Water security, risk and economic growth: insights from a dynamical systems model

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Abstract

Investments in the physical infrastructure, human capital, and institutions needed for water resources management have been noteworthy in the development of most civilisations. These investments affect the economy in two distinct ways: (i) by improving the factor productivity of water in multiple economic sectors, especially those that are water intensive such as agriculture and energy; and (ii) by reducing acute and chronic harmful effects of water-related hazards like floods, droughts, and water-related diseases. The need for capital investment to mitigate risks and promote economic growth is widely acknowledged, but prior conceptual work on the relationship between water-related investments and economic growth has focused on the productive and harmful roles of water in the economy independently. Here the two influences are combined using a simple, dynamical systems model of water-related investment, risk, and growth. In cases where initial water security is low, initial investment in water-related assets enables growth. Without such investment, losses due to water-related hazards exert a drag on economic growth and may create a poverty trap. The presence and location of the poverty trap is context-specific and depends on the exposure of productive water-related assets to water-related risk. Exogenous changes in water-related risk can potentially push an economy away from a growth path towards a poverty trap. Our investigation shows that an inverted-U-shaped investment relation between the level of investment in water security and the current level of water security leads to faster rates of growth than the alternatives that we consider here, and that this relation is responsible for the ‘S’-curve that is posited in the literature. These results illustrate the importance of accounting for environmental and health risks in economic models and offer insights for the design of robust policies for investment in water-related productive assets to manage risk, in the face of environmental change.

Index terms: 1880 (Water management); 4328 (Natural hazards: risk); 4336 (Natural hazards: economic impact of disasters); 4303 (Natural hazards: hydrology); 4430 (Complex systems).

Keywords: water security, risk, investment, economic growth, dynamical systems model

1 Introduction

Matching water availability and demand is amongst the most pressing environmental challenges in the twenty-first century [*Rockstrom et al.*, 2009; *Vörösmarty et al.*, 2010]. Environmental and economic constraints imposed by water scarcity can limit production and economic growth [*Dasgupta*, 2001; *Sachs et al.*, 2004]. At the same time, water-related natural hazards (e.g., floods, droughts, and, water-related diseases) pose risks to systems of agricultural and industrial production and human well-being [*Brown and Lall*, 2006; *Grey and Sadoff*, 2007].

An adequate, reliable supply of water is only one amongst many factors of production, but it is a crucial input for the development of many sectors of an economy, especially agriculture and energy [*Whittington et al.*, 2013]. In the United States, water-related infrastructure represents approximately 10–15 percent of total infrastructure capital [*Munnell*, 1992]. Earlier empirical studies that sought to quantify the contribution of capital investments in public infrastructure to economic growth encountered difficulties determining the direction of causality when using statistical regression to estimate reduced-form growth models [*Gramlich*, 1994; *Munnell*, 1992]. However, more recent work using structural growth models to account for the feedbacks between investment and growth in the wider economy has revealed more clearly the substantial contribution of infrastructure to growth in a dataset for 75 countries across a range of national incomes over the period 1965–1995 [*Esfahani and Ramírez*, 2003]. As progress is made toward water security, the ability of investments in water-related infrastructure to increase the factor productivity of water as an input in different sectors of the economy diminishes [*Barbier*, 2004].

At the same time the presence of water-related hazards has a detrimental effect of its own on economic growth. Depending on the value of assets at risk, and the ability of a country to invest in risk reduction, the primary objective of water-related investment may shift from directly increasing economic production to mitigating hazard-related losses. The mitigation of hazard-related losses increases human well-being directly and increases economic growth indirectly (for example through reduced water-related illnesses and increased labour productivity). The relationship between national wealth and vulnerability to natural hazards is complex and is linked to institutional and political processes as well as economic factors [*Noy*, 2009]. The number of human lives lost as a result of weather extremes has fallen dramatically over time in the United States and Europe [*Kellenberg and Mobarak*, 2011]; however financial losses have increased over the same period as a result of the increased value of property at risk [see *Hallegate*, 2012, for a review]. Investments in water-related infrastructure can alter the residual risk posed by water-related hazards and thus can create a dynamic interaction between investment, risk reduction and economic growth [cf. *Hallegatte and Ghil*, 2008; *Sivapalan and Blöschl*, 2015; *Viglione et al.*, 2014].

Two policy objectives emerge from this situation. First, there is a need to increase the upside potential associated with the availability of reliable water supplies of suitable quality for human consumption, agriculture, ecosystems, industry, and energy. Second, there is a need to reduce society's exposure to water-related risks. From a hydrological point of view the availability of water and water-related extremes (especially those due to floods and droughts) are component parts of hydrological variability, and adaptations are required to address each of these components of variability in order to reduce the resultant losses to economy and to society. The challenge to harness the productive aspects of water in the economy while simultaneously mitigating water-related losses leads to the idea of 'water security', which *Grey and Sadoff* [2007, p. 545] define as the "availability of an acceptable quantity and quality of water for health, livelihoods, ecosystems and production, coupled with an acceptable level of water-related risks to people, environments and economies". This definition of water security therefore includes both a country's natural, hydrological endowment and the changes in water security that human economic development may bring, which may be positive or negative.

Several approaches have been adopted in the literature to model investments in water resources [*Harou et al.*, 2009]. These models seek to integrate aspects of water resources systems with economic policies and outcomes at scales from household to multi-national. Approaches taken range from simulation of the impact of water-related losses on the economy [*Jonkman et al.*, 2008] to models which seek to optimize profits from water use under different scenarios [*Cai and Wang*, 2006].

The aim of this paper is to describe a conceptual model that accounts for the effect of investments on both (i) the increased productivity of water as an input to economic activities, and (ii) the reduction of losses arising from water-related risks. The purpose of presenting this conceptual model is to highlight the dual protective and productive nature of investments in the water sector and to examine the logical conclusions that arise from its assumptions. From these conclusions, we draw inferences that provide insights for decision-makers in the sector. Throughout the paper, discussions on investment in infrastructure encompass both the physical and institutional investments that are together needed to optimise outcomes. We define water investments to mean the financial funds required to both build, maintain, and operate water infrastructure (i.e., both capital and operational expenditures).

2 Model

2.1 Water-related growth and risk

Faced with the choice to allocate available capital to investment in water-related assets rather than to assets in other sectors, decisions should be made that simultaneously maximize the combined value of increased production due to water-related investments and of reduced water-related risks to the economy. Water-related capital comprises both natural capital necessary for the provision of 'ecosystem services', physical capital in the built infrastructure, and social and human capital embedded within institutions and information systems used for water management [*Dasgupta*, 2001]. We assume that if capital is invested in the water sector, the same capital will not be available for investment elsewhere during the same time period. Note that we do not assume that the right to allocate capital should be vested in one particular body, private or public; nor do we suppose that these decisions will necessarily take place in a coordinated manner. Rather, we assume that in pursuit of water security, it is rational for the nation to invest in water-related assets until the marginal cost of the investment is equal to the sum of (i) the marginal benefits to productive sectors of the economy, and (ii) the marginal benefits from reducing the risks of water-related hazards [*Grey and Sadoff*, 2007].

Let $K(t)$ be the total wealth at time t, and $N(t)$ be the amount of this wealth $K(t)$ allocated to water-related investments (Figure 1). We assume that growth occurs by capital accumulation and is proportional to the amount of capital already in existence. We focus in particular on the role of water security in growth. An amount of capital is allocated each year to investments that increase water security by both enhancing the productive capacity of the economy and reducing water-related hazards. That allocation is determined as a fraction of the available capital, moderated by the current state of water security (i.e., the amount of K allocated to N is calculated as a function of water security), which may be different in different countries. Losses occur each year both to national wealth and to water-related infrastructure at rates that are themselves proportional to the level of water security achieved. We note that the high fixed costs and investments in some forms of physical water infrastructure can be lumpy, and can occur on a range of time-scales, as can water-related losses. We suppose however that discrete investments or losses become smoothed when aggregated nationally and so we treat time as continuous in the model.

