partial_s(\partial_s u(t,s))\partial_s u(t,s)\ \mathrm{d}s \\ &\leq 2\|\partial_x u(t,\cdot)\|_{L^2(0,1)}\|\partial_{xx} u(t,\cdot)\|_{L^2(0,1)}.\end{aligned}$$

Thus

$$\|\partial_x u(t,\cdot)\|^2_{L^\infty(0,1)} \leq 2\|\partial_x u(t,\cdot)\|_{L^2(0,1)}\|\partial_{xx} u(t,\cdot)\|_{L^2(0,1)}. \tag{2.2.2.9}$$

Inserting this estimate into (2.2.2.8) we get

$$\partial_t \|\partial_x u(t,\cdot)\|^2_{L^2(0,1)} + \frac{\|\partial_{xx} u(t,\cdot)\|^2_{L^2(0,1)}}{1 + 2\|\partial_x u(t,\cdot)\|_{L^2(0,1)}\|\partial_{xx} u(t,\cdot)\|_{L^2(0,1)}} \leq 0.$$

Multiplying by $4\|\partial_x u(t,\cdot)\|^2_{L^2(0,1)}$ and using that for $a \geq 0$, we have $\frac{a^2}{1+a} \geq a - 1$:

$$\partial_t \|\partial_x u(t,\cdot)\|^4_{L^2(0,1)} + \|\partial_x u(t,\cdot)\|_{L^2(0,1)}\|\partial_{xx} u(t,\cdot)\|_{L^2(0,1)} \leq 1.$$

Using (2.2.2.9):

$$\partial_t \|\partial_x u(t,\cdot)\|^4_{L^2(0,1)} + \frac{1}{2}\|\partial_x u(t,\cdot)\|^2_{L^\infty(0,1)} \leq 1. \tag{2.2.2.10}$$

Now integrating in time:

$$\|\partial_x u(t,\cdot)\|^4_{L^2(0,1)} + \frac{1}{2}\int_0^t \|\partial_x u(t,\cdot)\|^2_{L^\infty}\ \mathrm{d}s \leq \|\partial_x u(0,\cdot)\|^4_{L^2(0,1)} + T.$$

Thus we have

$$\partial_x u \in L^\infty(0,T;L^2(0,1)), \tag{2.2.2.11}$$

$$\partial_x u \in L^2(0,T;L^\infty(0,1)). \tag{2.2.2.12}$$

Returning now to (2.2.2.7) we see that after integrating in time we can deduce that

$$\int_0^T \int_0^1 \frac{|\partial_{xx} u(t,x)|^2}{(1+(\partial_x u(t,x))_+)^2} \, \mathrm{d}x \, \mathrm{d}t \leq \|\partial_x u(0,\cdot)\|^2_{L^2(0,1)}. \tag{2.2.2.13}$$

If one considers

$$\int_0^1 |\partial_{xx} u(t,x)| \, \mathrm{d}x = \int_0^1 \frac{|\partial_{xx} u(t,x)|}{1+(\partial_x u(t,x))_+} (1+(\partial_x u(t,x))_+) \, \mathrm{d}x,$$

uses the Cauchy-Schwarz inequality, integrates in time and uses (2.2.2.11) and(2.2.2.13), one can show that

$$\partial_{xx} u \in L^2(0,T;L^1(0,1)), \tag{2.2.2.14}$$

which after using one-dimensional embeddings of Sobolev spaces gives

$$\partial_x u \in L^2(0,T,L^\infty(0,1)). \tag{2.2.2.15}$$

In particular, from this we can show that

$$\begin{aligned}
\|\partial_x u\|^4_{L^4(Q_T)} &\leq \int_0^T \int_0^1 |\partial_x u|^2 |\partial_x u|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \int_0^T \|\partial_x u(t,\cdot)\|^2_{L^\infty(0,1)} \|\partial_x u(t,\cdot)\|^2_{L^2(0,1)} \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \|\partial_x u\|^2_{L^2(0,T;L^\infty(0,1))} \|\partial_x u\|^2_{L^\infty(0,T;L^2(0,1))},
\end{aligned}$$

demonstrating that $\partial_x u \in L^4(0,T;L^4(0,1))$. From this it follows that

$$\begin{aligned}
\|1-(\partial_x u)_-\|_{L^4(0,T;L^4(0,1))} &\leq T^{\frac{1}{4}} + \|(\partial_x u)_-\|_{L^4(0,T;L^4(0,1))} \\
&\leq T^{\frac{1}{4}} + \|\partial_x u\|_{L^4(0,T;L^4(0,1))},
\end{aligned}$$

and thus that there is some constant C for which

$$\|1-(\partial_x u)_-\|_{L^4(0,T;L^4(0,1))} \leq C. \tag{2.2.2.16}$$

Finally, we use these results to obtain some regularity of the time derivative. Testing (2.2.2.6) against an arbitrary $\phi \in L^4(0,T;L^4(0,1))$ and applying (2.2.2.13) and (2.2.2.16) we see that

$$\begin{aligned}
\int_0^T\int_0^1 \partial_t u \cdot \phi \, \mathrm{d}x \, \mathrm{d}t &\leq \int_0^T\int_0^1 \left| \partial_{xx} u \left(\frac{\sqrt{1-(\partial_x u)_-}}{(1+(\partial_x u)_+)^2} - \frac{(\partial_x u)_-}{2\sqrt{1-(\partial_x u)_-}} \right) \right| |\phi| \, \mathrm{d}x \, \mathrm{d}t \\
&\leq c \int_0^T\int_0^1 \left| \frac{\partial_{xx} u}{1+(\partial_x u)_+} \right| |\phi| |1-(\partial_x u)_-| \, \mathrm{d}x \, \mathrm{d}t \\
&\leq c \left\| \frac{\partial_{xx} u}{1+(\partial_x u)_+} \right\|_{L^2(0,T;L^2(0,1))} \|\phi\|_{L^4(0,T;L^4(0,1))} \\
&\quad \times \|1-(\partial_x u)_-\|_{L^4(0,T;L^4(0,1))}.
\end{aligned}$$

From this, it follows that

$$\partial_t u \in L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(0,1)). \tag{2.2.2.17}$$

Using the previous estimates and applying them to the functions u_n we obtain the existence of a convergent subsequence (still denoted by u_n) and a function u such that

$$u_n \rightharpoonup u \text{ in } L^2(0,T;W^{2,1}(0,1)), \tag{2.2.2.18}$$

$$u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;W^{1,2}(0,1)), \tag{2.2.2.19}$$

$$\partial_t u_n \rightharpoonup \partial_t u \text{ in } L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(0,1)). \tag{2.2.2.20}$$

In one dimension we have the embedding $W^{2,1}(0,1) \subset\subset W^{1,p}(0,1)$ for all $p \in [1,\infty)$, and so by the Aubin−Lions lemma we have the stronger convergence

$$u_n \to u \text{ in } L^2(0,T;W^{1,p}(0,1)),$$

for any $p \in [1,\infty)$. By considering an arbitrary $\psi \in L^4(0,T;W^{1,4}_*(0,1))$ as the strong limit of functions ψ_n of the form

$$\psi_n(t,x) = \sum_{i=1}^n \beta_i(t)\phi_i(x),$$

it follows by considering $\psi = (\psi - \psi_n) + \psi_n$ that

$$\int_0^T \int_0^1 \partial_t u_n(t,x)\psi(t,x) \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_0^1 \frac{\partial_x u_n(t,x)\sqrt{1-(\partial_x u_n)_-}}{1+(\partial_x u_n(t,x))_+} \partial_x \psi(t,x) \, \mathrm{d}x \, \mathrm{d}t \to 0,$$

as $n \to \infty$ for all $\psi \in L^4(0,T;W_*^{1,4}(0,1))$.

By the convergence result for $\partial_t u_n$ we have that as $n \to \infty$,

$$\int_0^T \int_0^1 \partial_t u_n(t,x)\psi(t,x) \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_0^1 \partial_t u(t,x)\psi(t,x) \, \mathrm{d}x \, \mathrm{d}t$$

for all $\psi \in L^4(0,T;W_*^{1,4}(0,1))$. For the passage to the limit in the remaining term we note that by Minty's trick (see Lemma 2.13 in [56]) we need only check monotonicity of the map $f(\xi) = \frac{\xi\sqrt{1-(\xi)_-}}{1+(\xi)_+}$. That is, we aim to show that $(f(\xi) - f(\eta))(\xi - \eta) \geq 0$ for all $\xi, \eta \in \mathbb{R}$.

In the case where $\xi \geq \eta \geq 0$ this is seen from the fact that the function $\xi \mapsto \frac{\xi}{1+\xi}$ is increasing for $\xi > 0$. In the case where $\xi \geq 0 \geq \eta$ this comes from the fact that $f(\xi) \geq 0 \geq f(\eta)$. Finally in the case where $0 \geq \xi \geq \eta$ we see that $\xi - \eta\sqrt{1-\eta} > \eta(1 - \sqrt{1-\eta})$. Simple computations show that for $\eta \leq 0$ the right-hand side is nonnegative, and thus we have monotonicity as required.

To show that the initial data are satisfied, we consider any $\phi \in L^2(0,1)$ and compute as follows:

$$\begin{aligned}
\int_0^1 (u(t,x) - u_0(x))\phi(x) \, \mathrm{d}x &= \int_0^1 (u(t,x) - u_n(t,x))\phi(x) \, \mathrm{d}x \\
&\quad + \int_0^1 (u_n(t,x) - u_n(0,x))\phi(x) \, \mathrm{d}x \\
&\quad + \int_0^1 (u_n(0,x) - u_0(x))\phi(x) \, \mathrm{d}x \\
&\leq \Big(\|u - u_n\|_{C([0,T];L^2(0,1))} + \|u_n(t,\cdot) - u_n(0,\cdot)\| \\
&\quad + \|u_n(0,\cdot) - u_0(\cdot)\|_{L^2(0,1)} \Big) \|\phi\|_{L^2(0,1)}.
\end{aligned}$$

The first and third term in the above parentheses converge to zero as $n \to \infty$, while the second term converges to zero as $t \to 0$ by the fact that $u_n \in C([0,T];L^2(0,1))$. Thus, by first passing $n \to \infty$ and then $t \to 0$, we see that the initial condition is satisfied.

Next we tackle uniqueness. Note that as $u \in L^\infty(0,T;W^{1,2}(0,1))$ we in particular have that $u \in L^4(0,T;L^4(0,1))$, and by the Gagliardo-Nirenberg inequality we also have $\partial_x u \in L^4(0,T;L^4(0,1))$, which allows us to take u as a test funtion. So let u and v be two solutions to (2.2.2.1) and test against $u-v$ to obtain:

$$\int_0^1 \partial_t(u-v)(u-v) + \left(\frac{\partial_x u\sqrt{1-(\partial_x u)_-}}{1+(\partial_x u)_+} - \frac{\partial_x v\sqrt{1-(\partial_x v)_-}}{1+(\partial_x v)_+}\right)\partial_x(u-v)\ \mathrm{d}x = 0.$$

Consider the function $f(\xi) = \frac{\xi\sqrt{1-\xi_-}}{1+\xi_+}$. One can compute that

$$f'(\xi) = \frac{2(1-\xi_-)+(1-\xi_+)(-\xi_-)}{2(1+\xi_+)^2\sqrt{1-\xi_-}} \geq 0,$$

which gives strict monotonicity of the map f. Thus

$$\|u(t,\cdot)-v(t,\cdot)\|_{L^2(0,1)} \leq 0$$

for every $t \in [0,T]$. So for every $t \in [0,T]$, $u(t,x) = v(t,x)$ for almost every $x \in [0,1]$. Thus the solution is unique in the class of $L^4(0,T;W^{1,4}(0,1))$.

Finally we tackle the dependence upon the initial data. Note that $f'(\xi) \geq 1$ if $\xi < 0$, and $f'(\xi) = \frac{1}{(1+x)^2}$ if $\xi \geq 0$. Thus f is uniformly monotone on bounded intervals. However, we have $u \in L^2(0,T;W^{1,\infty}(0,1))$, so $f'(\partial_x u) \geq \frac{1}{1+\|\partial_x u\|_{L^\infty}^2}$, giving uniform monotonicity and hence continuous dependence upon the given data. $\square$

In this section we have considered the benefits of monotone operator theory, and shown that if the function K were to satisfy certain monotonicity, continuity, boundedness and coercivity properties then the problem (1.1.2.1) would be well-posed. In particular, we have considered the problem with a mixture of non-zero Dirichlet and Neumann boundary data. In later chapters, we shall only concern ourselves with zero Dirichlet boundary data for simplicity; the idea behind this section was to demonstrate that one can work with non-zero data in a reasonable way.

We have also considered how the Met Office's model behaves when we decouple the velocity and the temperature components, showing that in each case we can apply some form of monotone operator theory in order to deduce existence and uniqueness of weak solutions.

However, in these decoupled systems, we saw that when we were prescribing the temperature θ, we needed it to be in a function space which was different to the function space to which the velocity u belonged. If we were to attempt to look at the coupled system, we would see that this would be slightly reminiscent of systems exhibiting (p, q)-growth. In the next section, we look at what happens if we continue to prescribe monotonicity, but allow for the exponents appearing in the growth and coercivity conditions to be different.

Then later, in chapters 3 and 4, we shall drop the monotonicity assumption, and work more closely with the structure possessed by the Met Office's model.

2.3 Parabolic systems with (p, q)-growth

Work on systems of parabolic equations with so-called (p, q)- growth conditions has been carried out in, for example, [12] and [59]. Under monotonicity and suitable structural assumptions, the authors of [12] and [59] are able to prove existence and uniqueness of *weak energy* solutions to such systems and equations, provided that the exponents p and q are sufficiently close.

In this section we present a minor correction to section 6 of [12]. In that paper the authors apply a standard interpolation inequality after obtaining a Caccioppoli-style inequality in order to deduce some higher integrability result. However, the exponents used are incorrect. What we present here is a correction to this argument. We begin by stating the problem in question and any assumptions that we will use.

2.3.1 Summary

The paper [12] is concerned with proving existence and uniquess of weak solutions to the problem

$$\begin{cases} \partial_t u - \operatorname{div}(Df(Du)) = 0 & \text{in } Q_T, \\ u = g & \text{on } \partial_{\mathcal{P}} Q_T, \end{cases} \tag{2.3.1.1}$$

where $Q_T = \Omega \times (0,T)$ for some open and bounded $\Omega \subset \mathbb{R}^n$, $\partial_{\mathcal{P}} Q_T$ denotes the parabolic boundary of Q_T, $u : Q_T \to \mathbb{R}^m$ and $f : \mathbb{R}^{m\times n} \to \mathbb{R}_+$ is a C^2 integrand which satisfies the following growth conditions:

$$\begin{cases} |\xi|^p \leq f(\xi) \leq L(1+|\xi|)^q, \\ |D^2 f(\xi)| \leq L(1+|\xi|)^{q-2}, \\ \langle D^2 f(\xi)\eta, \eta\rangle \geq \nu|\xi|^{p-2}|\eta|^2, \end{cases} \tag{2.3.1.2}$$

where the exponents p and q satisfy

$$2 \leq p < q < p + \min\left\{1, \frac{4}{n}\right\}. \tag{2.3.1.3}$$

The purpose of this section is to correct this range of exponents from the claimed range (2.3.1.3) to the correct range

$$2 \leq p < q < p + \min\left\{1, \frac{4}{n+2}\right\}, \tag{2.3.1.4}$$

which is, except for the cases where $n = 1, 2$, more restrictive than the range (2.3.1.3) reported in [12]. We begin by making the following definition of weak solution:

Definition 2.3.1.1. A map

$$u \in L^p(0,T; W^{1,p}_g(\Omega,\mathbb{R}^m)) \cap L^q_{loc}(0,T; W^{1,q}_{loc}(\Omega,\mathbb{R}^m)) \cap C_w([0,T]; L^2(\Omega,\mathbb{R}^m))$$

with

$$\lim_{t\to 0_+} \int_\Omega (u(t,x) - g(0,x)) \cdot \varphi(x)\ \mathrm{d}x = 0$$

is defined as a weak solution of the parabolic system (2.3.1.1) with $g \in C([0,T], L^2(\Omega,\mathbb{R}^m))$, if and only if for all $\varphi \in C^\infty_0(Q_T)$ the following equality is satisfied:

$$\int_{Q_T} u \cdot \varphi_t - \langle Df(Du), D\varphi\rangle\ \mathrm{d}x = 0.$$

It is noted that this is a weaker notion of weak solution than the standard definition (in [12] it is termed a *weak energy solution*), but for ease of writing we stick to the terminology of weak solution. In the above definition we see the space C_w of weakly continuous functions, which is defined as follows:

Definition 2.3.1.2. Let X be a Banach space. A function $u \in L^\infty(0,T;X)$ is a member of the function space $C_w([0,T];X)$ of weakly continuous functions from $[0,T]$ to X if $u(t,\cdot) \in X$ for any $t \in [0,T]$ and the mapping $t \mapsto \langle \psi, u(t) \rangle_X$ is continuous for any $\psi \in X'$.

The following existence result is then claimed (see the first part of Theorem 2.6 in [12]):

Theorem 2.3.1.3. *Suppose that the integrand $f : \mathbb{R}^{mn} \to \mathbb{R}_+$ is C^2 and satisfies* (2.3.1.2) *with* (2.3.1.4) *holding and with*

$$\begin{cases} g \in L^{\frac{p'}{q-1}}(0,T;W^{1,\frac{p'}{q-1}}(\Omega,\mathbb{R}^m)) \cap C^0([0,T];L^2(\Omega,\mathbb{R}^m)), \\ g_t \in L^{p'}(0,T;W^{-1,p'}(\Omega,\mathbb{R}^m)). \end{cases}$$

Then there exists a weak solution of the system (2.3.1.1) *in the sense of Definition 2.3.1.1.*

The proof (or a sketch thereof) will be given after the correction to the paper, but we remark now the idea and the structure of said proof. One regularises the integrand f to $f_\epsilon(\xi) = f(\xi) + \epsilon|\xi|^q$. The derivative of this function now satisfies standard q-growth conditions (see Definition 2.1.3.1), from which we deduce the existence of a unique solution u_ϵ to the regularised problem using monotone operator theory. One then seeks to pass this parameter $\epsilon \to 0$ in the definition of weak solution in order to deduce existence of a weak solution to (2.3.1.1) with (2.3.1.2) and (2.3.1.4) holding. Passing to the limit requires the deduction of a local L^q estimate on Du_ϵ, and on obtaining a local strong convergence result on the sequence Du_ϵ in L^p.

2.3.2 Parabolic function spaces

There are a number of definitions and embedding results we need regarding parabolic function spaces before we can deduce the required L^q estimate. We refer for all the results to section 6 of [12], and the references cited therein. We begin by stating a standard interpolation inequality on a parabolic cylinder. The proof is based on a scaling argument, combined with the Gagliardo−Nirenberg interpolation inequality, which we include for the sake of completeness.

Lemma 2.3.2.1. *Let $\sigma \geq 1$, let $Q_\rho(z_0) = B_\rho(x_0) \times (t_0 - \rho^2, t_0) \subset \mathbb{R}^{n+1}$ with $n \geq 1$ and*

$$u \in L^\sigma(t_0 - \rho^2, t_0; W^{1,\sigma}(B_\rho(x_0))) \cap C^0(t_0 - \rho^2, t_0; L^2(B_\rho(x_0))).$$

Then $u \in L^{\frac{\sigma(n+2)}{n}}(Q_\rho(z_0))$ and there is a constant $c = c(n, \sigma)$ such that the following estimate holds:

$$\int_{Q_\rho(z_0)} |u|^{\frac{\sigma(n+2)}{n}} \,\mathrm{d}x \leq c \left(\int_{Q_\rho(z_0)} |Du|^\sigma + \left|\frac{u}{\rho}\right|^\sigma \,\mathrm{d}x \right) \left(\sup_{t \in (t_0 - \rho^2, t_0)} \int_{B_\rho(x_0)} |u(\cdot, t)|^2 \,\mathrm{d}x \right)^{\frac{\sigma}{n}}.$$

Proof. We begin with the following Gagliardo−Nirenberg type inequality, taken over a ball of radius 1 (see [24], Theorem 2.1, with the obvious modification for non-specified boundary data).

$$\|v\|_{L^p(B_1(x_0))} \leq c \|v\|^{1-\alpha}_{L^2(B_1(x_0))} \|v\|^\alpha_{W^{1,\sigma}(B_1(x_0))} \tag{2.3.2.1}$$

where $p = \frac{\sigma(n+2)}{n}$ and

$$\alpha = \left(\frac{1}{2} - \frac{n}{\sigma(n+2)} \right) \left(\frac{1}{2} + \frac{1}{n} - \frac{1}{\sigma} \right)^{-1} = \frac{n}{n+2}.$$

Raising both sides of (2.3.2.1) to the power p we obtain

$$\|v\|^p_{L^p(B_1(x_0))} \leq c \|v\|^{\frac{2\sigma}{n}}_{L^2(B_1(x_0))} \|v\|^\sigma_{W^{1,\sigma}(B_1(x_0))}. \tag{2.3.2.2}$$

Now consider the function u such that $v(t, x) = u(t, \rho x)$ and substitute this into (2.3.2.2) to get

$$\int_{B_\rho(x_0)} |u|^{\frac{\sigma(n+2)}{n}} \mathrm{d}x \leq c \left(\int_{B_\rho(x_0)} |u|^2 \,\mathrm{d}x \right)^{\frac{\sigma}{n}} \left(\int_{B_\rho(x_0)} \left|\frac{u}{\rho}\right|^\sigma + |Du|^\sigma \mathrm{d}x \right). \tag{2.3.2.3}$$

The remainder of the proof follows along the same lines as in DiBenedetto ([24], Proposition 3.1). One integrates both sides of the inequality in time over the range $t \in (t_0 - \rho^2, t_0)$ and uses Hölder's inequality to obtain the desired result. For the sake of clarity we compute the integral involving the derivative. We have that $|Dv(t,x)| = \rho^n |Du(t, \rho x)|$. Raising to the power σ and integrating over $B_\rho(x_0)$ before substituting $y = \rho x$ gives

$$\int_{B_1(x_0)} |Dv(t,x)|^\sigma \, \mathrm{d}x = \int_{B_\rho(x_0)} \rho^{n(\sigma-1)} |Du(t,y)|^\sigma \, \mathrm{d}y.$$

Performing similar computations for the other integrals results in being able to drop the powers of ρ appearing in the above expression. □

Next we state a well-known iteration lemma:

Lemma 2.3.2.2. *Let* $0 < \theta < 1$, $A, B > 0$ *and* $\alpha > 0$. *Then there exists a constant* $c = c(\alpha, \theta)$ *such that for any* $0 < r < \rho$ *and any non-negative bounded function* $\phi : [r, \rho] \to [0, \infty)$ *satisfying*

$$\phi(s) \le \theta \phi(t) + A(t-s)^{-\alpha} + B \quad \textit{for all } 0 < r \le s < t \le \rho,$$

we have

$$\phi(r) \le c[A(\rho - r)^{-\alpha} + B].$$

One of the main tools which we shall use in obtaining the local gradient estimate is that of fractional Sobolev spaces. We recall the definition below, before giving the corresponding notion of fractional Sobolev spaces in the context of time dependent functions. Then we end this subsection by stating two embedding theorems from Section 6 of [12], which will be used in the next subsection to obtain the desired estimate on the spatial gradient.

Definition 2.3.2.3. We say that a function $f \in W^{k,p}(\Omega, \mathbb{R}^m)$, with $1 \le p < \infty$ and $k \in \mathbb{N}_0$, lies in the fractional Sobolev space $W^{k+\alpha,p}(\Omega, \mathbb{R}^m)$ with $\alpha \in (0,1)$ if for any multi-index $\beta \in \mathbb{N}_0^n$ with $|\beta| = k$ we have

$$[D^\beta f]^p_{\alpha,p;\Omega} := \int_\Omega \int_\Omega \frac{|D^\beta f(x) - D^\beta f(y)|^p}{|x-y|^{n+\alpha p}} \, \mathrm{d}x \, \mathrm{d}y < \infty.$$

We endow the space $W^{k+\alpha,p}(\Omega,\mathbb{R}^m)$ with the norm

$$\|f\|_{W^{k+\alpha,p}(\Omega)} := \|f\|_{W^{k,p}(\Omega)} + \sum_{|\beta|=k} [D^\beta f]_{\alpha,p;\Omega}.$$

Definition 2.3.2.4. We say that a function $u \in L^p(0,T;W^{k,p}(\Omega,\mathbb{R}^m))$ is in the parabolic fractional Sobolev space $L^p(0,T;W^{k+\alpha,p}(\Omega,\mathbb{R}^m))$ if for any multi-index $\beta \in \mathbb{N}_0^n$ with $|\beta| = k$ we have

$$[D^\beta u]^p_{\alpha,0,p;Q_T} := \int_0^T \int_\Omega \int_\Omega \frac{|D^\beta u(t,x) - D^\beta u(t,y)|^p}{|x-y|^{n+\alpha p}} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t.$$

We endow the space $L^p(0,T;W^{k+\alpha,p}(\Omega,\mathbb{R}^m))$ with the following norm:

$$\|u\|_{k+\alpha,0,p;Q_T} := \|u\|_{L^p(0,T;W^{k,p}(\Omega;\mathbb{R}^m))} + \sum_{|\beta|=k} [D^\beta f]_{\alpha,0,p;Q_T}.$$

Lemma 2.3.2.5. *Let $B_\rho(x_o) \times (t_1,t_2) \subset \mathbb{R}^{1+n}$ be a general space-time cylinder with $\rho \leq 1$ and let λ, $\mu \in (0,1)$ and $1 < p, r < s < \infty$ be such that*

$$(s-p)\left(1 - \mu + \frac{n}{r}\right) \leq \lambda p.$$

Further, assume that $u \in L^p(t_1,t_2;W^{1+\lambda,p}(B_\rho(x_0))) \cap L^\infty(t_1,t_2;W^{\mu,r}(B_\rho(x_0)))$. Then for any $\theta \in (0,\rho)$ we have

$$Du \in L^s(B_\theta(x_0) \times (t_1,t_2)),$$

and the estimate

$$\|Du\|_{L^s(B_\theta(x_0)\times(t_1,t_2))} \leq c\|u\|^{\frac{p}{s}}_{L^p(t_1,t_2;W^{1+\lambda,p}(B_\rho(x_0)))} \sup_{t\in(t_1,t_2)} \|u(t,\cdot)\|^{\frac{s-p}{p}}_{W^{\mu,r}(B_\rho(x_0))}$$

holds with $c = c(n,\mu,\lambda,r,p,s,\frac{1}{\rho-\theta})$.

Finally we state two embeddings from Nikolskiĭ spaces (defined via finite differences) into fractional Sobolev spaces.

Lemma 2.3.2.6. *Let $k \in \mathbb{N}$, $\tilde{\Omega} \subset \Omega$, $\theta \in (0,1)$ and $0 \leq t_1 \leq t_2 \leq T$.*

1. *Assume that $u \in L^\infty(0,T; L^2(\Omega, \mathbb{R}^k))$ satisfies*

$$\sup_{t \in (t_1,t_2)} \int_{\tilde{\Omega}} |u(t, x + he_i) - u(t,x)|^2 \; \mathrm{d}x \leq M|h|^{2\theta}$$

for every $i \in \{1, \ldots, n\}$ and $h \in \mathbb{R}$ with $|h| \leq \min\{\mathrm{dist}(\tilde{\Omega}, \partial\Omega), A\}$, where $A, M > 0$. Then for every $\alpha \in (0,\theta)$ and $\mathcal{O} \subset\subset \tilde{\Omega}$ there exists a constant

$$c = c(n, \theta, \alpha, A, \mathrm{dist}(\mathcal{O}, \partial\tilde{\Omega}), \mathrm{dist}(\Omega, \tilde{\Omega}))$$

such that

$$\sup_{t \in (t_1,t_2)} [u(t,\cdot)]^2_{\alpha,p;\mathcal{O}} \equiv \sup_{t \in (t_1,t_2)} \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|u(t,x) - u(t,y)|^2}{|x-y|^{n+2\alpha}} \; \mathrm{d}x \; \mathrm{d}y \leq cM.$$

2. *Assume that $u \in L^p(Q_T, \mathbb{R}^k)$ satisfies*

$$\int_{t_1}^{t_2} \int_{\tilde{\Omega}} |u(t, x + he_i) - u(t,x)|^p \; \mathrm{d}x \; \mathrm{d}t \leq M|h|^{\theta p}$$

for every $i \in \{1, \ldots, n\}$ and $h \in \mathbb{R}$ with $|h| \leq \min\{\mathrm{dist}(\tilde{\Omega}, \partial\Omega), A\}$, where $A, M > 0$. Then for every $\gamma \in (0,\theta)$ and $\mathcal{O} \subset\subset \tilde{\Omega}$ there exists a constant

$$c = c(n, \theta, \gamma, A, \mathrm{dist}(\mathcal{O}, \partial\tilde{\Omega}), \mathrm{dist}(\Omega, \tilde{\Omega}))$$

such that

$$[u]^p_{\alpha,0,p;\mathcal{O}\times(t_1,t_2)} \equiv \int_{t_1}^{t_2} \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|u(t,x) - u(t,y)|^p}{|x-y|^{n+p\gamma}} \; \mathrm{d}x \; \mathrm{d}y \; \mathrm{d}t \leq cM.$$

2.3.3 Correction to "A Local L^q-Estimate for the Spatial Gradient"

The first result deduced in section 6 of [12] as a step towards obtaining an L^q−estimate on the spatial gradient is the following Caccioppoli-style inequality. In what follows, we define $\tau_{h,i}[u]$ to be the finite difference of the function $u \in L^1(Q_T)$. That is,

$$\tau_{h,i}[u](t,x) := u(t, x + he_i) - u(t,x),$$

where e_i is the unit vector in the i-th co-ordinate direction. This finite difference is only defined when $(t, x + he_i) \in Q_T$.

Lemma 2.3.3.1. *Let u be a weak solution to* (2.3.1.1) *with* (2.3.1.2) *and* (2.3.1.4) *holding. Then for any parabolic cylinder $Q_\rho(z_0) \subset\subset \Omega_T$, any $0 < |h| < \operatorname{dist}(B_\rho(x_0), \partial\Omega)$ and any $i \in \{1, \dots, n\}$, there holds*

$$\begin{aligned} \sup_{t\in(t_0-r^2,t_0)} \int_{B_r(x_0)} |\tau_{h,i}[u](t,\cdot)|^2 \, \mathrm{d}x + \int_{Q_r(z_0)} |\tau_{h,i}[Du]|^p \, \mathrm{d}x \\ \le \frac{c}{(\rho-r)^2} \int_{Q_\rho(z_0)} (|Du| + |\tau_{h,i}[Du]|)^{q-2} |\tau_{h,i}[u]|^2 + |\tau_{h,i}[u]|^2 \, \mathrm{d}x, \end{aligned} \tag{2.3.3.1}$$

with a constant $c = c(\nu, L, p, q)$, with $t_0 > r^2$ and with $\rho > r$.

Using this we begin to make the correction of the error in [12].

Lemma 2.3.3.2. *Let u be a weak solution to* (2.3.1.1) *with* (2.3.1.2) *and* (2.3.1.4) *holding. Then for any $\sigma \in [p,q)$, any cylinder $Q_\rho(z_0) \subset\subset \Omega_T$, any radius $r \in [\frac{\rho}{2}, \rho)$ and any*

$$s < S(\sigma) := p + \frac{4 - (n+2)(q-\sigma)}{n}, \tag{2.3.3.2}$$

the higher integrability estimate

$$\int_{Q_r(z_0)} |Du|^s \, \mathrm{d}x \, \mathrm{d}t \le c \left(\int_{Q_\rho(z_0)} |Du|^\sigma \, \mathrm{d}x \, \mathrm{d}t + M_{z_0,\rho} \right)^{\frac{n+2}{n}} \tag{2.3.3.3}$$

holds true with $c = c(n, \nu, L, p, q, \sigma, s, \rho, r)$ and with

$$M_{z_0,\rho} := 1 + \sup_{t\in(t_0-\rho^2,t_0)} \int_{B_\rho(x_0)} |u(t,\cdot)|^2 \, \mathrm{d}x + \int_{Q_\rho(z_0)} |u|^p \, \mathrm{d}x \, \mathrm{d}t. \tag{2.3.3.4}$$

Proof. We follow the proof of Lemma 6.8 in [12]. We start by choosing two radii ρ_1 and ρ_2 such that $r_1 := \frac{\rho+r}{2} \le \rho_1 < \rho_2 \le \frac{3\rho+r}{4} =: r_2$ and consider $0 < |h| < \frac{\rho-r}{4}$. This ensures that $x + he_i \in B_\rho(x_0)$ whenever $x \in B_{\rho_2}(x_0)$. Now we define

$$\alpha := 2 - \frac{(n+2)(q-\sigma)}{2} \tag{2.3.3.5}$$

and remark that in Lemma 6.8 of [12] the authors instead define $\alpha = 2 - \frac{(n+2)(q-\sigma)}{n}$. Note that $0 < \alpha < 2$ if and only if $0 < q - \sigma < \frac{4}{n+2}$. As this must hold for all $\sigma \in [p, q)$ we see that we in particular need $q - p < \frac{4}{n+2}$. Now we use this parameter α along with standard estimates regarding the finite difference $\tau_{h,i}$ to apply Hölder's inequality to the right-hand side of (2.3.3.1). We remark that

$$\frac{q-2}{\sigma} + \frac{\alpha}{\sigma} + \frac{\sigma - \alpha - (q-2)}{\sigma} = 1,$$

and that $\sigma - \alpha - (q-2) > 0$, so we can apply Hölder's inequality with the exponents $\frac{\sigma}{q-2}$, $\frac{\sigma}{\alpha}$ and $\frac{\sigma}{\sigma-\alpha-(q-2)}$. Doing so gives the following chain of inequalities:

$$\begin{aligned}
&\sup_{t\in(t_0-\rho_1^2,t_0)} \int_{B_{\rho_1}(x_0)} |\tau_{h,i}[u](t,\cdot)|^2 \,\mathrm{d}x + \int_{Q_{\rho_1}(x_0)} |\tau_{h,i}[Du]|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{c}{(\rho_2-\rho_1)^2} \int_{Q_{\rho_2}(z_0)} (1 + |Du| + |\tau_{h,i}|[Du]|)^{q-2} |\tau_{h,i}u|^\alpha |\tau_{h,i}u|^{2-\alpha} \,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{c}{(\rho_2-\rho_1)^2} \left(\int_{Q_{\rho_2}(z_0)} (1 + |Du| + |\tau_{h,i}Du|)^\sigma \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{q-2}{\sigma}} \\
&\quad \times \left(\int_{Q_{\rho_2}(z_0)} |\tau_{h,i}[u]|^\sigma \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{\alpha}{\sigma}} \left(\int_{Q_{\rho_2}(z_0)} |\tau_{h,i}[u]^{\frac{\sigma(2-\alpha)}{\sigma-\alpha-(q-2)}} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{\sigma-\alpha-(q-2)}{\sigma}} \\
&\le \frac{c|h|^\alpha}{(\rho_2-\rho_1)^2} \left(\int_{Q_\rho(z_0)} (1 + |Du|)^\sigma \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{q+\alpha-2}{\sigma}} \left(\int_{Q_{\rho_2}(z_0)} |\tau_{h,i}[u]^{\frac{\sigma(2-\alpha)}{\sigma-\alpha-(q-2)}} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{\sigma-\alpha-(q-2)}{\sigma}}.
\end{aligned}$$

We now compute:

$$\begin{aligned}
\frac{\sigma(2-\alpha)}{\sigma-\alpha-(q-2)} &= \frac{\sigma\frac{(n+2)(q-\sigma)}{2}}{\sigma - q + \frac{(n+2)(q-\sigma)}{2}} \\
&= \frac{\sigma(n+2)(q-\sigma)}{-2(q-\sigma) + (n+2)(q-\sigma)} \\
&= \frac{\sigma(n+2)}{n}.
\end{aligned}$$

We make the further remark that in the proof of Lemma 6.8 of [12] the authors obtain that $\frac{\sigma(2-\alpha)}{\sigma-\alpha-(q-2)} = \frac{\sigma(n+2)}{2}$ with their choice of α, and hence the application of Lemma 2.3.2.1 is only valid in the case $n = 2$. For $n = 2$ the result obtained here coincides with that obtained in [12]. Choosing α as in (2.3.3.5), we may apply Lemma 2.3.2.1 to obtain

$$\begin{aligned}\int_{Q_{\rho_2(z_0)}} |\tau_{h,i}[u]|^{\frac{\sigma(2-\alpha)}{\sigma-\alpha-(q-2)}} \, \mathrm{d}x \, \mathrm{d}t &\le c \left(\int_{Q_{\rho_2(z_0)}} |D[\tau_{h,i}u]|^\sigma + \left| \frac{\tau_{h,i}[u]}{\rho_2} \right|^\sigma \, \mathrm{d}x \, \mathrm{d}t \right) \\ &\quad \times \left(\sup_{t\in(t_0-\rho_2^2,t_0)} \int_{B_{\rho_2}(x_0)} |\tau_{h,i}(t,\cdot)|^2 \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{\sigma}{n}} \\ &\le c \left(\int_{Q_\rho(z_0)} (1+|Du|)^\sigma \, \mathrm{d}x \, \mathrm{d}t \right) \\ &\quad \times \left(\sup_{t\in(t_0-\rho_2^2,t_0)} \int_{B_{\rho_2}(x_0)} |\tau_{h,i}(t,\cdot)|^2 \, \mathrm{d}x \right)^{\frac{\sigma}{n}},\end{aligned}$$

where we have used that $\frac{h}{\rho_2} < 1$ in the final inequality. Inserting this into the previous chain of inequalities we get

$$\begin{aligned}&\sup_{t\in(t_0-\rho_1^2,t_0)} \int_{B_{\rho_1}(x_0)} |\tau_{h,i}[u](t,\cdot)|^2 \, \mathrm{d}x + \int_{Q_{\rho_1}(x_0)} |\tau_{h,i}[Du]|^p \, \mathrm{d}x \, \mathrm{d}t \\ &\le \frac{c|h|^\alpha}{(\rho_2-\rho_1)^2} \int_{Q_\rho(z_0)} |\tau_{h,i}[Du]|^p \, \mathrm{d}x \, \mathrm{d}t \\ &\quad \times \left(\sup_{t\in(t_0-\rho_2^2,t_0)} \int_{B_{\rho_2}(x_0)} |\tau_{h,i}[u](t,\cdot)|^2 \, \mathrm{d}x \right)^{\frac{\sigma-\alpha-(q-2)}{n}}.\end{aligned}$$

Again, we compute the exponent appearing in the right-hand side of the above inequality:

$$\begin{aligned}\frac{\sigma-\alpha-(q-2)}{n} &= \frac{\sigma + \frac{(n+2)(q-\sigma)}{2} - q}{n} \\ &= \frac{n(q-\sigma)}{2n} \\ &= \frac{q-\sigma}{2}.\end{aligned}$$

We note that the Hölder conjugate of $\frac{2}{q-\sigma}$ is $\frac{2}{2-(q-\sigma)}$, and that $q-\sigma < 2$. Therefore, we may apply Young's inequality to the right-hand side of the above inequality to see that

$$\sup_{t\in(t_0-\rho_1^2,t_0)} \int_{B_{\rho_1}(x_0)} |\tau_{h,i}[u](t,\cdot)|^2 \, \mathrm{d}x + \int_{Q_{\rho_1}(x_0)} |\tau_{h,i}[Du]|^p \, \mathrm{d}x \, \mathrm{d}t$$

$$
\begin{aligned}
&\leq \left(\frac{c|h|^{\alpha}}{(\rho_2-\rho_1)^2}\int_{Q_\rho(z_0)} |\tau_{h,i}[Du]|^p \, \mathrm{d}x \, \mathrm{d}t\right)^{\frac{2}{2-(q-\sigma)}} \\
&\quad + \frac{1}{2} \sup_{t\in(t_0-\rho_2^2,t_0)} \int_{B_{\rho_2}(x_0)} |\tau_{h,i}[u](t,\cdot)|^2 \, \mathrm{d}x.
\end{aligned}
$$

Now we can apply Lemma 2.3.2.2 to reabsorb the supremum from the right-hand side of the above inequality. Using that $r_2 - r_1 = \frac{\rho - r}{4}$ we see that we have:

$$
\begin{aligned}
&\sup_{t\in(t_0-r_1^2,t_0)} \int_{B_{r_1}(x_0)} |\tau_{h,i}[u](t,\cdot)|^2 \, \mathrm{d}x + \int_{Q_{r_1}(x_0)} |\tau_{h,i}[Du]|^p \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \left(\frac{c|h|^{\alpha}}{(\rho-r)^2}\int_{Q_\rho(z_0)} |\tau_{h,i}[Du]|^p \, \mathrm{d}x \, \mathrm{d}t\right)^{\frac{2}{2-(q-\sigma)}}.
\end{aligned}
$$

Now we may apply Lemma 2.3.2.6 to deduce that

$$
u \in L^p(t_0 - r_2^2, t_0; W^{1+\lambda,p}(B_{r_2}(x_0), \mathbb{R}^m)) \cap L^\infty(t_0 - r_2^2, t_0; W^{\mu,2}(B_{r_2}(x_0), \mathbb{R}^m)),
$$

along with the following estimates:

$$
[Du]^p_{\lambda,0,p;Q_{r_2}(z_0)} \leq c \left(\int_{Q_\rho(z_0)} (1+|Du|)^\sigma \, \mathrm{d}x \, \mathrm{d}t\right)^{\frac{2}{2-(q-\sigma)}}
$$

and

$$
\sup_{t\in(t_0-r_2^2,t_0)} [u(t,\cdot)]^2_{\mu,2;B_{r_2}(x_0)} \leq c \left(\int_{Q_\rho(z_0)} (1+|Du|)^\sigma \, \mathrm{d}x \, \mathrm{d}t\right)^{\frac{2}{2-(q-\sigma)}}.
$$

This holds for any

$$
\lambda < \frac{2\alpha}{p(2-(q-\sigma))} := \tilde{\lambda}
$$

and

$$
\mu < \frac{\alpha}{2-(q-\sigma)} := \tilde{\mu}.
$$

We remark that this is a valid choice of exponents for which to apply Lemma 2.3.2.6 as $\tilde{\mu} \leq 1$ (as $\frac{(n+2)(q-\sigma)}{2} > (q-\sigma)$). Note that $c = c(n, \nu, L, p, q, \sigma, \mu, \lambda, \rho, r)$. Then as in [12] we can bound

$$
\|u\|_{L^p(t_0-r_2^2,t_0;W^{1+\lambda,p}(B_{r_2}(x_0)))} \leq c \left(\int_{Q_\rho(z_0)} |Du|^\sigma \, \mathrm{d}x + M_{z_0,\rho}\right)^{\frac{2}{p(2-(q-\sigma))}}
$$

and

$$\sup_{t\in(t_0-r_2^2,t_0)} \|u(t,\cdot)\|_{W^{\mu,2}(B_{r_2}(x_0))} \leq c\left(\int_{Q_\rho(z_0)} |Du|^\sigma \,\mathrm{d}x + M_{z_0,\rho}\right)^{\frac{1}{2-(q-\sigma)}}.$$

We now apply Lemma 2.3.2.5, noting that we are allowed to do so as long as $s < S(\sigma)$ as

$$(S(\sigma)-p)\left(1-\tilde{\mu}+\frac{n}{2}\right)\frac{1}{\tilde{\lambda}p} = 1.$$

This, as $r < r_2$ gives us the estimate:

$$\int_{Q_r(z_0)} |Du|^s \,\mathrm{d}x\,\mathrm{d}t \leq c\left(\int_{Q_\rho(z_0)} |Du|^\sigma \,\mathrm{d}x\,\mathrm{d}t + M_{\rho,x_0}\right)^{(1+\frac{s-p}{2})\left(\frac{2}{2-(q-\sigma)}\right)},$$

where $c = c(n,\nu,L,p,q,\sigma,s,\rho,r)$. Now we estimate:

$$\begin{aligned}\left(1+\frac{s-p}{2}\right)\left(\frac{2}{2-(q-\sigma)}\right) &< \left(\frac{2}{2-(q-\sigma)}\right)\left(1+\frac{4-(n+2)(q-\sigma)}{2n}\right)\\ &= \left(\frac{1}{2-(q-\sigma)}\right)\left(\frac{(n+2)(2-(q-\sigma))}{n}\right)\\ &= \frac{n+2}{n},\end{aligned}$$

and use that $M_{\rho,x_0} \geq 1$ to obtain our final estimate

$$\int_{Q_r(z_0)} |Du|^s \,\mathrm{d}x\,\mathrm{d}t \leq c\left(\int_{Q_\rho(z_0)} |Du|^\sigma \,\mathrm{d}x\,\mathrm{d}t + M_{\rho,x_0}\right)^{\frac{n+2}{n}}.$$

All that remains to be checked is that this actually gives us some improvement in the integrability of Du. For this we need to show that $S(\sigma) > \sigma$ for all $\sigma \in [p,q)$. Once we have done this we can deduce the existence of a constant $\epsilon_0 > 0$ such that $S(\sigma) \geq \sigma + \epsilon_0$, and remark that in this case the constant ϵ_0 is determined by n, p and q, and can be chosen as, for example,

$$\epsilon_0 = \frac{S(p)-p}{2} = \frac{4-(n+2)(q-p)}{2n}. \tag{2.3.3.6}$$

In order to show this we first define the function

$$f(\sigma) := p - \sigma + \frac{4 - (n+2)(q-\sigma)}{n} (= S(\sigma) - \sigma),$$

and compute that

$$f'(\sigma) = \frac{2}{n} > 0,$$

so it suffices only to check that $f(p) > 0$. We easily see that:

$$f(p) = \frac{4 - (n+2)(q-p)}{n} > \frac{4 - (n+2)\left(\frac{4}{n+2}\right)}{n} = 0,$$

as required. □

2.3.4 A sketch of the remainder of the argument

We now sketch the remainder of the argument which is found in [12] sections 6 and 7.

