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Discrete Mathematics 197/198 (1999) 713–731

DISCRETE
MATHEMATICS

A characterisation of jointless Dowling geometries

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Received 9 July 1997; revised 18 December 1997; accepted 3 August 1998

Abstract

We use statistics of flats of small rank in order to characterise the jointless Dowling geometries defined by groups of order exceeding three and having rank greater than 3. In particular, we show that if the Tutte polynomial of a matroid is identical to the Tutte polynomial of a jointless Dowling geometry, then the matroid is indeed a jointless Dowling geometry. For rank 3 (and groups of order exceeding 3) this holds only if the order of the group is even. © 1999 Elsevier Science B.V. All rights reserved

1. Introduction

Characterising classes of matroids by their Tutte polynomials is particularly interesting due to the large number of applications of Tutte polynomials [7]. Only a few classes of matroids have been shown to be characterised by their Tutte polynomials. Examples of such classes of matroids are projective and affine geometries and Dowling lattices [4]. We will prove here that jointless Dowling lattices are also characterised by their Tutte polynomials.

Dowling lattices are group-theoretic generalisations of partition lattices [1,8]. They have, as well, an interpretation as matroids of gain graphs [14], and are, together with representable matroids, the only non-degenerate varieties [10]. Similarities between projective geometries and Dowling lattices have been studied in [1,2]. Dowling lattices form an infinite family of supersolvable tangential blocks [13].

The geometric characterisation of Dowling lattices [4] revolves around a special basis, the elements of which are called joints (these and other Dowling-lattice terms will be defined in Section 2). This was of central importance in [4] for the characterisation of Dowling lattices by statistics of flats of small rank. However, the group structure is reflected only in the lines that do not contain joints. Therefore it is natural to ask whether it is possible to obtain a similar characterisation, by statistics of flats of

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small rank, for the geometries obtained from Dowling lattices by deleting the joints. The geometries obtained in this way are the jointless Dowling geometries, and the present paper gives the desired characterisation of these geometries. Jointless Dowling geometries form an infinite family of non-supersolvable tangential blocks [13].

A *line* in a geometry is a flat of rank 2, a *plane* is a flat of rank 3. A *trivial line* is a line having only two points. A *trivial plane* is a plane having only three points. A *trivial rank- r flat* is a rank- r flat having only r points.

Let M be a geometry with point set \mathcal{P} . We say that $T \subseteq \mathcal{P}$ is *line-closed* if, for all $\{x_1, x_2\} \subseteq T$, the line $x_1 \vee x_2$ spanned by x_1 and x_2 is contained in T . The geometry M is *line-closed* if every line closed set in M is a flat of M .

A fact of fundamental importance in [1] is that Dowling lattices are line closed. Line closed geometries are characterised by their labelled points and lines. This means that if N is a geometry with the same rank and labelled points and lines as a line closed geometry M , then N is equal to M . By ‘labelled points and lines’ we mean labelled points and labelled lines. Line closed geometries were introduced by Halsey in [9], where he proved that supersolvable geometries are line closed. Dowling introduced in [8] the lattices that were later named Dowling lattices and proved that they are supersolvable. Therefore, a Dowling lattice is completely described by the list of its points and lines.

In Section 3 we define the concept of k -closure which generalises line-closure. We will prove that k -closed geometries are characterised by their flats of rank up to k . Furthermore we show that rank- r jointless Dowling geometries over a group G are 4-closed if $r \geq 4$. This will allow us to prove in Section 5 that jointless Dowling geometries are characterised by statistics of flats of small rank. The case of rank 3 will be considered in Section 4.

We will use standard matroid notation as in [12].

2. Jointless Dowling lattices

Let G be a finite group (written multiplicatively). The *rank- r Dowling lattice over G* , denoted $Q_r(G)$, is the geometry having the following types of points and lines. The points are *joints* p_1, p_2, \dots, p_r (which form a basis for $Q_r(G)$) and *internal points* λ_{ij} (for all $\lambda \in G$ and all i, j positive integers such that $1 \leq i < j \leq r$). The nontrivial lines are *coordinate lines* $p_i \vee p_j = \{p_i, p_j\} \cup \{\lambda_{ij} : \lambda \in G\}$ and *transversal lines* $\{\lambda_{ij}, \gamma_{jk}, (\lambda\gamma)_{ik}\}$ (for all $\lambda, \gamma \in G$ and all i, j, k positive integers with $1 \leq i < j < k \leq r$).

As in [1], Dowling lattices of rank $r \geq 4$ can be axiomatically defined through (D1)–(D3) in the following proposition. In the case $r = 3$, Dowling lattices based on quasigroups are characterised by these conditions.

Proposition 2.1. *A geometry M of rank $r \geq 4$ is a Dowling lattice if and only if M has a basis p_1, p_2, \dots, p_r satisfying the following axioms.*

(D1) *Each point of M lies on a line $p_i \vee p_j$.*

- (D2) No line $p_i \vee p_j$ is trivial.
 (D3) For points $x \in (p_i \vee p_j) \setminus \{p_i, p_j\}$ and $y \in (p_i \vee p_k) \setminus \{p_i, p_k\}$, the line $x \vee y$ is nontrivial.

In the case of a nontrivial group, the basis of M satisfying (D1)–(D3) is unique and consists of the joints. For the trivial group $\langle 1 \rangle$, the Dowling lattice $Q_r(\langle 1 \rangle)$ is the rank- r partition lattice Π_{r+1} . In this case, we refer to (D1)–(D3) as the axioms of Π_{r+1} . Dowling lattices can be considered as group-theoretic generalisations of partition lattices.

Let G be a group and let $Q_r(G)$ be the Dowling lattice of rank $r \geq 2$ based on G . Let p_1, \dots, p_r be the joints of $Q_r(G)$. Then $Q'_r(G) = Q_r(G) \setminus \{p_1, \dots, p_r\}$ is called a *jointless Dowling geometry*.

Note that the points of the jointless Dowling geometry $Q'_r(G)$ are the internal points λ_{ij} of the Dowling geometry $Q_r(G)$. The lines of $Q'_r(G)$ are the transversal lines of $Q_r(G)$ and the lines $l_{ij} = (p_i \vee p_j) \setminus \{p_i, p_j\}$.

The following proposition characterises the non-trivial flats of low rank in jointless Dowling geometries. The characterisation revolves around the lines l_{ij} . These special lines are uniquely determined by the first four properties given in the proposition. In Section 3 we will prove that the jointless Dowling lattices are characterised by their labelled flats of rank up to 4. The main result (Theorem 5.1) of this paper is that a geometry M having the same statistics of flats of rank up to 7 as a jointless Dowling geometry is itself a jointless Dowling geometry. In order to prove it we will label the flats of rank up to 4 of M as those of a jointless Dowling geometry. First we will label a collection of lines of M in such a way that the lines, planes and rank-4 flats of M are precisely as described in Proposition 2.2. Later we will label the points of M as those of a jointless Dowling geometry $Q'_r(G)$, proving that the labelled flats, of rank up to 4, of M and $Q'_r(G)$ coincide. With this we will conclude that M equals $Q'_r(G)$.

Proposition 2.2. *Let G be a group of order at least 4. The jointless Dowling geometry $Q'_r(G)$ has exactly one set of $\binom{r}{2}$ lines l_{ij} , with $1 \leq i < j \leq r$, satisfying the following four properties. (In each statement in this proposition, indices represented by distinct variables are unequal.)*

- (JD1) The lines l_{ij} partition the points of $Q'_r(G)$.
 (JD2) No line l_{ij} is trivial.
 (JD3) For all $x \in l_{ij}$ and $y \in l_{ik}$, the line $x \vee y$ is nontrivial. Furthermore, all non-trivial lines of $Q'_r(G)$ other than the lines l_{ij} are of this type.
 (JD4) For all i, j, k , the union of l_{ij} , l_{ik} and l_{jk} is a plane.