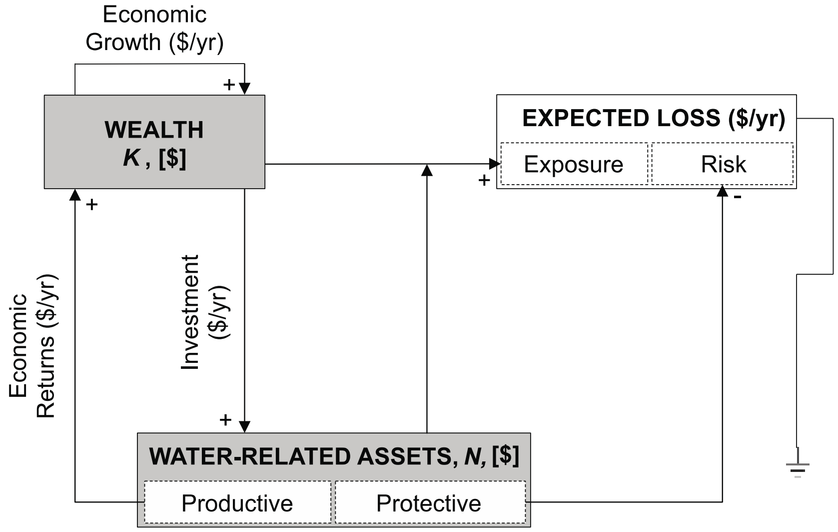

Figure 1. Schematic flow diagram depicting the relation between water, risk, and growth. Grey shaded boxes represent stores, arrows represent fluxes between stores with the direction of the relation between store size and flux indicated. The box labeled "Expected Loss" is not a store but is a diagnostic measure that is derived from the level of exposure (related to wealth) and risk (related to water-related assets). Exogenous risks (e.g., due to hydroclimatic variability) are assumed to be fixed and are not shown.

The relation between the two state variables, *K* and *N*, is shown graphically in Figure 1. The rate of change of *K* is given as:

$$\frac{dK}{dt} = rK\left(1 - \frac{K}{K_0}\right)\frac{N}{N_0} - sK - l_{\mathrm{e}}\,K\left(1 - \frac{N}{N_0}\right), \qquad (1)$$

where r is the rate of return on capital across the entire economy. The term s is the fraction of national wealth that is allocated to water-related investment at any given period in the model. At its simplest, s can be a constant; although in practice it is likely to be a function of the current state of water security, N/N_0. We evaluate several possible forms of this function in Section 2.4 below, and consider their effects on the resulting dynamics of water security and growth. The parameter, l_e, represents the expected loss to national wealth resulting from water-related hazards (e.g., floods, droughts, water-related disease). This parameter combines exposure to hazard and vulnerability to loss should a hazard arise.

The baseline parameters K_0 and N_0 represent, respectively, the level of wealth that would be possible in the absence of water-related constraints, and the level of investment in water-related assets required in order to be freed from water-related constraints. These terms limit what would otherwise be a process of exponential growth by supposing that there is a diminishing marginal return on water-related investment as growth is freed from its water constraint, and as a tolerable level of water-related risk is reached. The latter limit, N_0, incorporates elements of climate risk because the level of investment required to achieve water security depends on the natural endowment of the nation concerned. These parameters are assumed to change slowly relative to the rate of investment in water related assets and are therefore considered to be constant in any particular setting, but their values may change from one setting to another. They are not observable quantities, but serve to focus the analysis on the role of water in the economy while holding other factors constant.

2.2 Water-related risk

A second equation tracks the rate of change of N, the level of investment in water-related assets, including natural water-related assets:

$$\frac{dN}{dt} = sK - l_{\mathrm{w}} N \left(1 - \frac{N}{N_0}\right), \qquad (2)$$

where l_{w} is the fractional loss of water-related assets due to the effects of water-related risks. Equation 2 illustrates that the rate of investment in water-related infrastructure is proportional to a country's wealth and that the rate of investment in water-related infrastructure diminishes as water security is achieved. Equation 2 incorporates the simultaneous reduction in the stock of water-related assets caused by water-related damage to sector specific investments and natural assets.

Hydrological variability, including extremes such as flood and drought but also seasonal and interannual variability, is an important driver of the loss terms l_{e} and l_{w}. These terms represent losses to the economy in general as a result of water-related risks, and losses to water-related assets. It is through these loss terms that the negative economic effects of water-related risks are included in the model, and it is to be expected that these terms will vary from place to place depending on the hydroclimatic context. Where investments in water security have been made to mitigate the losses (i.e., as N approaches N_0), the impact of these potential losses on national wealth decreases.

We assume for simplicity that water related and non-water capital assets depreciate at the same rate, and that dK(t)/dt is net of depreciation. In practice, some water resources infrastructure assets have long economic lives, although some do not. To the extent that water-related assets depreciate at a slower rate than non-water related assets, in our model this would be manifest in a lower value of $\square_{\mathrm{w}}$.

The distinction between l_{e} and l_{w} is important in order to separate losses in the water sector from losses borne by the economy more generally. This separation permits discussion later of the resilience added to the system through the development of water infrastructure that reduces water-related hazards, and which is itself resilient to water-related risk. Resilience of water-related assets may be characterised as the ability to return to their former state following a disturbance, and may form an intrinsic feature of their engineering design. But resilience can also arise in systems which contain an (often institutional) adaptive capacity to maintain their function when faced with external perturbations [*Gunderson*, 2000]. Resilience may therefore also be achieved through the presence of strong institutions, or by financial means through the hedging of water-related risks using financial instruments such as insurance contracts and catastrophe bonds [*von Dahlen and von Peter*, 2012; *von Peter et al.*, 2012].

2.3 Non-dimensionalisation

The following canonical scales are applied in order to render the system in Equations 1 and 2 dimensionless: $\tau = tr$, $\alpha = K/K_0$, $\beta = N/N_0$, $\sigma = s/r$, $\lambda_e = l_e/r$, $\lambda_w = l_w/r$, and $\phi = N_0/K_0$. A detailed derivation of the non-dimensional equations is given in Supplementary Material. The scaling of time and the remaining rate parameters by the rate of return on water-related investment permits comparison between countries with different rates of return. Similarly the normalization of wealth and water-related investment by K_0 and N_0 respectively creates a set of equations in two new dimensionless variables, α and β, which represent, respectively, country wealth relative to its potential wealth if unrestricted by water availability and the level of investment in water-related assets relative to the level that would be required in order to reach $\alpha = 1$. The fraction of national wealth required to achieve water security, N_0/K_0 is defined as ϕ. The identification of the quantity β with the level of water security results from the definition that water security is achieved when the next available unit of currency is invested elsewhere. Therefore when water security is low (N is low compared with N_0), and as N approaches N_0, β approaches unity.

The dimensionless equations are:

$$\frac{d\alpha}{d\tau} = \alpha[(1-\alpha)\beta - \sigma - \lambda_e(1-\beta)], \text{ and} \quad (3)$$

$$\frac{d\beta}{d\tau} = \frac{\sigma}{\phi}\alpha - \lambda_w\beta(1-\beta). \quad (4)$$

The presentation of the system as dimensionless reduces the number of parameters to a minimum and frames the variables of interest in terms of a fractional contribution of investment in water-related assets to national wealth. We note that K_0 and N_0 are not intended to be observable quantities; they represent idealised states in which water-related factors do not constrain productivity and in which water-related risks are tolerable. These quantities are assumed to change slowly relative to the rate of investment in water-related assets. They are used in the non-dimensionalised model so that we can understand conceptually the dynamical interactions between water-related investment, risk, and growth. There are clearly many other factors that contribute to a country's wealth, but here we focus solely on the sensitivity of the economy to water security (including the possible situation in which the economy is not sensitive to water security).

2.4 Investment in water-related assets

The investment function σ, is a function of the level of water security, β. The function σ is a critical component of the model. We compare and contrast three possible forms, the most plausible of which is that current investment is a parabolic function of the investment to-date in water-related assets (i.e., an inverted-U; Figure 2), such that:

$$\sigma = 4\sigma_{max}\beta(1-\beta), \qquad (5)$$

where σ_{max} is the peak rate of investment (see the Supplementary Information for the derivation of this equation). The justification for this functional form is that in the early stage of a country's development an initially increasing fraction of national wealth is invested in water-related assets but that this fraction declines as water-related risks are reduced and water needs as a factor input to production are satisfied [cf. *Grey and Sadoff*, 2007]. *Barbier* [2004] also finds that the relation between growth and water utilization follows a concave-downwards ('inverted U-shaped') curve. At sub-optimal rates of water utilization, there remains an economic benefit from further investment in water-related infrastructure; above the optimal rate, such investment detracts from growth in the wider economy and results in diminished growth rates because capital is being poorly utilized in the water sector.

We also consider the alternative cases, shown in Figure 2, that water-related investment increases (dashed line in Figure 2) and decreases (dotted line in Figure 2) with investment to-date. The former possibility implies a reduced priority for water-related investment during the early phase of development; the latter alternative implies an early prioritisation of water-related investment at the expense of other sectors of the economy.

