

DISCONTINUOUS GALERKIN FINITE ELEMENT APPROXIMATION OF QUASILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS I: THE SCALAR CASE

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Abstract. We develop a one-parameter family of hp -version discontinuous Galerkin finite element methods, parameterised by $\theta \in [-1, 1]$, for the numerical solution of quasilinear elliptic equations in divergence-form on a bounded open set $\Omega \subset \mathbb{R}^d$, $d \geq 2$. In particular, we consider the analysis of the family for the equation $-\nabla \cdot \{\mu(x, |\nabla u|) \nabla u\} = f(x)$ subject to mixed Dirichlet–Neumann boundary conditions on $\partial\Omega$. It is assumed that μ is a real-valued function, $\mu \in C(\bar{\Omega} \times [0, \infty))$, and there exist positive constants m_μ and M_μ such that $m_\mu(t-s) \leq \mu(x, t)t - \mu(x, s)s \leq M_\mu(t-s)$ for $t \geq s \geq 0$ and all $x \in \Omega$. Using Brouwer's Fixed Point Theorem, for any value of $\theta \in [-1, 1]$ the corresponding method is shown to have a unique solution u_{DG} in the finite element space. If $u \in C^1(\Omega) \cap H^k(\Omega)$, $k \geq 2$, then, with discontinuous piecewise polynomials of degree $p \geq 1$, the error between u and u_{DG} , measured in the broken $H^1(\Omega)$ -norm, is $\mathcal{O}(h^{s-1}/p^{k-3/2})$, where $1 \leq s \leq \min\{p+1, k\}$.

Key words. hp -finite element methods, discontinuous Galerkin methods, quasilinear elliptic PDEs

AMS subject classifications. 65N12, 65N15, 65N30

1. Introduction. Let Ω be a bounded open set in \mathbb{R}^d , $d \geq 2$, with Lipschitz continuous boundary $\Gamma = \Gamma_D \cup \Gamma_N$, where Γ_D has positive $(d-1)$ -dimensional surface measure, and denote by $\nu = (\nu_1, \dots, \nu_d)^T$ the unit outward normal vector to Γ , defined almost everywhere on Γ . We consider the following elliptic boundary value problem:

$$-\nabla \cdot \{\mu(x, |\nabla u|) \nabla u\} = f \quad \text{in } \Omega, \quad (1.1)$$

$$u = g_D \quad \text{on } \Gamma_D, \quad (1.2)$$

$$\mu(x, |\nabla u|) \frac{\partial u}{\partial \nu} = g_N \quad \text{on } \Gamma_N, \quad (1.3)$$

where $f \in L^2(\Omega)$, $g_D \in H^{1/2}(\Gamma_D)$ and $g_N \in L^2(\Gamma_N)$.

In the second part of this work, see [21], we shall focus on the following non-Newtonian flow problem: given $\mathbf{f} \in [L^2(\Omega)]^d$, $\mathbf{g}_D \in [H^{1/2}(\Gamma_D)]^d$ and $\mathbf{g}_N \in [L^2(\Gamma_N)]^d$, find (\mathbf{u}, p) such that

$$-\nabla \cdot \{\mu(x, |e(\mathbf{u})|) e(\mathbf{u})\} + \nabla p = \mathbf{f} \quad \text{in } \Omega \quad (1.4)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (1.5)$$

$$\mathbf{u} = \mathbf{g}_D \quad \text{on } \Gamma_D, \quad (1.6)$$

$$\{\mu(x, |e(\mathbf{u})|) e(\mathbf{u}) - pI\} \cdot \nu = \mathbf{g}_N \quad \text{on } \Gamma_N, \quad (1.7)$$

where $\mathbf{u} = (u_1, \dots, u_d)^T$ is the velocity, p is the pressure, $\mathbf{f} = (f_1, \dots, f_d)^T$ is the applied force, I is the $d \times d$ identity matrix, $e(\mathbf{u})$ is the symmetric $d \times d$ strain tensor

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whose entries are

$$e_{ij}(\mathbf{u}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i, j = 1, \dots, d,$$

and $|e(\mathbf{u})|$ is the Frobenius norm of $e(\mathbf{u})$ defined by

$$|e(\mathbf{u})|^2 = e(\mathbf{u}) : e(\mathbf{u}) = \sum_{i,j=1}^d [e_{ij}(\mathbf{u})]^2.$$

In particular, if $X = \text{diag}(x_1, \dots, x_d)$ is a diagonal matrix, then the Frobenius norm $|X| = (X : X)^{1/2}$ of X is equal to the Euclidean norm $(\mathbf{x}^T \mathbf{x})^{1/2}$ of the vector $\mathbf{x} = (x_1, \dots, x_d)^T$ consisting of the diagonal entries of X .

When Γ_N is empty, the pressure p in problem (1.4)–(1.7) is only determined up to a constant; in that case, we supplement the problem with the condition

$$\int_{\Omega} p \, dx = 0. \quad (1.8)$$

We shall assume throughout that the function μ satisfies the following assumption.

(A) $\mu \in C(\bar{\Omega} \times [0, \infty))$ and there exist positive constants m_μ and M_μ such that

$$m_\mu(t-s) \leq \mu(x, t)t - \mu(x, s)s \leq M_\mu(t-s), \quad t \geq s \geq 0, \quad x \in \bar{\Omega}. \quad (1.9)$$

When μ satisfies (1.9), it follows from [4], Lemma 2.1 (for the case of $d = 2$, the case of $d > 2$ being analogous), that there exist positive constants C_1 and C_2 , $C_1 \geq C_2$, such that for all $d \times d$ real symmetric matrices Y and Z , and all $x \in \bar{\Omega}$,

$$|\mu(x, |Y|)Y - \mu(x, |Z|)Z| \leq C_1|Y - Z|, \quad (1.10)$$

$$C_2|Y - Z|^2 \leq (\mu(x, |Y|)Y - \mu(x, |Z|)Z) : (Y - Z). \quad (1.11)$$

By choosing $Y = \text{diag}(y_1, \dots, y_d)$ and $Z = \text{diag}(z_1, \dots, z_d)$, in particular, we deduce that (1.10) and (1.11) also hold when Y and Z are elements of \mathbb{R}^d where then $|\cdot|$ signifies the Euclidean norm on \mathbb{R}^d .

For the sake of notational simplicity we shall suppress the dependence of μ on x and write $\mu(t)$ instead of $\mu(x, t)$. In fact, in many physical applications μ is independent of x . For example, the Carreau law

$$\mu(t) = \mu_\infty + (\mu_0 - \mu_\infty)(1 + \lambda t^2)^{\frac{r-2}{2}},$$

where $\lambda > 0$, $1 < r \leq 2$ and $0 < \mu_\infty < \mu_0$ satisfies (1.9) with $m_\mu = \mu_\infty$ and $M_\mu = \mu_0$.

In recent years there has been considerable interest in discontinuous Galerkin finite element methods for the numerical solution of a wide range of partial differential equations. We shall not attempt to give an extensive survey of this area of research: the reader is referred to [11] for a detailed review. Discontinuous Galerkin Finite Element Methods (DGFEMs) were introduced in the early 1970s for the numerical solution of first-order hyperbolic problems (see [30, 26, 23, 24, 12, 13, 14]). Simultaneously, but quite independently, they were proposed as nonstandard schemes for the approximation of second-order elliptic equations [27, 32, 1]. The recent upsurge of interest in this class of techniques has been stimulated by the computational convenience of DGFEMs due to a high degree of locality, the need to approximate advection-dominated diffusion problems without excessive numerical stabilisation, the

necessity to accommodate high-order *hp*- and spectral element discretisations for first-order hyperbolic equations and advection-diffusion problems [17, 25], and the desire to handle nonlinear hyperbolic problems in a locally conservative manner and without auxiliary numerical stabilisation [10, 15]; see also [8, 9] for the error analysis of the local version of the DGFEM in the elliptic case, as well as [2], [5] and [28].

In the case of linear elliptic boundary value problems, two prominent techniques have emerged, referred to, respectively, as the symmetric and nonsymmetric interior penalty DGFEM (cf. [1], [29]). A common feature of the two methods is that the associated bilinear forms involve terms that penalise the jump $\llbracket u_h \rrbracket$ in the numerical solution u_h over internal faces e in the subdivision of Ω . For example, in the case of Poisson's equation $-\nabla \cdot \tau(u) = f$, with $\tau(u) = \nabla u$, the bilinear form $B_S(\cdot, \cdot)$ associated with the *symmetric* version of the interior penalty DGFEM includes the penalty term

$$\sum_{e \in \Omega} \int_e \llbracket u_h \rrbracket (\sigma_e \llbracket v_h \rrbracket - \langle \tau(v_h) \cdot \nu \rangle) \, ds,$$

where ν denotes a unit normal vector assigned to e and $\langle \tau(u_h) \cdot \nu \rangle$ is the arithmetic average of the values of $\tau(u_h) \cdot \nu$ on the two sides of the face e . The bilinear form $B_S(\cdot, \cdot)$ is then symmetric, and is also coercive if the positive penalty parameter σ_e is chosen to be sufficiently large, depending on the local mesh size and the local polynomial degree. In contrast, the bilinear form $B_{NS}(\cdot, \cdot)$ corresponding to the *nonsymmetric* version of the interior-penalty DGFEM includes the penalty term

$$\sum_{e \in \Omega} \int_e \llbracket u_h \rrbracket (\sigma_e \llbracket v_h \rrbracket + \langle \tau(v_h) \cdot \nu \rangle) \, ds.$$

The plus sign in front of $\langle \tau(u_h) \cdot \nu \rangle$ ensures that $B_{NS}(\cdot, \cdot)$ is coercive for any positive value of the penalty parameter σ_e , although this desirable feature is achieved at the expense of rendering the bilinear form nonsymmetric.

