Gauge/Gravity Duality and Non-Equilibrium Dynamics of Strongly Coupled Quantum Systems

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A thesis submitted for the degree of

Doctor of Philosophy

Trinity 2017

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Acknowledgements

Firstly, I would like to thank my supervisor Andrei Starinets for his invaluable support, guidance and for creating an inspiring and productive atmosphere during my DPhil studies in Oxford.

I would also like to express my gratitude to my collaborators Ville Keränen and Jonas Probst. Exchanging ideas with them in fruitful discussions was indispensable for the development of new insights which led to the results in this thesis.

I thank the European Research Council (ERC) and the Science and Technology Facilities Council (STFC) for financially supporting my doctoral studies. I am also grateful to Merton College for providing a supportive environment and a friendly community.

I am thankful to my friends and fellow students in Heidelberg, London and Oxford, in particular to Benjamin and Philipp, who inspired me through their passion for physics.

Finally, I would like to thank my family for years of care and support as well as Adela for her love and kindness.

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Abstract

This thesis explores applications of gauge/gravity duality to strongly coupled quantum field theories out of equilibrium. Within the framework of holography, it addresses both small deviations from equilibrium accessible via the effective theory of hydrodynamics, as well as far-from-equilibrium dynamics, which are studied using a first-principles approach. After an introduction into the duality and a presentation of the two approaches to out-of-equilibrium physics, we present our results on holographic fluids in the first part of the thesis. We derive new Kubo formulae for five second-order transport coefficients of relativistic non-conformal fluids and apply them to a class of non-conformal holographic field theories at infinite coupling. We find strong evidence that the Haack-Yarom identity, which is known to hold for conformal holographic fluids at infinite coupling, is universally satisfied by strongly coupled fluids regardless of conformal symmetry. We prove analytically that the identity is obeyed when taking into account leading non-conformal corrections and provide numerical evidence that the result holds beyond these leading corrections. In the second part of the thesis, we examine two proposals for the holographic dictionary for correlation functions in out-of-equilibrium states. We show that these proposals are equivalent on the level of two-point functions of operators dual to free scalar bulk fields. We then use one of the two non-equilibrium dictionaries to study the thermalisation of two-point functions of scalar operators and effective occupation numbers in a holographic model of quantum quenches. We find that both thermalise with a rate set by the lowest quasinormal mode of the final-state black hole in the gravitational dual.

Contents

Chapter 1

Introduction

Gauge/gravity duality is one of the most exciting developments in theoretical physics of the past twenty years. It bridges the gap between the seemingly distant fields of general relativity (GR) and quantum field theory (QFT), which represent two cornerstones of modern fundamental physics. GR describes the macroscopic realms of astronomy and black holes while QFTs, such as electroweak theory and quantum chromodynamics (QCD), accurately model phenomena on the microscopic scales of particle physics. Gauge/gravity duality was uncovered within the framework of string theory, where objects, called D-branes, were found to have two descriptions in different limits of a coupling parameter: one in terms of a quantum gauge theory and the other in terms of a gravitational theory. The twofold description led to the realisation that gauge and gravity theories are connected in an unexpected manner.

Gauge/string duality [1–4] is a realisation of the *holographic principle* [5–7] which states that $(d + 1)$-dimensional quantum gravity in certain spacetime regions possesses an equivalent description in terms of non-gravitational degrees of freedom located on the d-dimensional boundary of that region. Therefore, the field of gauge/string duality is also referred to as holography. Gauge/gravity duality is a remarkable realisation of the holographic principle in the sense that the boundary degrees of freedom can be captured by a QFT description. The best-understood and most widely studied examples of gauge/string duality identify a conformal field theory (CFT) in d dimensions with a gravity theory in $(d + 1)$-dimensional anti-de Sitter (AdS) space. These examples are collectively referred to as *AdS/CFT correspondence* and include the most famous and first-discovered realisation of gauge/string duality which identifies four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) with gauge group $SU(N)$ on the one side and type IIB string theory on an $AdS_5 \times S^5$ background with N units of five-form flux through the S^5 on the other side [1]. Starting from this original discovery, a wealth of similar equivalences was found and the holographic dictionary was developed. This dictionary links properties of the QFT, such as thermodynamic quantities, correlation functions and symmetries, to corresponding properties of the gravitational dual.

In its weak-coupling limit, string theory simplifies to supergravity and gauge/string duality reduces to *gauge/gravity duality*. Gauge/gravity duality has emerged as an invaluable tool for the study of strongly coupled gauge theories because the gravitational description is accessible in the strong-coupling regime of the QFT where traditional perturbative approaches fail. In its function as a computational tool, it does not rely on string theory being the fundamental theory of our universe. It merely uses string theory as a framework which allows for the simultaneous study of gauge and gravity theories. There exist a range of interesting physical systems which are thought to have a description in terms of strongly coupled field theories. Examples include the quark-gluon plasma created in relativistic heavy-ion collisions and certain condensed matter systems with strange metal phases. Gauge/gravity duality is to date the only ab-initio tool — i.e. without making simplifying assumptions which severely restrict the possible outcomes — to study certain strongly coupled quantum theories and therefore, in principle, offers a novel tool to approach strongly coupled quantum systems.

However, there are several obstacles that limit the applicability of gauge/gravity duality to strongly coupled physics in practice. Firstly, the QFT has to have a large rank N of the gauge group in order to gain control of the dual string theory. In the large-N limit, the dual string theory reduces to its classical limit and becomes tractable. Secondly, in the large-N limit, the holographic dictionary identifies each operator in the QFT with a field in the gravity dual. Therefore, the QFT has to have a discrete spectrum of operators in the strong-coupling limit so that the dual gravity becomes tractable. However, it is often sufficient that the field theory operators of interest correspond to a computationally tractable consistent truncation of the dual gravity theory. Thirdly, the applicability of gauge/gravity duality to real-world systems is often restricted by the presence of supersymmetry as well as conformal symmetry. However, this issue becomes less pronounced in finite-temperature systems, since a non-zero temperature breaks both symmetries. Moreover, there are examples of QFTs with holographic duals which are not conformal and where (part of) supersymmetry is broken [4].

Despite these challenges, gauge/gravity duality has been a valuable tool for the study of toy models of strongly-coupled systems. In holographic toy models modelling QCD in the confined phase — an approach referred to as AdS/QCD [8, 9] — a chiral phase transition and confinement have been found [10–14]. In the approach called AdS/CMT [15], gauge/gravity duality has also been successfully applied to condensed matter physics. For example, holographic models can describe superconducting phases at strong coupling which might be present in high-temperature superconductors [16, 17]. In addition, strongly coupled non-Fermi liquids have been constructed via holographic gravity duals [18, 19]. In general, if a mechanism has been understood in a holographic toy model, there is hope that it will also apply — at least qualitatively — to other strongly-coupled field theories. In particular, if a finding is valid for large classes of holographic QFTs, it is expected to apply to more general strongly coupled field theories. Results, which apply to large classes of theories independent of many of their details, are sometimes referred to as universal properties. The most important example of such a universal property is the famous result that any relativistic holographic fluid with a two-derivative gravity dual in the classical gravity limit has a shear viscosity over entropy density ratio of $\eta/s = 1/(4\pi)$ in the strong-coupling limit [20–22], which is indeed tantalisingly close to the value for the quark-gluon plasma observed in heavy ion collisions. We explore the possibility of a similar universal relation for strongly coupled fluids, the Haack-Yarom identity [23], in chapter 4.

The gravitational description is also well-suited for addressing questions of non-equilibrium physics since the bulk physics is classical at leading order in the large-N expansion and the real-time dynamics is reduced to solving differential equations with initial data specified by the initial state in question. For example, the study of gravitational shock waves has contributed to the understanding of rapid isotropisation and hydrodynamisation in heavy-ion collisions [24–26]. Nevertheless, there are open questions about how to address non-equilibrium physics in the framework of gauge/gravity duality. While the holographic dictionary for correlation functions of field theory operators is well developed in the Euclidean setting [2, 3], the Lorentzian case is much less clear. In time-independent settings, results can be obtained from the Euclidean prescription by means of analytic continuation. However, in time-dependent situations, correlation functions can have various different time orderings and there are two distinct suggestions for how to generalise the equilibrium dictionary. The dictionary due to Banks, Douglas, Horowitz and Martinec, on the one hand, requires the computation of bulk-to-bulk correlation functions and extrapolating them to the boundary [27–29]. The proposal due to Skenderis and van Rees, on the other hand, is similar to the familiar Euclidean dictionary and mimics the so-called Schwinger-Keldysh approach to non-equilibrium QFT [30, 31]. In chapter 5 we will address the issue of the seemingly conflicting proposals. We take a step towards its resolution and then use one of the two dictionaries to compute two-point functions in a holographic model of a quantum quench in chapter 6.

The thesis is structured as follows. Chapters 2 and 3 contain relevant background material and lay the foundations for the following chapters, which contain the results of this thesis. In chapter 2 we explain how gauge/gravity duality arises from two descriptions of branes in string theory valid in different limits of a coupling parameter. We explore various relevant aspects of gauge/gravity duality and discuss crucial entries in the holographic dictionary. In chapter 3 we introduce two approaches to physics out of equilibrium, the first-principles Schwinger-Keldysh formalism used in non-equilibrium QFT and the low-energy effective theory of hydrodynamics. We explore their overlap in the so-called linear-response regime and derive how Kubo formulae link the low-energy constants of hydrodynamics with correlation functions in the underlying microscopic theory. These insights are used in chapter 4, which is based on and has text overlap with ref. [32], where we study second-order hydrodynamic transport in strongly coupled non-conformal field theories with holographic gravity duals. We first derive new Kubo formulae for five second-order transport coefficients of general non-conformal fluids in $(3 + 1)$ dimensions. This result is independent of a dual holographic description. We then apply these Kubo formulae to a class of non-conformal holographic fluids and find evidence that the Haack-Yarom identity [23], which is known to be valid for conformal holographic fluids at infinite coupling, also holds in the non-conformal setting. This provides further evidence that the Haack-Yarom identity may be universally satisfied by strongly coupled fluids. In chapter 5, which is based on and has text overlap with ref. [33], we discuss the aforementioned two proposals for the holographic dictionary for correlation functions in out-of-equilibrium states. We show that these proposals are equivalent on the level of two-point functions of free scalar bulk fields in arbitrary asymptotically AdS spacetimes. In chapter 6, which is based on and has text overlap with ref. [34], we use one of these non-equilibrium holographic dictionaries to compute two-point functions of scalar operators in examples of dual spacetimes undergoing gravitational collapse. This allows us to address the problem of thermalisation following a quench in the dual strongly coupled field theory. From the two-point functions, we extract an effective occupation number in the gauge theory and study how it approaches the thermal Bose-Einstein distribution following the quench. We find that the two-point functions, as well as the effective occupation numbers, thermalise with a rate set by the lowest quasinormal mode of the dual final-state black hole. Chapter 7 concludes with a summary of our results and a discussion of possible future research directions.

Throughout this thesis we use the following conventions. We work in units where $c = \hbar = k_B = 1$. The metric is chosen to be mostly minus, i.e. $\eta_{\mu\nu} = \mathrm{diag}\,(-1, +1, \ldots, +1)$ for the Minkowski metric. We distinguish between bulk and boundary coordinates by using Roman letters $m, n, \ldots \in \{0, 1, \ldots, d\}$ for the indices of bulk coordinates and Greek letters $\mu, \nu, \ldots \in \{0, 1, \ldots, d-1\}$ for boundary coordinates. We denote the spatial directions of a point x by $\underline{x}$, i.e. $x = (x^0, \underline{x})$. In chapter 2 we denote string-worldsheet coordinates by σ^α, $\alpha \in \{0, 1\}$, and use capitals $M, N, \ldots \in \{0, 1, \ldots, D-1\}$ as indices for target-space coordinates. For the $(p+1)$-dimensional worldvolume of branes we use the letters $a, b, \ldots \in \{0, 1, \ldots, p\}$ as longitudinal coordinate indices as well as $i, j, \ldots \in \{p+1, \ldots, D-1\}$ for the transverse directions.

Chapter 2

Gauge/gravity duality

In this chapter we introduce gauge/gravity duality which underlies almost all results in this thesis. In section 2.1 we present a brief introduction into the string theory framework within which the duality was uncovered. We then discuss a heuristic derivation as well as several important aspects of gauge/gravity duality in section 2.2.

2.1 Superstring theory

In this section we give an insight into how string theory provides a common framework for the study of quantum gravity and gauge theories [35–40]. This framework allows us to find connections between these seemingly unrelated theories which ultimately lead to a heuristic derivation of the AdS/CFT correspondence. In favour of brevity, details and subtleties will be omitted in this section.

2.1.1 Closed strings and superstrings

The worldsheet of a string, parameterised by the coordinates $\sigma^\alpha = (\tau, \sigma)$, $0 \leq \sigma \leq \pi$, propagating in a flat target spacetime, is given by the Nambu-Goto action

$$S_{NG} = -T \int \mathrm{d}^2\sigma \sqrt{-\det\left(\partial_\alpha X^M \partial_\beta X^N \eta_{MN}\right)}\,, \tag{2.1}$$

where $X^M(\sigma)$, $M, N = 0, 1, \ldots, D$, describe the spacetime embedding of the string and η_{MN} is the flat Minkowskian spacetime metric. The only free parameter of the theory is the string tension $T = 1/(2\pi\alpha')$, which sets the string length scale $l_s = \sqrt{\alpha'}$. The

Nambu-Goto action is classically equivalent to the Polyakov action

$$S_P = -\frac{T}{2} \int \mathrm{d}^2\sigma \, \sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N \eta_{MN} \;, \tag{2.2}$$

where the worldsheet metric $\gamma_{\alpha\beta}$ is an auxiliary field whose equation of motion must be imposed in addition. Local Weyl and diffeomorphism invariance on the worldsheet can be used to gauge fix the worldsheet metric to be Minkowskian.[1] In this gauge the equations of motion for the coordinate embeddings $X^M(\sigma)$ in a flat target space become simple linear plane wave equations and we can therefore expand $X^M(\sigma)$ in plane waves. Different boundary conditions are imposed for open and closed strings in the σ-direction. We first consider closed strings, which obey periodic boundary conditions.

When we quantise the theory, the worldsheet scalars X^M are promoted to operators obeying canonical commutation relations. This is equivalent to the coefficients of the plane wave expansion becoming raising and lowering operators of the momentum-mode number operators. Cancellation of the Weyl anomaly in the quantum theory, i.e. preservation of conformal symmetry after quantisation, requires a critical number of dimensions, $D = 26$. We obtain a Fock space with infinite towers of states whose mass scale is set by the inverse string length scale $1/l_s$. The space of states contains a tachyon signalling an instability of the vacuum.

The instability can be remedied by adding a fermionic sector on the worldsheet such that the resulting theory is supersymmetric.[2] In this supersymmetric theory, cancellation of the Weyl anomaly requires the critical dimension of $D = 10$. The tachyon is removed from the spectrum by the spacetime supersymmetry-preserving GSO projection [48]. Supersymmetry can be implemented in five different ways leading to the five superstring theories known as *type IIA*, *type IIB*, *type I* and the two *heterotic* theories. A web of non-perturbative string dualities linking the five theories suggests that they might be different limits of a more fundamental underlying theory referred to as M-theory [49].

[1] Imposing conformal invariance on the worldsheet theory is therefore crucial to ensure that the system described by the Polyakov action (2.2) does not contain more degrees of freedom than the original system described by the Nambu-Goto action (2.1). Preserving conformal symmetry in the worldsheet theory on the quantum level proves to be a powerful tool.

[2] Supersymmetry can be implemented such that it is manifest either on the worldsheet in the Ramond-Neveu-Schwarz (RNS) formalism [41, 42] or in the target space in the Green-Schwarz (GS) formalism [43–47]. The two formalisms are equivalent, at least for ten-dimensional Minkowski space [39].

In this thesis we will consider type II theories only and, in particular, focus on type IIB. The bosonic subsector of massless excitations of the closed type II superstring contains the graviton g_{MN}, the Kalb-Ramond 2-form B_{MN} and the dilaton Φ in the so-called NS-NS sector as well as p-form fields C_p in the R-R sector.[3,4]

2.1.2 Low-energy effective actions

In the energy regime far below the string scale, $E \ll l_s^{-1}$, only the massless string modes are excited. The bosonic massless string modes can form coherent excitations which behave like classical fields and lead to non-trivial backgrounds around which we can study perturbative strings. These coherent excitations are analogous to coherent photons forming a laser beam.

To find the low-energy effective action governing the NS-NS background fields, we can include them as generalised couplings in the string worldsheet action (2.2) [50–52] (for a pedagogical discussion see [53]),

$$S = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \left(\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N G_{MN}(X) + \epsilon^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N B_{MN}(X) + \alpha'\sqrt{-\gamma}\Phi(X)R_\gamma\right) , \tag{2.3}$$

where $\epsilon^{\alpha\beta}$ is the anti-symmetric 2-tensor normalised such that $\epsilon^{01} = 1$ and R_γ is the Ricci scalar associated with the worldsheet metric.[5] To preserve Weyl invariance of the worldsheet theory on the quantum level, the beta-functionals $\beta(G)$, $\beta(B)$, $\beta(\Phi)$, which can be computed perturbatively in an α'-expansion, have to vanish. This leads to effective equations of motion for the classical background fields. A spacetime action that reproduces

[3] In type IIA the R-R sector contains a one- and a three-form, C_1 and C_3. In type IIB it contains a zero-, two- and a four-form, C_0, C_2 and C_4, where the field strength $F_5 = dC_4$ is restricted to be Hodge self-dual, $\star F_5 = F_5$.

[4] The sectors refer to the boundary conditions imposed on the fermionic fields on the closed string, which can be periodic (Neveu-Schwarz) or anti-periodic (Ramond) for both left- and right-moving fermions separately. States in the R-NS and NS-R sectors are spacetime fermions.

[5] If the dilaton is constant, $\Phi = \Phi_0$, the worldsheet integral for the dilaton coupling term can be performed explicitly resulting in the Euler density

$$\chi = 2 - 2g = \frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{-\gamma}\,R_\gamma\ , \tag{2.4}$$

where g is the genus of the worldsheet. This leads to a factor of $(e^{\Phi_0})^{2g-2}$ in the string partition function and we can interpret the expectation value of the dilaton as setting the string coupling, $g_s = e^{\langle\Phi\rangle}$.

the lowest-order equations as its Euler-Lagrange equations is called low-energy effective action.[6]

The target-space low-energy effective theories of type IIA and IIB superstrings, referred to as type IIA and IIB supergravity respectively, have an effective action of the form [40]

$$S_{II} = S_{NS} + S_R + S_F \ , \tag{2.5}$$

where S_{NS} governs the NS-NS background fields $G_{MN}(x)$, $B_{MN}(x)$ and $\Phi(x)$. The contribution S_R governs the R-R fields $C_p(x)$ and includes a Chern-Simons term while S_F refers to the fermionic part of the action.[7] In the so-called string frame the action reproducing the equations of motion derived from the vanishing of the worldsheet beta functionals is

$$S_{NS} = \frac{1}{2\tilde{\kappa}_{10}^2}\left[\int \mathrm{d}^{10}x\,\sqrt{-G}e^{-2\Phi}\left(R + 4(\partial\Phi)^2\right) - \frac{1}{2}\int e^{-2\Phi} H_3 \wedge \star H_3\right] \ , \tag{2.6}$$

where R is the Ricci scalar associated with the metric G_{MN} and $2\tilde{\kappa}_{10}^2 = (2\pi)^7\alpha'^4$.[8] The action governing the R-R sector and the Chern-Simons action are

$$S_{R,A} = -\frac{1}{4\tilde{\kappa}_{10}^2}\left[\int\left(F_2 \wedge \star F_2 + \tilde{F}_4 \wedge \star\tilde{F}_4\right) + \int B \wedge F_4 \wedge F_4\right] \tag{2.7}$$

for type IIA and

$$S_{R,B} = -\frac{1}{4\tilde{\kappa}_{10}^2}\left[\int\left(F_1 \wedge \star F_1 + \tilde{F}_3 \wedge \star\tilde{F}_3 + \frac{1}{2}\tilde{F}_5 \wedge \star\tilde{F}_5\right) + \int C_4 \wedge H_3 \wedge F_3\right] \tag{2.8}$$

in case of type IIB. The fields appearing in the action are related to the Kalb-Ramond B field and the R-R form fields C_p as

$$F_{p+1} = \mathrm{d}C_p \ , \qquad H_3 = \mathrm{d}B \ , \qquad \tilde{F}_3 = F_3 - C_0 H_3 \ , \tag{2.9a}$$

$$\tilde{F}_4 = \mathrm{d}A_3 - A_1 \wedge F_3 \ , \qquad \tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 \ . \tag{2.9b}$$

[6]The fundamental string does not couple to the p-form fields in the R-R sector in both type IIA and type IIB string theory. Therefore, the low-energy effective action for the R-R background fields cannot be determined with the sigma-model approach described above for the NS-NS sector. Instead, it can be derived using the low-energy limit of string scattering amplitudes [37–39].

[7]The type II supergravity actions possess the maximal possible amount of supersymmetry in ten dimensions. Their form is in fact completely fixed by supersymmetry [54].

[8]The exact value of the gravitational coupling is found by studying the low-energy limit of tree-level closed string scattering and relating it to graviton scattering in the low-energy effective supergravity theory [40].

For the type IIB effective action, the self-duality condition $\star\tilde{F}_5 = \tilde{F}_5$ has to be imposed as an additional constraint and does not follow from the equations of motion derived from the action (2.5).

In the following, we consider consistent truncations of both type IIA and type IIB supergravity for which only the metric G_{MN}, the dilaton Φ and one gauge potential C_{p+1} are non-vanishing. Shifting Φ to subtract its expectation value, $\Phi \to \langle\Phi\rangle + \Phi$, yields the string frame action

$$S = \frac{1}{2\tilde{\kappa}_{10}^2 g_s^2} \int \mathrm{d}^{10}x \sqrt{-G} \left[e^{-2\Phi} \left(R + 4(\partial\Phi)^2 \right) - \frac{1}{2(p+2)!} |F_{p+2}|^2 \right] , \tag{2.10}$$

where $|F_{p+2}|^2 = G_{M_1N_1} \cdots G_{M_{p+2}N_{p+2}} F_{p+2}^{M_1\cdots M_{p+2}} F_{p+2}^{N_1\cdots N_{p+2}}$. From the action (2.10), we can read off Newton's constant in this ten-dimensional theory,

$$16\pi G_{10} = 2\kappa_{10}^2 = 2\tilde{\kappa}_{10}^2 g_s^2 = (2\pi)^7 \alpha'^4 g_s^2 , \tag{2.11}$$

and the length scale associated with the gravitational coupling constant, the Planck length

$$l_P^8 = G_{10} \sim g_s^2 l_s^8 . \tag{2.12}$$

A rescaling of the metric, $g_{MN} = e^{-\Phi/2} G_{MN}$, yields the low-energy effective action in the so-called Einstein frame with the canonical Einstein-Hilbert term. Later in this section, we consider solutions to eq. (2.10) in the string frame.

2.1.3 p-branes and black branes

A natural generalisation of the Nambu-Goto action (2.1), the so-called Dirac action,

$$S_D = -\tau_p \int\limits_{\Sigma_{p+1}} \mathrm{d}^{p+1}\xi \sqrt{-\det\left(G_{ab}^*\right)} , \tag{2.13}$$

describes the propagation of $(p+1)$-dimensional extended objects, referred to as p-branes, in a flat background. The ξ^a, $a = 0, \ldots, p$, parameterise the worldvolume Σ_{p+1} of the extended object and $G_{ab}^* = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \eta_{\mu\nu}$ is the pullback of the spacetime metric. On dimensional grounds the coupling satisfies $\tau_p \sim l_s^{-(p+1)}$.

Supersymmetry introduces severe constraints on possible p-brane solutions. The supersymmetric p-brane solutions were studied as higher dimensional generalisations of the

Green-Schwarz superstring and only few such solutions have been found [55–58].[9] The addition of a Wess-Zumino term to the supersymmetric Dirac-type action, eq. (2.13), allows for a richer landscape of p-brane solutions. The $(p+1)$-form R-R fields in type II supergravity and superstring theory can be naturally sourced by the $(p+1)$-dimensional p-branes by means of the action[10]

$$S_{WZ} = \mu_p \int_{\Sigma_{p+1}} C^*_{p+1} \,, \tag{2.14}$$

where μ_p is the elementary charge and C^*_{p+1} the pullback of the $(p+1)$-form C_{p+1} to the worldsheet. Upon inclusion of the Wess-Zumino term, it turns out that each R-R $(p+1)$-form in type II theories has a corresponding supersymmetric p-brane solution and a magnetically dual $(6-p)$-brane [59].[11] The existence of these R-R charged states is required since the fundamental string is an NS-NS charged state and string dualities exchange states that are charged under NS-NS and R-R form fields [60–62].

Black p-branes are solitonic solutions of the low-energy supergravity equations of motion with $(p+1)$-dimensional translational symmetry and a singularity [63–66]. Reviews of black p-brane solutions of supergravity include refs. [67–70]. Black p-branes describe a condensate of p-branes which backreacts on the background. The extremal limit of black p-branes yields BPS solutions. They saturate the Bogomol'nyi-Prasad-Sommerfeld (BPS) bound, $M \geq Q$, where M is the mass and Q the charge [71, 72]. The extremal p-brane is assumed to be the strong-coupling incarnation of p-branes. In fact there is thought to be an exact correspondence between Green-Schwarz supersymmetric p-branes and extremal BPS p-brane solutions [59, 67].

We consider some p-brane solutions of type II supergravity in more detail. The equations of motion derived from the consistent truncation of the type II string-frame supergravity action (2.10) are solved by the background referred to as extremal p-brane solution

[9] The formidable obstacle that was overcome in [55] was to show that kappa symmetry can be generalised to worldvolume dimensions $d > 1+1$ which had previously been deemed impossible.

[10] The action $S_D + S_{WZ}$ is analogous to $m\int_\Sigma \mathrm{d}s + e\int_\Sigma A^*$ for an electrically charged particle of mass m on a trajectory Σ in an external field A with pull-back A^* to the trajectory.

[11] The NS-NS 2-form couples to the fundamental string which has the magnetically dual NS5-brane.

[66],

$$\mathrm{d}s^2 = G_{MN}\mathrm{d}x^M\mathrm{d}x^N = H_p(r)^{-\frac{1}{2}}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + H_p(r)^{\frac{1}{2}}\mathrm{d}x^i\mathrm{d}x^i \ , \tag{2.15a}$$

$$F_{p+2} = (1+\delta_{p,3}\star)\,\mathrm{d}(H_p(r)^{-1}) \wedge \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^p \ , \tag{2.15b}$$

$$e^\Phi(r) = H_p(r)^{\frac{3-p}{4}} \ , \qquad\qquad H_p(r) = 1 + \left(\frac{L_p}{r}\right)^{7-p} \ , \tag{2.15c}$$

where $\mu, \nu = 0, \ldots, p$ and $i = p+1, \ldots, 9$ as well as $r^2 = \sum_{i=p+1}^{9}(x^i)^2$. The parameter L_p is related to the number of branes, N, via the total Noether charge associated with the gauge field C_{p+1},

$$\mu_p N = Q = \frac{1}{2\kappa_{10}^2}\int_{S^{8-p}} \star F_{p+2} \quad \Rightarrow \quad \left(\frac{L_p}{l_s}\right)^{7-p} = (4\pi)^{\frac{5-p}{2}}\Gamma\left(\frac{7-p}{2}\right) g_s N \ . \tag{2.16}$$

This shows that the curvature scale L_p is large compared to the string length when $g_s N \gg 1$, which is necessary for the low-energy effective supergravity description to be valid. The extremal p-brane solution breaks ten-dimensional Lorentz symmetry as

$$SO(1,9) \longrightarrow SO(1,p) \times SO(9-p) \ . \tag{2.17}$$

The non-extremal black p-brane supergravity solution generalises eq. (2.15). An emblackening factor $f(r)$ enters the solution as

$$\begin{aligned}\mathrm{d}s^2 &= G_{MN}\mathrm{d}x^M\mathrm{d}x^N \\ &= H_p(r)^{-\frac{1}{2}}\left(-f(r)\mathrm{d}t^2 + \mathrm{d}\underline{x}^2\right) + H_p(r)^{\frac{1}{2}}\left(f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{8-p}^2\right) \ ,\end{aligned} \tag{2.18a}$$

$$f(r) = 1 - \left(\frac{r_H}{r}\right)^{7-p} \ , \tag{2.18b}$$

where $\underline{x} = (x_1, \ldots, x_p)$. The dilaton and the $(p+1)$-form are of the same form as in the extremal p-brane case (2.15). The parameter r_H plays the role of a horizon since $f(r_H) = 0$. These near-extremal solutions are non-BPS and, unlike the extremal solution, they do not satisfy the condition that their mass and charge are equal. The solutions (2.18) break $SO(1,p) \to SO(p)$ Lorentz symmetry in the longitudinal directions.

2.1.4 Open strings and Dp-branes

We now turn back to the superstring described by the supersymmetric generalisation of the Polyakov action, eq. (2.2), with worldsheet bosons $X^M(\sigma)$. For open strings, Dirichlet boundary conditions fixing the position of the string endpoint, X^μ = const, or Neumann boundary conditions fixing the derivative, $\partial_\sigma X^\mu = 0$, can be imposed at the string endpoints. If Dirichlet boundary conditions are imposed on a superstring endpoint in $9-p$ spatial dimension and Neumann boundary conditions in the remaining directions, the endpoint is fixed to lie in a p-dimensional spatial hyperplane which is referred to as Dirichlet-brane (or simply Dp-brane). It breaks ten-dimensional Lorentz symmetry in the same way as the extremal p-brane background (2.15), eq. (2.17). This hints at a deeper connection which we explore in subsection 2.1.5. Reviews of Dp-branes and their role in string theory can be found in refs. [73–77]. Quantising an open superstring with Neumann boundary conditions in the directions $a = 0, \ldots, p$ and Dirichlet boundary conditions in the remaining directions $i = p+1, \ldots, D-1$ yields a $U(1)$ gauge field A^a and transverse scalars ϕ^i in the massless bosonic sector.

2.1.5 DBI action and supersymmetric Yang-Mills theory

Dp-branes are by themselves dynamical objects and more than just hyperplanes on which open strings end. They constitute the UV completion of the p-branes introduced in subsection 2.1.3 [78]. They gravitate by coupling to closed strings in the NS-NS sector and they are charged under R-R $(p+1)$-form potentials.

The low-energy action describing both the dynamics of the bosonic massless worldvolume fields, the $U(1)$ gauge field A^a and the transverse scalars ϕ^i, as well as their coupling to the massless closed string background fields is given by the sum of a Dirac-Born-Infeld

(DBI) term [79–82][12] and a Chern-Simons term [83–86][13],

$$S_{DBI} = -T_p \int\limits_{\Sigma_{p+1}} \mathrm{d}^{p+1}\xi \, e^{-\Phi} \sqrt{-\det\left(G^*_{ab} + B^*_{ab} + 2\pi\alpha' F_{ab}\right)} \,, \tag{2.20a}$$

$$S_{CS} = \mu_p \int\limits_{\Sigma_{p+1}} \sum_{q=0}^{9} C^*_{q+1} \wedge \exp\left(B^* + 2\pi\alpha' F\right) \,. \tag{2.20b}$$

Here $G^*_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} G_{\mu\nu}$ is the pullback of the metric, B^*_{ab} the pullback of the Kalb-Ramond field and F_{ab} the field strength associated with the worldvolume gauge field A_a discovered in the quantisation of the open string in subsection 2.1.4. The integral in the Chern-Simons term picks out the right power from the exponential which is understood in terms of the wedge product. The Dp-brane tension is given by[14]

$$T_p = \tau_p g_s^{-1} = (2\pi)^{-p} \alpha'^{-\frac{p+1}{2}} g_s^{-1} \,. \tag{2.21}$$

The dilaton coupling in eq. (2.20a) dictates the $g_s^{-1} = e^{-\langle\Phi\rangle}$ factor in the brane tension (2.21), which signals that the Dp-branes are in fact non-perturbative objects. Reparameterisation invariance can be used to choose the static gauge, $\xi^a = X^a$, $a = 0, \ldots, p$. Then the scalars ϕ^i can be identified with the position of the brane in the transverse directions $X^i/(2\pi\alpha')$, $i = p+1, \ldots, 9$. For Dp-branes the brane charge μ_p is the same as the tension T_p and they are therefore BPS solutions like the extremal p-branes found in subsection 2.1.3.

[12] The action can be derived in a similar fashion as the low-energy effective supergravity action of the closed string sector. The Dirac action, eq. (2.13) is a direct generalisation of the Nambu-Goto action for a $(p+1)$-dimensional worldvolume. Both the dynamics of the gauge field and the fluctuation of the brane are captured by a mixture of the Dirac and the Born-Infeld action,

$$S = -\tau_p \int_{\Sigma_{p+1}} \mathrm{d}^{p+1}\xi \sqrt{\det\left(G^*_{ab} + (2\pi\alpha')F_{ab}\right)} \,, \tag{2.19}$$

where G^*_{ab} is the pullback of the flat target space metric. The coupling to a non-trivial background created by the massless closed-string excitation in the NS-NS and R-R sectors then proceeds analogously to coupling the fundamental string to a nontrivial background in eq. (2.3). Requiring the open string theory in the Dp-brane background to be conformally invariant leads to the equations of motion which are encoded in the DBI action (2.20a) [82].

[13] The Wess-Zumino term in eq. (2.14) is the leading contribution to the Chern-Simons term and omits more general terms.

[14] The normalisation has to be fixed by matching a scattering amplitude in the low-energy effective theory to the low-energy limit of a string scattering amplitude [40].

Saturation of the BPS bound is related to the fact that Dp-branes are non-perturbative states that preserve part of the supersymmetries of the theory. Of the 32 supercharges of the type II superstring theory 16 are broken. This is due to the open-string boundary conditions relating left- and right-moving fields on the worldsheet and thus leaving only a combination of the left- and right-handed supercharges unbroken. The supersymmetry algebra relates the supercharges to the momentum operator (and thus the mass) and the charge. The proof that BPS states preserve part of the supercharges uses this link [87].

The Wess-Zumino term (2.14) contained in the Chern-Simons action (2.20b) means that a Dp-brane is charged under a C_{p+1} gauge potential. In type IIA there are Dp-branes with p even while in type IIB there are Dp-branes with p odd.[15]

Expanding the DBI-action (2.20a) for small α' in a flat background with vanishing Kalb-Ramond field and constant dilaton, $g_s = e^{\Phi}$, yields

$$S_{DBI} = -(2\pi\alpha')^2 T_p \int_{\Sigma_{p+1}} \mathrm{d}^{p+1}\xi \left(\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}\partial_a\phi^i\partial_a\phi^i + \dots\right) , \tag{2.22}$$

the Abelian Yang-Mills action (plus scalars) with Yang-Mills coupling

$$g_{\mathrm{YM}}^2 = \frac{1}{(2\pi\alpha')^2 T_p} = (2\pi)^{p-2} g_s {\alpha'}^{\frac{p-3}{2}} . \tag{2.23}$$

Considering a stack of N coincident Dp-branes, an open string has N^2 possibilities to choose which of the branes to end on. This is captured by so-called Chan-Paton degrees of freedom which are localised on the string endpoints [88]. The Chan-Paton degrees of freedom, and therefore all states on the open string, form an adjoint representation of $U(N)$ with dimension N^2.[16] In particular, a $U(N)$ gauge field emerges from the N $U(1)$ gauge fields of the single branes [89, 90]. While the non-Abelian generalisation of the full DBI action (2.20a) is unknown, in the low-energy limit it becomes

$$S = -(2\pi\alpha')^2 T_p \int_{\Sigma_{p+1}} \mathrm{d}^{p+1}\xi \ \mathrm{Tr}\left(\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}\mathcal{D}_a\phi^i\mathcal{D}^a\phi^i - \frac{1}{4}\sum_{i\neq j}\left[\phi^i,\phi^j\right]^2 + \dots\right) , \tag{2.24}$$

[15] In addition to the R-R sector gauge potentials, the type II theories contain the electromagnetic duals of these potentials. A pair of dual potentials is related by $\mathrm{d}C_{p+1} = \star\mathrm{d}C_{7-p}$. Consequently, type IIA has gauge potentials C_1, C_3, C_5 and C_7 while type IIB contains C_0, C_2, C_4, C_6 and C_8 with C_4 having a self-dual field strength, $\mathrm{d}C_4 = \star\mathrm{d}C_4$.

[16] The Chan-Paton factors, which dress all open string states and capture the Chan-Paton degrees of freedom, are elements of a Lie algebra and the only algebra consistent with oriented open string scattering is $u(N)$ [89].

with the field strength $F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b]$, the gauge covariant derivative $\mathcal{D}_a\phi^i = \partial_a\phi^i + i\,[A_a, \phi^i]$ and $A_a = A_a^n T_n$ with $U(N)$ generators obeying $\mathrm{Tr}(T_m T_n) = \delta_{mn}$. We have only explicitly given the bosonic part of the low-energy action which is extended by the superstring supersymmetry to the $\mathcal{N} = 4$ maximally supersymmetric Yang-Mills (SYM) theory [48, 91, 92]. In the large N limit, the effective gauge theory coupling g_{eff} controlling the perturbative expansion of $\mathcal{N} = 4$ SYM is related to the t'Hooft coupling [93],

$$\lambda = g_{\text{YM}}^2 N \ , \tag{2.25}$$

with the Yang-Mills coupling g_{YM} as given in eq. (2.23). At an energy scale U the dimensionless effective coupling is

$$g_{\text{eff}} = \lambda U^{p-3} = g_{\text{YM}}^2 N U^{p-3} = (2\pi)^{p-2} g_s N \,(l_s U)^{p-3} \ , \tag{2.26}$$

where we recalled that $g_{\text{YM}}^2 \sim l_s^{p-3}$.

The coupling of the open string degrees of freedom on the brane to the closed string degrees of freedom in the bulk, i.e. their backreaction on the background, is controlled by [82][17]

$$\kappa_{10}^2 T_p N \sim g_s N \ , \tag{2.27}$$

omitting powers of the dimensionful scale l_s, where we have used eqs. (2.11) and (2.21). The perturbative picture for open strings on the Dp-brane neglecting backreaction on the background is valid when the effective dimensionless coupling is small, $g_s N \ll 1$.

Realising that Dp-branes and p-branes are two descriptions of the same object in opposite limits was the seminal contribution by Polchinski in ref. [78]. In this subsection we have considered Dp-branes as solitonic objects in perturbative type II string theory where the degrees of freedom associated with the brane are those of the quantised open string. We have seen in eq. (2.27) that this description breaks down when $g_s N \gtrsim 1$. An alternative description is in terms of black p-brane supergravity solutions which we studied in subsection 2.1.3. According to eq. (2.16), this description is valid for $g_s N \gg 1$.

[17] The matter backreaction of the N Dp-branes on the geometry is determined by Einstein's equations. The terms in the coupling follow from the κ_{10}^{-2} prefactor of the supergravity action (2.10), the string tension T_p in the DBI action (2.20a) and the factor of N from the trace over the Chan-Paton $U(N)$ generators.

2.2 The AdS/CFT correspondence

In this section we use our insights into D-branes in type II string theory from section 2.1 to examine D3-branes in more detail and present a heuristic derivation of the AdS/CFT correspondence, the best understood example of a gauge/gravity duality. Thereafter, we discuss various aspects of gauge/gravity duality and its applications, including a prescription to compute CFT correlation functions from gravity in AdS.

The seminal papers that introduced the AdS/CFT correspondence are refs. [1–3]. Reviews of the AdS/CFT correspondence and the broader field of holography include refs. [4, 15, 29, 94–103]. Recently, textbooks covering the subject and its applications to various fields such as condensed matter physics and heavy-ion collisions have been published [104–107].

2.2.1 Maldacena's decoupling argument

In this subsection we review Maldacena's heuristic derivation of the AdS/CFT correspondence conjecturing the equivalence of $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$ and type IIB string theory on $AdS_5 \times S^5$ [1].

We consider a stack of N extremal D3-branes in type IIB string theory. In the limit of $g_s N \gg 1$, the system is accurately described by closed strings in the corresponding p-brane supergravity background, eq. (2.15) with $p = 3$. The number of branes N and the curvature scale $L \equiv L_3$ of the supergravity background are related by eq. (2.16),

$$\left(\frac{L}{l_s}\right)^4 = 4\pi g_s N \ . \tag{2.28}$$

According to eq. (2.27), a perturbative open-string description is valid in the opposite limit of $\lambda = g_{\mathrm{YM}}^2 N = 2\pi g_s N \ll 1$. The case of $p = 3$ is special in the sense that, according to eq. (2.15), the dilaton Φ is constant allowing for small string coupling g_s everywhere. In addition, the t'Hooft coupling λ, eq. (2.25), is dimensionless and the brane worldsheet theory is conformal.

We consider the system at low energies while keeping all brane-worldvolume field-theory quantities fixed. The low-energy limit is accomplished by taking

$$\alpha' \to 0 \qquad \text{with} \qquad U \equiv \frac{r}{\alpha'} \qquad \text{kept fixed,} \tag{2.29}$$

where r is any distance [108, 109]. We thus take a simultaneous low-energy and near-horizon limit.[18]

In the parameter region $g_s N \gg 1$, in which we apply a closed-string description, closed strings move in the background (2.15) with $p = 3$. The near-horizon limit (2.29) entails $r/L \sim r/\sqrt{a'} = U\sqrt{a'} \to 0$, where the near-horizon geometry becomes

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\left(\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}z^2\right) + L^2 \mathrm{d}s^2_{\mathrm{S}^5}\ , \qquad\qquad \mu,\nu = 0,1,2,3\ . \tag{2.30}$$

We have changed the radial coordinate via $z = L^2/r$. This is $AdS_5 \times S^5$ with radius L of both AdS_5 and the S^5. On the other hand, in the asymptotic region $r \to \infty$, the geometry is simple ten-dimensional Minkowski space. In this region the low-energy limit removes all higher string excitations with masses set by the string scale from the spectrum. In contrast, excitations close to the horizon are blueshifted with respect to the asymptotic region such that near-horizon string-scale excitations survive the low-energy limit.[19] Nevertheless, near-horizon excitations decouple from the bulk dynamics due to the redshift.[20]

[18] The low-energy limit is taken in a way so that all considered energies become small compared to the string scale by keeping energies fixed while the string energy scale is sent to infinity, $\alpha' \to 0$. This is ensured by keeping U fixed in the limit (2.29). This can be seen in the following way: Separating one brane from the stack by a distance r corresponds to one of the worldvolume adjoint scalars acquiring an expectation value of $r/(2\pi\alpha')$. In order to keep the energy scale set by the expectation value fixed in the limit $\alpha' \to 0$, $U \equiv r/\alpha'$ has to be fixed.

