

The AdS/CFT Correspondence and Generalized Geometry



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À mes parents et à ma soeur.

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Statement of Originality

Chapters 2 and 3 are based mostly on the articles [2] and [3] written in collaboration with Jerome Gauntlett, Eran Palti, James Sparks, and Daniel Waldram.

Chapters 4, 5, and 6 are based on the article [4] with James Sparks.

Original contributions reported in this thesis include:

- The identification of the generalized holomorphic Killing vector field dual to the R-symmetry in every superconformal field theory that admits a holographic description as an AdS_5 solution of type IIB supergravity (section 3.3).
- The discovery of a canonical symplectic structure for such solutions and the related contact volume formulas (section 3.5).
- The definition of a notion of “generalized Sasakian geometry” and its reduction to a differential system in four dimensions (chapter 4).
- The extension of the procedure of volume minimization beyond the Sasaki-Einstein case (chapter 5).
- The new (if somewhat partial) solutions dual to massive deformations (chapter 6).

Abstract

The most general $\text{AdS}_5 \times Y$ solutions of type IIB string theory that are AdS/CFT dual to superconformal field theories in four dimensions can be fruitfully described in the language of generalized geometry, a powerful hybrid of complex and symplectic geometry. We show that the cone over the compact five-manifold Y is generalized Calabi-Yau and carries a generalized holomorphic Killing vector field ξ , dual to the R-symmetry. Remarkably, this cone always admits a symplectic structure, which descends to a contact structure on Y , with ξ as Reeb vector field. Moreover, the contact volumes of Y , which can be computed by localization, encode essential properties of the dual CFT, such as the central charge and the conformal dimensions of BPS operators corresponding to wrapped D3-branes. We then define a notion of “generalized Sasakian geometry”, which can be characterized by a simple differential system of three symplectic forms on a four-dimensional transverse space. The correct Reeb vector field for an AdS_5 solution within a given family of generalized Sasakian manifolds can be determined—without the need of the explicit metric—by a variational procedure. The relevant functional to minimize is the type IIB supergravity action restricted to the space of generalized Sasakian manifolds, which turns out to be just the contact volume. We conjecture that this contact volume is equal to the inverse of the trial central charge whose maximization determines the R-symmetry of the dual superconformal field theory. The power of this volume minimization is illustrated by the calculation of the contact volumes for a new infinite family of solutions, in perfect agreement with the results of a -maximization in the dual mass-deformed generalized conifold theories.

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Chapter 1

Introduction and results

One of the most significant advances in contemporary theoretical physics has been the realization that certain geometric solutions in string theory have completely equivalent descriptions as ordinary quantum field theories. Thanks to this gauge/gravity correspondence, many new insights have been obtained on both sides. The archetypal example is the $\text{AdS}_5 \times S^5$ solution of type IIB supergravity, with the round Einstein metric on S^5 , which corresponds to $\mathcal{N} = 4$ super Yang-Mills theory [5]. In fact, according to the AdS/CFT correspondence, *any* supersymmetric AdS_5 solution admits a dual description in terms of a four-dimensional superconformal field theory (SCFT). In this thesis we elucidate the geometric structure of such general solutions, and explain how it maps to important properties of SCFTs. We start by reviewing some key results about the well-studied class of Sasaki-Einstein solutions, which provides us with an outline of what we want to generalize.

1.1 Sasaki-Einstein solutions

Following on from the $\text{AdS}_5 \times S^5$ solution, a rich class of special solutions takes the form $\text{AdS}_5 \times Y_{\text{SE}}$, where Y_{SE} is a *Sasaki-Einstein* five-manifold, and the only non-trivial flux is the self-dual five-form [6, 7, 8, 9]. By definition, a Sasaki-Einstein manifold Y_{SE} is a compact manifold whose metric cone $C(Y_{\text{SE}}) \cong \mathbb{R}_+ \times Y_{\text{SE}}$ is Kähler and Ricci-flat, that is Calabi-Yau. The dual SCFTs have (at least) $\mathcal{N} = 1$ supersymmetry and can be understood as arising on a stack of D3-branes located at the apex of the cone. There has been much progress in understanding the AdS/CFT correspondence in this setting. For example, there are large sets of explicit Sasaki-Einstein

metrics [10, 11, 12], and there are also powerful constructions using toric geometry, whose corresponding dual SCFTs have been identified [13, 14, 15, 16].

Some essential properties of every SCFT with $\mathcal{N} = 1$ supersymmetry, such as the central charge and the conformal dimensions of chiral primary operators, are captured by its Abelian R-symmetry [17]. This R-symmetry manifests itself in the Sasaki-Einstein geometry as a canonical Killing vector field ξ , which is real holomorphic with respect to the complex structure on the Calabi-Yau cone, and is also the *Reeb vector field* associated with the contact structure on Y_{SE} descending from the symplectic structure of the cone. Recall that a *contact form* on a five-dimensional manifold is a one-form σ such that $\sigma \wedge d\sigma \wedge d\sigma$ is nowhere zero, that is a volume form, and the Reeb vector field is the unique vector field such that $\xi \lrcorner \sigma = 1$, $\xi \lrcorner d\sigma = 0$. The central charge of the field theory may then be expressed as the volume of Y_{SE} [18, 19], while the volumes of supersymmetric three-submanifolds give the conformal dimensions of chiral primary operators corresponding to wrapped D3-branes. Using symplectic geometry these volumes can be written as Duistermaat-Heckman integrals on the cone and hence evaluated by localization.

A very useful perspective [19, 20] is to regard Sasaki-Einstein metrics as critical points of the Einstein-Hilbert action restricted to a space of *Sasakian* metrics, whose cones are by definition Kähler but not necessarily Ricci-flat. More precisely, on the space of Sasakian metrics whose cones admit a nowhere-vanishing holomorphic $(3, 0)$ -form Ω , with homogeneity of degree three under the Euler vector field $r\partial_r$, the Einstein-Hilbert action precisely reduces to the *volume* functional. This volume actually depends only on the Reeb vector field, and a critical point (in fact a minimum) hence determines the unique Reeb vector field for which a Sasakian manifold is also Einstein, provided such a metric exists. This makes it possible to extract important geometric information without having to know the Sasaki-Einstein metric explicitly (which, apart from some special classes of solution, remains out of reach). This is extremely useful since there are now many existence results for Sasaki-Einstein metrics (for a review, see [21]), with vast classes of examples not known explicitly. Notwithstanding this ignorance, one can still compute the volumes of these solutions by volume minimization [19, 20] and compare them to BPS quantities in the dual SCFTs.

As first suggested in [20], the determination of the Reeb vector field by volume minimization corresponds to the determination of the R-symmetry of a four-dimensional $\mathcal{N} = 1$ SCFT by a -maximization [22]. The procedure involves constructing a trial R-symmetry, which mixes arbitrarily with the set of global (flavour) Abelian symmetries, and imposing anomaly cancellation constraints. The correct R-symmetry at the infrared fixed point is then the one which (locally) maximizes the trial central charge. A proof that the trial central charge function (appropriately interpreted) is equal to the inverse of the “off-shell” Sasakian volume function was presented in [23] for toric (that is $U(1)^3$ -invariant) Sasakian metrics, and very recently in [24] for general Sasakian metrics.

1.2 Generalization

The *most general* solutions of type IIB supergravity that are dual to four-dimensional $\mathcal{N} = 1$ SCFTs take the form $\text{AdS}_5 \times Y$, where Y is a compact Riemannian five-manifold which is not necessarily Sasaki-Einstein. The first detailed analysis of such solutions with generic fluxes activated was carried out in [25]. The conditions for supersymmetry boil down to a set of Killing spinor equations on Y for two spinors (when Y is Sasaki-Einstein there is only one such spinor). By analyzing these equations, a set of necessary and sufficient conditions for supersymmetry were established. In light of the progress summarized above for the Sasaki-Einstein case, it is natural to investigate the associated geometry of the cone over Y . This is the first goal of this thesis.

As we discuss in detail in chapter 3, the requirement of supersymmetry puts certain constraints on Y , which can be conveniently formulated in terms of a specific kind of generalized geometry on the cone $C(Y) \cong \mathbb{R}_+ \times Y$ (some aspects of generalized geometry are reviewed in chapter 2). This approach was already pursued in [26, 27]. By viewing $\text{AdS}_5 \times Y$ as a supersymmetric warped product $\mathbb{R}^{1,3} \times C(Y)$ (see figure 1.1), we will see that the cone admits two compatible generalized almost complex structures [28, 29]. One of these two generalized structures is actually integrable, which implies that the cone $C(Y)$ is *generalized Calabi-Yau*, in the sense of Hitchin [30]. The integrability of the second generalized structure is obstructed by the presence of the fluxes, but it nevertheless defines a *symplectic* structure on the

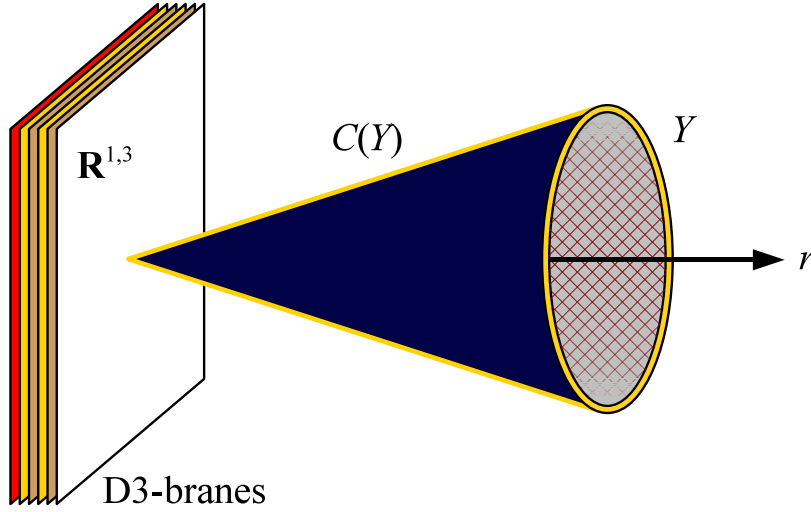


Figure 1.1: The $\text{AdS}_5 \times Y$ geometry generated by a stack of D3-branes can be viewed as a warped product $\mathbb{R}^{1,3} \times C(Y)$ of Minkowski space with the cone $C(Y) \cong \mathbb{R}_+ \times Y$.

cone [3] (provided the five-form flux sourced by D3-branes is non-vanishing). Precisely as in the Sasaki-Einstein case, one can define a canonical Killing vector field ξ which is also the Reeb vector field associated with the induced contact structure on Y . However, ξ is now *generalized holomorphic*, that is holomorphic with respect to the generalized complex structure (there is no complex structure in general). We briefly comment on some relations between generalized holomorphic objects and dual BPS operators. Remarkably, the volume formulas for the central charge and the conformal dimensions of chiral primary operators still hold in this generalized setting, but now in terms of *contact volumes* [2].¹ We obtain the following formulas for the central charge a and the conformal dimension $\Delta(\mathcal{O}_{\Sigma_3}) = 3R(\mathcal{O}_{\Sigma_3})/2$ of the chiral primary operator $\mathcal{O}_{\Sigma_3}$ dual to a D3-brane wrapped on a supersymmetric submanifold $\Sigma_3 \subset Y$:

$$\frac{a_{\mathcal{N}=4}}{a} = \frac{1}{(2\pi)^3} \int_Y \sigma \wedge d\sigma \wedge d\sigma, \quad \Delta(\mathcal{O}_{\Sigma_3}) = \frac{2\pi N \int_{\Sigma_3} \sigma \wedge d\sigma}{\int_Y \sigma \wedge d\sigma \wedge d\sigma}, \quad (1.1)$$

where $a_{\mathcal{N}=4} = N^2/4$ is the (large N) central charge for $SU(N)$ $\mathcal{N} = 4$ super Yang-Mills theory, and N is the quantized five-form flux through Y . We illustrate some of our results using the Pilch-Warner solution.

Since a -maximization applies in principle to every $\mathcal{N} = 1$ SCFT, it is clearly desirable to extend the procedure of volume minimization beyond the Sasaki-Einstein

¹In the Sasaki-Einstein case these contact volumes are equal to the Riemannian volumes defined by the metric. This is no longer true in the general case with fluxes.

case, to the most general supersymmetric AdS_5 solutions of type IIB supergravity. The motivation is the same as in the Sasaki-Einstein case: explicit solutions will form a very small subset of the space of all solutions, since finding them always relies on having a large amount of symmetry. On the other hand, one might hope that the development in this thesis will eventually lead to existence results, say for toric AdS_5 solutions with general fluxes. Then volume minimization will allow one to compute BPS quantities for these solutions.

In chapter 4 we express the constraints imposed on the cone $C(Y)$ by supersymmetry in a geometric form [4]. In the Sasaki-Einstein case, one can carry out a symplectic reduction of the Calabi-Yau cone to obtain a four-dimensional transverse Kähler-Einstein space which, in general, is only locally defined; constructing locally defined Kähler-Einstein spaces has been a profitable way to construct Sasaki-Einstein manifolds, see for instance [31]. Here we will show, using the formalism of generalized reduction developed in [32, 33], that for general Y there is an analogous reduction of the corresponding six-dimensional generalized Calabi-Yau cone geometry to a four-dimensional space, which is generalized Hermitian. This means that the four-dimensional geometry admits two compatible generalized almost complex structures, one of which is integrable.

After this reduction, the supersymmetry conditions turn into a system of equations on a four-dimensional transverse space for a *triple of orthogonal symplectic forms* $\{\omega_0, \omega_1, \omega_2\}$, a structure first studied in [34], and two functions h and $\hat{\Delta}$. More precisely, the symplectic forms satisfy

$$d\omega_i = 0 \quad \forall i \in \{0, 1, 2\} , \quad (1.2)$$

$$\omega_i \wedge \omega_j = 0 \quad \forall i \neq j , \quad (1.3)$$

$$\omega_0 \wedge \omega_0 = \alpha_1 \omega_1 \wedge \omega_1 = \alpha_2 \omega_2 \wedge \omega_2 \neq 0 , \quad (1.4)$$

where α_1 and α_2 are positive functions depending on h and $\hat{\Delta}$, together with the following differential conditions:

$$\omega_1 = \frac{1}{2} \mathcal{L}_{\mathcal{H}_h} \omega_2 , \quad \mathcal{L}_{\mathcal{H}_h} (\mathcal{L}_{\mathcal{H}_h} \omega_1) = \mathcal{L}_{\mathcal{H}_{e^{-4\hat{\Delta}}}} \omega_2 . \quad (1.5)$$

Here the notation $\mathcal{H}_h \equiv \omega_0^{-1} \lrcorner dh$ means the Hamiltonian vector field for the function h , with respect to the symplectic form ω_0 . This differential system defines what we call a

“*generalized Sasakian geometry*”.² In fact, there is in general also a special subspace of Y where h diverges and $\hat{\Delta}$ is constant, along which the geometry is simply Sasakian. This is the so-called *type-change locus* $\mathcal{T}$ and corresponds physically to the *mesonic moduli space* of the dual SCFT [35, 26] (for a Sasakian manifold $\mathcal{T}$ is of course the whole space, rather than a subspace). The rationale behind this definition is that when such a generalized Sasakian manifold satisfies an additional condition on the lengths of the three symplectic forms, it precisely provides a general supersymmetric AdS_5 solution of type IIB supergravity. This additional condition is effectively the *Einstein* equation, and so the resulting geometry can be called “*generalized Sasaki-Einstein*”. The reduction of the complicated supergravity equations to a simple system not only gives hope of finding new solutions but is also a significant step towards addressing existence and uniqueness questions.

In chapter 5 we explain how volume minimization works for generalized Sasakian manifolds [4]. We show that the type IIB supergravity action reduces, when restricted to a space of generalized Sasakian manifolds, to the *contact volume*, and that the latter is then a strictly convex function of the Reeb vector field. It follows that a supersymmetric AdS_5 solution that is in the same deformation class as a given generalized Sasakian manifold is obtained by minimizing the contact volume over a space of Reeb vector fields. As a concrete example, the critical Reeb vector field for which a *toric* generalized Sasakian manifold satisfies also the Einstein equation is obtained by minimizing the volume of a polytope, just as in the Sasakian case.

However, in contrast to the Sasakian case, generalized volume minimization requires not the standard holomorphic $(3,0)$ -form Ω , but rather the *pure spinor* Ω_- , which is a formal sum of one-, three-, and five-forms, to be homogeneous of degree three under the Euler vector field. This imposes additional constraints on the space of Reeb vector fields that is to be minimized over. Here, our current understanding of the space of Reeb vector fields for a deformation class of generalized Sasakian manifolds is not yet as developed as in the Sasakian case [19, 20]. We will nevertheless show in examples that this space is non-trivial, and that generalized volume minimization

²Note that we define a generalized Sasakian structure only in *dimension five*. Indeed, the definition is primarily motivated by the five-dimensional supersymmetry conditions we wish to study. It might be possible to extend the definition to manifolds of general dimension $2n - 1$, but we shall not comment further on this here.

agrees with computations in dual SCFTs. In fact, we go further and make the natural conjecture that the volume function is *equal* to the inverse of the trial central charge of the dual SCFT, again checking this is indeed true in examples. As a very simple illustration we recurrently refer to a so-called “ β -transform” of $\mathbb{C}^3 = C(S^5)$ by a bivector β , which is known to be dual to a certain marginal deformation of $\mathcal{N} = 4$ super Yang-Mills theory [36, 26, 27]. In chapter 6 we then study a new class of examples [4] obtained by massive deformation of generalized conifolds $C(L^{m,n,m})$ [12, 37]. After making some physically motivated assumptions on the geometry, we verify in this class of examples the equivalence of generalized volume minimization and a -maximization.

Chapter 2

Aspects of generalized geometry

We first recall a few relevant facts about generalized geometry, whose study was initiated in [30]. We refer the reader to [38] for an extensive mathematical introduction or to the review [39] for physicists. Generalized geometry can be seen as a hybrid of complex and symplectic geometries, to which it reduces in extreme cases. Many of the concepts from these classical geometries extend to generalized geometry, and one can for instance define “generalized vectors”, which consist of an ordinary vector and a differential one-form. Other important concepts, such as spinors, metrics, and almost complex structures, also generalize. The advantage of such a perspective is that the action of the Neveu-Schwarz B -field which is usually present in string theory can be naturally incorporated.

2.1 Generalized vectors

The essential idea of generalized geometry is to consider the *generalized tangent bundle* E over a manifold X , which is an extension of the tangent bundle TX by the cotangent bundle T^*X :

$$0 \longrightarrow T^*X \longrightarrow E \longrightarrow TX \longrightarrow 0 . \quad (2.1)$$

Sections of E , which we refer to as *generalized vectors*, may be written locally as $V = v + \nu \in \Gamma(E)$ with $v \in \Gamma(TX)$ and $\nu \in \Gamma(T^*X)$. More precisely, this extension is obtained by twisting with a *gerbe* [40]. A gerbe is simply a higher-degree version of a $U(1)$ -bundle with unitary connection: just as a $U(1)$ -bundle is determined topologically by its first Chern class in $H^2(X, \mathbb{Z})$, the topology of a gerbe is determined by a class in $H^3(X, \mathbb{Z})$. Consider an open cover $\{U_i\}$ of X together with a set of functions

$g_{ijk} : U_i \cap U_j \cap U_k \rightarrow U(1)$ defined on triple overlaps. These are required to satisfy $g_{ijk} = g_{jik}^{-1} = g_{ikj}^{-1} = g_{kji}^{-1}$, as well as the cocycle condition $g_{jkl}g_{ikl}^{-1}g_{ijl}g_{ijk}^{-1} = 1$ on quadruple overlaps. A *connective structure* on a gerbe is a collection of one-forms $\Lambda_{(ij)}$ defined on double overlaps $U_i \cap U_j$ satisfying $\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} = g_{ijk}^{-1}dg_{ijk}$ on triple overlaps. In going from one coordinate patch U_i to another U_j , the extension (2.1) is defined by the connective structure:

$$v_{(i)} + \nu_{(i)} = v_{(j)} + (\nu_{(j)} - v_{(j)} \lrcorner d\Lambda_{(ij)}) . \quad (2.2)$$

The generalized tangent bundle E is in fact isomorphic to $TX \oplus T^*X$. However, the isomorphism is not canonical but depends on a choice of splitting, defined by a *curving* B , which is a collection of two-forms $B_{(i)}$ on U_i satisfying

$$B_{(j)} - B_{(i)} = d\Lambda_{(ij)} . \quad (2.3)$$

It follows that, for any $V = v + \nu \in \Gamma(E)$,

$$v + (\nu - v \lrcorner B) \in \Gamma(TX \oplus T^*X) . \quad (2.4)$$

Thus the definition (2.2) of E can be viewed as encoding the patching by the two-form curving B . Note that (2.3) implies that $dB_{(j)} = dB_{(i)} = H$ is a global closed three-form on X , called the *curvature*, and, in cohomology, $H \in H^3(X, \mathbb{Z})$. The relevance of generalized geometry for string theory stems from the fact that the curving B adequately describes the Neveu-Schwarz B -field, and $H = dB$ its curvature.

Writing d for the real dimension of X , there is a natural $O(d, d)$ -invariant metric $\langle \cdot, \cdot \rangle$ on E , given by

$$\langle V, W \rangle \equiv \frac{1}{2}(v \lrcorner \mu + w \lrcorner \nu) , \quad (2.5)$$

with $V = v + \nu$, $W = w + \mu$, or in two-component notation,

$$\langle V, W \rangle = \frac{1}{2} \begin{pmatrix} v & \nu \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w \\ \mu \end{pmatrix} . \quad (2.6)$$

This metric is invariant under $O(d, d)$ -transformations acting on the fibres of E , defining a canonical $O(d, d)$ -structure. A general element $O \in O(d, d)$ may be written in terms of $d \times d$ matrices a , b , c , and d as

$$O = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \quad (2.7)$$

under which a general element $V \in \Gamma(E)$ transforms as

$$V = \begin{pmatrix} v \\ \nu \end{pmatrix} \mapsto OV = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v \\ \nu \end{pmatrix} . \quad (2.8)$$

The requirement that $\langle OV, OV \rangle = \langle V, V \rangle$ implies $a^T c + c^T a = 0$, $b^T d + d^T b = 0$ and $a^T d + c^T b = 1$. Note that the $GL(d)$ action on the fibres of TX and T^*X embeds as a subgroup of $O(d, d)$. Concretely, it maps

$$V \mapsto V' = \begin{pmatrix} a & 0 \\ 0 & a^{-T} \end{pmatrix} \begin{pmatrix} v \\ \nu \end{pmatrix} , \quad (2.9)$$

where $a \in GL(d)$. Given a two-form b , one also has the Abelian subgroup

$$e^b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \quad \text{such that} \quad V = v + \nu \mapsto V' = v + (\nu - v \lrcorner b) . \quad (2.10)$$

This is usually referred to as a B -transform. Given a bivector β , one can also define another Abelian subgroup of β -transforms:

$$e^\beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \quad \text{such that} \quad V = v + \nu \mapsto V' = (v + \beta \lrcorner \nu) + \nu . \quad (2.11)$$

Note that the patching (2.2) corresponds to a B -transform with the two-form $d\Lambda_{(ij)}$. Similarly, the splitting isomorphism between E and $TX \oplus T^*X$ defined by B (see (2.4)) is also a B -transform:

$$E \xrightleftharpoons[e^{-B}]{e^B} TX \oplus T^*X . \quad (2.12)$$

There is a natural bracket on generalized vectors known as the *Courant bracket*, which encodes the differentiable structure of E . It is defined as

$$[V, W] \equiv [v, w]_{\text{Lie}} + \mathcal{L}_v \mu - \mathcal{L}_w \nu - \frac{1}{2} d(v \lrcorner \mu - w \lrcorner \nu) , \quad (2.13)$$

where $[v, w]_{\text{Lie}}$ is the usual Lie bracket between vectors and $\mathcal{L}_v$ is the Lie derivative along v . The Courant bracket is invariant under the action of diffeomorphisms and closed B -transforms, giving an automorphism group which is a semi-direct product $\text{Diff}(X) \ltimes \Omega_{\text{closed}}^2(X)$. Note, however, that in string theory only B -transforms by the curvature of a unitary line bundle on X are gauge symmetries, as opposed to transforms by arbitrary closed two-forms, leading to a smaller automorphism group. Under an infinitesimal diffeomorphism generated by a vector field v and a B -transform

by $b = d\nu$, one has the *generalized Lie derivative* by $V = v + \nu$ on a generalized vector field $W = w + \mu$

$$\mathbb{L}_V W \equiv [v, w]_{\text{Lie}} + (\mathcal{L}_v \mu - w \lrcorner d\nu) . \quad (2.14)$$

This is also known as the Dorfman bracket $[V, W]_{\text{D}}$, the anti-symmetrization of which gives the Courant bracket (2.13). Note that since the metric $\langle \cdot, \cdot \rangle$ is invariant under $O(d, d)$ -transformations, its generalized Lie derivative vanishes. Given a particular choice of splitting (2.3) defined by B , the Courant bracket on E defines a Courant bracket on $TX \oplus T^*X$, known as the *twisted* Courant bracket. It is given by

$$\begin{aligned} [v + \nu, w + \mu]_H &\equiv e^B [e^{-B}(v + \nu), e^{-B}(w + \mu)] \\ &= [v + \nu, w + \mu] + w \lrcorner v \lrcorner H , \end{aligned} \quad (2.15)$$

where by an abuse of notation we are writing $v + \nu$ and $w + \mu$ for sections of $TX \oplus T^*X$, whereas above they were sections of E .

2.2 Generalized spinors

Given the metric $\langle \cdot, \cdot \rangle$, one can define $Spin(d, d)$ -spinors, which we call *generalized spinors*. Since the volume element of $\text{Cliff}(d, d)$ squares to one, one can take its ± 1 -eigenspaces to define two spin bundles $S_{\pm}(E)$ with opposite helicities and take spinors to be Majorana-Weyl. A section of $S_{\pm}(E)$ on U_i can be identified with a even- or odd-degree *polyform* $\Omega_{\pm} \in \Omega^{\text{even/odd}}(X)$ restricted to U_i , that is a formal sum of differential forms of various even or odd degrees: $\Omega_+ = \Omega_0 + \Omega_2 + \dots$ or $\Omega_- = \Omega_1 + \Omega_3 + \dots$ with Ω_p a p -form. The *Clifford action* of $V \in \Gamma(E)$ on such polyforms is given by

$$V \cdot \Omega_{\pm} \equiv v \lrcorner \Omega_{\pm} + \nu \wedge \Omega_{\pm} , \quad (2.16)$$

and it is easy to see that

$$(V \cdot W + W \cdot V) \cdot \Omega_{\pm} = 2\langle V, W \rangle \Omega_{\pm} , \quad (2.17)$$

as required for a Clifford algebra. Using this Clifford action, the B -transform (2.10) on spinors is given by

$$\Omega_{\pm} \mapsto e^b \Omega_{\pm} = (1 + b + \frac{1}{2} b \wedge b + \dots) \wedge \Omega_{\pm} . \quad (2.18)$$

The patching (2.2) of E then implies that

$$\Omega_{\pm}^{(i)} = e^{\mathrm{d}\Lambda_{(ij)}} \Omega_{\pm}^{(j)} . \quad (2.19)$$

Furthermore, a curving B also induces an isomorphism between $S_{\pm}(E)$ and $S_{\pm}(TX \oplus TX^*)$

$$S_{\pm}(E) \xrightleftharpoons[e^{-B}]{e^B} S_{\pm}(TX \oplus T^*X) . \quad (2.20)$$

If $\Omega_{\pm}$ is a section of $S_{\pm}(E)$, we will sometimes write $\Omega_{\pm}^B \equiv e^B \Omega_{\pm}$ for the corresponding section of $S_{\pm}(TX \oplus T^*X)$ defined by the curving B .

There is actually a slight subtlety in the relation between generalized spinors and polyforms. Given the embedding (2.9) in $O(d, d)$ of the $GL(d)$ action on the fibres of TX , one actually finds that the Clifford action (2.16) implies that on U_i we can identify $S_{\pm}(E)$ with $|\wedge^d T^*X|^{-1/2} \otimes \wedge^{\text{even/odd}} T^*X$: there is an additional factor of the determinant bundle $|\wedge^d T^*X|$. Since this bundle is trivial, generalized spinors can indeed be written as polyforms patched by (2.19), but there is no natural isomorphism to make this identification. The simplest solution, and one which will also allow us to incorporate the dilaton in a natural way, is to extend the $O(d, d)$ action to a conformal action $O(d, d) \times \mathbb{R}^+$. One can then define a family of spinor bundles $S_{\pm}^{(k)}(E)$ transforming with weight k under the conformal factor $\mathbb{R}^+$, that is with sections transforming as $\Omega_{\pm} \rightarrow \rho^k \Omega_{\pm}$ where $\rho \in \mathbb{R}^+$. If one embeds the $GL(d)$ action on TX in $O(d, d)$ as in (2.9) and, in addition, makes a conformal scaling by $\rho = \det a$ then sections of $S_{\pm}^{(-1/2)}(E)$ can be directly identified with polyforms patched by (2.19).

The real $Spin(d, d)$ -invariant spinor bilinear on sections Φ and Ψ of $S_{\pm}(E)$ is a top-form given by the *Mukai pairing*¹

$$\langle \Phi, \Psi \rangle \equiv [\Phi \wedge \lambda(\Psi)]|_{\text{top}} , \quad (2.21)$$

where the operator λ is defined as

$$\lambda(\Psi_p) \equiv (-1)^{[p/2]} \Psi_p , \quad (2.22)$$

with Ψ_p the p -form in Ψ and $[p/2]$ the integer part of $p/2$. The Mukai paring is invariant under B -transforms: $\langle e^b \Phi, e^b \Psi \rangle = \langle \Phi, \Psi \rangle$. For the case $d = 6$ of interest in

¹This will hopefully not be confused with the $O(d, d)$ -invariant metric $\langle \cdot, \cdot \rangle$ defined in (2.5).

this thesis, it is anti-symmetric. The usual action of the exterior derivative on the component forms of $\Omega_\pm$ is compatible with the patching (2.19) and defines an action

$$d : S_\pm(E) \rightarrow S_\mp(E) , \quad (2.23)$$

while the generalized Lie derivative on spinors is given by

$$\mathbb{L}_V \Omega_\pm = \mathcal{L}_v \Omega_\pm + d\nu \wedge \Omega_\pm = d(V \cdot \Omega_\pm) + V \cdot d\Omega_\pm . \quad (2.24)$$

Note that given a curving B , the operator on $\Omega_\pm^B \in S_\pm(TX \oplus T^*X)$ corresponding to d is the twisted differential d_H defined by

$$d_H \Omega_\pm^B \equiv e^B d(e^{-B} \Omega_\pm^B) = (d - H \wedge) \Omega_\pm^B , \quad (2.25)$$

where $H = dB$. Furthermore, one has

$$\mathbb{L}_V \Omega_\pm = e^{-B} (\mathbb{L}_{V^B} - v \lrcorner H \wedge) \Omega_\pm^B , \quad (2.26)$$

where $V^B \equiv e^B V = v + (\nu - v \lrcorner B)$.

A generalized spinor Ω is called *pure* if its annihilator space

$$L_\Omega \equiv \{V \in \Gamma(E_\mathbb{C}) : V \cdot \Omega = 0\} , \quad (2.27)$$

with $E_\mathbb{C} = E \otimes \mathbb{C}$ the complexification of the generalized tangent bundle, is *maximal isotropic*, which means that $\langle V, W \rangle = 0$ for any $V, W \in L_\Omega$ and that L_Ω has maximal dimension $2d$. At any given point on X a complex pure spinor $\Omega \in \Gamma(E \otimes \mathbb{C})$ takes the general form [38]

$$\Omega = \alpha \theta_1 \wedge \cdots \wedge \theta_k \wedge e^{-b+i\omega} , \quad (2.28)$$

where α is some complex function, θ_i are k complex one-forms, and b, ω are real two-forms. The integer k is called the *type* of the pure spinor. For example, the holomorphic $(3,0)$ -form $\Omega_{(3,0)}$ on a Calabi-Yau three-fold is a pure spinor that is everywhere of type $k=3$, with $b=\omega=0$. On the other hand, a symplectic form ω gives rise to a pure spinor $\exp(i\omega)$ that is everywhere of type $k=0$. Note that it is an important feature of generalized geometry that the type of a pure spinor can change along a distinguished sublocus of X , called the *type-change locus*. In this thesis we will want to replace the holomorphic $(3,0)$ -form on a Calabi-Yau three-fold by a pure spinor of type one on a dense open subset of X , but which can jump to type three.

2.3 Generalized metrics

A *generalized metric* G on E is the equivalent of a Riemannian metric g on TX . We have seen that there is a natural $O(d, d)$ -structure on E defined by the metric $\langle \cdot, \cdot \rangle$ in (2.5). The generalized metric G defines an $O(d) \times O(d)$ -substructure. It splits the generalized tangent bundle as $E = C_+ \oplus C_-$ such that the metric $\langle \cdot, \cdot \rangle$ gives a positive-definite metric on C_+ and a negative-definite metric on C_- , corresponding to the two $O(d)$ -structure groups. One can define G as a product structure on E , that is $G : E \rightarrow E$ with $G^2 = 1$ and $\langle GV, GW \rangle = \langle V, W \rangle$, such that $(1 \pm G)/2$ project onto $C_{\pm}$. In general G has the form

$$G = \begin{pmatrix} g^{-1}B & g^{-1} \\ g - Bg^{-1}B & -Bg^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \quad (2.29)$$

where g is a metric on X and B is a two-form. The patching of E implies that B satisfies (2.3), so that B may be identified with the curving of the gerbe used in the twisting of E . Thus the generalized metric G defines a particular splitting of E . In particular, we see from (2.29) that $G = e^{-B}G_0e^B$ where G_0 is a generalized metric on $TX \oplus T^*X$ defined by g .

The generalized metric G naturally encodes the Neveu-Schwarz fields g and B of string theory as the coset space $O(d, d)/O(d) \times O(d)$. The dilaton ϕ appears when one considers the conformal group $O(d, d) \times \mathbb{R}^+$, used to express the generalized spinors as true polyforms. To define an $O(d) \times O(d)$ -substructure in $O(d, d) \times \mathbb{R}^+$, one must give, in addition to G which gives the embedding in the $O(d, d)$ -factor, the embedding in the conformal factor $\rho \in \mathbb{R}^+$. Given the metric g we can define the generic embedding by $\rho = e^{2\phi}/\sqrt{g}$ for some positive function $e^{2\phi}$, which we identify as the dilaton. Note that ρ is by definition invariant under $O(d, d)$ and so one finds the conventional T-duality transformation of the dilaton under $O(d, d)$.

A generalized vector is called *generalized Killing* if it preserves the generalized metric under the generalized Lie derivative, that is $\mathbb{L}_V G = 0$, which implies [41]

$$\mathcal{L}_v g = 0, \quad \mathcal{L}_v B - dv = 0, \quad (2.30)$$

so that $\mathcal{L}_v H = 0$ where $H = dB$.

Given G we may decompose generalized spinors in $Spin(d, d)$ under $Spin(d) \times Spin(d)$. In fact one can go further. Using the projection $\pi : E \rightarrow TX$ the two

$Spin(d)$ groups can be determined and one can use the *Clifford map* to identify bispinors $\Phi = \eta_1 \otimes \bar{\eta}_2$, where $\eta_{1,2}$ are $Spin(d)$ -spinors, with generalized spinors Φ of $S_{\pm}(TX \oplus T^*X) \cong \wedge^{\text{even/odd}} T^*X$:

$$\Phi = \sum_p \frac{1}{p!} \Phi_{i_1 \dots i_p} \gamma^{i_1 \dots i_p} \longleftrightarrow \Phi = \sum_p \frac{1}{p!} \Phi_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} \quad (2.31)$$

where the sum is over p even/odd for the chiral spinors $\Phi_{\pm}$ respectively. The $Cliff(d, d)$ -action is realized via left- and right-multiplication by the $Spin(d)$ gamma matrices γ^i . We also note here the Fierz identity

$$\Phi = \frac{1}{n_d} \sum_p \frac{1}{p!} \text{Tr}(\Phi \gamma_{i_p \dots i_1}) \gamma^{i_1 \dots i_p}, \quad (2.32)$$

with $n_d = 2^{[d/2]}$. Generalized spinors may then be decomposed as bispinors of $Spin(d)$:

$$\Omega_{\pm} = e^{-\phi} e^{-B} \Phi_{\pm}. \quad (2.33)$$

In this expression, $\Phi_{\pm} \in \Gamma(S_{\pm}(TX \oplus T^*X))$ is mapped to a spinor $\Omega_{\pm} \in \Gamma(S_{\pm}(E))$ by a choice of curving B , and the factor of $e^{-\phi}$ appears because, as explained in the previous section, the polyforms are really sections of $S_{\pm}^{(-1/2)}(E)$ transforming with weight $-1/2$ under conformal rescalings.

