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Optimal Oil Extraction as a Multiple Real Option

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ABSTRACT

We study optimal oil extraction strategy and the value of an oil field using a multiple real option approach. Extracting a barrel of oil is similar to exercising a call option and optimal strategies lead to deferring production when oil prices are low and when volatility is high. We show that, in theory, the net present value of a country's oil reserves is increased significantly (by 100 percent, in the most extreme case) if production decisions are made conditional on oil prices. We also show that the marginal value of additional capacity is higher for countries with bigger resources and longer production horizons. We apply the model to Brazil and the U.A.E. in order to pin down two points of the global supply curve.

Keywords: Oil production ; Real Options ; Capacity Expansion ; Stochastic Optimization

JEL Classification: C61 ; Q30 ; Q43

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I. INTRODUCTION

In this paper we investigate the optimal oil extraction strategy of a small oil producer facing uncertain oil prices. We use a multiple real option approach. Extracting a barrel of oil is similar to exercising a call option, *i.e.* oil production can be modeled as the right to produce a barrel of oil with the payoff of the strategy depending on uncertain oil prices. Production is optimal if the payoff of extracting oil exceeds the value of leaving oil under the ground for later extraction (the continuation value). For an oil producer, the optimal extraction path corresponds to the optimal strategy of an investor holding a multiple real option with finite number of exercises (finite reserves of oil). At any single point in time, the oil producer is also limited in the number of options he can exercise, because of capacity constraints.

Our main contribution is to present the solution to the stochastic optimization problem as an exercise rule for a multiple real option and to solve the problem numerically using the Monte Carlo methods developed by Longstaff and Schwartz (2002), Rogers, and extended by Aleksandrov and Hambly (2008) and Bender (2008). The Monte Carlo regression method is flexible, it remains accurate even with increasing capacity constraints and with high-dimensionality problems (when there are several state variables, for instance when extraction costs are also stochastic and independent from oil prices).

We solve the real option problem for two countries with similar capacity profiles but vastly different reserves, Brazil and the U.A.E. and compute the threshold below which it is optimal to defer production (US\$ 73 for Brazil and US\$ 39 for the U.A.E., at 2000 constant prices). We also estimate the net present value of these countries' oil reserves and show that the difference with a similar calculation in a deterministic framework can exceed 20 percent. This result has important implications for oil production policy and for the design of macroeconomic policies that depend on inter-temporal and inter-generational equity

considerations. The model is flexible enough to include stochastic extraction costs, independent from oil prices, and we calibrate them for Brazil using Petrobras data. Extraction costs have the potential to reach high levels in the medium term (US\$ 18 in 2000 constant prices in one of our simulation) and to bring optimal production to a minimum. We also investigate the value of increasing capacity and show that the marginal value of additional capacity is higher for countries with higher resources. This is because a marginal increase in capacity generates additional optionality to extract oil. Such optionality has more value for countries with larger resources and longer horizons of production. The implication of our result is that project evaluation needs to be performed in the context of the overall oil strategy of a country.

Finally, we discuss what the world supply curve would look like if countries were fully optimizing production by maximizing expected NPV. Brazil and the U.A.E. are two important - and very different producers - that allow us to identify two important points of the theoretical global supply curve. Oil supply would be very elastic, production would be on average lower, and therefore prices would tend to be higher but much less volatile.

Section 2 provides a survey of the related literature on optimal production and real options, while Section 3 describes briefly the oil sector in the two countries used as applications. Sections 4, 5 and 6 cover the model formulation, solution method, and calibration of the oil price process. Section 7 presents the main results. Section 8 presents the global theoretical supply curve and section 9 concludes.

II. RELATED LITERATURE

A. Optimal Oil Production

The study of the economy of non-renewable resource extraction started with Hotelling (1931), who showed in a deterministic general equilibrium model that the price of the resource would grow at the rate of interest in competitive markets with constant extraction costs. General equilibrium models later included the effect of uncertainty in technology, the size of the resources, or the availability of substitutes. Partial equilibrium models in which the prices are given, but the decision to extract is a function of the stochastic price process, have a shorter history in the non-renewable resource literature, starting with Tourinho (1979a). Tourinho (1979a, 1979b) analyzed for the first time the valuation of resources in the context of a real ‘call’ option to exploit a field, using the Black and Scholes formula.

Paddock, Siegel and Smith (1988) later developed a model that became a popular approach for decisions on upstream oil investments, in which a company has the option to exploit a field before a given date (the time to expiration) at which the firm has to return the concession rights back to a national authority.¹ The model was able to take into consideration resource depletion when estimating the value of the underlying asset (the oil field). However, the issue of when to extract the resource after the field is developed is completely absent since the only decision to take is the optimal timing to develop the field.

Cherian *et al.* (1998) studied optimal production of a nonrenewable resource as a control problem in continuous time. The authors solved the Bellman nonlinear partial differential equation (PDE) numerically using the Markov chain approximation technique of Kushner

¹Real option models are surveyed in Dixit and Pindyck (1994) and, for applications to oil investments, in Dias (2004).

(1977) and Kushner and Dupuis (1992).² The cost of extraction in Cherian *et al.* (1998) is a function of time, the extraction rate and the total extracted amount, a model that fits well extraction costs for ore mining. However, a limitation is that the extraction rate is bounded from above by a constant, *i.e.* extraction capacity is constant over time. Another drawback of these numerical solutions to the Bellman PDEs is that they are feasible only for problems with low dimensionality, and it is therefore difficult to solve the problem with stochastic extraction costs.

Work close to ours in terms of modeling assumptions is Caldentey *et al.* (2006), who study the optimal operation of a copper mining project when the copper spot price follows a mean reverting stochastic process. The project is modeled as a collection of blocks (minimal extraction units) each with its own mineral composition and extraction costs. The authors are interested in maximizing the economic value of the project by controlling the sequence and rate of extraction as well as investing on costly capacity expansions. Our model is more general since we allow for multiple exercise (*i.e.* the company can choose to extract different volumes every period) and because in our model the firm is able to scale down production.

