New Journal of Physics
The open–access journal for physics

Work extraction from tripartite entanglement

Vincent Viguié[1,2], **Koji Maruyama**[1,3] **and Vlatko Vedral**[4,5]
[1] QOLS, Blackett Laboratory, Imperial College London,
London SW7 2BW, UK
[2] Département de Physique, Ecole Normale Supérieure de Lyon,
69364 Lyon, Cedex 07, France
[3] Laboratoire d'Information Quantique and QUIC, CP 165/59, Université Libre de Bruxelles, Avenue F D Roosevelt 50, 1050 Bruxelles, Belgium
[4] The School of Physics and Astronomy, University of Leeds,
Leeds LS2 9JT, UK
[5] Institut für Experimentalphysik, Universität Wien,
Boltzmanngasse 5 1090 Wien, Austria
E-mail: koji.maruyama@ulb.ac.be

New Journal of Physics **7** (2005) 195
Received 22 April 2005
Published 16 September 2005
Online at http://www.njp.org/
doi:10.1088/1367-2630/7/1/195

Abstract. It has recently been shown that the work extractable from correlated bipartite quantum systems under an appropriate protocol can be used to distinguish entanglement from classical correlation. A natural question is now whether it can be generalized to multipartite systems. In this paper, we devise a protocol to distinguish the GHZ, the W, and separable states in terms of the thermodynamically extractable work under local operations and classical communication, and compare the results with those obtained from Mermin's inequalities.

PII: S1367-2630(05)98843-5
1367-2630/05/010195+11$30.00

Contents

1. Introduction

Finding an efficient separability criterion for multipartite quantum systems and characterizing entanglement have been an important problem in the field of quantum information theory. Since Bell formulated the statistical irregularities that can be seen in the correlations between measurement outcomes from two distant, but entangled systems [1], various criteria and related notions have been discovered in the context of non-local correlations in quantum systems [2]–[5]. Much progress has been made also for the same problem in the multipartite setting. One of the most famous ones may be Mermin's inequality [6], which is a standard multipartite version of Bell's inequalities. Among others, there are some criteria, for example, in terms of the difference between the locally and globally extractable work [7, 8], the witness operator constructed with the Pauli operators [9], the realignment separability [10, 11], and a generalized partial transposition of the density matrix [12]. These criteria are very useful to get insights into multipartite entanglement and each of them has interesting properties in its own right. However, they assume global (or collective) operations on the whole system (or several subsystems), unlike Bell's and Mermin's inequalities.

On the other hand, it has recently been shown that it is possible to devise a protocol to extract more work via entangled systems from a heat bath thermodynamically than can be done from any separable state [13]. This is of interest because a physically useful quantity, locally extractable work (without global operation as in usual experimental setups), can be employed to test the existence of entanglement in bipartite quantum systems, leading to the idea of thermodynamical separability criteria. Throughout this paper, by extractable work we mean the (average) mechanical work one can extract from a heat bath via an ensemble of a physical state under a specific protocol. Thus, unlike in some other literature (such as [7]), it does not necessarily mean the maximum possible extractable work from a given state. Our attempt is to show that the extractable work under our protocol can be used in a separability criterion.

In this paper, we discuss whether the thermodynamical separability criteria can be generalized to multipartite systems, particularly tripartite ones, as even the simplest transition from bipartite to tripartite systems makes our problems much harder. It has been known that, unlike bipartite systems, there are two non-equivalent classes of entanglement, i.e. the GHZ and the W states, when three quantum subsystems have non-local correlations [14]. Suppose that we are given an ensemble of tripartite systems, which is in either the separable or the GHZ or the W states, where the GHZ state [15] can be in general written as

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0\,0\,0\rangle + |1\,1\,1\rangle \tag{1}$$

and the W state,

$$|\mathrm{W}\rangle = \frac{1}{\sqrt{3}}(|0\,0\,1\rangle + |0\,1\,0\rangle + |1\,0\,0\rangle), \tag{2}$$

where $|0\rangle$ and $|1\rangle$ are two orthogonal states of a two-level system, e.g. eigenstates of a Pauli spin operator σ_z. Our task here is to distinguish each of these two ensembles from separable states in terms of extractable work. By extractable work, particularly in the setting of correlated systems, we mean work that can be extracted thermodynamically from local heat baths under local operations and classical communication (LOCC). As our approach is similar to that of Bell's in the sense that we only assume local measurements, we shall compare our thermodynamical separability criterion for tripartite systems with Mermin's inequality.

