ASYMPTOTIC APPROXIMATION OF EIGENVALUES OF VECTOR EQUATIONS

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Abstract. A vectorial extension of the Keller-Rubinow method of computing asymptotic approximations of eigenvalues in bounded domains is presented. The method is applied to the problem of a multimode step-profile cylindrical optical fibre, including the effects of polarisation. A comparison of the asymptotic results with the exact eigenvalues is made when these are available, and the agreement is shown to be good.

Key words. Optical waveguide, optical fibre, large eigenvalue asymptotics, ray theory, quantisation conditions.

1. Introduction.

We present an asymptotic approximation to the eigenvalues of a multimode step-profile cylindrical optical fibre including polarisation effects. Our approximation is based on the *ray theory* ansatz and uses an extension of a method due to Keller and Rubinow [7] for the computation of eigenvalues in bounded domains. The formulas in [7] apply to scalar waves; our extension generalises the method so that it can be applied to vectorial waves.

A *cylindrical optical fibre* is a dielectric waveguide that consists (essentially) of two concentric cylinders made of different materials. The interior one is called the *core* of the fibre and the exterior one is called the *cladding*. The core has a larger refractive index n_1 than the cladding n_2, which accounts for the guiding property of the fibre. In the case of a step refractive profile, both n_1 and n_2 are assumed to be constants (see Figure 1.1).

To use the ray theory approximation, we assume that the *waveguide parameter*

$$V = \frac{2\,\pi\,a}{\lambda}\,\sqrt{n_1^2 - n_2^2} = k\,a\,\sqrt{n_1^2 - n_2^2}, \tag{1.1}$$

is much greater than unity, where a is the radius of the fibre core, λ is the wavelength of the light shone into the fibre and k is its wavenumber (in vacuo). This means that the fibre is multimode, which is often the case for long-distance communication fibres, as well as for fibres used in many other applications. We can then represent an electromagnetic wave by a *ray*, which is orthogonal to its wavefront. For simplicity we assume that the cladding of the fibre has infinite radius, although it is not too difficult to make the modifications necessary for finite cladding.

Rays in a cylindrical step-profile optical fibre can be completely characterised by two angles: the inclination angle ϕ with the axial direction and the skewness angle θ between the tangent to the core-cladding interface and the projection of the ray path on a core cross-section (see Figure 1.2).

The reflection and refraction of rays off the interface between the core and the cladding is governed by the usual laws of geometrical optics (see e.g. [2]). If $\phi < \phi_c = \cos^{-1}(n_2/n_1)$ then the ray is totally internally reflected at this interface (and is known as a *bound ray*). There is almost no loss of energy as the ray propagates down the fibre, and the field in the cladding decays exponentially with distance from the interface, corresponding to the rays in the cladding being complex or evanescent.

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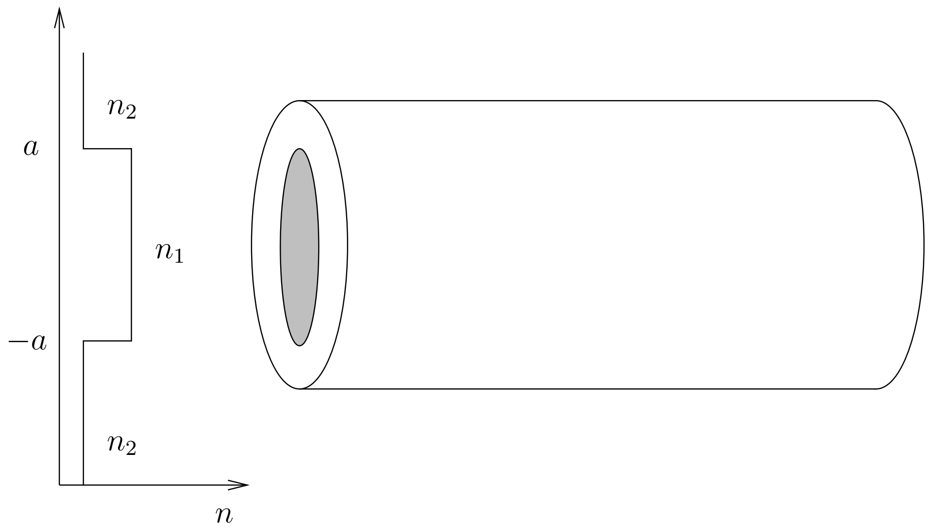

FIG. 1.1. *A step-profile circularly cylindrical optical fibre.*

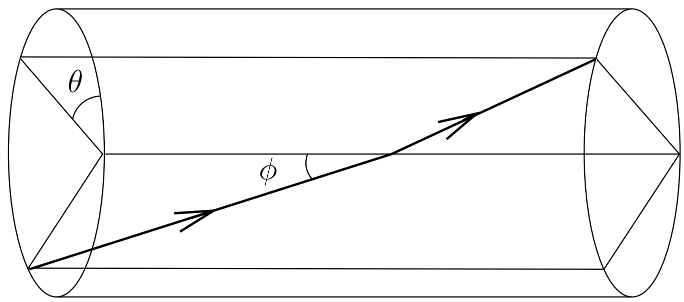

FIG. 1.2. *A ray reflecting from the core-cladding interface.*

If $\cos\phi < n_2/n_1$ and $\cos^2\phi + \sin^2\phi\cos^2\theta < n_2^2/n_1^2$ then the ray is refracted at the boundary between the core and the cladding, and the rays in the cladding are real and radiate energy to infinity.

In the intermediate case in which $\cos\phi < n_2/n_1$ but $\cos^2\phi + \sin^2\phi\cos^2\theta > n_2^2/n_1^2$ the rays are totally internally reflected at the boundary between the core and the cladding, but are complex only for a certain distance into the cladding (during which they are decaying exponentially), at which point (known as the *radiation caustic*) they become real and radiate energy to infinity. Such modes are known as *tunnelling modes*.

In Section 2 we present our vectorial extension of the Keller-Rubinow method and discuss how to apply it to the type of boundary conditions that are of importance for our fibres. We give a simple example of the use of the method in Section 3. In Section 4 we consider a multimode step-profile cylindrical optical fibre and compare our asymptotic results with exact eigenvalues. Finally, in Section 5, we present our conclusions.

2. Eigenvalues Relationships. Consider first the scalar Helmholtz equation in a bounded plane domain D:

$$\nabla^2\psi + k^2\psi = 0 \qquad \text{in } D, \tag{2.1}$$
$$\psi = 0 \qquad \text{on } \partial D. \tag{2.2}$$

Applying the ray ansatz

$$\psi \sim e^{iku}\sum_{n=0}^{\infty}\frac{A_n}{(ik)^n}, \tag{2.3}$$

leads to the eikonal equation

$$|\nabla u|^2 = 1 \tag{2.4}$$

and the recursive set of transport equations

$$2\nabla u\cdot\nabla A_0 + A_0\nabla^2 u = 0, \tag{2.5}$$
$$2\nabla u\cdot\nabla A_n + A_n\nabla^2 u = \nabla^2 A_{n-1}, \qquad n \geq 1. \tag{2.6}$$

The characteristics of (2.4) are the rays of geometrical optics, which are straight lines orthogonal to the wavefronts $u =$ constant. Its solution may be written as $u = u_0 + t$ where t is the distance along a ray measured in the direction of increasing u from the wavefront $u = u_0$. Since $\nabla u\cdot\nabla A = dA/dt$ the transport equations reduce to ordinary differential equations along rays; the solution of (2.5) may be written as

$$A_0 = a_0[\rho/(\rho + t)]^{1/2} \tag{2.7}$$

where ρ is the radius of curvature of the wavefront at $t = 0$. A curve on which $\rho+t = 0$ corresponds to a *caustic* of the rays. Locally the phase u is multivalued near a caustic, with the caustic corresponding to a branch point of u.