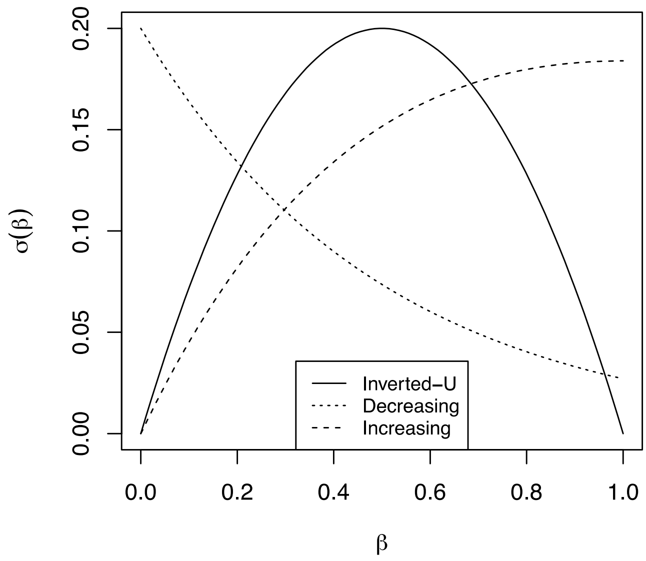

Figure 2. Rate of new investment in water-related assets, σ, as a function of current level of investment in water-related assets, β. Three possible functional forms are illustrated, including the parabolic inverted-U, and increasing and decreasing functions for comparison.

3 Results

3.1 Stability and stationary points

The non-linear system represented by Equations 3 and 4 has a stationary point at $\alpha = 0$, $\beta = 0$. This first stationary point represents a low-level equilibrium (i.e., a poverty trap) in which the absence of water-related investment prevents the economy from growing *and* the presence of destructive, unmitigated water-related risks continually strike the economy with disaster-related losses [cf. *Dasgupta*, 2001; *Sachs et al.*, 2004].

The direction field associated with the coupled model is shown in Figure 3. The upper right location in Figure 3 in the model's phase space contains a stationary point (at $\alpha = 1$, $\beta = 1$) providing that $\sigma = 0$ when $\beta = 1$. Note that Figure 3 is based on the inverted-U investment relation. In this situation, the upper equilibrium ($\alpha = 1$, $\beta = 1$) is the point at which water-related investment reaches the level required to prevent water-related constraints on growth and to ensure a tolerable level of water-related risk. The growth trajectory towards this point is in most cases S-shaped, with an initially-rapid rate of increase diminishing as growth proceeds. This system behaviour results from the diminishing marginal productivity of water-related investments assumed in the model. In this part of the model's phase space, growth is sufficiently rapid to provide capital for water-related investment at a sustainable rate, and water-related investments are able to sufficiently protect the economy from debilitating losses from water-related risks. As wealth increases, the value of wealth at risk increases in real terms [cf. *Hallegate*, 2012; *Kellenberg and Mobarak*, 2011], but the increase in wealth due to the growth in the wider economy is more than sufficient to cope with these water-related losses. In cases where $\sigma(1) > 0$, the point ($\alpha = 1$, $\beta = 1$) investment continues beyond the point at which water security is achieved.

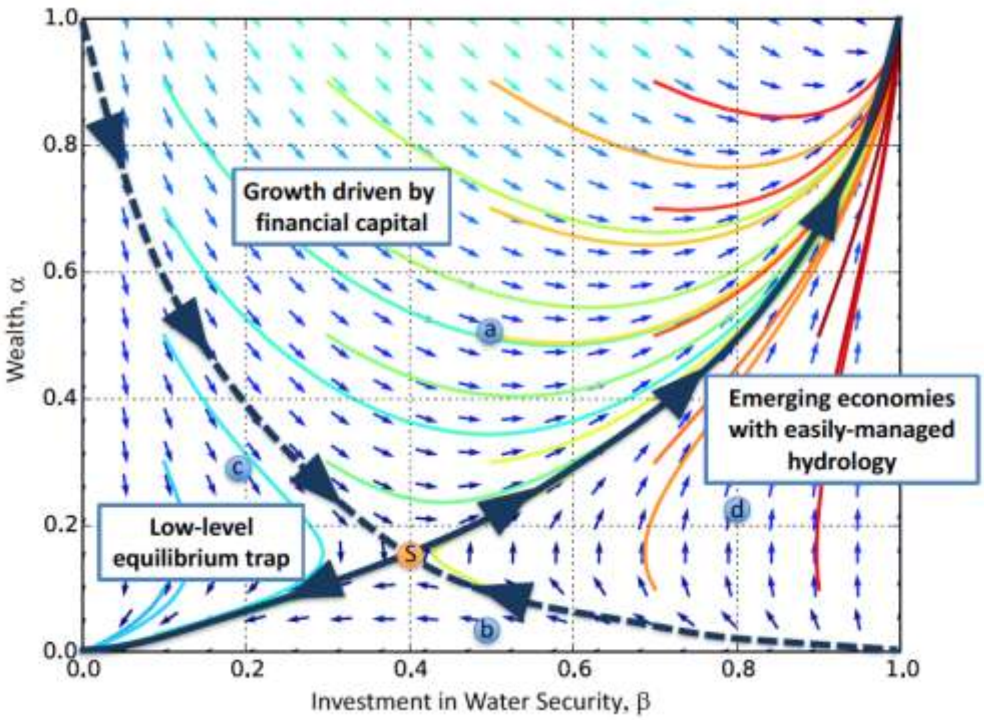

Figure 3. Direction field for the system represented in Equations 3 and 4. The variables α and β are defined in Section 2.3. Arrows indicate the direction of the rate of change points in the domain. The colour of the arrow indicates the total magnitude of the rate of change (red = rapid; blue = slow). Bold blue lines with large arrows represent the convergent trajectories (solid) and the separatrix (dashed). The unstable saddle point is marked with an 'S'; markers labelled 'a' to 'd' are referred to in the text. Other parameters used to create this plot are λ_ε=0.25; λ_ω=0.25; ϕ=0.5.

The two sets of trajectories discussed above are separated by a saddle point (or tipping point). Trajectories that begin above the separatrix indicated in Figure 3 converge upon a pathway of growth. The pathway to growth is context specific: different trajectories in the model phase space experience different combinations of water-related and non-water-related investment in order to drive growth. By contrast, trajectories that begin below the separatrix, which have neither sufficient investment nor a benign hydrological endowment, cannot sustain growth. Such instances experience rapid depletion of natural or financial resources, and independent, self-sustaining growth cannot be achieved. Without external investment, situations below the separatrix are drawn towards the poverty trap.

3.2 Trajectories for growth: water-constrained and investment-constrained pathways

Any point on Figure 3 will experience a trajectory of growth or decline which depends on its initial position (its 'water endowment' and its 'wealth endowment') and the context-dependent values of the function $\sigma(\beta)$, and the two parameters λ_e and λ_w. To explore these possibilities more fully, and to establish test cases for comparison with empirical data, numerical solutions to the non-linear system are presented for a series of different initial conditions (Figure 4).

3.2.1 Initial condition (a)

The prognosis for a country that begins at initial condition (a) in Figure 3 is shown in Figure 4a. This initial situation reflects moderate wealth and only moderate water endowment. There is sufficient national wealth that water-related investments are possible. These investments come at the expense of investment in other sectors of the economy and initially they are a drag on economic growth but without them further growth is hard to sustain. Once the initial investment is made, the amount of additional money required to mitigate water-related risk is still significant but can be obtained from the proceeds of growth in the wider economy, either through public or private investment or both. In practice, reductions in water use often coincide with economic growth, providing that appropriate investment in technology, infrastructure and institutions (broadly conceived) permits increased productivity [*Randall*, 1981].

The Colorado River offers an example of the trajectory described above. The region was characterized by both moderate wealth and a modest water endowment in 1902 when the Newlands Reclamation Act made irrigation a development priority for the Western USA. The river, whose hydrology was dominated by spring snowmelt, was surveyed by expeditions in the 1860s and early 1900s [*La Rue*, 1916; *Powell et al.*, 1879]. The results showed both irrigation and hydropower development opportunities. Private capital provided the first wave of investment in the Imperial Valley of Southern California in the late 1800s and early 1900s. Floods in 1905 overwhelmed the canals and spilled into the Salton Sea, demonstrating the downside risks of unmitigated hydro-climatic variability, and prompting farmers to petition the federal government for capital infusions to construct reservoirs and irrigation canals [*National Research Councils*, 2007].