We provide a multiple real option solution to the dynamic optimization problem. A real option model requires that : (i) the total number of oil barrels (or the total number of real options) is given and known;³ (ii) the oil producer is constrained by its production capacity on the amount of oil it can extract every year (*i.e.* on the number of options it can exercise every period). Production capacity does not need to be constant, but it has to be exogenous to the decision process. It is important to allow for increasing capacity - even if it is simply

²The standard numerical methods for PDEs - e.g. finite differences - assume smoothness of all functions entering the problem and therefore cannot be used

³See Pindyck (1978) for an equilibrium model of price and extraction in which exploratory activities can generate new reserve discoveries.

exogenous - since most of the oil producers with sizable resources have expansion plans. We also assume there is a minimum extraction capacity that is non-zero. This assumption captures the fact that some minimal extraction is often needed to finance the functioning of the firm, the transfers to the government, or even the spending of the government in countries where oil proceeds are the major source of revenue to the government.

B. Numerical Solutions for Real Options

Closed formulae for the price of early exercise options have not been derived yet, even for the simplest cases. In particular, there is no analytical method for solving multiple real option problems. The difficulty in any option model is to compute the *continuation value* (the expected value of delaying the extraction of a barrel). Following the analysis of Arrow, Blackwell and Girshick (1949), the problem was recognized and discussed as an abstract optimal stopping problem by Snell (1952), and the first application of the optimal stopping problem to finance appeared in Bensoussan (1984).

The literature has however suggested various analytical approximations and numerical methods. For most option pricing problems, three numerical methods are available: lattices, finite difference, and Monte Carlo methods. The first two approaches work best for simple options on a single underlying (a single state variable). When there are more state variables and the dimension of the problem increases, however, the Monte Carlo approach is preferred as the performance of the lattice and finite difference schemes is poor (the computational effort with these two methods grows exponentially with the number of state variables).

The first attempt to apply Monte Carlo techniques to American option pricing is due to Tilley (1993), while Broadie and Glasserman (1997) developed the first algorithm in which

the suggested lower and upper bound estimates are proved to converge to the true value. Their approach can also deal with high-dimensional American options, but the computational effort still grows exponentially with the number of possible exercise dates. There has been a renewal of interest in the recent years for these methods (Jaillet, Ronn and Tompaidis, 2004; Meinshausen and Hambly, 2004; Keppo, 2004; Barrera-Esteve et al., 2006; Bardou et al., 2007a and 2007b; Carmona and Touzi, 2008). The solution we use was developed in Aleksandrov and Hambly (2008) and Bender (2008) as an extension to the Monte Carlo method proposed by Longstaff and Schwartz (2002) and Tsitsiklis and van Roy (2001).

The method relies on approximating the value function by linear regression on a suitable space of basis functions (we will use polynomials of the oil price and of the logarithm of the oil price) and using this to determine the optimal stopping policy (see section A for more details). The fitted value from the regression gives an estimate for the continuation value. By construction this optimal stopping policy gives a *lower* bound for the option price - only the exact decision rule would give the maximum value and an approximation can only give a lower estimate. The method is comparatively easy to implement and for properly chosen regression functions gives a good estimate of the value function (see Glasserman, 2003). The method works well with two state variables (we will include stochastic extraction costs in section F) and with increasing capacity.

Rogers (2002), Haugh and Kogan (2001), and Andersen and Broadie (2001) developed a duality approach that provides an *upper* bound for the single exercise case. We use the extension by Aleksandrov and Hambly (2008) to calculate an upper bound for the option price in the multiple exercise case. The optimizing problem is re-written as one of a minimization over a space of martingales and provides a good fit to the continuation value (see section B). We show that the two methods we use for the lower bound and the upper

bound provide an accurate approximation of the solution of the problem faced by a small oil producer.

III. BRAZIL AND THE U.A.E.

We apply the model to two countries: Brazil and the U.A.E. The two countries were chosen to be representative of a non-OPEC country, with limited oil reserves and very high needs of oil revenues, and an OPEC country, with large reserves, and an intergenerational approach to managing its resources. Nonetheless, Brazil and the U.A.E. have roughly similar production capacities and ambitious expansion plans. The two countries are large producers and will allow us to pin down a rough approximate of the global supply curve (see last section).

In 2008, Brazil had 12.2 billion barrels of proven oil reserves, a number that fares well compared to that of the other non-OPEC countries, and that leaves Brazil with the second largest oil reserves in South America after Venezuela. The major oil fields are located offshore in the South-East, in the Rio de Janeiro state (Campos and Santos Basins). With 95 percent of crude oil produced by the state-controlled company Petroleos Brasil (Petrobras), and production licenses issued by the Petroleum Agency (ANP), the choice of extracting oil or leaving it under the ground in order to generate future income is an important policy choice. Discussions on the exploitation of the oil resources gained prominence recently after new oil reserves were discovered.

Brazil's production of oil reached 2.3 million barrels per day (bbl/d) in 2007 (Energy Information Administration, EIA). Brazil has been one of the fastest non-OPEC growing oil industry in the recent years, with numerous projects brought onstream. The government announced its intentions to exploit quickly new fields and to use the receipts as part of its

broader development strategy. However, since oil fields are offshore and in very deep water, the costs to develop new oil fields are very high in Brazil and will reduce significantly the NPV of the oil projects. Petrobras' expansion capacity investment plans amount to US\$ 78.2 billion over the next 5 years for Brazil (Petrobras, 2009), and speculative estimates of the costs of the new deep water fields have reached US\$ 100 billion.