Existence:

Consider a cylinder $Q_R(z_0) \subset\subset \Omega_T$ and set

$$I := \left\lceil \frac{2n(q-p)}{4 - (n+2)(q-p)} \right\rceil = \left\lceil \frac{q-p}{\epsilon_0} \right\rceil.$$

We define a sequence of exponents

$$\sigma_i = \frac{\sigma_{i-1} + S(\sigma_{i-1})}{2} = p + i\epsilon_0$$

and a sequence of radii

$$r_i = \frac{R}{2} + \frac{R}{2^{i+1}}$$

where $i \in \{1, \ldots, I\}$. Then clearly $\sigma_i \in (\sigma_{i-1}, S(\sigma_{i-1}))$, and $S(\sigma_I) > q$ follows from the fact that

$$p + \frac{i(4 - (n+2)(q-p))}{2n} > q \iff i > \frac{2n(q-p)}{4 - (n+2)(q-p)}.$$

Doing this we can iterate Lemma 2.3.3.2 I-times in order to obtain the bound

$$\int_{Q_{\frac{R}{2}}(z_0)} |Du|^q \, \mathrm{d}x \, \mathrm{d}t \leq c \left(\int_{Q_R(z_0)} |Du|^p \mathrm{d}x \, \mathrm{d}t + M_{z_0,R} \right)^{\left(\frac{n+2}{n}\right)^I} \tag{2.3.4.1}$$

with $c = c(n, \nu, L, p, q, R)$.

Now given $\epsilon \in (0, 1]$ we define $f_\epsilon(\xi) := f(\xi) + \epsilon|\xi|^q$ where f satisfies (2.3.1.2). One can show that Df_ϵ now satisfies standard q-growth conditions, and so from the monotone operator theory discussed in Subsections 2.1.2 and 2.1.3, we have the existence of a unique weak solution u_ϵ to the problem

$$\begin{cases} \partial_t u_\epsilon - \operatorname{div}(Df_\epsilon(Du_\epsilon)) = 0 & \text{in } Q_T, \\ u_\epsilon = g & \text{on } \partial_{\mathcal{P}} Q_T. \end{cases} \tag{2.3.4.2}$$

We now seek to pass $\epsilon \to 0$ in the following weak formulation:

$$\int_{Q_T} u_\epsilon \cdot \phi_t - \langle Df_\epsilon(Du_\epsilon), D\phi \rangle \mathrm{d}x \, \mathrm{d}t = 0 \tag{2.3.4.3}$$

for all $\phi \in C_0^\infty(\Omega, \mathbb{R}^m)$. This is done by using an energy estimate to deduce bounds on $\|u_\epsilon\|_{W^{1,p}}$ and $\|u_\epsilon\|_{L^\infty(t_0-\rho^2,t_0;L^2(\Omega,\mathbb{R}^m))}$ in terms of only the given boundary data and initial data. This allows us to deduce a local L^q bound on Du_ϵ which is uniform in ϵ, as we can apply Lemma 2.3.4.1 to each element Du_ϵ in our sequence. This local L^q bound is given on cylinders of a small enough radius, from which the bound follows on any compactly contained cylinder by a covering argument. Standard arguments are then applied in order to obtain the required convergence properties, which allow us to pass to the limit in (2.3.4.3), giving existence of the solution.

Uniqueness:

The regularity obtained on $\partial_t u$ in the above existence proof is insufficient to allow for testing against the function u in the weak formulation of the problem, meaning that the standard uniqueness argument fails to be applicable. Uniqueness is instead obtained by considering *variational solutions* and demonstrating that, provided g has sufficient regularity, there is a unique variational solution, before showing that the notions of variational solutions and weak solutions coincide. Variational solutions for systems of the form (2.3.1.1) are defined in [12] as follows:

Definition 2.3.4.1. Suppose that $f : \mathbb{R}^{mn} \to \mathbb{R}$ is a C^1 function satisfying the following growth and monotonicity assumptions:

$$\begin{cases} 0 \le f(\xi) \le L(1+|\xi|)^q, \\ \langle Df(\xi) - Df(\eta), \xi - \eta \rangle \ge \nu(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2 \end{cases} \tag{2.3.4.4}$$

for some $\nu \le 1 \le L$ and $\mu \in [0,1]$, and let the datum g have the following regularity:

$$\begin{cases} g \in L^{p'(q-1)}(0,T; W^{1,p'(q-1)}(\Omega, \mathbb{R}^m)) \cap C^0([0,T]; L^2(\Omega, \mathbb{R}^m)), \\ \partial_t g \in L^{p'}(0,T; W^{-1,p'}(\Omega, \mathbb{R}^m)). \end{cases} \tag{2.3.4.5}$$

Then a function $u \in L^p(0,T; W^{1,p}_g(\Omega, \mathbb{R}^m)) \cap C_w([0,T]; L^2(\Omega; \mathbb{R}^m))$ is termed a *variational solution* of (2.3.1.1) if and only if $u(\cdot, 0) = g(\cdot, 0)$ and the following variational inequality is satisfied for all

$$v \in L^p(0,T; W^{1,p}_g(\Omega, \mathbb{R}^m))$$

with

$$\partial_t v \in L^{p'}(0,T; W^{-1,p'}(\Omega, \mathbb{R}^m))$$

and all $\tau \in (0,T]$:

$$\begin{aligned} &\int_0^\tau \langle \partial_t v, v - u \rangle_{W^{1,p}(\Omega,\mathbb{R}^m)} \, \mathrm{d}t + \int_{\Omega_\tau} [f(Dv) - f(Du)] \, \mathrm{d}x \, \mathrm{d}t \\ &\qquad \ge \frac{1}{2} \|(v-u)(\cdot,\tau)\|^2_{L^2(\Omega;\mathbb{R}^m)} - \frac{1}{2} \|(v-g)(\cdot,0)\|^2_{L^2(\Omega;\mathbb{R}^m)}. \end{aligned} \tag{2.3.4.6}$$

With this definition in mind the following theorems are then proven (see Theorem 2.3 and 2.4 in [12]):

Theorem 2.3.4.2. *Suppose that the exponents p and q satisfy*

$$\max\{1, \frac{2n}{n+2}\} \le p < q < p+1. \tag{2.3.4.7}$$

Then there exists a variational solution to (2.3.1.1).

Theorem 2.3.4.3. *Suppose that g has instead the following higher regularity:*

$$\begin{cases} g \in L^{\frac{p}{p+1-q}}(0,T;W^{1,\frac{p}{p+1-q}}(\Omega,\mathbb{R}^m)) \cap C^0([0,T];L^2(\Omega,\mathbb{R}^m)), \\ \partial_t g \in L^{p'}(0,T;W^{-1,p'}(\Omega,\mathbb{R}^m)). \end{cases} \tag{2.3.4.8}$$

Then the variational solution is unique.

It is shown in [12] that all weak solutions are variational. The following theorem gives the converse; that all variational solutions are weak (also in [12]).

Theorem 2.3.4.4. *Suppose that f is C^2 and satisfies* (2.3.1.2) *with*

$$2 \leq p < q < p + \min\{1, \frac{4}{n+2}\},$$

and g satisfying (2.3.4.8). *Then any variational solution is also a weak solution.*

Combining this with the previous theorem gives uniqueness of the weak solution.

2.3.5 Closing remarks

We close this section by giving some remarks regarding the range of valid exponents and further remarks regarding the possible application of this to the model described by (1.1.2.1)–(1.1.2.7).

In [12] it is claimed that the range for which the result holds is $q - p < \min\left\{1, \frac{4}{n}\right\}$ for $n \geq 2$. This turns out not to be the case, and instead one requires the smaller range of $q - p < \min\left\{\frac{4}{n+2}, 1\right\}$. This range agrees with the range found in [11], where a single equation is considered as opposed to a system (that is, $m = 1$), and local L^∞ bounds on the spatial gradient are proven in place of the L^q bounds we have shown here.

The inclusion of the lower order terms and a constant in space right-hand side does not pose any problem to us in deducing the local L^q bound on Du_ϵ. When obtaining the Cacciopoli inequality, thanks to the fact that we have a constant right-hand side and a matrix satisfying $\langle Ax, x\rangle = 0$ for all $x \in \mathbb{R}^3$, the additional terms simply disappear when we choose the test function as in [12]. Passing to the limit in the term involving A follows from the weak convergence results which have already been obtained and using the fact that $p \geq 2$.

Chapter 3

Forward-Backward Systems of Equations Exhibiting (p, q)-like Growth Conditions

As has been discussed previously, the formulation of the nonlinearity given by (1.1.2.9) does not allow for the application of monotone operator theory. We now study higher-dimensional problems exhibiting similar growth conditions, which also do not satisfy a monotonicity condition. We begin this chapter by re-introducing the assumptions on the system which we will be studying. We will also define a *Young measure solution* to these systems. After showing existence, we consider the long-time behaviour of the Young measure solutions, showing that under suitable assumptions on the source term we have convergence (in some appropriate sense) to a particular Young measure solution of the corresponding steady-state equation. Finally, this chapter will end with a brief discussion about analogous results that hold for forward-backward systems of equations under additional assumptions. Content of Sections 3.1−3.3 can be found in [15].

3.1 Introduction of the systems under consideration

The systems which we shall be studying are of the following form:

$$\partial_t u - \operatorname{div}(a(Du)) - Bu = F, \tag{3.1.0.1}$$

with $u : [0,T] \times \Omega \to \mathbb{R}^m$ and $\Omega \subset \mathbb{R}^n$ a bounded open subset of $\mathbb{R}^n$. We define $Q_T = (0,T) \times \Omega$. We define real numbers p_i which satisfy, for each $i = 1, \dots, m$, the inequality $p_i > \max\{1, \frac{2n}{n+2}\}$. We also define $p := \min_{i=1,\dots,m}\{p_i\}$ and $q := \max_{i=1,\dots,m}\{p_i\}$, $\hat{q} = \max\{q, 2\}$, and impose

$$q - p < 1.$$

We also assume

$$F \in \bigtimes_{i=1}^{m} L^{p_i'}(Q_T) \tag{3.1.0.2}$$

and

$$Bu \cdot u \leq 0 \tag{3.1.0.3}$$

with $B \in \mathbb{R}^{m \times n}$ a constant matrix. The function u satisfies the initial condition

$$u(0, \cdot) = u_0(\cdot) \in L^2(\Omega; \mathbb{R}^m), \tag{3.1.0.4}$$

and boundary data of Dirichlet type

$$u|_{(0,T] \times \partial\Omega} = 0. \tag{3.1.0.5}$$

Here Du denotes the spatial derivative of u, and $a : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$ is a locally Lipschitz mapping with the following properties:

$$a(A) = K(A)A, \tag{3.1.0.6}$$

where $K : \mathbb{R}^{m\times n} \to \mathbb{R}$ is continuous, and

$$c_0 \sum_{i=1}^{m} (\mu_i^2 + |A_i|^2)^{\frac{p_i-2}{2}} \le K(A) \le c_1 \sum_{i=1}^{m} (\mu_i^2 + |A_i|^2)^{\frac{p_i-2}{2}} \tag{3.1.0.7}$$

with $0 < c_0 < c_1$, $p_i > 1$ for all $i = 1, \dots, m$, and μ_i is a constant defined by

$$\mu_i \begin{cases} \in \mathbb{R} \text{ if } p_i \ge 2 \\ \neq 0 \text{ if } 1 < p_i < 2, \end{cases} \tag{3.1.0.8}$$

and where $|A_i|$ denotes the Euclidean norm of the i-th row of the matrix A. Note that we make no assumptions here about the monotonicity of the map a. The constants μ_i are defined in this way to ensure that in the case that $1 < p_i < 2$, the expression $(\mu_i^2+|A_i|^2)^{\frac{p_i-2}{2}}$ has an upper bound, which shall prove useful during later energy estimates.

Definition 3.1.0.1. We say that a pair (u, ν), where u is a function

$$u \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^m)) \cap \bigtimes_{i=1}^{m} L^{p_i}(0, T; W_0^{1,p_i}(\Omega))$$

with

$$\partial_t u \in L^{q'}(0, T; W^{-1,q'}(\Omega; \mathbb{R}^m)),$$

and ν is a Young measure (defined below in Subsection 3.2.2) satisfying, for almost every $(t, x) \in Q_T$, the identity

$$Du(t, x) = \int_{\mathbb{R}^{m\times n}} \xi \, \mathrm{d}\nu_{t,x}(\xi),$$

is a *Young measure solution* to the problem described by (3.1.0.1), (3.1.0.4) and (3.1.0.5) if

$$\int_0^T \langle \partial_t u, \phi \rangle_{(W_0^{1,\hat{q}}(\Omega;\mathbb{R}^m), W^{-1,\hat{q}'}(\Omega;\mathbb{R}^m))} + \int_\Omega \langle \nu_{t,x}, a \rangle : D\phi \; - Bu \cdot \phi \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega F \cdot \phi \, \mathrm{d}x \, \mathrm{d}t$$

holds for all $\phi \in L^{\hat{q}}(0, T; W_0^{1,\hat{q}}(\Omega; \mathbb{R}^m))$, and the initial condition is satisfied in the sense that $u(0, x) = u_0(x)$ for almost every $x \in \Omega$.

Remark 3.1.0.2. *For the sake of clarity we make the following remark regarding the* *notation $\langle \nu, a\rangle$ in the case, as we have here, that $a : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$. In this case, by $\langle \nu, a\rangle$, we mean the matrix whose (i,j)-th entry is given by $\langle \nu, a_{i,j}\rangle$, where $a_{i,j} : \mathbb{R}^{m\times n} \to \mathbb{R}$ is the (i,j)-th entry of the matrix a. Thus, the expression $\langle \nu, a\rangle : D\phi$ is understood to be the Frobenius inner product between the two matrices $\langle \nu, a\rangle$ and $D\phi$.*

Before proceeding with the analysis we first need a significant number of preliminary statements and results, which we have collected below.

3.2 Preliminary Definitions and Results

3.2.1 Function Spaces

We begin by defining the following spaces, which will be of use to us later. In what follows we will let $p_1, \dots, p_m$ be real numbers such that $p_i > \max\{\frac{2n}{n+2}, 1\}$ for all $i = 1, \dots, m$. We also define $q := \max_{i=1,\dots,m}\{p_i\}$, $p := \min_{i=1,\dots,m}\{p_i\}$ and impose the conditions $p \geq \frac{2n}{n+2}$ and $q - p < 1$. We define, for $r \geq 1$, the following space:

$$E_r(\mathbb{R}^{m\times n};\mathbb{R}) := \left\{ g \in C(\mathbb{R}^{m\times n};\mathbb{R}) : \lim_{|A|\to\infty} \frac{g(A)}{1+|A|^r} \text{ exists} \right\}.$$

We shall shorten this to E_r for ease of notation. On the space E_r we consider the norm

$$\|g\|_{E_r} = \sup_{A\in\mathbb{R}^{m\times n}} \frac{|g(A)|}{1+|A|^r},$$

which makes $(E_r, \|\cdot\|_{E_r})$ a separable space (see, for example, [43] or [46]). During what follows we will need to consider spaces of the form $L^{\alpha'}(Q_T; E_r)$, where α' is the dual exponent of $\alpha > 1$. This is the space of functions $\gamma : Q_T \times \mathbb{R}^{m\times n} \to \mathbb{R}$ for which

$$\gamma(t,x,\cdot) \in E_r \text{ for almost every } (t,x) \in Q_T,$$

and for which

$$\int_0^T \int_\Omega \|\gamma(t,x,\cdot)\|_{E_r}^{\alpha'} \,\mathrm{d}x\,\mathrm{d}t < \infty.$$

We will at times also wish to consider the dual of this space. As E_r is separable we have that $(L^{\alpha'}(Q_T; E_r))'$ is isometrically isomorphic to $L^{\alpha}_{w}(Q_T; E'_r)$, the space of weakly measurable maps $\nu : Q_T \to E'_r$, equipped with the usual dual norm. For details of the isomorphism, see [29], Theorem 8.20.3. In particular we note that bounded sequences in this space have weakly-star convergent subsequences by the well-known Banach−Alaoglu Theorem.

We further remark that $C_0(\mathbb{R}^{m\times n}; \mathbb{R}) \subset E_{q-1}$, and thus that $E'_{q-1} \subset \mathcal{M}$, the space of finite Radon measures on $\mathbb{R}^{m\times n}$.

We shall also at times consider the space $E_r(\mathbb{R}^{m\times n}; \mathbb{R}^{m\times n})$, the space of matrix valued functions γ such that $\gamma_{i,j} \in E_r(\mathbb{R}^{m\times n}; \mathbb{R})$ for all $i = 1, \dots, m$, $j = 1, \dots, n$. We shall always write this space as $E_r(\mathbb{R}^{m\times n}; \mathbb{R}^{m\times n})$ in order to avoid confusion.

3.2.2 Young Measures

In this chapter we shall make use of the Fundamental Theorem of Young Measures. Below we state the Fundamental Theorem of Young Measures from [52], as well as some remarks from [3] which follow the statement of the Fundamental Theorem of Young Measures in that paper.

Definition 3.2.2.1. A map $\mu : \Omega \to \mathcal{M}(\mathbb{R}^m)$ is called weak-$*$ measurable if the function $x \mapsto \langle \mu_x, f\rangle$ for all $f \in C_0(\mathbb{R}^m)$.

Theorem 3.2.2.2. *Let $\Omega \subset \mathbb{R}^n$ be a subset of finite measure and let $z_j : \Omega \to \mathbb{R}^m$ be a sequence of measurable functions. Then there exists a subsequence z_{j_k} and a weak-$*$ measurable map $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$ such that the following holds.*

1. *$\nu_x \geq 0$, $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} = \int_{\mathbb{R}^m} \mathrm{d}\nu_x \leq 1$ for almost every $x \in \Omega$.*

2. *For all $f \in C_0(\mathbb{R}^m)$, $f(z_{j_k}) \overset{*}{\rightharpoonup} \bar{f}$, where*

$$\bar{f} = \langle \nu_x, f\rangle = \int_{\mathbb{R}^m} f \ \mathrm{d}\nu_x.$$

3. *Let $K \subset \mathbb{R}^m$ be compact. Then*

$$\operatorname{supp} \nu_x \subset K \textit{ if } \operatorname{dist}(z_{j_k}, K) \to 0 \textit{ in measure.}$$

4. *Furthermore one has*

$$\|\nu_x\|_{\mathcal{M}} = 1 \text{ for almost every } x \in \Omega \tag{3.2.2.1}$$

if and only if the sequence does not escape to infinity, i.e. if

$$\lim_{M\to\infty} \sup_k |\{|z_{j_k}| \geq M\}| = 0. \tag{3.2.2.2}$$

5. *If* (3.2.2.1) *holds, if $A \subset \Omega$ is measurable, if $f \in C(\mathbb{R}^m)$ and if $f(z_{j_k})$ is relatively weakly compact in $L^1(A)$ then*

$$f(z_{j_k}) \rightharpoonup f \text{ in } L^1(A),$$

where $f(x) = \langle \nu_x, f\rangle$.

6. *If* (3.2.2.1) *holds, then in the third point one can replace "if" by "if and only if".*

Definition 3.2.2.3. The map $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$ in Theorem 3.2.2.2 is called the Young measure generated by the sequence z_{j_k}.

The following remarks are from [3].

Remark 3.2.2.4. *In the event that A is bounded, the condition that $\{f(z_{j_k})\}$ is sequentially weakly relatively compact in $L^1(A)$ is equivalent to the existence of a non-negative continuous function $\psi\colon [0,\infty) \to \mathbb{R}$ satisfying $\frac{\psi(y)}{y} \to \infty$ as $y \to \infty$ such that*

$$\sup_k \int_A \psi(|f(z_{j_k})|)\, \mathrm{d}x < \infty.$$

Remark 3.2.2.5. *If Ω is bounded and z_j are uniformly bounded in $L^p(\Omega;\mathbb{R}^m)$ for some $p \in (1,\infty)$, then we obtain from the theorem the existence of a family of probability measures (ν_x) and a subsequence z_{j_k} such that*

$$f(z_{j_k}) \rightharpoonup \langle \nu, f\rangle$$

in $L^r(\Omega)$ *where* $f : \mathbb{R}^m \to \mathbb{R}$ *is a continuous function satisfying*

$$|f(\lambda)| \leq c(1 + |\lambda|^q)$$

with $1 < r < \frac{p}{q}$ *and* $q > 0$.

3.2.3 Miscellaneous statements

We recall the following result from Strauss [63] (cf. also Lions & Magenes [47], Lemma 8.1, Ch. 3, Sec. 8.4).

Lemma 3.2.3.1. *Suppose that* X *and* Y *are Banach spaces. Assume that the space* X *is reflexive and is continuously and densely embedded in the space* Y*; then,*

$$L^\infty(0,T;X) \cap C_w([0,T];Y) = C_w([0,T];X),$$

where $C_w([0,T];X)$ *and* $C_w([0,T];Y)$ *denote the spaces of weakly continuous functions from* $[0,T]$ *into* X *and* Y*, respectively.*

We shall also require the following consequence of the Arzelà–Ascoli theorem.

Lemma 3.2.3.2. *Let* $\{u^j\}$ *be a sequence of functions with the following properties:*

1. $\{u^j\}$ *is uniformly bounded in* $L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$ *and*

$$u^j \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^m));$$

2. $\{\partial_t u^j\}$ *is uniformly bounded in* $L^{s'}(0,T;W^{-1,s'}(\Omega;\mathbb{R}^m))$ *for some* $s \in [1,\infty)$*, and*

$$\partial_t u^j \rightharpoonup \partial_t u \text{ in } L^{q'}(0,T;W^{-1,s'}(\Omega;\mathbb{R}^m)).$$

Then, there is a subsequence $\{u^{j_k}\}$ *such that*

$$\int_\Omega u^{j_k}(t,x) \cdot w(x) \, \mathrm{d}x \to \int_\Omega u(t,x) \cdot w(x) \, \mathrm{d}x$$

uniformly for all $w \in L^2(\Omega;\mathbb{R}^m) \cap W_0^{1,s}(\Omega;\mathbb{R}^m)$ *as* $k \to \infty$.

Proof. We shall apply the Arzelà–Ascoli theorem. For $w \in L^2(\Omega;\mathbb{R}^m) \cap W_0^{1,s}(\Omega;\mathbb{R}^m)$, we define the family of functions $f^j : [0,T] \to \mathbb{R}$ by

$$f^j(t) = \int_\Omega u^j(t,x) \cdot w(x) \; \mathrm{d}x.$$

Since $\partial_t u^j \in L^{s'}(0,T;W^{-1,s'}(\Omega;\mathbb{R}^m))$, it automatically follows that

$$u^j \in C([0,T];W^{-1,s'}(\Omega;\mathbb{R}^m)),$$

whereby also $u^j \in C_w([0,T];W^{-1,s'}(\Omega;\mathbb{R}^m))$. As $u^j \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$, it then follows from Lemma 3.2.3.1 with $X = L^2(\Omega;\mathbb{R}^m)$ and $Y = L^2(\Omega;\mathbb{R}^m) + W^{-1,s'}(\Omega;\mathbb{R}^m) = (L^2(\Omega;\mathbb{R}^m) \cap W_0^{1,s}(\Omega;\mathbb{R}^m))'$, that $u^j \in C_w([0,T];L^2(\Omega;\mathbb{R}^m))$. We note here that the duality $(L^2(\Omega;\mathbb{R}^m) \cap W_0^{1,s}(\Omega;\mathbb{R}^m))' = L^2(\Omega;\mathbb{R}^m) + W^{-1,s'}(\Omega;\mathbb{R}^m)$ is a consequence of the Duality Theorem (cf. Theorem 2.7.1 in [6]), because $L^2(\Omega;\mathbb{R}^m) \cap W_0^{1,s}(\Omega;\mathbb{R}^m)$ is dense in both $L^2(\Omega;\mathbb{R}^m)$ and $W_0^{1,s}(\Omega;\mathbb{R}^m)$. Hence, $f^j \in C([0,T])$. We further have that

$$\begin{aligned}
|f^j(t)| &\le \int_\Omega |u^j(t,x)||w(x)| \; \mathrm{d}x \\
&\le \|u^j\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))} \|w\|_{L^2(\Omega;\mathbb{R}^m)} \\
&\le C\|w\|_{L^2(\Omega;\mathbb{R}^m)},
\end{aligned}$$

which gives the uniform boundedness of the sequence $\{f^j\}$ in $C([0,T])$ for each $w \in L^2(\Omega;\mathbb{R}^m) \cap W_0^{1,s}(\Omega;\mathbb{R}^m)$. For equicontinuity, note that, for all $w \in L^2(\Omega;\mathbb{R}^m) \cap W_0^{1,s}(\Omega;\mathbb{R}^m)$,

$$\begin{aligned}
|f^j(t+h) - f^j(t)| &= \int_\Omega (u^j(t+h,x) - u^j(t,x)) \cdot w(x) \; \mathrm{d}x \\
&= \int_t^{t+h} \int_\Omega \partial_t u^j(s,x) \cdot w(x) \; \mathrm{d}x \; \mathrm{d}s \\
&\le \|\partial_t u^j\|_{L^{s'}(0,T;W^{-1,s'}(\Omega;\mathbb{R}^m))} |h|^{\frac{1}{s}} \|w\|_{W^{1,s}(\Omega;\mathbb{R}^m)} \\
&\le C|h|^{\frac{1}{s}} \|w\|_{W^{1,s}(\Omega;\mathbb{R}^m)}.
\end{aligned}$$

Thus by the Arzelà–Ascoli theorem there is a subsequence $\{f^{j_k}\}$, which converges uniformly as $k \to \infty$ to some function $f \in C([0,T])$. We note that as this convergence is uniform and the sequence f^{j_k} is uniformly bounded we have that

$$\lim_{k\to\infty} \int_0^T \int_\Omega u^{j_k}(t,x) \cdot w(x) \ \mathrm{d}x \ \mathrm{d}t = \lim_{k\to\infty} \int_0^T f^{j_k}(t) \ \mathrm{d}t = \int_0^T f(t) \ \mathrm{d}t.$$

However, by the assumed weak convergence, we have that

$$\lim_{k\to\infty} \int_0^T \int_\Omega u^{j_k}(t,x) \cdot w(x) \ \mathrm{d}x \ \mathrm{d}t = \int_0^T \int_\Omega u(t,x) \cdot w(x) \ \mathrm{d}x \ \mathrm{d}t,$$

allowing us to identify

$$f(t) = \int_\Omega u(t,x) \cdot w(x) \ \mathrm{d}x, \qquad t \in [0,T].$$

Thus we have shown that, for all $w \in L^2(\Omega;\mathbb{R}^m) \cap W_0^{1,s}(\Omega;\mathbb{R}^m)$,

$$\int_\Omega u^{j_k}(t,x) \cdot w(x) \ \mathrm{d}x \to \int_\Omega u(t,x) \cdot w(x) \ \mathrm{d}x$$

uniformly in $C([0,T])$. As $L^2(\Omega;\mathbb{R}^m) \cap W_0^{1,s}(\Omega;\mathbb{R}^m)$ is dense in $L^2(\Omega;\mathbb{R}^m)$, it then follows that, for all $w \in L^2(\Omega;\mathbb{R}^m)$,

$$\int_\Omega u^{j_k}(t,x) \cdot w(x) \ \mathrm{d}x \to \int_\Omega u(t,x) \cdot w(x) \ \mathrm{d}x$$

uniformly, which in particular means that for all $t \in [0,T]$ we have $u^{j_k}(t,\cdot) \rightharpoonup u(t,\cdot)$ in the space $L^2(\Omega;\mathbb{R}^m)$. $\square$

Next, the following standard interpolation result.

Lemma 3.2.3.3. *Let $u : [0,T] \times \Omega \to \mathbb{R}^m$, where $\Omega \subset \mathbb{R}^n$ is open and bounded, satisfy*

$$u \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$$

for some $p \geq 1$. Then,

$$u \in L^{\frac{p(n+2)}{n}}(Q_T;\mathbb{R}^m).$$

In later subsections we will make use of the Vitali Convergence Theorem. We state it here, after first recalling the notion of uniform integrability of a family of functions.

Definition 3.2.3.4. Let $(X, \mathcal{F}, \mu)$ be a finite measure space. A set of functions $\Phi \subset L^1(\mu)$ is called uniformly integrable if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\int_E |f| \, \mathrm{d}\mu < \epsilon$$

whenever $f \in \Phi$ and $\mu(E) < \delta$.

In our applications of uniform integrability we will be considering finite measures, for which we can use the Vitali Convergence Theorem. This allows us to replace the assumption of a uniform upper bound on a sequence of functions with the uniform integrability of the sequence of functions under consideration:

Theorem 3.2.3.5. *Let $(X, \mathcal{F}, \mu)$ be a positive measure space and let $\{f_n\}_{n=1}^{\infty}$ be a family of functions on X. If the following four conditions hold:*

1. *The measure space $(X, \mathcal{F}, \mu)$ is a finite measure space, meaning that $\mu(X) < \infty$,*
2. *The family of functions $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable,*
3. *There is a function f such that $f_n \to f$ for μ-almost every point of X, and*
4. *The limiting function f satisfies $|f(x)| < \infty$ for μ-almost every $x \in X$,*

then the following two statements hold:

1. *The limiting function f is in L^1.*
2. *We have convergence of f_n to f also in the following sense:*

$$\lim_{n \to \infty} \int_X |f_n - f| \, \mathrm{d}\mu = 0.$$

3.3 Existence Theory and Examples

In this section we prove the existence of Young measure solutions to the systems given by (3.1.0.1)−(3.1.0.8). In the proof presented below, one can replace the finite element basis, consisting of continuous piecewise linear basis functions satisfying a homogeneous Dirichlet boundary condition on $\partial\Omega$, by an abstract Galerkin basis of $C_0^\infty(\Omega;\mathbb{R}^m)$ functions. This allows us to relax the assumption that Ω is a Lipschitz polytope, and allows us to instead consider those domains Ω which are instead open, bounded subsets of $\mathbb{R}^m$ with Lipschitz boundaries. We start with two definitions about families of subdivisions of domains:

Definition 3.3.0.6. The family of subdivisions $\mathcal{T} := \{\mathcal{T}_h\}_{h\in(0,h_0]}$ is said to be shape regular if there is a constant $c_0 > 0$ such that

$$\max_{\tau_h\in\mathcal{T}_h} \frac{\operatorname{diam}(\tau_h)^n}{|\tau_h|} \leq c_0 \quad \text{for all } \mathcal{T}_h \in \mathcal{T}.$$

Definition 3.3.0.7. The family of subdivisions $\mathcal{T} := \{\mathcal{T}_h\}_{h\in(0,h_0]}$ is said to be globally quasiuniform if there exists a constant $c > 1$, which is independent of h, such that for each fixed h, $\max_{\tau_h\in\mathcal{T}_h} |\tau_h| \leq c \min_{\tau_h\in\mathcal{T}_h} |\tau_h|$.

Suppose that $0 < h_0 \ll \operatorname{diam}(\Omega)$ and that $\mathcal{T} := \{\mathcal{T}_h\}_{h\in(0,h_0]}$ is a shape-regular family of subdivisions $\mathcal{T}_h$ of $\overline{\Omega}$ into closed simplexes Δ, where $h = \max_{\Delta\in\mathcal{T}_h} \operatorname{diam}(\Delta)$. Consider the finite element space

$$V^h := \{v^h \in W_0^{1,\infty}(\overline{\Omega}) : v^h|_\Delta \text{ is affine for all } \Delta \in \mathcal{T}_h\}.$$

By V_m^h we denote the space of m-component vector-valued functions, each of whose components lies in V^h, and we denote by $\mathcal{N}(h)$ the dimension of the space V_m^h. We shall assume that $\mathcal{T}$ is such that the $L^2(\Omega;\mathbb{R}^m)$ orthogonal projector P^h onto V_m^h is stable in $W_0^{1,\hat{q}}(\Omega;\mathbb{R}^m)$; i.e., there exists a positive constant C, independent of $h \in (0, h_0]$, such that

$$\|D(P^h\varphi)\|_{L^{\hat{q}}(\Omega;\mathbb{R}^{m\times n})} \leq C\|D\varphi\|_{L^{\hat{q}}(\Omega;\mathbb{R}^{m\times n})}, \tag{3.3.0.1}$$

for all $\varphi \in W_0^{1,\hat{q}}(\Omega;\mathbb{R}^m)$, where $\hat{q} := \max\{q, 2\}$. On globally quasiuniform subdivisions (3.3.0.1) is a consequence of a global inverse inequality and inequality (7) in [27]; we note, however, that (3.3.0.1) is in fact valid under less restrictive assumptions on the subdivision than quasiuniformity (see, for example, [20]).

Let $\{\phi_i^h\}_{i=1}^{\mathcal{N}(h)}$ be a basis for V_m^h. We shall seek an approximate solution $u^h \in V_m^h$ in the form

$$u^h(t,x) = \sum_{i=1}^{\mathcal{N}(h)} \alpha_i^h(t)\phi_i^h(x), \qquad t \in [0,T], \quad x \in \overline{\Omega},$$

satisfying

$$(\partial_t u^h(t), v^h) + (a(Du^h(t)), Dv^h) + (Bu^h, v^h) = (F, v^h) \qquad \forall\, v^h \in V_m^h, \tag{3.3.0.2a}$$

for all $t \in [0,T]$, and

$$u^h(0) = u_0^h, \tag{3.3.0.2b}$$

with $u_0^h \in V_m^h$ and $u_0^h \to u_0$ (strongly) in $L^2(\Omega;\mathbb{R}^m)$ as $h \to 0_+$. The system given by (3.3.0.2a) is a system of ODEs for the coefficients α_i^h, $i = 1,\dots,\mathcal{N}(h)$, and a solution to (3.3.0.2a), (3.3.0.2b) exists locally by Peano's Theorem on some interval $[0,T_h) \subset [0,T]$, thanks to the assumed (local Lipschitz) continuity of a. The existence theorem below is based on proving that u^h can be extended to the final time $T > 0$ for each $h \in (0,h_0]$, and that as $h \to 0_+$ a subsequence of the sequence of approximate solutions converges, in a sense to be made precise, to a Young measure solution of the problem.

3.3.1 Existence

Theorem 3.3.1.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is a Lipschitz polytope, let $u_0 \in L^2(\Omega;\mathbb{R}^m)$, assume that*

$$p := \min_{i=1,\dots,m} p_i > \max\left\{\frac{2n}{n+2}, 1\right\} \quad \textit{and} \quad q := \max_{i=1,\dots,m} p_i$$

satisfy $q - p < 1$, let $F \in \bigtimes_{i=1}^m L^{p_i'}(Q_T) \cap L^2(Q_T;\mathbb{R}^m)$, and suppose that $a : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ is a locally Lipschitz mapping satisfying the assumptions (3.1.0.6)–(3.1.0.8)*. Then, there exists a Young measure solution (u,ν) of the problem* (3.1.0.1) *with data given by* (3.1.0.4) *and* (3.1.0.5)*. Furthermore, there exists a subsequence (not indicated) of solutions $\{u^h\}$ to the semidiscrete problem* (3.3.0.2a), (3.3.0.2b) *such that*

$$
\begin{aligned}
u^h &\overset{*}{\rightharpoonup} u \ \textit{in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^m)) \\
Du_i^h &\rightharpoonup Du_i \ \textit{in } L^{p_i}(0,T;L^{p_i}(\Omega;\mathbb{R}^{m\times n})), \quad i=1,\dots,m, \\
a(Du^h) &\rightharpoonup \langle \nu, a\rangle \ \textit{in } L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n}), \\
\partial_t u^h &\rightharpoonup \partial_t u \ \textit{in } L^{\hat{q}'}(0,T;W^{-1,\hat{q}'}(\Omega;\mathbb{R}^m)), \quad \textit{where } \hat{q} := \max\{q,2\}, \\
u^h &\to u \ \textit{in } C_w([0,T];L^2(\Omega;\mathbb{R}^m)).
\end{aligned}
$$

Proof. As was noted above, (3.3.0.2a), (3.3.0.2b) is an initial-value problem for a system of ODEs for the coefficients α_i^h, and a solution exists locally by Peano's Theorem on some interval $[0,T_h) \subset [0,T]$, thanks to the assumed local Lipschitz continuity of a. We begin by showing that we can extend the numerical solution $u^h \in V_m^h$, defined on $[0,T_h)\times\overline{\Omega}$ up to time T for all $h \in (0,h_0]$, so that it is defined on the whole of $[0,T]\times\overline{\Omega}$. To this end, we take $v^h = u^h$ in (3.3.0.2a) which yields,

$$
\int_0^{T_h}\int_\Omega \partial_t u^h \cdot u^h + K(Du^h)|Du^h|^2 - Bu^h\cdot u^h \,\mathrm{d}x\,\mathrm{d}t = \int_0^{T_h}\int_\Omega F\cdot u^h \,\mathrm{d}x\,\mathrm{d}t. \qquad (3.3.1.1)
$$

We apply Hölder's inequality followed by Young's inequality to the RHS of (3.3.1.1):

$$
\begin{aligned}
\int_0^{T_h}\int_\Omega F\cdot u^h \,\mathrm{d}x\,\mathrm{d}t &\le \sum_{i=1}^m \|F_i\|_{L^{p_i'}(Q_{T_h})}\|u_i^h\|_{L^{p_i}(Q_{T_h})} \\
&\le \sum_{i=1}^m c\|F_i\|^{p_i'}_{L^{p_i'}(Q_{T_h})} + \epsilon\sum_{i=1}^m \|u_i^h\|^{p_i}_{L^{p_i}(Q_{T_h})} \\
&\le \sum_{i=1}^m c\|F_i\|^{p_i'}_{L^{p_i'}(Q_{T_h})} + c\epsilon\sum_{i=1}^m \int_0^{T_h}\int_\Omega |Du_i^h|^{p_i}\,\mathrm{d}x\,\mathrm{d}t \\
&\le \sum_{i=1}^m c\|F_i\|^{p_i'}_{L^{p_i'}(Q_{T_h})} + c\epsilon\sum_{p_i\ge 2}^m \int_0^{T_h}\int_\Omega |Du_i^h|^{p_i}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \sum_{p_i<2}\int_0^{T_h}\int_\Omega (|Du_i^h|^2)^{\frac{p_i}{2}}\,\mathrm{d}x\,\mathrm{d}t \\
&\le \sum_{i=1}^m c\|F_i\|^{p_i'}_{L^{p_i'}(Q_{T_h})} + c\epsilon\sum_{p_i\ge 2}^m \int_0^{T_h}\int_\Omega |Du_i^h|^{p_i}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + c\epsilon\sum_{p_i<2}\int_0^{T_h}\int_\Omega (\mu_i^2+|Du_i^h|^2)^{\frac{p_i}{2}}\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{m} c\|F_i\|^{p_i'}_{L^{p_i'}(Q_{T_h})} + c\epsilon \sum_{p_i\geq 2}^{m} \int_0^{T_h}\int_\Omega |Du_i^h|^{p_i} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + c\epsilon \sum_{p_i<2} \int_0^{T_h}\int_\Omega \frac{\mu_i^2 + |Du_i^h|^2}{(\mu_i^2+|Du_i^h|^2)^{\frac{2-p_i}{2}}} \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \sum_{i=1}^{m} c\|F_i\|^{p_i'}_{L^{p_i'}(Q_{T_h})} + c\epsilon \sum_{p_i\geq 2}^{m} \int_0^{T_h}\int_\Omega |Du_i^h|^{p_i} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + c\epsilon \sum_{p_i<2} \int_0^{T_h}\int_\Omega \frac{|Du^h|^2}{(\mu_i^2+|Du_i^h|^2)^{\frac{2-p_i}{2}}} + |\mu_i|^{p_i} \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

On the LHS of (3.3.1.1) we then insert the assumed bounds on K, use the fact that for $p_i \geq 2$ it holds that $|Du_i^h|^{p_i-2} \leq (\mu_i^2 + |Du_i^h|^2)^{\frac{p_i-2}{2}}$, and take ϵ to be sufficiently small in the above inequality to obtain the following energy inequality:

$$
\begin{aligned}
&\sup_{t\in[0,T_h]} \|u^h(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^m)} + \int_0^{T_h}\int_\Omega \left[\sum_{i=1}^{m} (\mu_i^2 + |Du_i^h|^2)^{\frac{p_i-2}{2}}\right] |Du^h|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq c\|u_0\|^2_{L^2(\Omega)} + c(\Omega, p_1, \dots, p_m) \sum_{i=1}^{m} \|F_i\|^{p_i'}_{L^{p_i'}(Q_{T_h})} + c\epsilon \sum_{p_i<2} |\mu_i|^{p_i} |Q_{T_h}|
\end{aligned}
\tag{3.3.1.2}
$$