The generalized metric also defines an action $\star_G$ on generalized spinors which is the analogue of the Hodge star. It is given by

$$\star_G \Omega_{\pm} = e^{-B} \star (e^B \Omega_{\pm}), \quad (2.34)$$

where λ is the operator defined in (2.22) and $\star$ denotes the ordinary Hodge star for the metric g . The *Mukai norm* of a pure spinor Ω is defined (here in the case $d = 6$) as

$$\|\Omega\|^2 \equiv i \frac{\langle \Omega, \bar{\Omega} \rangle}{\text{vol}}, \quad (2.35)$$

where vol denotes the Riemannian volume form of the metric g . In addition, we define the following convenient norms:

$$|\Omega|^2 \equiv \frac{\langle \Omega, \star \lambda(\bar{\Omega}) \rangle}{\text{vol}}, \quad |\Omega|_B^2 \equiv |e^B \Omega|^2. \quad (2.36)$$

2.4 Generalized complex structures

If $d = 2n$ one can also introduce a *generalized almost complex structure* on E . This is a map $\mathcal{J} : E \rightarrow E$ with $\mathcal{J}^2 = -1$ and $\langle \mathcal{J}V, \mathcal{J}W \rangle = \langle V, W \rangle$ and gives a decomposition

$$E_{\mathbb{C}} = L_{\mathcal{J}} \oplus \bar{L}_{\mathcal{J}}, \quad (2.37)$$

where $L_{\mathcal{J}}$ denotes the $+i$ -eigenspace of $\mathcal{J}$:

$$L_{\mathcal{J}} \equiv \{V \in \Gamma(E_{\mathbb{C}}) : \mathcal{J} \cdot V = iV\}. \quad (2.38)$$

Note that $L_{\mathcal{J}}$ is maximal isotropic, since for $V, W \in L_{\mathcal{J}}$ one finds $\langle V, W \rangle = \langle \mathcal{J}V, \mathcal{J}W \rangle = \langle iV, iW \rangle = -\langle V, W \rangle = 0$. This defines a $U(n, n) \subset O(2n, 2n)$ structure on E . Since by definition $\mathcal{J}$ preserves the metric $\langle \cdot, \cdot \rangle$ it is an element of $O(2n, 2n)$, but given that $\mathcal{J}^2 = -1$ this implies that $\langle V, \mathcal{J}W \rangle + \langle \mathcal{J}V, W \rangle = 0$, and so it can also be viewed as an element of the Lie algebra $\mathfrak{o}(2n, 2n)$. A generic $\mathcal{J}$ can be written locally as

$$\mathcal{J} = \begin{pmatrix} I & P \\ Q & -I^* \end{pmatrix}, \quad (2.39)$$

where I^* is the linear map on T^*X dual to the map I on TX , P is a bivector, and Q is a two-form. When the twisting (2.2) is trivial, so that $E = TX \oplus T^*X$, there are two canonical examples of generalized almost complex structures. The first is an ordinary almost complex structure I on TX , giving ²

$$\mathcal{J}_I = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix}, \quad (2.40)$$

and the second is a non-degenerate (stable) two-form ω , giving

$$\mathcal{J}_{\omega} = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}. \quad (2.41)$$

If $d\omega = 0$, this corresponds to a symplectic structure. A generalized almost complex structure $\mathcal{J}$ is *integrable* if L is closed under the Courant bracket: given $V, W \in L_{\mathcal{J}}$, then $[V, W] \in L_{\mathcal{J}}$. In the above two cases (2.40), (2.41), this reduces to integrability of I and the closure of ω , respectively. Viewing $\mathcal{J}$ as a Lie algebra element, one can define its action on generalized spinors via the Clifford action [42]. Explicitly, one has

$$\mathcal{J} \cdot = \frac{1}{2} \left(Q_{mn} dx^m \wedge dx^n \lrcorner + I^m_n [\partial_m \lrcorner, dx^n \wedge] + P^{mn} \partial_m \lrcorner \partial_n \lrcorner \right). \quad (2.42)$$

² Note that we have chosen the opposite sign in (2.40) compared with [38]. This is so that the $+i$ -eigenspace is identified with $T^{(1,0)} \oplus T^{*(0,1)}$.

Note that for any generalized vector V , one has $[\mathcal{J}\cdot, V\cdot] = (\mathcal{J}V)\cdot$.

A generalized almost complex structure $\mathcal{J}$ corresponds to (the conformal class of) a pure spinor Ω through the identification of the $+i$ -eigenspace $L_{\mathcal{J}}$ with the annihilator space L_{Ω} :

$$\mathcal{J} \leftrightarrow \Omega \quad \text{if} \quad L_{\mathcal{J}} = L_{\Omega} . \quad (2.43)$$

Notice that L_{Ω} is invariant under conformal rescalings $\Omega \mapsto f\Omega$, for any function f ; a generalized almost complex structure is therefore more precisely equivalent to the pure spinor line bundle generated by Ω . Integrability of $\mathcal{J}$ can be expressed as the condition $d\Omega = V\cdot\Omega$ for some $V \in \Gamma(E)$. If one can find a nowhere-vanishing globally-defined Ω then one has an $SU(n, n)$ -structure, and if in addition $d\Omega = 0$ then one has a *generalized Calabi-Yau* structure in the sense of Hitchin [30].³ For example, in the complex structure case (2.40) one has $\Omega = c\bar{\Omega}_{(n,0)}$, where $\Omega_{(n,0)}$ is the holomorphic $(n, 0)$ -form and c is a non-zero constant (the reason why the complex conjugate $\bar{\Omega}_{(n,0)}$ appears rather than $\Omega_{(n,0)}$ is directly related to the comment in footnote 2).

Given a curving B , one can define the corresponding generalized complex objects on $TX \oplus T^*X$. In particular, if $\mathcal{J}$ is the generalized almost complex structure for a pure spinor Ω , then the corresponding generalized almost complex structure on $TX \oplus T^*X$ is defined in terms of the annihilator of $\Omega^B = e^B\Omega$ and is given by $\mathcal{J}^B \equiv e^B\mathcal{J}e^{-B}$. In particular, integrability of $\mathcal{J}$ is equivalent to integrability of $\mathcal{J}^B$ using the twisted Courant bracket (2.15), or equivalently $d_H\Omega^B = V\cdot\Omega^B$.

A generalized vector V is called *generalized holomorphic* if it preserves the generalized complex structure, that is $\mathbb{L}_V\mathcal{J} = 0$. Equivalently, $\mathbb{L}_V$ preserves the spinor line bundle generated by the corresponding pure spinor Ω : $\mathbb{L}_V\Omega = f\Omega$ for some function f .

A generalized almost complex structure $\mathcal{J}$ defines a grading on generalized spinors. If $\Omega \in \Gamma(S_{\pm}(E))$ is a pure spinor corresponding to $\mathcal{J}$, one defines the canonical pure spinor line bundle $U^n \subset S_{\pm}(E)$ as sections of the form $\varphi = f\Omega$ for some function f . One can then define

$$U^{n-k} \equiv \wedge^k \bar{L}_{\mathcal{J}} \otimes U^n . \quad (2.44)$$

³Beware the existence of a different definition of generalized Calabi-Yau in [38] which requires *two* integrable generalized structures.

Elements of U^k have eigenvalues ik under the Lie algebra action of $\mathcal{J}$ given in (2.42). These bundles give a grading of the spinor bundles $S_{\pm}(E)$. A generalized vector $V \in \Gamma(E)$ acting on an element of U^k gives an element of $U^{k+1} \oplus U^{k-1}$. In particular, an annihilator of Ω acts by purely raising the level by one. If the generalized complex structure $\mathcal{J}$ is also integrable, the exterior derivative splits into the sum

$$d = \partial + \bar{\partial}, \quad (2.45)$$

where

$$C^{\infty}(U^k) \xrightarrow{\bar{\partial}} C^{\infty}(U^{k-1}). \quad (2.46)$$

A pair of generalized almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ are said to be *compatible* if

$$[\mathcal{J}_1, \mathcal{J}_2] = 0, \quad (2.47)$$

and the combination

$$G = -\mathcal{J}_1 \mathcal{J}_2 \quad (2.48)$$

is a generalized metric. If Ω_1 and Ω_2 are the corresponding pure spinors, the condition (2.47) is equivalent to $\langle \Omega_1, V \cdot \Omega_2 \rangle = \langle \bar{\Omega}_1, V \cdot \Omega_2 \rangle = 0$ for all $V \in \Gamma(E)$, or also to $\mathcal{J}_1 \cdot \Omega_2 = 0$, that is $\Omega_2 \in U_1^0$. An example of a pair of compatible pure spinors is (2.40) and (2.41), the compatibility condition being that $I_i^k \omega_{jk} = g_{ij}$ is positive definite. Note that this gives $\omega_{ij} = -g_{ik} I_j^k$, this mathematics convention differing by a sign from the usual physics convention. A pair of compatible generalized almost complex structures defines an $SU(n) \times SU(n)$ -structure. A *generalized Kähler* structure is an $SU(n) \times SU(n)$ -structure where both generalized structures are integrable, while for a *generalized Hermitian* structure only one need be integrable.

Note that an $SU(n) \times SU(n)$ -structure can equivalently be specified by a generalized metric and a pair of chiral $Spin(2n)$ -spinors. For example, for $d = 6$ a pair of chiral spinors η_+^1, η_+^2 can be used to construct an $SU(3) \times SU(3)$ -structure given by

$$\Omega_{\pm} = e^{-\phi} e^{-B} \eta_+^1 \otimes \bar{\eta}_{\pm}^2, \quad (2.49)$$

with $\eta_-^2 \equiv (\eta_+^2)^c$. This will play a central role in the following chapters. Similarly, for $d = 4$ a pair of chiral spinors η_+^1, η_+^2 give rise to an $SU(2) \times SU(2)$ -structure specified by two compatible pure spinors, but both of them consist of sums of even forms, since

now $(\eta_+^2)^c$ is a positive chirality spinor. That the spinors have the same chirality is necessary for them to be compatible in four dimensions [38].

2.5 Example: the generalized structures of Calabi-Yau cones

As a simple example, let us formulate the familiar case of a Calabi-Yau three-fold X in the language of generalized geometry. Here X is equipped with a pair of compatible pure spinors Ω_- and Ω_+ given by

$$\Omega_- = \bar{\Omega} , \quad (2.50)$$

$$\Omega_+ = \exp(i\omega) , \quad (2.51)$$

where Ω is a holomorphic $(3,0)$ -form and ω is the Kähler form.⁴ For example, taking $X = \mathbb{C}^3$ equipped with its flat metric, we have

$$\Omega = dz_1 \wedge dz_2 \wedge dz_3 , \quad \omega = \frac{i}{2} \sum_{i=1}^3 dz_i \wedge d\bar{z}_i , \quad (2.52)$$

where z_1, z_2, z_3 are standard complex coordinates on $\mathbb{C}^3$. In this case both pure spinors are closed, $d\Omega_- = d\Omega_+ = 0$, and thus the corresponding generalized almost complex structures are integrable. These are the same as in the above examples (2.40) and (2.41):

$$\mathcal{J}_- = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix} , \quad \mathcal{J}_+ = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix} , \quad (2.53)$$

respectively, where I denotes the integrable complex structure tensor on X and ω is the Kähler form (compatible and symplectic). Indeed, the compatibility condition gives $I^* \cdot \omega = 0$, which says that ω is a $(1,1)$ -form with respect to the complex structure I . From the expressions for $\mathcal{J}_-$ and $\mathcal{J}_+$ above, we obtain from (2.29) and (2.48) that

$$g_X = \omega(I, \cdot) , \quad B = 0 . \quad (2.54)$$

⁴ The complex conjugation in (2.50) is due to an unfortunate choice of conventions in the generalized geometry literature, see footnote 2 above.

Finally, for a *Ricci-flat* Kähler metric, the Mukai pairings (2.21) of the pure spinors (or equivalently their Mukai norms (2.35)) are equal:

$$\frac{i}{8}\langle\Omega_{-},\bar{\Omega}_{-}\rangle = \frac{i}{8}\Omega\wedge\bar{\Omega} = \frac{1}{3!}\omega^3 = \frac{i}{8}\langle\Omega_{+},\bar{\Omega}_{+}\rangle . \quad (2.55)$$

Without this condition, one instead has only a Kähler metric on a complex manifold with zero first Chern class, but the metric is *not* Ricci-flat.

For application to AdS_5 solutions of string theory, we are interested more specifically in *conical* geometries, as will be explained in detail in section 3.2. In the above context of Calabi-Yau three-folds, this means that by definition the Kähler metric takes the conical form $g_X = dr^2 + r^2 g_Y$, with g_Y a metric on a Sasaki-Einstein five-manifold Y , and that the holomorphic $(3,0)$ -form Ω is homogeneous of degree three under the *Euler* vector field $r\partial_r$, that is $\mathcal{L}_{r\partial_r}\Omega = 3\Omega$. Notice however that Ω_+ defined in (2.50) does not have a well-defined homogeneity under $r\partial_r$, since the Kähler form ω is homogeneous of degree two. Instead, the pure spinors associated with the Calabi-Yau cones appearing in the AdS/CFT correspondence take the rescaled forms [3]:

$$\begin{aligned} \Omega_- &= \bar{\Omega} , \\ \Omega_+ &= r^3 \exp\left(i\frac{\omega}{r^2}\right) . \end{aligned} \quad (2.56)$$

Both pure spinors are now homogeneous of degree three, and the resulting Riemannian metric is *conformal* to the cone metric: $g_X = r^{-2}(dr^2 + r^2 g_Y)$. Notice that g_X here is homogeneous of degree zero, and in fact setting $t = \log r \in (-\infty, \infty)$ we see that g_X is a *cylinder* over Y : $g_X = dt^2 + g_Y$. The crucial difference after the rescaling is that Ω_+ is no longer closed, and hence $\mathcal{J}_+$ is *not integrable*. This may sound peculiar, but these are the natural pure spinors associated with the geometry in AdS/CFT. Indeed, the lack of closure of Ω_+ may be understood as due to the presence of background fluxes on the cone. In the Sasaki-Einstein case discussed in this example, only the Ramond-Ramond five-form flux F_5 sourced by D3-branes is non-zero.

In this thesis we will study the generalization of the above geometric structure to the case where all the background fields of type IIB supergravity, including the B -field and all the components of the Ramond-Ramond fluxes, can be turned on.

Chapter 3

Supersymmetric AdS_5 solutions as generalized geometries

In this chapter we study the generalized geometry of the most general solutions of type IIB supergravity (the low energy limit of type IIB string theory) that are dual to superconformal field theories with $\mathcal{N} = 1$ supersymmetry. The ten-dimensional solutions of interest have the form $\text{AdS}_5 \times Y$, which can also be viewed using Poincaré coordinates as the warped product $\mathbb{R}^{1,3} \times X$ of Minkowski space-time with a six-dimensional cone $X \cong \mathbb{R}_+ \times Y$. As already mentioned, general solutions can involve background fluxes. The effect of the Neveu-Schwarz H -flux is nicely described by generalized geometry, and in particular there is an integrable generalized structure on the cone, in terms of which we can define a generalized Killing vector corresponding to the R-symmetry of the dual SCFT. However, the Ramond-Ramond fluxes F obstruct the integrability of the second compatible generalized structure. There is nevertheless a canonical symplectic structure on the cone, and we put it to good use by writing down formulas for the central charge and the conformal dimensions of certain BPS operators in terms of symplectic volumes. We illustrate our results with the Pilch-Warner solution.

3.1 Supersymmetric AdS_5 solutions of type IIB supergravity

After reviewing type IIB supergravity, we focus on the most general solutions that admit a dual SCFT with $\mathcal{N} = 1$ supersymmetry. Those solutions are of the form

$\text{AdS}_5 \times Y$, with AdS_5 the five-dimensional anti-de Sitter spacetime and Y a compact five-dimensional manifold, and satisfy some supersymmetry conditions, as first studied in [25] in terms of spinor bilinears.

Type IIB supergravity describes the dynamics of the ten-dimensional metric g_E (in Einstein frame), the dilaton ϕ , the Neveu-Schwarz (NS) B -field with curvature $H = dB$, and the Ramond-Ramond (RR) potential $C \equiv C_0 + C_2 + C_4$, written here as a polyform, that is a formal sum of a zero-form C_0 , a two-form C_2 , and a four-form C_4 . The corresponding RR flux is $F \equiv F_1 + F_3 + F_5 = (d - H \wedge) C$, or in components

$$F_1 = dC_0, \quad F_3 = dC_2 - HC_0, \quad F_5 = dC_4 - H \wedge C_2. \quad (3.1)$$

The five-form flux has the particularity that it is self-dual under the action of the Hodge star operator: $\star_{10} F_5 = F_5$. In terms of the convenient choice of fields

$$\begin{aligned} P_1 &\equiv \frac{i}{2} e^\phi F_1 + \frac{1}{2} d\phi, & Q_1 &\equiv -\frac{1}{2} e^\phi F_1, \\ G_3 &\equiv -ie^{\phi/2} F_3 - e^{-\phi/2} H, \end{aligned} \quad (3.2)$$

the equations of motion for the bosonic fields read [43, 25]

$$\begin{aligned} R_{MN} - \frac{1}{2} R g_{MN} &= P_M P_N^* + P_N P_M^* - g_{MN} |P_1|^2 \\ &\quad + \frac{1}{8} (G_{MP_1 P_2} G_N^{* P_1 P_2} + G_{NP_1 P_2} G_M^{* P_1 P_2}) - \frac{1}{4} g_{MN} |G_3|^2 \\ &\quad + \frac{1}{96} F_{MP_1 P_2 P_3 P_4} F_N^{P_1 P_2 P_3 P_4}, \\ D_M P^M &= -\frac{1}{24} G_{MNP} G^{MNP}, \\ D_P G^{MNP} &= P_P G^{*MNP} - \frac{i}{6} F^{MNP_1 P_2 P_3} G_{P_1 P_2 P_3}, \end{aligned} \quad (3.3)$$

where for a (complex) p -form A_p we write

$$|A_p|^2 \equiv \frac{1}{p!} g^{M_1 N_1} \dots g^{M_p N_p} A_{M_1 \dots M_p} \bar{A}_{N_1 \dots N_p}. \quad (3.4)$$

The covariant derivative D_M , with respect to local Lorentz transformations and local $U(1)$ transformations with gauge field Q_M , acts on a field A_p of charge q as

$$D_M A_p = (\nabla_M - iq Q_M) A_p. \quad (3.5)$$

Here P has charge 2 and G charge 1.

In order for a bosonic solution to preserve some supersymmetry, the fermionic fields of type IIB supergravity, namely the gravitino ψ_M and the gaugino λ , must be invariant under supersymmetry transformation, that is $\delta\psi_M = 0$ and $\delta\lambda = 0$. This implies the following “Killing spinor equations”:

$$\begin{aligned} D_M \epsilon - \frac{1}{96} (\Gamma_M^{P_1 P_2 P_3} G_{P_1 P_2 P_3} - 9 \Gamma^{P_1 P_2} G_{M P_1 P_2}) \epsilon^c \\ + \frac{i}{192} \Gamma^{P_1 P_2 P_3 P_4} F_{M P_1 P_2 P_3} \epsilon = 0 , \\ i \Gamma^M P_M \epsilon^c + \frac{i}{24} \Gamma^{P_1 P_2 P_3} G_{P_1 P_2 P_3} \epsilon = 0 . \end{aligned} \quad (3.6)$$

Here the ten-dimensional Dirac gamma matrices Γ^M satisfy $\{\Gamma^M, \Gamma^N\} = 2g^{MN}$ and generate the Clifford algebra $\text{Cliff}(1, 9)$, while $\epsilon = \epsilon_1 + i\epsilon_2$ is a complex combination of the two Majorana-Weyl spinors ϵ_i of type IIB supergravity, and ϵ^c is its charge-conjugate.

We are interested in the most general class of backgrounds that consist of the warped product of AdS_5 with a five-dimensional compact manifold Y :

$$g_E = e^{2\Delta} (g_{\text{AdS}} + g_Y) , \quad \text{or} \quad g_{MN} = e^{2\Delta} (g_{\mu\nu} + g_{mn}) . \quad (3.7)$$

The metric on AdS_5 is normalized to have unit radius, so that the Ricci tensor is $R_{\mu\nu} = -4g_{\mu\nu}$, which gives $R_{\text{AdS}} = -20$ for the Ricci scalar. In order to preserve the $SO(4, 2)$ symmetry of AdS_5 , all the fields have to be pullbacks of fields on the internal space Y (in particular, the warp factor Δ in (3.7) is a function on Y); the only exception is the five-form field strength, which satisfies the so-called Freund-Rubin ansatz

$$F_5 = f_5 (\text{vol}_{\text{AdS}} + \text{vol}_Y) , \quad \text{or} \quad F_{MNPQR} = f_5 (\varepsilon_{\mu\nu\lambda\rho\sigma} + \varepsilon_{mnpqr}) , \quad (3.8)$$

with f_5 a constant. In this thesis we will demand that $F_5 \neq 0$, or equivalently

$$f_5 \neq 0 . \quad (3.9)$$

Physically this corresponds to having non-vanishing D3-brane charge. It would be interesting to know whether or not all supersymmetric AdS_5 solutions of type IIB supergravity necessarily have this property.

The ten-dimensional gamma matrices Γ^M decompose accordingly:

$$\begin{aligned}\Gamma^\mu &= \rho^\mu \otimes 1 \otimes \sigma^3 , \\ \Gamma^m &= 1 \otimes \beta^m \otimes \sigma^1 .\end{aligned}\tag{3.10}$$

Here the ρ^μ generate $\text{Cliff}(1, 4)$, that is $\{\rho^\mu, \rho^\nu\} = 2\eta^{\mu\nu}$, and the β_m generate the Clifford algebra for g_Y , that is $\{\beta^m, \beta^n\} = 2g_Y^{mn}$. Equivalently, with respect to any orthonormal frame the corresponding $\hat{\beta}_m$ satisfy $\{\hat{\beta}_m, \hat{\beta}_n\} = 2\delta_{mn}$. We have chosen $\hat{\beta}_{12345} = +1$. In addition, σ^i are the usual Pauli matrices. Similarly, the supersymmetry spinor decomposes as

$$\epsilon = e^{\Delta/2} (\psi \otimes \xi_1 \otimes \theta + \psi^c \otimes \xi_2^c \otimes \theta) ,\tag{3.11}$$

where we assume that the spinor ψ satisfies the Killing spinor equation $\nabla_\mu \psi = \rho_\mu \psi / 2$ to ensure that supersymmetry is preserved on AdS_5 . Plugging this expression into the supersymmetry conditions (3.6) leads to the conditions for a supersymmetric AdS_5 background in terms of the two five-dimensional spinors ξ_1 and ξ_2 on Y [25]. There are two differential conditions:

$$\begin{aligned}(\nabla_m - \frac{i}{2} Q_m) \xi_1 + \frac{i}{4} (e^{-4\Delta} f_5 - 2) \beta_m \xi_1 + \frac{1}{8} e^{-2\Delta} G_{mnp} \beta^{np} \xi_2 &= 0 , \\ (\nabla_m + \frac{i}{2} Q_m) \xi_2 - \frac{i}{4} (e^{-4\Delta} f_5 + 2) \beta_m \xi_2 + \frac{1}{8} e^{-2\Delta} G_{mnp}^* \beta^{np} \xi_1 &= 0 ,\end{aligned}\tag{3.12}$$

and four algebraic conditions:

$$\begin{aligned}\beta^m \partial_m \Delta \xi_1 - \frac{1}{48} e^{-2\Delta} \beta^{mnp} G_{mnp} \xi_2 - \frac{i}{4} (e^{-4\Delta} f_5 - 4) \xi_1 &= 0 , \\ \beta^m \partial_m \Delta \xi_2 - \frac{1}{48} e^{-2\Delta} \beta^{mnp} G_{mnp}^* \xi_1 + \frac{i}{4} (e^{-4\Delta} f_5 + 4) \xi_2 &= 0 , \\ \beta^m P_m \xi_2 + \frac{1}{24} e^{-2\Delta} \beta^{mnp} G_{mnp} \xi_1 &= 0 , \\ \beta^m P_m^* \xi_1 + \frac{1}{24} e^{-2\Delta} \beta^{mnp} G_{mnp}^* \xi_2 &= 0 .\end{aligned}\tag{3.13}$$

Various spinor bilinears involving ξ_1 and ξ_2 were also introduced in [25], namely the following scalar bilinears:

$$\begin{aligned}A &\equiv \frac{1}{2} (\bar{\xi}_1 \xi_1 + \bar{\xi}_2 \xi_2) , & Z &\equiv \bar{\xi}_2 \xi_1 , \\ \sin \zeta &\equiv \frac{1}{2} (\bar{\xi}_1 \xi_1 - \bar{\xi}_2 \xi_2) , & S &\equiv \bar{\xi}_2^c \xi_1 ,\end{aligned}\tag{3.14}$$

the one-form bilinears:

$$\begin{aligned} K &\equiv \bar{\xi}_1^c \beta_{(1)} \xi_2 , & K_4 &\equiv \frac{1}{2} (\bar{\xi}_1 \beta_{(1)} \xi_1 - \bar{\xi}_2 \beta_{(1)} \xi_2) , \\ K_3 &\equiv \bar{\xi}_2 \beta_{(1)} \xi_1 , & K_5 &\equiv \frac{1}{2} (\bar{\xi}_1 \beta_{(1)} \xi_1 + \bar{\xi}_2 \beta_{(1)} \xi_2) , \end{aligned} \quad (3.15)$$

and the two-form bilinears:

$$V \equiv -\frac{i}{2} (\bar{\xi}_1 \beta_{(2)} \xi_1 - \bar{\xi}_2 \beta_{(2)} \xi_2) , \quad W \equiv -\bar{\xi}_2 \beta_{(2)} \xi_1 , \quad (3.16)$$

where $\beta_{(p)} \equiv \beta_{m_1 \dots m_p} dy^{m_1} \wedge \dots \wedge dy^{m_p} / p!$. It was shown in [25] that the supersymmetry conditions impose

$$A = 1 , \quad Z = 0 , \quad \sin \zeta = \frac{f_5}{4} e^{-4\Delta} . \quad (3.17)$$

Another key result of [25] is that $K_5^\#$, the vector dual to the one-form K_5 , is a Killing vector that preserves all of the fluxes. This was identified as corresponding to the R-symmetry of the dual SCFT. The Killing spinors ξ_1 and ξ_2 were used to introduce a canonical five-dimensional orthonormal frame in appendix B of [25], which is convenient for certain calculations. We will refer to that paper for further details.

Using the following equation of [25],

$$D(e^{6\Delta} W) = -e^{6\Delta} P \wedge W^* + \frac{f_5}{4} G , \quad (3.18)$$

and recalling the definition (3.2), we can obtain expressions for the two-form potentials B and C_2 in terms of the bilinear W introduced in (3.16) as well as two real closed two-forms that we call b_2 and c_2 :

$$B = -\frac{e^{2\Delta+\phi/2}}{\sin \zeta} \operatorname{Re} W + b_2 , \quad (3.19)$$

$$C_2 = -\frac{e^{2\Delta+\phi/2}}{\sin \zeta} (C_0 \operatorname{Re} W - e^{-\phi} \operatorname{Im} W) + c_2 . \quad (3.20)$$

Notice that $B - b_2$ is a globally defined two-form on Y , so that $H = dB$ is exact, and that $C_2 - c_2$ is also globally defined (up to large gauge transformations of C_0). Since the B -transform of b_2 by an exact form is a generalized diffeomorphism, and a gauge symmetry of string theory, we see that the physical information in b_2 is represented by its cohomology class in $H^2(X, \mathbb{R})$ (or more to the point in $H^2(Y, \mathbb{R})$). More precisely, large gauge transformations of the B -field, which correspond to tensoring the

underlying gerbe by a unitary line bundle on X , lead to the torus $H^2(X, \mathbb{R})/H^2(X, \mathbb{Z})$ (with suitable normalization). Turning on the two-form b_2 corresponds to giving vacuum expectation values to moduli (of the NS field B) and so is a symmetry of the supersymmetry equations. It is therefore left undetermined. In the field theory dual, the cohomology class of b_2 thus corresponds to a marginal deformation.

3.2 Reformulation as generalized geometries

The supersymmetric AdS_5 solutions described in the previous section can be advantageously reformulated in terms of generalized geometry, as in [26, 27]. The basic observation is that these solutions can be viewed as warped products of flat four-dimensional space with a six-dimensional manifold X , satisfying a set of supersymmetry conditions that imply the existence of a particular generalized geometry [28, 29].

We begin by rewriting the unit AdS_5 metric in a Poincaré patch as

$$g_{\text{AdS}} = \frac{dr^2}{r^2} + r^2 g_{\mathbb{R}^{1,3}} . \quad (3.21)$$

Switching from the Einstein frame to the string frame through $g_\sigma = e^{\phi/2} g_E$, we can consider (3.7) as a special case of a warped supersymmetric $\mathbb{R}^{1,3}$ solution of the form

$$g_\sigma = e^{2A} g_{\mathbb{R}^{1,3}} + g_X , \quad (3.22)$$

where the warp factor is given by

$$e^{2A} = e^{2\Delta + \phi/2} r^2 , \quad (3.23)$$

and

$$g_X = e^{2\Delta + \phi/2} \left(\frac{dr^2}{r^2} + g_Y \right) . \quad (3.24)$$

The six-dimensional manifold X , on which g_X is a metric, is a product $\mathbb{R}^+ \times Y$, where r may be interpreted as a coordinate on $\mathbb{R}^+$. This implies that X is non-compact. It thus follows that supersymmetric AdS_5 solutions are special cases of supersymmetric $\mathbb{R}^{1,3}$ solutions.

We now provide a map between the five-dimensional spinors $\xi_{1,2}$ appearing in the previous section and six-dimensional spinors $\eta_+^{1,2}$. We first use the $\text{Cliff}(5)$ gamma

matrices $\hat{\beta}_m$, with $m = 1, \dots, 5$, to construct Cliff(6) gamma matrices $\hat{\gamma}_i$, with $i = 1, \dots, 6$, via

$$\hat{\gamma}_m = \hat{\beta}_m \otimes \sigma_3, \quad \hat{\gamma}_6 = 1 \otimes \sigma_1, \quad (3.25)$$

These satisfy $\{\hat{\gamma}_i, \hat{\gamma}_j\} = 2\delta_{ij}$. The corresponding gamma matrices for the six-dimensional metric g_X will be denoted γ_i . We define the six-dimensional chirality operator to be

$$\tilde{\gamma} \equiv -i\hat{\gamma}_{123456} = 1 \otimes \sigma_2. \quad (3.26)$$

As usual with gamma matrix technology, we need to fix a consistent set of conventions for the various intertwiner operators (see for instance [44]). The A -, C -, and D -intertwiners in five dimensions operate as $A_5\beta_mA_5^{-1} = \beta_m^\dagger$, $C_5^{-1}\beta_mC_5 = \beta_m^T$, and $D_5^{-1}\beta_mD_5 = \beta_m^*$. We choose the D_6 intertwiner $D_6 = D_5 \otimes \sigma_2$, where $D_5 = C_5$ is the intertwiner of Cliff(5) discussed in more detail in appendix A of [25], and one checks $D_6^{-1}\gamma_i D_6 = -\gamma_i^*$. Taking $A_5 = 1$, we have $A_6 = 1$ and $\gamma_i^\dagger = \gamma_i$, so that $\bar{\eta} \equiv \eta^\dagger$ is just the Hermitian conjugate. If η_+ is a Weyl spinor, satisfying $\tilde{\gamma}\eta_+ = \eta_+$, then the conjugate spinor $\eta_- \equiv \eta_+^c \equiv D_6\eta_+^*$ satisfies $\tilde{\gamma}\eta_- = -\eta_-$.

To construct the relevant six-dimensional spinors we write

$$\xi_1 = \chi_1 + i\chi_2, \quad \xi_2 = \chi_1 - i\chi_2, \quad (3.27)$$

where $\bar{\chi}_1\chi_1 = \bar{\chi}_2\chi_2 = 1/2$ with $\bar{\chi} \equiv \chi^\dagger$. We then define

$$\begin{aligned} \eta_+^1 &= e^{A/2} \begin{pmatrix} \chi_1 \\ i\chi_1 \end{pmatrix}, & \eta_-^1 &= e^{A/2} \begin{pmatrix} -\chi_1^c \\ i\chi_1^c \end{pmatrix}, \\ \eta_+^2 &= e^{A/2} \begin{pmatrix} -\chi_2 \\ -i\chi_2 \end{pmatrix}, & \eta_-^2 &= e^{A/2} \begin{pmatrix} \chi_2^c \\ -i\chi_2^c \end{pmatrix}, \end{aligned} \quad (3.28)$$

where $\chi_i^c \equiv \tilde{D}_5\chi_i^*$ denotes five-dimensional charge conjugation.

After some detailed calculation we find that the five-dimensional supersymmetry equations (3.12)-(3.13) are equivalent to the following six-dimensional spinor equations:

$$\left(D_i - \frac{1}{4}H_i\right)\eta_+^1 + \frac{e^\phi}{8}\not{F}\gamma_i\eta_+^2 = 0, \quad (3.29)$$

$$\frac{1}{2}e^A\not{\partial}A\eta_+^1 - \frac{1}{8}e^{A+\phi}\not{F}\eta_+^2 = 0, \quad (3.30)$$

$$\not{D}\eta_+^1 + \left(\not{\partial}(2A - \phi) - \frac{1}{4}\not{H}\right)\eta_+^1 = 0, \quad (3.31)$$

as well as additional equations obtained by applying the rule:

$$\eta^1 \leftrightarrow \eta^2, \quad \not{F} \rightarrow -\not{F}^\dagger, \quad H \rightarrow -H. \quad (3.32)$$

In these equations we are using the notation that, for example,

$$H_i \equiv \frac{1}{2}H_{ijk}\gamma^{jk}, \quad \not{F} \equiv F_i\gamma^i + \frac{1}{3!}F_{ijk}\gamma^{ijk} + \frac{1}{5!}F_{ijklm}\gamma^{ijklm}. \quad (3.33)$$

These agree precisely with the general conditions derived in [29, 45] for an $\mathcal{N} = 1$ supersymmetric $\mathbb{R}^{1,3}$ background (with vanishing four-dimensional cosmological constant).

Using the two chiral $Spin(6)$ -spinors η_+^1 and η_+^2 we may define the bispinors

$$\Phi_+ \equiv \eta_+^1 \otimes \bar{\eta}_+^2, \quad \Phi_- \equiv \eta_+^1 \otimes \bar{\eta}_-^2. \quad (3.34)$$

As we explained in section 2.3, these bispinors may also be viewed via the Clifford map (2.31) as elements of $\Omega^*(X, \mathbb{C})$. We will therefore mainly tend to think of $\Phi_\pm$ as complex differential polyforms of even/odd degree. These generalized $Spin(6, 6)$ -spinors are in fact both *pure* spinors, and they are also *compatible*. They then define an $SU(3) \times SU(3)$ -structure on $TX \oplus T^*X$. In terms of $\Phi_\pm$, the spinor equations for a general supersymmetric $\mathbb{R}^{1,3}$ solution (not necessarily associated with an AdS_5 solution, but with vanishing four-dimensional cosmological constant) may be rewritten as [45] (see also [28])

$$d_H(e^{2A-\phi}\Phi_-) = 0, \quad (3.35)$$

$$d_H(e^{2A-\phi}\Phi_+) = e^{2A-\phi}dA \wedge \bar{\Phi}_+ + \frac{1}{16}e^{2A}[(|\eta_+^1|^2 - |\eta_+^2|^2)F + i(|\eta_+^1|^2 + |\eta_+^2|^2) \star \lambda(F)], \quad (3.36)$$

where from (2.22) we have $\lambda(F) = F_1 - F_3 + F_5$. Note that the Hodge star is with respect to the metric g_X , with positive orientation given by $-dr \wedge \text{vol}_Y$. The remaining Bianchi identities and equations of motion are (see equations (4.9)-(4.10) in [45])

$$dH = 0, \quad d_H F = \delta(\text{source}), \quad (3.37)$$

$$d(e^{4A-2\phi} \star H) - e^{4A} F_n \wedge \star F_{n+2} = 0, \quad (3.38)$$

$$(d + H \wedge)(e^{4A} \star F) = 0. \quad (3.39)$$

Here the higher-form analog of the Dirac delta function $\delta(\text{source})$ is the charge density of D-brane sources. The equation of motion for F can also be written as

$$d[e^{4A}e^{-B} \star \lambda(F)] = 0, \quad (3.40)$$

and follows from the supersymmetry equations. In fact, for AdS_5 solutions it was shown in [25] that supersymmetry implies *all* of the equations of motion and Bianchi identities $d_H F = 0$. In particular, this implies that there can be no D-branes sources in the background, that is $\delta(\text{source}) = 0$.