We also note here that even though our protocol is similar to that for Bell-type inequalities or related entropic inequalities, e.g. the ones shown in [16], ours is essentially different from them because the locality condition, i.e. space-like separated parties, is not considered as a relevant issue in the following discussion. The comparison with inequalities in [16] has been made in detail in [13]. Thus, rather than problems related to the local realism, our primary interest is whether a given state ρ is separable or entangled, i.e. whether ρ can be expressed as $\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B \cdots \otimes \rho_i^C$ or not.

2. The protocol for bipartite correlations

Before discussing the case of tripartite entanglement, we here sketch the protocol shown in [13] for the work-extraction from bipartite quantum systems. If we have an ensemble of two-level systems, either classical or quantum, we can extract work of amount of $kT \ln 2[1 - H(X)]$ from a heat reservoir thermodynamically [7, 17, 18], where k, T and $H(X)$ are the Boltzmann constant, the temperature of the heat bath and the Shannon entropy of a binary random variable X, respectively. The variable X corresponds to the outcome of measurement on the system and $H(X)$ can be written as $H(X) = -p_0 \log_2 p_0 - p_1 \log_2 p_1$, where p_0 and p_1 are the probabilities of the two outcomes. We will set $kT \ln 2 = 1$ for simplicity hereafter, and call the unit of work 'bit'. We will also write 'extracting work from quantum systems' in short, instead of 'extracting work thermodynamically from a heat bath via quantum systems'.

The extractable work from correlated pairs in [13], is a simple generalization of the above case. Suppose that two distant parties, Alice and Bob, have an ensemble of identically prepared pairs of quantum bits (qubits), which is described by a density operator ρ. Firstly, we define the extractable work $\xi_\rho(A(\theta), B(\theta'))$ when Alice and Bob chose θ and θ' as the directions of their (projective) measurement. After dividing the shared ensemble into groups of two pairs, Alice measures one of the two qubits in a group with the projector she chose and informs Bob of the outcome (see figure 1). Bob performs the same on his qubit of the other

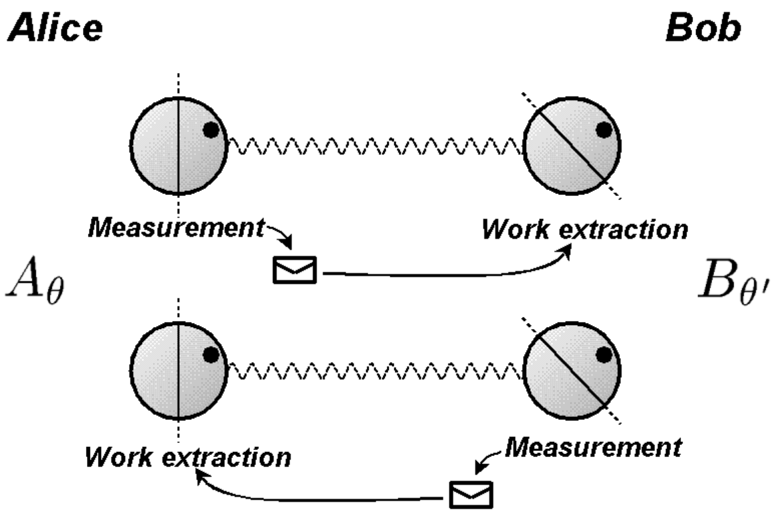

Figure 1. Schematic view of the protocol to extract work from correlated pairs. Dividing the whole ensemble into groups of two pairs, Alice and Bob use $A(\theta)$ and $B(\theta')$ for their measurement and work extraction. For a half of this ensemble, Alice measures her state with $A(\theta)$ and Bob maximizes the extractable work from his side along the direction of θ' by using Alice's measurement results. For the other half, they exchange their roles.

pair in the group. As a result of collective manipulations, they can extract $\xi_\rho(A(\theta), B(\theta')) = 1 - 1/2[H(A(\theta)|B(\theta')) - H(B(\theta')|A(\theta))]$ bits of work per pair at maximum.

