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Algebras of generalized dihedral type

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Abstract. We provide a complete classification of all algebras of generalized dihedral type, which are natural generalizations of algebras which occurred in the study of blocks of group algebras with dihedral defect groups. This gives a description by quivers and relations coming from surface triangulations.

§1. Introduction and main results

The representation theory of associative finite-dimensional algebras over an algebraically closed field K is divided into two disjoint classes, by the Tame and Wild Theorem of Drozd (see [10], [18]). The first class consists of the *tame algebras* for which the indecomposable modules occur in each dimension d in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the *wild algebras* whose representation theory comprises the representation theories of all algebras over K . Tame algebras include the *representation-finite algebras*, which have only finitely many isomorphism classes of indecomposable modules. Of the tame algebras of infinite representation type one distinguishes the *algebras of polynomial growth* for which the number of one-parameter families of indecomposable modules in each dimension d is bounded by d^m , for some positive integer m (depending only on the algebra) [53]. The representation theory of tame self-injective algebras of polynomial growth is currently well understood (see [3], [54], [55] for some general results). On the other hand, for tame algebras which are not of polynomial growth, there are few general tools, and it is open in general how to describe the basic algebras of tame self-injective algebras of non-polynomial growth.

As a step in this direction, we study in this paper a wide class of tame symmetric algebras related to tame blocks of group algebras, and to alge-

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bras constructed from surface triangulations. In the modular representation theory of finite groups, representation-infinite tame blocks occur only in characteristic 2, and their defect groups are dihedral, semidihedral, or quaternion 2-groups (see [5], [23]). In order to study such blocks, algebras of dihedral, semidihedral, and quaternion type have been introduced and investigated, they are defined over algebraically closed fields of arbitrary characteristic (see [23]).

Recently, cluster theory has led to new directions. Soon after the first paper appeared [34], cluster-algebraic structures in Teichmüller theory were defined and studied [37, 32]. These are based on the observation that in hyperbolic geometry an analog of the classical Ptolemy theorem holds. Inspired by this, the authors of [33] use ideas of combinatorial topology and hyperbolic geometry for cluster algebras. They introduce and study systematically cluster algebras ‘of topological origin’, based on surface triangulations. Recall that quiver mutations are essential for cluster theory. In [8] and [33] it was observed that certain quiver mutations can be interpreted in terms of arc flips. On the other hand, [12] introduced the notion of quiver with potential and extended the notion of ordinary quiver mutations. This has led to work of [44] which introduced quivers with potential for the cluster algebras of topological origin, and associated Jacobian algebras. In [45], Ladkani followed this up, and observed that the algebras are similar to algebras of quaternion type, as in [23].

Inspired by this we study in [28] a class of symmetric algebras which are defined in terms of surface triangulations, which we call *weighted surface algebras*. They are tame and we show that they are (with one exception) periodic as algebras, of period 4. We observe that most algebras of quaternion type occur in this setting. Furthermore, most algebras of dihedral type, and of semidihedral type occur naturally as degenerations of these weighted surface algebras. This suggests that blocks of tame representation type are part of a much wider context, which also connects with other parts of mathematics. This paper is part of a programme, exploring such a context.

As for blocks, it is natural to divide this programme into three parts. For tame blocks, those with dihedral defect groups have been studied extensively (see [6], [13], [14], [19], [20], [21], [22], [23], [24], [25], [48], [51] for some results on the structure of these blocks and their representations). In this paper, we investigate the general setting which includes such blocks. We will introduce algebras of generalized dihedral type, and we provide a complete classification. Before giving our definition, we recall basic definitions and

concepts.

Throughout, K will denote a fixed algebraically closed field. By an algebra we mean an associative, finite-dimensional K -algebra with an identity. For an algebra A , we denote by $\text{mod } A$ the category of finite-dimensional right A -modules. An algebra A is called *self-injective* if A_A is an injective module, or equivalently, the projective modules in $\text{mod } A$ are injective. Two self-injective algebras A and Λ are said to be *socle equivalent* if the quotient algebras $A/\text{soc}(A)$ and $\Lambda/\text{soc}(\Lambda)$ are isomorphic. Two socle equivalent self-injective algebras have the same non-projective indecomposable modules. Amongst self-injective algebras, there is the important class of *symmetric algebras* A for which there exists an associative, non-degenerate, symmetric, K -bilinear form $(-, -) : A \times A \rightarrow K$. Classical examples of symmetric algebras are blocks of group algebras of finite groups and the Hecke algebras of finite Coxeter groups. If A is symmetric and e is an idempotent of A then also eAe is symmetric. We call eAe an idempotent algebra of A .

Let A be an algebra. Given a module M in $\text{mod } A$, its *syzygy* is defined to be the kernel $\Omega_A(M)$ of a minimal projective cover of M in $\text{mod } A$. The syzygy operator Ω_A is an important tool to construct modules in $\text{mod } A$ and relate them. For A self-injective, it induces an equivalence of the stable module category $\underline{\text{mod}} A$, and its inverse is the shift of a triangulated structure on $\underline{\text{mod}} A$ [38]. A module M in $\text{mod } A$ is said to be *periodic* if $\Omega_A^n(M) \cong M$ for some $n \geq 1$, and if so the minimal such n is called the *period* of M .

An important combinatorial and homological invariant of the module category $\text{mod } A$ of an algebra A is its Auslander-Reiten quiver Γ_A . Recall that Γ_A is the translation quiver whose vertices are the isomorphism classes of indecomposable modules in $\text{mod } A$, the arrows correspond to irreducible homomorphisms, and the translation is the Auslander-Reiten translation $\tau_A = D \text{Tr}$. For A self-injective, we denote by Γ_A^s the stable Auslander-Reiten quiver of A , obtained from Γ_A by removing the isomorphism classes of projective modules and the arrows attached to them. A stable tube is a translation quiver Γ of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, for some $r \geq 1$, and call r the rank of Γ . We note that, for a symmetric algebra A , we have $\tau_A = \Omega_A^2$ (see [57, Corollary IV.8.6]). Hence the indecomposable modules in stable tubes are precisely the indecomposable periodic modules.

Let A be an algebra. We say that A is of *generalized dihedral type* if it satisfies the following conditions:

- (1) A is symmetric, indecomposable, and tame, with the Grothendieck group $K_0(A)$ of rank at least 2.
- (2) The stable Auslander-Reiten quiver Γ_A^s of A consists of the following components:
 - (i) stable tubes of ranks 1 and 3;
 - (ii) non-periodic components of the form $\mathbb{Z}\mathbb{A}_\infty^\infty$ or $\mathbb{Z}\tilde{\mathbb{A}}_n$. We assume that there is at least one such component.
- (3) Ω_A fixes all stable tubes of rank 3 in Γ_A^s .

We note that every algebra of generalized dihedral type is representation-infinite.

This is an algebraic definition, in terms of homological properties, and it is invariant under Morita equivalence. Blocks with dihedral defect groups (with at least two simple modules) are examples. Geometrically, and motivated by the results of [28] and [30], we introduce biserial weighted surface algebras, they are associated to triangulated surfaces with arbitrarily oriented triangles. As well we study distinguished idempotent algebras of these algebras with respect to families of 2-triangle disks (see Section 7). With this, the algebraic version and the geometric version are the same, which is the main result of this paper.

THEOREM 1. *Let A be a basic algebra over an algebraically closed field K . Then the following statements are equivalent:*

- (i) A is of generalized dihedral type.
- (ii) A is socle equivalent to an idempotent algebra $B(S, \vec{T}, \Sigma, m_\bullet)$ of a biserial weighted surface algebra $B(S, \vec{T}, m_\bullet)$, with respect to a collection Σ of 2-triangle disks of \vec{T} .

Moreover, if K is of characteristic different from 2, we may replace in (ii) “socle equivalent” by “isomorphic”.

The geometric version, that is, part (ii), provides a complete description of these algebras by quivers and relations (see Sections 5, 6, 7). The algebras

in this theorem are in particular socle deformations of the wider class of Brauer graph algebras which have attracted a lot of attention.

In fact, any Brauer graph algebra occurs as an idempotent algebra of some biserial weighted surface algebra, see [29]. Condition (3) in the definition of generalized dihedral type occurs naturally in the geometric version, it comes from degenerating relations around a triangle for a weighted surface algebra. This condition was not part of the original definition of algebras of dihedral type in [21], but it holds for blocks with dihedral defect groups.

The main restriction for blocks is that the Cartan matrix of a block must be non-singular, and that there are few simple modules and tubes of rank 3. This motivates the following definition: The algebra A is of strict dihedral type if it satisfies the above conditions (1), (2) and (3), the Cartan matrix of A is non-singular and in addition the number $\ell(A)$ of simple modules is at most 3, and Γ_A^s has $\ell(A) - 1$ stable tubes of rank 3. These include all non-local blocks of group algebras with dihedral defect groups.

We characterize these algebras among all algebras of generalized dihedral type.

THEOREM 2. *Let A be an algebra over an algebraically closed field K . Then the following statements are equivalent:*

- (i) *A is of strict dihedral type.*
- (ii) *A is of generalized dihedral type and the Cartan matrix C_A of A is non-singular.*

Moreover, we have the following consequences of Theorems 1 and 2.

COROLLARY 3. *Let A be an algebra of generalized dihedral type over an algebraically closed field K . Then A is a biserial algebra.*

COROLLARY 4. *Let A be an algebra of generalized dihedral type over an algebraically closed field K , with the Grothendieck group $K_0(A)$ of rank at least 4. Then the Cartan matrix C_A of A is singular.*

We note that every algebra of generalized dihedral type admits at least one non-periodic simple module (see Corollary 3.2). In the course of the proof of Theorem 1 we will also establish the following corollary.

COROLLARY 5. *Let A be an algebra of generalized dihedral type over an algebraically closed field K . Then the following statements are equivalent:*

- (i) All simple modules in $\text{mod } A$ are non-periodic.
- (ii) The Gabriel quiver Q_A of A is 2-regular.

There are only very few algebras of generalized dihedral type whose stable Auslander-Reiten quiver admits a non-periodic component of Euclidean type $\mathbb{Z}\tilde{A}_n$, they are described in Corollary 8.6.

In fact, we have the following consequences of Theorem 1 and the results established in [25, Sections 2 and 4].

COROLLARY 6. *Let A be an algebra of generalized dihedral type over an algebraically closed field K . Then the following statements are equivalent:*

- (i) A is of polynomial growth.
- (ii) Γ_A^s admits a component of Euclidean type $\mathbb{Z}\tilde{A}_n$.
- (iii) A is of strict dihedral type and Γ_A^s consists of one component of Euclidean type $\mathbb{Z}\tilde{A}_{1,3}$ or $\mathbb{Z}\tilde{A}_{3,3}$, at most two stable tubes of rank 3, and infinitely many stable tubes of rank 1.

For details on Euclidean components of types $\mathbb{Z}\tilde{A}_n$ and $\mathbb{Z}\tilde{A}_{p,q}$ we refer to [25].

COROLLARY 7. *Let A be an algebra of generalized dihedral type over an algebraically closed field K . Then the following statements are equivalent:*

- (i) A is a tame algebra of non-polynomial growth.
- (ii) Γ_A^s admits a component of the form $\mathbb{Z}A_\infty^\infty$.
- (iii) Γ_A^s consists of a finite number of stable tubes of rank 3, infinitely many stable tubes of rank 1, and infinitely many components of the form $\mathbb{Z}A_\infty^\infty$.

There are only finitely many stable tubes of rank 3 in the stable Auslander-Reiten quivers of biserial weighted surface algebras, explicitly we have the following corollary (see the end of Section 5).

COROLLARY 8. *Let A be a basic self-injective algebra which is socle equivalent to a biserial weighted surface algebra $B(S, \vec{T}, m_\bullet)$ of a directed triangulated surface (S, \vec{T}) . Then the number of stable tubes of rank 3 in the stable Auslander-Reiten quiver Γ_A^s of A is equal to the number of triangles in the triangulation T of the surface S .*

This paper is organized as follows. In Section 2 we recall the structure of stable Auslander-Reiten quivers of representation-infinite self-injective special biserial algebras. Section 3 contains a complete description of representation-infinite tame symmetric algebras for which all simple modules are periodic of period 3. In Section 4 we recall from [28] the definition of triangulation quivers and show that they arise naturally from orientation of triangles of triangulated surfaces. In Section 5 we define biserial weighted triangulation algebras and describe their basic properties. Section 6 is devoted to socle deformations of biserial weighted triangulation algebras and their basic properties. In Section 7 we introduce the idempotent algebras of biserial weighted surface algebras occurring in Theorem 1. Section 8 recalls known results on algebras of strict dihedral type and exhibits presentations of these algebras as biserial weighted surface algebras and their socle deformations. In Sections 9 and 10 we prove Theorems 1 and 2.

For general background on the relevant representation theory we refer to the books [1], [23], [52], [57].

§2. Special biserial algebras

A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$ consisting of a finite set Q_0 of vertices, a finite set Q_1 of arrows, and two maps $s, t : Q_1 \rightarrow Q_0$ which associate to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$. We denote by KQ the path algebra of Q over K whose underlying K -vector space has as its basis the set of all paths in Q of length ≥ 0 , and by R_Q the arrow ideal of KQ generated by all paths Q of length ≥ 1 . An ideal I in KQ is said to be *admissible* if there exists some $m \geq 2$ such that $R_Q^m \subseteq I \subseteq R_Q^2$. If I is an admissible ideal in KQ , then the quotient algebra KQ/I is called a bound quiver algebra, and is a finite-dimensional basic K -algebra. Moreover, KQ/I is indecomposable if and only if Q is connected. Every basic, indecomposable, finite-dimensional K -algebra A has a bound quiver presentation $A \cong KQ/I$, where $Q = Q_A$ is the *Gabriel quiver* of A and I is an admissible ideal in KQ (see [1, Section II.3]). For a bound quiver algebra $A = KQ/I$, we denote by e_i , $i \in Q_0$, the associated complete set of pairwise orthogonal primitive idempotents of A , and by $S_i = e_i A / e_i \text{rad } A$ (respectively, $P_i = e_i A$), $i \in Q_0$, the associated complete family of pairwise non-isomorphic simple modules (respectively, indecomposable projective modules) in $\text{mod } A$.

Following [56], an algebra A is said to be *special biserial* if A is isomorphic to a bound quiver algebra KQ/I , where the bound quiver (Q, I)

satisfies the following conditions:

- (a) each vertex of Q is a source and target of at most two arrows,
- (b) for any arrow α in Q there are at most one arrow β and at most one arrow γ with $\alpha\beta \notin I$ and $\gamma\alpha \notin I$.

Moreover, if I is generated by paths of Q , then $A = KQ/I$ is said to be a *string algebra* [7]. Every special biserial algebra is a biserial algebra [35], that is, the radical of any indecomposable non-uniserial projective, left or right, module is a sum of two uniserial modules whose intersection is simple or zero. Moreover, every representation-finite biserial algebra is special biserial, by [56, Lemma 2]. On the other hand, there are many biserial algebras which are not special biserial. In fact, it follows from the main result of [50] that the class of special biserial algebras coincides with the class of biserial algebras which admit simply connected Galois coverings. It has been proved in [59, Theorem 1.4] that every special biserial algebra is a quotient algebra of a symmetric special biserial algebra. We also mention that, if A is a self-injective special biserial algebra, then $A/\text{soc}(A)$ is a string algebra.

The following fact has been proved by Wald and Waschbüsch in [59] (see also [7] and [17] for alternative proofs).

PROPOSITION 2.1. *Every special biserial algebra is tame.*

We refer to [11] and [58] for the structure and tameness of arbitrary biserial algebras.

The following two theorems from [25, Theorems 2.1 and 2.2] describe the structure of the stable Auslander-Reiten quivers of representation-infinite self-injective special biserial algebras.

THEOREM 2.2. *Let A be a special biserial self-injective algebra. Then the following statements are equivalent:*

- (i) A is representation-infinite of polynomial growth.
- (ii) Γ_A^s admits a component of Euclidean type $\tilde{\mathbb{A}}_{p,q}$.
- (iii) There are positive integers m, p, q such that Γ_A^s is a disjoint union of m components of the form $\mathbb{Z}\tilde{\mathbb{A}}_{p,q}$, m components of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^p)$, m components of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^q)$, and infinitely many components of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau)$.

THEOREM 2.3. *Let A be a special biserial self-injective algebra. Then the following statements are equivalent:*

- (i) A is of non-polynomial growth.
- (ii) Γ_A^s admits a component of the form $\mathbb{Z}\mathbb{A}_\infty^\infty$.
- (iii) Γ_A^s is a disjoint union of a finite number of components of the form $\mathbb{Z}\mathbb{A}_\infty^\infty/(\tau^n)$ with $n > 1$, infinitely many components of the form $\mathbb{Z}\mathbb{A}_\infty^\infty$, and infinitely many components of the form $\mathbb{Z}\mathbb{A}_\infty^\infty/(\tau)$.

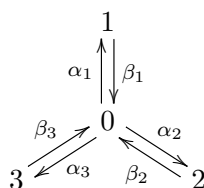
We have also the following fact from [29, Proposition 2.7].

PROPOSITION 2.4. *Let A be a symmetric special biserial algebra and e an idempotent of A . Then eAe is a symmetric special biserial algebra.*

§3. Tame periodic algebras of period three

Let A be an algebra and $A^e = A^{\text{op}} \otimes_K A$ the associated enveloping algebra. Then $\text{mod } A^e$ is the category of finite-dimensional A - A -bimodules. An algebra A is said to be *periodic* if $\Omega_{A^e}^d(A) \cong A$ in $\text{mod } A^e$ for some $d \geq 1$, and if so the minimal such d is called the *period* of A . It is known that if A is a periodic algebra of period d , then A is self-injective and every indecomposable non-projective module M in $\text{mod } A$ is periodic of period dividing d (see [57, Theorem IV.11.19]).