Using the ray ansatz (2.3) in the boundary condition (2.2) to give a boundary condition on the eikonal equation (2.4) we find that each time a ray is incident on the boundary a reflected ray is generated. The method of eigenvalue approximation developed by Keller (for a review see [6]) involves finding a family of ray congruences, each of which is defined on one copy of the domain D (or a subdomain of D), such that

$$\psi \sim \sum_{p=1}^{M} A_p e^{iku_p};$$

A_p and u_p are each viewed as different branches of two multiple-valued functions A and u, which are joined to one another at their boundaries (corresponding to the boundary of the domain ∂D and to caustics) in such a way as to make ∇u and $\nabla \log A$ continuous on the multi-sheeted space (which is called the covering space for ∇u). Since ψ is to be single-valued on D, the change in $iku + \log A$ around any closed path on the covering space must be an integer multiple of $2\pi i$, which we may write as

$$k\int_C \nabla u\cdot\mathbf{ds} = 2\pi m + i\int_C \nabla\log A\cdot\mathbf{ds}, \tag{2.8}$$

for all closed paths C. Since the integrals in (2.8) are the same for any two paths which can be continuously deformed into one another, there is a finite basis of independent paths, indexed by q say, each with its own quantum number m_q.

Often the change in $\log A$ can be determined by the number of times a ray crosses a caustic or meets the boundary. A local analysis reveals that as a curve passes from one sheet of the covering space to another through a caustic in the direction of increasing u the phase of A changes by $-\pi/2$ (see [6]) so that the change in $\log A$,

$$\Delta \log A = -\frac{i\pi}{2}.$$

With the homogeneous Dirichlet boundary condition (2.2) on ψ, the amplitude of the reflected ray is minus that of the incident ray, so that the amplitude changes sign as a curve passes from one sheet of the covering space to another via the boundary ∂D, so that its phase changes by π and

$$\Delta \log A = -i\pi.$$

If (2.2) is replaced with a homogeneous Neumann boundary condition on ψ, the amplitude of the reflected ray is equal to that of the incident ray, so that the amplitude is continuous as a curve passes from one sheet of the covering space to another via the boundary of D, and

$$\Delta \log A = 0.$$

Providing the change in $\log A$ at the boundary is independent of position on the boundary (as it is for these two boundary conditions), a_0 in (2.7) is independent of position on the wavefront, and the only changes in $\log A$ around a closed path occur at caustics and boundaries. In this case the quantisation conditions for a homogeneous Dirichlet boundary condition are

$$k \int_{C_q} \nabla u \cdot \mathbf{ds} = 2\pi \left(m_q + \frac{l_q}{4} + \frac{b_q}{2} \right).$$

where l_q is the number of times the curve C_q crosses a caustic in the positive direction, and b_q is the number of times it meets the boundary. The corresponding quantisation conditions for a homogeneous Neumann boundary condition are

$$k \int_{C_q} \nabla u \cdot \mathbf{ds} = 2\pi \left(m_q + \frac{l_q}{4} \right).$$

If the change in $\log A$ at the boundary depends on the position along the boundary then the theory needs to be modified in a non-trivial way as in [3]. Here we restrict our attention to situations in which the change in $\log A$ at the boundary is independent of position.

We will see how the quantisation conditions may be applied in a specific example shortly, but first let us see how the method should be modified when ψ is vectorial.

Consider the vectorial Helmholtz equation

$$\nabla^2 \boldsymbol{\psi} + k^2 \boldsymbol{\psi} = \mathbf{0} \qquad \text{in } D, \tag{2.9}$$

$$L_1 \boldsymbol{\psi} + L_2 \frac{\partial \boldsymbol{\psi}}{\partial n} = \mathbf{0} \qquad \text{on } \partial D, \tag{2.10}$$

where L_1 and L_2 are constant matrices. The ray ansatz

$$\boldsymbol{\psi} \sim e^{iku} \sum_{n=0}^{\infty} \frac{\mathbf{A}_n}{(ik)^n}, \tag{2.11}$$

may be applied as before, where u satisfies the eikonal equation, and each component of $\mathbf{A}$ satisfies the transport equations. Since each component of $\boldsymbol{\psi}$ satisfies the Helmholtz equation independently, each will undergo a phase change of $-\pi/2$ as we pass through a caustic in the direction of increasing u. The difference between the scalar and the vector cases occurs in the phase change at a boundary, where the components are all coupled.

Applying the boundary condition as a wave reflects gives[1]

$$L_1(\mathbf{A}_{\text{in}} + \mathbf{A}_{\text{out}}) + ik \sin\theta L_2(\mathbf{A}_{\text{in}} - \mathbf{A}_{\text{out}}) = \mathbf{0},$$

where θ is the angle the incoming ray makes with the tangent to the boundary. Hence

$$\mathbf{A}_{\text{out}} = P\mathbf{A}_{\text{in}}, \tag{2.12}$$

where

$$P = -(L_1 - ik\sin\theta L_2)^{-1}(L_1 + ik\sin\theta L_2).$$

In the vectorial case it is necessary that the vectorial amplitude is continuous across the joins of the covering space. This means that if we follow a path on the covering space which eventually brings us back to our original congruence, then the final amplitude which arises due to multiplication by the matrix P as we cross each join must be a multiple of the original amplitude, i.e. it is an eigenvalue of the product of these matrices. In the circular example we consider, there are only two congruences, and $\mathbf{A}_{\text{in}}$ is an eigenvector of P itself. When P has constant eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$ with corresponding eigenvalues λ_1 and λ_2, then we can write

$$\boldsymbol{\psi} \sim \mathbf{e}_j \sum_{p=1}^{M} A_p e^{iku_p},$$

and we are effectively back in the scalar case, with the change in $\log A$ at a caustic given by $-i\pi/2$, and the change in $\log A$ at the boundary ∂D given by $\log \lambda_j$, for $j = 1$, 2. Thus we have two families of eigenvalues, corresponding to the two eigenvectors, with the quantisation conditions

$$k \int_{C_q} \nabla u \cdot \mathbf{ds} = 2\pi \left(m_q + \frac{l_q}{4} + \frac{ib_q \log \lambda_j}{2\pi} \right), \qquad j = 1,\, 2. \tag{2.13}$$

Such a simple generalisation will work (without requiring the modifications of [3]) so long as P is independent of position on the boundary. Since P depends on θ this in particular requires that θ is the same for all rays in a family (as it is in the circular example we consider).

However, for many vectorial systems, including our example of an optical waveguide, the boundary condition depends on the orientation of the boundary through

[1] To achieve a non trivial balance here we assume that L_2 is $O(k^{-1}L_1)$.

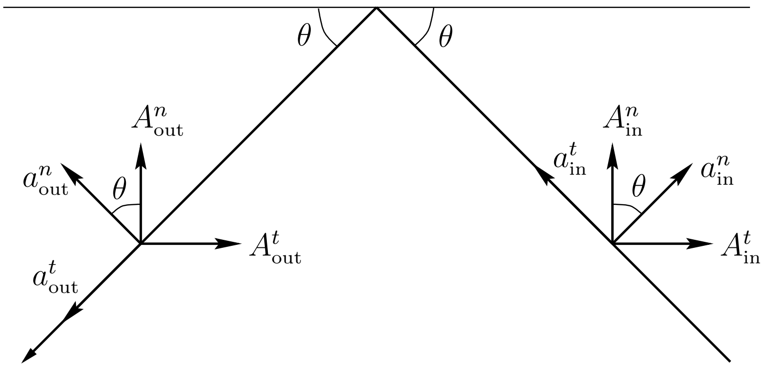

FIG. 2.1. *Polarisation diagram*

terms such as $\boldsymbol{\psi} \cdot \mathbf{n}$, which when written as (2.10) give L_1 and L_2 as functions of position. There is a second class of boundary conditions which we can treat simply, namely those for which $\boldsymbol{\psi}$ has dimension two and the boundary condition is independent of position when $\boldsymbol{\psi}$ is written in terms of components parallel and perpendicular to the boundary. In this case instead of working with polarisation relative to axes fixed in space, as in the example above, we need to work in terms of polarisation relative to axes determined by the ray directions.