Secondly, we consider a quantity $\Xi(\rho)$, which is an average work extractable when we set $\theta = \theta'$ and vary θ continuously over a great circle on the Bloch sphere

$$\Xi(\rho) := \frac{1}{2\pi}\int_0^{2\pi} \xi_\rho(A(\theta), B(\theta))\mathrm{d}\theta. \tag{3}$$

The great circle is the one that maximizes the integral. The integral in equation (3) can be taken over the whole Bloch sphere; nevertheless, it does not change the essential part of our discussion here.

In [13], it was shown that an inequality

$$\Xi(\rho) \leqslant \Xi(|0\,0\rangle) = 0.4427 \tag{4}$$

is a necessary condition for a two-dimensional bipartite state ρ to be separable, that is of the form, $\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B$. The state $|0\,0\rangle$ in the right-hand side (RHS) can be any pure product state $|\psi\psi'\rangle$.

If we integrate ξ_ρ over the whole surface of the Bloch sphere, then the condition (4) becomes

$$\Xi_{\mathrm{BS}}(\rho) \leqslant \Xi(|0\,0\rangle) = 0.2787, \tag{5}$$

where the subscript BS stands for the Bloch sphere. This condition, equation (5), has been shown to be more effective than the standard Bell–Clauser–Horne–Shimony–Holt (Bell–CHSH) inequalities [19] in detecting the inseparability of the two-qubit Werner state [20], which is a state in the form of $\rho_W = p|\Psi^-\rangle\langle\Psi^-| + (1-p)/4 \cdot I$.

3. Protocol for tripartite systems

Let us now consider the case of tripartite quantum systems. By generalizing the protocol in the previous section in a simple manner, we can have a necessary condition for separability, which is of the same form as equation (4), straightforwardly. That is, each one of three parties receives the outcomes of measurements from the other two, and we take conditional entropies such as $H(A(\theta)|B(\theta), C(\theta))$ instead of $H(A(\theta)|B(\theta))$ in the definition of ξ and Ξ. Then, the inequality (4) with $\Xi(|0\,0\,0\rangle)$ on the left-hand side (and a different numerical value) holds for all separable states of the form of $\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B \otimes \rho_i^C$. The proof is essentially the same as that in [13]. Thus, any violation of the inequality implies the existence of entanglement between at least two subsystems. Obviously, this extension can be applied to correlations between larger number of subsystems.

However, such an inequality tells little about the properties of multipartite entanglement. Thus, we would like to find a different protocol and focus on distinguishing the two inequivalent classes of tripartite entanglement: in this paper, we will devise a way to distinguish the GHZ state, the W state and separable states, in terms of the extractable work.

Figure 2 depicts our protocol to extract work from a tripartite system. With three parties (Alice, Bob and Charlie), we consider a protocol in which one extracts work along the direction of the u-axis after receiving information on the outcomes of the other two's measurements along the z and u axis, respectively. Namely, for a subensemble where Alice and Bob measure their qubits and Charlie extracts work from his, the (average) amount of work obtainable is

$$w_{\vec{z},\vec{u}}(\rho) = 1 - H(C(\vec{u})|A(\vec{z}), B(\vec{u})). \tag{6}$$

In this equation, we denote the direction of axis by a unit vector, such as $\vec{z}$. We have chosen two directions, $\vec{z}$ and $\vec{u}$, for two measurements in anticipation that the difference between the GHZ and the W states will be seen by varying the direction of $\vec{u}$, while keeping $\vec{z}$ fixed. We do not lose generality by choosing the direction of the work-extraction to be the same as one of the measurement axes, that is $\vec{u}$ in equation (6), as it is the right choice in detecting singlet-type bipartite entanglement. Also, as we focus on the GHZ and the W states, which are symmetric in terms of A, B and C, we will assume throughout the paper that it is Charlie who extracts work after Alice and Bob make measurements on their subsystems. Otherwise, we need to permute the roles of each party and take an average.