Recall that the preprojective algebra $P(\mathbb{D}_4)$ of Dynkin type \mathbb{D}_4 over K is the bound quiver algebra given by the quiver



and the relations $\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0$, $\beta_1\alpha_1 = 0$, $\beta_2\alpha_2 = 0$, $\beta_3\alpha_3 = 0$.

The following shows that there is a unique representation-infinite tame symmetric algebra such that all simple modules are periodic of period 3.

THEOREM 3.1. *Let A be a basic, indecomposable, tame symmetric algebra of infinite representation type over an algebraically closed field K . Then the following statements are equivalent:*

- (i) All simple modules in $\text{mod } A$ are periodic of period 3.
- (ii) A is a periodic algebra of period 3.
- (iii) K has characteristic 2 and A is isomorphic to the preprojective algebra $P(\mathbb{D}_4)$.

Proof. The implication (ii) \Rightarrow (i) is obvious (see [57, Theorem IV.11.19]). The implication (iii) \Rightarrow (ii) follows from [31, (2.10)]. Assume now that all simple modules in $\text{mod } A$ are periodic of period 3. Then it follows from [2, Theorem 1.2] and [26, Theorem 3.7] that A is socle equivalent to the preprojective algebra $P(\mathbb{D}_4)$. Since A is symmetric, applying [3, Proposition 6.8] and [4, Theorems 1 and Theorems 2], we conclude that K is of characteristic 2 and A is isomorphic to $P(\mathbb{D}_4)$. Hence the implication (i) \Rightarrow (iii) holds.

Since the stable Auslander-Reiten quiver of an algebra of generalized dihedral type has at least one non-periodic component, we have the following consequence, which will be essential for the proof of Theorem 1.

COROLLARY 3.2. *Let A be an algebra of generalized dihedral type. Then $\text{mod } A$ admits at least one non-periodic simple module.*

Proof. By Theorem 3.1, it is enough to show that every periodic simple module in $\text{mod } A$ has period 3. Suppose S is a periodic module in $\text{mod } A$. Then S belongs to a stable tube of Γ_A^s of rank 1 or 3. If S belongs to a stable tube of rank 3 then by the condition (3) S is of period 3. On the other hand, if S belongs to a stable tube of rank 1 then $\Omega_A^2(S) \cong S$, and then A is a local algebra of finite representation type, a contradiction.

REMARK 3.3. Suppose A is an algebra of generalized dihedral type and S is a non-periodic simple module in $\text{mod } A$. Then S belongs to a component of the form $\mathbb{Z}\mathbb{A}_\infty^\infty$ or $\mathbb{Z}\tilde{\mathbb{A}}_n$. This means that if $P(S)$ the projective cover of S , then $\text{rad } P(S)/S$ is a direct sum of two indecomposable modules (see [23, Section I.7]).

§4. Triangulation quivers of surfaces

We recall now the definition of triangulation quivers associated to directed triangulated surfaces, as introduced in [28], and we present two examples.

In this paper, by a *surface* we mean a connected, compact, 2-dimensional real manifold S , orientable or non-orientable, with or without boundary. It is well known that every surface S admits an additional structure of a finite 2-dimensional triangular cell complex, and hence a triangulation, by the deep Triangulation Theorem (see for example [9, Section 2.3]).

For a natural number n , we denote by D^n the unit disk in the n -dimensional Euclidean space \mathbb{R}^n , formed by all points of distance ≤ 1 from the origin. The boundary ∂D^n of D^n is the unit sphere S^{n-1} in \mathbb{R}^n , the points of distance 1 from the origin. Further, by an n -cell we mean a topological space, denoted by e^n , homeomorphic to the open disk $\text{int } D^n = D^n \setminus \partial D^n$. In particular, D^0 and e^0 consist of a single point, and $S^0 = \partial D^1$ consists of two points.

We refer to [39, Appendix] for some basic topological facts about cell complexes.

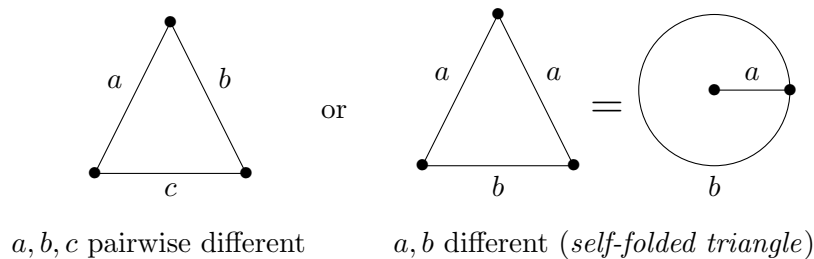
Let S be a surface. In this paper, by a *finite 2-dimensional triangular cell complex structure* on S we mean a family of continuous maps $\varphi_i^n : D_i^n \rightarrow X$, with $n \in \{0, 1, 2\}$ and $D_i^n = D^n$, for i in a finite index set, satisfying the following conditions:

- (1) Each φ_i^n restricts to a homeomorphism from the interior $\text{int } D_i^n$ to the n -cell $e_i^n = \varphi_i^n(\text{int } D_i^n)$ of S , and these cells are all disjoint and their union is S .
- (2) For each 2-dimensional cell e_i^2 , the set $\varphi_i^2(\partial D_i^2)$ is the union of k 1-cells and m 0-cells, with $k \in \{2, 3\}$ and $m \in \{1, 2, 3\}$.

Then the closure $\varphi_i^2(D_i^2)$ of a 2-cell e_i^2 is called a *triangle* of S , and the closure $\varphi_i^1(D_i^1)$ of a 1-cell e_i^1 is called an *edge* of S . The collection T of all triangles $\varphi_i^2(D_i^2)$ is said to be a *triangulation* of S .

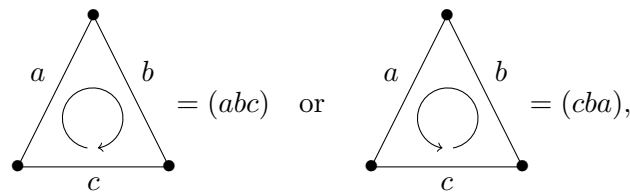
We assume that such a triangulation T of S has at least two different edges, or equivalently, there are at least two different 1-cells in the corresponding triangular cell complex structure on S . Then T is a finite collection

of triangles of the form

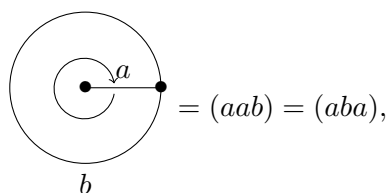


such that every edge of such a triangle in T is either the edge of exactly two triangles, or is the self-folded edge, or else lies on the boundary. A given surface S admits many finite 2-dimensional triangular cell complex structures, and hence triangulations. We refer to [9], [40], [41] for general background on surfaces and constructions of surfaces from plane models.

Let S be a surface and T a triangulation S . To each triangle Δ in T we may associate an orientation

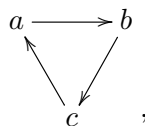


if Δ has pairwise different edges a, b, c , and

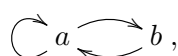


if Δ is self-folded, with the self-folded edge a , and the other edge b . Fix an orientation of each triangle Δ of T , and denote this choice by \vec{T} . Then the pair (S, \vec{T}) is said to be a *directed triangulated surface*. To each directed triangulated surface (S, \vec{T}) we associate the quiver $Q(S, \vec{T})$ whose vertices are the edges of T and the arrows are defined as follows:

- (1) for any oriented triangle $\Delta = (abc)$ in \vec{T} with pairwise different edges a, b, c , we have the cycle



- (2) for any self-folded triangle $\Delta = (aab)$ in \vec{T} , we have the quiver



- (3) for any boundary edge a in T , we have the loop



Then $Q = Q(S, \vec{T})$ is a triangulation quiver in the following sense (introduced independently by Ladkani in [46]). For the history see the Acknowledgements in [28].

A *triangulation quiver* is a pair (Q, f) , where $Q = (Q_0, Q_1, s, t)$ is a finite connected quiver and $f : Q_1 \rightarrow Q_1$ is a permutation on the set Q_1 of arrows of Q satisfying the following conditions:

- (a) every vertex $i \in Q_0$ is the source and target of exactly two arrows in Q_1 ,
- (b) for each arrow $\alpha \in Q_1$, we have $s(f(\alpha)) = t(\alpha)$,
- (c) f^3 is the identity on Q_1 .

For the quiver $Q = Q(S, \vec{T})$ of a directed triangulated surface (S, \vec{T}) , the pair (Q, f) is a triangulation quiver, where the permutation f on its set of arrows is defined as follows:

$$(1) \quad \begin{array}{ccc} a & \xrightarrow{\alpha} & b \\ & \searrow \gamma & \swarrow \beta \\ & c & \end{array} \quad f(\alpha) = \beta, f(\beta) = \gamma, f(\gamma) = \alpha,$$

for an oriented triangle $\Delta = (abc)$ in \vec{T} , with pairwise different edges a, b, c ,

$$(2) \quad \alpha \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} a \begin{array}{c} \xrightarrow{\beta} \\ \xleftarrow{\gamma} \end{array} b \quad f(\alpha) = \beta, f(\beta) = \gamma, f(\gamma) = \alpha,$$

for a self-folded triangle $\Delta = (aab)$ in \vec{T} ,

$$(3) \quad \alpha \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} a \quad f(\alpha) = \alpha,$$

for a boundary edge a of T .

We note that, if (Q, f) is a triangulation quiver, then Q is a 2-regular quiver. We will consider only the triangulation quivers with at least two vertices.

We would like to mention that different directed triangulated surfaces (even of different genus) may lead to the same triangulation quiver (see [28, Example 4.3]).

Let (Q, f) be a triangulation quiver. Then we have the involution $\bar{\cdot} : Q_1 \rightarrow Q_1$ which assigns to an arrow $\alpha \in Q_1$ the arrow $\bar{\alpha}$ with $s(\alpha) = s(\bar{\alpha})$ and $\alpha \neq \bar{\alpha}$. Then we obtain another permutation $g : Q_1 \rightarrow Q_1$ of the set Q_1 of arrows of Q such that $g(\alpha) = \bar{f(\bar{\alpha})}$ for any $\alpha \in Q_1$. We denote by $\mathcal{O}(g)$ the set of all g -orbits in Q_1 .

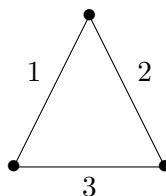
The following theorem and its consequence have been established in [28, Section 4] (see also Example 8.2 for the case with two vertices).

THEOREM 4.1. *Let (Q, f) be a triangulation quiver. Then there exists a directed triangulated surface (S, \vec{T}) such that $(Q, f) = (Q(S, \vec{T}), f)$.*

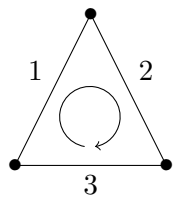
COROLLARY 4.2. *Let (Q, f) be a triangulation quiver. Then Q contains a loop α with $f(\alpha) = \alpha$ if and only if $(Q, f) = (Q(S, \vec{T}), f)$ for a directed triangulated surface (S, \vec{T}) with S having non-empty boundary.*

We will present now two examples of triangulation quivers associated to directed triangulated surfaces. Further examples may be found in [28].

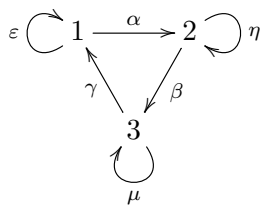
EXAMPLE 4.3. Let S consist of one triangle T



with the three pairwise different edges, forming the boundary of S , and consider the orientation \vec{T} of T

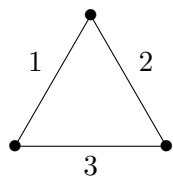


of T . Then the triangulation quiver $(Q(S, \vec{T}), f)$ is the quiver

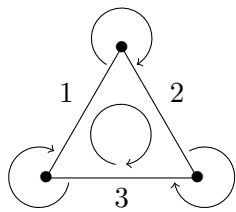


with f -orbits $(\alpha \beta \gamma)$, (ε) , (η) , (μ) . Observe that we have only one g -orbit $(\alpha \eta \beta \mu \gamma \varepsilon)$ of arrows in $Q(S, \vec{T})$. In particular, $|\mathcal{O}(g)| = 1$.

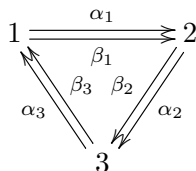
EXAMPLE 4.4. Let S be the sphere with the triangulation T



given by two unfolded triangles, and \vec{T} the following orientation



of triangles of T . Then the triangulation quiver $(Q(S, \vec{T}), f)$ is the quiver



with the f -orbits $(\alpha_1 \alpha_2 \alpha_3)$ and $(\beta_1 \beta_2 \beta_3)$. Then $\mathcal{O}(g)$ consists of one g -orbit $(\alpha_1 \beta_2 \alpha_3 \beta_1 \alpha_2 \beta_3)$. This triangulation quiver is called the *Markov quiver* (see [47], [49] for justification of this name).

§5. Biserial weighted triangulation algebras

Let (Q, f) be a triangulation quiver, so that we have two permutations $f : Q_1 \rightarrow Q_1$ and $g : Q_1 \rightarrow Q_1$ on the set Q_1 of arrows of Q such that f^3 is the identity on Q_1 and $g = \bar{f}$, where $\bar{\cdot} : Q_1 \rightarrow Q_1$ is the involution which assigns to an arrow $\alpha \in Q_1$ the arrow $\bar{\alpha}$ with $s(\alpha) = s(\bar{\alpha})$ and $\alpha \neq \bar{\alpha}$. For each arrow $\alpha \in Q_1$, we denote by $\mathcal{O}(\alpha)$ the g -orbit of α in Q_1 , and set $n_\alpha = n_{\mathcal{O}(\alpha)} = |\mathcal{O}(\alpha)|$. A function

$$m_\bullet : \mathcal{O}(g) \rightarrow \mathbb{N}^* = \mathbb{N} \setminus \{0\}$$

is said to be a *weight function* of (Q, f) . Write $m_\alpha = m_{\mathcal{O}(\alpha)}$ for $\alpha \in Q_1$. For any arrow $\alpha \in Q_1$, we have the oriented cycle

$$B_\alpha = \left(\alpha g(\alpha) \dots g^{n_\alpha-1}(\alpha) \right)^{m_\alpha}$$

of length $m_\alpha n_\alpha$. The triple (Q, f, m_\bullet) is said to be a *weighted triangulation quiver*.

Let (Q, f, m_\bullet) be a weighted triangulation quiver. We consider the quotient algebra

$$B(Q, f, m_\bullet) = KQ/J(Q, f, m_\bullet),$$

where $J(Q, f, m_\bullet)$ is the ideal in the path algebra KQ of Q over K generated by the elements:

- (1) $\alpha f(\alpha)$, for all arrows $\alpha \in Q_1$.
- (2) $B_\alpha - B_{\bar{\alpha}}$, for all arrows $\alpha \in Q_1$,

Then $B(Q, f, m_\bullet)$ is said to be a *biserial weighted triangulation algebra*. Let (S, \vec{T}) be a directed triangulated surface, $(Q(S, \vec{T}), f)$ the associated triangulation quiver, and m_\bullet a weight function of $(Q(S, \vec{T}), f)$. Then the biserial weighted triangulation algebra $B(Q(S, \vec{T}), f, m_\bullet)$ will be called a *biserial weighted surface algebra*, and denoted by $B(S, \vec{T}, m_\bullet)$.

REMARK 5.1. We note that the Gabriel quiver of a biserial weighted triangulation algebra $B(Q, f, m_\bullet)$ is the subquiver of the triangulation quiver (Q, f) obtained by removing the loops α fixed by g such that $m_\alpha = 1$. Namely, if α is such a loop then the element B_α occurring in the definition of $B(Q, f, m_\bullet)$ is equal to α and therefore, by condition (2) of that definition, α is in the square of the radical.

The following proposition describes basic properties of biserial weighted triangulation algebras.

PROPOSITION 5.2. *Let $B = B(Q, f, m_\bullet)$ be a biserial weighted triangulation algebra. Then the following statements hold.*

- (i) *B is finite-dimensional special biserial with $\dim_K B = \sum_{\mathcal{O} \in \mathcal{O}(g)} m_{\mathcal{O}} n_{\mathcal{O}}^2$.*
- (ii) *B is a tame symmetric algebra.*
- (iii) *B is an algebra of generalized dihedral type.*

Proof. We set $J = J(Q, f, m_\bullet)$. We recall that Q has at least two vertices.

(i) It follows from the definition and Remark 5.1 that B is a finite-dimensional special biserial algebra. Let i be a vertex of Q and $\alpha, \bar{\alpha}$ the two arrows in Q with source i . Then the indecomposable projective right B -module $P_i = e_i B$ at vertex i has dimension $\dim_K P_i = m_\alpha n_\alpha + m_{\bar{\alpha}} n_{\bar{\alpha}}$. Indeed, P_i has a basis given by e_i , the cosets $u + J$ of all initial proper subwords u of B_α and $B_{\bar{\alpha}}$, and $B_\alpha + J = B_{\bar{\alpha}} + J$. Hence we deduce that

$$\dim_K B = \sum_{\mathcal{O} \in \mathcal{O}(g)} m_{\mathcal{O}} n_{\mathcal{O}}^2.$$

(ii) It is well known (see [57, Theorem IV.2.2]) that B is symmetric if and only if there exists a K -linear form $\varphi : B \rightarrow K$ such that $\varphi(ab) = \varphi(ba)$ for all $a, b \in B$ and $\text{Ker } \varphi$ does not contain any non-zero one-sided ideal of

B (called a symmetrizing form). We observe that, for any vertex i of Q and the arrows $\alpha, \bar{\alpha}$ with source i , the indecomposable projective module $P_i = e_i B$ has one-dimensional socle generated by $B_\alpha + J = B_{\bar{\alpha}} + J$. Clearly, we have also $\text{top}(P_i) = S_i = \text{soc}(P_i)$. We define a required symmetrizing form $\varphi : B \rightarrow K$ by assigning to the coset $u + J$ of a path u in Q the following element of K

$$\varphi(u + J) = \begin{cases} 1 & \text{if } u = B_\alpha \text{ for an arrow } \alpha \in Q_1, \\ 0 & \text{otherwise,} \end{cases}$$

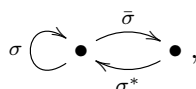
and extending to a K -linear form. It follows from Proposition 2.1 that B is tame.