For a supersymmetric $\mathbb{R}^{1,3}$ background, the spinor norms must satisfy

$$|\eta_+^1|^2 + |\eta_+^2|^2 = e^A c_+, \quad |\eta_+^1|^2 - |\eta_+^2|^2 = e^{-A} c_-, \quad (3.41)$$

where $c_{\pm}$ are constants. Upon squaring and subtracting these equations we obtain

$$\|\Phi_{\pm}\|^2 = \frac{1}{8} |\eta_+^1|^2 |\eta_+^2|^2 = \frac{1}{32} (e^{2A} c_+^2 - e^{-2A} c_-^2). \quad (3.42)$$

For the particular case of AdS_5 solutions the above equations simplify somewhat. In this case it is possible to fix the constant c_- in (3.42) by the scaling of $\Phi_{\pm}$ with r which, using (3.23), implies that $c_- = 0$ and hence $|\eta_+^1| = |\eta_+^2|$. This is consistent with the equation $Z = 0$ in (3.14), since from (3.28) we see that $|\eta_{\pm}^1| = |\eta_{\pm}^2|$ is equivalent to $\text{Re } Z = 0$. Notice that $c_- = 0$ is also a necessary condition in order to have supersymmetric probe branes [46]. The normalization that was used in [25] implies $|\eta_+^1|^2 = |\eta_+^2|^2 = e^A$ and hence $c_+ = 2$. One can actually go a little further. In [26] it was assumed that there was an $SU(2)$ -structure on the cone. In terms of the spinors η_i^+ this is equivalent to the condition that, in addition to $c_- = 0$, one has $\bar{\eta}_+^1 \eta_+^2 + \bar{\eta}_+^2 \eta_+^1 = 0$. However it is easy to see that this is equivalent to $\text{Im } Z = 0$, which again is required by supersymmetry on Y . Thus in fact all supersymmetric AdS_5 solutions necessarily satisfy the $SU(2)$ condition of [26].

Just as in (2.33), we can now define the following pure spinors of $S_{\pm}(E)$:

$$\Omega_{\pm} \equiv e^{2A-\phi} e^{-B} \Phi_{\pm}, \quad (3.43)$$

where we have also rescaled by $\exp(2A)$ for convenience. With these definitions we can write the supersymmetry equations for AdS_5 solutions in terms of a pair of pure spinors $\Omega_{\pm}$ with equal Mukai norms (2.35),

$$\|\Omega_-\|^2 = \|\Omega_+\|^2 = \frac{1}{8} e^{6A-2\phi}, \quad (3.44)$$

as two differential equations:

$$d\Omega_- = 0 , \quad (3.45)$$

$$d\Omega_+ = dA \wedge \bar{\Omega}_+ + \frac{i}{8} e^{3A} e^{-B} \star \lambda(F) . \quad (3.46)$$

We see that Ω_- is closed, and so the associated generalized almost complex structure, which we denote by $\mathcal{J}_-$, is integrable. Combined with the fact that the norm of Ω_- is nowhere vanishing, this means that the cone X is a *generalized Calabi-Yau* manifold in the sense of Hitchin [30].

However, the presence of the RR fluxes F on the right-hand side of (3.46) obstructs the integrability of the generalized almost complex structure $\mathcal{J}_+$ associated to Ω_+ . If it were integrable, we would have a generalized Kähler manifold. It is worth noting that the differential constraint (3.46) can be split as

$$d(e^{-A} \text{Re } \Omega_+) = 0 , \quad (3.47)$$

$$d(e^A \text{Im } \Omega_+) = \frac{1}{8} e^{4A} e^{-B} \star \lambda(F) , \quad (3.48)$$

and that in turn equation (3.48) can be rewritten as [47]

$$e^{-B} F = 8 \mathcal{J}_- \cdot d(e^{-3A} \text{Im } \Omega_+) = 8 d^{\mathcal{J}_-} (e^{-3A} \text{Im } \Omega_+) , \quad (3.49)$$

where $d^{\mathcal{J}_-} \equiv -[d, \mathcal{J}_- \cdot]$. The Bianchi identity (3.37) without sources then implies that

$$d(e^{-B} F) = 8 d d^{\mathcal{J}_-} (e^{-3A} \text{Im } \Omega_+) = 0 . \quad (3.50)$$

We now relate the pure spinors $\Omega_{\pm}$ to the spinor bilinears used in [25]. Using the definition of $\Omega_{\pm}$, the Fierz identity (2.32), and the results of section 3.1, we find the following expression for Ω_- :

$$\Omega_- = \theta \wedge e^{-b_- + i\omega_-} , \quad (3.51)$$

with

$$\begin{aligned} \theta &= -\frac{r^3}{8} e^{4\Delta} (iK + S d \log r) , \\ b_- &= \frac{e^{2\Delta + \phi/2}}{\sin \zeta \sin^2 2\bar{\phi}} (\cos^2 2\bar{\phi} d \log r \wedge \text{Im } K_3 - K_4 \wedge \text{Re } K_3) + b_2 , \\ \omega_- &= \frac{e^{2\Delta + \phi/2}}{\sin \zeta \sin^2 2\bar{\phi}} (\cos 2\bar{\theta} \cos 2\bar{\phi} d \log r \wedge \text{Re } K_3 + K_5 \wedge \text{Im } K_3) , \end{aligned} \quad (3.52)$$

where b_2 was introduced in (3.19). Note that b_- and ω_- are not uniquely defined since we can add any two-form that vanishes when wedged with θ . Here the angles $\bar{\theta}$ and $\bar{\phi}$ (which appear in appendix B of [25] without bars) are functions on the link Y that are related to the scalar spinor bilinears through

$$\sin \zeta = \cos 2\bar{\theta} \cos 2\bar{\phi}, \quad |S| = -\sin 2\bar{\theta} \cos 2\bar{\phi}. \quad (3.53)$$

Using the results of [25], we have the important result that θ is exact¹

$$\theta = d\left(-\frac{r^3}{24}e^{4\Delta}S\right) \equiv d(r^3\theta_0). \quad (3.54)$$

In addition, we see from the supersymmetry equation (3.48) that, given that we assume that $F_5 \neq 0$, the imaginary part of Ω_+ must have a scalar component and hence Ω_+ is of type zero (compare with (2.28)):

$$\Omega_+ = \alpha_+ e^{-b_+ + i\omega_+}. \quad (3.55)$$

In terms of the bilinears of [25] we find

$$\begin{aligned} \alpha_+ &= -i \frac{f_5}{32} e^{-A} r^4, \\ b_+ &= -\frac{e^{2\Delta+\phi/2}}{\sin \zeta} d \log r \wedge \text{Im } K_3 + b_2, \\ \omega_+ &= \frac{e^{2\Delta+\phi/2}}{\sin \zeta} (d \log r \wedge K_4 - V). \end{aligned} \quad (3.56)$$

3.2.1 Mesonic moduli space

A key point is that the type of Ω_- is generically one, but has the property that it can jump to three on the *type-change locus* $\mathcal{T} = \{\theta = 0\}$. This locus can be neatly parameterized through the angles $\bar{\theta}$ and $\bar{\phi}$. Since we assume that $f_5 \neq 0$, we have from (3.17) that $\sin \zeta$ is nowhere zero and then (3.53) implies that both $\cos 2\bar{\phi}$ and $\cos 2\bar{\theta}$ are nowhere zero. Using the expression for K in appendix B of [25], we see that

$$\sin 2\bar{\theta} = 0 \iff \theta_0 = 0, \quad (3.57)$$

$$\sin 2\bar{\theta} = \sin 2\bar{\phi} = 0 \iff \theta = 0. \quad (3.58)$$

¹The fact that θ is closed was essentially observed in [35], and in [26] it was also shown to be exact in the special cases of the Pilch-Warner and Lunin-Maldacena solutions.

The locus $\{\theta = 0\}$ is thus a sublocus of $\{\theta_0 = 0\}$. Notice that, where $\theta = 0$, Ω_- is not identically zero, as one might have naively expected from (3.51), but instead reduces to a finite, non-zero three-form. Indeed, the powers of $\sin 2\bar{\phi}$ in the denominator of b_- and ω_- are cancelled by those in K , K_3 , and K_4 .

The type-change locus $\mathcal{T} = \{\theta = 0\}$ can be given a physical interpretation as the Abelian *mesonic moduli space* of the dual SCFT. This follows from the study of the conditions for a D-brane configuration to be supersymmetric, which means that it must wrap a generalized calibrated cycle Σ in the cone X [35]. In the case of spacetime-filling D-branes, these conditions read, in our notations,

$$\begin{aligned} P_\Sigma[\text{Re } \Omega_+] \wedge e^{F_\Sigma}|_{\text{top}} &= 0 && \text{D-flatness ,} \\ P_\Sigma[(dx^m \wedge + \partial_m \lrcorner) \Omega_-] \wedge e^{F_\Sigma}|_{\text{top}} &= 0 && \text{F-flatness ,} \end{aligned} \quad (3.59)$$

where P_Σ is the pullback on the worldvolume of the brane, and F_Σ is the worldvolume field-strength (without the B -field to form a gauge-invariant combination, since our $\Omega_\pm$ are essentially B -transforms of the $\Psi_\pm$ in [35]).

Now the Abelian mesonic moduli space of the dual SCFT can be identified with the moduli space of a supersymmetric probe D3-brane in X , since strings stretched between this probe brane and the stack of D3-branes at the origin will appear as a meson. A spacetime-filling D3-brane is located at a point of X , and so the cycle Σ is zero-dimensional. The supersymmetry conditions thus become the scalar conditions

$$\begin{aligned} \text{Re } \Omega_+|_0 &= \text{Re } \alpha_+ = 0 && \text{D-flatness ,} \\ \partial_m \lrcorner \Omega_-|_0 &= \partial_m \lrcorner \theta = 0 && \text{F-flatness .} \end{aligned} \quad (3.60)$$

The D-flatness condition is trivially satisfied in our case since we saw in (3.56) that the zero-form part of Ω_+ is always purely imaginary. The F-flatness condition on the other hand coincides precisely with the type-change locus $\mathcal{T} = \{\theta = 0\}$.

Notice that in the Sasaki-Einstein case, the pure spinor Ω_- reduces to the holomorphic $(3, 0)$ -form Ω , which is everywhere of type three. Then the space of possible probe D3-brane configurations, and thus the mesonic moduli space of the dual SCFT, is the entire cone: $\mathcal{T} = X_{\text{CY}}$. In the generalized case however, the mesonic moduli space is restricted to a sublocus of X . We shall see examples of this in section 3.6 and in chapter 6.

3.3 Generalized Killing vector fields

In this section we identify the generalized Killing vector fields which correspond to the dilatation symmetry and the R-symmetry of the dual SCFT.

3.3.1 Dilatation symmetry

We begin with the Euler vector field $r\partial_r$, which corresponds to the dilatation symmetry. It immediately follows from (3.23), (3.28), and (3.34) that the pure spinors $\Phi_\pm$ are homogeneous of degree one under $r\partial_r$:

$$\mathcal{L}_{r\partial_r}\Phi_\pm = \Phi_\pm , \quad (3.61)$$

and therefore, since the B -field, the warp factor Δ , and the dilaton ϕ are pull-backs from Y , so that $\mathcal{L}_{r\partial_r}e^A = e^A$, the pure spinors $\Omega_\pm$ (3.43) are homogeneous of degree three:

$$\mathcal{L}_{r\partial_r}\Omega_\pm = 3\Omega_\pm . \quad (3.62)$$

This implies that $r\partial_r$ preserves the associated generalized structures $\mathcal{J}_\pm$:

$$\mathcal{L}_{r\partial_r}\mathcal{J}_\pm = 0 . \quad (3.63)$$

To see this, recall that $\mathcal{J}_\pm$ are defined by identifying their $+i$ -eigenspaces $L_{\mathcal{J}_\pm}$ with the annihilator spaces $L_{\Omega_\pm}$ of $\Omega_\pm$, and these are clearly preserved under the one-parameter family of diffeomorphisms generated by $r\partial_r$. It further follows that $\mathcal{L}_{r\partial_r}G = 0$, where G is the generalized metric $G = -\mathcal{J}_+\mathcal{J}_-$, so that $r\partial_r$ is *generalized Killing*. Moreover, equation (3.63) says that $r\partial_r$ is a *generalized holomorphic* vector field for the integrable generalized complex structure $\mathcal{J}_-$.² This clearly generalizes the standard result that the Euler vector $r\partial_r$ is Killing and holomorphic in the case where Y is Sasaki-Einstein and $X = C(Y)$ is Calabi-Yau.

3.3.2 R-symmetry

We next define the generalized vectors

$$\xi \equiv \mathcal{J}_-(r\partial_r) , \quad (3.64)$$

$$\eta \equiv \mathcal{J}_-(d\log r) . \quad (3.65)$$

²We shall not use this terminology for $\mathcal{J}_+$ since it is not integrable.

Note that the conical form (3.24) of the metric g_X and the fact that B has no component along dr implies that

$$G(d \log r) = e^{-2\Delta-\phi/2} r \partial_r, \quad G(r \partial_r) = e^{2\Delta+\phi/2} d \log r, \quad (3.66)$$

and hence we may also write

$$\xi = e^{2\Delta+\phi/2} \mathcal{J}_+(d \log r), \quad \eta = e^{-2\Delta-\phi/2} \mathcal{J}_+(r \partial_r). \quad (3.67)$$

In a fixed splitting of E , we may split ξ and η into vector and one-form parts, denoted by subscripts “v” and “f”, respectively:

$$\xi = \xi_v + \xi_f, \quad \eta = \eta_v + \eta_f. \quad (3.68)$$

By carrying out a calculation, presented in appendix A, we may then write these as bilinears constructed from the five-dimensional Killing spinors (3.15):

$$\begin{aligned} \xi_v &= K_5^\#, & \xi_f &= \xi_v \lrcorner b_2, \\ \eta_v &= e^{-2\Delta-\phi/2} \operatorname{Re} K_3^\#, & \eta_f &= \frac{4}{f_5} e^{4\Delta} K_4 + \eta_v \lrcorner b_2. \end{aligned} \quad (3.69)$$

As discussed in appendix A, it is the B -transform ξ^B of the generalized vector ξ that is naturally related to the bilinears of [25]. We have obtained (3.69) by performing an inverse B -transform using the expression for the B -field in terms of bilinears presented in (3.19). In particular, this is where the closed two-form b_2 appears.

In [25] it was shown that $K_5^\#$ is a Killing vector that preserves all the fluxes, and thus $K_5^\#$ was identified as being dual to the R-symmetry in the SCFT. In the generalized geometry we can show the stronger conditions that

$$\mathbb{L}_\xi \mathcal{J}_\pm = 0, \quad (3.70)$$

and hence the generalized vector ξ is *generalized holomorphic* and *generalized Killing*. Again, this clearly generalizes the result that in the Sasaki-Einstein case the vector field $\xi \equiv I(r \partial_r)$ is a holomorphic Killing vector field on the Calabi-Yau cone.

In fact it is straightforward to show $\mathbb{L}_\xi \Omega_- = -3i\Omega_-$ and hence $\mathbb{L}_\xi \mathcal{J}_- = 0$. Indeed since $d\Omega_- = 0$ and $r \partial_r - i\xi \in L_{\mathcal{J}_-} = L_{\Omega_-}$ annihilates Ω_- , using (2.24) and (3.62) we have

$$\mathbb{L}_\xi \Omega_- = d(\xi \cdot \Omega_-) = -\operatorname{id}(r \partial_r \cdot \Omega_-) = -i \mathcal{L}_{r \partial_r} \Omega_- = -3i\Omega_-. \quad (3.71)$$

To show that $\mathbb{L}_\xi \Omega_+ = 0$ and hence $\mathbb{L}_\xi \mathcal{J}_+ = 0$ requires a bit more work. We start by deriving a few useful results. Recall first that the compatibility condition $[\mathcal{J}_-, \mathcal{J}_+] = 0$ can be rephrased as $\mathcal{J}_- \cdot \Omega_+ = 0$, or $\Omega_+ \in U_-^0$. This implies that $d(e^{-3A} \text{Im } \Omega_+) = (\bar{\partial} + \partial)(e^{-3A} \text{Im } \Omega_+) \in U_-^1 \oplus U_-^{-1}$, and so we can write $e^{-B} F \equiv F^1 + F^{-1} \in U_-^1 \oplus U_-^{-1}$. Writing $r\partial_r \equiv r\partial_r^+ + r\partial_r^-$ with $r\partial_r^\pm : U_-^k \rightarrow U_-^{k\pm 1}$, we get from $r\partial_r \lrcorner (e^{-B} F) = 0$ that $r\partial_r^+ \lrcorner F^1 = r\partial_r^- \lrcorner F^{-1} = 0$. Using that the Lie algebra action of ξ on generalized spinors is given by $\xi \cdot = \mathcal{J}_-(r\partial_r) \cdot = [\mathcal{J}_-, r\partial_r \lrcorner]$, we can then calculate

$$\begin{aligned} \xi \cdot \mathcal{J}_- \cdot (e^{-B} F) &= \mathcal{J}_- \cdot r\partial_r \lrcorner \mathcal{J}_- \cdot (e^{-B} F) \\ &= i\mathcal{J}_- \cdot [(r\partial_r^+ + r\partial_r^-) \lrcorner (F^1 - F^{-1})] \\ &= i\mathcal{J}_- \cdot (r\partial_r^- \lrcorner F^1 - r\partial_r^+ \lrcorner F^{-1}) = 0, \end{aligned} \quad (3.72)$$

where in the last step we used that $r\partial_r^- \lrcorner F^1 - r\partial_r^+ \lrcorner F^{-1} \in U_-^0$. Next, using the supersymmetry constraint (3.48) we get

$$\begin{aligned} \mathcal{J}_- \cdot (e^{-B} F) &= 8\mathcal{J}_- \cdot \mathcal{J}_- \cdot d(e^{-3A} \text{Im } \Omega_+) \\ &= -8d(e^{-3A} \text{Im } \Omega_+) \\ &= 32e^{-3A} dA \wedge \text{Im } \Omega_+ - 8e^{-4A} d(e^A \text{Im } \Omega_+) \\ &= 32e^{-3A} dA \wedge \text{Im } \Omega_+ - e^{-B} \star \lambda(F). \end{aligned} \quad (3.73)$$

Notice that we can write down two independent annihilators of Ω_- :

$$\begin{aligned} Z_1^- &\equiv (1 - i\mathcal{J}_-)r\partial_r = r\partial_r - i\xi, \\ Z_2^- &\equiv (1 - i\mathcal{J}_-)d \log r = d \log r - i\eta, \end{aligned} \quad (3.74)$$

as well as two independent annihilators of Ω_+ :

$$\begin{aligned} Z_1^+ &\equiv (1 - i\mathcal{J}_+)r\partial_r = r\partial_r - ie^{2\Delta+\phi/2}\eta, \\ Z_2^+ &\equiv (1 - i\mathcal{J}_+)e^{2\Delta+\phi/2}d \log r = e^{2\Delta+\phi/2}d \log r - i\xi, \end{aligned} \quad (3.75)$$

and from the fact that these are null isotropic generalized vectors, $\langle Z_i^\pm, Z_j^\pm \rangle = 0$, we obtain the useful relations

$$\begin{aligned} \xi_v \lrcorner d \log r &= \eta_v \lrcorner d \log r = r\partial_r \lrcorner \xi_f = \xi_v \lrcorner \xi_f = r\partial_r \lrcorner \eta_f = \eta_v \lrcorner \eta_f = 0, \\ \xi_v \lrcorner \eta_f + \eta_v \lrcorner \xi_f &= 1. \end{aligned} \quad (3.76)$$

Acting on (3.73) with ξ gets rid of the left-hand side because of (3.72), and acting with $d \log r$ gets rid of the last term because $r \partial_r \lrcorner F = 0$ implies $\star \lambda(F) = d \log r \wedge \dots$. Then using the annihilator constraint $Z_2^+ \cdot \Omega_+ = 0$, we are left with

$$(\xi_v \lrcorner dA) d \log r \wedge \text{Im } \Omega_+ = 0, \quad (3.77)$$

from which we conclude, since the zero-form part of $\text{Im } \Omega_+$ is non-zero by (4.40), that $\xi_v \lrcorner dA = 0$ and so $\xi_v \lrcorner d(\Delta + \phi/4) = 0$.

We are now in a position to show that Ω_+ is preserved by ξ . From (3.46) we obtain

$$\begin{aligned} \mathbb{L}_\xi \Omega_+ &= \xi \cdot d\Omega_+ + d(\xi \cdot \Omega_+) \\ &= (\xi \cdot + i e^{2\Delta + \phi/2} d \log r \wedge) \left[dA \wedge \bar{\Omega}_+ + \frac{i}{8} e^{3A} e^{-B} \star \lambda(F) \right] \\ &\quad + i d \log r \wedge d e^{2\Delta + \phi/2} \wedge \Omega_+ \\ &= 2i e^{2\Delta + \phi/2} d \log r \wedge [d(\Delta + \phi/4) - dA] \wedge \Omega_+ = 0. \end{aligned} \quad (3.78)$$

This implies $\mathbb{L}_\xi \mathcal{J}_+ = 0$, and since we already know that $\mathbb{L}_\xi \mathcal{J}_- = 0$, we conclude that the generalized Reeb vector ξ preserves the generalized metric, $\mathbb{L}_\xi G = 0$, or in terms of the metric g and the two-form B ,

$$\mathcal{L}_{\xi_v} g = \mathcal{L}_{\xi_v} B - d\xi_f = 0. \quad (3.79)$$

Finally, using (3.48), (3.73), (3.75), and (3.47), we calculate

$$\begin{aligned} \mathbb{L}_\xi (e^{4A} e^{-B} \star \lambda(F)) &= d [\xi \cdot (e^{4A} e^{-B} \star \lambda(F))] \\ &= 4d [d e^{4\Delta + \phi} \wedge dr^2 \wedge e^{-A} \text{Re } \Omega_+] = 0. \end{aligned} \quad (3.80)$$

Then since $\mathcal{L}_{\xi_v} e^A = \mathbb{L}_\xi e^{-B} = 0$ and $\mathcal{L}_{\xi_v} g = 0$, this leads to $\mathcal{L}_{\xi_v} F = 0$, or equivalently

$$\mathbb{L}_\xi (e^{-B} F) = 0. \quad (3.81)$$

Thus, we have established that $\xi \equiv \mathcal{J}_-(r \partial_r)$ is a generalized holomorphic vector field, which moreover is generalized Killing for the generalized metric $G = -\mathcal{J}_- \mathcal{J}_+$, and also preserves the RR fluxes. This implies that ξ generates a full symmetry of the supergravity solution.

To conclude this section we note that when $f_5 \neq 0$ the vector field $\xi_v = K_5^\#$ is nowhere vanishing on Y . One can see this from the formula [25]

$$|K_5^\#|^2 = \sin^2 \zeta + |S|^2 , \quad (3.82)$$

and using (3.17). Thus ξ_v acts locally freely on Y and hence its orbits define a one-dimensional *foliation* of Y . This is again precisely as in the Sasaki-Einstein case (although in the Sasaki-Einstein case the norm of ξ_v is constant).

3.4 BPS operators and generalized holomorphic spinors

In the Sasaki-Einstein case, holomorphic functions on the Calabi-Yau cone with a definite scaling weight λ under the action of $r\partial_r$ also have a charge λ under the action of ξ (see for instance [48]). This stems from the intimate connection (via Kaluza-Klein reduction on the Sasaki-Einstein manifold) between (anti-)holomorphic functions on the cone and BPS operators in the dual CFT, in fact (anti-)chiral primary operators. For general AdS_5 solutions we might expect that the holomorphic functions should be replaced by polyforms and that the BPS condition of matching charges should be with respect to the generalized Lie derivative $\mathbb{L}$ discussed in chapter 2. We now derive such a result, leaving the detailed connection with Kaluza-Klein reduction on the internal space Y for future work.

Consider a generalized spinor Ψ satisfying

$$\Psi \in U_-^k , \quad \mathbb{L}_{r\partial_r} \Psi = \lambda \Psi , \quad (3.83)$$

for some k and λ . Then we obtain

$$\bar{\partial}_- \Psi = 0 , \quad (r\partial_r + i\xi) \cdot \Psi = 0 \quad \implies \quad \mathbb{L}_\xi \Psi = i \mathbb{L}_{r\partial_r} \Psi . \quad (3.84)$$

In other words, subject to the constraints (3.83), a generalized spinor is BPS if it is generalized holomorphic and is annihilated by $r\partial_r + i\xi$. To see this result, we first write $r\partial_r = (r\partial_r + i\xi)/2 + (r\partial_r - i\xi)/2$ and use (3.83) to deduce that

$$\begin{aligned} \partial_- [(r\partial_r + i\xi) \cdot \Psi] + (r\partial_r + i\xi) \cdot \partial_- \Psi &= 0 , \\ \bar{\partial}_- [(\partial_r - i\xi) \cdot \Psi] + (r\partial_r - i\xi) \cdot \bar{\partial}_- \Psi &= 0 . \end{aligned} \quad (3.85)$$

In obtaining this we used the fact that since $r\partial_r - i\xi$ is an annihilator of Ω_- it raises the level of Ψ and similarly $r\partial_r + i\xi$ lowers the level, while $\bar{\partial}_-$ and ∂_- raises and lowers it respectively. We then compute

$$\begin{aligned}\mathbb{L}_\xi \Psi &= i\mathbb{L}_{r\partial_r} \Psi - i \{ d[(r\partial_r + i\xi) \cdot \Psi] + (r\partial_r + i\xi) \cdot d\Psi \} \\ &= i\mathbb{L}_{r\partial_r} \Psi - i \{ \bar{\partial}_- [(r\partial_r + i\xi) \cdot \Psi] + (r\partial_r + i\xi) \cdot \bar{\partial}_- \Psi \} .\end{aligned}\quad (3.86)$$

In a similar way, given (3.83) we also have

$$\partial_- \Psi = 0 , \quad (r\partial_r - i\xi) \cdot \Psi = 0 \quad \implies \quad \mathbb{L}_\xi \Psi = -i\mathbb{L}_{r\partial_r} \Psi . \quad (3.87)$$

3.5 Contact volume formulas

In the previous sections we have shown how much of the geometric structure of Sasaki-Einstein solutions of type IIB supergravity can be “generalized” to the most general solutions admitting a dual description in terms of $\mathcal{N} = 1$ SCFT. We now show that part of this structure, namely the symplectic structure on the cone, is still present in the general solutions, exactly as in the Sasaki-Einstein case, that is without the need to be generalized. This symplectic structure descends to a contact structure on Y and the associated Reeb vector field is the generalized Killing vector ξ_v studied in section 3.3.2. A direct benefit is that the central charge and conformal dimensions of certain BPS operators in the dual superconformal field theory can be expressed neatly as contact volumes.

3.5.1 A canonical symplectic structure

We claim that the rescaled two-form

$$\omega \equiv e^{-2A} r^4 \omega_+ \quad (3.88)$$

defines a canonical symplectic structure on the cone $X = C(Y) \cong \mathbb{R}^+ \times Y$. To see this, we first observe that Y admits a contact structure defined by the one-form

$$\sigma \equiv \frac{4}{f_5} e^{4\Delta} K_4 . \quad (3.89)$$

Recall that for a one-form σ to be *contact*, the top-degree form $\sigma \wedge d\sigma \wedge d\sigma$ must be nowhere vanishing, and thus a volume form. Using (3.19) of [25], and results in

appendix B of [25], we can show that ³

$$\sigma \wedge d\sigma \wedge d\sigma = -\frac{128}{f_5^2} e^{8\Delta} \text{vol}_Y = -\frac{8}{\sin^2 \zeta} \text{vol}_Y . \quad (3.90)$$

We then observe that

$$\omega = \frac{1}{2} d(r^2 \sigma) , \quad (3.91)$$

which shows that ω is closed and non-degenerate, and hence defines a symplectic structure on the cone X . Alternatively, we can see the formula (3.91) for ω directly from the supersymmetry equation (3.47) on noting that $e^{-A}\Omega_+$ has scaling dimension two under $r\partial_r$. Furthermore, again using the results of appendix B of [25], we have

$$\xi_v \lrcorner \sigma = 1 , \quad \xi_v \lrcorner d\sigma = 0 , \quad (3.92)$$

which shows that ξ_v is also the unique Reeb vector field associated with the contact structure. Notice also that (3.91) implies that $r^2/2$ is precisely the Hamiltonian function for the Hamiltonian vector field ξ_v , that is $d(r^2/2) = -\xi_v \lrcorner \omega$. It is remarkable that these features, which are well-known in the Sasaki-Einstein case, are valid for all supersymmetric AdS_5 solutions (with $f_5 \neq 0$).

We remark here that although we have a symplectic structure, we do not quite have a Kähler structure, as in the Calabi-Yau case, but it is quite close. Using the last equation in (3.69) and the definition (3.89) we see that

$$\eta_f = \sigma + \eta_v \lrcorner b_2 , \quad (3.93)$$

and thus $(e^{b_2} \eta)|_f = \sigma$. Since $e^{b_2}(d \log r) = d \log r$ manifestly, and by definition $\eta \equiv \mathcal{J}_-(d \log r)$, we have, using (2.39),

$$\sigma = \mathcal{J}_-^{b_2}(d \log r)|_f = -(I_-^{b_2})^*(d \log r) . \quad (3.94)$$

This is precisely analogous to the formula for the contact form in the Sasakian case. We then have

$$d\mathcal{J}_-^{b_2} r^2 = -r^2 d \left(Q_-^{b_2} + \frac{1}{2} I_m^m \right) - (I_-^{b_2})^*(d(r^2)) , \quad (3.95)$$

³In terms of the orthonormal frame in appendix B of [25] we have $\text{vol}_Y = e^{12345}$.

where $d^{\mathcal{J}^-} \equiv [\mathcal{J}^-, d]$, and we use (2.42) for the action on generalized spinors. From this it follows that

$$\omega = \frac{1}{4} d d^{\mathcal{J}^{b_2}} r^2 + \frac{1}{4} d(r^2) \wedge d \left(Q_-^{b_2} + \frac{1}{2} I^m_m \right) . \quad (3.96)$$

Thus r^2 is almost a Kähler potential, for the b_2 -transformed complex structure $\mathcal{J}^{b_2}_- \equiv e^{b_2} \mathcal{J}_- e^{-b_2}$, except for the last term.

3.5.2 The central charge as the contact volume

Recall that in any four-dimensional CFT there are two central charges, usually called a and c , that are constant coefficients in the conformal anomaly

$$\langle T^\mu_\mu \rangle = \frac{1}{120(4\pi)^2} \left(c(\text{Weyl})^2 - \frac{a}{4}(\text{Euler}) \right) . \quad (3.97)$$

Here $T_{\mu\nu}$ denotes the stress-energy tensor, and “Weyl” and “Euler” denote certain curvature invariants for the background four-dimensional metric. For SCFTs, both a and c are related to the R-symmetry via [17]

$$a = \frac{3}{32} (3\text{Tr} R^3 - \text{Tr} R) , \quad c = \frac{1}{32} (9\text{Tr} R^3 - 5\text{Tr} R) . \quad (3.98)$$

Here the trace is over the fermions in the theory. For SCFTs with AdS_5 gravity duals, in fact $a = c$ holds necessarily in the large N limit [49]. The central charge of the SCFT is then inversely proportional to the dual five-dimensional Newton constant G_5 [49], obtained by Kaluza-Klein reduction on Y . The Newton constant, in turn, was computed in appendix E of [25], and is given by

$$G_5 = \frac{G_{10}}{V_5} = \frac{\kappa_{10}^2}{8\pi V_5} , \quad (3.99)$$

where G_{10} is the ten-dimensional Newton constant of type IIB supergravity, and we have defined

$$V_5 \equiv - \int_Y e^{8\Delta} \text{vol}_Y . \quad (3.100)$$

Using the relation (3.17), we may rewrite this as

$$V_5 = - \frac{f_5^2}{16} \int_Y \frac{1}{\sin^2 \zeta} \text{vol}_Y . \quad (3.101)$$

Importantly, the constant f_5 is quantized, being essentially the number of D3-branes N . Specifically, we have

$$N = \frac{1}{(2\pi l_s)^4 g_s} \int_Y dC_4 = \frac{1}{(2\pi l_s)^4 g_s} \int_Y (F_5 + H \wedge C_2) . \quad (3.102)$$

Using the Bianchi identity $DG = -P \wedge G^*$ and the result (3.18), one derives that $d(H \wedge C_2) = -(2/f_5)d[e^{6\Delta}\text{Im}(W^* \wedge G)]$ and so we can also write

$$N = \frac{1}{(2\pi l_s)^4 g_s} \int_Y \left(F_5 - \frac{2e^{6\Delta}}{f_5} \text{Im}[W^* \wedge G] \right) . \quad (3.103)$$

We may evaluate this expression in terms of the orthonormal basis of forms e^i introduced in appendix B of [25], and after some calculation we find

$$N = \frac{f_5}{(2\pi l_s)^4 g_s} \int_Y \frac{1}{\sin^2 \zeta} \text{vol}_Y = -\frac{V_5}{f_5 (\pi l_s)^4 g_s} . \quad (3.104)$$

Combining these formulas and using $2\kappa_{10}^2 = (2\pi)^7 l_s^8 g_s^2$ leads to the result

$$G_5 = \frac{8V_5}{\pi^2 f_5^2 N^2} . \quad (3.105)$$

Consider now the integral

$$I_{\text{DH}} \equiv \frac{1}{(2\pi)^3} \int_X e^{-r^2/2} \frac{\omega^3}{3!} . \quad (3.106)$$

This is the *Duistermaat-Heckman integral* for a symplectic manifold (X, ω) with Hamiltonian function $r^2/2$, which we have shown is the Hamiltonian for the Reeb vector field ξ_v . Using (3.90) and (3.91) we may rewrite

$$\frac{\omega^3}{3!} = -\frac{16}{f_5^2} e^{8\Delta} r^5 dr \wedge \text{vol}_Y . \quad (3.107)$$

Performing the integral over r in (3.106) allows us to rewrite the five-dimensional Newton constant as

$$G_5 = \frac{\pi I_{\text{DH}}}{2N^2} . \quad (3.108)$$

Since $I_{\text{DH}} = 1$ for the round five-sphere solution, we obtain the ratio $G_5/G_{S^5} = I_{\text{DH}}$. Recalling that this is, by AdS/CFT duality, the inverse ratio of central charges [49], we deduce the key result

$$\frac{a_{N=4}}{a} = \frac{1}{(2\pi)^3} \int_X e^{-r^2/2} \frac{\omega^3}{3!} = \frac{1}{(2\pi)^3} \int_Y \sigma \wedge d\sigma \wedge d\sigma . \quad (3.109)$$

Here $a_{\mathcal{N}=4} = N^2/4$ denotes the (large N) central charge of $\mathcal{N} = 4$ super Yang-Mills theory.

The formula (3.109) implies that the central charge depends only on the symplectic structure of the cone (X, ω) and the Reeb vector field ξ_v . This is perhaps surprising: one might have anticipated that the quantum numbers of quantized fluxes would appear explicitly in the central charge formula. However, recall from formulas (3.19), (3.20) that the two-form potentials $B - b_2$ and $C_2 - c_2$ are globally defined. This implies for example that the period of $H = dB$ through any three-cycle in Y is zero.

As discussed in [19], the Duistermaat-Heckman integral in (3.109) may be evaluated by *localization*. The integral localizes where $\xi_v = 0$, which is formally at the tip of the cone at $r = 0$. Unless the differentiable and symplectic structure is smooth here (which is only the case when $X \cup \{r = 0\}$ is diffeomorphic to $\mathbb{R}^6$), one needs to equivariantly resolve the singularity in order to apply the localization formula. Notice here that since ξ_v preserves all the structure on the compact manifold (Y, g_Y, σ) , the closure of its orbits defines an $U(1)^s$ action preserving all the structure, for some $s \geq 1$. Here we have used the fact that the isometry group of a compact Riemannian manifold is compact. Thus (X, ω) comes equipped with a $U(1)^s$ action.