[19] An excitation with energy ω_r at radius $r \ll L$ is blueshifted when taking the near-horizon limit, eq. (2.29), and appears in the asymptotic region as excitation with energy

$$\omega_\infty = \sqrt{-g_{00}}\,\omega_r \sim \frac{r}{L}\omega_r \sim U\sqrt{\alpha'}\,\omega_r \qquad \Rightarrow \qquad \sqrt{\alpha'}\,\omega_r \sim \frac{\omega_\infty}{U}\ , \tag{2.31}$$

i.e. $\sqrt{\alpha'}\omega_r$ remains fixed in the $\alpha' \to 0$ limit. Therefore near-horizon excitations with energies at the string scale survive the low-energy limit (2.29) as they are low-energy excitations from the perspective of an asymptotic observer.

[20] There is an effective potential barrier seen by the excitations in the 3-brane geometry, eq. (2.15). For example, for a minimally coupled massless scalar this potential barrier can be seen explicitly by bringing the equation of motion into Schrödinger form [110–113].

In the parameter region $g_s N \ll 1$, in which we apply an open-string description, the N D3-branes backreact weakly on the geometry as seen in eq. (2.27). The background metric remains flat and the interaction of the brane with the bulk degrees of freedom is captured by the non-Abelian generalisation of the DBI action (2.20a) and the Chern-Simons term (2.20b). The low energy-limit $\alpha' \to 0$ reduces the brane worldvolume theory to $\mathcal{N} = 4$ SYM, eq. (2.24). Interactions between brane and bulk degrees of freedom at energy scale ω are suppressed by $\omega^8 \kappa_{10}^2 \sim \omega^8 \alpha'^4$, eq. (2.11), such that the two sectors decouple. As described in section 2.1.2, the bulk sector reduces to type IIB supergravity.

In both descriptions the low-energy limit results in two decoupled sectors of which one is supergravity in a flat background. Assuming that the low-energy limit and the variation in the parameter $g_s N$ commute, Maldacena proposed that the near-horizon dynamics in the closed string description can be identified with the brane worldvolume dynamics in the open string picture. This proposal is the AdS/CFT correspondence claiming that $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ is equivalent to type IIB string theory on an $AdS_5 \times S^5$ background [1]. The parameters of the string theory, $L/\sqrt{\alpha'}$ and g_s, are linked to the gauge theory parameters, N and g_{YM}, via

$$2\pi g_s = g_{\mathrm{YM}}^2 \, , \qquad \frac{L^4}{\alpha'^2} = 2 g_{\mathrm{YM}}^2 N \, , \tag{2.32}$$

derived from eqs. (2.23) and (2.16). Equivalently, we could express the relations using the t'Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$, eq. (2.25), instead of the Yang-Mills coupling g_{YM}.

The correspondence is conjectured to hold independently of the parameters.[21] The string theory dual allows access to regions of the parameter space of the gauge theory that are inaccessible with traditional perturbative tools. Since $L^4/l_P^4 \sim N$ according to eqs. (2.12) and (2.32), the string theory becomes classical in the large N limit. For large $L^4/l_s^4 \sim \lambda$, massive string states decouple and the string theory reduces to supergravity. Therefore, classical type IIB supergravity on $AdS_5 \times S^5$ allows access to the large N and strong-coupling regime of the gauge theory. Including α' corrections to the supergravity

[21] Both sides of the correspondence exist for any parameter value, but capture the relevant degrees of freedom and their dynamics in different parameter regions. For example, we have a good understanding of the gauge theory for fixed N and small g_{YM} by means of perturbation theory. In the same parameter region the dual spacetime curvature length scale is below the string scale and we have a poor understanding of string theory in this regime.

action corresponds to considering $1/\lambda$ corrections in the field theory and gravity loop corrections in the perturbative parameter κ_{10}/L^4 to $1/N$ corrections.[22]

2.2.2 Top-down and bottom-up

The correspondence between $\mathcal{N} = 4$ SYM and type IIB string theory on $AdS_5 \times S^5$ is an example of *top-down* holography. The gauge theory arises as low-energy limit of the D3-brane worldvolume theory and is described by the action (2.24). Top-down refers to correspondences between gauge and gravity theories where both the open-string gauge theory description and the closed-string AdS gravity description are known explicitly. Their disadvantage is that, due to supersymmetry, they often contain more degrees of freedom than the realistic field theories, such as QCD, that one ultimately wants to investigate.

Studying the near-horizon decoupling limit of various branes configurations leads to different top-down models. In his original paper, ref. [1], Maldacena studied the near-horizon limits of M2- and M5-branes in eleven-dimensional supergravity as well as the D1/D5 brane system. In these systems the worldvolume field theory is conformal. The superconformal three-dimensional ABJM theory and its holographic dual were found in [114]. The near-horizon limit of Dp-branes was studied for various values of $p \neq 3$ in refs. [109, 115, 116], where particular attention must be paid to the non-constant dilaton corresponding to non-conformal worldvolume theories. Fundamental flavours are introduced in the D3/D7 [117, 118] and the D3/D5 [119] systems with D7 and D5 flavour branes added to the string theory configuration intersecting with the stack of D3 branes from Maldacena's construction. A prominent top-down holographic model for QCD is the Witten-Sakai-Sugimoto model [10–12]. More examples of top-down holographic models can be found in ref. [4] and references therein.

Top-down models are in contrast with *bottom-up* models, where the gauge theory microscopic (Lagrangian) description is unknown and the conjectured duality is not directly

[22]The α' corrections to the low-energy effective supergravity action modify the equations of motion such that worldsheet conformal symmetry is preserved beyond leading order in α'. The target space α' corrections correspond to string worldsheet loops. Quantum gravity loops refer to the effective target space description where the coupling is the dimensionless gravitational coupling $\kappa_{10}/L^4 \sim l_P^4/L^4 \sim 1/N$. The target space loops correspond to string worldsheet genus corrections via $l_P^4 = g_s l_s^4$.

deduced from string theory. In the spirit of the holographic principle (see subsection 2.2.8), one postulates that a gravity theory in some asymptotically AdS space should have an equivalent description in terms of a gauge theory on the boundary. The construction of bottom-up models is often guided by symmetries, which we explore in 2.2.4 in the context of holography. The advantage of bottom-up models is their simplicity and versatility compared to top-down models allowing us to focus on relevant features of the strongly coupled field theory we want to describe. Bottom-up models were discussed in refs. [13, 14] for holographic QCD and in refs. [16–19] for certain condensed matter systems. Other examples of bottom-up models with various applications can be found in refs. [15, 105].

2.2.3 The GKPW dictionary

In this subsection we describe how the dual AdS gravity description can be used to extract properties of a gauge theory in its strong-coupling regime. In order to get an idea, we consider a gravitational perturbation $h_{\mu\nu}$ in the original D3-brane setup. The perturbation couples to the stress-energy tensor $T^{\mu\nu}$ of the worldvolume field theory in the open-string description via the interaction described in the DBI action (2.20a). In the closed-string description the perturbation in the asymptotic region sets the boundary value for the gravitational perturbation in the near-horizon AdS throat region [2, 4]. The realisation that the same perturbation sets boundary values in the near-horizon AdS region in the closed-string description and that it sources a corresponding operator in the open-string description led Gubser, Klebanov and Polyakov [2] as well as Witten [3] to propose the *GKPW formula* identifying the Euclidean generating functionals of the gravity theory and the dual CFT as

$$\left\langle e^{\int \mathrm{d}^d x\, \phi_{(0)}(x)\mathcal{O}(x)} \right\rangle_{\mathrm{CFT}} = Z_{\mathrm{CFT}}[\phi_{(0)}(x)] = Z_{\mathrm{gravity}}[\phi_{(0)}(x)] = \int\limits_{\phi\to\phi_{(0)}} \mathcal{D}\phi\, e^{-S_E[\phi]} \,. \tag{2.33}$$

The field $\phi_{(0)}$ acts as source for the corresponding operator in the CFT generating functional Z_{CFT} and as AdS boundary value for the bulk fields in the supergravity generating functional Z_{gravity}. This is the reason why the gauge theory is sometimes referred to as living on the AdS boundary. In the supergravity large-N limit, the GKPW formula simplifies to

$$Z_{\text{CFT}}[\phi_{(0)}(x)] \underset{N\to\infty}{\approx} e^{-S_E[\phi\to\phi_{(0)}]} \,, \tag{2.34}$$

where $S_E[\phi \to \phi_{(0)}]$ is the supergravity action evaluated on a bulk solution with boundary asymptotics determined by $\phi_{(0)}$.[23] The generating functional of connected CFT correlators $W_{\text{CFT}}[\phi_{(0)}]$ reduces to the on-shell supergravity action,

$$W_{\text{CFT}}\left[\phi_{(0)}\right] \equiv -\ln Z_{\text{CFT}}\left[\phi_{(0)}(x)\right] \underset{N\to\infty}{\approx} S_E\left[\phi \to \phi_{(0)}\right] \,. \tag{2.35}$$

2.2.4 Field-operator map

The GKPW formula (2.33) establishes a correspondence between fields in the gravity theory on AdS and operators in the dual CFT. In the large-N limit, multi-trace operators decouple and disappear from the CFT spectrum [4]. The remaining single-trace operators map to the spectrum of classical strings in the AdS background.[24] In the limit of large t'Hooft coupling λ, only the chiral primary operators in the CFT are protected from acquiring large anomalous dimensions while all other operators decouple.[25] This corresponds to the decoupling of the towers of massive string states in the dual string theory when L^2/α' becomes large. Only the massless string modes contribute in the large-λ limit.

The link between bulk fields in the supergravity theory and operators in the dual CFT through eq. (2.33) links large gauge transformations in the bulk theory to bosonic symmetry transformations in the boundary gauge theory [3, 4]. For example, the isometry group $SO(2,4) \times SO(6)$ of $AdS_5 \times S^5$, which can be identified by embedding AdS_5 into $\mathbb{R}^6$ [4], maps to the conformal symmetry $SO(2,4)$ and the R-symmetry $SU(4) \cong SO(6)$ of $\mathcal{N} = 4$ SYM. Due to the internal S^5 space, the gravity theory contains infinite Kaluza-Klein (KK) towers of fields. The compactified spectrum contains the AdS_5 metric fluctuations dual to the CFT stress tensor and an $SU(4)$ gauge field dual to the CFT R-symmetry current.

[23] We will discuss boundary asymptotics of bulk fields in subsection 2.2.4 and how a unique bulk solution is fixed in subsection 2.2.5.

[24] For fixed λ and large N, the string coupling $g_s \sim \lambda/N$ is small such that target-space loops are suppressed.

[25] The scaling dimension of chiral primary operators is determined by R-symmetry and can therefore not acquire quantum corrections [4, 120].

The dilaton arising from the KK reduction is dual to the supersymmetric completion of the $\mathrm{Tr}(F^2)$ operator in the SYM theory. Together with the other lowest KK modes, these fields give rise to $\mathcal{N}=8$ supergravity in five dimensions which is thought to be a consistent truncation of the ten-dimensional type IIB supergravity theory [4].

In order to understand the map between bulk fields and boundary operators in more detail, we consider a bulk scalar ϕ in AdS_{d+1},[26]

$$\mathrm{d}s^2 = g_{mn}\mathrm{d}x^m\mathrm{d}x^n = \frac{L^2}{z^2}\left(\mathrm{d}z^2 + \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\right) \ , \tag{2.36}$$

with the action

$$S = -\frac{1}{2}\int \mathrm{d}^{d+1}x\,\sqrt{-g}\left(g^{mn}\partial_m\phi\partial_n\phi + m^2\phi^2\right) \ . \tag{2.37}$$

The field ϕ is dual to a scalar operator $\mathcal{O}$.[27] The equation of motion,

$$\partial_m\left(\sqrt{-g}g^{mn}\partial_m\phi\right) - \sqrt{-g}\,m^2\phi = 0, \tag{2.38}$$

is a second-order differential equation with two independent solutions which asymptote the boundary at $z=0$ according to $\sim z^{\Delta_\pm}$, where $\Delta_\pm$ are the larger and smaller roots of the equation $\Delta(\Delta-d) = L^2m^2$ satisfying $\Delta_- = d-\Delta_+$. The full solution can therefore be expanded near the boundary at $z=0$ as

$$\phi(z,x) = \phi_{(0)}(x)z^{d-\Delta_+} + \ldots + \phi_{(+)}(x)z^{\Delta_+} + \ldots \ , \tag{2.39}$$

where $\phi_{(0)}$ and $\phi_{(+)}$ are the two modes referred to as leading and sub-leading, respectively.[28] In addition, $\phi_{(0)}$ is also called the boundary value of the solution ϕ. Applying the GKPW formula (2.33) shows that the normalisable mode determines the expectation value of the dual operator, $\langle\mathcal{O}\rangle \sim \phi_{(+)}$ [121–123].

[26] If a top-down geometry contains an internal space X, ϕ is the AdS_{d+1}-component of a field that has been decomposed in orthogonal functions on the internal space (e.g. after decomposition in spherical harmonics in case of $X = S^5$).

[27] By studying non-dynamical gravity and choosing a fixed background metric in (2.37), we ignore the backreaction of the scalar on the metric. From the perspective of field theory, we ignore that turning on a source for the scalar operator $\mathcal{O}$ breaks conformal invariance.

[28] The leading and sub-leading modes are also referred to as non-normalisable and normalisable modes since the Klein-Gordon norm evaluated on the leading mode diverges while it is finite when evaluated on the sub-leading mode.

We can determine the scaling dimension Δ of the dual operator $\mathcal{O}$ by applying the relation between isometries in the bulk gravitational theory and the conformal symmetry transformations in the boundary theory. The boundary scale transformation $x^\mu \to x'^\mu = \lambda x^m$ is related to the isometry $(z, x^\mu) \to (z', x'^\mu) = (\lambda z, \lambda x^\mu)$ of the AdS_{d+1} metric (2.36). Under this isometry the bulk scalar field transforms as

$$\phi(z,x) \to \phi'(z',x') = \phi_{(0)}(\lambda x)\,(\lambda z)^{d-\Delta_+} + \ldots + \phi_{(+)}(\lambda x)\,(\lambda z)^{\Delta_+} + \ldots\ . \tag{2.40}$$

From $\phi'(z',x') = \phi(z,x)$ we deduce the mass dimensions of $\phi_{(0)}$ and $\phi_{(+)}$ as

$$\phi_{(0)}(\lambda x) = \lambda^{-(d-\Delta_+)}\phi_{(0)}(x) \qquad \Rightarrow \qquad \left[\phi_{(0)}\right] = d - \Delta_+\ , \tag{2.41a}$$

$$\phi_{(+)}(\lambda x) = z^{-\Delta_+}\phi_{(+)}(x) \qquad \Rightarrow \qquad \left[\phi_{(+)}\right] = \Delta_+\ . \tag{2.41b}$$

These are indeed the values expected for the source and the expectation value of an operator with scaling dimension $\Delta = \Delta_+$, $\mathcal{O}(\lambda x) = \lambda^{-\Delta}\mathcal{O}(x)$. The dual operator $\mathcal{O}$ therefore is[29]

$$\text{relevant if} \qquad \frac{d}{2} \le \Delta < d \qquad \Leftrightarrow \qquad -\frac{d^2}{4} \le m^2L^2 < 0\ , \tag{2.43a}$$

$$\text{marginal if} \qquad \Delta = d \qquad \Leftrightarrow \qquad m^2L^2 = 0\ , \tag{2.43b}$$

$$\text{irrelevant if} \qquad \Delta > d \qquad \Leftrightarrow \qquad m^2L^2 > 0\ . \tag{2.43c}$$

The *Breitenlohner-Freedman bound* [125, 126], stating that scalar fields in AdS are only stable if their mass satisfies

$$m^2L^2 \ge -\frac{d^2}{4}\ , \tag{2.44}$$

is equivalent to the bound that m must satisfy to ensure that Δ_+ and Δ_- are both real.

[29]There is a unitarity bound for the scaling dimension of scalar operators in d-dimensional QFTs: $\Delta \ge (d-2)/2$ [4, 105]. In the range of operator dimensions $(d-2)/2 \le \Delta < d/2$, the operator expectation value cannot be identified with the sub-leading mode as $\Delta_+ \ge d/2$. However, in the narrow range of AdS scalar masses

$$-\frac{d^2}{4} < m^2L^2 \le -\frac{d^2}{4} + 1\ , \tag{2.42}$$

there is a so-called alternative quantisation identifying $\phi_{(+)}$ with the source and $\phi_{(0)}$ with the expectation value of the dual operator [124]. Choosing the alternative quantisation in the range (2.42) allows for dual operators with dimension $(d-2)/2 \le \Delta < d/2$ which cannot be realised in standard quantisation.

A similar near-boundary analysis as the one we have presented for a scalar in AdS can be performed for other bulk fields. As an example, we consider a massless bulk $U(1)$ gauge field with boundary asymptotics

$$A_m(z,x) = A_{(0)m}(x)z^{\Delta_-} + \dots A_{(+)m}(x)z^{\Delta_+} + \dots \, , \tag{2.45}$$

Gauge invariance of the bulk field translates to a conservation equation for dual operator, $\partial_\mu \langle J^\mu \rangle = 0$, i.e. the dual operator is a conserved $U(1)$ current. The expectation value of the conserved current is related to the sub-leading mode of the bulk gauge field, $\langle J^\mu \rangle \sim A^\mu_{(+)}$. Under the scaling isometry $(z, x^\mu) \to (z', x'^\mu) = (\lambda z, \lambda x^\mu)$ the gauge field transforms as $A'_m(z', x') = (\partial x^m / \partial x'^n) A_n(z,x) = \lambda^{-1} A_m(z,x)$ and a near-boundary analysis of the equations of motion shows that

$$\left[A_{(0)}\right] = 1 \, , \qquad\qquad \left[A_{(+)}\right] = d - 1 \, , \tag{2.46}$$

as is expected for the source and expectation value of a conserved $U(1)$ current J^μ.

For bulk metric fluctuations, the scaling dimensions of the leading and sub-leading modes, 0 and d, match the scaling dimension of the source and the expectation value of the stress-energy tensor $T^{\mu\nu}$ of the dual field theory.

2.2.5 Holographic dictionary

The holographic dictionary contains entries which describe how properties of a holographic QFT can be extracted from the dual gravitational theory. One crucial entry covers correlation functions following from the GKPW formula (2.33).

In order to compute correlation functions using the GKPW formula in the large-N supergravity limit, eq. (2.34), we have to compute the on-shell supergravity action. Fixing the leading mode to be ϕ_0 provides one of the two boundary conditions to make the solution of the second-order equations of motion unique. In the Euclidean case and for static solutions in the Lorentzian setting, the other boundary condition is fixed by requiring the solution to be regular at the horizon $z \to \infty$. For time-dependent solutions in a black-hole background, the two independent solutions are an in-going and an out-going wave at the horizon. A prescription for determining causal correlation functions, e.g. the retarded two-point function, is to impose in-going boundary conditions at the horizon [127].[30,31] Imposing two boundary conditions uniquely fixes the solution and determines the sub-leading mode in terms of the leading mode, i.e. the expectation value in terms of the source. Higher-point connected correlators can be determined by taking functional derivatives of the one-point function with respect to the source. In order to obtain two-point functions, it is sufficient to solve the linearised bulk equation of motion since we only have to take one functional derivative of the expectation value with respect to the source. For example, when imposing in-going boundary conditions on a scalar ϕ at the horizon, the retarded two-point function G_R is determined by $\langle\phi\rangle \sim -G_R\phi_{(0)} + \mathcal{O}(\phi_{(0)}^2)$, i.e. $G_R \sim -\lim_{\phi_{(0)}\to 0}(\delta\langle\phi\rangle/\delta\phi_{(0)})$.

The plane-wave solutions to the linearised equations of motion of a field in a black hole background with in-going boundary conditions at the horizon and Dirichlet boundary conditions $\phi_{(0)} = 0$ at the boundary are referred to as the field's *quasinormal modes*. As a consequence of $\langle\phi\rangle \sim -G_R\phi_0 + \mathcal{O}(\phi_{(0)}^2)$, these quasinormal modes appear as poles of the retarded correlator in momentum space [127, 129].

We briefly mention other prominent entries in the holographic dictionary. As they will not be used in this thesis, we will not discuss them further. Expectation values of Wilson line operators in the field theory are determined by the area of the bulk minimal surface whose boundary ends on the Wilson line [130]. The field theory entanglement entropy between a spatial subregion A of the boundary and its complement is derived from the minimal volume bulk region ending on the boundary subregion A [131–133]. Degrees of freedom dual to fundamental flavours are introduced in the gravity via D-branes extending along the radial direction in AdS [117–119].

2.2.6 UV/IR relation, holographic renormalisation and RG flows

The radial coordinate in the bulk dual AdS gravity can be interpreted as corresponding to the energy scale of the gauge theory. This connection is called the *UV/IR relation* [134] since the asymptotic region of the bulk close to the AdS boundary is related to the UV

[30] A generalisation for obtaining other correlation functions was proposed in [128].

[31] We discuss two- and higher-point causal correlation functions, as well as other correlators in the real-time setting, in more detail in section 3.1.

of the dual field theory. The UV/IR relation can be motivated as follows [109]: In the string theory D-brane setup described in subsection 2.2.1, separating one of the branes from the stack by a distance r in the radial direction corresponds to a worldvolume scalar acquiring a Higgs expectation value of $r/(2\pi\alpha')$.[32] A large radial distance leads to a large expectation value, which has dimensions of energy and therefore corresponds to an energy scale. Another piece of evidence for the UV/IR relation is that UV divergences in the field theory are related to boundary (long-distance) divergences in the bulk [134, 135].

The issue of near-boundary divergences in holographic QFTs is tackled with the machinery of *holographic renormalisation* [122, 123, 136–138]. Computing the on-shell supergravity action, which is the generating functional for connected CFT correlators according to the GKPW formula (2.33) at large N, requires the following steps. First, the theory is regulated by introducing a near-boundary cut-off $z = \epsilon$ resulting in the regularised action S_{reg}. Then all divergences are cancelled by an appropriate counterterm action S_{ct} defined on the cut-off surface so that the subtracted action $S_{\text{sub}} = S_{\text{reg}} + S_{\text{ct}}$ is finite in the $\epsilon \to 0$ limit. Finally, the cut-off can be removed and the renormalised action, $S_{\text{ren}} = \lim_{\epsilon\to 0} S_{\text{sub}}$, is finite. The GKPW formula is modified such that functional derivatives of the original action with respect to the boundary value of the bulk field are replaced with derivatives of the subtracted action S_{sub} with respect to the field on the cut-off surface before taking the limit to remove the cut-off.

We can break conformal invariance of a holographic QFT by introducing a source for a relevant operator which triggers an RG flow. From a holographic perspective, this corresponds to turning on a non-zero boundary value for a bulk field which backreacts on the metric. Close to the boundary, the bulk geometry remains unaffected (asymptotically AdS) while deeper in the bulk backreaction leads to a deformation of the metric corresponding to the breaking of conformal symmetry. Therefore, this mechanism allows for a geometric interpretation of the RG flow where the effect of the relevant perturbation at a certain energy scale is captured by the deformation of the metric at some radius z [139]. Holography allows for the study of flows to IR fixed points, where the geometry deep in the

[32] Recall that the radial coordinate r is related to the coordinate z in (2.36) via $z = L^2/r$. The AdS boundary is located at $r \to \infty$.

bulk recovers the AdS form, and flows without IR fixed points, where the bulk geometry in the deep interior is not AdS.

There are a few prominent examples of top-down RG flows, for which the dual field theory flow is explicitly known. The Leigh-Strassler flow from $\mathcal{N} = 4$ at the UV fixed point to an IR $\mathcal{N} = 1$ fixed point [140] is dual to the FGWP (Freedman, Gubser, Warner, Pilch) flow, a solution of $\mathcal{N} = 8$ five-dimensional supergravity [141].[33] It was proved in refs. [139, 141], that all QFTs in even dimensions with holographic duals obey the C-theorem. The C-theorem states that there is a function of the coupling constants which is non-increasing along RG flows and only stationary at fixed points.[34] The proof relies on a connection between the monotonicity of the field-theory C-function and the null-energy condition in gravity. Holographic top-down RG flows within ten-dimensional supergravity are the Klebanov-Strassler flow [143], the Maldacena-Núñez flow [144] and the Polchinski-Strassler flow [145]. We will study a class of bottom-up RG flows in section 4.3.

2.2.7 Finite temperature and chemical potential

Holography provides a deep connection between the thermodynamic properties of black holes and thermal states of QFTs. The laws of black hole thermodynamics are not only seen as analogies between thermodynamics and black hole physics that might have some quantum gravity interpretation (in terms of black hole microstates), but the thermodynamics of black holes in AdS backgrounds can be linked directly to the thermodynamics of the dual gauge theory on the boundary.

The near-horizon limit of the non-extremal black 5-brane geometry, eq. (2.18), is the AdS_5-Schwarzschild black brane times S^5,

$$ds^2 = G_{MN}\mathrm{d}x^M\mathrm{d}x^N = \frac{L^2}{z^2}\left(-f(z)\mathrm{d}t^2 + \mathrm{d}\underline{x}^2 + \frac{\mathrm{d}z^2}{f(z)}\right) + L^2\mathrm{d}\Omega_5^2\,, \tag{2.47a}$$

$$f(z) = 1 - \frac{z^4}{z_H^4}\,, \tag{2.47b}$$

[33] $\mathcal{N} = 8$ five-dimensional supergravity is expected to be a consistent truncation of type IIB supergravity on $AdS_5 \times S^5$ to five dimensions.

[34] Zamolodchikov proved the theorem for two-dimensional field theories with a C-function that equals the conformal anomaly at the fixed points [142].

where $z_H = L^2/r_H$. In Euclidean time, $\tau = it$, vanishing of the conical deficit angle at the horizon z_H requires the time coordinate to be compact with $\tau \sim \tau + \pi z_H$. A Euclidean QFT with compactified time and period β is equivalent to the real-time QFT at finite temperature $\beta = T^{-1}$ [146, 147]. Since the boundary time is compactified with the same period, the Euclidean black-brane solution corresponds to a QFT at finite temperature $T = 1/(\pi z_H)$, which is also the Hawking temperature of the black brane [105].

According to the GKPW formula (2.33), the on-shell supergravity action is equal to β times the free energy of the theory in the classical supergravity limit [10]. From the free energy F, we can determine other thermodynamical quantities such as the entropy,

$$S = -\frac{\partial F}{\partial T}\,, \tag{2.48}$$

which turns out to coincide with the Bekenstein-Hawking entropy [148, 149], $S_{\text{BH}} = A/(4\pi G)$, of the black brane [150]. We can switch the thermodynamical ensemble by adding appropriate terms to the gravity action, which implement the necessary Legendre transformation [15].

Considering the global AdS-Schwarzschild solution in contrast to the Poincaré patch (2.47), Witten found a phase transition interpreted as a deconfinement transition [3, 10]. In global AdS the Euclidean Schwarzschild solution has a boundary topology of $S^1 \times S^3$. The thermal and the spatial circle have radii β and β', respectively. As a function of the dimensionless ratio β'/β, the thermodynamically preferred bulk solution switches from thermal AdS at low temperatures (small β'/β) to the global AdS-Schwarzschild black hole at high temperatures (large β'/β). This is the Hawking-Page transition at the transition temperature $T_{\text{HP}} = (d-1)/(2\pi L)$ in d boundary dimensions [151]. The large-N gauge theory has a free energy of order 1, a 'confined' phase, for $T < T_{\text{HP}}$ and a free energy of order N^2, an 'unconfined' phase, for $T > T_{\text{HP}}$. The Poincaré patch with boundary topology $S^1 \times \mathbb{R}^3$ is recovered from global AdS as $\beta' \to \infty$ and the dual flat field theory is therefore always in the high temperature phase at any finite temperature. This is not unexpected, since all finite temperature states are equivalent in a conformal field theory without any other scale to compare the temperature to. Consequently, there cannot be a phase transition at any finite temperature. However, there is a sign of a deconfinement transition at $T = 0$ in the Poincaré patch. In the low temperature phase at $T = 0$, there is an infinite energy cost to introduce a free external quark, which is interpreted as a sign of confinement, while it is finite in the high temperature phase at any non-zero temperature.[35,36]

In presence of a conserved charge, we can also introduce a finite chemical potential. For the example of a conserved global $U(1)$ charge density J^0, the conserved current J^μ is dual to a gauge field A_m in the bulk. The charge density is sourced by the boundary value of the A_t component of the bulk gauge field, which corresponds to a chemical potential μ in the boundary theory. To study thermodynamics at finite temperature, a black hole solution has to be found in the bulk with the time component of the gauge field satisfying $A_t(z,x) = \mu + \mathcal{O}(z)$. Such black hole solutions exist, for example, for the bottom-up Einstein-Maxwell model [15] or the top-down probe D7-brane model [152, 153].

2.2.8 Gauge/gravity duality and the holographic principle

The holographic principle is a term that has been coined for the idea that the yet to be determined quantum theory of gravity of our universe, when considering certain spacetime regions, admits an equivalent description in terms of degrees of freedom living on the lower dimensional boundary of this spacetime region [5–7]. It is motivated by the seminal result of Bekenstein and Hawking [148, 149] that black holes admit a thermodynamic description with a non-extensive entropy proportional to the area of the horizon. This finding suggests that a black hole as quantum mechanical object can be captured completely by degrees of freedom living on its boundary. The holographic principle states that the theory that governs the boundary degrees of freedom is a non-gravitational quantum theory. Gauge/gravity dualities are the first known realisations of that principle and the boundary dual description of the bulk string theory is in terms of a gauge theory. For this reason,

[35] The (free) energy cost of adding a free, static quark is measured by the expectation value of a temporal Wilson line operator (the Polyakov loop $\mathcal{P}$), $\langle \mathcal{P} \rangle = \exp(-\beta F)$. If the Polyakov loop expectation value is zero, the free energy cost is infinite, while it is finite for a non-zero expectation value. The expectation value of the Wilson line in a holographic theory is computed via a bulk minimal surface [130, 135].

[36] The cost of introducing an external heavy quark does not provide a confinement criterion for the Hawking-Page transition in global AdS with finite boundary volume [10]. In global AdS the expectation value of the Polyakov loop $\mathcal{P}$ is always zero.

gauge/gravity duality is often referred to as holography and field theories with a gravity dual are called holographic.

Chapter 3

Non-equilibrium dynamics

In this chapter we introduce two formalisms which can be used to analyse field theories in out-of-equilibrium states. We discuss a first-principles approach, the so-called Schwinger-Keldysh formalism, for the study of quantum fields in arbitrary states in section 3.1. Subsequently, we study the low-energy and close-to-equilibrium regime of quantum field theories via an effective theory referred to as hydrodynamics in section 3.2. The low-energy constants of the effective theory are the so-called transport coefficients. In section 3.3 we use the Schwinger-Keldysh formalism to show how transport coefficients in the effective hydrodynamic theory are linked to correlation functions in the underlying microscopic field theory via so-called Kubo formulae.

There are two reasons for introducing these concepts. First, we will use Kubo formulae in chapter 4 to compute transport coefficients in strongly coupled fluids via gauge/gravity duality. Second, the Schwinger-Keldysh formalism lies at the heart of the Skenderis-van Rees prescription, a holographic dictionary for correlation functions in non-equilibrium states, which we will introduce in chapter 5.

3.1 Non-equilibrium quantum field theory

In this section we study the so-called Schwinger-Keldysh [154, 155] formalism which is prominent in the study of non-equilibrium QFTs [156–161]. In subsection 3.1.1 the formalism is introduced and applied to the problem of computing two-point correlation functions. We then explore initial states and focus on states which can be prepared via Euclidean path integrals in subsection 3.1.2. This set of states is shown to contain the ground state and the thermal equilibrium state. We also derive how the fluctuation-dissipation theorem is recovered in thermal equilibrium.

3.1.1 The Schwinger-Keldysh formalism

In this subsection we will use the Schwinger-Keldysh closed-time-contour formalism to study two-point correlation functions. We choose this application because the study of two-point correlators will be of prime interest in chapters 5 and 6. The formalism is of interest for the study of non-equilibrium physics as, in contrast to the standard Feynman path integral which computes transition amplitudes from initial to final states, it does not presuppose a final state. We will see in chapter 5 that the Skenderis-van Rees prescription is a holographic dictionary for non-equilibrium states that closely resembles the Schwinger-Keldysh formalism.

We illustrate the formalism for a simple scalar field operator in the Heisenberg picture,

$$\Phi(x) = \Phi(t, \underline{x}) \ . \tag{3.1}$$

We choose a set of field operator eigenstates at time t_0,

$$\Phi(t_0, \underline{x})|\phi\rangle = \phi(\underline{x})|\phi\rangle \ , \tag{3.2}$$

and further consider two composite operators, $\mathcal{O}_i(x) = \mathcal{O}_i[\Phi(x)]$, $i = 1, 2$. Our aim is to compute the two-point function

$$\langle \mathcal{O}_2(x_2)\mathcal{O}_1(x_1)\rangle = \mathrm{Tr}\left(\rho\, \mathcal{O}_2(x_2)\mathcal{O}_1(x_1)\right) \tag{3.3}$$

in the state characterised by the density matrix ρ.[1]

Using the time evolution operator $U(t_2, t_1) = \mathcal{T} \exp\left(-i \int_{t_1}^{t_2} \mathrm{d}t\, H(t)\right)$, we can write

$$\mathcal{O}_i(x_i) = U(t_0, t_i)\mathcal{O}_i(t_0, \underline{x}_i)U(t_i, t_0) \ . \tag{3.4}$$

[1] In the Heisenberg picture the density matrix ρ is time-independent. In a pure state it factorises, $\rho = |\psi\rangle\langle\psi|$. It is normalised such that $\mathrm{Tr}\,\rho = 1$. Despite not being explicit in the notation $\langle \ldots \rangle$, it will always be clear from the context in which state ρ the expectation value is taken.

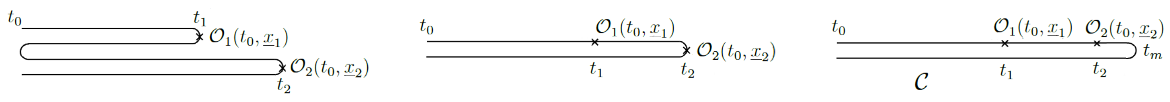


Figure 3.1: Different time contours displaying the time evolution in eq. (3.5). Figure 3.1c is the domain $\mathcal{C}$ of the time argument of the fields in the path integral for the correlation function (3.5) and for the generating functional (3.7).

With the notation $\mathrm{d}\psi = \prod_{\underline{x}} \mathrm{d}\psi(\underline{x})$ and $\mathcal{D}\varphi \propto \prod_x \mathrm{d}\varphi(x)$, the two-point function becomes

$$\begin{aligned}
&\langle \mathcal{O}_2(x_2)\mathcal{O}_1(x_1)\rangle \qquad (3.5)\\
&\quad = \mathrm{Tr}\left(\rho\, U(t_0,t_2)\mathcal{O}_2(t_0,\underline{x}_2)U(t_2,t_0)U(t_0,t_1)\mathcal{O}_1(t_0,\underline{x}_1)U(t_1,t_0)\right)\\
&\quad = \int \mathrm{d}\psi_1\,\mathrm{d}\psi_2\,\langle\psi_1|\rho|\psi_2\rangle\langle\psi_2|U(t_0,t_2)\mathcal{O}_2(t_0,\underline{x}_2)U(t_2,t_1)\mathcal{O}_1(t_0,\underline{x}_1)U(t_1,t_0)|\psi_1\rangle\\
&\quad = \int \mathrm{d}\psi_1\,\mathrm{d}\psi_2\,\langle\psi_2|\rho|\psi_1\rangle \int \mathcal{D}\varphi\,\Psi_2^*[\varphi(t_0^-,\underline{x})]\Psi_1[\varphi(t_0^+,\underline{x})]e^{iS_{\mathcal{C}}[\varphi]}\mathcal{O}_2\left[\varphi(x_2)\right]\mathcal{O}_1\left[\varphi(x_1)\right]\ .
\end{aligned}$$

We have employed the usual time-slicing argument [162, 163] to obtain the path integral form of the correlator and have introduced the wavefunctional $\Psi[\varphi] = \langle\varphi|\psi\rangle$.[2] Figure 3.1 illustrates the time evolution in the three lines of eq. (3.5) and shows how the closed time contour in the path integral arises. The field φ in the path integral has domain $\mathcal{C}\times\mathbb{R}^3$, where the closed time contour $\mathcal{C}$ runs along the time contour from t_0 to some arbitrary time $t_m > t_i$, $i = 1, 2$, (upper part of the contour) and back (lower part of the contour) as shown in figure 3.1c. By $t^\pm$ we denote times on the upper and lower part of the contour, respectively. The notation $S_{\mathcal{C}}[\varphi]$ indicates the domain of the time coordinate. In the path integral the operators $\mathcal{O}_i$, which are functions of the field operator, are evaluated on the classical spatial field configuration $\varphi(t_i,\underline{x}_i)$. All information about the initial state is contained in the functional matrix $\langle\psi_2|\rho|\psi_1\rangle$. We will consider initial states more closely in subsection 3.1.2.

[2]The wavefunctionals project the boundary values of the classical field φ in the path integral on the spatial field configurations $\psi_i(\underline{x})$. We could equivalently write

$$\int \mathcal{D}\varphi\,\Psi_2^*[\varphi(t_0^-,\underline{x})]\Psi_1^*[\varphi(t_0^+,\underline{x})](\ldots) = \int_{\varphi(t_0^+,\underline{x})=\psi_1(\underline{x})}^{\varphi(t_0^-,\underline{x})=\psi_2(\underline{x})} \mathcal{D}\varphi\,(\ldots)\,. \qquad (3.6)$$

For a discussion of wavefunctionals in QFT see e.g. [164–166].

We can define a generating functional of correlation functions,

$$Z_\rho[J] = \int \mathrm{d}\psi_1\,\mathrm{d}\psi_2\,\langle\psi_2|\rho|\psi_1\rangle \int_{\varphi(t_0^+,\underline{x})=\psi_1(\underline{x})}^{\varphi(t_0^-,\underline{x})=\psi_2(\underline{x})} \mathcal{D}\varphi\,\exp\left(iS_{\mathcal{C}}[\varphi] + i\int_{\mathcal{C}} \mathrm{d}^4x\,J(x)\varphi(x)\right)\,, \quad (3.7)$$

where the source term is defined on the domain $\mathcal{C}\times\mathbb{R}^3$ and the turning point of the contour, t_m, is moved to infinity. The source on the complex time contour allows us to extract two-point functions with different time orderings.[3] To simplify notation we introduce $\varphi_\pm$ and $J_\pm$ to refer to φ and J with their time argument on the upper and lower part of the contour respectively.[4] The generating functional then reads

$$\begin{aligned} Z_\rho[J_\pm] = \int \mathrm{d}\psi_1\,\mathrm{d}\psi_2\,\langle\psi_2|\rho|\psi_1\rangle \int_{\varphi_+(t_0,\underline{x})=\psi_1(\underline{x})}^{\varphi_-(t_0,\underline{x})=\psi_2(\underline{x})} \mathcal{D}\varphi_+\,\mathcal{D}\varphi_-\,\delta(\varphi_+(t_m,\underline{x}) - \varphi_-(t_m,\underline{x})) \\ \times \exp\left[i\int_{t_0}^{t_m} \mathrm{d}^4x\,(\mathcal{L}[\varphi_+] - \mathcal{L}[\varphi_-] + J_+\varphi_+ - J_-\varphi_-)\right]\,. \quad (3.9)\end{aligned}$$

Defining the generating functional of connected correlation functions $\langle\ldots\rangle_c$ as

$$W_\rho[J_\pm] = \ln Z_\rho[J_\pm]\,, \quad (3.10)$$

the Wightman two-point correlator G_+ and the time-ordered Feynman correlator G_F can be expressed as

$$G_+(x_1,x_2) = \langle\Phi(x_1)\Phi(x_2)\rangle_c = \left.\frac{(-i)^2\delta^2 W_\rho[J_\pm]}{\delta J_-(x_1)\,\delta J_+(x_2)}\right|_{J_\pm=0}\,, \quad (3.11a)$$

$$G_F(x_1,x_2) = \langle\mathcal{T}\Phi(x_1)\Phi(x_2)\rangle_c = \left.\frac{(-i)^2\delta^2 W_\rho[J_\pm]}{\delta J_+(x_1)\,\delta J_+(x_2)}\right|_{J_\pm=0}\,. \quad (3.11b)$$

The Wightman function G_- is defined as $G_-(x_1,x_2) \equiv G_+(x_2,x_1)$. We can formally define the operator Φ on the closed time path $\mathcal{C}$ in figure 3.1 and introduce the notation $\Phi_\pm(t,\underline{x}) = \Phi(t^\pm,\underline{x})$. This notation is purely a bookkeeping device for ordering in correlation functions.

[3]As usual, the path integral computes time-ordered correlation functions. In this case time ordering is along the complex time path $\mathcal{C}$. For two field insertions in the upper (lower) part of the contour, the path integral results in the time-ordered (anti-time-ordered) two-point function. Two insertions on opposite sides of the contour results in the Wightman two-point function.

[4]Explicitly, we define

$$\varphi_\pm(t,\underline{x}) \equiv \varphi(t^\pm,\underline{x})\,, \qquad J_\pm(t,\underline{x}) \equiv J(t^\pm,\underline{x})\,. \quad (3.8)$$

Denoting the time-ordering operator along the path $\mathcal{C}$ as $\mathcal{T}_{\mathcal{C}}$, we can rewrite the two-point correlators in eq. (3.11) as

$$G_+(x_1,x_2) = \langle \mathcal{T}_{\mathcal{C}}\, \Phi_-(x_1)\Phi_+(x_2)\rangle_c \,, \qquad G_F(x_1,x_2) = \langle \mathcal{T}_{\mathcal{C}}\, \Phi_+(x_1)\Phi_+(x_2)\rangle_c \,. \tag{3.12}$$

The retarded two-point function G_R, which will play a significant role in the following discussion, is related to the Feynman and Wightman two-point functions by

$$\begin{aligned} G_R(x_1,x_2) &\equiv -i\,\Theta\big(x_1^0 - x_2^0\big)\,\langle[\Phi(x_1),\Phi(x_2)]\rangle_c \\ &= -i\,\Theta\big(x_1^0 - x_2^0\big)\,\big(G_+(x_1,x_2) - G_-(x_1,x_2)\big) \\ &= -iG_F(x_1,x_2) + iG_-(x_1,x_2)\,. \end{aligned} \tag{3.13}$$

We obtain the retarded two-point function from the generating functional (3.9) by combining the relations given in eq. (3.11), resulting in

$$G_R(x_1,x_2) = \left.\frac{(-i)\delta^2 W_\rho[J_\pm]}{\delta J_+(x_1)\,\delta J_+(x_2)}\right|_{J_\pm=0} + \left.\frac{i\delta^2 W_\rho[J_\pm]}{\delta J_+(x_1)\,\delta J_-(x_2)}\right|_{J_\pm=0} . \tag{3.14}$$

Changing the basis for the operators $\Phi_\pm$,

$$\Phi_a \equiv \Phi_+ - \Phi_- \,, \qquad \Phi_r \equiv (\Phi_+ + \Phi_-)/2 \,, \tag{3.15}$$

and analogously introducing the sources J_a, J_r and the fields φ_a, φ_r in the generating functional (3.9), we can express the retarded correlator as[5]

$$G_R(x_1,x_2) = -i\langle \Phi_r(x_1)\Phi_a(x_2)\rangle_c = \frac{(-i)^3\delta^2 W_\rho[J_r,J_a]}{\delta J_a(x_1)\,\delta J_r(x_2)} \,. \tag{3.17}$$

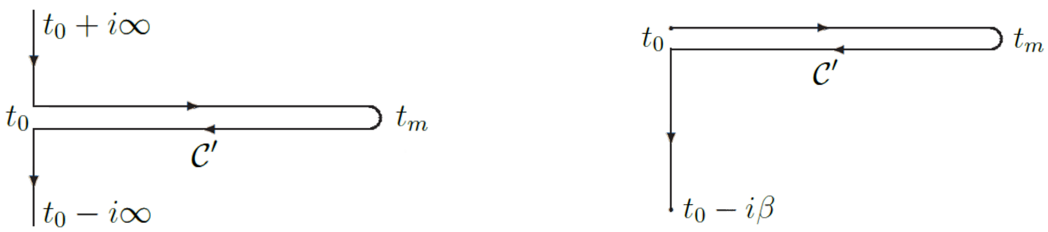
The retarded correlator $G_R(x_1,x_2)$ is causal, i.e. it vanishes for $x_1^0 - x_2^0 < 0$. We will see in section 3.3 that it encodes the (causal) response of a system to small external perturbations. When we Fourier transform the retarded correlator in a translation-invariant state,

$$G_R(x_1 - x_2) = \int \frac{d\omega\, d^3\underline{k}}{(2\pi)^4}\, e^{-i\omega(x_1^0 - x_2^0) + i\underline{k}\cdot(\underline{x}_1 - \underline{x}_2)} G_R(\omega,\underline{k}) \,, \tag{3.18}$$

[5]Note that the source term in the generating functional (3.9) becomes

$$i\int_{t_0}^{t_m} d^4x\,(J_+\varphi_+ - J_-\varphi_-) = i\int_{t_0}^{t_m} d^4x\,(J_a\varphi_r + J_r\varphi_a) \tag{3.16}$$

after the change of variables (3.15).