We now show that a quantity below, $W(\rho)$, achieves our task. We assume that the direction $\vec{z}$ is given and fixed for everyone of three parties throughout our discussion. Instead of varying $\vec{u}$ continuously, we average the extractable work over three orthogonal directions in the Bloch sphere, $(\vec{x}, \vec{y}, \vec{z})$, for $\vec{u}$. This is mainly because of our limited resource for numerical computation, but it does not change our main task at all. With only the direction of $\vec{z}$ fixed, those of $\vec{x}$ and $\vec{y}$ cannot be determined. We use ϕ to specify the angle between $\vec{x}$ and a certain predetermined direction in the space. Then, the averaged work can be written as

$$W_\phi(\rho) = \frac{1}{3} \sum_{\vec{u} \in \{\vec{x},\vec{y},\vec{z}\}} w_{\vec{z},\vec{u}}(\rho) \tag{7}$$

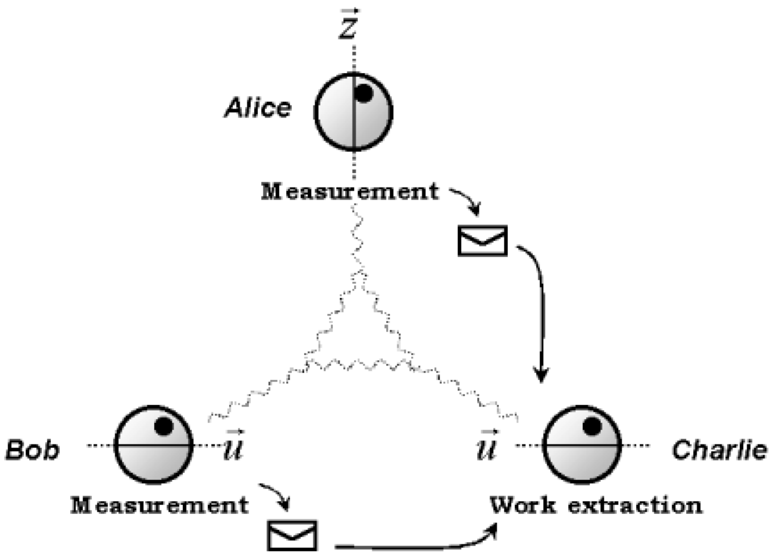

Figure 2. Work extraction from a tripartite system. Alice and Bob measure their particle along the direction of $\vec{z}$ and $\vec{u}$, respectively, where Bob varies the direction of $\vec{u}$, choosing one of the three orthogonal directions $\{\vec{x}, \vec{y}, \vec{z}\}$ at each round. Charlie extracts work along $\vec{u}$ after receiving measurement results from Alice and Bob. They exchange their roles cyclically after three rounds; however, we do not consider the exchange of roles as the states of our interest, here, are symmetric with respect to three parties.

and we take the maximum value of $W_\phi(\rho)$ over ϕ to remove the ϕ-dependence as

$$W(\rho) = \max_\phi W_\phi(\rho). \tag{8}$$

We will analyse how we can use $W(\rho)$ to distinguish three classes of correlations in tripartite systems.

3.1. Product states

With a product state, $|\psi^A\psi^B\psi^C\rangle$, the extractable work depends only on the subsystem from which we extract work, regardless of the measurements on the other two. Thus, $W(|\psi^A\psi^B\psi^C\rangle)$ will be determined solely by the geometric relation between the state of the system and the choice of axes. $W_\phi(|\psi^A\psi^B\psi^C\rangle)$ takes its maximum value of $1/3$ when the Bloch vector representing the state coincides with one of the axes, i.e. $W(|\psi^A\psi^B\psi^C\rangle) \leqslant 1/3$ for a given $\vec{z}$.

3.2. Separable states

We now show that $1/3$ is indeed the upper bound for the extractable work W from any separable state. That is, an inequality,

$$W(\rho) \leqslant \tfrac{1}{3} \tag{9}$$

is a necessary condition for a tripartite state ρ to be separable. This condition is very similar to the thermodynamical separability criterion for bipartite correlations [13]; however, $W(\rho)$ is an average of work over only three directions as in equation (7), while it was over a great circle on the Bloch sphere (or the whole surface of the sphere) in [13]. This results in a difference in the efficiency of detecting weakly entangled states: the condition (9) is less efficient than the criterion in [13], in this sense.