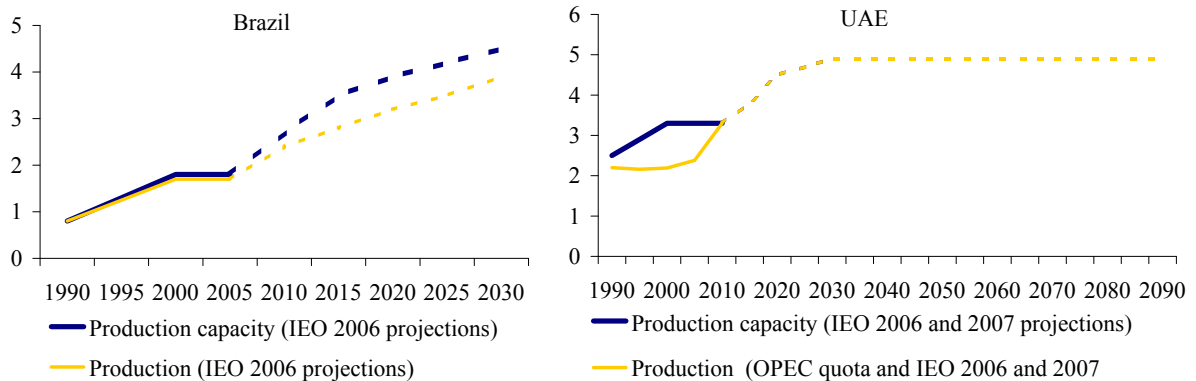


Figure 1. Oil production and capacity - Brazil and the U.A.E.

The United Arab Emirates (U.A.E.) holds the sixth largest proven oil reserves in the world (estimated at 97.8 billion barrels, around 8.5 percent of total world reserves⁴) and is an OPEC member. In 2006, the U.A.E. produced 2.5 million barrels per day of crude oil but the government plans to expand capacity so as to make the U.A.E. a 'swing' producer, which would give the country a clearer negotiating power within OPEC. Projects that are under development are the Upper Zakum's project to increase Zakum production capacity by 200,000 bbl/d in 2009, the Nasr and Umm Loulou project (190,000 bbl/d in 2010), the Qusahwira and Bida-al Qemzan project (90,000 bbl/d in 2010), and the Sahil Asab and Shah project (60,000 bbl/d in 2010) for a total increase of more than half-a-million barrels

⁴Oil and Gas Journal, January 2007

per day by 2010 (Energy Information Administration). The plans to scale up the oil sector in the U.A.E. will cost around US\$ 20 billion in investment projects (IMF, 2007).

IV. MODEL FORMULATION

We present here the multiple real option optimization problem. We consider an economy in discrete time defined up to a finite time horizon of T years at which we assume reserves will be depleted.⁵ The maximum yearly capacity for extraction (the production capacity) is k_t (in billion barrels per year), $t = 1, 2, \dots, T$. k_t typically varies with time.⁶ The optimal extraction strategy maximizes the discounted cash flows from oil sales $V_t^{*,m,k}$ at time t , subject to the capacity constraints $\mathbf{k} = \{k_0, k_1, k_2, \dots, k_T\}$ and to the total oil reserves constraint m . If the oil producer decides to extract a barrel of oil at time t , the profit is $S_t - c_t$, where S_t is the price of oil and c_t the extraction cost. Profits are always positive when the producer decides to extract oil since she is not forced to produce making losses.

We assume that oil prices follow a discrete Markov chain process $(X_t)_{t=0,1,\dots,T} \in \mathbb{R}^d$ (we come back to the discussion on oil prices in section VI). We write $h_t(X_t)$ for the payoff from the extraction of one unit of oil at time t when the oil price is X_t (the argument of h_t is kept implicit in later formulations). We assume that the payoff is non-negative: $h_t(x) \geq 0$ for all $x \in \mathbb{R}^d, t = 0, \dots, T$. The decision maker has the opportunity to extract up to k_t units, that is to receive $k_t h_t$, at time t .

We define an extraction policy $\pi_{\mathbf{k}}$ to be a set of ‘stopping’ times (*i.e.* times at which the real options are exercised) $\{\tau_i\}_{i=1}^m, \tau_1 \leq \tau_2 \leq \dots \leq \tau_m$, such that the number of exercises (*i.e.*

⁵In several of our applications, we assume that countries extract every year a minimum amount of barrels, in which case extraction occurs indeed over a finite horizon.

⁶ k_t must be a multiple of the discretization unit that we use and that represents one real option to extract oil

annual production) is lower than production capacity each year: $\#\{j : \tau_j = s\} \leq k_s$. The value of the policy $\pi_{\mathbf{k}}$ at time t is then given by

$$V_t^{\pi_{\mathbf{k}},m} = \mathbb{E}_t \left[\sum_{i=1}^m h_{\tau_i}(X_{\tau_i}) B(t, \tau_i) \right],$$

where $B(t, \tau_i)$ is the discount factor used to value a payoff at time τ_i in terms of cash at time t .

Definition 1. *The value function is defined as*

$$V_t^{*,m} = \sup_{\pi_{\mathbf{k}}} V_t^{\pi_{\mathbf{k}},m} = \sup_{\pi_{\mathbf{k}}} \mathbb{E}_t \left[\sum_{i=1}^m h(X_{\tau_i}) B(t, \tau_i) \right].$$

The corresponding optimal policy is $\pi^* = \{\tau_1^*, \tau_2^*, \dots, \tau_m^*\}$.

Two alternative but equivalent formulations are needed to explain the numerical solution used. The first formulation is used to compute the lower bound of the estimate of the oil field (the value function) and uses dynamic programming and regression methods. The second formulation is used to derive the dual formulation of the problem, and is the basis for calculating the upper bound as a the minimum over a set of martingales. The formulations are presented in the Appendix.