From (3.3.1.2) we obtain also the following inequality:

$$
\begin{aligned}
&\sup_{t\in[0,T_h]} \|u^h(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^m)} + \int_0^{T_h}\int_\Omega \sum_{i=1}^{m} |Du_i^h|^{p_i} \mathrm{d}x\,\mathrm{d}t \\
&\leq c(\Omega, p_1, \dots, p_m) \left(\|u_0^h\|^2_{L^2(\Omega)} + \sum_{i=1}^{m} \|F_i\|^{p_i'}_{L^{p_i'}(Q_T;\mathbb{R}^m)} + (1+\epsilon)|Q_T| \sum_{p_i<2} |\mu_i|^{p_i} \right),
\end{aligned}
\tag{3.3.1.3}
$$

which shows that the right-hand side is bounded independently of h, and therefore we can extend our solution from $[0, T_h)$ to the whole of the time interval $[0, T]$ as T_h cannot have been the maximal existence time. Thus it holds that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|u^h(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^m)} + \int_0^{T}\int_\Omega \sum_{i=1}^{m} |Du_i^h|^{p_i} \mathrm{d}x\,\mathrm{d}t \\
&\leq c(\Omega, p_1, \dots, p_m) \left(\|u_0^h\|^2_{L^2(\Omega;\mathbb{R}^m)} + \sum_{i=1}^{m} \|F_i\|^{p_i'}_{L^{p_i'}(Q_T;\mathbb{R}^m)} + (1+\epsilon)|Q_T| \sum_{p_i<2} |\mu_i|^{p_i} \right).
\end{aligned}
\tag{3.3.1.4}
$$

This gives bounds for u^h in the product space $\bigtimes_{i=1}^m L^{p_i}(0,T; W_0^{1,p_i}(\Omega))$, and also in $L^\infty(0,T; L^2(\Omega;\mathbb{R}^m))$, which are independent of h. Hence there is a function u for which, up to a subsequence, we have

$$u^h \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^m)), \tag{3.3.1.5}$$

and

$$u^h \rightharpoonup u \text{ in } \bigtimes_{i=1}^m L^{p_i}(0,T;W_0^{1,p_i}(\Omega)). \tag{3.3.1.6}$$

We recall that we have defined $\hat{q} := \max\{q,2\}$ and $q := \max_{i=1,\dots,m} p_i$. In order to pass to the limit in the weak formulation, we do two things. We first show that $a(Du^h)$ is bounded in $L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n})$, and then apply the Fundamental Theorem of Young Measures to complete the passage to the limit in the sense of Young measure solutions. Secondly, we obtain bounds on the time derivative in order to deduce the existence of a subsequence for which we have weak convergence to a limit. One has the following bound using the equivalence of norms:

$$|a(Du^h)| \le C \sum_{i=1}^m \sum_{k=1}^m (\mu_i^2 + |Du_i^h|^2)^{\frac{p_i-2}{2}} |Du_k^h|.$$

Now we seek to bound each of these terms appearing in the above sum. Clearly for the case $i=k$ we have an expression of the form $(\mu_i^2+|Du_i^h|^2)^{\frac{p_i-1}{2}}$ appearing, giving boundedness of this term in $L^{p_i'}(Q_T)$ by the previously obtained energy estimate (3.3.1.2). In the case $i \neq k$ we proceed by using Hölder's inequality together with (3.3.1.2) in order to obtain the required bounds. We begin with the terms for which $p_i \ge 2$:

$$\begin{aligned}
&\int_0^T \int_\Omega ((\mu_i^2 + |Du_i^h|^2)^{\frac{p_i-2}{2}} |Du_k^h|)^{p_i'} \,\mathrm{d}x \,\mathrm{d}t \\
&\quad = \int_0^T \int_\Omega (\mu_i^2 + |Du_i^h|^2)^{\frac{p_i'(p_i-2)}{4}} |Du_k^h|^{p_i'} \\
&\qquad\qquad \times (\mu_i^2 + |Du_i^h|^2)^{\frac{p_i'(p_i-2)}{4}} \,\mathrm{d}x \,\mathrm{d}t \\
&\quad \le \left(\int_0^T \int_\Omega (\mu_i^2 + |Du_i^h|^2)^{\frac{p_i-2}{2}} |Du_k^h|^2 \,\mathrm{d}x \,\mathrm{d}t \right)^{\frac{p_i'}{2}} \\
&\qquad \times \left(\int_0^T \int_\Omega (\mu_i^2 + |Du_i^h|^2)^{\frac{p_i}{2}} \,\mathrm{d}x \,\mathrm{d}t \right)^{1-\frac{p_i'}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\leq c(\Omega, p_1, \ldots, p_m)\left(\|u_0\|^2_{L^2(\Omega)} + \sum_{i=1}^m \|F_i\|^{p_i'}_{L^{p_i'}(Q_T)}\right) \\
&\quad \times (|\mu_i|^{p_i}|Q_T| + \|Du_i^h\|^{p_i}_{L^{p_i}(Q_T;\mathbb{R}^n)})^{1-\frac{p_i'}{2}} \\
&\leq c(\Omega, p_1, \ldots, p_m)\left(\|u_0\|^2_{L^2(\Omega;\mathbb{R}^m)} + \sum_{i=1}^m \|F_i\|^{p_i'}_{L^{p_i'}(Q_T;\mathbb{R}^m)}\right), \\
&\quad \times \left(|\mu_i|^{p_i}|Q_T| + \|u_0\|^2_{L^2(\Omega;\mathbb{R}^m)} + (1+\epsilon)|Q_T|\sum_{p_i<2}|\mu_i|^{p_i}\right)^{1-\frac{p_i'}{2}}.
\end{aligned}
$$

Now for the terms in which $1 < p_i < 2$:

$$
\begin{aligned}
&\int_0^T \int_\Omega ((\mu_i^2 + |Du_i^h|^2)^{\frac{p_i-2}{2}}|Du_k^h|)^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad = \int_0^T \int_\Omega (\mu_i^2 + |Du_i^h|^2)^{\frac{p_i-2}{2}}(\mu_i^2 + |Du_i^h|^2)^{\frac{p_i-2}{2}}|Du_k^h|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad \leq |\mu_i|^{p_i-2}\int_0^T \int_\Omega (\mu_i^2 + |Du_i^h|^2)^{\frac{p_i-2}{2}}|Du_k^h|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad \leq |\mu_i|^{p_i-2}c(\Omega, p_1, \ldots, p_m)\left(\|u_0\|^2_{L^2(\Omega;\mathbb{R}^m)} + \sum_{i=1}^m \|F_i\|^{p_i'}_{L^{p_i'}(Q_T;\mathbb{R}^m)} + (1+\epsilon)|Q_T|\sum_{p_i<2}|\mu_i|^{p_i}\right).
\end{aligned}
$$

Thus we have bounded each term in the sum in $L^{p_i'}(Q_T;\mathbb{R}^{m\times n})$ (if $p_i \geq 2$) or in $L^2(Q_T;\mathbb{R}^{m\times n})$ (if $1 < p_i < 2$), so in particular we have the bound in $L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n})$. That is, $a(Du^h)$ is uniformly bounded in $L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n})$, meaning in particular that there is a subsequence (still indexed only by h) and a $\chi \in L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n})$ such that

$$
a(Du^h) \rightharpoonup \chi
$$

in $L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n})$. Using similar estimates to those above we can also show that

$$
\partial_t u^h \in L^{\hat{q}'}(0,T;W^{-1,\hat{q}'}(\Omega;\mathbb{R}^m)) \tag{3.3.1.7}
$$

with bounds independent of h.

Now let $\psi \in L^{\hat{q}}(0,T;W_0^{1,\hat{q}}(\Omega;\mathbb{R}^m))$ be the strong limit of the sequence of functions $\psi^{\mathcal{N}(h),l}(t,x) = \sum_{i=1}^{\mathcal{N}(l)} \beta_i^l(t)\phi_i^h(x)$ in the $L^{\hat{q}}(0,T;W_0^{1,\hat{q}}(\Omega;\mathbb{R}^m))$ norm as $l \to \infty$ and $h \to 0$, with $\beta_i^l \in C([0,T])$ and $\phi_i^h \in V_m^h$ (such a limit necessarily exists by density). Then we have that:

$$
\begin{aligned}
&\int_0^T \int_\Omega \partial_t u^h \cdot \psi + a(Du^h) : D\psi - Bu^h \cdot \psi - F^h \cdot \psi \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_\Omega \partial_t u^h \cdot \psi^l + a(Du^h) : D\psi^{\mathcal{N}(l)} - Bu^h \cdot \psi^{\mathcal{N}(h),l} - F^h \cdot \psi^{\mathcal{N}(l)} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \partial_t u^h \cdot (\psi - \psi^{\mathcal{N}(h),l}) + a(Du^h) : (D\psi - D\psi^{\mathcal{N}(h),l}) \\
&\qquad\qquad - Bu^h \cdot (\psi - \psi^{\mathcal{N}(h),l}) - F^h \cdot (\psi - \psi^{\mathcal{N}(h),l}) \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_\Omega \partial_t u^h \cdot (\psi - \psi^{\mathcal{N}(h),l}) + a(Du^h) : (D\psi - D\psi^{\mathcal{N}(h),l}) \\
&\qquad\qquad - Bu^h \cdot (\psi - \psi^{\mathcal{N}(h),l}) - F^h \cdot (\psi - \psi^{\mathcal{N}(h),l}) \,\mathrm{d}x\,\mathrm{d}t \\
&\le \|\partial_t u^h\|_{L^{\hat{q}'}(0,T;W^{-1,\hat{q}'}(\Omega;\mathbb{R}^m))} \|\psi - \psi^{\mathcal{N}(h),l}\|_{L^{\hat{q}}(0,T;W_0^{1,\hat{q}}(\Omega;\mathbb{R}^m))} \\
&\quad + \|a(Du^h)\|_{L^{\hat{q}'}(0,T;L^{\hat{q}'}(\Omega;\mathbb{R}^{m\times n}))} \|D\psi - D\psi^{\mathcal{N}(h),l}\|_{L^{\hat{q}}(0,T;L^{\hat{q}}(\Omega;\mathbb{R}^{m\times n}))} \\
&\quad + \left(\sum_{i=1}^m \|F_i\|_{L^{p_i'}(Q_T;\mathbb{R}^m)}\right) \|\psi - \psi^{\mathcal{N}(h),l}\|_{L^{\hat{q}}(0,T;W_0^{1,\hat{q}}(\Omega;\mathbb{R}^m))} \\
&\quad + c\|u\|_{L^{\hat{q}'}(Q_T;\mathbb{R}^m)} \|\psi - \psi^{\mathcal{N}(h),l}\|_{L^{\hat{q}}(0,T;W_0^{1,\hat{q}}(\Omega;\mathbb{R}^m))} \\
&\le c(\|u_0\|_{L^2(\Omega;\mathbb{R}^m)}, \mu_i, T, |\Omega|, n, F) \\
&\quad \times \big(\|\psi - \psi^{\mathcal{N}(h),l}\|_{L^{\hat{q}}(0,T;W_0^{1,\hat{q}}(\Omega;\mathbb{R}^m))} + \|D\psi - D\psi^{\mathcal{N}(h),l}\|_{L^{\hat{q}}(0,T;L^{\hat{q}}(\Omega;\mathbb{R}^{m\times n}))}\big),
\end{aligned}
$$

which is seen to converge to zero in the limit as $h \to 0$ ($\mathcal{N}(h) \to \infty$). Thus we have

$$
\int_0^T \int_\Omega \partial_t u^h \cdot \psi + a(Du^h) : D\psi - Bu^h \cdot \psi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega F^h \cdot \psi \,\mathrm{d}x\,\mathrm{d}t, \tag{3.3.1.8}
$$

for all $\psi \in L^{\hat{q}}(0,T;W_0^{1,\hat{q}}(\Omega;\mathbb{R}^m))$.

The above results allow passage to the limit in the time derivative term in the usual sense. That is to say that there exists a subsequence (still indexed by h) for which $\partial_t u^h \rightharpoonup \partial_t u$ in $L^{\hat{q}'}(0,T;W^{-1,\hat{q}'}(\Omega;\mathbb{R}^m))$, meaning that

$$
\lim_{h\to 0} \int_0^T \langle \partial_t u^h, \phi \rangle \,\mathrm{d}t = \int_0^T \langle \partial_t u, \phi \rangle \,\mathrm{d}t, \tag{3.3.1.9}
$$

for all $\phi \in L^{\hat{q}}(0,T;W^{1,\hat{q}}(\Omega;\mathbb{R}^m))$.

For the term

$$\int_0^T \int_\Omega Bu^h \cdot \psi \, \mathrm{d}x \, \mathrm{d}t,$$

note that by Lemma 3.2.3.3 it holds that u^h is uniformly bounded in $L^{\frac{p(n+2)}{n}}(Q_T;\mathbb{R}^m)$, and that $\frac{p(n+2)}{n} > 2 \geq \hat{q}'$. Therefore it holds that $u^h \rightharpoonup u$ in $L^{\hat{q}'}(Q_T;\mathbb{R}^m)$. In order to complete the passage to the limit in the second term we invoke Theorem 3.2.2.2. In order to do this we need to check that the subsequence Du^{h_k} given by Theorem 3.2.2.2 satisfies, for every $R > 0$,

$$\lim_{s\to\infty} \sup_k |\{(t,x) \in ([0,T] \times \Omega) \cap Q_R(0) : \ |Du^{h_k}(t,x)| \geq s\}| = 0.$$

In order to establish this we set

$$A_{k,s,R} := \{(t,x) \in ([0,T] \times \Omega) \cap Q_R(0) : \ |Du^{h_k}(t,x)| \geq s\},$$

and apply Chebyshev's inequality:

$$\begin{aligned} |A_{k,s,R}| &\leq \frac{1}{s^p} \int_{A_{k,s,R}} |Du^{h_k}|^p \, \mathrm{d}x \, \mathrm{d}t \\ &\leq \frac{c||u_0||^2_{L^2(\Omega)} + (n-1)|Q_T|}{s^p}, \end{aligned}$$

where we have used (3.3.1.2) along with the fact that for $r_1 > r_2$, we have $|s|^{r_2} \leq 1 + |s|^{r_1}$ to go from the first line to the second. The resulting bound is independent of k and R, and so by passing $s \to \infty$ we show that the desired condition is satisfied.

Applying Theorem 3.2.2.2, we therefore deduce the existence of a parameterised family of Young measures $(\nu_{t,x})$ and a further subsequence (which we still index by h) for which

$$a(Du^h) \rightharpoonup \langle \nu_{t,x}, a \rangle \text{ in } L^r(Q_T;\mathbb{R}^{m\times n}),$$

where $1 < r < \frac{p}{q-1}$ (cf. Remark 3.1.0.2, and see the remarks following the statement of Theorem 3.2.2.2, combined with the fact that $a(A) \leq c(1 + |A|^{q-1})$ from the definition). By the uniqueness of weak limits there is then a subsequence of u^h (still indexed by h)

such that

$$a(Du^h) \rightharpoonup \langle \nu_{t,x}, a \rangle \text{ in } L^{\hat{q}'}(Q_T; \mathbb{R}^{m\times n}).$$

This allows us to complete the passage to the limit and obtain the existence of a function

$$u \in \bigtimes_{i=1}^{n} L^{p_i}(0,T; W_0^{1,p_i}(\Omega))$$

with

$$\partial_t u \in L^{\hat{q}'}(0,T; W^{-1,\hat{q}'}(\Omega; \mathbb{R}^m))$$

satisfying

$$\int_0^T \langle \partial_t u, \phi \rangle + \int_\Omega \langle \nu_{t,x}, a \rangle : D\phi - Bu \cdot \phi \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega F \cdot \phi \, \mathrm{d}x \, \mathrm{d}t \tag{3.3.1.10}$$

for all $\phi \in L^{\hat{q}}(0,T; W_0^{1,\hat{q}}(\Omega; \mathbb{R}^m))$. It remains to be shown that the initial condition is satisfied. By Lemma 3.2.3.2 it follows that we can extract a further subsequence h_l along which we have that

$$\int_\Omega u^{h_l}(t,x) \cdot \varphi(x) \, \mathrm{d}x \to \int_\Omega u(t,x) \cdot \varphi(x) \, \mathrm{d}x, \tag{3.3.1.11}$$

uniformly in $C([0,T])$, for all $\varphi \in L^2(\Omega; \mathbb{R}^m)$. Let us decompose

$$\begin{aligned}
&\int_\Omega (u_0(x) - u(0,x)) \cdot \varphi(x) \, \mathrm{d}x \\
&\quad = \int_\Omega (u_0(x) - u^{h_l}(0,x)) \cdot \varphi(x) \, \mathrm{d}x + \int_\Omega (u^{h_l}(0,x) - u^{h_l}(t,x)) \cdot \varphi(x) \, \mathrm{d}x \\
&\qquad + \int_\Omega (u^{h_l}(t,x) - u(t,x)) \cdot \varphi(x) \, \mathrm{d}x + \int_\Omega (u(t,x) - u(0,x)) \cdot \varphi(x) \, \mathrm{d}x \\
&\quad =: I_l + II_l(t) + III_l(t) + IV(t).
\end{aligned}$$

The term I_l converges to zero as $h_l \to 0_+$ by the assumed strong convergence in $L^2(\Omega; \mathbb{R}^m)$ of the discretized initial condition $u_0^{h_l}$ to u_0, and the term $III_l(t)$ converges to zero, uniformly in $C([0,T])$ as $h_l \to 0_+$ by (3.3.1.11). Hence, for any $\varepsilon > 0$, there exists an

$h_l^* = h_l^*(\varepsilon, \varphi) \in (0, h_0]$, such that

$$\left| \int_\Omega (u_0(x) - u^{h_l^*}(0,x)) \cdot \varphi(x) \, \mathrm{d}x \right| < \frac{1}{4}\varepsilon \quad \text{and} \quad \max_{t \in [0,T]} \left| \int_\Omega (u^{h_l^*}(t,x) - u(t,x)) \cdot \varphi(x) \, \mathrm{d}x \right| < \frac{1}{4}\varepsilon.$$

Consequently, for any $\varepsilon > 0$, there exists an $h_l^* \in (0, h_0]$, such that

$$\begin{aligned} &\left| \int_\Omega (u_0(x) - u(0,x)) \cdot \varphi(x) \, \mathrm{d}x \right| \\ &\quad < \frac{1}{2}\varepsilon + \left| \int_\Omega (u^{h_l^*}(0,x) - u^{h_l^*}(t,x)) \cdot \varphi(x) \, \mathrm{d}x \right| + \left| \int_\Omega (u(t,x) - u(0,x)) \cdot \varphi(x) \, \mathrm{d}x \right|. \end{aligned}$$

With h_l^* fixed, we now pass to the limit $t \to 0_+$ in this inequality. The term $II_l(t)$ converges to zero as $t \to 0_+$ because $u^{h_l^*}$, as a solution to the Galerkin equation (3.3.0.2a), is continuous and satisfies the discretized initial condition. The term $IV(t)$ converges to as $t \to 0_+$ since by Lemma 3.2.3.2 we have that $u \in C_w([0,T]; L^2(\Omega; \mathbb{R}^m))$. Thus, by passing to the limit $t \to 0_+$ in the last inequality we deduce that, for any $\varphi \in L^2(\Omega; \mathbb{R}^m)$ and any $\varepsilon > 0$,

$$\left| \int_\Omega (u_0(x) - u(0,x)) \cdot \varphi(x) \, \mathrm{d}x \right| < \frac{1}{2}\varepsilon.$$

Taking $\varphi(x) = u_0(x) - u(0,x)$ and letting $\varepsilon \to 0_+$ then yields that $u_0(x) = u(0,x)$ for almost every $x \in \Omega$.

Finally we must show that

$$Du(t,x) = \int_{\mathbb{R}^{m \times n}} \xi \, \mathrm{d}\nu_{t,x}(\xi) \tag{3.3.1.12}$$

holds for almost every $(t,x) \in Q_T$ (cf. Remark 3.1.0.2). This follows from the fact that $Du^h \rightharpoonup Du$ in $\bigtimes_{i=1}^m L^{p_i}(0,T; W_0^{1,p_i}(\Omega))$, but, letting f denote the identity function, we have that $f(Du^h) \rightharpoonup \langle \nu, f \rangle$ in $L^r(Q_T; \mathbb{R}^{m \times n})$ for $1 < r < p$ (cf. Remark 3.1.0.2 and see the remark following the statement of Theorem 3.2.2.2). From the uniqueness of the weak limits the claimed equality holds true, completing the proof of the result. □

We have the following Lemma, which gives us a way of interpreting the Young measures ν as elements of the dual space of a separable Banach space.

Lemma 3.3.1.2. *The Young measures ν arising in the proof of Theorem 3.3.1.1 are elements of the space $L^\alpha_w(Q_T; E'_{q-1})$ for $\alpha \in (1, \frac{p}{q-1})$.*

Proof. We compute:

$$\begin{aligned}
\|\nu\|_{L^\alpha_w(Q_T;E'_{q-1})} &= \sup_{\|\gamma\|_{L^{\alpha'}(Q_T;E_{q-1})}=1} \int_0^T \int_\Omega \langle \nu_{t,x}, \gamma(t,x,\cdot)\rangle \, \mathrm{d}x \, \mathrm{d}t \\
&= \sup_{\|\gamma\|_{L^{\alpha'}(Q_T;E_{q-1})}=1} \int_0^T \int_\Omega \left\langle \nu_{t,x}, \frac{\gamma(t,x,\cdot)(1+|\cdot|^{q-1})}{1+|\cdot|^{q-1}} \right\rangle \mathrm{d}x \, \mathrm{d}t \\
&\leq \sup_{\|\gamma\|_{L^{\alpha'}(Q_T;E_{q-1})}=1} \int_0^T \int_\Omega \sup_{A\in\mathbb{R}^{m\times n}} \left| \frac{\gamma(t,x,A)}{1+|A|^{q-1}} \right| \langle \nu_{t,x}, 1+|\cdot|^{q-1}\rangle \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \left(\int_0^T \int_\Omega |\langle \nu_{t,x}, 1+|\cdot|^{q-1}\rangle|^\alpha \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{\alpha}}.
\end{aligned}$$

As the function $\zeta(A) = 1 + |A|^{q-1} \in E_{q-1}$, we have that $\zeta(Du^h) \rightharpoonup \langle \nu, \zeta\rangle$ in the space $L^\alpha(Q_T)$, where Du^h is the sequence from the proof of Theorem 3.3.1.1 which generates the Young measure ν (this follows from the remarks made after the statement of Theorem 3.2.2.2). We then therefore have that

$$\left(\int_0^T \int_\Omega |\langle \nu_{t,x}, 1+|\cdot|^{q-1}\rangle|^\alpha \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{\alpha}} \leq \liminf_{h\to 0} \left(\int_0^T \int_\Omega |1+|Du^h(t,x)|^{q-1}|^\alpha \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{\alpha}}.$$

By the fact that $\alpha < \frac{p}{q-1}$, the fact that we are on a domain of finite measure, and using the energy estimates obtained for the functions Du^h we see that the right-hand side of the above inequality is finite. $\square$

We make some comments regarding the result proven above in Theorem 3.3.1.1. There exist results in the literature (see for example [23] and [37]) for which the definition of a Young measure solution includes extra regularity on the time derivative $\partial_t u$ and some extra spatial regularity, and thus is a stronger notion of solution than the one presented here. However, in these papers extra structure of the non-linearity is assumed, whereby the authors can express it as the gradient of a (non-convex) function (that is, they solve $\partial_t u - \operatorname{div}(D\psi(Du)) = 0$ where $\psi \in C^1$).

Comparisons should be drawn to the work of Bögelein, Duzaar and Marcellini in [12] in the case where $p_i \geq 2$ for all $i = 1, \ldots, m$. The nonlinear term described in the above model gives the problem (3.1.0.1) a p, q-growth flavour, which is seen as follows. The nonlinearity described in [12] is the derivative of a convex function f satisfying the following growth conditions:

$$\begin{cases} |x|^p \leq f(\xi) \leq C(1 + |\xi|^q), \\ |D^2 f(\xi)| \leq L(1 + |\xi|)^{q-2}, \\ \langle D^2 f(\xi)\eta, \eta\rangle \geq \nu|\xi|^{p-2}|\eta|^2. \end{cases}$$

The convexity of the function f, combined with the above growth estimates, allow us to deduce the following bounds on Df:

$$c|\xi|^{p-1} \leq |Df(\xi)| \leq C|\xi|^{q-1}.$$

Under these assumptions the existence of a weak energy solution was obtained with Dirichlet boundary data provided that $2 \leq p \leq q < p + \min\{1, \frac{4}{n+2}\}$.

The form taken by the mapping a in Theorem 3.3.1.1 yields the following bounds:

$$c(|\xi|^{p-1} - 1) \leq |a(\xi)| \leq C(1 + |\xi|^{q-1}),$$

meaning that one can view the problem as being a non-monotone version of the (p, q)-growth problems studied in [12].

3.3.2 Examples and the Met Office problem

The growth conditions used in this subsection were chosen to be similar to those used in (1.1.2.9):

Example 3.3.2.1. In the case where we have $n = 1$, $m = 3$ and the function K is given by $K(\xi) = \sqrt{\xi_1^2 + \xi_2^2 + |\xi_3|}$ then we are in the regime $\partial_z\theta < 0$ from (1.1.2.9). One has the existence of constants c_0 and c_1 such that

$$c_0(|\xi_1| + |\xi_2| + \sqrt{|\xi_3|}) \leq K(\xi) \leq c_1(|\xi_1| + |\xi_2| + \sqrt{|\xi_3|})$$

through bounding and using the equivalence of norms on $\mathbb{R}^3$. Thus in this case we have a Young measure solution. Recall from previous sections that this formulation of K leads to a non-monotone operator. In the appendicies we show that this formulation leads to the operator $a(\xi) = K(\xi)\xi$ being a locally Lipschitz map from $\mathbb{R}^3$ to $\mathbb{R}^3$.

3.4 Long-time behaviour of Young measure solutions

In this section we shall discuss the long-time behaviour of the Young measure solutions arising in Theorem 3.3.1.1. We start by considering a nonlinear ordinary differential inequality and its relation to a nonlinear ODE. Following that, we show the existence of a global in time Young measure solution, and under suitable assumptions on F, show convergence to a particular Young measure solution of the corresponding steady-state problem. For this section, while we shall be assuming that $p_i \geq 2$ for all $i = 1, \dots, m$, and that we are taking $\mu_i = 0$ for all $i = 1, \dots, m$, we shall nevertheless aim to give results whose proofs are as general as possible, with the aim to cover other cases in the future.

3.4.1 Chini-type differential inequalities

Consider (3.3.0.2a) with the particular choice of $\phi = (0, \dots, 0, u_i^h, 0, \dots, 0)^T$, where the i-th component of ϕ is the i-th component of the vector u^h. This gives

$$\int_\Omega \partial_t u_i^h(t,x) u_i^h(t,x) + K(Du^h(t,x))|Du_i^h(t,x)|^2 \, \mathrm{d}x = \int_\Omega F_i(t,x) u_i^h(t,x) \, \mathrm{d}x.$$

Using the lower growth bounds on K, applying the Sobolev inequality, and applying Hölder's inequality to the right-hand side, the above equation yields the inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_i^h(t,\cdot)\|_{L^2(\Omega)}^2 + c\|u_i^h(t,\cdot)\|_{L^2(\Omega)}^{p_i} \leq C\|F_i(t,\cdot)\|_{L^2(\Omega)}\|u_i^h(t,\cdot)\|_{L^2(\Omega)}. \tag{3.4.1.1}$$

Setting $x_i^h(t) := \|u_i^h(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^m)}$ and $f_i(t) := \|F_i(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^m)}$, and dividing through by $x_i^h(t)$, we see that (3.4.1.1) becomes the following nonlinear differential inequality:

$$\frac{\mathrm{d}x_i^h}{\mathrm{d}t}(t) + x_i^h(t)^{p_i-1} \leq f_i(t), \tag{3.4.1.2}$$

subject to the conditions that $x_i^h(t) \geq 0$ for all $t \geq 0$ and $x_i^h(0) = x_{i,0}^h = \|u_i^h(0,\cdot)\|_{L^2(\Omega)}$. We also consider the associated nonlinear ODE

$$\frac{\mathrm{d}y_i^h}{\mathrm{d}t}(t) + y_i^h(t)^{p_i-1} = f_i(t), \tag{3.4.1.3}$$

with $y_i^h(0) = x_{i,0}^h$ and $f_i(t) \geq 0$ for all $t \geq 0$. When p_i is an integer *and* $p_i \geq 2$, (3.4.1.3) is a particular case of *Chini's equation.* In general one cannot currently write down the solution of Chini's equation explicitly, but in certain cases we aim to study the long-time behaviour of solutions to such inequalities.

The aim of studying (3.4.1.2) is to determine conditions on f_i which will ensure that the function x_i^h converges to zero as $t \to \infty$. In particular, for studying the long-time behaviour of the Young measure solutions we will be interested in obtaining decay estimates which are independent of h and depend only on the initial condition, the source term, and t. We begin the analysis by showing that, regardless of the choice of f, a function x_i^h satisfying (3.4.1.2) and a function y_i^h satisfying (3.4.1.3) will also satisfy $x_i^h(t) \leq y_i^h(t)$ for all $t > 0$.

Lemma 3.4.1.1. *If x_i^h satisfies* (3.4.1.2) *and y_i^h satisfies* (3.4.1.3), *with $y_i^h(0) \geq x_{i,0}^h$ then $x_i^h(t) \leq y_i^h(t)$ for all $t \geq 0$.*

Proof. Combining (3.4.1.2) and (3.4.1.3) we see that

$$(x_i^h - y_i^h)'(t) + x_i^h(t)^{p_i-1} - y_i^h(t)^{p_i-1} \leq 0.$$

We multiply this inequality by $(x_i^h - y_i^h)_+$, the positive part of $x_i^h - y_i^h$, to get

$$(x_i^h - y_i^h)'(t)(x_i^h(t) - y_i^h(t))_+ + (x_i^h(t)^{p_i-1} - y_i^h(t)^{p_i-1})(x_i^h(t) - y_i^h(t))_+ \leq 0. \tag{3.4.1.4}$$

We compute that

$$\begin{aligned} \left[\frac{1}{2}(x_i^h - y_i^h)_+^2\right]'(t) &= (x_i^h - y_i^h)_+(t)\chi_{[0,\infty)}(x_i^h(t) - y_i^h(t))(x_i^h - y_i^h)'(t) \\ &= (x_i^h - y_i^h)'(t)(x_i^h - y_i^h)_+(t), \end{aligned}$$

which means that (3.4.1.4) becomes

$$\left[\frac{1}{2}(x_i^h - y_i^h)_+^2\right]'(t) + (x_i^h(t)^{p_i-1} - y_i^h(t)^{p_i-1})(x_i^h(t) - y_i^h(t))_+ \leq 0. \tag{3.4.1.5}$$

As the term $(x_i^h(t)^{p_i-1} - y_i^h(t)^{p_i-1})(x_i^h(t) - y_i^h(t))_+$ is non-negative, we can drop it from (3.4.1.5), resulting in

$$\left[\frac{1}{2}(x_i^h - y_i^h)_+^2\right]'(t) \leq 0. \tag{3.4.1.6}$$

Integrating both sides of this inequality and using that $x_i^h(0) \leq y(0)$ we see that

$$(x_i^h(t) - y_i^h(t))_+^2 \leq 0,$$

which implies that

$$x_i^h(t) \leq y_i^h(t)$$

for all $t \geq 0$, as required. $\square$

Therefore in order to obtain decay estimates for $x_i^h(t)$ in the limit as $t \to \infty$ it suffices to obtain decay estimates for $y_i^h(t)$. Before considering the decay we first show that we have uniqueness of the solution to (3.4.1.3) subject to a given initial condition.

Lemma 3.4.1.2. *The equation* (3.4.1.3)*, subject to the initial condition*

$$y_i^h(0) = \|u_i^h(0, \cdot)\|_{L^2(\Omega)} \geq 0$$

and with $f_i(t) \geq 0$ *and* $f_i(t) \in L^{p_i'}(0, \infty)$*, possesses a unique solution on* $(0, \infty)$*.*

Proof. By Peano's Theorem we have the existence of a solution to (3.4.1.3) on the interval $(0, \varepsilon)$ for some $\varepsilon > 0$. Now we seek to show that we can bound independently of ε, showing that ε cannot be the maximal existence time. Multiplying (3.4.1.3) by $y_i^h(t)$ and integrating, and then by applying Young's inequality, we can show that

$$\begin{aligned} y_i^h(\varepsilon)^2 + \|y_i^h\|_{L^{p_i}(0,\varepsilon)} &\leq c\left(y_i^h(0)^2 + \|f_i\|_{L^{p_i'}(0,\varepsilon)}^{p_i'}\right) \\ &\leq c\left(y_i^h(0)^2 + \|f_i\|_{L^{p_i'}(0,\infty)}^{p_i'}\right). \end{aligned}$$

Thus ε cannot be the maximal existence time, and so we can extend to a solution on the whole of the positive real line.

We let y_1 and y_2 be two solutions to (3.4.1.3). Note that we can repeat the calculations done in the proof of Lemma (3.4.1.1), testing against $(y_1 - y_2)_+(t)$ and using that $(y_1(t)^{q-1} - y_2(t)^{q-1})(y_1 - y_2)_+(t) \geq 0$, to obtain that $y_1(t) \leq y_2(t)$ for all $t \geq 0$. By symmetry it follows that $y_1(t) \geq y_2(t)$ for all $t \geq 0$. $\square$

Lemma 3.4.1.3. *Under the assumptions of Lemma 3.4.1.2, it holds that $y_i^h(t) \geq 0$ for all $t \geq 0$.*

Proof. Note that the constant function $x_i^h(t) \equiv 0$ is a solution of (3.4.1.2) with $x_i^h(0) = 0$. That $y_i^h(t) \geq 0$ for all $t \geq 0$ now follows from Lemma 3.4.1.1. $\square$

Now we wish to discuss the decay rates of the solutions to these equations. We start off by deriving suitable energy estimates and from these deducing regularity of y_i^h.

Proposition 3.4.1.4. *Let $f_i \in L^{p_i'}(0,\infty) \cap L^2(0,\infty)$ with $f_i' \in L^2(0,\infty)$. Then, there exist constants $c_1, c_2 > 0$ depending on f_i and $y_i^h(0)$ such that the solution y_i^h of* (3.4.1.3) *satisfies*

$$\|y_i^h\|_{L^r(0,\infty)} \leq c_1(\|f_i\|_{L^{p_i'}(0,\infty)}, y_i^h(0))$$

for all $r \in [p_i, \infty]$, and

$$\left\|\frac{\mathrm{d}y_i^h}{\mathrm{d}t}\right\|_{L^2(0,\infty)} \leq c_2(\|f_i\|_{L^2(0,\infty)}, y_i^h(0)).$$

Proof. Multiplying (3.4.1.3) by $y_i^h(t)$ and integrating we see that:

$$\frac{1}{2}y_i^h(t)^2 + \int_0^t y_i^h(s)^{p_i}\ \mathrm{d}s = \int_0^t f_i(s)y_i^h(s)\ \mathrm{d}s + \frac{1}{2}y_i^h(0)^2.$$

Applying Young's inequality and rearranging, we obtain:

$$y_i^h(t) + \int_0^t y_i^h(s)^{p_i}\ \mathrm{d}s \leq c\left(\int_0^t f_i(s)^{p_i'}\ \mathrm{d}s + y_i^h(0)^2\right) \leq c\left(\|f_i\|^{p_i'}_{L^{p_i'}(0,\infty)} + y_i^h(0)^2\right),$$

using here the assumption that $f_i \in L^{p_i'}(0,\infty)$. Therefore, taking the supremum over

$t \in (0, \infty)$, we have that

$$y_i^h \in L^\infty(0, \infty), \tag{3.4.1.7}$$

and

$$y_i^h \in L^{p_i}(0, \infty). \tag{3.4.1.8}$$

Note that by (3.4.1.7) and (3.4.1.8), it follows from interpolation results on Lebesgue spaces that

$$y_i^h \in L^r(0, \infty) \tag{3.4.1.9}$$

for all $r \in [p_i, \infty]$. Returning to (3.4.1.3) and testing instead by $\frac{\mathrm{d}y_i^h}{\mathrm{d}t}$ we see that

$$\int_0^t \left(\frac{\mathrm{d}y_i^h}{\mathrm{d}t}(s)\right)^2 + \frac{1}{p_i}\frac{\mathrm{d}}{\mathrm{ds}}\left(y_i^h(s)^{p_i}\right)\ \mathrm{d}s = \int_0^t f_i(s)\frac{\mathrm{d}y_i^h}{\mathrm{d}t}(s)\ \mathrm{d}s.$$

Applying Young's inequality and using the assumption that $f_i \in L^2(0, \infty)$ we see that:

$$\left\|\frac{\mathrm{d}y_i^h}{\mathrm{d}t}\right\|^2_{L^2(0,\infty)} + \|y_i^h\|^q_{L^\infty(0,\infty)} \le c\left(\|f_i\|^2_{L^2(0,\infty)} + y_i^h(0)^{p_i}\right),$$

from which it follows that

$$\frac{\mathrm{d}y_i^h}{\mathrm{d}t} \in L^2(0, \infty), \tag{3.4.1.10}$$

as required. □

Proposition 3.4.1.5. *In addition to the assumptions of Proposition 3.4.1.4, let* $f_i \in C([0, \infty))$ *with* $\lim_{t\to\infty} f_i(t) = 0$*. Then,* $\lim_{t\to\infty} y_i^h(t) = 0$*.*

Proof. Consider the following, which is obtained through the Fundamental Theorem of Calculus, where $\alpha = \frac{p_i+2}{2}$:

$$y_i^h(t)^\alpha = y_i^h(0)^\alpha + \int_0^t \alpha y_i^h(s)^{\alpha-1}\frac{\mathrm{d}y_i^h}{\mathrm{d}t}(s)\ \mathrm{d}s.$$

Through estimating as before we see that

$$\begin{aligned}\left|\int_0^t y_i^h(s)^{\alpha-1}\frac{\mathrm{d}y_i^h}{\mathrm{d}t}(s)\ \mathrm{d}s\right| &\leq \int_0^t y_i^h(s)^{\alpha-1}\left|\frac{\mathrm{d}y_i^h}{\mathrm{d}t}(s)\right|\ \mathrm{d}s \\ &\leq \left(\int_0^t y_i^h(s)^{2(\alpha-1)}\ \mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^t\left|\frac{\mathrm{d}y_i^h}{\mathrm{d}t}(s)\right|^2\ \mathrm{d}s\right)^{\frac{1}{2}} \\ &\leq \left(\int_0^\infty y_i^h(s)^{p_i}\ \mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^\infty\left|\frac{\mathrm{d}y_i^h}{\mathrm{d}t}(s)\right|^2\ \mathrm{d}s\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore, since $y_i^h \geq 0$, we have that ${y_i^h}^{\alpha-1}\frac{\mathrm{d}y_i^h}{\mathrm{d}t} \in L^1(0,\infty)$, and consequently

$$\lim_{t\to\infty}\int_0^t y_i^h(s)^{\alpha-1}\frac{\mathrm{d}y_i^h}{\mathrm{d}t}(s)\ \mathrm{d}s$$

exists. Thus $\lim_{t\to\infty} y_i^h(t)^\alpha$ exists and so $\lim_{t\to\infty} y_i^h(t)$ also exists as $y_i^h(t) \geq 0$ for all $t \geq 0$ and the map $s \mapsto s^\alpha$ is convex with positive derivative if $s > 0$.

We can then take $t \to \infty$ in (3.4.1.3) to see that

$$\lim_{t\to\infty}\frac{\mathrm{d}y_i^h}{\mathrm{d}t}(t) + \lim_{t\to\infty} y_i^h(t)^{p_i-1} = 0. \tag{3.4.1.11}$$

Recall that $y_i^h(t) \geq 0$ for all $t \in [0,\infty)$. It must then hold that

$$\lim_{t\to\infty}\frac{\mathrm{d}y_i^h}{\mathrm{d}t}(t) \leq 0.$$

If $\lim_{t\to\infty}\frac{\mathrm{d}y_i^h}{\mathrm{d}t}(t) = c < 0$, then it would follow that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\frac{\mathrm{d}y_i^h}{\mathrm{d}t}(t) - c| < \varepsilon$ for all $t > \delta$. Choosing in particular $\varepsilon = -\frac{c}{2}$ we see that there exists a $\delta_c > 0$ such that $\frac{\mathrm{d}y_i^h}{\mathrm{d}t}(t) < \frac{c}{2} < 0$ for all $t > \delta_c$. This contradicts the fact that $y_i^h \geq 0$, as well as (3.4.1.7). Therefore $\lim_{t\to\infty} y'(t) = 0$, and so by (3.4.1.11), $\lim_{t\to\infty} y_i^h(t) = 0$. $\square$

Next we show that the family of functions y_i^h converges strongly in the appropriate space as we pass $h \to 0$.

Lemma 3.4.1.6. *Let $y_i : [0,\infty) \to [0,\infty)$ be a solution of the nonlinear differential equation*

$$y_i'(t) + y_i(t)^{p_i-1} = f_i(t)$$

with the initial condition $y_i(0) = \|u_{i,0}\|_{L^2(\Omega;\mathbb{R}^m)}$ and $f_i(t) \geq 0$ for all $t \in [0,\infty)$. Then we have that $\|y_i - y_i^h\|_{L^\infty(0,\infty)} \to 0$ as $h \to 0$.

Proof. We have that

$$(y_i - y_i^h)'(t) + y_i(t)^{p_i-1} - y_i^h(t)^{p_i-1} = 0.$$

Multiplying by $y_i - y_i^h$ results in

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((y_i - y_i^h)^2\right)(t) + (y_i(t)^{p_i-1} - y_i^h(t)^{p_i-1})(y_i(t) - y_i^h(t)) = 0.$$

Note that both y_i and y_i^h are non-negative functions. It follows that if $y_i^h(t) \geq y_i(t)$, then $y_i(t)^{p_i-1} \geq y_i^h(t)^{p_i-1}$, and vice-versa. Therefore, the expression

$$(y_i(t)^{p_i-1} - y_i^h(t)^{p_i-1})(y_i(t) - y_i^h(t))$$

is always non-negative. Bounding this term in the above equality below by 0 and integrating in time we see that

$$(y_i(t) - y_i^h(t))^2 \leq (y_i(0) - y_i^h(0))^2.$$

The result now follows from the assumed strong convergence of our initial condition u_0^h to u_0 in $L^2(\Omega;\mathbb{R}^m)$. □

3.4.2 Asymptotic behaviour

In this subsection we discuss the long-time behaviour of the previously constructed Young measure solutions. Such long-time behaviour has already been studied in [23] and [65] among others, but under the assumption that there exists a C^1-function ψ for which $a = D\psi$, and that $p = q = 2$. As usual, we aim to show that our time-dependent Young measure solution converges to some limiting pair of function and Young measure, and then identify this as a solution of a corresponding steady-state problem. We begin with the definition of a global Young measure solution.

Definition 3.4.2.1. We say that the pairing (u, ν), where u is a function

$$u \in L^\infty(0,T; L^2(\Omega; \mathbb{R}^m)) \cap \bigtimes_{i=1}^{m} L^{p_i}(0,T; W_0^{1,p_i}(\Omega)) \text{ for any } T > 0$$

with

$$\partial_t u \in L^{q'}(0,T; W^{-1,q'}(\Omega; \mathbb{R}^m)) \text{ for any } T > 0,$$

and $(\nu_{t,x})_{(t,x)\in Q_\infty}$ is a family of probability measures satisfying, for almost all $(t,x) \in Q_T$ for any $T > 0$, the identity

$$Du(t,x) = \int_{\mathbb{R}^{m\times n}} \xi \, \mathrm{d}\nu_{t,x}(\xi),$$

is a *global Young measure solution* to the problem described by (3.1.0.1)−(3.1.0.5) if, for any $T > 0$, we have that

$$\int_0^T \langle \partial_t u, \phi \rangle + \int_\Omega \langle \nu_{t,x}, a \rangle : D\phi \, - Bu \cdot \phi \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega F \cdot \phi \, \mathrm{d}x \, \mathrm{d}t \tag{3.4.2.1}$$

holds for all $\phi \in L^q(0,T; W_0^{1,q}(\Omega; \mathbb{R}^m))$, and the initial condition is satisfied in the sense that $u(0,x) = u_0(x)$ for almost every $x \in \Omega$.