(a) Vacuum state (b) Thermal equilibrium state

Figure 3.2: Time contours for path integral in the vacuum state and in the thermal equilibrium state with temperature $T = \beta^{-1}$.

the condition that $G_R(x_1 - x_2)$ vanishes for $x_1^0 - x_2^0 < 0$ implies that $G_R(\omega, \underline{k})$ cannot have poles in the upper half of the complex ω plane.

When we go beyond linear response to external perturbations, we will see in section 3.3 that the response of the system is captured by the so-called *fully retarded* n-point correlator [167–170],

$$G_{ra\ldots a}(x, y_1 \ldots, y_{n-1}) \equiv (-i)^{n-1} \langle \Phi_r(x)\Phi_a(y_1) \ldots \Phi_a(y_{n-1})\rangle_c = \frac{(-i)^{2n-1}\delta^n W_\rho[J_r, J_a]}{\delta J_a(x)\, \delta J_r(y_1) \ldots \delta J_r(y_{n-1})} \,. \tag{3.19}$$

The fully retarded n-point correlator encodes the causal response to n-th order in an external source and reduces to the ordinary retarded two-point function (3.17) for $n = 2$.[6] For $n = 1$ we obtain the one-point function $G_r(x) = \langle \Phi_r(x)\rangle = \langle \Phi(x)\rangle$.

3.1.2 Initial states and Euclidean path integrals

Of particular interest are initial states ρ that can be prepared via a Euclidean path integral so that the generating functional (3.7) becomes a path integral along a complex time contour. The non-equilibrium generalisation of the holographic dictionary for correlation functions called the Skenderis-van Rees prescription, which we will introduce in chapter 5, requires the initial state to be in this class. Two important examples of such states are the vacuum and the thermal equilibrium states, which we will consider in detail.

[6]The fully retarded correlator (3.19) can be written as a sum of nested commutators of the operator Φ with appropriate Heaviside Θ-functions to ensure causality [157, 168].

Ground state: The ground state density matrix is $\rho_0 = |0\rangle\langle 0|$. The inner product $\langle\phi|0\rangle$, the ground state wavefunctional, can be expressed as[7]

$$\Psi_0[\phi] \equiv \langle\phi|0\rangle \propto \int_{\varphi_E(0,\underline{x})=\phi_0(\underline{x})}^{\varphi_E(\infty,\underline{x})=\phi(\underline{x})} \mathcal{D}\varphi_E \, \exp\left(-\int_{t=0}^{t=\infty} \mathrm{d}^4x \, \mathcal{L}_E[\varphi_E]\right) , \tag{3.21}$$

where $\mathcal{L}_E$ is the Euclidean Lagrangian and ϕ_0 is an arbitrary reference state with non-zero overlap with the ground state. The ground state generating functional can therefore be expressed as[8]

$$Z_0[J] = \int \mathcal{D}\varphi \, e^{iS_{\mathcal{C}'}[\varphi]+i\int_{\mathcal{C}} \mathrm{d}^4x\, J\varphi} , \tag{3.23}$$

where $\mathcal{C}'$ denotes the complex time contour in figure 3.2a. The initial state is prepared by the Euclidean part of the path integral (which corresponds to the imaginary part of the time contour).

Thermal equilibrium state: The thermal equilibrium density matrix is $\rho = \exp(-\beta H)$, which can be interpreted as time evolution operator in imaginary time. We can prepare the state by a Euclidean path integral and the generating functional, eq. (3.7), becomes[9]

$$Z_\beta[J] = \int \mathcal{D}\varphi \, e^{iS_{\mathcal{C}'}[\varphi]+i\int_{\mathcal{C}} \mathrm{d}^4x\, J\varphi} , \tag{3.25}$$

[7] The derivation considers the amplitude

$$\lim_{\tau\to\infty} \langle\phi|e^{-\tau H}|\phi_0\rangle = \lim_{\tau\to\infty} \sum_n e^{-\tau E_n} \langle\phi|n\rangle\langle n|\phi_0\rangle = \langle\phi|0\rangle\langle 0|\phi_0\rangle \tag{3.20a}$$

$$\Rightarrow \quad \langle\phi|0\rangle \propto \lim_{\tau\to\infty} \langle\phi|e^{-i(-i\tau)H}|\phi_0\rangle \propto \int_{\varphi_E(0,\underline{x})=\phi_0(\underline{x})}^{\varphi_E(\infty,\underline{x})=\phi(\underline{x})} \mathcal{D}\varphi_E \, \exp\left(-\int_{t=0}^{t=\infty} \mathrm{d}^4x \, \mathcal{L}_E[\varphi_E]\right) , \tag{3.20b}$$

where we have made use of the fact that $\exp(-\tau H)$ can be interpreted as time evolution operator in imaginary time.

[8] The Euclidean path integral (3.21) can be written as path integral in imaginary time,

$$\Psi_0[\phi] \propto \int_{\varphi(0,\underline{x})=\phi_0(\underline{x})}^{\varphi(-i\infty,\underline{x})=\phi(\underline{x})} \mathcal{D}\varphi \, \exp\left(i\int_{t=0}^{t=-i\infty} \mathrm{d}^4x \, \mathcal{L}[\varphi]\right) . \tag{3.22}$$

[9] The amplitude $\langle\psi_2|\rho|\psi_1\rangle$ can be written as a path integral,

$$\langle\psi_2|e^{-\beta H}|\psi_1\rangle \propto \int_{\varphi_E(0,\underline{x})=\psi_2(\underline{x})}^{\varphi_E(\beta,\underline{x})=\psi_1(\underline{x})} \mathcal{D}\varphi_E \, \exp\left(-\int_{t=0}^{t=\beta} \mathrm{d}^4x \, \mathcal{L}_E[\varphi_E]\right) . \tag{3.24}$$

where $\mathcal{C}'$ is the complex time contour in figure 3.2b. The field obeys periodic boundary conditions at the endpoints of the contour, $\varphi(t_0, \underline{x}) = \varphi(t_0 - i\beta, \underline{x})$.

A signature for equilibrium physics is the emergence of a *fluctuation-dissipation relation* linking the system's response to external perturbations encoded by the retarded correlator G_R, eq. (3.13), to the fluctuations encoded by the Wightman function G_-, eq. (3.11). The relation can be derived from the KMS (Kubo-Martin-Schwinger) condition which is valid in thermal equilibrium [159].[10] The fluctuation-dissipation relation is[11]

$$\operatorname{Im} G_F(k) = (1 + 2n_\beta(\omega)) \operatorname{Im} G_R(k) , \tag{3.28}$$

where the occupation number, $n_\beta(\omega) = \left(e^{\beta\omega} - 1\right)^{-1}$, is the Bose-Einstein distribution. For correlators of a hermitian operator, the relation can be rewritten as

$$G_-(\omega, \underline{k}) = -2n_\beta(\omega) \operatorname{Im} G_R(\omega, \underline{k}) . \tag{3.29}$$

We use the fluctuation-dissipation theorem in chapter 6 to define an effective occupation number out of equilibrium.

3.2 Hydrodynamics

In this section we introduce the effective theory referred to as hydrodynamics. In subsection 3.2.1 the basic assumptions are laid out. We first study ideal hydrodynamics and the connection to thermodynamics in subsection 3.2.2 before we turn to dissipative hydrodynamics in subsection 3.2.3 and consider the transport coefficients that arise in first- and second-order hydrodynamics for conformal and non-conformal fluids. In subsection 3.2.4 we solve the linearised conservation equations in dissipative hydrodynamics and interpret the resulting fluctuations.

[10] The KMS condition,

$$G_+(x,y)|_{x^0=t_0} = G_-(x,y)|_{x^0=t_0-i\beta} , \tag{3.26}$$

with G_+ and G_- as given in eq. (3.11), is derived using the fact that the thermal density matrix can be interpreted as time evolution operator in imaginary time.

[11] The Fourier transform of the two-point function $G(x)$, $G \in \{G_+, G_-, G_F, G_R\}$, is

$$G(k) = \int \mathrm{d}^4x\, e^{-ikx} G(x) . \tag{3.27}$$

3.2.1 Basic assumptions of hydrodynamics

In an interacting field theory, hydrodynamics [171–173] is a low-energy effective theory for slowly varying fluctuations around global thermal equilibrium. The basic assumption is that the only relevant degrees of freedom are the expectation values of global charge densities whose dynamics are governed by the conservation equations for the corresponding current densities, the so-called fluid equations of motion. The expectation values are assumed to be constant on patches which are large compared to the scale of the underlying microscopic field theory, but small compared to macroscopic thermodynamic scales.

The so-called constitutive relations express the current densities entering the fluid equations in terms of the charge densities. Since hydrodynamics describes slowly-varying fluctuations of the charge densities, the gradients of these densities are small by definition. Consequently, this allows for a gradient expansion of the current densities. As common for effective theories, all tensor structures composed of current densities which are compatible with the underlying symmetries are included at each order in the expansion. Each such tensor structure is accompanied by a free parameter, which has to be determined from the underlying microscopic theory, commonly referred to as transport coefficient.

We consider relativistic hydrodynamics by studying the hydrodynamic regime of a Poincaré-invariant quantum field theory in (3 + 1)-dimensions, $x^\mu = (t, x, y, z)$, on a background with metric $g_{(0)\mu\nu}$.[12] For simplicity we consider a fluid with no additional conserved charges beyond the ones following from Poincaré invariance. The charge densities associated with translation symmetry are the energy density T^{00} and the momentum densities T^{0i}, $i = 1, 2, 3$.[13] The fluid equations of motion are

$$\nabla_\mu \langle T^{\mu\nu} \rangle = 0 \,. \tag{3.30}$$

The constitutive relations determine the expectation values $\langle T^{\mu\nu} \rangle$ in terms of the charge densities $\langle T^{0\mu} \rangle$.[14]

[12] Despite focusing on a flat background metric later on, we introduce a general background metric $g_{(0)\mu\nu}$ here since it allows us to consider perturbations of the hydrodynamic stress tensor around flat Minkowski space by setting $g_{(0)\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and then expanding the constitutive relations in $h_{\mu\nu}$.

[13] The charges associated with Lorentz symmetry, $\mathcal{M}^{\mu\nu\lambda} = x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda}$ are identically conserved due to the symmetry of the stress-energy tensor.

[14] The constitute relations can also be expressed in terms of alternative fields, such the temperature T

3.2.2 Ideal hydrodynamics

The thermal equilibrium state of an uncharged fluid is fully characterised by the expectation values of the conserved energy and momentum densities, $\langle T^{0\mu}\rangle$, or equivalently by their conjugate variables, the temperature and the fluid velocity (T, u^μ). The velocity is restricted by the condition $u^\mu u_\mu = -1$. As hydrodynamics describes systems which deviate slightly from global thermal equilibrium, the parameters are promoted to slowly varying fields $T(x)$, $u^\mu(x)$. We can use these fields as fluid variables instead of the expectation values of the conjugate charge densities.

The constitutive relations express the conserved currents $\langle T^{\mu\nu}\rangle$ in terms of the hydrodynamic variables usually constructed as a derivative expansion. To zeroth order in gradients $\mathcal{O}(\partial^0)$, interactions between patches of local equilibrium are neglected resulting in the constitutive relation for an ideal fluid,

$$\langle T^{\mu\nu}\rangle = [\epsilon(x) + p(x)]\, u^\mu(x) u^\nu(x) + p(x) g^{\mu\nu}_{(0)}(x) + \mathcal{O}(\partial)\ , \tag{3.31}$$

with local 4-velocity u^μ, energy density ϵ and pressure p. The local equilibrium quantities $\epsilon(x)$ and $p(x)$ are linked by the fluid's equation of state and satisfy the usual thermodynamic relations. The entropy current in an ideal fluid is conserved,[15]

$$\nabla_\mu (s u^\mu) = 0\ . \tag{3.33}$$

3.2.3 Dissipative hydrodynamics

In order to include dissipation, we need to go beyond the leading order in the gradient expansion of the current densities. We add the symmetric tensor $\Pi^{\mu\nu}$ to the constiutive relation (3.31), which is built from the fluid variables as well as $g_{(0)\mu\nu}$ and contains all higher derivative corrections. The constitutive relations for an uncharged relativistic fluid

and fluid velocity u^μ, after a change of variables.

[15]To deduce this, observe that the longitudinal projection of the stress-energy conservation equations, eq. (3.30), lead to

$$\nabla_\mu \left[(\epsilon + p) u^\mu\right] = u^\mu \nabla_\mu p\ . \tag{3.32}$$

Applying the hydrodynamic relations $\epsilon = Ts - p$ and $\mathrm{d}p = s\mathrm{d}T$, the entropy conservation (3.33) follows [173].

are known up to third order in gradients [172, 174, 175]. Up to second order, seventeen independent tensor structures can be constructed from the fluid variables and $g_{(0)\mu\nu}$. Each of these tensor structures is multiplied by a transport coefficient when included in the derivative expansion of the energy-momentum tensor. These transport coefficients are the free input parameters of the effective hydrodynamic description and they can only be obtained from the underlying microscopic theory. In order to compute their values we have to match the hydrodynamic result for appropriate correlators of $T^{\mu\nu}$ with the corresponding result in the underlying microscopic theory.

The fluid variables can be modified by derivative terms as long as $\langle T^{\mu\nu}\rangle$ remains unchanged. In the so-called Landau frame we use this freedom to ensure $u_\mu(x)\langle T^{\mu\nu}(x)\rangle = -\epsilon(x)u^\nu(x)$, i.e. $u_\mu \Pi^{\mu\nu} = 0$ to all orders in the derivative expansion [171–173]. We define the projection to symmetric, traceless tensors that are transverse to the fluid motion as

$$\Delta^{\mu\nu} \equiv u^\mu u^\nu + g_{(0)}^{\mu\nu} \, , \tag{3.34}$$

$$A^{<\mu\nu>} \equiv \frac{1}{2}\Delta^{\mu\rho}\left(A_{\rho\sigma} + A_{\sigma\rho}\right)\Delta^{\sigma\nu} - \frac{1}{3}\Delta^{\mu\nu}\left(\Delta^{\rho\sigma}A_{\rho\sigma}\right) \, , \tag{3.35}$$

the shear tensor as

$$\sigma^{\mu\nu} \equiv 2\nabla^{<\mu}u^{\nu>} \, , \tag{3.36}$$

and the vorticity tensor as

$$\Omega^{\mu\nu} \equiv \frac{1}{2}\Delta^{\mu\rho}\left(\nabla_\rho u_\sigma - \nabla_\sigma u_\rho\right)\Delta^{\sigma\nu} \, . \tag{3.37}$$

Employing these definitions, the constitutive relation for $\langle T^{\mu\nu}\rangle$ can then be written as

$$\langle T^{\mu\nu}\rangle = \epsilon u^\mu u^\nu + p\Delta^{\mu\nu} + \Pi^{\mu\nu}_{\text{conf.}} + \Pi^{\mu\nu}_{\text{non-conf.}} + \mathcal{O}(\partial^3) \, , \tag{3.38}$$

where

$$\begin{aligned} \Pi^{\mu\nu}_{\text{conf.}} \equiv& -\eta\,\sigma^{\mu\nu} \\ &+ \eta\,\tau_\pi\left[\left(u^\lambda\nabla_\lambda\sigma\right)^{<\mu\nu>} + \frac{1}{3}\sigma^{\mu\nu}\left(\nabla\cdot u\right)\right] \\ &+ \kappa\left[R^{<\mu\nu>} - 2u_\lambda R^{\lambda<\mu\nu>\kappa}u_\kappa\right] \\ &+ \lambda_1\sigma_\lambda{}^{<\mu}\sigma^{\nu>\lambda} + \lambda_2\sigma_\lambda{}^{<\mu}\Omega^{\nu>\lambda} - \lambda_3\Omega_\lambda{}^{<\mu}\Omega^{\nu>\lambda} \end{aligned} \tag{3.39}$$

is present in conformal and non-conformal fluids and was first derived in ref. [174], and where

$$\begin{aligned}\Pi^{\mu\nu}_{\text{non-conf.}} \equiv & -\zeta\,\Delta^{\mu\nu}\,(\nabla\cdot u) \\ & + \eta\,\tau_\pi^*\,\frac{1}{3}\sigma^{\mu\nu}\,(\nabla\cdot u) + \kappa^*\,2u_\lambda R^{\lambda<\mu\nu>\kappa}u_\kappa + \lambda_4 \nabla^{<\mu}\log s\nabla^{\nu>}\log s \\ & + \Big(\zeta\,\tau_\Pi\,u^\lambda\nabla_\lambda\,(\nabla\cdot u) + \xi_1\,\sigma^{\kappa\lambda}\sigma_{\kappa\lambda} + \xi_2\,(\nabla\cdot u)^2 + \xi_3\,\Omega^{\kappa\lambda}\Omega_{\kappa\lambda} \\ & + \xi_4\,\Delta_\kappa{}^\lambda\,(\nabla_\lambda\log s)\,\Delta^{\kappa\rho}\,(\nabla_\rho\log s) + \xi_5\,R + \xi_6\,u^\kappa u^\lambda R_{\kappa\lambda}\Big)\,\Delta^{\mu\nu}\end{aligned} \tag{3.40}$$

was constructed in ref. [172] and vanishes for conformal fluids.

3.2.4 Hydrodynamic equations in flat Minkowski space

In this subsection we consider hydrodynamic modes in conformal and non-conformal dissipative hydrodynamics in a flat background, $g_{(0)\mu\nu} = \eta_{\mu\nu}$. These modes are solutions to the linearised hydrodynamic equations, $\nabla_\mu\langle T^{\mu\nu}\rangle = 0$. The constitutive relations for the stress tensor is eq. (3.38) with $\Pi^{\mu\nu}_{\text{non-conf.}}$ only contributing in the non-conformal case, complemented with an equation of state, $p = p(\epsilon)$, which is fixed to $p = \epsilon/3$ in the conformal case.

We go to the local rest frame of the fluid in global equilibrium so that the equilibrium fluid velocity is $u^\mu = (1, \underline{0})$ and consider fluctuations of the hydrodynamic variables

$$\epsilon(x) = \bar\epsilon + \delta\epsilon(x)\,, \qquad u^\mu(x) = (1, \underline{v}(x))(1 + \underline{v}(x)^2)^{-1/2}\,. \tag{3.41}$$

Here $\bar\epsilon$ is the energy density in global thermal equilibrium. The fluctuations $\delta\epsilon$ and $\underline{v}$ are assumed to be small and we denote $\mathcal{O}(\delta) = \mathcal{O}(\delta\epsilon, \underline{v})$.[16] Further denoting the equilibrium stress tensor by

$$\overline{T}^{\mu\nu} \equiv \langle T^{\mu\nu}\rangle\,[\delta\epsilon = \underline{v} = 0]\;, \tag{3.43}$$

[16]The pressure fluctuations $\delta p(x)$ are determined by $\delta\epsilon$ via the equation of state,

$$p(x) = p(\epsilon(x)) = p(\bar\epsilon) + \left.\frac{\partial p}{\partial \epsilon}\right|_{\bar\epsilon} \delta\epsilon(x) + \mathcal{O}(\delta^2) \equiv \bar p + \delta p(x)\,. \tag{3.42}$$

the off-shell stress tensor of the perturbed fluid, eq. (3.38), assumes the following form to first order in fluctuations,

$$\langle T^{\mu\nu}\rangle\,[\delta\epsilon,\underline{v}] \equiv \overline{T}^{\mu\nu} + \delta T^{\mu\nu} = \overline{T}^{\mu\nu} + \left[\frac{\partial \overline{T}^{\mu\nu}}{\partial\,\delta\epsilon}\delta\epsilon + \frac{\partial \overline{T}^{\mu\nu}}{\partial v^i}v^i\right] + \mathcal{O}(\delta^2)\ , \tag{3.44}$$

where we defined

$$\frac{\partial \overline{T}^{\mu\nu}}{\partial\,\delta\epsilon} \equiv \frac{\partial\langle T^{\mu\nu}\rangle}{\partial\,\delta\epsilon}\bigg|_{\delta\epsilon=\underline{v}=0} \qquad \text{and} \qquad \frac{\partial \overline{T}^{\mu\nu}}{\partial v^i} \equiv \frac{\partial\langle T^{\mu\nu}\rangle}{\partial v^i}\bigg|_{\delta\epsilon=\underline{v}=0}\ . \tag{3.45}$$

The equations of motion for the fluid variables $\delta\epsilon$ and $\underline{v}$ can be obtained by linearising the conservation equation, eq. (3.30), around equilibrium.

We assume that the fluctuations $\delta\epsilon$ and $\underline{v}$ (and consequently all stress tensor components $\langle T^{\mu\nu}\rangle$) only depend on the coordinates (t,z) preserving an $SO(2)$ subgroup of the $SO(1,3)$ symmetry. The indices α, β, γ denote the (x,y)-directions orthogonal to the (t,z)-plane. The stress tensor components can then be classified according to representations of the residual $SO(2)\subseteq SO(1,3)$ symmetry orthogonal to the (t,z)-plane:

- Scalar: T^{tt}, T^{tz}, T^{zz}, $\sum_\alpha T^{\alpha\alpha}$,
- Vector: $T^{t\alpha}$, $T^{\alpha z}$,
- Tensor: $T^{\alpha\beta} - \frac{1}{2}\delta^{\alpha\beta}\sum_\gamma T^{\gamma\gamma}$.

The fluctuations of their expectations values, $\delta T^{\mu\nu}$, are functions of the energy density and fluid velocity via the constitutive relations, e.g. $\delta T^{tt} = \delta\epsilon + \mathcal{O}(\delta^2)$, $\delta T^{tz} = (\overline{\epsilon}+\overline{p})v^z + \mathcal{O}(\delta^2)$ and $\delta T^{t\alpha} = (\overline{\epsilon}+\overline{p})v^\alpha + \mathcal{O}(\delta^2)$. The tensor component does not fluctuate at the linear level.[17] We now solve the linearised hydrodynamic equations for scalar and vector fluctuations in the conformal and non-conformal case using the constitutive relation (3.38).

Conformal fluid: The two scalar equations of motion are

$$\partial_z\delta\epsilon + \overline{w}\partial_t v^z = 0\ , \qquad v_s^2\partial_z\delta\epsilon - \frac{4}{3}\eta\left(1-\tau_\pi\partial_t\right)\partial_z^2 v^z + \overline{w}\partial_t v^z = 0\ , \tag{3.46}$$

where $\overline{w} = \overline{\epsilon} + \overline{p}$ is the equilibrium enthalpy and the speed of sound is

$$v_s^2 = \frac{\partial p}{\partial\epsilon}\bigg|_{\epsilon=\overline{\epsilon}}\ . \tag{3.47}$$

[17] Due to symmetries the tensor fluctuation has to be $\delta T^{\alpha\beta} - \frac{1}{2}\delta^{\alpha\beta}\sum_\gamma \delta T^{\gamma\gamma} \propto v^\alpha v^\beta = \mathcal{O}(\delta^2)$ which is quadratic in fluctuations.

Employing a plane-wave ansatz, $\{\delta\epsilon, v^z\} \propto \{\widetilde{\delta\epsilon}, \widetilde{v}^z\} e^{-i\omega t + ikz}$, we can write the equations of motion in matrix form as

$$\underline{0} = \underline{M}(\omega, k) \begin{pmatrix} \widetilde{\delta\epsilon} \\ \widetilde{v}^z \end{pmatrix} \ . \tag{3.48}$$

The equation has non-trivial solutions only if $\det \underline{M}(\omega, k) = 0$, which leads to the dispersion relation

$$\omega(k) = \pm v_s k - i\Gamma k^2 \pm \frac{\Gamma}{v_s} \left(v_s^2 \tau_\pi - \frac{\Gamma}{2} \right) k^3 + \mathcal{O}(k^4) \ , \tag{3.49}$$

where we have introduced the notation $\Gamma = \frac{4}{3}\frac{\eta}{\overline{\epsilon}+\overline{p}}$. This is a dissipative longitudinal sound mode. The transverse fluctuations obey

$$0 = \overline{w}\partial_t v^\alpha - \eta \left(1 - \tau_\pi \partial_t\right) \partial_z^2 v^\alpha \ , \tag{3.50}$$

which leads to a hydrodynamic dispersion relation for purely damped shear modes,

$$\omega(k) = -i\frac{\eta}{\overline{\epsilon} + \overline{p}} k^2 + \mathcal{O}(k^4) \ , \tag{3.51}$$

with the shear viscosity η appearing at leading order.

Non-conformal fluid: For the non-conformal fluid we only consider first-order corrections in the constitutive relation for the stress tensor when deriving linearised equations of motion for the hydrodynamic fluctuations. In the scalar channel, the second equation in eq. (3.46) is modified to

$$0 = v_s^2 \partial_z \delta\epsilon - \frac{4\eta + 3\zeta}{3} \partial_z^2 v^z + \overline{w}\partial_t v^z \ . \tag{3.52}$$

Using a plane-wave ansatz in the resulting equations leads to the dispersion relation

$$\omega(k) = \pm v_s k - i\frac{4\eta + 3\zeta}{3(\overline{\epsilon} + \overline{p})} k^2 + \mathcal{O}(k^3) \ , \tag{3.53}$$

i.e. the sound mode dissipation is affected by the bulk viscosity ζ. The transverse channel is unaffected at this order in the hydrodynamic expansion.

3.3 Transport coefficients and Kubo formulae

In section 3.2 we have seen that the hydrodynamic constitutive relations for the stress-energy tensor contain constants that cannot be determined from hydrodynamics alone. They enter the dispersion relations of hydrodynamic modes, e.g. eqs. (3.49), (3.51) and (3.53). These transport coefficients depend on the underlying microscopic theory. Introducing the same small external perturbations of the background both in the microscopic theory and its effective description helps us to identify the hydrodynamic transport coefficients with correlation functions of the conserved currents in the underlying theory. These relations are called Kubo formulae [176]. A suitable quantity to match is the response $\langle T^{\mu\nu}\rangle$ of a fluid in equilibrium to an external metric perturbation around flat space, $g_{(0)\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$. We use the results from this section to compute Kubo formulae in section 4.2.

3.3.1 External perturbations in hydrodynamics

We first study how the expectation value $\langle T^{\mu\nu}\rangle$ in hydrodynamics responds to the perturbation of the background. As in the case of a flat background, eq. (3.41), we can consider the fluid variables of the perturbed fluid in the equilibrium rest frame, where they take the form

$$\epsilon(x) = \bar{\epsilon} + \delta\epsilon(x)\ , \qquad u^\mu(x) = (1, \underline{v})\left(-g_{(0)tt} - 2g_{(0)ti}v^i - g_{(0)ij}v^i v^j\right)^{-1/2} \tag{3.54}$$

in the presence of the background metric $g_{(0)\mu\nu}$. In static equilibrium the energy density is $\bar{\epsilon}$ and the fluid is at rest, $u^\mu = (1, \underline{0})$. Using the fluctuations $\delta\epsilon(x)$ and $\underline{v}(x)$ as fluid variables, we can conveniently study a second expansion in fluctuations around static global equilibrium sourced by $h_{\mu\nu}$ in addition to the hydrodynamic gradient expansion. By analogy with eq. (3.43), we write the equilibrium stress tensor without sources as

$$\overline{T}^{\mu\nu} \equiv \langle T^{\mu\nu}\rangle\left[\delta\epsilon = \underline{v} = h = 0\right]\ . \tag{3.55}$$

In consequence, to first order in fluctuations, $\mathcal{O}(\delta)$, the off-shell stress tensor of the perturbed fluid, eq. (3.38), reads

$$\langle T^{\mu\nu}\rangle\,[\delta\epsilon,\underline{v};h] = \overline{T}^{\mu\nu} + \left[\frac{\partial\overline{T}^{\mu\nu}}{\partial\,\delta\epsilon}\delta\epsilon + \frac{\partial\overline{T}^{\mu\nu}}{\partial v^i}v^i\right] + \frac{\partial\overline{T}^{\mu\nu}}{\partial h_{\rho\sigma}}h_{\rho\sigma} + \mathcal{O}(\delta^2)\ , \tag{3.56}$$

using the definitions from eq. (3.45) as well as

$$\frac{\partial\overline{T}^{\mu\nu}}{\partial h_{\rho\sigma}} \equiv \left.\frac{\partial\langle T^{\mu\nu}\rangle}{\partial h_{\rho\sigma}}\right|_{\delta\epsilon=\underline{v}=h=0}\ . \tag{3.57}$$

We obtain the equations of motion for the fluid variables $\delta\epsilon$ and $\underline{v}$ in the presence of the linear metric perturbation $h_{\rho\sigma}$ by linearising the fluid equations, eq. (3.30), around equilibrium. Using the definitions

$$\delta T^{\mu\nu}_{(\delta\epsilon,\underline{v})} \equiv \frac{\partial\overline{T}^{\mu\nu}}{\partial\,\delta\epsilon}\delta\epsilon + \frac{\partial\overline{T}^{\mu\nu}}{\partial v^i}v^i\ , \qquad \delta T^{\mu\nu}_{(h)} \equiv \frac{\partial\overline{T}^{\mu\nu}}{\partial h_{\rho\sigma}}h_{\rho\sigma}\ , \tag{3.58}$$

these equations take the form

$$\partial_\mu\delta T^{\mu\nu}_{(\delta\epsilon,\underline{v})} = -\partial_\mu\delta T^{\mu\nu}_{(h)} - \delta\Gamma^{\mu}_{\mu\rho}\overline{T}^{\rho\nu} - \delta\Gamma^{\nu}_{\mu\rho}\overline{T}^{\mu\rho} + \mathcal{O}(\delta^2)\ . \tag{3.59}$$

Since we only consider hydrodynamic modes sourced by the background fluctuations $h_{\rho\sigma}$ and not the usual free (i.e. unsourced) hydrodynamic modes, we impose that $\delta\epsilon = \underline{v} = 0$ for $h = 0$ as boundary conditions.

The Kubo formulae we derive in section 4.2 link transport coefficients to three-point functions. To this end we have to determine the response of the on-shell stress tensor up to second order in metric perturbations h. If we turn on a set of perturbations h such that the right hand side of eq. (3.59) vanishes at $\mathcal{O}(h)$, the hydrodynamic equations are solved by $\delta\epsilon(h), \underline{v}(h) = \mathcal{O}(h^2)$. In that case, we find the following expressions for the on-shell stress tensor to second order in background fluctuations $\mathcal{O}(h^2)$,

$$\begin{aligned}\langle T^{\mu\nu}\rangle\,[h] = \overline{T}^{\mu\nu} &+ \frac{\partial\overline{T}^{\mu\nu}}{\partial h_{\rho\sigma}}h_{\rho\sigma} + \frac{1}{2}\frac{\partial^2\overline{T}^{\mu\nu}}{\partial h_{\rho\sigma}\partial h_{\kappa\lambda}}h_{\rho\sigma}h_{\kappa\lambda}\\ &+ \left(\frac{\partial\overline{T}^{\mu\nu}}{\partial\,\delta\epsilon}\delta\epsilon(h) + \frac{\partial\overline{T}^{\mu\nu}}{\partial v^i}v^i(h)\right) + \mathcal{O}(h^3)\ .\end{aligned} \tag{3.60}$$

This expression simplifies even further if we focus on the transverse-tensor component $\langle T^{xy}(t,z)\rangle$ and only consider perturbations of the form $h_{\mu\nu}(t,z)$. Symmetry then dictates, that the transverse-tensor component is independent of the scalars $\delta\epsilon$, v^z and the transverse-vector components v^x, v^y to first order $\mathcal{O}(\delta)$,

$$\frac{\partial \overline{T}^{xy}}{\partial\, \delta\epsilon} = \frac{\partial \overline{T}^{xy}}{\partial v^i} = 0\ . \tag{3.61}$$

The resulting expression for the transverse-tensor component of the stress tensor up to second order in background perturbations h is

$$\langle T^{xy}(t,z)\rangle\,[h] = \overline{T}^{xy} + \frac{\partial \overline{T}^{xy}}{\partial h_{\rho\sigma}} h_{\rho\sigma} + \frac{1}{2}\frac{\partial^2 \overline{T}^{xy}}{\partial h_{\rho\sigma}\partial h_{\kappa\lambda}} h_{\rho\sigma} h_{\kappa\lambda} + \mathcal{O}(h^3)\ . \tag{3.62}$$

In summary, if we consider metric perturbations $h_{\mu\nu}(t,z)$ that do not source fluid fluctuations to linear order, the transverse-tensor component of the stress tensor does not receive contributions from the fluid fluctuations to second order. Therefore, we can determine the transverse-tensor component up to second order in h without solving the fluid equations.

3.3.2 External perturbations in microscopic field theories

In this subsection we want to use the formalism developed in section 3.1 to study the response of the one-point correlator $\langle T^{xy}\rangle$ to second order in an external metric perturbation h around flat space, $g_{(0)\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$. The external source $h_{\mu\nu}$ couples to the stress tensor of the theory. In subsection 3.3.1 we studied the response to such a perturbation in the hydrodynamic effective theory and obtained the result (3.62) for the transverse-tensor component when considering subsets of metric perturbations that do not source fluid fluctuations to first order. Now, we consider the same perturbation in the microscopic theory.

We consider the field theory in the thermal equilibrium state with temperature $T = \beta^{-1}$. Denoting the field content of the theory collectively by φ, the generating functional on connected correlators, introduced in eq. (3.10), is

$$W_\beta[h] = \ln \int \mathcal{D}\varphi\, e^{iS_{\mathcal{C}'}[\varphi]}\ , \qquad S_{\mathcal{C}'}[\varphi] = \int_{\mathcal{C}'} \mathrm{d}^4x\, \sqrt{-g_{(0)}}\mathcal{L}[\varphi, g_{(0)}]\ , \tag{3.63}$$

where the contour $\mathcal{C}'$ is given in figure 3.2b. The one-point function of the stress tensor in the presence of the perturbation is

$$\langle T^{\mu\nu}(x)\rangle[h] = \frac{-2i}{\sqrt{-g_0}}\frac{\delta W_\beta[h]}{\delta h_{\mu\nu}} . \tag{3.64}$$

The one-point function (3.64) can now be expanded in the background metric perturbation h which is only turned on on the real part of the time contour. Using the notation we developed in subsection 3.1.1[18] this expansion yields

$$\begin{aligned}&\langle T^{\mu\nu}(t,\underline{x})\rangle[h] \\ &= \int \mathcal{D}\varphi_+\,\mathcal{D}\varphi_-\,\mathcal{D}\varphi_E\,T_r^{\mu\nu}(t,\underline{x})\exp\left[i\int_{t_0}^t \mathrm{d}^4x\,\left(\mathcal{L}[\varphi_+]-\mathcal{L}[\varphi_-]\right)-\int_0^\beta \mathrm{d}^4x\,\mathcal{L}_E[\varphi_E]\right] \\ &\quad\times\left(1+\frac{i}{2}\int_{t_0}^t \mathrm{d}^4y\,T_a^{\gamma\lambda}(y)h_{\gamma\lambda}(y)-\frac{1}{8}\int_{t_0}^t \mathrm{d}^4y\,\mathrm{d}^4z\,T_a^{\gamma\lambda}(y)T_a^{\rho\sigma}(z)h_{\gamma\lambda}(y)h_{\rho\sigma}(z)+\mathcal{O}(h^3)\right) .\end{aligned} \tag{3.67}$$

We can express this result in terms of the fully retarded correlator, eq. (3.19), of the energy-momentum tensor in flat space in a finite temperature state,

$$\begin{aligned}&G_{ra\ldots a}^{\mu\nu,\alpha_1\beta_1,\ldots,\alpha_{n-1}\beta_{n-1}}(x,y_1\ldots,y_{n-1}) \\ &\qquad= (-i)^{n-1}\left\langle \mathcal{T}_\mathcal{C} T_r^{\mu\nu}(x)T_a^{\alpha_1\beta_1}(y_1)\ldots T_a^{\alpha_{n-1}\beta_{n-1}}(y_{n-1})\right\rangle_c \\ &\qquad= \left.\frac{(-i)^{n-1}(-2i)^n\,\delta^n W_\beta[h_a,h_r]}{\delta h_{a\mu\nu}(x)\,\delta h_{r\alpha_1\beta_1}(y_1)\ldots\delta h_{r\alpha_{n-1}\beta_{n-1}}(y_{n-1})}\right|_{h=0} .\end{aligned} \tag{3.68}$$

We introduced sources $h_\pm$ on the upper and lower part of the contour in the generating functional and $h_a = h_+ - h_-$, $h_r = (h_+ + h_-)/2$. The resulting expression is [170][19]

$$\begin{aligned}\langle T^{\mu\nu}(x)\rangle[h] = G_r^{\mu\nu}(x) &- \frac{1}{2}\int_{t_0}^\infty \mathrm{d}^4y\,G_{ra}^{\mu\nu,\gamma\lambda}(x,y)h_{\gamma\lambda}(y) \\ &+\frac{1}{8}\int_{t_0}^\infty \mathrm{d}^4y\,\mathrm{d}^4z\,G_{raa}^{\mu\nu,\gamma\lambda,\rho\sigma}(x,y,z)h_{\gamma\lambda}(y)h_{\rho\sigma}(z)+\mathcal{O}(h^3) .\end{aligned} \tag{3.69}$$

[18] We formally introduce $T_\pm^{\mu\nu}$, the stress tensor with time argument on the upper/lower part of the complex time contour $\mathcal{C}'$ in figure 3.2b, to keep track of time ordering of correlators and denote

$$T_a^{\mu\nu} = T_+^{\mu\nu} - T_-^{\mu\nu} , \qquad T_r^{\mu\nu} = (T_+^{\mu\nu} + T_-^{\mu\nu})/2 . \tag{3.65}$$

We also write

$$\varphi_\pm(t,\underline{x}) \equiv \varphi(t^\pm,\underline{x}) , \qquad \varphi_E(t,\underline{x}) = \varphi(-it,\underline{x}) . \tag{3.66}$$

Identification of the fields φ_+, φ_- and φ_E at the turning points of the contour is implicit.

[19] The path integral (3.67) leads to disconnected correlators while $G_{ra\ldots a}$ is defined in terms of a connected correlator in eq. (3.68). The disconnected pieces can be shown to cancel at each order in the expansion in terms of h.

The correlators are computed in a finite-temperature state which is translation-invariant. Therefore we can Fourier transform after taking $t_0 \to -\infty$ and obtain

$$\langle T^{\mu\nu}(x=0)\rangle = G_r^{\mu\nu}(0) - \frac{1}{2}\int \frac{\mathrm{d}^4p}{(2\pi)^4} G_{ra}^{\mu\nu,\gamma\lambda}(p)h_{\gamma\lambda}(p) + \frac{1}{8}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{d}^4p}{(2\pi)^4} G_{raa}^{\mu\nu,\gamma\lambda,\rho\sigma}(q,p)h_{\gamma\lambda}(q)h_{\rho\sigma}(p) + \mathcal{O}(h^3)\,. \quad (3.70)$$

Matching the hydrodynamic response of $\langle T^{xy}\rangle$ to the perturbation h, eq. (3.62), with the response of the microscopic field theory, eq. (3.70), yields the Kubo formulae we derive in section 4.2. The appearance of the (fully) retarded correlators in eq. (3.70) explains the effectiveness of holography to study transport properties of strongly coupled holographic fluids. The holographic dictionary provides a simple prescription to determine retarded correlators of boundary operators from the bulk dual [127].

3.3.3 Kubo formula for the shear viscosity

Before we develop more complicated examples of Kubo formulae in section 4.2, we consider an example that portrays how we can bring together the results of subsections 3.3.1 and 3.3.2 to link a first-order transport coefficient to a retarded two-point function.

A fluid at rest in flat space is perturbed by a metric fluctuation $h_{xy}(t,z)$. The perturbation does not couple to longitudinal sound or transverse shear fluctuations to linear order and therefore the fluid remains at rest, $\delta\epsilon, v^i = \mathcal{O}(h^2)$. Therefore, the transverse-tensor component of the stress tensor responds as

$$\langle T^{xy}\rangle[h] = (-p - \eta\partial_t)h_{xy}(t,z) + \mathcal{O}(h^2,\partial^2)\,. \quad (3.71)$$

Choosing a plane-wave perturbation of the form $h_{xy} = \epsilon H_{xy}\exp(-i\omega t + ikz)$, results in

$$\langle T^{xy}(x=0)\rangle[h] = (-p + i\omega\eta)\epsilon H_{xy} + \mathcal{O}(\epsilon^2,\omega^2,k^2)\,. \quad (3.72)$$

From the microscopic result, eq. (3.69), we obtain[20]

$$\langle T^{xy}(x=0)\rangle[h] = -G_{ra}^{xy,xy}(\omega,0,0,k)\epsilon H_{xy} + \mathcal{O}(\epsilon^2,\omega^2,k^2)\,. \quad (3.73)$$

[20] We use that the perturbation h_{xy} is accompanied by $h_{yx} = h_{xy}$ as required by symmetry of the metric and that $G_{ra}^{xy,yx} = G_{ra}^{xy,xy}$ by symmetry of the stress tensor.