Rather than attempt to describe this in general, we focus here on the special case where the solution is *toric*: that is, there is a $U(1)^3$ action on Y under which σ , and hence ω under the lift to X , is invariant. Notice that we do not necessarily require that the full supergravity solution is invariant under $U(1)^3$ (we shall illustrate this in the next section with the Pilch-Warner solution, where σ and the metric are invariant under $U(1)^3$, but the G_3 -flux is invariant only under a $U(1)^2$ subgroup). For the arguments that follow, it is only σ , and hence ω , that we need to be invariant under a maximal dimension torus $U(1)^3$. In fact any such symplectic toric cone is also an affine toric variety. This implies that there is a (compatible) complex structure on X , and that the $U(1)^3$ action complexifies to a holomorphic $(\mathbb{C}^*)^3$ action with a dense open orbit. There is then always a symplectic toric resolution (X', ω') of (X, ω) , obtained by toric blow-up. In physics language, this is because one can realize (X, ω) as a gauged linear sigma model, and one obtains (X', ω') by simply turning on generic Fayet-Iliopoulos parameters. One can also describe this in terms of moment maps as follows. The image of a symplectic toric cone under the moment map $\mu : X \rightarrow \mathbb{R}^3$

is a strictly convex rational polyhedral cone (see [19]). Choosing a toric resolution (X', ω') then amounts to choosing any simplicial resolution $\mathcal{P}$ of this polyhedral cone. Here $\mathcal{P}$ is the image of $\mu' : X' \rightarrow \mathbb{R}^3$. Assuming the fixed points of ξ_v are all isolated, the localization formula is then simply [19]

$$\frac{1}{(2\pi)^3} \int_X e^{-r^2/2} \frac{\omega^3}{3!} = \sum_{\text{vertices } p \in \mathcal{P}} \prod_{i=1}^3 \frac{1}{\langle \xi_v, u_i^p \rangle} . \quad (3.110)$$

Here u_i^p , $i = 1, 2, 3$, are the three edge vectors of the moment polytope $\mathcal{P}$ at the vertex point p , and $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product on $\mathbb{R}^3$ (where we regard ξ_v as being an element of the Lie algebra $\mathbb{R}^3$ of $U(1)^3$). The vertices of $\mathcal{P}$ precisely correspond to the $U(1)^3$ fixed points of the symplectic toric resolution $X' = X_{\mathcal{P}}$ of X . Thus, remarkably, these results of [19] hold in general, even when there are non-trivial fluxes turned on and X is not Calabi-Yau.

3.5.3 Conformal dimensions of BPS branes

A supersymmetric D3-brane wrapped on a three-submanifold $\Sigma_3 \subset Y$ manifests itself as a BPS particle in AdS_5 . The quantum field Ψ whose excitations give rise to this particle state then couples, in the usual way in AdS/CFT , to a dual chiral primary operator $\mathcal{O}_{\Sigma_3}$ in the boundary SCFT. More precisely, there is an asymptotic expansion of Ψ near the AdS_5 boundary,

$$\Psi \sim \Psi_0 r^{\Delta-4} + A_{\Psi} r^{-\Delta} , \quad (3.111)$$

where Ψ_0 acts as the source for $\mathcal{O}_{\Sigma_3}$ and $\Delta = \Delta(\mathcal{O}_{\Sigma_3})$ is the conformal dimension of $\mathcal{O}_{\Sigma_3}$. In [50], following [51], it was argued that the vacuum expectation value A_{Ψ} of $\mathcal{O}_{\Sigma_3}$ in a given asymptotically AdS_5 background may be computed from $\exp(-S_E)$, where S_E is the on-shell Euclidean action of the D3-brane wrapped on $\Sigma_4 = \mathbb{R}^+ \times \Sigma_3$, where $\mathbb{R}^+$ is the radial direction parameterized by r . In particular, via the second term in (3.111) this identifies the conformal dimension Δ with the coefficient of the logarithmically divergent part of the on-shell Euclidean action of the D3-brane wrapped on Σ_4 . We refer to section 2.3 of [50] for further details.

We are thus interested in the on-shell Euclidean action of a supersymmetric D3-brane wrapped on $\Sigma_4 = \mathbb{R}^+ \times \Sigma_3$. The condition of supersymmetry is equivalent to

a generalized calibration condition, namely equation (3.16) of [46]. In our notation and conventions, this calibration condition reads

$$\text{Re} \left[-i\Phi_+ \wedge e^{\mathcal{F}} \right] |_{\Sigma_4} = \frac{|\eta_+^1|^2}{8} \sqrt{\det(h + \mathcal{F})} dx_1 \wedge \cdots \wedge dx_4 . \quad (3.112)$$

Here h is the induced (string frame) metric on Σ_4 , and $\mathcal{F} = F_\Sigma - B$, with F_Σ the worldvolume field-strength, satisfying

$$d\mathcal{F} = -H |_{\Sigma_4} . \quad (3.113)$$

Recalling from section 3.2 that $|\eta_+^1|^2 = e^A$, we may then substitute for Φ_+ in terms of Ω_+ using (3.43) and (3.55) to obtain

$$\text{Re} \left[-i\Phi_+ \wedge e^{\mathcal{F}} \right] |_{\Sigma_4} = \frac{f_5}{64} e^{A+\phi} \left[d \log r \wedge \sigma \wedge d\sigma - e^{-4A} r^4 (F_\Sigma - b_+)^2 \right] |_{\Sigma_4} , \quad (3.114)$$

where, as in (3.56),

$$b_+ = -\frac{e^{2\Delta+\phi/2}}{\sin \zeta} d \log r \wedge \text{Im } K_3 + b_2 . \quad (3.115)$$

Here b_2 is a closed two-form, whose gauge-invariant information is contained in its cohomology class in $H^2(X, \mathbb{R})/H^2(X, \mathbb{Z})$. In writing b_+ in (3.114) we have chosen a particular representative two-form for the class of b_2 in $H^2(X, \mathbb{R})/H^2(X, \mathbb{Z})$. Then under any gauge transformation of b_+ (induced from a B -transform of Ω_+), the world-volume field-strength F_Σ transforms by precisely the opposite gauge transformation restricted to Σ_4 , so that the quantity $F_\Sigma - b_+$ is gauge invariant on Σ_4 . We now choose the worldvolume gauge field F_Σ to be

$$F_\Sigma = b_2 |_{\Sigma_4} , \quad (3.116)$$

so that (3.114) becomes simply

$$\text{Re} \left[-i\Phi_+ \wedge e^{\mathcal{F}} \right] |_{\Sigma_4} = \frac{f_5}{64} e^{A+\phi} d \log r \wedge \sigma \wedge d\sigma |_{\Sigma_4} . \quad (3.117)$$

In fact, there is a slight subtlety in (3.116). If the cohomology class of $b_2/(2\pi l_s)^2 |_{\Sigma_4}$ in $H^2(\Sigma_4, \mathbb{R})$ is not integral, then the choice (3.116) is not possible as F_Σ is the curvature of a unitary line bundle. Having said this, notice that $H^2(\Sigma_4, \mathbb{R}) \cong H^2(\Sigma_3, \mathbb{R})$, and thus in particular that if $H^2(\Sigma_3, \mathbb{R}) = 0$ then every closed $b_2 |_{\Sigma_4}$ is exact, and thus may be gauge-transformed to zero on Σ_4 . Then (3.116) simply sets $F_\Sigma = 0$. For every

example of a supersymmetric Σ_3 that we are aware of, this is indeed the case. In any case, we shall assume henceforth that the choice (3.116) is possible.

The calibration condition (3.112) for a D3-brane with worldvolume Σ_4 and with gauge field (3.116) is thus

$$\frac{f_5}{8} d \log r \wedge \sigma \wedge d\sigma|_{\Sigma_4} = e^{-\phi} \sqrt{\det(h - B + b_2)} dx_1 \wedge \cdots \wedge dx_4. \quad (3.118)$$

Notice the right hand side is precisely the Dirac-Born-Infeld Lagrangian, up to the D3-brane tension $\tau_3 = 1/(2\pi)^3 l_s^4 g_s$. From (3.118), and the comments above on the scaling dimension $\Delta(\mathcal{O}(\Sigma_3))$ of the dual operator $\mathcal{O}(\Sigma_3)$, we thus deduce

$$\Delta(\mathcal{O}(\Sigma_3)) = -\frac{\tau_3 f_5}{8} \int_{\Sigma_3} \sigma \wedge d\sigma, \quad (3.119)$$

where the sign is just arising from a convenient choice of orientation. Using (3.104) and (3.90) we have

$$f_5 = -\frac{8(2\pi l_s)^4 g_s N}{\int_Y \sigma \wedge d\sigma \wedge d\sigma}, \quad (3.120)$$

and hence

$$\Delta(\mathcal{O}(\Sigma_3)) = \frac{2\pi N \int_{\Sigma_3} \sigma \wedge d\sigma}{\int_Y \sigma \wedge d\sigma \wedge d\sigma}. \quad (3.121)$$

This is our final formula for the conformal dimension of the chiral primary operator dual to a BPS D3-brane wrapped on Σ_3 . Since we may write

$$\int_{\Sigma_3} \sigma \wedge d\sigma = \int_{\Sigma_4} e^{-r^2/2} \frac{\omega^2}{2!}, \quad (3.122)$$

we see that it depends only on the symplectic structure of (X, ω) and the Reeb vector field ξ_v . This may again be evaluated by localization, having appropriately resolved the tip of the cone Σ_4 .

3.6 Example: the Pilch-Warner solution

In this section we illustrate the general results derived so far with the Pilch-Warner solution of type IIB supergravity [52, 53]. Some aspects of the generalized complex geometry of this solution have already been discussed in [26]. Recall that the Pilch-Warner solution is dual to a Leigh-Strassler fixed point theory [54] which is obtained

by giving a mass to one of the three chiral superfields (in $\mathcal{N} = 1$ language) of $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory and following the resulting renormalization group flow to the IR fixed point theory (see section 6.1 for more detail). This latter theory is an $\mathcal{N} = 1$ $SU(N)$ gauge theory with two adjoint fields Z_a , $a = 1, 2$, which form a doublet under an $SU(2)$ flavour symmetry, and a quartic superpotential. Since the superpotential has scaling dimension three, this fixes $\Delta(Z_a) = 3/4$, implying that the IR theory is strongly coupled. The mesonic moduli space is $\text{Sym}^N \mathbb{C}^2$.

The Pilch-Warner supergravity solution [52] was rederived in [25], and we shall use some of the results from that reference too. We have $Y = S^5$ with non-trivial metric

$$g_Y = \frac{1}{9} \left[6d\vartheta^2 + \frac{3\cos^2\vartheta}{1+\sin^2\vartheta}(\sigma_1^2 + \sigma_2^2) + \frac{3\sin^2 2\vartheta}{2(1+\sin^2\vartheta)^2}\sigma_3^2 + 4 \left(d\varphi + \frac{\cos^2\vartheta}{1+\sin^2\vartheta}\sigma_3 \right)^2 \right], \quad (3.123)$$

where $0 \leq \vartheta \leq \pi/2$, $0 \leq \varphi \leq 2\pi$, and σ_i , $i = 1, 2, 3$, are left-invariant one-forms on $SU(2)$ (denoted with hats in [25]). The dilaton ϕ and axion C_0 are constant, while the warp factor is

$$e^{4\Delta} = \frac{f_5}{4}(1 + \sin^2\vartheta). \quad (3.124)$$

There are also non-trivial NS and RR three-form fluxes given by (recall (3.2))

$$G_3 = \frac{(2f_5)^{1/2}}{3^{3/2}} e^{2i\varphi} \cos\vartheta \left(d\varphi \wedge d\vartheta - \frac{i\sin 2\vartheta}{2(1+\sin^2\vartheta)} d\varphi \wedge \sigma_3 - \frac{\cos^2\vartheta}{(1+\sin^2\vartheta)^2} d\vartheta \wedge \sigma_3 \right) \wedge (\sigma_2 - i\sigma_1). \quad (3.125)$$

We introduce the Euler angles (α, β, γ) on $SU(2)$ (as in [25]), so that

$$\begin{aligned} \sigma_1 &= -\sin\gamma d\alpha - \cos\gamma \sin\alpha d\beta, \\ \sigma_2 &= \cos\gamma d\alpha - \sin\gamma \sin\alpha d\beta, \\ \sigma_3 &= d\gamma - \cos\alpha d\beta. \end{aligned} \quad (3.126)$$

In terms of these coordinates, the R-symmetry vector ξ_v is⁴

$$\xi_v = \partial_\psi = \frac{3}{2}\partial_\varphi - 3\partial_\gamma. \quad (3.127)$$

⁴Note that this, more conventional, normalization of ψ differs from the corresponding coordinate in [25] by a factor of three.

Using the explicit formulas in [25], it is easy to show that the contact form is

$$\sigma = -\frac{2}{3} (\cos 2\vartheta d\varphi + \cos^2 \vartheta \sigma_3) . \quad (3.128)$$

The Pilch-Warner solution is toric, in the sense that both σ and the metric are invariant under shifts of φ , β and γ . However, notice that the G_3 -flux in (3.125) is not invariant under shifts of φ , thus breaking this $U(1)^3$ symmetry to only a $U(1)^2$ symmetry of the full supergravity solution. This is expected, since the dual field theory described above has only an $SU(2) \times U(1)_R \supset U(1)^2$ global symmetry.

On $Y = S^5$ there are precisely three invariant circles under the $U(1)^3$ action, where two of the $U(1)$ actions degenerate, namely at $\{\vartheta = \frac{\pi}{2}\}$, $\{\vartheta = 0, \alpha = 0\}$, and $\{\vartheta = 0, \alpha = \pi\}$. A set of 2π -period coordinates on $U(1)^3$ are

$$\varphi_1 = \varphi, \quad \varphi_2 = -\frac{1}{2}(\varphi + \gamma - \beta), \quad \varphi_3 = -\frac{1}{2}(\varphi + \gamma + \beta) . \quad (3.129)$$

These restrict to coordinates on the above three invariant circles, respectively. On $X \cong \mathbb{R}^6 \setminus \{0\}$ we also have three corresponding moment maps

$$\mu_1 = \frac{r^2}{3} \sin^2 \vartheta, \quad \mu_2 = \frac{2r^2}{3} \cos^2 \vartheta \cos^2 \frac{\alpha}{2}, \quad \mu_3 = \frac{2r^2}{3} \cos^2 \vartheta \sin^2 \frac{\alpha}{2}, \quad (3.130)$$

so that $\omega = d(r^2\sigma)/2 = \sum_{i=1}^3 d\mu_i \wedge d\varphi_i$. It follows that the image of the moment map (the space spanned by the μ_i coordinates) is the cone $(\mathbb{R}_+)^3$, where the three invariant circles map to the three generating rays $u_1 = (1, 0, 0)$, $u_2 = (0, 1, 0)$, $u_3 = (0, 0, 1)$. The Reeb vector (3.127) is then computed in this basis to be

$$\xi_v = \frac{3}{2}\partial_{\varphi_1} + \frac{3}{4}\partial_{\varphi_2} + \frac{3}{4}\partial_{\varphi_3} . \quad (3.131)$$

Since the symplectic structure is smooth at $r = 0$, we may evaluate the Duistermaat-Heckman integral (3.109) by localization without having to resolve X at $r = 0$. In the case at hand, we have the single fixed point at $r = 0$, and from (3.131) one obtains the known result for the central charge a_{PW} of the SCFT dual to the Pilch-Warner solution:

$$\frac{a_{\mathcal{N}=4}}{a_{\text{PW}}} = \frac{1}{\xi_{v1}\xi_{v2}\xi_{v3}} = \frac{32}{27} . \quad (3.132)$$

The key point about this calculation is that we performed it knowing only the symplectic structure and the Reeb vector field ξ_v .

We may similarly compute the conformal dimensions of the operators $\det Z_a$, using (3.121), by interpreting them as arising from a BPS D3-brane wrapped on the three-spheres at $\alpha = 0$ and $\alpha = \pi$, respectively. It is simple to check these indeed satisfy the calibration condition (3.118) and are thus supersymmetric. Using (3.122) and localization at $r = 0$ implies that (3.122) is equal to $1/\xi_{v1}\xi_{v2}$, $1/\xi_{v1}\xi_{v3}$, respectively, which in both cases is $8/9$. The formula (3.121) thus gives $\Delta(\det Z_a) = 3N/4$, or equivalently $\Delta(Z_a) = 3/4$, which is indeed the correct result.

Next recall from subsection 3.2.1 that the mesonic moduli space should be the locus $\{\theta = 0\}$, where the complex one-form is $\theta = d(r^3\theta_0) = -d(r^3e^{4\Delta}S)/24$. This is the locus where $\sin 2\bar{\theta} = \sin 2\bar{\phi} = 0$. For the Pilch-Warner solution, we may easily compute

$$\sin 2\bar{\theta} = -\frac{\sqrt{3}\sin^2\vartheta}{\sqrt{1+3\sin^4\vartheta}}, \quad \cos 2\bar{\phi} = \frac{\sqrt{1+3\sin^4\vartheta}}{1+\sin^2\vartheta}. \quad (3.133)$$

Thus, as discussed in [26], the mesonic moduli space is equivalent to $\vartheta = 0$, which is a codimension two submanifold in $\mathbb{R}^6$ diffeomorphic to $\mathbb{R}^4$. Moreover, this is $\mathbb{C}^2$ in the induced complex structure, and we thus see explicit agreement with the field theory Abelian mesonic moduli space.

Finally, although the Pilch-Warner solution is generalized complex, rather than complex, we note that one can nevertheless define a natural complex structure [55]. The relation between this integrable complex structure and the generalized geometry has been discussed in [26]. Let us conclude this section by elucidating this connection. One can introduce the following complex coordinates [26] in terms of the angular variables (3.129):

$$\begin{aligned} s_1 &= r^{3/2} \sin \vartheta e^{-i\varphi_1}, \\ s_2 &= r^{3/4} \cos \vartheta \cos \frac{\alpha}{2} e^{i\varphi_2}, \\ s_3 &= r^{3/4} \cos \vartheta \sin \frac{\alpha}{2} e^{i\varphi_3}. \end{aligned} \quad (3.134)$$

This makes $\mathbb{R}^6 \cong \mathbb{C}^3$. However, because of the minus sign in the phase of the first coordinate in (3.134), the corresponding integrable complex structure, which we call I_* , is *not* the unique complex structure that is compatible with the toric structure of the solution: the latter instead has complex coordinates $\bar{s}_1, s_2, s_3$. Also, the Reeb vector field ξ_v is *not* given by $I_*(r\partial_r)$. This makes the physical significance of this

complex structure rather unclear. Nevertheless, one can show that I_* does in fact come from an $SU(3)$ -structure defined by a Killing spinor η_* . Following [26], we define

$$2\hat{a}\eta_* = \eta_+^1 + i\eta_+^2 = e^{A/2} \begin{pmatrix} \xi_2 \\ i\xi_2 \end{pmatrix}, \quad (3.135)$$

where by definition we require $\bar{\eta}_*\eta_* = 1$. It is then convenient to define $\hat{a} \equiv |\hat{a}|e^{iz}$, where $|\hat{a}|^2 = e^A|\xi_2|^2/2 = e^A(1 - \sin \zeta)/2$. We then introduce the bilinears corresponding to the $SU(3)$ -structure defined by η_* :

$$J_* \equiv -i\bar{\eta}_*\gamma_{(2)}\eta_*, \quad \Omega_* \equiv \bar{\eta}_*^c\gamma_{(3)}\eta_* . \quad (3.136)$$

We compute that $d\Omega_* = 0$, implying that the corresponding complex structure I_* is integrable, and moreover that

$$e^{2iz}\Omega_* = -e^{2i\alpha}\frac{\sqrt{2}f_5^{3/2}}{9e^{3A}}ds_1 \wedge ds_2 \wedge ds_3, \quad (3.137)$$

implying that (3.134) are indeed complex coordinates for this complex structure. We also compute

$$J_* = -\frac{e^{2A}}{r^2} \left[d\log r \wedge \frac{2}{3} \left(d\varphi + \frac{\cos^2 \vartheta}{1 + \sin^2 \vartheta} \sigma_3 \right) + \frac{d(\cos^2 \vartheta \sigma_3)}{3(1 + \sin^2 \vartheta)} \right]. \quad (3.138)$$

Chapter 4

Generalized Sasakian geometry

In this chapter we propose a generalization of the concept of a Sasakian manifold. We saw in chapter 3 that the requirement of supersymmetry for AdS_5 solutions of type IIB supergravity may be expressed in terms of a pair of compatible generalized structures with pure spinors Ω_- and Ω_+ . It consists in the differential constraints (3.45), (3.47), and (3.50):

$$\begin{aligned} d\Omega_- &= 0, \\ d(e^{-A} \text{Re } \Omega_+) &= 0, \quad dd^{\mathcal{J}_-}(e^{-3A} \text{Im } \Omega_+) = 0, \end{aligned} \quad (4.1)$$

together with the equal Mukai norm condition (3.44):

$$\|\Omega_-\|^2 = \|\Omega_+\|^2. \quad (4.2)$$

Here the function A is a conformal factor, $\mathcal{J}_-$ is the (integrable) generalized complex structure associated with Ω_- , and $d^{\mathcal{J}_-} \equiv [\mathcal{J}_-, d]$. We will define the concept of a *generalized Sasakian* manifold Y by imposing the constraints (4.1) on the cone $X \cong \mathbb{R}_+ \times Y$, while relaxing the condition (4.2) on the norms of $\Omega_{\pm}$. This definition reduces to the definition of a Sasakian manifold¹ in the case with only the five-form flux. Although these conditions resulted from the supergravity analysis of section 3.2, here we shall argue that they are rather natural from a purely geometric point of view, and follow carefully the consequences of each condition. The closure of Ω_- implies of course that the cone is generalized Calabi-Yau [30], in the sense of Hitchin. The generalized Darboux theorem of [38] then allows us to put Ω_- locally into a normal

¹Strictly speaking this gives a Sasakian manifold which is *transversely Fano*, as defined for example in [21].

form, which determines a symplectic foliation of X . The second pure spinor Ω_+ is instead related to the background RR fluxes. This structure is not integrable, but it nevertheless provides a symplectic structure on the cone. After reduction along the Euler and Reeb vector fields, the compatibility condition between Ω_- and Ω_+ leads to a system involving a symplectic triple on the transverse space to the Reeb foliation. More precisely, this is true away from the type-change locus along which the generalized Sasakian structure becomes Sasakian. The RR fluxes also satisfy a Bianchi identity, the third equation in (4.1), which gives an additional differential constraint.

4.1 Generalized Calabi-Yau structure

In this section we consider an integrable generalized complex structure $\mathcal{J}_-$ on a six-manifold X that is associated with a closed pure spinor Ω_- :

$$d\Omega_- = 0. \quad (4.3)$$

According to Hitchin's definition [30], this makes X a generalized Calabi-Yau manifold. We will also choose Ω_- to be of odd type k . Then Ω_- will generally have the lowest odd type possible, that is $k = 1$, over a dense open subset $X_0 \subset X$, but at special loci $\mathcal{S}$ the type may change to $k = 3$. As explained in section 3.2, this is the case of interest for application to AdS_5 solutions of type IIB string theory. For example, as already mentioned, a Calabi-Yau three-fold is everywhere of type $k = 3$.

In the remainder of this chapter we focus almost exclusively on the dense open set $X_0 \subset X$ where Ω_- has type one. The limit points of X_0 are then by assumption of type three, and one can view these as imposing certain *boundary conditions* on the various type-one objects on X_0 that we study. In fact we shall not study these boundary conditions in detail here, since for our purposes it will be sufficient to know simply the local conditions on X_0 , together with the fact that certain structures are in fact defined globally on X .

The most general algebraic form for a closed polyform of type $k = 1$ is [38]

$$\theta \wedge e^{-b_- + i\omega_-}, \quad (4.4)$$

with θ a complex one-form and b_- , ω_- real two-forms. By the *generalized Darboux theorem* [38], this structure is locally equivalent, via a diffeomorphism and a closed B -transform (2.10), to the direct sum of a complex structure of complex dimension one and a symplectic structure of real dimension four. More precisely, for any point in X_0 there is a neighbourhood with a symplectic foliation that is isomorphic to an open set in $\mathbb{C} \times \mathbb{R}^4$, with transverse complex coordinate $z = x + iy$ and real coordinates $\{x_1, y_1, x_2, y_2\}$ on the symplectic leaves.² The appropriate leaf-preserving diffeomorphism φ is such that the pull-back of the two-form ω_- to each leaf is the standard Darboux symplectic form ω_0 :

$$\varphi^* \omega_-|_{\mathbb{R}^4 \times \{\text{pt}\}} = \omega_0 \equiv dx_1 \wedge dy_1 + dx_2 \wedge dy_2 . \quad (4.5)$$

The freedom to shift the exponent in (4.4) by a two-form whose wedge product with θ vanishes allows us to trade b_- for a closed two-form b_0 , and obtain³

$$\varphi^* [\theta \wedge \exp(-b_- + i\omega_-)] = d\bar{z} \wedge e^{-b_0 + i\omega_0} . \quad (4.6)$$

We dispose of b_0 by a closed B -transform, and take the resulting polyform as the definition of Ω_- in this open neighbourhood:

$$\Omega_- \equiv d\bar{z} \wedge e^{i\omega_0} . \quad (4.7)$$

In the application to physics, the above closed B -transform will also act on the compatible pure spinor Ω_+ introduced in section 4.2, and will be reabsorbed into its definition. Notice that such closed B -transforms are symmetries of the supergravity equations, but that globally only *integer-period* closed B -transforms are symmetries of string theory.

The generalized structure corresponding to (4.7) combines a standard complex structure I_0 on the complex leaf space with a symplectic structure on the leaves (recall the standard examples in (2.53)). In the coordinate basis $\{\partial_z, \partial_{\bar{z}}, \partial_{x_1}, \partial_{y_1}, \dots, \partial_{x_2}, \partial_{y_2}\}$

²When we introduce the compatible pure spinor Ω_+ in section 4.2, we shall see that there is another foliation by orbits of ∂_z . In fact the latter will turn out to be a *global* vector field on X , not simply a local vector field in a neighbourhood of a point in X_0 , and on Y this will reduce to a Reeb foliation (see section 4.3).

³The reason for choosing the *anti*-holomorphic one-form $d\bar{z}$ is to align with the sign conventions in section 2, see (2.40) and (2.50).

of the (complexified) generalized tangent bundle it is given by

$$\mathcal{J}_- = \left(\begin{array}{c|c} I_0 & 0_2 \\ \hline 0_2 & -I_0^* \\ \hline 0_4 & \omega_0^{-1} \\ \hline -\omega_0 & 0_4 \end{array} \right) .$$

The two-form ω_0 gives an isomorphism between the tangent and cotangent spaces of the leaves, as $\omega_0 : (\partial_{x_a}, \partial_{y_a}) \mapsto (dy_a, -dx_a)$, $a = 1, 2$, with inverse $\omega_0^{-1} : (dx_a, dy_a) \mapsto (-\partial_{y_a}, \partial_{x_a})$. The group action of $\mathcal{J}_-$, viewed as an element of $O(6, 6)$, is

$$\begin{aligned} \mathcal{J}_-(\partial_z) &= i\partial_z, & \mathcal{J}_-(\partial_{\bar{z}}) &= -i\partial_{\bar{z}}, \\ \mathcal{J}_-(dz) &= -idz, & \mathcal{J}_-(d\bar{z}) &= id\bar{z}, \\ \mathcal{J}_-(\partial_{x_a}) &= dy_a, & \mathcal{J}_-(\partial_{y_a}) &= -dx_a, \\ \mathcal{J}_-(dx_a) &= \partial_{y_a}, & \mathcal{J}_-(dy_a) &= -\partial_{x_a}. \end{aligned} \tag{4.8}$$

On the other hand, $\mathcal{J}_-$ may also be regarded as an element of the Lie algebra $\mathfrak{o}(6, 6)$, and the algebra action of $\mathcal{J}_-$ on differential forms is then defined via the Clifford action as $\mathcal{J}_- \cdot \equiv -I_0^* \cdot -\omega_0 \wedge + \omega_0^{-1} \lrcorner$, with the bivector $\omega_0^{-1} \equiv \partial_{y_1} \wedge \partial_{x_1} + \partial_{y_2} \wedge \partial_{x_2}$.

Example: β -transform of $\mathbb{C}^3$

Let $\{z_1, z_2, z_3\}$ be standard complex coordinates on $\mathbb{C}^3$, which is the complex structure associated with the pure spinor

$$\Omega = dz_1 \wedge dz_2 \wedge dz_3. \tag{4.9}$$

If we deform as in (2.11) by a bivector ⁴

$$\beta = z_2 \partial_{z_2} \wedge z_1 \partial_{z_1} + \text{c.p.}, \tag{4.10}$$

where “c.p.” means the cyclic permutations of pairs of indices $\{1, 2, 3\}$, we obtain

$$\begin{aligned} e^\beta \Omega &= d(z_1 z_2 z_3) + dz_1 \wedge dz_2 \wedge dz_3 \\ &= d(z_1 z_2 z_3) \wedge \exp \left(\frac{dz_1 \wedge dz_2}{3z_1 z_2} + \text{c.p.} \right). \end{aligned} \tag{4.11}$$

⁴ More generally, the deformation complex of a generalized structure on a complex manifold M is $\oplus_{p+q=2} H^p(M, \wedge^q T_{1,0})$. If M is a *compact* Calabi-Yau manifold, only $H^1(M, T_{1,0})$, whose elements are ordinary complex deformations, is non-vanishing. There is therefore no bivector $\beta \in H^0(M, \wedge^2 T_{1,0})$ that can be used to deform it. However, as observed by Wijnholt [56], for Calabi-Yau manifolds X that are cones over regular Sasaki-Einstein manifolds Y with Kähler-Einstein base M , one can consider elements $\beta \in H^0(M, \wedge^2 T_{1,0})$ and then holomorphically extend these over the entire cone to obtain a non-commutative deformation. In general, there might be obstructions in $\oplus_{p+q=3} H^p(M, \wedge^q T_{1,0})$ to the integrability of such deformations. For the $\mathbb{CP}^2$ base of $\mathbb{C}^3$, Gualtieri showed that the obstructions vanish [38].

This deformed pure spinor is of type three on the locus $\{d(z_1 z_2 z_3) = 0\}$, corresponding to the union of the three complex lines $\{z_i = z_j = 0 \mid i, j \in \{1, 2, 3\}, i < j\}$, but is otherwise of type one as shown by the expression on the second line.

We now assume that $X \cong \mathbb{R}_+ \times Y$, with Y compact, and introduce an appropriate homogeneity property under the Euler vector field $r\partial_r$, where r is a coordinate on $\mathbb{R}_+$. For a Calabi-Yau cone, recall that the holomorphic $(3, 0)$ -form Ω is required to be homogeneous of degree three under $r\partial_r$, that is $\mathcal{L}_{r\partial_r}\Omega = 3\Omega$. Following section 3.2, we thus similarly impose this condition on the *polyform* Ω_- :

$$\mathcal{L}_{r\partial_r}\Omega_- = 3\Omega_- . \quad (4.12)$$

This gives in general separate conditions on each of the one-, three-, and five-form components of Ω_- . Recall that this implies that $r\partial_r$ is generalized holomorphic, that is $\mathcal{L}_{r\partial_r}\mathcal{J}_- = 0$, as explained below (3.63). Next, we define the generalized Reeb vector $\xi = \xi_v + \xi_f$ and the generalized contact form $\eta = \eta_v + \eta_f$ as

$$\xi \equiv \mathcal{J}_-(r\partial_r) , \quad \eta \equiv \mathcal{J}_-(d \log r) . \quad (4.13)$$

From the fact that the complex combination $r\partial_r - i\xi$ annihilates Ω_- , it follows that Ω_- has a definite charge under ξ (see (3.71)):

$$\mathbb{L}_\xi \Omega_- = -3i\Omega_- , \quad (4.14)$$

and so ξ is generalized holomorphic as well, that is $\mathbb{L}_\xi \mathcal{J}_- = 0$.

This homogeneity requirement leads to the following results, which for clarity we present as a proposition [4]. Of course, these are to be understood as local expressions, defined in the coordinate patch in which Ω_- takes the form (4.7).

Proposition 1.

- a) *The complex coordinate z can be expressed in terms of the radial coordinate r , a real function h , and a phase ψ as:*

$$z = r^3 e^{3h} e^{3i\psi} . \quad (4.15)$$

Here $\partial_r h = \partial_r \psi = 0$, which means that h and ψ are pullbacks from (a neighbourhood in) Y . Moreover, h depends only on the symplectic (leaf) coordinates.

b) The Euler vector field takes the form

$$r\partial_r = 3(z\partial_z + \bar{z}\partial_{\bar{z}}) - \mathcal{H}_\varphi , \quad (4.16)$$

where we define the Hamiltonian vector field as $\mathcal{H}_\varphi \equiv \omega_0^{-1} \lrcorner d\varphi$, which is tangent to the symplectic leaves, and the function φ also depends only on the symplectic coordinates.

Proof.

Consider the general ansatz $dz = \mu_0 d \log r + \nu_1$, with μ_0 a function and ν_1 a one-form such that $\partial_r \lrcorner \nu_1 = 0$. The one-form part of the homogeneity condition (4.12) is

$$\mathcal{L}_{r\partial_r} dz = 3dz , \quad (4.17)$$

which leads to $3\nu_1/\mu_0 = d \log(\mu_0/r^3)$. Since $\partial_r \lrcorner \nu_1 = 0$, we can write $\mu_0 = 3r^3 e^{3h} e^{3i\psi}$ with h and ψ real functions, independent of r . But $dz = d(r\partial_r \lrcorner dz)/3 = d\mu_0/3$ and so $z = \mu_0/3 + c = r^3 e^{3h} e^{3i\psi} + c$, with c a constant which we may set to zero by shifting the origin of z . This proves a), except for the last statement that h is independent of z .

To show b), notice first that it is clear from the condition (4.17) and its complex conjugate that $r\partial_r$ has to contain the term $3(z\partial_z + \bar{z}\partial_{\bar{z}})$. Now the three-form part of the homogeneity condition (4.12), $dz \wedge \mathcal{L}_{r\partial_r} \omega_0 = 0$, can only be satisfied non-trivially by a term of $\mathcal{L}_{r\partial_r} \omega_0$ proportional to $dz \wedge d\bar{z}$. But since the action of the Lie derivative on ω_0 will leave one symplectic component intact in every term, there is no such term in $\mathcal{L}_{r\partial_r} \omega_0$, and we must then have

$$\mathcal{L}_{r\partial_r} \omega_0 = 0 . \quad (4.18)$$

This fixes $r\partial_r$ up to a Hamiltonian vector field $\mathcal{H}_\varphi$ tangent to the leaves such that $\mathcal{H}_\varphi \lrcorner \omega_0 = -\tilde{d}\varphi$, with $\tilde{d}$ the exterior derivative along the symplectic leaves and $\varphi = \varphi(z, \bar{z}, x_a, y_a)$ an arbitrary real function.

To show that φ is independent of z , we use the homogeneity of Ω_- under the generalized Reeb vector ξ , which in terms of generalized Darboux coordinates reads

$$\xi = 3i(z\partial_z - \bar{z}\partial_{\bar{z}}) + \tilde{d}\varphi = \partial_\psi + \tilde{d}\varphi . \quad (4.19)$$

This implies in particular that $d\bar{z} \wedge d(\tilde{d}\varphi) = 0$ and so $d(\tilde{d}\varphi) = 0$, which means that locally we can write $\tilde{d}\varphi = d\tilde{\varphi}$, with $\tilde{\varphi} = \tilde{\varphi}(x_a, y_a)$. We can then set $\varphi = \tilde{\varphi}$.

Similarly, the generalized contact form reads

$$\eta = \frac{i}{6} d \log \frac{\bar{z}}{z} - \mathcal{H}_h = d\psi - \mathcal{H}_h, \quad (4.20)$$

with the Hamiltonian vector field $\mathcal{H}_h \equiv \omega_0^{-1} \lrcorner dh$. A priori, η contains an additional term $I_0^* \cdot dh$, but the condition $d\bar{z} \wedge (\eta_v \lrcorner \omega_0 + d \log r - i\eta_f) = 0$ from the fact that $d \log r - i\eta$ annihilates Ω_- gives $d\bar{z} \wedge I_0^* \cdot dh = 0$, and so $\partial_z h = 0$. The function h is thus a function on the symplectic leaves, $h = h(x_a, y_a)$. $\square$

4.2 Compatible structure of symplectic type

We now introduce a second generalized structure $\mathcal{J}_+$ on X , with an associated pure spinor Ω_+ . We require that $\mathcal{J}_-$ and $\mathcal{J}_+$ are compatible, which means that they should commute and define a positive definite generalized metric via

$$G \equiv -\mathcal{J}_- \mathcal{J}_+. \quad (4.21)$$

We assume that Ω_+ is *everywhere of type zero*:

$$\Omega_+ \equiv \alpha_+ e^{-b_+ + i\omega_+}, \quad (4.22)$$

with α_+ a nowhere-vanishing complex function, and b_+ , ω_+ real two-forms. As explained in section 3.2, in the context of string theory solutions this assumption is equivalent to the background having non-zero five-form flux sourced by the D3-brane charge.