So, let us now consider the boundary condition

$$L_1 \begin{pmatrix} \psi^t \\ \psi^n \end{pmatrix} + L_2 \begin{pmatrix} \frac{\partial \psi^t}{\partial n} \\ \frac{\partial \psi^n}{\partial n} \end{pmatrix} = 0 \qquad \text{on } \partial D. \tag{2.14}$$

where ψ^n and ψ^t are the normal and tangential components of $\boldsymbol{\psi}$, and the matrices L_1 and L_2 are independent of position on the boundary. Then, as before

$$\begin{pmatrix} A^t_{\text{out}} \\ A^n_{\text{out}} \end{pmatrix} = P \begin{pmatrix} A^t_{\text{in}} \\ A^n_{\text{in}} \end{pmatrix}, \tag{2.15}$$

where

$$P = -(L_1 - ik \sin\theta L_2)^{-1}(L_1 + ik \sin\theta L_2).$$

However, now we are not looking for eigenfunctions of P, since the basis of normal and tangential directions to the boundary is different for rays meeting the boundary at different points. Instead we use a basis for $\boldsymbol{\psi}$ relative to the ray direction (i.e. a polarisation). If a^n_{in} and a^t_{in} represent the components of $\mathbf{A}_{\text{in}}$ orthogonal to the incoming ray direction and in the direction of it, then (see Figure 2.1)

$$\begin{pmatrix} a^n_{\text{in}} \\ a^t_{\text{in}} \end{pmatrix} = \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} A^t_{\text{in}} \\ A^n_{\text{in}} \end{pmatrix} = \text{Rot}\left(\frac{\pi}{2} + \theta\right) \begin{pmatrix} A^t_{\text{in}} \\ A^n_{\text{in}} \end{pmatrix}.$$

Similarly

$$\begin{pmatrix} a^n_{\text{out}} \\ a^t_{\text{out}} \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} A^t_{\text{out}} \\ A^n_{\text{out}} \end{pmatrix} = \text{Rot}\left(\frac{\pi}{2} - \theta\right) \begin{pmatrix} A^t_{\text{out}} \\ A^n_{\text{out}} \end{pmatrix}.$$

Thus the transformation of the amplitude at the boundary is given by

$$\begin{pmatrix} a^n_{\text{out}} \\ a^t_{\text{out}} \end{pmatrix} = \text{Rot}\left(\theta + \frac{\pi}{2}\right) P \,\text{Rot}\left(\theta - \frac{\pi}{2}\right) \begin{pmatrix} a^n_{\text{in}} \\ a^t_{\text{in}} \end{pmatrix} = Q \begin{pmatrix} a^n_{\text{in}} \\ a^t_{\text{in}} \end{pmatrix}, \tag{2.16}$$

say, where Q is independent of position on the boundary. Since the polarisation of a ray is constant along it, the ray amplitudes are now given by eigenfunctions of products of Q. Again, in the case of a circle they are given by eigenfunctions of Q itself; suppose these are given by $\mathbf{e}_1$ and $\mathbf{e}_2$ with corresponding eigenvalues λ_1 and λ_2. Then, if we write $\mathbf{e}_i = (e_i^1, e_i^2)$ and

$$\boldsymbol{\psi} \sim \sum_{p=1}^{M} (e_i^1 \nabla u_p + e_i^2 (\nabla u_p)^\perp) A_p e^{iku_p},$$

where $(\nabla u_p)^\perp$ is the rotation of ∇u_p by $\pi/2$, then we are effectively back in the scalar case; since ∇u is independent of distance along a ray t, the amplitude A again satisfies the transport equations, with the change in $\log A$ at a caustic given by $-i\pi/2$, and the change in $\log A$ at the boundary ∂D given by $\log \lambda_j$, for $j = 1$, 2.

Our example of an optical waveguide, considered in Section 4, does not have a boundary condition of the form (2.14), but rather complicated continuity conditions between wavefields in the interior of the domain D and exterior to it. However, these conditions can be used to determine the matrix P in (2.15), from which the quantisation conditions can be determined.

3. A simple example. To illustrate the ideas of the previous section we consider a simple artificial example of boundary coupled Helmholtz equations in a circle of radius a, before we go on to consider the more complicated real problem of determining the eigenmodes of a step-profile optical fibre.

Consider the coupled system

$$\nabla^2\phi + k^2\phi = 0 \qquad \text{in } r < a, \tag{3.1}$$

$$\nabla^2\psi + k^2\psi = 0 \qquad \text{in } r < a, \tag{3.2}$$

$$\phi = \alpha\psi + \beta\frac{\partial\psi}{\partial r} \qquad \text{on } r = a, \tag{3.3}$$

$$\frac{\partial\phi}{\partial r} = \gamma\psi + \delta\frac{\partial\psi}{\partial r} \qquad \text{on } r = a; \tag{3.4}$$

where r is the radial distance, and α, β, γ and δ are constant. The ray congruences for this problem are of course identical to those for the scalar Helmholtz equation on a circle, as described in [6, 7]. A ray tangent to a concentric circle of radius $b < a$ will produce a reflected ray which is also tangent to this concentric circle, which in turn produces another such ray, etc. The ray congruences comprise two families of rays tangent to the circle of radius b: the outward directed and inward directed anticlockwise travelling rays, as shown in Figure 3.1. The outgoing rays reflect into the incoming rays at the outer boundary $r = a$, and the incoming rays become outgoing rays as they pass through the caustic at $r = b$. Each family of rays determines one branch of u in the annulus $b < r < a$, and they are joined together at the boundaries $r = b$ and $r = a$ to produce the covering space, which is topologically a torus.

Applying the boundary conditions (3.3), (3.4) as a ray reflects gives the matrix P of (2.12) as[2]

$$P = \begin{pmatrix} \frac{-\gamma+(\alpha+\delta)ik\sin\theta+\beta k^2\sin^2\theta}{(\gamma+(\alpha-\delta)ik\sin\theta+\beta k^2\sin^2\theta)} & \frac{-2(\alpha\delta-\beta\gamma)ik\sin\theta}{(\gamma+(\alpha-\delta)ik\sin\theta+\beta k^2\sin^2\theta)} \\ \frac{2ik\sin\theta}{(\gamma+(\alpha-\delta)ik\sin\theta+\beta k^2\sin^2\theta)} & \frac{2ik\sin\theta\gamma+(\alpha+\delta)ik\sin\theta-\beta k^2\sin^2\theta}{(\gamma+(\alpha-\delta)ik\sin\theta+\beta k^2\sin^2\theta)} \end{pmatrix},$$

[2] In using the ray approximation here it is necessary to assume that β is $O(k^{-1})$ and γ is $O(k)$.