Consequently, from eqs. (3.72) and (3.73) we find the relation

$$G_{ra}^{xy,xy}(\omega, 0, 0, k) = p - i\omega\eta + \mathcal{O}(\omega^2, k^2) \ , \tag{3.74}$$

from which we derive the Kubo relation

$$\eta = -\lim_{\omega \to 0} \frac{1}{\omega} \operatorname{Im} G_{ra}^{xy,xy}(\omega, \underline{k} = 0) \ . \tag{3.75}$$

Chapter 4

Hydrodynamics of non-conformal holographic fluids

4.1 Introduction and summary

We introduced the theory of hydrodynamics [171, 173] as the low-energy effective description of slowly varying fluctuations around the thermal equilibrium state in section 3.2. We discussed the constitutive relations, which express the current densities in terms of the charge densities. For the stress-energy tensor of an uncharged relativistic fluid in $3+1$ dimensions, eq. (3.38), the constitutive relation contains two transport coefficients at first order, the shear viscosity η and the bulk viscosity ζ, as well as further fifteen coefficients at second order [172, 174]. Ten of the latter fifteen coefficients, as well as the bulk viscosity, are absent in conformal fluids and only appear in the non-conformal case. We introduced Kubo formulae in section 3.3 as relations linking transport coefficients in the hydrodynamic effective theory to correlation functions in the underlying microscopic field theory. Kubo relations are known for the two first-order coefficients as well as for the five second-order coefficients present in conformal fluids [170] and for six of the ten second-order coefficients that only present for non-conformal fluids [178].

The theory of hydrodynamics has a wide range of applications in current research. For example, the early-stage evolution of the so-called quark-gluon plasma (QGP), created in heavy-ion collisions at particle colliders such as RHIC, has been successfully described applying hydrodynamics [179–183]. The applicability of hydrodynamics has even been shown to extend to very early stages in the evolution of the QGP during which the gradients of the fluid variables are not small, thus seemingly invalidating the hydrodynamic gradient expansion. Since first-order hydrodynamics suffers from superluminal modes violating causality, hydrodynamic simulations of the QGP cannot be restricted to the leading order in the gradient expansion but have to include second-order terms [174]. This necessitates knowledge of the first- and second-order transport coefficients. The microscopic correlation functions linked to the transport coefficients by means of Kubo formulae cannot be computed perturbatively as the temperature of the QGP is close to the confinement scale of QCD and therefore in the intermediate-coupling regime. Exacerbating the situation even further, lattice methods are unsuitable for the computation of the real-time correlation functions due to problems with analytic continuation, while indirect methods avoiding real-time correlators are prone to large uncertainties [184].

Gauge/gravity duality [1–4, 127, 128] provides a tool allowing for the computation of real-time correlation functions in interacting field theories at strong coupling, i.e. in a regime where other suitable methods are currently lacking. By means of the holographic duality, the problem of calculating correlation functions of the stress-energy tensor at strong coupling in the hydrodynamic expansion translates into solving classical Einstein's equations in AdS backgrounds perturbatively in small momenta. Using the link provided by Kubo formulae between transport coefficients and correlation functions in the underlying microscopic theory, transport coefficients of the strongly-coupled theory in the hydrodynamic regime can therefore be determined from the solutions to Einstein's equations in AdS [22, 185, 186]. In addition to these practical applications, the hydrodynamic interpretation of solutions to Einstein gravity extends the known analogy between black-hole mechanics and thermodynamics to a correspondence between gravity and fluids, which are only thermally equilibrated in local patches which are small compared to macroscopic scales [187].

Making contact with experiment such as heavy ion collisions and testing predictions for the QGP, which were made using the holographic duality, is facing a formidable obstacle. Even if it may be possible in principle, it is currently unknown how to construct the exact holographic dual of theories realised in nature, such as QCD. A way to circumvent this obstacle is to consider so-called universal properties, which are insensitive to the microscopic details of the field theory and are therefore potentially shared by large classes of strongly-coupled QFTs. In this manner, we can try to approach strongly coupled field theories without knowing their exact gravity dual. The famous result that the ratio of the shear viscosity η and the entropy density s obeys the Kovtun-Son-Starinets relation, $\eta/s = 1/4\pi$, for any strongly-coupled theory with a two-derivative gravity dual and preserved spatial isotropy is the outstanding example of such a universal relation determined by means of the holographic duality [20–22, 188–194]. Recent experimental results from heavy ion collision allow for estimates for the value of η/s which is indeed close to the proposed universal value [195–201].

There exists a promising candidate for a universally satisfied relation between transport coefficients at strong coupling beyond the famous $\eta/s = 1/4\pi$. This candidate is the so-called Haack-Yarom identity,

$$H \equiv 2\eta\tau_\pi - 4\lambda_1 - \lambda_2 = 0\ . \tag{4.1}$$

It links second-order hydrodynamic transport coefficients and was first encountered in ref. [203]. Haack and Yarom showed in ref. [23] that it is satisfied for conformal holographic theories with a two-derivative gravity dual and with any number of $U(1)$-charges at finite density. Further evidence has been amassed that the relation (4.1) is indeed universally satisfied at strong coupling. The combination H also vanishes for the non-conformal fluids which are holographically dual to the Chamblin-Reall background [208] and compactified D4-branes [212].[1] Leading corrections to second-order transport away from the strict infinite-coupling limit have been shown to preserve the Haack-Yarom identity in planar $\mathcal{N} = 4$ SYM [204] and in the hypothetical fluid dual to Gauss-Bonnet gravity [205–207]. A further encouraging hint towards the possible universality of the Haack-Yarom identity,

[1] However, the Chamblin-Reall background as well as the compactified D4-branes are special cases. The transport coefficients of the non-conformal fluids dual to these backgrounds are entirely dictated by those of the conformal fluids dual to higher dimensional AdS spaces [116]. The Chamblin-Reall background can simply be viewed as an analytic continuation of higher-dimensional AdS compactified on a torus [209]. Also the compactified D4-brane background can be viewed as a compactification of AdS and one can essentially borrow the higher dimensional AdS/CFT dictionary [115, 221]. The second-order transport coefficients of the conformal d-dimensional fluid dual to AdS_{d+1} in arbitrary dimensions, determined in refs. [210, 211] and listed in appendix F of our paper, ref. [32], satisfy the Haack-Yarom identity and the relation is preserved by the dimensional reduction. In addition, both backgrounds do not admit an asymptotically AdS region and therefore do not have an obvious UV fixed point.

eq. (4.1), is that the coefficients entering the relation can be determined via Kubo formulae without considering sound perturbations, which would couple to the non-universal matter content and thus likely lead to results which are sensitive to the model details [202].

In this chapter we want to address the open question whether the Haack-Yarom identity holds more generally in holographic theories without conformal symmetry. To this end, we first derive Kubo formulae for five combinations of second-order transport coefficients,

$$\kappa\,, \quad \eta\,\tau_\pi + \kappa^*\,, \quad \lambda_1 + \frac{\kappa^*}{2}\,, \quad \lambda_2\,, \quad \lambda_3 - 2\kappa^*\,, \tag{4.2}$$

featuring the non-conformal coefficient κ^* in addition to the five coefficients already present in conformal fluids. The combination H, eq. (4.1), can be determined from the coefficients in eq. (4.2) through linear combination. We apply the Kubo formulae to a large class of non-conformal holographic models dual to renormalisation-group (RG) flows in asymptotically AdS_5 triggered by an arbitrary scalar operator of dimension $\Delta = 3$. As a by-product of the investigation into the Haack-Yarom identity, we find that another relation between second-order coefficients,

$$\tilde{H} \equiv 2\eta\tau_\pi - 2\,(\kappa - \kappa^*) - \lambda_2 = 0\,, \tag{4.3}$$

is satisfied for any field theory that falls into the large class of holographic models under consideration. We prove analytically that the Haack-Yarom identity (4.1) remains satisfied in this class of models when including leading non-conformal corrections to the transport coefficients. Subsequently, we study two specific families of RG flows within the larger class of models and provide strong numerical evidence that H indeed vanishes along flows in both families even beyond leading non-conformal corrections.

This chapter, which is based on ref. [32], is structured as follows: in section 4.2 we derive Kubo formulae for the five second-order coefficients (4.2) valid for any uncharged relativistic fluid in 3+1 dimensions regardless of conformal symmetry. The derivation of the Kubo formulae considers the response of the tensor component of the stress-energy tensor to external shear perturbations up to second order. We explain how the formulae can be applied to QFTs with a gravity dual. The holographic ansatz requires that we solve Einstein's equations up to second order in amplitudes and momenta for the bulk metric fluctuations sourced by shear perturbations of the boundary metric. In section 4.3 we introduce the class of non-conformal holographic models whose transport properties we subsequently study. These models are four-dimensional RG flows with a conformal UV fixed point triggered by an operator of scaling dimension $\Delta = 3$. The holographically dual description of such a flow is provided by Einstein-scalar theory in asymptotically AdS_5 with the scalar mass fixed by the scaling dimension of the dual operator. Beyond the mass term, the scalar potential is arbitrary and it encodes details of the dual RG flow. To allow for the study of finite-temperature states in the field theory, we consider black-brane ansätze and derive the equations of motion for both the background and metric perturbations. In section 4.4 we solve the equations of motion for the background perturbations around an arbitrary black-brane background as far as possible without specifying the scalar potential beyond the mass term. The results therefore apply to all holographic RG flows in the class introduced above. In section 4.5 we present the key analytic results on second-order transport in the models under consideration. From the general solution for background perturbations found in the preceding section, we extract expressions for the five combinations of transport coefficients, eq. (4.2), in terms of the background. It follows from these expressions that the transport coefficients obey the relation $\tilde{H} = 0$, eq. (4.3), regardless of the exact form of the bulk black-brane background and of the details of the scalar potential. We also provide an analytical proof that the Haack-Yarom identity, $H = 0$, eq. (4.1), is satisfied up to leading order in non-conformal corrections. Section 4.6 contains numerical results for two families of scalar potentials within the larger class of potentials under consideration. The first family flows to a conformal IR fixed point while the second family flows to a non-conformal IR which is dual to the Chamblin-Reall background. We find that the individual transport coefficients entering the combination H, eq. (4.1), vary significantly along the flows while H itself vanishes within the limits of numerical accuracy. We end section 4.6 applying relations between second-order transport coefficients that have been derived from the requirement of positivity of the local entropy production allowing us to determine three more coefficients. In section 4.7 we summarise our results and provide suggestions for directions of future research. Some technical details of our calculations are provided in appendices A.1–A.3.

4.2 Kubo formulae for non-conformal second-order hydrodynamics

We introduced the effective theory of hydrodynamics in section 3.2 and reviewed how linear response can be used to infer transport coefficients from the underlying microscopic theory via Kubo relations in section 3.3. In this section we derive Kubo formulae for the five second-order transport coefficients (4.2) in subsection 4.2.1. We outline how these formulae can be applied to field theories with holographic duals in subsection 4.2.2.

4.2.1 Derivation of Kubo formulae

We studied the response of the hydrodynamic stress-energy tensor of a fluid to small perturbations of the background metric in subsection 3.3.1. We concluded that the response takes a particularly simple form, eq. (3.62), in the special case that the background perturbations do not source fluctuations of the fluid variables at first order. A particularly simple subset of such metric perturbations was considered for conformal fluids in ref. [224] and is given by

$$\{h_{xy}(t,z), h_{tx}(z), h_{ty}(z), h_{xz}(t), h_{yz}(t)\} \ . \tag{4.4}$$

For this set of perturbations, the perturbed on-shell response of $\langle T^{xy}\rangle$ up to second order in the metric perturbations, $\mathcal{O}(h^2)$, reads[2]

$$\begin{aligned}\langle T^{xy}\rangle = &\left[-\bar{p} - \eta\,\partial_t - \frac{\kappa}{2}\partial_z^2 + \left(\eta\,\tau_\pi - \frac{\kappa}{2} + \kappa^*\right)\partial_t^2\right] h_{xy}(t,z)\\ &+ \left[\bar{p}\,h_{xz}\,h_{yz} + \eta\left(h_{xz}\,\partial_t h_{yz} + \partial_t h_{xz}\,h_{yz}\right) + \left(\lambda_1 - \eta\,\tau_\pi - \frac{\kappa^*}{2}\right)\partial_t h_{xz}\,\partial_t h_{yz}\right.\\ &\qquad \left. + \left(\frac{\kappa}{2} - \eta\,\tau_\pi - \kappa^*\right)\left(h_{xz}\,\partial_t^2 h_{yz} + \partial_t^2 h_{xz}\,h_{yz}\right)\right]\\ &+ \left[-\bar{p}\,h_{tx}\,h_{ty} + \left(\frac{\lambda_3}{4} - \frac{\kappa^*}{2}\right)\partial_z h_{tx}\,\partial_z h_{ty} - \frac{\kappa}{2}\left(h_{tx}\,\partial_z^2 h_{ty} + \partial_z^2 h_{tx}\,h_{ty}\right)\right]\\ &+ \left[\frac{1}{2}\eta\,\tau_\pi - \frac{\lambda_2}{4} + \frac{\kappa^*}{2}\right]\left(\partial_z h_{tx}\,\partial_t h_{yz} + \partial_z h_{ty}\,\partial_t h_{xz}\right) + \mathcal{O}(h^3, \partial^3)\ ,\end{aligned} \tag{4.5}$$

[2] Parts of the expression (4.5) have appeared in the literature before: the linear response to the tensor perturbation h_{xy} was computed in ref. [174] and the quadratic response to the transverse-vector perturbations was derived in the conformal case in ref. [170]. The full response for the case of a non-conformal fluid first appeared in our paper, ref. [32].

where $\bar{p}$ is the pressure in global equilibrium. To derive eq. (4.5) we made use of the constitutive relation (3.38). Despite not restricting to conformal fluids, the only non-conformal coefficient featuring in eq. (4.5) is κ^*. We observe that the response of $\langle T^{xy}\rangle$ to the perturbations (4.4) gives us access to the five linear combinations of second-order transport coefficients in eq. (4.2).[3] From those five combinations, we can obtain $H = 2\eta\,\tau_\pi - 4\lambda_1 - \lambda_2$ via linear combination. All five coefficients (4.2) can be obtained by separately considering three sets of plane-wave excitations: $\{h_{xz}(t), h_{yz}(t)\}$, $\{h_{tx}(z), h_{ty}(z)\}$ and $\{h_{ty}(z), h_{xz}(t)\}$. No additional coefficients can be extracted from the response to h_{xy}.

We consider the three sets of plane-wave perturbations one by one: First, we perturb the metric by

$$\frac{1}{2}h_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \epsilon\left(H^{(b)}_{xz}e^{-iq_0 t}\mathrm{d}x\,\mathrm{d}z + H^{(b)}_{yz}e^{-ip_0 t}\mathrm{d}y\,\mathrm{d}z\right)\ , \tag{4.6}$$

with amplitudes $H^{(b)}_{xz}$ and $H^{(b)}_{yz}$. According to the third line in eq. (4.5), the response of the tensor component of the stress tensor reads

$$\begin{aligned}\langle T^{xy}(x)\rangle = \Bigg[&\bar{p} - i\,(q_0+p_0)\,\eta - q_0 p_0\left(\lambda_1 - \eta\,\tau_\pi - \frac{\kappa^*}{2}\right)\\ &- \left(q_0^2+p_0^2\right)\left(\frac{\kappa}{2} - \eta\,\tau_\pi - \kappa^*\right)\Bigg]\epsilon^2 H^{(b)}_{xz}H^{(b)}_{yz}e^{-i(q_0+p_0)t} + \mathcal{O}(\epsilon^3,\partial^3)\ .\end{aligned} \tag{4.7}$$

Second, we turn on the metric perturbation

$$\frac{1}{2}h_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \epsilon\left(H^{(b)}_{tx}e^{iq_z z}\mathrm{d}t\,\mathrm{d}x + H^{(b)}_{ty}e^{ip_z z}\mathrm{d}t\,\mathrm{d}y\right)\ , \tag{4.8}$$

with plane-wave amplitudes $H^{(b)}_{tx}$ and $H^{(b)}_{ty}$. According to the fourth line in eq. (4.5), this perturbation sources the response

$$\langle T^{xy}(x)\rangle = \left[-\bar{p} - q_z p_z\left(\frac{\lambda_3}{4} - \frac{\kappa^*}{2}\right) + \left(q_z^2+p_z^2\right)\frac{\kappa}{2}\right]\epsilon^2 H^{(b)}_{tx}H^{(b)}_{ty}e^{i(q_z+p_z)z} + \mathcal{O}(\epsilon^3,\partial^3)\ . \tag{4.9}$$

Third, we perturb the metric by

$$\frac{1}{2}h_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \epsilon\left(H^{(b)}_{ty}e^{ip_z z}\mathrm{d}t\,\mathrm{d}y + H^{(b)}_{xz}e^{-iq_0 t}\mathrm{d}x\,\mathrm{d}z\right)\ , \tag{4.10}$$

[3] Note that if we wanted to extract all fifteen second-order coefficients we would have to turn on metric perturbations in the scalar sound channel which would necessarily source fluctuations of $\delta\epsilon$ and $\underline{v}$.

with plane-wave amplitudes $H^{(b)}_{ty}$ and $H^{(b)}_{xz}$. According to the final line in eq. (4.5), this perturbation sources the response

$$\langle T^{xy}(x)\rangle = q_0 p_z \left(\frac{1}{2}\eta\,\tau_\pi - \frac{\lambda_2}{4} + \frac{\kappa^*}{2}\right)\epsilon^2 H^{(b)}_{ty} H^{(b)}_{xz} e^{-iq_0t+ip_zz} + \mathcal{O}(\epsilon^3,\partial^3)\ . \tag{4.11}$$

We compare eqs. (4.7), (4.9), (4.11) to the expression (3.70) for the microscopic stress-tensor response in terms of the fully retarded correlators (3.68) derived in subsection 3.3.2. Considering the low-momentum expansion of the fully retarded three-point correlators $G^{xy,xz,yz}_{raa}$, $G^{xy,tx,ty}_{raa}$, $G^{xy,ty,xz}_{raa}$ we obtain the following Kubo formulae:[4]

$$\kappa = \partial^2_{q_z} G^{xy,tx,ty}_{raa}(q,p)\big|_{q=p=0}\ , \tag{4.12a}$$

$$\eta\tau_\pi + \kappa^* = \frac{\kappa}{2} + \frac{1}{2}\,\partial^2_{q_0} G^{xy,xz,yz}_{raa}(q,p)\big|_{q=p=0}\ , \tag{4.12b}$$

$$\lambda_1 + \frac{\kappa^*}{2} = (\eta\tau_\pi + \kappa^*) - \partial_{q_0}\partial_{p_0} G^{xy,xz,yz}_{raa}(q,p)\big|_{q=p=0}\ , \tag{4.12c}$$

$$\lambda_2 = 2\,(\eta\tau_\pi + \kappa^*) - 4\,\partial_{q_0}\partial_{p_z} G^{xy,tx,xz}_{raa}(q,p)\big|_{q=p=0}\ , \tag{4.12d}$$

$$\lambda_3 - 2\kappa^* = -4\,\partial_{q_z}\partial_{p_z} G^{xy,tx,ty}_{raa}(q,p)\big|_{q=p=0}\ . \tag{4.12e}$$

4.2.2 Second-order transport via holography

For the remainder of this chapter, we will study second-order transport in strongly-coupled non-conformal QFTs with holographic gravity duals in asymptotically AdS_5 [4, 105, 107]. As we have seen in subsection 4.2.1, in order to extract the transport coefficients (4.2), we have to study the response of the tensor component of the stress-energy tensor $\langle T^{xy}\rangle$ to second order $\mathcal{O}(\epsilon^2)$ in the perturbations (4.6), (4.8) and (4.10) of the background metric and match the result with the corresponding response (4.7), (4.9) and (4.11) in the effective hydrodynamic description [22, 224, 225]. The response of the stress tensor in a holographic theory can be determined by considering the field-theory background metric perturbations as boundary sources for the dynamical bulk metric g_{mn} in the dual asymptotic AdS space. The backreaction of the boundary perturbations can be computed order by order in ϵ.

[4] The expressions for the Kubo formulae do not exactly agree with ref. [170] since the Fourier-transformed three-point functions are defined with opposite signs for the momenta. The shear viscosity in our convention is given by $\eta = i\,\partial_{q_0} G^{xy,xz,yz}_{raa}(q,p)\big|_{q=p=0}$ in disagreement with eq. (21) in ref. [170] by an extra minus sign. In addition, eqs. (22) and (23) in ref. [170] should not contain a factor of 2 in agreement with their eq. (26).

An encouraging hint towards the possible universality of the Haack-Yarom identity (4.1) is that the coefficients entering the relation can be determined without considering sound perturbations. Sound perturbations necessarily couple to the non-universal matter content rather than the universal gravity sector common to all holographic theories and thus likely lead to results which are sensitive to the model details [202].

The stress-energy tensor $\langle T^{\mu\nu}\rangle$ of a holographic field theory is given up to rescaling by the quasi-local gravity stress tensor $\mathcal{T}^{\mu\nu}$ of the dual AdS bulk. The quasi-local stress tensor quantifies the response of the renormalised on-shell bulk gravity action to changes $\delta\gamma_{\mu\nu}$ of the induced AdS boundary metric [138]. The off-shell response is given by

$$\delta S^{\text{ren}} = -\frac{1}{16\pi G_N}\int \mathrm{d}^5x\sqrt{-g}\,\mathrm{EOM}^{mn}\delta g_{mn} + \frac{1}{2}\int\limits_{\partial AdS_5} \mathrm{d}^4x\sqrt{-\gamma}\,\mathcal{T}^{\mu\nu}\delta\gamma_{\mu\nu}\,, \qquad (4.13)$$

where EOM_{mn} denote Einstein's equations in the bulk. Therefore, the tensor component of Einstein's equations, EOM^{xy}, must be satisfied to order $\mathcal{O}(\epsilon^2)$ for $(2/\sqrt{-\gamma})\,(\delta S^{\text{ren}}/\delta\gamma_{xy})$ to yield the correct quasi-local stress tensor $\mathcal{T}^{xy}$ up to and including $\mathcal{O}(\epsilon^2)$ [224].[5]

In summary, we apply the following steps to extract the transport coefficients (4.2) in a holographic theory. We consider the boundary metric perturbations (4.6), (4.8) and (4.10) which source a response of the bulk metric. We solve Einstein's equations for the fluctuations of the bulk metric to second order in amplitudes $\mathcal{O}(\epsilon^2)$ for the xy-component and to leading order in the other components. We can then extract the quasi-local stress tensor component $\mathcal{T}^{xy}$ and infer expressions for the transport coefficients by comparing it to the general hydro results (4.7), (4.9) and (4.11).

4.3 A class of non-conformal holographic models

In this section we introduce a certain class of non-conformal field theories in 3+1 dimensions with a holographic gravity dual, a subclass of the so-called holographic RG flows [95, 139, 141, 143, 144, 226–230], which we study for the remainder of this chapter. In this

[5] We can make sure that the tensor component of Einstein's equations with upper indices is satisfied to second order $\mathcal{O}(\epsilon^2)$, $\mathrm{EOM}^{xy} = \mathcal{O}(\epsilon^3)$, by solving the usual form of Einstein's equations with lower indices, EOM_{mn}, to leading order $\mathcal{O}(\epsilon)$ and the tensor component EOM_{xy} to second order, $\mathcal{O}(\epsilon^2)$, since the bulk metric is diagonal to leading order, $g_{mn} \propto \delta_{mn} + \mathcal{O}(\epsilon)$.

class the flows are triggered at the conformal UV fixed point by a relevant deformation with a scalar operator $O(x)$ of scaling dimension $\Delta = 3$.[6] The deformation takes the form

$$\int \mathrm{d}^4x \sqrt{-g_{(0)}}\, \Lambda(x) O(x) \ , \tag{4.14}$$

where the scalar source Λ is a dimensionful quantity setting an energy scale. Since we consider thermal states, Λ can be compared to the temperature T and the ratio Λ/T is the only dimensionless parameter of the setup. The holographic dual description of flows triggered by such a deformation is Einstein gravity in 4+1 dimensions coupled to a scalar,

$$S = \frac{1}{16\pi G_N} \int \mathrm{d}^5x \sqrt{-g} \left[R - \frac{1}{2} (\partial\phi)^2 - V(\phi) \right] \ . \tag{4.15}$$

The scalar potentials take the form

$$V(\phi) = \frac{1}{L^2} \left[-12 - \frac{3}{2}\phi^2 + \mathcal{O}(\phi^4) \right] \ , \tag{4.16}$$

where the constant term encodes the cosmological constant. The quadratic term is fixed by the relation $m^2 L^2 = \Delta(\Delta - 4) = -3$ between the scalar mass m and the scaling $\Delta = 3$ of the dual scalar operator O. The requirement that the bulk spacetime be asymptotically AdS_5 entails that the metric asymptotes to

$$\mathrm{d}s^2 \to \frac{L^2}{\zeta^2} \left(\mathrm{d}\zeta^2 + \mathrm{d}x \cdot \mathrm{d}x \right) \tag{4.17}$$

and that the scalar ϕ vanishes in the near-boundary region. The near-boundary region is dual to the UV fixed point. The leading mode of the scalar near the boundary is $\phi \sim \Lambda\zeta$, where Λ sources the dual operator O according to eq. (4.14).

4.3.1 Equations of motion for the gravitational background

The holographic description of thermal states in the QFT feature black-brane solutions to the Einstein-scalar five-dimensional action, eq. (4.15), which are homogeneous and isotropic in Euclidean three-space [1, 10]. These solutions take the form

$$\mathrm{d}s^2 = g^{(0)}_{mn} \mathrm{d}x^m \mathrm{d}x^n = e^{2A(u)} \left[-f(u)\mathrm{d}t^2 + \mathrm{d}\underline{x}^2 \right] + \frac{L^2}{4u^2 f(u)} \mathrm{d}u^2 \ , \tag{4.18}$$

[6] We only consider operators of dimension $\Delta = 3$ because the explicit form of the counterterms of the holographic renormalisation procedure have been computed for this case, see appendix C of our paper, ref. [32], for details. More general Einstein-scalar models can be renormalised with the procedure developed in ref. [231].

where a convenient gauge has been chosen for the radial coordinate u and where we denoted the field-theory directions as $x^\mu = (t, x, y, z)$. The emblackening factor $f(u)$, which breaks boost invariance, has a simple zero at the black-brane horizon. The horizon position can be set to $u = 1$ using the residual scaling symmetry inherited from the conformal UV fixed point. The Hawking temperature T and entropy density s of the solution are given by

$$T = \left.\frac{-f'(u)\, e^{A(u)}}{2\pi L}\right|_{u=1} , \qquad s = \left.\frac{e^{3A(u)}}{4G_N}\right|_{u=1} . \tag{4.19}$$

All dimensionful quantities can be expressed in units of the Hawking temperature. The equations of motion following from the action (4.15) with the ansatz (4.18) read

$$\phi'' + \left(4A' + \frac{1}{u}\right)\phi' + \frac{f'}{f}\phi' - \frac{L^2}{4u^2 f}\left(\frac{\mathrm{d}V}{\mathrm{d}\phi}\right) = 0 , \tag{4.20a}$$

$$A'' + \frac{1}{u}A' + \frac{1}{6}(\phi')^2 = 0 , \tag{4.20b}$$

$$f'' + \left(4A' + \frac{1}{u}\right) f' = 0 , \tag{4.20c}$$

$$6A'f' + f\left[24\,(A')^2 - (\phi')^2\right] + \frac{L^2}{2u^2}V = 0 , \tag{4.20d}$$

with primes denoting derivatives with respect to the holographic coordinate u. The equations (4.20) are not completely independent since the constraint equation (4.20d) follows from the other three equations via

$$\left(\frac{\mathrm{d}}{\mathrm{d}u} + \frac{2}{u}\right)(4.20\text{d}) = -2f\frac{\mathrm{d}\phi}{\mathrm{d}u}(4.20\text{a}) + (48fA' + 6f')\,(4.20\text{b}) + 6A'(4.20\text{c}) . \tag{4.21}$$

In the conformal case, when the scalar source Λ is turned off and the scalar ϕ vanishes, the solution to the equations of motion (4.20) is given by the AdS_5-black brane,

$$A(u) = \frac{1}{2}\log\left[\frac{(\pi T L)^2}{u}\right] , \qquad f(u) = 1 - u^2 . \tag{4.22}$$

4.3.2 Equations of motion for metric perturbations

After deriving the equations of motion governing the gravitational background in subsection 4.3.1, we now turn to the bulk metric perturbations sourced by the boundary metric fluctuations discussed in section 4.2, which we need to consider in order to extract the transport coefficients (4.2). The field-theory metric of a holographic QFT sets the AdS boundary values for the bulk metric g_{mn} [3], whose change in response to the boundary metric fluctuations encodes the response of the QFT stress tensor $\langle T^{\mu\nu}\rangle$ [122, 138]. Choosing a radial gauge $g_{u\mu}=0$, the perturbed bulk metric takes the form

$$\mathrm{d}s^2 = g_{mn}\mathrm{d}x^m\mathrm{d}x^n = g^{(0)}_{mn}\mathrm{d}x^m\mathrm{d}x^n + \epsilon\, g^{(1)}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \epsilon^2\, g^{(2)}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathcal{O}(\epsilon^3)\,, \tag{4.23}$$

where $\epsilon\, g^{(1)}_{\mu\nu}$ and $\epsilon^2\, g^{(2)}_{\mu\nu}$ encode the first- and second-order response and bulk backreaction to the boundary metric perturbations $h_{\mu\nu}$. They depend on which set of perturbations, (4.6), (4.8) or (4.10), we turn on. Since we only consider transverse-vector perturbations, scalar fluctuations are not sourced at order $\mathcal{O}(\epsilon)$ and at order $\mathcal{O}(\epsilon^2)$ they do not affect $g^{(2)}_{xy}$.[7]

We now consider in turn the three sets of boundary metric perturbations, (4.6), (4.8) and (4.10). First, the metric perturbation (4.6) corresponds to[8]

$$\frac{1}{2}g^{(1)}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = e^{2A(u)}\left[H^{(b)}_{xz}e^{-iq_0t}\,H^{(1t)}(u,q_0)\,\mathrm{d}x\,\mathrm{d}z + H^{(b)}_{yz}e^{-ip_0t}\,H^{(1t)}(u,p_0)\,\mathrm{d}y\,\mathrm{d}z\right] \tag{4.24}$$

in the bulk, where $H^{(1t)}$ is normalised to one at the boundary, $H^{(1t)}(u=0)=1$. At $\mathcal{O}(\epsilon)$ Einstein's equations read

$$H^{(1t)\prime\prime}(u,\omega) + \left(\frac{1}{u} + 4A' + \frac{f'}{f}\right)H^{(1t)\prime}(u,\omega) + \frac{e^{-2A}L^2\,\omega^2}{4u^2f^2}H^{(1t)}(u,\omega) = 0\,. \tag{4.25}$$

At second order $\mathcal{O}(\epsilon^2)$, we only need to compute the backreaction on the xy-component of the metric, which can be suitably parameterised as

$$g^{(2)}_{xy} = e^{2A(u)}\,H^{(b)}_{xz}\,H^{(b)}_{yz}\,e^{-ip_0t-iq_0t}H^{(2tt)}(u,q_0,p_0)\,. \tag{4.26}$$

The fluctuation $H^{(2tt)}$ is not directly sourced by a deformation of the boundary metric itself since $H^{(2tt)}(u=0)=0$, but it is excited via backreaction of the directly sourced

[7]Scalar fluctuations would be sourced at order $\mathcal{O}(\epsilon)$ if we considered metric perturbations in the scalar sound channel. Avoiding the scalar fluctuations simplifies the computation significantly but only allows us to extract five combinations of second-order coefficients listed in eq. (4.2). It is less likely to find universal behaviour among the remaining coefficients, since the model-specific scalar is non-universal.

[8]The xz- and yz-components of the bulk metric fluctuations are a priori independent. However, since they turn out to obey the same equations of motion as well as the same boundary conditions at the AdS boundary and at the horizon, we use the same notation $H^{(1t)}$ for both components. The boundary conditions are covered in more detail in subsection 4.4.2.

components on the bulk metric at sub-leading order, resulting in the equation of motion

$$H^{(2tt)\prime\prime}(u,q_0,p_0)+\left(\frac{1}{u}+4A'+\frac{f'}{f}\right)H^{(2tt)\prime}(u,q_0,p_0)+\frac{e^{-2A}L^2(q_0+p_0)^2}{4u^2f^2}H^{(2tt)}(u,q_0,p_0)$$
$$=\frac{e^{-2A}L^2q_0p_0}{4u^2f^2}H^{(1t)}(u,q_0)H^{(1t)}(u,p_0)+H^{(1t)\prime}(u,q_0)H^{(1t)\prime}(u,p_0)\ . \quad (4.27)$$

Second, we consider the metric perturbation (4.8), which corresponds to[9]

$$\frac{1}{2}g^{(1)}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu=e^{2A(u)}\left[H^{(b)}_{tx}e^{iq_zz}\,H^{(1z)}(u,q_z)\,\mathrm{d}t\,\mathrm{d}x+H^{(b)}_{ty}e^{ip_zz}\,H^{(1z)}(u,p_z)\,\mathrm{d}t\,\mathrm{d}y\right]\ , \quad (4.28)$$

in the bulk, where $H^{(1z)}$ is normalised to one at the boundary, $H^{(1z)}(u=0)=1$. At first order $\mathcal{O}(\epsilon)$, Einstein's equations read

$$H^{(1z)\prime\prime}(u,\omega)+\left(\frac{1}{u}+4A'\right)H^{(1z)\prime}(u,\omega)-\frac{e^{-2A}L^2\,\omega^2}{4u^2f}H^{(1z)}(u,\omega)=0\ . \quad (4.29)$$

At second order $\mathcal{O}(\epsilon^2)$, we only need to compute the backreaction on the xy-component of the metric, which can be suitably parameterised as

$$g^{(2)}_{xy}=e^{2A(u)}\,H^{(b)}_{tx}\,H^{(b)}_{ty}\,e^{iq_zz+ip_zz}H^{(2zz)}(u,q_z,p_z)\ . \quad (4.30)$$

The fluctuation $H^{(2zz)}$ is not itself directly sourced by a deformation of the boundary metric since $H^{(2zz)}(u=0)=0$, but is excited via backreaction of the directly sourced components on the bulk metric at sub-leading order, resulting in the equation of motion

$$H^{(2zz)\prime\prime}(u,q_z,p_z)+\left(\frac{1}{u}+4A'+\frac{f'}{f}\right)H^{(2zz)\prime}(u,q_z,p_z)-\frac{e^{-2A}L^2\,(q_z+p_z)^2}{4u^2f}H^{(2zz)}(u,q_z,p_z)$$
$$=\frac{e^{-2A}L^2q_zp_z}{4u^2f^2}H^{(1z)}(u,q_z)H^{(1z)}(u,p_z)-\frac{1}{f}H^{(1t)\prime}(u,q_z)H^{(1z)\prime}(u,p_z)\ . \quad (4.31)$$

Third, we consider the metric perturbation (4.10), which corresponds to

$$\frac{1}{2}g^{(1)}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu=e^{2A(u)}\left[H^{(b)}_{ty}e^{ip_zz}\,H^{(1z)}(u,p_z)\,\mathrm{d}t\,\mathrm{d}y+H^{(b)}_{xz}e^{-iq_0t}\,H^{(1t)}(u,q_0)\,\mathrm{d}x\,\mathrm{d}z\right]\ . \quad (4.32)$$

[9] The tx- and ty-components of the bulk metric fluctuations are a priori independent. However, since they turn out to obey the same equations of motion as well as the same boundary conditions at the AdS boundary and at the horizon, we use the same notation $H^{(1z)}$ for both components. The boundary conditions are discussed in more detail in subsection 4.4.2.

The equations of motion at order $\mathcal{O}(\epsilon)$ turn out to be eqs. (4.25) and (4.29) for $H^{(1t)}$ and $H^{(1z)}$ respectively.[10] At second order $\mathcal{O}(\epsilon^2)$, we only need to compute the backreaction on the xy-component of the metric, which can be suitably parameterised as

$$g_{xy}^{(2)} = e^{2A(u)}\, H_{ty}^{(b)}\, H_{xz}^{(b)}\, e^{-iq_0 t + ip_z z} H^{(2tz)}(u, q_0, p_z)\ . \tag{4.33}$$

The fluctuation $H^{(2tz)}$ is not itself directly sourced by a deformation of the boundary metric since $H^{(2tz)}(u=0) = 0$, but is excited via backreaction of the directly sourced components on the bulk metric at sub-leading order, resulting in the equation of motion

$$\begin{aligned} &H^{(2tz)\prime\prime}(u, q_0, p_z) + \left(\frac{1}{u} + 4A' + \frac{f'}{f}\right) H^{(2tz)\prime}(u, q_0, p_z) \\ &- \frac{e^{-2A} L^2 \left(-q_0^2 + f\, p_z^2\right)}{4u^2 f^2} H^{(2tz)}(u, q_0, p_z) = \frac{e^{-2A} L^2 q_0 p_z}{4u^2 f^2} H^{(1t)}(u, q_0) H^{(1z)}(u, p_z)\ . \end{aligned} \tag{4.34}$$

4.4 Solving Einstein's equations

In the previous section we introduced the class of holographic RG flows triggered by a scalar operator of dimension $\Delta = 3$. In subsection 4.3.1 we derived the equations of motion for static black-brane bulk solutions of these models corresponding to finite-temperature states in the boundary field theory. In subsection 4.3.2 we considered the bulk fluctuations sourced by the boundary metric perturbations (4.6), (4.8) and (4.10). We derived the equations of motion for the leading response as well as the backreaction on the xy-component. In order to determine the response of the boundary theory stress tensor $\langle T^{xy}\rangle$ to these perturbations, we have to find the solutions to these equations with appropriate boundary conditions. The field-theory stress tensor then follows from the near-boundary expansion of the bulk solutions [122, 232].

In order to keep the discussion general, we proceed to solve the equations of motion for the bulk metric fluctuations as far as possible for an arbitrary black-brane background without specifying the scalar potential (4.16) beyond the mass term. Consequently, the results of this section apply to all holographic RG flows triggered by a scalar operator

[10] $H^{(1t)}$ and $H^{(1z)}$ are indeed the same functions as the perturbations in eqs. (4.24) and (4.28). Both $H^{(1t)}$ and $H^{(1z)}$ are normalised to one at the boundary and obey the same boundary conditions at the horizon as the perturbations in eqs. (4.24) and (4.28). We will discuss boundary conditions in more detail in subsection 4.4.2.

of scaling dimension $\Delta = 3$. In subsection 4.4.1 we consider the near-horizon and near-boundary expansions of a generic background solution. In subsection 4.4.2 we first find local solutions for the bulk fluctuations near the horizon and the boundary in a hydrodynamic expansion in momenta imposing appropriate boundary conditions. Then we try to solve the connection problem of finding a global solution for the fluctuations. We analytically find such global solutions in all but five cases. For four of those cases, we express the solutions in terms of integrals over the background.[11]

4.4.1 Local background solutions

Near-horizon analysis We have derived the second-order equations of motion (4.20a)–(4.20c) and the first-order constraint equation (4.20d) for the background fields $A(u)$, $f(u)$ and $\phi(u)$. The constraint equation (4.20d) is redundant, eq. (4.21). Therefore, it does not impose constraints on the coefficients of the local expansions of $A(u)$, $f(u)$ and $\phi(u)$ at the horizon. Since the fields obey second-order equations, there are six integration constants. Three of the constants are fixed by the regularity condition that A and ϕ are constant at the horizon and that f vanishes. Consequently, there are three remaining undetermined near-horizon modes $\{A_H, f_H, \phi_H\}$, which enter the near-horizons expansions as

$$A(u) = A_H + \sum_{k\geq 1} b_k^A \,(1-u)^k \ , \tag{4.35a}$$

$$f(u) = (1-u)\left[f_H + \sum_{k\geq 1} b_k^f \,(1-u)^k\right] \ , \tag{4.35b}$$

$$\phi(u) = \phi_H + \frac{L^2\, V'(\phi_H)}{4 f_H}\,(1-u) + \sum_{k\geq 2} b_k^\phi \,(1-u)^k \ . \tag{4.35c}$$

The coefficients $\left\{b_k^A, b_k^f, b_k^\phi\right\}$ only depend on the modes $\{A_H, f_H, \phi_H\}$ and the form of the scalar potential $V(\phi)$.

Near-boundary expansion Similarly, we can expand the background fields $A(u)$, $f(u)$ and $\phi(u)$ near the boundary $u = 0$ and solve the equations of motion (4.20) order by order

[11] The one metric fluctuation, for which we could not find an integral form for the solution of the connection problem, cancels out in the ultimate expression for $\langle T^{\mu\nu}\rangle$ presented in subsection 4.5.1.

in u. As near the horizon, there are in general six integration constants, two of which are fixed by the condition that the spacetime be asymptotically AdS_5,

$$A(u) = -\frac{1}{2}\log(u) + \mathcal{O}\left(u^0\right) \ , \qquad\qquad f(u) = 1 + \mathcal{O}(u) \ . \tag{4.36}$$

The remaining four near-boundary modes remain undetermined. When we solve the equations (4.20) order by order near the boundary, we find that the equations fail to fix the three expansion coefficients of $A(u)$, $f(u)$ and $\phi(u)$ whenever we encounter one of the integration constants. Instead, at these orders, either only two of the equations of motion are independent or one of them fixes a term in the scalar potential that is required for the spacetime to be asymptotically AdS_5.

Solving the equations of motion order by order, the near-boundary expansions read

$$A(u) = \frac{1}{2}\log\left(\frac{A_b}{u}\right) - \frac{\phi_L^2}{24}u + \sum_{k\geq 2} c_k^A u^k \ , \tag{4.37a}$$

$$f(u) = 1 + f_b\, u^2 + u^2 \sum_{k\geq 1} c_k^f u^k \ , \tag{4.37b}$$

$$\phi(u) = \phi_L \sqrt{u} + \phi_{SL}\, u^{3/2} + u^{3/2} \sum_{k\geq 1} c_k^\phi u^k \ , \tag{4.37c}$$

with the coefficients $\left\{c_k^A, c_k^f, c_k^\phi\right\}$ fully determined by the near-boundary modes $\{A_b, f_b, \phi_L, \phi_{SL}\}$ and by the form of the scalar potential $V(\phi)$. The equations of motion also imply that the scalar potential must satisfy[12]

$$4!\,\frac{\mathrm{d}^4 V}{\mathrm{d}\phi^4} = -\frac{1}{12L^2} \ . \tag{4.38}$$

No further constraints on the potential can arise at higher orders, since the three independent equations must fix the three series coefficients of $A(u)$, $f(u)$ and $\phi(u)$ at each of the following orders. The four near-boundary modes already account for all free integration constants.

[12] Potentials derived from superpotentials W whose Taylor series leads to $L^2\, V = -12-(3/2)\,\phi^2+\mathcal{O}\left(\phi^4\right)$, i.e. $L\, W = -\,(3/2)-\phi^2/8+\mathcal{O}\left(\phi^4\right)$, automatically meet the condition (4.38). This condition was overlooked in ref. [209].