We prove that such a solution exists. In [65] it is assumed that a is a Lipschitz map, a property which is exploited in order to show the existence of a global Young measure solution. As we are instead only assuming that a is locally Lipschitz, we use a similar method to that found in [57].

Theorem 3.4.2.2. *Let the assumptions of Theorem 3.3.1.1 hold, with the additional assumptions that* $F \in \bigtimes_{i=1}^{m} L^{p_i'}(Q_\infty) \cap L^2(Q_\infty; \mathbb{R}^m)$. *There exists a global Young measure solution to the problem described by* (3.1.0.1)−(3.1.0.5).

Proof. Let $\{T_r\}_{r\in\mathbb{N}}$ be a sequence of real numbers satisfying $\lim_{r\to\infty} T_r = \infty$. For the sake of brevity, we shall define:

$$G_r := L^\infty(0,T_r;L^2(\Omega;\mathbb{R}^m)),$$

$$H_r := \bigtimes_{i=1}^{m} L^{p_i}(0,T_r;W_0^{1,p_i}(\Omega)),$$

and

$$I_r = L^{q'}(0,T_r;W^{-1,q'}(\Omega;\mathbb{R}^m)).$$

For each fixed T_r, Theorem 3.3.1.1 yields the existence of a function $u^r \in G_r \cap H_r$ with $\partial_t u^r \in I_r$ and a measure $\nu^r \in L^{\frac{p}{p-q+1}+\epsilon}(Q_{T_k};E_{q-1})$ with $\langle \nu^r, a\rangle \in L^{q'}(Q_{T_r})$ such that

$$\int_0^{T_r} \langle \partial_t u^r, \phi\rangle + \int_\Omega \langle \nu^r_{t,x}, a\rangle : D\phi - Bu^r \cdot \phi \,\mathrm{d}x\,\mathrm{d}t = \int_0^{T_r}\int_\Omega F\cdot\phi\,\mathrm{d}x\,\mathrm{d}t$$

for all $\phi \in L^q(0,T_r;W_0^{1,q}(\Omega))$. Note that for a fixed $T < T_r$ we have that (u^r,ν^r) is a Young measure solution on Q_T (this follows by the fact that $C_0^\infty(Q_T) \subset C_0^\infty(Q_{T_r})$, and density).

Fixing $T = T_r$ we then have that u^r is bounded uniformly in the space $G_r \cap H_r$. Then there is a subsequence $\{k_r(j)\}_{j=1}^\infty$ of r and a function $u_r \in G_r \cap H_r$ such that

$$u^{k_r(j)} \rightharpoonup u_r \text{ in } H_r,$$

and

$$u^{k_r(j)} \overset{*}{\rightharpoonup} u_r \text{ in } G_r.$$

In addition, by Theorem 3.3.1.1 there is, for each $u^{k_r(j)}$ a sequence $h_{k_r(j)}(l)$ and a corresponding sequence of functions $u^{h_{k_r(j)}(l)}$ such that $u^{h_{k_r(j)}(l)} \rightharpoonup u^{k_r(j)}$ in H_r, $u^{h_{k_r(j)}(l)} \overset{*}{\rightharpoonup} u^{k_r(j)}$ in G_r, $\partial_t u^{h_{k_r(j)}(l)} \rightharpoonup \partial_t u^{k_r(j)}$ in I_r, and such that $u^{h_{k_r(j)}(l)} \to u^{k_r(j)}$ in $C_w([0,T_r];L^2(\Omega;\mathbb{R}^m))$. There is also a Young measure $\nu^{k_r(j)}$ for which it holds that $a(Du^{h_{k_r(j)}(l)}) \rightharpoonup \langle \nu^{k_j(r)}, a\rangle$ in $L^{q'}(Q_{T_r};\mathbb{R}^{m\times n})$ (cf. Remark 3.1.0.2). Now there is a subsequence $\{h_{k_r(j+1)}(l)\}_{l=1}^\infty \subset \{h_{k_r(j)}(l)\}_{l=1}^\infty$ along which it holds that $u^{h_{k_r(j+1)}(l)} \rightharpoonup u^{k_r(j+1)}$ in H_r, $u^{h_{k_r(j+1)}(l)} \overset{*}{\rightharpoonup} u^{k_r(j+1)}$ in G_r, $\partial_t u^{h_{k_r(j+1)}(l)} \rightharpoonup \partial_t u^{k_r(j+1)}$ in I_r, and such that we have $u^{h_{k_r(j+1)}(l)} \to u^{k_r(j+1)}$ in

$C_w([0,T_r];L^2(\Omega;\mathbb{R}^m))$ and also $a(Du^{h_{k_r(j+1)}(l)}) \rightharpoonup \langle \nu^{k_r(j+1)}, a\rangle$. Continuing in this fashion we see that there is a subsequence $\{u^{h_{k_r(j)}(l)}\}_{l=1}^\infty \subset \{u^{h_{k_r(j-1)}(l)}\}_{l=1}^\infty \subset \cdots \subset \{u^{h_r(l)}\}_{l=1}^\infty$ along which $u^{h_{k_r(j)}(l)} \rightharpoonup u^{k_r(j)}$ in H_r, $u^{h_{k_r(j)}(l)} \overset{*}{\rightharpoonup} u^{k_r(j)}$ in G_r and $u^{h_{k_r(j)}(l)} \to u^{k_r(j)}$ in $C_w([0,T];L^2(\Omega;\mathbb{R}^m))$, and $a(Du^{h_{k_r(j)}(l)}) \rightharpoonup \langle \nu^{k_r(j)}, a\rangle$. Now take the diagonal subsequence $h_{k_r(j)}(j)$. This gives, for each fixed r, the existence of a function u_r and a measure ν_r for which we have

$$u^{h_{k_r(j)}(j)} \rightharpoonup u_r \text{ in } H_r,$$

$$u^{h_{k_r(j)}(j)} \overset{*}{\rightharpoonup} u_r \text{ in } G_r,$$

and

$$u^{h_{k_r(j)}(j)} \to u_r \text{ in } C_w([0,T];L^2(\Omega;\mathbb{R}^m)),$$

and

$$a(Du^{h_{k_r(j)}(j)}) \rightharpoonup \langle \nu_r, a\rangle \text{ in } L^{q'}(Q_{T_r};\mathbb{R}^{m\times n}).$$

Now considering $T = T_{r+1}$ we see that as the subsequence $u^{h_{k_r(j)}(j)}$ is uniformly bounded in the space $G_r \cap H_r$, there is a subsequence $\{h_{k_{r+1}(j)}(j)\}_{j=1}^\infty$ of $\{h_{k_r(j)}(j)\}_{j=1}^\infty$ and a function $u_{r+1} \in G_{r+1} \cap H_{r+1}$ and a measure ν_{r+1} such that

$$u^{h_{k_{r+1}(j)}(j)} \rightharpoonup u_{r+1} \text{ in } H_{r+1}$$

and

$$u^{h_{k_{r+1}(j)}(j)} \overset{*}{\rightharpoonup} u_{r+1} \text{ in } G_{r+1},$$

$$u^{h_{k_{r+1}(j)}(j)} \to u_{r+1} \text{ in } C_w([0,T_{r+1}];L^2(\Omega;\mathbb{R}^m)),$$

and

$$a(Du^{h_{k_{r+1}(j)}(j)}) \rightharpoonup \langle \nu_{r+1}, a\rangle \text{ in } L^{q'}(Q_{T_{r+1}};\mathbb{R}^{m\times n}).$$

Note that in particular we have the convergence $u^{h_{k_{r+1}(j)}(j)} \rightharpoonup u_{r+1}$ in H_r, $u^{h_{k_{r+1}(j)}(j)} \overset{*}{\rightharpoonup} u_{r+1}$ in G_r (this follows from the fact that if we take a function $\phi \in L^q(0,T_r;W_0^{1,q}(\Omega;\mathbb{R}^m))$ and extend it by zero in time to $(0,T_{r+1})$ then the resulting function ψ is in the space $L^q(0,T_{r+1};W_0^{1,q}(\Omega;\mathbb{R}^m))$), as well as $a(Du^{h_{k_{r+1}(j)}(j)}) \rightharpoonup \langle \nu_{r+1}, a\rangle$ in $L^{q'}(Q_{T_r};\mathbb{R}^{m\times n})$. So by the uniqueness of weak limits we see that $u_{r+1}|_{(0,T_r)\times\Omega} = u_r$ and $\nu_{r+1}|_{(0,T_r)\times\Omega} = \nu_r$.

Continuing in this fashion, we see that for every $r \in \mathbb{N}\backslash\{0\}$ there is a susbequence $\{h_{k_r(j)}(j)\}_{j=1}^{\infty} \subset \{h_{k_{r-1}(j)}(j)\}_{j=1}^{\infty} \subset \cdots$ and a function $u_r \in G_r \cap H_r$ and a measure ν_r such that $u^{h_{k_r(j)}(j)} \rightharpoonup u_r$ in the space H_r and $u^{h_{k_r(j)}(j)} \overset{*}{\rightharpoonup} u_r$ in the space G_r and such that $a(Du^{h_{k_r(j)}(j)}) \rightharpoonup \langle \nu_r, a\rangle$ in $L^{q'}(Q_{T_r})$, and such that for all $l = 1, \ldots, r-1$, $u_r|_{(0,T_l)\times\Omega} = u_l$ and $\nu_r|_{(0,T_l)\times\Omega} = \nu_l$. Now we take the diagonal subsequence $h_{k_r(r)}(r)$. This gives the existence of a function u and a measure ν such that for any $T > 0$ we have $u^{h_{k_r(r)}(r)} \rightharpoonup u$ in the space $\bigtimes_{i=1}^{m} L^{p_i}(0,T;W_0^{1,p_i}(\Omega))$ and $u^{h_{k_r(r)}(r)} \overset{*}{\rightharpoonup} u$ in $L^{\infty}(0,T;L^2(\Omega;\mathbb{R}^m))$, and $a(Du^{h_{k_r(r)}(r)}) \rightharpoonup \langle \nu, a\rangle$ in $L^{q'}(Q_T;\mathbb{R}^{m\times n})$.

For the time derivatives we have (for r sufficiently large)

$$\begin{aligned}
\|\partial_t u^{h_{k_r(r)}(r)}\|_{L^{q'}(0,T;W^{-1,q'}(\Omega))} &= \sup_{\|\phi\|_{L^{q'}(0,T;W_0^{-1,q'}(\Omega))}=1} \int_0^T \int_\Omega \partial_t u^{h_{k_r(r)}(r)} \cdot \phi \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \|\langle \nu^{h_{k_r(r)}(r)}, a\rangle\|_{L^{q'}(Q_T;\mathbb{R}^{m\times n})} + C\|u^{h_{k_r(r)}(r)}\|_{L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))} \\
&\quad + \|F\|_{L^{p'}(Q_T\mathbb{R}^m)}.
\end{aligned}$$

This is bounded independently of r by previous estimates, and so there is a subsequence converging weakly-star to $\partial_t u$. Thus we can pass to the limit in the weak formulation to see that for any $T > 0$, the following identity is satisfied:

$$\int_0^T \int_\Omega \partial_t u \cdot \phi + \langle \nu_{t,x}, a\rangle : D\phi - Bu\cdot\phi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega F\cdot\phi \,\mathrm{d}x\,\mathrm{d}t$$

for all $\phi \in L^q(0,T_k;W_0^{1,q}(\Omega;\mathbb{R}^m))$. In addition, it follows from Lemma 3.2.3.2 that for every $w \in L^2(\Omega;\mathbb{R}^m)$,

$$\int_\Omega u^{h_{k_r(r)}(r)}(t,x)\cdot w(x)\,\mathrm{d}x \to \int_\Omega u(t,x)\cdot w(x)\,\mathrm{d}x.$$

The satisfaction of the initial condition follows immediately from the fact that all elements of the sequence $u^{h_{k_r(r)}(r)}$ satisfy the initial condition.

Finally, for each $T > 0$ we have by Theorem 3.2.2.2 that $Du^{h_{k_r(r)}(r)} \rightharpoonup \langle \nu, id\rangle$ in $L^{\hat{r}}(Q_T;\mathbb{R}^{m\times n})$ for any $1 < \hat{r} < p$, where id denotes the identity function (cf. Remark

3.1.0.2). Then by the uniqueness of weak limits it holds that

$$Du(t,x) = \int_{\mathbb{R}^{m\times n}} \xi \, \mathrm{d}\nu_{t,x}(\xi)$$

for almost every $(t,x) \in Q_T$ and for every $T > 0$, which completes the proof. $\square$

Remark 3.4.2.3. *We note that in the case in which $p \geq 2$ and $\mu_i = 0$ for all $i = 1,\dots,m$ the bounds coming from Theorem 3.3.1.1 are independent of T and we in fact have that*

$$u \in L^\infty(0,\infty; L^2(\Omega;\mathbb{R}^m)) \cap \bigtimes_{i=1}^{m} L^{p_i}(0,\infty; W_0^{1,p_i}(\Omega)),$$

$$\partial_t u \in L^{q'}(0,\infty; W^{-1,q'}(\Omega;\mathbb{R}^m)),$$

and

$$\langle \nu, a\rangle \in L^{q'}(Q_\infty; \mathbb{R}^{m\times n}).$$

However, although the measure ν is in the space $L^\alpha_w(Q_T; E'_{q-1})$ for any $T > 0$, the bounds cannot be made independent of T.

Now we study one particular aspect of the long-time behaviour of the function u, making use of results from the previous subsection.

Proposition 3.4.2.4. *Let F be such that $\lim_{t\to\infty} \|F(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^m)} \to 0$ and let the assumptions of Proposition 3.4.1.5 hold with $f(t) = \|F(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^m)}$. Then we have that $\lim_{t\to\infty} \|u(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^m)} = 0$.*

Proof. We have by Lemma 3.2.3.2 that, for every $T > 0$, $u \in C_w([0,T]; L^2(\Omega;\mathbb{R}^m))$ and

$$\int_\Omega u^{h_{k_r(r)}(r)}(t,x)\cdot w(x) \, \mathrm{d}x \to \int_\Omega u(t,x)\cdot w(x) \, \mathrm{d}x$$

uniformly in $C([0,T])$ as $r \to \infty$ and for every $w \in L^2(\Omega;\mathbb{R}^m)$. Letting $y = (y_1,\dots,y_m)$ and $y^h = (y^h_1,\dots,y^h_m)$, with y_i and y^h_i be as defined in the previous subsection, we compute:

$$
\begin{aligned}
\int_\Omega u(t,x)\cdot w(x)\ \mathrm{d}x &= \lim_{r\to\infty}\int_\Omega u^{h_{k_r(r)}(r)}(t,x)\cdot w(x)\ \mathrm{d}x \\
&\le \lim_{r\to\infty}\|u^{h_{k_r(r)}(r)}(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^m)}\|w\|_{L^2(\Omega;\mathbb{R}^m)} \\
&\le \lim_{r\to\infty}|y^{h_{k_r(r)}(r)}(t)|_2\|w\|_{L^2(\Omega;\mathbb{R}^m)} \\
&= |y(t)|_2\|w\|_{L^2(\Omega;\mathbb{R}^m)}
\end{aligned}
$$

Thus, taking the supremum over functions $w \in L^2(\Omega;\mathbb{R}^m)$ with $\|w\|_{L^2(\Omega;\mathbb{R}^m)} \le 1$, we see that

$$
\|u(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^m)} \le |y(t)|_2.
$$

From Proposition 3.4.1.5 we have that $\lim_{t\to\infty}|y(t)|_2 = 0$, and thus we have the claimed result. □

We have the following definition of a Young measure solution to the corresponding steady-state problem. We include this here for completeness, but we remark that for the particular cases with which we are concerning ourselves, we shall not meet such a Young measure solution.

Definition 3.4.2.5. We say that a pair $(\tilde{u},\tilde{\nu})$ is a steady-state Young measure solution of (3.1.0.1)–(3.1.0.8), with $\tilde{u} \in \bigtimes_{i=1}^m W_0^{1,p_i}(\Omega)$ and $\tilde{\nu} = (\tilde{\nu}_x)_{x\in\Omega}$ a Young measure if the following are satisfied:

1. For all $\phi \in W_0^{1,q}(\Omega;\mathbb{R}^m)$ we have

$$
\int_\Omega \langle\tilde{\nu}_x, a\rangle : D\phi(x) - B\tilde{u}(x)\cdot\phi(x)\ \mathrm{d}x = \int_\Omega \tilde{F}(x)\cdot\phi(x)\ \mathrm{d}x,
$$

2. For almost every $x \in \Omega$ we have that

$$
D\tilde{u}(x) = \int_{\mathbb{R}^{m\times n}} \xi\ \mathrm{d}\tilde{\nu}_x(\xi).
$$

Remark 3.4.2.6. *In this subsection we will in fact only study the behaviour under suitable decay properties of F. It is hoped that in the future we will be able to deal with right-hand* *sides which do not decay to zero.*

We follow the ideas in [65], Lemma 1.6.2. Given the Young measure solution pair (u, ν) from Theorem 3.3.1.1 we take a strictly monotonic increasing divergent sequence t_k of positive real numbers and define a sequence (u^k, ν^k), where $u^k(t, x) = u(t + t_k, x)$ and $\nu^k_{t,x} = \nu_{t+t_k,x}$.

Lemma 3.4.2.7. *There exists a subsequence of k (not re-labelled), and a function*

$$\tilde{u} \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^m)) \cap \bigtimes_{i=1}^{m} L^{p_i}(0, T; W_0^{1,p_i}(\Omega))$$

with

$$\partial_t \tilde{u} \in L^{q'}(0, T; W^{-1,q'}(\Omega; \mathbb{R}^m)),$$

for which we have the following convergence results:

$$\begin{aligned} u^k &\rightharpoonup \tilde{u} \text{ in } \bigtimes_{i=1}^{m} L^{p_i}(0, T; W_0^{1,p_i}(\Omega)), \\ u^k &\overset{*}{\rightharpoonup} \tilde{u} \text{ in } L^\infty(0, T; L^2(\Omega; \mathbb{R}^m)), \\ \partial_t u^k &\rightharpoonup \partial_t \tilde{u} \text{ in } L^{q'}(0, T; W^{-1,q'}(\Omega; \mathbb{R}^m) \end{aligned}$$

for any $T > 0$. Furthermore there is a further subsequence of k (not re-labelled) and a parameterised family of probability measures $\tilde{\nu} = \{\tilde{\nu}_{t,x}\}$ such that

$$\begin{aligned} \nu^k &\overset{*}{\rightharpoonup} \tilde{\nu} \text{ in } L^\alpha_w(Q_T; E'_{q-1}), \\ \langle \nu^k, a \rangle &\rightharpoonup \langle \tilde{\nu}, a \rangle \text{ in } L^{q'}(Q_T; \mathbb{R}^{m\times n}) \end{aligned}$$

for any $T > 0$.

Proof. It is easily seen that

$$\|u^k\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))} \leq \|u\|_{L^\infty(0,T+t_k;L^2(\Omega;\mathbb{R}^m))},$$

and

$$\|u_i^k\|_{L^{p_i}(0,T;W_0^{1,p_i}(\Omega))} \leq \|u_i\|_{L^{p_i}(0,T+t_k;W_0^{1,p_i}(\Omega))}$$

for all $i = 1, \dots, m$. Also we have

$$
\begin{aligned}
&\|\partial_t u^k\|_{L^{q'}(0,T;W^{-1,q'}(\Omega;\mathbb{R}^m))} \\
&\quad = \|\partial_t u\|_{L^{q'}(t_k,T+t_k;W^{-1,q'}(\Omega;\mathbb{R}^m))} \\
&\quad = \sup_{\phi \in L^q(t_k,T+t_k;W_0^{1,q}(\Omega;\mathbb{R}^m))} \frac{\int_{t_k}^{T+t_k}\int_\Omega \partial_t u(t,x)\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t}{\|\phi\|_{L^q(t_k,T+t_k;W_0^{1,q}(\Omega;\mathbb{R}^m))}} \\
&\quad \le \sup_{\phi \in L^q(t_k,T+t_k;W_0^{1,q}(\Omega;\mathbb{R}^m))} \frac{\|\langle \nu, a\rangle\|_{L^{q'}(Q_{T+t_k};\mathbb{R}^{m\times n})}\|\phi\|_{L^q(t_k,T+t_k;W_0^{1,q}(\Omega;\mathbb{R}^m))}}{\|\phi\|_{L^q(t_k,T+t_k;W_0^{1,q}(\Omega;\mathbb{R}^m))}} \\
&\quad = \|\langle \nu, a\rangle\|_{L^{q'}(Q_{T+t_k};\mathbb{R}^{m\times n})}.
\end{aligned}
$$

In addition,

$$
\|\langle \nu^k, a\rangle\|_{L^{q'}(Q_T;\mathbb{R}^{m\times n})} \le \|\langle \nu, a\rangle\|_{L^{q'}(Q_{T+t_k};\mathbb{R}^{m\times n})}.
$$

Thus we must identify the weak limit of $\langle \nu^k, a\rangle$. To do this we show that there is a subsequence of the measures converging weakly-star to some measure $\tilde{\nu}$. In what follows, a superscript j will denote an element of the sequence defined by $\nu^j_{t,x} = \nu_{t+t_j,x}$, whereas a superscript $h_{k_r(r)}(r)$ will denote an element of the diagonal sequence constructed in the proof of Theorem 3.4.2.2. We compute:

$$
\begin{aligned}
\|\nu^j\|_{L^\alpha(Q_T;E'_{q-1})} &\le \left(\int_0^T \int_\Omega |\langle \nu^j_{t,x}, 1 + |\cdot|^{q-1}\rangle|^\alpha \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{\alpha}} \\
&= \left(\int_0^T \int_\Omega |\langle \nu_{t+t_j,x}, 1 + |\cdot|^{q-1}\rangle|^\alpha \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{\alpha}} \\
&= \left(\int_{t_j}^{T+t_j} \int_\Omega |\langle \nu_{t,x}, 1 + |\cdot|^{q-1}\rangle|^\alpha \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{\alpha}} \\
&\le \liminf_{r\to\infty} \left(\int_{t_j}^{T+t_j} \int_\Omega |Du^{h_{k_r(r)}(r)}|^{(q-1)\alpha} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{\alpha}} + |Q_T|^{\frac{1}{\alpha}},
\end{aligned}
$$

which can be bounded independently of j. Thus there is a subsequence j_m and a measure $\tilde{\nu}$ such that for any $T > 0$ we have $\nu \in L^\alpha_w(Q_T; E'_{q-1})$ and

$$
\nu^{j_m} \overset{*}{\rightharpoonup} \tilde{\nu} \text{ in } L^\alpha_w(Q_T; E'_{q-1}).
$$

As before this then allows us to identify the weak limit of $\langle \nu^j, a\rangle$. The fact that $\tilde{\nu}$ is a

Young measure follows from weak-$*$ lower-semicontinuity, combined with the fact that the measure generated in the proof of Theorem 3.3.1.1 is a Young measure. $\square$

Lemma 3.4.2.8. *The function $\tilde{u}$ and the measure $\tilde{\nu}$ defined in the statement of Lemma 3.4.2.7 satisfy*

$$D\tilde{u}(t,x) = \int_{\mathbb{R}^{m\times n}} \xi \ \mathrm{d}\tilde{\nu}_{t,x}(\xi)$$

for almost every $(t,x) \in Q_T$.

Proof. From the proof of Theorem 3.4.2.2 it follows that for each k we have

$$Du(t+t_k,x) = \int_{\mathbb{R}^{m\times n}} \xi \ \mathrm{d}\nu_{t+t_k,x}(\xi)$$

for almost every $(t,x) \in Q_T$. By the convergence results established in Lemma 3.4.2.7 we have that for every $\phi \in C_0^\infty(Q_T;\mathbb{R}^{m\times n})$ the following two limits hold:

$$\int_0^T \int_\Omega Du(t+t_k,x) : \phi(t,x) \ \mathrm{d}x \ \mathrm{d}t \to \int_0^T \int_\Omega D\tilde{u}(t,x) : \phi(t,x) \ \mathrm{d}x \ \mathrm{d}t,$$

and

$$\int_0^T \int_\Omega \left(\int_{\mathbb{R}^{m\times n}} \xi \ \mathrm{d}\nu_{t+t_k,x}(\xi) \right) : \phi(t,x) \ \mathrm{d}x \ \mathrm{d}t \to \int_0^T \int_\Omega \left(\int_{\mathbb{R}^{m\times n}} \xi \ \mathrm{d}\tilde{\nu}_{t,x}(\xi) \right) : \phi(t,x) \ \mathrm{d}x \ \mathrm{d}t,$$

where we have used that the identity function componentwise is in E_{q-1} in order to obtain the convergence of the term involving the measures. Thus by the uniqueness of the weak limits and the fact that T was chosen arbitrarily we have the desired equality. $\square$

Now we are in a position to study $\tilde{\nu}$. We have the following proposition, which was inspired by Theorems 1.3.9 and 1.3.10 in [65].

Proposition 3.4.2.9. *The measure $\tilde{\nu}$ satisfies*

$$\int_0^T \int_\Omega \langle \tilde{\nu}_{t,x}, a \rangle : D\phi(t,x) \ \mathrm{d}x \ \mathrm{d}t = 0$$

for all $\phi \in L^q(0,T;W_0^{1,q}(\Omega;\mathbb{R}^m))$.

Proof. We begin by considering $\phi \in C_0^\infty([0,T]; W_0^{1,q}(\Omega;\mathbb{R}^m))$ and perform the following compuations:

$$\begin{aligned}
\int_0^T \int_\Omega \langle \tilde{\nu}_{t,x}, a\rangle : D\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t &= \lim_{k\to\infty} \int_0^T \int_\Omega \langle \nu^k, a\rangle : D\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{k\to\infty} \int_{t_k}^{T+t_k} \int_\Omega \langle \nu_{t,x}, a\rangle : D\phi(t-t_k,x)\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{k\to\infty}\lim_{r\to\infty} \int_{t_k}^{T+t_k} \int_\Omega a(Du^{h_{k_r(r)}(r)}(t,x)) : D\phi(t-t_k,x)\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{k\to\infty}\lim_{r\to\infty} \int_{t_k}^{T+t_k} \int_\Omega F(t,x)\cdot\phi(t-t_k,x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \lim_{k\to\infty}\lim_{r\to\infty} \int_{t_k}^{T+t_k} \int_\Omega \partial_t u^{h_{k_r(r)}(r)}(t,x)\cdot\phi(t-t_k,x)\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{k\to\infty}\lim_{r\to\infty} \int_{t_k}^{T+t_k} \int_\Omega F(t,x)\cdot\phi(t-t_k,x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \lim_{k\to\infty}\lim_{r\to\infty} \int_\Omega u^{h_{k_r(r)}(r)}(T+t_k,x)\cdot\phi(T,x)\,\mathrm{d}x \\
&\quad + \lim_{k\to\infty}\lim_{r\to\infty} \int_\Omega u^{h_{k_r(r)}(r)}(t_k,x)\cdot\phi(0,x)\,\mathrm{d}x \\
&\quad - \lim_{k\to\infty}\lim_{r\to\infty} \int_{t_k}^{T+t_k} \int_\Omega u^{h_{k_r(r)}(r)}(t,x)\cdot\partial_t\phi(t-t_k,x)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

As we have chosen $\phi \in C([0,T]; W_0^{1,q}(\Omega;\mathbb{R}^m))$, the second and third integrals on the final line above are zero. For the first integral on the final line above, we estimate

$$\begin{aligned}
\left| \int_{t_k}^{T+t_k} \int_\Omega F(t,x)\cdot\phi(t-t_k,x)\,\mathrm{d}x\,\mathrm{d}t \right| &\le \|F\|_{L^2((t_k,T+t_k)\times\Omega;\mathbb{R}^m)} \|\phi\|_{L^2(Q_T;\mathbb{R}^m)} \\
&\le C\|F\|_{L^\infty(t_k,T+t_k;L^2(\Omega;\mathbb{R}^m))} \|\phi\|_{L^2(Q_T;\mathbb{R}^m)}.
\end{aligned}$$

Using the assumed properties of F, we have that the right-hand side of the above inequality converges to zero as $k \to \infty$. Finally, for the fourth integral, we perform the limit passage in r using the weak-$*$ convergence of $u^{h_{k_r(r)}(r)}$ to u in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$ for any $T > 0$, and we estimate

$$\begin{aligned}
\left| \int_{t_k}^{T+t_k} \int_\Omega u(t,x)\cdot\partial_t\phi(t-t_k,x)\,\mathrm{d}x\,\mathrm{d}t \right| &\le \|u\|_{L^\infty(t_k,T+t_k;L^2(\Omega;\mathbb{R}^m))} \|\partial_t\phi\|_{L^2(Q_T;\mathbb{R}^m)} \\
&\le \liminf_{r\to\infty} \|u^{h_{k_r(r)}(r)}\|_{L^\infty(t_k,T+t_k;L^2(\Omega;\mathbb{R}^m))} \|\partial_t\phi\|_{L^2(Q_T;\mathbb{R}^m)}
\end{aligned}$$

$$\leq \sup_{t\in(t_k,T+t_k)} |y(t)|_2 \|\partial_t\phi\|_{L^2(Q_T;\mathbb{R}^m)},$$

where $y(t)$ is, as in the previous subsection, a continuous non-negative function whose each component y_i satisfies $y_i'(t) + y_i(t)^{p_i-1} = \|F_i(t,\cdot)\|_{L^2(\Omega)}$ subject to $y_i(0) = \|u_{i,0}\|_{L^2(\Omega)}$. By results from the previous subsection it therefore holds that $\lim_{k\to\infty}\sup_{t\in(t_k,T+t_k)} |y(t)|_2 = 0$. Therefore it holds that

$$\int_0^T \int_\Omega \langle \tilde{\nu}_{t,x}, a\rangle : D\phi(t,x) \, \mathrm{d}x \, \mathrm{d}t = 0$$

for all $\phi \in C_0^\infty([0,T]; W_0^{1,q}(\Omega;\mathbb{R}^m))$.

We now extend to $L^q(0,T;W_0^{1,q}(\Omega;\mathbb{R}^m))$ functions. The set of linear combinations of $C_0^\infty([0,T];W_0^{1,q}(\Omega;\mathbb{R}^m))$ functions is dense in $L^q(0,T;W_0^{1,q}(\Omega;\mathbb{R}^m))$. Therefore, given for any $\phi \in L^q(0,T;W_0^{1,q}(\Omega;\mathbb{R}^m))$ there exists a sequence $\Phi_v(t,x) := \sum_{w=1}^v \phi_w(t,x)$ converging stongly to ϕ in the space $L^q(0,T;W_0^{1,q}(\Omega;\mathbb{R}^m))$, where $\phi_w \in C_0^\infty([0,T];W_0^{1,q}(\Omega;\mathbb{R}^m))$. Then we have

$$\begin{aligned}\int_0^T \int_\Omega \langle \tilde{\nu}_{t,x}, a\rangle : D\phi \, \mathrm{d}x \, \mathrm{d}t &= \int_0^T \int_\Omega \langle \tilde{\nu}_{t,x}, a\rangle : (D\phi - D\Phi_v) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_0^T \int_\Omega \langle \tilde{\nu}_{t,x}, a\rangle : D\Phi_v \, \mathrm{d}x \, \mathrm{d}t \\ &\leq \|\langle \tilde{\nu}, a\rangle\|_{L^{q'}(Q_T;\mathbb{R}^m)} \|D\phi - D\Phi_v\|_{L^q(Q_T;\mathbb{R}^m)},\end{aligned}$$

which converges to 0 as $v \to \infty$. $\square$

Combining this with Proposition 3.4.2.4, we obtain the following result:

Theorem 3.4.2.10. *Let $p \geq 2$, let $\mu_i = 0$ for $i = 1,\ldots,m$ and let (u,ν) be a global Young measure solution to* (3.1.0.1)−(3.1.0.5)*, with the mapping a satisfying* (3.1.0.6) − (3.1.0.8)*, and with*

$$F \in L^2(Q_\infty;\mathbb{R}^m) \cap C([0,\infty);L^2(\Omega;\mathbb{R}^m)) \cap \bigtimes_{i=1}^m L^{p_i}(Q_\infty)$$

and $\|F(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^m)} \to 0$ as $t \to \infty$. Then, for any strictly monotone increasing divergent *sequence of positive real numbers* $\{t_j\}_{j\in\mathbb{N}}$ *there is a subsequence (not relabelled), and a measure* $\tilde{\nu}$*, for which the sequence* (u^j, ν^j) *converges weakly star to* $(0, \tilde{\nu})$*, where* $\tilde{\nu}$ *satisfies*

$$\int_0^T \int_\Omega \langle \tilde{\nu}_{t,x}, a\rangle : D\phi \,\mathrm{d}x\,\mathrm{d}t = 0$$

for all $\phi \in L^q(0,T; W_0^{1,q}(\Omega;\mathbb{R}^m))$ *and for any* $T > 0$*.*

3.5 Extensions to more general nonlinearities

There are nonlinearities that one can consider which are of a more general form than those we have considered up until this point. For such nonlinearities there hold similar results to those found up until now in this chapter, and we now take some time to state these. The proofs of these statements will be almost identical to their analogues appearing previously in this chapter, and as such the proofs will be omitted where necessary.

In this section we will consider maps $a : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ satisfying the growth conditions

$$a(\xi)\cdot\xi \geq c|\xi|^p, \tag{3.5.0.2}$$

$$|a(\xi)| \leq C(1 + |\xi|^{q-1}), \tag{3.5.0.3}$$

where, as before, $p > \max\{1, \frac{2n}{n+2}\}$, $q \geq 2$ (for convenience) and $p \leq q < p+1$.

At times we will also impose the existence of a function $\psi \in C^1(\mathbb{R}^{m\times n})$ such that

$$a(\xi) = \nabla\psi(\xi). \tag{3.5.0.4}$$

By the growth assumptions (3.5.0.2) and (3.5.0.3) we see that we then have that ψ satisfies

$$c_1|\xi|^p - c_2 \leq \psi(\xi) \leq C(1 + |\xi|^q). \tag{3.5.0.5}$$

3.5.1 Existence results

We start with the analogue to Theorem 3.3.1.1:

Theorem 3.5.1.1. *Let the assumptions of Theorem 3.3.1.1 hold, but with* (3.5.0.2) *and* (3.5.0.3) *in place of* (3.1.0.6), (3.1.0.7) *and* (3.1.0.8). *Also, let* $F \in L^2(Q_T;\mathbb{R}^m) \cap L^{p'}(Q_T;\mathbb{R}^m)$. *Then there exists a Young measure solution to* (3.1.0.1)−(3.1.0.5), *but with the space of test functions being given by* $L^{\frac{p}{p-q+1}}(0,T;W_0^{1,\frac{p}{p-q+1}}(\Omega;\mathbb{R}^m))$, *and*

$$u \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)),$$

with

$$\partial_t u \in L^{\frac{p}{q-1}}(0,T;W^{-1,\frac{p}{q-1}}(\Omega;\mathbb{R}^m)).$$

Proof. The proof is mostly identical to the proof of Theorem 3.3.1.1. The only differences appear in the energy estimates and in the space in which we can bound the sequence $a(Du^h)$. The energy estimate we obtain is

$$\|u^h\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))} + \|u^h\|^p_{L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))} \le C\left(\|F\|^{p'}_{L^{p'}(Q_T;\mathbb{R}^m)} + \|u_0\|^2_{L^2(\Omega;\mathbb{R}^m)}\right).$$

Using the growth condition (3.5.0.2) we see that we can make the following computations:

$$\begin{aligned}
\|a(Du^h)\|_{L^{\frac{p}{q-1}}(Q_T;\mathbb{R}^{m\times n})} &\le \|1 + |Du^h|^{q-1}\|_{L^{\frac{p}{q-1}}(Q_T;\mathbb{R}^{m\times n})} \\
&\le |Q_T|^{\frac{q-1}{p}} + \|Du^h\|^{q-1}_{L^p(Q_T;\mathbb{R}^{m\times n})} \\
&\le |Q_T|^{\frac{q-1}{p}} + C\left(\|F\|^{p'}_{L^{p'}(Q_T;\mathbb{R}^m)} + \|u_0\|^2_{L^2(\Omega;\mathbb{R}^m)}\right)^{q-1}.
\end{aligned}$$

From Theorem 3.2.2.2 it then follows that there is a subsequence $\{h_j\}$ and a parameterised family of probability measures ν such that

$$a(Du^h) \rightharpoonup \langle \nu, a\rangle$$

in the space $L^{\frac{p}{q-1}}(Q_T;\mathbb{R}^{m\times n})$ (cf. Remark 3.1.0.2). □

Next, an immediate analogue to Proposition 3.5.1.1 under an additional assumption:

Proposition 3.5.1.2. *Let the assumptions of Proposition 3.5.1.1 hold. In addition let* (3.5.0.4) *and* (3.5.0.5) *hold, and let* $u_0 \in W_0^{1,q}(\Omega;\mathbb{R}^m)$. *Then there exists a Young measure* *solution* (u, ν) *in the same sense as in Theorem 3.5.1.1, and the function* u *has the additional regularity*

$$u \in H^1(0, T; L^2(\Omega; \mathbb{R}^m)) \cap L^\infty(0, T; W_0^{1,p}(\Omega; \mathbb{R}^m)) \cap C([0, T]; L^2(\Omega; \mathbb{R}^m)).$$

Proof. Due to the similarity with previous results, all we do here is state the energy estimates at the Galerkin level that we would use to complete the proof (which follows in the same way as Theorem 3.3.1.1 and Proposition 3.5.1.1).

Note that in the Galerkin equation we can test against $\partial_t u^h$, and use the structural assumption (3.5.0.4) to write

$$a(Du^h) : D\partial_t u^h = \partial_t \psi(Du^h),$$

which, after applying (3.5.0.5), results in the following energy estimate:

$$\|\partial_t u^h\|^2_{L^2(Q_T;\mathbb{R}^m)} + \|Du^h\|^p_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^{m\times n}))} \leq c\left(\|F\|^2_{L^2(Q_T;\mathbb{R}^m)} + |\Omega| + \|Du^h(0,\cdot)\|^q_{L^q(\Omega;\mathbb{R}^{m\times n})}\right).$$

Combining this with the energy estimate that arises from testing against u^h instead of $\partial_t u^h$, we see that u^h has the desired regularity, with bounds uniform in h (and T). The rest of the proof then proceeds as in Proposition 3.5.1.1. □

3.5.2 Asymptotic behaviour

In the interests of brevity we highlight three of the main results which one can prove. We start by stating a theorem regarding the existence of global Young measure solutions:

Theorem 3.5.2.1. *Let the assumptions of Proposition 3.5.1.2 hold, and assume further that* $F \in L^2(Q_\infty; \mathbb{R}^m) \cap L^{p'}(Q_\infty; \mathbb{R}^m)$. *Then there is a global Young measure solution* (u, ν) *of the problem* (3.1.0.1)$-$(3.1.0.5) *in the sense that*

$$u \in H^1(0, \infty; L^2(\Omega; \mathbb{R}^m)) \cap L^\infty(0, \infty; W_0^{1,p}(\Omega; \mathbb{R}^m)) \cap C([0, \infty); L^2(\Omega; \mathbb{R}^m))$$

and

$$\nu \in L^\alpha_w(Q_T; E'_{q-1})$$

for $\alpha \in (1, \frac{p}{q-1})$, *and*

$$\int_0^T \int_\Omega \partial_t u \cdot \phi + \langle \nu_{t,x}, a \rangle : D\phi \text{ d}x \text{ d}t = \int_0^T \int_\Omega F \cdot \phi \text{ d}x \text{ d}t$$

is satisfied for every $\phi \in L^{\frac{p}{p-q+1}}(0,T; W_0^{1,\frac{p}{p-q+1}}(\Omega;\mathbb{R}^m))$ *for any* $T > 0$.

Proof. The proof is similar to the proof of Theorem 3.4.2.2, and so is omitted. □

As before, defining $(u^k(t,x), \nu^k_{t,x}) := (u(t+t_k), \nu_{t+t_k,x})$ we see that the following holds:

Proposition 3.5.2.2. *There exists a subsequence of* k *(not relabelled), a function*

$$\tilde{u} \in H^1(0,\infty; L^2(\Omega;\mathbb{R}^m)) \cap L^\infty(0,\infty; W_0^{1,p}(\Omega;\mathbb{R}^m)) \cap C([0,\infty); L^2(\Omega;\mathbb{R}^m))$$

and a Young measure

$$\tilde{\nu} \in L^\alpha_w(Q_T; E'_{q-1})$$

such that

$$\begin{aligned} &u^k \rightharpoonup \tilde{u} \text{ in } H^1(0,\infty; L^2(\Omega;\mathbb{R}^m)) \\ &u^k \overset{*}{\rightharpoonup} \tilde{u} \text{ in } L^\infty(0,\infty; W_0^{1,p}(\Omega;\mathbb{R}^m)) \\ &\nu^k \overset{*}{\rightharpoonup} \tilde{\nu} \text{ in } L^\alpha_w(Q_T; E'_{q-1}) \text{ for any } T > 0. \end{aligned}$$

Finally we can show an analogue of Theorem 3.4.2.10.

Proposition 3.5.2.3. *Let* $p \geq 2$ *and let* (u,ν) *be a global Young measure solution to* (3.1.0.1)−(3.1.0.5), *with the mapping* a *satisfying* (3.1.0.6) − (3.1.0.8), *and with*

$$F \in L^2(Q_\infty;\mathbb{R}^m) \cap C([0,\infty); L^2(\Omega;\mathbb{R}^m)) \cap L^{p'}(Q_\infty;\mathbb{R}^m)$$

and $\|F(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^m)} \to 0$ *as* $t \to \infty$. *Then, for any strictly monotone increasing divergent sequence of positive real numbers* $\{t_j\}_{j\in\mathbb{N}}$ *there is a subsequence (not relabelled), and a* *measure $\tilde{\nu}$ for which the sequence (u^j, ν^j) converges weakly star to $(0, \tilde{\nu})$, where $\tilde{\nu}$ satisfies*

$$\int_0^T \int_\Omega \langle \tilde{\nu}_{t,x}, a \rangle : D\phi \, \mathrm{d}x \, \mathrm{d}t = 0$$

for all $\phi \in L^q(0, T; W_0^{1,q}(\Omega; \mathbb{R}^m))$ and for any $T > 0$

3.6 Regularisations

To bring this chapter to a close we consider a higher-order regularisation of the problems considered up until this point, and show that given a suitable choice of regularisation and additional boundary condition we can obtain weak solutions. However, as we pass our regularisation parameter to zero we will still obtain Young measure solutions.