The following are the three other types of nontrivial planes of $Q'_r(G)$.

- (JD5) For any point y in l_{hi} , the set $l_{jk} \cup y$ is a plane.
 (JD6) For points x in l_{ij} , y in l_{jk} , and z in l_{si} , the set $(x \vee y) \cup z$ is a plane.
 (JD7) For each pair $h, i \in \{j, k, m, n\}$, let y_{hi} be a point in l_{hi} , and let P be the set of these six points. If P satisfies the axioms of Π_4 , then P is a plane.

The following are the nontrivial rank-4 flats of $Q'_r(G)$.

- (JD8) The union of six lines l_{ij} whose indices i, j are the 2-subsets of a 4-element subset of $\{1, \dots, r\}$ is a rank-4 flat.
- (JD9) Let P be a plane as described in (JD4) and y be a point on the line l_{ij} . Then $P \cup y$ is a rank-4 flat if neither i nor j are indices of any of the lines in P .
- (JD10) Let x be a point in l_{jk} and y a point in l_{mn} . Then the set $l_{hi} \cup x \cup y$ is a rank-4 flat.
- (JD11) Let x and y be points in l_{ij} and l_{ik} respectively. Then the set $l_{mn} \cup (x \vee y)$ is a rank-4 flat.
- (JD12) For all $w \in l_{hi}$, $x \in l_{hj}$, $y \in l_{km}$ and $z \in l_{ns}$, the set $(w \vee x) \cup y \cup z$ is a rank-4 flat.
- (JD13) For all $w \in l_{hi}$, $x \in l_{hj}$, $y \in l_{km}$, $z \in l_{kn}$, the set $(w \vee x) \cup (y \vee z)$ is a rank-4 flat.
- (JD14) For each pair h, i of elements of $\{j, k, m, n\}$, let x_{ij} be a point of l_{ij} so that the six points x_{ij} satisfy the axioms of Π_4 . Then for all $y \in l_{st}$, the set $\{x_{ij}\} \cup y$ is a rank-4 flat.
- (JD15) For each pair h, i of elements of $\{j, k, m, n, s\}$ let y_{hi} be a point of l_{hi} so that the ten points y_{hi} satisfy the axioms of Π_5 (the partition lattice of rank 4). Then these ten points form rank-4 flat.

Proof. Conditions (JD1)–(JD3) are straightforward to prove from the definitions of Dowling and jointless Dowling geometries. If x is a point in l_{ij} , then $l_{ik} \vee x$ is a flat as described in (JD4), that is $p_i \vee p_j \vee p_k$. If y_{jk}, y_{jm}, y_{jn} are points in l_{jk}, l_{jm} and l_{jn} , respectively, then $(y_{jk} \vee y_{jm}) \vee y_{jn}$ is a plane P . The points y_{jk}, y_{jm}, y_{jn} constitute a basis of P that satisfies the axioms of Π_4 . The proof of the remaining conditions follows the same idea, that is, if F is a flat of rank s and x is a point not in F , then $F \vee x$ is a flat of rank $s + 1$.

Given $x \in l_{ij}$ and $y \in l_{ik}$, the line $x \vee y$ is contained in the plane $l_{ij} \cup l_{ik} \cup l_{jk}$. Therefore $x \vee y$ is a 3-point line with the third point in l_{jk} . Fix a point x in l_{ij} and define the function ϕ from l_{ik} to l_{jk} where $\phi(y)$ is the third point on the line $x \vee y$ (that is ϕ is a projection). Thus all the lines l_{st} have the same cardinality. Since $Q'_r(G)$ has $\binom{r}{2}g$ points (where g is the order of G) and there are $\binom{r}{2}$ lines in the given collection, all those lines have g points. Assume that $\{l'_{st}\}$ is a collection of lines satisfying (JD1)–(JD4). As above, the lines l'_{st} have $g > 3$ points, thus l'_{st} can not be a line of the type described in JD2). Therefore l'_{st} is in $\{l_{ij}\}$. Since there are $\binom{r}{2}$ lines in each collection $\{l'_{st}\}$ and $\{l_{ij}\}$, both collections are equal. \square

3. k -closure

The concept of line-closure is crucial to the proof that the Dowling geometries are characterised by a small collection of counting facts [4]. Unfortunately it is not possible to use line-closure in order to characterise jointless Dowling geometries.

Proposition 3.1. *The only line-closed jointless Dowling geometries are $Q'_2(G)$, $Q'_3(G)$ and $Q'_r(\langle 1 \rangle)$, where G is any group and $r \geq 2$ is a positive integer.*

Proof. Note that $Q'_2(G)$ has rank 2, therefore it is line-closed. The geometry $Q'_r(\langle 1 \rangle)$ is line-closed because it is isomorphic to $Q_{r-1}(\langle 1 \rangle)$.

Recall that $Q'_3(G)$ has only coordinate and transversal lines and no trivial lines if $g \geq 3$. Note that the transversal lines of $Q'_r(G)$ and $Q_r(G)$ coincide. To distinguish joins in $Q_r(G)$ from those in $Q'_r(G)$, we will use \vee for joins in $Q_r(G)$ and \vee' for those in $Q'_r(G)$. We will use Cl to denote the closure in $Q_r(G)$ and Cl' to denote the closure in $Q'_r(G)$.

Let P be the set of points of $Q_3(G)$, $B = \{p_1, p_2, p_3\}$ and $P' = P \setminus B$ (the set of points of $Q'_3(G)$). Let $T \subseteq P'$ be line-closed in $Q'_3(G)$. If $r(T) \leq 2$, then it is clearly a flat of $Q'_3(G)$. Thus we can assume that T has rank 3. Therefore, there are points x and y in T that lie in two distinct coordinate lines. Since T is line-closed, it contains the 3-point line $x \vee' y$ and it has at least one point in each coordinate line. Since the rank of T is 3, it has a point that is not on the line $x \vee' y$. Thus there is a coordinate line $p_i \vee p_j$ that contains two points of T . The line $p_i \vee' p_j = (p_i \vee p_j) \setminus \{p_i, p_j\}$ is contained in T (because T is line-closed). Note that T has a point not in $p_i \vee' p_j$ and the transversal lines of $Q'_3(G)$ are the same as the transversal lines of $Q_3(G)$. Therefore, $T = P'$ and $Q'_3(G)$ is line-closed.

Now let $r \geq 4$ and $g \geq 2$. We will prove that $Q'_r(G)$ is not line-closed. Let A be the internal points on the coordinate lines $p_1 \vee p_2$ and $p_3 \vee p_4$. More formally, $A = \{\lambda_{12} : \lambda \in G\} \cup \{\lambda_{34} : \lambda \in G\} = [(p_1 \vee p_2) \setminus \{p_1, p_2\}] \cup [(p_3 \vee p_4) \setminus \{p_3, p_4\}]$. If λ, λ' are elements of G (not necessarily distinct), then $\lambda_{12} \vee \lambda'_{34} = \lambda_{12} \vee' \lambda'_{34}$ is a trivial line contained in A . Furthermore, if $\lambda \neq \lambda'$, then $\lambda_{12} \vee' \lambda'_{12} = (p_1 \vee p_2) \setminus \{p_1, p_2\} \subseteq A$. That $\lambda_{34} \vee' \lambda'_{34}$ is contained in A is proved in an analogous way. So A is a line-closed subset of $Q'_r(G)$. Note that if $\lambda \neq \lambda'$, then $\lambda_{12} \vee \lambda'_{12} = p_1 \vee p_2$ and $\lambda_{34} \vee \lambda'_{34} = p_3 \vee p_4$. Thus the closure of A contains the union of the lines $p_1 \vee p_2$ and $p_3 \vee p_4$, and, in particular, it contains the joints p_1 and p_3 , hence the line $p_1 \vee p_3$. So $(p_1 \vee p_3) \setminus \{p_1, p_3\} \subseteq \text{Cl}(A) \setminus \{p_1, \dots, p_r\} = \text{Cl}'(T)$, but $(p_1 \vee p_3) \setminus \{p_1, p_3\}$ is not contained in A . Thus $A \neq \text{Cl}'(A)$ and A is not a flat of $Q'_r(G)$. Therefore, $Q'_r(G)$ is not line-closed.