Proof. The conditional entropy in equation (6) can be written as

$$H(C(\vec{u})|A(\vec{z}), B(\vec{u})) = \sum_{i,j=\{0,1\}} p(A^i_{\vec{z}}, B^j_{\vec{u}}) H(C(\vec{u})|A^i_{\vec{z}}, B^j_{\vec{u}}), \tag{10}$$

where $A^i_{\vec{z}}$ ($B^j_{\vec{u}}$) means that Alice (Bob) obtained the outcome i (j) by the measurement along $\vec{z}$ ($\vec{u}$) and $p(A^i_{\vec{z}}, B^j_{\vec{u}})$ is the probability of a joint event of $A^i_{\vec{z}}$ and $B^j_{\vec{u}}$. Omitting the directions of axes for simplicity, we can write equation (10) as

$$H(C|A, B) = \sum_{i,j} p_{ij} H(C|A^i, B^j). \tag{11}$$

If the state ρ is separable, i.e. $\rho = \sum_n p_n \rho^A_n \otimes \rho^B_n \otimes \rho^C_n$, then the density operator for Charlie after Alice and Bob obtained i and j becomes

$$\rho^C_{ij} = \frac{1}{p_{ij}} \sum_n p_n \mathrm{Tr}(P^i_z \rho^A_n) \mathrm{Tr}(P^j_u \rho^B_n) \rho^C_n, \tag{12}$$

where P^i_z (P^j_u) is a projection operator for outcome i (j) along the direction of $\vec{z}$ ($\vec{u}$) and $p_{ij} = \sum_n p_n \mathrm{Tr}(P^i_z \rho^A_n) \mathrm{Tr}(P^j_u \rho^B_n)$. Therefore,

$$\begin{aligned} H(C|A, B) &= \sum_{i,j} H\left(\frac{1}{p_{ij}} \sum_n p_n \mathrm{Tr}(P^i_z \rho^A_n) \mathrm{Tr}(P^j_u \rho^B_n) \mathrm{Tr}(P^0_u \rho^C_n)\right) \\ &\geqslant \sum_n \sum_{i,j} p_n \mathrm{Tr}(P^i_z \rho^A_n) \mathrm{Tr}(P^j_u \rho^B_n) H(\rho^C_n) \\ &= \sum_n p_n H(\rho^C_n) \end{aligned} \tag{13}$$

because of the concavity of the Shannon entropy. If we were sure that we had a tripartite product state, $\rho^A_k \otimes \rho^B_k \otimes \rho^C_k$, the conditional entropy would be

$$H(C^k|A^k, B^k) = H\left(\frac{1}{p^k_{ij}} \mathrm{Tr}(P^i_z \rho^A_k) \mathrm{Tr}(P^j_u \rho^B_k) \rho^C_k\right) = H(\rho^C_k), \tag{14}$$

where $p^k_{ij} = \mathrm{Tr}(P^i_z \rho^A_k) \mathrm{Tr}(P^j_u \rho^B_k)$.

Since $W(\rho_k) = \max\,[1 - (1/3) \sum_{\vec{u}} H(C^k|A^k, B^k)] \leqslant 1/3$ for all separable states ρ_k, combining equations (13) and (14), we have

$$W(\rho) = \max\left[1 - \frac{1}{3} \sum_{\vec{u}} H(C|A, B)\right] \leqslant \max\left[1 - \frac{1}{3} \sum_n \sum_{\vec{u}} p_n H(C^n|A^n, B^n)\right] \leqslant \frac{1}{3}, \tag{15}$$

thus equation (9). As $W(\rho)$ still has a dependence on the direction of $\vec{z}$, even a highly entangled state may not violate the inequality (9); however, once it is violated it is surely a manifestation of entanglement. □

3.3. GHZ and W states

Let us discuss how we can distinguish the GHZ, and the W states, using the above criterion. To this end, we make use of the remaining variable, i.e. the direction of $\vec{z}$.