The penalty terms for the symmetric and the nonsymmetric versions of the interior penalty DGFEMs are particular incarnations of the more general expression

$$\sum_{e \in \Omega} \int_e \llbracket u_h \rrbracket (\sigma_e \llbracket v_h \rrbracket + \theta \langle \tau(v_h) \cdot \nu \rangle) \, ds,$$

with $\theta = -1$ corresponding to the symmetric and $\theta = 1$ to the nonsymmetric case; we remark that $\theta = 0$ corresponds to the so-called incomplete interior penalty method studied by Sun and Wheeler, cf. [31, 16]. The relative merits of these methods have been widely discussed in the literature (see, for example, [20] for a comparison in the context of duality-based *a posteriori* error estimation).

The purpose of this paper and its companion-article [21] is to formulate and analyse the natural extensions to quasilinear elliptic PDEs of interior penalty *hp*-DGFEM. To the best of our knowledge, our paper is the first attempt in this direction. For the *a priori* error analysis of the *h*-version local discontinuous Galerkin finite element approximation of (1.1)–(1.3) and (1.4)–(1.7), we refer to the articles of Bustinza and Gatica [7] and Gatica, González and Meddahi [18], respectively.

The paper is structured as follows. In Section 2 we formulate the *hp*-version discontinuous Galerkin finite element approximation to (1.1)–(1.3). By using a corollary of Brouwer's Fixed Point Theorem, we show that the discrete problem has a unique solution u_{DG} in the finite element space. Section 3 discusses the error analysis of

the method in the broken $H^1(\Omega)$ -norm. For sufficiently large values of the positive penalty parameter σ_e involved in the definition of the method, depending on the local mesh size and the local polynomial degree, the semilinear form associated with the method is uniformly monotone; together with the Lipschitz continuity of the semilinear form with respect to its first argument, this then leads to precisely the same h -optimal and mildly p -suboptimal rate of convergence in the broken $H^1(\Omega)$ -norm as in the case of a linear elliptic PDE approximated by the interior penalty DGFEM, cf. [22]. More precisely, if $u \in C^1(\Omega) \cap H^k(\Omega)$, $k \geq 2$, then, using discontinuous piecewise polynomials of degree $p \geq 1$, the error between u and u_{DG} , measured in the broken $H^1(\Omega)$ -norm, is $\mathcal{O}(h^{s-1}/p^{k-3/2})$. A similar result will be proved in the companion-paper [21] for (1.4)–(1.7). Section 4 is devoted to numerical experiments. We close with a brief discussion of some open problems in Section 5.

2. Finite element spaces. Let us suppose for simplicity that Ω is a bounded open polyhedral domain in \mathbb{R}^d , and let \mathcal{T} be a subdivision of Ω into disjoint open element domains κ such that $\bar{\Omega} = \bigcup_{\kappa \in \mathcal{T}} \bar{\kappa}$, where \mathcal{T} is regular or 1-irregular, i.e., each face of κ in \mathcal{T} has at most one hanging node, the barycenter of the face. We assume that the family of subdivisions \mathcal{T} is shape-regular (see, for example, pp. 61, 113, and Remark 2.2, p.114, in [6]) and each $\kappa \in \mathcal{T}$ is an affine image of a fixed master element $\hat{\kappa}$; i.e., $\kappa = F_\kappa(\hat{\kappa})$ for all $\kappa \in \mathcal{T}$, where $\hat{\kappa}$ is either the open unit simplex or the open unit hypercube in \mathbb{R}^d . For a nonnegative integer k , we denote by $\mathcal{P}_k(\hat{\kappa})$ the set of polynomials of total degree k on $\hat{\kappa}$. When $\hat{\kappa}$ is the unit hypercube, we also consider $\mathcal{Q}_k(\hat{\kappa})$, the set of all tensor-product polynomials on $\hat{\kappa}$ of degree k in each coordinate direction. To each $\kappa \in \mathcal{T}$ we assign a nonnegative integer p_κ (local polynomial degree) and a nonnegative integer s_κ (local Sobolev index), collect the p_κ , s_κ and F_κ in the vectors $\mathbf{p} = \{p_\kappa : \kappa \in \mathcal{T}\}$, $\mathbf{s} = \{s_\kappa : \kappa \in \mathcal{T}\}$ and $\mathbf{F} = \{F_\kappa : \kappa \in \mathcal{T}\}$, respectively, and consider the finite element space

$$S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F}) = \{v \in L^2(\Omega) : v|_\kappa \circ F_\kappa \in \mathcal{R}_{p_\kappa}(\hat{\kappa}) \quad \forall \kappa \in \mathcal{T}\},$$

where \mathcal{R} is either \mathcal{P} or \mathcal{Q} .

We shall suppose that the polynomial degree vector \mathbf{p} , with $p_\kappa \geq 1$ for each $\kappa \in \mathcal{T}$, has *bounded local variation*, i.e., there exists a constant $\rho \geq 1$ such that, for any pair of elements κ and κ' which share a $(d-1)$ -dimensional face,

$$\rho^{-1} \leq p_\kappa / p_{\kappa'} \leq \rho.$$

We assign to the subdivision \mathcal{T} the broken Sobolev space of composite index \mathbf{s} ,

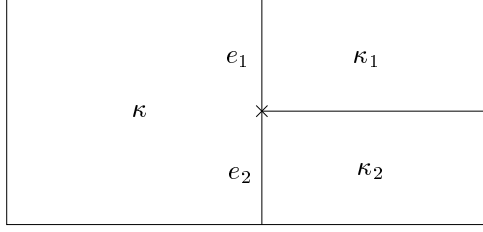
$$H^{\mathbf{s}}(\Omega, \mathcal{T}) = \{v \in L^2(\Omega) : v|_\kappa \in H^{s_\kappa}(\kappa) \quad \forall \kappa \in \mathcal{T}\},$$

equipped with the broken Sobolev norm and corresponding seminorm, respectively,

$$\|v\|_{\mathbf{s}, \mathcal{T}} = \left(\sum_{\kappa \in \mathcal{T}} \|v\|_{H^{s_\kappa}(\kappa)}^2 \right)^{\frac{1}{2}}, \quad |v|_{\mathbf{s}, \mathcal{T}} = \left(\sum_{\kappa \in \mathcal{T}} |v|_{H^{s_\kappa}(\kappa)}^2 \right)^{\frac{1}{2}}.$$

When $s_\kappa = s$ for all $\kappa \in \mathcal{T}$, we shall write $H^s(\Omega, \mathcal{T})$, $\|v\|_{s, \mathcal{T}}$ and $|v|_{s, \mathcal{T}}$.

Let us consider the set \mathcal{E} of all open $(d-1)$ -dimensional faces (open edges when $d = 2$ or open faces when $d = 3$) of all elements $\kappa \in \mathcal{T}$. Given that \mathcal{T} may be irregular, since hanging nodes are permitted in the DGFEM, \mathcal{E} will be understood to contain the *smallest* common $(d-1)$ -dimensional faces of neighbouring elements (cf. Figure 2.1). Further, we denote by \mathcal{E}_{int} the set of all e in \mathcal{E} that are contained

FIG. 2.1. *Hanging node* \times and faces $e_1, e_2 \in \mathcal{E}_{\text{int}}$.

in Ω , we let $\Gamma_{\text{int}} = \{x \in \Omega : x \in e \text{ for some } e \in \mathcal{E}_{\text{int}}\}$ and we introduce the set \mathcal{E}_{D} of $(d-1)$ -dimensional boundary faces contained in the subset Γ_{D} of Γ . Implicit in these definitions is the assumption that \mathcal{T} respects the decomposition of Γ in the sense that each $e \in \mathcal{E}$ that lies on Γ belongs to the interior of exactly one of Γ_{D} and Γ_{N} .

Suppose that e is a $(d-1)$ -dimensional face of an element $\kappa \in \mathcal{T}$; then, the following *inverse inequalities* hold: there exists a positive constant C_3 , independent of the discretisation parameters, such that

$$\|w\|_{L^2(e)}^2 \leq C_3 \frac{p_\kappa^2}{h_e} \|w\|_{L^2(\kappa)}^2 \quad \text{and} \quad \|\nabla w\|_{L^2(e)}^2 \leq C_3 \frac{p_\kappa^2}{h_e} \|\nabla w\|_{L^2(\kappa)}^2 \quad (2.1)$$

for all $w \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$. Here, h_e is the diameter of the face e . Due to our assumption that the subdivision \mathcal{T} is shape-regular, if $e \subset \partial\kappa$ then h_e in these inequalities can be replaced by h_κ , the diameter of κ , at the expense of altering the constant C_3 .

Given that $e \in \mathcal{E}_{\text{int}}$, there exist indices i and j such that $i > j$ and κ_i and κ_j share the face e ; we define the (element-numbering-dependent) jump of $v \in H^1(\Omega, \mathcal{T})$ across e and the mean value of v on e by

$$[[v]]_e = v|_{\partial\kappa_i \cap e} - v|_{\partial\kappa_j \cap e} \quad \text{and} \quad \langle v \rangle_e = \frac{1}{2} (v|_{\partial\kappa_i \cap e} + v|_{\partial\kappa_j \cap e}),$$

respectively. If there is no danger of confusion, the subscript e will be suppressed. Additionally, we associate with the face e the unit normal vector ν which points from κ_i to κ_j .