4.4.2 Solutions for metric perturbations

Boundary conditions In order to extract the boundary retarded correlator of the stress tensor, we impose incoming-wave boundary conditions on the time-dependent metric perturbations at the horizon [127, 128]. Time-independent perturbations, in contrast, are required to be regular at the horizon. In the following, expressions in Fraktur denote momenta in units of the Hawking temperature (4.19),

$$\mathfrak{w} \equiv \frac{L\,\omega}{2 f_H e^{A_H}} = \frac{\omega}{4\pi T} \,. \tag{4.39}$$

The independent first-order metric perturbations $H^{(1t)}$, the incoming-wave solution to eq. (4.25), and $H^{(1z)}$, the regular solution to eq. (4.29), can be written as

$$H^{(1t)}(u,\omega) = (1-u)^{-i\mathfrak{w}}\, K^{(1t)}(u,\omega) \,, \tag{4.40a}$$

$$H^{(1z)}(u,\omega) = (1-u)\, K^{(1z)}(u,\omega) \,, \tag{4.40b}$$

such that the $K^{(1\alpha)}$, $\alpha \in \{t,z\}$, are analytic at the black-hole horizon, $u=1$, and unit-normalised at the boundary, $u=0$. The solutions for the second-order perturbations $H^{(2tt)}$, $H^{(2zz)}$ and $H^{(2tz)}$ obeying eqs. (4.27), (4.31) and (4.34) can be written as

$$H^{(2tt)}(u,q_0,p_0) = (1-u)^{-i\mathfrak{q}_0 - i\mathfrak{p}_0}\, K^{(2tt)}(u,q_0,p_0) \,, \tag{4.41a}$$

$$H^{(2zz)}(u,q_z,p_z) = K^{(2zz)}(u,q_z,p_z) \,, \tag{4.41b}$$

$$H^{(2tz)}(u,q_0,p_z) = (1-u)^{-i\mathfrak{q}_0}\, K^{(2tz)}(u,q_0,p_z) \,, \tag{4.41c}$$

such that all $K^{(2\beta)}$, $\beta \in \{tt,zz,tz\}$, are analytic at the horizon and vanish at the boundary.

Hydrodynamic gradient expansion In addition to the expansion in small perturbations, studying the hydrodynamic regime allows us to solve the equations of motion for the metric perturbations in a small-momentum expansion. Therefore, we expand the metric perturbations in series of the form

$$K^{(1\alpha)}(u,\omega) = K_0^{(1\alpha)}(u) + K_1^{(1\alpha)}(u)\,\omega + K_2^{(1\alpha)}(u)\,\omega^2 + \mathcal{O}(\mathfrak{w}^3) \,, \tag{4.42}$$

$$\begin{aligned} K^{(2\beta)}(u,q,p) = {} & K_{(0,0)}^{(2\beta)}(u) + \left[K_{(1,0)}^{(2\beta)}(u)\,q + K_{(0,1)}^{(2\beta)}(u)\,p\right] \\ & + \left[K_{(2,0)}^{(2\beta)}(u)\,q^2 + K_{(1,1)}^{(2\beta)}(u)\,q\,p + K_{(0,2)}^{(2\beta)}(u)\,p^2\right] + \left(\mathcal{O}(\mathfrak{q},\mathfrak{p})\right)^3 \,, \end{aligned}$$

with $\alpha \in \{t, z\}$ and $\beta \in \{tt, zz, tz\}$. In the following we use the label a for the metric perturbation and j for the order in the hydrodynamic expansion, i.e. $j \in \{0, 1 \ldots\}$ for $a \in \{1t, 1z\}$ and $j \in \{(0,0), (1,0), (0,1) \ldots\}$ for $a \in \{2tt, 2zz, 2tz\}$. When expanding the equations of motion, eqs. (4.25), (4.29), (4.27), (4.31) and (4.34), in momenta, many of the $K_j^{(a)}$ turn out to vanish or to be related to each other. $K_1^{(1z)}$ is the solution to a linear and homogeneous equation which obeys regularity at the horizon and vanishes at the boundary. Therefore, $K_1^{(1z)}$ vanishes identically. The same is true for $K_{(1,0)}^{(2zz)}$, $K_{(0,1)}^{(2zz)}$, $K_{(0,0)}^{(2tz)}$, $K_{(1,0)}^{(2tz)}$, $K_{(0,1)}^{(2tz)}$, $K_{(2,0)}^{(2tz)}$ and $K_{(0,2)}^{(2tz)}$. Moreover, $K_{(1,0)}^{(2tt)}$ and $K_{(0,1)}^{(2tt)}$ are in fact the same since they obey the same equation of motion and satisfy identical boundary conditions. The same is true for $K_{(2,0)}^{(2tt)}$ and $K_{(0,2)}^{(2tt)}$ as well as for $K_{(2,0)}^{(2zz)}$ and $K_{(0,2)}^{(2zz)}$. All these relations follow from symmetry properties of the setup and from the fact that we only consider a limited set of metric perturbations (4.6), (4.8) and (4.10).

Local solutions In analogy to the gravitational background functions $A(u)$, $f(u)$ and $\phi(u)$ in subsection 4.4.1, we can perform local analyses of the metric perturbations $K_j^{(a)}$ near the horizon and near the boundary. Since they obey second-order equations, two integration constants enter the local near-horizon and near-boundary solutions for each of the $K_j^{(a)}$. Since the requirement of analyticity at the horizon fixes one of the two constants, the local solution near the horizon only depends on one near-horizon mode $Z_j^{(a)}$,

$$K_j^{(a)}(u) = Z_j^{(a)} + \sum_{s\geq 1} \lambda_{j,s}^{(a)} \, (1-u)^s \ . \tag{4.43}$$

Near the boundary, the equations of motion (4.25), (4.29), (4.27), (4.31) and (4.34) require the $K_j^{(a)}$ to take the form

$$K_j^{(a)} = X_j^{(a)} + k_{j,1}^{(a)}\, u + Y_j^{(a)}\, u^2 + \sum_{s\geq 3} k_{j,s}^{(a)}\, u^s + \log u \sum_{s\geq 2} l_{j,s}^{(a)}\, u^2 \ , \tag{4.44}$$

with leading and sub-leading modes $X_j^{(a)}$ and $Y_j^{(a)}$. Due to boundary conditions on the metric perturbations $H^{(a)}$, discussed when we introduced the $K_j^{(a)}$ in eqs. (4.40) and (4.41), the leading modes are given by

$$\begin{aligned} &X_0^{(1\alpha)} = 1 \ , \quad X_{j\geq 1}^{(1\alpha)} = 0 \ , && \alpha \in \{t, z\} \ , \\ &X_j^{(2\beta)} = 0 \ , && \beta \in \{tt, zz, tz\} \ . \end{aligned} \tag{4.45}$$

Global solutions It turns out that we can solve the equations of motion for some of the $K_j^{(a)}(u)$ analytically without further specifying the background functions $A(u)$, $f(u)$ and $\phi(u)$ as well as the scalar potential $V(\phi)$. Firstly, employing the constraint (4.20d), we find that $\partial_u K_0^{(1t)}$ and $\partial_u K_{(0,0)}^{(2tt)}$ satisfy linear first-order equations without a source term. Imposing the boundary conditions of analyticity at the horizon and the values of 1 and 0 at the boundary, respectively, the solutions are found to be

$$K_0^{(1t)} = 1 \,, \qquad\qquad K_{(0,0)}^{(2tt)} = 0 \,. \tag{4.46}$$

This finding removes the source terms from the equations for $K_{(1,0)}^{(2tt)}$ and $K_{(2,0)}^{(2tt)}$, which both vanish at the boundary. The unique solution under these boundary conditions is

$$K_{(1,0)}^{(2tt)} = 0 \,, \qquad\qquad K_{(2,0)}^{(2tt)} = 0 \,. \tag{4.47}$$

Using both the constraint (4.20d) as well as eq. (4.20c), we can solve the equations for $K_0^{(1z)}$, $K_{(0,0)}^{(2zz)}$, and $K_1^{(1t)}$ consecutively finding

$$K_0^{(1z)} = \frac{f(u)}{1-u} \,, \quad K_{(0,0)}^{(2zz)} = 1 - f(u) \,, \tag{4.48}$$

$$K_1^{(1t)} = -\frac{i}{4\pi T} \log\left(\frac{f(u)}{1-u}\right) \,. \tag{4.49}$$

We can furthermore derive an identity among the $K_j^{(a)}$ by comparing the equations of motion and boundary conditions that are satisfied by the expressions on both sides of

$$K_{(2,0)}^{(2zz)} = -\,(1-u)\,K_2^{(1z)} + K_{(1,1)}^{(2tz)} \,. \tag{4.50}$$

In consequence, only five of the original 24 $K_j^{(a)}$ remain unknown:

$$K_2^{(1z)} \,, \quad K_2^{(1t)} \,, \quad K_{(1,1)}^{(2tt)} \,, \quad K_{(1,1)}^{(2zz)} \,, \quad K_{(1,1)}^{(2tz)} \,. \tag{4.51}$$

While we were unable to make further progress on $K_2^{(1z)}$, the residual gauge symmetry in metric perturbations at order $\mathcal{O}(\epsilon^2)$ leads to equations for the other four remaining $K_j^{(a)}(u)$ taking a particularly convenient form:

$$\frac{\mathrm{d}}{\mathrm{d}u}\left[u\, f(u)\, e^{4A(u)}\, \frac{\mathrm{d}}{\mathrm{d}u} K_j^{(a)}(u)\right] = u\, f(u)\, e^{4A(u)}\, \Upsilon_j^{(a)}(u) \,, \tag{4.52}$$

where

$$K_j^{(a)} \in \left\{ K_2^{(1t)},\ K_{(1,1)}^{(2tt)},\ K_{(1,1)}^{(2zz)},\ K_{(1,1)}^{(2tz)} \right\} . \tag{4.53}$$

Since these four $K_j^{(a)}(u)$ vanish at the boundary and since the regularity condition (4.43) forces the expression in brackets in (4.52) to vanish at the horizon, we can integrate eq. (4.52) and obtain

$$K_j^{(a)}(u) = \int_0^u \mathrm{d}v \, \frac{1}{v\, f(v)\, e^{4A(v)}} \int_1^v \mathrm{d}w \, w\, f(w)\, e^{4A(w)}\, \Upsilon_j^{(a)}(w) \,. \tag{4.54}$$

Expanding the integral near the boundary yields the following expression for the sub-leading modes of the $K_j^{(a)}$:

$$\begin{aligned} Y_j^{(a)} = \frac{1}{8\pi^2 T^2} \left(\frac{f_H^2\, e^{2A_H}}{4A_b} \right) \Bigg\{ \pm \left(1 - \frac{\phi_L^2}{8} \right) \\ - \int_0^1 \mathrm{d}w \left[4w\, f(w)\, e^{4A(w)}\, \frac{\Upsilon_j^{(a)}(w)}{A_b\, L^2} \mp \frac{1}{w^2} \left(1 - \frac{\phi_L^2}{12} w \right) \right] \Bigg\} \,, \end{aligned} \tag{4.55}$$

where the upper signs refer to $K_j^{(a)} \in \left\{ K_{(1,1)}^{(2tt)},\ K_{(1,1)}^{(2zz)},\ K_{(1,1)}^{(2tz)} \right\}$ and the lower signs to $K_j^{(a)} = K_2^{(1t)}$. More details on the computation can be found in appendix A.1 which also contains explicit expressions for the $\Upsilon_j^{(a)}$ in eq. (A.1). Taking the conformal limit, $\phi \to 0$, the integrals (4.55) can be solved to yield

$$\left(Y_2^{(1t)},\ Y_{(1,1)}^{(2tt)},\ Y_{(1,1)}^{(2zz)},\ Y_{(1,1)}^{(2tz)} \right) \xrightarrow{\phi\to 0} \left(\frac{-5+4\log 2}{32\pi^2 T^2},\ \frac{1-\log 2}{8\pi^2 T^2},\ 0,\ \frac{1}{8\pi^2 T^2} \right) . \tag{4.56}$$

4.5 Analytical results for second-order transport

In this section we present our analytical results on second-order transport in strongly-coupled fluids dual to holographic RG flows triggered in the UV by a scalar operator of scaling dimension $\Delta = 3$. First, we find explicit expressions for the five transport coefficients (4.2) in terms of the gravitational dual in subsection 4.5.1. From these expressions we derive that the combination $\tilde{H}$, eq. (4.3), always vanishes. In subsection 4.5.2 we prove that the Haack-Yarom identity $H = 0$, eq. (4.1), which is known to hold for conformal holographic fluids [23], remains satisfied when taking into account leading non-conformal corrections.

4.5.1 Formulae for transport coefficients

In order to determine the five combinations of transport coefficients from eq. (4.2), we have to compare the effective hydrodynamic result for the expectation value of the stress tensor $\langle T^{\mu\nu}\rangle$, eq. (4.5), with the corresponding microscopic result. For holographic theories, the stress tensor can be extracted from the near-boundary expansion of the dual bulk field, the metric, after renormalisation of the bulk action [122, 232]. Details of the holographic renormalisation procedure can be found in appendix C of our paper, ref. [32]. Ultimately, we can express the transport coefficients in terms of the background near-boundary modes $\{A_b, f_b, \phi_L, \phi_{SL}\}$, the temperature T, which sets the only dimensionful scale, and the sub-leading modes, $\left\{Y_2^{(1t)}, Y_{(1,1)}^{(2tt)}, Y_{(1,1)}^{(2zz)}, Y_{(1,1)}^{(2tz)}\right\}$, of hydro-expanded metric perturbations (4.51) for which we found the integral-form solutions (4.55). The expressions can be simplified by employing the relation

$$f'(u) = 2 f_b\, A_b^2\, u^{-1} e^{-4A(u)} \ , \tag{4.57}$$

which we obtain by imposing the condition of AdS asymptotic behaviour, eq. (4.37), on the first-order equation of motion for $f'(u)$, eq. (4.20c). We find

$$A_b^2 = -\left(\frac{4\pi G_N L}{f_b}\right) s\, T \tag{4.58}$$

by evaluating eq. (4.57) at the black-brane horizon and employing the expressions (4.19) for the Hawking temperature and entropy density. Using the results of our holographic renormalisation, which we performed in appendix C of our paper, ref. [32], we can reexpress the leading and sub-leading modes of the bulk scalar in terms of field-theory quantities,

$$\Lambda = \frac{\sqrt{A_b}}{L}\phi_L \ , \qquad\qquad \langle O\rangle = \frac{L^3}{8\pi G_N}\left(\frac{A_b}{L^2}\right)^{3/2}\phi_{SL} + \mathcal{O}(\epsilon^2) \ , \tag{4.59}$$

where $\langle O\rangle$ is the expectation value of the scalar operator O dual to ϕ and Λ is its source.

We now turn to presenting an expression for the field-theory stress tensor $\langle T^{\mu\nu}\rangle$ as derived via holography but expressed purely in terms of the field-theory quantities T, s, Λ and $\langle O\rangle$. At leading order $\mathcal{O}(\epsilon^0)$, the stress tensor turns out to take the ideal-fluid diagonal form,

$$\overline{T}^{\mu\nu} = \mathrm{diag}\left(\bar{\epsilon}, \bar{p}, \bar{p}, \bar{p}\right) \ , \tag{4.60}$$

with the energy density and pressure given by[13]

$$\bar{\epsilon} = \frac{3}{4}\, sT - \frac{1}{4}\Lambda \langle O\rangle \ , \qquad\qquad \bar{p} = \frac{1}{4}\, sT + \frac{1}{4}\Lambda \langle O\rangle \ . \tag{4.61}$$

To compare the microscopic result to the effective hydrodynamic expressions (4.7), (4.9) and (4.11), we have to determine the response of the transverse-tensor component of the stress tensor $\langle T^{xy}\rangle$ to each of the perturbations (4.6), (4.8) and (4.10). In agreement with the hydro results, the leading deviations from the unperturbed case appear at second order $\mathcal{O}(\epsilon^2)$ and take the expected form. This provides a non-trivial check for our calculations. The fact that the second-order responses (4.7) and (4.9) reproduce the same pressure as the zeroth-order response relies on some of our analytical results for the bulk metric perturbations, eqs. (4.46) and (4.48). From the response (4.7) to the perturbation (4.6), we recover the universal value for the shear viscosity over entropy density ratio [21, 22, 188–194],

$$\eta = \frac{1}{4\pi} s \ , \tag{4.62}$$

which provides a further non-trivial check of our computations. We extract the following expressions for the five combinations of second-order transport coefficients (4.2):

$$\kappa = -\frac{2}{f_b}\, Y^{(2tz)}_{(1,1)}\, sT \ , \tag{4.63a}$$

$$\eta\, \tau_\pi + \kappa^* = \frac{1}{f_b}\left(\frac{1}{32\pi^2 T^2} + Y^{(1t)}_2 - Y^{(2tz)}_{(1,1)}\right)\, sT \ , \tag{4.63b}$$

$$\lambda_1 + \frac{\kappa^*}{2} = \frac{1}{f_b}\left(\frac{1}{32\pi^2 T^2} + Y^{(1t)}_2 - Y^{(2tz)}_{(1,1)} + Y^{(2tt)}_{(1,1)}\right)\, sT \ , \tag{4.63c}$$

$$\lambda_2 = \frac{2}{f_b}\left(\frac{1}{32\pi^2 T^2} + Y^{(1t)}_2 + Y^{(2tz)}_{(1,1)}\right)\, sT \ , \tag{4.63d}$$

$$\lambda_3 - 2\kappa^* = \frac{4}{f_b}\, Y^{(2zz)}_{(1,1)}\, sT \ . \tag{4.63e}$$

These expressions only feature the four sub-leading modes $\left\{Y^{(1t)}_2, Y^{(2tt)}_{(1,1)}, Y^{(2zz)}_{(1,1)}, Y^{(2tz)}_{(1,1)}\right\}$, for which we succeeded in finding integral-form expressions given in eq. (4.55). The sub-leading mode $Y^{(1z)}_2$, for which we could not find a closed- or integral-form expression, drops out of eq. (4.63) thanks to the relation (4.50).

[13] In the conformal case of $\phi = 0$, the expressions (4.61) for the energy density and pressure simplify resulting in $\bar{\epsilon} = 3\bar{p} = \frac{3\pi^3 L^3}{16 G_N} T^4$ for generic holographic models and $\bar{\epsilon} = 3\bar{p} = \frac{3\pi^2}{8} N^2 T^4$ in the special case of $\mathcal{N} = 4$ SYM [150]. In particular, the hydrodynamic stress tensor is traceless in the conformal case.

The eqs. (4.63) allow for a straightforward computation of the five combinations of second-order coefficients for an arbitrary background characterised by the functions $A(u)$, $f(u)$ and $\phi(u)$ solving the background equations of motion (4.20) given a scalar potential $V(\phi)$, eq. (4.16). From the background we can extract the near-boundary and near-horizon modes $\{A_b, f_b, \phi_L, \phi_{SL}\}$ and $\{A_H, f_H, \phi_H\}$ using eqs. (4.35) and (4.37) and perform the integrals (4.55) to obtain the four $Y_j^{(a)}$ entering eqs. (4.63). In the conformal limit $\phi \to 0$, where the non-conformal coefficient $\kappa^\star$ vanishes, the analytically accessible values for the $Y_j^{(a)}$, eq. (4.56), reproduce the known values for the five coefficients [174, 187],

$$\{\ \kappa,\ \eta\tau_\pi,\ \lambda_1,\ \lambda_2,\ \lambda_3\ \} = \left(\frac{s}{8\pi^2 T}\right)\{\ 2,\ 2-\log 2,\ 1,\ -2\log 2,\ 0\ \}\ , \tag{4.64}$$

with $s/T = \pi^2 N^2 T^2/2$ for $\mathcal{N}=4$ specifically.

From eq. (4.63) we find that a particular combination of coefficients,

$$\tilde{H} \equiv 2\eta\,\tau_\pi - 2\,(\kappa-\kappa^*) - \lambda_2 = 0\ , \tag{4.65}$$

is independent of the sub-leading modes $Y_j^{(a)}$ and therefore holds at arbitrary temperature for all holographic fluids dual to an RG flow triggered by an operator of scaling dimension $\Delta = 3$. The result (4.65) depends on the global solutions for metric perturbations, (4.46)–(4.50), and thus cannot simply be a consequence of Ward identities, which only constrain the UV near-boundary solutions [122, 136]. The transport coefficients entering eq. (4.65) have been computed for the conformal planar $\mathcal{N}=4$ SYM theory (at infinite coupling [20, 174, 187] and including leading coupling corrections [204, 233–238]) as well as for the non-conformal theory dual to the Chamblin-Reall background [208]. At infinite coupling, $\tilde{H}$ vanishes in both cases. However, $\tilde{H}$ receives non-vanishing leading-order finite-coupling corrections in $\mathcal{N}=4$ SYM.

4.5.2 The Haack-Yarom identity and non-conformal corrections

In this subsection we provide a proof that the Haack-Yarom identity $H=0$, eq. (4.1), which is known to hold for conformal strongly coupled holographic fluids, is still satisfied when taking into account leading non-conformal corrections triggered by a relevant operator of dimension $\Delta = 3$. Using eq. (4.63), we can express H as

$$H = -\frac{4}{f_b}\left(\frac{1}{32\pi^2 T^2} + Y_2^{(1t)} + Y_{(1,1)}^{(2tt)}\right) \tag{4.66}$$

with $Y_2^{(1t)}$ and $Y_{(1,1)}^{(2tt)}$ given by (4.55) in combination with eq. (A.1). We can eliminate the dependence of $Y_2^{(1t)}$ and $Y_{(1,1)}^{(2tt)}$ on the background function $A(u)$ employing the relation (4.57),

$$A(u) = \frac{1}{4}\log\left(\frac{2 f_b\, A_b^2}{u\, f'(u)}\right) \; . \tag{4.67}$$

Consequently, $Y_2^{(1t)}$ and $Y_{(1,1)}^{(2tt)}$ as well as H only depend on the near-boundary and near-horizon modes f_b and f_H and on $f(u)$ itself through a complicated integral of a rational function of $f(u)$ and its derivatives. Nevertheless, close to the conformal UV fixed point, we can expand the integrand in the deviation $\delta f(u)$ from the conformal case. To linear order in the deviation, H takes the form

$$\begin{aligned} H = \frac{1}{2\pi^2 T^2}\Bigg(&-\frac{1}{4 f_b} + \frac{1}{4} + \\ &+ \int_0^1 \mathrm{d}w \left[P(u)\delta f''(u) + Q(u)\delta f'(u) + \left(Q'(u) - P''(u)\right)\delta f(u)\right]\Bigg) \; , \end{aligned} \tag{4.68}$$

with

$$P(u) \equiv \frac{(1+u)\log(1+u)}{8u^2} \; , \tag{4.69}$$

$$Q(u) \equiv \frac{u\left(1+2u-3u^2\right) - (1+u)^2 (2-3u)\log(1+u)}{8u^3\left(1-u^2\right)} \; . \tag{4.70}$$

Using integration by parts as well as the near-boundary and near-horizon asymptotics of the function $f(u)$,

$$f(u\to 0) \sim 1 + (-1+\delta f_b)\, u^2 \; , \qquad f(u \to 1) \sim (2+\delta f_H)\,(1-u) \; , \tag{4.71}$$

we find that H vanishes to linear order in $\delta f(u)$. This completes the proof that, in a strongly coupled holographic fluid, leading non-conformal corrections triggered by an operator of dimension $\Delta = 3$ preserve the Haack-Yarom identity.

4.6 Numerical results for second-order transport

After having presented our analytical results on second-order transport in non-conformal holographic fluids in the previous section, we now turn to our numerical results. In subsection 4.6.1 we present numerical results for the leading non-conformal deviation of the five second-oder transport coefficients (4.2) from their conformal values. These deviations are common to all flows triggered by an operator of scaling dimension $\Delta = 3$ since they only depend on the mass term of the scalar potential. In subsection 4.6.2 we introduce two families of RG flows within the larger class of flows we considered so far. These two families flow to conformal and non-conformal IR fixed points, respectively. In subsection 4.6.3 we numerically compute the second-order coefficients along these flows and provide evidence that the Haack-Yarom identity is satisfied within both families. In subsection 4.6.4 we use three relations among second-order coefficients derived from the requirement of positivity of the local entropy production in order to extend our numerical results from four to seven second-order coefficients.

4.6.1 Leading non-conformal correction to transport coefficients

At high temperatures T compared to the scalar source Λ, the scalar ϕ can be treated as a perturbation from the conformal UV fixed point, which is dual to the AdS_5-black brane geometry with vanishing scalar (4.22). The background remains unchanged at linear order in the scalar perturbation ϕ. The leading backreaction on $A(u)$ and $f(u)$ occurs at quadratic order $\mathcal{O}(\phi^2)$ and can be determined analytically, which we show explicitly in appendix A.2. In particular, the result is independent of the scalar potential beyond the mass term and is therefore the same for all flows triggered by an operator of dimension $\Delta = 3$. In order to determine the transport coefficients in this quadratic approximation using eqs. (4.63), we have to compute the effect of the backreaction on the intregrals $Y_j^{(a)}$, eq. (4.55). We plug the expressions for the backreacted geometry (eqs. (A.10), (A.11), (A.17), (A.19), and (A.20)) into the integrals (4.55) and evaluate them numerically. The results are common to all non-conformal holographic RG flows triggered by an operator of scaling dimension $\Delta = 3$.

Given the identities $H = 0$ and $\tilde{H} = 0$, eqs. (4.1) and (4.65), which we proved to hold for leading non-conformal corrections, only three of the five coefficients (4.2) are independent at this order. We give the expressions for κ, λ_2, and $\lambda_1 + \lambda_3/4$, [14]

$$\kappa = 2\left(\frac{s}{8\pi^2 T}\right)\left(1 - 4.5979 \cdot 10^{-3}\,(\Lambda/T)^2\right) + \mathcal{O}\left((\Lambda/T)^4\right) , \tag{4.72a}$$

$$\lambda_2 = -2\log 2\left(\frac{s}{8\pi^2 T}\right)\left(1 + 4.9253 \cdot 10^{-3}\,(\Lambda/T)^2\right) + \mathcal{O}\left((\Lambda/T)^4\right) , \tag{4.72b}$$

$$\lambda_1 + \lambda_3/4 = \left(\frac{s}{8\pi^2 T}\right)\left(1 + 2.5506 \cdot 10^{-3}\,(\Lambda/T)^2\right) + \mathcal{O}\left((\Lambda/T)^4\right) , \tag{4.72c}$$

while the other two combinations can be determined from these three via

$$\eta\tau_\pi + \kappa^* = \kappa + \lambda_2/2 , \qquad \lambda_1 + \kappa^*/2 = \kappa/2 . \tag{4.73}$$

While we only provide the first five digits in eq. (4.72), the numerical integration can be done to very high accuracy. We estimate the numerical error to be smaller than 10^{-14} based on checking the identity $H = 0$ numerically.

4.6.2 Two families of holographic RG flows

In this subsection we introduce two families of RG flows: the first has conformal IR fixed points while the second flows to non-conformal IR regions.

The one-parameter family $V_{(1)}$ of potentials we consider first was studied in ref. [239] and is derived from quartic superpotentials of the form

$$L\,W = -\frac{3}{2} - \frac{\phi^2}{8} + \frac{\phi^4}{16\phi_m^2} , \tag{4.74}$$

where ϕ_m is the free parameter. The potentials therefore take the form

$$\begin{aligned} V_{(1)} &= 8\left[\left(\frac{\partial W}{\partial \phi}\right)^2 - \frac{2}{3}W^2\right] \\ &= \frac{1}{L^2}\left[-12 - \frac{3}{2}\phi^2 - \frac{1}{12}\phi^4 + \frac{6+\phi_m^2}{12\phi_m^4}\phi^6 - \frac{1}{48\phi_m^4}\phi^8\right] . \end{aligned} \tag{4.75}$$

They have a maximum at $\phi = 0$ and the free parameter ϕ_m encodes the location of the minimum of the potential. In the vicinity of the minimum, the potential can be expanded

[14] The coefficient λ_3 vanishes in conformal holographic theories at infinite coupling [187]. However, unlike κ^*, it does not vanish for generic conformal fluids [170, 224].

as

$$L^2\,V_{(1)} = -12\frac{L^2}{L_{\mathrm{IR}}^2} + \frac{m_{\mathrm{IR}}^2 L^2}{2}\,(\phi-\phi_m)^2 + \mathcal{O}\left((\phi-\phi_m)^3\right)\ . \tag{4.76}$$

This minimum results in a second asymptotically AdS_5-region, in addition to the near-boundary region, corresponding to the IR fixed point. In the IR the AdS radius and the scalar mass are given by

$$L_{\mathrm{IR}} \equiv \left(1+\frac{\phi_m^2}{24}\right)^{-1} L\ , \qquad\qquad m_{\mathrm{IR}}^2 L^2 = 12 + \frac{\phi_m^2}{3}\ . \tag{4.77}$$

Therefore, the potential $V_{(1)}$ encodes a flow triggered by a scalar operator of scaling dimension $\Delta = 3$ from a conformal UV fixed point to a conformal IR fixed point. At the IR fixed point, the number of degrees of freedom is smaller by a factor of $(L_{\mathrm{IR}}/L)^{3/2}$ compared to the UV [136, 240]. The scalar operator deforming the CFT in the IR has scaling dimension [2, 3]

$$\begin{aligned}\Delta_{\mathrm{IR}} &= 2 + 2\sqrt{1+\frac{m_{\mathrm{IR}}^2 L_{\mathrm{IR}}^2}{4}} \\ &= 4 + \frac{48}{24+\phi_m^2} \in (4,6)\ .\end{aligned} \tag{4.78}$$

If ϕ_m is large, the potential $V_{(1)}$, eq. (4.75), becomes effectively quartic and the operator O becomes marginally irrelevant in the IR,

$$V_{(1)} \xrightarrow{\phi_m\to\infty} \frac{1}{L^2}\left[-12 - \frac{3}{2}\phi^2 - \frac{1}{12}\phi^4\right]\ , \qquad\qquad \Delta_{\mathrm{IR}} \xrightarrow{\phi_m\to\infty} 4\ . \tag{4.79}$$

The number of degrees of freedom in the IR approaches zero. In the opposite limit of small ϕ_m, the IR operator becomes more irrelevant and the number of degrees of freedom in the IR increases towards the UV value.

Second, we consider another one-parameter family of potentials $V_{(2)}$ defined by

$$\begin{aligned}V_{(2)} &= \frac{1}{L^2}\left[-12 - \left(\frac{3}{2}-\frac{1}{\gamma^2}\right)\phi^2 + \frac{2}{\gamma^4}\left(1-\cosh(\gamma\phi)\right)\right] \\ &= \frac{1}{L^2}\left[-12 - \frac{3}{2}\phi^2 - \frac{1}{12}\phi^4 - \frac{\gamma^2}{360}\phi^6 + \mathcal{O}(\phi^8)\right]\ .\end{aligned} \tag{4.80}$$

Since the potential is monotonically decreasing for any value of the free parameter γ and does therefore not have a minimum, it corresponds to a flow from a conformal UV CFT to a non-conformal IR.

The asymptotic form of the potential for large scalar ϕ, $L^2 V_{(2)} \to -e^{\gamma\phi}/\gamma^4$, has a finite-temperature solution, the so-called Chamblin-Reall background [209, 241]. Therefore, in the case of small temperatures corresponding to large values ϕ_H at the horizon, the Chamblin-Reall background is the near-horizon geometry of the flow. In this regime of large ϕ_H, the temperature T and the entropy density s are given by [209]

$$\log(LT) = \left(\frac{\gamma}{2} - \frac{1}{3\gamma}\right)\phi_H + (\text{const in } \phi_H) \ , \tag{4.81a}$$

$$\log(4G_N s) = -\frac{\phi_H}{\gamma} + (\text{const in } \phi_H) \ , \tag{4.81b}$$

implying the following IR-asymptotics for the speed of sound:

$$c_s^2 = \frac{\mathrm{d}\bar{p}}{\mathrm{d}\bar{\epsilon}} = \frac{\mathrm{d}\log T}{\mathrm{d}\log s} \xrightarrow{T\to 0} \frac{1}{3} - \frac{\gamma^2}{2} \ . \tag{4.82}$$

Black-brane solutions of $V_{(2)}$ are only stable if the speed of sound squared is positive, $c_s^2 > 0$, implying the condition $|\gamma| < \sqrt{2/3}$.[15] The potentials $V_{(1)}$ and $V_{(2)}$ agree in the limit of $\phi_m \to \infty$ and $\gamma \to 0$, where they both assume the form given in eq. (4.79).

We employed the method developed in ref. [209] to construct the backgrounds for both families of potentials $V_{(1)}$ and $V_{(2)}$ numerically. We summarise the most important steps in appendix A.3. We plot the speed of sound squared c_s^2 along the RG flows for a few values of the respective parameters Δ_{IR} and γ in figure 4.1

4.6.3 Second-order coefficients along examples of RG flows

In the previous subsection 4.6.2 we introduced two families of holographic RG flows. In this subsection we present numerical results for the second-order transport coefficients (4.2) along flows in both of these one-parameters families. In each family we consider around 20 values for the parameter for 40 different temperatures. The transport coefficients are computed using the expressions (4.63) featuring integrals of the gravity backgrounds, which we construct numerically employing the method developed in ref. [209].

The crucial finding of this subsection is that the Haack-Yarom identity $H = 0$, eq. (4.1), is satisfied along both families of flows within the limits of numerical accuracy estimated

[15] This bound implies that, below a certain temperature threshold, the top-down *GPPZ*-flow does not have stable black-brane solutions. Its superpotential is given by $W = -\frac{3}{4}\left(1 + \cosh(\phi/\sqrt{3})\right)$ and the potential therefore approaches $V \to -(3/8)\exp(2\phi/\sqrt{3})$ at large ϕ.

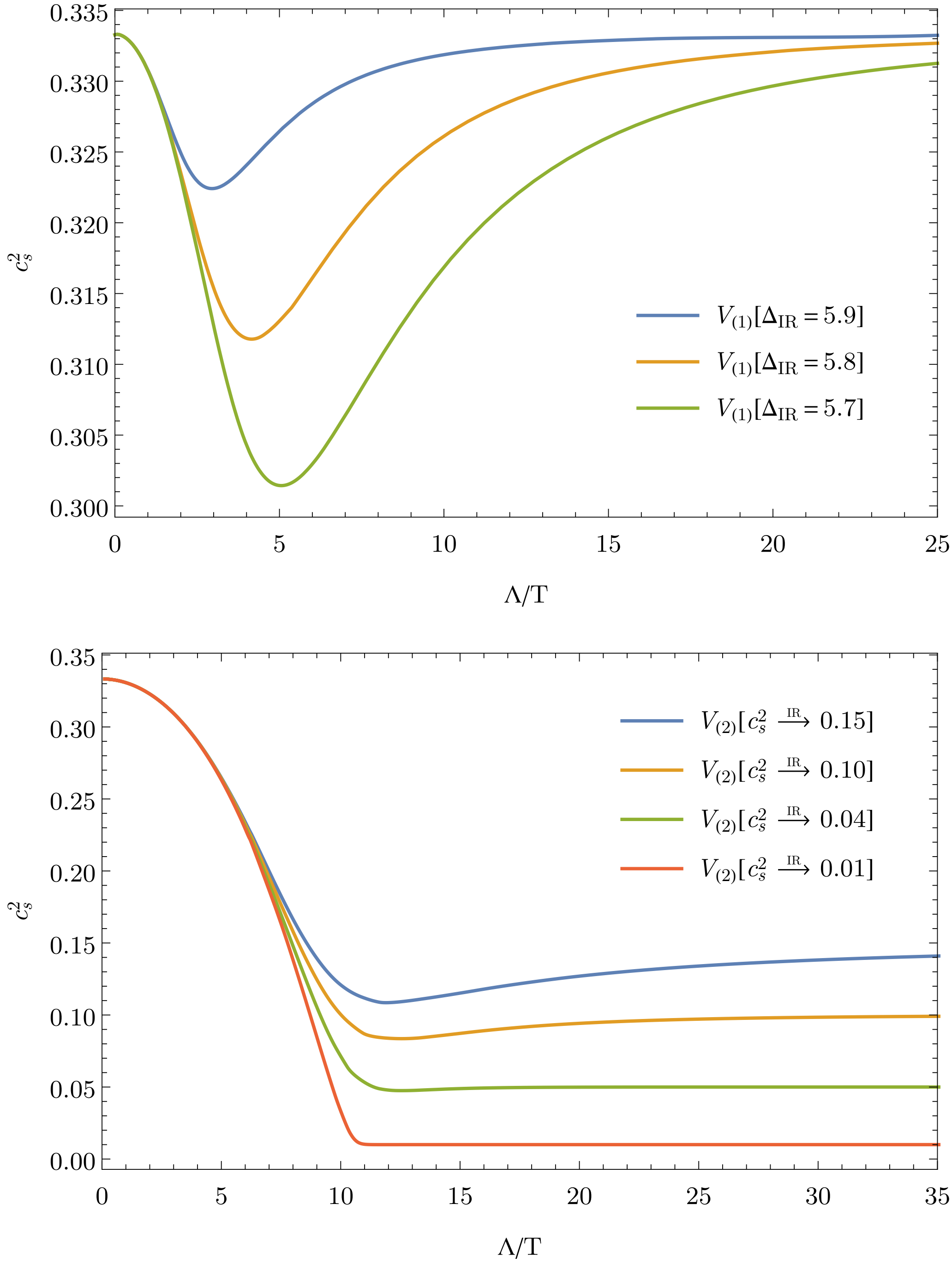


Figure 4.1: The speed of sound squared c_s^2 as a function of the scalar source Λ in units of temperature T. In both plots, each curve corresponds to a different holographic RG flow with the speed of sound squared taking the conformal value of $1/3$ in the UV at large temperatures or small scalar source. In the upper plot, the speed of sound squared is plotted for the potential $V_{(1)}$ for three values of the IR operator dimension Δ_{IR}, eq. (4.78). In the lower plot, the speed of sound squared is plotted for the potential $V_{(2)}$ with four different parameter values γ, which set the IR speed of sound according to eq. (4.82).

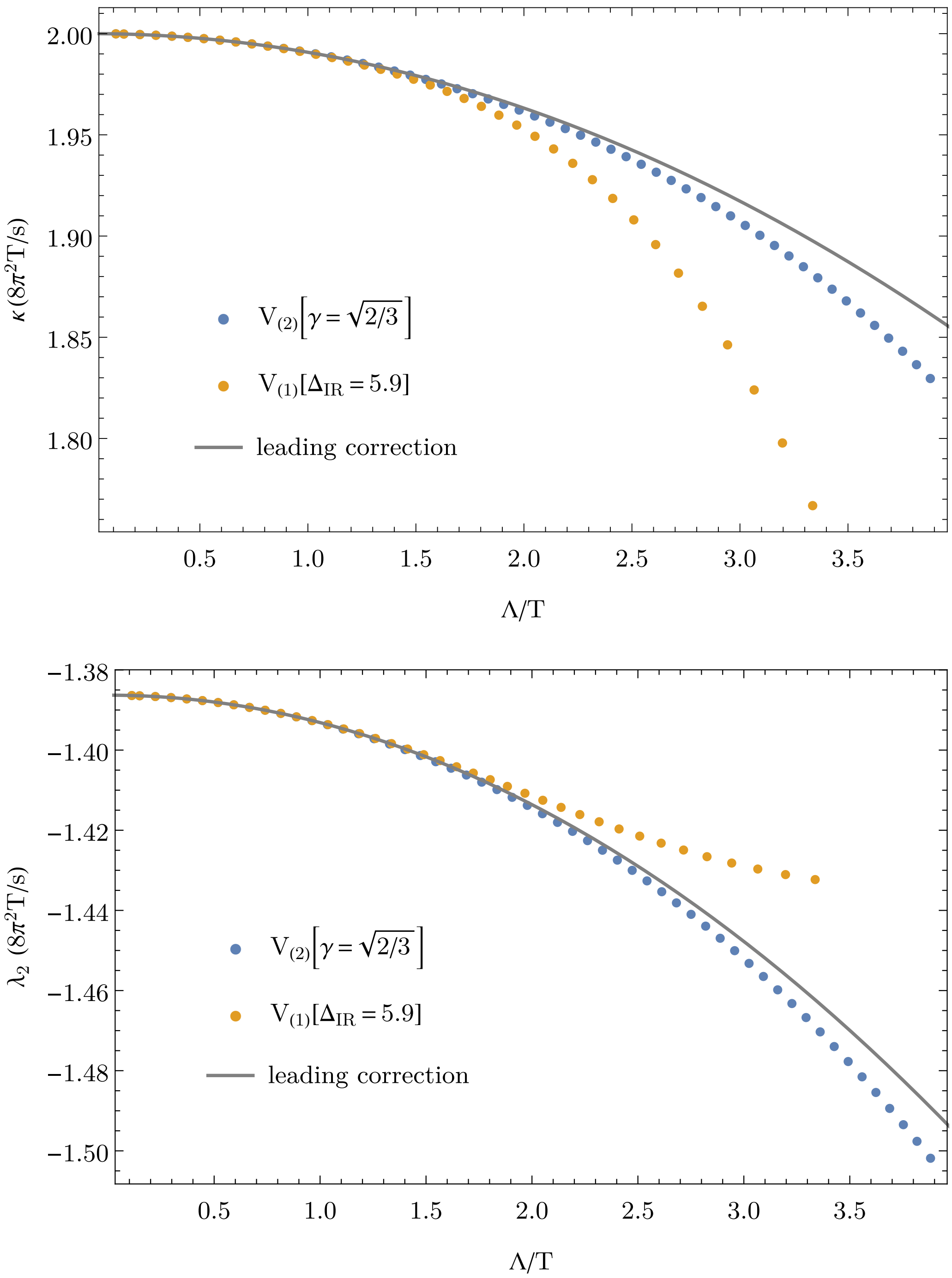
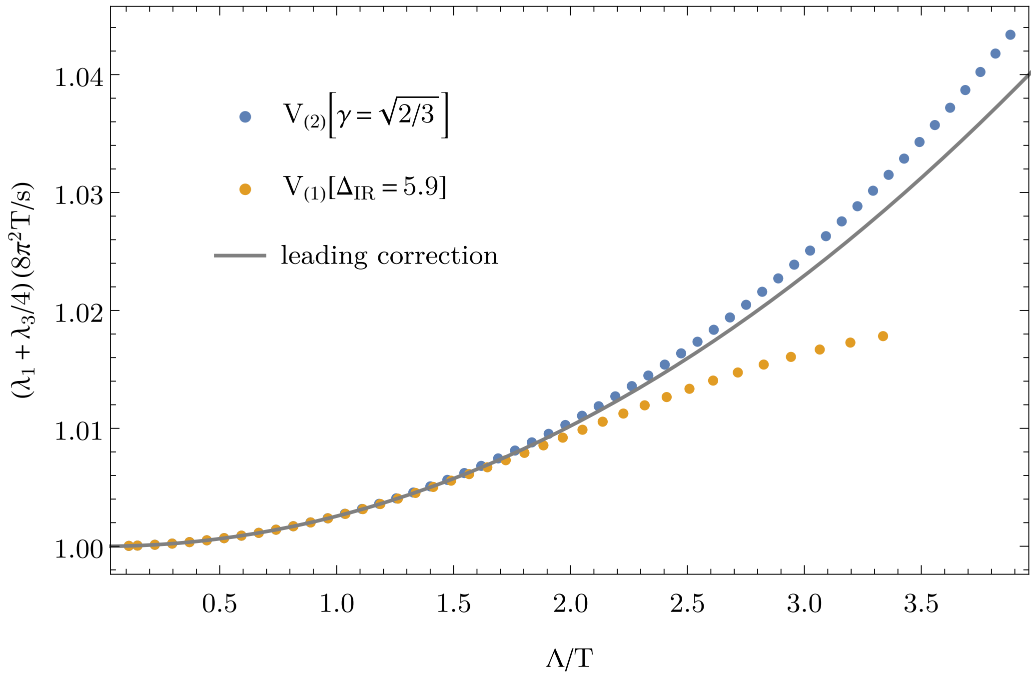


Figure 4.2: The second-order coefficients κ and λ_2 in units of $s/\left(8\pi^2T^2\right)$ plotted as functions of the scalar source Λ in units of the temperature T. We plot the numerical results for the coefficients for $V_{(1)}$ with the parameter ϕ_m fixed by $\Delta_{\mathrm{IR}} = 5.9$ and for $V_{(2)}$ with $\gamma = \sqrt{2/3}$. For smaller values of Δ_{IR} and γ, the two curves corresponding to the two potentials approach each other and coincide for $\Delta_{\mathrm{IR}} \to 4$ and $\gamma = 0$, where the potentials $V_{(1)}$ and $V_{(2)}$ agree. The grey solid lines show the leading non-conformal corrections (4.72) obtained in subsection 4.6.1.