V. NUMERICAL SOLUTION

In this section we explain the solution method using a simple example with only one real option (*i.e.* only one barrel to extract). In this case the value function can be written as

$$V_t^* = \sup_{t \leq \tau \leq T} \mathbb{E}_t \left[B_t \frac{h_\tau}{B_\tau} \right], \quad (1)$$

which is equivalent to the dynamic programming equations

$$\text{for } t = T, \quad V_T^*(X_T) = h_T(X_T), \quad (2)$$

$$\forall t < T, \quad V_t^*(X_t) = \max \left\{ h_t(X_t), \mathbb{E}_t \left[\frac{B_t}{B_{t+1}} V_{t+1}^*(X_{t+1}) \right] \right\}, \quad (3)$$

where h_t is the payoff from extracting at time t . B_t is the discount factor. If r is a constant risk-free rate, $B_t = e^{rt}$. $\frac{h_T}{B_T}$ is the payoff discounted to time zero. The expected payoff from deferring production (continuation) is called the *continuation value* and is defined as $C_t^*(X_t)$ by

$$\forall t < T, \quad C_t^*(X_t) = \mathbb{E}_t \left[\frac{B_t}{B_{t+1}} V_{t+1}^*(X_{t+1}) \right], \quad (4)$$

Since there are no analytical solutions for such problems, the literature has resorted to several numerical methods. We discuss here how we estimated a lower bound of the value function using a regression approach whereas the upper bound was obtained thanks to the Doob-Meyer decomposition.

A. Lower Bound

The Least Squares Monte Carlo (LSM) method to estimate lower bounds for multiple real option value functions was suggested by Longstaff and Schwartz (2002) and, independently, by Tsitsiklis and Roy (2001). The method relies on approximating the continuation value using linear regressions on polynomials of the state variable (in our case the logarithm of the oil price).

American option problems can be solved backwards: if exercise did not occur until the last period T , the decision rule is: extract oil if the current oil price S_T is greater than the current extraction cost c_T and make profits $S_T - c_T$. At time $T - 1$, the decision to exercise

will depend on whether the discounted value of $\max(S_T - c_T, 0)$ (the continuation value) is higher than the immediate payoff $\max(S_{T-1} - c_{T-1}, 0)$. The LSM method consists in estimating the continuation value at $T - 1$ using a regression of the discounted value of $\max(S_T - c_T, 0)$ (which is known for each Monte Carlo path) on functions of the state variables S_{T-1} and c_{T-1} . Each simulation corresponds to one observation in the regression.

Going backwards, we can calculate the continuation value for all periods. The coefficients of the regression⁷, applied to an ‘actual’ price path provides us with an estimate of the continuation value and the decision rule. The intuition behind this method is that the continuation value must be a function of the state variable and therefore can be approximated using simple functions of the oil price. The method provides a lower-bound estimate of the value function, since only the ‘exact’ optimal policy delivers the maximum value of the oil field, which is the value function. The estimated continuation value yields an exercise rule: it is optimal to extract a barrel if and only if the current oil price is greater than the continuation value of the barrel.

The precision of this method depends on the choice of the regressors, but we show later in the paper that our numerical solution is accurate. The Monte Carlo simulations are used here at two stages. During the first stage, a set of paths is simulated to determine the optimal extraction rule using the regressions. During the second stage, having determined the stopping rule, a second set of paths is simulated to calculate the value function given that stopping rule.

Let $\psi_1, \psi_2, \dots, \psi_k, \psi_i : \mathbb{R}^d \rightarrow \mathbb{R}$ be the basis functions (functions of the oil price in our model) used for regression. We use backward induction and at each step the method approximates the continuation value with a linear combination of the basis functions.

⁷The coefficients are calculated for each period t as the decision rule changes when approaching expiration.

Definition 2. For all times $t \in \{0, 1, 2, \dots, T\}$, at each point of the space set define an approximation to the continuation value by

$$\hat{C}_t(x) = \sum_{i=1}^k c_{t,i} \psi_i(x). \quad (5)$$

Let $\psi = (\psi_1, \psi_2, \dots, \psi_k)$ and $\bar{c}_t = (c_{t,1}, c_{t,2}, \dots, c_{t,k})$. If n paths of the Markov chain are simulated, an estimation for the regression coefficients would be

$$\bar{c}_t = \arg \min_{c \in \mathbb{R}^k} \sum_{j=1}^n \left(C_t^{(j)} - \sum_{i=1}^k c_{t,i} \psi_i(X_t^{(j)}) \right)^2, \quad (6)$$

where

$$C_t^{(j)} = \frac{B_t}{B_{t+1}} V_{t+1}^{(j)}. \quad (7)$$

This is a least square problem and explicit formulas exist for the coefficients.

$$\bar{c}_t = \Psi^{-1} v \quad (8)$$

$$\Psi_{l,p} = \sum_{j=1}^n \psi_l(X_t^{(j)}) \psi_p(X_t^{(j)}) \quad (9)$$

$$v_l = \sum_{j=1}^n \psi_l(X_t^{(j)}) C_t^{(j)} \quad (10)$$

We approximate the continuation value using the explanatory variables $X, X^2, \exp(X)$,

and a constant, where X is the logarithm of the oil price. Once the coefficients

$c_{t,1}, c_{t,2}, \dots, c_{t,k}$ are obtained the continuation value can be approximated at any point using the current oil price. By simulating then a new set of paths, we can find a lower bound for the value function.

B. Upper Bound

The method we use to obtain an upper-bound estimate is based on the idea of Rogers (2002) and Haugh and Kogan (2001) to rely on a dual representation of the value function. The continuation value function is

$$C_t^*(X_t) = \mathbb{E}_t \left[\frac{B_t}{B_{t+1}} V_{t+1}^*(X_{t+1}) \right] \leq V_t^*(X_t), \quad (11)$$

and as a result V_t^*/B_t is a supermartingale, since $\mathbb{E}_t[V_{t+1}^*(X_{t+1})/B_{t+1}] \leq V_t^*(X_t)/B_t$.