The extractable work from a GHZ state can reach 1 bit, which is the maximum possible violation of the inequality (9). This maximum is attained when we choose $\vec{z}$ to be perpendicular to $|0\rangle$ in the Bloch sphere (such as $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$), the state after Alice's measurement still has maximum (bipartite) entanglement. Because of the perfect correlation between B and C, Charlie can extract maximum work, 1 bit per one set of three subsystems.

We have obtained the value of the minimum extractable work from the GHZ as 0.1619 bits numerically. This occurs when $\theta = 1.5560$ and $\phi = 0.5600$, for example, where θ and ϕ are the azimuthal and the longitudinal angles in the Bloch sphere for the direction of Alice's measurement basis $\vec{z}$. That is, the direction denoted by $\theta = 0$ is that of $|0\rangle$ and ϕ is a relative phase between $|0\rangle$ and $|1\rangle$ in their superpositions.

The maximum work extractable from $|\mathrm{W}\rangle$ is 7/9, which is attained when the $\vec{z}$-direction is parallel to $|0\rangle$. If the outcome of Alice's measurement is 0, which occurs with probability 2/3, the other two systems, B and C, will still have a perfect correlation along the three orthogonal directions, being in the state $(1/\sqrt{2})(|01\rangle + |1\,0\rangle)$ and thus Charlie can get 1 bit of work. If Alice's outcome is 1, then Charlie can obtain only 1/3 bits on average as the remaining two systems are in a product state $|1\,1\rangle$. Therefore, the average extractable work will be $(2/3)\cdot 1 + (1/3)\cdot(1/3) = 7/9$ bits. The minimum work from the W state was obtained numerically as 0.1696 bits. This is achieved when $\theta = 0.7169$ and $\phi = \pi/4$ for the direction of $\vec{z}$.

Now we can distinguish the GHZ and the W states, looking at the maximum and the minimum values of W after varying the direction of $\vec{z}$. The maximum and minimum extractable work from the GHZ and the W states have the following relationships:

$$\tfrac{1}{3} < \max_{\vec{z}} W(|\mathrm{W}\rangle) < \max_{\vec{z}} W(|\mathrm{GHZ}\rangle) \qquad \text{and} \qquad \min_{\vec{z}} W(|\mathrm{GHZ}\rangle) < \min_{\vec{z}} W(|\mathrm{W}\rangle), \tag{16}$$

although the difference between the two minima is rather small. Therefore, when the state of a given ensemble is one of the three possibilities, GHZ, W and separable states, as we have assumed above, it is possible to specify the state by examining the range of extractable work. If $W(\rho) \leqslant 1/3$ always holds regardless of the direction of $\vec{z}$, then it is in a separable state due to the above proposition. On the other hand, if $\max_{\vec{z}} W(\rho) > \max_{\vec{z}} W(|\mathrm{W}\rangle) = 7/9$, the state ρ necessarily is in the GHZ class. Even if ρ was a mixed state, it contains states in the GHZ class as its components if the relation $\max_{\vec{z}} W(\rho) > 7/9$ holds.

Suppose that entanglement exists only in two subsystems out of three, such as $\sigma = \sigma^{AB} \otimes \sigma^{C}$, where $\sigma^{AB} = |\Psi^-\rangle\langle\Psi^-|$ is a maximally entangled state. If we wish to discriminate such bipartite entanglement in tripartite systems, we need to examine the extractable work from each party, instead of simply computing the average W. In this example of state σ, Alice and Bob can obtain the maximum work (1 bit), regardless of the outcome of Charlie's measurement, because of the perfect correlation between A and B. However, no information from Alice and Bob can be useful to maximize the work Charlie can extract. Thus, he can have only $1/3$ bits of work at maximum after averaging over three directions. If one of the subsystems is disentangled from the other two, the extractable work from this site cannot exceed $1/3$, which is precisely what has been proved in the preceding subsection. By comparing the extractable work from each site, we can distinguish tripartite correlation with bipartite entanglement.