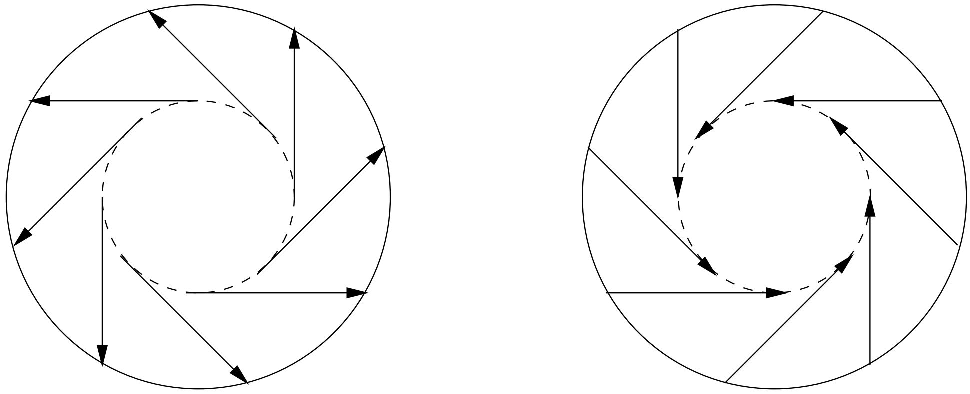

FIG. 3.1. *The two congruences of rays in a circular domain.*

with eigenvalues

$$\lambda_{1,2} = \frac{-\gamma + \beta k^2 \sin^2\theta \pm ik\sin\theta\left((\alpha-\delta)^2 + 4\gamma\beta\right)^{1/2}}{\gamma + (\alpha-\delta)ik\sin\theta + \beta k^2\sin^2\theta}. \tag{3.5}$$

Now, there are two linearly independent paths on a torus, so there are two quantum conditions. Following [7] we choose the first path as the caustic itself. Since the rays are all tangent to this path, the change in u as we traverse it is simply its length, which is $2\pi b$. The path does not cross the caustic or meet the boundary, so that the first quantum condition is simply

$$2\pi kb = 2\pi m_1, \qquad m_1 \in \mathbb{Z}. \tag{3.6}$$

For the second path we also follow [7] and choose a ray from the caustic to the boundary, its reflection back to the caustic, and the arc of the caustic between them (see Figure 3.2). The change in u along this path is again equal to its length, which is

$$2\left((a^2-b^2)^{1/2} - b\cos^{-1}(b/a)\right).$$

The path crosses the caustic once, and meets the boundary once. Hence the quantum condition is

$$2k\left((a^2-b^2)^{1/2} - b\cos^{-1}(b/a)\right) = 2\pi\left(m_2 + \frac{1}{4}\right) + i\log\lambda_j, \qquad j = 1,\, 2, \tag{3.7}$$

where $\lambda_{1,2}$ are given by (3.5) with $\cos\theta = b/a$. Equations (3.6) and (3.7) are two conditions which determine the radius of the caustic b and the eigenvalue k.

The exact solution is easily found for this simple example, and the eigenvalue condition is found to be

$$\frac{J_{m_1}(k)}{kJ'_{m_1}(k)} = \frac{(\alpha-\delta) \pm \left((\alpha-\delta)^2 + 4\gamma\beta\right)^{1/2}}{2\gamma}, \tag{3.8}$$

where J_m is the Bessel function of the first kind. A comparison of the exact eigenvalues with the approximate ones is given in Table 3.1.

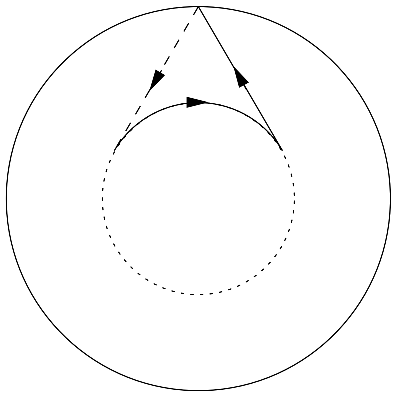

FIG. 3.2. *A closed curve on the toroidal covering space comprising a ray from the caustic to the boundary, its reflection back to the caustic, and the arc of the caustic between them.*

m_1	m_2	λ	$+/-$	Exact	Approx	Rel. Error
0	0	λ_2	$-$	2.17950	2.14491	0.0159
0	1	λ_1	$+$	2.65204	2.61165	0.0152
0	1	λ_2	$-$	5.03321	5.03161	0.0003
0	2	λ_1	$+$	6.04191	6.04123	0.0001
0	2	λ_2	$-$	7.95688	7.96667	0.0012
1	0	λ_2	$-$	3.47976	3.44886	0.0089
1	1	λ_1	$+$	4.21418	4.20564	0.0020
1	1	λ_2	$-$	6.42015	6.41982	0.0001
1	2	λ_1	$+$	7.64602	7.65664	0.0014
1	2	λ_2	$-$	9.39261	9.40234	0.0010
2	0	λ_2	$-$	4.67335	4.62500	0.0103
2	1	λ_1	$+$	5.63278	5.65494	0.0034
2	1	λ_2	$-$	7.72842	7.72136	0.0009
2	2	λ_1	$+$	9.13867	9.16189	0.0025
2	2	λ_2	$-$	10.7665	10.7722	0.0005
3	0	λ_2	$-$	5.81732	5.74432	0.0125
3	1	λ_1	$+$	6.97902	7.03109	0.0075
3	1	λ_2	$-$	8.98940	8.97251	0.0019
3	2	λ_1	$+$	10.5614	10.5970	0.0034
3	2	λ_2	$-$	12.0978	12.0978	0.0000

TABLE 3.1

Comparison of approximate and exact eigenvalues for (3.1)–(3.4) with $\alpha = 1$, $\beta = 0.1$, $\gamma = 10$ and $\delta = 1$. The column λ indicates which eigenvalue of P is used in the approximation, while the column $+/-$ indicates which sign is taken in the exact solution (3.8). We see that λ_1 corresponds to taking the plus sign in (3.8), and λ_2 corresponds to taking the minus sign. The relative error is equal to the absolute value of the exact eigenvalue minus the approximation divided by the exact eigenvalue.

4. Eigenvalues of a circularly cylindrical optical fibre. We now consider the problem of determining the eigenfunctions and eigenvalues associated with an optical waveguide. The electric and magnetic fields **E** and **H** in the fibre must satisfy

Maxwell's equations

$$\text{curl } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \qquad \text{curl } \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0, \tag{4.1}$$
$$\text{div } \mathbf{D} = 0, \qquad \text{div } \mathbf{B} = 0;$$

together with the constitutive relations

$$\mathbf{B} = \mu \mathbf{H}, \qquad \mathbf{D} = \epsilon \mathbf{E};$$

where, since we are assuming a step profile in the refractive index of the fibre, the permittivity μ and permeability ϵ will be assumed constant in each of the core (occupying $r < a$) and the cladding (which we take to occupy $r > a$, with radiation/decay conditions at infinity). Modifications can be made for a finite cladding by using complex rays as in [4]; the correction to the eigenvalues is exponentially small for bound and tunnelling modes. It is usual for optical fibres for μ to be the same in the core and the cladding, and for the guiding property to be due to variations in ϵ; however, the analysis is no harder if both μ and ϵ are allowed to vary, and we treat this more general case.

We look for modes which are time harmonic with (real) frequency ω, and propagate in the z-direction with wavenumber β. Thus we write the fields in phasor notation as

$$\mathbf{E}(x,y,z,t) = \text{Re}\{\mathbf{E}'(x,y)\, e^{i\,(\beta z - \omega t)}\}, \quad \mathbf{H}(x,y,z,t) = \text{Re}\{\mathbf{H}'(x,y)\, e^{i(\beta z - \omega t)}\}. \tag{4.2}$$

Then Maxwell's equations are satisfied providing (dropping the primes without fear of confusion)

$$\nabla^2 \mathbf{E} + k_j^2 \mathbf{E} = \mathbf{0}, \qquad \nabla^2 \mathbf{H} + k_j^2 \mathbf{H} = \mathbf{0}, \qquad \text{div } \mathbf{E} = \text{div } \mathbf{H} = 0, \qquad j = 1,\, 2, \tag{4.3}$$

where ∇^2 is the two-dimensional Laplacian in the xy-plane, and $k_j^2 = \kappa_j^2 - \beta^2$, with $\kappa_j = \omega\sqrt{\mu_j \epsilon_j} = k n_j$, where k is the wavenumber in vacuo and n_j is the refractive index; we use the subscript 1 to denote the core and 2 to denote the cladding. Bound rays correspond to $\kappa_2 \le \beta \le \kappa_1$ (so that k_1 is real and k_2 is imaginary), while *leaky rays* (refracting and tunnelling rays) correspond to $0 \le \beta < \kappa_2$ (so that both k_1 and k_2 are real). Note that we need only solve (4.3) for $\mathbf{E}$ or $\mathbf{H}$ since the other can then be determined from Maxwell's equations (4.1). In particular, if we solve (4.3) for $\mathbf{E}$ then $\mathbf{H}$ is given by

$$\omega\mu H_x = -\beta E_y - i\frac{\partial E_z}{\partial y}, \qquad \omega\mu H_y = i\frac{\partial E_z}{\partial x} + \beta E_x, \qquad i\omega\mu H_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}; \tag{4.4}$$

where $\mathbf{H} = (H_x, H_y, H_z)$ and $\mathbf{E} = (E_x, E_y, E_z)$.