According to the Doob-Meyer decomposition, a supermartingale can be written as the sum of a martingale M_t^* and an increasing process D_t^* , both vanishing at $t = 0$:

$$\frac{V_t^*}{B_t} = V_0^* + M_t^* - D_t^*,$$

We call M_t^* the optimal martingale. We use the following theorem, taken from Rogers (2002), according to which maximizing the value function is equivalent to minimizing over a set of integrable martingales.

Theorem 1. *The value function V_0^* at time zero is given by*

$$V_0^* = \inf_{M \in H_0^1} \mathbb{E} \left[\sup_{0 \leq t \leq T} \left(\frac{h_t}{B_t} - M_t \right) \right], \quad (12)$$

where H_0^1 is the space of martingales M , for which $\sup_{0 \leq t \leq T} |M_t|$ is integrable and such that $M_0 = 0$. The infimum is attained by taking $M = M^*$.

Hence, by constructing an approximation to the optimal martingale M_t^* , we can find an upper bound for the value function (only the achieved minimum would yield the exact value function). More details are available in Rogers (2002) and Aleksandrov and Hambly (2008).

VI. OIL PRICE MODEL

A variety of oil price models have been suggested in the literature, but the mean-reverting (one factor) model, as in Schwartz (1997), has been the model of choice:

$$dS_t = \kappa(\mu - \ln S_t)S_t dt + \sigma S_t dW_t \quad (13)$$

where σ is the volatility of oil prices. Defining $X_t = \ln S_t$ and applying Ito's Lemma, the log price process is a Ornstein-Uhlenbeck stochastic process:

$$dX_t = \kappa(\alpha - X_t)dt + \sigma dW_t \text{ with } \alpha = \mu - \frac{\sigma^2}{2\kappa} \quad (14)$$

The process is mean-reverting towards α , and the speed of convergence is κ . We use a discretization of that process:

$$X_t - X_{t-1} = a + bX_{t-1} + \sigma Z \quad (15)$$

where Z is white noise. We estimated the parameters on 'real' oil prices⁸ on yearly data for the period 1957 - 2008.

Table 1. Parameters of the oil price process (OLS on yearly data for the period 1957 - 2008)

Price process	a	b	σ
coefficients	0.183	-0.047	0.26

VII. RESULTS

We report here our results for Brazil and U.A.E. and investigate the dependence of optimal policies on the model parameters. We present five types of results:

⁸Deflated by the US CPI, with index 100 in 2000.

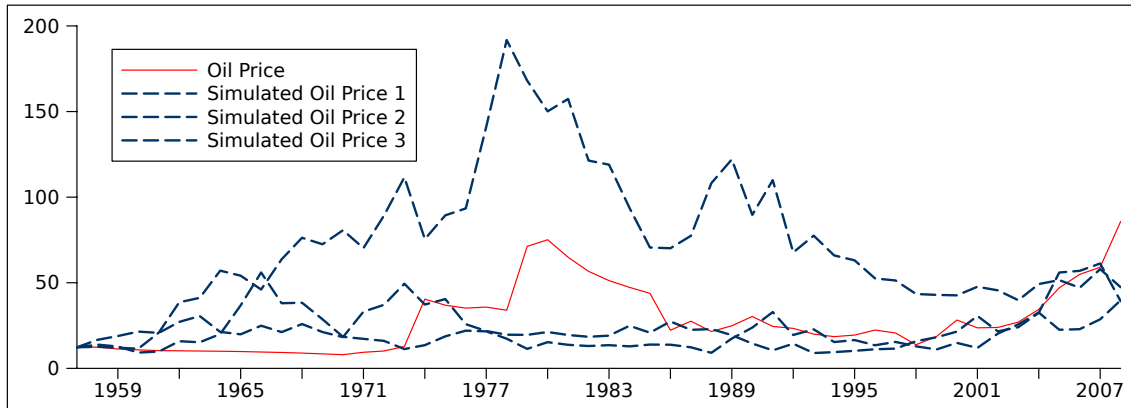


Figure 2. Oil price - historical and simulations.

1. The model parameters we used to determine the optimal policy are assumptions of the model. We observe the changes in optimal policy when these parameters change, *for a given path of the oil price*. We are interested in the dependence of optimal policy on:
 - Volatility of oil prices;
 - Interest rates;
 - Oil reserves;
 - Long-term mean of the oil price;
2. We compare the performance, in terms of discounted cash flows, of the optimal extraction policy with that of a policy of constant extraction rates. We find that the value function (*i.e.* the discounted cash flow for the optimal policy) is around twice higher than the discounted cash flow obtained with constant extraction policies for the most extreme case.⁹
3. Brazil's production is at par with that of the U.A.E. in spite of having much smaller confirmed reserves. Is this optimal for Brazil? We compare the decision rule of for

⁹We also calculate the upper and lower bound of the value function and show that the bounds around the value function are tight, demonstrating the effectiveness of the numerical method.

Brazil with that of the U.A.E. and show that Brazil should only produce when oil prices are high, at US\$ 73, whereas the U.A.E. should produce as soon as the oil price exceeds US\$ 39.

4. We estimate the value in terms of discounted cash flows of an additional capacity unit, for a given volume of reserves. The result can be fed into a model of Net Present Value of an investment project when assessing the optimality of capacity expansion.
5. We extend the model to allow for extraction costs, which assumed to follow a random walk with drift. We use data from Petrobras to estimate the parameters of the stochastic process and show how optimal policy is affected by these extraction costs.

We assume in all calculations that the extraction decisions are made on a yearly basis. The smallest unit that is used to change production is 0.2 billion barrels per year, equivalent to 0.55 million barrels per day. We also make the assumption that each country extracts at least 0.5 billion barrels per year to finance operations and ensure a minimum of revenues.