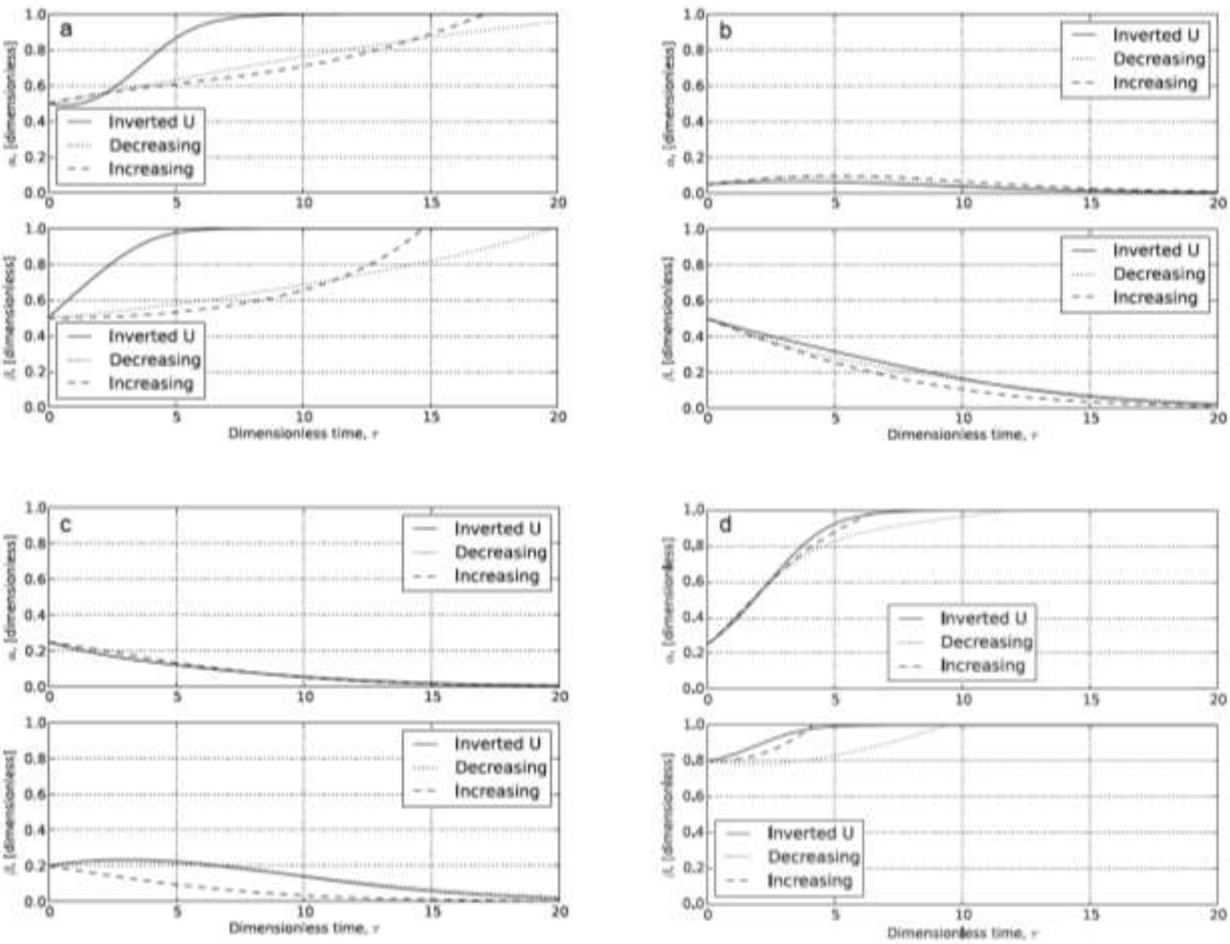

Figure 4. Trajectories for typical countries. Numerical solutions were obtained using the Livermore Solver for Ordinary Differential Equations [LSODE; *Hindmarsh*, 1983], which automatically switches between predictor-corrector and backward differentiation methods according to the numerical properties of the problem being solved. Lines indicate dimensionless country wealth (α) and dimensionless water security (β) in upper and lower plots respectively. Solid lines use inverted-U relation for the function σ whereas dotted and dashed lines use decreasing and increasing σ relation respectively. Parameter values as in Figure 3.

The Bureau of Reclamation, a federal agency in the USA, has since constructed more than 180 projects, supporting 31 million people across 17 states in the Western US, at a cost over $20 billion. These costs have been borne primarily by taxpayers throughout the United States [*GAO*, 2014]. These investments have created significant reservoir storage capacity on the river that has buffered climate variability and reduced the risks of shortages in the lower basin. This has enabled the development of 4.5 million acres of irrigation, hydropower generation, and the growth of major cities and economic activity in Denver, Los Angeles and Phoenix [*US Bureau of Reclamation*, 2012]. These water-related investments have been combined with a system of urban and rural infrastructure (e.g., roads) to foster regional economic growth.

This deployment of financial capital to enhance water security has also come with a significant cost to environmental systems and natural capital. Upstream dams and diversions have reduced the Colorado Delta in Mexico to less than 10% of its historic area, leading to bi-national investments between the US and Mexico to restore flows and habitat. Climate change and competition for water have required on-going investments in infrastructure, information and institutions to sustain growth [*US Bureau of Reclamation*, 2012]. Whilst it is neither possible nor desirable to predict the potential future wealth of nations using our model, comparison of outcomes depicted in Figure 4a it does indicate that had the investment rule been different (i.e., had the initial rate of investment in water-related assets been slower), the rate of economic growth in situations analogous to initial condition (a) might have been lower than observed.

3.2.2 Initial condition (b)

The situation for a country that begins at initial condition (b) in Figure 3 is shown in Figure 4b. Here, the level of water-related investment is identical to the situation in (a), perhaps owing to a similar level of unmitigated hydro-climatic variability or to absolute water scarcity. However, the level of initial wealth is much lower. In this case the lack of wealth seriously constrains mitigation of water-related losses and, after a period of stagnation, the economy is drawn into the poverty trap. Such an example trajectory is rarely seen in reality: these are locations with extreme water scarcity and variability, where large-scale commercial agriculture is unlikely ever to have been viable. Under these conditions, growth is restricted by the presence of a poverty trap, and infrastructure investment is constrained by a lack of wealth so only an exogenous injection of wealth (e.g., through foreign investment or the discovery of mineral resources) can shift the economy to a growth trajectory.

3.2.3 Initial condition (c)

The trajectory followed by a country that begins at initial condition (c) in Figure 3 is shown in Figure 4c. This initial condition, in which both the initial water endowment *and* wealth are low, represents the most perilous case. With vulnerable water resources and only modest wealth, an initial phase of investment increases factor productivity and reduces losses from water-related risks but this investment depletes wealth to the point where growth is insufficient to compensate for the continued losses due to unmitigated water-related risks.

The portion of the Indus River Basin located in Pakistan provides an example of this pathway. Characterized by significant poverty, a low water endowment, and high water-related risks, Pakistan is a lower-middle income nation, with $1,360 per capita GNI, but with 30% of the national population below the national poverty line [2013 data; *World Development Indicators*, 2015]. The

Indus Basin Irrigation System (IBIS) is the largest contiguous irrigation system in the world, with canals totaling over 60,000 km in length used to irrigate up to 16.45 million hectares. Irrigation contributes 22% of Pakistan's GDP, employs 54% of its labour force, and generates 70% of its export earnings [*Yu et al.*, 2013]. However, between 1951 and 2016, population growth and reservoir sedimentation caused per capita water stocks to decline from 5,260 to 1,032 m^3 per person, a figure that is among the lowest in the world. Only 10% of Pakistan's hydropower capacity has been developed, despite severe power load-shedding and blackouts in both urban and rural areas.

Geopolitical factors exert an important influence, as Pakistan is entirely dependent on Indus Basin waters, with water security an 'existential challenge' [*Briscoe*, 2010]. Independence and the Partition of India in 1947 resulted in 90% of the Basin's irrigable lands being in downstream Pakistan. On the other hand, the Basin's existing and potential hydropower facilities are in the headwaters, located mostly (but not only) in upstream India. The 1960 Indus Waters Treaty (IWT) allocated the flows of the three eastern Basin tributaries – the Ravi, Beas, and Sutlej – to India and the flows of the three western tributaries – the Indus, Jhelum and Chenab – to Pakistan, allowing hydropower development and minor diversions in India. India contributed to the financing of the 'replacement works' Pakistan needed to divert flows from the western rivers to the areas formerly irrigated by the eastern rivers.

In the Pakistan Indus, long-standing water-related risks include high rainfall and runoff variability (with frequent floods and droughts), dependence on upstream snowmelt for a significant proportion of Indus flows, inadequate sanitation, and soil salinity. Settlements in the extensive floodplains have suffered many severe floods, including in 1953, 1973, 1975, and 2010. The 2010 Pakistan floods affected 20 million people and caused an estimated 2,500 deaths, two million hectares of lost crops, and a 6% reduction in national GDP [*Ali*, 2013]. Rising groundwater levels due to inadequate irrigation management have caused salinity and waterlogging across large areas of the Indus floodplain, reducing crop yields. These risks and opportunities have prompted on-going efforts to increase agricultural productivity, manage floods, build infrastructure for energy production, and adapt to the impacts of climate change on water availability and variability.