With these notations and $\theta \in [-1, 1]$, we introduce the semilinear form

$$\begin{aligned} B(w, v) &= \sum_{\kappa \in \mathcal{T}} \int_{\kappa} \mu(|\nabla w|) \nabla w \cdot \nabla v \, dx \\ &\quad - \int_{\Gamma_{\text{D}}} \mu(|\nabla w|) \frac{\partial w}{\partial \nu} v \, ds - \int_{\Gamma_{\text{int}}} \langle \mu(|\nabla w|) \frac{\partial w}{\partial \nu} \rangle [[v]] \, ds \\ &\quad + \theta \int_{\Gamma_{\text{D}}} \mu(h^{-1}|w - g_{\text{D}}|) \frac{\partial v}{\partial \nu} (w - g_{\text{D}}) \, ds + \theta \int_{\Gamma_{\text{int}}} \langle \mu(h^{-1}|[w]|) \frac{\partial v}{\partial \nu} \rangle [[w]] \, ds \\ &\quad + \int_{\Gamma_{\text{D}}} \sigma w v \, ds + \int_{\Gamma_{\text{int}}} \sigma [[w]] [[v]] \, ds, \end{aligned} \quad (2.2)$$

and the linear functional

$$\ell(v) = \sum_{\kappa \in \mathcal{T}} \int_{\kappa} f v \, dx + \int_{\Gamma_{\text{D}}} \sigma g_{\text{D}} v \, ds + \int_{\Gamma_{\text{N}}} g_{\text{N}} v \, ds. \quad (2.3)$$

Here, $h^{-1}|_e = h_e^{-1}$ for all $e \in \Gamma_D \cup \Gamma_{\text{int}}$. Let $\kappa \in \mathcal{T}$ and let e be a $(d-1)$ -dimensional face of $\partial\kappa$. The *discontinuity penalisation parameter* σ , featuring in $B(\cdot, \cdot)$ and $\ell(\cdot)$ above, is defined by

$$\sigma|_e = \sigma_e = \alpha \frac{\langle p^2 \rangle_e}{h_e} \quad \text{for } e \in \Gamma_D \cup \Gamma_{\text{int}}, \quad (2.4)$$

with the convention that if $e \subset \Gamma_D$, and thereby $e \subset \partial\kappa \cap \Gamma_D$ for some $\kappa \in \mathcal{T}$, then $\langle p^2 \rangle_e = p_\kappa^2$. Here α is a positive constant whose size will be fixed later on. We shall see that, at least for the purposes of the analysis pursued here, $\alpha \geq (1 + |\theta|)^2 C_1^2 C_2^{-1} C_3 C_d$ with $C_d = 2^d d + 2d$ and $\theta \in [-1, 1]$ will suffice.

The *hp*-DGFEM approximation of problem (1.1)–(1.3) is: find $u_{\text{DG}} \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$ such that

$$B(u_{\text{DG}}, v) = \ell(v) \quad \forall v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}). \quad (2.5)$$

REMARK 2.1. *In (2.2), the role of the fourth and fifth integral (namely those that are multiplied by θ) is, respectively, to weakly and approximately enforce the Dirichlet boundary condition $u = g_D$ on Γ_D and the continuity condition $\llbracket u \rrbracket = 0$ on Γ_{int} satisfied by the analytical solution u . The choice of the factors $\mu(h^{-1}|w - g_D|)$ and $\mu(h^{-1}|\llbracket w \rrbracket|)$ appearing in the corresponding integrands is, in principle, arbitrary. However, in the present context our choice has been guided by the following three requirements:*

- (a) *when the problem is linear we would like our scheme to collapse to a standard hp-DGFEM scheme (cf. [1], [2] or [29], for example);*
- (b) *in our analysis, we wish to make use of the monotonicity condition (A). This, in turn dictates that the arguments of μ in the two relevant terms should be multiples of $|w - g_D|$ and $|\llbracket w \rrbracket|$, respectively;*
- (c) *while any multiple of $|w - g_D|$ and $|\llbracket w \rrbracket|$ would have been appropriate, for reasons of scaling we have chosen to use $h^{-1}|w - g_D|$ and $h^{-1}|\llbracket w \rrbracket|$, so that the terms $\mu(h^{-1}|w - g_D|)$ and $\mu(h^{-1}|\llbracket w \rrbracket|)$ resemble $\mu(|\nabla w|)$.*

Before embarking on the proof of the existence and uniqueness of solutions to (2.5), we shall make some preparatory observations.

Let us consider $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$ equipped with the norm $\|\cdot\|_{1,h}$ defined by

$$\|v\|_{1,h} = \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla v|^2 dx + \int_{\Gamma_D} \sigma v^2 ds + \int_{\Gamma_{\text{int}}} \sigma \llbracket v \rrbracket^2 ds \right)^{1/2},$$

induced by the inner product $(\cdot, \cdot)_{1,h}$, where

$$(w, v)_{1,h} = \sum_{\kappa \in \mathcal{T}} \int_{\kappa} \nabla w \cdot \nabla v dx + \int_{\Gamma_D} \sigma w v ds + \int_{\Gamma_{\text{int}}} \sigma \llbracket w \rrbracket \llbracket v \rrbracket ds.$$

The next two lemmas stem, respectively, from (1.10) and (1.11).

LEMMA 2.2. *The semilinear form $B(\cdot, \cdot)$ is Lipschitz-continuous in its first argument in the sense that*

$$|B(w_1, v) - B(w_2, v)| \leq C_4 \|w_1 - w_2\|_{1,h} \|v\|_{1,h} \quad \forall w_1, w_2, v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}), \quad (2.6)$$

where $C_4 = \max\{C_1, 1\} + C_1(C_3 C_d \alpha^{-1})^{1/2}$, $\theta \in [-1, 1]$ and $\alpha > 0$.

Proof. Using the fact that $|\frac{\partial v}{\partial \nu}| = |\nabla v \cdot \nu| \leq |\nabla v|$, we have that

$$\begin{aligned}
|B(w_1, v) - B(w_2, v)| &\leq \sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\mu(|\nabla w_1|) \nabla w_1 - \mu(|\nabla w_2|) \nabla w_2| |\nabla v| \, dx \\
&+ \int_{\Gamma_D} |\mu(|\nabla w_1|) \nabla w_1 - \mu(|\nabla w_2|) \nabla w_2| |v| \, ds \\
&+ \int_{\Gamma_{\text{int}}} \langle \mu(|\nabla w_1|) \nabla w_1 - \mu(|\nabla w_2|) \nabla w_2 \rangle \llbracket v \rrbracket \, ds \\
&+ |\theta| \int_{\Gamma_D} |\mu(h^{-1}|w_1 - g_D|)(w_1 - g_D) - \mu(h^{-1}|w_2 - g_D|)(w_2 - g_D)| |\nabla v| \, ds \\
&+ |\theta| \int_{\Gamma_{\text{int}}} |\mu(h^{-1} \llbracket w_1 \rrbracket) \llbracket w_1 \rrbracket - \mu(h^{-1} \llbracket w_2 \rrbracket) \llbracket w_2 \rrbracket| \langle |\nabla v| \rangle \, ds \\
&+ \int_{\Gamma_D} \sigma |w_1 - w_2| |v| \, ds + \int_{\Gamma_{\text{int}}} \sigma \llbracket w_1 - w_2 \rrbracket \llbracket v \rrbracket \, ds \\
&\equiv T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7.
\end{aligned} \tag{2.7}$$

Let $w = w_1 - w_2$. Applying (1.10) and the Cauchy-Schwarz inequality, it follows that

$$T_1 \leq C_1 \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla w|^2 \, dx \right)^{1/2} \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla v|^2 \, dx \right)^{1/2}. \tag{2.8}$$

For T_6 and T_7 , we have

$$T_6 \leq \left(\int_{\Gamma_D} \sigma |w|^2 \, ds \right)^{1/2} \left(\int_{\Gamma_D} \sigma |v|^2 \, ds \right)^{1/2}, \tag{2.9}$$

$$T_7 \leq \left(\int_{\Gamma_{\text{int}}} \sigma \llbracket w \rrbracket^2 \, ds \right)^{1/2} \left(\int_{\Gamma_{\text{int}}} \sigma \llbracket v \rrbracket^2 \, ds \right)^{1/2}. \tag{2.10}$$

For T_2 , (1.10) and the Cauchy-Schwarz inequality yield the bound

$$T_2 \leq C_1 \int_{\Gamma_D} |\nabla w| |v| \, ds \leq C_1 \left(\int_{\Gamma_D} \sigma^{-1} |\nabla w|^2 \, ds \right)^{1/2} \left(\int_{\Gamma_D} \sigma |v|^2 \, ds \right)^{1/2}. \tag{2.11}$$

Hence, using the second of the inverse inequalities (2.1) and recalling the definition of the penalty parameter σ_e on $e \subset \Gamma_D$, we have that

$$T_2 \leq C_1 (C_3 \alpha^{-1} 2d)^{1/2} \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla w|^2 \, dx \right)^{1/2} \left(\int_{\Gamma_D} \sigma |v|^2 \, ds \right)^{1/2}, \tag{2.12}$$

where, $2d$ denotes the maximum number of faces an element may possess which lie on the boundary of the computational domain Ω . Analogously,

$$\begin{aligned}
T_3 &\leq C_1 \int_{\Gamma_{\text{int}}} \langle |\nabla w| \rangle \llbracket v \rrbracket \, ds \\
&\leq C_1 \left(\int_{\Gamma_{\text{int}}} \sigma^{-1} \langle |\nabla w| \rangle^2 \, ds \right)^{1/2} \left(\int_{\Gamma_{\text{int}}} \sigma \llbracket v \rrbracket^2 \, ds \right)^{1/2}.
\end{aligned}$$

Let us write

$$\int_{\Gamma_{\text{int}}} \sigma^{-1} \langle |\nabla w|^2 \rangle^2 ds = \sum_{e \in \mathcal{E}_{\text{int}}} \sigma_e^{-1} \int_e \langle |\nabla w|^2 \rangle^2 ds,$$

and, for $e \in \mathcal{E}_{\text{int}}$, let κ and κ' be the two elements that share e . Then,

$$\begin{aligned} \int_e \langle |\nabla w|^2 \rangle^2 ds &\leq \frac{1}{2} \int_e |\nabla w|_{\kappa}|^2 ds + \frac{1}{2} \int_e |\nabla w|_{\kappa'}|^2 ds \\ &\leq C_3 \frac{p_{\kappa}^2}{2h_e} \int_{\kappa} |\nabla w|^2 dx + C_3 \frac{p_{\kappa'}^2}{2h_e} \int_{\kappa'} |\nabla w|^2 dx \\ &\leq C_3 \frac{\langle p^2 \rangle|_e}{h_e} \max \left\{ \int_{\kappa} |\nabla w|^2 dx, \int_{\kappa'} |\nabla w|^2 dx \right\}. \end{aligned}$$