3.6.1 The regularisation

In this section we study systems of the following form:

$$\partial_t u^\epsilon - \mathrm{div}(a(Du^\epsilon)) + \epsilon \Delta(\Delta u^\epsilon) = 0, \tag{3.6.1.1}$$

with some initial condition

$$u^\epsilon(0, x) = u_0(x) \tag{3.6.1.2}$$

On the nonlinearity a we assume that there is a function $\psi \in C^1$ (note the lack of convexity assumption) for which we can write $a = \nabla\psi$, and we assume that there are constants existing for which the following growth rates hold:

$$|a(\xi)| \leq C(1 + |\xi|^{q-1}), \tag{3.6.1.3}$$

$$a(\xi) \cdot \xi \geq C|\xi|^p, \tag{3.6.1.4}$$

and

$$c|\xi|^p \leq \psi(\xi) \leq C(1 + |\xi|^q), \tag{3.6.1.5}$$

where as in previous sections, p and q are related by the inequality $q - p < 1$, and in this section we take $q \geq p \geq 2$. On the initial data we assume that $u_0^\epsilon \in H_0^2(\Omega;\mathbb{R}^m) \cap W_0^{1,q}(\Omega;\mathbb{R}^m)$, and that it converges strongly to some function $u_0 \in W_0^{1,q}(\Omega;\mathbb{R}^m)$ as $\epsilon \to 0$. We impose the boundary conditions

$$u^\epsilon(t,x) = Du^\epsilon(t,x) \cdot \mathbf{n} = 0 \text{ for } (x,t) \in \partial\Omega \times (0,T), \tag{3.6.1.6}$$

where $\mathbf{n}$ is the outward unit normal to Ω.

3.6.2 Existence

We first define the appropriate notion of solution:

Definition 3.6.2.1. We say that a function

$$u^\epsilon \in L^\infty(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)) \cap L^\infty(0,T;H^2(\Omega;\mathbb{R}^m)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^m))$$
$$\cap\, C([0,T];L^p(\Omega;\mathbb{R}^m))$$

is a weak solution of the system described by (3.6.1.1)–(3.6.1.6) if for all $\phi \in C_0^\infty(Q_T;\mathbb{R}^m)$ we have that

$$\int_0^T \int_\Omega \partial_t u^\epsilon \cdot \phi + a(Du^\epsilon) : D\phi + \epsilon \Delta u^\epsilon \cdot \Delta\phi \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{3.6.2.1}$$

and the initial condition is satisfied in the sense that $u^\epsilon(0,x) = u_0^\epsilon(x)$ for almost every $x \in \Omega$.

We then have the following result, giving the existence of a weak solution to the system (3.6.1.1)–(3.6.1.6).

Theorem 3.6.2.2. *There exists a weak solution to* (3.6.1.1)–(3.6.1.6) *in the sense of Definition 3.6.2.1.*

Proof. We employ the Galerkin method once more, only this time we shall use a complete orthogonal basis of $C_0^\infty(\Omega;\mathbb{R}^m)$ functions, as opposed to a finite element basis. That is, we shall write

$$u^{\epsilon,k}(t,x) := \sum_{l=1}^k \alpha^{\epsilon,l}(t)\phi^l(x),$$

where $\{\phi^k\}_{k=1}^{\infty}$ is a countable dense subset of $C_0^{\infty}(\Omega;\mathbb{R}^m)$ where all of the functions ϕ^k are pairwise orthogonal, and where the functions $\alpha^{\epsilon,l}$ satisfy

$$\int_{\Omega} \partial_t u^{\epsilon,k}(t,x)\cdot\phi^l(x) + a(Du^{\epsilon,k}(t,x)) : D\phi^l(x) + \epsilon\Delta u^{\epsilon,k}(t,x)\cdot\Delta\phi^l(x)\ \mathrm{d}x = 0.$$

As the details remain pretty much the same as in previous occasions in which we have applied this method, we shall omit the majority of the details. Let $u^{\epsilon,k}$ be a Galerkin approximation. Testing the equation against $u^{\epsilon,k}$ we get

$$\int_{\Omega} \partial_t u^{\epsilon,k}\cdot u^{\epsilon,k} + a(Du^{\epsilon,k}) : Du^{\epsilon,k} + \epsilon\Delta u^{\epsilon,k}\cdot\Delta u^{\epsilon,k}\ \mathrm{d}x\ \mathrm{d}t = 0, \tag{3.6.2.2}$$

which leads to the energy estimate

$$\|u^{\epsilon,k}\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))} + \|u^{\epsilon,k}\|^p_{L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))} + \epsilon\|D^2u^{\epsilon,k}\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n\times m}))} \leq \|u_0\|^2_{L^2(\Omega;\mathbb{R}^m)}. \tag{3.6.2.3}$$

There is therefore a function

$$u^\epsilon \in L^2(0,T;H_0^2(\Omega;\mathbb{R}^m)) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m)) \cap L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$$

such that $u^{\epsilon,k} \overset{*}{\rightharpoonup} u^\epsilon$ in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$, and

$$u^{\epsilon,k} \rightharpoonup u^\epsilon \text{ in } L^2(0,T;H_0^2(\Omega;\mathbb{R}^m)) \cap L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)).$$

From the energy estimates and the growth rate assumed on a it follows that we can bound

$$\|a(Du^{\epsilon,k})\|_{L^{\frac{p}{q-1}}(Q_T;\mathbb{R}^{m\times n})} \leq C(|Q_T|, \|u_0\|_{L^2(\Omega;\mathbb{R}^m)}),$$

and as such that is a subsequence of $u^{\epsilon,k}$ (not relabelled) and a function $\chi \in L^{\frac{p}{q-1}}(Q_T;\mathbb{R}^{m\times n})$ such that $a(Du^{\epsilon,k}) \rightharpoonup \chi$ in $L^{\frac{p}{q-1}}(Q_T;\mathbb{R}^{m\times n})$. For the time derivative, testing our Galerkin equation against

$$\phi \in L^2(0,T;H_0^2(\Omega;\mathbb{R}^m)) \cap L^{\frac{p}{p-q+1}}(0,T;W_0^{1,\frac{p}{p-q+1}}(\Omega;\mathbb{R}^m))$$

functions allows us to bound the sequence of time derivatives uniformly:

$$\partial_t u^{\epsilon,k} \in L^2(0,T;H^{-2}(\Omega;\mathbb{R}^m)) \cap L^{\frac{p}{q-1}}(0,T;W^{-1,\frac{p}{q-1}}(\Omega;\mathbb{R}^m)),$$

and as such we get that

$$\partial_t u^{\epsilon,k} \rightharpoonup \partial_t u^{\epsilon} \text{ in } L^2(0,T;H^{-2}(\Omega;\mathbb{R}^m)) \cap L^{\frac{p}{q-1}}(0,T;W^{-1,\frac{p}{q-1}}(\Omega;\mathbb{R}^m)).$$

However by the assumption on the initial condition, as well as the remark that

$$a(Du^{\epsilon,k}) : D\partial_t u^{\epsilon,k} = \partial_t \psi(Du^{\epsilon,k}),$$

we can test our equation against $\partial_t u^{\epsilon,k}$ to get

$$\int_0^T \int_\Omega \partial_t u^{\epsilon,k} \cdot \partial_t u^{\epsilon,k} + \partial_t \psi(Du^{\epsilon,k}) + \frac{\epsilon}{2}\partial_t(|\Delta u^{\epsilon,k}|^2) \, \mathrm{d}x \, \mathrm{d}t = 0, \tag{3.6.2.4}$$

from which we obtain the following estimate

$$\begin{aligned}\|\partial_t u^{\epsilon,k}\|^2_{L^2(Q_T;\mathbb{R}^m)} + \|Du^{\epsilon,k}\|^p_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^{m\times n}))} + \epsilon\|u^{\epsilon,k}\|^2_{L^\infty(0,T;H^2_0(\Omega;\mathbb{R}^m))} \\ \leq C\left(|\Omega| + \|Du_0^{\epsilon,k}\|^q_{L^q(\Omega;\mathbb{R}^{m\times n})} + \epsilon\|u_0^{\epsilon,k}\|^2_{H^2(\Omega;\mathbb{R}^m)}\right).\end{aligned}$$

From this it follows that we can take a subsequence (again not relabelled) such that

$$\begin{aligned}\partial_t u^{\epsilon,k} &\rightharpoonup \partial_t u^{\epsilon} \text{ in } L^2(Q_T;\mathbb{R}^m),\\ u^{\epsilon,k} &\overset{*}{\rightharpoonup} u^{\epsilon} \text{ in } L^\infty(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)),\\ u^{\epsilon,k} &\overset{*}{\rightharpoonup} u^{\epsilon} \text{ in } L^\infty(0,T;H_0^2(\Omega;\mathbb{R}^m)).\end{aligned}$$

Also, by applying the Aubin-Lions Lemma, using that $u^{\epsilon,k} \in L^\infty(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$ and $\partial_t u^{\epsilon,k} \in L^2(0,T;H^{-2}(\Omega;\mathbb{R}^m))$ to get that there is a subsequence (not relabelled) such that

$$u^{\epsilon,k} \to u^{\epsilon} \text{ in } C([0,T];L^p(\Omega;\mathbb{R}^m)).$$

We now recall some facts about embedding of Sobolev spaces. For $n \geq 3$ we have that $H_0^2(\Omega;\mathbb{R}^m)$ is compactly contained in $W_0^{1,\alpha}(\Omega;\mathbb{R}^m)$ for any $1 \leq \alpha < 2^* := \frac{2n}{n-2}$. For $n = 1, 2$ we have that $H^2(\Omega;\mathbb{R}^m)$ is compactly contained in $W^{1,2}(\Omega;\mathbb{R}^m)$. Now applying the Aubin−Lions Lemma we see that there is a subsequence (not re-labelled) for which we have $Du^{\epsilon,k} \to Du^\epsilon$ strongly in $L^2(0,T;L^2(\Omega;\mathbb{R}^m))$. Therefore we may extract a further subsequence (which we once again do not re-label) such that we have $Du^{\epsilon,k} \to Du^\epsilon$ for almost every point (t,x). By the continuity of the map a this then implies that $a(Du^{\epsilon,k}) \to a(Du^\epsilon)$ almost everywhere. We also have by the boundedness of $a(Du^{\epsilon,k})$ in $L^{q'}(Q_T;\mathbb{R}^{m\times n})$ that the family of function $\{a(Du^{\epsilon,k})\}_{k\in\mathbb{N}}$ is uniformly integrable. The set Ω is bounded and so Q_T is a finite measure space. Finally by the growth rates on a and fact that u^ϵ is in some Sobolev space we see that $|a(Du^\epsilon)| < \infty$ almost everywhere. Therefore we may apply Theorem 3.2.3.5 to see that $a(Du^{\epsilon,k}) \to a(Du^\epsilon)$ in $L^1(Q_T;\mathbb{R}^{m\times n})$. By uniqueness of weak limits we must therefore have that $\chi = a(Du^\epsilon)$, and therefore passing to the limit we have that

$$\int_0^T \int_\Omega \partial_t u^\epsilon \cdot \phi + a(Du^\epsilon) : D\phi + \epsilon \Delta u^\epsilon \cdot \Delta\phi \,\mathrm{d}x\,\mathrm{d}t = 0$$

for all $\phi \in L^2(0,T;H_0^2(\Omega;\mathbb{R}^m)) \cap L^{\frac{p}{p-q+1}}(0,T;W_0^{1,\frac{p}{p-q+1}}(\Omega;\mathbb{R}^m))$. Verifying that the initial data is satisfied is done similarly to previous existence results. □

3.6.3 Passing the regularisation parameter to zero

We now pass $\epsilon \to 0$, showing the existence of a Young measure solution with higher regularity than was obtained in Theorem 3.3.1.1.

Proposition 3.6.3.1. *There is a subsequence ϵ_k, a function*

$$u \in L^\infty(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^m))$$

and a parameterised family of Young measures ν such that the pair (u,ν) is a Young measure solution to

$$\partial_t u - \operatorname{div}(a(Du)) = 0,$$

with

$$u(0,x) = u_0(x)$$

with $u_0 \in W_0^{1,q}(\Omega; \mathbb{R}^m)$ *and*

$$u(t,x) = 0 \textit{ for } (t,x) \in [0.T] \times \partial\Omega,$$

with (u, ν) *being such that*

$$u^{\epsilon_k} \rightharpoonup u \textit{ in } L^\infty(0,T; W_0^{1,p}(\Omega; \mathbb{R}^m)) \cap H^1(0,T; L^2(\Omega; \mathbb{R}^m))$$

and the measure ν *is generated by* Du^{ϵ_k}*.*

Proof. We start with the term involving the Laplacian. We have that

$$\begin{aligned}
\left|\int_0^T \int_\Omega \epsilon \Delta u^\epsilon \cdot \Delta\phi \;\mathrm{d}x\;\mathrm{d}t\right| &= \left|\int_0^T \int_\Omega \sqrt{\epsilon}\Delta u^\epsilon \cdot \sqrt{\epsilon}\Delta\phi \;\mathrm{d}x\;\mathrm{d}t\right| \\
&\leq \sqrt{\epsilon}\|u^\epsilon\|_{L^2(0,T;H_0^2(\Omega;\mathbb{R}^m))}\sqrt{\epsilon}\|\phi\|_{L^2(0,T;H_0^2(\Omega;\mathbb{R}^m))} \\
&\leq \sqrt{\epsilon}\liminf_{k\to\infty}\|u^{\epsilon,k}\|_{L^2(0,T;H_0^2(\Omega;\mathbb{R}^m))}\sqrt{\epsilon}\|\phi\|_{L^2(0,T;H_0^2(\Omega;\mathbb{R}^m))} \\
&\leq C\liminf_{k\to\infty}\left(\|Du_0^{\epsilon,k}\|^{\frac{q}{2}}_{L^q(\Omega;\mathbb{R}^{m\times n})} + \sqrt{\epsilon}\|u_0^{\epsilon,k}\|_{H_0^2(\Omega;\mathbb{R}^m)}\right) \\
&\quad \times \sqrt{\epsilon}\|\phi\|_{L^2(0,T;H_0^2(\Omega;\mathbb{R}^m)))} \\
&\leq C\sqrt{\epsilon}\|\phi\|_{L^2(0,T;H_0^2(\Omega;\mathbb{R}^m)))},
\end{aligned}$$

from which it follows that for all $\phi \in L^2(0,T; H_0^2(\Omega; \mathbb{R}^m)))$ we have that

$$\lim_{\epsilon\to 0}\int_0^T \int_\Omega \epsilon\Delta u^\epsilon \cdot \Delta\phi \;\mathrm{d}x\;\mathrm{d}t = 0.$$

Passage to the limit in the other terms is carried out similarly to in the proof of Theorem 3.3.1.1, just with different regularity.

We have in particular that the sequence Du^ϵ is uniformly bounded in the space $L^p(Q_T; \mathbb{R}^{m\times n}))$. This, combined with the growth rates assumed on a, gives that $a(Du^\epsilon)$ is uniformly bounded in the space $L^{\frac{p}{q-1}}(Q_T; \mathbb{R}^{m\times n}))$, and so there is a subsequence of u^ϵ, denoted by u^{ϵ_k} and a function $\chi \in L^{\frac{p}{q-1}}(Q_T; \mathbb{R}^{m\times n}))$ for which $a(Du^{\epsilon_k}) \rightharpoonup \chi$ in

$L^{\frac{p}{q-1}}(Q_T;\mathbb{R}^{m\times n}))$. Once more we appeal to the remarks made following the statement of Theorem 3.2.2.2 in order to identify this limit. The sequence of functions Du^{ϵ_k} generates a family of Young measures $\nu_{t,x}$ such that $a(Du^{\epsilon_k}) \rightharpoonup \langle \nu, a\rangle$ in $L^{\alpha}(Q_T;\mathbb{R}^{m\times n}))$, where $1 < \alpha < \frac{p}{q-1}$. Therefore by the uniqueness of weak limits we have that $\chi = \langle \nu, a\rangle$. As in the proof of Theorem 3.3.1.1, we can also consider the sequence $f(Du^{\epsilon_k}$ where f is the identity function to show that $Du(t,x) = \langle \nu_{t,x}, \xi\rangle$.

Note that as Du^{ϵ} is uniformly bounded in $L^{\infty}(0,T;L^p(\Omega;\mathbb{R}^{m\times n})))$ we have that there is a subsequence (not further relabelled) and a function u such that

$$u^{\epsilon_k} \overset{*}{\rightharpoonup} u \text{ in } L^{\infty}(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)).$$

Similar calculations to before then give that $\partial_t u^{\epsilon_k} \rightharpoonup \partial_t u$ in $L^2(Q_T;\mathbb{R}^m))$, and so by the Aubin-Lions Lemma it follows that there is a further subsequence (not further relabelled) such that $u^{\epsilon_k} \to u$ in the space $C([0,T];L^2(\Omega;\mathbb{R}^m)))$.

In addition to this we have that for all $\phi \in L^{\frac{p}{p-q+1}}(0,T;W_0^{1,\frac{p}{p-q+1}}(\Omega;\mathbb{R}^m)))$ we have that

$$\int_0^T \int_{\Omega} \partial_t u \cdot \phi + \langle \nu_{t,x}, a\rangle : D\phi \, \mathrm{d}x \, \mathrm{d}t = 0.$$

Showing that the initial condition is satisfied is done in the same way as before and will not be detailed again here. □

As we have seen, this gives Young measure solutions which have a higher regularity than those obtained in the proof of Theorem 3.3.1.1. Regularising the system in this way could provide us with a selection criteria for Young measure solutions (at least, in the event that the solutions constructed during the proof of Theorem 3.3.1.1 do not also possess this highger regularity). However, regularising the system of PDEs in the way in which we have done so here may not always have physical relevance.

Chapter 4

Numerical construction of Young measure solutions

In this chapter we shall discuss an algorithm which shall allow for the numerical construction of certain Young measure solutions which appeared in the previous chapter. The main references for this chapter shall be [35] and [36], in which the authors develop and analyse numerical schemes to compute measure-valued solutions for systems of hyperbolic conservation laws. In this chapter we show that their main algorithm can be adapted for our systems of parabolic equations. Content in this chapter can be found in [15]. Before proceeding to the first section of this chapter, we first make a remark. Throughout this chapter, we shall be taking subsequences to obtain a Young measure solutions, for which we have not shown any uniqueness result. As taking different subsequences could result in different Young measure solutions, and we have no notion of uniqueness, this makes the results in this chapter difficult to use for actual computations. What follows in this chapter is therefore another method to approximate Young measure solutions. We use the term “construction” as opposed to “computation” to indicate this, and we keep in mind that we are not computing a particular Young measure solution, nor are we approximating a unique Young measure solution.

4.1 Preliminary definitions and results

We collect here some results which shall be useful to us in the remainder of this chapter. The majority of the results discussed in this section will deal with ways of expressing Young measures which are more suitable to computations than what has already been considered.

4.1.1 Young measures

We discuss some properties of Young measures which will be found to be useful when we consider the numerical computation of the Young measure solutions. These results and definitions can be found in (or are inspired by) Appendix A of [35].

Definition 4.1.1.1. We let $\mathcal{P}(\mathbb{R}^m)$ denote the set of probability measures on $\mathbb{R}^m$, and define the subset $\mathcal{P}^p(\mathbb{R}^m)$ (for $1 \leq p < \infty$) to be the set of probability measures μ for which $\langle \mu, |\xi|^p \rangle < \infty$. On this subspace, we define the *p-Wasserstein metric* W_p as

$$W_p(\mu, \rho) := \inf \left\{ \int_{\mathbb{R}^m \times \mathbb{R}^m} |\xi - \eta|^p \, \mathrm{d}\pi(\xi, \eta) \;:\; \pi \in \Pi(\mu, \rho) \right\}^{\frac{1}{p}},$$

where $\Pi(\mu, \rho)$ is the set of probability measures on $\mathbb{R}^m \times \mathbb{R}^m$ with marginals μ and ρ:

$$\Pi(\mu, \rho) := \{\pi \in \mathcal{P}(\mathbb{R}^m \times \mathbb{R}^m) : \; \pi(B \times \mathbb{R}^m) = \mu(B), \; \pi(\mathbb{R}^m \times B) = \rho(B)\},$$

where B is any Borel set.

We let $(X, \mathcal{F}, P)$ be a probability space and let $u : X \times Q_T \to \mathbb{R}^m$ be a random field (that is, a jointly measurable function). Given this, we can define the law of u as follows:

$$\mu_{t,x}(B) := P(\{\omega \in X : \; u(w; t, x) \in B\}), \tag{4.1.1.1}$$

for Borel subsets $B \subset \mathbb{R}^m$. It is clear that $\mu(\mathbb{R}^m) = 1$. We have the following result, which is a standard result in probability theory:

Lemma 4.1.1.2. *Let* $(\omega; t, x) \in X \times Q_T \mapsto u(\omega; t, x) \in \mathbb{R}^m$ *be a random field; then,* (4.1.1.1) *is equivalent to*

$$\langle \mu_{t,x}, g\rangle = \int_X g(u(\omega; t, x))\, \mathrm{d}P(\omega) \qquad \textit{for a.e. } (t, x) \in Q_T,$$

for every continuous r*-component vector function* g*, defined on* $\mathbb{R}^m$*, which is such that* $\int_X |g(u(\omega; t, x))|\, \mathrm{d}P(\omega)$ *is finite for a.e.* $(t, x) \in Q_T$*.*

Proof. As the desired equality is understood componentwise, it suffices to prove it component-by-component. We shall therefore assume in the argument below that $r = 1$, i.e., that g is a continuous mapping from $\mathbb{R}^m$ into $\mathbb{R}$ such that $\int_X |g(u(\omega; t, x))|\, \mathrm{d}P(\omega)$ is finite for a.e. $(t, x) \in Q_T$. Let us first consider the case when the function $g : \mathbb{R}^m \to \mathbb{R}$ is nonnegative, and take a sequence of simple functions g_n defined on $\mathbb{R}^m$ by

$$g_n(\xi) = \sum_{k=1}^{n2^n+1} b_k^n \chi_{A_k^n}(\xi), \qquad \xi \in \mathbb{R}^m, \quad n = 1, 2, \dots,$$

where A_k^n, $k = 1, \dots, n$, are disjoint measurable subsets of $\mathbb{R}^m$, n is a positive integer, and $b_k^n \in \mathbb{R}$, $k = 1, \dots, n$, are such that $g_n(\xi)$ increases to $g(\xi)$. This can be achieved by defining

$$A_k^n := g^{-1}\left(\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)\right), \qquad 1 \leq k \leq n2^n,$$

and

$$A_{n,n2^n+1} := g^{-1}([n, \infty)).$$

Observe that $g_n(\xi) \leq g(\xi)$ for all $\xi \in \mathbb{R}^m$. Further, for a given $\xi \in \mathbb{R}^m$ we have that $g_n(\xi) = n$ for $g(\xi) \geq n$ whereas for $g(\xi) < n$ we have that

$$g_n(\xi) = \frac{1}{2^n}\lfloor 2^n g(\xi)\rfloor,$$

where $\lfloor y\rfloor$ denotes the greatest integer $\leq y$. As $\lfloor 2^{n+1}g(\xi)\rfloor \geq 2\lfloor 2^n g(\xi)\rfloor$, it follows that $g_{n+1}(\xi) \geq g_n(\xi)$ for all $\xi \in \mathbb{R}^m$. Furthermore, as $\lfloor 2^n g(\xi)\rfloor \geq 2^n g(\xi) - 1$, we have that $g_n(\xi) \geq g(\xi) - 2^{-n}$ as soon as $n > g(\xi)$. Since $g(\xi) \geq g_n(\xi)$ it follows that $\lim_{n\to\infty} g_n(\xi) = g(\xi)$ for all $\xi \in \mathbb{R}^m$.

Now, let $\mathcal{F}_k^n := \{\omega \in X : \ u(\omega; t, x) \in A_k^n\}$. We observe that

$$\chi_{\mathcal{F}_k^n}(\omega) = \begin{cases} 1 & \text{for } \omega \in \mathcal{F}_k^n \\ 0 & \text{for } \omega \notin \mathcal{F}_k^n \end{cases} = \begin{cases} 1 & \text{for } u(\omega; t, x) \in A_k^n \\ 0 & \text{for } u(\omega; t, x) \notin A_k^n \end{cases} = \chi_{A_k^n}(u(\omega; t, x)).$$

Hence we have that

$$\begin{aligned} \langle \mu_{t,x}, g \rangle &= \int_{\mathbb{R}^m} g(\xi)\, \mathrm{d}\mu_{t,x}(\xi) = \lim_{n\to\infty} \int_{\mathbb{R}^m} g_n(\xi)\, \mathrm{d}\mu_{t,x}(\xi) = \lim_{n\to\infty} \sum_{k=1}^{n2^n+1} b_k^n \mu_{t,x}(A_k^n) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n2^n+1} b_k^n P(\mathcal{F}_k^n) = \lim_{n\to\infty} \int_X \sum_{k=1}^{n2^n+1} b_k^n \chi_{\mathcal{F}_k^n}(\omega)\, \mathrm{d}P(\omega) \\ &= \lim_{n\to\infty} \int_X \sum_{k=1}^{n2^n+1} b_k^n \chi_{A_k^n}(u(\omega; t, x))\, \mathrm{d}P(\omega) \\ &= \lim_{n\to\infty} \int_X g_n(u(\omega; t, x))\, \mathrm{d}P(\omega) = \int_X g(u(\omega; t, x))\, \mathrm{d}P(\omega), \end{aligned}$$

where we have applied the Monotone Convergence Theorem in the first and last lines to exchange the limit with the integral over $\mathbb{R}^m$. From here the extension to any, not necessarily nonnegative, function $g : \mathbb{R}^m \to \mathbb{R}$ such that $\int_X |g(u(\omega; t, x))|\, \mathrm{d}P(\omega)$ is finite for a.e. $(t, x) \in Q_T$, is straightforward: we decompose g as $g = g^+ - g^-$, where $g^+ := \frac{1}{2}(|g| + g)$ and $g^- := \frac{1}{2}(|g| - g)$ are (respectively) the positive and negative part of g, and apply the above argument to g^+ and g^- separately, noting that both are nonnegative on $\mathbb{R}^m$. □

Remark 4.1.1.3. *We can easily extend this to include functions of the form $g(t, x, u(\omega, t, x))$ which are continuous in the third variable and measurable in the first two.*

We have the following proposition, which shows that the law $\mu_{t,x}$ defined above is in fact a Young measure. This is Proposition A.2 from [35]:

Proposition 4.1.1.4. *If $u : X \times Q_T \to \mathbb{R}^m$ is jointly measurable then the law μ defines a Young measure.*

Proof. The proof is the same as that in [35]. □

Finally we consider the following result adapted from [35], found therein as Proposition A.3:

Proposition 4.1.1.5. *For every Young measure $\{\sigma_z\}_{z\in\Omega} \subset \mathcal{P}^{q-1}$ on $\mathbb{R}^m$ there exists a probability space $(X, \mathcal{F}, P)$ with $P \in \mathcal{P}^{q-1}$ and a Borel measurable function $u : X \times \Omega \to \mathbb{R}^m$ such that u has law σ. In particular we may choose the probability space to be the Lebesgue measure on the interval $[0, 1)$ with the Borel σ-algebra.*

Proof. The proof of this result relies on using the characterization given in Lemma 4.1.1.2. In [35] a proof is supplied with $g \in C_0(\mathbb{R}^m)$; we extend this result below to cover our particular class of Young measures. We give the proof in the case of scalar-valued functions u; the extension to vector-valued functions follows by a component-wise argument.

To extend Proposition 2 in Appendix 1 of [35] from $g \in C_0(\mathbb{R})$ to $g \in C(\mathbb{R})$ with $\int_X |g(u(\omega; t, x))| \, \mathrm{d}P(\omega) < \infty$ for a.e. $(t, x) \in Q_T$, let $\varphi \in C_0^\infty(\mathbb{R})$ be such that $0 \leq \varphi \leq 1$, $\varphi(\xi) \equiv 1$ for $|\xi| \leq 1$ and $\varphi(\xi) \equiv 0$ for $|\xi| \geq 2$. Let $g_k(x) := g(x)\varphi(x/k)$. Clearly, g_k is a sequence of functions in $C_0(\mathbb{R})$, $g_k(\xi) = g(\xi)$ if $|\xi| \leq k$, and g_k converges pointwise to g on $\mathbb{R}$ as $k \to \infty$. Furthermore, $|g_k(x)| \leq |g(x)|$ for all $x \in \mathbb{R}$. Thus, thanks to the Dominated Convergence Theorem, we have that

$$\int_X g(u(\omega; x)) \, \mathrm{d}P(\omega) = \lim_{k\to\infty} \int_X g_k(u(\omega; x)) \, \mathrm{d}P(\omega) = \lim_{k\to\infty} \int_{\mathbb{R}} g_k(\xi) \, \mathrm{d}\nu_x(\xi) = \int_{\mathbb{R}} g(\xi) \, \mathrm{d}\nu_x(\xi).$$

We remark that progressing from the left-hand side of the second equality above to the right-hand side of that equality is precisely the result of *Proposition 2* in Appendix 1 of [35]. □

4.1.2 Miscellaneous statements

Finally, we state and prove a standard lemma regarding independent and identically distributed random variables, and their images under measurable functions.

Lemma 4.1.2.1. *Let $(X, \mathcal{F}, P)$ be a probability space and let $f : X \to \mathbb{R}^m$ be a measurable function. If two random variables on X, Y_1 and Y_2, are independent and identically distributed, then the random variables $f(Y_1)$ and $f(Y_2)$ are independent and identically distributed.*

Proof. We first prove that the random variables $f(Y_1)$ and $f(Y_2)$ are identically distributed. We let $\mathcal{A} \in \mathbb{R}^m$ be a measurable set. It then follows from the measurability of f

that

$$\begin{aligned} P(f(Y_1) \in \mathcal{A}) &= P(Y_1 \in f^{-1}(\mathcal{A})) \\ &= P(Y_2 \in f^{-1}(\mathcal{A})) \\ &= P(f(Y_2) \in \mathcal{A}). \end{aligned}$$

To show independence, we let $\mathcal{A}_1$ and $\mathcal{A}_2$ be measurable sets and calculate

$$\begin{aligned} P\left((f(Y_1) \in \mathcal{A}_1) \cap (f(Y_2) \in \mathcal{A}_2)\right) &= P\Big((Y_1 \in f^{-1}(\mathcal{A}_1)) \cap (Y_2 \in f^{-1}(\mathcal{A}_2))\Big) \\ &= P(Y_1 \in f^{-1}(\mathcal{A}_1))P(Y_2 \in f^{-1}(\mathcal{A}_2)) \\ &= P(f(Y_1) \in \mathcal{A}_1)P(f(Y_2) \in \mathcal{A}_2). \end{aligned}$$

□

4.2 Time discretisation

During the proof of Theorem 3.3.1.1, Peano's Theorem was utilised in order to show that one could solve the Galerkin equation (3.3.0.2a). As the eventual goal of this chapter is to be able to compute these Young measure solutions explicitly, one would ideally wish for a numerical scheme to approximate the solutions to (3.3.0.2a). We discuss such a scheme in this section.

4.2.1 Convergence of a backwards Euler method

The scheme we employ is the usual backward Euler method. Starting from our discretised initial condition u_0^h, we inductively define a sequence u_{i+1}^h as the solutions to the following time-independent problems:

$$\int_\Omega \frac{u_{i+1}^h - u_i^h}{\Delta t} \cdot \phi^h + a(Du_{i+1}^h) : D\phi^h - Bu_{i+1}^h \cdot \phi^h \, \mathrm{d}x = \int_\Omega F_{i+1} \cdot \phi^h \, \mathrm{d}x \tag{4.2.1.1}$$

for all $\phi^h \in V_m^h$. Here, Δt denotes the time step, chosen such that the sequence of points $0 = t_0, t_1, \ldots, t_N = T$ with $t_i = i\Delta t$ form a partition of the interval $[0, T]$ on which we wish to solve the system. Clearly, $T = N\Delta t$. We must also discretise our right-hand side F. We define

$$F_i(x) := \frac{1}{\Delta t}\int_{t_{i-1}}^{t_i} F(t,x)\ \mathrm{d}t,$$

and the function

$$\tilde{F}_{\Delta t}(t,x) := F_i(x),\ \text{ for } t \in (t_{i-1}, t_i].$$

Before starting with the convergence analysis of this scheme we must first check that the scheme is well defined, meaning that given u_i^h, there necessarily exists a function u_{i+1}^h satisfying (4.2.1.1). The existence of such a function u_{i+1}^h will require the use of the following proposition (see, for example, Corollary 1.1 on p.279 of [40]).

Proposition 4.2.1.1. *Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function, and suppose that there exists a ρ such that $\langle g(\xi), \xi\rangle \geq 0$ for $|\xi| \geq \rho$. Then there exists a ξ_0 with $|\xi_0| \leq \rho$ such that $g(\xi_0) = 0$.*

We use this proposition to prove the following result, showing that given u_i^h we can determine a function u_{i+1}^h satisfying (4.2.1.1):

Proposition 4.2.1.2. *For a given function F_{i+1} and a function $u_i^h \in V_m^h$ there exists a function $u_{i+1}^h \in V_m^h$ satisfying*

$$\int_\Omega u_{i+1}^h \cdot \phi^h + \Delta t\, a(Du_{i+1}^h) : D\phi^h - \Delta t\, Bu_{i+1}^h \cdot \phi^h\ \mathrm{d}x = \int_\Omega (\Delta t\, F_{i+1} + u_i^h)\cdot \phi^h\ \mathrm{d}x. \quad (4.2.1.2)$$

Proof. We follow [57] and apply Proposition 4.2.1.1 to the function g whose components g_j are given by

$$\begin{aligned} g_j(\alpha^h) = \Delta t \int_\Omega a(\alpha_1^h D\phi_1^h + \ldots + \alpha_{\mathcal{N}(h)}^h D\phi_{\mathcal{N}(h)}^h) : D\phi_j^h\ \mathrm{d}x \\ + \int_\Omega (I - \Delta t\, B)(\alpha_1^h \phi_1^h + \ldots + \alpha_{\mathcal{N}(h)}^h \phi_{\mathcal{N}(h)}^h)\cdot \phi_j^h - (\Delta t\, F_{i+1} + u_i^h)\cdot \phi_j^h\ \mathrm{d}x, \end{aligned}$$

where $\alpha^h = (\alpha_1^h, \dots, \alpha_{\mathcal{N}(h)}^h)$ and we set

$$u_{i+1}^h := \sum_{j=1}^{\mathcal{N}(h)} \alpha_j^h \phi_j^h$$

for $\phi_j^h \in \mathcal{B}^h$. Clearly the function g_j is continuous for each $j = 1, \dots, \mathcal{N}(h)$. Taking the inner product of g and α we get

$$\begin{aligned}
\langle g(\alpha), \alpha \rangle = \Delta t &\int_\Omega a(\alpha_1^h D\phi_1^h + \cdots + \alpha_{\mathcal{N}(h)}^h D\phi_{\mathcal{N}(h)}^h) : (\alpha_1^h D\phi_1^h + \cdots + \alpha_{\mathcal{N}(h)}^h D\phi_{\mathcal{N}(h)}^h)\,\mathrm{d}x \\
&+ \int_\Omega (I - \Delta t\, B)(\alpha_1^h \phi_1^h + \cdots + \alpha_{\mathcal{N}(h)}^h \phi_{\mathcal{N}(h)}^h) \cdot (\alpha_1^h \phi_1^h + \cdots + \alpha_{\mathcal{N}(h)}^h \phi_{\mathcal{N}(h)}^h)\,\mathrm{d}x \\
&- \int_\Omega (\Delta t\, F_{i+1} + u_i^h) \cdot (\alpha_1^h \phi_1^h + \cdots + \alpha_{\mathcal{N}(h)}^h \phi_{\mathcal{N}(h)}^h)\,\mathrm{d}x \\
= \Delta t &\int_\Omega a(Du_{i+1}^h) : Du_{i+1}^h + (u_{i+1}^h)^2\,\mathrm{d}x - \int_\Omega (\Delta t\, F_{i+1} + u_i^h) \cdot u_{i+1}^h\,\mathrm{d}x \\
\geq c_1 \Delta t &\sum_{j=1}^m \|u_{i+1,j}^h\|_{L^{p_j}(\Omega)}^{p_j} + \frac{1}{2}\|u_{i+1}^h\|_{L^2(\Omega;\mathbb{R}^m)}^2 - \frac{1}{2}\|u_i^h\|_{L^2(\Omega;\mathbb{R}^m)}^2 \\
&- c_2(|\Omega|, p_1, \dots, p_m)\Delta t \sum_{j=1}^m \varepsilon^{\frac{-1}{p_j - 1}} \|F_{i+1,j}\|_{L^{p_j{}'}(\Omega)}^{p_j{}'} \\
&- c_2(|\Omega|, p_1, \dots, p_m)\varepsilon \Delta t \sum_{j=1}^m \|u_{i+1,j}^h\|_{L^{p_j}(\Omega)}^{p_j},
\end{aligned}$$

where ε is such that $c_1 - c_2\varepsilon > 0$. The final line of the above inequality is larger than zero provided that we take $|\alpha|$ to be sufficiently large. Thus by Proposition 4.2.1.1 we have a solution of (4.2.1.2) as required. □

We can now discuss the convergence analysis. We begin with the following estimate, which is reminiscent of the energy estimates obtained in the proof of Theorem 3.3.1.1.

Lemma 4.2.1.3. *There exists a positive constant c, independent of h and Δt, such that the functions u_i^h, $i = 1, \dots, N$, satisfy, for all $h \in (0, h_0]$,*

$$\begin{aligned}
\max_{i=1,\dots,N} \|u_i^h\|_{L^2(\Omega;\mathbb{R}^m)}^2 + \sum_{i=1}^N \int_\Omega |u_i^h - u_{i-1}^h|^2\,\mathrm{d}x + \Delta t \sum_{i=1}^N \int_\Omega a(Du_i^h) : Du_i^h\,\mathrm{d}x \\
\leq c(\|u_0^h\|_{L^2(\Omega;\mathbb{R}^m)}^2 + \|F\|_{L^2(Q_T;\mathbb{R}^m)}^2).
\end{aligned} \tag{4.2.1.3}$$

Proof. We begin by choosing $\phi^h = u^h_{i+1}$ in (4.2.1.1), and noting the following identity:

$$\frac{(a-b)\cdot a}{\Delta t} = \frac{|a|^2 - |b|^2}{2\Delta t} + \frac{|a-b|^2}{2\Delta t} \qquad \forall\, a, b \in \mathbb{R}^m.$$

Using this in (4.2.1.1) gives

$$\int_\Omega \frac{|u^h_{i+1}|^2 - |u^h_i|^2}{2\Delta t} + \frac{|u^h_{i+1} - u^h_i|^2}{2\Delta t} + a(Du^h_{i+1}) : Du^h_{i+1} + Bu^h_{i+1}\cdot u^h_{i+1}\,\mathrm{d}x = \int_\Omega F_{i+1}\cdot u^h_{i+1}\,\mathrm{d}x.$$

Now we use that $Bv\cdot v \geq 0$ for all $v \in \mathbb{R}^m$, multiply by $2\Delta t$ and sum through $i = 0, \dots, M$ for some $M < N$ to get

$$\begin{aligned}\int_\Omega |u^h_{M+1}|^2\,\mathrm{d}x + \sum_{i=0}^{M}\int_\Omega |u^h_{i+1} - u^h_i|^2\,\mathrm{d}x + 2\Delta t\sum_{i=0}^{M}\int_\Omega a(Du^h_{i+1}) : Du^h_{i+1}\,\mathrm{d}x \\ \leq \int_\Omega |u^h_0|^2\,\mathrm{d}x + 2\sum_{i=0}^{M}\int_\Omega\left(\int_{t_i}^{t_{i+1}} F(t,x)\,\mathrm{d}t\right)\cdot u^h_{i+1}\,\mathrm{d}x \\ \leq \int_\Omega |u^h_0|^2\,\mathrm{d}x + 2\|F\|^2_{L^2(Q_T;\mathbb{R}^m)} + \frac{1}{2}\max_{i=1,\dots,N}\|u^h_i\|^2_{L^2(\Omega;\mathbb{R}^m)}.\end{aligned} \tag{4.2.1.4}$$

By omitting the second and the third term from the left-hand side of the inequality (4.2.1.4) noting the independence of its right-hand side of M, and then taking the maximum over $M = 1, \dots, N-1$, yields the desired bound on the first term on the left-hand side of the inequality (4.2.1.3). We then return with that bound to the inequality (4.2.1.4) to further estimate its right-hand side from above, whilst omitting the first term from the left-hand side of (4.2.1.4); thus we arrive at the desired bound on the second and third term on the left-hand side of the inequality (4.2.1.3). The expression on the right-hand side of (4.2.1.3) is clearly independent of Δt. We remark that the right-hand side of the above ineuqality is also independent of h by the assumed strong convergence of u^h_0 to u_0 in $L^2(\Omega;\mathbb{R}^m)$. □

Note that the functions u^h_i, $i = 0, \dots, N$, are only defined for the discrete times $\{t_0 = 0, \dots, t_i = i\Delta t, \dots, t_N = T\}$, as opposed to being defined on the whole interval $[0, T]$. We therefore define the following two interpolation functions:

$$u^h_{\Delta t}(t) := \frac{t - t_{i-1}}{\Delta t}u^h_i + \frac{t_i - t}{\Delta t}u^h_{i-1} \quad \text{for } t \in [t_{i-1}, t_i],$$

which is continuous and piecewise linear in time, and

$$\tilde{u}^h_{\Delta t}(t) = u^h_i \quad \text{for } t \in [t_{i-1}, t_i],$$

which is piecewise constant in time. With respect to these two new functions, (4.2.1.1) can be expressed as:

$$\int_\Omega \partial_t u^h_{\Delta t} \cdot \phi^h + a(D\tilde{u}^h_{\Delta t}) : D\phi^h - B\tilde{u}^h_{\Delta t} \cdot \phi^h \, \mathrm{d}x = \int_\Omega \tilde{F}_{\Delta t} \cdot \phi^h \, \mathrm{d}x \tag{4.2.1.5}$$

for all $\phi^h \in V^h_m$. We aim now to obtain suitable bounds on the functions defined above, which will allow us to complete a passage to the limit in Δt, possibly along some subsequence.

Lemma 4.2.1.4. *There exists a positive constant c, independent of h and Δt, such that, for all $h \in (0, h_0]$,*

$$\|\tilde{u}^h_{\Delta t}\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))} \leq c(\|u^h_0\|^2_{L^2(\Omega;\mathbb{R}^m)} + \|F\|^2_{L^2(Q_T;\mathbb{R}^m)}),$$

$$\|\tilde{u}^h_{\Delta t} - \tilde{u}^h_{\Delta t}(\cdot - \Delta t)\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^m))} \leq c(\|u^h_0\|^2_{L^2(\Omega;\mathbb{R}^m)} + \|F\|^2_{L^2(Q_T;\mathbb{R}^m)}),$$

$$\|a(D\tilde{u}^h_{\Delta t})\|^{\hat{q}'}_{L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n})} \leq c\left(1 + \|u^h_0\|^2_{L^2(\Omega;\mathbb{R}^m)} + \|F\|^2_{L^2(Q_T;\mathbb{R}^m)} + \sum_{i=1}^m \|F_i\|^{p_i'}_{L^{p_i'}(Q_T)}\right),$$

$$\sum_{i=1}^m \|D(\tilde{u}^h_{\Delta t})_i\|^{p_i}_{L^{p_i}(0,T;L^{p_i}(\Omega;\mathbb{R}^n))} \leq c\left(1 + \|u^h_0\|^2_{L^2(\Omega;\mathbb{R}^m)} + \sum_{i=1}^m \|F_i\|^{p_i'}_{L^{p_i'}(Q_T)}\right).$$

We recall that $\hat{q} := \max\{q, 2\}$. Furthermore, the right-hand sides of these inequalities are bounded above by a positive constant, independent of $h \in (0, h_0]$ and Δt.