After a half century of relative stability, the IWT has come under strain as both countries intensify basin development to meet the growing demand for irrigation and power. Pakistan has recently given high priority to managing perceived geopolitical risks to its Indus water resources, invoking the IWT to address concerns over two recent hydropower projects upstream in India: a 'difference' in the Baglihar case on the Chenab river, with a verdict by a 'neutral expert' in 2007; and a 'dispute' in the Kishenganga case on the Jhelum river, with a ruling by the Court of

Arbitration in 2013. India is planning many hydropower projects on the rivers allocated to Pakistan, which Pakistan views as seriously threatening its water security. These water security risks could be overcome if the Indus again became a catalyst for international cooperation between India and Pakistan [*Briscoe*, 2010]. The situation in the Indus Basin therefore underscores the importance of investment in both institutional *and* physical assets needed to optimise outcomes.

3.2.4 Initial condition (d)

The situation for a country that begins at initial condition (d) in Figure 3 is shown in Figure 4d. This initial state complements (a) in the sense that its trajectory is one of growth, but the starting point is one of relatively low water-related risk, yet little wealth. Such a system might be typical of the eastern United States in the mid-nineteenth century, or of a north-western European country in the mid-eighteenth century. During the initial phases of growth, the reduction of water-related risks is not a priority; the economy can grow without constraints imposed by water scarcity or water-related risks. The economy is not encumbered by the need to invest heavily in water-related infrastructure and can instead allocate more capital to opportunities in other sectors. The lower level of investment in water-related infrastructure that is needed can be made from the proceeds of growth, providing that the losses from water-related risks are not so severe that they drag the growth trajectory across the separatrix towards the poverty trap.

The Rhine exemplifies this situation during its sustained development path. The pathway to water security in the Rhine can be traced back to land reclamation of the Rhine-Meuse delta starting 800–1100 AD. The reclaimed land was highly productive, population increased and cities developed, protected by embankments from flooding. The industrial revolution in the 19th Century and the subsequent period of intensive demographic and economic development led to new waves of development in the basin, accompanied by increased vulnerability to flooding [*Cioc*, 2002]. Industrial activities in the Ruhr area in Germany increased the need for water transport and, as ships grew in size, long stretches of the Rhine were modified and even canalized, resulting in narrower and deeper channels. Hydropower was developed to power industry. The subsequent emergence of systemic risks increasingly required international cooperation. Water quality became an issue in the wake of rapid industrial development in the Rhine basin after the Second World War. The severe floods of 1993 and 1995 demonstrated the importance of extreme weather events and climate change as a serious risk to the region’s continued growth, requiring both traditional and ecological infrastructure. A new approach has been developed in the Netherlands where water security risks are potentially existential, using vulnerability analysis to define Adaptation Tipping Points (ATP) to indicate whether current water management strategies will continue to be effective under different climate change scenarios [*Haasnoot et al.*, 2013]. Initial investments favoured navigation to facilitate industrial activity; the income generated by these investments – both water-related and non-water related – eventually provided the wealth to address the industrial pollution generated by this growth.

3.3 Sensitivity to investment relation

While the broad behaviour in the model is similar for each of the investment relations presented in Figure 2, the detailed trajectories and the rate at which water security is achieved differ considerably. The adoption of an inverted-U relation, in which investment is greatest in the mid-stages of development, leads to rapid growth in both water security and the broader economy that outpaces the trajectory followed under either of the alternative increasing and decreasing investment relations. Our findings for initial conditions (b) and (c) – in which a low-level equilibrium is attained – confirm that, in situations with poor initial water endowment, strategies without early investment in water-related infrastructure can result in accelerated economic decline (Figure 4b,c).

We note also that the S-curve postulated by *Grey and Sadoff* [2007] arises in the model only for growth scenarios typified by initial conditions (a) and (d) in Figure 3 and that, moreover, it is present only when the inverted-U investment relation is adopted. This finding suggests that the presence of an S-curve for water is a result of the assumption of an inverted-U investment relation rather than a feature inherent to the dynamics of water security and growth. We find that, in the model, the adoption of an inverted-U investment relation leads to the most rapid rates of growth away from the poverty trap, and reduces the rate of decline in economies that are heading towards the low-level equilibrium. Whilst our analysis cannot remove the need for careful appraisal of individual investments in water-related assets, it does illuminate the specific connection between the investment relation pursued and the type of growth that might be expected to result.

4 Discussion

4.1 Understanding the dynamics of water's poverty trap

The presence of a tipping point in the system (marked in Figure 3) has been described as a low-level equilibrium trap, or a poverty trap, in the literature on water security [*Grey and Sadoff*, 2007; *World Bank Water Demand Research Team*, 1993] and other literatures linking environment and economics refer to similar behaviour [*Bonds et al.*, 2010; *Dasgupta*, 2001]. A key debate within development economics concerns the prevalence or likelihood of poverty traps in reality, although we note that the majority of work to date has concerned poverty traps at the household rather than at the macroeconomic level [*Dasgupta*, 2001; *Sachs*, 2005]. Our model shows that although a poverty trap forms an inevitable part of the dynamics of the system represented by Equations 3 and 4, the presence or absence of a poverty trap in Figure 3 is critically dependent on the individual country's context.

In our model, the location of the tipping point is sensitive to the values of the four parameters, ϕ, σ_{max}, λ_e and λ_w/λ_e. The sensitivity of the tipping point location to the parameter ϕ is intuitively straightforward (shown in Figure 5a): the greater the investment required to achieve water security, the larger the fraction of the model phase space occupied by situations that lead towards a poverty trap. The value of ϕ might change as a result of exogenous changes in hydro-climatic variability, but it may also be reduced through technological innovations that reduce the cost of water security provision.

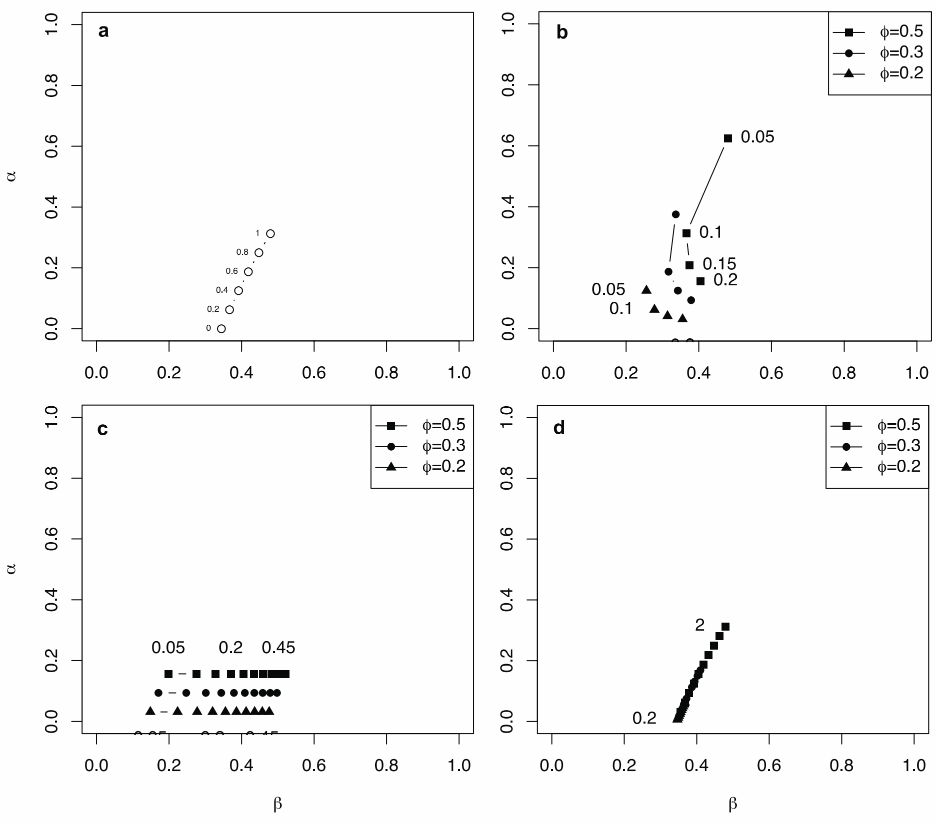

Figure 5 Sensitivity of saddle point location to model parameters. Each figure marks the location of the saddle point: (a) sensitivity of saddle point location to variations in the parameter ϕ over the range 0–1; (b) sensitivity to σ_{max} over the range 0.05–0.2; the filled squares, circles, and triangles show the saddle point location for values of ϕ fixed at 0.5, 0.3, and 0.2, respectively; (c,d) sensitivity to λ_e and λ_w/λ_e respectively; symbols as (b).