Definition 3.2. Let M be a geometry with point set P and let k be a positive integer. We say that $T \subseteq P$ is k -closed if for all $\{x_1, \dots, x_k\} \subseteq T$, $x_1 \vee \dots \vee x_k \subseteq T$. A geometry M is k -closed if every k -closed set in M is a flat of M .

Note that for every geometry M and for each positive integer k , every flat F of M is k -closed.

By an argument analogous to that in the proof of Proposition 3.1, the jointless Dowling geometry $Q'_r(G)$ is not 3-closed when $r \geq 4$ and G is a group of order at least 2. If the order of G is one, then $Q'_r(G)$ is isomorphic to $Q_{r-1}(G)$, thus it is line-closed and, therefore, it is 3-closed. If $r \geq 3$, then $Q'_r(G)$ is trivially 3-closed.

Proposition 3.3. *For all $r \geq 4$ and all groups G , the jointless Dowling geometry $Q'_r(G)$ is 4-closed.*

Proof. The geometry $Q'_r(\langle 1 \rangle)$ is line closed because it is isomorphic to $Q_{r-1}(\langle 1 \rangle)$. Therefore $Q'_r(\langle 1 \rangle)$ is 4-closed.

Let G be a group of order $g \geq 2$ and T' be a 4-closed set in $Q'_r(G)$ and $T = T' \cup \{p_i, p_j : (p_i \vee p_j) \setminus \{p_i, p_j\} \subseteq T'\}$. We claim that T is line-closed in $Q_r(G)$. If $r(T') = 1$, then both T and T' consist of one point. If $r(T') = 2$, then T is a coordinate line or a transversal line or a trivial line depending on whether T' contains two points in a coordinate line. Assume that $r(T') \geq 3$ and let x, y be two points in T . We need to prove that the line $x \vee y$ is contained in T . This can be done by considering the following cases: $x = p_i, y = p_j$; $x = p_i, y = \lambda_{ij}$ for some $\lambda \in G$; $x = p_i, y = \lambda_{jm}$ where i, j, m are distinct; $x = (\lambda_1)_{ij}, y = (\lambda_2)_{ij}$; $x = (\lambda_1)_{ij}, y = (\lambda_2)_{im}$, where i, j, m are distinct, $x = (\lambda_1)_{ij}, y = (\lambda_2)_{km}$, where i, j, k, m are distinct. In each case checking is easy. Thus T is line-closed, therefore it is a flat of $Q_r(G)$. Since $T' = T \setminus \{p_1, \dots, p_r\}$ we conclude that T' is a flat of $Q'_r(G)$. \square

As one can see in the proof of Proposition 3.3, the fact that $Q_r(G)$ is line-closed has central importance in proving that $Q'_r(G)$ is 4-closed.

Proposition 3.4. *Let M and N be geometries of rank r such that their rank- $(k+1)$ truncations are isomorphic. If either M or N is k -closed, then N and M are isomorphic.*

Proof. Without loss of generality we can assume that N is k -closed. Let the isomorphism between the truncations $T^{r-k-1}(N)$ and $T^{r-k-1}(M)$ be given by the bijection ϕ from $E(N)$ to $E(M)$. Thus, for all x_1, \dots, x_k in $E(N)$, we have $y \in x_1 \vee \dots \vee x_k$ if and only if $\phi(y) \in \phi(x_1) \vee \dots \vee \phi(x_k)$. Let F be a flat of M . Then F is k -closed in M and so $\phi^{-1}(F)$ is k -closed in N . Therefore, $\phi^{-1}(F)$ is a flat of N since N is k -closed. Thus, ϕ is a bijective strong map between matroids of the same rank. It follows that M and N are isomorphic [11]. \square

4. The rank-3 case

By Proposition 3.1, $Q'_3(G)$ is line-closed. Thus these geometries are characterised by their points and lines, that is, if N is a geometry with the same labelled points and lines as $Q'_3(G)$, then N is equal to $Q'_3(G)$.

We say that a flat F of rank k in a geometry is *maximal* if F is of maximum cardinality among all flats of rank k . F is *submaximal* if the only flats of rank k of cardinality greater than the cardinality of F are maximal.

Note that if the order of G exceeds three, then the jointless Dowling geometry $Q'_3(G)$ has exactly three maximal lines and these are disjoint. In the remainder of the paper we will assume that G is a group of order $g > 3$.

The *Tutte polynomial* of a matroid M on the ground set S is defined by

$$t(M; x, y) = \sum_{X \subseteq S} (x-1)^{r(S)-r(X)} (y-1)^{|X|-r(X)}.$$

From $t(M; x, y)$ one can determine, among other things, the rank of M and the number of copoints. When M is a rank-3 geometry we can express the number of lines of each cardinality as a linear combination of the coefficients of the Tutte polynomial. For information about the Tutte polynomial, see [6,7].

Theorem 4.1. *Let M be a geometry. If $t(M; x, y) = t(Q'_3(G); x, y)$ for a group G of even order g , then M is a jointless Dowling geometry.*

Proof. Since $t(M; x, y) = t(Q'_3(G); x, y)$, we have that M has three maximal lines (with g points); let these be l_a, l_b, l_c . All other lines of M have exactly three points. We will prove that the maximal lines do not intersect and that M is a jointless Dowling geometry.

Assume that the intersection of the three maximal lines of M is a single point and let $l_a = \{q, a_1, \dots, a_{g-1}\}$, $l_b = \{q, b_1, \dots, b_{g-1}\}$, $l_c = \{q, c_1, \dots, c_{g-1}\}$. There are two points that are not in any of the maximal lines. Let those points be q_1, q_2 . Note that the line $q \vee q_1$ cannot contain any point, other than q , in the maximal lines. So $q \vee q_1 = \{q, q_1, q_2\}$.

Note that since q_1 is not on any of the three maximal lines and there are no trivial lines, all lines through q_1 are 3-point lines. Thus each line through q_1 other than $\{q, q_1, q_2\}$ contains one point from two of the sets $\{a_1, a_2, \dots, a_{g-1}\}$, $\{b_1, b_2, \dots, b_{g-1}\}$, and $\{c_1, c_2, \dots, c_{g-1}\}$. Furthermore, each point in these three sets is in such a 3-point line through q_1 . Thus the lines through q_1 pair off the elements in these three sets. However, this is impossible since there are an odd number (specifically, $3g-3$) of elements in these sets. This contradiction shows that there is no point q common to l_a, l_b and l_c .

If exactly two of the maximal lines intersect, let $l_a = \{q, a_1, \dots, a_{g-1}\}$, $l_b = \{q, b_1, \dots, b_{g-1}\}$ and $l_c = \{c_1, \dots, c_g\}$. There is one point that is not in any of the maximal lines, let it be q_1 . For every i , the line $q \vee c_i$ cannot contain any point from any of the sets $l_a \setminus \{q\}$, $l_b \setminus \{q\}$. Since $q \vee c_i$ is a 3-point line, it is equal to $\{q, q_1, c_i\}$. However, this is not possible for two different values of i .

If exactly two pairs of maximal lines intersect, let the maximal lines of M be $l_a = \{q_1, a_1, \dots, a_{g-1}\}$, $l_b = \{q_1, b_1, \dots, b_{g-2}, q_2\}$, $l_c = \{q_2, c_1, \dots, c_{g-1}\}$. Let q_3, q_4 be the two points that are not in any of the maximal lines. As above, for every i , the line $q_1 \vee c_i$ is $\{q_1, q_3, c_i\}$ or $\{q_1, q_4, c_i\}$. This is impossible because $g-1 \geq 3$.