The next condition we wish to impose is an appropriate homogeneity condition under the Euler vector field $r\partial_r$. Recall that an ordinary metric on $X = \mathbb{R}_+ \times Y$ is said to be *conical* if it is homogeneous of degree two under $r\partial_r$, and moreover $r\partial_r$ is orthogonal to all tangent vectors in Y . Such a metric then takes the form $dr^2 + r^2 g_Y$. At the end of chapter 2 we explained that in applying generalized geometry to $\text{AdS}_5 \times Y$ backgrounds, the metric g_X defined by the generalized metric (2.29) is *conformal* to the cone metric over Y , via $g_X = r^{-2}(dr^2 + r^2 g_Y)$. In fact this is the metric of a *cylinder* over Y , which is characterized by $\mathcal{L}_{r\partial_r} g_X = 0$ and $r\partial_r$ being

orthogonal to the base Y of the cylinder. It is then natural to extend these conditions to the generalized metric G by imposing

$$\mathcal{L}_{r\partial_r} G = 0, \quad \text{and} \quad G(r\partial_r) = e^{2\hat{\Delta}} d \log r, \quad (4.23)$$

with $\hat{\Delta}$ a real homogeneous function of degree zero, $\mathcal{L}_{r\partial_r} \hat{\Delta} = 0$.⁵ It is straightforward to show that these conditions are equivalent to the two-form B in (2.29) being *basic* with respect to $r\partial_r$, that is $\mathcal{L}_{r\partial_r} B = r\partial_r \lrcorner B = 0$, and that the metric on X takes the form

$$g_X = e^{2\hat{\Delta}} \left(\frac{dr^2}{r^2} + g_Y \right), \quad (4.24)$$

where g_Y is the metric on the compact space Y . Thus g_X is in general conformal to a cylinder metric, with $e^{2\hat{\Delta}}$ being an invariant conformal factor. The Riemannian volume form on X is, with a sign convention chosen to match that of chapter 3,

$$\text{vol}_X \equiv -\sqrt{g_X} d \log r \wedge d^5 y = -e^{6\hat{\Delta}} d \log r \wedge \text{vol}_Y. \quad (4.25)$$

The homogeneity condition $\mathcal{L}_{r\partial_r} G = 0$ together with $\mathcal{L}_{r\partial_r} \mathcal{J}_- = 0$ imply that $\mathcal{J}_+$ must be invariant under $r\partial_r$. As for Ω_- , we may thus similarly impose the following homogeneity condition on Ω_+ :

$$\mathcal{L}_{r\partial_r} \Omega_+ = 3\Omega_+. \quad (4.26)$$

From a purely geometrical point of view, it would now be natural to impose that $\mathcal{J}_+$ is also integrable. The manifold would then be *generalized Kähler*, in the sense of [38]. This is the case, for instance, in the topological string and in purely Neveu-Schwarz solutions of type II string theories (a short and highly incomplete list of references is [57, 58, 59]). For general AdS₅ solutions of type IIB string theory, however, the presence of background RR fluxes on the cone is an obstruction to the integrability of $\mathcal{J}_+$. As we pointed out in section 2.5, this is true even for Sasaki-Einstein solutions. Thus imposing the closure of Ω_+ would be too strong in our context. We instead impose the weaker differential conditions

$$d(e^{-A} \text{Re } \Omega_+) = 0, \quad d\mathcal{J}_-(e^{-3A} \text{Im } \Omega_+) = 0. \quad (4.27)$$

⁵The function $\hat{\Delta}$ is related to the warp factor Δ and the dilaton ϕ in chapter 3 through $\hat{\Delta} = \Delta + \phi/4$. The presence of the dilaton is due to the transition from the Einstein frame g_E to the string frame g_σ , which is carried out by a Weyl rescaling $g_\sigma = e^{\phi/2} g_E$.

Here e^A is a homogeneous function of degree one, $\mathcal{L}_{r\partial_r}e^A = e^A$, and $d\mathcal{J}^- \equiv [\mathcal{J}^-, d]$.

The first constraint in (4.27) will ensure that the cone X is *symplectic*, as we shall prove below. Clearly, this is a natural geometric condition to impose. The presence of the e^{-A} factor sets the homogeneous degree of this symplectic form to two, again as we would expect for a *symplectic cone*.

The second constraint in (4.27) is physically none other than the Bianchi identity, $d(e^{-B}F) = 0$, for the RR fluxes of type IIB supergravity. These fluxes can be encapsulated in the odd polyform $F \equiv F_1 + F_3 + F_5$, where F_p is a p -form. From a geometric point of view, we simply *define* this polyform directly in terms of the imaginary part of Ω_+ as

$$e^{-B}F \equiv 8d\mathcal{J}^-(e^{-3A}\text{Im}\Omega_+) . \quad (4.28)$$

Again, as part of the homogeneity conditions, we impose that the RR fluxes are basic with respect to the foliation defined by $r\partial_r$:

$$\mathcal{L}_{r\partial_r}F = 0 , \quad r\partial_r \lrcorner F = 0 . \quad (4.29)$$

These conditions now explain the presence of e^{-3A} in the definition (4.28): its homogeneity property is required to balance the degree three of Ω_+ .

For the five-form component F_5 , the condition (4.29) implies that $F_5 = f_5\text{vol}_Y$, where a priori f_5 is a homogeneous function of degree zero. The final condition that we impose is that f_5 is a (non-zero) *constant*. From the string theory point of view, this follows since in type IIB supergravity the full RR five-form is self-dual, and thus of the form (recall (3.8))

$$f_5(\text{vol}_{\text{AdS}} + \text{vol}_Y) . \quad (4.30)$$

The Bianchi identity is $dF_5 = H \wedge F_3$, but the right-hand side vanishes since by construction H and F_3 are three-forms on Y . Thus f_5 is necessarily constant. This constant is then necessarily non-zero if Ω_+ is everywhere type zero, as we have already assumed, and as shown in Proposition 2 below.

This is the full set of conditions that we shall impose. The motivation largely came from the fact that these conditions are implied by supersymmetry, as shown in chapter 3. However, hopefully the above discussion also motivates these as natural

geometric conditions. As we mentioned in the introduction of this chapter, this is not quite the full set of conditions required for a supersymmetric AdS₅ solution. In fact the structure we have now defined is in some sense the generalized version of Kähler cones, which by definition are cones over Sasakian manifolds, and we analogously call the base Y a *generalized Sasakian manifold*. The following proposition [4] summarizes the consequences of the above conditions, which also justify our use of terminology:

Proposition 2.

- a) *The function e^A is related to the radial function r and the conformal factor $\hat{\Delta}$ through*

$$e^A = r e^{\hat{\Delta}} . \quad (4.31)$$

- b) *The generalized Reeb vector ξ preserves the generalized metric as well as the RR fluxes:*

$$\mathbb{L}_\xi G = 0 , \quad \mathbb{L}_\xi(e^{-B}F) = 0 . \quad (4.32)$$

- c) *The cone is symplectic with symplectic form*

$$\omega = \frac{1}{2} d(r^2 \sigma) , \quad (4.33)$$

where the contact one-form associated with the Reeb vector field ξ_v is

$$\sigma = \eta_{\text{f}} - \eta_{\text{v}} \lrcorner b_2 = d\psi + \mathcal{H}_h \lrcorner b_2, \quad (4.34)$$

with b_2 a closed two-form.

- d) *The pure spinor $\Omega_+ \equiv \alpha_+ \exp(-b_+ + i\omega_+)$ can be expressed as*

$$\alpha_+ = -i \frac{f_5}{32} r^3 e^{-\hat{\Delta}} , \quad (4.35)$$

$$\omega_+ = \frac{e^{2\hat{\Delta}}}{r^2} \omega , \quad (4.36)$$

$$\begin{aligned} b_+ &= e^{2\hat{\Delta}} d \log r \wedge \eta_{\text{v}} \lrcorner \omega_+ + b_2 \\ &= -e^{4\hat{\Delta}} d \log r \wedge \mathcal{H}_h \lrcorner \omega_{\text{T}} + b_2 . \end{aligned} \quad (4.37)$$

Here we have defined $\omega_{\text{T}} \equiv d\sigma/2$, which is the symplectic form on the transverse space to the Reeb foliation descending from ω . Notice that Ω_+ being type zero implies that $f_5 \neq 0$.

In particular, the vector part ξ_v of the generalized Reeb vector ξ , defined via the integrable generalized complex structure $\mathcal{J}_-$ in (4.13), is indeed the Reeb vector field for the contact structure induced by the symplectic form ω on the cone. Thus Y is a contact manifold, and this contact structure is in some sense compatible with the generalized complex structure $\mathcal{J}_-$. The generalized Reeb vector is also *generalized Killing*, $\mathbb{L}_\xi G = 0$. These properties all mimic those of Kähler cones, or equivalently Sasakian manifolds. We give some examples of these generalized structures in section 4.5 below.

Proof of a).

The definition (4.28) of the RR fluxes can be rewritten as [47]

$$d(e^A \text{Im } \Omega_+) = \frac{1}{8} e^{4A} e^{-B} \star \lambda(F), \quad (4.38)$$

with $\lambda(F) \equiv F_1 - F_3 + F_5$. The Hodge star operator on X can be written as $\star F_p \equiv (-1)^p F_p \lrcorner \text{vol}_X$. Since $F_5 = f_5 \text{vol}_Y$, the one-form part of (4.38) immediately gives

$$d(e^A \text{Im } \alpha_+) = -\frac{f_5}{8} e^{4(A-\hat{\Delta})} d \log r. \quad (4.39)$$

Since e^A and $e^{\hat{\Delta}}$ are homogeneous of degree one and zero respectively, we deduce that $e^A \text{Im } \alpha_+ = -\gamma r^4$, with γ a constant. We set $\gamma = f_5/32$ by shifting A by a constant appropriately and obtain

$$e^{4A} = r^4 e^{4\hat{\Delta}}, \quad \text{Im } \alpha_+ = -\frac{f_5}{32} r^3 e^{-\hat{\Delta}}. \quad (4.40)$$

The first equation establishes a), and the second will be used in the proof of d). $\square$

Proof of b). The proof given in section 3.3.2 goes through without modification. $\square$

Proof of c).

From $d(e^{-A} \text{Re } \Omega_+) = 0$ and $\mathcal{L}_{r\partial_r} \alpha_+ = 3\alpha_+$ (since Ω_+ is homogeneous degree three), we obtain

$$\text{Re } \alpha_+ = 0, \quad d(e^{-A} \text{Im } \alpha_+ \omega_+) = 0, \quad db_+ \wedge \omega_+ = 0. \quad (4.41)$$

The second equation, combined with the fact that the Mukai pairing of Ω_+ is nowhere vanishing, $\langle \Omega_+, \bar{\Omega}_+ \rangle = -(4i/3) |\alpha_+|^2 \omega_+^3 \neq 0$, implies that the two-form

$$\omega \equiv e^{-2\hat{\Delta}} r^2 \omega_+ \quad (4.42)$$

is closed and non-degenerate, and hence symplectic. The justification for the presence of e^{-A} in the differential condition for $\text{Re } \Omega_+$ is that it leads to a symplectic form ω which is homogeneous of degree two, as usual for a symplectic cone. We may thus *globally* write $\omega \equiv d(r^2\sigma)/2$ for a real one-form σ , called the contact form, which is basic with respect to $r\partial_r$, $r\partial_r \lrcorner \sigma = r\partial_r \lrcorner d\sigma = 0$. Comparison with the annihilator constraint $r\partial_r \lrcorner \omega_+ = e^{2\hat{\Delta}}(\eta_f - \eta_v \lrcorner b_+)$ arising from $(r\partial_r - ie^{2\hat{\Delta}}\eta) \cdot \Omega_+ = 0$ leads to $\sigma = \eta_f - \eta_v \lrcorner b_+$. From (3.76) and the annihilator constraint $\xi_v \lrcorner \omega_+ = -e^{2\hat{\Delta}}d\log r$, we obtain

$$\xi_v \lrcorner \sigma = 1, \quad \xi_v \lrcorner d\sigma = 0, \quad (4.43)$$

as expected for a contact form σ and its associated unique Reeb vector field ξ_v .

It remains to show that $\eta_v \lrcorner b_+ = \eta_v \lrcorner b_2$, with b_2 a closed two-form. The fact that (3.75) annihilate Ω_+ gives $e^{2\hat{\Delta}}\eta_v \lrcorner \omega_+ = r\partial_r \lrcorner b_+$ and $\xi_v \lrcorner b_+ = \xi_f$, while the homogeneity of Ω_+ under $r\partial_r$ and ξ implies the conditions $\mathcal{L}_{r\partial_r} b_+ = 0$ and $\mathcal{L}_{\xi_v} b_+ = d\xi_f$. This allows us to write the general ansatz

$$b_+ = d\log r \wedge e^{2\hat{\Delta}}\eta_v \lrcorner \omega_+ + b_2, \quad (4.44)$$

where b_2 is a real two-form with $r\partial_r \lrcorner b_2 = 0$ and $r\partial_r \lrcorner db_2 = \xi_v \lrcorner db_2 = 0$. Since $\eta_v \lrcorner d\log r = 0$, this shows $\eta_v \lrcorner b_+ = \eta_v \lrcorner b_2$. From the term in $d\log r$ in $db_+ \wedge \omega = 0$, we get $d(e^{2\hat{\Delta}}\eta_v \lrcorner \omega_+) \wedge d\sigma + 2db_2 \wedge \sigma = 0$, and contracting with ξ_v gives $db_2 = 0$.

Recall that in section 4.1 we performed a *closed* B -transform of Ω_- by b_0 to put it into the product form (4.7) of a complex and a symplectic structure. This B -transform will similarly act on Ω_+ , and we consider that b_0 has been reabsorbed into the definition of b_+ , and more precisely in its closed part b_2 . $\square$

Proof of d).

Statement $d)$ is obtained from (4.40), (4.41), (4.42), and (4.44).

Note also that the condition $r\partial_r \lrcorner b_2 = 0$ gives $3(z\partial_z + \bar{z}\partial_{\bar{z}}) \lrcorner b_2 = \mathcal{H}_\varphi \lrcorner b_2$, while the annihilator constraint $\xi_v \lrcorner b_2 = \xi_f$ gives $3i(z\partial_z - \bar{z}\partial_{\bar{z}}) \lrcorner b_2 = d\varphi$, from which we conclude that b_2 can be expressed as

$$b_2 = (d\log r + dh) \wedge \mathcal{H}_\varphi \lrcorner \tilde{b}_2 + d\psi \wedge d\varphi + \tilde{b}_2, \quad (4.45)$$

where $\tilde{b}_2$ is the part of b_2 along the symplectic leaves defined by the $\mathcal{J}_-$ foliation. $\square$

4.3 Generalized reduction

In the Sasakian case one can consider the symplectic reduction of the Calabi-Yau cone metric with respect to the R-symmetry Killing vector ξ (or alternatively a holomorphic quotient with respect to $r\partial_r - i\xi$). Generically ξ does not define a $U(1)$ -fibration and the four-dimensional reduced space is not a manifold. Nonetheless, locally one can consider the geometry on the transversal section to the foliation formed by the orbits of ξ in the Sasaki-Einstein space. The result of the reduction is that this four-dimensional geometry is Kähler-Einstein. Thus locally one can always write the Sasaki-Einstein metric as

$$g_{\text{SE}} = \xi^b \otimes \xi^b + g_{\text{KE}} , \quad (4.46)$$

where $\xi^b \equiv g_Y(\cdot, \xi)$ and g_{KE} is a Kähler-Einstein metric.

The existence of the generalized holomorphic vectors $r\partial_r$ and ξ in the generic case suggests one can make an analogous generalized reduction to four dimensions. In this section, we show that this is indeed the case following the theory of generalized quotients developed in [32, 33]. We apply their formalism to our particular case, showing that there is a generalized Hermitian structure on the local transversal section, and giving the conditions satisfied by the corresponding reduced pure spinors.

There are two different ways we can view the generalized reduction, mirroring the complex quotient and the symplectic reduction in the Sasaki-Einstein case. In the first case, we take the quotient of X by the complex Lie group generated by $r\partial_r - i\xi_v$. In the second case, we first restrict to the zero-level set of the moment map $\mu = \log r$ before taking the quotient by the Lie algebra generated by ξ_v alone. As in the Sasaki-Einstein case, both methods lead to the same reduced structure on the reduced space $M^{\text{red}} = X/\text{span}\{r\partial_r - i\xi_v\} = \mu^{-1}(0)/\text{span}\{\xi_v\}$.

In order to construct the reduced pure spinors, first note that the reduction gives a splitting of the generalized tangent space $E = E_K \oplus K^G$ (see section 4 of [3] for more detail) such that the $O(6,6)$ metric $\langle \cdot, \cdot \rangle$ factors into an $O(2,2)$ metric on E_K and an $O(4,4)$ metric on K^G . Thus we can similarly decompose sections of the spinor bundles $S_{\pm}(E)$ into spinors of $Spin(2,2) \times Spin(4,4) \subset Spin(6,6)$. In particular, generic sections $\Omega_{\pm}$ in $S_{\pm}(E)$ can be written as

$$\Omega_{\pm} = \Theta_{\pm} \otimes \tilde{\Omega}_+ \oplus \Theta_{\mp} \otimes \tilde{\Omega}_- . \quad (4.47)$$

It is then the spinor components of $\tilde{\Omega}_{\pm}$ in $S_{\pm}(K_G)$ which correspond to the reduced pure spinors.

To make this explicit we need a basis for the $Spin(6, 6)$ gamma matrices reflecting this decomposition. We first introduce coordinates adapted to the reduction. Let u^m with $m = 1, \dots, 4$ be coordinates on the transversal section to the R-symmetry foliation. This means that $\xi_v \lrcorner du^m = 0$ and, in particular, the metric decomposes as

$$g_Y = \frac{\xi_v^b \otimes \xi_v^b}{\|\xi_v\|^2} + g_{mn}^{\text{red}} du^m du^n, \quad (4.48)$$

in analogy to (4.46). The reduction structure already defines a natural basis on E_K given by

$$\hat{f}_1 = r \partial_r, \quad f^1 = d \log r, \quad \hat{f}_2 = \xi, \quad f^2 = \eta, \quad (4.49)$$

and satisfying $2\langle f^i, \hat{f}_j \rangle = \delta^i_j$ and $\langle f^i, f^j \rangle = \langle \hat{f}_i, \hat{f}_j \rangle = 0$. We can then define an orthogonal basis on K^G given by

$$\hat{e}_m = e^{-b_2} \partial_{u^m} - \tilde{\eta}_m \xi, \quad e^m = du^m - \eta^m \xi \quad (4.50)$$

where $\tilde{\eta}_m = 2\langle \eta, e^{-b_2} \partial_{u^m} \rangle$ and $\eta^m = 2\langle \eta, du^m \rangle = \eta_v \lrcorner du^m$. This basis again satisfies $2\langle e^m, \hat{e}_n \rangle = \delta^m_n$ and $\langle e^m, e^n \rangle = \langle \hat{e}_m, \hat{e}_n \rangle = 0$.

Given such a basis we can then write a generic $Spin(6, 6)$ -spinor using the standard raising and lowering operator construction. Consider the polyform $\Omega^{(0)} = e^{-b_2} \in \Gamma(S_+(E))$. It is easy to see that we have the Clifford actions

$$\hat{f}_i \cdot \Omega^{(0)} = \hat{e}_m \cdot \Omega^{(0)} = 0, \quad (4.51)$$

for all i and m . Thus we can regard $\Omega^{(0)}$ as a ground state for the lowering operators $(\hat{f}_i, \hat{e}_m)$. A generic spinor is then given by acting with the anti-commuting raising operators (f^i, e^m) . Acting with the e^m first, we see that a generic (non-chiral) spinor has the form

$$\Omega = e^{-b_2} \tilde{\Omega}_0 + f^1 \cdot e^{-b_2} \tilde{\Omega}_1 + f^2 \cdot e^{-b_2} \tilde{\Omega}_2 + f^1 \cdot f^2 \cdot e^{-b_2} \tilde{\Omega}_3, \quad (4.52)$$

where $\tilde{\Omega}_i$ are polyforms in du^m , and $e^{-b_2} \tilde{\Omega}_i$ transform as a $Spin(4, 4)$ -spinor under the Clifford action of $(e^m, \hat{e}_m)$.

We can now write the supersymmetry pure spinors $\Omega_{\pm}$ in the form (4.52). Requiring that $r\partial_r - i\xi$ and $d\log r - i\eta$ annihilate Ω_- while $r\partial_r - ie^{2\hat{\Delta}}\eta$ and $d\log r - ie^{-2\hat{\Delta}}\xi$ annihilate Ω_+ we find that the only possibility is

$$\begin{aligned}\Omega_- &= r^3 e^{-3i\psi} (d\log r \wedge -i\eta) e^{-b_2} \Omega_1 , \\ \Omega_+ &= r^3 (1 + ie^{2\hat{\Delta}} d\log r \wedge \eta) e^{-b_2} \Omega_2 ,\end{aligned}\tag{4.53}$$

where Ω_1 and Ω_2 are both even polyforms in du^m . We have introduced factors of r^3 and $e^{-3i\psi}$ so that Ω_1 and Ω_2 are independent of the r and ψ coordinates. In general, they are only locally defined.

We can immediately deduce that

$$d\varphi = 0 .\tag{4.54}$$

Indeed, Ω_- has no terms in $d\log r \wedge d\psi$, whereas given the form of b_2 in (4.45) the right-hand side contains a term in $d\log r \wedge d\psi \wedge d\varphi$. Recalling (4.16) and (4.19), we see that this gives

$$r\partial_r = 3(z\partial_z + \bar{z}\partial_{\bar{z}}) , \quad \xi = \xi_v = 3i(z\partial_z - \bar{z}\partial_{\bar{z}}) ,\tag{4.55}$$

which means that the foliation determined by $r\partial_r$ and ξ_v coincides with the complex transverse space of the local foliation defined by $\mathcal{J}_-$. Since by definition $r\partial_r$ and ξ_v are both global vector fields on X , it follows that ∂_z , which was initially defined only as a local vector field in X_0 , is in fact also a global vector field on X . Henceforth we shall use the term foliation only with respect to the Reeb foliation defined by ξ_v , which is a *global foliation* of Y . The above comments also imply that $b_2 = \tilde{b}_2$ is a two-form on the four-dimensional transverse space to the Reeb foliation, or more precisely it is basic with respect to this foliation.

The pair of reduced pure spinors turns out to be

$$\begin{aligned}\Omega_1 &= 3e^{3h} \exp(b_2 + i\omega_0) , \\ \Omega_2 &= -i \frac{f_5}{32} e^{-\hat{\Delta}} \exp\left(ie^{2\hat{\Delta}}\omega_T\right) ,\end{aligned}\tag{4.56}$$

where the symplectic form ω_T on the transverse reduced space is

$$\omega_T \equiv \frac{1}{2} d\sigma = \frac{1}{2} \mathcal{L}_{\mathcal{H}_h} b_2 .\tag{4.57}$$

The corresponding generalized structures are

$$\mathcal{J}_1 = \begin{pmatrix} -\omega_0^{-1}b_2 & \omega_0^{-1} \\ -\omega_0 - b_2\omega_0^{-1}b_2 & b_2\omega_0^{-1} \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & e^{-2\hat{\Delta}}\omega_T^{-1} \\ -e^{2\hat{\Delta}}\omega_T & 0 \end{pmatrix} \quad (4.58)$$

The generalized structure $\mathcal{J}_1$ is integrable since we have $d\Omega_1 = 3dh \wedge \Omega_1$.

The compatibility of $\mathcal{J}_-$ and $\mathcal{J}_+$ reduces to the compatibility of $\mathcal{J}_1$ and $\mathcal{J}_2$ [32], which thus define a generalized metric G_T on the transverse space with the following transverse metric g_T and B -field B_T :

$$\begin{aligned} g_T &= e^{2\hat{\Delta}}\omega_T b_2^{-1}\omega_0, \\ B_T &= e^{4\hat{\Delta}}\omega_T b_2^{-1}\omega_T = -\omega_0 b_2^{-1}\omega_0 - b_2. \end{aligned} \quad (4.59)$$

The compatibility condition $\mathcal{J}_1 \cdot \Omega_2 = 0$ is most easily analyzed by first performing a B -transform by $-b_2$ to put $\mathcal{J}_1$ in the standard symplectic form

$$e^{-b_2}\mathcal{J}_1 e^{b_2} = \begin{pmatrix} 0 & \omega_0^{-1} \\ -\omega_0 & 0 \end{pmatrix}. \quad (4.60)$$

The equivalent compatibility condition $(e^{-b_2}\mathcal{J}_1 e^{b_2}) \cdot e^{-b_2}\Omega_2 = 0$ then gives

$$b_2 \wedge \omega_0 = b_2 \wedge \omega_T = \omega_0 \wedge \omega_T = 0, \quad (4.61)$$

$$e^{4\hat{\Delta}}\omega_T^2 = b_2^2 - \omega_0^2. \quad (4.62)$$

Note that $\omega_0 \wedge \omega_T = 0$ is already implied by $b_2 \wedge \omega_0 = 0$ and the fact that $\omega_T = \mathcal{L}_{\mathcal{H}_h} b_2/2$.

4.4 Differential system

In this section we present the full set of conditions for a generalized Sasakian geometry reduced on the space transverse to the Reeb foliation. We will see that this amounts to a simple differential system for three orthogonal symplectic forms on this transverse space. The only supersymmetry condition from chapter 3 that we have not imposed is the equality of the norms of Ω_+ and Ω_- . For Kähler cones, this condition is equivalent to the Einstein equation, and indeed directly leads to the Monge-Ampère equation in this case. Imposing this condition in the generalized setting thus leads to a supersymmetric AdS₅ solution, which in our terminology would be *generalized Sasaki-Einstein*.⁶

⁶Although here Einstein is meant to indicate that the Einstein equations of supergravity are satisfied, rather than g_Y is an Einstein metric, which in general it is not.

4.4.1 Bianchi identity

The condition in (4.27) on the imaginary part of Ω_+ corresponds to the physical requirement that away from any source the RR fluxes must satisfy the Bianchi identity $d(e^{-B}F) = 0$. To obtain the physical RR fluxes, we need to undo the closed B -transform by b_2 that we performed at the very beginning in section 4.1 to put Ω_- into the local form of a complex/symplectic product (4.7).⁷ We then obtain the following explicit formulas for the fluxes:

$$F_1 = -\frac{f_5}{4} (\mathcal{H}_h \lrcorner \mathcal{L}_{\mathcal{H}_h} \omega_T - \mathcal{H}_{e^{-4\hat{\Delta}}} \lrcorner b_2) , \quad (4.63)$$

$$e^{-(B-b_2)} F|_3 = \frac{f_5}{4} [\sigma \wedge \mathcal{L}_{\mathcal{H}_h} \omega_T + 2(\mathcal{H}_h - \mathcal{H}_{\hat{\Delta}}) \lrcorner \omega_T^2] , \quad (4.64)$$

$$e^{-(B-b_2)} F|_5 = -\frac{f_5}{2} \sigma \wedge \omega_T^2 . \quad (4.65)$$

The Bianchi identity then gives one new condition:

$$\mathcal{L}_{\mathcal{H}_h}(\mathcal{L}_{\mathcal{H}_h} \omega_T) = \mathcal{L}_{\mathcal{H}_{e^{-4\hat{\Delta}}}} b_2 . \quad (4.66)$$

4.4.2 Einstein condition

By definition, the Mukai pairings $\langle \Omega_-, \bar{\Omega}_- \rangle$ and $\langle \Omega_+, \bar{\Omega}_+ \rangle$ are nowhere-vanishing top-degree forms on X , and as such they must be proportional:

$$\langle \Omega_-, \bar{\Omega}_- \rangle = e^f \langle \Omega_+, \bar{\Omega}_+ \rangle \quad \text{or} \quad \|\Omega_-\|^2 = e^f \|\Omega_+\|^2 , \quad (4.67)$$

with f a real function independent of r , such that e^f is homogeneous of degree zero under $r\partial_r$. This leads to a corresponding relation between the “lengths” of ω_T and ω_0 . The calculation here is again most easily carried out in terms of the reduced pure spinors. Because of the factor of $e^{2\hat{\Delta}}$ in the decomposition of Ω_+ , the proportionality condition (4.67) becomes

$$\langle \Omega_1, \bar{\Omega}_1 \rangle = e^f e^{2\hat{\Delta}} \langle \Omega_2, \bar{\Omega}_2 \rangle , \quad (4.68)$$

which gives

$$\left(\frac{96}{f_5} \right)^2 e^{6h} \omega_0 \wedge \omega_0 = e^{4\hat{\Delta}+f} \omega_T \wedge \omega_T . \quad (4.69)$$

⁷It is a curious fact that without this transform we obtain in particular $e^{-B}F|_5 = 0$.

Note that combining this condition with the compatibility condition (4.62) we get

$$b_2 \wedge b_2 = \left[1 + \left(\frac{96}{f_5} \right)^2 e^{6h-f} \right] \omega_0 \wedge \omega_0 , \quad (4.70)$$

which implies that b_2 is also non-degenerate, and hence a symplectic form on the transverse space to the Reeb foliation.

Let us compare again with the standard Kähler setting (see section 2.5). For a Kähler cone with metric $dr^2 + r^2 g_Y$ and trivial canonical bundle, so that (Y, g_Y) is a transversely Fano Sasakian manifold, the equal norm condition (4.67) becomes

$$\frac{i}{8} \Omega \wedge \bar{\Omega} = \frac{e^f}{3!} \omega^3 . \quad (4.71)$$

The Ricci-form is $\rho = i\partial\bar{\partial}f$ and the Ricci scalar is then $R = -\Delta_X f$, where Δ_X denotes the Laplacian on X . When f is constant, the Kähler metric is Ricci-flat and hence Calabi-Yau, which means that (Y, g_Y) is Sasaki-Einstein. Moreover, (4.71) immediately leads to the Monge-Ampère equation for such a metric. We thus refer to the condition that f is a constant, which we can set to zero by rescaling, as the *Einstein condition*:

$$f = 0 . \quad (4.72)$$

More physically, adding this condition to the definition of generalized Sasakian geometry implies that our structure satisfies all the supersymmetry conditions for an AdS_5 solution of type IIB supergravity as shown in section 3.2, and in particular the Einstein equation. For such a solution, the physical dilaton ϕ is defined by the norms of the pure spinors as in (3.44):

$$\|\Omega_-\|^2 = \|\Omega_+\|^2 \equiv \frac{1}{8} e^{6A-2\phi} . \quad (4.73)$$

This allows us compute an expression for the volume form on Y in terms of the contact volume. Using

$$\langle \Omega_+, \bar{\Omega}_+ \rangle = -i \frac{4}{3} |\alpha_+|^2 \omega_+^3 = -i \left(\frac{f_5}{32} \right)^2 e^{4\hat{\Delta}} r^6 d \log r \wedge \sigma \wedge d\sigma^2 , \quad (4.74)$$

and $\text{vol}_X = -e^{6\hat{\Delta}} d \log r \wedge \text{vol}_Y$, this gives

$$\text{vol}_Y = -\frac{f_5^2}{128} e^{-8\Delta} \sigma \wedge d\sigma^2 , \quad (4.75)$$

where $\Delta \equiv \hat{\Delta} - \phi/4$, in agreement with (3.90).

4.4.3 Symplectic triple

We have now reduced our definition of a *generalized Sasakian* geometry to a simple differential system on the transverse space to the Reeb foliation of a contact manifold [4]. More precisely, this system holds on $Y_0 = X_0|_{\{r=1\}}$, the open dense subset where Ω_- has type one. The compatibility condition, the Bianchi identity and the proportionality of the norms of the pure spinors boil down to a system of algebraic and differential equations for three transverse orthogonal symplectic forms ω_0 , $\omega_1 \equiv \omega_T$, and $\omega_2 \equiv b_2$:

$$d\omega_i = 0 \quad \forall i \in \{0, 1, 2\} , \quad (4.76)$$

$$\omega_i \wedge \omega_j = 0 \quad \forall i \neq j , \quad (4.77)$$

which induce the same orientation:

$$\omega_0 \wedge \omega_0 = \alpha_1 \omega_1 \wedge \omega_1 = \alpha_2 \omega_2 \wedge \omega_2 \quad \text{nowhere zero} , \quad (4.78)$$

where the positive proportionality functions are

$$\alpha_1 = \left(\frac{f_5}{96} \right)^2 e^{4\hat{\Delta} - 6h + f} , \quad \alpha_2 = \left[1 + \left(\frac{96}{f_5} \right)^2 e^{6h - f} \right]^{-1} . \quad (4.79)$$

This is called a “symplectic triple” in [34] and can be chosen as an orthogonal basis for the space Λ^+ of positively oriented two-forms on the transverse leaf space of the Reeb foliation (see for example [60]). There are also the following differential conditions:

$$\omega_1 = \frac{1}{2} \mathcal{L}_{\mathcal{H}_h} \omega_2 , \quad \mathcal{L}_{\mathcal{H}_h} (\mathcal{L}_{\mathcal{H}_h} \omega_1) = \mathcal{L}_{\mathcal{H}_{e^{-4\hat{\Delta}}}} \omega_2 , \quad (4.80)$$

where $\mathcal{H}_h = \omega_0^{-1} \lrcorner dh$ for the real function h and similarly for $e^{-4\hat{\Delta}}$.

Altogether, this set of conditions characterizes what we have called a “generalized Sasakian structure”, at least on the dense open subset $Y_0 \subset Y$. As mentioned at the beginning, the type-change locus points that are limit points of Y_0 in Y effectively lead to boundary conditions on the above structure, which degenerates at these limit points. We shall not analyse this in generality in this thesis, but rather comment only in examples (see [32, 33] for preliminary mathematical studies). Notice that, nevertheless, the contact structure and Reeb foliation are defined globally on Y .

To obtain a *generalized Sasaki-Einstein* manifold, we must also impose the Einstein condition

$$f = 0 . \quad (4.81)$$

Note that the system for a triple of symplectic forms looks very similar to a hyper-Kähler structure on the transverse space. We can indeed define three almost complex structures

$$I \equiv \sqrt{\frac{\alpha_1}{\alpha_2}} \omega_1^{-1} \omega_2 , \quad J \equiv \sqrt{\alpha_2} \omega_2^{-1} \omega_0 , \quad K \equiv \frac{1}{\sqrt{\alpha_1}} \omega_0^{-1} \omega_1 , \quad (4.82)$$

which satisfy the hyper-Kähler relations

$$I^2 = J^2 = K^2 = IJK = -1 . \quad (4.83)$$

This implies that the reduced transverse space carries an $SU(2)$ -structure, in resonance with our discussion in section 3.2.

However, the fact that the symplectic forms here have different lengths means that these almost complex structures are not integrable. Thus there is no (natural) integrable complex structure on this transverse space. This is the key difference from Sasakian geometry, where the corresponding transverse space is Kähler, and hence both symplectic and complex.

4.5 Example: β -transform of Kähler cones

To make sure that our definition is not vacuous, we now present an explicit class of examples of generalized Sasakian manifolds. There are two important points here. First, these give a large family of such geometries that have varying Reeb vector fields and contain a generalized Sasaki-Einstein geometry (with $f = 0$) as a special case. Second, we will see that these are in a very precise sense generalizations of Kähler cones that are not Ricci-flat. Indeed, our strategy will be to perform a β -transform of the complex and symplectic structures of a cone that is Kähler but *not* Ricci-flat in general. Perhaps the most important issue that our work raises is to understand better this space of generalized Sasakian structures, or more pressingly the associated space of Reeb vector fields in a given deformation class. We will content ourselves here with showing that there are non-trivial examples, with non-trivial spaces of Reeb

vector fields. This will be sufficient to show that the generalized volume minimization we define in the next chapter is indeed a non-trivial problem in general.

Consider a Kähler cone metric on $\mathbb{C}^3$ that is a cone with respect to the *weighted* Euler vector field $r\partial_r = \sum_{i=1}^3 \xi_i r_i \partial_{r_i}$, where the weights $\xi_i \in \mathbb{R}_+$ are the components of the Reeb vector field, $\xi = \sum_i \xi_i \partial_{\phi_i}$.⁸ The holomorphic $(3,0)$ -form is

$$\Omega = dz_1 \wedge dz_2 \wedge dz_3, \quad (4.84)$$

with standard complex coordinates $z_i = r_i \exp(i\phi_i)$, while the Kähler form is as usual $\omega = i\partial\bar{\partial}r^2/2$. A natural choice [21] for the Kähler potential in this case is $r^2 = \sum_i r_i^{2/\xi_i}$, which gives

$$\omega = \sum_i \frac{r_i^{2/\xi_i}}{\xi_i^2} d \log r_i \wedge d\phi_i = \frac{i}{2} \sum_i \frac{r_i^{2/\xi_i-2}}{\xi_i^2} dz_i \wedge d\bar{z}_i. \quad (4.85)$$

We then have

$$\frac{i}{8} \Omega \wedge \bar{\Omega} = e^f \frac{\omega^3}{3!}, \quad (4.86)$$

where the real function f is given by

$$e^{f/2} = \xi_1 \xi_2 \xi_3 r_1^{1-1/\xi_1} r_2^{1-1/\xi_2} r_3^{1-1/\xi_3}. \quad (4.87)$$

Note that the homogeneity condition $\mathcal{L}_{r\partial_r} \Omega = 3\Omega$ implies that $\xi_1 + \xi_2 + \xi_3 = 3$.