As we have mentioned above, the thermodynamical separability criterion for bipartite systems is able to detect more inseparability of the Werner state of two qubits than the Bell–CHSH inequalities. Let us consider a Werner-type state, in analogy of the bipartite Werner state, with tripartite systems [21]: we define the GHZ–Werner-type state as

$$\rho_W^{\mathrm{GHZ}} = p|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + \frac{1-p}{8} \cdot I \tag{17}$$

and the W–Werner-type state as

$$\rho_W^{\mathrm{W}} = p|\mathrm{W}\rangle\langle\mathrm{W}| + \frac{1-p}{8} \cdot I. \tag{18}$$

These states are also referred to as isotropic states [22]. The inequality (9) turns out to be violated when $p > 0.6521$ with the GHZ–Werner-type state, and when $p > 0.6981$ with the W–Werner-type state. Are these criteria better than other schemes of multipartite-entanglement detection? In [21], the GHZ–Werner-type state with $p \geqslant 0.3226$ has been shown to be distillable, thus entangled, where $p = (8f - 1)/7$ with $f = \langle\mathrm{GHZ}|\rho_W^{\mathrm{GHZ}}|\mathrm{GHZ}\rangle$, while we have no corresponding data for the W–Werner-type state.

Let us compare equation (9) with Mermin's inequality [6], the multipartite version of Bell–CHSH inequalities. Mermin's inequality for a tripartite system can be written as

$$\langle\mathcal{B}_3\rangle = \mathrm{Tr}(\rho\mathcal{B}_3) \leqslant 2, \tag{19}$$

where the operator $\mathcal{B}_3$ is defined as (by omitting the $\otimes$ sign)

$$\mathcal{B}_3 = (\sigma_1^A\sigma_{2'}^B + \sigma_{1'}^A\sigma_2^B)\sigma_3^C + (\sigma_1^A\sigma_2^B - \sigma_{1'}^A\sigma_{2'}^B)\sigma_{3'}^C. \tag{20}$$

In equation (20), $\sigma_i^x = \vec{a}_i \cdot \vec{\sigma}^x$ are measurement operators at site $x \in \{A, B, C\}$ and $\vec{a}_i$ are unit vectors representing the direction of the measurement.

The GHZ state, equation (1), can violate the inequality (19) maximally as $\langle\mathcal{B}_3\rangle_{\mathrm{GHZ}} = 4$. This means that the GHZ–Werner-type state, equation (17), can violate equation (19) when $p > 1/2$ since

$$\langle\mathcal{B}_3\rangle_{\mathrm{GHZ-Werner}} = \mathrm{Tr}(\rho_W^{\mathrm{GHZ}}\mathcal{B}_3) = p\langle\mathcal{B}_3\rangle_{\mathrm{GHZ}}. \tag{21}$$

Similarly, the W–Werner-type state, equation (18), can violate the inequality when $p > 0.6566$ as the maximum value of $\langle\mathcal{B}_3\rangle_{\mathrm{W-Werner}}$ is 3.046 [23]. Therefore, the condition (9) is less effective in detecting the entanglement in both the GHZ–Werner-type and the W–Werner-type states: these are a class of weakly entangled states that violate Mermin's inequality, but do not violate equation (9). The comparison with Mermin's inequality is summarized in table 1.

Table 1. The comparison of the thermodynamical separability criterion for tripartite systems, equation (9), and Mermin's inequality, equation (19), in detecting the inseparability of the Werner-type states. This table shows the minimum values of p to violate the inequalities, i.e. the smaller value, the more detection of inseparability.

	Thermodynamical separability criterion (9)	Mermin's inequality (19)
GHZ–Werner-type state	$p > 0.6521$	$p > 0.5$
W–Werner-type state	$p > 0.6981$	$p > 0.6566$

The situation does not change much even if we average the extractable work by varying the $\vec{u}$ over the whole Bloch sphere. The inequality (9) can be violated by the GHZ–Werner-type state when $p > 0.8392$ and by the W–Werner-type state when $p > 0.9057$, after choosing $\vec{z}$ optimally. The reason why equation (9) is not as effective as Mermin's inequality is yet unclear, while a similar thermodynamical criterion is more effective in the case of bipartite entanglement as shown in [13].