Assume that the intersection of every pair of maximal lines is non-empty. Let $l_a = \{q_1, a_1, \dots, a_{g-2}, q_2\}$, $l_b = \{q_2, b_1, \dots, b_{g-2}, q_3\}$, $l_c = \{q_1, c_1, \dots, c_{g-2}, q_3\}$. Let q_4, q_5, q_6 be the three points not in any of the maximal lines. For each i , the line $q_1 \vee b_i$ is $\{q_1, b_i, q_j\}$ for j in $\{4, 5, 6\}$. Thus $g-2$ is at most 3. Since g is even, it is equal to

four. This would imply that one of the lines $q_1 \vee q_j$ (for $4 \leq j \leq 6$) is trivial, which is a contradiction.

Therefore l_a, l_b and l_c are pairwise disjoint. Then $l_a \cup l_b \cup l_c$ has $3g$ points. Since M has $3g$ points we have that every point of M is in exactly one of the maximal lines. Let $l_a = l_{12}$, $l_b = l_{13}$ and $l_c = l_{23}$.

Let $\mathcal{N}_1 = \{l_{12}, l_{13}, E(M)\}$; this is a modular cut of M . Therefore, it determines a single-element extension M_1 of M . Note that M_1 is simple and has rank 3. Let p_1 be the point added to M in order to obtain M_1 . Define a modular cut of M_1 by $\mathcal{N}_2 = \{(l_{12} \cup p_1), l_{23}, E(M_1)\}$. Let M_2 be the single-element extension of M_1 determined by it, where $E(M_2) = E(M_1) \cup p_2$. Observe that M_2 is a rank-3 simple matroid. Let $\mathcal{N}_3 = \{l_{13} \cup p_1, l_{23} \cup p_2, E(M_2)\}$, which is a modular cut of M_2 . Let M_3 be the extension of M_2 determined by \mathcal{N}_3 , where $E(M_3) = E(M_2) \cup p_3$. Note that M_3 has rank 3 and is simple.

Note that $\{p_1, p_2, p_3\}$ is a basis of M_3 and for all i, j such that $1 \leq i < j \leq 3$, the line $p_i \vee p_j$ is equal to $l_{ij} \cup \{p_i, p_j\}$. Given i, j, k and points x in $(p_i \vee p_j) \setminus \{p_i, p_j\}$, y in $(p_i \vee p_k) \setminus \{p_i, p_k\}$, the line $x \vee y$ is non-trivial because M has no trivial lines. By Proposition 2.1, M_3 is a Dowling geometry defined by a quasigroup G of order g . Since $M = M_3 \setminus \{p_1, p_2, p_3\}$ we conclude that M is a jointless Dowling geometry. \square

When g is odd, there are geometries M such that $t(M; x, y) = t(Q'_3(G); x, y)$, where g is the order of G , yet M is not a jointless Dowling geometry. The following example of such a geometry $M(g)$ is due to Bonin [3]. Recall from [5] (Proposition 7.11) that the Tutte polynomial of a rank-3 geometry is completely determined by its number of points and lines. Thus we need only to construct a geometry on $3g$ points for which there are three lines having g points, g^2 lines having three points, and no other lines. The $3g$ points $M(g)$ lie on four lines through a fixed point q , namely $l_a = \{q, a_1, a_2, \dots, a_{g-1}\}$, $l_b = \{q, b_1, b_2, \dots, b_{g-1}\}$, $l_c = \{q, c_1, c_2, \dots, c_{g-1}\}$ and $l_q = \{q, q_1, q_2\}$. Let $\{a_i, b_j, c_k\}$ be a line if and only if $k + j$ and i are congruent modulo g where $i, j, k \in \{1, 2, \dots, g-1\}$. Let $\{q_1, a_i, b_i\}$, $\{q_2, a_i, c_i\}$ and $\{q_2, b_i, c_{g-i}\}$ be lines if $i < g/2$; if $i > g/2$, then let $\{q_2, a_i, b_i\}$, $\{q_1, a_i, c_i\}$ and $\{q_1, b_i, c_{g-i}\}$ be lines. An easy congruence argument shows that any two points from distinct sets $l_a \setminus \{q\}$, $l_b \setminus \{q\}$, $l_c \setminus \{q\}$ and $l_q \setminus \{q\}$ are on precisely one line. Basic counting shows that the number of lines of each cardinality is as needed. Note that M is not a jointless Dowling geometry since the maximal lines of M intersect at a common point.

5. Geometries of rank at least four

Our main result is the following theorem which characterises jointless Dowling geometries by statistics of flats of rank up to 7. It can be regarded as an analogue of Theorem 3.4 in [4].

Theorem 5.1. *Let M be a rank r geometry ($r > 3$) and $g > 3$ an integer. The geometry M is the jointless Dowling lattice $Q'_r(G)$ defined by a group G of order g if and only if M has the following properties:*

1. *The geometry M has $\binom{r}{2}g$ points.*
2. *The geometry M has $\binom{r}{2}$ lines with g points, $\binom{r}{3}g^2$ lines with 3 points, no other nontrivial lines.*
3. *The geometry M has $\binom{r}{3}$ planes with $3g$ points, and no other planes with $2g$ or more points.*
4. *The geometry M has $\binom{r}{4}$ rank-4 flats with $6g$ points, no other rank-4 flats with $5g$ or more points and a total of $\binom{r}{4} + \binom{r}{3}\binom{r-3}{2}g + [\binom{r}{2}\binom{r-2}{3} + \frac{1}{2}\binom{r}{2}\binom{r-2}{2}\binom{r-4}{2}]g^2 + [\frac{1}{2}\binom{r}{3}\binom{3}{r-3} + \frac{1}{2}\binom{r}{3}\binom{r-3}{2}\binom{r-5}{2} + \binom{r}{5}\binom{r-4}{2} + \binom{r}{5} + \frac{1}{4!}\binom{r}{2}\binom{r-2}{2}\binom{r-4}{2}\binom{r-6}{2}]g^4$ rank-4 flats.*
5. *The geometry M has $\binom{r}{5}$ rank-5 flats with $10g$ points, $\binom{r}{4}\binom{r-4}{2}g$ rank-5 flats with $6g + 1$ points and no other rank-5 flats with $6g$ or more points.*
6. *The geometry M has $\binom{r}{6}$ rank-6 flats with $15g$ points, and no other rank-6 flats with $14g$ or more points.*
7. *The geometry M has $\binom{r}{7}$ rank-7 flats with $21g$ points, and no other rank-7 flats with $21g - 1$ or more points.*

Note that Theorem 5.1 implies that if M_1, M_2 are two rank- r geometries for which (1)–(7) hold, then there are groups G_1, G_2 of order g such that M_i is isomorphic to $Q'_r(G_i)$ ($i = 1, 2$). However, G_1 might not be isomorphic to G_2 . This is because the counting information we have assumed reflects only the order of the group, not its structure.

In this section M will denote a rank r geometry satisfying (1)–(7) in Theorem 5.1.

This section is largely devoted to the proof of Theorem 5.1, which is presented through a series of propositions. We will use Proposition 5.2 to prove in Proposition 5.4 that the maximal rank-4 flats containing four maximal planes are the disjoint union of their maximal lines. In Proposition 5.6 we prove that the same holds for maximal rank-5 flats containing five maximal rank-4 flats. In Proposition 5.8 it will be proved that every maximal rank-5 flat contains precisely five maximal rank-4 flats. Proposition 5.9–5.13 are used to prove in Proposition 5.14 that M is the disjoint union of its maximal lines. This is of central importance in proving Proposition 5.17 that is, the flats of M of ranks 2–4 are precisely as described in conditions (JD1)–(JD15) in Proposition 2.2. Finally, we will use a version of Desargues' theorem (Theorem 5.22) to construct a group G of order g and prove that one can label the points of M as in the jointless Dowling geometry $Q'_r(G)$. Note that this approach is different from the one used at the end of Section 4, in which we defined modular cuts in order to add the joints.