Proof. The first two bounds follow from the bounds on the first two terms in (4.2.1.3), while the fourth inequality is proved by an argument that is analogous to the proof of the bound on the second term appearing on the left-hand side of (3.3.1.4). To prove the third statement, we begin by noting that since $a(\xi) : \xi \geq 0$ for all $\xi \in \mathbb{R}^{m\times n}$, we have that

$$\begin{aligned} |a(Du^h_i)|^{q'} &= |a(Du^h_i) : a(Du^h_i)|^{\frac{q'}{2}} \\ &= \left|\left(a(Du^h_i) : Du^h_i\right) K(Du^h_i)\right|^{\frac{q'}{2}} \end{aligned}$$

$$= \left(a(Du_i^h) : D\tilde{u}_{\Delta t}^h\right)^{\frac{q'}{2}} |K(Du_i^h)|^{\frac{q'}{2}}.$$

First suppose that $q > 2$; then, $\hat{q} = q$ and $\hat{q}' = q' \in (1, 2)$; hence, by Young's inequality,

$$|a(Du_i^h)|^{\hat{q}'} = |a(Du_i^h)|^{q'} \leq \frac{q'}{2} a(Du_i^h) : Du_i^h + \frac{2-q'}{2} |K(Du_i^h)|^{\frac{q'}{2-q'}}.$$

Therefore, by (4.2.1.3) and (3.1.0.7), we have that

$$\begin{aligned}
\int_0^T \int_\Omega |a(D\tilde{u}_{\Delta t}^h)|^{\hat{q}'} \, \mathrm{d}x \, \mathrm{d}t &= \int_0^T \int_\Omega |a(D\tilde{u}_{\Delta t}^h)|^{q'} \, \mathrm{d}x \, \mathrm{d}t = \Delta t \sum_{i=1}^N \int_\Omega |a(Du_i^h)|^{q'} \, \mathrm{d}x \\
&\leq \frac{q'}{2} \Delta t \sum_{i=1}^N \int_\Omega a(Du_i^h) : Du_i^h \, \mathrm{d}x + \frac{2-q'}{2} \Delta t \sum_{i=1}^N \int_\Omega |K(Du_i^h)|^{\frac{q'}{2-q'}} \, \mathrm{d}x \\
&\leq c(q) \left(\|u_0^h\|^2_{L^2(\Omega;\mathbb{R}^m)} + \|F\|^2_{L^2(\Omega;\mathbb{R}^m)} \right) \\
&\quad + c(q) \Delta t \sum_{i=1}^N \int_\Omega \left(c_1 \sum_{j=1}^m \left(|\mu_j|^2 + |Du_{i,j}^h|^2 \right)^{\frac{p_j - 2}{2}} \right)^{\frac{q'}{2-q'}} \mathrm{d}x \\
&\leq c(q) \left(\|u_0^h\|^2_{L^2(\Omega;\mathbb{R}^m)} + \|F\|^2_{L^2(Q_T;\mathbb{R}^m)} \right) \\
&\quad + c(c_1, q, m) \Delta t \sum_{i=1}^N \int_\Omega \left[\sum_{j=1}^m |\mu_j|^{\frac{(p_j-2)q}{q-2}} + \sum_{\substack{j=1 \\ p_j \geq 2}}^m |Du_{i,j}^h|^{\frac{(p_j-2)q}{q-2}} \right] \mathrm{d}x.
\end{aligned}$$

As $q := \max\{p_1, \ldots, p_m\} > 2$, for any $j \in \{1, \ldots, m\}$ such that $p_j \geq 2$ we have that $\frac{(p_j-2)q}{q-2} < p_j$, from which it follows that

$$\begin{aligned}
\int_0^T \int_\Omega |a(D\tilde{u}_{\Delta t}^h)|^{\hat{q}'} \, \mathrm{d}x \, \mathrm{d}t &\leq c(q) \left(\|u_0^h\|^2_{L^2(\Omega;\mathbb{R}^m)} + \|F\|^2_{L^2(Q_T;\mathbb{R}^m)} \right) \\
&\quad + c(c_1, q, m) \, T \, |\Omega| \sum_{j=1}^m |\mu_j|^{\frac{(p_j-2)q}{q-2}} \\
&\quad + c(q, m, c_1, p_1, \ldots, p_m, |\Omega|) \, \Delta t \sum_{i=1}^N \int_\Omega \left(\sum_{j=1}^m |Du_{i,j}^h|^{p_j} \right) \mathrm{d}x.
\end{aligned}$$

The final term on the right-hand side of this inequality is then bounded analogously to the corresponding term in the proof of Theorem 3.3.1.1, and as such we omit the details (cf. (3.3.1.4)):

$$\Delta t \sum_{i=1}^N \int_\Omega \left(\sum_{j=1}^m |Du_{i,j}^h|^{p_j} \mathrm{d}x \right) \leq c \Big(1 + \|u_0^h\|^2_{L^2(\Omega;\mathbb{R}^m)} + \sum_{i=1}^m \|F_i\|^{p_i'}_{L^{p_i'}(Q_T)} \Big).$$

Substituting this into the right-hand side of the previous inequality completes the proof of the lemma in the case when $\hat{q} = q > 2$. If $\hat{q} = 2$ (and therefore $\hat{q}' = 2$), then $p_j \in (1, 2]$ for all $j = 1, \dots, m$. It then follows from (3.1.0.7) that

$$0 \le K(A) \le c_1 \sum_{i=1}^{m} |\mu_i|^{p_i - 2} \qquad \forall\, A \in \mathbb{R}^{m \times n},$$

whereby

$$|a(Du_i^h)|^{\hat{q}'} = |a(Du_i^h)|^2 = \left(a(Du_i^h) : D\tilde{u}_{\Delta t}^h\right) K(Du_i^h) \le c_1 \left(\sum_{i=1}^{m} |\mu_i|^{p_i-2}\right) \left(a(Du_i^h) : D\tilde{u}_{\Delta t}^h\right).$$

Hence, by Lemma 4.2.1.3,

$$\begin{aligned}
\int_0^T \int_\Omega |a(D\tilde{u}_{\Delta t}^h)|^{\hat{q}'} \,\mathrm{d}x\,\mathrm{d}t &= \Delta t \sum_{i=1}^{N} \int_\Omega |a(Du_i^h)|^{q'} \,\mathrm{d}x \\
&\le c_1 \left(\sum_{i=1}^{m} |\mu_i|^{p_i-2}\right) \Delta t \sum_{i=1}^{N} \int_\Omega a(Du_i^h) : Du_i^h \,\mathrm{d}x \\
&\le c(c_1, p_1, \dots, p_m, \mu_1, \dots, \mu_m, m, T) \left(\|u_0^h\|^2_{L^2(\Omega;\mathbb{R}^m)} + \|F\|^2_{L^2(\Omega;\mathbb{R}^m)}\right).
\end{aligned}$$

The final assertion in the statement of the lemma follows from the assumed strong convergence of u_0^h to u_0 as $h \to 0_+$. $\square$

Next, we prove the required uniform bounds on the sequence $\{u_{\Delta t}^h\}$.

Lemma 4.2.1.5. *There exists a positive constant c, independent of h and Δt, such that, for all $h \in (0, h_0]$,*

$$\|u_{\Delta t}^h\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))} \le c(\|u_0^h\|^2_{L^2(\Omega;\mathbb{R}^m)} + \|F\|^2_{L^2(Q_T;\mathbb{R}^m)});$$

furthermore, with $\hat{q} := \max\{q, 2\}$,

$$\|\partial_t u_{\Delta t}^h\|_{L^{\hat{q}'}(0,T;W^{-1,\hat{q}'}(\Omega;\mathbb{R}^m))} \le c$$

and

$$(\Delta t)^{-1} \|\tilde{u}_{\Delta t}^h - \tilde{u}_{\Delta t}^h(\cdot - \Delta t)\|_{L^{\hat{q}'}(0,T;W^{-1,\hat{q}'}(\Omega;\mathbb{R}^m))} \le c.$$

Proof. The first inequality is just a restatement of the bound on the first term in the inequality (4.2.1.3) in terms of $u^h_{\Delta t}$, and the third inequality is a restatement of the second inequality in terms of $\tilde{u}^h_{\Delta t}$. For the second inequality, the argument proceeds along the same lines as the proof of (3.3.1.7). For each $\varphi \in L^{\hat{q}}(0,T;W^{1,\hat{q}}_0(\Omega;\mathbb{R}^m))$ we have, using the stability in $W^{1,\hat{q}}_0(\Omega;\mathbb{R}^m)$ of the $L^2(\Omega;\mathbb{R}^m)$ orthogonal projector $P^h : L^2(\Omega;\mathbb{R}^m) \to V^h_m$ (cf. (3.3.0.1)), that

$$\begin{aligned}
\int_0^T (\partial_t u^h_{\Delta t}, \varphi)\,\mathrm{d}t = \int_0^T (\partial_t u^h_{\Delta t}, P^h\varphi)\,\mathrm{d}t &= \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_\Omega \frac{u^h_{i+1} - u^h_i}{\Delta t} \cdot P^h\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&= -\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_\Omega a(Du^h_{i+1}) : DP^h\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} (F_{i+1} - Bu^h_{i+1}) \cdot P^h\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^T (a(D\tilde{u}^h_{\Delta t}(t)), DP^h\varphi)\,\mathrm{d}t + \int_0^T (\tilde{F}_{\Delta t} - B\tilde{u}^h_{\Delta t}, P^h\varphi)\,\mathrm{d}t \\
&\le \|a(D\tilde{u}^h_{\Delta t})\|_{L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n})} \|D\varphi\|_{L^{\hat{q}}(Q_T;\mathbb{R}^{m\times n})} \\
&\quad + \left(\|\tilde{F}_{\Delta t}\|_{L^2(Q_T;\mathbb{R}^m)} + |B|\|\tilde{u}^h_{\Delta t}\|_{L^2(Q_T;\mathbb{R}^m)}\right)\|\varphi\|_{L^2(Q_T;\mathbb{R}^m)} \\
&\le c(\|D\varphi\|_{L^{\hat{q}}(Q_T;\mathbb{R}^{m\times n})} + \|\varphi\|_{L^{\hat{q}}(Q_T;\mathbb{R}^m)}) \le c\|\varphi\|_{L^{\hat{q}}(0,T;W^{1,\hat{q}}(\Omega;\mathbb{R}^m))},
\end{aligned}$$

where in the transition to the last line we have used the third and the first bound from Lemma 4.2.1.4, together with the assumed strong convergence of u^h_0 to u_0 in $L^2(\Omega;\mathbb{R}^m)$ to deduce that the constant c is independent of h and Δt. Dividing through both sides of the above inequality by $\|\varphi\|_{L^{\hat{q}}(0,T;W^{1,\hat{q}}(\Omega;\mathbb{R}^m))}$, taking the supremum over all $\varphi \in L^{\hat{q}}(0,T;W^{1,\hat{q}}_0(\Omega;\mathbb{R}^m))$ and recalling the definition of the norm of the dual space, we obtain the desired inequality. $\square$

Now we study the convergence of the functions $\tilde{u}^h_{\Delta t}$ in the following Lemma.

Lemma 4.2.1.6. *There exists a subsequence of Δt, labelled Δt_k, and a function $u^h \in C([0,T];V^h_m)$ such that*

$$u^h_{\Delta t_k} \to u^h \text{ in } C([0,T];L^2(\Omega;\mathbb{R}^m)),$$

and

$$\partial_t u^h_{\Delta t_k} \rightharpoonup \partial_t u^h \text{ in } L^{\hat{q}'}(0,T;W^{-1,\hat{q}'}(\Omega;\mathbb{R}^m))$$

for any fixed $h \in (0, h_0]$*, as* $k \to \infty$*.*

Proof. We begin by noting that, for any Δt and any $h \in (0, h_0]$, the functions $u^h_{\Delta t}$ are continuous in both t and x. By the first two bounds in Lemma 4.2.1.5 there exists a subsequence of Δt, labelled Δt_k, and a function $u^h \in L^\infty(0,T;V^h_m)$ such that $u^h_{\Delta t_k} \overset{*}{\rightharpoonup} u^h$ in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$, and $\partial_t u^h_{\Delta t_k} \rightharpoonup \partial_t u^h$ in $L^{\hat{q}'}(0,T;W^{-1,\hat{q}'}(\Omega;\mathbb{R}^m))$. By Lemma 3.2.3.2, $u^h \in C_w([0,T];L^2(\Omega;\mathbb{R}^m))$ and $u^h_{\Delta t_k} \to u^h$ in $C_w([0,T];L^2(\Omega;\mathbb{R}^m))$. As both $u^h_{\Delta t}(t,\cdot)$ and $u^h(t,\cdot)$ belong to the finite-dimensional space V^h_m for all $t \in [0,T]$, it follows that $u^h \in C([0,T];L^2(\Omega;\mathbb{R}^m))$ and $u^h_{\Delta t_k} \to u^h$ in $u^h \in C([0,T];L^2(\Omega;\mathbb{R}^m))$.. $\square$

Next, we study the convergence of the sequence of functions $\{\tilde{u}^h_{\Delta t}\}$ with $h \in (0, h_0]$ fixed.

Lemma 4.2.1.7. *There exists a function* $\tilde{u}^h \in L^\infty(0,T;V^h_m)$ *and a subsequence* Δt_k *such that*

$$\tilde{u}^h_{\Delta t_k} \rightharpoonup \tilde{u}^h \text{ in } \bigtimes_{j=1}^{m} L^{p_j}(0,T;W^{1,p_j}_0(\Omega)) \text{ and } \tilde{u}^h_{\Delta t_k} \to \tilde{u}^h \text{ in } L^r(0,T;L^2(\Omega;\mathbb{R}^m)),$$

for any $r \in [1,\infty)$ *and fixed* $h \in (0, h_0]$*, as* $k \to \infty$*. Furthermore, along this subsequence we have* $a(D\tilde{u}^h_{\Delta t_k}) \to a(D\tilde{u}^h)$ *in* $L^s(Q_T;\mathbb{R}^{m\times n})$ *for any* $s \in [1, \hat{q}')$ *and* $a(D\tilde{u}^h_{\Delta t_k}) \rightharpoonup a(D\tilde{u}^h)$ *in* $L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n})$*, for any fixed* $h \in (0, h_0]$*, as* $k \to \infty$*.*

Proof. By the first and fourth bound from Lemma 4.2.1.4, there exists a $\tilde{u}^h \in L^\infty(0,T;V^h_m)$ and a subsequence Δt_k such that $\tilde{u}^h_{\Delta t_k} \overset{*}{\rightharpoonup} \tilde{u}^h$ in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$ and

$$\tilde{u}^h_{\Delta t_k} \rightharpoonup \tilde{u}^h \text{ in } \bigtimes_{j=1}^{m} L^{p_j}(0,T;W^{1,p_j}_0(\Omega)),$$

for any fixed $h \in (0, h_0]$, as $k \to \infty$.

By applying Theorem 1 from [28] (with $X = \bigtimes_{j=1}^m W^{1,p_j}_0(\Omega;\mathbb{R}^m)$, $B = L^2(\Omega;\mathbb{R}^m)$ and $Y = W^{-1,\hat{q}'}(\Omega;\mathbb{R}^m)$ and $r = 1$, there), we deduce that $\{\tilde{u}^h_{\Delta t_k}\}$ is relatively compact in $L^p(0,T;L^2(\Omega;\mathbb{R}^m))$, for each fixed $h \in (0, h_0]$. As this sequence is also bounded in

$L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$, by interpolation it follows that $\{\tilde{u}^h_{\Delta t_k}\}$ is also relatively compact in $L^r(0,T;L^2(\Omega;\mathbb{R}^m))$ for each fixed $h \in (0,h_0]$ and all $r \in [1,\infty)$.

Hence, in particular, $\{\tilde{u}^h_{\Delta t_k}\}$ is relatively compact in $L^1(0,T;L^1(\Omega;\mathbb{R}^m))$, for each fixed $h \in (0,h_0]$. As $\tilde{u}^h_{\Delta t_k}(t,\cdot) \in V^h_m$ for all $t \in [0,T]$, and V^h_m is finite-dimensional, for each fixed $h \in (0,h_0]$, by norm equivalence in finite-dimensional spaces it follows that $\{\tilde{u}^h_{\Delta t_k}\}$ is relatively compact in $L^1(0,T;W^{1,1}_0(\Omega;\mathbb{R}^m))$, for each fixed $h \in (0,h_0]$. Therefore $D\tilde{u}^h_{\Delta t_k} \to D\tilde{u}^h$ in $L^1(Q_T;\mathbb{R}^{m\times n})$ as $k \to \infty$, for each fixed $h \in (0,h_0]$. Thus we can extract a subsequence of k (not relabelled) such that $D\tilde{u}^h_{\Delta t_k} \to D\tilde{u}^h$ a.e. on Q_T as $k \to \infty$, for each fixed $h \in (0,h_0]$. As a is continuous, we then have that $a(D\tilde{u}^h_{\Delta t_k}) \to a(D\tilde{u}^h)$ a.e. on Q_T as $k \to \infty$, for each fixed $h \in (0,h_0]$. Since by the third inequality from Lemma 4.2.1.4 the sequence $\{a(D\tilde{u}^h_{\Delta t_k})\}$ is weakly compact in $L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n})$ and hence in particular also in $L^1(Q_T;\mathbb{R}^{m\times n})$, for each fixed $h \in (0,h_0]$, it follows by Vitali's convergence theorem that $a(D\tilde{u}^h_{\Delta t_k}) \to a(D\tilde{u}^h)$ strongly in $L^1(Q_T;\mathbb{R}^{m\times n})$ as $k \to \infty$, for each fixed $h \in (0,h_0]$, and therefore also strongly in $L^s(Q_T;\mathbb{R}^{m\times n})$ as $k \to \infty$, for each fixed $h \in (0,h_0]$ and any $s \in [1,\hat{q}')$. By the uniqueness of weak limits it then follows that $a(D\tilde{u}^h_{\Delta t_k}) \rightharpoonup a(D\tilde{u}^h)$ weakly in $L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n})$, for any fixed $h \in (0,h_0]$, as $k \to \infty$. □

We now show that the limit functions defined above are equal almost everywhere by showing that they are equal in an appropriate function space.

Lemma 4.2.1.8. *The limiting functions u^h and $\tilde{u}^h$ are equal in the space $L^2(Q_T;\mathbb{R}^m))$.*

Proof. We apply the triangle inequality to see that

$$\begin{aligned}\|u^h - \tilde{u}^h\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^m))} \leq \|u^h - u^h_{\Delta t_{k_l}}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^m))} + \|u^h_{\Delta t_{k_l}} - \tilde{u}^h_{\Delta t_{k_l}}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^m))} \\ + \|\tilde{u}^h_{\Delta t_{k_l}} - \tilde{u}^h\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^m))}.\end{aligned}$$

The first and last terms converge to zero in the limit as $\Delta t_{k_l} \to 0$ by the previously obtained convergence results. We now show that the second term also converges to zero in the limit.

$$
\begin{aligned}
\|u^h_{\Delta t_{k_l}} - \tilde{u}^h_{\Delta t_{k_l}}\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^m))} &= \int_0^T \int_\Omega |u^h_{\Delta t_{k_l}} - \tilde{u}^h_{\Delta t_{k_l}}|^2 \ \mathrm{d}x \ \mathrm{d}t \\
&= \sum_{i=1}^N \int_\Omega \int_{t_{i-1}}^{t_i} \left| \frac{t - t_{i-1} - \Delta t_{k_l}}{\Delta t_{k_l}} u^h_i + \frac{t_i - t}{\Delta t_{k_l}} u^h_{i-1} \right|^2 \ \mathrm{d}t \ \mathrm{d}x \\
&= \frac{1}{(\Delta t_{k_l})^2} \sum_{i=1}^N \int_\Omega \int_{t_{i-1}}^{t_i} (t - t_{i-1} - \Delta t_{k_l})^2 |u^h_i|^2 + (t_i - t)^2 |u^h_{i-1}|^2 \\
&\qquad\qquad + 2(t - t_{i-1} - \Delta t_{k_l})(t_i - t) u^h_i \cdot u^h_{i-1} \ \mathrm{d}t \ \mathrm{d}x \\
&= \frac{1}{(\Delta t_{k_l})^2} \sum_{i=1}^N \int_\Omega \Bigg[\frac{\Delta t_{k_l}}{3} (t_i^2 + t_i t_{i-1} + t_{i-1}^2) - t_i t_{i-1} \Delta t_{k_l} \\
&\qquad\qquad + (\Delta t_{k_l})^3 - (\Delta t_{k_l})^2 (t_i + t_{i-1} - 2t_{i-1}) \Bigg] |u^h_i|^2 \\
&\qquad\qquad + \frac{(\Delta t_{k_l})^3}{3} |u^h_{i-1}|^2 \\
&\qquad\qquad + 2 \Bigg(\frac{(\Delta t_{k_l})}{2} (t_i + t_{i-1})^2 - \frac{(t_i^3 - t_{i-1}^3)}{3} \\
&\qquad\qquad\qquad - t_i t_{i-1} \Delta t_{k_l} - \frac{(\Delta t_{k_l})^3}{2} \Bigg) u^h_i \cdot u^h_{i-1} \ \mathrm{d}x \\
&= \frac{\Delta t_{k_l}}{3} \sum_{i=1}^N \int_\Omega |u^h_i - u^h_{i-1}|^2 \ \mathrm{d}x \\
&\leq \frac{\Delta t_{k_l}}{3} \|u^h_0\|^2_{L^2(\Omega;\mathbb{R}^m)}.
\end{aligned}
$$

This gives the required convergence to zero, and thus we see that $u^h = \tilde{u}^h$. □

We now show that the limit function satisfies the initial condition.

Lemma 4.2.1.9. *The limiting function u^h satisfies the initial condition in the following sense:*

$$\lim_{t \to 0_+} \|u^h(t, \cdot) - u^h_0(\cdot)\|_{L^2(\Omega;\mathbb{R}^m)} = 0,$$

for each fixed $h \in (0, h_0]$.

Proof. By Lemma 4.2.1.6, for any $\epsilon > 0$ there is an $\ell \in \mathbb{N}$ such that

$$\|u^h(t, \cdot) - u^h_{\Delta t_{k_l}}(t, \cdot)\|_{L^2(\Omega;\mathbb{R}^m)} < \epsilon$$

for all $l \geq \ell$ and all $t \in [0,T]$. Furthermore, by the triangle inequality:

$$\|u^h(t,\cdot) - u_0^h(\cdot)\|_{L^2(\Omega;\mathbb{R}^m)} \leq \|u^h(t,\cdot) - u^h_{\Delta t_{k_l}}(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^m)} + \|u^h_{\Delta t_{k_l}}(t,\cdot) - u_0^h(\cdot)\|_{L^2(\Omega;\mathbb{R}^m)}$$

for any $l \in \mathbb{N}$. Hence, in particular, for $l = \ell$ (fixed),

$$\|u^h(t,\cdot) - u_0^h(\cdot)\|_{L^2(\Omega;\mathbb{R}^m)} \leq \|u^h(t,\cdot) - u^h_{\Delta t_{k_\ell}}(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^m)} + \epsilon.$$

As $u^h_{\Delta t_{k_\ell}}$ is a continuous function in t, passage to the limit in this inequality, using that $\lim_{t\to 0_+} u^h_{\Delta t_{k_l}}(t,x) = u_0^h(x)$, implies that, for each $\epsilon > 0$,

$$0 \leq \limsup_{t\to 0_+} \|u^h(t,\cdot) - u_0^h(\cdot)\|_{L^2(\Omega;\mathbb{R}^m)} \leq \epsilon.$$

Thus we have the asserted result. $\square$

We also need to consider the convergence of the approximating source terms $\tilde{F}_{\Delta t}$:

Lemma 4.2.1.10. *We have that* $\lim_{\Delta t\to 0_+} \|\tilde{F}_{\Delta t}(\cdot,x) - F(\cdot,x)\|_{L^r(0,T)}$ *for all* $r \in [1,\infty)$ *and a.e.* $x \in \Omega$.

Proof. The proof of this is standard and follows from the Lebesgue Differentiation Theorem. $\square$

Finally let us complete the passage to the limit in (4.2.1.5). By Proposition 4.2.1.6 and Lemma 4.2.1.7 we see that the limit function u^h satisfies

$$\int_0^T \int_\Omega \partial_t u^h \cdot \Phi^h + a(Du^h) : D\Phi^h - Bu^h \cdot \Phi^h \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega F \cdot \Phi^h \,\mathrm{d}x\,\mathrm{d}t$$

where $\Phi^h(t,x) = \sum_{i=1}^{\mathcal{N}(h)} \beta_i^h(t)\phi_i^h(x)$ for $\beta_i^h \in C([0,T];\mathbb{R})$ and $\phi_i^h \in \mathcal{B}^h$. Now one can proceed as in the proof of Theorem 3.3.1.1, obtaining the required bounds on $a(Du^h)$ and completing the passage to the limit via the Fundamental Theorem of Young Measures.

4.2.2 Uniqueness and continuous dependence on the discretised initial condition

In this subsection we consider the continuous dependence of u^h on the initial conditions in the event that a is also a locally Lipschitz function, a property which will be important to us later. Note that in the proof of Theorem 3.3.1.1, the map a being locally Lipschitz was sufficient for us to obtain a unique solution to the system of ODEs for the coefficients of the functions u^h. We start by considering two functions, u_0^h and v_0^h, and we let u_1^h and v_1^h be the solutions to (4.2.1.2) guaranteed by Proposition 4.2.1.2. Subtracting the resulting equations satisfied by u_1^h from the equation satisfied by v_1^h and choosing as the test function $\phi^h = u_1^h - v_1^h$, it follows that

$$\int_\Omega |u_1^h - v_1^h|^2 + \Delta t(a(Du_1^h) - a(Dv_1^h)) : (Du_1^h - Dv_1^h) \,\mathrm{d}x = \int_\Omega (u_0^h - v_0^h) \cdot (u_1^h - v_1^h) \,\mathrm{d}x,$$

which after applying Young's inequality to the right-hand side and absorbing terms, gives

$$\|u_1^h - v_1^h\|_{L^2(\Omega)}^2 + 2\Delta t \int_\Omega (a(Du_1^h) - a(Dv_1^h)) : (Du_1^h - Dv_1^h) \,\mathrm{d}x \leq \|u_0^h - v_0^h\|_{L^2(\Omega)}^2.$$

As the functions u_1^h and v_1^h are expressed in terms of a finite basis, we see through the equivalence of norms and Lemma 4.2.1.3 that we in fact have an L^∞ bound on u_1^h and v_1^h, and on their spatial derivatives. Therefore, letting $K_{Lip,loc}$ be the local Lipschitz constant of a, we can make use of the local Lipschitz condition to see that

$$\|u_1^h - v_1^h\|_{L^2(\Omega;\mathbb{R}^m)}^2 (1 - C(h, K_{Lip,loc})\Delta t) \leq \|u_0^h - v_0^h\|_{L^2(\Omega;\mathbb{R}^m)}^2.$$

By choosing Δt to be sufficiently small we have that

$$0 < 1 - C(h, K_{Lip,loc})\Delta t < 1,$$

and thus have the desired continuous dependence on the initial data for a single time step. By iterating this estimate we obtain that for each $i = 1, \dots, N$ the following estimate holds:

$$\|u_i^h - v_i^h\|^2_{L^2(\Omega;\mathbb{R}^m)}(1 - C(h, K_{Lip,loc})\Delta t)^i \le \|u_0^h - v_0^h\|^2_{L^2(\Omega;\mathbb{R}^m)}. \tag{4.2.2.1}$$

Note that we may take the local Lipschitz constant of a to be independent of u_i^h and v_i^h by Lemma 4.2.1.3. To transfer this dependence on the initial condition to the limiting functions u^h and v^h we consider the functions $\tilde{u}^h_{\Delta t}$ and $\tilde{v}^h_{\Delta t}$ defined, as before, as

$$\tilde{u}^h_{\Delta t}(t) = u_i^h \text{ for } t \in [t_{i-1}, t_i],$$

and

$$\tilde{v}^h_{\Delta t}(t) = v_i^h \text{ for } t \in [t_{i-1}, t_i].$$

It then follows that

$$\sup_{t\in[0,T]} \|\tilde{u}^h_{\Delta t} - \tilde{v}^h_{\Delta t}\|^2_{L^2(\Omega)}(1 - C(h, K_{Lip,loc})\Delta t)^N \le \|u_0^h - v_0^h\|^2_{L^2(\Omega)}. \tag{4.2.2.2}$$

Recall that Δt and N are related via the identity $T = N\Delta t$. Using this and basic results regarding sequences of real numbers it follows that the sequence

$$(1 - C(h, K_{Lip,loc})\Delta t)^N = \left(1 - \frac{TC(h, K_{Lip,loc})}{N}\right)^N$$

converges as $N \to \infty$ to $\mathrm{e}^{-TC(h,K_{Lip,loc})}$. We define sequences $u^h_{\Delta t}$ and $v^h_{\Delta t}$ as before:

$$u^h_{\Delta t} = \frac{t - t_{i-1}}{\Delta t} u_i^h + \frac{t_i - t}{\Delta t} u^h_{i-1} \text{ for } t \in [t_{i-1}, t_i],$$

and analogously

$$v^h_{\Delta t} = \frac{t - t_{i-1}}{\Delta t} v_i^h + \frac{t_i - t}{\Delta t} v^h_{i-1} \text{ for } t \in [t_{i-1}, t_i].$$

Let Δt_k be a sequence such that $u^h_{\Delta t_k} \to u^h$ and $v^h_{\Delta t_k} \to v^h$ in $C([0,T]; L^2(\Omega))$ and $k \to \infty$. We then have

$$
\begin{aligned}
\|u^h - v^h\|_{C([0,T];L^2(\Omega))} &\le \|u^h - u^h_{\Delta t_k}\|_{C([0,T];L^2(\Omega))} \\
&\quad + \|u^h_{\Delta t_k} - v^h_{\Delta t_k}\|_{C([0,T];L^2(\Omega))} + \|v^h_{\Delta t_k} - v^h\|_{C([0,T];L^2(\Omega))} \\
&\le \|u^h - u^h_{\Delta t_k}\|_{C([0,T];L^2(\Omega))} \\
&\quad + 2 \sup_{t\in[0,T]} \|\tilde{u}^h_{\Delta t_k} - \tilde{v}^h_{\Delta t_k}\|_{L^2(\Omega))} + \|v^h_{\Delta t_k} - v^h\|_{C([0,T];L^2(\Omega))} \\
&\le \|u^h - u^h_{\Delta t_k}\|_{C([0,T];L^2(\Omega))} \\
&\quad + \frac{2}{\left(1 - \frac{TC(h,K_{Lip,loc})}{N_k}\right)^{\frac{N_k}{2}}} \|u^h_0 - v^h_0\|_{L^2(\Omega)} + \\
&\quad + \|v^h_{\Delta t_k} - v^h\|_{C([0,T];L^2(\Omega))}.
\end{aligned}
$$

Now passing $k \to \infty$ we see that

$$
\|u^h - v^h\|_{C([0,T];L^2(\Omega))} \le 2\mathrm{e}^{\frac{TC(h,K_{Lip,loc})}{2}} \|u^h_0 - v^h_0\|_{L^2(\Omega)}.
$$

We have therefore shown the following proposition:

Proposition 4.2.2.1. *Let all of the assumptions of Theorem 3.3.1.1 hold, with the additional assumption that the initial condition* $u^h_0 \in \bigtimes_{i=1}^m W^{1,p_i}_0(\Omega)$. *Then there is a unique solution to* (3.3.0.2a) *which depends continuously on the choice of discretised initial data, and which can be constructed numerically by the approximations considered in this subsection.*

The above proposition in particular implies that the map S^h_t sending the initial condition u^h_0 to $S^h_t u^h_0 = u^h(t,\cdot)$ is a continuous map from $\mathbb{R}^m$ to $\mathbb{R}^m$ for all t, and by the equivalence of norms in finite-dimensional vector spaces, so is the map sending the initial condition u^h_0 to $Du^h(t,\cdot)$. Recall that we are writing the functions u^h in terms of a finite element basis

$$
u^h(t,x) = \sum_{i=1}^{\mathcal{N}(h)} \alpha^h_i(t)\phi^h_i(x).
$$

The above considerations lead to us being able to view the map from $\mathbb{R}^{\mathcal{N}(h)}$ to $\mathbb{R}^{\mathcal{N}(h)}$ which sends the vector $\underline{\alpha}^h(0)$ of coefficients of the discretised initial condition in terms of the finite-element basis to the vector $\underline{\alpha}^h(t)$ of coefficients of u^h in the same finite-element basis as a continuous function. To see this, let $\underline{\alpha}^h$ and $\underline{\beta}^h$ be the coefficients of u^h and v^h respectively. By expressing

$$u^h(t,x) - v^h(t,x) = \sum_{i=1}^{\mathcal{N}(h)} (\alpha_i^h(t) - \beta_i^h(t))\phi_i^h(x),$$

it follows that

$$\begin{aligned}
&\max_{t\in[0,T]} \left(\sum_{i=1}^{\mathcal{N}(h)} |\alpha_i^h(t) - \beta_i^h(t)|^2 \min_{i=1,\ldots,\mathcal{N}(h)} \int_\Omega |\phi_i^h(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \\
&\quad \le \max_{t\in[0,T]} \left(\sum_{i=1}^{\mathcal{N}(h)} |\alpha_i^h(t) - \beta_i^h(t)|^2 \int_\Omega |\phi_i^h(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \\
&\quad = \max_{t\in[0,T]} \left(\int_\Omega \sum_{i=1}^{\mathcal{N}(h)} |\alpha_i^h(t) - \beta_i^h(t)|^2 |\phi_i^h(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \\
&\quad = \max_{t\in[0,T]} \left(\int_\Omega \sum_{i=1}^{\mathcal{N}(h)} |\alpha_i^h(t) - \beta_i^h(t)|^2 |\phi_i^h(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \\
&\quad \le \|u^h - v^h\|_{C([0,T];L^2(\Omega))} \\
&\quad \le C(T,h,K_{Lip,loc}) \|u_0^h - v_0^h\|_{L^2(\Omega)} \\
&\quad \le C(T,h,K_{Lip,loc}) \max_{i=1,\ldots,\mathcal{N}(h)} \|\phi_i^h\|_{L^2(\Omega)} \left(\sum_{i=1}^{\mathcal{N}(h)} |\alpha_i^h(0) - \beta_i^h(0)|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Thus the map $\underline{\alpha}^h(0) \mapsto \underline{\alpha}^h(t)$ is continuous for all $t \in [0,T]$. Now we can get the continuous dependence on the initial condition for each $z \in \Omega$:

$$\begin{aligned}
|u^h(t,x) - v^h(t,x)| &\le \sum_{i=1}^{\mathcal{N}(h)} |\alpha_i^h(t) - \beta_i^h(t)||\phi_i^h(x)| \\
&\le C\left(T,h,K_{Lip,loc}, \max_{i=1,\ldots,\mathcal{N}(h)} \|\phi_i^h\|_{L^2(\Omega)}, \min_{i=1,\ldots,\mathcal{N}(h)} \|\phi_i^h\|_{L^2(\Omega)} \right) \\
&\quad \times \left(\sum_{i=1}^{\mathcal{N}(h)} |\alpha_i^h(0) - \beta_i^h(0)| \right) \sum_{i=1}^{\mathcal{N}(h)} |\phi_i^h(x)|.
\end{aligned}$$

4.3 An overview of the algorithm

The main idea behind the algorithm presented in [36] is to generate (randomly) a large set of initial data, evolve all of these forward (separately) under some solution operator, and look at the limiting spread of values as the size of the amplitude of the randomisation is sent to zero. In [35] and [36] this is used to compute measure-valued solutions to systems of hyperbolic conservation laws in the form of the following two algorithms. The first is called **Algorithm 4.1** in [36]:

Algorithm A: Let the initial data for an underlying time-dependent PDE be given as a Young measure σ.

Step 1: Let $u_0 : \Omega \to L^p(\mathbb{R}^m)$ be a random field on a probability space $(X, \mathcal{F}, P)$ with law σ, meaning that $\sigma(E) = P(u_0(\omega) \in E)$.

Step 2: Evolve the initial random field by applying a suitable numerical scheme, with solution map $\mathbf{S}_t^{\Delta}$, to the initial data $u_0(\omega)$ for every $\omega \in X$, obtaining an approximate random field $u^{\Delta}(\omega, \cdot, t) := \mathbf{S}_t^{\Delta} u_0(\omega; \cdot)$.

Step 3: Define the approximate measure-valued solution ν^{Δ} as the law of u^{Δ} with respect to P, that is, for all Borel sets $E \subset \mathbb{R}^N$,

$$\nu_{x,t}^{\Delta}(E) = P(u(\omega; x, t) \in E).$$

Note that we should expect a difference compared with [35] and [36] in what we are trying to do, as our Young measures appearing are generated by sequences of gradients. Next, we need a method to approximate the random field $u_0(\omega; x)$. This is **Algorithm 4.3** in [36].

Algorithm B: Let Δ denote a parameter coming from some numerical scheme (in [35] and [36] a finite difference scheme is utilised, and so Δ would correspond to some vector of grid sizes) and let $M \in \mathbb{N}$. Let σ^{Δ} be the initial Young measure.

Step 1: From some probability space $(X, \mathcal{F}, P)$ draw M independent and identically distributed random fields $u_0^{\Delta,1}, \ldots, u_0^{\Delta,M}$ all with the same law σ^{Δ}.

Step 2: For each k and for a fixed $\omega \in X$ approximate the solution to the PDE using the solution operator with initial data $u_0^{\Delta,k}(\omega)$. Denote $u^{\Delta,k}(\omega; \cdot, t) = \mathbf{S}_t^{\Delta} u_0^{\Delta,k}(\omega; \cdot)$.

Step 3: Define the approximate measure-valued solution by

$$\nu_{x,t}^{\Delta,M} := \frac{1}{M}\sum_{k=1}^{M} \delta_{u^{\Delta,k}(\omega;x,t)}.$$

Results from [36] (see **Theorem 4.5**, **Theorem 5.1** and **Corollary 5.4**) then guarantee convergence to a measure-valued solution as one passes to the limit (diagonally). In this chapter we go into more depth regarding these algorithms and their adaptation to the systems of parabolic PDEs in which we are interested.

4.4 Application to systems of parabolic PDEs

We now give the necessary changes to be made to the algorithm described above so that it can be applied to the problem:

$$\partial_t u - \operatorname{div}(a(Du)) - Bu = F \tag{4.4.0.3}$$

with

$$F \in \bigtimes_{i=1}^{m} L^{p_i'}(Q_T), \tag{4.4.0.4}$$

and

$$Bu \cdot u \leq 0, \tag{4.4.0.5}$$

and a a locally Lipschitz map satisfying (3.1.0.6)−(3.1.0.8). We impose a zero Dirichlet boundary condition as before, and consider the initial data to be given by a function $u_0 \in \bigtimes_{i=1}^{m} W_0^{1,p_i}(\Omega;\mathbb{R}^m)$.

4.4.1 Alterations to Algorithm A and Algorithm B

Below we give the analogues of Algorithm A and Algorithm B which are needed for our parabolic problem. We here make the additional restriction that we only consider atomic intial data: that is, where our initial data is given by a function, rather than some Young measure. As has already been discussed at the start of this chapter, repeatedly taking subsequences means that the results are hard to use for actual computation, and as such these are not really computable algorithms, but we use the word in order to be consistent with terminology used in [35] and [36].

Algorithm C: Let the initial datum for an underlying time-dependent PDE be given as a function u_0.

Step 1: Let $Y : X \to \bigtimes_{i=1}^{m} W_0^{1,p_i}(\Omega;\mathbb{R}^m)$ be a random field on a probability space $(X, \mathcal{F}, P)$, and discretise Y by a finite element approximation of random fields Y^h, such that $\|Y^h(\omega;\cdot)\|_{L^2(\Omega)} \leq 1$ P-a.s and for every $\omega \in X$, $Y^h(\omega, x) = 0$ for all $x \in \partial\Omega$. Discretize u_0 by some finite element approximation u_0^h, and then perturb these discretisations by defining

$$u_0^{h,\epsilon}(\omega;x) := u_0^h(x) + \epsilon Y^h(\omega;x),$$

and let $\sigma^{h,\epsilon}$ be the law of $u_0^{h,\epsilon}$, meaning that $\sigma^{h,\epsilon}(E) = P(u_0^{h,\epsilon}(\omega) \in E)$.

Step 2: Evolve the initial random field by applying a suitable numerical scheme, with solution map $\mathbf{S}_t$, to the initial data $u_0^{h,\epsilon}(\omega)$ for every $\omega \in \Omega$, obtaining an approximate random field $u^{h,\epsilon}(\omega, t, \cdot) := \mathbf{S}_t u_0^{h,\epsilon}(\omega;\cdot)$.

Step 3: Define the approximate Young measure solution $\mu^{h,\epsilon}$ as the law of $u^{h,\epsilon}$ with respect to P, that is, for all Borel sets $E \subset \mathbb{R}^m$,

$$\mu_{t,x}^{h,\epsilon}(E) = P(u^{h,\epsilon}(\omega;t,x) \in E),$$

and define $\nu^{h,\epsilon}$ to be the law of $Du^{h,\epsilon}$ with respect to P.

As before we need a method to approximate the random field $u_0^{\epsilon}(\omega;x)$, as well as approximate the measures $\mu^{h,\epsilon}$ and $\nu^{h,\epsilon}$.

Algorithm D: Let h denote the grid size parameter and let $M \in \mathbb{N}$. Let $\sigma^{h,\epsilon}$ be the initial Young measure, chosen so that as $h \to 0$, we have that $\sigma^{h,\epsilon}$ converges weakly-$*$ to σ^{ϵ}.

Step 1: From some probability space (X, F, P) draw M independent and identically distributed random fields $Y^{h,1}, \ldots, Y^{h,M}$ such that $\|Y^{h,i}(\omega)\|_{L^2(\Omega)} \leq 1$ for P-a.e $\omega \in X$ and for all $i = 1, \ldots, M$, and such that for all $\omega \in X$ and all $i = 1, \ldots, m$, $Y^{h,i}(\omega;x) = 0$

for all $x \in \partial\Omega$, and such that $u_0^{h,1,\epsilon}, \ldots, u_0^{h,M,\epsilon}$ all have the same law $\sigma^{h,\epsilon}$, where

$$u_0^{h,k,\epsilon}(\omega; x) = u_0^h(x) + \epsilon Y^{h,k}(\omega; x).$$

We make explicit here the fact that $\|u_0^{h,k,\epsilon}\|_{L^2(\Omega)} \leq \|u_0^h\|_{L^2(\Omega)} + \epsilon$, so that the bound is independent of k, ϵ, and h (as u_0^h is chosen to converge strongly to u_0 in $L^2(\Omega; \mathbb{R}^m)$).

Step 2: For each k and for a fixed $\omega \in \Omega$ approximate the solution to the PDE using the solution operator with initial data $u_0^{h,k,\epsilon}(\omega)$. Denote $u^{h,k,\epsilon}(\omega; t, \cdot) = \mathbf{S}_t u_0^{h,k,\epsilon}(\omega; \cdot)$.

Step 3: Define the approximate Young measure solution $(\mu_{t,x}^{h,M,\epsilon}, \nu_{t,x}^{h,M,\epsilon})$, where

$$\mu_{\omega;t,x}^{h,M,\epsilon} := \frac{1}{M} \sum_{k=1}^{M} \delta_{u^{h,k,\epsilon}(\omega;t,x)},$$

and

$$\nu_{\omega;t,x}^{h,M,\epsilon} := \frac{1}{M} \sum_{k=1}^{M} \delta_{Du^{h,k,\epsilon}(\omega;t,x)}.$$

4.4.2 Applying the algorithm to construct Young measure solutions

In what follows we will show that the algorithms described above converge to a solution of the system of PDEs under consideration as we pass $h, \epsilon \to 0$ and $M \to \infty$. We will split this proof up into a number of sections, but for ease of reading we first give an outline:

1. Firstly we shall construct the weak formulation which we wish to study, and discuss passage to the limit in M.

2. In the second step we shall pass $h \to 0$ and $\epsilon \to 0$.

3. In the third step we show that the limiting measures can be interpreted as a Young measure solution of the system under consideration.