(iii) First we show that B is representation-infinite. Let $B' = B/\text{soc}(B)$. Then B' is an algebra of the form KQ/I , where I is the ideal in KQ generated by the elements of the forms $\alpha f(\alpha)$ and B_α , for all arrows $\alpha \in Q_1$. Moreover, B is representation-infinite if and only if B' is representation-infinite. We note that $KQ/I = KQ'/I'$, where $Q' = Q_B = Q_{B'}$, with Q_B the Gabriel quiver of B and $Q_{B'}$ the Gabriel quiver of B' , and I' is the ideal in KQ' generated by the elements of the forms $\alpha f(\alpha)$, with $\alpha, f(\alpha)$ in Q'_1 , and B_γ , for all arrows $\gamma \in Q'_1$. In particular, we conclude that $B' = KQ'/I'$ is a string algebra. It follows from [56, Theorem 1] that B' is representation-infinite if and only if (Q', I') admits a primitive walk. For each arrow $\alpha \in Q_1$, we denote by α^{-1} the formal inverse of α and set $s(\alpha^{-1}) = t(\alpha)$ and $t(\alpha^{-1}) = s(\alpha)$. By a walk in Q we mean a sequence $w = \alpha_1 \dots \alpha_n$, where α_i is an arrow or the inverse of an arrow in Q , satisfying the conditions: $t(\alpha_i) = s(\alpha_{i+1})$ and $\alpha_{i+1} \neq \alpha_i^{-1}$ for any $i \in \{1, \dots, n-1\}$. Moreover, w is said to be a bipartite walk if, for any $i \in \{1, \dots, n-1\}$, exactly one of α_i and α_{i+1} is an arrow. A walk $w = \alpha_1 \dots \alpha_n$ in Q with $s(\alpha_1) = t(\alpha_n)$ is called closed. A closed walk w in Q is called a primitive walk if $w \neq v^r$ for any closed walk v in Q and positive integer r . We claim that for any arrow $\alpha \in Q_1$, there is a bipartite primitive walk $w(\alpha)$ in Q containing the arrow α . Since Q is a 2-regular quiver, we have two involutions $\bar{\cdot} : Q_1 \rightarrow Q_1$ and $\ast : Q_1 \rightarrow Q_1$. The first involution assigns to each arrow $\alpha \in Q_1$ the arrow $\bar{\alpha}$ with $s(\alpha) = s(\bar{\alpha})$ and $\alpha \neq \bar{\alpha}$. The second involution assigns to each arrow $\alpha \in Q_1$ the arrow α^\ast with $t(\alpha) = t(\alpha^\ast)$ and $\alpha \neq \alpha^\ast$. Consider the bijection $h : Q_1 \rightarrow Q_1$ such that $h(\alpha) = \bar{\alpha}^\ast$ for any arrow $\alpha \in Q_1$. Clearly, h has finite order. In particular, for any arrow $\alpha \in Q_1$, there exists a minimal positive integer r such that $h^r(\alpha) = \alpha$. Then

the required bipartite primitive walk $w(\alpha)$ is of the form

$$\alpha(\alpha^*)^{-1}h(\alpha)(h(\alpha)^*)^{-1} \dots h^r(\alpha)(h^{r-1}(\alpha)^*)^{-1}.$$

Observe now that, if σ is an arrow in Q_1 with $m_\sigma n_\sigma = 1$, then σ is a loop and Q admits a subquiver of the form



with $f(\sigma) = \bar{\sigma}$, $f(\bar{\sigma}) = \sigma^*$, $f(\sigma^*) = \sigma$. Moreover, for such a subquiver, the path $\sigma^* \bar{\sigma}$ is not in I' , because $\bar{\sigma} = g(\sigma^*)$ and $\sigma^* \neq g(\bar{\sigma})$. Take an arrow $\alpha \in Q_1$ which is not a loop. We denote by $w(\alpha)_0$ the primitive walk in Q' obtained from $w(\alpha)$ by removing all loops σ (respectively, the inverse loops σ) such that $m_\sigma n_\sigma = 1$. Then $w(\alpha)_0$ is a primitive walk in (Q', I') , that is, $w(\alpha)_0$ does not contain a subpath v such that v or v^{-1} belongs to I' . Therefore, B' is representation-infinite, and hence B is representation-infinite.

Since B is a representation-infinite special biserial symmetric algebra, the structure of the stable Auslander-Reiten quiver Γ_B^s of B is described in Theorems 2.2 and 2.3. Hence, in order to prove that B is an algebra of generalized dihedral type, it remains to show that, if \mathcal{T} is a stable tube of rank $r \geq 2$ in Γ_B^s , then \mathcal{T} is of rank 3 and Ω_B fixes \mathcal{T} . By the general theory of special biserial algebras, the stable tubes of ranks at least 2 consist entirely of string modules (see [7], [17], [59]). Moreover, it follows from [7, Section 3] that the mouth of stable tubes of ranks at least 2 are formed by the uniserial string modules given by the arrows of the Gabriel quiver Q_B of B . We recall that the Gabriel quiver Q_B of B is obtained from the quiver Q by removing all loops α with $m_\alpha n_\alpha = 1$. Further, if α is a loop in Q_1 with $m_\alpha n_\alpha = 1$, then $P_{s(\alpha)} = e_{s(\alpha)}B$ is a uniserial module and $\text{rad } P_{s(\alpha)} = \bar{\alpha}B$. On the other hand, if α is an arrow in Q_1 with $m_\alpha n_\alpha \geq 2$ and $m_{\bar{\alpha}} n_{\bar{\alpha}} \geq 2$, then the indecomposable projective module $P_{s(\alpha)} = e_{s(\alpha)}B$ is not uniserial, $\text{rad } P_{s(\alpha)} = \alpha B + \bar{\alpha}B$, and $\alpha B \cap \bar{\alpha}B = \text{soc}(P_{s(\alpha)})$ is the one-dimensional space generated by $B_\alpha = B_{\bar{\alpha}}$.

Let α be an arrow in Q_1 , and $U(\alpha) = \alpha B$. Then $U(\alpha)$ is a uniserial module such that $P_{s(\alpha)}/U(\alpha)$ is isomorphic to $U(f^{-1}(\alpha))$. We also note that, if α is a loop with $m_\alpha n_\alpha = 1$, then $U(\alpha) = \alpha B = B_\alpha B$ is simple and $P_{s(\alpha)}/U(\alpha) = U(\alpha^*)$ for the unique arrow α^* in Q_1 with $t(\alpha^*) = t(\alpha)$ and

$\alpha^* \neq \alpha$. We have the canonical short exact sequence in $\text{mod } B$

$$0 \rightarrow U(f(\alpha)) \hookrightarrow P_{t(\alpha)} \xrightarrow{\pi_\alpha} U(\alpha) \rightarrow 0$$

with π_α being the projective cover, and hence $\Omega_B(U(\alpha)) = U(f(\alpha))$. In particular, we conclude that $\Omega_B(U(\alpha)) \cong U(\alpha)$ if $f(\alpha) = \alpha$, and $U(\alpha)$ is a periodic module of period 3 if $f(\alpha) \neq \alpha$. Since B is a symmetric algebra we have $\tau_B = \Omega_B^2$. Hence, if $f(\alpha) = \alpha$, the module $U(\alpha)$ forms the mouth of a stable tube in Γ_B^s of rank 1. On the other hand, if $f(\alpha) \neq \alpha$, then the modules $U(\alpha)$, $U(f(\alpha))$, $U(f^2(\alpha))$ form the mouth of a stable tube in Γ_B^s of rank 3, and the Auslander-Reiten translation τ_B acts on these modules as follows

$$\tau_B U(\alpha) = U(f^2(\alpha)), \quad \tau_B U(f^2(\alpha)) = U(f(\alpha)), \quad \tau_B U(f(\alpha)) = U(\alpha).$$

It follows from [7, Section 3] that the uniserial modules $U(\alpha)$, $\alpha \in Q_1$, are the only string modules in $\text{mod } B$ lying on the mouths of stable tubes in Γ_B^s . Therefore, the stable tubes in Γ_B^s are of ranks 1 and 3, and Ω_B fixes all stable tubes of rank 3 in Γ_B^s . Summing up, we conclude that B is an algebra of generalized dihedral type.

Proof of Corollary 8. Let A be a basic self-injective algebra which is socle equivalent to a biserial weighted surface algebra $B = B(S, \vec{T}, m_\bullet)$ of a directed triangulated surface (S, \vec{T}) . Since the stable Auslander-Reiten quivers Γ_A^s and Γ_B^s are isomorphic, we may assume that $A = B$. We observe now that there is a bijection between the triangles in T and the f -orbits of length 3 in the associated triangulation quiver $(Q(S, \vec{T}), f)$, defined in Section 4. Further, it follows from the final part of the above proof of Proposition 5.2, that the mouth of stable tubes of rank 3 in Γ_B^s are formed by the uniserial modules associated to the arrows of f -orbits of length 3 in $(Q(S, \vec{T}), f)$. Therefore, the statement of Corollary 8 follows.

§6. Socle deformed biserial weighted triangulation algebras

In this section we introduce socle deformations of biserial weighted triangulation algebras occurring in the characterization of algebras of generalized dihedral type.

For a positive integer d , we denote by $\text{alg}_d(K)$ the affine variety of associative K -algebra structures with identity on the affine space K^d . The general linear group $\text{GL}_d(K)$ acts on $\text{alg}_d(K)$ by transport of the structures, and the $\text{GL}_d(K)$ -orbits in $\text{alg}_d(K)$ correspond to the isomorphism

classes of d -dimensional algebras (see [42] for details). We identify a d -dimensional algebra A with the point of $\text{alg}_d(K)$ corresponding to it. For two d -dimensional algebras A and B , we say that B is a *degeneration* of A (A is a *deformation* of B) if B belongs to the closure of the $\text{GL}_d(K)$ -orbit of A in the Zariski topology of $\text{alg}_d(K)$.

Geiss' Theorem [36] says that if A and B are two d -dimensional algebras, A degenerates to B and B is a tame algebra, then A is also a tame algebra (see also [11]). We will apply this theorem in the following special situation.

PROPOSITION 6.1. *Let d be a positive integer, and $A(t)$, $t \in K$, be an algebraic family in $\text{alg}_d(K)$ such that $A(t) \cong A(1)$ for all $t \in K \setminus \{0\}$. Then $A(1)$ degenerates to $A(0)$. In particular, if $A(0)$ is tame, then $A(1)$ is tame.*

A family of algebras $A(t)$, $t \in K$, in $\text{alg}_d(K)$ is said to be *algebraic* if the induced map $A(-) : K \rightarrow \text{alg}_d(K)$ is a regular map of affine varieties.

Let (Q, f) be a triangulation quiver. A vertex $i \in Q_0$ is said to be a *border vertex* of (Q, f) if there is a loop α at i with $f(\alpha) = \alpha$, which we call a *border loop*. We denote by $\partial(Q, f)$ the set of all border vertices of (Q, f) , and call it the *border* of (Q, f) . Observe that, if (S, \vec{T}) is a directed triangulated surface with $(Q(S, \vec{T}), f) = (Q, f)$, then the border vertices of (Q, f) correspond bijectively to the boundary edges of the triangulation T of S . Hence, the border $\partial(Q, f)$ of (Q, f) is not empty if and only if the boundary ∂S of S is not empty. A function

$$b_\bullet : \partial(Q, f) \rightarrow K$$

is said to be a *border function* of (Q, f) .

Let (Q, f) be a triangulation quiver, and assume that the border $\partial(Q, f)$ of (Q, f) is not empty. Let $m_\bullet : \mathcal{O}(g) \rightarrow \mathbb{N}^*$ be a weight function and $b_\bullet : \partial(Q, f) \rightarrow K$ a border function of (Q, f) . We consider the quotient algebra

$$B(Q, f, m_\bullet, b_\bullet) = KQ/J(Q, f, m_\bullet, b_\bullet),$$

where $J(Q, f, m_\bullet, b_\bullet)$ is the ideal in the path algebra KQ of Q over K generated by the elements:

- (1) $\alpha f(\alpha)$, for all arrows $\alpha \in Q_1$ which are not border loops,
- (2) $\alpha^2 - b_{s(\alpha)} B_\alpha$, for all border loops $\alpha \in Q_1$,
- (3) $B_\alpha - B_{\bar{\alpha}}$, for all arrows $\alpha \in Q_1$.

Then $B(Q, f, m_\bullet, b_\bullet)$ is said to be a *socle deformed biserial weighted triangulation algebra*. Moreover, if $(Q, f) = (Q(S, \vec{T}), f)$ for a directed triangulated surface (S, \vec{T}) , then $B(Q(S, \vec{T}), f, m_\bullet, b_\bullet)$ is said to be a *socle deformed biserial weighted surface algebra*, and is denoted by $B(S, \vec{T}, m_\bullet, b_\bullet)$.

PROPOSITION 6.2. *Let (Q, f) be a triangulation quiver with $\partial(Q, f)$ not empty, m_\bullet, b_\bullet weight and border functions of (Q, f) , $\bar{B} = B(Q, f, m_\bullet, b_\bullet)$, and $B = B(Q, f, m_\bullet)$. Then the following statements hold.*

- (i) \bar{B} is finite-dimensional biserial with $\dim_K \bar{B} = \sum_{\mathcal{O} \in \mathcal{O}(g)} m_{\mathcal{O}} n_{\mathcal{O}}^2$.
- (ii) \bar{B} is a symmetric algebra.
- (iii) \bar{B} is socle equivalent to B .
- (iv) \bar{B} degenerates to B .
- (v) \bar{B} is a tame algebra.
- (vi) \bar{B} is an algebra of generalized dihedral type.

Proof. Write $\bar{J} = J(Q, f, m_\bullet, b_\bullet)$.

(i) Let i be the vertex of Q and $\alpha, \bar{\alpha}$ the two arrows in Q with source i . Then the indecomposable projective right \bar{B} -module $P_i = e_i \bar{B}$ at the vertex i has a basis given by e_i , the cosets $u + \bar{J}$ of all initial proper subwords u of B_α and $B_{\bar{\alpha}}$, and $B_\alpha + \bar{J} = B_{\bar{\alpha}} + \bar{J}$, and hence $\dim_K P_i = m_\alpha n_\alpha + m_{\bar{\alpha}} n_{\bar{\alpha}}$. Then we obtain

$$\dim_K \bar{B} = \sum_{\mathcal{O} \in \mathcal{O}(g)} m_{\mathcal{O}} n_{\mathcal{O}}^2.$$

Clearly, \bar{B} is a biserial algebra.

(ii) We define a symmetrizing form $\bar{\varphi} : \bar{B} \rightarrow K$ of $\bar{B} = KQ/\bar{J}$ by assigning to the coset $u + \bar{J}$ of a path u in Q the following element

$$\bar{\varphi}(u + \bar{J}) = \begin{cases} 1 & \text{if } u = B_\alpha \text{ for an arrow } \alpha \in Q_1, \\ b_i & \text{if } u = \alpha^2 \text{ for some border loop } \alpha \in Q_1, \\ 0 & \text{otherwise,} \end{cases}$$

and extending to a K -linear form.

(iii) The algebras $B/\text{soc}(B)$ and $\bar{B}/\text{soc}(\bar{B})$ are isomorphic to the algebra KQ/I , where I is the ideal in KQ generated by the elements $\alpha f(\alpha)$ and B_α , for all arrows α in Q_1 . Hence B and \bar{B} are socle equivalent.

(iv) For each $t \in K$, consider the quotient algebra $\bar{B}(t) = KQ/\bar{J}^{(t)}$, where $\bar{J}^{(t)}$ is the ideal in KQ generated by the elements:

- (1) $\alpha f(\alpha)$, for all arrows $\alpha \in Q_1$ which are not border loops,
- (2) $\alpha^2 - tb_{s(\alpha)}B_{\bar{\alpha}}$, for all border loops $\alpha \in Q_1$,
- (3) $B_\alpha - B_{\bar{\alpha}}$, for all arrows $\alpha \in Q_1$.

Then $\bar{B}(t)$, $t \in K$, is an algebraic family in the variety $\text{alg}_d(K)$, with $d = \dim_K \bar{B}$, such that $\bar{B}(t) \cong \bar{B}(1) = \bar{B}$ for all $t \in K^*$ and $\bar{B}(0) \cong B$. Then it follows from Proposition 6.1 that \bar{B} degenerates to B .

(v) \bar{B} is a tame algebra, since it is socle equivalent to the tame algebra B .

(vi) Since B and \bar{B} are socle equivalent, their stable Auslander-Reiten quivers Γ_B^s and $\Gamma_{\bar{B}}^s$ are isomorphic as translation quivers. Moreover, $\tau_B = \Omega_B^2$ and $\tau_{\bar{B}} = \Omega_{\bar{B}}^2$, because B and \bar{B} are symmetric. Finally, we observe that the actions of the syzygy operators Ω_B and $\Omega_{\bar{B}}$ on the uniserial modules $U(\alpha)$, for $\alpha \in Q_1$ with $f(\alpha) \neq \alpha$, forming the mouth of stable tubes of rank 3 in $\Gamma_B^s = \Gamma_{\bar{B}}^s$ coincide. Therefore, \bar{B} is an algebra of generalized dihedral type.

Let (Q, f) be a triangulation quiver, α an arrow in Q_1 , and $i = s(\alpha)$. We define also the path

$$A_\alpha = \begin{cases} (\alpha g(\alpha) \dots g^{n_\alpha-1}(\alpha))^{m_\alpha-1} \alpha g(\alpha) \dots g^{n_\alpha-2}(\alpha), & \text{if } n_\alpha \geq 2, \\ \alpha^{m_\alpha-1}, & \text{if } n_\alpha = 1 \text{ and } m_\alpha \geq 2, \\ e_i, & \text{if } n_\alpha = 1 \text{ and } m_\alpha = 1, \end{cases}$$

in Q of length $m_\alpha n_\alpha - 1$ from $i = s(\alpha)$ to $t(g^{n_\alpha-2}(\alpha))$. Hence, $B_\alpha = A_\alpha g^{n_\alpha-1}(\alpha)$.