In [4] the authors proved that if a geometry M' has the same statistics of flats of rank up to 7 as a Dowling geometry, then M' is itself a Dowling geometry. In order to prove it they exhibit a basis of M' that satisfies the conditions (D1)–(D3) in Proposition 2.1. Such a basis is formed by points which are intersection of three maximal planes contained in a maximal rank-4 flat. In fact, in a Dowling

geometry, the intersection of three maximal planes contained in a maximal rank-4 flat is a joint.

Proposition 5.2. *The intersection of three distinct maximal planes contained in a rank-4 flat is empty.*

Proof. Let F be a rank-4 flat containing the distinct maximal planes P_1, P_2, P_3 . Assume that $P_1 \cap P_2 \cap P_3 \neq \emptyset$. As shown in [4], $P_1 \cap P_2 \cap P_3$ does not contain a line. Thus there is a point x such that $P_1 \cap P_2 \cap P_3 = \{x\}$.

For i, j distinct let $l_{ij} = P_i \cap P_j$ and m_{ij} be the number of points of $l_{ij} \setminus \{x\}$. Note that l_{ij} is a flat properly contained in P_i . Thus l_{ij} is a point or a line, hence it has at most g points and so $m_{ij} \leq g - 1$. Since $P_i \setminus (l_{ij} \cup l_{ik}) = P_i \setminus [P_i \cap (P_j \cup P_k)]$ and $l_{ij} \cap l_{ik} = P_i \cap P_j \cap P_k = \{x\}$, we have that the union of the three planes P_1, P_2 and P_3 is

$$\{x\} \dot{\cup} (l_{12} - x) \dot{\cup} (l_{13} - x) \dot{\cup} (l_{23} - x) \dot{\cup} [P_1 \setminus (l_{12} \cup l_{13})] \dot{\cup} [P_2 \setminus (l_{12} \cup l_{23})] \\ \dot{\cup} [P_3 \setminus (l_{13} \cup l_{23})].$$

So the cardinality of $P_1 \cup P_2 \cup P_3$ is

$$1 + m_{12} + m_{13} + m_{23} + (3g - m_{12} - m_{13} - 1) + (3g - m_{12} - m_{23} - 1) \\ + (3g - m_{13} - m_{23} - 1),$$

that is, $9g - m_{12} - m_{13} - m_{23} - 2$ which is at most $6g$ (F has at most $6g$ points). Thus $3g - 2 \leq m_{12} + m_{13} + m_{23} \leq 3(g - 1) = 3g - 3$, and we have a contradiction. Therefore $P_1 \cap P_2 \cap P_3 = \emptyset$. \square

Proposition 5.3. *Each rank-4 flat contains at most four maximal planes. If a rank-4 flat contains four maximal planes, then it is maximal.*

Proof. Let F be a rank-4 flat and P_1, P_2, P_3, P_4 be four maximal planes in F . By Proposition 5.2, for all i, j, k , we have $P_i \cap P_j \cap P_k = \emptyset$ (here and below distinct subscripts are unequal). Thus

$$6g \geq |F| \geq |P_1 \cup P_2 \cup P_3 \cup P_4| = \sum_{i=1}^4 |P_i| - \sum_{1 \leq i < j \leq 4} |P_i \cap P_j| \geq 12g - 6g = 6g.$$

The last inequality is justified because $P_i \cap P_j$ is a flat of rank at most 2. Thus F and $P_1 \cup P_2 \cup P_3 \cup P_4$ have $6g$ points, therefore both sets are equal and F is a maximal rank-4 flat. Moreover $\sum_{1 \leq i < j \leq 4} |P_i \cap P_j| = 6g$. Then for all i, j with $1 \leq i < j \leq 4$, we have that $P_i \cap P_j$ has g points and $l_{km} = P_i \cap P_j$ is a maximal line. Note that the lines l_{km} and l_{kj} are disjoint.

Assume that F contains another maximal plane P_5 . For $i = 1, 2, 3, 4$ let $l_{i5} = P_i \cap P_5$. Note that, $l_{i5} \cap l_{j5} = P_i \cap P_j \cap P_5 = \emptyset$. As above $l_{i5} = P_i \cap P_5$ is a maximal line. Thus

$3g = |P_5| \geq \sum_{i=1}^4 |P_i \cap P_5| = 4g$, which is a contradiction. Therefore F contains at most four maximal planes. If F contains four maximal planes, then it is a maximal rank-4 flat. \square

The proofs of Propositions 5.4, 5.5, 5.7 and 5.8 are very similar to the proofs of the analogous propositions in [4]. Only a sketch of the proof of Proposition 5.7 is given; for details see [4].

Proposition 5.4. *If a rank-4 flat F contains four maximal planes P_1, P_2, P_3 and P_4 , then F is the disjoint union of the six maximal lines $l_{ij} = P_i \cap P_j$ and these are the only maximal lines of F . The plane P_i is $l_{jk} \cup l_{jm} \cup l_{km}$ (i, j, k, m , distinct). Each 3-point line of F is in some plane P_i . Therefore, F contains at most $4g^2$ submaximal lines.*

Proof. As in the proof of Proposition 5.3, the lines l_{ij} are maximal and F is a maximal rank-4 flat. By Proposition 5.2 they are disjoint. Since the cardinality of F (namely $6g$) is equal to the sum of the cardinalities of the lines l_{ij} , F is the disjoint union of these lines. For each i let j, k, m be such that i, j, k, m are distinct. Since $l_{jk} \cup l_{jm} \cup l_{km}$ is a set of cardinality $3g$ contained in P_i (which has cardinality $3g$ as well), P_i equals $l_{jk} \cup l_{jm} \cup l_{km}$.

The points of a 3-point line are in different maximal lines, therefore two of these points are in lines l_{ij}, l_{ik} for distinct i, j, k . The lines l_{ij} and l_{ik} are contained in P_m (with i, j, k, m distinct). Thus two of the points are in P_m and so is the given 3-point line. The same argument shows that every maximal line is contained in a maximal plane. It follows that the six lines l_{ij} are the only maximal lines in F . \square

Let us consider the case $r=4$. Since M has $4g^2$ lines with three points, given x in l_{ij} and y in l_{ik} (with i, j, k distinct), by Proposition 5.4, the line $x \vee y$ has three points and the third point is in l_{jk} . Note that, since the only planes with cardinality $2g$ or more are maximal, the rank of $l_{ij} \cup l_{km}$ is four when i, j, k, m are distinct. If P is a plane of M that contains no maximal line, then P consists of six points one in each maximal line of M , the points in l_{12}, l_{13} and l_{14} are a basis of P which satisfies the axioms for Π_4 . One can check that in the case $r=4$ conditions (JD1)–(JD15) in Proposition 2.2 hold for the lines l_{12}, \dots, l_{34} . In fact the lines and planes of M are exactly the flats as described in (JD1)–(JD15). We will prove that this holds for $r>4$.