After a β -transform (2.11) by $\beta = \gamma(\partial_{\phi_1} \wedge \partial_{\phi_2} + \text{c.p.})$ on the associated pair of pure spinors (2.56) (multiplied by $1/8$ and $-i/8$ respectively to agree with the conventions in section 3.2; see also appendix B), we get

$$\begin{aligned} e^\beta \Omega_- &= \frac{\gamma}{8} d(\bar{z}_1 \bar{z}_2 \bar{z}_3) \wedge \exp \left[\frac{1}{3\gamma} \frac{d\bar{z}_1 \wedge d\bar{z}_2}{\bar{z}_1 \bar{z}_2} + \text{c.p.} \right], \\ e^\beta \Omega_+ &= -i \frac{r^3}{8} \exp \left[\frac{i}{r^2} \omega - \frac{\gamma}{r^4} \left(\frac{r_1^{2/\xi_1} r_2^{2/\xi_2}}{\xi_1^2 \xi_2^2} d \log r_1 \wedge d \log r_2 + \text{c.p.} \right) \right]. \end{aligned} \quad (4.88)$$

The exponent of $e^\beta \Omega_-$ can be put in the generalized Darboux form by shifting by a two-form proportional to $d(\bar{z}_1 \bar{z}_2 \bar{z}_3)$. This then gives

$$\begin{aligned} z &= \frac{\gamma}{8} z_1 z_2 z_3 = \frac{\gamma}{8} r_1 r_2 r_3 e^{i(\phi_1 + \phi_2 + \phi_3)}, \\ \omega_0 &= dx_1 \wedge dy_1 + dx_2 \wedge dy_2, \\ b_0 &= -\frac{\xi_1 \xi_2 \xi_3}{3\gamma} dx_1 \wedge dx_2 + \frac{3\gamma}{\xi_1 \xi_2 \xi_3} dy_1 \wedge dy_2, \end{aligned} \quad (4.89)$$

⁸We use the same notation for the first two components of the Reeb vector field as for the five-dimensional spinors $\xi_{1,2}$ introduced in (3.11) but this will hopefully not lead to ambiguities.

with the symplectic coordinates

$$\begin{aligned} x_1 &= \log \frac{r_1^{1/\xi_1}}{r_3^{1/\xi_3}} , & y_1 &= \frac{\xi_1 \xi_2 \xi_3}{3\gamma} \left(\frac{\phi_3}{\xi_3} - \frac{\phi_2}{\xi_2} \right) , \\ x_2 &= \log \frac{r_2^{1/\xi_2}}{r_3^{1/\xi_3}} , & y_2 &= \frac{\xi_1 \xi_2 \xi_3}{3\gamma} \left(\frac{\phi_1}{\xi_1} - \frac{\phi_3}{\xi_3} \right) . \end{aligned} \quad (4.90)$$

The two-form b_2 is the difference of the part of b_+ that is independent of r , which we call b'_2 , and b_0 : $b_2 = b'_2 - b_0$.

We can obtain the contact one-form, and so $\omega_T = d\sigma/2$, by contracting the Euler vector field $r\partial_r$ with ω in (4.85):

$$\sigma = \sum_i \frac{r_i^{2/\xi_i}}{\xi_i r^2} d\phi_i . \quad (4.91)$$

It is then straightforward to verify that the generalized Sasakian conditions compiled in the last section are satisfied for all values of the Reeb vector field ξ . However, the Einstein condition does not hold in general, and we rather have

$$\|e^\beta \Omega_-\|^2 = e^f \|e^\beta \Omega_+\|^2 . \quad (4.92)$$

We have thus constructed a family of explicit generalized Sasakian geometries which contains a generalized Sasaki-Einstein geometry with $f = 0$.

Chapter 5

Volume Minimization

In this chapter we show that the Reeb vector field ξ for a generalized Sasaki-Einstein manifold can be determined by a (finite-dimensional) variational problem on a space of generalized Sasakian manifolds. Given that generalized Sasaki-Einstein manifolds provide $\text{AdS}_5 \times Y$ solutions of type IIB supergravity, the relevant functional to minimize is an action whose Euler-Lagrange equations are the equations of motion for the type IIB bosonic fields on the five-dimensional compact space Y . We rewrite this functional in terms of pure spinors and show that, when restricted to a space of generalized Sasakian manifolds, it reduces to the contact volume (corresponding to the central charge of the dual SCFT). This result precisely generalizes the volume minimization introduced in [19] for Sasaki-Einstein manifolds.

5.1 Supergravity action

We now construct an effective action for the bosonic fields on Y in a supersymmetric AdS_5 background in type IIB supergravity. The Euler-Lagrange equations for this action give rise to the equations of motion satisfied by the fields.

5.1.1 Five-dimensional action

Let us first analyze the Einstein equation in (3.3). Recall that under a Weyl rescaling $g = e^{2\alpha}\bar{g}$ in D dimensions, the Ricci tensor and the Ricci scalar transform as

$$\begin{aligned} R_{MN} &= \bar{R}_{MN} + (D-2)[-\bar{\nabla}_M \partial_N \alpha + \partial_M \alpha \partial_N \alpha] \\ &\quad - [\bar{\nabla}^2 \alpha + (D-2)|d\alpha|^2] \bar{g}_{MN} , \end{aligned} \tag{5.1}$$

$$R = e^{-2\alpha} [\bar{R} - 2(D-1)\bar{\nabla}^2 \alpha - (D-2)(D-1)|d\alpha|^2] , \tag{5.2}$$

where $\bar{\nabla}$ denotes the Levi-Civita connection for $\bar{g}$, and the indices are contracted with $\bar{g}$. Defining $\bar{g} = g_{\text{AdS}} + g_Y$ we then have $\bar{R}_{MN} = R_{\mu\nu} + R_{mn}$, and the Ricci scalar is $\bar{R} = R_{\text{AdS}} + R_Y$. The Freund-Rubin ansatz (3.8) gives

$$F_{MP_1P_2P_3P_4} F_N^{P_1P_2P_3P_4} = 4! f_5^2 (-g_{\mu\nu} + g_{mn}) . \quad (5.3)$$

Using the above formulas, the Einstein equation then splits as

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} R_Y g_{\mu\nu} &= - \left[10 + 8 \nabla^2 \Delta + 28 |\text{d}\Delta|^2 + |P_1|^2 + \frac{e^{-4\Delta}}{4} |G_3|^2 + \frac{e^{-8\Delta} f_5^2}{4} \right] g_{\mu\nu} , \\ R_{mn} - \frac{1}{2} R_Y g_{mn} &= 8 (\nabla_m \partial_n \Delta - \partial_m \Delta \partial_n \Delta) + P_m P_n^* + P_m P_n^* \\ &\quad + \frac{e^{-4\Delta}}{8} (G_{mp_1p_2} G_n^{*p_1p_2} + G_{np_1p_2} G_m^{*p_1p_2}) \\ &\quad - \left[10 + \nabla^2 \Delta + 28 |\text{d}\Delta|^2 + |P_1|^2 + \frac{e^{-4\Delta}}{4} |G_3|^2 - \frac{e^{-8\Delta} f_5^2}{4} \right] g_{mn} , \end{aligned}$$

which gives the Ricci scalar on Y

$$R_Y = \frac{100}{3} + \frac{8}{3} (8 \nabla^2 \Delta + 37 |\text{d}\Delta|^2) + 2 |P_1|^2 - \frac{e^{-4\Delta}}{6} |G_3|^2 - \frac{5}{6} e^{-8\Delta} f_5^2 . \quad (5.4)$$

The consistency between the two parts of the Einstein equation requires

$$\frac{e^{-4\Delta}}{8} |G_3|^2 + \frac{e^{-8\Delta} f_5^2}{4} - 4 = \nabla^2 \Delta + 8 |\text{d}\Delta|^2 . \quad (5.5)$$

For later reference, note that multiplying the right-hand side by $e^{8\Delta}$ and integrating by parts over Y gives zero, and so we have ¹

$$\int d^5 y \sqrt{g_Y} e^{8\Delta} \left(\frac{e^{-4\Delta}}{8} |G_3|^2 + \frac{e^{-8\Delta} f_5^2}{4} - 4 \right) = 0 . \quad (5.6)$$

The equations for P_1 and G_3 can be rewritten as

$$\begin{aligned} e^{-8\Delta} D_m (e^{8\Delta} P^m) &= - \frac{e^{-4\Delta}}{24} G_{mnp} G^{mnp} , \\ e^{-8\Delta} D_p (e^{4\Delta} G^{mnp}) &= e^{-4\Delta} P_p G^{*mnp} - i \frac{e^{-8\Delta}}{6} f_5 \varepsilon^{mnp_1p_2p_3} G_{p_1p_2p_3} . \end{aligned} \quad (5.7)$$

In terms of real fields (recall (3.2)) this reads

$$\nabla_m (e^{8\Delta+2\phi} \partial^m C_0) = - \frac{e^{4\Delta+\phi}}{6} F_{mnp} H^{mnp} , \quad (5.8)$$

$$\nabla_m (e^{8\Delta} \partial^m \phi) = e^{8\Delta+2\phi} |F_1|^2 + \frac{1}{2} e^{4\Delta+\phi} |F_3|^2 - \frac{1}{2} e^{4\Delta-\phi} |H|^2 , \quad (5.9)$$

$$\nabla_p (e^{4\Delta+\phi} F^{mnp}) = - \frac{f_5}{6} \varepsilon^{mnp_1p_2p_3} H_{p_1p_2p_3} , \quad (5.10)$$

$$\nabla_p (e^{4\Delta-\phi} H^{mnp}) = e^{4\Delta+\phi} \partial_p C_0 F^{mnp} + \frac{f_5}{6} \varepsilon^{mnp_1p_2p_3} F_{p_1p_2p_3} . \quad (5.11)$$

¹This result can also be obtained by imposing the equation of motion for G_3 , or by combining the equation of motion for the warp factor Δ and the Einstein equations.

All of these equations of motion can be derived from the variation of the following effective action on Y :²

$$\begin{aligned}
S_{\text{IIB}} = & \int_Y d^5y \sqrt{g_Y} e^{8\Delta} \left(R_Y - 20 + 72|d\Delta|^2 - \frac{1}{2}|d\phi|^2 - \frac{1}{2}e^{-4\Delta-\phi}|H|^2 \right. \\
& \left. - \frac{1}{2}e^{2\phi}|F_1|^2 - \frac{1}{2}e^{-4\Delta+\phi}|F_3|^2 + \frac{1}{2}e^{-8\Delta}f_5^2 \right) \\
& + f_5 \int_Y H \wedge C_2 .
\end{aligned} \tag{5.12}$$

This is the action with which we shall work. Notice in particular the final Chern-Simons-type term.

5.1.2 On-shell action and central charge

We will now show that our action S_{IIB} reduces *on shell*, that is when supersymmetry and the equations of motion of type IIB supergravity are imposed, to the contact volume of Y . For a supersymmetric solution, this is the inverse central charge of the dual SCFT, as shown in [2, 3]. Notice that going on shell corresponds to imposing the generalized Sasakian conditions together with the Einstein condition. This is therefore stronger than the restriction to generalized Sasakian manifolds which is appropriate for our variational problem. We will see how to implement this in the next section.

When the metric is on shell, that is when we impose the Einstein equation and hence (5.4), the action reduces to

$$S_{\text{IIB}}(g_Y \text{ on shell}) = \int_Y d^5y \sqrt{g_Y} e^{8\Delta} \left(\frac{40}{3} - \frac{2}{3}e^{-4\Delta}|G_3|^2 - \frac{1}{3}e^{-8\Delta}f_5^2 \right) + f_5 \int_Y H \wedge C_2 .$$

The Chern-Simons term can be rewritten on shell as

$$\begin{aligned}
f_5 \int_Y H \wedge C_2 &= \frac{f_5}{2!3!} \int_Y d^5y \sqrt{g_Y} H_{mnp} C_{qr} \varepsilon^{mnpqr} \\
&= -\frac{1}{2} \int_Y d^5y \sqrt{g_Y} C_{mn} \nabla_p (e^{4\Delta+\phi} F^{mnp}) \\
&= \frac{1}{2} \int_Y d^5y \sqrt{g_Y} e^{4\Delta+\phi} \nabla_p C_{mn} F^{mnp} ,
\end{aligned} \tag{5.13}$$

²Notice that to obtain a canonical Einstein term $\sqrt{g'}R'$, one has to rescale the metric as $g_Y = e^{-16\Delta/3}g'$.

where the second equality uses the equation of motion (5.10) contracted into C_{mn} . On the other hand, we have

$$\begin{aligned}
e^{4\Delta}|G|^2 &= e^{4\Delta+\phi}|F_3|^2 + e^{4\Delta-\phi}|H|^2 \\
&= e^{4\Delta+\phi}\nabla_m C_{np} F^{mnp} - C_0 \frac{e^{4\Delta+\phi}}{3} F_{mnp} H^{mnp} \\
&\quad + 2e^{8\Delta+2\phi}\partial_m C_0 \partial^m C_0 - 2\nabla_m (e^{8\Delta}\partial^m \phi) \\
&= e^{4\Delta+\phi}\nabla_m C_{np} F^{mnp} + 2\nabla_m [e^{8\Delta}(e^{2\phi}C_0\partial^m C_0 - \partial^m \phi)] , \quad (5.14)
\end{aligned}$$

where we have used (5.9) in going from the first line to the second, and (5.8) from the second to the last. When integrated over Y , the total divergence vanishes so that the Chern-Simons term gives on shell

$$f_5 \int_Y H \wedge C_2 = \int_Y d^5 y \sqrt{g_Y} \frac{e^{4\Delta}}{2} |G_3|^2 . \quad (5.15)$$

Using also (5.6) we obtain finally

$$S_{\text{IIB}}(\text{on shell}) = 8 \int_Y d^5 y \sqrt{g_Y} e^{8\Delta} . \quad (5.16)$$

For a supersymmetric solution we also have (4.75), and hence obtain the result that the supersymmetric on-shell S_{IIB} is proportional to the contact volume of Y :

$$S_{\text{IIB}}(\text{on shell}) = -\frac{f_5^2}{16} \int_Y \sigma \wedge d\sigma \wedge d\sigma . \quad (5.17)$$

5.2 Restriction to generalized Sasakian manifolds

In order to set up the variational problem, we would like to obtain an expression for S_{IIB} when it is not necessarily fully on shell, in the sense that the generalized Sasakian conditions are imposed but the Einstein condition is lifted. This is analogous to the computations of the Einstein-Hilbert action restricted to a space of Sasakian metrics in [19, 20], and indeed generalizes these computations to general backgrounds with all fluxes activated. Following the latter references, we first need to rewrite S_{IIB} as an integral over a finite segment of the six-dimensional cone X , and express the integrand in terms of the pure spinors $\Omega_{\pm}$.

Before carrying the computation, we should begin by clarifying how we relate the fields in the action (5.12) to the generalized Sasakian structures we have defined

in chapter 4. A generalized Sasakian structure involves choosing compatible pure spinors $\Omega_{\pm}$ on the cone X , and these in particular then define a Riemannian metric g_X of the form (4.24) and B -field that is basic with respect to $r\partial_r$, thus leading to a metric g_Y , B -field and scalar function $\hat{\Delta}$ on Y . The RR fluxes F are then defined in terms of the generalized structure via (4.28). Since the Bianchi identity $d(e^{-B}F) = 0$ is part of our definition of generalized Sasakian structure, we may hence introduce RR potentials C . This then defines all the quantities in the action (5.12), except for the warp factor Δ and dilaton ϕ . Instead the generalized Sasakian structure gives us a function $\hat{\Delta}$; we shall give the relation between these functions below.

We also make some mild topological assumptions, which conveniently bypass some of the subtleties involved in defining integrals of forms that are not gauge invariant.³ It is convenient to assume that $b_1(Y) = 0$, so that $F_1 = dC_0$ holds for a *globally* defined potential C_0 on Y . This is a necessary condition in the Sasaki-Einstein case, by Myers' theorem [61], and every known supersymmetric AdS_5 solution also satisfies this condition. Without this assumption, one has to be a little more careful about global issues in the integrations by parts that will follow. In fact we have tacitly already assumed that the B -field is a globally defined two-form in writing the original supersymmetry conditions in the form (4.1). This is in fact a mild assumption, since in section 3.2 it was shown that the differential form H is always exact for any supersymmetric AdS_5 solution. More precisely, the difference $B - b_2$, which is what we shall integrate by parts below, may be expressed in terms of globally-defined spinor bilinears, see (3.19). This leaves the possibility of adding to B a discrete torsion B -field, which we shall again suppress. In any case, as we have defined a generalized Sasakian structure, B is a globally-defined two-form on Y , since both $\Omega_{\pm}$ were defined as global differential forms. More generally there can also be a topological twisting by a gerbe, on which B is a curving (see chapter 2). A similar comment applies also to the RR potential C_2 . Of course, we are then only interested in generalized Sasakian structures with these global properties also, since a continuous deformation of such a structure cannot change the topological class of these objects.

We begin by rewriting the Chern-Simons term. By a succession of integrations by parts, bearing in mind the above comments that C_0 , B and C_2 are all global forms

³This is really just to avoid such issues entirely: we do not believe the following assumptions are necessary.

on Y , we obtain

$$f_5 \int_Y H \wedge C_2 = f_5 \int_Y e^{-(B-b_2)} F|_5 - F_5 - \frac{1}{2} d [C_0(B-b_2)^2] . \quad (5.18)$$

The integral on Y of the exact term vanishes on using Stokes' theorem. Using also the formulas $F_5 = f_5 \text{vol}_Y$ and $e^{-(B-b_2)} F|_5 = -(f_5/2) \sigma \wedge \omega_T^2 = (16/f_5) e^{8\Delta} \text{vol}_Y$ from (4.65) and (4.75), we get

$$f_5 \int_Y H \wedge C_2 = \int_Y (16e^{8\Delta} - f_5^2) \text{vol}_Y . \quad (5.19)$$

This agrees with the calculation (3.104).⁴ Inserting this form of the Chern-Simons term into S_{IIB} then gives

$$S_{\text{IIB}} = \int_Y d^5 y \sqrt{g_Y} e^{8\Delta} \left(R_Y - 4 + 72 |d\Delta|^2 - \frac{1}{2} |d\phi|^2 - \frac{1}{2} e^{-4\Delta-\phi} |H|^2 - \frac{1}{2} (e^{2\phi} |F_1|^2 + e^{-4\Delta+\phi} |F_3|^2 + e^{-8\Delta} f_5^2) \right) . \quad (5.20)$$

We now want to write the action S_{IIB} , expressed in (5.12) and (5.20) in terms of the warped metric $e^{2\Delta} g_Y$, as an integral on the cone X with metric g_X , or rather its truncation at $r = 1$, which we call $X_1 \cong [0, 1] \times Y$. The metrics on X and Y are related through (4.24):

$$g_X = e^{2\hat{\Delta}} r^{-2} (dr^2 + r^2 g_Y) . \quad (5.21)$$

Note that the metric g_E in (3.7) is in the Einstein frame, whereas in the application of generalized geometry to type IIB the metric g_X is in the string frame. The two are hence related by the Weyl rescaling $g_X = e^{\phi/2} g_E$, which introduces the dilaton ϕ . This then implies $\hat{\Delta} = \Delta + \phi/4$, relating the generalized Sasakian function $\hat{\Delta}$ to this particular combination of the physical fields Δ and ϕ . Using that $r^2 \bar{R}_X = R_Y - 20$ for a metric $\bar{g}_X = dr^2 + r^2 g_Y$, and performing a Weyl rescaling by $e^{2\hat{\Delta}} r^{-2}$, we get

$$R_Y - 20 = e^{2\hat{\Delta}} \left(R_X + 10 \nabla^2 \hat{\Delta} - 20 |d\hat{\Delta}|^2 - 20 e^{-2\hat{\Delta}} \right) . \quad (5.22)$$

⁴Remember that in string theory the five-form flux $F_5 + H \wedge C_2$ is quantized (3.102):

$$\int_Y F_5 + H \wedge C_2 = (2\pi l_s)^4 g_s N .$$

Since $F_5 + H \wedge C_2 = dC_4$, the potential C_4 is an example of a necessarily non-globally-defined RR potential. This is true of course even in the Einstein case. The vanishing of the integral of the exact term in (5.18) implies that $e^{-(B-b_2)} F|_5$ satisfies the same quantization condition.

The functional can now be written as an integral over X_1 :

$$S_{\text{IIB}} = 6 \int_{X_1} r^6 dr d^5 y \sqrt{g_X} e^{4\Delta - \phi} \left(R_X - \frac{1}{2} |H|^2 - 16e^{-2\hat{\Delta}} \right. \\ \left. + 12|dA|^2 - 16dA \cdot d\phi + 4|d\phi|^2 - \frac{1}{2} e^{2\phi} |F|^2 \right), \quad (5.23)$$

where $|F|^2 = |F_1|^2 + |F_3|^2 + |F_5|^2$.

A general formula appeared in [62] (following [63]) for the combination $R_X - H^2/2$ of the Ricci scalar on X and the kinetic term of the H -flux. These are defined via the generalized metric (2.48) associated to a pair of compatible pure spinors Φ and Ψ with equal norms, $\|\Phi\|^2 = \|\Psi\|^2 \equiv e^{6A-2\phi}/8$ (as in (4.73)). In our notation (recall in particular the norms defined in (2.36)), the expression in [62] reads ⁵

$$R_X - \frac{1}{2} H^2 = 32e^{2\phi-6A} \left[|d\Phi|_B^2 + e^{2A} |d(e^{-A} \text{Re } \Psi)|_B^2 + e^{-2A} |d(e^A \text{Im } \Psi)|_B^2 \right. \\ \left. + 32 \left| \frac{\langle \Psi, d\Phi \rangle}{\text{vol}_X} \right|^2 + 32 \left| \frac{\langle \bar{\Psi}, d\Phi \rangle}{\text{vol}_X} \right|^2 \right] \\ + 28dA^2 + 4d\phi^2 - 20dA \cdot d\phi + 10\nabla^2 A - 4\nabla^2 \phi \\ + 4(d\phi - 2dA) \cdot (u_R^1 + u_R^2) - 2\nabla^m (u_R^1 + u_R^2)_m + 4[(u_R^1)^2 + (u_R^2)^2], \quad (5.24)$$

where the one-forms $u_R^{1,2} \equiv (u_m^{1,2} + u_m^{*1,2})dx^m$ on X can be expressed as

$$u_m^1 = \frac{\langle \gamma_m^B \bar{\Phi}, d\Phi \rangle}{2\langle \Phi, \bar{\Phi} \rangle} + e^A \frac{\langle \gamma_m^B \bar{\Psi}, d(e^{-A} \text{Re } \Psi) \rangle}{\langle \Psi, \bar{\Psi} \rangle}, \\ u_m^2 = \frac{\langle \bar{\Phi} \gamma_m^B, d\Phi \rangle}{2\langle \Phi, \bar{\Phi} \rangle} + e^A \frac{\langle \Psi \gamma_m^B, d(e^{-A} \text{Re } \Psi) \rangle}{\langle \Psi, \bar{\Psi} \rangle}. \quad (5.25)$$

Here we have defined (omitting the Clifford map slashes)

$$\gamma_m^B \Phi_k \equiv e^{-B} (\gamma_m e^B \Phi_k) = e^{-B} [(dx^m \wedge + g^{mn} \partial_n \lrcorner) e^B \Phi_k], \\ \Phi_k \gamma_m^B \equiv e^{-B} (e^B \Phi_k \gamma_m) = (-1)^k e^{-B} [(dx^m \wedge - g^{mn} \partial_n \lrcorner) e^B \Phi_k]. \quad (5.26)$$

Now recall that without imposing the Einstein condition (4.72) our pure spinors $\Omega_{\pm}$ do not have equal norms, but satisfy instead $\|\Omega_{-}\|^2 = e^f \|\Omega_{+}\|^2$. We thus choose $\Phi = e^{-f/2} \Omega_{-}$ and $\Psi = \Omega_{+}$. The pure spinor Φ is not closed, but nevertheless defines an integrable generalized almost complex structure since

$$d\Phi = -\frac{1}{2} df \wedge \Phi. \quad (5.27)$$

⁵There is a typographical error in (C.3) of [62]: the term $+22(dA)^2$ should read $+28(dA)^2$. We thank Luca Martucci for communications about this point.

Note that whenever Φ is integrable, the second line in (5.24) vanishes by compatibility.

When the differential constraints (4.1) on $\Omega_{\pm}$ are taken into account many terms cancel and we are left with

$$\begin{aligned} R_X - \frac{1}{2}H^2 &= -\frac{1}{2}e^{2\phi}|F|^2 + 28|dA|^2 + 4|d\phi|^2 - 20dA \cdot d\phi + 10\nabla^2 A - 4\nabla^2 \phi \\ &\quad + (4dA - 2d\phi) \cdot df + \nabla^2 f . \end{aligned} \quad (5.28)$$

As a check on this result, consider the case where Y is Sasakian, rather than generalized Sasakian, so that X is Kähler. In this case $\Delta = \phi = H = 0$, $F = 4\text{vol}_Y$, and this gives the correct result that $R_X = 20 + \nabla^2 f$, where f is the Ricci potential for the corresponding Kähler cone metric.⁶

In the expression for S_{IIB} in (5.23) the Ricci scalar is multiplied by $e^{4\hat{\Delta}-2\phi}$ and integrated over X_1 . The integration of $\nabla^2 f$ over r can be performed trivially since f is independent of r , and then integrating by parts we see that the two terms in the second line of (5.28) cancel each other. Similar cancellations also happen after integrating the Laplacians of A and ϕ by parts and we are left with

$$S_{\text{IIB}} = 6 \int_{X_1} r^6 dr d^5 y \sqrt{g_X} e^{4\Delta-\phi} \left(24e^{-2\hat{\Delta}} - e^{2\phi}|F|^2 \right) .$$

Using the expressions (4.63) for the RR fluxes and the generalized Sasakian conditions, we find

$$\begin{aligned} |F|^2 &= 16e^{8\Delta}e^{-10\hat{\Delta}} \\ &\quad + \frac{f_5^2 e^{-4\hat{\Delta}}}{4\text{vol}_X} d \log r \wedge d\psi \wedge d \left[\frac{e^{4\hat{\Delta}}}{4} (\omega_{\text{T}} \wedge \mathcal{H}_{e^{-4\hat{\Delta}}} \lrcorner b_2 - \mathcal{H}_h \lrcorner (\omega_{\text{T}} \wedge \mathcal{L}_{\mathcal{H}_h} \omega_{\text{T}})) \right] . \end{aligned} \quad (5.29)$$

The second term produces an exact term in S_{IIB} , which vanishes on using Stokes' theorem when integrated over Y . In fact this step, although correct, is a little cavalier: notice that the above formula is really valid only on the dense open set $Y_0 \subset Y$ where the integrable structure is of type one. Thus strictly speaking we end up with an integral over an infinitesimal boundary around the type-change locus after applying Stokes' theorem. One can then check that the integrand is smooth as one approaches

⁶The factor of 20 arises here because R_X is the Ricci scalar not of the Kähler cone metric, but rather of the corresponding cylindrical metric that is related to it by a conformal factor of r^2 .

the type-change locus and thus this integral is indeed zero. To see this, we note that the three-form in square brackets in (5.29) may be rewritten as

$$\frac{e^{4\hat{\Delta}}}{4} \left(\omega_T \wedge \frac{4}{f_5} F_1 - (\mathcal{H}_h \lrcorner \omega_T) \wedge \mathcal{L}_{\mathcal{H}_h} \omega_T \right). \quad (5.30)$$

From the form of Ω_+ given in Proposition 2 d), which recall is a global polyform on X , we see that $e^{\hat{\Delta}}$ and $\mathcal{H}_h \lrcorner \omega_T$ are in fact everywhere smooth on Y . Moreover, ω_T lifts to a global smooth two-form on Y , since it is $d\sigma/2$ with σ the contact one-form. This demonstrates that the above three-form is in fact a smooth three-form on Y , not just on Y_0 . On the other hand, the function h itself certainly diverges along the type-change locus.

We thus finally obtain that for generalized Sasakian manifolds Y , the action functional is proportional to the contact volume:

$$S_{\text{IIB}}|_{\text{gen. Sasakian}} = 8 \int_Y d^5 y \sqrt{g_Y} e^{8\Delta} = -\frac{f_5^2}{16} \int_Y \sigma \wedge d\sigma \wedge d\sigma. \quad (5.31)$$

This allows us to define a functional Z which is the action S_{IIB} restricted to a space of generalized Sasakian manifolds, normalized such that it gives exactly the contact volume of Y divided by the volume of the round metric on S^5 :

$$\begin{aligned} Z &\equiv -\frac{2}{f_5^2 \pi^3} S_{\text{IIB}}|_{\text{gen. Sasakian}} = -\frac{16}{f_5^2 \pi^3} \int_Y d^5 y \sqrt{g_Y} e^{8\Delta} \\ &= \frac{1}{(2\pi)^3} \int_Y \sigma \wedge d\sigma \wedge d\sigma = \frac{1}{\pi^3} \int_Y \sigma \wedge \frac{\omega_T^2}{2!}. \end{aligned} \quad (5.32)$$

Defining the *contact volume* of a $(2n-1)$ -dimensional manifold Y_{2n-1} whose transverse space carries a symplectic form $\omega_T = d\sigma/2$ by

$$\text{Vol}_\sigma(Y_{2n-1}) \equiv \int_{Y_{2n-1}} \sigma \wedge \frac{\omega_T^{n-1}}{(n-1)!}, \quad (5.33)$$

we can simply write

$$Z = \frac{\text{Vol}_\sigma(Y)}{\text{Vol}(S^5)}. \quad (5.34)$$

Note that in the case of Sasakian manifolds, for which the warp factor vanishes, $\Delta = 0$, the notion of contact volume coincides with the ordinary notion of Riemannian volume, so for instance $\text{Vol}_\sigma(S^5) = \text{Vol}(S^5)$.

5.3 Volume minimization: summary

We are now in a position to outline the procedure of volume minimization for generalized Sasakian manifolds [4].

In the previous section we have shown that if we restrict the action (5.12) for a space of supergravity fields on Y to generalized Sasakian structures, it becomes precisely the *contact volume* Z (5.34). The contact volume, in turn, depends only on the Reeb vector field ξ , that is $Z = Z(\xi)$. A general proof of this statement, which supersedes the proofs in [19], may be found in appendix D. The Reeb vector field ξ for which a generalized Sasakian manifold is also Einstein is then a critical point of the contact volume Z over the Reeb vector fields of a space of generalized Sasakian structures. As also shown in appendix D, Z is strictly convex, and thus such a critical point is necessarily a *minimum*. Provided we work within a deformation class of generalized Sasakian structures, implying that the space of Reeb vector fields we are minimizing over is path-connected, then this minimum will be a global minimum. Clearly, all these statements generalize the results of [19] to general supersymmetric AdS_5 solutions of type IIB string theory (with the only constraint that the background has non-zero D3-brane charge, $f_5 \neq 0$).

The main technical difference to the Sasakian case, which is currently also a deficiency, is that we do not yet have a good understanding of the space of generalized Sasakian structures, and thus the corresponding space of Reeb vector fields over which we are to vary Z . In the next chapter we shall make some reasonable assumptions about this, based on physical arguments in some particular examples, and show that the geometric result above indeed agrees with the field theory *a*-maximization computation (see next section). It should be noted, however, that even in Sasakian geometry there is currently no general understanding of the deformation space. In fact a global picture may not even be necessary, depending on what one wants to show. For example, one of the motivations for [19] was to prove that the on-shell Z is an *algebraic number*, since this is a definite prediction of *a*-maximization in field theory. As pointed out in [2], this follows in the general case for *quasi-regular* Reeb vector fields, which by definition generate a $U(1)$ action on Y , since then the on-shell Z is a rational number, again as expected from field theory. What about irregular critical Reeb vector fields? Since the Reeb vector field also generates an

isometry, and the isometry group of Y is necessarily compact, it follows that such a Reeb vector field lies in the Lie algebra $\mathfrak{t}$ of some torus $\mathbb{T}$ of rank at least two that acts isometrically on Y . If we assume that there is at least a one-parameter family of deformations of generalized Sasakian structures away from such a critical point, with Reeb vector fields defining a curve in $\mathfrak{t}$, then the Duistermaat-Heckman formula for the contact volume in [19] implies that the critical Reeb vector field ξ_* is algebraic, and hence that $Z(\xi_*)$ is also algebraic, as desired. To see this, one notes that there is then always a nearby generalized Sasakian structure with Reeb vector field ξ_0 that is quasi-regular, and thus one can apply the Duistermaat-Heckman formula to the total space of the associated complex line bundle over the orbifold $Y/U(1)_0$, where ξ_0 defines the action of $U(1)_0$ on Y . This formula is then a rational function of the Reeb vector field with rational coefficients determined by certain Chern classes and weights, and thus setting its derivative to zero will give polynomial equations for ξ_* with rational coefficients. We refer to [19] for the details.

In fact the only case over which there is complete control is the case of *toric* Sasakian structures. In this setting the paper [20] provides a complete description. It is worth contrasting this situation with the corresponding case in generalized geometry. Thus, as in [2], we define a *toric* generalized Sasakian manifold to be a generalized Sasakian manifold for which the symplectic structure on the cone is invariant under $\mathbb{T} \cong U(1)^3$. We also assume that the corresponding Reeb vector field lies in the Lie algebra of this torus.⁷ Notice that this does not imply that the whole structure is invariant under $U(1)^3$ (for example, the Pilch-Warner solution discussed in section 3.6 is a non-trivial solution with fluxes which is toric in this sense, but for which only a $U(1)^2 = U(1) \times U(1)_R$ subgroup preserves the fluxes). In any case, in this setting there is a moment map μ under which the image of the cone X is a strictly convex rational polyhedral cone $\mathcal{C}^* \subset \mathfrak{t}^* \cong \mathbb{R}^3$. This is a set of the form

$$\mathcal{C}^* = \{y \in \mathfrak{t}^* \mid \langle y, v_a \rangle \geq 0, a = 1, \dots, d\} \subset \mathbb{R}^3, \quad (5.35)$$

where the integer vectors $v_a \in \mathbb{Z}^3$, $a = 1, \dots, d$, are the inward normal vectors to the $d \geq 3$ faces of the polyhedral cone $\mathcal{C}^*$. The Reeb vector field ξ then defines a hyperplane $\{\langle y, \xi \rangle = 1/2\}$ in $\mathbb{R}^3$ that cuts $\mathcal{C}^*$ in a compact convex two-dimensional

⁷The cases where this is not true form a finite and uninteresting list [64].

polytope, and the contact volume is simply the Euclidean volume of this polytope, as a function of ξ . Thus the minimization problem we want to solve involves minimizing this volume over an appropriate space of Reeb vector fields ξ . As explained in [21], ξ lies necessarily in the interior $\mathcal{C}_{\text{Int}}$ of the dual polyhedral cone $\mathcal{C} \subset \mathfrak{t}$ since $\mu(\xi) = r^2/2$. However, in the Sasakian case, the condition that the holomorphic volume form Ω has charge three then further restricts ξ to lie in the intersection of $\mathcal{C}_{\text{Int}}$ with a hyperplane. This then leads to a well-defined volume minimization problem, with a unique (finite) critical point ξ_* .

In the toric generalized setting, almost everything said above remains true. Thus a toric generalized Sasaki-Einstein solution is similarly obtained by minimizing the same two-dimensional polytope volume that appears above. The difference is that the space of Reeb vector fields over which one minimizes is in general more constrained. This is related to the fact that it is now the closed pure spinor Ω_- on the cone that is required to have charge three under the Reeb vector field, as part of our definition of generalized Sasakian, so $\mathcal{L}_{r\partial_r}\Omega_- = 3\Omega_-$, or equivalently $\mathcal{L}_\xi\Omega_- = -3i\Omega_-$. Since Ω_- is in general a polyform, the minimization is usually going to be over a smaller space.

We shall see some examples of precisely this in chapter 6. Although the generalized geometry in these examples is not under good control, fortunately the *physical* interpretation is, and this allows us to determine the constraints on the Reeb vector field and apply volume minimization. But even simpler examples are provided by the β -transforms, to which we now turn.

5.3.1 Example: β -transform of $\mathbb{C}^3$

For concreteness, let us return to the class of generalized Sasakian manifolds presented in section 4.5. Recall that this arises from a family of generalized Kähler cone structures on $\mathbb{C}^3$ with Reeb vector fields in $\mathbb{R}_+^3$. In fact these are toric, in the above sense, and here $\mathbb{R}_+^3 = \mathcal{C}_{\text{Int}}$. We can then calculate the contact volume as a function of the Reeb vector field ξ :

$$Z = \frac{1}{8\pi^3} \int_Y \sigma \wedge d\sigma \wedge d\sigma = \frac{1}{\xi_1 \xi_2 \xi_3} . \quad (5.36)$$

The homogeneity condition $\mathcal{L}_{r\partial_r}\Omega_- = 3\Omega_-$ imposes that the components of the Reeb vector field $\xi = \sum_i \xi_i \partial_{\phi_i}$ satisfy

$$\xi_1 + \xi_2 + \xi_3 = 3 . \quad (5.37)$$

Notice that everything is independent of the parameter γ appearing in the bivector β . We see immediately that Z is minimized for $\xi_1 = \xi_2 = \xi_3 = 1$, at which point $Z = 1$ so that the contact volume of the generalized Sasaki-Einstein manifold Y is equal to the volume of the five-sphere, $\text{Vol}_\sigma(Y) = \text{Vol}(S^5) = \pi^3$. Given the definition (4.87) of the function f , this then indeed corresponds to the Einstein condition $f = 0$. We have thus reproduced the result that the β -transform of $\mathbb{C}^3$ does not change the supergravity central charge [36]. Of course this is physically fairly obvious, since it corresponds to a marginal “beta deformation” [54], but the important point is that we have reproduced this in a non-trivial way using generalized geometry.