On recalling from the definition of σ that

$$\sigma_e = \alpha \frac{\langle p^2 \rangle|_e}{h_e} \quad \text{for } e \in \mathcal{E}_{\text{int}},$$

we have that

$$\sum_{e \in \mathcal{E}_{\text{int}}} \sigma_e^{-1} \int_e \langle |\nabla w|^2 \rangle^2 ds \leq C_3 \alpha^{-1} \sum_{e \in \mathcal{E}_{\text{int}}} \max_{\{\kappa : e \subset \partial \kappa\}} \int_{\kappa} |\nabla w|^2 dx.$$

Thanks to our assumption that no face e of any element $\kappa \in \mathcal{T}$ contains more than one hanging node, it follows that no element κ can have more than $2d \cdot 2^{d-1} = 2^d d$ faces if $\hat{\kappa}$ is the d -dimensional hypercube, or more than $(d+1)d$ faces if $\hat{\kappa}$ is the d -dimensional simplex. On writing $c_d = \max\{2^d d, (d+1)d\} = 2^d d$, we then have that

$$\sum_{e \in \mathcal{E}_{\text{int}}} \sigma_e^{-1} \int_e \langle |\nabla w|^2 \rangle^2 ds \leq C_3 \alpha^{-1} c_d \sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla w|^2 dx,$$

and hence

$$T_3 \leq C_1 (C_3 \alpha^{-1} c_d)^{1/2} \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla w|^2 dx \right)^{1/2} \left(\int_{\Gamma_{\text{int}}} \sigma \llbracket v \rrbracket^2 ds \right)^{1/2}. \quad (2.13)$$

For T_4 , we have, in exactly the same way as for T_2 (only, exchanging v and w),

$$T_4 \leq |\theta| C_1 (C_3 \alpha^{-1} 2d)^{1/2} \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla v|^2 dx \right)^{1/2} \left(\int_{\Gamma_D} \sigma |w|^2 ds \right)^{1/2}. \quad (2.14)$$

For T_5 , in the same way as for T_3 (only, exchanging v and w),

$$T_5 \leq |\theta| C_1 (C_3 \alpha^{-1} c_d)^{1/2} \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla v|^2 dx \right)^{1/2} \left(\int_{\Gamma_{\text{int}}} \sigma \llbracket w \rrbracket^2 ds \right)^{1/2}. \quad (2.15)$$

Substituting the bounds on T_1, \dots, T_7 into (2.7), recalling that $w = w_1 - w_2$, and collecting the constants, we deduce that

$$|B(w_1, v) - B(w_2, v)| \leq C_4 \|w_1 - w_2\|_{1,h} \|v\|_{1,h} \quad \forall w_1, w_2, v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}), \quad (2.16)$$

where $C_4 = \max\{C_1, 1\} + C_1(C_3C_d\alpha^{-1})^{1/2}$ and $C_d = c_d + 2d = 2^d d + 2d$. \square

We note, in particular, that if $\alpha \geq (1 + |\theta|)^2 C_1^2 C_2^{-1} C_3 C_d$, as we shall assume from now on, then $C_4 \leq \max\{C_1, 1\} + C_2^{1/2}$. Hence we may set $C_4 = \max\{C_1, 1\} + C_2^{1/2}$.

LEMMA 2.3. *Suppose that $\theta \in [-1, 1]$ and $\alpha \geq (1 + |\theta|)^2 C_1^2 C_2^{-1} C_3 C_d$; then, the semilinear form $B(\cdot, \cdot)$ is uniformly monotone in the sense that*

$$B(w_1, w_1 - w_2) - B(w_2, w_1 - w_2) \geq \frac{1}{2} \min\{C_2, 1\} \|w_1 - w_2\|_{1,h}^2 \quad \forall w_1, w_2 \in S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F}). \quad (2.17)$$

Proof. Using (1.10) and (1.11) and writing $w = w_1 - w_2$, we have that

$$\begin{aligned} & B(w_1, w_1 - w_2) - B(w_2, w_1 - w_2) \\ & \geq C_2 \sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla w|^2 dx - C_1(1 + |\theta|) \left(\int_{\Gamma_D} \sigma^{-1} |\nabla w|^2 ds \right)^{1/2} \left(\int_{\Gamma_D} \sigma |w|^2 ds \right)^{1/2} \\ & \quad - C_1(1 + |\theta|) \left(\int_{\Gamma_{\text{int}}} \sigma^{-1} |\nabla w|^2 ds \right)^{1/2} \left(\int_{\Gamma_{\text{int}}} \sigma [w]^2 ds \right)^{1/2} \\ & \quad + \int_{\Gamma_D} \sigma |w|^2 ds + \int_{\Gamma_{\text{int}}} \sigma [w]^2 ds. \end{aligned}$$

In exactly the same way as in the case of terms T_2 and T_3 in the proof of Lemma 2.2, we have

$$\left(\int_{\Gamma_D} \sigma^{-1} |\nabla w|^2 ds \right)^{1/2} \leq (C_3 \alpha^{-1} 2d)^{1/2} \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla w|^2 dx \right)^{1/2}$$

and

$$\left(\int_{\Gamma_{\text{int}}} \sigma^{-1} |\nabla w|^2 ds \right)^{1/2} \leq (C_3 \alpha^{-1} c_d)^{1/2} \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla w|^2 dx \right)^{1/2}.$$

Therefore,

$$\begin{aligned} & B(w_1, w_1 - w_2) - B(w_2, w_1 - w_2) \geq C_2 \sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla w|^2 dx \\ & \quad - ((1 + |\theta|)^2 C_1^2 C_3 \alpha^{-1} 2d)^{1/2} \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla w|^2 dx \right)^{1/2} \left(\int_{\Gamma_D} \sigma |w|^2 ds \right)^{1/2} \\ & \quad - ((1 + |\theta|)^2 C_1^2 C_3 \alpha^{-1} c_d)^{1/2} \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla w|^2 dx \right)^{1/2} \left(\int_{\Gamma_{\text{int}}} \sigma [w]^2 ds \right)^{1/2} \\ & \quad + \int_{\Gamma_D} \sigma |w|^2 ds + \int_{\Gamma_{\text{int}}} \sigma [w]^2 ds. \end{aligned}$$

Applying Cauchy's inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ to the second and third terms on the right-hand side and recalling that $C_d = c_d + 2d$, we have

$$\begin{aligned} & B(w_1, w_1 - w_2) - B(w_2, w_1 - w_2) \geq C_2 \left(1 - \frac{(1 + |\theta|)^2 C_1^2 C_3 C_d}{2C_2 \alpha} \right) \sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla w|^2 dx \\ & \quad + \frac{1}{2} \int_{\Gamma_D} \sigma |w|^2 ds + \frac{1}{2} \int_{\Gamma_{\text{int}}} \sigma [w]^2 ds. \end{aligned}$$

Thus, on selecting α such that $\alpha \geq (1 + |\theta|)^2 C_1^2 C_2^{-1} C_3 C_d$, we deduce (2.17). \square

Now we are ready to show the existence of a unique solution to (2.5). We shall make use of the following corollary to Brouwer's Fixed Point Theorem (see, [19], p. 105).

PROPOSITION 2.4. *Let H be a finite-dimensional Hilbert space whose scalar product and norm are denoted, respectively, by (\cdot, \cdot) and $|\cdot|$. Let P be a continuous mapping from H into H with the following property: there exists $\xi > 0$ such that*

$$(P(w), w) > 0 \quad \forall w \in H \text{ with } |w| = \xi. \quad (2.18)$$

Then, there exists an element u in H such that

$$|u| \leq \xi, \quad P(u) = 0. \quad (2.19)$$

THEOREM 2.5. *Suppose that $\theta \in [-1, 1]$ and $\alpha \geq (1 + |\theta|)^2 C_1^2 C_2^{-1} C_3 C_d$; then, there exists a unique element u_{DG} in $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$ such that (2.5) holds.*

Proof. As a first step in our argument, we shall rewrite the numerical method (2.5) as a nonlinear operator equation $P(u) = 0$ on $H \equiv S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$. We shall do so by exploiting the Riesz Representation Theorem from Hilbert space theory.

It is a straightforward matter to show that $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$ is a finite-dimensional Hilbert space with the norm $\|\cdot\|_{1,h}$ induced by the inner product $(\cdot, \cdot)_{1,h}$. Let us consider a second norm, $\|\cdot\|_{1,h}^*$, on $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$ defined by

$$\|v\|_{1,h}^* = \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} (|\nabla v|^2 + v^2) \, dx + \int_{\Gamma_{\text{int}}} \sigma [v]^2 \, ds + \int_{\Gamma_{\text{D}}} \sigma v^2 \, ds + \int_{\Gamma_{\text{N}}} v^2 \, ds \right)^{1/2}.$$

Since $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$ has finite dimension, the norms $\|\cdot\|_{1,h}$ and $\|\cdot\|_{1,h}^*$ are equivalent on $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$; that is, there exists a positive constant c_δ , dependent on the dimension δ of $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$, such that

$$\|v\|_{1,h} \leq \|v\|_{1,h}^* \leq c_\delta \|v\|_{1,h} \quad \forall v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}). \quad (2.20)$$

Given any w in $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$, consider the linear functional

$$\psi_w : v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}) \mapsto \psi_w(v) = B(w, v) - \ell(v) \in \mathbb{R}.$$

The fact that $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$ is finite-dimensional implies that ψ_w is a bounded linear functional on $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$. The actual bound of $\psi_w(v)$ is easily established: by the Cauchy-Schwarz inequality,

$$\begin{aligned} |\ell(v)| &\leq \left(\int_{\Omega} f^2 \, dx \right)^{1/2} \left(\int_{\Omega} v^2 \, dx \right)^{1/2} + \left(\int_{\Gamma_{\text{D}}} \sigma g_{\text{D}}^2 \, ds \right)^{1/2} \left(\int_{\Gamma_{\text{D}}} \sigma v^2 \, ds \right)^{1/2} \\ &\quad + \left(\int_{\Gamma_{\text{N}}} g_{\text{N}}^2 \, ds \right)^{1/2} \left(\int_{\Gamma_{\text{N}}} v^2 \, ds \right)^{1/2}, \end{aligned}$$

and therefore

$$\begin{aligned} |\ell(v)| &\leq \left(\int_{\Omega} f^2 \, dx + \int_{\Gamma_{\text{D}}} \sigma g_{\text{D}}^2 \, ds + \int_{\Gamma_{\text{N}}} g_{\text{N}}^2 \, ds \right)^{1/2} \\ &\quad \times \left(\int_{\Omega} v^2 \, dx + \int_{\Gamma_{\text{D}}} \sigma v^2 \, ds + \int_{\Gamma_{\text{N}}} v^2 \, ds \right)^{1/2}, \end{aligned}$$

which yields that

$$|\ell(v)| \leq \left(\int_{\Omega} f^2 dx + \int_{\Gamma_D} \sigma g_D^2 ds + \int_{\Gamma_N} g_N^2 ds \right)^{1/2} \|v\|_{1,h}^*.$$