Proposition 5.5. *Each rank-5 flat contains at most five maximal rank-4 flats.*

Proof. Let F be a rank-5 flat and let T_1, T_2, \dots, T_t be the maximal rank-4 flats in F . Let i, j be positive integers such that $1 \leq i < j \leq t$. Since $|T_i \cup T_j| = 12g - |T_i \cap T_j| \leq |F| \leq 10g$, the cardinality of $T_i \cap T_j$ is at least $2g$. Since the maximal lines have g points and the only planes with cardinality at least $2g$ are maximal, we have that $T_i \cap T_j$ is a maximal plane. If $T_1 \cap T_i = T_1 \cap T_j$, then $T_1 \cup T_i \cup T_j$ has cardinality $12g$. But this is impossible

because the cardinality of F is at most $10g$. Thus T_1 is a rank-4 flat that contains the $t - 1$ maximal planes $T_1 \cap T_2, \dots, T_1 \cap T_t$. By Proposition 5.3 we have that $t - 1 \leq 4$, therefore $t \leq 5$. \square

Proposition 5.6. *Let F be a maximal rank-5 flat containing five maximal rank-4 flats T_1, T_2, T_3, T_4, T_5 . Then F is the disjoint union of the ten maximal lines $l_{ij} = T_i \cap T_m \cap T_n$ (i, j, k, m, n distinct) and these are the only maximal lines of F . The maximal rank-4 flat T_n is the union of the six maximal lines l_{ij} such that i and j are distinct from n . All submaximal lines in F lie in planes of the form $l_{ij} \cup l_{ik} \cup l_{jk}$. Hence there are at most $10g^2$ submaximal lines in F .*

Proof. Each T_i contains four maximal planes, the intersection of T_i with the other four maximal rank-4 flats. Thus Proposition 5.4 applies to each T_i . Let $l_{23}, l_{24}, l_{25}, l_{34}, l_{35}, l_{45}$ be the maximal lines of T_1 . We may assume that whenever $\{i, j, m, n\} = \{2, 3, 4, 5\}$, we have $T_1 \cap T_m \cap T_n = l_{ij} = l_{ji}$. For $m = 2, 3, 4, 5$, we have $T_1 \cap T_m = l_{ij} \cup l_{in} \cup l_{jn}$ where i, j, m, n are distinct. Considering T_2 , we note that $T_1 \cap T_2 = l_{34} \cup l_{35} \cup l_{45}$ and that T_2 contains three other maximal lines disjoint from them. We label those lines such that

$$T_2 \cap T_3 \cap T_4 = l_{15}, \quad T_2 \cap T_3 \cap T_5 = l_{14}, \quad T_2 \cap T_4 \cap T_5 = l_{13}.$$

The four maximal planes contained in T_2 are correctly labelled (that is for all $m = 1, 3, 4, 5$, the intersection of T_2 and T_m is $l_{ij} \cup l_{in} \cup l_{jn}$). Finally, we label the lines in T_3 such that $l_{12} = T_3 \cap T_4 \cap T_5$. Thus the five maximal rank four flats contained in F are the union of the required lines.

Given any two of these maximal lines, there is a maximal rank-4 flat that contains both and, by Proposition 5.4, the intersection of the two given lines is empty. Thus $|T_1 \cup T_2 \cup T_3| = |F|$ and $F = T_1 \cup T_2 \cup T_3$. Therefore, F is the disjoint union of the 10 maximal lines l_{12}, \dots, l_{45} .

Let l be a non-trivial line. Then one of the maximal rank-4 flats T_j contains l and we can apply Proposition 5.4 to conclude that if l is maximal, then $l \in \{l_{12}, \dots, l_{45}\}$ and if l is submaximal, then it is contained in a plane of the form $l_{ij} \cup l_{ik} \cup l_{jk}$ (with i, j, k distinct). \square

When $r = 5$, M contains five maximal rank-4 flats, therefore Proposition 5.6 applies for M . One can easily check that the flats of M of ranks 2–4 are precisely those described in conditions (JD1)–(JD15) in Proposition 2.2.

Proposition 5.7. *Each maximal rank-4 flat is in exactly $r - 4$ maximal rank-5 flats.*

Proof. The first part of the proof consists of proving that every maximal rank-4 flat is contained in at most $r - 4$ maximal rank-5 flats. Let F be a maximal rank-4 flat contained in the maximal rank-5 flats F_1, \dots, F_r . Then each $A_{ij} = F_i \vee F_j$ is a maximal rank-6 flat such that the set $L_{ij} = A_{ij} - (F_i \cup F_j)$ has cardinality g . The sets L_{ij} are pairwise disjoint. There are exactly $6g + t(10g - 6g) + \binom{t}{2}g$ points in the union of

F, F_1, \dots, F_t and the $\binom{r}{2}$ sets L_{ij} . Recall that the number of points in M is $\binom{r}{2}g$, and thus it follows that $t \leq r - 4$.

Let

$$S_1, \dots, S_{\binom{r}{2}}$$

be all the maximal rank-4 flats of M and m_i be the number of maximal rank-5 flats in which S_i is contained. There are $\binom{r}{2}g - 6g - 4gm_i$ submaximal rank-5 flats that contain S_i . Since at most $\binom{r}{4}\binom{r-4}{2}g$ maximal rank-4 flats are contained in submaximal rank-5 flats, we have

$$\sum_{i=1}^{\binom{r}{2}} \left(\binom{r}{2}g - 6g - 4gm_i \right) \leq \binom{r}{4} \binom{r-4}{2} g.$$

By the first part of the proof $m_i \leq r - 4$, and we have

$$\binom{r}{4} \left(\binom{r}{2}g - 6g - 4g(r-4) \right) \leq \binom{r}{4} \binom{r-4}{2} g.$$

We can see, using simple algebra, that the expressions in both sides of this inequality are, in fact, equal. Therefore $m_i = r - 4$.

Proposition 5.8. *Each maximal rank-5 flat contains precisely five maximal rank-4 flats.*

Proof. By Proposition 5.5 each maximal rank-5 flat contains at most five maximal rank-4 flats. Therefore, there are at most $5\binom{r}{5}$ pairs (F, F') where F is a maximal rank-5 flat and F' is a maximal rank-4 flat contained in F . By Proposition 5.7, the number of such pairs is $\binom{r}{4}(r-4)$. Since $\binom{r}{4}(r-4) = 5\binom{r}{5}$, we have that each maximal rank-5 flat contains precisely five maximal rank-4 flats. \square

Therefore Proposition 5.6 applies to all maximal rank-5 flats.

Let F be a maximal rank-4 flat and let F_5, \dots, F_r be the $r - 4$ maximal rank-5 flats that contain F . Since F_5 contains five maximal rank-4 flats, by Proposition 5.6 the ten maximal lines of F_5 can be labelled as $l_{12}, l_{13}, \dots, l_{45}$ so that $F = \bigcup_{i,j \in \{1,2,3,4\}} l_{ij}$ and three maximal lines form a maximal plane if and only if their subscripts come from a 3-subset of $\{1, 2, 3, 4, 5\}$. By Proposition 5.6, this labelling can be extended to a labelling of the maximal lines of each F_i , for $i \geq 5$, so that three maximal lines in F_i form a maximal plane if and only if their subscripts come from a 3-subset of $\{1, 2, 3, 4, i\}$. It follows from this labelling that six maximal lines in F_i form a maximal rank-4 flat if and only if their subscripts come from a 4-subset of $\{1, 2, 3, 4, i\}$. For $5 \leq i < j \leq r$ let $F_{ij} = l_{12} \vee l_{13} \vee l_{1i} \vee l_{1j}$.