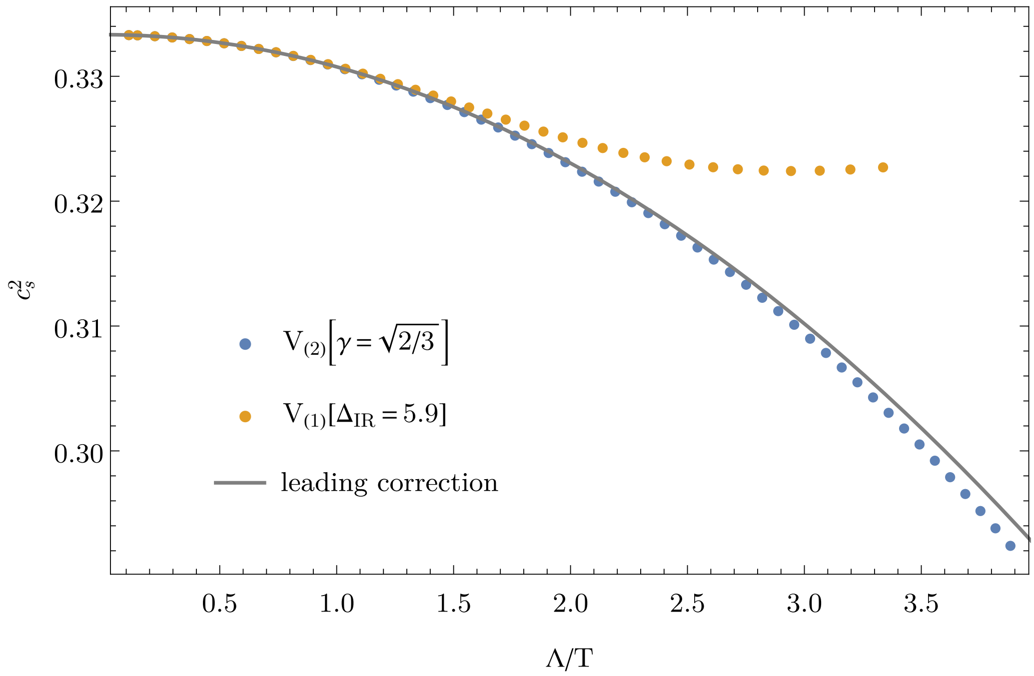


Figure 4.3: The second-order coefficient $\lambda_1 + \lambda_3/4$ in units of $s/\left(8\pi^2 T^2\right)$ and the speed of sound squared c_s^2 plotted as a function of the scalar source Λ in units of the temperature T. We plot the numerical results for $V_{(1)}$ with the parameter ϕ_m fixed by $\Delta_{\mathrm{IR}} = 5.9$ and for $V_{(2)}$ with $\gamma = \sqrt{2/3}$. For smaller values of Δ_{IR} and γ, the two curves corresponding to the two potentials approach each other and coincide for $\Delta_{\mathrm{IR}} \to 4$ and $\gamma = 0$, where the potentials $V_{(1)}$ and $V_{(2)}$ agree. The grey solid lines show the leading non-conformal corrections (4.72) obtained in subsection 4.6.1.

to be of the order of 10^{-5}. In contrast, the individual transport coefficients differ significantly from their conformal values. This result provides evidence that the identity, which is known to hold for conformal holographic fluids [23, 204–207] and, as we proved in subsection 4.5.2, close to the conformal holographic fixed point, is satisfied by all holographic fluids at infinite coupling regardless of conformal symmetry.

Together with the identity $\tilde{H} = 0$, eq. (4.3), which we proved to be satisfied analytically for all holographic RG flows triggered by an operator of scaling dimension $\Delta = 3$ in subsection 4.5.1, the Haack-Yarom identity implies that only three of the five combinations of transport coefficients, eq. (4.2), are independent. We plot the three combinations κ, λ_2 and $\lambda_1 + \lambda_3/4$ as well as the speed of sound squared c_s^2 as functions of Λ in units of the temperature T in figures 4.2 and 4.3. The coefficients are plotted for the largest considered parameters, i.e. $\Delta_{\mathrm{IR}} = 5.9$ for $V_{(1)}$ and $\gamma = \sqrt{2/3}$ for $V_{(2)}$. The curves for other parameters lie between these two extremal cases and smoothly interpolate between them. The two potentials $V_{(1)}$ and $V_{(2)}$ coincide for $\Delta_{\mathrm{IR}} = 4$ and $\gamma = 0$. In the figures we also include the leading non-conformal corrections of the transport coefficients, which we determined in subsection 4.6.1. The agreement of curves for $V_{(1)}$ and $V_{(2)}$ with the leading non-conformal correction at larger temperature, i.e. at small Λ/T, provides a check for our computations.

Comparing the figures 4.2 and 4.3 to figure 4.1, there is a notable difference in the x-axis range of values for Λ/T. We can observe the speed of sound in the deep IR in figure 4.1, i.e. the reversion back to the conformal value in the case of $V_{(1)}$ and the approach to the value corresponding to the appropriate Chamblin-Reall background in the case of $V_{(2)}$. In contrast, the behaviour of the transport coefficients in the deep IR is not apparent in our numerical results in figures 4.2 and 4.3. Nevertheless, the asymptotics of the coefficients in the IR are known analytically since they are determined by the IR geometry: the coefficients approach the conformal values in the case of $V_{(1)}$ and the values corresponding to the Chamblin-Reall background in the case of $V_{(2)}$. The reason for the smaller range of values for Λ/T lies in the choice of the radial bulk coordinate for the numerics. For the computation of thermodynamic quantities, we chose the scalar ϕ itself as radial coordinate, which we also used to construct the background according to ref. [209], see appendix A.3 for details. However, since the scalar vanishes in the UV, using it as a coordinate leads

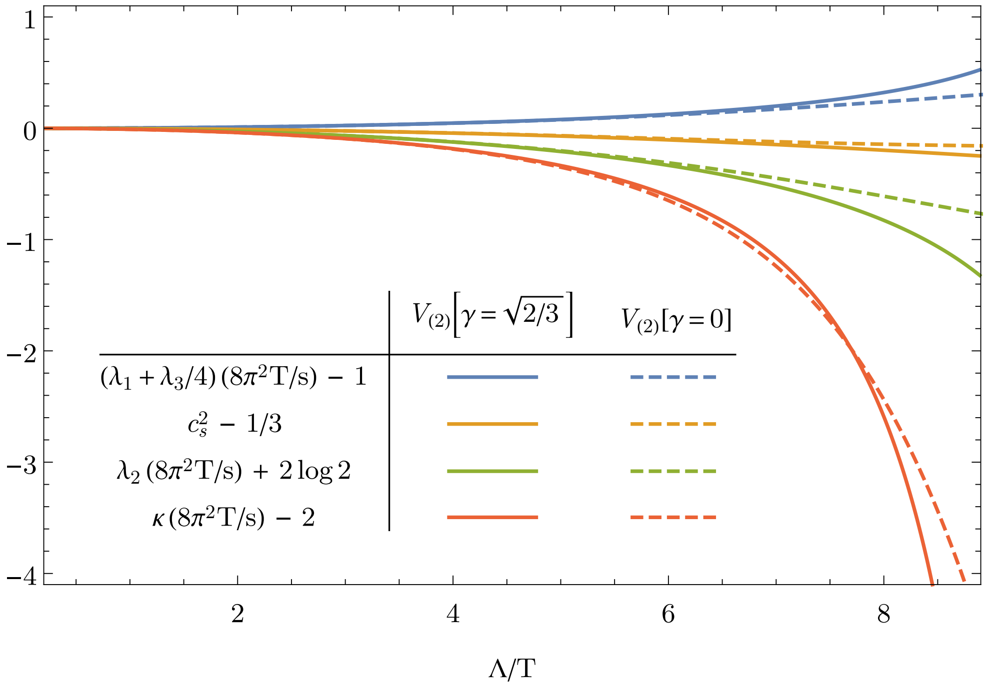


Figure 4.4: Numerical results for the deviation of the second-order coefficients $\lambda_1+\lambda_3/4$, λ_2, κ as well as the speed of sound squared c_s^2, measured in units of $s/\left(8\pi^2 T\right)$, from their conformal values (4.64) as a function of Λ/T. The solid line shows refers to $V_{(2)}$ with $\gamma=\sqrt{2/3}$ and the dashed line to $V_{(2)}$ with $\gamma=0$. For other values of γ, the results interpolate smoothly between these two extremal curves.

to unstable numerics in the UV. In addition, it prohibits us from reverting back to the conformal case by taking the limit of $\phi\to 0$ and it is unsuitable for a perturbative study as in 4.6.1. Consequently, we used the u-coordinate, eq. (4.18), for our study of metric fluctuations. While it avoids the issues of the ϕ-coordinate when studying the conformal limit and the UV, it renders numerics unstable in the deep IR, where a large range $\phi\in(0,\phi_H)$ is mapped to the interval $u\in(0,1)$. Therefore, reliable numerical results for the transport coefficients are available only for a relatively small range of Λ/T. A numerical study of transport in the deep IR is left for future work.

From figure 4.1 we can conclude that the IR becomes dominant in the case of $V_{(2)}$ for $\Lambda/T\gtrsim 10$. The same is true in the case of $V_{(1)}$ when $\Delta_{\text{IR}}\lesssim 5.2$. For these instances it is possible to obtain reliable numerical results for larger values of Λ/T. Figure 4.4 shows the deviations of the coefficients κ, λ_2 and $\lambda_1+\lambda_3/4$ as well as c_s^2 from their conformal values as a function of Λ/T for $V_{(2)}$ with $\gamma=\sqrt{2/3}$ and $\gamma=0$. The curves for other values of

γ smoothly interpolate between these two extremal cases. In the case of $V_{(1)}$, the results closely resemble those for $V_{(2)}$ with $\gamma = 0$ since $V_{(1)} \xrightarrow{\Delta_{\mathrm{IR}}\to 4} V_{(2)}|_{\gamma=0}$..

4.6.4 Employing relations from the entropy production

Patches of local equilibrium in a fluid obey the second law of thermodynamics [173]. This requires the existence of an entropy current whose divergence is non-negative when the equations of motion are satisfied. A sufficient condition, which ensures that the second law is obeyed, is to demand that at each order in the entropy current's gradient expansion only terms that always lead to non-negative entropy production can appear. Imposing this condition, refs. [172, 222] found five equalities relating second-order coefficients. The same relations were found in ref. [223] by coupling the fluid to external sources. Written in our conventions these equalities can be found in ref. [178]. In particular, they determine κ^*, ξ_5, and $\xi_3 + \xi_6$ in terms of κ and λ_3 as

$$\kappa^* = \kappa - \frac{T}{2}\frac{d\kappa}{dT} \,, \qquad \xi_5 = \frac{1}{2}\left(c_s^2 T\frac{d\kappa}{dT} - c_s^2\kappa - \frac{\kappa}{3}\right) \,, \tag{4.83}$$
$$\xi_3 + \xi_6 = \left[\frac{1}{3}\left(1 - 3c_s^2\right) + \frac{T}{12}\left(1 - 6c_s^2\right)\frac{d}{dT} + \frac{T^2 c_s^2}{4}\frac{d^2}{dT^2}\right]\kappa + \left[\frac{1}{12}\left(1 - 9c_s^2\right) + \frac{T c_s^2}{4}\frac{d}{dT}\right]\lambda_3 \,.$$

Making use of these relations, we can extend our numerical results for second-order transport coefficients to all five conformal coefficients κ, $\eta\tau_\pi$, λ_1, λ_2, λ_3 and to three non-conformal coefficients κ^*, ξ_5, $\xi_3 + \xi_6$. They are shown in figure 4.5.

4.7 Summary and outlook

In this chapter we studied the hydrodynamic regime of non-conformal strongly coupled holographic field theories, in particular transport properties at second order. We derived a set of Kubo formulae, which are particularly suitable for the holographic study of transport, valid for any uncharged non-conformal fluid in (3+1) dimensions. Using these Kubo formulae we determined expressions for five second-order coefficients in strongly coupled field theories dual to holographic RG flows triggered by a scalar operator of scaling dimension $\Delta = 3$. From these expressions, we derived the relation

$$\tilde{H} = 2\eta\tau_\pi - 2\left(\kappa - \kappa^*\right) - \lambda_2 = 0 \,, \tag{4.84}$$

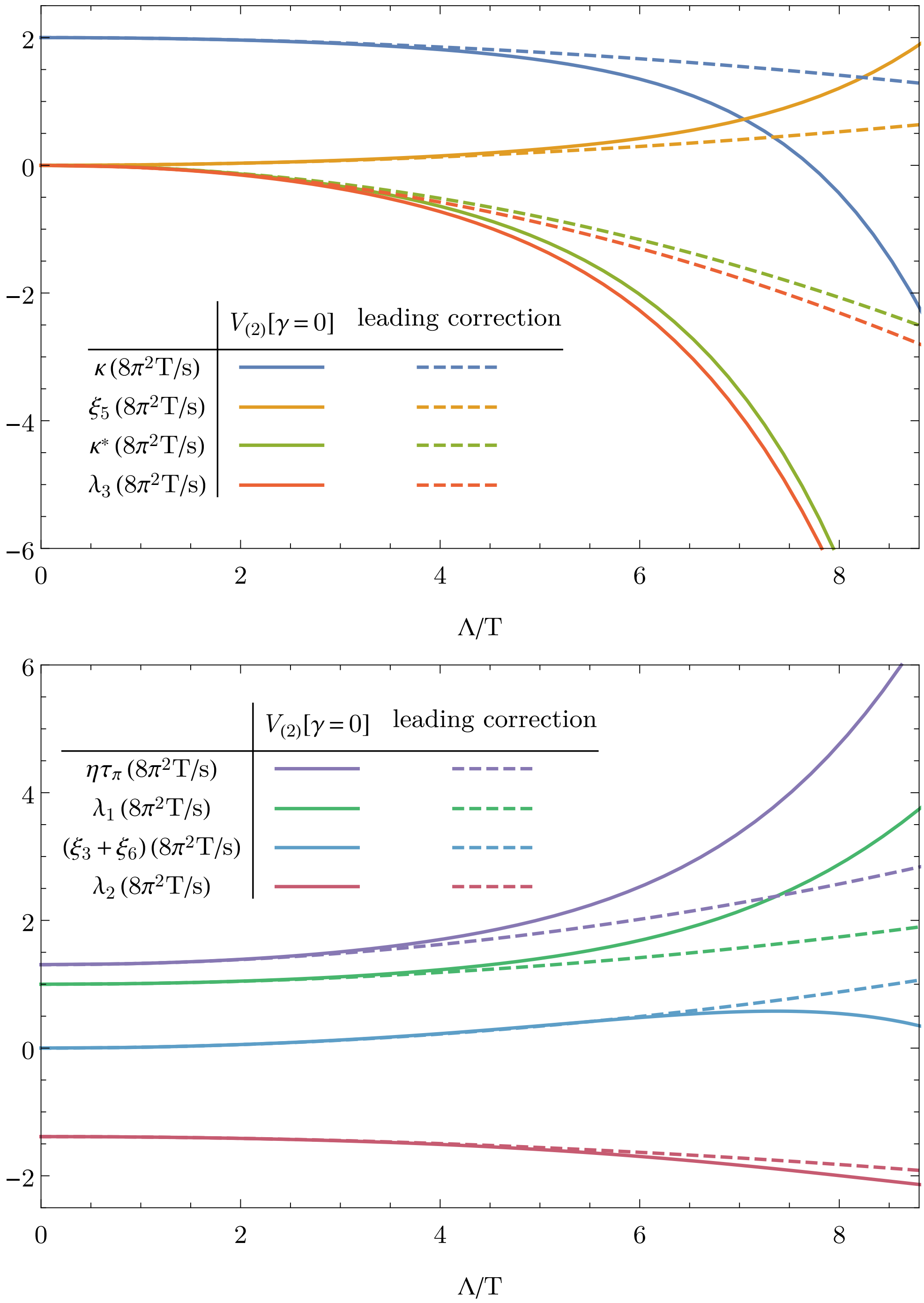


Figure 4.5: All five conformal second-order coefficients κ, $\eta\tau_\pi$, λ_1, λ_2, λ_3 and three non-conformal second-order coefficients κ^*, ξ_5, $\xi_3 + \xi_6$ in units of $s/\left(8\pi^2 T\right)$ versus Λ in units of the temperature T. The solid line shows our numerical results for $V_{(2)}$ with $\gamma = 0$, the dashed line describes the leading non-conformal corrections from subsection 4.6.1. Three of the eight second-order coefficients were obtained applying constraints derived from the positivity of the local entropy production as described in subsection 4.6.4.

and proved that the Haack-Yarom identity,

$$H = 2\eta\tau_\pi - 4\lambda_1 - \lambda_2 = 0 \,, \tag{4.85}$$

which is known to hold for conformal strongly-coupled holographic fluids, is also satisfied when taking into account leading non-conformal corrections. We verified numerically that the identity even holds beyond leading-order corrections for two one-parameter families of flows within the class of flows triggered by an operator of dimension $\Delta = 3$. This result suggests that the Haack-Yarom identity is indeed satisfied by all strongly coupled fluids regardless of conformal symmetry [204–208, 212].

A number of open questions arises from the results of this chapter. Firstly, we only computed five of the fifteen second-order transport coefficients in non-conformal fluids. The Kubo formulae we derived only allow for access to these five coefficients since we restricted to transverse vector and tensor perturbations of the metric. Considering scalar sound perturbations and deriving a set of appropriate Kubo formulae should permit the computation of the remaining ten coefficients. The technical obstacle to overcome is that scalar perturbations of the boundary metric couple to bulk fluctuations of the theory-specific matter content. If the obstacle is overcome and all fifteen second-order coefficients are computed, one could verify that the relations among second-order transport coefficients derived from the requirement of positivity of the local entropy production (three of which we employed in subsection 4.6.4) are satisfied [172, 222, 223].

Second, for technical reasons our numerical results for transport coefficients in subsections 4.6.3 and 4.6.4 did not extend to the deep IR. As previously explained, choosing the scalar ϕ as radial coordinate instead of our choice of the coordinate u would permit us to study the IR and to verify that the numerical results agree with the expected values for the two families of RG flows we considered. For the first family, the second-order coefficient should revert back to their conformal values in the IR while, for the second family, they should approach the values they take in the non-conformal theories dual to the Chamblin-Reall background.

Third, an interesting open question is whether the proof we provided in subsection 4.5.2, that the Haack-Yarom identity is still satisfied when taking into account leading non-conformal corrections triggered by an operator of scaling dimension $\Delta = 3$, can be generalised to relevant operators of arbitrary scaling dimension $2 < \Delta < 4$. The obstacle to overcome is that different set of holographic counterterms is required to renormalise the theory. The set of relevant metric perturbations remains unchanged.

Fourth, the findings of this paper might have implications for an ongoing investigation into whether fluids with simply gravity duals obey a conjectured principle of minimal dissipation meaning that these fluids minimise local entropy production [204, 242]. The shear viscosity over entropy density ratio, η/s, appears as coefficient of the leading contribution to entropy production. The ratio takes the universal value of $1/4\pi$ in a large class of holographic theories [21, 22, 188–194] and this universal value is conjectured to serve as a lower bound for the ratio more generally [21]. If this lower bound holds, it would imply that the holographic theories minimise entropy production at first order. Two terms appear in the entropy production of conformal fluids at second order [172]. The first term vanishes if $2\lambda_1 = \kappa$, which is true for conformal holographic fluids [174]. The coefficient of the second term is unknown. Less is known about entropy production at second order for non-conformal holographic fluids. Nevertheless, within an effective action approach to adiabatic hydrodynamics, it was derived that perfect fluids that do not produce entropy must satisfy the relations $H = 0$ and $2\lambda_1 = \kappa - \kappa^*$ [243], which are indeed equivalent to $H = 0$ and $\tilde{H} = 0$. However, the Haack-Yarom identity $H = 0$ is not trivial and it seems to require either infinite coupling or adiabaticity. It is violated in certain weakly coupled theories in the kinetic regime [244] as well as in the hypothetical dual of Gauss-Bonnet gravity when corrections away from the infinite coupling limit are taken into account [204, 206, 207].[16] Further evidence for the principle of minimal dissipation would be provided if the relations $H = 0$ and $2\lambda_1 = \kappa - \kappa^*$ were shown to lead to cancellations in the local entropy production for non-conformal fluids at second order analogous to the conformal case.

[16]Consistently, the Haack-Yarom identity cannot be derived from the generalised Onsager relations which are applicable to any uncharged conformal fluid [245].

Chapter 5

The holographic dictionary out of equilibrium

5.1 Introduction and summary

Gauge/gravity duality relates the dynamics of strongly-coupled quantum field theories out of equilibrium to semiclassical gravitational dynamics [246, 247]. The gravitational description is particularly well-suited for addressing questions of non-equilibrium physics since the bulk physics is classical at leading order in the $1/N$ expansion and the real-time dynamics is reduced to solving differential equations with initial data determined by the field theory initial state.

An interesting topic to explore in the context of non-equilibrium dynamics is the question of thermalisation. While a pure initial state cannot become thermal under unitary time evolution, observables can assume, or at least approach, the value they take in a thermal equilibrium state. This process differs qualitatively between local one-point and non-local multi-point correlation functions [248]. Even when expectation values of localised observables have already assumed their thermal values, the quantum correlations between operators at separate points can still be non-thermal. In this and the following chapter, we explore how non-local correlation functions approach their thermal values when initialised in an out-of-equilibrium state in order to gain a better understanding of thermalisation in the field theories with gravitational duals.

When studying non-equilibrium dynamics in field theories with holographic duals, we first attempt to understand the correct dictionary between bulk and boundary quantities.

Building on earlier work [128, 249, 250], Skenderis and van Rees constructed a dictionary for boundary theory correlation functions by constructing a holographic version of the Schwinger-Keldysh real-time formalism, which we introduced in the context of non-equilibrium QFT in section 3.1 [30, 31]. In their prescription, which we refer to as *SvR prescription*, the initial state of the boundary theory is prepared by a path integral on a Euclidean manifold M_∂, while the bulk state is prepared by the path integral over the bulk Euclidean manifolds M whose boundary is M_∂.[1] Correlation functions are then obtained by taking functional derivatives of the on-shell action on a glued manifold consisting of the Euclidean manifold M and a Lorentzian section corresponding to the real-time evolution. The prescription closely resembles the GKPW formula (2.33).

Another a priori independent dictionary between bulk and boundary correlation functions was suggested and identified as possibly distinct by Banks, Douglas, Horowitz and Martinec and is referred to as *BDHM prescription* [27].[2] They observed that connected two-point functions are suppressed in the $1/N$ expansion compared to their disconnected counterparts. Thus, from the bulk point of view, they are quantum mechanical and the prescription proposes to extract them in the following way. To leading order in $1/N$, a classical bulk solution for the metric and all other bulk fields is found. Fluctuations around these classical backgrounds are then treated quantum mechanically.[3] In particular, one must specify a state for these fluctuations.[4] The BDHM prescription proposes that boundary correlators are obtained as boundary limits of bulk correlation functions of these fluctuations. For scalar fields and in Euclidean time, the BDHM prescription is equivalent to the GKPW formula (2.33) as was shown in ref. [253].

In this chapter, which is based on ref. [33], we will discuss the BDHM dictionary and the SvR prescription for real-time correlation functions in detail in section 5.2. Subsequently, in section 5.3 we will show that the two prescriptions are equivalent for two-point

[1] In the supergravity limit the path integral restricts to one bulk manifold.

[2] The prescription was implicitly used in ref. [134] and later explicitly in refs. [29, 251].

[3] See e.g. ref. [252] for a review of the semiclassical approximation in the context of quantum field theory.

[4] In a large part of the literature, the problem of specifying a state in the bulk has been bypassed by considering retarded correlation functions, which are independent of the state in the quadratic approximation in bulk fluctuations.

correlators of boundary theory operators dual to free scalar fields in arbitrary asymptotically AdS spacetimes. In the concrete example we study in chapter 6, which belongs to the class of models covered by the equivalence proof in this chapter, we use the BDHM version of the AdS/CFT dictionary to compute two-point functions in a far-from-equilibrium state. In section 5.4 we summarise the results of this chapter and present an outlook.

5.2 Two holographic dictionaries out of equilibrium

In subsection 2.2.3 we introduced the GKPW formula (2.33) which determines the generating functional of correlation functions of holographic Euclidean field theories in terms of bulk quantities. In subsection 2.2.5 we explained how retarded correlation functions in a holographic Lorentzian field theory in thermal equilibrium can be determined from the bulk by introducing in-going boundary conditions on the bulk solutions at the horizon. In this section we discuss the SvR and BDHM prescriptions that generalise the holographic dictionary to correlation functions in out-of-equilibrium states.

5.2.1 The Skenderis-van Rees prescription

The familiar version of the AdS/CFT dictionary for correlation functions, the GKPW formula (2.33), identifies the boundary generating functional of connected correlators $W_{\text{CFT}}\left[\phi_{(0)}\right]$ limit with the Euclidean on-shell action of the bulk theory, eq. (2.35), in the large-N limit. Then, correlation functions of the boundary operator $\mathcal{O}$ are obtained by differentiating the generating functional of connected correlators with respect to the boundary value $\phi_{(0)}$ of the bulk field dual to the operator $\mathcal{O}$,

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle_{\text{CFT}} = \left. \frac{C_n \, \delta^n W_{\text{CFT}}\left[\phi_{(0)}\right]}{\delta\phi_{(0)}(x_1) \ldots \delta\phi_{(0)}(x_n)} \right|_{\phi_{(0)}=0} . \tag{5.1}$$

The constant C_n is a coefficient related to the normalisation of the CFT operators and $\phi_{(0)}$ is the coefficient of the non-normalisable solution ϕ of the classical bulk equations of motion, i.e. $\phi_{(0)}(x) = \lim_{z\to 0} z^{-\Delta_-} \phi(x,z)$.

For the case of real-time out-of-equilibrium correlation functions, where analytic continuation to a Euclidean spacetime is not available, the situation is less clear. In particular,

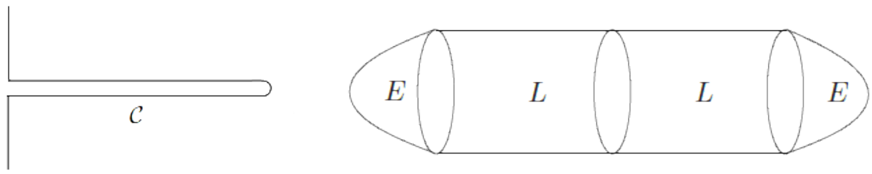


Figure 5.1: Left: The complex time contour for Schwinger-Keldysh correlation functions in non-equilibrium QFT. Right: A corresponding bulk manifold with two Lorentzian pieces corresponding to the real time evolution and two Euclidean caps corresponding to the imaginary part of the time contour which prepares the initial state.

it is not obvious how to generalise the GKPW dictionary (5.1) to such situations. In general, the boundary value $\phi_{(0)}$ does not uniquely fix a bulk solution which is required to compute the on-shell bulk action. In thermal equilibrium the retarded correlator can be obtained by fixing the bulk solution with in-going boundary conditions at the event horizon [127]. The out-of-equilibrium version of (5.1) was developed further for retarded two-point correlation functions in ref. [254] building previous work in ref. [193] and on the fact that the retarded correlator encodes the causal response of the one-point function to a small external perturbation, as we have seen in subsection 3.3.2. Nevertheless, the generalisation to other correlation functions, such as Wightman correlation functions, is less clear. For a class of states whose wavefunctionals can be obtained as path integrals on Euclidean manifolds, a generalisation of the GKPW dictionary was proposed by Skenderis and van Rees [30, 31] who constructed a holographic version of the Schwinger-Keldysh generating functional, which we introduced in the context of non-equilibrium QFT in section 3.1.[5]

Building on earlier work [128, 249, 250], Skenderis and van Rees suggested that the bulk dual of the field theory Schwinger-Keldysh generating functional, eq. (3.9), is given by the bulk path integral [30, 31]

$$Z[J_+, J_-] = \int \mathcal{D}\phi_+ \mathcal{D}\phi_- \, e^{iS[\phi_+]-iS[\phi_-]}\Psi^*[\phi_-(t_i)]\Psi[\phi_+(t_i)]\delta(\phi_+(t_m) - \phi_-(t_m)) \,, \qquad (5.2)$$

where the turning point of the complex time contour is denoted by t_m. The boundary sources are, as usual, identified as the asymptotic values of bulk fields

$$\phi_\pm(x, z) = z^{\Delta_-} J_\pm(x) + \dots \qquad (5.3)$$

[5] See also [128, 249, 250] for earlier work in this direction.

and $\phi(x,z)$ collectively denotes all the bulk fields. As before, the subscript in $\phi_\pm$ denotes whether the time argument of the field is on the upper or lower section of the closed time contour. Using the semiclassical approximation in the bulk, the path integral (5.2) becomes

$$Z[J_+, J_-] \approx e^{iS[\phi_+]-iS[\phi_-]}\Psi^*[\phi_-(t_i)]\Psi[\phi_+(t_i)] \; , \tag{5.4}$$

where the bulk fields $\phi_\pm$ satisfy the saddle point condition

$$\frac{\delta}{\delta\phi_\pm}\Bigg[iS[\phi_+] - iS[\phi_-] + \log\Psi^*[\phi_-(t_i)] + \log\Psi[\phi_+(t_i)]\Bigg] = 0 \; . \tag{5.5}$$

Furthermore, Skenderis and van Rees considered the case where the bulk initial state wavefunctional Ψ is obtained from a path integral over Euclidean manifolds, that can be glued to the Lorentzian part of the bulk spacetime. The semiclassical approximation reduces this path integral to a single Euclidean manifold. Then eq. (5.5) gives rise to the equations of motion for all the bulk fields and continuity conditions for the fields at the gluing surfaces between the Euclidean and Lorentzian manifolds at time t_i. The resulting bulk spacetime is sketched on the right in figure 5.1. The two Euclidean caps prepare the initial state wavefunctionals Ψ and Ψ^*. The Lorentzian sections correspond to the upper and lower sections of the complex time contour plotted on the left. The bulk fields and their first derivatives have to be continuous at the matching surfaces.[6]

5.2.2 The Banks-Douglas-Horowitz-Martinec prescription

The BDHM prescription directly relates correlation functions in the bulk to correlation functions of the dual boundary operators. For a scalar field Φ of mass m in AdS_{d+1}, which is dual to a scalar operator $\mathcal{O}$ of dimension $\Delta_+ = d/2 + \sqrt{d^2/4 + L^2m^2}$, the Euclidean version of the dictionary is

$$\langle\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\rangle_{\rm CFT} = \tilde{C}_n \lim_{z\to 0} z^{-n\Delta_+}\langle\Phi(x_1,z)\ldots\Phi(x_n,z)\rangle_{\rm bulk} \; , \tag{5.6}$$

where $\tilde{C}_n$ is a constant related to the normalisation of the CFT operators.

[6] The details of the gluing procedure on the matching surfaces can be found in ref. [31].

The BDHM dictionary (5.6) can be straightforwardly generalised to real-time out-of-equilibrium correlation functions. One chooses the CFT state and correlator of interest and obtains the correlator from a corresponding bulk correlation function via[7]

$$\langle \mathcal{O}(x) \ldots \rangle_{\mathrm{CFT}} \sim \lim_{z \to 0} z^{-\Delta_+} \langle \Phi(x, z) \ldots \rangle_{\mathrm{bulk}} \,. \tag{5.7}$$

For example, for the boundary time-ordered two-point function $\langle \mathcal{T}\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle$, the substitution (5.7) results in the bulk correlation function

$$\langle \mathcal{T}\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_{\mathrm{CFT}} = \tilde{C}_2 \lim_{z_1 \to 0} \lim_{z_2 \to 0} z_1^{-\Delta_+} z_2^{-\Delta_+} \langle \mathcal{T}\Phi(x_1, z_1)\Phi(x_2, z_2)\rangle_{\mathrm{bulk}} \,. \tag{5.8}$$

Therefore, the calculation of the boundary two-point function becomes a standard problem of quantum field theory in curved spacetime.

This prescription leads to correct results for correlation functions in the vacuum and in thermal equilibrium, as shown for example in Appendix A of ref. [255] for the BTZ black hole. Issues such as choosing in-going versus out-going boundary conditions and the problem of boundary terms from the horizon (see e.g. ref. [127]) do not exist in this formalism.

5.2.3 Two-point functions as initial value problems

In the BDHM framework, where we have to compute bulk correlation functions, it is natural to view the time evolution of correlation functions as an initial value problem. The state $|\psi\rangle$ of the system is time-independent in the Heisenberg picture and acts as an initial condition. In this and the following chapter, we will only consider free scalar fields in the bulk. When the one-point function of the scalar vanishes, this is sufficient for the purpose of calculating connected scalar two-point functions to leading order in the $1/N$ expansion.[8] We consider a free scalar field with action

$$S = -\frac{1}{2} \int \mathrm{d}^d x \, \mathrm{d}z \, \sqrt{-g} \left(\partial_\mu \Phi \partial^\mu \Phi + m^2 \Phi^2 \right) \tag{5.9}$$

[7] The mapping between bulk and boundary quantum states is unknown in most cases. However, it is is known for states that can be prepared via a Euclidean path integral.

[8] If the one-point function $\langle \Phi \rangle$ is non-vanishing, one should include bulk Feynman/Witten diagrams, where Φ mixes with the graviton and other bulk fields, which results in contributions to the two-point function that are not suppressed by powers of $1/N$ compared to the free result. Thus, we will assume in the following that the one-point function $\langle \Phi \rangle$ vanishes. In our explicit example in chapter 6, this follows from conformal symmetry of the initial state, and from the $\Phi \to -\Phi$ symmetry of the action (5.9).

in a fixed asymptotically AdS_{d+1} background. In the Heisenberg picture, the bulk scalar quantum field satisfies its Heisenberg equation of motion,

$$(\Box - m^2)\Phi(x) = 0\,. \tag{5.10}$$

The bulk Wightman and retarded two-point correlators, introduced in eqs. (3.11) and (3.13),

$$G_+(x_1, z_1; x_2, z_2) = \langle \Phi(x_1, z_1)\Phi(x_2, z_2)\rangle\ , \tag{5.11a}$$

$$G_R(x_1, z_1; x_2, z_2) = -i\Theta(x_1^0 - x_2^0)\,\langle [\Phi(x_1, z_1), \Phi(x_2, z_2)]\rangle\ , \tag{5.11b}$$

satisfy

$$(\Box_{1/2} - m^2)G_+(x_1, z_1; x_2, z_2) = 0\ , \tag{5.12a}$$

$$(\Box_{1/2} - m^2)G_R(x_1, z_1; x_2, z_2) = \frac{\delta^d(x_2 - x_1)\delta(z_1 - z_2)}{\sqrt{-g}}\ , \tag{5.12b}$$

where the Laplacian $\Box_{1/2}$ can be applied with respect to either (x_1, z_1) or (x_2, z_2). We can solve the equations of motion for the Wightman correlator, eq. (5.12a), using that the retarded correlator is a causal Green's function of the Klein-Gordon operators according to eq. (5.12b). Consequently, for $t_1 < t_2 < t_3$ the Wightman two-point function can be written as [254, 256, 257]

$$G_+(x_3, z_3; x_1, z_1) = \int\limits_{t_2=\text{const}} \mathrm{d}^{d-1}\underline{x}_2\, \mathrm{d}z_2\, G_+(x_2, z_2; x_1, z_1) \overleftrightarrow{D}^{t_2} G_R(x_3, z_3; x_2, z_2)\ , \tag{5.13}$$

where $D^t = \sqrt{-g}g^{t\mu}\partial_\mu$ and the integral is over all bulk spatial coordinates on a constant time slice. Therefore, given the two-point function $G_+(x_2, x_1)$, we can obtain it at some later time t_3 by using eq. (5.13). The same procedure can also be used to propagate the other argument of the Wightman function forward in time. Thus, if we know $G_+(x_2, x_1)$ and its first time derivative on some initial time slice, we can use the equations of motion to evolve it forwards in time. For free bulk fields, this implies that the specification of the initial state is identical to specifying the two-point correlation function and its first time derivatives as initial data.

An important realisation is that the retarded correlator G_R of a free field does not depend on the quantum state of the system. This can be understood as follows. Since the operators $\Phi(x,z)$ commute at equal time it follows that $G_R(x_1,z_1;x_2,z_2) = 0$ for $t_1 = t_2$. The first time derivatives of G_R are proportional to canonical commutators, e.g. $\partial_{t_2} G_R|_{t_1=t_2} \sim \langle[\Phi(x_1,z_1),\Pi(x_2,z_2)]\rangle|_{t_1=t_2} = i\delta(\underline{x}_1-\underline{x}_2)\delta(z_1-z_2)$. Thus, also the first time derivatives of G_R on an equal time slice are independent of the state. This means that the initial data for G_R is independent of the state and, since G_R satisfies the second order differential equation (5.12b), it is independent of the state at all times.[9] Therefore, all the information about the bulk quantum state that enters eq. (5.13) is captured by the initial data for the Wightman function G_+.

5.3 Equivalence of the SvR and BDHM prescriptions

In this section we outline the proof that the SvR prescription from subsection 5.2.1 and the BDHM dictionary, introduced in subsection 5.2.2, are equivalent on the level of two-point functions for free scalar bulk fields with an action given by (5.9) in arbitrary asymptotically AdS_{d+1} spacetimes. The idea behind the proof is the following. The bulk solutions in the SvR prescriptions obey equations of motion determined by the saddle point condition (5.5) as well as boundary conditions determined by the initial state and the matching conditions on the gluing surfaces of the bulk spacetime. Via the SvR dictionary, these become equations of motion and boundary conditions for the boundary theory correlation functions. Since the boundary conditions uniquely fix a solution to the equations of motion, it is sufficient to show that the correlation functions derived via the BDHM prescription, eq. (5.7), obey the same equations of motion and boundary conditions. We will derive the equations of motion and boundary conditions for a free bulk scalar in a state prepared via a Euclidean path integral and refer to our publication, ref. [33], for the proof that the BDHM correlators obey the same equations and boundary conditions.

The SvR prescription applies to initial states that can be prepared as a path integral over Euclidean manifolds. Therefore, the wavefunctional of the scalar field in the

[9]The correlator G_R still depends on the metric of the bulk spacetime. G_R does not depend on the bulk quantum state of the scalar field while, since it depends on the spacetime in question, it depends on the boundary CFT state.

semiclassical approximation is given by the on-shell Euclidean action

$$\Psi[\varphi_i] \approx \frac{1}{\sqrt{\mathcal{N}}} e^{-S_E[\varphi_E]}\Big|_{\varphi_E(t_i,\underline{x},z)=\varphi_i(\underline{x},z)} = \frac{1}{\sqrt{\mathcal{N}}} e^{-\frac{1}{4}\int \varphi_i K \varphi_i} \,, \tag{5.14}$$

where

$$\int \varphi_i K \varphi_i = \int \mathrm{d}^{d-1}\underline{x}_1\, \mathrm{d}z_1\, \mathrm{d}^{d-1}\underline{x}_2\, \mathrm{d}z_2\, \varphi_i(\underline{x}_1, z_1) K(\underline{x}_1, z_1; \underline{x}_2, z_2) \varphi_i(\underline{x}_2, z_2) \,. \tag{5.15}$$

We discussed wavefunctionals encoding initial states in detail in subsection 3.1.2. Here we used the fact that the on-shell action can be written as a boundary term at the gluing surface and that it is a quadratic functional of φ_i after applying the Euclidean equations of motion.[10] The last equality in (5.14) defines K in terms of the Euclidean manifold and the action functional. This wavefunctional is defined on an initial Cauchy slice, i.e. on the spatial slice $t = t_i$. Above, $\mathcal{N}$ is a normalisation factor chosen so that the norm of the wavefunctional is one, i.e.

$$\mathcal{N} = \int \mathrm{d}\varphi_i\, e^{-\frac{1}{2}\int \varphi_i K \varphi_i} \,, \tag{5.16}$$

where the integral over field configurations $\mathrm{d}\varphi_i = \prod_{\underline{x},z} \mathrm{d}\varphi_i(\underline{x}, z)$ integrates over spatial configurations at fixed time $t = t_i$. The kernel $K(\underline{x}_1, z_1; \underline{x}_2, z_2)$ is related to the initial equal-time two-point function of the bulk field Φ via

$$\begin{aligned} G(\underline{x}_1, z_1; \underline{x}_2, z_2) &= \langle\psi|\Phi(t_1, \underline{x}_1, z_1)\Phi(t_2, \underline{x}_2, z_2)|\psi\rangle|_{t_1=t_2=t_i} \\ &= \frac{1}{\mathcal{N}} \int \mathrm{d}\varphi_i\, e^{-\frac{1}{2}\int \varphi_i K \varphi_i} \varphi_i(\underline{x}_1, z_1)\varphi_i(\underline{x}_2, z_2) \\ &= K^{-1}(\underline{x}_1, z_1; \underline{x}_2, z_2) \,. \end{aligned} \tag{5.17}$$

Thus, K is simply the inverse of the initial equal-time two-point function and is implicitly determined in terms of G by

$$\int \mathrm{d}^{d-1}\underline{x}\, \mathrm{d}z\, G(\underline{x}_1, z_1; \underline{x}, z) K(\underline{x}, z; \underline{x}_2, z_2) = \delta^{d-1}(\underline{x}_1 - \underline{x}_2)\delta(z_1 - z_2) \,. \tag{5.18}$$

In the following, we will not need the explicit form of the kernel K but only that it is a real function which satisfies eq. (5.18).

[10] The reader unfamiliar with this fact should consult Problem 3, in Section F of ref. [258].