We consider the high-frequency limit in which $k_1,\, k_2 \to \infty$ with k_1/k_2 bounded. Usually, the frequency ω is known, and the axial wavenumber β is an eigenvalue, which we aim to determine. Using the ray ansatz

$$\mathbf{E}(x,y) \sim \exp(ik_j u(x,y)) \sum_{r=0}^{\infty} \frac{\mathbf{A}_r(x,y)}{(ik_j)^r}, \tag{4.5}$$

in each of the core and the cladding we obtain as usual at leading order the eikonal equation

$$|\nabla u|^2 = 1, \tag{4.6}$$

and at higher orders the system of transport equations

$$(\nabla^2 u)\mathbf{A}_0 + 2(\nabla u \cdot \nabla)\mathbf{A}_0 = 0, \tag{4.7}$$

$$(\nabla^2 u)\mathbf{A}_r + 2(\nabla u \cdot \nabla)\mathbf{A}_r = \nabla^2 \mathbf{A}_{r-1}, \qquad r \geq 1. \tag{4.8}$$

At the boundary between the core and the cladding we must impose the continuity of $\mathbf{E} \wedge \mathbf{n}$, $\epsilon \mathbf{E} \cdot \mathbf{n}$, $\mathbf{H} \wedge \mathbf{n}$, and $\mu \mathbf{H} \cdot \mathbf{n}$. Thus the boundary conditions written in terms of E_x, E_y, E_z etc will vary around the boundary (as the normal varies), but they are invariant when written in terms of the normal and tangential components of $\mathbf{E}$ and $\mathbf{H}$.

In the high frequency limit we can determine the reflection and transmission coefficients to leading order by considering reflection at a flat interface. To this end let us consider a wave (projected into the xy-plane) propagating in the direction $(-\cos\theta, \sin\theta)$ striking a plane boundary $y = 0$ (see Figure 4.1). Then, denoting the incident, reflected and transmitted fields by $\mathbf{E}^i$, $\mathbf{E}^r$ and $\mathbf{E}^t$ respectively we have

$$\begin{aligned}
\mathbf{E}^i(x,y) &= (I_1, I_2, I_3) \quad e^{ik_1(-x\cos\theta + y\sin\theta)}, \\
\mathbf{E}^r(x,y) &= (R_1, R_2, R_3)\, e^{ik_1 u_1(x,y)}, \\
\mathbf{E}^t(x,y) &= (T_1, T_2, T_3) \;\; e^{ik_2 u_2(x,y)},
\end{aligned}$$

where u_1, u_2 and the complex constants R_1, R_2, R_3, T_1, T_2 and T_3 and determined in terms of the complex constants I_1, I_2, I_3 by the continuity of E_x, ϵE_y, E_z, H_x, μH_y and H_z. It is easily seen that $u_1 = -x\cos\theta - y\sin\theta$, while u_2 must satisfy the eikonal equation subject to $u_2(x, 0) = -(k_1/k_2)x\cos\theta$, giving

$$u_2(x,y) = -\frac{xk_1\cos\theta}{k_2} + y\sqrt{1 - \frac{k_1^2\cos^2\theta}{k_2^2}}, \tag{4.9}$$

corresponding to the transmission of a wave at an angle θ^* to the tangent to the interface, where $k_2\cos\theta^* = k_1\cos\theta$, which is of course just Snel's law. If $\theta < \cos^{-1}(k_2/k_1)$ then θ^* is complex and the transmitted ray is evanescent (we have total internal reflection), with u_2 given by

$$k_2 u_2(x,y) = -xk_1\cos\theta + iy\sqrt{k_1^2\cos^2\theta - k_2^2}, \tag{4.10}$$

The boundary conditions now give us six equations:

$$I_1 + R_1 = T_1, \tag{4.11}$$

$$\epsilon_1 I_2 + \epsilon_1 R_2 = \epsilon_2 T_2, \tag{4.12}$$

$$I_3 + R_3 = T_3, \tag{4.13}$$

$$-\mu_2\beta(I_2 + R_2) + k_1\mu_2\sin\theta(I_3 - R_3) = -\mu_1\beta T_2 + i\mu_1\sqrt{k_1^2\cos^2\theta - k_2^2}\, T_3, \tag{4.14}$$

$$\beta(I_1 + R_1) + k_1\cos\theta(I_3 + R_3) = \beta T_1 + k_1\cos\theta\, T_3, \tag{4.15}$$

$$k_1\mu_2\cos\theta(I_2 + R_2) + k_1\mu_2\sin\theta(I_1 - R_1) = k_1\mu_1\cos\theta T_2 + i\mu_1\sqrt{k_1^2\cos^2\theta - k_2^2}\, T_1. \tag{4.16}$$

However, these conditions are not all independent (we see immediately that (4.15) follows from (4.11) and (4.13)). Extra equations arise from the constraint that $\mathbf{E}$ and

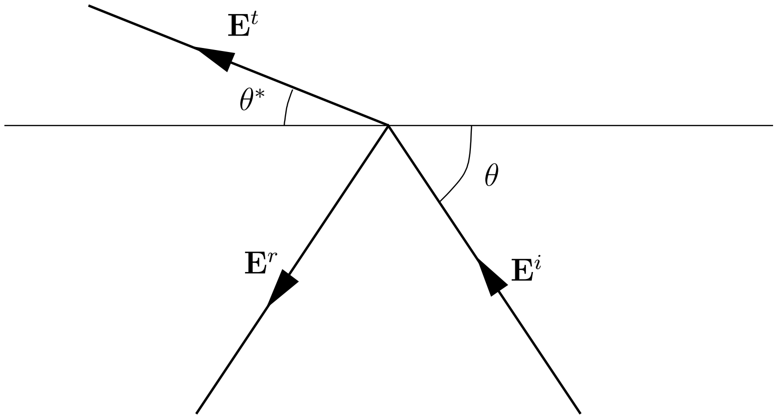

FIG. 4.1. *Reflection and transmission of a ray at a planar interface.*

H are divergence free so that each of **E** and **H** is orthogonal to its wavenumber vector (note that this also imposes a constraint on the incoming wave (I_1, I_2, I_3)):

$$-k_1 I_1 \cos\theta + k_1 I_2 \sin\theta + \beta I_3 = 0, \tag{4.17}$$

$$-k_1 R_1 \cos\theta - k_1 R_2 \sin\theta + \beta R_3 = 0, \tag{4.18}$$

$$-k_1 T_1 \cos\theta + i T_2 \sqrt{k_1^2 \cos^2\theta - k_2^2} + \beta T_3 = 0. \tag{4.19}$$

Using (4.17)-(4.19) we can easily show that (4.12) follows from (4.14) and (4.16). Thus, between them (4.11)-(4.16) and (4.17)-(4.19) provide six equations for our six unknowns R_1, R_2, R_3, T_1, T_2 and T_3 in terms of I_1 and I_2. Eliminating I_3, R_3 and T_3 and using (4.11)-(4.14) as our independent equations we have

$$\begin{aligned} R_1 - T_1 &= -I_1, \\ R_2 - \epsilon T_2 &= -I_2, \\ \sin\theta R_2 + i(\cos^2\theta - K^2)^{1/2} T_2 &= \sin\theta I_2 \\ \cos\theta(\mu R_2 - T_2) - \sin\theta \mu R_1 - i(\cos^2\theta - K^2)^{1/2} T_1 &= -\sin\theta \mu I_1 - \cos\theta \mu I_2, \end{aligned}$$

where $K = k_2/k_1$, $\epsilon = \epsilon_2/\epsilon_1$, $\mu = \mu_2/\mu_1$, with solution