4. Finally we shall show that the initial condition is satisfied.

Remark 4.4.2.1. *To avoid confusion we make the following fact clear here. The parameter ω taken from the probability space X is intended to represent the "seed" in the* *numerical algorithm. Different seeds ω will give rise to different approximating solutions. What will be shown during the second step is that one can pass to the limit in M (along a subsequence) to obtain convergence to a quantity which is independent of $\omega \in X$ for P-a.e $\omega \in X$.*

Remark 4.4.2.2. *In reality, in order to make this algorithm able to be implemented we should include the time-stepping procedure. However, as we have to make a choice as to the order in which we pass limits (there is a calculational step in which we cannot interchange the limits in M and h), we simply choose to pass $\Delta t \to 0$ first.*

Step 1: Passing $M \to \infty$:

We shall use the semidiscrete numerical scheme discussed in Section 3.3, augmented with the time-stepping procedure discussed in Section 4.2. By Lemma 4.2.2.1 we have that, for any $\omega \in X$, any $\epsilon > 0$, any positive integer k and any $h \in (0, h_0]$ and $\Delta t = T/N$ for any $N \in \mathbb{N}$, we can construct, via the time-stepping procedure from Section 4.2, a function $u^{h,k,\epsilon}_{\Delta t}(\omega; \cdot, \cdot)$, which for any $h \in (0, h_0]$ and any $\Delta t > 0$ (sufficiently small, for a given fixed $h \in (0, h_0]$) uniquely solves the fully discrete scheme and depends continuously on the discretized initial condition $u^{h,k,\epsilon}_0$. Therefore we may assume that the passage to the limit $\Delta t \to 0_+$ has already been made for $\omega \in X$ and $h \in (0, h_0]$ fixed, and we shall therefore consider the limiting function $u^{h,k,\epsilon}(\omega; \cdot, \cdot)$ resulting from the passage to the limit $\Delta t \to 0_+$ instead of $u^{h,k,\epsilon}_{\Delta t}(\omega; \cdot, \cdot)$. In particular we remark that as we are at this stage considering finitely many perturbations of the discretized initial condition, the existence of a suitable subsequence $\Delta t_j \to 0_+$, which works for each $k \in \{1, \dots, M\}$, is straightforward.

For each initial datum $u^{h,k,\epsilon}_0(\omega)$ (for a fixed $\omega \in X$) we have the following energy estimate, obtained (for each fixed $\omega \in X$ and $k \in \{1, \dots, M\}$, analogously to those found in the proof of Theorem 3.3.1.1 (see, in particular, (3.3.1.4)):

$$\begin{aligned}\|u^{h,k,\epsilon}(\omega)\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))} &+ \sum_{i=1}^m \|Du^{h,k,\epsilon}_i(\omega)\|^{p_i}_{L^{p_i}(0,T;W^{1,p_i}_0(\Omega;\mathbb{R}^n))} \\ &\leq c(\Omega, p_1, \dots, p_m) \left(\|u_0\|^2_{L^2(\Omega;\mathbb{R}^m)} + \sum_{i=1}^m \|F_i\|^{p_i'}_{L^{p_i'}(Q_T)} + |Q_T| \sum_{p_i<2} |\mu_i|^{p_i} \right).\end{aligned}$$

Here Du_i denotes the spatial derivative of the i-th component of the vector u. Similarly to the analogous bounds from Theorem 3.3.1.1, we can bound

$$\|a(Du^{h,k,\epsilon})\|_{L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n})} \leq C$$

and

$$\|\partial_t u^{h,k,\epsilon}\|_{L^{\hat{q}'}(0,T;W^{-1,\hat{q}'}(\Omega;\mathbb{R}^m))} \leq C$$

with C in both cases being independent of h, k and ϵ (recall that we have defined $\hat{q} = \max\{q, 2\}$). This follows by the assumption that the perturbed initial conditions are bounded uniformly in h, k and ϵ. Summing over $k = 1, \ldots, M$ and dividing by M we see that the functions $u^{h,k,\epsilon}$ satisfy, for all $\psi \in L^q(0, T; V_m^h)$ the equation:

$$\begin{aligned}\int_0^T \int_\Omega \frac{1}{M}\sum_{k=1}^{M} \partial_t u^{h,k,\epsilon}(\omega;t,x)\cdot\psi(t,x) + \langle \nu_{\omega;t,x}^{h,M,\epsilon}, a\rangle : D\psi(t,x)\ \mathrm{d}x\ \mathrm{d}t \\ -\int_0^T \int_\Omega \frac{1}{M}\sum_{k=1}^{M} Bu^{h,k,\epsilon}(\omega;t,x)\cdot\psi(t,x)\ \mathrm{d}x\ \mathrm{d}t = \int_0^T\int_\Omega F\cdot\psi\ \mathrm{d}x\ \mathrm{d}t.\end{aligned} \tag{4.4.2.1}$$

This is the weak formulation which we shall be considering. An integration by parts in time results in the equation:

$$\begin{aligned}\int_0^T \int_\Omega \frac{1}{M}\sum_{k=1}^{M} u^{h,k,\epsilon}(\omega;t,x)\cdot(\partial_t\psi(t,x) + B^T\psi(t,x))\ \mathrm{d}x\ \mathrm{d}t \\ -\int_0^T\int_\Omega \langle \nu_{\omega;t,x}^{h,M,\epsilon}, a\rangle : D\psi(t,x) - F(t,x)\cdot\psi(t,x)\ \mathrm{d}x\ \mathrm{d}t = 0\end{aligned} \tag{4.4.2.2}$$

for all $\psi \in W^{1,1}(0,T;V_m^h)$. We now wish to let $M \to \infty$. In order to take this limit we need to prove some convergence results for the first two terms in (4.4.2.2). This is accomplished by proving an analogue of *Theorem 4.9* from [35].

We let $u_0^{h,\epsilon}(\omega;x) = u_0^h(x) + \epsilon Y^h(\omega;x)$ be some initial condition which is independent of the initial conditions $u_0^{h,1,\epsilon}, \ldots, u_0^{h,M,\epsilon}$, and has the same distribution as these initial conditions (meaning in particular that the law of $u_0^{h,\epsilon}$ is given by $\sigma^{h,\epsilon}$), satisfies $\|Y^h(\omega;\cdot)\|_{L^2(\Omega)} \leq 1$ for P-almost every $\omega \in X$, and is such that for all $\omega \in X$ we have $Y^h(\omega;x) = 0$ when $x \in \partial\Omega$. We evolve this according to the previous numerical scheme to obtain a function $u^{h,\epsilon}(\omega;t,x)$. We denote the law of $u^{h,\epsilon}$ by $\mu^{h,\epsilon}$, and similarly denote the law of $Du^{h,\epsilon}$ by $\nu^{h,\epsilon}$. From the proof of Theorem 3.3.1.1 we see that the following regularity holds:

$$u^{h,\epsilon} \in L^\infty\left(X; \bigtimes_{i=1}^{m} L^{p_i}(0,T;W_0^{1,p_i}(\Omega))\right) \bigcap L^\infty(X\times(0,T);L^2(\Omega;\mathbb{R}^m)), \tag{4.4.2.3}$$

$$\partial_t u^{h,\epsilon} \in L^\infty(X;L^{\hat{q}'}(0,T;W^{-1,\hat{q}'}(\Omega;\mathbb{R}^m))) \tag{4.4.2.4}$$

$$a(Du^{h,\epsilon}) \in L^\infty(X;L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n})). \tag{4.4.2.5}$$

All of the norms of the above functions in their respective spaces can be bounded independently of h and ϵ.

We now also consider the function $\langle \nu^{h,M,\epsilon}_{\omega;t,x}, a\rangle$ for a fixed $\omega \in X$ (cf. Remark 3.1.0.2). We remark that by the bounds obtained during the proof of Theorem 3.3.1.1 we can bound this function in the $L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n})$ norm, and as such there is a subsequence M_n and a function $\chi \in L^{\hat{q}'}(Q_T)$ such that $\langle \nu^{h,M_n,\epsilon}_{\omega}, a\rangle \overset{*}{\rightharpoonup} \chi$. We aim to identify this function χ later, using the (by now) usual strategy of identifying the limit in some weaker space and appealing to the uniqueness of weak limits.

Recall that everything done up until now has been for a fixed $\omega \in X$. As such, the limiting function χ still depends on the parameter ω. In order to show that the limit is independent of ω we have the following result.

Theorem 4.4.2.3. *Let $\gamma : Q_T\times\mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ be such that $\gamma \in L^{\alpha'}(Q_T;E_{\frac{p}{\alpha}}(\mathbb{R}^{m\times n};\mathbb{R}^{m\times n}))$ with $\alpha \in (1, \frac{p}{q-1})$. Along a subsequence M_n we have*

$$\int_0^T\int_\Omega \langle \mu^{h,M_n,\epsilon}_{\omega;t,x}, \gamma(t,x,\cdot)\rangle \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega \langle \mu^{h,\epsilon}_{t,x}, \gamma(t,x,\cdot)\rangle \,\mathrm{d}x\,\mathrm{d}t,$$

and

$$\int_0^T\int_\Omega \langle \nu^{h,M_n,\epsilon}_{\omega;t,x}, \gamma(t,x,\cdot)\rangle \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega \langle \nu^{h,\epsilon}_{t,x}, \gamma(t,x,\cdot)\rangle \mathrm{d}x\,\mathrm{d}t,$$

in $L^2(X)$ as $n\to\infty$. In particular, for each γ there is a subsequence of M_n such that for P-a.e $\omega \in X$ the convergence happens pointwise.

Proof. The proof of this theorem is mostly the same as the proof of Theorem 11 of [35], found therein in Appendix B. We consider only the family of measures $\nu^{h,M_n,\epsilon}$, the other family is dealt with similarly. Given a random field $\eta : X \to L^1(Q_T)$ we define its expectation with respect to the probability measure P as

$$\mathbb{E}(\eta) := \int_X \eta(\omega) \, \mathrm{d}P(\omega).$$

Similarly to [35] we define, for $k = 1, \ldots, M$, the following two quantities:

$$G_{i,j}(\omega) = \int_0^T \int_\Omega \gamma_{i,j}(t, x, Du^{h,\epsilon}(\omega; t, x)) \, \mathrm{d}x \, \mathrm{d}t,$$

and

$$G_{i,j}^k(\omega) = \int_0^T \int_\Omega \gamma_{i,j}(t, x, Du^{h,k,\epsilon}(\omega; t, x)) \, \mathrm{d}x \, \mathrm{d}t.$$

We need to show that $G_{i,j}^1, \ldots, G_{i,j}^{M_n}$ are independent and identically distributed. We start by remarking that as the initial conditions $u_0^{h,1,\epsilon}, \ldots, u_0^{h,M_n,\epsilon}$ are independent and identically distributed (in the variable $\omega \in X$), and as the solution operator $\mathbf{S}_t$ mapping an initial condition $\tilde{u}_0$ to $\tilde{u}(t, \cdot)$ is a continuous map (see Proposition 4.2.2.1), the functions $u^{h,k,\epsilon}$ are all independent and identically distributed as well (see Lemma 4.1.2.1). Furthermore, by the remarks following the statement of Theorem 4.2.2.1, the functions $Du^{h,k,\epsilon}$ are also independent and identically distributed.

Recall that the functions $\gamma_{i,j}$ are measurable, and in particular are continuous in their third component and satisfy the growth rate coming from the space E_{q-1}. This implies that the functions $\gamma_{i,j}(t, x, Du^{h,k,\epsilon}(\omega; t, x))$ are independent and identically distributed in $\omega \in X$. From Tonelli's Theorem it then follows that the functions $G_{i,j}^k$ are measurable in ω, and thus by Lemma 4.1.2.1 are also independent and identically distributed in $\omega \in X$.

We now compute the following quantity

$$\mathbb{E}\left(\left(\mathbb{E}(G_{i,j}(\omega)) - \frac{1}{M_n} \sum_{k=1}^{M_n} G_{i,j}^k(\omega)\right)^2\right),$$

which is the $L^2(X)$ error with respect to the probability measure P. As in [35], using the fact that the random variables $G_{i,j}^k$ are independent and identically distributed, we can reduce this to

$$\mathbb{E}\left(\left(\mathbb{E}(G_{i,j}(\omega)) - \frac{1}{M_n}\sum_{k=1}^{M_n} G_{i,j}^k(\omega)\right)^2\right) = \frac{1}{M_n}(\mathbb{E}((G_{i,j})^2) - (\mathbb{E}(G_{i,j}))^2). \tag{4.4.2.6}$$

We then bound the terms on the right-hand side using the energy estimates corresponding to the numerical scheme, as well as the fact that we can choose our initial data to have a bounded $L^2(\Omega;\mathbb{R}^m)$ norm for a.e $\omega \in X$. We show the calculation for $\mathbb{E}((G_{i,j})^2)$; the calculation for the other term follows in a similar way. Let $\alpha \in (1, \frac{p}{q-1})$.

$$\begin{aligned}
\mathbb{E}((G_{i,j})^2) &= \int_X \left(\int_0^T \int_\Omega \gamma_{i,j}(t,x,Du^{h,\epsilon}(\omega;t,x))\ \mathrm{d}x\ \mathrm{d}t\right)^2\ \mathrm{d}P(\omega) \\
&\leq \int_X \|\gamma_{i,j}\|^2_{L^{\alpha'}(Q_T;E_{\frac{p}{\alpha}})} \|1 + |Du^{h,\epsilon}(\omega)|^{\frac{p}{\alpha}}\|^2_{L^\alpha(Q_T;\mathbb{R}^{m\times n})}\ \mathrm{d}P(\omega).
\end{aligned}$$

We justify this by noting that

$$\begin{aligned}
&\left|\int_0^T \int_\Omega \gamma_{i,j}(t,x,Du^{h,\epsilon}(\omega;t,x))\ \mathrm{d}x\ \mathrm{d}t\right| \\
&\quad\leq \int_0^T \int_\Omega \frac{|\gamma_{i,j}(t,x,Du^{h,\epsilon}(\omega;t,x))|(1 + |Du^{h,\epsilon}(\omega;t,x)|^{\frac{p}{\alpha}})}{1 + |Du^{h,\epsilon}(\omega;t,x)|^{\frac{p}{\alpha}}}\ \mathrm{d}x\ \mathrm{d}t \\
&\quad\leq \int_0^T \int_\Omega \|\gamma_{i,j}(t,x,\cdot)\|_{E_{\frac{p}{\alpha}}} (1 + |Du^{h,\epsilon}(\omega;t,x)|)^{\frac{p}{\alpha}}\ \mathrm{d}x\ \mathrm{d}t \\
&\quad\leq \|\gamma_{i,j}\|_{L^{\alpha'}(Q_T;E_{\frac{p}{\alpha}})} \|1 + |Du^{h,\epsilon}(\omega)|^{\frac{p}{\alpha}}\|_{L^\alpha(Q_T)}.
\end{aligned}$$

Given two real numbers a and b satisfying $0 \leq |a| \leq b$ it then follows that $a^2 \leq b^2$, meaning that we can square both sides of the above inequality and then integrate in ω to deduce the stated bound on $\mathbb{E}((G_{i,j})^2)$. Then using the fact that $\alpha < \frac{p}{q-1}$ we can bound the norm $\|1 + |Du^{h,\epsilon}(\omega)|^{\frac{p}{\alpha}}\|_{L^\alpha(Q_T)}$ independently of h, ϵ and ω by using the energy estimates obtained through the proof of Theorem 3.3.1.1.

Thus as P is a probability measure we have that $\mathbb{E}((G_{i,j})^2)$ is bounded, and so the right-hand side of (4.4.2.6) converges to zero as $M_n \to \infty$. Then, by applying Lemma 4.1.1.2, we see that:

$$\mathbb{E}(G_{i,j}(\omega)) = \int_X \int_0^T \int_\Omega \gamma_{i,j}(t,x,Du^{h,\epsilon}(\omega;t,x))\ \mathrm{d}x\ \mathrm{d}t\ \mathrm{d}P(\omega)$$

$$= \int_0^T \int_\Omega \left(\int_X \gamma_{i,j}(t,x,Du^{h,\epsilon}(\omega;t,x))\ \mathrm{d}P(\omega) \right)\ \mathrm{d}x\ \mathrm{d}t$$
$$= \int_0^T \int_\Omega \langle \nu^{h,\epsilon}_{t,x}, \gamma_{i,j}(t,x,\cdot)\rangle\ \mathrm{d}x\ \mathrm{d}t.$$

Repeating these calculations for all components $\gamma_{i,j}$ gives us the convergence of

$$\int_0^T \int_\Omega \langle \nu^{h,M_n,\epsilon}_{\omega;t,x}, \gamma(t,x,\cdot)\rangle \mathrm{d}x\ \mathrm{d}t \to \int_0^T \int_\Omega \langle \nu^{h,\epsilon}_{t,x}, \gamma(t,x,\cdot)\rangle\ \mathrm{d}x\ \mathrm{d}t,$$

in $L^2(X)$ as $M_n \to \infty$, from which we deduce the existence of a subsequence (not relabelled) converging for P-almost every $\omega \in X$ using standard results in measure theory. □

What we would like to conclude is that by the above theorem we have, for $P-$almost every $\omega \in X$, the weak-star convergence of $\nu^{h,M_n,\epsilon}_\omega$ to $\nu^{h,\epsilon}$. This does not follow immediately, as a-priori there is *no reason* why the subsequence of M_n along which we have pointwise convergence should be *independent* of the function γ (a point not clarified in [35] or [36]). To get around this we note that as $E_{\frac{p}{\alpha}}$ is separable (see [46]), the space $L^{\alpha'}(Q_T; E_{\frac{p}{\alpha}})$ is also separable for our choice of α. Thus, the space $E_{\frac{p}{\alpha}}(\mathbb{R}^{m\times n};\mathbb{R}^{m\times n})$ is also separable, and therefore there is a countable dense subset $\{\gamma_\mathcal{J}\}_{\mathcal{J}=1}^\infty \subset E_{\frac{p}{\alpha}}(\mathbb{R}^{m\times n};\mathbb{R}^{m\times n})$.

We fix $\gamma_1 \in L^{\alpha'}(Q_T; E_{\frac{p}{\alpha}}(\mathbb{R}^{m\times n};\mathbb{R}^{m\times n}))$. Then there is a subsequence $M_{n_1(j)} \subset M_n$ and a set $A_1 \subset X$ such that $P(A_1) = 1$, and we have the pointwise, and $L^2(X)$, convergence holding along the subsequence for all $\omega \in A_1$. Fixing γ_2 we obtain a subsequence $M_{n_2(j)} \subset M_{n_1(j)}$ and a set A_2 with $P(A_2) = 1$ such that the pointwise, and $L^2(X)$, convergence results hold along the subsequence $M_{n_2(j)}$ for all $\omega \in A_1 \cap A_2$. We continue like this and choose a diagonal subsequence $M_m = M_{n_m(m)}$, along which (by construction) we have the pointwise convergence for all $\gamma_\mathcal{I}$ in the countable dense subset and for all $\omega \in A := \bigcap_{i=1}^\infty A_i$.

Given a function $\gamma \in L^{\alpha'}(Q_T; E_{\frac{p}{\alpha}}(\mathbb{R}^{m\times n};\mathbb{R}^{m\times n}))$, there is a subsequence $\{\gamma_\mathcal{I}\}_{\mathcal{I}=1}^\infty \subset \{\gamma_\mathcal{J}\}_{\mathcal{J}=1}^\infty$ for which $\gamma_\mathcal{I}$ converges strongly to γ in $L^{\alpha'}(Q_T; E_{\frac{p}{\alpha}}(\mathbb{R}^{m\times n};\mathbb{R}^{m\times n}))$ as $\mathcal{I} \to \infty$. Let $\omega \in A$ (which, by standard results in measure theory, satisfies $P(A) = 1$). Then

$$\left| \int_0^T \int_\Omega \langle \nu^{h,M_m,\epsilon}_{\omega;t,x}, \gamma(t,x,\cdot)\rangle\ \mathrm{d}x\ \mathrm{d}t - \int_0^T \int_\Omega \langle \nu^{h,\epsilon}_{t,x}, \gamma(t,x,\cdot)\rangle\ \mathrm{d}x\ \mathrm{d}t \right|$$
$$= \left| \int_0^T \int_\Omega \langle \nu^{h,M_m,\epsilon}_{\omega;t,x}, \gamma(t,x,\cdot) - \gamma_\mathcal{I}(t,x,\cdot) + \gamma_\mathcal{I}(t,x,\cdot)\rangle\ \mathrm{d}x\ \mathrm{d}t \right.$$

$$
\begin{aligned}
&\qquad - \int_0^T \int_\Omega \langle \nu^{h,\epsilon}_{t,x}, \gamma(t,x,\cdot) - \gamma_{i_j}(t,x,\cdot) + \gamma_{i_j}(t,x,\cdot)\rangle \, \mathrm{d}x \, \mathrm{d}t \Big| \\
&\le \left| \int_0^T \int_\Omega \langle \nu^{h,M_m,\epsilon}_{\omega;t,x}, \gamma_{\mathcal{I}}(t,x,\cdot) \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_\Omega \langle \nu^{h,\epsilon}_{t,x}, \gamma_{\mathcal{I}}(t,x,\cdot)\rangle \, \mathrm{d}x \, \mathrm{d}t \right| \\
&\qquad + \left| \int_0^T \int_\Omega \langle \nu^{h,M_m,\epsilon}_{\omega;t,x} - \nu^{h,\epsilon}_{t,x}, \gamma(t,x,\cdot) - \gamma_{\mathcal{I}}(t,x,\cdot)\rangle \, \mathrm{d}x \, \mathrm{d}t \right|.
\end{aligned}
$$

For a fixed $\mathcal{I}$ the first term converges to zero as $m \to \infty$. For the second term we compute:

$$
\begin{aligned}
&\left| \int_0^T \int_\Omega \langle \nu^{h,M_m,\epsilon}_{\omega;t,x} - \nu^{h,\epsilon}_{t,x}, \gamma(t,x,\cdot) - \gamma_{\mathcal{I}}(t,x,\cdot)\rangle \, \mathrm{d}x \, \mathrm{d}t \right| \\
&\quad = \left| \int_0^T \int_\Omega \left\langle \nu^{h,M_m,\epsilon}_{\omega;t,x} - \nu^{h,\epsilon}_{t,x}, \frac{(\gamma(t,x,\cdot) - \gamma_{\mathcal{I}}(t,x,\cdot))(1 + |\cdot|^{\frac{p}{\alpha}})}{1 + |\cdot|^{\frac{p}{\alpha}}} \right\rangle \, \mathrm{d}x \, \mathrm{d}t \right| \\
&\quad \le \|\gamma - \gamma_{\mathcal{I}}\|_{L^{\alpha'}(Q_T; E_{\frac{p}{\alpha}})} \left(\frac{1}{M_m} \sum_{k=1}^{M_m} \|1 + |Du^{h,k,\epsilon}(\omega;t,x)|^{\frac{p}{\alpha}}\|_{L^\alpha(Q_T)} \right. \\
&\qquad\qquad \left. + \|1 + |Du^{h,\epsilon}(\omega;t,x)|^{\frac{p}{\alpha}}\|_{L^\infty(X; L^\alpha(Q_T))} \right),
\end{aligned}
$$

noting that by previous calculations the term in the parantheses is bounded independently of M_m, h, ϵ and ω. Therefore passing $M_m \to \infty$ and then $\mathcal{I} \to \infty$ results in the desired convergence.

The terms involving the measures $\mu^{h,\epsilon}$ are treated similarly, using that we can obtain similar bounds on the functions $u^{h,k,\epsilon}$ as on their derivatives, and that the identity function is in the space $E_{\frac{p}{\alpha}}$. Thus we have that for all $\psi \in W^{1,1}(0,T; V^h_m)$ the following identity is satisfied:

$$
\int_0^T \int_\Omega \langle \mu^{h,\epsilon}_{t,x}, \xi\rangle \cdot (\partial_t \psi(t,x) - B^T \psi(t,x)) - \langle \nu^{h,\epsilon}_{t,x}, a\rangle : D\psi(t,x) + F(t,x) \cdot \psi(t,x) \, \mathrm{d}x \, \mathrm{d}t = 0.
$$

Step 2: Passing $h, \epsilon \to 0$:

We now move on to discuss passing to the limit in the remaining two parameters. By the aforementioned weak convergence and lower-semicontinuity of the norm, we have that $\|\langle \nu^{h,\epsilon}, a\rangle\|_{L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n})} \le \liminf_{n\to\infty} \|\langle \nu^{h,M_n,\epsilon}, a\rangle\|_{L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n})} \le C$, independently of h and ϵ, implying that there is a subsequence h_j and a function $\chi \in L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n})$ for which $\langle \nu^{h_j,\epsilon}, a\rangle \rightharpoonup \chi$ in $L^{\hat{q}'}(Q_T;\mathbb{R}^{m\times n})$. We need to identify what this limit is.

Consider the function $\tilde{a}(\xi) = 1 + |\xi|^{\frac{p}{\alpha}}$. We can apply the same calculations as carried out in step 2 to obtain that $\langle \nu^{h,M_l,\epsilon}, \tilde{a}\rangle \rightharpoonup \langle \nu^{h,\epsilon}, \tilde{a}\rangle$ in $L^\alpha(Q_T)$ as $l \to \infty$. In particular:

$$\|\langle \nu^{h,\epsilon}, \tilde{a}\rangle\|_{L^\alpha(Q_T)} \leq \liminf_{l\to\infty} \|\langle \nu^{h,M_l,\epsilon}, \tilde{a}\rangle\|_{L^\alpha(Q_T)},$$

which is uniformly bounded by the growth rate of $\tilde{a}$ and the energy estimates, independently of l, h and ϵ. We now compute

$$\begin{aligned}
\|\nu^{h,\epsilon}\|_{L^\alpha(Q_T;E'_{\frac{p}{\alpha}})} &= \sup_{\|\gamma\|_{L^{\alpha'}(Q_T;E_{\frac{p}{\alpha}})}=1} \int_{Q_T} \langle \nu^{h,\epsilon}_{t,x}, \gamma(t,x,\cdot)\rangle \,\mathrm{d}x \,\mathrm{d}t \\
&= \sup_{\|\gamma\|_{L^{\alpha'}(Q_T;E_{\frac{p}{\alpha}})}=1} \int_{Q_T} \left\langle \nu^{h,\epsilon}_{t,x}, \frac{\gamma(t,x,\cdot)(1+|\cdot|^{q-1})}{1+|\cdot|^{q-1}} \right\rangle \,\mathrm{d}x \,\mathrm{d}t \\
&\leq \|\langle \nu^{h,\epsilon}, \tilde{a}\rangle\|_{L^\alpha(Q_T)},
\end{aligned}$$

which is uniformly bounded as before. Thus there is a subsequence of the measures indexed by (h_j, ϵ) and a measure $\nu^\epsilon \in L^\alpha(Q_T; E'_{\frac{p}{\alpha}})$ such that for every $\gamma \in E_{\frac{p}{\alpha}}$ we have that

$$\langle \nu^{h_j,\epsilon}, \gamma\rangle \rightharpoonup \langle \nu^\epsilon, \gamma\rangle$$

in $L^\alpha(Q_T; \mathbb{R}^{m\times n})$ as $j \to \infty$. In particular this results in:

$$\langle \nu^{h_j,\epsilon}, a\rangle \rightharpoonup \langle \nu^\epsilon, a\rangle$$

in $L^\alpha(Q_T; \mathbb{R}^{m\times n})$. Thus we have identified the limit χ, and by uniqueness of weak limits we have that $\langle \nu^\epsilon, a\rangle \in L^{\hat{q}'}(Q_T; \mathbb{R}^{m\times n})$. Similar calculations hold for the family of measures $\mu^{h,\epsilon}$. Passing $\epsilon \to 0$ is now done similarly, and we note that one can indeed take a diagonal subsequence of (h, ϵ) and pass to the limit along this diagonal subsequence to obtain the existence of two Young measures μ and ν satisfying, for all $\psi \in L^{\hat{q}}(0,T; W^{1,\hat{q}}_0(\Omega; \mathbb{R}^m))$, the identity

$$\int_0^T \int_\Omega \langle \mu_{t,x}, \xi\rangle \cdot (\partial_t \psi(t,x) - B^{\mathrm{T}}\psi(t,x)) - \langle \nu_{t,x}, a\rangle : D\psi(t,x) + F(t,x)\cdot\psi(t,x) \,\mathrm{d}x \,\mathrm{d}t = 0. \tag{4.4.2.7}$$

Step 3: Interpretation as a Young measure solution:

We now give a relation between the measures μ and ν. Recall that $\mu^{h,\epsilon}$ is the law of the random variable $u^{h,\epsilon}(\omega;t,x)$, and that $u^{h,\epsilon}$ is written as a sum of finite element basis functions over the domain Ω. Therefore we can take a derivative in space, and get:

$$D\langle \mu^{h,\epsilon}_{t,x}, \xi\rangle = \int_X Du^{h,\epsilon}(\omega;t,x)\ \mathrm{d}P(\omega) = \langle \nu^{h,\epsilon}, \xi\rangle.$$

Using the previously obtained weak-star convergence of $\mu^{h,\epsilon}$ and $\nu^{h,\epsilon}$, the regularity of the functions $u^{h,\epsilon}$ coming from (4.4.2.3)$-$(4.4.2.5), and the fact that X is a space of finite P-measure, we see that by passing $h, \epsilon \to 0$ along subsequences, we get the existence of a function

$$u \in L^\infty\left(X; \bigtimes_{i=1}^m L^{p_i}(0,T; W_0^{1,p_i}(\Omega;\mathbb{R}^m))\right)$$

such that

$$\langle \mu_{t,x}, \xi\rangle = \int_X u(\omega;t,x)\ \mathrm{d}P(\omega)$$

and

$$\langle \nu_{t,x}, \xi\rangle = \int_X Du(\omega;t,x)\ \mathrm{d}P(\omega).$$

This comes from the fact that, for example,

$$\langle \mu^{h,\epsilon}, \xi\rangle \rightharpoonup \langle \mu, \xi\rangle,$$

and we have that

$$\int_0^T \int_\Omega \int_X u^{h,\epsilon}(\omega;t,x)\cdot\phi(\omega;t,x)\ \mathrm{d}P(\omega)\ \mathrm{d}x\ \mathrm{d}t \to \int_0^T \int_\Omega \int_X u(\omega;t,x)\cdot\phi(\omega;t,x)\ \mathrm{d}P(\omega)\ \mathrm{d}x\ \mathrm{d}t$$

for all $\phi \in L^1(X; L^q(0,T; W_0^{1,q}(\Omega;\mathbb{R}^m)))$. However, as X has finite P-measure, we can choose test functions which are constant in the parameter ω to see that

$$\int_0^T \int_\Omega \left(\int_X u^{h,\epsilon}(\omega;t,x)\ \mathrm{d}P(\omega)\right)\cdot\phi(t,x)\ \mathrm{d}x\ \mathrm{d}t \to \int_0^T \int_\Omega \left(\int_X u(\omega;t,x)\ \mathrm{d}P(\omega)\right)\cdot\phi(t,x)\ \mathrm{d}x\ \mathrm{d}t$$

for all $\phi \in L^q(0,T;W_0^{1,q}(\Omega;\mathbb{R}^m))$, which gives the desired representation for $\langle \mu_{t,x}, \xi\rangle$ (the corresponding term for $\langle \nu_{t,x}, \xi\rangle$ follows similarly).

Now we define the function

$$U(t,x) := \int_X u(\omega;t,x)\ \mathrm{d}P(\omega) = \langle \mu_{t,x}, \xi\rangle.$$

This is locally integrable over Q_T and thus we can speak about its distributional derivative DU. If we can demonstrate that $DU(t,x) = \langle \nu_{t,x}, \xi\rangle$ in a suitable sense, then we will have shown that the pairing (U,ν) is a Young measure solution of the system under consideration (the desired regularity follows from using the lower-semi-continuity properties of the norm and the previously obtained regularity of the functions $u^{h,\epsilon}$ in (4.4.2.3)−(4.4.2.5)). What must first be shown is that it makes sense to speak of DU as a function, rather than as a distribution. To that end, note that the function $u \in \bigtimes_{i=1}^m L^{p_i}(0,T;W_0^{1,p_i}(\Omega))$ satisfies $u|_{\partial\Omega} = 0$, and so we can extend the function u (and hence the function U) by zero from $[0,T]\times\overline{\Omega}$ to the whole of $[0,T]\times\mathbb{R}^n$ so that the extended function (still denoted by u) belongs to $\bigtimes_{i=1}^m L^{p_i}(0,T;W^{1,p_i}(\mathbb{R}^n))$. We consider the difference quotient of U, defined by

$$D_j^\delta U = \frac{U(t,x+\delta e_j) - U(t,x)}{\delta},$$

where e_j is the unit vector in the j-th co-ordinate direction. Then, denoting the i-th component of the vector $U \in \mathbb{R}^m$ by U_i, it follows that

$$\begin{aligned}
\|D_j^\delta U_i(t,\cdot)\|_{L^{p_i}(\Omega)} &= \left\| \frac{U_i(t,\cdot+\delta e_j) - U_i(t,\cdot)}{\delta} \right\|_{L^{p_i}(\Omega)} \\
&= \left\| \int_X \frac{u_i(\omega;t,\cdot+\delta e_j) - u_i(\omega;t,\cdot)}{\delta}\,\mathrm{d}P(\omega) \right\|_{L^{p_i}(\Omega)} \\
&\le \int_X \left\| \frac{u_i(\omega;t,\cdot+\delta e_j) - u_i(\omega;t,\cdot)}{\delta} \right\|_{L^{p_i}(\Omega)} \mathrm{d}P(\omega) \\
&\le \int_X \|D_j u_i(\omega;t,\cdot)\|_{L^{p_i}(\Omega)}\,\mathrm{d}P(\omega),
\end{aligned}$$

for all $\delta \in (0,1)$, and $j = 1, \dots, n$. In the transition to the last line we have used that, by Jensen's inequality,

$$
\begin{aligned}
\left\| \frac{u_i(\omega; t, \cdot + \delta e_j) - u_i(\omega; t, \cdot)}{\delta} \right\|_{L^{p_i}(\Omega)}^{p_i} &= \left\| \int_0^1 D_j u_i(\omega; t, \cdot + s\delta) \, \mathrm{d}s \right\|_{L^{p_i}(\Omega)}^{p_i} \\
&\leq \int_0^1 \| D_j u_i(\omega; t, \cdot + s\delta) \|_{L^{p_i}(\Omega)}^{p_i} \, \mathrm{d}s \\
&= \int_0^1 \int_\Omega |D_j u_i(\omega; t, x + s\delta)|^{p_i} \, \mathrm{d}x \, \mathrm{d}s \\
&\leq \int_0^1 \int_\Omega |D_j u_i(\omega; t, x)|^{p_i} \, \mathrm{d}x \, \mathrm{d}s \\
&= \| D_j u_i(\omega; t, \cdot) \|_{L^{p_i}(\Omega)}^{p_i}.
\end{aligned}
$$

Therefore, by a standard characterization of Sobolev functions in terms of difference quotients, $D_j U_i(t, \cdot) \in L^{p_i}(\Omega)$, $j = 1, \dots, n$, for almost every $t \in [0,T]$. By integrating over $t \in [0,T]$ in the above inequality it follows that $D_j U_i \in L^{p_i}(Q_T)$, $i = 1, \dots, n$, and in particular $U \in \bigtimes_{i=1}^m L^{p_i}(0,T; W_0^{1,p_i}(\Omega; \mathbb{R}^m))$, as required. By an analogous argument,

$$
U \in L^\infty(0,T; L^2(\Omega; \mathbb{R}^m)),
$$

and

$$
\partial_t U \in L^{\hat{q}'}(0,T; W^{-1,\hat{q}'}(\Omega; \mathbb{R}^m)),
$$

thanks to the regularity results

$$
u(\omega; \cdot, \cdot) \in L^\infty(0,T; L^2(\Omega; \mathbb{R}^m)), \qquad \text{for a.e. } \omega \in X,
$$

and

$$
\partial_t u(\omega; \cdot, \cdot) \in L^{\hat{q}'}(0,T; W^{-1,\hat{q}'}(\Omega; \mathbb{R}^m)), \qquad \text{for a.e. } \omega \in X,
$$

which follow from the regularity properties of $u^{h,\epsilon}$ stated in (4.4.2.3)–(4.4.2.5) using the weak lower-semi-continuity of the norm function.

Next, we define

$$
V(t,x) := \langle \nu_{t,x}, \xi \rangle = \int_X Du(\omega; t, x) \, \mathrm{d}P(\omega),
$$

and let φ be a test function. We then compute, using Fubini's theorem,

$$\begin{aligned}
\int_0^T \int_\Omega DU(t,x) \cdot \varphi(t,x) \,\mathrm{d}x \,\mathrm{d}t &= -\int_0^T \int_\Omega U(t,x) \cdot D\varphi(t,x) \,\mathrm{d}x \,\mathrm{d}t \\
&= -\int_0^T \int_\Omega \int_X u(\omega;t,x) \cdot D\varphi(t,x) \,\mathrm{d}P(\omega) \,\mathrm{d}x \,\mathrm{d}t \\
&= -\int_X \int_0^T \int_\Omega u(\omega;t,x) \cdot D\varphi(t,x) \,\mathrm{d}x \,\mathrm{d}t \,\mathrm{d}P(\omega) \\
&= \int_X \int_0^T \int_\Omega Du(\omega;t,x) \cdot \varphi(t,x) \,\mathrm{d}x \,\mathrm{d}t \,\mathrm{d}P(\omega) \\
&= \int_0^T \int_\Omega \int_X Du(\omega;t,x) \cdot \varphi(t,x) \,\mathrm{d}P(\omega) \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_0^T \int_\Omega V(t,x) \cdot \varphi(t,x) \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}$$

Step 4: Attainment of the initial condition:

Defining $U^{h,\epsilon}(t,x) := \int_X u^{h,\epsilon}(\omega;t,x)\,\mathrm{d}P(\omega)$, it follows from (4.4.2.3)$-$(4.4.2.5) that $\partial_t U^{h,\epsilon} \in L^{\hat{q}'}(0,T;W^{-1,\hat{q}'}(\Omega;\mathbb{R}^m))$, and $U^{h,\epsilon} \in \bigtimes_{i=1}^m L^{p_i}(0,T;W_0^{1,p_i}(\Omega;\mathbb{R}^m)) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$. It follows from Lemma 3.2.3.2 that there is a subsequence (h_j,ϵ_j) of (h,ϵ) along which

$$\int_\Omega U^{h_j,\epsilon_j}(t,x) \cdot \varphi(x)\,\mathrm{d}x \to \int_\Omega U(t,x) \cdot \varphi(x)\,\mathrm{d}x \tag{4.4.2.8}$$

uniformly for all $\varphi \in L^2(\Omega;\mathbb{R}^m)$. We let $\varphi \in L^2(\Omega;\mathbb{R}^m)$ and compute:

$$\begin{aligned}
\int_\Omega (U(0,x) - u_0(x)) \cdot \varphi(x)\,\mathrm{d}x &= \int_\Omega (U(0,x) - U^{h_j,\epsilon_j}(0,x)) \cdot \varphi(x)\,\mathrm{d}x \\
&\quad + \int_\Omega (U^{h_j,\epsilon_j}(0,x) - U^{h_j,\epsilon_j}(t,x)) \cdot \varphi(x)\,\mathrm{d}x \\
&\quad + \int_\Omega \left(U^{h_j,\epsilon_j}(t,x) - \int_X u_0^{h_j,\epsilon_j}(\omega,x)\,\mathrm{d}P(\omega) \right) \cdot \varphi(x)\,\mathrm{d}x \\
&\quad + \int_\Omega \left(\int_X u_0^{h_j,\epsilon_j}(\omega,x)\,\mathrm{d}P(\omega) - u_0(x) \right) \cdot \varphi(x)\,\mathrm{d}x \\
&=: I_j + II_j(t) + III_j(t) + IV_j.
\end{aligned} \tag{4.4.2.9}$$

The term I_j converges to zero as $j \to \infty$ by the uniform convergence in $C([0,T])$ stated in (4.4.2.8). The term $II_j(t)$ converges to zero as $t \to 0_+$ as it holds that $u^{h_j,\epsilon_j}(\omega;t,x) \to$

$u^{h_j,\epsilon_j}(\omega,0,x)$ by continuity of $t \in [0,T] \mapsto u^{h_j,\epsilon_j}(\omega;t,x)$ for each $\omega \in X$ and $x \in \Omega$, and we can apply the Dominated Convergence Theorem to interchange this limit with the integral over X that appears in the definition of U^{h_j,ϵ_j}. The term $III_j(t)$ also converges to zero as $t \to 0_+$ as we have that the numerical functions u^{h_j,ϵ_j} satisfy the initial condition, and we may once again apply the Dominated Convergence Theorem to interchange the limit with the integral over X. Finally for the term IV_j we compute:

$$\begin{aligned}\left|\int_X u_0^{h_j,\epsilon_j}(\omega,x)\,\mathrm{d}P(\omega) - u_0(x)\right| &= \left|\int_X u_0^{h_j} + \epsilon_j Y^{h_j}(\omega;x) - u_0(x)\,\mathrm{d}P(\omega)\right| \\ &\leq |u_0^h(x) - u_0(x)| + \epsilon_j \int_X |Y^{h_j}(\omega,x)|\,\mathrm{d}P(\omega).\end{aligned}$$

Integrating this over Ω we get

$$\begin{aligned}\int_\Omega \left|\int_X u_0^{h_j,\epsilon_j}(\omega,x)\,\mathrm{d}P(\omega) - u_0(x)\right| \mathrm{d}x &\leq \|u_0^{h_j} - u_0\|_{L^1(\Omega;\mathbb{R}^m)} \\ &\quad + \epsilon_j \int_X \|Y^{h_j}(\omega,\cdot)\|_{L^1(\Omega;\mathbb{R}^m)}\,\mathrm{d}P(\omega) \\ &\leq \|u_0^{h_j} - u_0\|_{L^2(\Omega;\mathbb{R}^m)} \\ &\quad + \epsilon_j |\Omega|^{\frac{1}{2}} \int_X \|Y^{h_j}(\omega,\cdot)\|_{L^2(\Omega;\mathbb{R}^m)}\,\mathrm{d}P(\omega) \\ &\leq \|u_0^{h_j} - u_0\|_{L^2(\Omega;\mathbb{R}^m)} + \epsilon_j |\Omega|^{\frac{1}{2}},\end{aligned}$$

which converges to zero as $j \to \infty$ by the assumed strong convergence of our discretized initial condition u_0^h to u_0 in $L^2(\Omega;\mathbb{R}^m)$. In view of this, we see that passing to the limit $t \to 0_+$ in (4.4.2.9) with $j \geq 1$ kept fixed before passing $j \to \infty$ we deduce that

$$\int_\Omega (U(0,x) - u_0(x)) \cdot \varphi(x)\,\mathrm{d}x = 0 \qquad \forall\, \varphi \in L^2(\Omega;\mathbb{R}^m).$$

Thus we have that $U(0,x) = u_0(x)$ for almost every $x \in \Omega$, showing that the initial condition is satisfied.