The solution ϕ of the SvR equations of motion (5.5) for the scalar field with action (5.9) obeys the saddle point condition

$$\frac{\delta}{\delta\phi_\pm}\Big(iS[\phi_+] - iS[\phi_-] - \frac{1}{4}\int \phi_+ K\phi_+ - \frac{1}{4}\int \phi_- K\phi_-\Big) = 0\ , \tag{5.19}$$

where

$$\int \phi_\pm K\phi_\pm = \int \mathrm{d}^{d-1}\underline{x}_1\, \mathrm{d}z_1\, \mathrm{d}^{d-1}\underline{x}_2\, \mathrm{d}z_2\, \phi_\pm(t_i,\underline{x}_1,z_1)K(\underline{x}_1,z_1;\underline{x}_2,z_2)\phi_\pm(t_i,\underline{x}_2,z_2)\ . \tag{5.20}$$

The variation of (5.19) with respect to $\phi_\pm$ away from the endpoints of the contour leads to the bulk equation of motion

$$(\Box - m^2)\phi_\pm(x,z) = 0\ . \tag{5.21}$$

The variation of $\phi_\pm$ at the initial time t_i gives an initial condition

$$D^t\phi_\pm(t,\underline{x},z)|_{t=t_i} = \mp\frac{i}{2}\int \mathrm{d}^{d-1}\underline{x}_1\, \mathrm{d}z_1 \phi_\pm(t_i,\underline{x}_1,z_1)K(\underline{x}_1,z_1;\underline{x},z)\ , \tag{5.22}$$

with $D^t = \sqrt{-g}g^{t\mu}\partial_\mu$. Variation at the turning point of the contour, $t = t_m$, gives

$$D^t\phi_+(x,z)|_{t=t_m} = D^t\phi_-(x,z)|_{t=t_m}\ . \tag{5.23}$$

The delta functional in the original path integral (5.2) leads to an additional condition at $t = t_m$, which is the continuity of the fields

$$\phi_+(x,z)|_{t=t_m} = \phi_-(x,z)|_{t=t_m}\ . \tag{5.24}$$

Thus, in order to obtain the SvR generating functional (5.4), we have to solve the equations of motion (5.21) for ϕ_+ and ϕ_- with the boundary conditions (5.22), (5.23) and (5.24).

As in Euclidean-time AdS/CFT, the most general solution to the bulk equations of motion, satisfying the boundary conditions

$$\phi_\pm(x,z) = z^{\Delta_-}J_\pm(x) + \dots\ , \tag{5.25}$$

can be written in terms of bulk-to-boundary propagators $K_{\pm\pm}$,

$$\phi_\pm(x,z) = \int \mathrm{d}^d y\, K_{\pm+}(x,z;y)J_+(y) + \int \mathrm{d}^d y\, K_{\pm-}(x,z;y)J_-(y)\ , \tag{5.26a}$$

where y is integrated over the AdS boundary. In the SvR prescription at infinite N, the boundary theory correlation functions are determined by taking functional derivatives of the on-shell action. Therefore, the boundary two-point functions of interest are given by

$$\langle \mathcal{T}\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = -2i\nu K^{(1)}_{++}(x_1,x_2)\ , \qquad \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = -2i\nu K^{(1)}_{-+}(x_1,x_2)\ , \tag{5.27a}$$

$$\langle \mathcal{O}(x_2)\mathcal{O}(x_1)\rangle = -2i\nu K^{(1)}_{+-}(x_1,x_2)\ , \quad \langle \overline{\mathcal{T}}\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = -2i\nu K^{(1)}_{--}(x_1,x_2)\ , \tag{5.27b}$$

where $\overline{\mathcal{T}}$ is the inverse time-ordering operator and

$$K_{\alpha\beta}(x,z;y) = z^{\Delta_-}\delta_{\alpha\beta}\delta^{d-1}(x-y) + \ldots + z^{\Delta_+}K^{(1)}_{\alpha\beta}(x,y) + \ldots\ , \tag{5.28}$$

in order to obey the boundary condition (5.25). The equations of motion (5.21) for ϕ_+ and ϕ_- with the boundary conditions (5.22), (5.23) and (5.24) translate into conditions for the $K_{\pm\pm}$ via eq. (5.26).

Claim: The $K_{\pm\pm}$ are given by boundary limits of the following bulk correlators:

$$K_{++}(x,z;y) = 2i\nu \lim_{z'\to 0} z'^{-\Delta_+} G_F(x,z;y,z')\ , \tag{5.29a}$$

$$K_{-+}(x,z;y) = 2i\nu \lim_{z'\to 0} z'^{-\Delta_+} G_+(x,z;y,z')\ , \tag{5.29b}$$

$$K_{+-}(x,z;y) = 2i\nu \lim_{z'\to 0} z'^{-\Delta_+} G_-(x,z;y,z')\ , \tag{5.29c}$$

$$K_{--}(x,z;y) = 2i\nu \lim_{z'\to 0} z'^{-\Delta_+} G_{\overline{F}}(x,z;y,z')\ , \tag{5.29d}$$

where we denote $\nu = (\Delta - \Delta_-)/2 = \sqrt{d^2/4 + L^2m^2}$ and G_F, $G_{\overline{F}}$ and $G_\pm$ are the bulk Feynman, anti-time-ordered and Wightman correlators computed in the state $|\psi\rangle$.

We refer to our publication, ref. [33], for the proof of this claim. Assuming its correctness, the SvR relations for two-point correlators, eq. (5.27), become

$$\langle \mathcal{T}\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = (2\nu)^2 \lim_{z,z'\to 0} (zz')^{-\Delta_+} G_F(x,z;y,z')\ , \tag{5.30a}$$

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = (2\nu)^2 \lim_{z,z'\to 0} (zz')^{-\Delta_+} G_+(x,z;y,z')\ , \tag{5.30b}$$

$$\langle \mathcal{O}(x_2)\mathcal{O}(x_1)\rangle = (2\nu)^2 \lim_{z,z'\to 0} (zz')^{-\Delta_+} G_-(x,z;y,z')\ , \tag{5.30c}$$

$$\langle \overline{\mathcal{T}}\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = (2\nu)^2 \lim_{z,z'\to 0} (zz')^{-\Delta_+} G_{\overline{F}}(x,z;y,z')\ , \tag{5.30d}$$

which are the relations postulated by the BDHM dictionary, eq. (5.7). Consequently, the two prescriptions are equivalent for the set of operators we considered.

5.4 Summary and outlook

In this chapter we studied different versions of the AdS/CFT dictionary for computing out-of-equilibrium two-point correlation functions. In the first version, developed by Skenderis and van Rees in refs. [30, 31], one constructs a holographic version of the Schwinger-Keldysh generating functional. This procedure amounts to calculating the on-shell action for solutions of the bulk equations of motion in a spacetime that is obtained by gluing together Euclidean and Lorentzian spacetimes to construct the Schwinger-Keldysh contour. In this formalism, correlation functions are obtained by taking functional derivatives of the on-shell action. The second version of the AdS/CFT dictionary we discussed was a non-equilibrium version of the BDHM prescription [27]. In this dictionary, boundary correlation functions are obtained from bulk correlation functions with the operator replacement $\mathcal{O}(x) \to \lim_{z\to 0} z^{-\Delta_+}\Phi(x,z)$. In section 5.3 we explicitly showed that the two dictionaries are equivalent on the level of two-point functions for free bulk scalar fields by showing that the bulk-to-boundary propagators entering the BDHM dictionary satisfy all the equations of motion and boundary conditions obeyed by the propagators entering the bulk solution in the SvR prescription. Since a solution of the SvR prescription is uniquely fixed by the boundary conditions, the two dictionaries result in the same two-point correlation functions in the boundary CFT. We believe that the equivalence might hold beyond two-point functions, but we have not constructed a proof of this statement. One approach might be to generalise the path integral approach of ref. [253], which proved the equivalence for Euclidean correlation functions. We leave this problem for future work.

Chapter 6

Two-point correlators in holographic far-from-equilibrium dynamics

6.1 Introduction and summary

The AdS/CFT duality provides a novel tool for the study of non-equilibrium real-time dynamics of strongly coupled QFTs in terms of semiclassical gravity. An interesting question to study in non-equilibrium QFT is how fast, if at all, field theory observables relax towards thermal equilibrium from a non-equilibrium initial state. This process is called thermalisation. The time scale at which a given quantity approaches the value that it assumes in a thermal state (with temperature determined by the energy of the initial non-equilibrium state) is called the thermalisation time associated with the observable. Using the AdS/CFT dictionary, eq. (2.33), one-point functions of boundary field theory operators can be extracted using the asymptotics of the classical bulk solutions as described in subsection 2.2.4. Studying the dynamics of the one-point function of the stress-energy tensor in non-equilibrium states has lead to interesting results in the dual field theory, such as the fast thermalisation of classes of out-of-equilibrium initial conditions and early applicability of a hydrodynamic description in strongly coupled $\mathcal{N} = 4$ SYM at large N [24–26]. The dynamics of scalar one-point functions acting as a superfluid order parameter was studied holographically after a quench across the superfluid phase transition in refs. [259–263]. Turbulence and vortex formation of strongly coupled superfluids was explored in refs. [264, 265]. It has been noted that expectation values of these local operators approach their thermal values with rates dictated by the lowest quasinormal mode of the corresponding bulk field [260, 266]. Phrased in field theory language, one-point functions thermalise on the same time scale for far-from-equilibrium initial states and for small perturbations of thermal equilibrium.

The one-point functions are expectation values of physical quantities in the quantum state in question. In quantum mechanical systems, the measured values of the observables fluctuate around the expectation value. These fluctuations are quantified by higher-point correlation functions. For example, the temperature one measures in a subregion of a system has fluctuations that can be quantified by the stress-tensor two-point function [267]. More concretely, two-point functions determine, for example, particle creation rates such as the photon creation rate of the quark-gluon plasma, studied in the holographic context in refs. [268, 269]. As an even simpler example, the two-point function of an operator $\mathcal{O}$ quantifies the response of a detector coupled to the operator $\mathcal{O}$, as is familiar, for example, from the Unruh effect [270]. From the point of view of out-of-equilibrium physics in condensed matter systems, one can even directly measure real space two- and multi-point correlation functions and how they approach their thermal values under time evolution in ultracold atomic gases [271, 272]. For these reasons, multi-point correlation functions are interesting quantities to study.

The thermalisation time scale of non-local observables such as two-point functions of local operators, spacelike Wilson loops and entanglement entropy [273–278] depends crucially on the length scale l associated with the observable, such as the distance between the points of the two point correlation function. In many cases the thermalisation time increases linearly with the length scale, $t_{therm} \propto l$. Thus, it seems that the thermalisation time scale of the non-local observables behaves very differently from that of local observables. Our main claim in this chapter is that there is actually no difference between the thermalisation time scales of local and nonlocal observables, at least for two-point correlation functions in momentum space. We will provide evidence that the two-point functions, after being Fourier transformed with respect to their spatial coordinates, approach the thermal two-point functions with a rate set by the lowest quasinormal mode in thermal equilibrium. This result suggests a unified picture of thermalisation time scales being determined by quasinormal modes.

Our analysis mainly focuses on AdS_3-Vaidya spacetime,

$$\mathrm{d}s^2 = g_{mn}\mathrm{d}x^m\mathrm{d}x^n = \frac{L^2}{z^2}\Big[-\big[1-\Theta(v)(2\pi T_f)^2z^2\big]\,\mathrm{d}v^2 - 2\mathrm{d}v\,\mathrm{d}z + \mathrm{d}x^2\Big]\,, \tag{6.1}$$

which provides a simple example of a collapsing spacetime modelling a homogeneous quench in the dual field theory, which can be thought of as living on the boundary at $z = 0$. It has been argued to be a reasonably good approximation to many more realistic out-of-equilibrium states [279–282]. The coordinate v in eq. (6.1) is a null time coordinate that reduces at the boundary to the boundary theory time coordinate denoted by t. For $v < 0$ the AdS_3-Vaidya geometry reduces to AdS_3 and for $v > 0$ to the BTZ black hole. In the dual field theory, the spacetime (6.1) corresponds to initialising the field theory in the vacuum state at times $t < 0$. At $t = 0$, the field theory is kicked out of equilibrium which is manifested on the AdS side by a shell of matter that falls along a lightlike geodesic and collapses into a black hole. The Hawking temperature of the final black hole state is denoted as T_f and can be identified with the field theory temperature once thermal equilibrium has been reached at late times. In Vaidya spacetime, the expectation value of the energy momentum tensor becomes immediately thermal after $t = 0$, while the two-point correlation functions take more time to thermalise. Thus, even though the black hole forms suddenly at $v = 0$ in the null coordinate system, non-local observables in the field theory have memory of the non-thermal initial state.

A question worth addressing is why one should consider two-point correlation functions at all. As a motivation, we recall some of the applications of Wightman two-point functions. Firstly, while one-point functions in a quantum system give average values of observable quantities, Wightman two-point functions quantify fluctuations of the value of the observable in question.[1] Secondly, Wightman two-point functions quantify particle production and absorption rates. For example, the photon production rate of the quark-gluon plasma is proportional to the current operator Wightman two-point function at leading order in the electromagnetic coupling [146, 268, 269]. Thirdly, an occupation number for out-of-equilibrium physics in the context of holography was introduced in ref. [254]. The computation of this quantity requires the knowledge of the Wightman

[1]For example, the measured values of an observable A has the variance $\Delta A^2 = \langle A^2\rangle - \langle A\rangle^2$.

two-point function.[2] The definition was inspired by the fluctuation-dissipation relation in a thermal state and has already been used earlier in the context of non-equilibrium quantum field theory [283]. It is well-defined even out of equilibrium and without well-defined particle states. We will calculate this quantity in AdS_3-Vaidya and study how it approaches the thermal Bose-Einstein distribution. Finally, one can even directly measure real space two- and multi-point correlation functions and how they approach their thermal values under time evolution in ultracold atomic gases [271, 272].

This chapter, which is based on ref. [34], is organised as follows: in section 6.2 we briefly explain how scalar two-point functions can be calculated by solving the bulk Klein-Gordon equation and using the BDHM dictionary. In section 6.3 we present numerical results for the boundary theory Wightman function in AdS_3-Vaidya spacetime for a scalar field corresponding to an operator with scaling dimension $\Delta = 3/2$ and show numerical evidence for the quasinormal decay of the correlation functions towards their thermal values. In section 6.4 we provide an explanation for the quasinormal mode decay that is expected to apply to AdS-Vaidya spacetimes in higher dimensions. We substantiate this explanation by a concrete computation in the case of AdS_3-Vaidya. In section 6.5 we extract an effective occupation number from the boundary Wightman function and study how it approaches the thermal Bose-Einstein distribution. In section 6.6 we present our conclusions.

6.2 Numerical calculation of the Wightman function

We will study two-point correlation functions of a minimally coupled scalar field Φ with action (5.9) and mass $m^2 = -3/4$ in the background defined by the metric (6.1). In the BDHM version of the AdS/CFT dictionary [27, 28], which we introduced in subsection 5.2.2, the boundary theory Wightman function G_+^{CFT} is given by

$$G_+^{\text{CFT}}(x_1, x_2) = 2\pi \lim_{z_1,z_2\to 0} (z_1 z_2)^{-\Delta_+} G_+(x_1, z_1; x_2, z_2) \,, \tag{6.2}$$

[2] Originally in [254], the occupation number was written in terms of the Feynman and the retarded correlator, but the same formula can be written in a simpler way in terms of the Wightman function.

where the correlator on the right hand side is the bulk Wightman function from eq. (3.11). Another version of the out-of-equilibrium dictionary in AdS/CFT is the Skenderis-van Rees dictionary [31], where one constructs a bulk spacetime for the Schwinger-Keldysh contour. We showed in chapter 5 that these two dictionaries are identical at the level of scalar two-point functions. Thus, it does not matter which one we use here and we choose the one more convenient to us.

The bulk Wightman function satisfies the Klein-Gordon equation (5.12a), which we restate here,

$$(\Box_{1/2} - m^2) G_+(x_1, z_1; x_2, z_2) = 0 \,, \tag{6.3}$$

with respect to both of its arguments. The initial data for solving these equations is fixed by the correlations in the initial state of the scalar field. In AdS-Vaidya spacetime, a natural initial state for the bulk scalar is the AdS ground state. This corresponds to the dual field theory being prepared in the vacuum state. The main difficulty in solving eq. (6.3) is that the correlators have singularities at short Lorentzian distances. A numerical strategy for solving the differential equations, based on the method of Green's functions, was presented in detail in our paper, ref. [33]. Here, we will introduce a simpler way of solving the problem. We have compared the two methods and find that they give the same results within the limits of numerical accuracy.

In what follows, we will consider the bulk- and boundary-theory two-point function after performing a Fourier transform with respect to the boundary spatial coordinate x,

$$G_+(v_1, z_1; v_2, z_2; k) = \int_{-\infty}^{\infty} \mathrm{d}x \, e^{ikx} \, G_+(v_1, x_1, z_1; v_2, x_2, z_2) \,, \tag{6.4}$$

as allowed by spatial translation invariance of both the theory and the initial state. Moreover, we consider the subtracted Wightman function

$$\delta G_+(v_1, z_1; v_2, z_2; k) = G_+(v_1, z_1; v_2, z_2; k) - G_+^{\text{th}}(v_1, z_1; v_2, z_2; k) \,, \tag{6.5}$$

where G_+^{th} is the Wightman function in the black hole background in the Hartle-Hawking state which reduces to the thermal Wightman function on the boundary. Taking the boundary limit as in eq. (6.2), the subtracted correlator δG_+ measures the difference of

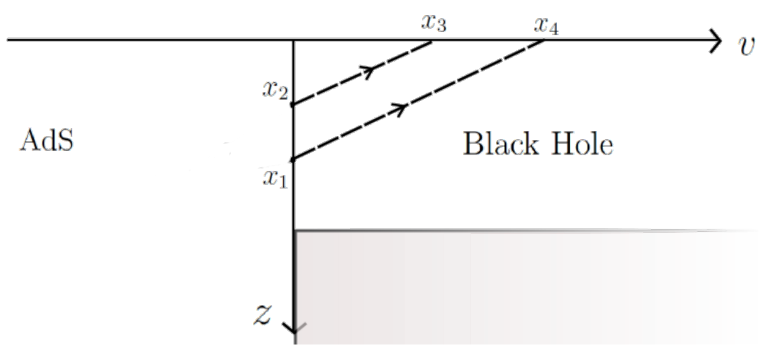


Figure 6.1: The propagation formula (6.9) visualised in diagrammatic form. The grey shaded area is the interior of the apparent horizon. The initial data is given on the $v = 0$ surface and is propagated into the black hole region using retarded propagators denoted by dashed lines. As we are ultimately interested in boundary theory propagators, we take the points x_3 and x_4 to be located on the boundary.

G_+^{CFT} from its thermal value. We have to solve eq. (6.3) with the initial condition that the correlator at early times (that is, for $v_1 < 0$ and $v_2 < 0$) is given by the AdS vacuum Wightman function since the initial state we are interested in is the AdS_3 ground state for the scalar field. The subtracted correlator δG_+ is finite on the $v = 0$ surface even though both G_+^{AdS} and G_+^{BTZ} have lightcone divergences.[3] Therefore, $\delta G_\pm(0, z_1; 0, z_2; k)$ can be used as initial data for the computation of the time evolution of $\delta G_\pm(v_1, z_1; v_2, z_2; k)$ in the Vaidya background as explained in subsection 5.2.3.

The equations of motion (6.3) can be solved by using the propagation with Green's functions, eq. (5.13), for both of the arguments of δG_+, leading to

$$\begin{aligned}\delta G_+(v_2, z_2; &v_1, z_1; k) \\ &= \int \mathrm{d}z\, \mathrm{d}z'\, G_R^{\mathrm{BTZ}}(v_2, z_2; 0, z'; k) G_R^{\mathrm{BTZ}}(v_1, z_1; 0, z; k) \delta\tilde{G}_+(z', z, k)\ . \end{aligned} \tag{6.6}$$

We introduced the notation

$$\delta\tilde{G}_+(z', z, k) \equiv \left(\frac{1}{z^2 z'^2} - \frac{2\partial_z}{z z'^2} - \frac{2\partial_{z'}}{z^2 z'} + \frac{4\partial_z \partial_{z'}}{z z'}\right) \delta G_+(0, z'; 0, z; k)\ , \tag{6.7}$$

which comprises all information about the initial state entering eq. (6.6). Taking the boundary limit (6.2) and using the definition of the bulk-to-boundary propagator G_R^{bb},

$$G_R^{bb}(v; v', z'; k) = \sqrt{2\pi} \lim_{z \to 0} z^{-\Delta} G_R^{\mathrm{BTZ}}(v, z; v', z'; k)\ , \tag{6.8}$$

[3] Both the AdS and the BTZ Wightman two-point functions are known analytically and can be found in our paper [34].

the boundary CFT correlator becomes

$$\delta G_+^{\mathrm{CFT}}(v_2; v_1; k) = \int \mathrm{d}z\, \mathrm{d}z'\, G_R^{bb}(v_2; 0, z'; k) G_R^{bb}(v_1; 0, z; k) \delta \tilde{G}_+(z', z, k) , \tag{6.9}$$

where the bulk-to-boundary propagator is given by

$$\begin{aligned} &G_R^{bb}(v; 0, z_1; k) \\ &= \sqrt{2\pi} \Bigg[\left(1 + \frac{k^2}{\pi^2 T_f^2}\right) \frac{z_1^{\frac{3}{2}}}{8} \Theta\left(z_a^* - z_1\right) \\ &\qquad \times {}_2F_1\left(\frac{3}{4} - \frac{ik}{4\pi T_f}, \frac{3}{4} + \frac{ik}{4\pi T_f}, 2, 1 - (\cosh v - z_1 \sinh v)^2\right) \\ &\qquad - \frac{\sqrt{z_a^*}}{1 + \cosh v} \delta\left(z_a^* - z_1\right) \Bigg] . \end{aligned} \tag{6.10}$$

with

$$z_a^*(v_2 - v_1, z_2) = \tanh \frac{v_2 - v_1}{2} , \tag{6.11}$$

marking the location of the lightcone. Figure 6.1 illustrates eq. (6.9), which we can integrate numerically while evaluating the δ-function contributions analytically. The resulting one- and two-dimensional integrals are performed using Mathematica's NIntegrate.[4]

6.3 Thermalisation of the Wightman function

The solid blue curve in figure 6.2 shows how the subtracted Fourier-transformed boundary Wightman function

$$\delta G_+^{\mathrm{CFT}} = G_+^{\mathrm{CFT}} - G_+^{\mathrm{CFT(thermal)}} , \tag{6.12}$$

with both of its time arguments t_1 and t_2 are set equal and then evolved forwards. The spatial momentum is chosen to take the value $k = 2\pi T_f$. Similar results have been obtained for other values of k. The Wightman function approaches thermal equilibrium in a damped oscillating fashion as can be observed in figure 6.2. We have found that the damped oscillation is well approximated by

$$\delta G_+^{\mathrm{CFT}}(t, t; k) \approx \Big(A_1 \cos(2kt + \alpha_1) + B_1\Big) e^{-4\pi t T_f \Delta} , \tag{6.13}$$

[4]Numerical convergence of the method was studied in our paper, ref. [33].

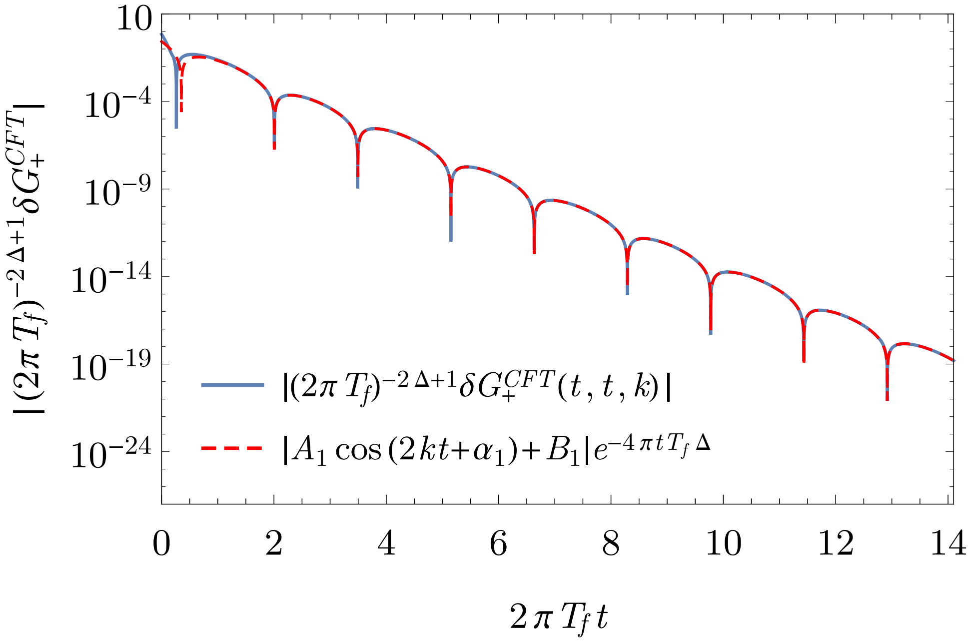


Figure 6.2: The boundary theory equal-time Wightman function as a function of time with the thermal Wightman function subtracted, as defined in eq. (6.12). The blue solid curve correspond to the value of the correlator while the red dashed curve is a fit to the lowest quasinormal mode of the BTZ black hole in the form (6.13). We have fixed $k = 2\pi T_f$.

where (A_1, B_1, α_1) are constants that depend on the value of k. The red dashed curve in figure 6.2 shows a fit of the form (6.13) together with the numerically calculated boundary theory Wightman function (blue solid curve), with (A_1, B_1, α_1) chosen in order to obtain a good fit at late times. The exponential decay rate and oscillation frequency in eq. (6.13) are determined by twice the lowest BTZ quasinormal mode, $\omega_*^{(0)} = \pm k - 2i\pi T_f \Delta$.

6.4 Quasinormal decay in AdS-Vaidya

In this section, we present an argument for the appearance of the quasinormal decay of the Wightman function (6.13) that we observe in the numerical calculation in section 6.3. The argument consists of two steps presented in the subsections 6.4.1 and 6.4.2. First, in subsection 6.4.1 we remind the reader that smooth scalar field configurations in AdS-Vaidya spacetime are expected to decay for $v > 0$ with a rate set by the lowest quasinormal mode of the BTZ black hole. For the case of AdS_3-Vaidya and $\Delta = 3/2$, we show this explicitly. We point out which parts of this argument apply to higher dimensions and which parts lack a generalisation to higher dimensions. In higher-dimensional cases, the quasinormal decay of smooth initial data is still expected to hold based on numerical simulations (see e.g. ref. [284]). As a second part of the argument, in subsection 6.4.2 we assume the quasinormal decay of smooth initial data and show how this implies the observed quasinormal decay of the Wightman function.

6.4.1 Quasinormal decay of smooth scalar field configurations

Consider solving the Klein-Gordon equation of motion $(\Box - m^2)\phi = 0$ in the AdS-Vaidya background. Upon Fourier transforming with respect to the boundary spatial coordinates, the scalar field becomes a function of the variables (v, z, k). Given the profile $\phi(v = 0, z, k) = \phi_0(z, k)$ of the solution at time $v = 0$, the solution at a later time $v > 0$ is given by

$$\phi(v, z, k) = \int_{v'=0} \mathrm{d}z' \, \phi_0(z', k) \overleftrightarrow{D}^{v'} G_R(v, z; v', z'; k) \; , \tag{6.14}$$

where $G_R(v, z; v', z'; k)$ is the retarded propagator in the static black hole background and $D^v = \sqrt{-g} g^{vz} \partial_z$. The reason why we can use the retarded propagator of the static black hole is that the retarded propagator in the $v \geq 0$ region does not depend on the form of the spacetime for $v < 0$. We refer the reader to ref. [33] for a detailed explanation of this point.

Since we are interested in boundary theory quantities in the end, we consider the boundary limit $\phi^b(v, k) = \sqrt{2\pi} \lim_{z \to 0} z^{-\Delta} \phi(v, z, k)$. In this limit, eq. (6.14) becomes

$$\phi^b(v, k) = \int_{v'=0} \mathrm{d}z' \, \phi_0(z', k) \overleftrightarrow{D}^{v'} G_R^{bb}(v; v', z'; k) \; . \tag{6.15}$$

When the two points $(0, v)$ and (z', v') have a large timelike separation, the retarded propagator decays exponentially according to the quasinormal modes $\omega_*^{(n)}$ of the black hole background,

$$G_R^{bb}(v; v', z'; k) \approx \sum_n c_n(z') e^{-i\omega_*^{(n)}(v - v')} \; . \tag{6.16}$$

This follows from a definition of the quasinormal modes as the poles of the retarded propagator in Fourier space. Furthermore, the leading term in eq. (6.16) at late times is given by the lowest quasinormal mode

$$G_R^{bb}(v;v',z';k) \approx c_0(z')e^{-i\omega_*^{(0)}(v-v')} + c.c.\ , \tag{6.17}$$

where the complex conjugate term follows from the fact that the lowest quasinormal mode is degenerate with the quasinormal mode $-(\omega_*^{(0)})^*$.

For the case of the AdS_3-Vaidya spacetime, G_R^{bb} is the BTZ retarded bulk-to-boundary propagator, which is analytically known for a $\Delta = 3/2$ scalar field and given in eq. (6.10).[5] The lowest quasinormal mode contribution is given by

$$G_R^{bb}(v;v',z';k) \approx \left(\frac{(4\pi T_f z')^{\frac{3}{2}}\Gamma\left(-\frac{ik}{2\pi T_f}\right) e^{-ik(v-v')}}{(1-2\pi T_f z')^{\frac{3}{2}+\frac{ik}{2\pi T_f}}\Gamma\left(-\frac{1}{2}-\frac{ik}{2\pi T_f}\right)} + c.c. \right) e^{-\frac{3}{2}2\pi T_f(v-v')}\ . \tag{6.18}$$

A shortcoming of the quasinormal mode expansion (6.16) is that it is not valid when the points $(0,v)$ and (z',v') are lightlike separated. We will denote the point where this happens as $z' = z_a^*$. At late times, z_a^* is near the black hole horizon $z_a^* \approx 1/(2\pi T_f)$. The way we will proceed is to split the z' integral in eq. (6.15) into a region near the lightcone $z' \in (z_a^* - \zeta, z_a^*)$ and a region away from the lightcone $z' \in (0, z_a^* - \zeta)$, where ζ is taken to be a small positive real number,

$$\begin{aligned} \phi^b(v,k) = &\int_0^{z_a^*-\zeta} \mathrm{d}z' \phi_0(z',k) \overleftrightarrow{D}^{v'} G_R^{bb}(v;v'=0,z';k) \\ &+ \int_{z_a^*-\zeta}^{z_a^*} \mathrm{d}z' \phi_0(z',k) \overleftrightarrow{D}^{v'} G_R^{bb}(v;v'=0,z';k)\ . \end{aligned} \tag{6.19}$$

In the first line of eq. (6.19), the integral covers the region away from the lightcone singularity, in which case we can use the quasinormal mode expansion inside the integral. Thus, the first integral in eq. (6.19) decays at late times with a rate set by the lowest quasinormal mode. However, it is not obvious how the second line of eq. (6.19) behaves at late times. In the appendix of ref. [34] we show that for AdS_3-Vaidya spacetime and $\Delta = 3/2$, the integral near the lightcone can be analytically computed and is shown to decay with a rate set by the lowest quasinormal mode, as long as the initial data $\phi_0(z',k)$

[5] The expansion in quasinormal modes (6.16) in this case is studied in the appendix of our paper, ref. [34], and the first few terms are given explicitly.

is a smooth function of z' near the black hole horizon. Thus, we are led to the late time behavior

$$\phi^b(v,k) \approx a e^{-i(\omega_r + i\omega_i)v} + b e^{-i(-\omega_r + i\omega_i)v} \, , \tag{6.20}$$

where we denote the pair of lowest quasinormal modes as $\omega_*^{(0)} = \pm\omega_r + i\omega_i$.

Showing that the integral near the lightcone singularity in eq. (6.19) decays in time with a rate set by the lowest quasinormal mode also in the case of higher dimensions is out of the scope of this thesis. The most straightforward way to convince oneself that eq. (6.20) is true more generally is to solve the equation of motion of the scalar field numerically. For example, for a massless scalar in the AdS$_5$-Schwarzschild spacetime such a numerical calculation confirming the quasinormal decay of smooth initial data can be found in ref. [284].[6] In what follows, we assume that the quasinormal decay (6.20) holds for smooth initial data. Thus, the rest of this section can be applied to any AdS_d-Vaidya spacetime if this assumption holds. In particular, we have shown that it holds for the AdS_3-Vaidya case for $\Delta = 3/2$.

6.4.2 Quasinormal decay of the Wightman function

Consider the subtracted Wightman correlator (6.12) in the Vaidya spacetime. For $v_1 \geq 0$ and $v_2 \geq 0$, this quantity satisfies the Klein-Gordon equation with respect to both of its arguments. Furthermore, at $v_1 = v_2 = 0$, δG_+ is a smooth function of z_1 and z_2, thus providing smooth initial data for the time evolution. Note that while both G_+^{AdS} and G_+^{th} have singularities at lightlike separation, their difference does not. This is because in a quantum field theory, the leading short distance singularities of correlation functions are independent of the state for any reasonable state. Alternatively, one can check this explicitly in AdS_3 where the functions are known analytically.

In order to obtain the boundary Wightman function (6.2) in the black hole region, we can time evolve $\delta G_+(v = 0, z; v' = 0, z'; k)$ applying eq. (6.20) as it provides smooth

[6] The relevant calcuation in [284] is phrased as solving the linearized Einstein's equations in a certain channel, but the resulting equation can be shown to be equivalent to the massless Klein-Gordon equation.

initial data on the hypersurface of the infalling shell. We can first time evolve $(v = 0, z)$ to (v_1, z_1) to get

$$\lim_{z_1\to 0} z_1^{-\Delta}\delta G_+(v_1, z_1; v' = 0, z'; k) \propto f(z')e^{(\omega_i - i\omega_r)v_1 - ig(z')} + c.c. \;, \tag{6.21}$$

where $f(z')$ and $g(z')$ are some real valued smooth functions of z', and we have taken the limit $z_1 \to 0$ anticipating the fact that we want to obtain the boundary theory correlator in the end. We can take eq. (6.21) as initial data to be evolved forwards in time from $(v' = 0, z')$ to (v_2, z_2), which results in

$$\begin{aligned}
&\lim_{z_1,z_2\to 0}(z_1 z_2)^{-\Delta}\delta G_+(v_1, z_1; v_2, z_2; k) \\
&\qquad \propto e^{(\omega_i - i\omega_r)v_1}\Big(A_2 e^{(\omega_i - i\omega_r)v_2 + i\alpha_2} + B_2 e^{(\omega_i + i\omega_r)v_2 + i\gamma_2}\Big) + c.c. \\
&\qquad = 2e^{\omega_i(v_1+v_2)}\Big(A_2\cos(\omega_r(v_1+v_2) - \alpha_2) + B_2\cos(\omega_r(v_1 - v_2) + \gamma_2)\Big) \;,
\end{aligned} \tag{6.22}$$

where $(A_2, B_2, \alpha_2, \gamma_2)$ are constants that depend on the momentum. Thus, we are lead to conclude that $\delta G_+^{\rm CFT}$ decays with the lowest quasinormal mode in the way we observed from the numerical calculation in eq. (6.13).

6.5 Effective occupation numbers

In this section, we change gear and consider some concrete information that can be extracted from the Wightman functions, namely the occupation numbers in the boundary field theory. A strategy for obtaining an effective occupation number for non-equilibrium systems was presented in the context of holography in ref. [254], inspired by the fluctuation dissipation relation in thermal equilibrium (3.29). The same definition was earlier used in non-equilibrium quantum field theory [283]. Here, we will calculate the effective occupation number in the system at hand, the 1+1 dimensional conformal field theory dual to the AdS_3-Vaidya collapse. The effective occupation number is defined as

$$n_{\rm eff}(\omega, k, t) = -\frac{G_-(\omega, k, t)}{2\,{\rm Im}\,G_R(\omega, k, t)} \;, \tag{6.23}$$

where we have performed a Wigner transform of both the Wightman and the retarded two-point function,

$$G(\omega, k, t) = \int {\rm d}\tau\, e^{i\omega\tau} G(t + \tau/2, t - \tau/2; k) \;, \tag{6.24}$$

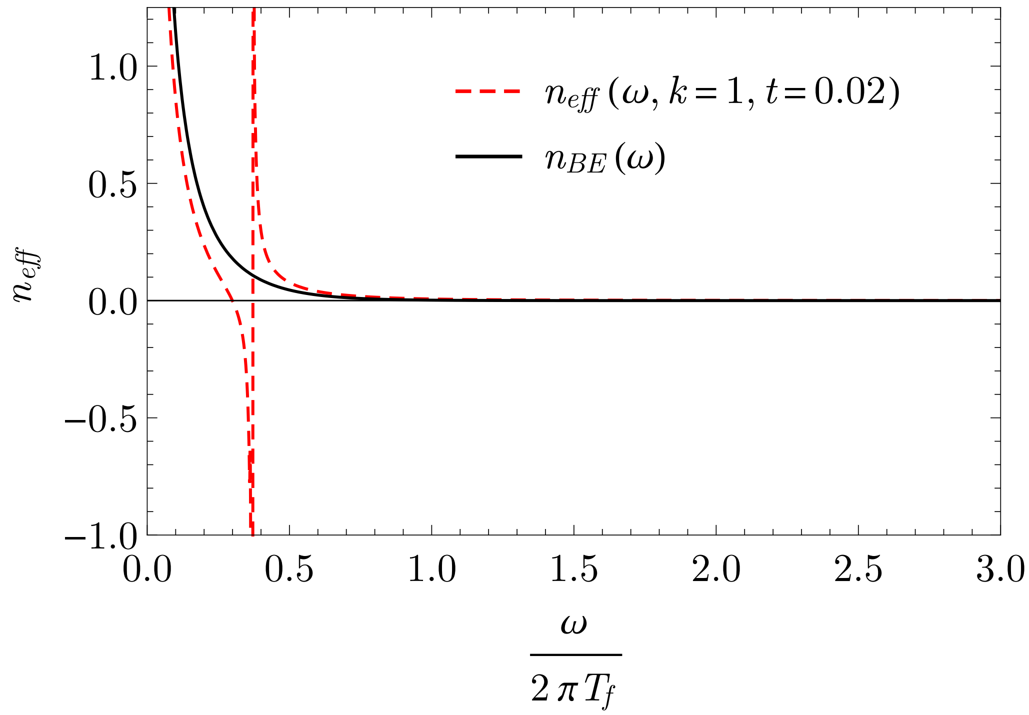


Figure 6.3: The red dashed curve is the effective occupation number n_{eff} for average time $t = 0.02/(2\pi T_f)$ as a function of energy ω and at momentum $k = 2\pi T_f$, while the black solid curve is the Bose-Einstein distribution corresponding to the final temperature T_f.

i.e. a Fourier transform with respect to the time difference $\tau = t_1 - t_2$. Due to the fluctuation-dissipation relation, eq. (3.29), the effective occupation number (6.23) reduces to the Bose-Einstein distribution

$$n_{\text{BE}}(\omega) = \frac{1}{e^{\beta\omega} - 1} \tag{6.25}$$

in thermal equilibrium. Thus, n_{eff} provides a generalisation of the equilibrium occupation number. We calculate the effective occupation number by first computing $G(t + \tau/2, t - \tau/2; k)$ for a range of discretised values of t and τ and then numerically performing the Fourier transform with respect to τ.

As an example, we show the effective occupation number at a fixed average time $t = 0.02/(2\pi T_f)$ in figure 6.3. The first thing to note is that the effective occupation number differs from the thermal one. This is not very surprising as we are considering a non-equilibrium quenched state in the CFT. Secondly, the occupation number has poles as a function of ω. This is not quite as surprising as it might seem at first. Even in thermal equilibrium $n_{\rm BE}$ has a pole at $\omega = 0$. The quantity that is relevant for counting the number of occupied states is the combination $\Omega(\omega)n_{\rm BE}(\omega)$, where $\Omega(\omega)$ is the density of states. For a free boson QFT in thermal equilibrium, one can show that the number of occupied particle states with energy less than E is given by the integral

$$N(E) = \int \mathrm{d}^{d-1}x \int_{k^0\geq 0}^{k^0\leq E} \frac{\mathrm{d}^d k}{(2\pi)^d}\, 2\omega\rho(\omega,k)n_{\rm BE}(\omega)\ , \tag{6.26}$$

where $\rho(\omega,k) = -2\,\mathrm{Im}\,G_R(\omega,k)$ is the thermal spectral function of the free boson.[7] This example suggests that the quantity that appears in counting occupied states is the combination $\rho\cdot n$. In the following, we will therefore plot the combination $\rho(\omega,k,t)n_{\rm eff}(\omega,k,t)$, where ρ is the non-equilibrium spectral function. The time evolution of ρ has been studied before in ref. [254]. The combination $\rho\cdot n_{\rm eff}$ has no poles, as all the poles in $n_{\rm eff}$ coincide with the zeros of ρ. Thus, the positions of the poles in $n_{\rm eff}$ correspond to energies where the effective density of states vanishes.

In figure 6.4 $\rho\cdot n_{\rm eff}$ is plotted for different values of time as a function of ω. At very early times, it starts from being very close to zero. This is indeed expected as the field theory is prepared in the vacuum state, so no states should be occupied. As time progresses, it first oscillates to a negative value, then later becomes positive. At the intermediate times near $t = 0$, $\rho\cdot n_{\rm eff}$ is smaller than the thermal value (black curve) for small ω, while it is larger than the thermal value for large ω. This suggests that after the quench, the low energy states are under-occupied and the high energy states over-occupied as compared to thermal equilibrium. In the regions where it is negative, we do not expect it to have an interpretation in terms of an actual occupation number.

Figure 6.5 shows $\rho\cdot n_{\rm eff}$ as a function of time for fixed values of ω and k. This shows the same basic features as figure 6.4. At late times, $\rho\cdot n_{\rm eff}$ approaches a constant value which coincides with the thermal value. Figure 6.6 shows a plot of the difference of the effective occupation number and the thermal Bose-Einstein distribution. The effective occupation number decays towards the thermal distribution in a form that is well approximated by

$$n_{\rm eff} - n_{\rm BE} \approx e^{-4\pi t T_f \Delta}[A_3\cos(2(\omega+k)t+\alpha_3) + B_3] \tag{6.27}$$

[7] The time evolution of the spectral function in AdS-Vaidya was studied in ref. [285].