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = P_r \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \qquad \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = P_t \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \tag{4.20}$$

where

$$P_r = \begin{pmatrix} \frac{\mu\sin\theta - i(\cos^2\theta - K^2)^{1/2}}{\mu\sin\theta + i(\cos^2\theta - K^2)^{1/2}} & \frac{(\epsilon\mu - 1)\sin(2\theta)}{(\mu\sin\theta + i(\cos^2\theta - K^2)^{1/2})(\epsilon\sin\theta + i(\cos^2\theta - K^2)^{1/2})} \\ 0 & \frac{\epsilon\sin\theta - i(\cos^2\theta - K^2)^{1/2}}{\epsilon\sin\theta + i(\cos^2\theta - K^2)^{1/2}} \end{pmatrix}, \tag{4.21}$$

and

$$P_t = \begin{pmatrix} \frac{2\mu\sin\theta}{\mu\sin\theta + i(\cos^2\theta - K^2)^{1/2}} & \frac{(\epsilon\mu - 1)\sin(2\theta)}{(\mu\sin\theta + i(\cos^2\theta - K^2)^{1/2})(\epsilon\sin\theta + i(\cos^2\theta - K^2)^{1/2})} \\ 0 & \frac{2\sin\theta}{\epsilon\sin\theta + i(\cos^2\theta - K^2)^{1/2}} \end{pmatrix}. \tag{4.22}$$

As in Section 2 we need to translate this to the amplitudes of the components of **E** normal and orthogonal to the ray directions rather than normal and orthogonal to the reflecting boundary. If i_1 is the amplitude normal to the ray direction, and i_2 the amplitude tangential to it (see Figure 4.2), and similarly for r_1 and r_2, then the transformation of the amplitude at the boundary is given by

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = Q \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} \quad \text{where} \quad Q = \operatorname{Rot}\left(\theta + \frac{\pi}{2}\right) P_r \operatorname{Rot}\left(\theta - \frac{\pi}{2}\right). \tag{4.23}$$

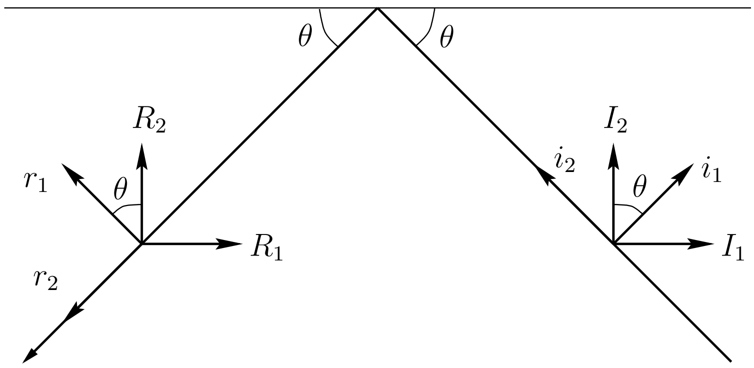

FIG. 4.2. *Polarisation diagram*

Since we are again dealing with a circular domain (the cross section of the fibre), the ray congruences are identical to those of Section 3, and we may choose the same two independent paths on the toroidal covering space. Thus the quantum conditions are

$$2\pi k_1 b = 2\pi m_1, \tag{4.24}$$

$$k_1 \left(a^2 - b^2\right)^{1/2} = k_1 b\theta + \left(m_2 + \frac{1}{4} + \frac{i \log \lambda_j}{2\pi}\right)\pi, \qquad j = 1, 2, \tag{4.25}$$

with $\cos\theta = b/a$, where the eigenvalues λ_1 and λ_2 of Q are given by

$$\lambda_1 = r + is, \qquad \lambda_2 = r - is, \tag{4.26}$$

where

$$r = \frac{K^2 - \epsilon\mu + (1 + \epsilon\mu - 2K^2)\cos^2\theta}{(\mu \sin\theta + i(\cos^2\theta - K^2)^{1/2})(\epsilon \sin\theta + i(\cos^2\theta - K^2)^{1/2})}, \tag{4.27}$$

$$s = \frac{\sin\theta \left[((\epsilon + \mu)^2 + 4K^2(K^2 - 1 - \epsilon\mu))\cos^2\theta - K^2(\epsilon - \mu)^2\right]^{1/2}}{(\mu \sin\theta + i(\cos^2\theta - K^2)^{1/2})(\epsilon \sin\theta + i(\cos^2\theta - K^2)^{1/2})}. \tag{4.28}$$

For bound and tunnelling rays (for which we have evanescent rays in the cladding) λ_j is of modulus one and k_1 is real; reflection at the boundary introduces only a phase shift in the ray (note that r and s are not real). Bound rays are those for which k_2 (and therefore K) is imaginary ($\kappa_2 < \beta < \kappa_1$). Tunnelling rays are those for which $K^2 - \cos^2\theta < 0$ so that u_2 is complex until the radiation caustic at $r = b/k_2 > a$.

4.1. Limiting cases. We can solve (4.24), (4.25) explicitly in two asymptotic limits, corresponding to *very skew rays* and *meridional bound rays.*

If the rays are very skew, we have that $\theta \ll 1$. Then

$$b \sim a\left(1 - \frac{\theta^2}{2} + \cdots\right)$$

so that

$$k_1 a \sim m_1\left(1 + \frac{\theta^2}{2} + \cdots\right).$$

Equation (4.25) gives

$$k_1 a\left(\theta - \frac{\theta^3}{6} + \cdots\right) \sim k_1 a\theta\left(1 - \frac{\theta^2}{2} + \cdots\right) + \left(m_2 + \frac{1}{4} + \frac{\log \lambda_j}{2\pi}\right)\pi,$$

while

$$r \sim -1 - \frac{i(\epsilon+\mu)\theta}{\sqrt{1-K^2}} + \cdots, \qquad s \sim -\frac{\theta\left[(\epsilon+\mu)^2 - 4K^2\right]^{1/2}}{\sqrt{1-K^2}} + \cdots,$$

so that

$$\lambda_1 \sim -1 - \frac{i\theta}{\sqrt{1-K^2}}\left(\epsilon + \mu + \left[(\epsilon+\mu)^2 - 4K^2\right]^{1/2}\right) + \cdots,$$
$$\lambda_2 \sim -1 - \frac{i\theta}{\sqrt{1-K^2}}\left(\epsilon + \mu - \left[(\epsilon+\mu)^2 - 4K^2\right]^{1/2}\right) + \cdots,$$

Hence

$$k_1\theta^3 \sim 3\left(m_2 + \frac{3}{4}\right)\pi,$$

so that

$$k_1 a \sim m_1 + \frac{m_1^{1/3}}{2}\left[3\pi\left(m_2 + \frac{3}{4}\right)\right]^{2/3} + \cdots.$$

This equation is valid both for bound and tunnelling rays (the only possible ones in this regime) and is the same that was obtained by Keller and Rubinow [7] for the scalar Helmholtz equation with Dirichlet boundary conditions in the *whispering gallery limit.* Note that in this limit both eigenvalues λ_1 and λ_2 tend to -1, so that the eigenvalues for both polarisations are close together.