In 2008, Brazil had 12.2 billion barrels of proven oil reserves and U.A.E. had 97.8 billion barrels. We will use these numbers, adjusted for a recovery rate of 65 percent, for the total extractable amounts of oil when computing the threshold at which the countries should produce oil. Given the large difference in the proven total reserves, the production horizon of U.A.E. is much longer: for Brazil the time horizon chosen is 15 years, which leaves us with 25 units of 0.2 billion barrels per year (after the minimal extraction rate of 0.5 billion barrels per year are subtracted). For the U.A.E. we consider a time horizon of 100 years and 240 extractable units.

The EIA provides us with projections of production capacity for the next twenty years (see

tables 2 and 3), and we assume capacity is constant after 2030. The calibration used for the numerical results is that of table 4.

Table 2. Oil production capacity outlook 2010-2030 (Million Barrels per Day). Source: EIA, System for the Analysis of Global Energy Markets (2006);

Country	2010	2015	2020	2025	2030
United Arab Emirates	3.3	3.5	3.9	4.4	4.6
Brazil	2.7	3.5	3.9	4.2	4.5

Table 3. Oil production capacity outlook 2010-2030 (Billion Barrels per Year). Source: EIA, System for the Analysis of Global Energy Markets (2006);

Country	2010	2015	2020	2025	2030
United Arab Emirates	1.2	1.3	1.4	1.6	1.7
Brazil	1.0	1.3	1.4	1.5	1.6

Table 4. Calibration for Oil production capacity outlook 2010-2110 (One unit equals 0.2 billion barrels).

Year	1-5	5 -10	10-15	15-20	20-100
Extractable Units	3	4	5	6	6

A. Dependence of the Optimal Policy on the Model Parameters: the case of Brazil

We show figure 3 the optimal policy for Brazil for a particular path of the oil price. The oil price is relatively low at the beginning and therefore the oil producer, knowing the volatility and trend of oil prices, decides to defer production. In 2020 (the model starts in 2010), when oil prices rise above 60 US dollars (in 2000 terms, *i.e.* around 75 US dollars of 2008¹⁰), the optimal strategy is to produce at full capacity. Oil reserves are almost depleted by 2025. The model predicts very volatile production plans, since as soon as the price exceeds a certain threshold value (see discussion later), production at full capacity is optimal. This result could be reversed with high and increasing marginal extraction costs but we abstract from extraction costs until section F.