By the norm-equivalence (2.20), we then have that

$$|\ell(v)| \leq C_5 \|v\|_{1,h} \quad \forall v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}), \quad (2.21)$$

where

$$C_5 = c_{\delta} \left(\int_{\Omega} f^2 dx + \int_{\Gamma_D} \sigma g_D^2 ds + \int_{\Gamma_N} g_N^2 ds \right)^{1/2}.$$

On the other hand, by (2.6) with $w_1 = w$ and $w_2 = 0$, we have that

$$|B(w, v)| \leq C_4 \|w\|_{1,h} \|v\|_{1,h} \quad \forall v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}),$$

and therefore,

$$|\psi_w(v)| \leq (C_4 \|w\|_{1,h} + C_5) \|v\|_{1,h} \quad \forall v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}).$$

Since the linear functional ψ_w is bounded (and therefore continuous) on $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$, by virtue of the Riesz Representation Theorem, there exists $P(w)$ in $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$ such that

$$\psi_w(v) = (P(w), v)_{1,h} \quad \forall v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}). \quad (2.22)$$

As w passes through $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$, (2.22) defines the mapping

$$w \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}) \mapsto P(w) \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$$

of $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$ into itself.

Next, we show that the mapping $w \mapsto P(w)$ is Lipschitz continuous (and, thereby, continuous) in the norm $\|\cdot\|_{1,h}$, uniformly in the mesh size $h = \max h_{\kappa}$ and the polynomial degree vector \mathbf{p} . Clearly,

$$(P(w_1) - P(w_2), v)_{1,h} = B(w_1, v) - B(w_2, v).$$

Hence, by Lemma 2.2,

$$\begin{aligned} |(P(w_1) - P(w_2), v)_{1,h}| &= |B(w_1, v) - B(w_2, v)| \\ &\leq C_4 \|w_1 - w_2\|_{1,h} \|v\|_{1,h} \end{aligned} \quad (2.23)$$

for all w_1, w_2, v in $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$, and therefore,

$$\begin{aligned} \|P(w_1) - P(w_2)\|_{1,h} &= \sup_{v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})} \frac{|(P(w_1) - P(w_2), v)_{1,h}|}{\|v\|_{1,h}} \\ &\leq C_4 \|w_1 - w_2\|_{1,h} \end{aligned}$$

for all $w_1, w_2 \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$. Thus, P is a Lipschitz continuous mapping of (the Hilbert space) $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$ into itself, with Lipschitz constant C_4 independent of the discretisation parameters.

In order to apply Proposition 2.4, it remains to show that there exists $\xi > 0$ such that $(P(w), w)_{1,h} > 0$ for all w in $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$ with $\|w\|_{1,h} = \xi$. Clearly,

$$(P(w), w)_{1,h} = \psi_w(w) = B(w, w) - \ell(w).$$

Now, Lemma 2.3, with $w_1 = w$, $w_2 = 0$ and $\alpha \geq (1 + |\theta|)^2 C_1^2 C_2^{-1} C_3 C_d$, implies that

$$B(w, w) \geq \frac{1}{2} \min\{C_2, 1\} \|w\|_{1,h}^2. \quad (2.24)$$

Combining (2.24) and (2.21) gives

$$\begin{aligned} (P(w), w) &= B(w, w) - \ell(w) \\ &\geq \frac{1}{2} \min\{C_2, 1\} \|w\|_{1,h}^2 - C_5 \|w\|_{1,h} \quad \forall w \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}). \end{aligned}$$

In particular, if $\xi \in \mathbb{R}$ and $\xi > 2C_5 / \min\{C_2, 1\}$, then, for any $w \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$ such that $\|w\|_{1,h} = \xi$, we have that $(P(w), w)_{1,h} > 0$. By applying Proposition 2.4 we deduce the *existence* of a solution u_{DG} to (2.5) in $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$.

To prove the uniqueness of the solution to (2.5), suppose that u_{DG} and u'_{DG} are two solutions to (2.5) in $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$. Then,

$$B(u_{\text{DG}}, v) - B(u'_{\text{DG}}, v) = 0 \quad \forall v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}),$$

and thereby also

$$B(u_{\text{DG}}, u_{\text{DG}} - u'_{\text{DG}}) - B(u'_{\text{DG}}, u_{\text{DG}} - u'_{\text{DG}}) = 0.$$

On applying (2.17) with $w_1 = u_{\text{DG}}$, $w_2 = u'_{\text{DG}}$ and $\alpha \geq (1 + |\theta|)^2 C_1^2 C_2^{-1} C_3 C_d$, we get

$$B(u_{\text{DG}}, u_{\text{DG}} - u'_{\text{DG}}) - B(u'_{\text{DG}}, u_{\text{DG}} - u'_{\text{DG}}) \geq \frac{1}{2} \min\{C_2, 1\} \|u_{\text{DG}} - u'_{\text{DG}}\|_{1,h}^2. \quad (2.25)$$

Since the left-hand side of this inequality is equal to 0 and $\|\cdot\|_{1,h}$ is a norm on $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$, we deduce that $u_{\text{DG}} - u'_{\text{DG}} = 0$, which establishes the uniqueness of the solution to (2.5). \square

3. Error analysis of the method. We recall the following approximation result for the finite element space $S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$.

LEMMA 3.1. *Suppose that $\kappa \in \mathcal{T}$ is a d -simplex or d -parallelepiped of diameter h_κ . Suppose further that $u|_\kappa \in H^{k_\kappa}(\kappa)$, $k_\kappa \geq 0$, for $\kappa \in \mathcal{T}$. Then, there exists a sequence $z_{p_\kappa}^{h_\kappa}$ in $\mathcal{R}_{p_\kappa}(\kappa)$, $p_\kappa = 1, 2, \dots$, such that for $0 \leq q \leq k_\kappa$,*

$$\|u - z_{p_\kappa}^{h_\kappa}\|_{H^q(\kappa)} \leq C \frac{h_\kappa^{s_\kappa - q}}{p_\kappa^{k_\kappa - q}} \|u\|_{H^{k_\kappa}(\kappa)},$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u and the discretisation parameters.

Proof. For the proof, see Lemma 4.5 in [3] for $d = 2$; when $d > 2$ the argument is completely analogous. \square

Given $u \in H^2(\Omega, \mathcal{T})$, we now define $\Pi_p^h u \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F})$ by

$$(\Pi_p^h u)|_\kappa = z_{p_\kappa}^{h_\kappa}(u|_\kappa).$$

Then, assuming that $u|_\kappa \in H^{k_\kappa}(\kappa)$, $k_\kappa \geq 2$, for $\kappa \in \mathcal{T}$, and writing

$$\eta = u - \Pi_p^h u,$$

by virtue of Lemma 3.1 we have that

$$\|\eta\|_{L^2(\kappa)}^2 \leq C \frac{h_\kappa^{2s_\kappa}}{p_\kappa^{2k_\kappa}} \|u\|_{H^{k_\kappa}(\kappa)}^2 \quad \text{and} \quad \|\nabla \eta\|_{L^2(\kappa)}^2 \leq C \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-2}} \|u\|_{H^{k_\kappa}(\kappa)}^2,$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u and the discretisation parameters.

The multiplicative trace inequality asserts the existence of a positive constant $C = C(d)$ such that

$$\|\eta\|_{L^2(\partial\kappa)}^2 \leq C(d) \left(\|\eta\|_{L^2(\kappa)} \|\nabla \eta\|_{L^2(\kappa)} + h_\kappa^{-1} \|\eta\|_{L^2(\kappa)}^2 \right).$$

Hence,

$$\|\eta\|_{L^2(\partial\kappa)}^2 \leq C \frac{h_\kappa^{2s_\kappa-1}}{p_\kappa^{2k_\kappa-1}} \|u\|_{H^{k_\kappa}(\kappa)}^2 \quad \text{and} \quad \|\nabla \eta\|_{L^2(\partial\kappa)}^2 \leq C \frac{h_\kappa^{2s_\kappa-3}}{p_\kappa^{2k_\kappa-3}} \|u\|_{H^{k_\kappa}(\kappa)}^2,$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u and the discretisation parameters. Since \mathcal{T} is shape-regular and the polynomial degree vector \mathbf{p} has bounded local variation, it then follows, using (2.4), that

$$\begin{aligned} \sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla \eta|^2 dx + \int_{\Gamma_D} \sigma^{-1} |\nabla \eta|^2 ds + \int_{\Gamma_{\text{int}}} \sigma^{-1} \langle |\nabla \eta| \rangle^2 ds + \int_{\Gamma_D} \sigma \eta^2 ds + \int_{\Gamma_{\text{int}}} \sigma [\![\eta]\!]^2 ds \\ \leq C \sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|_{H^{k_\kappa}(\kappa)}^2, \end{aligned} \quad (3.1)$$

and therefore, also,

$$\begin{aligned} \|\eta\|_{1,h} &= \left(\sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla \eta|^2 dx + \int_{\Gamma_D} \sigma \eta^2 ds + \int_{\Gamma_{\text{int}}} \sigma [\![\eta]\!]^2 ds \right)^{1/2} \\ &\leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|_{H^{k_\kappa}(\kappa)}^2 \right)^{1/2}, \end{aligned} \quad (3.2)$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u and the discretisation parameters.