The above result presumably extends to general β -transforms of toric Kähler cones, which could be treated as in [26, 27]. As explained above, the minimization problem involves the volume function of precisely the same polytope.

5.4 Relation to a -maximization

As mentioned in the introduction, volume minimization corresponds to a -maximization in the dual $\mathcal{N} = 1$ SCFT. The equivalence of the two procedures has been *proven* for the case of toric Sasakian manifolds in [23], and in a very interesting and recent paper for non-toric Sasakian manifolds as well [24]. In this section we briefly review the relation, and make a more general conjecture.

In [2] it was shown that for a general solution of type IIB supergravity of the form $\text{AdS}_5 \times Y$, with Y a generalized Sasaki-Einstein manifold, the contact volume of Y is related to the central charge a of the dual SCFT by the simple formula

$$\frac{\text{Vol}_\sigma(Y)}{\text{Vol}(S^5)} = \frac{a_{\mathcal{N}=4}}{a} , \quad (5.38)$$

where $a_{\mathcal{N}=4} = N^2/4$ is the central charge for $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$ at large N . Moreover, it was shown in [3] that the Reeb vector field corresponds to the R-symmetry of the dual $\mathcal{N} = 1$ SCFT.

Just as the contact volume is determined by the Reeb vector field, so the central charge a is completely determined by the R-symmetry through [17, 65]

$$a = \frac{3}{32} (3\text{Tr } R^3 - \text{Tr } R) . \quad (5.39)$$

Here the trace is over the fermions in the theory. More precisely, one typically computes this quantity in a UV theory that has a Lagrangian description and is believed to flow to an interacting superconformal fixed point in the IR, and then uses 't Hooft anomaly matching. For some time a major problem was identifying the correct global symmetry in such a UV description that becomes the R-symmetry in the IR. This was solved by Intriligator and Wecht in the beautiful paper [22]. The result is that, among the set of potential R-symmetries that are free of ABJ anomalies, the correct R-symmetry is that which (locally) maximizes the central charge. That is, one maximizes the trial central charge function over all admissible R-symmetries:

$$a_{\text{trial}} = \frac{3}{32} (3\text{Tr } R_{\text{trial}}^3 - \text{Tr } R_{\text{trial}}) . \quad (5.40)$$

Of course, this immediately resembles Z -minimization, where one varies the contact volume as a function of the Reeb vector field. Indeed, even the condition that the superpotential has R-charge two is analogous to the condition that Ω has scaling dimension three: both are immediate consequences of the supersymmetry parameters having a canonical (non-zero) R-charge.

But in general even the dimensions of the spaces of trial R-charges and trial Reeb vector fields are different. However, in [23] it was shown in the *toric Sasakian* case that one can effectively perform the field theory a -maximization in two steps, the first step resolving the mixing with global baryonic symmetries. The upshot of this is that one obtains trial R-charges which are then functions of the Reeb vector field; that is, the field theory trial R-charges satisfy the well-established AdS/CFT formula [66, 67, 68, 69]

$$R(\Phi) = \frac{\pi \text{Vol}(\Sigma_3)}{3 \text{Vol}(Y_{\text{SE}})} . \quad (5.41)$$

Here Φ is a chiral matter field which is “dual” to a supersymmetric three-subspace $\Sigma_3 \subset Y$, and the Riemannian volumes agree, in this Sasakian case, with the contact volumes, which are thus functions of the trial Reeb vector field. More geometrically, in the Abelian mesonic moduli space, $\Phi = 0$ defines a conical divisor in X , which is then a cone over Σ_3 . It is a non-trivial and striking fact that the trial R-charges defined this way satisfy the field theory anomaly cancellation conditions, for *any* choice of trial Reeb vector field. The authors of [23] then proved that

$$Z(\xi) = \frac{a_{\mathcal{N}=4}}{a_{\text{trial}}} , \quad (5.42)$$

holds as a relation between *functions*, with the right hand side understood also as a function of the Reeb vector field, as described above.

It is then natural to conjecture that the relation (5.42) still holds for generalized Sasakian manifolds. Of course, in general there would also be some analogue of the baryonic mixing to resolve in the dual field theory. However, in the examples we shall study in the next chapter there is no such mixing as there are no baryonic symmetries, and the functions will agree on the nose.⁸ We also note that, although the Abelian mesonic moduli space in the field theory is only a subspace of X in general (namely the type-change locus of Ω_-), it is nevertheless still true in examples that one can match chiral matter fields Φ with supersymmetric three-subspaces Σ_3 , and that (5.41) still holds, but now in terms of contact volumes (recalling that $\Delta = 3R/2$ for chiral primaries, we see this is in fact (3.121)). This was verified for the explicit Pilch-Warner solution in section 3.6, and we shall see it is also true of the new examples in the next chapter.

⁸It is a straightforward exercise to check that this is also the case in the β -transform example, but this is somewhat trivial.

Chapter 6

Massive deformation of generalized conifolds

In this chapter we present new examples of superconformal field theories whose dual geometries are generalized Sasaki-Einstein. They are obtained by massive deformations of quiver gauge theories describing the worldvolume theories of a stack of D3-branes located at so-called “generalized conifold” singularities [70]. The simplest such example is the suspended pinch point singularity [7], but this generalizes to an infinite family of generalized conifolds which are cones over the $L^{m,n,m}$ Sasaki-Einstein orbifolds [12, 37]. The massive deformation induces an renormalization group flow, and the field theory analysis suggests that these theories flow to interacting superconformal fixed points in the infrared [54]. The Abelian mesonic moduli spaces of the corresponding SCFTs are not (N symmetrized copies of) the original Calabi-Yau singularities, but rather only a subspace. Given the identification [35, 26] between the mesonic moduli space and the type-change locus $\mathcal{T}$ of Ω_- in X , this means that the dual supergravity solutions are indeed necessarily *generalized* Sasaki-Einstein. Notice that these theories must have a dual AdS₅ type IIB description, since they have been obtained by deformation of a Sasaki-Einstein background of type IIB. Although we do not know the supergravity solutions explicitly, we will show that with some reasonable assumptions about their geometry, we have enough information to perform the generalized Z -minimization described in the previous chapter, and hence compute geometrically the central charge of the dual SCFTs and the conformal dimensions of certain chiral primary operators. We then show that these agree with the dual field theory a -maximization computations, and moreover that the quantities even agree off

shell, as we conjectured in (5.42).

6.1 Massive deformation of super Yang-Mills theory

Before considering massive deformations of generalized conifolds, we start by looking at a simple well-known example in order to acquire some geometric intuition.

One way to deform $\mathcal{N} = 4$ super Yang-Mills theory is by giving a mass to one of its three chiral superfields Φ_i with $i = 1, 2, 3$ (in $\mathcal{N} = 1$ language), which are all in the adjoint representation of $SU(N)$. The corresponding superpotential deformation is thus ¹

$$W_{m\text{SYM}} = \Phi_1[\Phi_2, \Phi_3] + \frac{m_1}{2}\Phi_1^2. \quad (6.1)$$

The resulting theory flows to an infrared fixed point with $\mathcal{N} = 1$ supersymmetry, as argued by Leigh and Strassler [54]. After integrating out the massive field Φ_1 by putting it on shell, $\Phi_1 = -[\Phi_2, \Phi_3]/m_1$, we obtain a quartic superpotential:

$$W_{m\text{SYM}} = \lambda_1[\Phi_2, \Phi_3]^2, \quad (6.2)$$

with $\lambda_1 = -1/(2m_1)$. The requirement that the superpotential has R-charge two gives, with the notation R_i for the R-charge of the chiral superfield Φ_i ,

$$R_1 = R_2 + R_3 = 1. \quad (6.3)$$

The ABJ anomaly for the R-symmetry then vanishes automatically. The trial central charge is

$$a_{\text{trial}} = \frac{27N^2}{32}R_2R_3. \quad (6.4)$$

A local maximum is obtained for $R_2 = R_3 = 1/2$, which gives

$$\frac{a_{\mathcal{N}=4}}{a_{m\text{SYM}}} = \frac{32}{27}. \quad (6.5)$$

Of course, in this example a -maximization is somewhat redundant, since the global $SU(2)$ symmetry at the fixed point requires in any case that $R_2 = R_3$.

¹An overall trace is always implicit in these formulas.

The dual geometry is the Pilch-Warner solution [52, 53] already discussed in section 3.6. It involves a non-trivial metric (3.123) on S^5 , as well as three- and five-form fluxes. It follows that topologically $X = C(S^5) \cong \mathbb{R}^6$. Although the solution is generalized complex, rather than complex, it is nevertheless convenient to write it in terms of complex coordinates on $\mathbb{R}^6 \cong \mathbb{C}^3$. This structure is essentially inherited from that of the original solution before mass deformation, which is $\mathbb{C}^3$ with its flat Calabi-Yau metric. The complex coordinates z_i , effectively get rescaled (as the R-symmetry changes), and in polar coordinates they have weights ξ_i and are given by

$$\begin{aligned} z_1 &= r^{\xi_1} \sin \vartheta e^{i\phi_1} , & \xi_1 &= 3/2 , \\ z_2 &= r^{\xi_2} \cos \vartheta \cos \frac{\alpha}{2} e^{i\phi_2} , & \xi_2 &= 3/4 , \\ z_3 &= r^{\xi_3} \cos \vartheta \sin \frac{\alpha}{2} e^{i\phi_3} , & \xi_3 &= 3/4 . \end{aligned}$$

The closed pure spinor Ω_- is given by

$$\Omega_- = \sqrt{3} \frac{f_5}{96} d\bar{z}_1^2 \wedge e^{-b_- + i\omega_-} , \quad (6.6)$$

with the rather complicated expression

$$\begin{aligned} -b_- + i\omega_- &= -2i\sqrt{\frac{2f_5}{3}} \frac{1}{3r^3(r^3 + |z_1|^2)} \left[-\frac{r^3(r^3 + |z_1|^2)}{\bar{z}_1} d\bar{z}_2 d\bar{z}_3 \right. \\ &\quad - z_1^2 \bar{z}_1 d\bar{z}_2 d\bar{z}_3 + \frac{\bar{z}_1^2}{2} (2z_1 dz_2 dz_3 - z_3 dz_2 dz_1 + z_2 dz_3 dz_1) \\ &\quad + r^{3/2} \left(\frac{1}{2} (z_1 \bar{z}_2^2 - \bar{z}_1 z_3^2) dz_2 d\bar{z}_3 - \frac{1}{2} (z_1 \bar{z}_3^2 - \bar{z}_1 z_2^2) dz_3 d\bar{z}_2 \right. \\ &\quad \left. + \frac{1}{4} (|z_2|^2 + |z_3|^2) dz_1 (\bar{z}_3 d\bar{z}_2 - \bar{z}_2 d\bar{z}_3) \right. \\ &\quad \left. \left. - \frac{1}{2} (\bar{z}_1 z_2 z_3 + z_1 \bar{z}_2 \bar{z}_3) (dz_2 d\bar{z}_2 - dz_3 d\bar{z}_3) \right) \right] . \quad (6.7) \end{aligned}$$

Notice that $z \propto \bar{z}_1^2$ corresponds to the superpotential $W_{m\text{SYM}}$ in (6.2), provided we identify the complex coordinate z_1 with the scalar component of the chiral superfield $\Phi_1 = -[\Phi_2, \Phi_3]/m_1$. Indeed, this is generally expected from the observation that the condition $dz = 0$ reproduces the F-term equations of the theory on the worldvolume of a probe D3-brane [35, 26] (see subsection 3.2.1). Thus the type-change locus $\mathcal{T} = \{dz = 0\}$ of the pure spinor Ω_- always corresponds to the mesonic moduli space of the SCFT. Here $\mathcal{T} = \{z_1 = 0\}$ is a copy of $\mathbb{C}^2 \subset \mathbb{R}^6$, on which Ω_- reduces to a three-form

$$\Omega_-|_{\mathcal{T}} = i \frac{\sqrt{2} f_5^{3/2}}{72} d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 . \quad (6.8)$$

Notice that in such expressions we do not mean a pullback to $\mathcal{T}$, but rather a restriction of the bundle of forms to $\mathcal{T}$.

After shifting the exponent by a suitable two-form proportional to $d\bar{z}_1$ to put Ω_- in the generalized Darboux form, we obtain

$$\omega_0 = \frac{1}{3} \sqrt{\frac{f_5}{6}} \left[\frac{1}{\bar{z}_1^2} (2\bar{z}_1 d\bar{z}_2 \wedge d\bar{z}_3 - \bar{z}_2 d\bar{z}_1 \wedge d\bar{z}_3 + \bar{z}_3 d\bar{z}_1 \wedge d\bar{z}_2) + \text{c.c.} \right], \quad (6.9)$$

while the expression for b_0 is rather complicated and we thus omit it. The symplectic form on $X \cong \mathbb{R}^6$ is

$$\omega = \frac{1}{2} \sum_i dr_i^2 \wedge \frac{d\phi_i}{\xi_i}, \quad (6.10)$$

with $r_1 = r \sin \vartheta$, $r_2 = r \cos \vartheta \cos(\alpha/2)$, $r_3 = r \cos \vartheta \sin(\alpha/2)$. We have explicitly verified that all the conditions enunciated in subsection 4.4.3 for a generalized Sasaki-Einstein solution are indeed satisfied by this Pilch-Warner solution, which is thus also a further check on our equations.

Of course, in this case we know the explicit solution and hence the Reeb vector field. However, we may now show how to recover some of these results *without* using the full solution, which is in fact quite complicated. The key observation is that (6.10) describes the standard symplectic structure on $\mathbb{R}^6$, as observed in [3]. In order to perform Z -minimization, we let the Reeb vector field $\xi = \sum_i \xi_i \partial_{\phi_i} \in \mathbb{R}_+^3$ be arbitrary in the expression (6.10) for the symplectic form, which then leads to the contact volume

$$Z(\xi) = \frac{1}{\xi_1 \xi_2 \xi_3}. \quad (6.11)$$

Note that this is the same contact volume function (5.36) as for the β -transform of Kähler cones on $\mathbb{C}^3$, since in both cases the symplectic structure on $\mathbb{R}^6 \cong \mathbb{C}^3$ is the standard one. The generalized holomorphy condition $\mathcal{L}_\xi \Omega_- = -3i\Omega_-$ gives constraints on the Reeb vector field. In particular, the three-form condition gives $\xi_1 + \xi_2 + \xi_3 = 3$, which is easily deduced by looking at the homogeneity of $\Omega_-|_{\mathcal{T}}$ in (6.8), while, in contrast to the β -transform example, the one-form condition $\mathcal{L}_\xi d\bar{z}_1^2 = -3i d\bar{z}_1^2$ gives the *additional* condition $\xi_1 = 3/2$, as does the five-form condition. We thus have the constraints

$$\xi_1 + \xi_2 + \xi_3 = 3, \quad \xi_1 = \frac{3}{2}. \quad (6.12)$$

Minimizing Z under these constraints indeed gives the correct Reeb vector field $\xi_* = (3/2, 3/4, 3/4)$. Using the relation between the Reeb vector field and the R-charges, $\xi_i/3 = R_i/2$, we see that the conditions (6.12) and (6.3) match and that the conjecture (5.42) indeed holds:

$$Z = \frac{a_{\mathcal{N}=4}}{a_{\text{trial}}} . \quad (6.13)$$

6.2 Suspended pinch point

Before turning on massive deformations, we first review the gauge theory on N D3-branes probing the suspended pinch point singularity [7].

The suspended pinch point (SPP) is a non-isolated hypersurface singularity given by

$$X_{\text{SPP}} = \{u^2v = wz\} \subset \mathbb{C}^4 , \quad (6.14)$$

where u, v, w, z are complex coordinates on $\mathbb{C}^4$. All such hypersurface singularities are Calabi-Yau (or, more precisely, *Gorenstein*), in the sense that they admit a nowhere zero holomorphic $(3, 0)$ -form Ω on the locus of smooth points. This particular singularity is also toric, meaning that there is a holomorphic action of $\mathbb{T}_{\mathbb{C}} = (\mathbb{C}^*)^3$ with a dense open orbit. It may thus be rewritten in the language of toric geometry, reviewed very briefly in section 5.3 (we also refer the reader to [20], where the suspended point singularity is discussed in further detail). In particular, the image of X_{SPP} under the moment map for *any* choice of toric Kähler metric on X_{SPP} is given by a polyhedral cone $\mathcal{C}^*$ in $\mathbb{R}^3$ of the form (5.35), where the inward-pointing normal vectors are

$$\begin{aligned} v_0 &= (1, 1, 0) , & v_1 &= (1, 2, 0) , & v_2 &= (1, 1, 1) , \\ v_3 &= (1, 0, 1) , & v_4 &= (1, 0, 0) . \end{aligned} \quad (6.15)$$

Here we have used the fact that for any toric Gorenstein singularity one can conveniently set the first component of the normal vectors to 1 by an appropriate $SL(3; \mathbb{Z})$ transformation of the torus. They are thus of the form $v_a = (1, w_a)$, where $w_a \in \mathbb{Z}^2$.

Figure 6.1 shows the *toric diagram*, which is the convex hull of the $\{w_a\}$ in $\mathbb{R}^2$, or equivalently is the projection of the *dual cone* $\mathcal{C}$ to the plane $e_1 = 1$. The four external

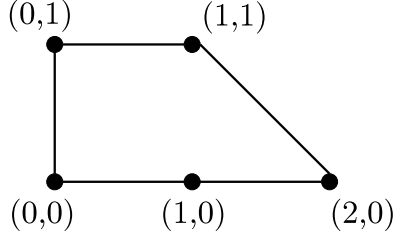


Figure 6.1: Toric diagram for the suspended pinch point. The four external vertices correspond to toric divisors $D_\alpha = C(Y_\alpha)$, with $\alpha = 1, 2, 3, 4$.

vertices correspond to four torus-invariant divisors $D_\alpha = C(\Sigma_\alpha)$, $\alpha = 1, 2, 3, 4$, which are cones over three-subspaces $\Sigma_\alpha \subset Y_{\text{SPP}}$. It is the additional vertex point $w_0 = (1, 0)$ on the interior of an external edge that signifies that X_{SPP} is not an isolated singularity (in fact there is an A_1 singularity running out of $u = v = w = z = 0$, at every non-zero value of v). The relation between the toric and algebraic descriptions is obtained as usual by noting that the normal vectors satisfy $\sum_{\alpha=1}^4 Q_\alpha v_\alpha = 0$, with the $U(1)_B$ charge vector $Q = (-1, 2, -2, 1)$. We may then associate complex coordinates Z_α to each divisor $C(\Sigma_\alpha)$, in terms of which we construct $U(1)_B$ -invariant monomials as

$$u = Z_1 Z_4, \quad v = Z_2 Z_3, \quad w = Z_1^2 Z_2, \quad z = Z_3 Z_4^2. \quad (6.16)$$

These generate *all* such invariants, and satisfy our original algebraic equation $u^2 v = w z$. Indeed, being holomorphic functions on X_{SPP} of definite charge under the torus action, they define lattice points inside the cone $\mathcal{C}^*$, and then precisely generate its lattice points over $\mathbb{Z}_{\geq 0}$. Thus with this interpretation we also have

$$u = (1, 0, -1), \quad v = (0, 0, 1), \quad w = (0, 1, 0), \quad z = (2, -1, -1), \quad (6.17)$$

being the generators of $\mathcal{C}^*$. We shall need these formulas later.

It was only recently that an explicit Calabi-Yau cone metric was constructed on X_{SPP} [12, 37]. Indeed the corresponding Sasaki-Einstein orbifold metric on Y_{SPP} is one of these $L^{p,q,r}$ spaces, namely $L^{1,2,1}$. However, before this metric was known (and in fact known to exist), the Reeb vector field ξ and hence volumes of Y_{SPP} and its supersymmetric toric subspaces Σ_α were computed using volume minimization in the

original paper [20]. These are given by

$$\text{Vol}(Y) = \frac{\pi}{2\xi_1} \sum_{\alpha} \text{Vol}(\Sigma_{\alpha}) , \quad (6.18)$$

$$\text{Vol}(\Sigma_{\alpha}) = 2\pi^2 \frac{(v_{\alpha-1}, v_{\alpha}, v_{\alpha+1})}{(\xi, v_{\alpha-1}, v_{\alpha})(\xi, v_{\alpha}, v_{\alpha+1})} , \quad (6.19)$$

where (u, v, w) denotes the determinant of the 3×3 matrix whose rows are u , v , and w . With the choice of normal vectors in Figure 6.1 we obtain

$$\begin{aligned} \text{Vol}(\Sigma_1) &= \frac{2\pi^2}{\xi_3(2\xi_1 - \xi_2 - \xi_3)} , & \text{Vol}(\Sigma_4) &= \frac{2\pi^2}{\xi_2\xi_3} , \\ \text{Vol}(\Sigma_2) &= \frac{2\pi^2}{(\xi_1 - \xi_3)(2\xi_1 - \xi_2 - \xi_3)} , & \text{Vol}(\Sigma_3) &= \frac{2\pi^2}{\xi_2(\xi_1 - \xi_3)} , \end{aligned} \quad (6.20)$$

$$Z = \frac{2\xi_1 - \xi_3}{8\xi_2\xi_3(\xi_1 - \xi_3)(2\xi_1 - \xi_2 - \xi_3)} . \quad (6.21)$$

In the basis in which the normal vectors to $\mathcal{C}^*$ all have their first component equal to one, the holomorphic three-form Ω satisfies $\mathcal{L}_{\partial/\partial\phi_1}\Omega = i\Omega$ and $\mathcal{L}_{\partial/\partial\phi_{2,3}}\Omega = 0$, so the homogeneity condition requires the first component of the Reeb vector field ξ to be equal to three [20]:

$$\xi_1 = 3 . \quad (6.22)$$

Then it is straightforward to check that Z has a (global) minimum for

$$\xi_* = \left(3, \frac{3 + \sqrt{3}}{2}, 3 - \sqrt{3} \right) . \quad (6.23)$$

Recall that in the Sasaki-Einstein case the contact volume Z reduces to the Riemannian volume of Y , relative to that of the round metric on S^5 , and so we have

$$\text{Vol}(Y_{\text{SPP}}) = \frac{2\pi^3}{3\sqrt{3}} , \quad \text{Vol}(\Sigma_{1,4}) = \frac{2\pi^2}{3} , \quad \text{Vol}(\Sigma_{2,3}) = \frac{4\pi^2}{3 + 3\sqrt{3}} . \quad (6.24)$$

The gauge theory on N D3-branes at such a singularity was first studied by Morrison and Plesser [7] and Uranga [70]. This is of quiver form, with the *quiver diagram* shown in Figure 6.2. Here the three nodes represent three $U(N)$ gauge groups, and the arrows represent bifundamental chiral superfields. More precisely, a field Φ_{ij} connecting the i^{th} node to the j^{th} node is in the fundamental representation of $U(N)_i$ and the anti-fundamental of $U(N)_j$; the field Φ_{33} is in the adjoint representation of $U(N)_3$. The superpotential is

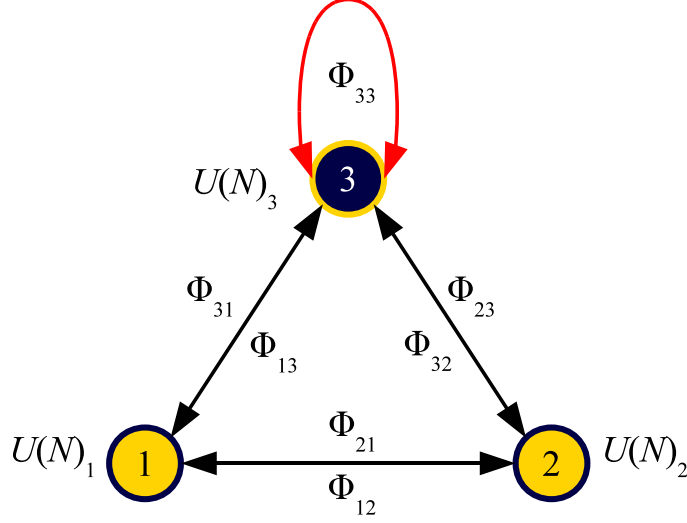


Figure 6.2: Quiver diagram for the gauge theory on N D3-branes probing the suspended pinch point. There are three $U(N)_i$ gauge groups, with six bifundamental fields Φ_{ij} and one adjoint field Φ_{33} .

$$W_{\text{SPP}} = \Phi_{12}\Phi_{21}\Phi_{13}\Phi_{31} - \Phi_{23}\Phi_{32}\Phi_{21}\Phi_{12} + \Phi_{33}(\Phi_{32}\Phi_{23} - \Phi_{31}\Phi_{13}) . \quad (6.25)$$

Focusing on the Abelian theory with $N = 1$, we find the following F-term and D-term conditions:

$$\begin{aligned} \Phi_{23}\Phi_{32} &= \Phi_{13}\Phi_{31} , & \Phi_{33} &= \Phi_{12}\Phi_{21} , \\ |\Phi_{21}|^2 - |\Phi_{12}|^2 + |\Phi_{31}|^2 - |\Phi_{13}|^2 &= 0 , & U(1)_1 , \\ |\Phi_{21}|^2 - |\Phi_{12}|^2 + |\Phi_{23}|^2 - |\Phi_{32}|^2 &= 0 , & U(1)_2 . \end{aligned} \quad (6.26)$$

Notice here that we have neglected the branch of solutions to the F-term equations in which $\Phi_{23} = \Phi_{32} = \Phi_{13} = \Phi_{31} = 0$, for which then Φ_{33} , Φ_{12} and Φ_{21} are left unconstrained by the F-terms; imposing also the D-terms on this branch leads to a copy of $\mathbb{C}^2$, which exists precisely because the singularity is not isolated and fractional branes can move along the residual singularity. Ignoring this branch, we can construct the following $U(1)_{1,2}$ -invariant monomials in the fields, which then generate the top-dimensional irreducible component of the mesonic moduli space:

$$\begin{aligned} u &= \Phi_{23}\Phi_{32} = \Phi_{13}\Phi_{31} , & v &= \Phi_{33} = \Phi_{12}\Phi_{21} , \\ w &= \Phi_{13}\Phi_{32}\Phi_{21} , & z &= \Phi_{12}\Phi_{23}\Phi_{31} . \end{aligned} \quad (6.27)$$

We see that these indeed satisfy the suspended pinch point hypersurface relation $u^2v = wz$.

By comparing the expressions for u, v, w, z in terms of the coordinates Z_α associated with the three-subspaces Σ_α (6.16), and in terms of the gauge theory fields Φ_{ij} (6.27), we deduce that the vanishing locus of a field Φ_{ij} is associated with a divisor $D_\alpha = C(\Sigma_\alpha)$ as in Table 6.1.

3-subspace	Fields	Q_B	R-charge
Σ_1	Φ_{32}, Φ_{13}	-1	$1/\sqrt{3}$
Σ_2	Φ_{21}	2	$1 - 1/\sqrt{3}$
Σ_3	Φ_{12}	-2	$1 - 1/\sqrt{3}$
Σ_4	Φ_{31}, Φ_{23}	1	$1/\sqrt{3}$
$\Sigma_2 \cup \Sigma_3$	Φ_{33}	0	$2 - 2/\sqrt{3}$

Table 6.1: Divisors $D_\alpha = C(\Sigma_\alpha)$, fields, and charges for the SPP theory.

We now perform a -maximization for the superconformal fixed point of this theory, at large N . The requirement that the superpotential has R-charge two gives

$$\begin{aligned} R_{12} + R_{21} + R_{23} + R_{32} &= 2, \\ R_{23} + R_{32} &= R_{13} + R_{31}, \quad R_{12} + R_{21} = R_{33}. \end{aligned} \quad (6.28)$$

Using this, we can see that anomaly cancellation is then automatically satisfied. The trial central charge is then

$$a_{\text{trial}} = \frac{9N^2}{32} \left[3 + \sum_{i,j} (R_{ij} - 1)^3 \right], \quad (6.29)$$

which is locally maximized for

$$R_{23,32,13,31} = \frac{1}{\sqrt{3}}, \quad R_{12,21} = 1 - \frac{1}{\sqrt{3}}, \quad R_{33} = 2 - \frac{2}{\sqrt{3}}. \quad (6.30)$$

This gives

$$\frac{a_{\mathcal{N}=4}}{a_{\text{SPP}}} = \frac{2}{3\sqrt{3}}. \quad (6.31)$$

We now wish to compare this with Z -minimization already performed. The R-charge of a dibaryonic operator $\mathcal{B}_\alpha = \det \Phi_{ij}$ arising from wrapping a D3-brane over Σ_α is computed using the AdS/CFT formula (5.41). Using the toric volumes above, we can see that the conditions on the R-charges are equivalent to the condition $\xi_1 = 3$, and that the contact volume Z is equal to the inverse of the central charge, where one

takes the trial R-charges to be functions of the trial Reeb vector ξ using the volume formula (5.41):

$$Z = \frac{a_{\mathcal{N}=4}}{a_{\text{trial}}} . \quad (6.32)$$

Of course, this was proven in generality by Butti and Zaffaroni [23].

Massive deformation

After this summary of the suspended pinch point theory, we now turn to its massive deformation [4]. We thus consider deforming the theory by adding a mass term for the adjoint field:

$$W_{m\text{SPP}} = W_{\text{SPP}} + \frac{m}{2} \Phi_{33}^2 .$$

Integrating out the massive field by imposing its equation of motion, $\Phi_{33} = (\Phi_{31}\Phi_{13} - \Phi_{32}\Phi_{23})/m$, we are left with a quartic superpotential

$$W_{m\text{SPP}} = \Phi_{12}\Phi_{21}\Phi_{13}\Phi_{31} - \Phi_{23}\Phi_{32}\Phi_{21}\Phi_{12} - \lambda_{33}(\Phi_{32}\Phi_{23} - \Phi_{31}\Phi_{13})^2 , \quad (6.33)$$

with $\lambda = 1/(2m)$. Neglecting as before the uninteresting branch of the moduli space (which the reader may check is a copy of $\mathbb{C}$), we obtain the F-terms

$$\Phi_{13}\Phi_{31} = \Phi_{23}\Phi_{32} , \quad \Phi_{12}\Phi_{21} = 0 . \quad (6.34)$$

The D-terms are the same as for the SPP theory, and we may similarly construct the gauge-invariant monomials

$$p = \Phi_{23}\Phi_{32} = \Phi_{13}\Phi_{31} , \quad q = \Phi_{12}\Phi_{21} , \quad s = \Phi_{13}\Phi_{32}\Phi_{21} , \quad t = \Phi_{12}\Phi_{23}\Phi_{31} \quad (6.35)$$

The F-term condition $q = 0$ also enforces that either s or t vanishes. The moduli space is thus $\{u, s, t = 0\} \cup \{u, t, s = 0\} \simeq \mathbb{C}^2 \cup_{\mathbb{C}} \mathbb{C}^2$, that is two copies of $\mathbb{C}^2$ intersecting over $\mathbb{C}$. The a -maximization computation below suggests the existence of a non-trivial interacting IR fixed point for this theory, and then the fact that this mesonic moduli space is *not* a three-fold implies that the dual type IIB description must involve a *generalized* Sasaki-Einstein manifold.

The R-charges at the putative IR fixed point can be determined by a -maximization. The condition that the superpotential has R-charge two gives

$$R_{12} + R_{21} = 1 , \quad R_{23} + R_{32} = 1 , \quad R_{13} + R_{31} = 1 . \quad (6.36)$$

The condition of vanishing ABJ anomaly is then automatically satisfied. The trial central charge is

$$a_{\text{trial}} = \frac{27N^2}{32} (R_{12}R_{21} + R_{13}R_{31} + R_{23}R_{32}) . \quad (6.37)$$

A local maximum is obtained when all the R-charges are equal to $1/2$, which gives

$$\frac{a_{\mathcal{N}=4}}{a_{m\text{SPP}}} = \frac{32}{81} . \quad (6.38)$$

Numerically, this is slightly less than the central charge for the SPP theory,

$$\frac{a_{m\text{SPP}}}{N^2} = \frac{81}{128} \approx 0.63 < \frac{a_{\text{SPP}}}{N^2} = \frac{3\sqrt{3}}{8} \approx 0.65 . \quad (6.39)$$

This is then consistent with the a -theorem, $a_{\text{IR}} < a_{\text{UV}}$, which in turn is based on the intuition that we are integrating out degrees of freedom when flowing to the IR.

One of the new results in this thesis is that we now have some understanding of the dual Z -minimization to perform on the gravity side. However, to apply this we need to make *two assumptions*, which are motivated by our previous examples. Firstly, we assume that the symplectic structure of X is left unchanged by the massive deformation. This ensures that the toric diagram remains the same as for the original SPP singularity. This is true of the explicit Pilch-Warner solution, which is the IR fixed point of a similar massive deformation of $\mathcal{N} = 4$ super Yang-Mills. It would certainly be nice to understand better the physical significance of this. The second condition is easier to justify. Here we assume that the homogeneity condition on the pure spinor Ω_- for the putative IIB dual requires

$$\xi_3 = 3/2 . \quad (6.40)$$

The reason for this is that the one-form part of the pure spinor Ω_- is precisely related to the scalar part of the superpotential. Hence $\Omega_- \propto d\bar{v}^2$, where recall that in the Abelian moduli space of the original SPP theory $v = \Phi_{33}$, and we deformed by the mass term $m\Phi_{33}^2/2$. This is indeed precisely what happens for the Pilch-Warner solution, as we reviewed in section 6.1. In the basis we have chosen one immediately sees from (6.17) that the one-form part of the homogeneity condition $\mathcal{L}_\xi dv^2 = 3idv^2$ gives precisely $\xi_3 = 3/2$.

With the homogeneity condition $\xi_1 = 3$, the function Z then reads

$$Z = \frac{1}{2\xi_2(9 - 2\xi_2)} , \quad (6.41)$$

which is minimized at $\xi_2 = 9/4$. Using again (5.41), we verify the equivalence of the Z and a functions:

$$Z = \frac{a_{\mathcal{N}=4}}{a_{\text{trial}}} . \quad (6.42)$$

The contact volumes of $Y_{m\text{SPP}}$ and the subspaces Σ_α after mass deformation are

$$\text{Vol}_\sigma(Y_{m\text{SPP}}) = \frac{32\pi^3}{81} , \quad \text{Vol}_\sigma(\Sigma_\alpha) = \frac{16\pi^2}{27} , \quad \forall \alpha = 1, 2, 3, 4 . \quad (6.43)$$

We also see that the AdS/CFT formula (5.41), which was shown to hold also for generalized geometries in subsection 3.5.3 provided the volumes are interpreted as contact volumes, gives the correct result that the R-charge of each bifundamental field is $1/2$. That is,

$$R(\Phi_{ij}) = \frac{\pi \text{Vol}_\sigma(\Sigma_\alpha)}{3 \text{Vol}_\sigma(Y_{m\text{SPP}})} = \frac{1}{2} , \quad (6.44)$$

which acts as a further check on this result.

We have thus predicted the existence of a supersymmetric AdS_5 solution of type IIB supergravity, with the same topology and toric symplectic structure as X_{SPP} , a Reeb vector field which in the above basis is $\xi_* = (3, 9/4, 3/2)$, a pure spinor Ω_- with one-form component proportional to $d\bar{v}^2$, where v is the complex-valued function on X_{SPP} specified above, and with a corresponding type-change locus $\mathcal{T} = \mathbb{C}^2 \cup_{\mathbb{C}} \mathbb{C}^2$. This is a substantial amount of information about this solution. In fact, this is essentially as much as one knows about toric Calabi-Yau solutions for which we only know that there exists a Sasaki-Einstein metric via the existence result of [71]. Our results then show that the central charges and R-charges of chiral fields computed from such a supergravity solution match those of the dual field theory using AdS/CFT.

6.3 Generalized conifolds

Having studied the SPP theory and its massive deformation in detail, we turn now to a simple infinite family of generalizations of this example. Since the details are similar, we shall be more brief.

We begin with the generalized conifolds described by the hypersurface [70]

$$X_{m,n} = \{u^n v^m = wz\} \subset \mathbb{C}^4. \quad (6.45)$$

These are again also toric, and provided $\gcd(m,n) = 1$ the corresponding polyhedral cone $\mathcal{C}^*$ has primitive normal vectors

$$v_1 = (1, n, 0), \quad v_2 = (1, m, 1), \quad v_3 = (1, 0, 1), \quad v_4 = (1, 0, 0). \quad (6.46)$$

The toric diagram is shown in Figure 6.3.