¹⁰US CPI inflation between 2000 and 2008 was 25 percent

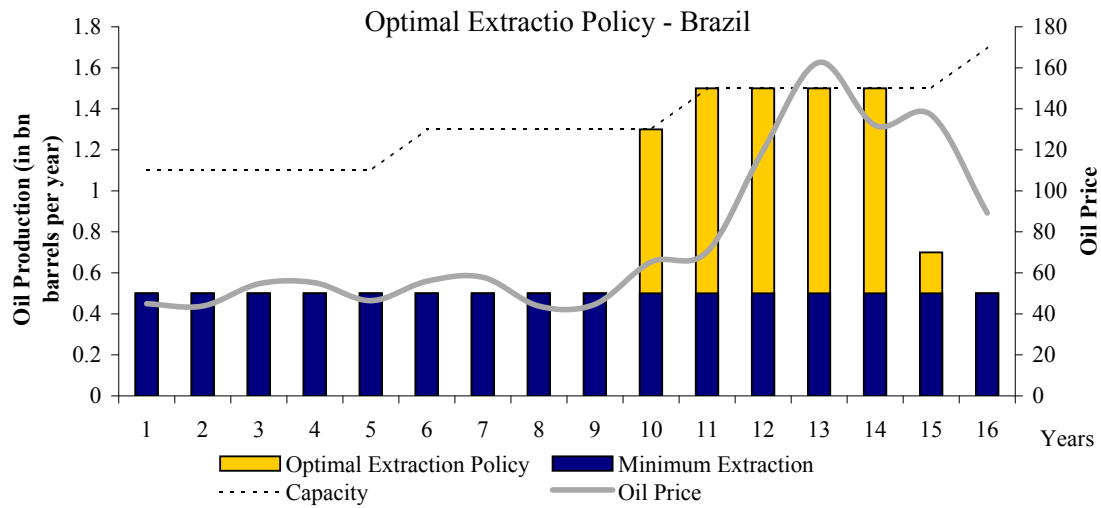


Figure 3. Optimal extraction policy

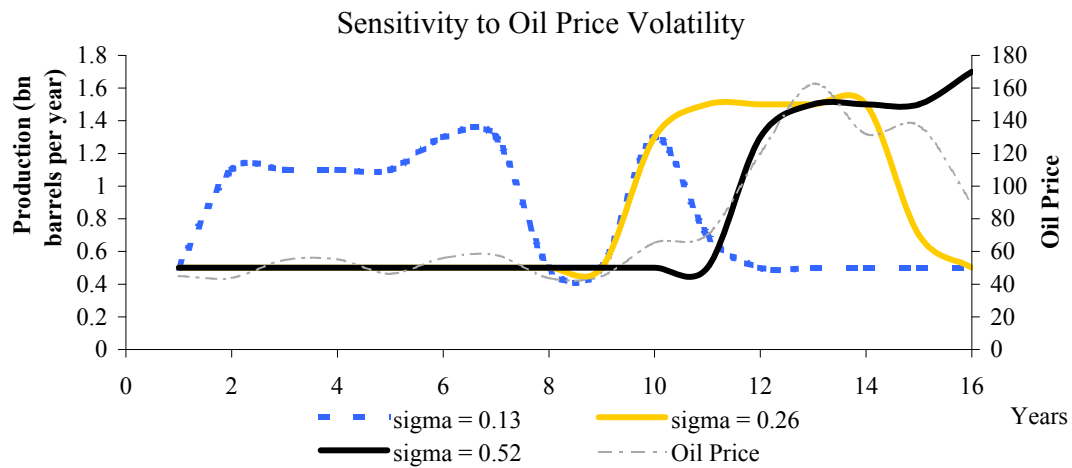


Figure 4. Optimal extraction policy - dependence on oil price volatility

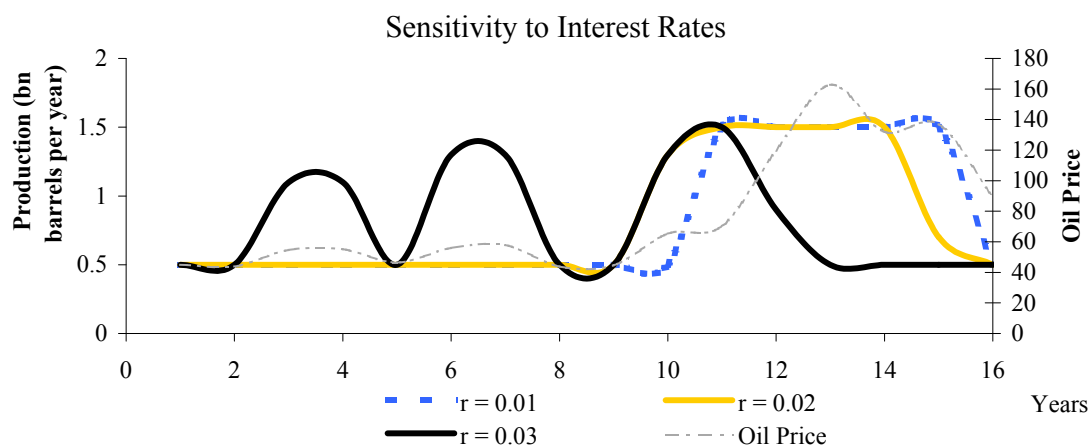


Figure 5. Optimal extraction policy - dependence on real interest rates

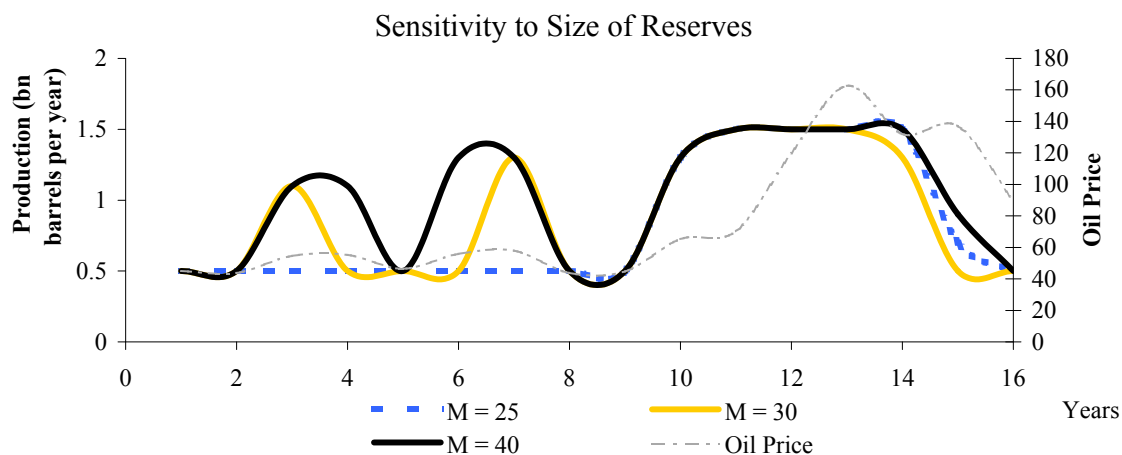


Figure 6. Optimal extraction policy - dependence on oil reserves

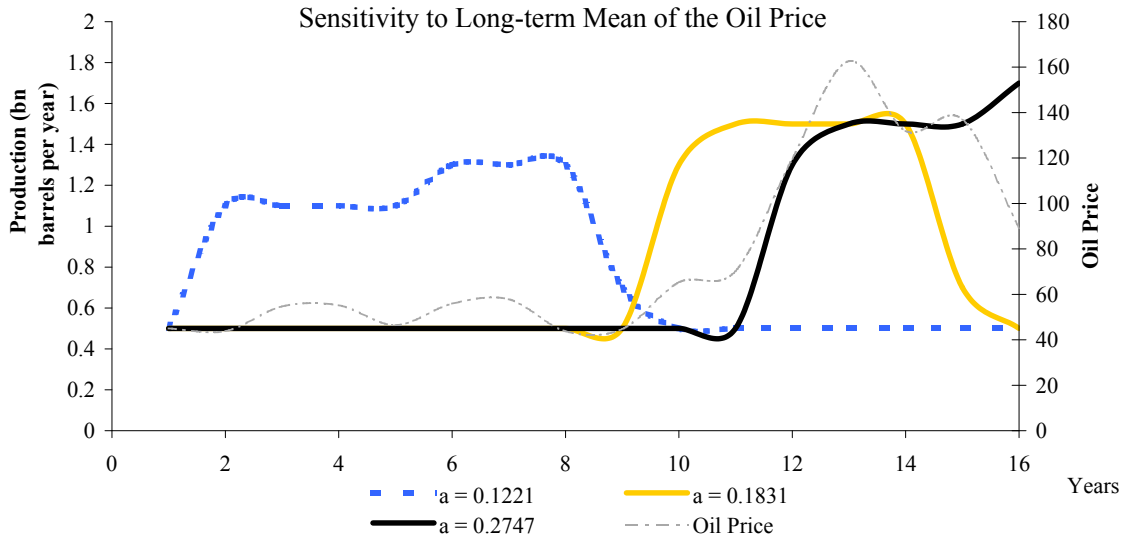


Figure 7. Optimal extraction policy - dependence on long term mean

Figure 4 allows us to discuss the dependence of oil extraction on the volatility of the oil price, for a given *realization* of the oil price path. We show optimal extraction for a volatility parameter of the oil price process equal to that estimated using historical data ($\sigma = 0.26$), and for a volatility twice smaller ($\sigma = 0.13$) and twice larger ($\sigma = 0.52$). As expected, high volatility makes it optimal to defer production. When volatility is perceived to be low, there are no incentives to defer production and a large share of reserves are depleted with the first 7 years. On the other hand, when volatility is expected to be high the threshold at which it is worth producing is high as the producer is hoping for higher payoffs in the future. As a result, with our particular realization of oil prices, production is deferred to the last 4 years during which oil prices are at the highest.