Our next result is concerned with the consistency of the semilinear form $B(\cdot, \cdot)$.

LEMMA 3.2. *Suppose that $u|_\kappa \in H^{k_\kappa}(\kappa)$, $k_\kappa \geq 2$, for $\kappa \in \mathcal{T}$. Assume, further, that $\theta \in [-1, 1]$ and $\alpha \geq (1 + |\theta|)^2 C_1^2 C_2^{-1} C_3 C_d$; then, the semilinear form $B(\cdot, \cdot)$, satisfies the following bound:*

$$|B(u, v) - B(\Pi_p^h u, v)| \leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|_{H^{k_\kappa}(\kappa)}^2 \right)^{1/2} \|v\|_{1,h} \quad \forall v \in S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F}), \quad (3.3)$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u , v , and the discretisation parameters.

Proof. We proceed in much the same way as in the proof of Lemma 2.2:

$$\begin{aligned}
|B(u, v) - B(\Pi_p^h u, v)| &\leq \sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\mu(|\nabla u|) \nabla u - \mu(|\nabla \Pi_p^h u|) \nabla \Pi_p^h u| |\nabla v| dx \\
&+ \int_{\Gamma_D} |\mu(|\nabla u|) \nabla u - \mu(|\nabla \Pi_p^h u|) \nabla \Pi_p^h u| |v| ds \\
&+ \int_{\Gamma_{\text{int}}} \langle \mu(|\nabla u|) \nabla u - \mu(|\nabla \Pi_p^h u|) \nabla \Pi_p^h u \rangle \llbracket v \rrbracket ds \\
&+ |\theta| \int_{\Gamma_D} |\mu(h^{-1}|u - g_D|)(u - g_D) - \mu(h^{-1}|\Pi_p^h u - g_D|)(\Pi_p^h u - g_D)| |\nabla v| ds \\
&+ |\theta| \int_{\Gamma_{\text{int}}} |\mu(h^{-1} \llbracket u \rrbracket) \llbracket u \rrbracket - \mu(h^{-1} \llbracket \Pi_p^h u \rrbracket) \llbracket \Pi_p^h u \rrbracket| \langle \nabla v \rangle ds \\
&+ \int_{\Gamma_D} \sigma |u - \Pi_p^h u| |v| ds + \int_{\Gamma_{\text{int}}} \sigma \llbracket u - \Pi_p^h u \rrbracket \llbracket v \rrbracket ds \\
&\equiv T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7.
\end{aligned} \tag{3.4}$$

Using (1.10), the Cauchy–Schwarz inequality and (3.2) with $\eta = u - \Pi_p^h u$, we get

$$\begin{aligned}
T_1 &\leq C_1 \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla(u - \Pi_p^h u)|^2 dx \right)^{1/2} \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla v|^2 dx \right)^{1/2} \\
&\leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{H^{k_\kappa}(\kappa)}^2 \right)^{1/2} \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla v|^2 dx \right)^{1/2} \quad \forall v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}), \tag{3.5}
\end{aligned}$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u , v and the discretisation parameters.

For T_2 , (1.10) and the Cauchy–Schwarz inequality yield the bound

$$\begin{aligned}
T_2 &\leq C_1 \int_{\Gamma_D} |\nabla(u - \Pi_p^h u)| |v| ds \\
&\leq C_1 \left(\int_{\Gamma_D} \sigma^{-1} |\nabla(u - \Pi_p^h u)|^2 ds \right)^{1/2} \left(\int_{\Gamma_D} \sigma |v|^2 ds \right)^{1/2}. \tag{3.6}
\end{aligned}$$

Hence, using the bound on the second term on the left-hand side of (3.1), we have that

$$T_2 \leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{H^{k_\kappa}(\kappa)}^2 \right)^{1/2} \left(\int_{\Gamma_D} \sigma |v|^2 ds \right)^{1/2} \quad \forall v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}), \tag{3.7}$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u , v and the discretisation parameters.

Analogously,

$$\begin{aligned}
T_3 &\leq C_1 \int_{\Gamma_{\text{int}}} \langle |\nabla(u - \Pi_p^h u)| \rangle |[[v]]| \, ds \\
&\leq C_1 \left(\int_{\Gamma_{\text{int}}} \sigma^{-1} \langle |\nabla(u - \Pi_p^h u)| \rangle^2 \, ds \right)^{1/2} \left(\int_{\Gamma_{\text{int}}} \sigma [[v]]^2 \, ds \right)^{1/2} \\
&\leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|_{H^{k_\kappa}(\kappa)}^2 \right)^{1/2} \left(\int_{\Gamma_{\text{int}}} \sigma [[v]]^2 \, ds \right)^{1/2} \quad \forall v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}), \quad (3.8)
\end{aligned}$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u, v and the discretisation parameters.

For T_4 , (1.10) and the Cauchy–Schwarz inequality yield the bound

$$\begin{aligned}
T_4 &\leq |\theta| C_1 \int_{\Gamma_D} |u - \Pi_p^h u| |\nabla v| \, ds \\
&\leq |\theta| C_1 \left(\int_{\Gamma_D} \sigma |u - \Pi_p^h u|^2 \, ds \right)^{1/2} \left(\int_{\Gamma_D} \sigma^{-1} |\nabla v|^2 \, ds \right)^{1/2}. \quad (3.9)
\end{aligned}$$

Hence, using the second of the two inverse inequalities in (2.1) in the second factor on the right-hand side of (3.9), the bound on the fourth term on the left-hand side of (3.1) and recalling the definition of the penalty parameter σ_e on $e \subset \Gamma_D$, we have that

$$T_4 \leq |\theta| C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|_{H^{k_\kappa}(\kappa)}^2 \right)^{1/2} \left(\sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla v|^2 \, dx \right)^{1/2} \quad \forall v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}), \quad (3.10)$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u, v and the discretisation parameters. Analogously,

$$T_5 \leq |\theta| C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|_{H^{k_\kappa}(\kappa)}^2 \right)^{1/2} \left(\sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla v|^2 \, dx \right)^{1/2} \quad \forall v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}), \quad (3.11)$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u, v and the discretisation parameters.

Proceeding in the same manner, we obtain the following bounds on T_6 and T_7 :

$$\begin{aligned}
T_6 &\leq \left(\int_{\Gamma_D} \sigma |u - \Pi_p^h u|^2 \, ds \right)^{1/2} \left(\int_{\Gamma_D} \sigma |v|^2 \, ds \right)^{1/2} \\
&\leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|_{H^{k_\kappa}(\kappa)}^2 \right)^{1/2} \left(\int_{\Gamma_D} \sigma |v|^2 \, ds \right)^{1/2} \quad \forall v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}), \quad (3.12)
\end{aligned}$$

$$\begin{aligned}
T_7 &\leq \left(\int_{\Gamma_{\text{int}}} \sigma [[u - \Pi_p^h u]]^2 \, ds \right)^{1/2} \left(\int_{\Gamma_{\text{int}}} \sigma [[v]]^2 \, ds \right)^{1/2} \\
&\leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|_{H^{k_\kappa}(\kappa)}^2 \right)^{1/2} \left(\int_{\Gamma_{\text{int}}} \sigma [[v]]^2 \, ds \right)^{1/2} \quad \forall v \in S^{\mathbf{P}}(\Omega, \mathcal{T}, \mathbf{F}), \quad (3.13)
\end{aligned}$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u , v and the discretisation parameters. Substituting the bounds on T_1, \dots, T_7 into (3.4) we arrive at (3.3). \square

Now we are ready to prove the main result of the paper.

THEOREM 3.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral domain, $\mathcal{T} = \{\kappa\}$ a shape-regular subdivision of Ω into d -simplexes or d -parallelepipeds, and \mathbf{p} a polynomial degree vector of bounded local variation. Suppose, further, that $\theta \in [-1, 1]$, $\alpha \geq (1 + |\theta|)^2 C_1^2 C_2^{-1} C_3 C_d$ and assign to each face $e \in \mathcal{E}_{\text{int}} \cup \mathcal{E}_D$ the positive real number σ_e defined by (2.4) where h_e is the diameter of e . Then, assuming that $u \in C^1(\Omega)$ and $u|_\kappa \in H^{k_\kappa}(\kappa)$, $k_\kappa \geq 2$, for $\kappa \in \mathcal{T}$, the solution $u_{\text{DG}} \in S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F})$ of (2.5) satisfies the error bound*

$$\|u - u_{\text{DG}}\|_{1,h}^2 \leq C \sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|_{H^{k_\kappa}(\kappa)}^2,$$

with $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant independent of u and the discretisation parameters.

Proof. Let us write

$$u - u_{\text{DG}} = (u - \Pi_p^h u) + (\Pi_p^h u - u_{\text{DG}}) \equiv \eta + \xi.$$

Note that since $u \in C^1(\Omega) \cap H^2(\Omega, \mathcal{T})$, we have that $B(u, v) = \ell(v)$ for all v in $S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F})$; in particular, $B(u, \xi) = \ell(\xi)$. We begin by estimating ξ . Recalling (2.17), (2.5) and (3.3), we have that

$$\begin{aligned} \frac{1}{2} \min\{C_2, 1\} \|\xi\|_{1,h}^2 &= \frac{1}{2} \min\{C_2, 1\} \|\Pi_p^h u - u_{\text{DG}}\|_{1,h}^2 \\ &\leq B(\Pi_p^h u, \Pi_p^h u - u_{\text{DG}}) - B(u_{\text{DG}}, \Pi_p^h u - u_{\text{DG}}) \\ &= B(\Pi_p^h u, \xi) - \ell(\xi) = B(\Pi_p^h u, \xi) - B(u, \xi) \\ &\leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|_{H^{k_\kappa}(\kappa)}^2 \right)^{1/2} \|\xi\|_{1,h}. \end{aligned}$$