Proposition 5.9. *F_{ij} is a maximal rank-5 flat.*

Proof. Let $A = l_{12} \vee l_{13} \vee l_{1i}$ and $B = l_{12} \vee l_{13} \vee l_{1j}$. Note that A consists of the six maximal lines of F_i having subscripts in the 4 set $\{1, 2, 3, i\}$, therefore A is a maximal rank-4 flat. Similarly B is a maximal rank-4 flat contained in F_j . Thus A and B are maximal rank-4 flats meeting in the maximal plane $l_{12} \vee l_{13}$ and such that $F_{ij} = A \vee B$. By semimodularity, the rank of F_{ij} is five. Since $|F_{ij}| \geq |A \cup B| = |A| + |B| - |A \cap B| = 9g$ and the only rank-5 flats with cardinality $9g$ or more are maximal, we have that F_{ij} is a maximal rank-5 flat. \square

Let l_{ij} be the missing line in F_{ij} . That is, F_{ij} is the disjoint union of the ten maximal lines l_{km} with k, m in $\{1, 2, 3, i, j\}$. We want to argue that the $\binom{5}{2}$ maximal lines l_{ij} partition the points of M . Since any two of these maximal lines occur in one of the pairs F_{ij} and F_i, F_{ij} and F_k, F_{ij} and F_{ik} , and F_{ij} and F_{km} , with one line in the first flat and the other in the second, it suffices to show that the intersections of these flats are as small as possible. (Here and below we assume that distinct subscripts are unequal.) Specifically we need to show

$$F_{ij} \cap F_i = l_{12} \vee l_{13} \vee l_{1i} = F_{ij} \cap F_{ik},$$

and

$$F_{ij} \cap F_k = l_{12} \vee l_{13} = F_{ij} \cap F_{km}.$$

Proposition 5.10. *The intersection of F_{ij} and F_i is the maximal rank-4 flat $l_{12} \vee l_{13} \vee l_{1i}$, that is, the disjoint union of the six maximal lines l_{km} with k, m in $\{1, 2, 3, i\}$.*

Proof. This is immediate since F_{ij} and F_i are unequal rank-5 flats that both contain the rank-4 flat $l_{12} \vee l_{13} \vee l_{1i}$. \square

Proposition 5.11. *The intersection of F_{ij} and F_k is the maximal plane $l_{12} \vee l_{13}$.*

Proof. The intersection of F_{ij} and F_k has rank 3 or 4, if it has rank 3, then it is equal to $l_{12} \vee l_{13}$. In order to prove that the rank of $F_{ij} \cap F_k$ is not 4, it is enough to prove that $r(F_{ij} \vee F_k)$ is not 6. If $F_{ij} \vee F_k$ had rank 6, then $|5g| \geq |F_{ij} \vee F_k| \geq |F_{ij} \cup F_k| = 20g - |F_{ij} \cap F_k|$. Thus $|F_{ij} \cap F_k| \geq 5g$, which would force $F_{ij} \cap F_k$ to be a maximal rank-4 flat in F_k . This is impossible because every maximal rank-4 flat in F_k is the union of the lines l_{st} with subscripts in a 4-subset of $\{1, 2, 3, 4, k\}$. \square

Proposition 5.12. *The intersection of F_{ij} and F_{ik} is the maximal rank-4 flat $l_{12} \vee l_{13} \vee l_{1i}$.*

Proof. Just note that F_{ij} and F_{ik} are two distinct rank-5 flats containing the rank-4 flat $l_{12} \vee l_{13} \vee l_{1i}$. \square

The proof of Proposition 5.13 is similar to the proof of Proposition 5.11.

Proposition 5.13. *The intersection of F_{ij} and F_{km} equals the maximal plane $l_{12} \vee l_{13}$.*

Proposition 5.14. *The geometry M is the disjoint union of the maximal lines l_{ij} , with $1 \leq i < j \leq r$.*

Proof. Using the previous four propositions it follows that the lines l_{ij} are pairwise disjoint. Therefore the cardinality of $\bigcup_{1 \leq i < j \leq r} l_{ij}$ equals $\sum_{1 \leq i < j \leq r} |l_{ij}| = \binom{r}{2}g$ which is the number of points of M . \square

The following proposition can be easily checked.

Proposition 5.15. *Let s be such that $3 \leq s \leq 7$, and let \mathcal{L} be a set of maximal lines. The union of the lines in \mathcal{L} is a maximal rank- s flat if and only if their subscripts form an s -subset of $\{1, \dots, r\}$.*

Proposition 5.16. *Every submaximal line is contained in a maximal plane $l_{ij} \vee l_{ik}$.*

Proof. Let y_1 and y_2 be two points in a submaximal (3-point) line. By Proposition 5.14, M is the disjoint union of the maximal lines l_{ij} ($1 \leq i < j \leq r$). Thus there are i, j, k, m (not necessarily distinct) such that $y_1 \in l_{ij}$ and $y_2 \in l_{km}$. Therefore, the given line is contained in a maximal rank-4 flat, and, by Proposition 5.6 (rank 5 case) it is contained in a maximal plane. \square

Proposition 5.17. *The flats of M of ranks 2–4 are precisely as specified in conditions (JD1)–(JD15) of Proposition 2.2.*

Proof. By Proposition 5.15 the maximal flats of ranks 2–4 are exactly those described in (JD1), (JD4) and (JD8) in Proposition 2.2. The condition (JD3) follows easily from Proposition 5.16 using that there are $\binom{r}{3}g^2$ lines with three points. All sets of points described in (JD5)–(JD15) are contained in the union of a set of maximal lines as in Proposition 5.15. Therefore, those sets of points are contained in a maximal flat of rank at most 7, it is easy to check that they are flats.

We wish to prove now that there are no other flats of rank 2, 3 or 4. It is easy by considering that a flat of rank s is generated by a flat F of rank $s - 1$ and a point not in F . In particular, we have to prove that given distinct indices i, j, k, m, n, t, v, w and points $x_{ij}, x_{km}, x_{nt}, x_{vw}$ in $l_{ij}, l_{km}, l_{nt}, l_{vw}$, the set $\{x_{ij}, x_{km}, x_{nt}, x_{vw}\}$ is a rank-4 flat. Only there we use the total number of rank-4 flats. \square

We will give a geometric construction based on M that produces a group G of order g so that M and $Q'_r(G)$ have, up to labelling, the same points and lines. In other words, the truncations to rank 3 of M and $Q'_r(G)$ are isomorphic. From this, Propositions 2.2, and 5.17, it follows that M and $Q'_r(G)$ have, up to relabelling, the same flats of ranks 2–4. From this, Theorem 4.1 follows immediately since $Q'_r(G)$ is 4-closed.

Proposition 5.18. *Let N be a geometry of rank $r \geq 4$ such that*

- (JD1) *there are $\binom{r}{2}$ disjoint lines l_{ij} ($1 \leq i < j \leq r$) such that every point lies in exactly one of these lines,*
- (JD2) *no line l_{ij} is trivial,*
- (JD3) *for all $x \in l_{ij}$ and all $y \in l_{ik}$ the line $x \vee y$ is a 3-point line with the third point in l_{jk} , and*
- (JD4) *for all $1 \leq i < j < k \leq r$, the union of the lines l_{ij} , l_{ik} and l_{jk} , is a plane.*

Then there is a group G such that N has the same (labelled) points and lines as $Q'_r(G)$.

From now on N will be a geometry as in Proposition 5.18, and we will call the lines in (JD1) *coordinate lines* and the planes in (JD4) *coordinate planes*. As in [1] one easily shows that all the coordinate lines have the same number of points.

Definition 5.19. Let two triplets of points a, b, c and a', b', c' be given. We say that the triangles a, b, c and a', b', c' are *in perspective from i* if there exists j, k, m such that $a, a' \in l_{ij}$, $b, b' \in l_{ik}$ and $c, c' \in l_{im}$.

Definition 5.20. Given three points a, b, c , we say that the triangle a, b, c is *in perspective from i with the coordinate plane $l_{jk} \vee l_{jm}$* if $a \in l_{ij}$, $b \in l_{ik}$, $c \in l_{im}$.

The concept of perspective from i is based on the concept of perspective from the joint p_i in a Dowling lattice. The concept of perspective from a line is analogous to that in Dowling lattices (see [1]).

Definition 5.21. Given three points a, b, c we say that the triangle a, b, c and the coordinate plane $l_{jk} \vee l_{jm}$ are *in perspective from a line* if $(a \vee b) \wedge l_{jk}$, $(a \vee c) \wedge l_{jm}$, $(b \vee c) \wedge l_{km}$ are collinear points.

The following can be regarded as a Desargues' theorem for jointless Dowling lattices (Fig. 1).