PROPOSITION 6.3. *Let (Q, f) be a triangulation quiver with non-empty border $\partial(Q, f)$, and let m_\bullet and b_\bullet be weight and border functions of (Q, f) . Assume that K has characteristic different from 2. Then the algebras $B(Q, f, m_\bullet, b_\bullet)$ and $B(Q, f, m_\bullet)$ are isomorphic.*

Proof. Since K has characteristic different from 2, for any vertex $i \in \partial(Q, f)$ there exists a unique element $a_i \in K$ such that $b_i = 2a_i$. Then there exists an isomorphism of K -algebras $h : B(Q, f, m_\bullet) \rightarrow B(Q, f, m_\bullet, b_\bullet)$ such that

$$h(\alpha) = \begin{cases} \alpha & \text{for any arrow } \alpha \in Q_1 \text{ which is not a border loop,} \\ \alpha - a_{s(\alpha)}A_{\bar{\alpha}} & \text{for any border loop } \alpha \in Q_1. \end{cases}$$

We note that, if $\alpha \in Q_1$ is a border loop, then the following equalities

$$\alpha A_{\bar{\alpha}} = B_{\alpha} = B_{\bar{\alpha}} = A_{\bar{\alpha}} \alpha, \quad (\alpha - a_{s(\alpha)} A_{\bar{\alpha}})^2 = 0$$

hold in $B(Q, f, m_{\bullet}, b_{\bullet})$.

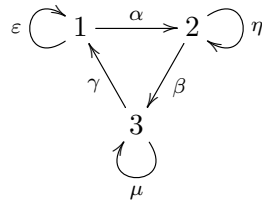
The following proposition is a special case of [29, Theorem 5.3].

PROPOSITION 6.4. *Let A be a basic, indecomposable, symmetric algebra with the Grothendieck group $K_0(A)$ of rank at least 2 which is socle equivalent to a biserial weighted triangulation algebra $B(Q, f, m_{\bullet})$.*

- (i) *If $\partial(Q, f)$ is empty then A is isomorphic to $B(Q, f, m_{\bullet})$.*
- (ii) *Otherwise A is isomorphic to $B(Q, f, m_{\bullet}, b_{\bullet})$ for some border function b_{\bullet} of (Q, f) .*

The following example shows that a socle deformed biserial weighted triangulation algebra need not be isomorphic to a biserial weighted triangulation algebra.

EXAMPLE 6.5. Let (Q, f) be the triangulation quiver



with f -orbits $(\alpha\beta\gamma)$, (ε) , (η) , (μ) , considered in Example 4.3. Then $\mathcal{O}(g)$ consists of one g -orbit $(\alpha\eta\beta\mu\gamma\varepsilon)$. Let $m_{\bullet} : \mathcal{O}(g) \rightarrow \mathbb{N}$ be the weight function with $m_{\mathcal{O}(\alpha)} = 1$. Then the associated biserial weighted triangulation algebra $B = B(Q, f, m_{\bullet})$ is given by the above quiver and the relations

$$\begin{array}{lll} \alpha\beta = 0, & \varepsilon^2 = 0, & \alpha\eta\beta\mu\gamma\varepsilon = \varepsilon\alpha\eta\beta\mu\gamma, \\ \beta\gamma = 0, & \eta^2 = 0, & \beta\mu\gamma\varepsilon\alpha\eta = \eta\beta\mu\gamma\varepsilon\alpha, \\ \gamma\alpha = 0, & \mu^2 = 0, & \gamma\varepsilon\alpha\eta\beta\mu = \mu\gamma\varepsilon\alpha\eta\beta. \end{array}$$

Observe that the border $\partial(Q, f)$ of (Q, f) is the set $Q_0 = \{1, 2, 3\}$ of vertices of Q , and ε, η, μ are the border loops. Take now a border function

$b_\bullet : \partial(Q, f) \rightarrow K$. Then the associated socle deformed biserial weighted triangulation algebra $\bar{B} = B(Q, f, m_\bullet, b_\bullet)$ is given by the above quiver and the relations

$$\begin{aligned} \alpha\beta &= 0, & \varepsilon^2 &= b_1\varepsilon\alpha\eta\beta\mu\gamma, & \alpha\eta\beta\mu\gamma\varepsilon &= \varepsilon\alpha\eta\beta\mu\gamma, \\ \beta\gamma &= 0, & \eta^2 &= b_2\eta\beta\mu\gamma\varepsilon\alpha, & \beta\mu\gamma\varepsilon\alpha\eta &= \eta\beta\mu\gamma\varepsilon\alpha, \\ \gamma\alpha &= 0, & \mu^2 &= b_3\mu\gamma\varepsilon\alpha\eta\beta, & \gamma\varepsilon\alpha\eta\beta\mu &= \mu\gamma\varepsilon\alpha\eta\beta. \end{aligned}$$

Assume that K has characteristic 2 and b_\bullet is non-zero, say $b_1 \neq 0$. We claim that the algebras B and \bar{B} are not isomorphic. Suppose that the algebras B and \bar{B} are isomorphic. Then there is an isomorphism $h : B \rightarrow \bar{B}$ of K -algebras such that $h(e_i) = e_i$ for any $i \in \{1, 2, 3\}$ (see Theorem 2.9 in [43]). In particular, we conclude that $h(\varepsilon) \in e_1\bar{B}e_1$. Observe that the K -vector space $e_1\bar{B}e_1$ has the basis $\varepsilon, \alpha\eta\beta\mu\gamma, \alpha\eta\beta\mu\gamma\varepsilon = \varepsilon\alpha\eta\beta\mu\gamma$. Hence $h(\varepsilon) = u_1\varepsilon + u_2\alpha\eta\beta\mu\gamma + u_3\alpha\eta\beta\mu\gamma\varepsilon$ for some $u_1 \in K^*$ and $u_2, u_3 \in K$. Since K is of characteristic 2 and $(\text{rad } \bar{B})^7 = 0$, we conclude that the following equalities hold in \bar{B}

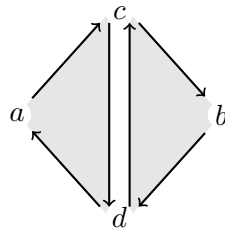
$$0 = h(\varepsilon^2) = h(\varepsilon)^2 = u_1^2\varepsilon^2 + u_1u_2\varepsilon\alpha\eta\beta\mu\gamma + u_1u_2\alpha\eta\beta\mu\gamma\varepsilon = u_1^2b_1\varepsilon\alpha\eta\beta\mu\gamma,$$

and hence $u_1^2b_1 = 0$, a contradiction. This shows that the algebras B and \bar{B} are not isomorphic. We also note that \bar{B} is a biserial but not a special biserial algebra.

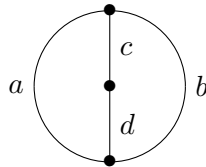
§7. Idempotent algebras of biserial weighted triangulation algebras

The aim of this section is to introduce the idempotent algebras of biserial weighted surface algebras occurring in Theorem 1.

Let (Q, f) be a triangulation quiver. A *2-triangle disk* in (Q, f) is a subquiver D of (Q, f) of the form



where the shaded triangles describe f -orbits. We note that D may be obtained as follows: Take the triangulation quiver associated to the following triangulation of the disk

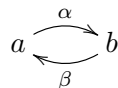


with boundary edges a and b , and the coherent orientation of triangles (acd) , (bdc) , then remove the loops given by a and b .

Let (Q, f) be a triangulation quiver, $m_\bullet : \mathcal{O}(g) \rightarrow \mathbb{N}^*$ a weight function of (Q, f) , and Σ be a collection of 2-triangle disks in (Q, f) . We denote by e_Σ the idempotent of the algebra $B(Q, f, m_\bullet)$ which is the sum of all primitive idempotents corresponding to all vertices of (Q, f) excluding the 2-cycle vertices of the 2-triangle disks from Σ , that is the vertices c, d in the above diagram. We define

$$B(Q, f, \Sigma, m_\bullet) := e_\Sigma B(Q, f, m_\bullet) e_\Sigma$$

and we call this the *idempotent algebra* of $B(Q, f, m_\bullet)$ with respect to Σ . We note that if Σ is empty then $B(Q, f, \Sigma, m_\bullet) = B(Q, f, m_\bullet)$. On the other hand, if Σ is not empty, then every 2-triangle disk D from Σ is replaced in the quiver of $B(Q, f, \Sigma, m_\bullet)$ by the 2-cycle



with $\alpha\beta = 0$ and $\beta\alpha = 0$ in $B(Q, f, \Sigma, m_\bullet)$. If (Q, f) is the triangulation quiver $(Q(S, \vec{T}), f)$ associated to a directed triangulated surface (S, \vec{T}) , then the idempotent algebra $B(Q(S, \vec{T}), f, \Sigma, m_\bullet)$ is called the *idempotent biserial weighted surface algebra*, with respect to Σ , and denoted by $B(S, \vec{T}, \Sigma, m_\bullet)$.

The following proposition describes basic properties of these idempotent algebras.

PROPOSITION 7.1. *Let (Q, f) be a triangulation quiver, $m_\bullet : \mathcal{O}(g) \rightarrow \mathbb{N}^*$ a weight function of (Q, f) , Σ a non-empty collection of 2-triangle disks in (Q, f) , and $B = B(Q, f, \Sigma, m_\bullet)$ the associated idempotent algebra. Then the following statements hold.*

- (i) B is finite-dimensional special biserial.
- (ii) B is a tame symmetric algebra.
- (iii) B is an algebra of generalized dihedral type.
- (iv) The Cartan matrix C_B of B is singular.
- (v) For each 2-cycle $a \begin{smallmatrix} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{smallmatrix} b$ in the Gabriel quiver Q_B given by a 2-triangle disk from Σ , the uniserial modules αB and βB are periodic of period 2 such that $\Omega_B(\alpha B) = \beta B$ and $\Omega_B(\beta B) = \alpha B$, and hence they lie on the mouth of two different stable tubes of Γ_B of rank 1.

Proof. Statements (i) and (ii) follow from Propositions 2.1 and 2.4.

Let $a \begin{smallmatrix} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{smallmatrix} b$ be a 2-cycle in Q_B given by a 2-triangle disk from Σ .

Then we have $\alpha\beta = 0$ and $\beta\alpha = 0$ in B , and direct checking shows that $\Omega_B(\alpha B) = \beta B$ and $\Omega_B(\beta B) = \alpha B$. Clearly, αB and βB are periodic modules of period 2 lying on the mouth of two different stable tubes of Γ_B of rank 1. Moreover, we have $[e_a B] = [e_b B]$ in $K_0(B)$, and hence C_B is singular. This proves statements (iv) and (v). Finally, the statement (iii) follows from Proposition 5.2 and its proof. We note that every arrow in Q_B which does not belong to a 2-cycle given by a 2-triangle disk from Σ belongs to an f -orbit of length 1 or 3.

Let (Q, f) be a triangulation quiver with non-empty border $\partial(Q, f)$, $m_\bullet : \mathcal{O}(g) \rightarrow \mathbb{N}^*$ a weight function of (Q, f) , $b_\bullet : \partial(Q, f) \rightarrow K$ a border function of (Q, f) , and Σ a collection of 2-triangle disks in (Q, f) . We denote also by e_Σ the idempotent in the socle deformed biserial weighted triangulation algebra $B(Q, f, m_\bullet, b_\bullet)$ being the sum of all primitive idempotents corresponding to all vertices of (Q, f) except the 2-cycle vertices of the 2-triangle disks from Σ . Then

$$B(Q, f, \Sigma, m_\bullet, b_\bullet) = e_\Sigma B(Q, f, m_\bullet, b_\bullet) e_\Sigma$$

is said to be the *idempotent algebra* of $B(Q, f, m_\bullet, b_\bullet)$, with respect to Σ .

The following proposition describes basic properties of the idempotent algebras introduced above.

PROPOSITION 7.2. *Let (Q, f) be a triangulation quiver with $\partial(Q, f)$ non-empty, m_\bullet, b_\bullet weight and border functions of (Q, f) , Σ a non-empty collection of 2-triangle disks in (Q, f) , and $B = B(Q, f, \Sigma, m_\bullet)$, $\bar{B} = B(Q, f, \Sigma, m_\bullet, b_\bullet)$. Then the following statements hold.*

- (i) \bar{B} is finite-dimensional biserial.
- (ii) \bar{B} is a symmetric algebra.
- (iii) \bar{B} is socle equivalent to B .
- (iv) \bar{B} degenerates to B .
- (v) \bar{B} is a tame algebra.
- (vi) \bar{B} is an algebra of generalized dihedral type.
- (vii) The Cartan matrix $C_{\bar{B}}$ of \bar{B} is singular.

Moreover, if K is of characteristic different from 2, then

- (viii) \bar{B} is isomorphic to B .

Proof. The required statements follow from Propositions 6.2, 6.3, 7.1, and their proofs.

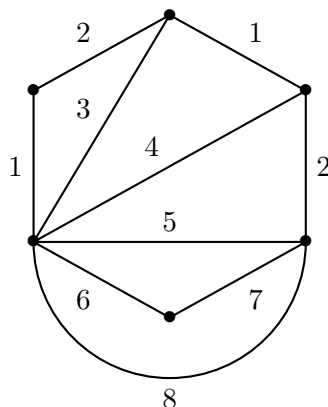
A biserial quiver is a pair (Q', f') where Q' is a 2-regular quiver and f' is a permutation of the arrows such that $s(f'(\alpha)) = t(\alpha)$ (see Definition 2.2 of [29]). Then a biserial quiver algebra is an algebra of the form KQ'/J where J is the ideal of KQ' generated by all the elements of the form (1) and (2) as in the definition of a weighted triangulation algebra. Hence a weighted triangulation algebra is a special case of a biserial quiver algebra. Moreover, an idempotent algebra $B(Q, f, \Sigma, m_\bullet)$ is a biserial quiver algebra with quiver Q' obtained from Q by the process as described in the beginning of Section 7. With this, the following becomes a special case of Theorem 5.3 in [29].

PROPOSITION 7.3. *Let A be a basic, indecomposable, symmetric algebra with the Grothendieck group $K_0(A)$ of rank at least 2 which is socle equivalent to the idempotent algebra $B(Q, f, \Sigma, m_\bullet)$ for a non-empty collection Σ of 2-triangle disks and a weight function m_\bullet of (Q, f) .*

- (i) *If $\partial(Q, f)$ is empty then A is isomorphic to $B(Q, f, \Sigma, m_\bullet)$.*

(ii) Otherwise A is isomorphic to $B(Q, f, \Sigma, m_\bullet, b_\bullet)$ for some border function b_\bullet of (Q, f) .

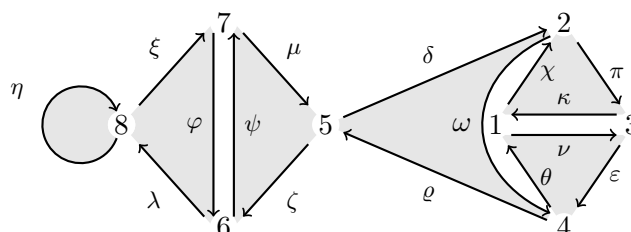
EXAMPLE 7.4. Let S be the connected sum $\mathbb{T} \# D$ (see [9, Section 3.1]) of the torus \mathbb{T} and the disk $D = D^2$, and T be the following triangulation of S



Consider the following orientation \vec{T} of triangles in T

$$(1\ 2\ 3), (1\ 3\ 4), (2\ 4\ 5), (5\ 6\ 7), (8\ 7\ 6).$$

Then the associated triangulation quiver $(Q(S, \vec{T}), f)$ is of the form



where the shaded triangles denote f -orbits. We set $Q = Q(S, \vec{T})$. The set $\mathcal{O}(g)$ of g -orbits of (Q, f) consists of the g -orbits

$$\mathcal{O}(\eta) = (\eta\ \xi\ \mu\ \delta\ \pi\ \varepsilon\ \varrho\ \zeta\ \lambda), \quad \mathcal{O}(\omega) = (\omega\ \theta\ \chi), \quad \mathcal{O}(\kappa) = (\kappa\ \nu), \quad \mathcal{O}(\varphi) = (\varphi\ \psi).$$

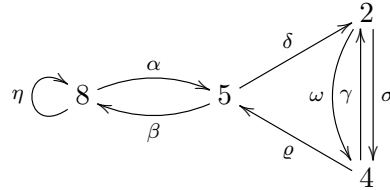
Hence a weight function $m_\bullet : \mathcal{O}(g) \rightarrow \mathbb{N}^*$ is given by four positive natural numbers

$$m = m_{\mathcal{O}(\eta)}, \quad n = m_{\mathcal{O}(\omega)}, \quad p = m_{\mathcal{O}(\kappa)}, \quad q = m_{\mathcal{O}(\varphi)}.$$