For clarity we state the following theorem which encapsulates the result proven in steps 1-4:

Theorem 4.4.2.4. *Algorithms C and D together converge to a Young measure solution* *of* (4.4.0.3) *in the event that the initial measure is a Dirac mass, $\delta_{u_0(x)}$, for any initial datum $u_0 \in L^2(\Omega)$.*

Remark 4.4.2.5. *If we have a (nonatomic) measure-valued initial datum σ, Steps 1–4 above provide an outline of an algorithm for proving the existence of a Young measure solution, assuming that all of the M "perturbations" to σ can be chosen to be bounded, independent of M. This requires being able to find a random variable u_0 whose law is given by the initial Young measure σ. The existence of such a random variable is guaranteed by Proposition 4.1.1.5.*

Remark 4.4.2.6. *In [35] and [36], in their analysis of semidiscrete schemes the authors are able to interchange the order in which the limits of the spatial discretization parameter and the parameter ϵ tending to zero and $M \to \infty$ are taken. For us here the passages to the limits in $h \to 0_+$, $\epsilon \to 0_+$ and $\Delta t \to 0_+$ are all interchangeable, but the passage in M is problematic from this point of view. The reason for this is that, while the limits obtained in the proof of Theorem 4.4.2.3 are independent of h, computing these limits in the first place required us to use the independence and the "identicalness" of the distributions of the functions $u^{h,k,\epsilon}$. This required that the solution operator $\boldsymbol{S}_t$ was measurable. However, for the model problem under consideration here, we were only able to show continuity of this operator on the functions $u^{h,k,\epsilon}$ for each fixed h, and were unable to transfer this property to the limiting function as we pass $h \to 0_+$, meaning that within the context of the algorithms, we* must *take a limit passage in M before the limit passage in h. If instead one could show that the operator $\boldsymbol{S}_t$ was measurable when considered as a mapping from the nondiscretized initial condition $u_0 \in L^2(\Omega; \mathbb{R}^m)$, we would then be able to interchange the limits in any way we desired to, as this property would be sufficient for the independence and "identicalness" of the distributions to carry over to the limiting functions as we take $h \to 0_+$.*

Finally, we bring this chapter to a close by mentioning that we can apply this same algorithm to systems of the form (4.4.0.3) – (4.4.0.5), with initial data $u_0 \in \bigtimes_{i=1}^{m} W_0^{1,p_i}(\Omega)$, and with a satisfying (3.5.0.2) and (3.5.0.3), in order to construct Young measure solutions.

Chapter 5

Closing remarks and further study

During the course of this thesis we have seen some of the problems associated with systems of nonlinear partial differential equations which are forward-backward in nature. In the absence of any monotonicity property, existence results for *Young measure solutions* were obtained, and long-time behaviour of the constructed solutions was investigated. We also briefly considered what were to happen should we still lack a monotonicity condition, but have slightly stronger structural assumptions in place.

We then discussed the numerical construction of said Young measure solutions, looking at ways in which one could express the solutions as Monte-Carlo averages of Dirac measures which are centred at the solution of a discrete problem.

Recently, along with Matteo Croci, this author has been attempting to implement the numerical algorithm outlined in Chapter 4. For clarity of authorship we make the following statement:

Remark 5.0.2.7. *All of the code for this implementation was written by Matteo Croci. In particular, Figures 5.1–5.4 were generated by him.*

5.1 Implementation for a model problem

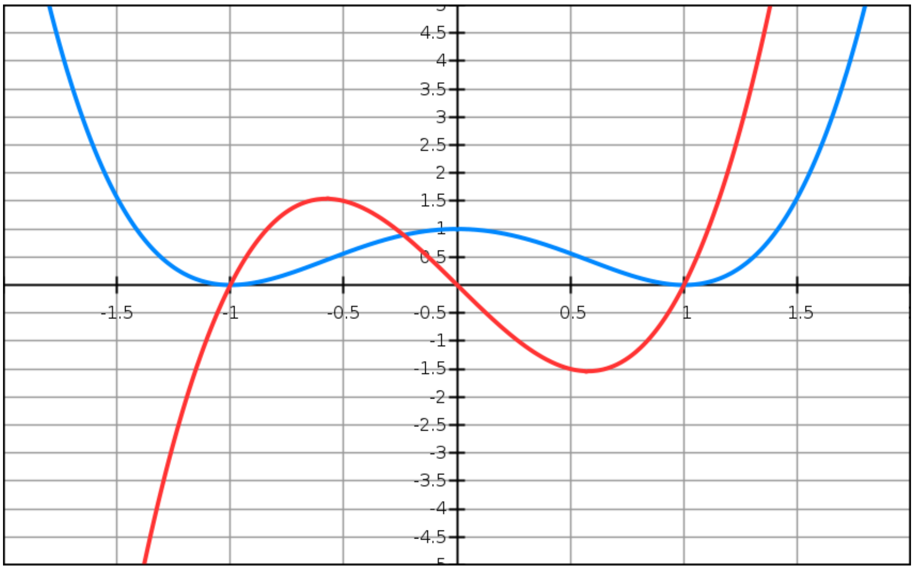
We begin by discussing the implementation for the following model problem, where the lack of monotonicity and resulting forward-backward structure are clear:

$$\partial_t u - \partial_x(f'(\partial_x u)) = 1, \tag{5.1.0.1}$$

$$u(t, 0) = u(t, 1) = 0, \tag{5.1.0.2}$$

$$u(0, x) = \begin{cases} 0.035x & \text{if } x \in [0, \frac{1}{2}], \\ 0.035 - 0.035x & \text{if } x \in [\frac{1}{2}, 1], \end{cases} \tag{5.1.0.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is given by $f(\xi) = (\xi - 1)^2(\xi + 1)^2$ (so $f'(\xi) = 4(\xi^2 - 1)\xi$). Here, $u : [0, T] \times (0, 1) \to \mathbb{R}$. We plot below a graph of f in blue and f' in red.



One can show that for each ξ such that $|\xi| < \frac{2}{\sqrt{3}}$ there exists an $\eta \in \mathbb{R}$ such that $(f'(\xi) - f'(\eta))(\xi - \eta) < 0$. Our choice of initial data is such that the initial gradient $\partial_x u(0, x)$ is in this non-monotone region of f'. For this problem one should therefore expect non-uniqueness of solution, and indeed that is what one sees in the numerics.

Below we discuss the numerical experiments. It is worth remarking here that the intention is not for this to be a thorough in-depth analysis; rather, at this stage the author seeks only to present a brief overview.

We give here a short explanation of the sampling method and random field generation method utilised in the implementation. In order to perturb the initial condition, Gaussian *Matérn fields* were chosen. Matérn fields can be viewed as being solutions of the following linear elliptic PDE [19]:

$$(\mathcal{I} - \kappa^{-2}\Delta_x)^k w(x,\omega) = \eta \dot{W}(x,\omega) \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega,$$

where, in the notation of Chapter 4, $x \in \Omega$, $\omega \in X$ (recall that $(X, \mathcal{F}, P)$ is a probability space), $k > \frac{n}{4}$ and η is some scaling factor. W here represents a spatial Gaussian white noise in $\mathbb{R}^m$. Such fields are, for example, almost-surely continuous (see [1], section 2.1.3). Once obtained, we use this Matérn field w to perturb our initial condition u_0^h, obtaining

$$u_0^{h,\epsilon}(\omega; x) := u_0^h(x) + \epsilon w(\omega; x).$$

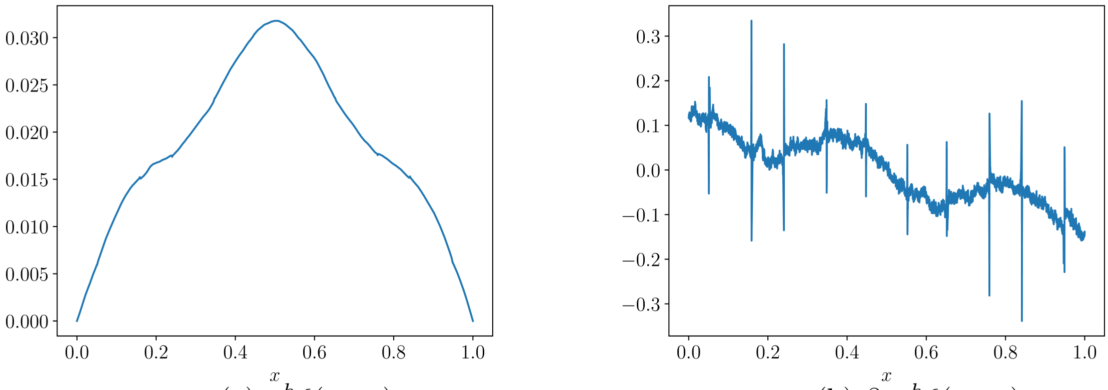
The perturbed initial condition is then evolved forwards in time using the backwards Euler method outlined in Chapter 4 to obtain an approximate solution $u^{h,\epsilon}(\omega; t, x)$.

Recall the result of Lemma 4.1.1.2, which asserted that if $\mu_{t,x}$ is the law of a random variable $u(\omega; t, x)$, then for every continuous vector function g one has that

$$\langle \mu_{t,x}, g \rangle = \int_X g(u(\omega; t, x)) \; \mathrm{d}P(\omega)$$

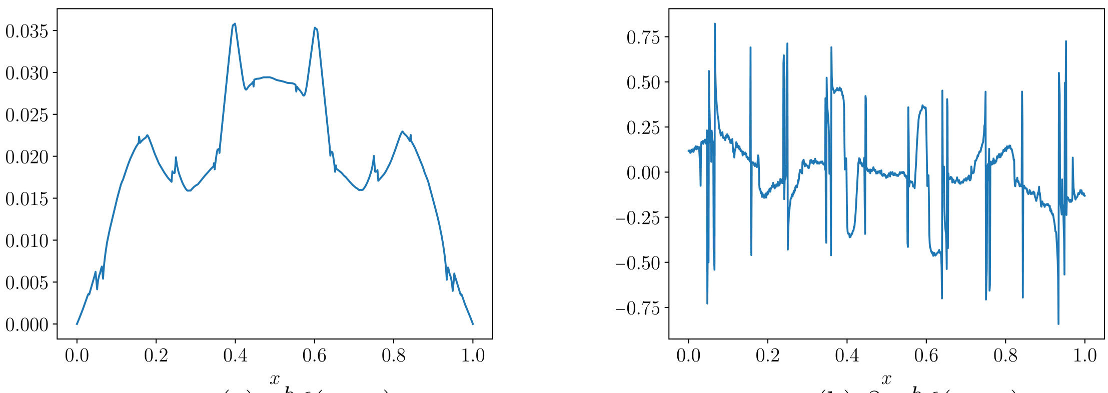
for almost every $(t, x) \in Q_T$. Therefore, in order to compute expressions of the form $\langle \mu_{t,x}, g \rangle$, it suffices to compute $\int_X g(u(\omega; t, x)) \; \mathrm{d}P(\omega)$ (and similarly for the measures ν which relate to the spatial derivative of u). Computationally, this is done via a Monte Carlo method with a specific sampling technique, details of which can be found in [19].

We now perform some numerical experiments. All of the experiments were done with 10,000 Monte Carlo samples. Solutions were evolved to a steady state, and the solutions and the derivatives were then plotted. We give some examples below. The plots below give the numerical approximation to the expected value of $u^{h,\epsilon}$ and $\partial_x u^{h,\epsilon}$, computed using a Monte Carlo procedure.



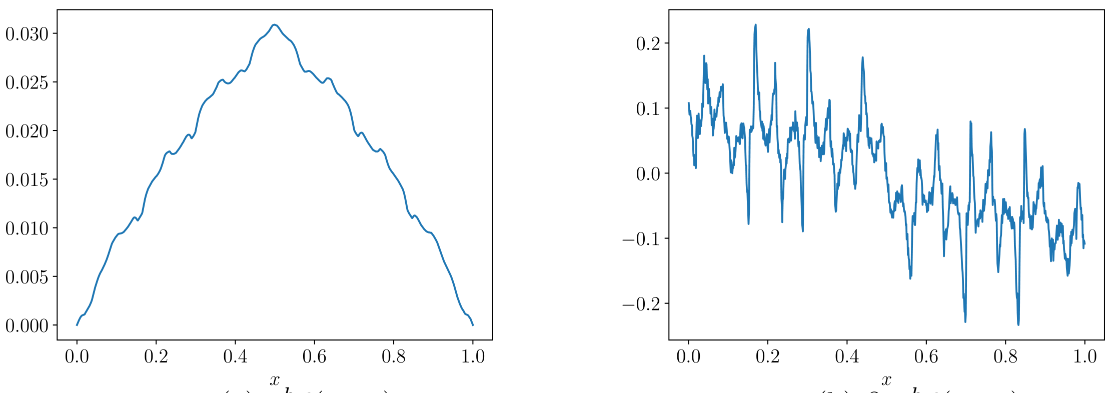
(a) $u^{h,\epsilon}(\infty, x)$ (b) $\partial_x u^{h,\epsilon}(\infty, x)$

Figure 5.1: Plots of $u^{h,\epsilon}(\infty, \cdot)$ and $\partial_x u^{h,\epsilon}(\infty, \cdot)$ with 8000 mesh elements, a time-step of $\Delta t = 0.001$, and the perturbation size of the initial datum being $\epsilon = 1 \times 10^{-6}$.



(a) $u^{h,\epsilon}(\infty, x)$ (b) $\partial_x u^{h,\epsilon}(\infty, x)$

Figure 5.2: Plots of $u^{h,\epsilon}(\infty, \cdot)$ and $\partial_x u^{h,\epsilon}(\infty, \cdot)$ with 1000 mesh elements, a time-step of $\Delta t = 0.001$, and the perturbation size of the initial datum being $\epsilon = 1 \times 10^{-10}$.



(a) $u^{h,\epsilon}(\infty, x)$ (b) $\partial_x u^{h,\epsilon}(\infty, x)$

Figure 5.3: Plots of $u^{h,\epsilon}(\infty, \cdot)$ and $\partial_x u^{h,\epsilon}(\infty, \cdot)$ with 1000 mesh elements, a time-step of $\Delta t = 0.000125$, and the perturbation size of the initial datum being $\epsilon = 1 \times 10^{-6}$.

We see different behaviour as the parameters are varied, meaning that picking out a convergent subsequence numerically is challenging. What is interesting to note is that regardless of the parameters chosen, the steady state plot of $a(\partial_x u^{h,\epsilon})$ is always the following:

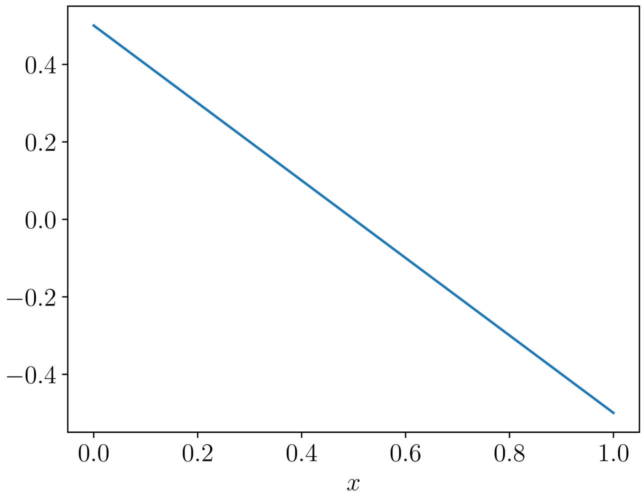


Figure 5.4: Plots of $a(\partial_x u^{h,\epsilon}(\infty,\cdot))$ with 10000 mesh elements, a time-step of $\Delta t = 0.00001$, and the perturbation size of the initial datum being $\epsilon = 1 \times 10^{-8}$.

That this plot is the same for every set of parameters that were chosen is interesting for two reasons. Firstly, in conjunction with the plots for $u^{h,\epsilon}(\infty,\cdot)$ and $\partial_x u^{h,\epsilon}(\infty,\cdot)$ in Figures 5.1−5.3, it suggests that there are multiple solutions. Secondly, this steady state was reached with a right-hand side which does not decay to zero as $t \to \infty$, thereby suggesting that the long-time behaviour results obtained in Chapter 3 could be extended to include right-hand sides which, rather than decaying to zero, instead converge to some non-zero function. Such results would require further work done beyond this thesis.

5.2 Implementation for the Met Office model

When this algorithm was implemented for systems of the form (1.1.1.1), with $n = 1$, $m = 2$, $B = 0$, $F = 1$, $u(0,x) = (u_1(0,x), u_2(0,x))^T$ with

$$u_1(0,x) = \begin{cases} 0.035x & \text{if } x \in [0, \frac{1}{2}], \\ 0.035 - 0.035x & \text{if } x \in [\frac{1}{2}, 1], \end{cases} \tag{5.2.0.4}$$

$$u_2(0,x) = \begin{cases} -0.01x & \text{if } x \in [0, \frac{1}{2}], \\ -0.01 + 0.01x & \text{if } x \in [\frac{1}{2}, 1], \end{cases} \tag{5.2.0.5}$$

and

$$a(\xi) = \sqrt{\xi_1^2 + |\xi_2|}\,\xi,$$

what was seen did not mirror expectations. Altering the grid size, time step, and the size of the perturbation did not change the solution in any meaningful way. Indeed, we saw the same solution appearing in both components regardless of the values of parameters that were chosen. We hypothesise some reasons as to why this is the case.

It may indeed be the case that there is no uniqueness for these Young measure solutions, but some element of the code results in the same solution being picked out every time. It could be that the parameters were simply not chosen sufficiently exhaustively, and that for some possibly very specific set of parameters, one could in fact see a completely different solution being approximated. What could also be the case is that we simply do not understand sufficiently well what it means for the function a defined above to be non-monotone, or indeed whether or not monotonicity is the correct property to consider in this simplified one-dimensional model. The main difficulty in choosing the initial condition comes from the fact that, because a is not expressible as $a = \nabla\psi$ for some C^1 function ψ, the failure to satisfy the monotonicity condition cannot be understood in terms of a real-valued potential ψ. This in turn means that the failure to satisfy the monotonicity condition cannot be understood in terms of the lack of convexity of a function ψ, and must instead be shown to fail through finding specific pairs of values for which the monotonicity condition is not satisfied. This approach to showing that monotonicity does not hold creates problems when choosing our initial condition.

Take, for example, the problem described by (5.1.0.1)−(5.1.0.3) described at the start of this chapter. The initial condition we have chosen satisfies $|\partial_x u(0,x)| = 0.035$. Below we plot the function $\gamma(\xi) := (a(\xi) - a(0.035))(\xi - 0.035)$ on the vertical axis with the argument ξ on the horizontal axis.

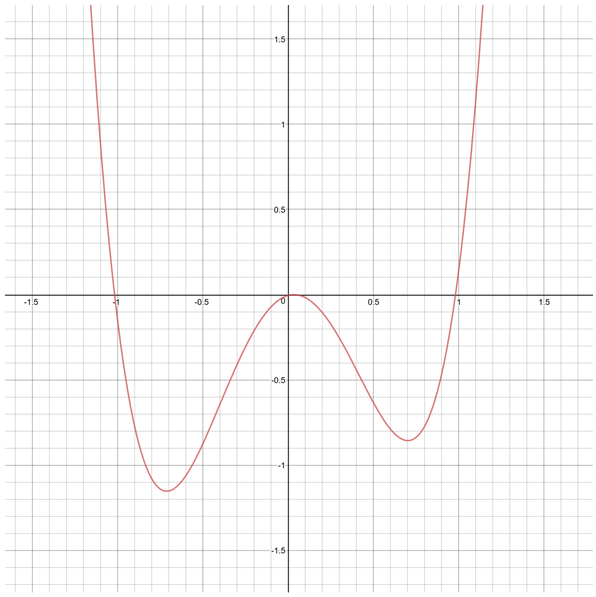
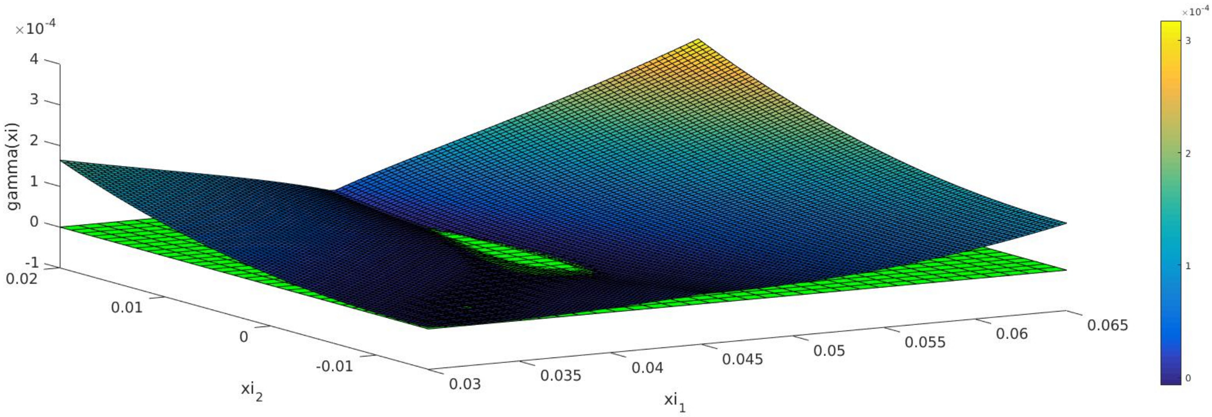


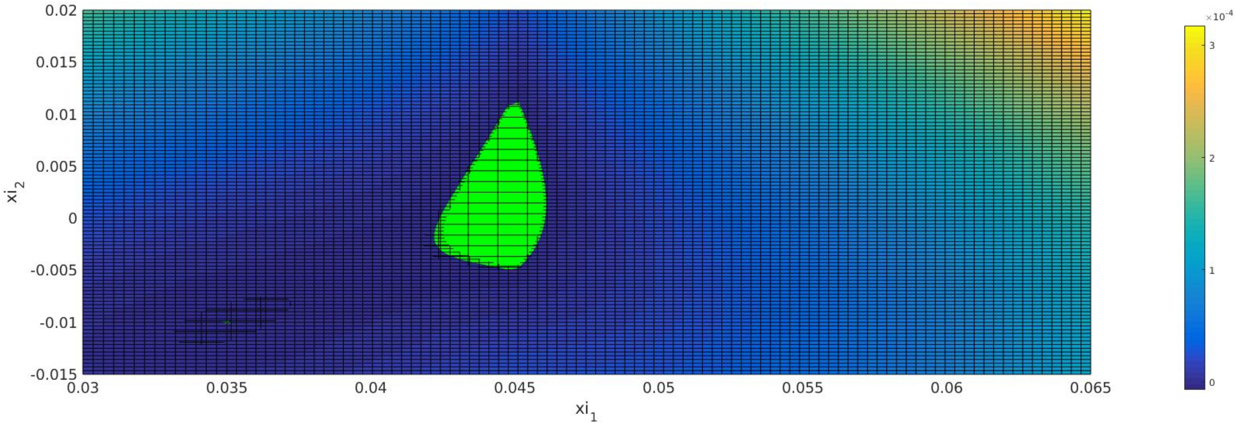
Figure 5.5: A plot of ξ on the horizontal axis against $\gamma(\xi)$ on the vertical axis.

Clearly it holds that $\gamma(0.035) = 0$, and from Figure 5.5 it is clear that γ has a local maximum at $\xi = 0.035$, meaning that for all ξ in a sufficiently small neighbourhood of 0.035, one has that $a((\xi) - a(0.035))(\xi - 0.035) \leq 0$, with equality only if $\xi = 0.035$. In some sense this is then a sensible initial condition to have chosen in order to demonstrate the bad behaviour and lack of monotonicity; if one moves away from the initial gradient by a small amount, then one must do so "non-monotonically".

The same is not true for the Met Office type model. We take the initial condition described by (5.2.0.4)−(5.2.0.5), and plot, for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ in a neighbourhood of $\eta := (0.035, -0.01)$, the value of the inner product $\gamma(\xi) := (a(\xi) - a(\eta)) \cdot (\xi - \eta)$. The colourbar on the right of the below plot indicates the value of γ, and the bright green area is the ξ_1-ξ_2 plane (included so as to make it easier to see where $\gamma < 0$.



We also include the projection to the ξ_1-ξ_2 plane:



As can be seen from the plot (as well as direct calculations), it is not the case that γ has a local maximum at $\xi = \eta$; note that the point $\eta = (0.035, -0.01)$ does not lie in the green region. This means that there is no neighbourhood $\mathcal{N}$ of η in which it is true that $(a(\xi) - a(\eta)) \cdot (\xi - \eta) \leq 0$ for all $\xi \in \mathcal{N}$. While there is indeed one set in which this inequality holds, it is not a set which contains the point η itself; a distinct difference between this Met Office style model and the one-dimensional equation (5.1.0.1).

Whether this is enough to be able to say for sure that there is a gap in the theory remains unclear. It could be that the model used by the Met Office falls into an analogue of the cases considered in [23, 44, 45], wherein there are regions of the non-monotone operator which are somehow well-behaved for certain initial data, or it could be that there is some element of the numerical algorithm whose implementation results in finding the same solution every time, or simply that the parameters chosen were such that the same solution was observed. What we saw in the experiments could also be an artifact of the fact that the Met Office model had been simplified down to a one-dimensional system of equations, and it may be that in one space dimension, monotonicity is not the correct notion to be considering. Answering such questions requires further study beyond that carries out in this thesis.

A further possible direction of study would be to consider fluid models where the stress-strain relation is non-monotone. That is, systems of equations of the form

$$\partial_t u - \sigma \mathrm{div}(a(\epsilon(u))) + (u \cdot D)u + D\pi = F \tag{5.2.0.6}$$

$$\mathrm{div}(u) = 0 \tag{5.2.0.7}$$

$$u|_{[0,T]\times\partial\Omega} = 0 \tag{5.2.0.8}$$

$$u(0,x) = u_0(x), \tag{5.2.0.9}$$

where $a : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ is a non-monotone mapping, $\epsilon(u) := \frac{Du+Du^T}{2}$ is the symmetric gradient, and π represents the pressure. Non-monotone stress-strain relations have been observed occuring physically (see, for example, [39] and [48]), but to this author's knowledge there are no results concerning fluid models of the above type with non-monotone stress-strain relations.

Appendix A

Properties of K from (1.1.2.9)

A.1 Non-monotonicity of the operator $K(\xi)\xi$

We present here the required calculations to show that the operator $K(\xi)\xi$ from (1.1.2.9) is non-monotone. Firstly we show the non-monotonicity in the case where $K(\xi) = \sqrt{\xi_1^2 + \xi_2^2 - \xi_3}$ and $\xi_3 \leq 0$.

Take $\xi = (0.035, 0, -0.01)$ and $\eta = (0.05, 0, 0)$. Then one computes:

$$\begin{aligned}
\langle K(\xi)\xi - K(\eta)\eta, \xi - \eta\rangle &= K(\xi)(|\xi|^2 - \xi \cdot \eta) + K(\eta)(|\eta|^2 - \xi \cdot \eta) \\
&= \sqrt{\frac{35^2 + 10000}{1000000}} \left(\frac{35^2 + 100 - (35)(50)}{1000000}\right) \\
&\qquad + \frac{5}{100}\left(\frac{25}{10000} - \frac{(35)(5)}{100000}\right) \\
&= \frac{\sqrt{11225}}{1000}\left(\frac{1325 - 1750}{1000000}\right) + \frac{50}{1000}\left(\frac{2500 - 1750}{1000000}\right) \\
&= \frac{-425\sqrt{11225} + 37500}{10^9} \\
&< \frac{-(425)(105) + 37500}{10^9} \\
&= \frac{37500 - 44625}{10^9} \\
&< 0.
\end{aligned}$$

Now for the case where $K(\xi) = \frac{(\xi_1^2+\xi_2^2)^{\frac{3}{2}}}{\xi_1^2+\xi_2^2+\xi_3}$ and $\xi_3 \geq 0$, take $\xi = (-0.2, -0.1, 0.2)$ and

$\eta = (-0.1, 0, 0.5)$. Then one computes:

$$\begin{aligned}
\langle K(\xi)\xi - K(\eta)\eta, \xi - \eta\rangle &= K(\xi)(|\xi|^2 - \xi\cdot\eta) + K(\eta)(|\eta|^2 - \xi\cdot\eta) \\
&= 4\left(\frac{5}{100}\right)^{\frac{3}{2}}\left(\frac{-21}{100}\right) + \frac{1}{510}\left(\frac{7}{50}\right) \\
&= \frac{1}{500}\left(\frac{-21(5^{\frac{3}{2}})}{50} + \frac{7}{51}\right) \\
&< 0.
\end{aligned}$$

A.2 The vector field $K(\xi)\xi$ is not conservative

Next is the proof of the fact that the vector field $K(\xi)\xi$ cannot be expressed as the derivative of a C^1 function. If this were to be the case, then, denoting $F(\xi) = K(\xi)\xi$, we would have that the line integral of F over any closed path is equal to zero.

Consider the circle C parameterised by $r(\theta) = (1+\cos(\theta), 0, \sin(\theta))$. Then

$$\frac{dr}{d\theta}(\theta) = (-\sin(\theta), 0, \cos(\theta)),$$

and:

$$\begin{aligned}
\oint_C F(\xi)\cdot \mathrm{d}\xi &= \int_0^{2\pi} F(r(\theta))\cdot\frac{\mathrm{d}r}{\mathrm{d}\theta}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} K(r(\theta))r(\theta)\cdot\frac{\mathrm{d}r}{\mathrm{d}\theta}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} K(r(\theta))(-\sin(\theta))\,\mathrm{d}\theta \\
&= \int_0^{\pi} \frac{|1+\cos(\theta)|^3(-\sin(\theta))}{(1+\cos(\theta))^2+\sin(\theta)}\,\mathrm{d}\theta + \int_\pi^{2\pi}\sqrt{(1+\cos(\theta))^2-\sin(\theta)}(-\sin(\theta))\,\mathrm{d}\theta \\
&= \int_0^{\pi} \frac{|1+\cos(\theta)|^3(-\sin(\theta))}{(1+\cos(\theta))^2+\sin(\theta)} + \sqrt{(1+\cos(\theta))^2+\sin(\theta)}(\sin(\theta))\,\mathrm{d}\theta \\
&= \int_0^{\pi} (\sin(\theta))\left(\frac{-(1+\cos(\theta))^3}{(1+\cos(\theta))^2+\sin(\theta)} + \sqrt{(1+\cos(\theta))^2+\sin(\theta)}\right)\,\mathrm{d}\theta.
\end{aligned}$$

We study the expression in the large parantheses above. On the interval $(0,\pi)$ we have that:

$$\frac{-(1+\cos(\theta))^3}{(1+\cos(\theta))^2+\sin(\theta)} + \sqrt{(1+\cos(\theta))^2+\sin(\theta)} > 0$$

if and only if

$$((1+\cos(\theta))^2+\sin(\theta))^{\frac{3}{2}} > (1+\cos(\theta))^3,$$

which is true for $\theta \in (0, \pi)$ as the function $s \mapsto |s|^{\frac{3}{2}}$ is increasing for $s > 0$, and $\sin(\theta) > 0$ on the interval $(0, \pi)$. Thus the integral is larger than zero, and F is not conservative.

“He still liked mathematics as much as ever, but he had no desire to make a profession of research in the field. (…) It was fine as an amateur endeavour, but he had neither the personality nor the drive to stake his whole life on it, which he well knew.”

HARUKI MURAKAMI, 1Q84

Bibliography

[1] P. Abrahamsen. *A Review of Gaussian Ransom Fields and Correlation Functions.* Norweigen Computing Centre, 2nd edition, 1997.

[2] F. Andreu, V. Caselles, and J. M. Mazón. A parabolic quasilinear problem for linear growth functionals. *Rev. Mat. Iberoamericana*, 18(1):135–185, 2002.

[3] J. M. Ball. A version of the fundamental theorem for Young measures. In *PDEs and Continuum Models of Phase Transitions*, volume 344 of *Lecture Notes of Physics*, pages 207–215. Springer-Verlag, 1989.

[4] R. J. Beare and M. J. P. Cullen. Validating weather and climate models at small Rossby numbers: including a boundary layer. *Quarterly Journal of the Royal Meteorological Society*, 142(700):2636–2645, 2016.

[5] G. Belletini, G. Fusco, and N. Guglielmi. A concept of solution and numerical experiments for forward-backward diffusion equations. *Discrete and Continuous Dynamical Systems*, 16(4):783 – 842, 2006.

[6] J. Bergh and J. Löfström. *Interpolation Spaces: An Introduction*, volume 223 of *Grundlehren der mathematischen Wissenschaften.* Springer, 1976.

[7] M. Bertsch, F. Smarrazzo, and A. Tesei. Pseudo-parabolic regularization of forward–backward parabolic equations: Power-type nonlinearities. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2016(712):51–80.

[8] M. Bertsch, F. Smarrazzo, and A. Tesei. On a class of forward–backward parabolic equations: Properties of solutions. *SIAM Journal on Mathematical Analysis*, 49(3):2037–2060, 2017.

[9] M. Bildhauer and M.Fuchs. Twodimensional anisotropic variational problems. *Calc. Var.*, 16:177–186, 2003.

[10] M. Bildhauer and M.Fuchs. Higher integrability of the gradient for vectorial minimizers of decomposable variational integrals. *Manuscripta mathematica*, 123:269–283, 2007.

[11] V. Bögelein, F. Duzaar, and P. Marcellini. Parabolic equations with p, q-growth. *J. Math. Pures Appl*, 100:535–563, 2013.

[12] V. Bögelein, F. Duzaar, and P. Marcellini. Parabolic systems with p, q–growth: A variational approach. *Arch. Rational Mech. Anal.*, 210:219–267, 2013.

[13] V. Bögelein, F. Duzaar, and P. Marcellini. Existence of evolutionary variational solutions via the calculus of variations. *Journal of Differential Equations*, 256:3912–3942, 2014.

[14] Y. Brenier, C. De Lellis, and L. Székelyhidi Jr. Weak-strong uniqueness for measure-valued solutions. *Communications in Mathematical Physics*, 305:351–361, 2011.

[15] M. Caddick, M. Cullen, and E. Süli. Numerical approximation of young measure solutions to parabolic systems of forward-backward type. In preparation.

[16] Y. Cao and Y. Nie. Blow-up of solutions of the nonlinear Sobolev equation. *Applied Mathematics Letters*, 28:1–6, 2014.

[17] M. Carozza, J. Kristensen, and A. Passarelli di Napoli. Higher differentiability of minimizers of convex variational integrals. *Ann. I. H. Poincaré*, 28:395–411, 2011.

[18] C. Carstensen and T. Roubíček. Numerical approximation of Young measures in non-convex variational problems. *Numer. Math.*, 84:395–415, 2000.

[19] M. Croci, M. B. Giles, M. E. Rognes, and P. E. Farrell. Efficient white noise sampling and coupling for multilevel Monte Carlo with non-nested meshes. *ArXiv e-prints*, March 2018. arXiv:1803.04857.

[20] M. Crouzeix and V. Thomée. The stability in L_p and W_p^1 of the L_2-projection onto finite element function spaces. *Math. Comp.*, 48(178):521–532, 1987.

[21] G. Cupini, M. Guidorzi, and E. Mascolo. Regularity of minimizers of vectorial integrals with $p-q$ growth. *Nonlinear Analysis*, 54:591–616, 2003.

[22] G. Cupini, P. Marcellini, and E. Mascolo. Existence and regularity for elliptic equations under p,q-growth. *Adv. Differential Equations*, 19(7/8):693–724, 2014.

[23] S. Demoulini. Young measure solutions for a nonlinear parabolic equation of forward–backward type. *SIAM J. Math. Anal.*, 27:376–403, 1996.

[24] E. DiBenedetto. *Degenerate Parabolic Equations.* Universitext. Springer-Verlag, 1993.

[25] R. J. DiPerna. Measure-valued solutions to conservation laws. *Archive of Rational Mechanics and Anaysis*, 88:223–270, September 1985.

[26] R. J. DiPerna and A. J. Majda. Oscillations and concentrations in weak solutions of the incompressible fluid equations. *Comm. Math. Phys.*, 108(4):667–689, 1987.

[27] J. Douglas, Jr., T. Dupont, and L. Wahlbin. The stability in L^q of the L^2-projection into finite element function spaces. *Numer. Math.*, 23:193–197, 1974/75.

[28] M. Dreher and A. Jüngel. Compact families of piecewise constant functions in $L^p(0,T;B)$. *Nonlinear Anal.*, 75(6):3072–3077, 2012.

[29] R. E. Edwards. *Functional Analysis: Theory and Applications.* Holt, Rinehart and Winston, 1965.

[30] L. Esposito, F. Leonetti, and G. Mingione. Higher integrability for minimizers of integral functionals with p,q growth. *Journal of Differential Equations*, 157:414–438, 1999.

[31] L. Esposito, F. Leonetti, and G. Mingione. Regularity for minimizers of functionals with $p-q$ growth. *Nonlinear Differential Equations and Applications*, 6:133–148, 1999.

[32] L. Esposito, F. Leonetti, and G. Mingione. Regularity results for minimizers of irregular integrals with p, q growth. *Forum Mathematicum*, 14:245–272, 01 2002.

[33] L. Esposito, F. Leonetti, and G. Mingione. Sharp regularity for functions with p, q growth. *Journal of Differential Equations*, 204:5–55, 2004.

[34] E. Feireisl, P. Gwiazda, A. Świerczewska-Gwiazda, and E. Wiedemann. Dissipative measure-valued solutions to the compressible Navier-Stokes system. *Calc. Var.*, 55(6):141–161, 2016.

[35] U. S. Fjordholm, R. Käppeli, S. Mishra, and E. Tadmor. Construction of approximate entropy measure valued solutions for hyperbolic systems of conservation laws. *Foundations of Computational Mathematics*, 17(3):763–827, 2017.

[36] U. S. Fjordholm, S. Mishra, and E. Tadmor. On the computation of measure-valued solutions. *Acta Numerica*, 25:567–679, 2016.

[37] J. Frehse and M. Specovius-Neugebauer. Hölder continuous Young measure solutions to coercive non-monotone parabolic systems in two space dimensions. *Applicable Analysis*, 90(1):67–84, 2011.

[38] J.R. Garrat. *The Atmospheric Boundary Layer*. Cambridge University Press, second edition, 1992.

[39] Luca Giomi, Tanniemola B. Liverpool, and M. Cristina Marchetti. Sheared active fluids: Thickening, thinning, and vanishing viscosity. *Phys. Rev. E*, 81:051908, May 2010.

[40] V. Girault and P.-A. Raviart. *Finite element methods for Navier–Stokes equations: Theory and algorithms*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986.

[41] P. Guidotti. A backward–forward regularization of the Perona–Malik equation. *Journal of Differential Equations*, 252(4):3226 – 3244, 2012.

[42] P. Gwiazda, A. Swierczewska-Gwiazda, and E. Wiedemann. Weak-strong uniqueness for measure-valued solutions of some compressible fluid models. *Nonlinearity*, 28:3873–3890, 2015.

[43] N. Hungerbühler. *Young measures and nonlinear PDEs.* Habilitation thesis, University of Alabama, 1999.

[44] S. Kim and B. Yan. Convex Integration and Infinitely Many Weak Solutions to the Perona–Malik Equation in all dimensions. *SIAM Journal on Mathematical Analysis*, 47(4):2770–2794, 2015.

[45] S. Kim and B. Yan. On Lipschitz solutions for some forward–backward parabolic equations. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, 35(1):65 – 100, 2018.

[46] D. Kinderlehrer and P. Pedregal. Gradient Young measures generated by sequences in Sobolev spaces. *The Journal of Geometric Analysis*, 4(1), 1994.

[47] J.-L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications. Vol. I.* Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[48] Aurore Loisy, Jens Eggers, and Tanniemola Liverpool. Active suspensions have nonmonotonic flow curves and multiple mechanical equilibria. *Physical Review Letters*, 6 2018.

[49] P. Marcellini. Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions. *Arch. Rational Mech. Anal.*, 105:267–284, 1989.

[50] P. Marcellini. Regularity and existence of solutions of elliptic equations with p, q–growth conditions. *Journal of Differential Equations*, 90(1):1–30, 1991.

[51] C. Mascia, A. Terracina, and A. Tesei. Two-phase entropy solutions of a forward–backward parabolic equation. *Archive for Rational Mechanics and Analysis*, 194(3):887–925, Dec 2009.

[52] Stefan Müller. Variational models for microstructure and phase transitions. Max-Planck-Institut für Mathematik in den Naturwissenschaften, 1998.

[53] I. V. Nezhinskaya. Boundary estimate for the gradient of a solution to the Dirichlet problem for p, q–nonlinear equations. *Journal of Mathematical Sciences*, 120(2), 2004.

[54] I. V. Nezhinskaya. On solvability of the boundary-value problem for p, q–nonlinear elliptic and parabolic equations. *Journal of Mathematical Sciences*, 124(3), 2004.

[55] I. V. Nezhinskaya. Estimate for the gradient of the solution to the Neumann problem for p, q–nonlinear equations. *Journal of Mathematical Sciences*, 132(4), 2006.

[56] T. Roubíček. *Nonlinear Partial Differential Equations with Applications*, volume 153 of *International Series of Numerical Mathematics*. Birkhäuser, second edition, 2013.

[57] L. Simon. Application of monotone type operators to nonlinear pdes. http://www.cs.elte.hu/∼simonl/Nonlinear%20PDE's_book_hyperref.pdf.

[58] T. Singer. Parabolic equations with p, q-growth: The subquadratic case. *The Quarterly Journal of Mathematics*, 66(2):707–742, 2015.

[59] T. Singer. Existence of weak solutions of parabolic systems with p, q-growth. *Manuscripta Mathematica*, 151(1):87–112, 2016.

[60] F. Smarrazzo. On a class of equations with variable parabolicity direction. *Discrete & Continuous Dynamical Systems*, 22(3):729–758, 2008.

[61] F. Smarrazzo and A. Tesei. Degenerate regularization of forward–backward parabolic equations: The regularized problem. *Archive for Rational Mechanics and Analysis*, 204(1):85–139, Apr 2012.

[62] F. Smarrazzo and A. Tesei. Degenerate regularization of forward–backward parabolic equations: the vanishing viscosity limit. *Mathematische Annalen*, 355(2):551–584, Feb 2013.

[63] W. A. Strauss. On continuity of functions with values in various Banach spaces. *Pacific J. Math.*, 19:543–551, 1966.

[64] R. B. Stull. *An introduction to boundary layer meteorology.* Kluwer Academic Publishers, 1988.

[65] B. L. T. Thanh. *Forward–Backward Parabolic Equations.* PhD thesis, Sapienza University of Rome, 2013.

[66] B. L. T. Thanh, A. Dao Nguyen, and J. Ildefonso Diaz. Critical case for the viscous Cahn-Hilliard equation. *Electronic Journal of Differential Equations*, 2017:1–8, 07 2017.

[67] B. L. T. Thanh, F. Smarrazzo, and A. Tesei. Passage to the limit over small parameters in the viscous Cahn–Hilliard equation. *Journal of Mathematical Analysis and Applications*, 420(2):1265 – 1300, 2014.

[68] B. L. T. Thanh, F. Smarrazzo, and A. Tesei. Sobolev regularization of a class of forward–backward parabolic equations. *Journal of Differential Equations*, 257(5):1403 – 1456, 2014.

[69] B. L. T. Thanh and S. Tozza. On the Cahn–Hilliard regularization of a Perona–Malik type equation. *Advances in Mathematical Sciences and Applications*, 24:543–567, 01 2014.