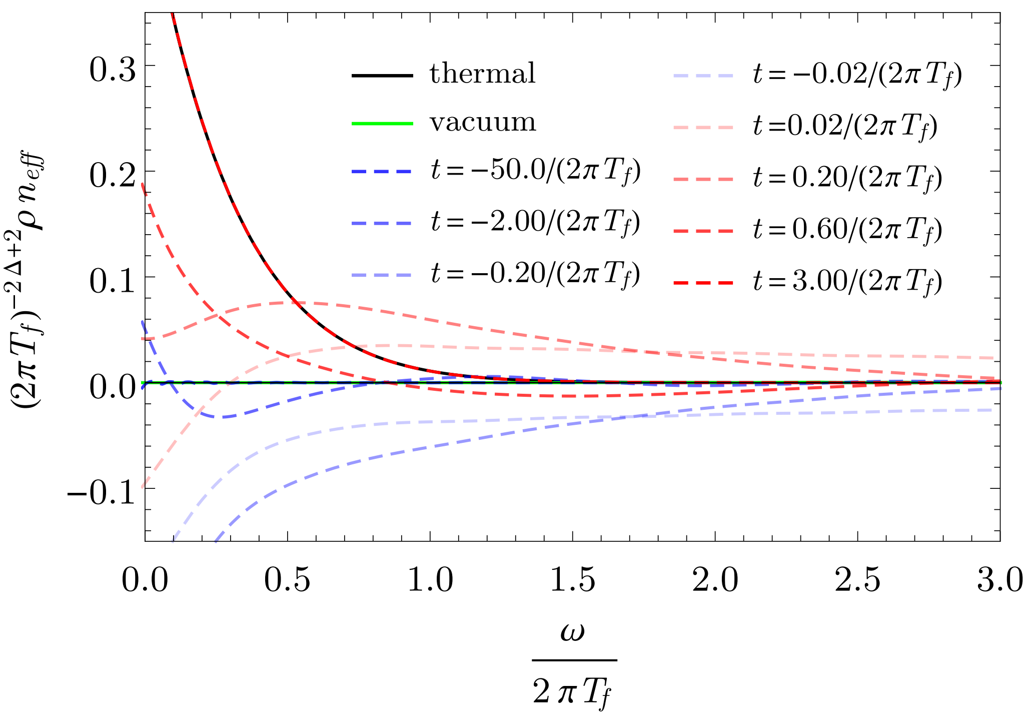


Figure 6.4: The combination $\rho(\omega, k, t) \cdot n_{\text{eff}}(\omega, k, t)$, which counts occupied states, is plotted as a function of ω for different times at momentum $k = 2\pi T_f$. Initially, at large negative times, no states are occupied and $\rho \cdot n_{\text{eff}}$ vanishes. Around the time of the quench, $\rho \cdot n_{\text{eff}}$ oscillates until it finally settles to its thermal value.

at late times, where (A_3, B_3, α_3) are constants, which depend on the frequency and momentum. This is the same exponential rate at which the equal-time Wightman function decays towards equilibrium, the lowest quasinormal mode of the scalar field. In contrast, the oscillation frequency of $n_{\text{eff}} - n_{\text{BE}}$ differs from the real part of the lowest quasinormal mode due to the Wigner transform with respect to relative time.

6.6 Summary and outlook

In AdS_3-Vaidya spacetime, we have presented numerical and analytical evidence that the boundary theory Wightman two-point functions thermalise with the rate set by the lowest quasinormal mode of the corresponding bulk field in the BTZ background. This is not too surprising from the point of view of the bulk problem as the quasinormal modes appear as solutions of the same equation of motion that the bulk Wightman function satisfies. A crucial observation is that the difference between the thermal BTZ bulk two-point function

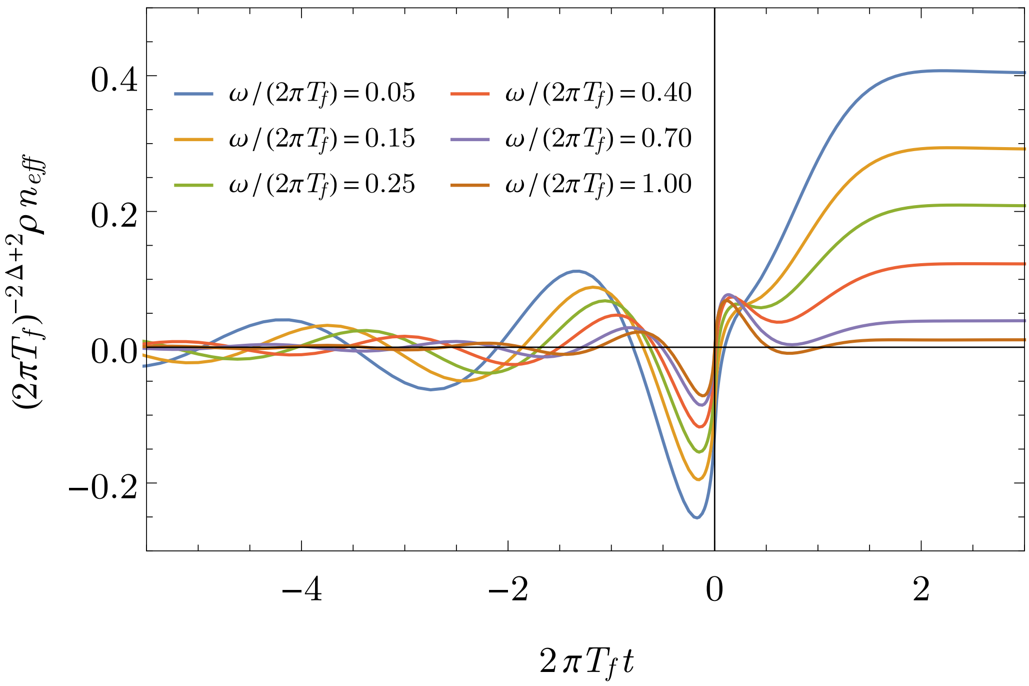


Figure 6.5: The quantity $\rho(\omega, k, t) \cdot n_{\text{eff}}(\omega, k, t)$, which counts occupied states, is plotted as a function of t for different values of ω and fixed $k = 2\pi T_f$. It starts oscillating well before the quench at $t = 0$ since it is sensitive to all times due to the Wigner transform in eq. (6.24). The largest changes happen at the time of the quench before the quantity settles down to its thermal value with $n_{\text{eff}} \to n_{\text{BE}}$ at late times.

and the bulk two-point function in the Vaidya spacetime satisfies the Klein-Gordon equation for $v \geq 0$ while it has no singularities. Therefore, one can expect it to decay at later times like any smooth scalar perturbation of the final black hole, i.e. with the lowest quasinormal mode. This argument is expected to hold also in higher-dimensional AdS$_d$-Vaidya spacetime, and for arbitrary values of the mass m. Other examples of holographic settings where the two-point function, starting far from thermal equilibrium, relaxes towards equilibrium according to the lowest quasinormal mode are studied in refs. [286, 287].

The big picture that has emerged from the study of non-equilibrium one-point functions in holography is that, as long as there is black hole formation in the bulk, the subsequent dynamics of the one-point functions is controlled by the lowest quasinormal mode [260, 266]. Our results suggest that the same picture applies to two-point correlation functions. In the presence of a horizon, they decay towards equilibrium with a rate set by the lowest quasinormal mode. Often, non-equilibrium two-point functions are calculated

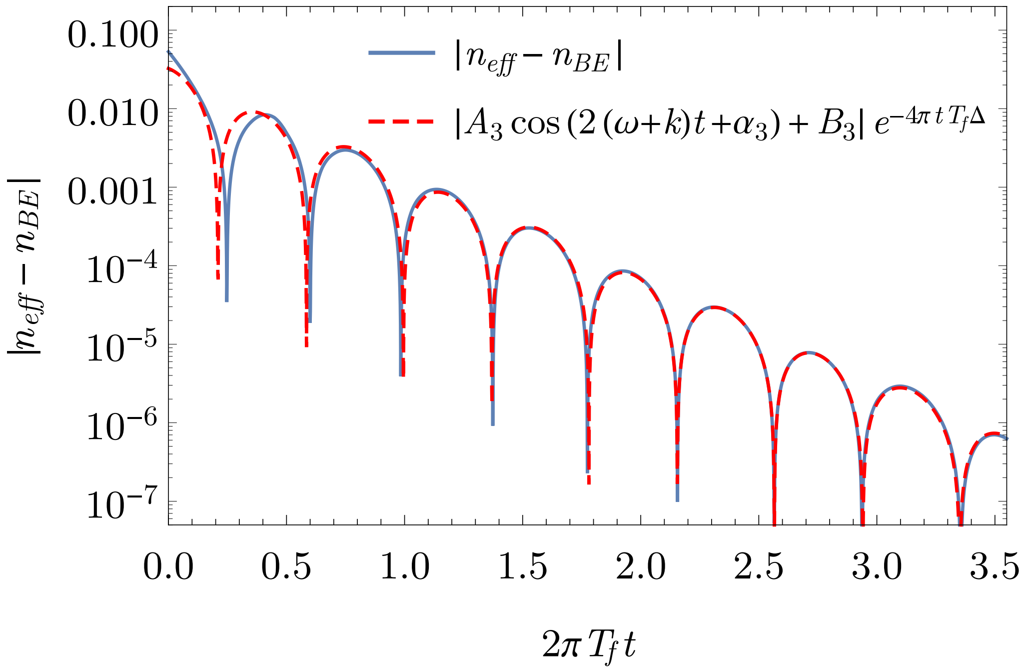


Figure 6.6: The absolute difference between the effective occupation number $n_{\text{eff}}(\omega, k, t)$ and the thermal Bose-Einstein distribution $n_{\text{BE}}(\omega)$ is plotted as a function of t with $k = 4\pi T_f$ and $\omega = 4\pi T_f$. At late times, the difference shrinks exponentially with the time scale set by the lowest quasinormal mode of the black hole background.

in the geodesic approximation, in which case this behaviour is not completely obvious. In ref. [34] however, we found numerical evidence that one indeed also finds the quasinormal decay in the geodesic approximation after performing the Fourier transform to momentum space.

The quasinormal decay we find in Wightman functions should be contrasted with what one finds for the boundary retarded correlation function [254, 288, 289]. The retarded correlation function $G_R(t_1, t_2; k)$ decays exponentially, approximately with the lowest quasinormal mode, when the later time argument t_1 is taken to the black hole region $t_1 > 0$. On the other hand, when both of the points in the retarded correlator are taken to the black hole region, $t_1 > 0$ and $t_2 > 0$, the retarded correlator in the Vaidya spacetime becomes immediately thermal. This is not what happens for the Wightman function, which instead approaches its thermal value exponentially in the time variable $t_1 + t_2$.

Finally, we would like to comment on the general thermalisation pattern in holography in a more speculative vein. In the study of two-point functions in the geodesic approximation (and Wilson loops and entanglement entropy), it has been found that the correlators with short spacelike separation thermalise earlier than the correlators with large spacelike separation. As we have shown, the momentum space two-point functions decay towards thermal equilibrium with a rate that is independent of momentum in the AdS_3-Vaidya example. Naively, one might have expected that the faster thermalisation of the short distance correlators would imply faster thermalisation of the large momentum correlators. The AdS_3-Vaidya example we have studied in this paper shows that this expectation is incorrect. Here, all momenta thermalise on the same time scale of $1/(\pi T_f \Delta)$ set by the lowest quasinormal mode, which in the AdS_3 case is independent of momentum. Furthermore, it was shown in ref. [290] that for momentum space two-point functions in an AdS_5 non-equilibrium black hole, the largest momenta thermalise the slowest. This is the opposite of what one would expect from combining results of the geodesic approximation with the naive expectation that large momentum corresponds to short distances. Our picture of the thermalisation rate being dictated by the lowest quasinormal mode is consistent with the results of ref. [290] since, for AdS_5-Schwarzschild black holes, the imaginary part of the lowest scalar quasinormal mode approaches zero as $k^{-1/3}$, signalling slow thermalisation of the large momentum correlator. We believe that the apparent difference from the geodesic approximation results could follow from the incorrect expectation that large momentum always corresponds to short distance in the correlation function.[8] We leave a detailed study of this problem for future work.

[8] In a Fourier transform of a function $f(x)$, the high momenta are of course sensitive to the short-scale features of the function $f(x)$. But when one applies this to the two-point function $\int dx\, e^{ikx} \langle \mathcal{O}(x,t_1)\mathcal{O}(0,t_2)\rangle$, the relevant short-scale features need not be located near $x=0$, when there are many scales in the problem. Indeed, from the geodesic approximation one learns that the function $\langle \mathcal{O}(x,t_1)\mathcal{O}(0,t_2)\rangle$ has sharp features at $x \propto t_1 + t_2$, which at late times is certainly not at short distances, but can give a sizeable contribution to the large momentum Fourier transform.

Chapter 7

Conclusions

This thesis explored applications of gauge/gravity duality to strongly coupled quantum field theories in out-of-equilibrium states. In chapter 2 we described how the duality arises in the framework of string theory and discussed the holographic dictionary, which translates between the gauge and gravity sides of the duality. In chapter 3 we introduced two approaches to physics away from thermal equilibrium, the low-energy effective theory of hydrodynamics and the first-principles Schwinger-Keldysh formalism used in non-equilibrium QFT. We explored their overlap in the so-called linear-response regime and showed how Kubo formulae link the low-energy constants of hydrodynamics with correlation functions in the underlying microscopic theory.

In chapter 4 we studied second-order hydrodynamic transport in strongly coupled non-conformal field theories with holographic gravity duals in asymptotically anti-de Sitter space. We first derived new Kubo formulae for five second-order transport coefficients of non-conformal fluids in $(3+1)$ dimensions. We applied them to holographic RG flows induced by scalar operators of dimension $\Delta = 3$. For general background solutions of the dual bulk geometry, we found explicit expressions for the five transport coefficients at infinite coupling and showed that a specific combination, $\tilde{H} = 2\eta\tau_\pi - 2\left(\kappa - \kappa^*\right) - \lambda_2$, always vanishes. We proved analytically that the Haack-Yarom identity $H = 2\eta\tau_\pi - 4\lambda_1 - \lambda_2 = 0$, which is known to be true for conformal holographic fluids at infinite coupling, also holds when taking into account leading non-conformal corrections. The numerical results we obtained for two specific families of RG flows suggest that H vanishes regardless of conformal symmetry. It would be desirable to find a more general proof for the relations

$\tilde{H} = 0$ and $H = 0$ for holographic field theories at strong coupling. This could elevate the two relations to equal footing with the famous $\eta/s = 1/(4\pi)$ result, which holds for a very general class of holographic models.

In chapter 5 we introduced and explored two versions of the AdS/CFT dictionary for computing out-of-equilibrium correlation functions. The SvR prescription constructs a holographic version of the Schwinger-Keldysh generating functional. Correlation functions are obtained by taking functional derivatives of the on-shell action. The BDHM prescription obtains field theory correlation functions from the corresponding bulk correlation functions by extrapolation to the boundary. We outlined the proof that the two dictionaries are equivalent on the level of two-point functions for free bulk scalar fields. The equivalence might hold beyond two-point functions, but we have not constructed a proof of this statement. One approach might be to generalise the path integral approach of ref. [253]. In addition, the proof might be extendable to other fields, such as gauge fields and gravity fluctuations, where the issue of gauge fixing has to be overcome. We leave these problems for future work.

In chapter 6, we presented numerical and analytical evidence that the boundary theory Wightman two-point functions of a free scalar in AdS_3-Vaidya spacetime thermalise with the rate set by the lowest quasinormal mode of the corresponding bulk field in the BTZ background. It was noted previously that, as long as a black hole forms in the bulk, the subsequent dynamics of the one-point function is controlled by the lowest quasinormal mode. Our results suggest that the same applies to two-point correlation functions. In the presence of a horizon, they decay towards equilibrium with a rate set by the lowest quasinormal mode. We expect that a generalisation of our computations to higher dimensions, to more realistic models of gravitational collapse and to interacting fields is a big challenge. In higher-dimensional AdS-Vaidya, the Fourier-transformed retarded correlators in black hole backgrounds are not known analytically. Moreover, our approach to computing the two-point correlators is tailored to the Vaidya setting and it is not clear how to generalise it to other, more realistic, collapsing spacetimes. A different extension of our work would be to see how bulk interactions affect the thermalisation process. It seems feasible that interactions of the bulk field could be treated perturbatively.

Appendix A

A class of non-conformal models

A.1 Sub-leading modes of metric perturbations

The expressions $\Upsilon_j^{(a)}$ appearing in eq. (4.52), which are functionals of the background functions $f(u)$ and $A(u)$, are given by

$$\Upsilon_{(1,1)}^{(2tt)}(u) = \frac{L^2}{4f(u)^2}\left(\frac{1}{u^2\, e^{2A(u)}} - \frac{f'(u)^2}{f_H^2\, e^{2A_H}}\right) , \tag{A.1a}$$

$$\Upsilon_{(1,1)}^{(2zz)}(u) = \frac{L^2}{4u^2\, f(u)\, e^{2A(u)}}\,(2 - f(u)) , \tag{A.1b}$$

$$\Upsilon_{(1,1)}^{(2tz)}(u) = \frac{L^2}{4u^2\, f(u)\, e^{2A(u)}} , \tag{A.1c}$$

$$\Upsilon_{2}^{(1t)}(u) = -\frac{L^2}{4f(u)}\left\{\frac{1}{u^2\, f(u)\, e^{2A(u)}} + \frac{f + 2\,(1-u)\,f' - \log\left(\frac{1-u}{f}\right)\left[\frac{f}{u} + 4\,(1-u)\,A'\,f + (1-u)\,f'\right]}{(1-u)^2\, f_H^2\, e^{2A_H}}\right\} . \tag{A.1d}$$

For any of the expressions in eq. (A.1), the near-boundary expansion of the integrand of the inner integral in eq. (4.54) takes the form

$$w\, f(w)\, e^{4A(w)}\, \Upsilon_j^{(a)}(w) = \pm\frac{A_b\, L^2}{4}\left(\frac{1}{w^2} - \frac{\phi_L^2}{12}\frac{1}{w}\right) + \mathcal{O}(w^0) , \tag{A.2}$$

with the upper sign referring to $a \in \{2tt, 2zz, 2tz\}$, $j = (1,1)$, and lower sign to $a = 1t$, $j = 2$. Consequently, the inner integral in eq. (4.54) can be expanded near the boundary as

$$\int_1^v \mathrm{d}w\, w\, f(w)\, e^{4A(w)}\, \Upsilon_j^{(a)} = \mp\frac{A_b\, L^2}{4}\left(\frac{1}{v} + \frac{\phi_L^2}{12}\log v\right) + c_j^{(a)} + \mathcal{O}(v) , \tag{A.3}$$

with $c_j^{(a)}$ given by

$$c_j^{(a)} = \mp \frac{A_b L^2}{4} + \int_1^0 \mathrm{d}w \left[w\, f\, e^{4A}\, \Upsilon_j^{(a)} \mp \frac{A_b L^2}{4w^2} \left(1 - \frac{\phi_L^2}{12} w\right)\right] , \tag{A.4}$$

which is independent of the radial coordinate v. Expanding the remaining part of the integrand of the outer integral in eq. (4.54) as

$$\frac{1}{v\, f(v) e^{4A(v)}} = \frac{v}{A_b^2} \left(1 + \frac{\phi_L^2}{6} v + \mathcal{O}(v^2)\right) , \tag{A.5}$$

we obtain the following near-boundary expansion of the $K_j^{(a)}$:

$$K_j^{(a)} = \mp \frac{L^2}{4A_b} u \left(1 + \frac{\phi_L^2}{24}\, u \log u\right) + \frac{1}{2A_b^2} \left(c_j^{(a)} \mp \frac{A_b\, L^2\, \phi_L^2}{32}\right) u^2 + o(u^2)\, . \tag{A.6}$$

The sub-leading modes are therefore given by

$$Y_j^{(a)} = \frac{1}{2A_b^2} \left(c_j^{(a)} \mp \frac{A_b\, L^2\, \phi_L^2}{32}\right) , \tag{A.7}$$

which entails eq. (4.55) using the definition of $c_j^{(a)}$ as well as eq. (4.19) for the Hawking temperature.

A.2 Leading backreaction of the scalar on AdS-black branes

In this appendix we derive the perturbative leading-order backreaction of a relevant scalar with scaling dimension $2 < \Delta < 4$ on the AdS_5-black brane background geometry. The results are used in subsection 4.6.1 for the special case of $\Delta = 3$ to compute the leading non-conformal correction to second-order transport coefficients. For general Δ, the scalar potential (4.16) takes the form

$$V(\phi) = -\frac{12}{L^2} + \frac{\Delta\,(\Delta - 4)}{2L^2} \phi^2 + \mathcal{O}(\phi^3)\, . \tag{A.8}$$

At zeroth order in non-conformal corrections, in the case of $\phi = 0$, Einstein's equations (4.20) are solved by the AdS_5-black brane, given by eqs. (4.18) and (4.22). To obtain the first-order correction, we solve the scalar equation of motion, eq. (4.20a), around the black-brane background in linearised form. This yields the regular solution

$$\begin{aligned}\phi(u) = \delta\phi(u) &\equiv \phi_H\, {}_2F_1(1-\Delta/4, \Delta/4; 1; 1-1/u^2) \\ &= \delta\phi_L\, u^{(4-\Delta)/2}\, {}_2F_1(1-\Delta/4, 1-\Delta/4; 2-\Delta/2; u^2) \\ &\quad + \delta\phi_{SL}\, u^{\Delta/2}\, {}_2F_1(\Delta/4, \Delta/4; \Delta/2; u^2)\end{aligned} \tag{A.9}$$

with the leading and sub-leading near-boundary modes, $\delta\phi_L$ and $\delta\phi_{SL}$, given by

$$\delta\phi_L = \phi_H \frac{\Gamma(\Delta/2-1)}{\Gamma(\Delta/4)^2}\,, \qquad \delta\phi_{SL} = \phi_H \frac{\tan(\pi\Delta/4)}{2\pi}\frac{\Gamma(\Delta/4)^2}{\Gamma(\Delta/2)}\,. \tag{A.10}$$

The functions $A(u)$ and $f(u)$, eq. (4.22), remain unmodified at first order in the scalar ϕ since the scalar does not backreact on the background at first order as can be seen from the equations of motion (4.20). At second order, the scalar backreacts on the geometry while it remains unmodified itself.[1] When solving the perturbative problem order-by-order, we have to impose boundary conditions on the scalar at each order. After imposing regularity, one free parameter remains corresponding to the single free physical parameter T/Λ. We choose to hold ϕ_H fixed when going beyond the leading order by imposing that the higher-order corrections to the scalar vanish at the horizon. This however implies that T/Λ receives corrections at each higher order. Alternative boundary conditions could be chosen to fix A_H or T/Λ.

We now turn to the backreaction of ϕ on the background, characterised by the functions $A(u)$ and $f(u)$. We separate $A(u)$ into a non-perturbative part and $\tilde{A}(u)$, which vanishes at the boundary, by writing

$$\begin{aligned}A(u) &= \frac{1}{2}\log\left(\frac{A_b}{u}\right) + \tilde{A}(u) \\ &= \frac{1}{2}\log\left(\frac{A_b}{u}\right) + \int_0^u \mathrm{d}v\, \tilde{A}'(v)\,.\end{aligned} \tag{A.11}$$

The near-boundary mode A_b can be determined from the global solution of ϕ. As we will see in appendix A.3, in terms of the dimensionful coordinate ζ, eq. (A.32), $\phi \sim (\Lambda\zeta)^{4-\Delta}$

[1] More generally, the scalar receives corrections to the linearised solution (A.9) at odd orders in the perturbation while the backreaction of the geometry appears at even orders.

and $A \sim \log(L/\zeta)$ near the boundary, from which we can infer that

$$A \sim \log\left(\Lambda L \phi^{-1/(4-\Delta)}\right) . \tag{A.12}$$

If we switch back to the u-coordinate, this implies

$$A = \frac{1}{2}\log\left(\left(\Lambda\, L\, \phi_L^{-1/(4-\Delta)}\right)^2 u^{-1}\right) + \mathcal{O}(u) , \tag{A.13}$$

such that we can extract the near-boundary mode A_b as

$$A_b = \left(\Lambda\, L\, \phi_L^{-1/(4-\Delta)}\right)^2 . \tag{A.14}$$

Plugging the result back into eq. (A.11) and evaluating $A(u)$ at the horizon $u = 1$, we find that the horizon mode A_H and the Hawking temperature, eq. (4.19), are given by

$$A_H = \log\left(\Lambda\, L\, \phi_L^{-1/(4-\Delta)}\right) + I , \qquad T = \Lambda\left(\frac{f_H\, e^I}{2\pi\, \phi_L^{1/(4-\Delta)}}\right) , \tag{A.15}$$

employing the definition

$$I \equiv \int_0^1 \mathrm{d}u\, \tilde{A}'(u) . \tag{A.16}$$

The relations (A.15) determine A_H and T/Λ in terms of ϕ_L and $\tilde{A}(u)$ and thus, in our perturbative treatment, in terms of the expansion parameter ϕ_H.

We now turn to computing the backreaction of the scalar on the background at quadratic order $\mathcal{O}(\phi_H^2)$,

$$\tilde{A}(u) = \delta A(u) , \qquad f(u) = 1 - u^2 + \delta f(u) , \tag{A.17}$$

where δA and δf capture the deviation from the conformal case. From the equations of motion (4.20b) and (4.20d) we find that δA and δf satisfy

$$\delta A'' + \frac{1}{u}\delta A' = -\frac{1}{6}\left(\delta\phi'\right)^2 , \tag{A.18a}$$

$$\delta f' - \frac{2}{u}\delta f = -4\left(2 - u^2\right)\delta A' - \frac{\Delta\,(4-\Delta)}{12u}\left(\delta\phi\right)^2 - \frac{u\left(1-u^2\right)}{3}\left(\delta\phi'\right)^2 . \tag{A.18b}$$

For the spacetime to remain asymptotically AdS_5, we require that δA vanishes at the boundary. Using this boundary condition, we integrate eq. (A.18a) twice to find

$$\delta A(u) = -\int_0^u \mathrm{d}v\, \frac{1}{v}\int_0^v \mathrm{d}w\, w\frac{1}{6}\left(\delta\phi'(v)\right)^2 \xrightarrow{u\to 0} -\frac{\phi_L^2}{24}\, u^{4-\Delta} , \tag{A.19}$$

where the near-boundary behaviour is in agreement with eq. (4.37a). Using the condition that δf must vanish at the horizon $u = 1$, we can also integrate eq. (A.18b) to obtain

$$\delta f(u) = -u^2 \int_1^u \mathrm{d}v\, \frac{1}{v^2}\Big[4\left(2-v^2\right)\delta A'(v) + \frac{\Delta\,(4-\Delta)}{12v}\,(\delta\phi(v))^2 + \frac{v\,(1-v^2)}{3}\,(\delta\phi'(v))^2\Big]\,. \tag{A.20}$$

From eq. (A.18b) we find that the horizon mode $f_H = -f'(u=1)$ is modified as

$$\begin{aligned}\delta f_H &= \frac{\Delta\,(4-\Delta)}{12}\phi_H^2 + 4\,\delta A'(u=1) \qquad\qquad (A.21)\\ &= \frac{\Delta\,(4-\Delta)}{12}\phi_H^2\left(1 - \frac{\Delta\,(4-\Delta)}{8}\int_0^1 \mathrm{d}v\, v^{-5}\left[{}_2F_1(2-\Delta/4, 1+\Delta/4; 2; 1-1/v^2)\right]^2\right)\,.\end{aligned}$$

The temperature T and entropy density s can be obtained from eq. (A.15),

$$4G_N s = e^{3A_H} = \frac{(\Lambda\, L)^3}{\phi_L^{3/(4-\Delta)}}\left(1 + 3\delta I + \mathcal{O}(\phi_H^4)\right)\,, \tag{A.22a}$$

$$T/\Lambda = \frac{(2+\delta f_H)\,(1+\delta I) + \mathcal{O}(\phi_H^4)}{2\pi\phi_L^{1/(4-\Delta)}}\,, \tag{A.22b}$$

where we used the definition $\delta I \equiv \int_0^1 \mathrm{d}u\, \delta A'(u)$. Through the leading near-boundary mode

$$\phi_L = \delta\phi_L + \mathcal{O}(\phi_H^3) = \frac{\Gamma(\Delta/2-1)}{\Gamma(\Delta/4)^2}\phi_H\left(1 + \mathcal{O}(\phi_H^2)\right)\,, \tag{A.23}$$

the third-order correction to the scalar enters the expressions in eq. (A.22) at the same order as δf_H and δI, such that

$$\left(\frac{\Lambda}{\pi T}\right)^{4-\Delta} = \frac{\Gamma(\Delta/2-1)}{\Gamma(\Delta/4)^2}\phi_H\left(1 + \mathcal{O}(\phi_H^2)\right)\,. \tag{A.24}$$

In contrast, the speed of sound is independent of the sub-leading corrections to ϕ_L and it is given by[2]

$$c_s^2 = \frac{\mathrm{d}\bar{p}}{\mathrm{d}\bar{\epsilon}} = \frac{\mathrm{d}\log T}{\mathrm{d}\log s} = \frac{1}{3}\left[1 - (4-\Delta)\,\delta f_H\right] + \mathcal{O}\left(\phi_H^4\right)\,. \tag{A.25}$$

[2]The speed of sound squared c_s^2 is in agreement with ref. [291], where it was computed using a different radial coordinate. The deviation to the conformal value 1/3 is negative for all Δ, $2 < \Delta < 4$.

A.3 Numerical construction of RG-flow geometries

In this appendix we describe how we numerically construct solutions to the action (4.15) for the two families of scalar potentials introduced in subsection 4.6.2. As proposed in ref. [209], we choose the scalar ϕ as radial coordinate assuming that it increases monotonically from the boundary, where it vanishes, to the horizon, where it takes the value ϕ_H. We define the function $B(\phi)$ via

$$e^{B(\phi)} \equiv \frac{L}{2u}\frac{\mathrm{d}u}{\mathrm{d}\phi}\,, \tag{A.26}$$

such that we can express the metric ansatz (4.18) as

$$\mathrm{d}s^2 = g^{(0)}_{mn}\mathrm{d}x^m\mathrm{d}x^n = e^{2A(\phi)}\left[-f(\phi)\mathrm{d}t^2 + \mathrm{d}\underline{x}^2\right] + \frac{e^{2B(\phi)}}{f(\phi)}\mathrm{d}\phi^2\,. \tag{A.27}$$

Due to a residual scaling symmetry inherited from the UV conformal fixed point, we can set the value of the scalar source to $\Lambda = 1/L$. All physical observables only depend on the dimensionless ratio T/Λ.

In the ϕ-coordinate the equations of motion (4.20) for the gravitational background take the form

$$4\frac{\mathrm{d}A}{\mathrm{d}\phi} - \frac{\mathrm{d}B}{\mathrm{d}\phi} + \frac{1}{f}\left(\frac{\mathrm{d}f}{\mathrm{d}\phi}\right) - \frac{e^{2B}}{f}\left(\frac{\mathrm{d}V}{\mathrm{d}\phi}\right) = 0\,, \tag{A.28a}$$

$$\frac{\mathrm{d}^2A}{\mathrm{d}\phi^2} - \left(\frac{\mathrm{d}A}{\mathrm{d}\phi}\right)\left(\frac{\mathrm{d}B}{\mathrm{d}\phi}\right) + \frac{1}{6} = 0\,, \tag{A.28b}$$

$$\frac{\mathrm{d}^2f}{\mathrm{d}\phi^2} + \left[4\frac{\mathrm{d}A}{\mathrm{d}\phi} - \frac{\mathrm{d}B}{\mathrm{d}\phi}\right]\frac{\mathrm{d}f}{\mathrm{d}\phi} = 0\,, \tag{A.28c}$$

$$6\left(\frac{\mathrm{d}A}{\mathrm{d}\phi}\right)\left(\frac{\mathrm{d}f}{\mathrm{d}\phi}\right) + f\left[24\left(\frac{\mathrm{d}A}{\mathrm{d}\phi}\right)^2 - 1\right] + 2e^{2B}V = 0\,. \tag{A.28d}$$

This system of equations contains the first-order equation (A.28a) for B, two second-order equations (A.28b)–(A.28c) for A and f as well as the first-order constraint (A.28d). As we have observed before using the u-coordinate in eq. (4.21), there is a redundancy in the equations (A.28):

$$\left(\frac{\mathrm{d}}{\mathrm{d}\phi} - 2\frac{\mathrm{d}B}{\mathrm{d}\phi}\right)(\text{A.28d}) = -2f(\text{A.28a}) + \left(48f\frac{\mathrm{d}A}{\mathrm{d}\phi} + 6\frac{\mathrm{d}f}{\mathrm{d}\phi}\right)(\text{A.28b}) + 6\frac{\mathrm{d}A}{\mathrm{d}\phi}(\text{A.28c})\,. \tag{A.29}$$

As the equations of motion (A.28) contain one first-order and two second-order independent equations, there are five integration constants in each of the local near-boundary and hear-horizon expansions.

Near the horizon $\phi = \phi_H$, where A and B are regular and f has a simple pole, the expansions of the fields depends on three near-horizon modes $\left\{A_H, f_H^\phi, \phi_H\right\}$. The expansions are given by

$$A(\phi) = A_H - \frac{1}{3}\frac{V(\phi_H)}{V'(\phi_H)}(\phi - \phi_H) + \sum_{k\geq 2} b_k^{A,\phi}(\phi - \phi_H)^k \ , \tag{A.30a}$$

$$f(\phi) = (\phi - \phi_H)\left[f_H^\phi + \sum_{k\geq 1} b_k^{f,\phi}(\phi - \phi_H)^k\right] \ , \tag{A.30b}$$

$$B(\phi) = \frac{1}{2}\log\left(\frac{f_H^\phi}{V'(\phi_H)}\right) + \sum_{k\geq 1} b_k^B(\phi - \phi_H)^k \ , \tag{A.30c}$$

where all coefficients $b_k^{A,\phi}$, $b_k^{f,\phi}$, b_k^B are fixed in terms of the near-horizon modes and the scalar potential $V(\phi)$ via the equations of motion (A.28). We can relate the mode f_H^ϕ to the corresponding mode f_H in the u-coordinate by plugging the expansion of $\phi(u)$, eq. (4.35c), into eqs. (A.30) and comparing the resulting expansions to the corresponding ones in eqs. (4.35) while, at the same time, ensuring that eq. (A.26) is obeyed. From the near-horizon expansion (4.35c) of $\phi(u)$ and eq. (A.26), we find that

$$f_H = -\frac{L^2\, V'(\phi_H)}{2}\,\frac{e^{B(\phi_H)}}{L} \ . \tag{A.31}$$

To impose the condition that the spacetime be asymptotically AdS_5, it is convenient to switch to a radial coordinate ζ such that the metric (4.18) takes the form

$$\mathrm{d}s^2 = g^{(0)}_{mn}\mathrm{d}x^m\mathrm{d}x^n = e^{2A}\left[-f\,\mathrm{d}t^2 + \mathrm{d}\underline{x}^2\right] + \frac{L^2}{\zeta^2 f}\mathrm{d}\zeta^2 \ . \tag{A.32}$$

In order for the metric to approach AdS_5, eq. (4.17), in the limit of $\zeta \to 0$, A and f have to display the following near-boundary behaviour,

$$A \sim \log\left(\frac{L}{\zeta}\right) \ , \qquad\qquad f \sim 1 \ . \tag{A.33}$$

Recalling that the scalar behaves as $\phi \sim \Lambda\,\zeta$ near the boundary, we find that

$$A = -\log\phi + \log(\Lambda\, L) + o(1) \ , \qquad\qquad f = 1 + o(1) \ . \tag{A.34}$$

Setting the operator source to $\Lambda = 1/L$ and imposing the boundary conditions (A.34) leads to the following near-horizon behaviour for the functions $A(\phi)$, $f(\phi)$ and $B(\phi)$:

$$A(\phi) = -\log\phi + \mathcal{O}(\phi^2)\,,\quad f(\phi) = 1 + \mathcal{O}(\phi^4)\,,\quad B(\phi) = \log\left(\frac{L}{\phi}\right) + \mathcal{O}(\phi^2)\,. \tag{A.35}$$

Plugging the near-boundary expansions of $A(u)$ and $\phi(u)$, eq. (4.37), into eq. (A.34), we observe

$$A_b = \left(\frac{\Lambda\, L}{\phi_L}\right)^2 = \frac{1}{\phi_L^2}\,, \tag{A.36}$$

which is consistent with eq. (A.14) for $\Delta = 3$ and $\Lambda = 1/L$.

To solve the connection problem of finding a global solution to the equations of motion (A.28) with the near-horizon and near-boundary asymptotics determined by (A.30) and (A.35), we employ a method developed in ref. [209]. Defining $G(\phi) \equiv A'(\phi)$, one can show using the equations of motion (A.28) that $G(\phi)$ satisfies a decoupled non-linear second-order equation,

$$\frac{G'(\phi)}{G + V/\left(3V'(\phi)\right)} = \frac{\mathrm{d}}{\mathrm{d}\phi}\log\left(\frac{G'(\phi)}{G(\phi)} + \frac{1}{6G(\phi)} - 4G(\phi) - \frac{G'(\phi)}{G + V/\left(3V'(\phi)\right)}\right)\,. \tag{A.37}$$

Solutions of this equation obeying regularity at the horizon can be constructed numerically. Integrating the equations of motion (A.28) numerically, imposing the boundary conditions (A.35) at the boundary and requiring that $f(\phi)$ vanishes at the horizon, we find the following expressions for $A(\phi)$, $B(\phi)$ and $f(\phi)$:

$$A(\phi) = -\log\phi + \int_0^\phi \mathrm{d}\varphi\left(G(\varphi) + \frac{1}{\varphi}\right)\,, \tag{A.38a}$$

$$B(\phi) = \log\left(\frac{L}{\phi}\right) + \int_0^\phi \mathrm{d}\varphi\left(\frac{G'(\varphi) + 1/6}{G(\varphi)} + \frac{1}{\varphi}\right)\,, \tag{A.38b}$$

$$f(\phi) = \frac{\int_\phi^{\phi_H}\mathrm{d}\varphi\,\exp\left[-4A(\varphi) + B(\varphi)\right]}{\int_0^{\phi_H}\mathrm{d}\varphi\,\exp\left[-4A(\varphi) + B(\varphi)\right]}\,. \tag{A.38c}$$

The single free physical parameter $T/\Lambda = T\,L$ depends on the free parameter ϕ_H.

In section 4.3 we found it more convenient to work in the u-coordinate, eq. (4.18), when dealing with perturbations of the gravitational background. We are therefore interested in finding the global solutions $A(u)$, $f(u)$, $\phi(u)$ in the u-coordinate. We can extract these solutions from the global solutions (A.38) in the following manner. First, we expand the solutions (A.38) near the horizon to find the horizon modes A_H and f_H employing the relation (A.31). Then we use the result in the near-horizon expansions (4.35) and match this near-horizon expansion with the near-boundary expansion (4.37) of $A(u)$, $f(u)$, and $\phi(u)$ at an intermediate value of the radial coordinate, $0 < u < 1$. We can verify that the solution indeed satisfies the correct boundary conditions by checking that the relation (A.36) is satisfied. This should be guaranteed by the fact that we derived the near-horizon solution from the global solution (A.38) of the connection problem.

From eqs. (A.30a) and (A.35), we find

$$G(\phi_H) = -\frac{1}{3}\frac{V(\phi_H)}{V'(\phi_H)}\,, \qquad \phi\, G(\phi) = \phi\, A'(\phi) \xrightarrow{\phi\to 0} -1\,, \tag{A.39}$$

which prove useful in simplifying (A.31) to yield

$$\begin{aligned} f_H &= -\frac{L^2\,V'(\phi_H)}{2}\,\frac{e^{B(\phi_H)}}{L} \\ &= \left(\frac{L^2\,V(\phi_H)}{6G(\phi_H)}\right)\frac{1}{\phi_H}\exp\left\{\lim_{\phi\to 0}\left[\log\left(\frac{-G(\phi_H)}{-G(\phi)}\right)+\log\left(\frac{\phi_H}{\phi}\right)\right]+\int_0^{\phi_H}\mathrm{d}\varphi\frac{1}{6G(\varphi)}\right\} \\ &= -\frac{L^2\,V(\phi_H)}{6}\,\exp\left\{\int_0^{\phi_H}\mathrm{d}\varphi\,\frac{1}{6G(\varphi)}\right\}\,. \end{aligned} \tag{A.40}$$

Using eq. (4.19), the Hawking temperature T and the entropy density s are given by

$$T\,L = \frac{f_H\,e^{A_H}}{2\pi} = -\frac{L^2\,V(\phi_H)}{12\pi}\,\frac{1}{\phi_H}\exp\left\{\int_0^{\phi_H}\mathrm{d}\varphi\left(G(\varphi)+\frac{1}{\varphi}+\frac{1}{6G(\varphi)}\right)\right\}\,, \tag{A.41a}$$

$$4G_N s = e^{3A_H} = \frac{1}{\phi_H^3}\exp\left\{3\int_0^{\phi_H}\mathrm{d}\varphi\left(G(\phi_H)+\frac{1}{\varphi}\right)\right\}\,. \tag{A.41b}$$

Taking the limit $\phi_H \to 0$ and using $V(0) = -12/L^2$, we find the UV asymptotics

$$T\,L \xrightarrow{\phi_H\to 0} \frac{1}{\pi\phi_H}\exp\left\{\lim_{\phi_H\to 0}\left(\int_0^{\phi_H}\mathrm{d}\varphi\,G(\varphi)\right)\right\}\,, \tag{A.42a}$$

$$4G_N s \xrightarrow{\phi_H\to 0} \frac{1}{\phi_H^3}\exp\left\{3\lim_{\phi_H\to 0}\left(\int_0^{\phi_H}\mathrm{d}\varphi\,G(\varphi)\right)\right\}\,. \tag{A.42b}$$

The equation of motion (A.37) has regular singular points at $\phi = 0$ and $\phi = \phi_H$ and thus the integrals in these expressions do not vanish in the limit $\phi_H \to 0$ since $G(\phi)$ does not behave smoothly in this limit. By comparing with the perturbative high-temperature solution, eqs. (A.10) and (A.22), we find

$$T/\Lambda = T\,L \xrightarrow{\phi_H\to 0} \frac{1}{\pi\phi_H}\frac{\Gamma(3/4)^2}{\sqrt{\pi}}\,, \qquad 4G_N s \xrightarrow{\phi_H\to 0} \left(\frac{\Gamma(3/4)^2}{\phi_H\sqrt{\pi}}\right)^3\,. \tag{A.43a}$$

At the end of this appendix, we provide some details on our numerical computations. To compute the integrals (A.38)–(A.41), we used a near-horizon expansion to eleventh order in the region $\phi_H - \phi < 10^{-2}$. A check of the numerical accuracy was performed by comparing the dependence of the near-horizon modes A_H and f_H on ϕ_H^4, which is common to all solutions as it is dictated by the quadratic term of the potential (4.38). We expanded the local near-boundary and near-horizon solutions (4.35) and (4.37) to sixteen orders beyond the boundary mode and determined the boundary modes A_b, f_b, ϕ_L, ϕ_{SL} by inverting the near-boundary expansions and matching them with the near-horizon expansion at $u = 0.5$. The numerical error due to the truncation of the expansion beyond sixteen orders is approximately $0.5^{17} \sim 8 \cdot 10^{-6}$. All solutions we found numerically satisfy the relations (4.58) and (A.36) linking horizon and boundary modes with an error smaller than 10^{-5}. The somewhat tedious inversion of the near-horizon series comes with the great benefit of simplifying the background integrals (4.55) significantly as they now simply involve two integrals over the local near-horizon and near-boundary solutions.

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