Therefore,

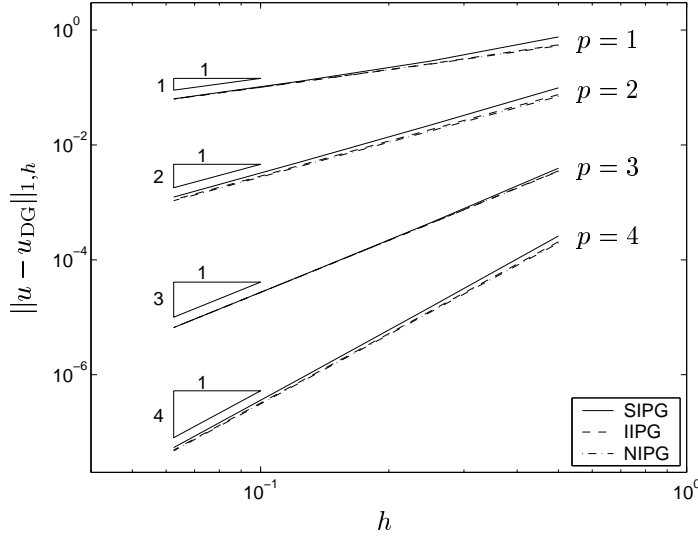
$$\|\xi\|_{1,h} \leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|_{H^{k_\kappa}(\kappa)}^2 \right)^{1/2},$$

which, by the triangle inequality and (3.2), gives

$$\|u - u_{\text{DG}}\|_{1,h} \leq \|\xi\|_{1,h} + \|\eta\|_{1,h} \leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|_{H^{k_\kappa}(\kappa)}^2 \right)^{1/2}, \quad (3.14)$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u and the discretisation parameters. \square

4. Numerical experiments. In this section we present a series of numerical experiments to highlight the practical performance of the interior penalty DG method introduced and analysed in this article for the numerical approximation of the quasi-linear elliptic boundary value problem (1.1)–(1.3). For simplicity, we restrict ourselves to two-dimensional model problems, i.e. $d = 2$; additionally, we note that throughout this section we select the constant appearing in the discontinuity penalisation parameter σ defined in (2.4) as follows: $\alpha = 10$.

FIG. 4.1. *Example 1. Convergence of the DGFEM with h -refinement.*

4.1. Example 1. In this first example we take $\Omega \subset \mathbb{R}^2$ to be the square domain $(-1, 1)^2$ with $\Gamma_D = [-1, 1] \times \{-1\} \cup \{1\} \times [-1, 1]$ and $\Gamma_N = [-1, 1] \times \{1\} \cup \{-1\} \times [-1, 1]$. Furthermore, we set the nonlinear diffusion coefficient as follows:

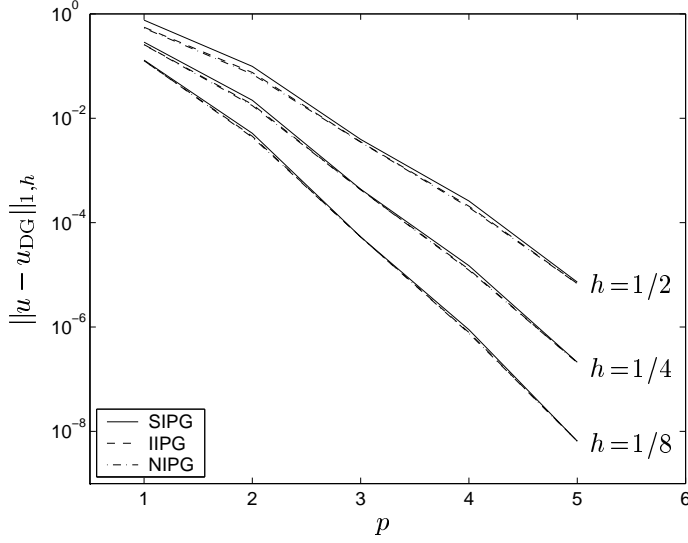
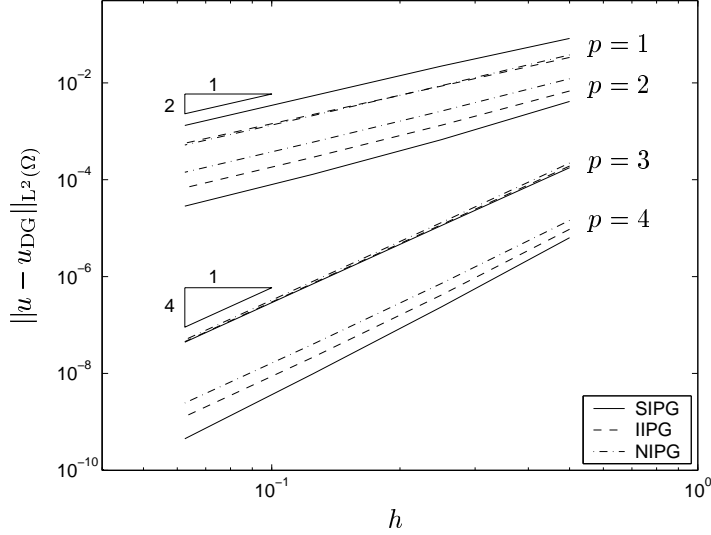
$$\mu(x, |\nabla u|) = 2 + \frac{1}{1 + |\nabla u|}; \quad (4.1)$$

the Dirichlet and Neumann boundary conditions, g_D and g_N , respectively, and the forcing function f are then chosen so that the analytical solution to (1.1)–(1.3) is given by

$$u(x, y) = \cos(\pi x/2) \cos(\pi y/2), \quad (4.2)$$

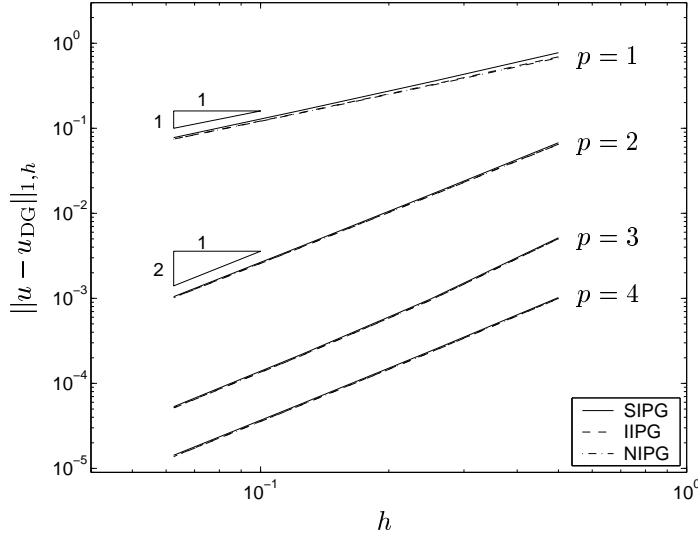
cf. [7]. We remark that a simple calculation verifies that the coefficient μ defined in (4.1) satisfies our assumption **(A)** with $m_\mu = 2$ and $M_\mu = 3$.

We investigate the asymptotic convergence of the hp -DGFEM (2.5) on a sequence of successively finer uniform square meshes for $\mathbf{p} \equiv p = 1, 2, 3, 4$ for different choices of θ . Here, we consider the three most popular choices for θ : $\theta = -1$ corresponding to the symmetric interior penalty method (SIPG), $\theta = 0$ corresponding to the incomplete interior penalty method (IIPG), and $\theta = 1$ corresponding to the nonsymmetric interior penalty method (NIPG). To this end, in Figure 4.1, we present a comparison of the DG-norm $\|\cdot\|_{1,h}$ of the error with the mesh function h for each polynomial degree and each value of θ . Here, we observe that $\|u - u_{DG}\|_{1,h}$ converges to zero, for each fixed p , at the rate $\mathcal{O}(h^p)$ as the mesh is refined, in agreement with Theorem 3.3. In particular, we note that the error in the DG-norm is fairly insensitive to the choice of θ ; indeed, while an increase in the size of θ leads to a decrease in $\|u - u_{DG}\|_{1,h}$, the convergence lines are almost indistinguishable as the mesh is refined, especially in the case of odd-order polynomial degrees. Secondly, we investigate the convergence of the hp -DGFEM with p -enrichment on a fixed mesh. Since the analytical solution (4.2) is a real analytic function, we expect to see exponential rates of convergence as p increases. In Figure 4.2, we plot the DG-norm of the error against p on three different

FIG. 4.2. *Example 1. Convergence of the DGFEM with p -refinement.*FIG. 4.3. *Example 1. Convergence of the DGFEM with h -refinement.*

square meshes for each value of θ . In each case, we observe that on a linear-log scale, the convergence plots become straight lines as the spectral order p is increased, thereby indicating exponential convergence in p .

Finally, in Figure 4.3 we plot the $L^2(\Omega)$ -norm of the error against h for each p and each θ . Here, we observe that, for each of the three choices of θ , the error in the $L^2(\Omega)$ -norm behaves like $\mathcal{O}(h^{p+1})$ for odd p and like $\mathcal{O}(h^p)$ for even p . We remark that, in the case of a linear elliptic partial differential equation, the SIPG scheme ($\theta = -1$) is optimally accurate for both odd and even order polynomial degrees, cf. [20], for example; though both the IIPG and NIPG still exhibit the same lack of optimality when p is even in this case. This loss of optimality of the SIPG

FIG. 4.4. *Example 2. Convergence of the DGFEM with h -refinement.*

scheme for the numerical approximation of the quasilinear elliptic partial differential equation (1.1)–(1.3) when p is even is attributed to the loss of adjoint consistency of the interior penalty method (2.5). By this we mean that the integral-mean-value linearisation of the semilinear form $B(\cdot, \cdot)$ in its first argument is a bilinear form that fails to be adjoint consistent with the bilinear form which arises from the integral-mean-value linearisation of the semilinear form in the (standard) weak formulation of the boundary value problem.

However, we remark that numerical experiments indicate that the SIPG method does not suffer from this sub-optimality for even p when the method is employed for the numerical approximation of semilinear elliptic partial differential equations.