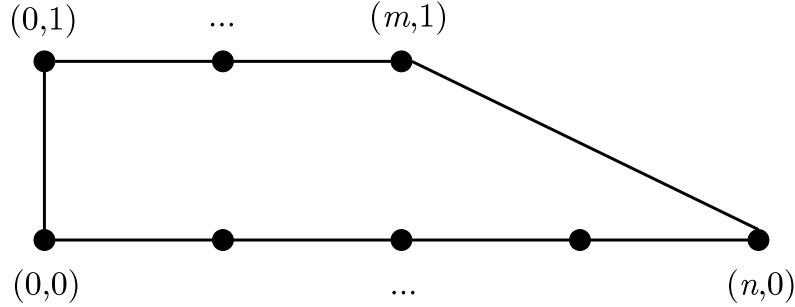


Figure 6.3: Toric diagram for the generalized conifolds $X_{m,n} = C(L^{m,n,m})$.

The dual cone $\mathcal{C}$ is also generated by four primitive vectors, namely

$$u = (1, 0, -1), \quad v = (0, 0, 1), \quad w = (0, 1, 0), \quad z = (n, -1, m - n), \quad (6.47)$$

which correspond to four holomorphic functions on $X_{m,n}$ with definite charge under the torus. It is again an elementary exercise to check that these generate over $\mathbb{Z}_{\geq 0}$ all lattice points in $\mathcal{C}^*$. Notice that we have $\sum_{\alpha=1}^4 Q_{\alpha} v_{\alpha} = 0$, with the $U(1)_B$ charge vector $Q = (-m, n, -n, m)$. Writing Z_{α} , $\alpha = 1, 2, 3, 4$, as coordinates on $\mathbb{C}^4$, then the $U(1)_B$ invariants are spanned by the four functions

$$u = Z_1 Z_4, \quad v = Z_2 Z_3, \quad w = Z_1^n Z_2^m, \quad z = Z_3^m Z_4^n, \quad (6.48)$$

which then satisfy $u^n v^m = wz$. Again, this certainly requires $\gcd(m,n) = 1$.

These generalized conifolds are cones over the Sasaki-Einstein orbifolds $L^{m,n,m}$. The SCFT dual to N D3-branes probing $X_{m,n} = C(L^{m,n,m})$ was studied in [16]. Again, this is the IR limit of a quiver gauge theory, now with $N_g = m + n$ $U(N)$

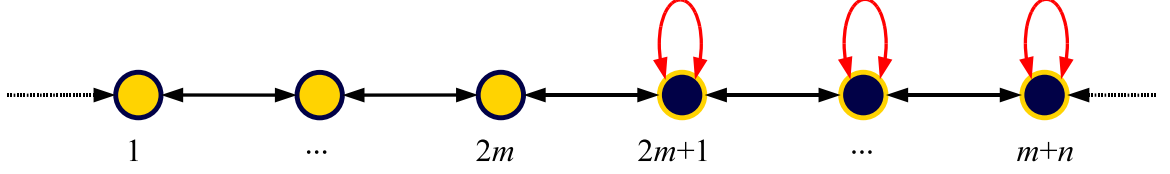


Figure 6.4: Quiver diagram for the gauge theory on N D3-branes probing $X_{m,n} = C(L^{m,n,m})$. The dashed arrows at both extremities are identified.

gauge group factors, the last $n - m$ of which have an adjoint field. The quiver is shown in Figure 6.4, and the superpotential is

$$W_{L^{m,n,m}} = \sum_{i=1}^{2m} (-)^i \Phi_{i,i-1} \Phi_{i-1,i} \Phi_{i,i+1} \Phi_{i+1,i} + \sum_{i=2m+1}^{n-m} \Phi_{i,i} (\Phi_{i,i+1} \Phi_{i+1,i} - \Phi_{i,i-1} \Phi_{i-1,i}) ,$$

where the index i is defined modulo N_g . Notice here that to each torus-invariant divisor $D_\alpha = C(\Sigma_\alpha)$, with Σ_α a three-subspace of the orbifold $Y = L^{m,n,m}$, we can associate a set of the bifundamental fields Φ_{ij} . Geometrically, the relation is that $\{\Phi_{ij} = 0\}$ is the divisor D_α in the mesonic moduli space, which contains $X_{m,n}$. These fields have multiplicities $n_\alpha = |(v_{\alpha-1}, v_\alpha, v_{\alpha+1})|$ [16], giving here $n_1 = n_4 = n$ and $n_2 = n_3 = m$, see Table 6.2.

After adding mass terms of the form $\sum_{i=2m+1}^{m+n} m_i \Phi_{i,i}^2 / 2$ and integrating out the massive fields, we obtain

$$\begin{aligned} W_{mL^{m,n,m}} &= \sum_{i=1}^{2m} (-)^i \Phi_{i,i-1} \Phi_{i-1,i} \Phi_{i,i+1} \Phi_{i+1,i} \\ &\quad - \sum_{i=2m+1}^{n-m} \lambda_i (\Phi_{i,i+1} \Phi_{i+1,i} - \Phi_{i,i-1} \Phi_{i-1,i})^2 , \end{aligned} \quad (6.49)$$

with $n - m$ complex coupling constants $\lambda_i = 1/2m_i$. The corresponding F-term equations give rise to

$$\begin{aligned} v &= \Phi_{i,i+1} \Phi_{i+1,i} = 0 && \text{for all odd } i < 2m , \\ u &= \Phi_{j,j+1} \Phi_{j+1,j} && \text{for all } j \neq i , \end{aligned} \quad (6.50)$$

where again we focus on the branch of the moduli space which does not correspond to moving fractional branes along the residual singularity. In addition, we find the following gauge-invariant monomials

$$w = \Phi_{12} \cdots \Phi_{m+n,1} , \quad z = \Phi_{1,m+n} \cdots \Phi_{21} , \quad (6.51)$$

3-subspace	Fields	Q_B	Multiplicity
Σ_1	$\Phi_{j+1,j}$	$-m$	n
Σ_2	$\Phi_{i,i+1}$	n	m
Σ_3	$\Phi_{i+1,i}$	$-n$	m
Σ_4	$\Phi_{j,j+1}$	m	n

Table 6.2: Divisors $D_\alpha = C(\Sigma_\alpha)$, fields, charges, and multiplicities for the $L^{m,n,m}$ theories. Here, as in (6.50), the index i is odd and smaller than $2m$, while the index j covers the remainder.

which then satisfy $u^n v^m = wz$. As an illustration, consider the $L^{1,3,1}$ theory: the F-terms lead to

$$u = \Phi_{23}\Phi_{32} = \Phi_{34}\Phi_{43} = \Phi_{41}\Phi_{14}, \quad v = \Phi_{12}\Phi_{21} = 0, \quad (6.52)$$

and we can construct the $U(1)_{1,2,3}$ -invariant monomials

$$w = \Phi_{12}\Phi_{23}\Phi_{34}\Phi_{41}, \quad z = \Phi_{14}\Phi_{43}\Phi_{32}\Phi_{21}, \quad (6.53)$$

which satisfy $u^3 v = wz$.

In the general case the condition $v = 0$ implies that either w or z vanishes. The moduli space is thus again two copies of $\mathbb{C}^2$, intersecting over $\mathbb{C}$.

We next perform a -maximization for the IR fixed point of the massive deformation. The requirement that the superpotential has R-charge 2 gives

$$R[\Sigma_1] + R[\Sigma_4] = 1, \quad R[\Sigma_2] + R[\Sigma_3] = 1, \quad (6.54)$$

where the field-divisor map is given in Table 6.2. The ABJ anomaly is then automatically satisfied. The trial central charge function is

$$\begin{aligned} a_{\text{trial}} &= \frac{27N^2}{32} \sum_{i=1}^{N_g} R_{i,i+1} R_{i+1,i} \\ &= \frac{27N^2}{32} (mR[\Sigma_2]R[\Sigma_3] + nR[\Sigma_1]R[\Sigma_4]), \end{aligned} \quad (6.55)$$

which is locally maximized when all R-charges are equal to $1/2$, at which point the central charge is

$$\frac{a_{\mathcal{N}=4}}{a_{mL^{m,n,m}}} = \frac{32}{27N_g} = \frac{32}{27(m+n)}. \quad (6.56)$$

In accord with the a -theorem, the central charge of the infrared theory is strictly smaller than the central charge of the original theory given in [16], for all values of m and n :

$$a_{L^{m,n,m}} = \frac{27N^2}{16} m^2 n^2 \left[(2m-n)(2n-m)(m+n) + 2(m^2 - mn + n^2)^{3/2} \right]^{-1} \quad (6.57)$$

Finally, we turn to the dual Z -minimization problem. Again, the homogeneity condition for Ω_- leads to $\xi_1 = 3$ and $\xi_3 = 3/2$, the latter condition again coming from the expectation that the one-form part of Ω_- is proportional to $d\bar{v}^2$, precisely as for the Pilch-Warner solution and our discussion of the SPP theory. The Z function is then

$$Z = \frac{4N_g}{3\xi_2(3N_g - 2\xi_2)} , \quad (6.58)$$

which is minimized for $\xi_2 = 3N_g/4$. The volume of $L^{m,n,m}$ after mass deformation and the volumes of the subspaces Σ_α are

$$\text{Vol}(mL^{m,n,m}) = \frac{32\pi^3}{27N_g} , \quad \text{Vol}(\Sigma_\alpha) = \frac{16\pi^2}{9N_g} . \quad (6.59)$$

Using again (5.41) in terms of contact volumes, we can verify that the conjectured relation between Z and a_{trial} indeed holds.

Chapter 7

Discussion

In this thesis we developed a deeper understanding of general supersymmetric $\text{AdS}_5 \times Y$ solutions of type IIB supergravity using generalized geometry. The cone over Y is generalized Calabi-Yau and carries a generalized holomorphic vector field which corresponds to the R-symmetry of the dual SCFT. The key point is that there is a generalized complex structure on the cone, which becomes an ordinary complex structure on a special type-change locus. Physically, this locus is the Abelian mesonic moduli space of the dual field theory. In particular, there is an underlying contact structure, and the associated Reeb vector field, dual to the R-symmetry, is generalized holomorphic, generalized Killing, and related to $r\partial_r$ via the integrable generalized complex structure $\mathcal{J}_-$.

We identified a relationship between “BPS polyforms”, that is polyforms with equal R-charge and scaling dimension, and generalized holomorphic polyforms that should be worth exploring further. In particular, we would like to make a precise connection between such objects and the spectrum of chiral operators in the SCFT via Kaluza-Klein reduction on Y . It would also be interesting to relate the symplectic volume of a generalized Calabi-Yau cone to some generalized index counting generalized holomorphic objects. Encouragingly, such a generalization has been shown to work in the context of topological strings [72].

We showed that with the assumption that the five-form flux sourced by D3-branes does not vanish the cone is actually symplectic, and we obtained contact volume formulas for the central charge of the dual SCFT and the conformal dimensions of operators dual to BPS wrapped D3-branes. This assumption is not very restrictive since all known solutions satisfy it. However, in principle there could exist AdS_5

solutions where all the D3-branes are replaced say by fractional D5-branes. It would be interesting to determine whether such solutions do indeed exist.

In chapter 4 we introduced the notion of “generalized Sasakian geometry”, which shares many properties with Sasakian geometry. It consists of a pair of compatible pure spinors on a generalized Calabi-Yau cone satisfying certain rather natural differential conditions. Away from the type-change locus, the transverse space to the Reeb foliation, instead of being Kähler as in the Sasakian case, is endowed with a triple of orthogonal symplectic forms satisfying a system of differential equations. By analogy with the Sasaki-Einstein case (see for example [31]) this perspective could be useful for constructing new explicit solutions.

Most known $\text{AdS}_5 \times Y$ solutions of type IIB supergravity with fluxes are actually part of continuous families of solutions containing a Sasaki-Einstein solution. For example, starting with a toric Sasaki-Einstein solution one can construct new β -transformed solutions using the techniques of [36], which corresponds to the addition of an exactly marginal deformation in the dual SCFT. It has also been shown numerically in [73] that the Pilch-Warner solution (or more precisely its $\mathbb{Z}_2$ -orbifold) is part of a family that includes the conifold solution. In fact, the study of the conformal manifolds of CFTs [54, 74] reveals the existence of many other deformations whose dual geometries are still to be discovered. The fact that the mesonic moduli spaces of these deformed CFTs are in general not three-folds indicates that the dual cones must be generalized geometric. It would be exciting to find a systematic way to match geometric deformations of the cone over Y to conformal deformations of the dual CFT.

Generalizing the results of [19], we then proved in chapter 5 that the action for the bosonic supergravity fields is equal, when restricted to a space of generalized Sasakian structures, to the underlying contact volume, and thus depends only on the Reeb vector field. This implies that the Reeb vector field of a supersymmetric AdS_5 solution is obtained by minimizing the contact volume over a space of Reeb vector fields under which the pure spinor Ω_- has charge three. Since at the critical point this contact volume is equal to the inverse central charge of the dual field theory [2], this is conjecturally the geometric counterpart of a -maximization in four-dimensional $\mathcal{N} = 1$ SCFTs.

One important open issue is to gain a better understanding of the space of generalized Sasakian structures. However, even for the space of Sasakian manifolds, which has been studied since the 1960s, there is still no general understanding. The only complete description [20] covers the case of toric Sasakian manifolds. We believe that a similar level of understanding should be achievable for generalized toric Sasakian manifolds, but leave this for future work. Notice that, in any case, our definition of generalized Sasakian geometry reduces to that of Sasakian geometry when the generalized complex structure is complex, and that the β -transform of a Kähler cone is a cone over a generalized Sasakian manifold.

In addition, we have not investigated the type-change locus $\mathcal{T}$ in any detail here. It is necessary to understand what the constraints are on the type-change locus, and to classify the types of boundary conditions associated with the structures introduced in section 4. We note that this is very much an open problem in generalized geometry, for which there are currently only some very preliminary results [32, 33].

In chapter 6 we bypassed most of the above open issues by focusing on some examples for which we have a fairly good understanding of the physics on the SCFT side of the correspondence. This allowed us to predict the existence of supersymmetric AdS_5 solutions of type IIB supergravity, with the same topology and toric symplectic structure as the cones over $L^{m,n,m}$. Although we do not fully know the pure spinor Ω_- , we made some reasonable assumptions about the generalized geometry based on the dual field theories and on the Pilch-Warner solution that these solutions generalize, and thereby determined the type-change locus to be $\mathcal{T} = \mathbb{C}^2 \cup_{\mathbb{C}} \mathbb{C}^2$. We were then able to compute the critical Reeb vector field and hence the contact volumes. Using the formulas in section 3.5, we found perfect agreement with the central charges and R-charges of chiral primary fields computed via a -maximization in the dual SCFTs obtained by massive deformation. Perhaps the main issue raised here is why the toric symplectic structure is preserved after the renormalization group flow triggered by the massive deformation. In fact, the field theory interpretation of the contact or symplectic structure, which exists whenever the solution has non-zero D3-brane charge, is still mysterious.

Appendix A

Generalized Killing vector field

In this appendix we derive an expression for the generalized vector ξ in terms of the bilinears introduced in [25]. We also use the results of [25] to give an alternative proof that ξ preserves both generalized structures, that is $\mathbb{L}_\xi \mathcal{J}_\pm = 0$, and is thus generalized holomorphic and generalized Killing.

The projections of ξ onto the vector and form parts (in a fixed trivialization of E) are denoted ξ_v , ξ_f , respectively. It will be convenient to introduce $\xi^B \equiv e^B \xi = \xi_v + (\xi_f - \xi_v \lrcorner B)$. Since Z_1^- in (3.74) annihilates Ω_- , using the definition (3.43) we deduce that

$$r \partial_r \lrcorner \Phi_- = i (\xi_v \lrcorner \Phi_- + \xi_f^B \wedge \Phi_-) . \quad (\text{A.1})$$

To proceed we use (3.28) to write

$$\Phi_- \equiv \eta_+^1 \otimes \bar{\eta}_-^2 = e^A \chi_1 \bar{\chi}_2^c \otimes (\sigma_3 + i\sigma_1) . \quad (\text{A.2})$$

Since $\Phi_- = \sum_{\text{odd } p} \Phi_{i_1 \dots i_p} \gamma^{i_1 \dots i_p} / p!$ we have

$$v \lrcorner \Phi_- = \frac{1}{2} \{v^i \gamma_i, \Phi_-\} , \quad \nu \wedge \Phi_- = \frac{1}{2} [\nu \gamma^i, \Phi_-] . \quad (\text{A.3})$$

Hence, using the Clifford algebra decomposition (3.25) and metric (3.24) we have

$$\begin{aligned} r \partial_r \lrcorner \Phi_- &= \frac{1}{2} \{e^{\Delta+\phi/4} \hat{\gamma}_6, \Phi_-\} = \frac{1}{2} e^{A+\Delta+\phi/4} \chi_1 \bar{\chi}_2^c \otimes \{\sigma_1, \sigma_3 + i\sigma_1\} \\ &= i e^{A+\Delta+\phi/4} \chi_1 \bar{\chi}_2^c \otimes 1 . \end{aligned} \quad (\text{A.4})$$

On the other hand using (3.76) we have

$$\begin{aligned}
\xi_v \lrcorner \Phi_- + \xi_f^B \wedge \Phi_- &= \frac{1}{2} \{ e^{\Delta+\phi/4} \xi_v^m \beta_m \otimes \sigma_3, \Phi_- \} + \frac{1}{2} [e^{-\Delta-\phi/4} \xi_f^B{}_m \beta^m \otimes \sigma_3, \Phi_-] \\
&= e^{\Delta+\phi/4} v_m^+ \beta^m \otimes \sigma_3 \Phi_- + e^{\Delta+\phi/4} v_m^- \Phi_- \beta^m \otimes \sigma_3 \\
&= e^{A+\Delta+\phi/4} (v_m^+ \beta^m (\chi_1 \bar{\chi}_2^c) + v_m^- (\chi_1 \bar{\chi}_2^c) \beta^m) \otimes 1 \\
&\quad - e^{A+\Delta+\phi/4} (v_m^+ \beta^m (\chi_1 \bar{\chi}_2^c) - v_m^- (\chi_1 \bar{\chi}_2^c) \beta^m) \otimes \sigma_2 , \tag{A.5}
\end{aligned}$$

where recall that $\{\beta_m, \beta_n\} = 2g_{Ymn}$ and we have defined

$$v_m^\pm \equiv \frac{1}{2} (\xi_{vm} \pm e^{-2\Delta-\phi/2} \xi_f^B{}_m) . \tag{A.6}$$

To satisfy (A.1) we thus require

$$v_m^+ \beta^m (\chi_1 \bar{\chi}_2^c) = v_m^- (\chi_1 \bar{\chi}_2^c) \beta^m = \frac{1}{2} \chi_1 \bar{\chi}_2^c , \tag{A.7}$$

which implies

$$v_m^+ \beta^m \chi_1 = \frac{1}{2} \chi_1 , \quad v_m^- \beta^m \chi_2 = \frac{1}{2} \chi_2 , \tag{A.8}$$

or equivalently

$$v_m^+ = \frac{\bar{\chi}_1 \beta_m \chi_1}{2 \bar{\chi}_1 \chi_1} , \quad v_m^- = \frac{\bar{\chi}_2 \beta_m \chi_2}{2 \bar{\chi}_2 \chi_2} . \tag{A.9}$$

Hence, given the normalization $\bar{\chi}_1 \chi_1 = \bar{\chi}_2 \chi_2 = 1/2$, we deduce that, in terms of the bilinears defined in (3.15),

$$\xi_v = K_5^\# , \quad \xi_f^B = e^{2\Delta+\phi/2} \text{Re} K_3 . \tag{A.10}$$

A similar calculation using

$$r \partial_r \lrcorner \Phi_+ = i e^{2\Delta+\phi/2} (\eta_v \lrcorner \Phi_- + \eta_f^B \wedge \Phi_+) , \tag{A.11}$$

leads to

$$\eta_v = e^{-2\Delta-\phi/2} \text{Re} K_3^\# , \quad \eta_f^B = K_5 . \tag{A.12}$$

Using the expression for the B -field given in (3.19), we obtain the expressions for ξ_f and η_f given in (3.69).

In [25] it was shown that K_5 is a Killing one-form, so that its dual vector field $K_5^\#$ (with respect to g_Y) is a Killing vector field. In fact $K_5^\#$ generates a full symmetry of

the supergravity solution, in that all bosonic fields (warp factor, dilaton, NS three-form H , and RR fluxes) are preserved under the Lie derivative along $\xi_v = K_5^\#$. However, importantly, the Killing spinors ξ_1 and ξ_2 are not invariant under ξ_v . In [25] it was shown that

$$\mathcal{L}_{\xi_v} S = -3iS , \quad (\text{A.13})$$

where $S \equiv \bar{\xi}_2^c \xi_1$. Notice that, since ξ_v preserves all of the bosonic fields, one may take the Lie derivative of the Killing spinor equations (3.12)-(3.13) for ξ_1, ξ_2 along ξ_v , showing that $\{\mathcal{L}_{\xi_v} \xi_i\}$ satisfy the same equations as the $\{\xi_i\}$. It thus follows that

$$\mathcal{L}_{\xi_v} \xi_i = i\mu \xi_i , \quad (\text{A.14})$$

where μ is a constant. Now (A.13) implies that $2\mu = -3$, and thus

$$\mathcal{L}_{\xi_v} \xi_i = -\frac{3i}{2} \xi_i . \quad (\text{A.15})$$

One can also derive this last equation directly from the Killing spinor equations (3.12)-(3.13) of [25]. It thus follows that

$$\mathcal{L}_{\xi_v} \Phi_+ = 0 , \quad \mathcal{L}_{\xi_v} \Phi_- = -3i\Phi_- . \quad (\text{A.16})$$

Next, using the following equation of [25],

$$D(e^{6\Delta} W) = -e^{6\Delta} P \wedge W^* + \frac{f_5}{4} G , \quad (\text{A.17})$$

where W is the two-form bilinear defined in (3.16), we can show that

$$K_5^\# \lrcorner \left(\frac{4}{f_5} e^{6\Delta+\phi/2} \text{Re } W \right) = e^{2\Delta+\phi/2} \text{Re } K_3 , \quad (\text{A.18})$$

and furthermore that

$$d(e^{2\Delta+\phi/2} \text{Re } K_3) = \xi_v \lrcorner H . \quad (\text{A.19})$$

To see the latter, we can derive an expression for the left-hand side using, amongst other things, (3.18), (3.38) and (B.10) of [25], and an expression for the right-hand side using equation (3.38) and (B.8) of [25]. Using these results we can deduce that

$$\mathcal{L}_{K_5^\#} B = d(K_5^\# \lrcorner b_2) , \quad \mathcal{L}_{K_5^\#} C_2 = d(K_5^\# \lrcorner c_2) , \quad (\text{A.20})$$

where b_2, c_2 were introduced in (3.19), (3.20), respectively.

From (A.19) we have $d\xi_f^B = \xi_v \lrcorner H$ and we deduce that

$$\mathbb{L}_{\xi^B} \Phi_+ = \xi_v \lrcorner H \wedge \Phi_+ , \quad \mathbb{L}_{\xi^B} \Phi_- = -3i\Phi_- + \xi_v \lrcorner H \wedge \Phi_- . \quad (\text{A.21})$$

Since (A.19) is also equivalent to $d\xi_f = \mathcal{L}_{\xi_v} B$ we deduce that

$$\mathbb{L}_{\xi} \Omega_+ = 0 , \quad \mathbb{L}_{\xi} \Omega_- = -3i\Omega_- , \quad (\text{A.22})$$

and hence $\mathbb{L}_{\xi} \mathcal{J}_{\pm} = 0$. Finally, it is also interesting to point out that

$$(\mathbb{L}_{\xi^B} - \xi_v \lrcorner H \wedge) F = 0 , \quad \text{or equivalently} \quad \mathbb{L}_{\xi}(e^{-B} F) = 0 . \quad (\text{A.23})$$

Appendix B

The Sasaki-Einstein case

Here we discuss the special case in which the compact five-manifold Y is Sasaki-Einstein. Setting $G = P = Q = 0$, $f_5 = 4e^{4\Delta}$, and $\xi_2 = 0$, the Killing spinor equations (3.12)-(3.13) reduce to

$$\nabla_m \xi_1 + \frac{i}{2} \beta_m \xi_1 = 0 . \quad (\text{B.1})$$

In terms of appendix B of [25] we choose $\bar{\theta} = \bar{\phi} = 0$ and $e^{2i\bar{\alpha}} = -1$ (these angles had no bars on them in [25]). We then have the equalities

$$\begin{aligned} \eta &= \frac{1}{2} \bar{\xi}_1 \beta_{(1)} \xi_1 = K_5 = e^1 , \\ \omega_{\text{KE}} &= \frac{i}{2} \bar{\xi}_1 \beta_{(2)} \xi_1 = -V = e^{25} + e^{43} , \\ \Omega_{\text{KE}} &= \frac{1}{2} \bar{\xi}_1 \beta_{(2)} \xi_1^c = (e^2 + ie^5) \wedge (e^4 + ie^3) , \end{aligned} \quad (\text{B.2})$$

and

$$\begin{aligned} d\eta &= 2\omega_{\text{KE}} , \\ d\Omega_{\text{KE}} &= 3i\eta \wedge \Omega_{\text{KE}} . \end{aligned} \quad (\text{B.3})$$

Observe that

$$\eta \wedge \frac{1}{2!} \omega_{\text{KE}}^2 = -e^{12345} = -\text{vol}_Y . \quad (\text{B.4})$$

Next using the map (3.28) between five- and six-dimensional spinors, we obtain

$$\begin{aligned} i\bar{\eta}_+^1 \gamma_{(2)} \eta_+^1 &= r(d \log r \wedge e^1 + \omega_{\text{KE}}) \equiv \frac{1}{r} \omega_{\text{CY}} , \\ -i\bar{\eta}_+^{1c} \gamma_{(3)} \eta_+^1 &= r(d \log r - ie^1)(e^2 - ie^5)(e^4 - ie^3) \equiv \frac{1}{r^2} \bar{\Omega}_{\text{CY}} . \end{aligned} \quad (\text{B.5})$$

It is worth noting that

$$\frac{\omega_{\text{CY}}^3}{3!} = r^6 e^{123456} = r^6 d \log r \wedge \eta \wedge \frac{\omega_{\text{KE}}^2}{2!} . \quad (\text{B.6})$$

We also find, directly from (3.43),

$$\begin{aligned} \Omega_- &= \frac{1}{8} \bar{\Omega}_{\text{CY}} , \\ \Omega_+ &= -\frac{i r^3}{8} \exp \left(\frac{i}{r^2} \omega_{\text{CY}} \right) . \end{aligned} \quad (\text{B.7})$$

A useful check is that these expressions agree with those obtained from the general expressions obtained in section 3.2.

We can also write down the corresponding reduced structures Ω_1 and Ω_2 , as defined in section 4.3, on the Kähler-Einstein space. We find

$$\Omega_1 = \frac{1}{8} e^{3i\psi} \bar{\Omega}_{\text{KE}} , \quad \Omega_2 = -\frac{i}{8} e^{i\omega_{\text{KE}}} , \quad (\text{B.8})$$

where ψ is the coordinate defined via $\xi_v = K_5^\# = \partial_\psi$.

Appendix C

The hazards of dimensional reduction

In section 5.1 we derived the equations of motion on Y for the bosonic fields of type IIB supergravity from ten-dimensional equations, and then constructed an action Z whose variation led to those equations. An alternative strategy to obtain Z would have been to dimensionally reduce the type IIB action on $\text{AdS}_5 \times Y$, where the “reduction” is along the AdS_5 direction. However, this approach is complicated by ambiguities inherent to the self-duality of F_5 , and to the lack of proper normalization of the Chern-Simons term [75]. In this appendix we outline the relation between these two approaches.

Even though the field equations of type IIB supergravity cannot be derived directly from the variation of an action, one can have recourse to a pseudo-action that leads to equations of motion that match the type IIB equations only when supplemented by the self-duality condition $\star F_5 = F_5$. With $\tilde{F}_5 \equiv F_5 + \frac{1}{2}d(B \wedge C_2)$, this pseudo-action reads [76]

$$S_{\text{IIB}}^{10} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g_E} \left(R_E - 2|P_1|^2 - \frac{1}{2}|G_3|^2 - \frac{1}{4}|\tilde{F}_5|^2 \right) - \frac{1}{4\kappa_{10}^2} \int dC_4 \wedge H \wedge C_2 . \quad (\text{C.1})$$

After a Weyl rescaling $\bar{g} = g_{\text{AdS}} + g_Y = e^{-2\Delta} g_E$ this becomes

$$S_{\text{IIB}}^{10} = \frac{1}{2\kappa_{10}^2} \int_{\text{AdS}_5 \times Y} d^{10}x \sqrt{\bar{g}} e^{8\Delta} \left(\bar{R} - 18e^{-8\Delta} \nabla_M (e^{8\Delta} \partial^M \Delta) + 72|d\Delta|^2 - 2|P_1|^2 - \frac{1}{2}e^{-4\Delta}|G_3|^2 - \frac{1}{4}e^{-8\Delta}|\tilde{F}_5|^2 \right) - \frac{1}{4\kappa_{10}^2} \int dC_4 \wedge H \wedge C_2 . \quad (\text{C.2})$$

Splitting the integral into a part over AdS_5 and a part over Y , one finds

$$S_{\text{IIB}}^{10} = \frac{\text{Vol}_{\text{AdS}}}{2\kappa_{10}^2} \left[\int_Y d^5y \sqrt{g_Y} e^{8\Delta} \left(R(g_Y) - 20 + 72|\text{d}\Delta|^2 - 2|P_1|^2 - \frac{1}{2}e^{-4\Delta}|G_3|^2 \right) - \frac{1}{2} \int_Y \left(d^5y \sqrt{g_Y} \frac{|\tilde{F}_5|^2}{2} + f_5 H \wedge C_2 \right) \right]. \quad (\text{C.3})$$

Note that the prefactor Vol_{AdS} here is infinite. Pragmatically, one can simply discard this prefactor in order to obtain an action for the bosonic fields on Y , although in a more systematic treatment this should be regularized holographically following Henningson and Skenderis [49]. The term $|\tilde{F}_5|^2$ should be understood only symbolically, since the diabolic self-duality property makes it vanish. Naively, one might be tempted to formally set $|\tilde{F}_5|^2 = f_5^2$. Comparing with the action Z in (5.12), we conclude that a factor of -2 is missing in front of the second line of (C.3). A similar factor was already pointed out by Belov and Moore [75].

Appendix D

The contact volume functional

In this appendix we consider the contact volume

$$\text{Vol}_\sigma(Y) \equiv \int_Y \sigma \wedge \frac{\omega_T^2}{2!} = \frac{1}{8} \int_Y \sigma \wedge d\sigma \wedge d\sigma = \int_Y \text{vol}_\sigma \quad (\text{D.1})$$

as a functional on an appropriate space of contact structures on a fixed five-manifold Y . Thus here σ is a contact one-form on Y . We begin by showing that this volume depends only on the unique Reeb vector field ξ that is associated with σ . As we explain, this is analogous to the statement in symplectic geometry that the symplectic volume depends only on the cohomology class of the symplectic form. We then compute the first and second derivatives of the contact volume. In particular, provided one considers only deformations of the Reeb vector field that preserve σ , then the volume functional is *strictly convex*. These results generalize those of [19, 71] for Sasakian manifolds to general contact manifolds. Of course, the results that follow hold in arbitrary odd dimension, with appropriate replacements of dimension-dependent constants.

Consider a fixed contact one-form σ on Y , and a one-parameter family of deformations σ_t , with $\sigma_0 = \sigma$ and $t \in (-\epsilon, \epsilon) \subset \mathbb{R}$. We Taylor-expand $\sigma_t = \sigma + t\sigma' + \mathcal{O}(t^2)$, with a similar expansion of the Reeb vector field $\xi_t = \xi + t\xi' + \mathcal{O}(t^2)$. Since by definition $\sigma_t(\xi_t) = 1$, $\xi_t \lrcorner d\sigma_t = 0$, one immediately deduces the first-order equations

$$\sigma'(\xi) = -\sigma(\xi') , \quad \xi \lrcorner d\sigma' = -\xi' \lrcorner d\sigma . \quad (\text{D.2})$$

We now compute

$$\begin{aligned}
\int_Y \text{vol}_{\sigma_t} - \int_Y \text{vol}_{\sigma} &= \frac{t}{8} \left[\int_Y \sigma' \wedge (d\sigma)^2 + 2 \int_Y \sigma \wedge d\sigma \wedge d\sigma' \right] + \mathcal{O}(t^2) \\
&= \frac{3t}{8} \int_Y \sigma' \wedge (d\sigma)^2 + \mathcal{O}(t^2) \\
&= -\frac{3t}{8} \int_Y \sigma(\xi') \sigma \wedge (d\sigma)^2 + \mathcal{O}(t^2) , \tag{D.3}
\end{aligned}$$

where in going from the first to the second line we have integrated the second term by parts and used Stokes' theorem, and in going from the second to the third line we have used the first equation in (D.2). In particular, if we consider deformations of the contact structure that leave fixed the Reeb vector field, then by definition $\xi' = 0$ and the contact volume is invariant. Thus we may regard the contact volume as a functional of ξ , as opposed to σ , and we have then shown that the first derivative of the contact volume is

$$d\text{Vol}_{\sigma}[\xi'] = -3 \int_Y \sigma(\xi') \text{vol}_{\sigma} . \tag{D.4}$$

Of course, this result reproduces that in [19], but here we have used only contact geometry. In the special case in which ξ generates a $U(1)$ action on Y , the quotient $Y/U(1)$ is a symplectic orbifold and the contact volume is (proportional to) the symplectic volume of $Y/U(1)$. Deformations of the contact structure that leave ξ invariant are then deformations of the symplectic structure on $Y/U(1)$ that leave the cohomology class fixed, which thus preserve the volume. More generally, such deformations leave fixed the *basic* cohomology class of the symplectic structure on the leaf space of the Reeb foliation.

We next deform again the contact form and the Reeb vector field as $\sigma_t = \sigma + t\sigma'' + \mathcal{O}(t^2)$ and $\xi_t = \xi + t\xi'' + \mathcal{O}(t^2)$, and similarly compute

$$\begin{aligned}
\left. \frac{d}{dt} \int_Y \sigma_t(\xi') \sigma_t \wedge (d\sigma_t)^2 \right|_{t=0} &= 8 \int_Y \sigma''(\xi') \text{vol}_{\sigma} + \int_Y \sigma(\xi') \sigma'' \wedge (d\sigma)^2 \\
&\quad + 2 \int_Y \sigma(\xi') \sigma \wedge d\sigma \wedge d\sigma'' , \\
&= -24 \int_Y \sigma(\xi') \sigma(\xi'') \text{vol}_{\sigma} \\
&\quad + 8 \int_Y \sigma''(\xi') \text{vol}_{\sigma} - 2 \int_Y d(\sigma(\xi')) \wedge \sigma'' \wedge \sigma \wedge d\sigma . \tag{D.5}
\end{aligned}$$

Here we have used precisely the same steps as when computing the first derivative in (D.3). To deal with the last line, we write

$$d(\sigma(\xi')) = \mathcal{L}_{\xi'}\sigma - \xi' \lrcorner d\sigma , \quad (\text{D.6})$$

using Cartan's formula. We now also impose that our original deformation vector field ξ' preserves the initial contact one-form, so $\mathcal{L}_{\xi'}\sigma = 0$. This means that ξ' is in the Lie algebra of *strict* contact deformations of σ . Notice that a similar assumption was also made in [19], where the space of Sasakian metrics considered had a fixed isometry group, with the Reeb vector field varied in the Lie algebra of this group. Focusing on the last line in (D.5), we then have

$$\begin{aligned} 8 \int_Y \sigma''(\xi') \text{vol}_\sigma - 2 \int_Y d(\sigma(\xi')) \wedge \sigma'' \wedge \sigma \wedge d\sigma &= \int_Y \sigma \wedge \xi' \lrcorner [(d\sigma)^2 \wedge \sigma''] \\ &= \int_Y \sigma(\xi') \sigma'' \wedge (d\sigma)^2 , \\ &= -8 \int_Y \sigma(\xi') \sigma(\xi'') \text{vol}_\sigma . \end{aligned} \quad (\text{D.7})$$

Altogether we have thus shown that the second derivative of the contact volume is

$$d^2 \text{Vol}_\sigma[\xi', \xi''] = 12 \int_Y \sigma(\xi') \sigma(\xi'') \text{vol}_\sigma , \quad (\text{D.8})$$

thus showing that $\text{Vol}_\sigma(Y)$ is strictly convex. Again, notice that this formula reproduces that in [19].

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