4. Summary

We have generalized the thermodynamical separability criterion to tripartite quantum systems and shown that it can be used to distinguish different classes of tripartite entanglement, the GHZ and the W, in terms of thermodynamically extractable work under LOCC. We have also found that the criterion for tripartite systems is less effective in detecting the Werner-type entanglement than that for bipartite systems. Although it is not perfectly clear if the Werner-type states above are the proper counterpart of the bipartite Werner state we should compare with, it appears that the separability criterion we have obtained here is not as efficient as other criteria, such as Mermin's inequality, in detecting weak entanglement. Nevertheless, it is capable of distinguishing the GHZ, the W and separable states with physical quantity. Our results are of interest in their own right in obtaining more intuitive insight into the entanglement from thermodynamic point of view. Also the difference between extractable work from GHZ and W states in equation (16) may lead to an interesting physical or information processing process, in which the W state is more useful than the GHZ state, unlike most of the known processes.

As one can see that the protocol to obtain the criterion, equation (9), has not been particularly optimized yet, there could be a smarter and nicer way to distinguish entangled states from separable ones. Such a method would give us a different ordering of the GHZ and the W states as some known criteria actually do so [8, 9], but could provide more detailed pictures of the structure of multipartite entanglement. We hope that this research direction will be useful to acquire more fundamental understanding of (not only tripartite) entanglement with respect to thermodynamics.

Acknowledgments

We gratefully acknowledge Č Brukner for fruitful discussions. V Viguié is grateful to P Holdsworth for arranging his stay in London. K Maruyama is a Boursier de l'ULB, and is supported by the European Union through the project RESQ IST-2001-37559, the Action de Recherche Concertée de la Communauté Française de Belgique, and the IUAP program of the Belgian Federal Government under grant V-18. V Vedral acknowledges funding from EPSRC and the European Union.

References

[1] Bell J S 1964 *Physics* **1** 195
[2] Horodecki M, Horodecki P and Horodecki R 1996 *Phys. Lett.* A **223** 1
[3] Horodecki M and Horodecki P 1999 *Phys. Rev.* A **59** 4206
[4] Nielsen M A 2001 *Phys. Rev. Lett.* **86** 5184
[5] Terhal B M 2000 *Phys. Lett.* **271** 319
[6] Mermin N D 1990 *Phys. Rev. Lett.* **65** 1838
[7] Oppenheim J, Horodecki M, Horodecki P and Horodecki R 2002 *Phys. Rev. Lett.* **89** 180402
[8] Horodecki M, Horodecki P, Horodecki R, Oppenheim J, Sen(De) A, Sen U and Synak-Radtke B 2005 *Phys. Rev.* A **71** 062307
[9] Tóth G 2004 *Phys. Rev.* A **69** 052327
[10] Rudolph O 2002 *Preprint* quant-ph/0202121
Rudolph O 2003 *Phys. Rev.* A **67** 032312
[11] Chen K and Wu L A 2003 *Quant. Inform. Comput.* **3** 193
[12] Chen K and Wu L A 2002 *Phys. Lett.* A **306** 14
[13] Maruyama K, Morikoshi F and Vedral V 2005 *Phys. Rev.* A **71** 012108
[14] Dür W, Vidal G and Cirac J I 2000 *Phys. Rev.* A **62** 062314
[15] Greenberger D M, Horne M A and Zeilinger A 1989 *Bell's Theorem, Quantum Theory, and Conceptions of the Universe* ed M Kafatos (Dordrecht: Kluwer) p 69
[16] Schumacher B W 1991 *Phys. Rev.* A **44** 7047
[17] von Neumann J 1955 *Mathematical Foundations of Quantum Mechanics* (English translation by R T Beyer) (Princeton, NJ: Princeton University Press)
[18] Bennett C H 1982 *Int. J. Theor. Phys.* **21** 905
[19] Clauser J F, Horne M A, Shimony A and Holt R A 1969 *Phys. Rev. Lett.* **23** 880
[20] Werner R F 1989 *Phys. Rev.* A **40** 4277
[21] Murao M, Plenio M B, Popescu S, Vedral V and Knight P L 1998 *Phys. Rev.* A **57** R4075
[22] Rains E M 1999 *Phys. Rev.* A **60** 173
[23] Cabello A 2002 *Phys. Rev.* A **65** 032108