On the other hand, for meridional bound rays, we have $m_1 \ll k_1 a$, $\theta \sim \pi/2$ and

$$r \sim \frac{K^2 - \epsilon\mu}{(\epsilon + i\sqrt{-K^2})(\mu + i\sqrt{-K^2})} + O\left(\left(\theta - \frac{\pi}{2}\right)^2\right),$$
$$s \sim \frac{(\epsilon-\mu)\sqrt{-K^2}}{(\epsilon + i\sqrt{-K^2})(\mu + i\sqrt{-K^2})} + O\left(\left(\theta - \frac{\pi}{2}\right)^2\right).$$

If we take k_2 (and therefore K) to be positive imaginary, writing $K = i\gamma$, then $i\sqrt{-K^2} = i\gamma$, and

$$\lambda_1 \sim \frac{-\gamma^2 - \epsilon\mu + i\gamma(\epsilon - \mu)}{(\epsilon + i\gamma)(\mu + i\gamma)} = -\frac{\epsilon - i\gamma}{\epsilon + i\gamma},$$
$$\lambda_2 \sim \frac{-\gamma^2 - \epsilon\mu - i\gamma(\epsilon - \mu)}{(\epsilon + i\gamma)(\mu + i\gamma)} = -\frac{\mu - i\gamma}{\mu + i\gamma}.$$

Then

$$\log \lambda_1 = i\left(\pi - 2\tan^{-1}\gamma/\epsilon\right), \tag{4.29}$$
$$\log \lambda_2 = i\left(\pi - 2\tan^{-1}\gamma/\mu\right). \tag{4.30}$$

Now,

$$K^2 = \frac{k_2^2}{k_1^2} = \frac{k_1^2 + \omega^2(\epsilon_2\mu_2 - \epsilon_1\mu_1)}{k_1^2} = \frac{k_1^2 - k^2(n_1^2 - n_2^2)}{k_1^2},$$

where k is the wavenumber of the incoming light in vacuo. Hence

$$\gamma = \frac{\sqrt{k^2(n_1^2 - n_2^2) - k_1^2}}{k_1}.$$

Now, since $b \sim -a(\theta - \pi/2)$ as $\theta \to \pi/2$, the eigenvalue relation is

$$k_1 a \sim m_1\frac{\pi}{2} + \left(m_2 + \frac{1}{4} + \frac{i\log\lambda_j}{2\pi}\right)\pi,$$

giving

$$k_1 \sim M + \frac{1}{a}\tan^{-1}\frac{\sqrt{k^2(n_1^2 - n_2^2) - M^2}}{M\epsilon} + \cdots,$$
$$k_1 \sim M + \frac{1}{a}\tan^{-1}\frac{\sqrt{k^2(n_1^2 - n_2^2) - M^2}}{M\mu} + \cdots,$$

with $M = \pi(2m_1 + 4m_2 - 1)/(4a)$.

Keller and Rubinow obtained M as the leading-order term for k_1 in their *bouncing ball limit* for the scalar Helmholtz equation with Dirichlet boundary conditions (see [7]).

4.2. Exact Solutions. For the case of cylindrical optical fibres with a step refractive profile, it is possible to compute exact solutions. Indeed, these solutions can be found in several places [1, 8, 9]. It is easiest to work with radial, azimuthal and axial components of **E** and **H**, denoted with a subscript r, θ and z respectively. For bound rays we have

$$E_r = e^{im_1\theta}\left(AJ_{m_1-1}(k_1 r) + BJ_{m_1+1}(k_1 r)\right) \qquad \text{in } r < a,$$
$$E_\theta = e^{im_1\theta}\left(iAJ_{m_1-1}(k_1 r) - iBJ_{m_1+1}(k_1 r)\right) \qquad \text{in } r < a,$$
$$E_r = e^{im_1\theta}\left(CK_{m_1-1}(wr) + DK_{m_1+1}(wr)\right) \qquad \text{in } r > a,$$
$$E_\theta = e^{im_1\theta}\left(iCK_{m_1-1}(wr) - iDK_{m_1+1}(wr)\right) \qquad \text{in } r > a;$$

with E_z is determined by the constraint that **E** is divergence free and **H** given by (4.4), where J_m is the Bessel function of the first kind, K_m is the modified Bessel function, and $k_2 = iw$, $w > 0$. Imposing the continuity of $\mathbf{E} \wedge \mathbf{n}$, $\epsilon\mathbf{E} \cdot \mathbf{n}$, $\mathbf{H} \wedge \mathbf{n}$, and $\mu\mathbf{H} \cdot \mathbf{n}$ as before gives the eigenvalue relationship as

$$\left[\frac{\epsilon_1 J'_{m_1}(ak_1)}{ak_1\, J_{m_1}(ak_1)} + \frac{\epsilon_2 K'_{m_1}(aw)}{aw\, K_{m_1}(aw)}\right]\left[\frac{\mu_1 J'_{m_1}(ak_1)}{ak_1\, J_{m_1}(ak_1)} + \frac{\mu_2 K'_{m_1}(aw)}{aw\, K_{m_1}(aw)}\right]$$
$$= \frac{\beta^2 m_1^2 \omega^2 (\epsilon_1\mu_1 - \epsilon_2\mu_2)^2}{a^4 k_1^4 w^4} = \left(\frac{m_1\,\beta}{\omega}\right)^2 \left(\frac{V}{(ak_1)(aw)}\right)^4, \qquad (4.31)$$

where V is the waveguide parameter of equation (1.1). In expression (4.31), m_1 is exactly equivalent to m_1 in equation (4.24). There is no direct equivalence, though, to the constant m_2 in (4.25); (4.31) simply has several solutions for each m_1 and we have to order them in increasing order.

4.2.1. Asymptotic approximation of the exact solution. As a check on our asymptotic solution, let us show how it arises via a direct asymptotic expansion of (4.31). Using the Debye expansion of the Bessel functions

$$\frac{J'_\nu(\nu z)}{J_\nu(\nu z)} \sim -\frac{(z^2-1)^{1/2}}{z} \tan\left(\nu(z^2-1)^{1/2} - \nu\cos^{-1}(1/z) - \pi/4\right),$$
$$\frac{K'_\nu(\nu z)}{K_\nu(\nu z)} \sim -\frac{(1+z^2)^{1/2}}{z}$$

as $\nu \to \infty$ with z fixed, the exact expression (4.31) is

$$\left[\frac{\epsilon_1(a^2k_1^2 - m_1^2)^{1/2}}{a^2k_1^2} \tan\left((a^2k_1^2 - m_1^2)^{1/2} - m_1\cos^{-1}\left(\frac{m_1}{ak_1}\right) - \frac{\pi}{4}\right) + \frac{\epsilon_2(a^2w^2 + m_1^2)^{1/2}}{a^2w^2}\right]$$
$$\times\left[\frac{\mu_1(a^2k_1^2 - m_1^2)^{1/2}}{a^2k_1^2} \tan\left((a^2k_1^2 - m_1^2)^{1/2} - m_1\cos^{-1}\left(\frac{m_1}{ak_1}\right) - \frac{\pi}{4}\right) + \frac{\mu_2(a^2w^2 + m_1^2)^{1/2}}{a^2w^2}\right]$$
$$= \frac{\beta^2 m_1^2 \omega^2(\epsilon_1\mu_1 - \epsilon_2\mu_2)^2}{a^4k_1^4w^4}.$$

Writing $b = m_1/k_1$, $\cos\theta = b/a$, $K = iw/k_1$ and using $\omega^2(\epsilon_1\mu_1 - \epsilon_2\mu_2) = k_1^2 - k_2^2 = k_1^2(1-K^2)$ and $\beta^2(1-\epsilon\mu) = k_1^2(\epsilon\mu - K^2)$ this is

$$\left[(a^2-b^2)^{1/2}\tan\left(k_1(a^2-b^2)^{1/2} - k_1 b\,\theta - \frac{\pi}{4}\right) - \frac{\epsilon(-a^2K^2+b^2)^{1/2}}{K^2}\right]$$
$$\times\left[(a^2-b^2)^{1/2}\tan\left(k_1(a^2-b^2)^{1/2} - k_1 b\,\theta - \frac{\pi}{4}\right) - \frac{\mu(-K^2a^2+b^2)^{1/2}}{K^2}\right]$$
$$= \frac{(\epsilon\mu - K^2)b^2(1-K^2)}{K^4}.$$