Then the associated biserial weighted triangulation algebra $B(Q, f, m_\bullet)$ is given by the quiver Q and the relations

$$\begin{aligned} \eta^2 &= 0, \quad \xi\varphi = 0, \quad \varphi\lambda = 0, \quad \lambda\xi = 0, \quad \zeta\psi = 0, \quad \psi\mu = 0, \quad \mu\zeta = 0, \quad \delta\omega = 0, \\ \omega\rho &= 0, \quad \rho\delta = 0, \quad \chi\pi = 0, \quad \pi\kappa = 0, \quad \kappa\chi = 0, \quad \theta\nu = 0, \quad \nu\varepsilon = 0, \quad \varepsilon\theta = 0, \\ (\chi\omega\theta)^n &= (\nu\kappa)^p, \quad (\pi\varepsilon\rho\zeta\lambda\eta\xi\mu\delta)^m = (\omega\theta\chi)^n, \quad (\kappa\nu)^p = (\varepsilon\rho\zeta\lambda\eta\xi\mu\delta\pi)^m, \\ (\theta\chi\omega)^n &= (\rho\zeta\lambda\eta\xi\mu\delta\pi\varepsilon)^m, \quad (\delta\pi\varepsilon\rho\zeta\lambda\eta\xi\mu)^m = (\zeta\lambda\eta\xi\mu\delta\pi\varepsilon\rho)^m, \\ (\psi\varphi)^q &= (\lambda\eta\xi\mu\delta\pi\varepsilon\rho\zeta)^m, \quad (\varphi\psi)^q = (\mu\delta\pi\varepsilon\rho\zeta\lambda\eta\xi)^m, \\ (\xi\mu\delta\pi\varepsilon\rho\zeta\lambda\eta)^m &= (\eta\xi\mu\delta\pi\varepsilon\rho\zeta\lambda)^m. \end{aligned}$$

The triangulation quiver (Q, f) contains two 2-triangle disks: D_1 , given by the arrows $\chi, \pi, \kappa, \nu, \varepsilon, \theta$, and D_2 , given by the arrows $\lambda, \xi, \varphi, \psi, \mu, \zeta$. We note that D_1 is not a full subquiver of Q . Let $\Sigma = \{D_1, D_2\}$. Then the idempotent algebra $B(Q, f, \Sigma, m_\bullet)$ is given by the quiver Q_Σ of the form

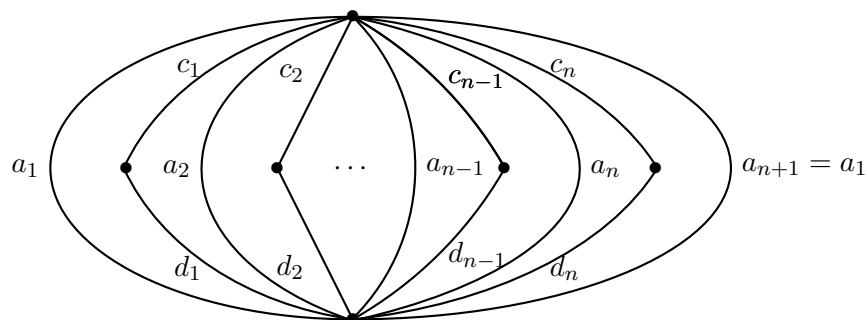


with $\alpha = \xi\mu$, $\beta = \zeta\lambda$, $\gamma = \theta\chi$, $\sigma = \pi\varepsilon$, and the relations

$$\begin{aligned} \eta^2 &= 0, \quad \alpha\beta = 0, \quad \beta\alpha = 0, \quad \delta\omega = 0, \quad \omega\rho = 0, \quad \rho\delta = 0, \\ \gamma\sigma &= 0, \quad \sigma\gamma = 0, \quad (\sigma\rho\beta\eta\alpha\delta)^m = (\omega\gamma)^n, \quad (\gamma\omega)^n = (\rho\beta\eta\alpha\delta\sigma)^m, \\ (\delta\sigma\rho\beta\eta\alpha)^m &= (\beta\eta\alpha\delta\sigma\rho)^m, \quad (\alpha\delta\sigma\rho\beta\eta)^m = (\eta\alpha\delta\sigma\rho\beta)^m. \end{aligned}$$

The border $\partial(Q, f)$ of (Q, f) consists of the vertex 8, and hence a border function $b_\bullet : \partial(Q, f) \rightarrow K$ is given by an element $b \in K$. Then the socle deformed biserial weighted triangulation algebra $B(Q, f, m_\bullet, b_\bullet)$ is obtained from $B(Q, f, m_\bullet)$ by replacing the relation $\eta^2 = 0$ by the relation $\eta^2 = b(\eta\xi\mu\delta\pi\varepsilon\rho\zeta\lambda)^m$. Similarly, the algebra $B(Q, f, \Sigma, m_\bullet, b_\bullet)$ is obtained from the algebra $B(Q, f, \Sigma, m_\bullet)$ by replacing the relation $\eta^2 = 0$ by the relation $\eta^2 = b(\eta\alpha\delta\sigma\rho\beta)^m$.

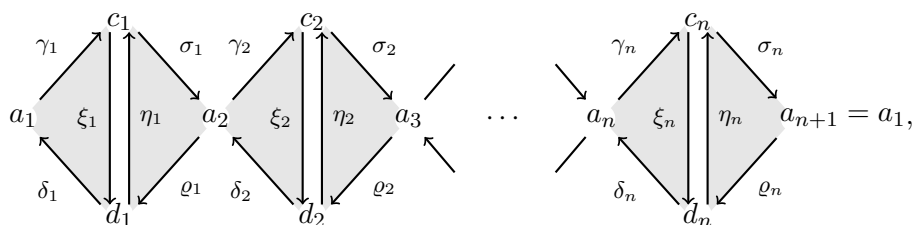
EXAMPLE 7.5. Let $n \geq 2$ be a natural number. Consider the following triangulation $T(n)$ of the sphere $\mathbb{S} = S^2$ in \mathbb{R}^3



with two pole vertices and n vertices lying on the equator. Moreover, let $\overrightarrow{T(n)}$ be the coherent orientation of the triangles of $T(n)$

$$(a_i c_i d_i), \quad (d_i c_i a_{i+1}), \quad \text{for } i \in \{1, \dots, n\}.$$

Then the associated triangulation quiver $(Q(\mathbb{S}, \overrightarrow{T(n)}), f)$ is of the form



where the shaded triangles denote f -orbits. The set $\mathcal{O}(g)$ of g -orbits consists of the 2-cycle orbits $\mathcal{O}(\xi_i) = (\xi_i \eta_i)$, $i \in \{1, \dots, n\}$, and the two n -cycles

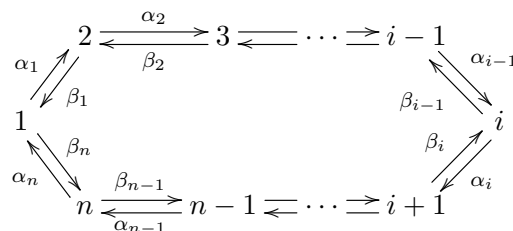
$$\mathcal{O}(\gamma_1) = (\gamma_1 \sigma_1 \gamma_2 \sigma_2 \dots \gamma_n \sigma_n), \quad \mathcal{O}(\varrho_n) = (\varrho_n \delta_n \dots \varrho_2 \delta_2 \varrho_1 \delta_1).$$

Let $m_\bullet : \mathcal{O}(g) \rightarrow \mathbb{N}^*$ be a weight function, and

$$p = m_{\mathcal{O}(\gamma_1)}, \quad q = m_{\mathcal{O}(\varrho_n)}.$$

Observe that the triangulation quiver $(Q(\mathbb{S}, \overrightarrow{T(n)}), f)$ is formed by 2-triangle disks D_1, D_2, \dots, D_n . Take $\Sigma = \{D_1, D_2, \dots, D_n\}$. Then the associated

idempotent algebra $B(Q(\mathbb{S}, \overrightarrow{T(n)}), f, \Sigma, m_\bullet)$ is given by the quiver



(with i replacing a_i), and the relations

$$\alpha_i \beta_i = 0, \quad \beta_i \alpha_i = 0, \quad (\alpha_i \alpha_{i+1} \dots \alpha_{i-1})^p = (\beta_{i-1} \beta_{i-2} \dots \beta_i)^q,$$

for $i \in \{1, \dots, n\}$, where $\alpha_0 = \alpha_n$, $\beta_0 = \beta_n$.

§8. Algebras of strict dihedral type

Algebras of dihedral type were introduced and studied in [21] and [23]. We introduce the following refinement.

Assume A is an algebra. We say that A is of *strict dihedral type* if it satisfies the following conditions:

- (1) A is symmetric, indecomposable, and tame.
- (2) The stable Auslander-Reiten quiver Γ_A^s of A consists of the following components:
 - (i) stable tubes of ranks 1 and 3;
 - (ii) non-periodic components of the form $\mathbb{Z}\mathbb{A}_\infty^\infty$ or $\mathbb{Z}\tilde{\mathbb{A}}_n$. We assume that there is at least one such component.
- (3) Ω_A fixes all stable tubes of rank 3 in Γ_A^s .
- (4) The number $\ell(A)$ of isomorphism classes of simple modules in $\text{mod } A$ is two or three.
- (5) The number of stable tubes of rank 3 in Γ_A^s is equal to $\ell(A) - 1$.
- (6) The Cartan matrix C_A of A is non-singular.

An algebra A with $\ell(A) > 1$ is of dihedral type if and only if it satisfies conditions (1) to (5). Then we have the following consequence of [21, Theorem 1.4].

THEOREM 8.1. *Let K be an algebraically closed field of characteristic 2, G a finite group, and B be a block of KG whose defect groups are dihedral 2-groups. Then B is an algebra of strict dihedral type.*

We refer to [23, Tables] for a complete description of algebras of dihedral type, by quivers and relations.

In the rest of this section we will exhibit presentations of all algebras of strict dihedral type as biserial weighted triangulation algebras and their socle deformations. Note that a loop of the triangulation quiver need not be a loop in the Gabriel quiver, so that the triangulation quiver allows us to unify the description of these algebras.

EXAMPLE 8.2. Let (Q, f) be the triangulation quiver

$$\alpha \curvearrowright 1 \begin{matrix} \xrightarrow{\beta} \\ \xleftarrow{\gamma} \end{matrix} 2 \curvearrowright \eta$$

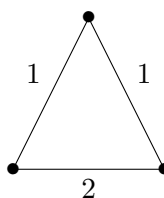
with $f(\alpha) = \beta$, $f(\beta) = \gamma$, $f(\gamma) = \alpha$, and $f(\eta) = \eta$. Then we have $g(\alpha) = \alpha$, $g(\beta) = \eta$, $g(\eta) = \gamma$, and $g(\gamma) = \beta$. Hence, $\mathcal{O}(g)$ consists of the two g -orbits $\mathcal{O}(\alpha)$ and $\mathcal{O}(\beta)$. Let $m_\bullet : \mathcal{O}(g) \rightarrow \mathbb{N}^*$ be a weight function, and let $r = m_\alpha$ and $s = m_\beta$. The border $\partial(Q, f)$ of (Q, f) consists of the vertex 2 and η is the unique border loop. Take a border function $b_\bullet : \partial(Q, f) \rightarrow K$, and set $b = b_2$. Then the associated socle deformed biserial weighted triangulation algebra

$$\Lambda(r, s, b) = B(Q, f, m_\bullet, b_\bullet)$$

is given by the above quiver and the relations

$$\alpha\beta = 0, \beta\gamma = 0, \gamma\alpha = 0, \alpha^r = (\beta\eta\gamma)^s, (\gamma\beta\eta)^s = (\eta\gamma\beta)^s, \eta^2 = b(\eta\gamma\beta)^s.$$

Clearly, if $b = 0$, then $\Lambda(r, s, 0) = B(Q, f, m_\bullet)$. We note that the family $\Lambda(r, s, b)$, $r, s \in \mathbb{N}^*$, $b \in K$, coincides with the families $D(2\mathcal{A})$ and $D(2\mathcal{B})$ from [23, Tables]. Finally, we note that the considered triangulation quiver (Q, f) is the triangulation quiver $(Q(S, \vec{T}), f)$ of the self-folded triangle $S = T$ of the form



with 2 being the boundary edge and $\vec{T} = (1 \ 1 \ 2)$.

EXAMPLE 8.3. Let (Q, f) be the triangulation quiver

$$\alpha \circlearrowleft 1 \xrightleftharpoons[\gamma]{\beta} 2 \xrightleftharpoons[\eta]{\delta} 3 \circlearrowright \xi$$

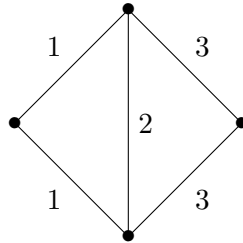
with $f(\alpha) = \beta$, $f(\beta) = \gamma$, $f(\gamma) = \alpha$, $f(\xi) = \eta$, $f(\eta) = \delta$, $f(\delta) = \xi$. Then we have $g(\alpha) = \alpha$, $g(\xi) = \xi$, $g(\beta) = \delta$, $g(\delta) = \eta$, $g(\eta) = \gamma$, and $g(\gamma) = \beta$. Hence $\mathcal{O}(g)$ consists of the three g -orbits $\mathcal{O}(\alpha)$, $\mathcal{O}(\beta)$, and $\mathcal{O}(\xi)$. Let $m_\bullet : \mathcal{O}(g) \rightarrow \mathbb{N}^*$ be a weight function, and $r = m_\alpha$, $s = m_\beta$, $t = m_\xi$. Observe that the border $\partial(Q, f)$ of (Q, f) is empty. The associated biserial weighted triangulation algebra

$$\Gamma(r, s, t) = B(Q, f, m_\bullet)$$

is given by the above quiver and the relations

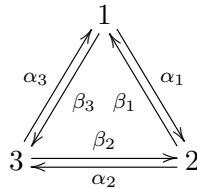
$$\begin{aligned} \alpha\beta &= 0, & \beta\gamma &= 0, & \gamma\alpha &= 0, & \delta\xi &= 0, & \xi\eta &= 0, & \eta\delta &= 0, \\ \alpha^r &= (\beta\delta\eta\gamma)^s, & (\gamma\beta\delta\eta)^s &= (\delta\eta\gamma\beta)^s, & \xi^t &= (\eta\gamma\beta\delta)^s. \end{aligned}$$

We note that the family $\Gamma(r, s, t)$, $r, s, t \in \mathbb{N}^*$, coincides with the families $D(3\mathcal{A})_1$, $D(3\mathcal{B})_1$, $D(3\mathcal{D})_1$ from [23, Tables]. Finally, we observe that the considered triangulation quiver (Q, f) is the triangulation quiver $(Q(S, \vec{T}), f)$ for the triangulation T



of the surface $S = \mathbb{P} \# \mathbb{P}$ and $\vec{T} = (1 \ 1 \ 2)(3 \ 3 \ 2)$.

EXAMPLE 8.4. Let (Q, f) be the triangulation quiver



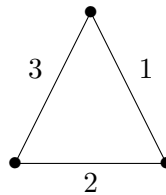
with $f(\alpha_1) = \alpha_2$, $f(\alpha_2) = \alpha_3$, $f(\alpha_3) = \alpha_1$, $f(\beta_1) = \beta_3$, $f(\beta_3) = \beta_2$, $f(\beta_2) = \beta_1$. Then we have $g(\alpha_1) = \beta_1$, $g(\beta_1) = \alpha_1$, $g(\alpha_2) = \beta_2$, $g(\beta_2) = \alpha_2$, $g(\alpha_3) = \beta_3$, and $g(\beta_3) = \alpha_3$. Hence $\mathcal{O}(g)$ consists of the three g -orbits $\mathcal{O}(\alpha_1)$, $\mathcal{O}(\alpha_2)$, $\mathcal{O}(\alpha_3)$. Let $m_\bullet : \mathcal{O}(g) \rightarrow \mathbb{N}^*$ be a weight function, and $m_1 = m_{\alpha_1}$, $m_2 = m_{\alpha_2}$, $m_3 = m_{\alpha_3}$. Then the associated biserial weighted triangulation algebra

$$\Omega(m_1, m_2, m_3) = B(Q, f, m_\bullet)$$

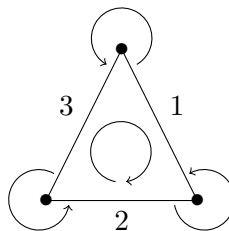
is given by the above quiver and the relations

$$\begin{aligned} \alpha_1\alpha_2 = 0, \quad \alpha_2\alpha_3 = 0, \quad \alpha_3\alpha_1 = 0, \quad \beta_1\beta_3 = 0, \quad \beta_3\beta_2 = 0, \quad \beta_2\beta_1 = 0, \\ (\alpha_1\beta_1)^{m_1} = (\beta_3\alpha_3)^{m_3}, \quad (\beta_1\alpha_1)^{m_1} = (\alpha_2\beta_2)^{m_2}, \quad (\beta_2\alpha_2)^{m_2} = (\alpha_3\beta_3)^{m_3}. \end{aligned}$$

We note that the family $\Omega(m_1, m_2, m_3)$, $m_1, m_2, m_3 \in \mathbb{N}^*$, coincides with the family $D(3\mathcal{K})$ from [23, Tables]. Finally, the considered triangulation quiver (Q, f) is the triangulation quiver $(Q(\mathbb{S}, \vec{T}), f)$ associated to the following triangulation T of the sphere \mathbb{S}



and the orientation \vec{T} of two triangles in T



The following theorem is a complete classification of the basic algebras of strict dihedral type, established in [21] and [23].

THEOREM 8.5. *Let A be a basic algebra over an algebraically closed field K , with the Grothendieck group $K_0(A)$ of rank at least 2. The following statements are equivalent:*

- (i) A is an algebra of strict dihedral type.
- (ii) A is isomorphic to an algebra of one of the forms $\Lambda(r, s, b)$, $\Gamma(r, s, t)$, or $\Omega(m_1, m_2, m_3)$.

Moreover, we have the following consequence of the results established in [25, Section 4].

COROLLARY 8.6. *Let A be a basic algebra of strict dihedral type over an algebraically closed field K . Then the following statements are equivalent:*

- (i) A is of polynomial growth.
- (ii) A is isomorphic to an algebra of one of the forms $\Lambda(1, 1, 0)$, $\Lambda(1, 1, 1)$, $\Gamma(1, 1, 1)$, or $\Omega(1, 1, 1)$.