Theorem 5.22. *If a triangle a, b, c is in perspective from i with the coordinate plane $l_{jk} \vee l_{jm}$, then it is in perspective from a line with the given coordinate plane.*

Proof. The points a, b, c span a plane P . The plane P meets $l_{jk} \vee l_{jm}$ in three points (namely $q = (a \vee b) \cap l_{jk}$, $p = (a \vee c) \cap l_{jm}$, $s = (b \vee c) \cap l_{km}$). Therefore these points lie on a line (the intersection of the two planes). \square

Corollary 5.23. *Let a, b, c and a', b', c' be two triangles in perspective from i and such that $a \vee b$, $a' \vee b'$ and $b \vee c$, $b' \vee c'$ are pairs of intersecting lines. Thus a, b, c and a', b', c' are in perspective from a line and, in particular, $a \vee c$ and $a' \vee c'$ meet in a point.*

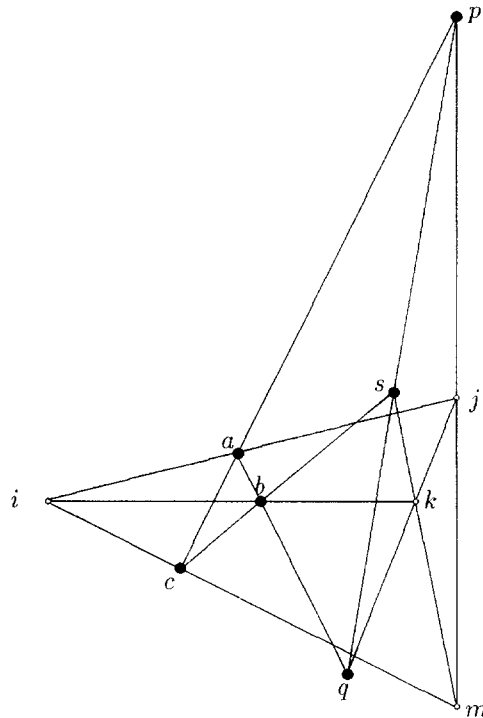


Fig. 1. Desargues' theorem for jointless Dowling lattices.

As in [1] we define a binary operation in each coordinate line of N , and prove that this defines a group G and that it is possible to label the points of N as in $Q'_r(G)$. The binary operation is defined as follows (see Fig. 2).

Fix a point 1 in l_{ij} and a point d in l_{ik} . Let p be the third point in the line $1 \vee d$ (note that p is in l_{jk}). Let a, c be arbitrary points in l_{ij} . These are the points that we will multiply. Let d' be the third point in $a \vee p$ and q the third point on $c \vee d$. Then ac is defined to be the third point on the line $q \vee d'$.

By Corollary 5.23 the binary operation defined on the lines l_{ij} depends only on the points of l_{ij} and not on the choice of auxiliary points on other lines. As in [1] this operation defines a group, with identity 1 , on the points of each coordinate line. Groups on different coordinate lines are isomorphic by projection. Let G be the group defined on the points of every coordinate line. In order to label the points of N as in $Q'_r(G)$ fix a point 1 in l_{1j} (for $j \geq 2$). For each $j > 2$, let 1_{2j} be the third point on the line $1_{12} \vee 1_{1j}$. Label the points of l_{12} labelling the group G . For each λ in G and each $j > 2$, label as λ_{1j} the third point on the line $\lambda_{12} \vee 1_{2j}$. For $2 \leq i < j \leq n$ label as λ_{ij} the third point on the line $1_{1i} \vee \lambda_{1j}$. Using Corollary 5.23 one shows that the 3-point lines of N are labelled as those of $Q'_r(G)$. For details see [1].

This proves Proposition 5.18 and therefore Theorem 5.1.

In [6] a more general version of the following proposition is proved.

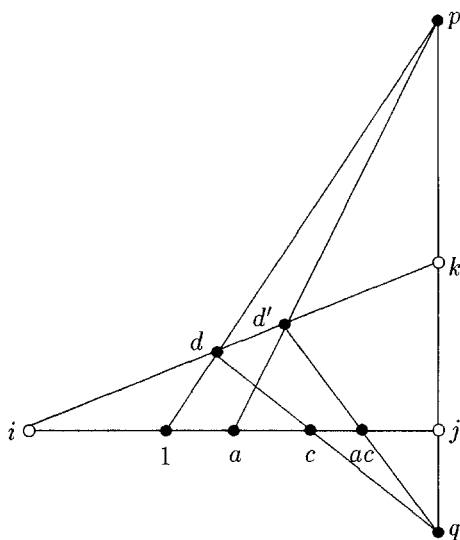


Fig. 2. Binary operation in the line l_{ij} .

Proposition 5.24. *For a rank- r matroid R and any integer i with $0 \leq i \leq r$, let c_i be the largest cardinality among rank- i flats of R . Then for each i with $1 \leq i \leq r$ and each j with $c_{i-1} < j \leq c_i$, we can express the number of flats of R having rank i and cardinality j as a linear combination of the coefficients of the Tutte polynomial of R .*

By Proposition 5.24 and Theorem 5.1 the following holds.

Corollary 5.25. *Let M be a matroid such that $T(Q'_r(G); x, y) = T(M; x, y)$ for a group G of order g . Thus there exists a group G_1 , uniquely determined by the matroid structure of M , such that G_1 has order g and M is isomorphic to the jointless Dowling lattice $Q'_r(G_1)$.*

This proves that jointless Dowling geometries are characterised both by their Tutte polynomial and by the counting information given in Theorem 5.1.

Acknowledgements

I thank Professor Joseph E. Bonin at George Washington University for introducing me to the problem and for several helpful discussions. I thank my advisor D.J.A. Welsh for his patience, support and guidance. I am grateful to Eric Bartels for his help writing this paper. The work was supported by the Consejo Nacional de Ciencia y Tecnología of Mexico through Ph.D. scholarship 22059.

References

- [1] M.K. Bennett, K.P. Bogart, J.E. Bonin, The geometry of Dowling lattices, *Adv. Math.* 103 (1994) 131–161.
- [2] J.E. Bonin, Automorphisms of Dowling lattices and related geometries, *Combin. Probab. Comput.* 4 (1995) 1–9.
- [3] J.E. Bonin, Private communication.
- [4] J.E. Bonin, W.P. Miller, Characterizing geometries by numerical invariants, Preprint.
- [5] T.H. Brylawski, A decomposition for combinatorial geometries, *Trans. AMS* 171 (1972) 235–282.
- [6] T.H. Brylawski, The Tutte polynomial. Part I: General theory, in: A. Barlotti (Ed.), *Matroid Theory and Its Applications*, C.I.M.E., Liguori, Naples, 1980, pp. 125–175.
- [7] T.H. Brylawski, J. Oxley, The Tutte polynomial and its Applications, in: N. White (Ed.), *Matroid Applications*, Cambridge University Press, Cambridge, 1992, pp. 123–225.
- [8] T.A. Dowling, A class of geometric lattices based on finite groups, *J. Combin. Theory B* 14 (1973) 61–86.
- [9] M.D. Halsey, Line-closed combinatorial geometries, *Discrete Math.* 65 (1987) 245–248.
- [10] J. Kahn, J. Kung, Varieties of combinatorial geometries, *Trans. Amer. Math. Soc.* 271 (1982) 485–499.
- [11] J.P.S. Kung, Strong maps, in: N.L. White (Ed.), *Theory of Matroids*, Cambridge University Press, Cambridge, 1986, pp. 224–253.
- [12] J.G. Oxley, *Matroid Theory*, Oxford University Press, Oxford, 1992.
- [13] G. Whittle, Dowling group geometries and the critical problem, *J. Combin. Theory Ser. B* 47 (1989) 80–92.
- [14] T. Zaslavsky, Biased graphs. I. Bias, balance, and gains, *J. Combin. Theory Ser. B* 47 (1989) 32–52.