To relate this to the asymptotic approximation (4.24)-(4.25) we let

$$k_1(a^2-b^2)^{1/2} - k_1 b\theta - \frac{\pi}{4} = \frac{i\log\lambda}{2}$$

Then, since

$$\tan\left(\frac{i\log\lambda}{2}\right) = \frac{\lambda^{-1/2} - \lambda^{1/2}}{i(\lambda^{-1/2} + \lambda^{1/2})} = \frac{1-\lambda}{i(1+\lambda)}$$

we have

$$\left[i(a^2-b^2)^{1/2}(1-\lambda)+\frac{\epsilon(-a^2K^2+b^2)^{1/2}(1+\lambda)}{K^2}\right]$$
$$\times\left[i(a^2-b^2)^{1/2}(1-\lambda)+\frac{\mu(-K^2a^2+b^2)^{1/2}(1+\lambda)}{K^2}\right]$$
$$=\frac{(\epsilon\mu-K^2)b^2(1-K^2)(1+\lambda)^2}{K^4},$$

which is a quadratic equation in λ, with solutions λ_1 and λ_2 given by (4.26)-(4.28).

4.2.2. Weakly-guiding approximation. It is common in the engineering literature to employ the so-called weakly-guiding approximation (see, for example, [1, 9]) which consists in simplifying (4.31), for $m_1>0$, by assuming a posteriori that $n_2\sim n_1$. Since for most fibres the difference between n_1 and n_2 occurs by varying ϵ (so that $\mu_1=\mu_2$ anyway), assuming additionally that $\epsilon_1=\epsilon_2$ means that the left-hand side of (4.31) is a perfect square, so that the weakly guiding approximation gives

$$\frac{J'_{m_1}(ak_1)}{ak_1\,J_{m_1}(ak_1)}+\frac{K'_{m_1}(aw)}{aw\,K_{m_1}(aw)}=\pm\frac{m_1\,V^2}{(ak_1)^2\,(aw)^2},\tag{4.32}$$

since $\beta/\kappa_1\sim 1$ with the above assumption. This can be further simplified to

$$\frac{J_{m_1\pm1}(ak_1)}{ak_1\,J_{m_1}(ak_1)}\pm\frac{K_{m_1\pm1}(aw)}{aw\,K_{m_1}(aw)}=0.\tag{4.33}$$

In addition to this algebraic simplification, the weakly-guiding approximation provides a classification of the bound modes of (4.33) into the so-called EH and HE modes, depending on the ratio of E_z/H_z (for $m_1=0$, the solutions of (4.31) are TE and TM modes, that is modes with either E_z or H_z equal to zero, respectively). For an EH mode, take the positive sign in (4.33) and the negative sign for an HE mode.

In the Table 4.1 we compare the exact eigenvalue to our asymptotic approximation and to the weakly-guiding approximation.

5. Discussion. We have extended the Keller-Rubinow method to systems of equations to obtain asymptotic approximations to their eigenvalues in the limit of large eigenvalue. In fact, such large-eigenvalue asymptotics is often quite accurate even for the first few eigenvalues, as can be seen in tables 3.1 and 4.1.

We have applied the method to the problem of determining the eigenvalues of a step-profile optical fibre. As well as giving good agreement with the exact eigenvalues, the asymptotic approximation allows for a systematic classification of the eigenvalues according to which eigenvalue of the matrix Q in (4.23) is used in the eigenvalue relation (4.25). For the eigenvalues we have examined, this classification corresponds exactly to the usual EH or HE classification, which (slightly arbitrarily) depends on whether the polarisation of the eigenfunction is predominantly transverse magnetic or transverse electric.

For the optical fibre we have considered only real eigenvalues, which in the ray approximation are the bound and tunnelling modes (tunnelling modes have an exponentially small imaginary component in the limit of large eigenvalue). This restriction is due to the fact that we took the cladding to be of infinite extent for simplicity, in which case the refracting (and tunnelling) eigenmodes form a continuous spectrum with the eigenfunctions growing at infinity. In reality the cladding has a jacket at a

m_1	m_2	λ	b/a	Exact	Ray	Rel. Error	Weak	Rel. Error	Mode
0	1	λ_2	0	3.47700	3.56272	0.0246			TE 01
0	1	λ_1	0	3.68549	3.78101	0.0259			TM 01
0	2	λ_2	0	6.33103	6.37707	0.0073			TE 02
0	2	λ_1	0	6.69080	6.75039	0.0089			TM 02
1	0	λ_1	0.443	2.25772	2.43051	0.0765	2.18452	0.0324	HE 11
1	1	λ_2	0.211	4.73274	4.78613	0.0113	4.65441	0.0166	EH 11
1	1	λ_1	0.192	5.20516	5.28617	0.0156	4.99665	0.0401	HE 12
1	2	λ_2	0.131	7.63130	7.66211	0.0040	7.56661	0.0085	EH 12
1	2	λ_1	0.123	8.09993	8.16202	0.0077	7.76420	0.0415	HE 13
2	0	λ_1	0.555	3.60319	3.74709	0.0399	3.47699	0.0350	HE 21
2	1	λ_2	0.341	5.86716	5.89572	0.0049	5.77402	0.0159	EH 21
2	1	λ_1	0.303	6.59374	6.67642	0.0125	6.33103	0.0398	HE 22
2	2	λ_2	0.227	8.79509	8.80609	0.0012	8.72903	0.0075	EH 22
2	2	λ_1	0.214	9.35062	9.43982	0.0095	9.04628	0.0325	
3	0	λ_1	0.620	4.83685	4.97335	0.0282	4.65441	0.0377	HE 31
3	1	λ_2	0.432	6.94907	6.9571	0.0012	6.85600	0.0134	EH 31
3	1	λ_1	0.380	7.88902	7.98313	0.0119	7.56661	0.0409	HE 32
3	2	λ_2	0.305	9.83206	9.82197	0.0010	9.81532	0.0017	EH 32
5	0	λ_1	0.697	7.16925	7.31144	0.0198	6.85600	0.0437	
5	1	λ_2	0.556	8.99060	8.96731	0.0026	8.93898	0.0057	

TABLE 4.1

Comparison of asymptotic, exact, and weakly guiding eigenvalues for ak_1 of an optical fibre with $\mu = 1$, $n_1 = 2.5$, $n_2 = 1.5$ and $V = 10$. The column headed λ indicates whether λ_1 or λ_2 is taken in (4.25)*, while that headed Mode indicates whether the mode is EH or HE (i.e. whether the plus or minus sign is taken in the weakly guiding approximation* (4.32)*).*

finite distance, at which the rays are reflected with an intensity loss. The inclusion of the jacket means that the refracted and tunnelling modes form a discrete spectrum, which is then amenable to our method of approximation. The ray congruences are mode complicated in this case, and we defer the analysis to a later work.

A main restriction on the analysis we have presented is that the matrix Q (or at least its eigenvalues) is independent of position on the boundary. This means that θ must be independent of position on the boundary, so that the rays in a ray congruence must all meet the boundary at the same angle. Even for rectangular waveguides this is not the case.

When Q varies around the boundary the amplitude of the rays is not constant around the boundary, and the problem is much more complicated. The condition of single-valuedness of the solution now leads to a delay equation for the amplitude $\boldsymbol{A}$. Even in the scalar case considered in [3] this equation is hard to solve; in the vector case it is doubly so.

One geometry in which Q is again independent of position which is of particular interest in optics is the so called *all-dielectric coaxial waveguide* which has recently been proposed by Ibanescu *et al* [5] as a way of overcoming problems of polarisation rotation and pulse broadening. This fibre consist of several concentric layers of different thickness of various dielectric materials. We believe that our extension of the Keller-Rubinow method could also handle this geometry.

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