§9. Proof of Theorem 1

The aim of this section is to prove Theorem 1. The implication (ii) \Rightarrow (i) of Theorem 1 follows from work done so far. Namely, assume A is socle equivalent to $B(S, T, \Sigma, m_\bullet)$, the idempotent algebra of $B(S, T, m_\bullet)$ with respect to a set Σ of 2-triangle disks. Recall that we can view the quiver $Q(S, T)$ as a triangulation quiver (Q, f) , and with this, A is socle equivalent to $B(Q, f, \Sigma, m_\bullet)$. We apply Proposition 7.3 which shows that either A is isomorphic to $B(Q, f, \Sigma, m_\bullet)$, or else A is isomorphic to $B(Q, f, \Sigma, m_\bullet, b_\bullet)$ for some border function b_\bullet . Therefore A is of generalized dihedral type, either by Proposition 7.1, or by Proposition 7.2. That is, part (i) of Theorem 1 follows.

Hence it remains to show that (i) implies (ii). We split the proof into several steps.

LEMMA 9.1. *Let $A = KQ/I$ be a symmetric bound quiver algebra, and i, j two different vertices in Q . Assume that the simple module S_i in $\text{mod } A$ is periodic of period 3. Then the following statements hold.*

- (i) *The number of arrows in Q from i to j is the same as the number of arrows in Q from j to i .*
- (ii) *No path of length 2 between i to j occurs in a minimal relation of I .*

Proof. This follows from [23, Lemmas IV.1.5 and IV.1.7].

LEMMA 9.2. *Let $A = KQ/I$ be a bound quiver algebra of generalized dihedral type with $|Q_0| \geq 3$, and $i \in Q_0$ a vertex such that $|s^{-1}(i)| \geq 2$ and S_i is a periodic module in $\text{mod } A$. Then, for any vertex j in Q connected to i by an arrow, the simple module S_j is a periodic module in $\text{mod } A$.*

Proof. By Lemma 9.1 we know $m := |s^{-1}(i)| = |t^{-1}(i)|$, and we assume $m \geq 2$. Let $\alpha_1, \dots, \alpha_m$ be the arrows in Q starting at i , and let $j_r := t(\alpha_r)$.

Since S_i has period 3, there exists an exact sequence in $\text{mod } A$

$$0 \rightarrow S_i \xrightarrow{d_3} P_i \xrightarrow{d_2} \bigoplus_{r=1}^m P_{j_r} \xrightarrow{d_1} P_i \xrightarrow{d_0} S_i \rightarrow 0$$

with d_0 the canonical epimorphism, d_3 the canonical monomorphisms, and where d_1 is defined as

$$d_1(x_1, \dots, x_m) := \sum_{r=1}^m \alpha_i x_i$$

for any $(x_1, \dots, x_m) \in \bigoplus_{j=1}^r P_{j_r}$.

Since $\bigoplus_{r=1}^m P_{j_r}$ is the injective hull of $\text{Ker } d_1 \cong \Omega^2(S_i) \cong \Omega^{-1}(S_i)$, we deduce that the m arrows ending at i have starting vertices j_1, \dots, j_m .

(1) We claim that the j_r are pairwise distinct: If not, say $j_2 = j_1$, then we have double arrows from i to j_1 and also from j_1 to i . By assumption, $|Q_0| \geq 3$ and Q is connected, and therefore there is a vertex $k \neq i, j_1$ and some arrow between k and one of i or j_1 . Then A has a quotient algebra $K\Delta$ where Δ is the wild quiver of the form

$$\bullet \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} \bullet \longrightarrow \bullet,$$

or its opposite quiver, which is a contradiction since A is tame.

(2) We have $\text{Ker } d_1 \cong \Omega^{-1}(S_i) \cong P_i/S_i$, and this is a cyclic A -module. So it is of the form $(\beta_1, \dots, \beta_m)A$ with $\beta_r \in P_{j_r}$. We know that the j_r are distinct, that the arrows ending at i have starting vertices j_r , and that P_i/S_i is isomorphic to $(\beta_1, \dots, \beta_m)A$ with β_r in P_{j_r} . This implies that β_r is not in $\text{rad}^2(P_{j_r})$, so we can take the β_r as the arrows ending at i . Now we have

$$\sum_{r=1}^m \alpha_r x_r = 0 \Leftrightarrow (x_1, \dots, x_r) = (\beta_1, \dots, \beta_m)a \text{ (some } a \in A).$$

Suppose now that one of the simple modules S_{j_1}, \dots, S_{j_m} , say S_{j_1} , is not periodic. We set $j = j_1$, $\alpha = \alpha_1$, and $\beta = \beta_1$.

(3) There is an exact sequence in $\text{mod } A$

$$0 \rightarrow \Omega_A(\alpha A) \rightarrow P_j \xrightarrow{d} \alpha A \rightarrow 0$$

with $d(x) = \alpha x$ for $x \in P_j$, and $\Omega_A(\alpha A) = \{x \in P_j \mid \alpha x = 0\}$. If $x \in \Omega_A(\alpha A)$ then $(x, 0, \dots, 0) \in \text{Ker } d_1$ and hence $(x, 0, \dots, 0) = (\beta, \beta_2, \dots, \beta_m)a$ for $a \in A$ so that $x = \beta a$ and therefore $\Omega_A(\alpha A) \subseteq \beta A$.

We will now show that $\Omega_A(\alpha A) = \beta A$. Note first that $\text{rad } P_j / \Omega_A(\alpha A)$ has a simple socle (since αA has a simple socle). Since S_j is not periodic, the quotient $\text{rad } P_j / S_j$ is a direct sum of two non-zero modules. That is, there are submodules M and N of $\text{rad } P_j$ such that $\text{rad } P_j = M + N$ and $M \cap N = S_j$.

We know (by (1)) that S_i is a direct summand of $\text{rad } P_j / \text{rad}^2 P_j$ with multiplicity one, so we may assume that S_i is a direct summand of $M / \text{rad}^2 P_j$. Now $\beta = e_j \beta e_i \notin \text{rad}^2 P_j$ and hence $\beta A \subseteq M$.

We have now $S_j \subseteq \Omega_A(\alpha A) \subseteq M \subseteq \text{rad } P_j$ and since $\text{rad } P_j / S_j = (M / S_j) \oplus (N / S_j)$, we conclude that $\text{rad } P_j / \Omega_A(\alpha A) = M / \Omega_A(\alpha A) \oplus N / S_j$. Now $N / S_j \neq 0$ and $\text{rad } P_j / \Omega_A(\alpha A)$ has a simple socle, so it follows that $M / \Omega_A(\alpha A) = 0$ and therefore $\Omega_A(\alpha A) = \beta A = M$.

Now $\beta A = \beta A \oplus 0 \oplus \dots \oplus 0 \oplus 0$ is contained in $(\beta, \beta_2, \dots, \beta_m)A$. In particular, there is $y \in A$ such that $\beta = \beta y$ and $\beta_r y = 0$ for any $r \in \{2, \dots, m\}$. Let $z = e_i y e_i$, then $\beta = \beta z$ and $\beta_r z = 0$. Since $\beta_2 \neq 0$, we see that z is a non-invertible element of the local algebra $e_i A e_i$. But then $e_i - z$ is an invertible element of $e_i A e_i$ and $\beta(e_i - z) = 0$, which contradicts the fact that $\beta \neq 0$. Hence S_{j_1} and similarly all simple modules S_{j_2}, \dots, S_{j_m} are periodic.

PROPOSITION 9.3. *Let $A = KQ/I$ be a bound quiver algebra of generalized dihedral type, with $|Q_0| \geq 3$, and let i be a vertex of Q . Then the following statements are equivalent:*

- (i) S_i is a periodic module in $\text{mod } A$.
- (ii) $|s^{-1}(i)| = 1$.
- (iii) $|t^{-1}(i)| = 1$.
- (iv) There are unique arrows $i \xrightleftharpoons[\alpha]{\beta} j$ adjacent to the vertex i , and $\beta\alpha = 0$.

Proof. First we show that for any simple periodic module S_i we have $|s^{-1}(i)| = |t^{-1}(i)| = 1$. By definition of an algebra of generalized dihedral type and Corollary 3.2, $\text{mod } A$ admits a non-periodic simple module S_k . If t is a vertex connected to k by an arrow and if S_t is periodic then S_t does not satisfy the assumption of Lemma 9.2, therefore $|s^{-1}(t)| = |t^{-1}(t)| = 1$. If S_t is not periodic we consider similarly all vertices j which are connected to t by an arrow. If S_j is periodic then we apply Lemma 9.2, else we repeat the process. The quiver is connected and after finitely many steps we have reached all vertices, and the claim follows. Hence (i) implies (ii) and (iii).

Assume now that $|s^{-1}(i)| = 1$. Let β be the arrow in Q with source i and $j = t(\beta)$. Then $\text{rad } P_i = \beta A$ which has simple top, and hence the quotient $\text{rad } P_i / S_i$ is indecomposable. But then S_i is a periodic module of period 3, because A is of generalized dihedral type. Consider an exact sequence in $\text{mod } A$

$$0 \rightarrow S_i \xrightarrow{d_3} P_i \xrightarrow{d_2} P_j \xrightarrow{d_1} P_i \xrightarrow{d_0} S_i \rightarrow 0$$

which gives rise to a minimal projective resolution of S_i in $\text{mod } A$. Then $P_i / S_i = \Omega_A^{-1}(S_i) = \Omega_A^2(S_i)$ is a submodule of P_j of the form αA for a unique arrow α in Q from j to i . Therefore, we have in Q only two arrows

$$i \begin{matrix} \xrightarrow{\beta} \\ \xleftarrow{\alpha} \end{matrix} j$$

containing the vertex i . Moreover, from the proof of Lemma 9.2, we have $\beta\alpha = 0$. Hence, (ii) implies (i) and (iv). Similarly, (iii) implies (i) and (iv). Obviously, (iv) implies (ii) and (iii). This finishes the proof.

LEMMA 9.4. *Let $A = KQ/I$ be a bound quiver algebra of generalized dihedral type with $|Q_0| \geq 3$, i a vertex of Q with $|s^{-1}(i)| = 1 = |t^{-1}(i)|$, and*

$$i \begin{matrix} \xrightarrow{\beta} \\ \xleftarrow{\alpha} \end{matrix} j$$

be the unique arrows in Q adjacent to i . Then $\alpha\beta$ does not occur in a minimal relation of A .

Proof. Assume that $\alpha\beta$ occurs in a minimal relation of A . Since $\alpha \notin \text{soc}(A)$, we have $\alpha(\text{rad } A) \neq 0$, and consequently $\alpha(\text{rad } A)$ is not contained

in $\alpha(\text{rad } A)^2$. Observe also that $\alpha(\text{rad } A) = \alpha\beta A$. Then Q contains a subquiver

$$i \begin{matrix} \xrightarrow{\beta} \\ \xleftarrow{\alpha} \end{matrix} j \begin{matrix} \xrightarrow{\gamma} \\ \xleftarrow{\delta} \end{matrix} k$$

such that $\alpha\beta$ and $\gamma\delta$ occur in the same minimal relation of A . It follows from Proposition 9.3 that S_i is a periodic module of period 3 and $\beta\alpha = 0$ (see also the proof of Lemma 9.2). We also note that S_j is not periodic because $|s^{-1}(j)| \geq 2$. This implies that the heart $\text{rad } P_j/S_j$ is a direct sum of two non-zero submodules. We have two cases to consider.

(1) Assume that $|s^{-1}(j)| = 2 = |t^{-1}(j)|$. Then we have $\text{rad } P_j = \alpha A + \gamma A$ and $\alpha A \cap \gamma A = S_j$. On the other hand, we have a minimal relation in A of the form

$$\alpha\beta + \alpha x + a\gamma\delta + \gamma y = 0$$

for some $a \in K^*$ and $x \in e_i(\text{rad } A)^2$, $y \in e_k(\text{rad } A)^2$. Moreover, $x = \beta z$ for some $z \in e_j(\text{rad } A)e_j$. Hence, we get

$$\alpha\beta(1 - z) + \gamma(a\delta + y) = 0.$$

Since $1 - z$ is an invertible element of A , we obtain an equality

$$\alpha\beta = \gamma\delta',$$

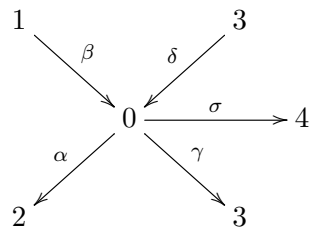
where $\delta' \in e_k(\text{rad } A)e_j \setminus e_k(\text{rad } A)^2e_j$. This implies that $\alpha\beta$ generates $\text{soc}(P_j)$, because $\alpha A \cap \gamma A = S_j = \text{soc}(P_j)$. Let $\varphi : A \rightarrow K$ be a symmetrizing K -linear form. Then $\varphi(\alpha\beta) = \varphi(\beta\alpha) = \varphi(0) = 0$. Thus $K\alpha\beta$ is a non-zero ideal in A which is contained in $\text{Ker } \varphi$, a contradiction.

(2) Assume that $|s^{-1}(j)| \geq 3$. Then there is an arrow σ in Q with $s(\sigma) = j$, different from α and γ . We set $l = t(\sigma)$. Observe that $l \notin \{i, k\}$, because otherwise A admits a quotient algebra isomorphic to the path algebra of the wild quiver of the form

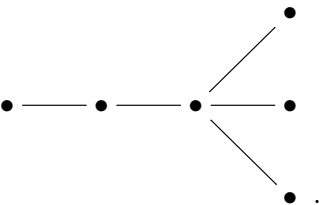
$$\bullet \begin{matrix} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{matrix} \bullet \longrightarrow \bullet.$$

Recall that S_i is a periodic module of period 3. Then it follows from Lemma 9.1 that the paths $\beta\gamma$, $\beta\sigma$, $\delta\alpha$ do not occur in minimal relations of A , and $\beta\gamma$, $\beta\sigma$, $\delta\alpha$ are non-zero paths. We also note that $\delta\gamma \neq 0$ because $\delta\alpha\beta \neq 0$. Similarly, $\alpha\beta\sigma \neq 0$ forces $\delta\sigma \neq 0$. Let B be the quotient algebra of A by the ideal generated by all paths of length 2 except $\beta\gamma$, $\beta\sigma$, $\delta\alpha$, $\delta\gamma$,

$\delta\sigma$. Then there is a Galois covering $F : R \rightarrow R/G = B$, with a finitely generated free group G , such that the locally bounded K -category R admits a full convex subcategory Λ isomorphic to the bound quiver algebra $C = K\Delta/L$, where Δ is the quiver



and L is the ideal in $K\Delta$ generated by $\beta\alpha$. Then C is a wild concealed algebra of the tree type $\tilde{\mathbb{D}}_4$



Applying [15, Proposition 2] and [16, Theorem] we conclude that B is a wild algebra. This is a contradiction, because B is a quotient algebra of the tame algebra A .

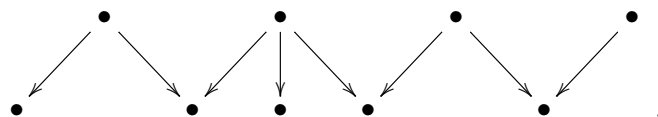
PROPOSITION 9.5. *Let $A = KQ/I$ be a bound quiver algebra of generalized dihedral type with $|Q_0| \geq 3$. Then the following conditions are satisfied.*

- (Q1) *For each vertex $i \in Q_0$, we have $|s^{-1}(i)| = |t^{-1}(i)|$ and this 1 or 2.*
- (Q2) *For each vertex $i \in Q_0$ with $|s^{-1}(i)| = 2$, $\text{rad } P_i/S_i = U \oplus V$ with U, V indecomposable modules.*

Proof. Let $I_0 = \{i \in Q_0 \mid |s^{-1}(i)| = 1\}$. Consider the idempotent $\varepsilon = 1_A - \sum_{i \in I_0} e_i$ and the associated idempotent algebra $B = \varepsilon A \varepsilon$. We note that B is a tame algebra (see [16, Theorem]), because A is a tame algebra. Clearly, $B = A$ if I_0 is empty. For each vertex $i \in I_0$, we have in Q_0 unique arrows

$$i \begin{matrix} \xrightarrow{\beta_i} \\ \xleftarrow{\alpha_i} \end{matrix} i^*$$

containing the vertex i . Moreover, by Proposition 9.3, the simple module S_i is periodic. Since $|Q_0| \geq 3$, applying Proposition 9.3 again, we conclude that $|s^{-1}(i^*)| \geq 2$ and $|t^{-1}(i^*)| \geq 2$. It follows also from Lemma 9.1 that $\alpha_i \beta_i$ does not occur in a minimal relation of A . Therefore, B is the bound quiver algebra KQ^*/I^* , where Q^* is the quiver obtained from Q by replacing all 2-cycles $i \xrightleftharpoons[\alpha_i]{\beta_i} i^*$, for $i \in I_0$, by the loops η_i at i^* and keeping all other arrows of Q , and $I^* = \varepsilon I \varepsilon$. We note that for any vertex k of Q^* we have $|s^{-1}(k)| \geq 2$ and $|t^{-1}(k)| \geq 2$. We claim that $|s^{-1}(k)| = 2 = |t^{-1}(k)|$ for any vertex k of Q^* . Suppose that $|s^{-1}(j)| \geq 3$ or $|t^{-1}(j)| \geq 3$ for a vertex j of Q^* . Without loss of generality, we may assume that $|s^{-1}(j)| \geq 3$. Consider the quotient algebra $D = B/(\text{rad } B)^2$. Then D admits a Galois covering $F : R \rightarrow R/G = D$, with a finitely generated free group G , such that the locally bounded K -category R contains a full convex subcategory Λ isomorphic to the path algebra $K\Delta$ of the wild quiver Δ of type $\tilde{\mathbb{E}}_7$



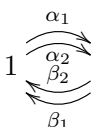
Applying [15, Proposition 2] and [16, Theorem] again, we conclude that B is a wild algebra, a contradiction. Therefore, indeed $|s^{-1}(k)| = 2 = |t^{-1}(k)|$ for any vertex $k \in Q_0^*$.

This shows that A satisfies (Q1). But then it follows from Proposition 9.3 and Remark 3.3 that A satisfies also (Q2).