The parameter σ_{max} exerts a strong control on the location of the tipping point. When investment in water security is high, the tipping point moves lower in the phase space indicating that a poverty trap is less likely (Figure 5b). It is notable that the effect of varying this parameter depends itself on the value of ϕ: the greater the required investment the more sensitive is the model's response to altering σ_{max}.

Reducing the losses to the economy caused by water-related hazards (λ_e) lowers the likelihood of seeing a poverty trap (Figure 5c). Moreover, the location of the tipping point is also sensitive to the ratio λ_w/λ_e (Figure 5d), which describes the relative resilience of water-related investments compared with risks faced by capital in the wider economy. Although low values of ϕ, σ_{max}, λ_e, and λ_w/λ_e do not eliminate the poverty trap completely, they may move it to a location in the phase space where it can safely be ignored. We note in particular that the dependence of the location of the poverty trap on λ_w/λ_e is not an obvious consequence of the model formulation, but it is intuitively plausible. This finding suggests that investment in protecting productive water infrastructure from natural hazards, or investing in the first place in water-related infrastructure that is resilient to natural hazards, may pay disproportionately high dividends. In practice such physical investments may also require institutional adaptive capacity through the use of insurance and reinsurance contracts and catastrophe bonds to distribute risks among groups large enough to bear significant inter-annual losses. These results may argue for expediting the rehabilitation of water-related assets following damage by natural hazards to provide a more robust infrastructure and institutional platform to promote economic growth [cf. *Noy*, 2009].

4.2 Response to external drivers and policy interventions

Although the trajectories considered in Section 3.2 were permitted to evolve freely through time given a prescribed initial condition, it is instructive also to consider the effect of planned and unplanned interventions in the model state variables. Such disturbances include climate change, which may lead to an exogenous change in water availability or a change in variability or predictability of runoff. Such a change would correspond to a reduction in the value of β, leading to a requirement for greater investment to maintain the same level of risk and, in some cases, moving the system closer to the poverty trap in Figure 3. It is particularly important to pay close attention to climate change adaptation measures in those countries that lie closest to the water-related poverty trap.

An increase in water demand (without a corresponding increase in supply) might likely have a similar effect on β. In circumstances where water demand were to rise due to a decline in infrastructure, leakage or inefficiency it is likely that this would draw the growth trajectory downwards. However, if demand were to increase as a result of economically-productive growth the additional wealth generated would have a countervailing effect. By contrast, a technology- or policy-driven reduction in water demand would lead to an exogenous *increase* in β, and move the country away from the poverty trap.

We note that it is also possible to experience exogenous shifts in GDP due, for example, to the onset of war or the discovery of valuable natural resources. Wars may cause otherwise water-secure countries to descend into water insecurity; the discovery of natural resources may increase access to external capital and may permit increased water-related investments, which could place the country on a trajectory of growth. Whether such growth can be sustained depends on the amount of external capital available relative to the cost of the necessary investment in water-related infrastructure. Some trajectories could require a substantial initial commitment of national wealth to the goal of reducing water-related risks before returns to productivity materialize. Historical studies from the Netherlands, the United Kingdom, and Germany have documented the substantial public investment in water-related infrastructure in the early nineteenth century, which was stimulated by both the requirements of private industries for water and power, and a pressing need to improve public health [*Brown*, 1988; *Geels*, 2005; 2006; *Groote et al.*, 1999; *Hassan*, 1985].

It is important to consider the optimal route away from the poverty trap in Figure 3. The fastest route away from the poverty trap is on a trajectory perpendicular to the separatrix, in which investments in water-related assets are combined with investments in other sectors to stimulate broader economic growth. In other words, water-related investment on its own is not the optimal route to growth as there will almost always be other productive investment opportunities elsewhere in the economy that will be more or less exposed to water-related risks depending on the differing industrial mix in each country. Nonetheless, growth without adequate provision for sustainable water resources management will leave a fragile economy vulnerable to water-related risks. Policy interventions that stand the greatest chance of success include a combination of broad-based measures to simulate growth in national wealth *and* directed investments in water security (infrastructure, institutions and information systems). Indeed such a combination is more likely to stimulate sustained growth than either form of intervention alone.

4.3 Relevance for decision-makers

The combination of productive returns and mitigated water-related risk justifies prioritized investment in water-related assets in relation to investments that do not achieve these benefits. First, such assets bring productivity benefits to individuals, communities and private enterprise. Second, the model highlights the risk reduction brought about by investments in water-related infrastructure assets and the resulting reduction in losses due to natural hazards and disease. Our model begins with these assumptions and offers a number of insights for decision-makers about the design of an optimal investment program in water security. The model supports the intuitive notion that there are conditions in which investments in water-related infrastructure can accelerate economic growth, and that it is necessary to invest more in an economy subject to greater water-related risks. In the face of continued water-related losses, ongoing investment is necessary to maintain the asset base because it remains exposed to water-related hazards.

Guided by these findings, a well-designed program of investment in water-related assets should maximize the difference between the sum of productivity gains and benefits from reductions in water-related risk and the costs of the investments. Two examples of productive multipurpose investments are (i) municipal piped distribution systems that provide improved water supplies for industry and health benefits to households and (ii) reservoirs that generate hydropower, provide water for irrigation, and protect against flood risks. Investments in risk reduction include improved household sanitation and wastewater treatment. Physical investments must always be accompanied by investments in human capital and institutions needed to manage assets and allocate water to different users. The clearest metric of water-related risk is given by λ_e and λ_w, the fraction of growth which is lost per year due to water-related losses. Countries at greatest risk of experiencing a water-related poverty trap will be those with high values of λ_e and λ_w. This is particularly the case where the investment required to achieve water security, ϕ, is high (i.e., water resources per capita are low or unpredictably variable), and where investment per capita in mitigating water-related risks is insufficient to counteract these drags on growth. Special attention should be paid to the resilience of water-related investments in the face of increasing hydro-climatic risk, to enable opportunities elsewhere in the economy to flourish with added resilience to external shocks.

5 Conclusion

In this paper we have presented a dynamical systems model that illustrates the link between national wealth, water-related productivity and losses from water-related hazards. The model consists of two coupled non-linear differential equations that track country wealth and investment in water-related capital respectively. The model reveals that wealthy countries that have limited water-related constraints can experience growth that is unconstrained by hydrology with relatively low investments in water-related infrastructure. By contrast, countries that have more challenging hydrological conditions can experience many pathways to growth, but trajectories with sustainable growth typically require a significant fraction of national wealth to be invested in infrastructure and institutions to increase the productivity of water as a factor input to other sectors of the economy and to reduce water-related risks. Whether investment in growth is sustainable in the face of water-related risk is context dependent and differs according to social and environmental factors. Countries that lack wealth and are confronted with poor water endowments and extreme hydrological variability are most likely to descend into a low-level equilibrium or poverty trap, the location of which is controlled by context-specific social and environmental factors. The model reveals that the location of the poverty trap also depends on the resilience of productive assets to hazard-related loss. We believe that these findings provide important insights for the design of robust policies for investment in water-related productive assets and risk management.

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Tables

Table 1. Notation

Quantity	Definition	Units
$K(t)$	Total country wealth at time t	[$]
$N(t)$	Total investment in water-related assets at time t	[$]
t	Time	[yr]
r	Annual rate of return on investment	[-]
s	Fraction of national wealth invested in water-related assets annually	[-]
l_e	Fraction of national wealth exposed to water-related risks	[-]
l_w	Fraction of water-related assets exposed to water-related risks	[-]
K_0	Potential wealth when unrestricted by water-related factors	$
N_0	Investment in water-related assets required to achieve K_0	$
α	Dimensionless wealth (K/K_0)	[-]
β	Dimensionless investment in water-related assets (N/N_0)	[-]
ϕ	Fraction of national wealth invested in water-related assets when water secure (N_0/K_0)	[-]
$\sigma(\beta)$	Scaled investment function $s(N/N_0)$/r	[-]
σ_{max}	Peak value of investment function	[-]
λ_e, λ_w	Scaled loss functions, l_e/r, and l_w/r	[-]

Figure 1.