4.2. Example 2. In this second example, we investigate the performance of the hp -DGFEM (2.5) for a problem with a non-smooth solution. To this end, let Ω be the L-shaped domain $(-1, 1)^2 \setminus [0, 1) \times (-1, 0]$, with $\Gamma_D = \partial\Omega$,

$$\mu(x, |\nabla u|) = 1 + e^{-|\nabla u|^2}, \quad (4.3)$$

and select g_D and f so that the analytical solution to (1.1)–(1.3) is given by

$$u(x, y) = \cos(\pi y/2) \chi(x) x^{2.5}, \quad (4.4)$$

where $\chi : \mathbb{R} \rightarrow \mathbb{R}$ denotes the characteristic function of the interval $(0, 1) \subset \mathbb{R}$, cf. [7]. Again, as in the above example, the coefficient μ defined in (4.3) satisfies our assumption **(A)** with $m_\mu = 1 - \sqrt{2/e}$ and $M_\mu = 1$. The analytical solution given by (4.4) contains a singularity along the line $x = 0$; in particular, we note that u lies in the Sobolev space $H^{3-\varepsilon}(\Omega)$, for any $\varepsilon > 0$.

In this example we again consider the convergence of the hp -DGFEM (2.5) on a sequence of successively finer uniform square meshes for $p = 1, 2, 3, 4$ and $\theta = -1, 0, 1$. To this end, in Figure 4.4 we plot the DG-norm of the error against h for each p and each θ . Here, we observe that for each of the three methods considered the error, $\|u - u_{DG}\|_{1,h}$, converges to zero at the optimal rate $\mathcal{O}(h^{\min(p+1, k)-1})$ predicted by

p	SIPG ($\theta = -1$)		IIPG ($\theta = 0$)		NIPG ($\theta = 1$)	
	$\ u - u_{\text{DG}}\ _{1,h}$	k	$\ u - u_{\text{DG}}\ _{1,h}$	k	$\ u - u_{\text{DG}}\ _{1,h}$	k
1	7.745e-1	-	6.927e-1	-	6.737e-1	-
2	6.749e-2	3.52	6.505e-2	3.41	6.463e-2	3.38
3	5.163e-3	6.34	5.033e-3	6.31	5.017e-3	6.30
4	1.021e-3	5.63	9.994e-4	5.62	9.949e-4	5.62
5	3.813e-4	4.41	3.731e-4	4.41	3.715e-4	4.41
6	1.759e-4	4.24	1.722e-4	4.24	1.715e-4	4.24
7	9.242e-5	4.18	9.044e-5	4.18	9.005e-5	4.18
8	5.327e-5	4.13	5.218e-5	4.12	5.198e-5	4.11
9	3.304e-5	4.06	3.237e-5	4.05	3.225e-5	4.05
10	2.170e-5	3.99	2.130e-5	3.97	2.125e-5	3.96

TABLE 4.1

Example 2. Convergence of the DGFEM with p -refinement.

Theorem 3.3 as h tends to zero for each fixed p . Again, as in the previous example we see that the size of the error is insensitive to the choice of θ , though an increase in θ does again lead to a marginal decrease in $\|u - u_{\text{DG}}\|_{1,h}$.

Finally, we investigate the asymptotic behaviour of the proposed methods with p -enrichment. In Table 4.1 we show the the DG-norm of the error and the computed rate of convergence k for the SIPG, IIPG and NIPG schemes on a uniform square mesh consisting of twelve elements. Here, we observe that the DG-norm of the error converges to zero at (approximately) the rate $\mathcal{O}(p^{-4})$ for all three methods considered; this is twice the (optimal) rate predicted by Theorem 3.3. However, this behaviour is due to the presence of the singularity in u arising on an inter-element boundary, rather than in the interior of an element κ in the mesh. Indeed, by employing approximation results in terms of weighted Sobolev norms, *a priori* error bounds which reflect this order-doubling in the rate of convergence of the DGFEM with p -refinement may be established, cf. [22].

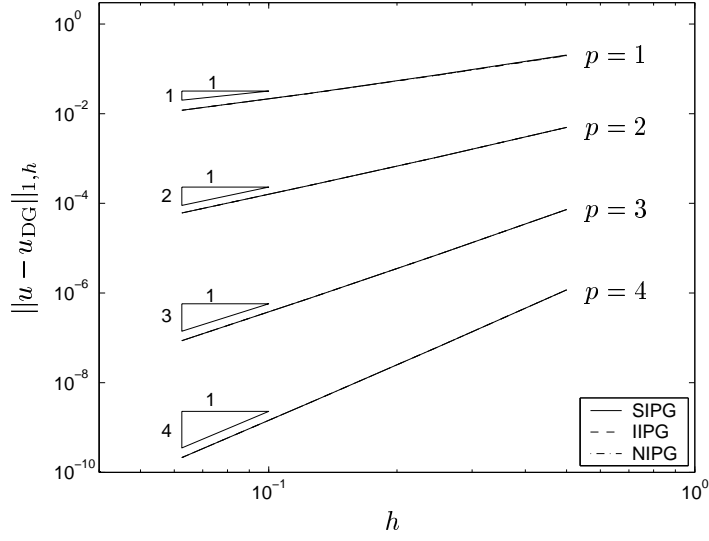
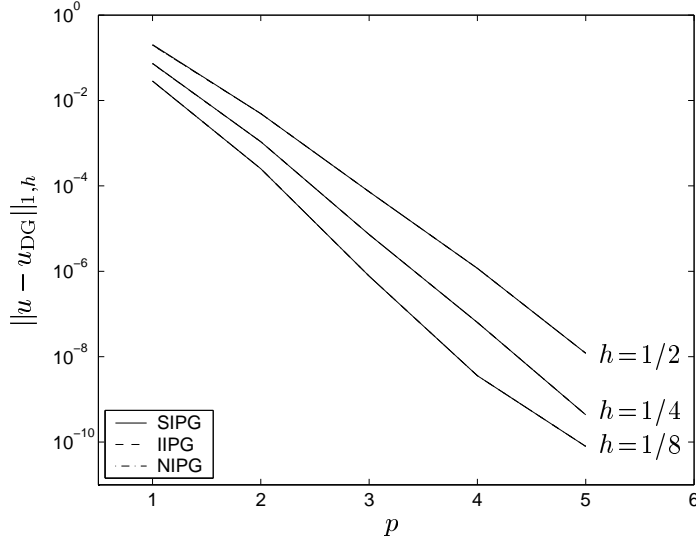
4.3. Example 3. In this final example, we consider a problem for which the structural hypothesis (1.9) on the coefficient μ is violated. To this end, we set $\Omega = (0, 1)^2$, with $\Gamma_D = \partial\Omega$ and

$$\mu(x, |\nabla u|) = |\nabla u|^{r-2}, \quad 1 < r < \infty. \quad (4.5)$$

Choosing $r = 3$, we select g_D and f so that the analytical solution to (1.1)–(1.3) is given by

$$u(x, y) = e^{xy}. \quad (4.6)$$

In Figure 4.5 we plot the DG-norm of the error against the mesh function h for $p = 1, 2, 3, 4$ and $\theta = -1, 0, 1$. As in Example 1, we again observe that $\|u - u_{\text{DG}}\|_{1,h}$ converges to zero, for each fixed p , at the rate $\mathcal{O}(h^p)$ as the mesh is refined, for each choice of θ ; this is in agreement with the optimal rate predicted by Theorem 3.3, even though the underlying hypotheses on μ no longer hold. As in the previous examples, we note that the error is relatively insensitive to changes in θ . Furthermore, we note that the $L^2(\Omega)$ -norm of the error behaves in an analogous manner as in Example 1, for a fixed polynomial degree as the mesh is refined; i.e. the $L^2(\Omega)$ -norm of the error in hp -DGFEM converges to zero at the optimal rate $\mathcal{O}(h^{p+1})$ for odd p as h tends to

FIG. 4.5. *Example 3. Convergence of the DGFEM with h -refinement.*FIG. 4.6. *Example 3. Convergence of the DGFEM with p -refinement.*

zero, but at only the rate $\mathcal{O}(h^p)$ when p is even. For brevity, these numerical results are omitted.

Finally, in Figure 4.6 we plot $\|u - u_{\text{DG}}\|_{1,h}$ against the polynomial degree p for each value of θ on a linear-log scale. Given that the analytical solution (4.6) is a real analytic function, we again observe exponential convergence of the error in the hp -DGFEM as p is enriched for each fixed h and each fixed θ , cf. Example 1.

5. Concluding remarks. In this article we have developed a one-parameter family of hp -version discontinuous Galerkin finite element methods for the numerical solution of quasilinear elliptic equations in divergence-form on a bounded open

polyhedral domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$. We then considered the analysis of the methods for the equation $-\nabla \cdot \{\mu(x, |\nabla u|) \nabla u\} = f(x)$ subject to mixed Dirichlet–Neumann boundary conditions on $\partial\Omega$, under the assumption that μ is a real-valued function, $\mu \in C(\bar{\Omega} \times [0, \infty))$, and there exist positive constants m_μ and M_μ such that $m_\mu(t-s) \leq \mu(t)t - \mu(s)s \leq M_\mu(t-s)$ for $t \geq s \geq 0$.

The discrete problem was shown to have a unique solution u_{DG} in the finite element space for any value of the parameter $\theta \in [-1, 1]$. If $u \in C^1(\Omega) \cap H^k(\Omega)$, $k \geq 2$, then, with discontinuous piecewise polynomials of degree $p \geq 1$, the error between u and u_{DG} , measured in the broken H^1 -norm, was proved to be $\mathcal{O}(h^{s-1}/p^{k-3/2})$, where $1 \leq s \leq \min\{p+1, k\}$. The theoretical results were illustrated by numerical experiments. Provided that the structural hypothesis (1.9) on the nonlinearity is retained, the theoretical results of the paper are easily extended to quasilinear elliptic and parabolic problems containing lower order terms. In the absence of hypothesis (1.9), however, — as would be the case with nonlinearities such as $\mu(t) = (1+t^2)^{\frac{r-2}{2}}$ or $\mu(t) = |t|^{r-2}$, $1 < r < \infty$, — either one or both of the uniform Lipschitz continuity and the uniform monotonicity of the semilinear form $B(\cdot, \cdot)$ is violated. The analysis is then much more complicated and is the subject of our current research.

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