LEMMA 9.6. *The statements of Propositions 9.5 and 9.3 also hold when $|Q_0| = 2$.*

Proof. Suppose $Q_0 = \{1, 2\}$. It follows from Corollary 3.2 that at least one of the simple modules, say S_1 , is non-periodic, and then $(\text{rad } P_1)/S_1 = U_1 \oplus V_1$ with two non-zero summands. Hence there are at least two arrows starting at 1 and two arrows ending at 1. That is, Q contains one of the following quivers

$$(a) \quad \alpha \circlearrowleft 1 \begin{matrix} \xrightarrow{\beta} \\ \xleftarrow{\gamma} \end{matrix} 2.$$

(b)  . This is then all of Q since A is tame.

Note that since A is tame, it cannot have a connected quiver with two loops at some vertex.

Consider first the quiver (b). If also S_2 is non-periodic then Proposition 9.5 follows directly, and Proposition 9.3 is vacuously true. So assume for a contradiction that S_2 is periodic, and then it has period three. We have an exact sequence

$$0 \rightarrow S_2 \rightarrow P_2 \rightarrow P_1 \oplus P_1 \rightarrow P_2 \rightarrow S_2 \rightarrow 0$$

and it follows that in the Grothendieck group $K_0(A)$ we have

$$[P_1] = [P_2] - [S_2].$$

We may take the arrows α_1, α_2 so that $\alpha_1 A \cap \alpha_2 A = S_1$, and then $[\alpha_1 A] + [\alpha_2 A] = [P_1]$.

As well we know that $(\alpha_1, \alpha_2)A \cong P_2/S_2$ in mod A . We have an exact sequence

$$0 \rightarrow (\alpha_1, \alpha_2)A \rightarrow \alpha_1 A \oplus \alpha_2 A \rightarrow C \rightarrow 0.$$

The first term has composition factors $[P_2] - [S_2]$ and the second term has composition factors $[P_1]$ and these are equal. Hence $C = 0$ and P_2/S_2 is a direct sum, a contradiction. This shows that Proposition 9.3 holds as well in this case.

Consider now the quiver (a). Assume S_2 is not periodic. Then the quiver Q has a loop at both vertices, and we have Propositions 9.5 and 9.3. So assume now that S_2 is periodic and we know that $|s^{-1}(2)| = |t^{-1}(2)|$ and this is 1 or 2. We are done if we show that it is equal to 1. Assume this is false, then we have an exact sequence

$$0 \rightarrow S_2 \rightarrow P_2 \rightarrow P_1 \oplus P_2 \rightarrow P_2 \rightarrow S_2 \rightarrow 0,$$

and it follows that $[P_1] = [P_2] - 2[S_2]$. Therefore the Cartan matrix is non-singular. This contradicts [23, Lemma VI.1.1 and Theorem VI.8.2].

From now on until the end of this section we assume that $A = KQ/I$ is a bound quiver algebra of generalized dihedral type. Then A satisfies the conditions (Q1) and (Q2) of Proposition 9.5.

NOTATION 9.7. We say that a vertex $i \in Q_0$ is a 1-vertex if $|s^{-1}(i)| = 1$ and a 2-vertex otherwise. We denote by I_0 the set of all 1-vertices of Q . For any 2-vertex i there are arrows α, β starting at i such that

$$\alpha A \cap \beta A = \text{soc}(e_i A)$$

We fix a set of arrows which satisfy this.

For such a choice of arrows, if $\alpha, \beta \in Q_1$ start at i then $\text{rad } P_i/S_i = U_i \oplus V_i$, where $U_i = \alpha A/S_i$ and $V_i = \beta A/S_i$.

For an arrow $\gamma \in Q_1$ and $i = t(\gamma)$, we set

$$R_\gamma = \{x \in e_i A \mid \gamma x = 0\}.$$

We note that R_γ is isomorphic to $\Omega_A(\gamma A)$, and we will always take this as an identification. The following lemma from [23, Lemma VI.1.1] provides another description of the hearts of indecomposable projective modules associated to 2-vertices.

LEMMA 9.8. *Let i be a vertex of Q at which two arrows γ and δ end. Then $\text{rad } P_i = R_\gamma + R_\delta$, $R_\gamma \cap R_\delta = S_i$, and hence $(\text{rad } P_i)/S_i = (R_\gamma/S_i) \oplus (R_\delta/S_i)$.*

If we modify some arrows but keep the intersection condition of Notation 9.7 then the collection of modified arrows also satisfies Lemma 9.8.

As well we have $(\text{rad } P_i)/S_i = \alpha A/S_i \oplus \beta A/S_i$, the direct sum of two indecomposable modules. The Krull-Schmidt Theorem gives that R_γ/S_i is isomorphic to one of $\alpha A/S_i$ or $\beta A/S_i$.

NOTATION 9.9. (1) We define a map $f : Q_1 \cup I_0 \rightarrow Q_1 \cup I_0$ by

$$f(\gamma) = \begin{cases} \alpha & \text{if } R_\gamma/S_i \cong (\alpha A)/S_i, \\ i & \text{if } R_\gamma = S_i, \end{cases}$$

for an arrow $\gamma \in Q_1$, and for $i \in I_0$, we define

$$f(i) = \alpha,$$

where α is a unique arrow in Q_1 with source i .

This is a permutation of $Q_1 \cup I_0$. We note that if no double arrows start at vertex i then $R_\gamma/S_i \cong (\alpha A)/S_i$ if and only if $R_\gamma = \alpha' A$ where $\alpha - \alpha' \in (\text{rad } A)^2$.

If there are no double arrows then the map f is the same as the map π in [23, VI.1.2].

(2) There is also the permutation of Q_1 which describes the composition series of modules generated by arrows. It is called π^* in [23, VI.1.3], and we will denote it by g .

We define the permutation $g : Q_1 \rightarrow Q_1$ as follows

$$g(\gamma) = \begin{cases} \alpha & \text{if } t(\gamma) \in I_0 \text{ and } \alpha \in Q_1 \text{ with } s(\alpha) = t(\gamma), \\ \delta & \text{if } t(\gamma) \notin I_0 \text{ and } \delta \in Q_1 \setminus \{f(\gamma)\} \text{ with } s(\delta) = t(\gamma). \end{cases}$$

(3) The permutation f describes the action of Ω_A on modules in $\text{mod } A$ generated by arrows. In fact, it follows from [23, Theorem IV.4.2] that any Auslander-Reiten sequence in $\text{mod } A$ with the right term αA , for $\alpha \in Q_1$, has indecomposable middle term. In our setting, these modules occur at mouths of stable tubes, and stable tubes have rank 1 or 3, which means that the modules generated by arrows are periodic of period at most three (with respect to Ω_A). Furthermore, by Lemma 9.8 we know that for each arrow α , also $\Omega(\alpha A)$ is generated by an arrow (and hence also $\Omega^2(\alpha A)$ in the case when αA has period three).

We summarize the possibilities for the cycles of the permutation f of $Q_1 \cup I_0$.

- (i) If α is an arrow occurring in an f -cycle of a vertex in I_0 that αA has period three. This follows from Proposition 9.3.

Suppose $\alpha : i \rightarrow j$ is an arrow whose f -cycle consists of arrows.

- (ii) If αA has period one then α is a loop fixed by f ,
- (iii) Suppose αA has period two. Then we have an exact sequence

$$0 \rightarrow \alpha A \rightarrow e_i A \rightarrow e_j A \rightarrow \alpha A \rightarrow 0$$

and hence $\Omega(\alpha A)$ is generated by an arrow $j \rightarrow i$. In particular, if $j \neq i$ this can only occur if there is an arrow $j \rightarrow i$. The f -cycle is then either (α) or $(\alpha \beta)$,

(iv) Suppose αA has period three. Then there is an exact sequence

$$0 \rightarrow \alpha A \rightarrow e_i A \rightarrow e_k A \rightarrow e_j A \rightarrow \alpha A \rightarrow 0$$

and $\Omega(\alpha A)$ is generated by an arrow $\delta : j \rightarrow k$, and $\Omega^2(\alpha A)$ is generated by an arrow $\gamma : k \rightarrow i$. If the f -cycle contains a loop then the other two arrows lie on a subquiver $a \rightleftarrows b$. Otherwise the three arrows form a triangular subquiver with three different vertices.

The next lemma is a variation of [23, Lemma VI.1.4.4]. Consider a 2-vertex i of Q , there are either four distinct arrows adjacent to i , or else three when one of them is a loop. This holds since Q is connected, with at least three vertices.

LEMMA 9.10. *Assume $i \in Q_0$ is a vertex at which two arrows α, β start and two arrows γ, δ end, and $f(\gamma) = \alpha$ and $f(\delta) = \beta$. Then the following statements hold.*

- (i) *Suppose γ, δ are not fixed by f . Then there are arrows α' and β' with $\alpha' A = R_\gamma$ and $\beta' A = R_\delta$, such that $\gamma \alpha' = 0$ and $\delta \beta' = 0$, and $\alpha' A \cap \beta' A \subseteq \text{soc}(e_i A)$. If α, β are not double arrows we may assume $\alpha - \alpha' \in (\text{rad } A)^2$ and $\beta - \beta' \in (\text{rad } A)^2$.*
- (ii) *Suppose $f(\gamma) = \gamma$ so that $\gamma = \alpha$, and δ is not a loop. Then there are arrows α' and β' with $\alpha - \alpha' \in (\text{rad } A)^2$ and $\beta - \beta' \in (\text{rad } A)^2$ such that $(\alpha')^2$ lies in $\text{soc}(e_i A)$, $\delta \beta' = 0$, and $\alpha' A \cap \beta' A \subseteq \text{soc}(e_i A)$.*

We have also the following lemma from [23, IV.1.4.5] for loops fixed by g .

LEMMA 9.11. *Assume α is a loop at i in Q_1 fixed by g . Then, for any choice of arrows γ ending at i and β starting at i , one has $\gamma \alpha = 0$ and $\alpha \beta = 0$.*

In Lemma 9.10 we have $R_\gamma = \alpha' A$ and $\alpha' A / S_i \cong \alpha A / S_i = U_i$. One would like to know when this necessarily implies that $\alpha' A \cong \alpha A$.

LEMMA 9.12. *Let α be an arrow in Q with $j = t(\alpha)$ a 2-vertex, $i = s(\alpha)$, and $\alpha A / S_i = U_i$. Then $\dim_K \text{Ext}_A^1(U_i, S_i) = 2$ if $\Omega_A(\alpha A) = \gamma A$ for an arrow γ in Q from j to i , and $\dim_K \text{Ext}_A^1(U_i, S_i) = 1$ otherwise.*

Proof. There is a commutative diagram in $\text{mod } A$ with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_A(\alpha A) & \longrightarrow & P_j & \longrightarrow & \alpha A \longrightarrow 0 \\ & & \downarrow & & \downarrow \text{id} & & \downarrow p \\ 0 & \longrightarrow & \Omega_A(U_i) & \longrightarrow & P_j & \longrightarrow & U_i \longrightarrow 0 \end{array}$$

with p the canonical epimorphism, and hence a short exact sequence of the form

$$0 \rightarrow \Omega_A(\alpha A) \rightarrow \Omega_A(U_i) \rightarrow S_i \rightarrow 0.$$

We also note that $\Omega_A(\alpha A) \cong R_\alpha$ has a simple top and must be generated by an arrow ρ starting at j . Applying $\text{Hom}_A(-, S_i)$ to the lower exact sequence of the above diagram we obtain an exact sequence of K -vector spaces

$$0 \rightarrow \text{Hom}_A(U_i, S_i) \rightarrow \text{Hom}_A(P_j, S_i) \rightarrow \text{Hom}_A(\Omega_A(U_i), S_i) \rightarrow \text{Ext}_A^1(U_i, S_i) \rightarrow 0,$$

and hence an isomorphism of K -vector spaces

$$\text{Ext}_A^1(U_i, S_i) \cong \text{Hom}_A(\Omega_A(U_i), S_i),$$

because $\text{top}(U_i) = \text{top}(P_j)$. Further, we see that $\dim_K \text{Hom}_A(\Omega_A(U_i), S_i) = 1$ if the top of $\Omega_A(\alpha A)$ is not isomorphic to S_i , and $\dim_K \text{Hom}_A(\Omega_A(U_i), S_i) = 2$ if S_i is the top of $\Omega_A(\alpha A)$. But S_i is the top of $\Omega_A(\alpha A)$ if and only if $\Omega_A(\alpha A) = \rho A$ for an arrow ρ in Q from j to i . This proves the claim.

COROLLARY 9.13. *Let α be an arrow in Q from i to j and suppose there is no arrow in Q from j to i . Then $\alpha A \cong \alpha' A$ for any arrow α' such that $\alpha A/S_i \cong \alpha' A/S_i$.*

Proof. It follows from Lemma 9.12 that $\text{Ext}_A^1(U_i, S_i)$ is one-dimensional, where $U_i = \alpha A/S_i$. Hence any two indecomposable modules which are extensions of U_i by S_i are isomorphic.

Next, we want to show that A is special biserial, with the exception as described in Proposition 9.15. The following lemma which is slightly more general, will be used several times for the proof.

For a vertex $i \in Q_0$, $u \in e_i A e_i$ is called a *normalized unit* if $u - e_i \in \text{rad } e_i A e_i$.

LEMMA 9.14. *Assume Q has a subquiver*

$$k \xrightarrow{\alpha} j \xrightarrow{\gamma} i \xrightarrow{\delta} l.$$

Assume that

(i) $\Omega_A(\delta A) = \gamma A$ and $\Omega_A(\alpha A) = \delta' A$, in particular $\delta\gamma = 0$ and $\alpha\delta' = 0$.

(ii) $\delta A \cong \delta' A$.

Then there are normalized units $u, v \in e_i A e_i$ such that $\delta = \delta v u$ and $\delta' = \delta' u v$, and moreover $(\delta' u)\gamma = 0$ and $\alpha(\delta' u) = 0$. In particular, $\delta' - \delta' u \in (\text{rad } A)^2$.

Proof. Let $\psi : \delta A \rightarrow \delta' A$ be an isomorphism in $\text{mod } A$. We have

$$\psi(\delta) = \delta' u, \quad \psi^{-1}(\delta') = \delta v,$$

for some $u, v \in e_i A e_i$. Then

$$\delta = \psi^{-1}\psi(\delta) = \psi^{-1}(\delta' u) = \psi^{-1}(\delta')u = \delta v u,$$

and similarly $\delta' = \delta' u v$. Hence $\delta(e_i - v u) = 0$ and therefore $(e_i - v u) \in \Omega(\delta A) = \gamma A$. Write $e_i - v u = \gamma z$ for some $z \in A$. Hence $e_i - v u$ lies in the radical of $e_i A e_i$ and is therefore nilpotent. This implies that $v u$ is a unit in $e_i A e_i$ and so are u, v . We may assume that u, v are normalized, so that u is of the form $e_i + u'$ with u' in the radical. Clearly $\alpha(\delta' u) = 0$. As well $0 = \psi(0) = \psi(\delta\gamma) = \psi(\delta)\gamma = (\delta' u)\gamma$. Since u is normalized, we have $\delta' - \delta' u \in (\text{rad } A)^2$.

PROPOSITION 9.15. *The algebra A is special biserial, except possibly that squares of loops may be non-zero in the socle.*

Proof. (I) We prove this first when Q is not the Markov quiver (see Example 4.4).

First we show that for suitable choice of arrows, the condition on paths of length two of the definition holds. That is, we must show that for suitable choice, the product of two arrows along a cycle of f , in Notation 9.9, is zero.

(1) Assume $i \in I_0$, then the cycle of f containing i clearly has length three. Then, by Proposition 9.3, we may assume that it is of the form

$$(i \gamma \delta)$$



and $\gamma\delta = 0$.

From now, we need to consider only arrows adjacent to 2-vertices.

(2) Consider a fixed point $\alpha \in Q_1$ of f . Then α is a loop. We have the setting as in part (ii) of Lemma 9.10, and we may assume that α^2 lies in the socle of A . Then we can write

$$\alpha^2 = b\omega_i$$

where ω_i generates the socle of $P_i = e_i A$ and $b \in K$. Moreover, we see from this directly that $\Omega_A^2(\alpha A) \cong \alpha A$.

(3) Consider a loop γ which is not fixed by f . With the notation of Lemma 9.10(ii), one of α, β is equal to γ and $\alpha \neq \gamma$ and therefore $\beta = \gamma$.

Since γ is fixed by g , we have by Lemma 9.11 that $\gamma\alpha = 0$, and $\Omega(\gamma A) \cong \alpha A$. Let $j = t(\alpha)$. This is a 2-vertex $\neq i$ and $\Omega(\alpha A)$ is generated by an arrow which must end at i since the period is ≤ 3 . It must be an arrow δ' with $\alpha\delta' = 0$ and $\delta' - \delta \in (\text{rad } A)^2$. Now again by Lemma 9.11 we have even $\delta\gamma = 0$ and $\Omega(\alpha A) \cong \Omega^{-1}(\gamma A) \cong \delta A$.

Now we have the hypotheses of Lemma 9.14 which shows that we can assume also that $\alpha\delta = 0$.

We are left to consider cycles of f on arrows which do not contain any loops.

(4) Consider such a cycle of length three, say $(\gamma \alpha \delta)$. This must then pass through three distinct vertices. By assumption, Q is not the Markov quiver, so at most one of the three arrows can be part of a double arrow. So we may label such that α, δ are not double arrows. Then we may apply part (i) of Lemma 9.10 which gives that we may replace α (and β with the labelling there) and assume $\gamma\alpha = 0$ and $\Omega(\gamma A) = \alpha A$. Say α ends at j so that δ starts at j . Then j must be a 2-vertex and δ is not part of a double arrow. So we may assume $\alpha\delta = 0$ and $\Omega(\alpha A) = \delta A$.

The period is three, so $\Omega(\delta A) \cong \gamma A$ and $\Omega(\delta A) = \gamma' A$ for an arrow γ' starting at $t(\delta)$. Now we use Lemma 9.14 again. This shows that we may assume $\delta\gamma = 0$.