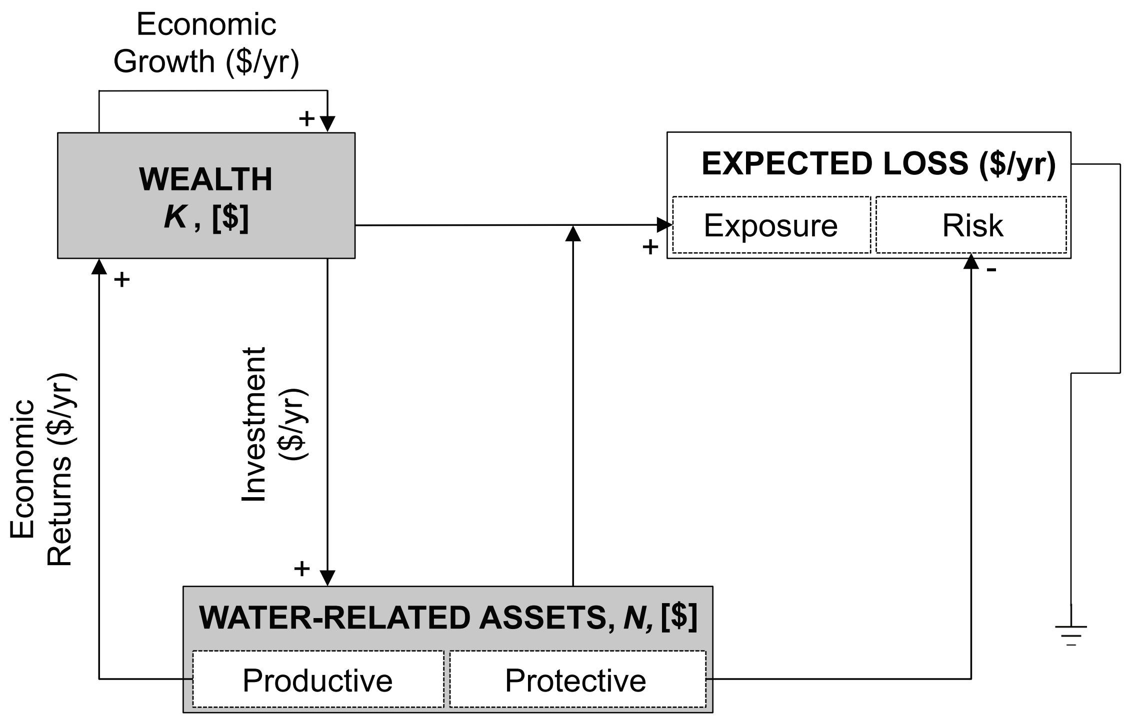
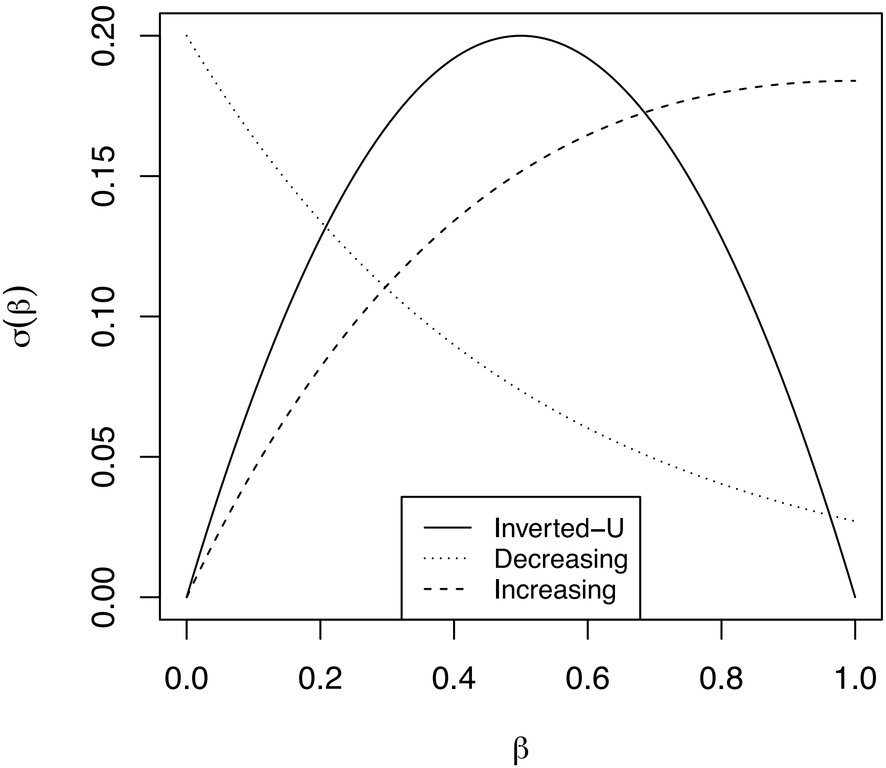

Economic
Growth ($/yr)
+
WEALTH
K , [$]
EXPECTED LOSS ($/yr)
Exposure
Risk
+
+
-
Economic
Returns ($/yr)
Investment
($/yr)
+
WATER-RELATED ASSETS, N, [$]
Productive
Protective

Figure 2.