(5) Assume now that α is not a loop and αA has Ω_A -period 2. Then, by part (3) in Notation 9.9, we have $\Omega_A(\alpha A) = \beta A$ for an arrow β with $\alpha\beta = 0$. Then we must have $\Omega_A(\beta A) \cong \alpha A$ and also $\Omega(\beta A) = \alpha' A$ for some arrow $\alpha' : i \rightarrow j$.

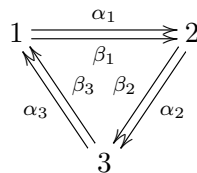
We apply the Lemma 9.14 again, this gives that we may assume $\alpha\beta = 0$ and $\beta\alpha = 0$.

Let α be an arrow starting at a 2-vertex, and let n_α be the size of the g -orbit of α . Then there is a maximal $m = m_\alpha \geq 1$ such that $B_\alpha := (\alpha g(\alpha) \dots g^{n_\alpha-1}(\alpha))^m \neq 0$. This generates the socle of αA . The parameters n_α and m_α are constant on g -orbits since A is symmetric. Whenever $i \in Q_0$ is a 2-vertex and γ, δ are the arrows starting at i , there is a non-zero scalar such that

$$B_\gamma = c_\delta B_\delta$$

The algebra is symmetric, and by a standard argument one can scale some arrows suitably and obtain that any such scalar c_δ is equal to 1.

(II) Now we prove the Proposition for an algebra A of generalized dihedral type where the quiver is as follows



We choose arrows α_i, β_i such that $e_i A = \alpha_i A + \beta_i A$ with $\alpha_i A \cap \beta_i A = \text{soc}(e_i A)$. For this quiver, every module generated by an arrow must have Ω -period three.

Then we use Lemma 9.8, and get identifications, that is

$$\{\alpha_i A/S_i, \beta_i A/S_i\} = \{R_{\alpha_{i-1}}/S_i, R_{\beta_{i-1}}/S_i\}$$

(taking i modulo 3).

(1) We start with α_1, β_1 . We may assume $R_{\alpha_1}/S_2 \cong \alpha_2 A/S_2$ and $R_{\beta_1}/S_2 \cong \beta_2 A/S_2$. By Lemma 9.8 we may assume α_2 generates R_{α_1} and β_2 generates R_{β_1} .

(2) The same reasoning gives that we may assume $R_{\alpha_2} = \alpha_3 A$ and $R_{\beta_2} = \beta_3 A$.

So far we have $\Omega(\alpha_i A) = \alpha_{i+1} A$ and $\Omega(\beta_i A) = \beta_{i+1} A$ for $i \in \{1, 2\}$. The period of $\alpha_i A$ is 3, and therefore $\Omega(\alpha_3 A) \cong \alpha_1 A$. As well $\Omega(\alpha_3 A) = \alpha'_1 A$ where α'_1 is an arrow from 1 to 2. Similarly, $\beta_1 A \cong \Omega(\beta_3 A) = \beta'_1 A$ for an arrow β'_1 from 1 to 2.

We apply Lemma 9.8 which shows that, without loss of generality, $\alpha'_1 = \alpha_1$ and $\beta'_1 = \beta_1$.

Now we can get socle relations, similarly as in the general case. In total the relations are

$$\begin{aligned} \alpha_1\alpha_2 &= 0, & \beta_1\beta_2 &= 0, & (\alpha_1\beta_2\alpha_3\beta_1\alpha_2\beta_3)^m &= (\beta_1\alpha_2\beta_3\alpha_1\beta_2\alpha_3)^m, \\ \alpha_2\alpha_3 &= 0, & \beta_2\beta_3 &= 0, & (\alpha_2\beta_3\alpha_1\beta_2\alpha_3\beta_1)^m &= (\beta_2\alpha_3\beta_1\alpha_2\beta_3\alpha_1)^m, \\ \alpha_3\alpha_1 &= 0, & \beta_3\beta_1 &= 0, & (\alpha_3\beta_1\alpha_2\beta_3\alpha_1\beta_2)^m &= (\beta_3\alpha_1\beta_2\alpha_3\beta_1\alpha_2)^m. \end{aligned}$$

Here $m \geq 1$, and this defines the weight function.

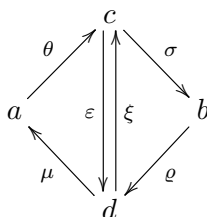
We will now construct a biserial weighted triangulation algebra such that the idempotent algebra associated to an appropriate set of 2-triangle disks is isomorphic to A .

We fix A , and then (Q, f) is fixed. We denote by $\Delta = \Delta(Q, f)$ the family of all 2-cycles $a \xrightleftharpoons[\beta]{\alpha} b$ in Q_1 with $\alpha\beta = 0$ and $\beta\alpha = 0$.

We define now a triangulation quiver (\tilde{Q}, \tilde{f}) with $\tilde{Q} = (\tilde{Q}_0, \tilde{Q}_1, \tilde{s}, \tilde{t})$. This is obtained from (Q, f) as follows.

(I) For each vertex $i \in I_0$ and the arrows $i \xrightleftharpoons[\delta_i]{\gamma_i} i^*$ adjacent to i , we create in \tilde{Q}_1 a loop η_i at i , and set $\tilde{f}(\eta_i) = \gamma_i$, $\tilde{f}(\gamma_i) = \delta_i$, and $\tilde{f}(\delta_i) = \eta_i$.

(II) We replace each 2-cycle $a \xrightleftharpoons[\beta]{\alpha} b$ in Δ by a 2-triangle disk in \tilde{Q} of the form



with $\tilde{f}(\theta) = \varepsilon$, $\tilde{f}(\varepsilon) = \mu$, $\tilde{f}(\mu) = \theta$, $\tilde{f}(\sigma) = \varrho$, $\tilde{f}(\varrho) = \xi$, $\tilde{f}(\xi) = \sigma$.

(III) We keep in \tilde{Q}_1 all arrows ω of Q_1 which do not belong to 2-cycles in Δ and set $\tilde{f}(\omega) = f(\omega)$.

We note that (\tilde{Q}, \tilde{f}) is a triangulation quiver with $|\tilde{Q}_0| = |Q_0| + 2|\Delta|$. We denote by $\tilde{\Sigma}$ the family of all 2-triangle disks in (\tilde{Q}, \tilde{f}) created from the

2-cycles in Δ . Observe also that the border $\partial(\tilde{Q}, \tilde{f})$ of (\tilde{Q}, \tilde{f}) is given by the sources (targets) of all loops ν in Q_1 with $f(\nu) = \nu$.

We denote by $\tilde{g} : \tilde{Q}_1 \rightarrow \tilde{Q}_1$ the permutation induced by \tilde{f} , and by $\mathcal{O}(\tilde{g})$ the set of all \tilde{g} -orbits in \tilde{Q}_1 .

Let $\tilde{\mathcal{O}}(\gamma)$ be the \tilde{g} -orbit of an arrow γ of \tilde{Q} . These orbits are as follows:

- (IV) For each vertex $i \in I_0$, we have the loop η_i with $\tilde{g}(\eta_i) = \eta_i$.
- (V) For each 2-triangle disk created from a cycle in Δ we have the orbit of size 2 consisting of ε and ξ , as in the above subquiver.
- (VI) For each 2-triangle disk in (\tilde{Q}, \tilde{f}) created by a 2-cycle $(\alpha \beta)$ from Δ , we have $\tilde{g}^{-1}(\theta) = g^{-1}(\alpha)$, $\tilde{g}(\theta) = \sigma$, $\tilde{g}(\sigma) = g(\alpha)$, $\tilde{g}^{-1}(\varrho) = g^{-1}(\beta)$, $\tilde{g}(\varrho) = \mu$, $\tilde{g}(\mu) = g(\beta)$. That is, we obtain for example the \tilde{g} -orbit of $g^{-1}(\alpha)$ by replacing α with θ, σ , and keeping the rest, and similarly we replace β by ρ, μ to obtain the \tilde{g} -orbit of $g^{-1}(\beta)$.
- (VII) If $\gamma \in Q_1$ and $\mathcal{O}(\gamma)$ does not contain an arrow from a cycle in Δ then $\tilde{\mathcal{O}}(\gamma) = \mathcal{O}(\gamma)$.

In particular, we have $|\mathcal{O}(\tilde{g})| = |\mathcal{O}(g)| + |I_0| + |\Delta|$, where $\mathcal{O}(g)$ is the set of all g -orbits in Q_1 . We also note that two arrows γ and δ in $Q_1 \cap \tilde{Q}_1$ belong to the same g -orbit in Q_1 if and only if they belong to the same \tilde{g} -orbit in \tilde{Q}_1 .

We set $\tilde{n}_\gamma = |\tilde{\mathcal{O}}(\gamma)|$. For each arrow $\delta \in Q_1$, we had already defined n_δ to be the length of the g -orbit $\mathcal{O}(\delta)$ of δ in Q_1 . Clearly, $\tilde{n}_\delta \geq n_\delta$ for any arrow $\delta \in \tilde{Q}_1 \cap Q_1$, but in general we may have $\tilde{n}_\delta > n_\delta$.

We shall define now a suitable weight function $\tilde{m}_\bullet : \mathcal{O}(\tilde{g}) \rightarrow \mathbb{N}^*$.

(1) For an arrow in each of the new orbits of \tilde{g} , we set $\tilde{m} = 1$, that is $\tilde{m}_{\eta_i} = 1$ and $\tilde{m}_\varepsilon = 1 = \tilde{m}_\xi$.

(2) Any other orbit of \tilde{g} contains $\gamma \in Q_1$ and $\tilde{\mathcal{O}}(\gamma) \cap Q_1 = \mathcal{O}(\gamma)$. Set $\tilde{m}_\rho = m_\gamma$ for any arrow ρ in this \tilde{g} -orbit.

The border $\partial(\tilde{Q}, \tilde{f})$ consists of the sources of all loops whose square is in the socle. From the presentation of A we have that if ν is a loop with ν^2 is in the socle then there is $\tilde{b}_i \in K$, with $i = \tilde{s}(\nu)$, such that

$$\nu^2 = \tilde{b}_i(\nu g(\nu) \dots g^{n_\nu-1}(\nu))^{m_\nu}.$$

This define a border function $\tilde{b}_\bullet : \partial(\tilde{Q}, \tilde{f}) \rightarrow K$.

We have (\tilde{Q}, \tilde{f}) which is a triangulation quiver. We also have \tilde{m}_\bullet and \tilde{b}_\bullet . With these data, let $B = B(\tilde{Q}, \tilde{f}, \tilde{m}_\bullet, \tilde{b}_\bullet)$ be the associated socle deformed biserial weighted triangulation algebra, and $B(\tilde{Q}, \tilde{f}, \tilde{\Sigma}, \tilde{m}_\bullet, \tilde{b}_\bullet)$ be its idempotent algebra with respect to the family $\tilde{\Sigma}$ of 2-triangle disks in (\tilde{Q}, \tilde{f}) . Then it follows from the above that A is isomorphic to $B(\tilde{Q}, \tilde{f}, \tilde{\Sigma}, \tilde{m}_\bullet, \tilde{b}_\bullet)$. We also note that $B(\tilde{Q}, \tilde{f}, \tilde{\Sigma}, \tilde{m}_\bullet, \tilde{b}_\bullet)$ is socle equivalent to the idempotent algebra $B(\tilde{Q}, \tilde{f}, \tilde{\Sigma}, \tilde{m}_\bullet)$ of the biserial weighted triangulation algebra $B(\tilde{Q}, \tilde{f}, \tilde{m}_\bullet)$. Moreover, by Theorem 4.1, (\tilde{Q}, \tilde{f}) is the triangulation quiver $(Q(S, \vec{T}), f)$ associated to a directed triangulated surface (S, \vec{T}) . This completes the proof that (i) implies (ii), and hence the proof of Theorem 1.

§10. Proof of Theorem 2

Let A be an algebra. For a module M in $\text{mod } A$, we denote by $[M]$ the image of M in the Grothendieck group $K_0(A)$ of A . Hence, for two modules M, N in $\text{mod } A$, we have $[M] = [N]$ if and only if M and N have the same simple composition factors including the multiplicities.

PROPOSITION 10.1. *Let $B = B(Q, f, m_\bullet)$ be a biserial weighted triangulation algebra and suppose that the Cartan matrix C_B of B is non-singular. Then B is an algebra of strict dihedral type.*

Proof. By Proposition 5.2, we only have to show that $|Q_0| = 2$ or 3 and that the number of stable tubes of rank 3 in Γ_B^s is $|Q_0| - 1$. In fact we will show the first part, the second part will follow.

We apply Theorem 1. We know that A is special biserial, and we have a permutation f on $Q_1 \cup I_0$ describing the zero relations of length two and 1-vertices of Q . This permutation has cycles of length ≤ 3 , and every arrow belongs to a unique cycle. In the following, we exploit the exact sequences from Notation 9.9.

(1) Assume (for a contradiction) that f has a 2-cycle $(\alpha \beta)$ and $\alpha\beta = 0$ and $\beta\alpha = 0$. Then $j = t(\alpha) \neq s(\alpha) = i$, for otherwise A would be local. Then we have an exact sequence

$$0 \rightarrow \alpha A \rightarrow P_i \rightarrow P_j \rightarrow \alpha A \rightarrow 0.$$

It follows that $[P_i] = [P_j]$ and C_A is singular, a contradiction. So no such cycle exists.

(2) Consider a 2-vertex i . Suppose there are two f -cycles of length three passing through i , and let $\alpha, \bar{\alpha}$ be the arrows starting at i . Then we have

exact sequences

$$\begin{aligned} 0 \rightarrow \alpha A \rightarrow P_i \rightarrow P_k \rightarrow P_j \rightarrow \alpha A \rightarrow 0, \\ 0 \rightarrow \bar{\alpha} A \rightarrow P_i \rightarrow P_s \rightarrow P_t \rightarrow \bar{\alpha} A \rightarrow 0. \end{aligned}$$

Here, the cycle of α passes through i, j, k (where two of these vertices may be equal), and similarly the cycle of $\bar{\alpha}$ passes through vertices i, t, s . Note that $[\alpha A] + [\bar{\alpha} A] = [P_i]$. Hence, if we take the direct sum of these sequences, we get an identity for composition factors of projective modules, namely

$$2[P_i] + [P_k] + [P_s] = [P_j] + [P_t] + 2[P_i],$$

and $[P_k] + [P_s] = [P_j] + [P_t]$. Hence

$$\{[P_k], [P_s]\} = \{[P_j], [P_t]\}.$$

If $k = s$ then also $j = t$. But then to have C_A non-singular it follows that $k = s = j = t$ and we get a contradiction with Q being 2-regular. In particular, this shows that Q is not the Markov quiver, considered in Example 4.4 .

Hence $k \neq s$ and then $j \neq t$. It follows that either $k = j$ and $s = t$, or $k = t$ and $s = j$. In the first case, these f -cycles give a 2-regular subquiver with three vertices and two loops which is then all of Q , and Q is the quiver with three vertices, considered in Example 8.3. In the second case, these f -cycles give a quiver with three vertices and no loops, which also is 2-regular and hence all of Q , and then Q is the quiver with three vertices considered in Example 8.4.

(3) Assume i is a 2-vertex where both f -cycles through i contain a vertex in I_0 , then obviously Q has three vertices. Suppose one of the f -cycles through i contains a 1-vertex, say j , but not the other. Then we have exact sequences

$$0 \rightarrow \alpha A \rightarrow P_i \rightarrow P_j \rightarrow P_j \rightarrow \alpha A \rightarrow 0$$

and

$$0 \rightarrow \bar{\alpha} A \rightarrow P_i \rightarrow P_k \rightarrow P_r \rightarrow \bar{\alpha} A \rightarrow 0.$$

with $j = t(\alpha)$ and $r = t(\bar{\alpha})$. Taking the direct sum gives now the identity for composition factors of projective modules

$$[P_j] + [P_k] = [P_j] + [P_r],$$

and $[P_k] = [P_r]$, which implies $k = r$. That is, the cycle of $\bar{\alpha}$ has a loop and the quiver has three vertices.

Now assume at most one 3-cycle of f passes through any 2-vertex. Then since Q is connected, there can only be one 3-cycle and otherwise fixed points of f , and then Q has at most three vertices. We can have the quiver with two vertices and one or two loops. We note that the quiver with three vertices and three loops fixed by f , considered in Example 4.3, is not possible. If so, then g consists of just one cycle which passes through each vertex twice, and it follows that all projectives have the same composition factors, and C_A is singular.

The second statement follows in each possible case, by counting the cycles of f of length three.

REMARK 10.2. Given conditions (1), (3) and (4) of the definition of strict dihedral type, then condition (5) implies that (2) holds. This is proved in [22]. The final argument in the above proof uses that (2) implies (5). We note that the classification of algebras of dihedral type is not sufficient to prove Theorem 2.

We may complete now the proof of Theorem 2. Since every algebra of strict dihedral type is of generalized dihedral type and with non-singular Cartan matrix, the implication (i) \Rightarrow (ii) holds. Assume now that A is an algebra of generalized dihedral type and the Cartan matrix C_A is non-singular. It follows from Theorem 1 that A is socle equivalent to an algebra of the form $B(Q, f, m_\bullet, \Sigma)$. Then, applying Proposition 7.3, we conclude that A is isomorphic to an algebra of the form $B(Q, f, m_\bullet, \Sigma, b_\bullet)$. Since the Cartan matrix C_A is non-singular, it follows from Propositions 7.1 and 7.2 that Σ is empty. Then A is socle equivalent to $B = B(Q, f, m_\bullet)$, by Proposition 6.2. Moreover, the Cartan matrices C_A and C_B coincide, because A and B are socle equivalent symmetric algebras. In particular, C_B is non-singular. Applying Proposition 10.1 we conclude that B is an algebra of strict dihedral type. But then A is an algebra of strict dihedral type, again because A is socle equivalent to B (see also Proposition 6.4 and Theorem 8.5).

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