0.00
0.05
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0.15
0.20
σ(β)
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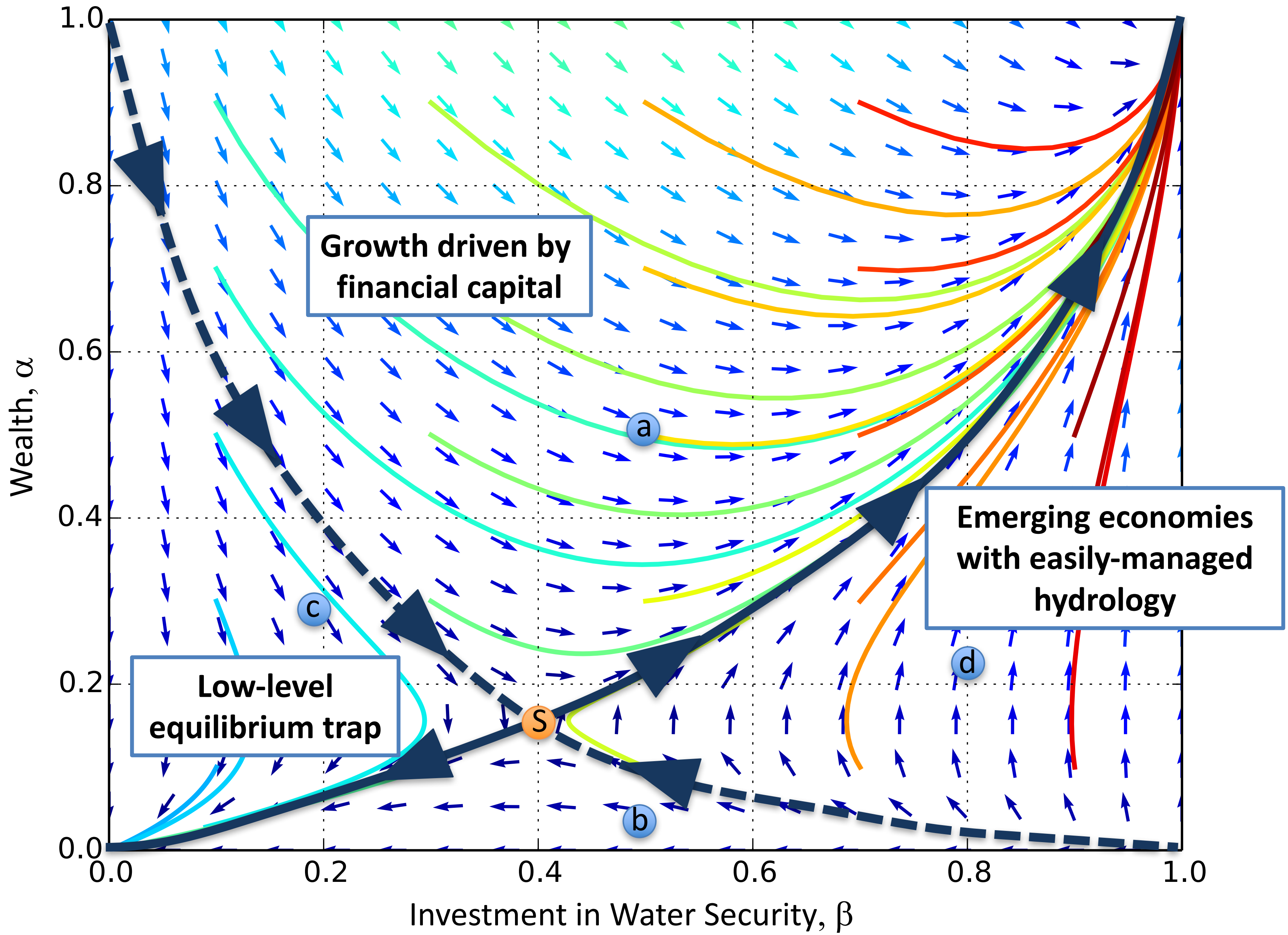
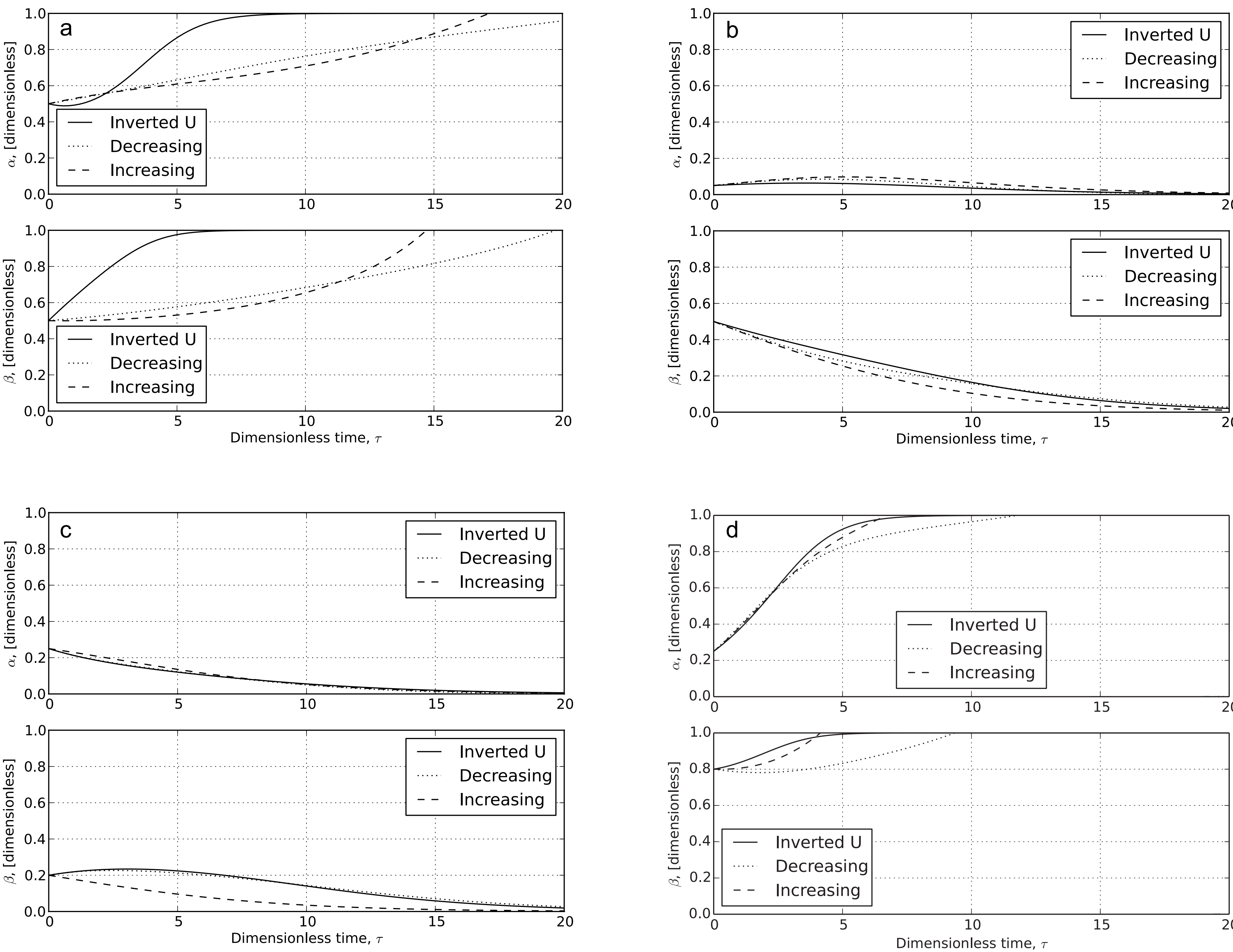
0.2
0.4
0.6
0.8
1.0
β
Inverted−U
Decreasing
Increasing

Figure 3.



Growth driven by financial capital
Emerging economies with easily-managed hydrology
Low-level equilibrium trap
a
b
c
d
S
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
Wealth, α
Investment in Water Security, β

Figure 4.


a
b
c
d
α, [dimensionless]
β, [dimensionless]
Dimensionless time, τ
Inverted U
Decreasing
Increasing
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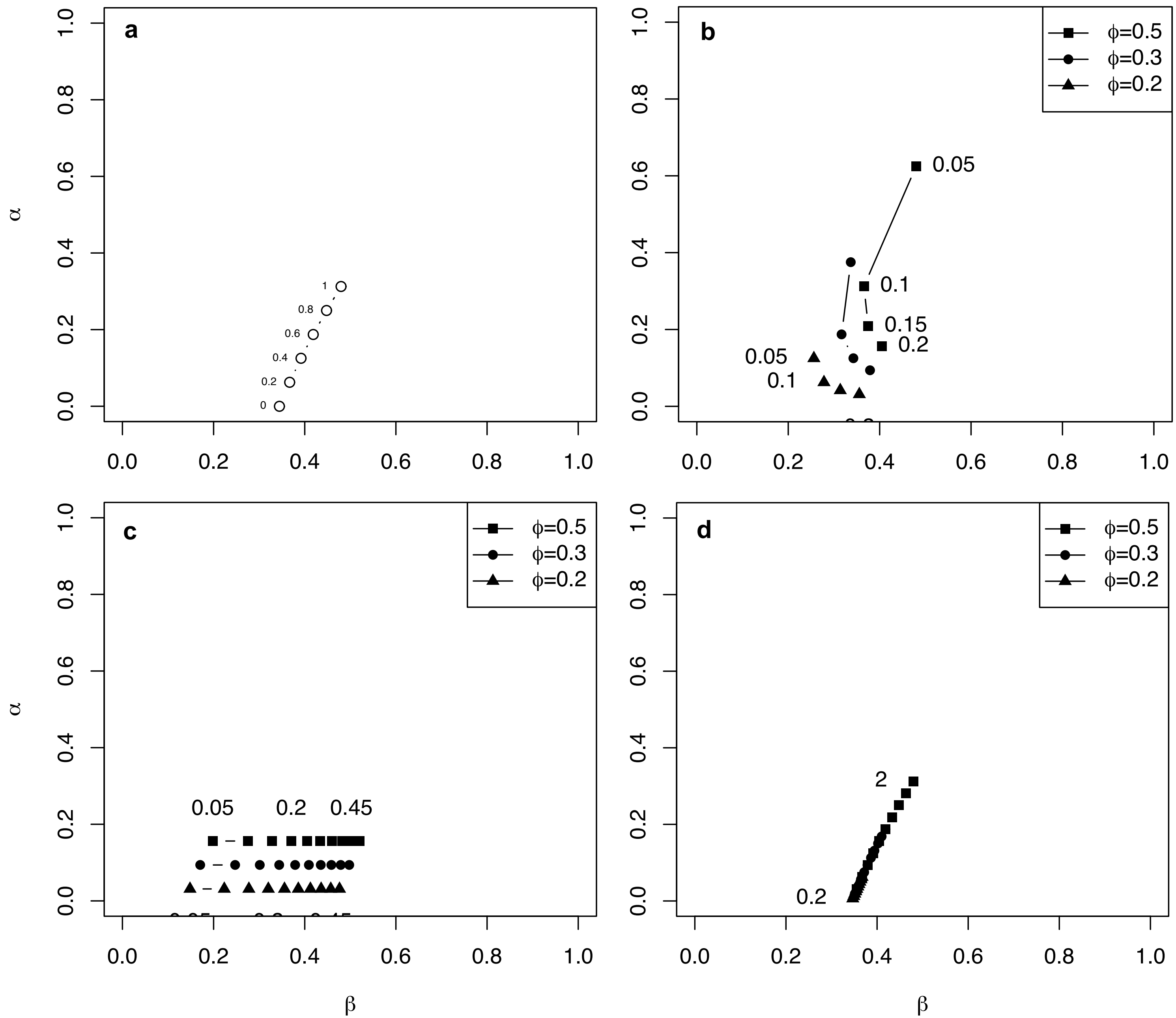
0.6
0.8
1.0
0
5
10
15
20

Figure 5.


a
b
c
d
α
β
φ=0.5
φ=0.3
φ=0.2
0.05
0.1
0.15
0.2
0.45
2