Dependence on the real interest rate is presented in figure 5. We consider three values of the real interest rate: 1, 2 and 3 percent. As usual, production occurs earlier at higher interest rates. For a discount rate of 2 percent, oil production at full capacity is optimal for Brazil at relatively high oil prices (US\$ 73 of 2008).

Finally, we show the dependence of optimal extraction on the volume of recoverable oil reserves. The amount of oil that is known to be in the ground or ‘proven reserve’ varies as new fields are being explored. Furthermore, recovery rates from extraction have improved with technological advances and therefore the volume of recoverable reserves can change. We show in figure 6, for Brazil, how production would change if it had access to 12.5, 13.5 or 15.5 billion barrels. The profiles of production are not fundamentally changed.

B. Precision of the Solution

The numerical solution method can be affected by two types of errors:

- Monte Carlo errors that depend on the number of paths;
- Errors coming from the approximation of the continuation values.

We can reduce the Monte Carlo error by increasing the number of paths. The size of the Monte Carlo error, which depends on the number of paths and the standard deviation of the estimates, can also be computed. It is more complicated however to assess the accuracy of our approximation to the continuation value for a chosen set of basis functions. We use the lower and the upper bounds to construct a confidence interval for the value function. We compare the upper and lower bound approaches for our baseline model.¹¹ Figure 8 shows that the relative difference between the upper and the lower bound does not exceed 3%, which confirms the accuracy of the numerical approximations used.

¹¹The baseline model use the estimated parameters ($a = 0.183126$, $b = -.04708$ and $\sigma = 0.26$), the real interest rate r is assumed at 2 percent, and the time horizon is 100 years. Oil reserves are those provided by the IEA, and production capacity is that of table 3 and table 4.

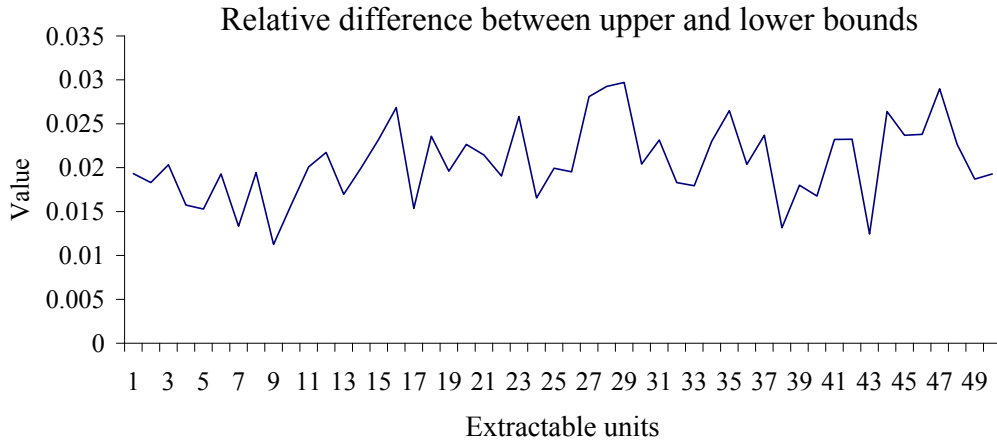


Figure 8. Relative difference between the upper and lower bounds for the value functions

C. Optimal Strategy as an Improvement over Constant Extraction Policy

In this section, we compare the value of an oil field under different production policies. The value functions are computed for different minimum extraction rates, since this is an important determinant of the value of the oil field. *A priori*, minimum extraction rates should be zero, for countries that can borrow from financial markets. However, many oil producers crucially depend on contemporaneous oil surpluses to finance government expenditure and current account deficits, and as a result, it may be more realistic to assume some minimum extraction rates. We compute the value function for three different minimum extraction rates, of 0, 0.5 and 0.7 billion barrels per year.¹²

We also compute the value of the oil field under a policy of constant extraction rate (a linear extraction model). The value of the oil field under linear extraction is calculated by averaging the net present value of cash flows received from the sales of oil if the extraction was done at a constant rate over time.

¹²To be consistent with the previous sections we use 0.2 billion barrels as a unit in all models.

We show in figure 9 the inter-temporal value of an oil field as a function of its size (in units of 0.2 billion barrels) under the two extreme assumptions of zero minimum extraction rate and linear extraction. For instance, a 10 billion barrels oil field is worth US\$ 357.9 billion under a constant extraction policy, but is worth US\$ 835.4 billion under the optimal policy with zero minimum extraction rate. The benefits from following an optimal strategy would therefore reach 133% in terms of net present value, a dramatic improvement. This proportion is approximately constant for different values of the oil reserves.

Assuming that a minimum amount of 0.5 billion barrels per year is produced every year, the gains from optimizing the remaining production in function of oil prices still reach 25.6 percent (above the linear extraction value). If the minimum extraction rate is 0.7 billion barrels per year, the gains from optimizing would be 22.5 percent.

Overall, the model predicts that the value of an oil field for which production is decided in function of oil prices far exceeds the linear extraction value computed in a deterministic setting. The economic and policy implications of our results for Brazil and the U.A.E. are different. In recent years, OPEC countries tended to vary production with oil prices, because OPEC had a policy to stabilize prices. This policy is also in line with maximizing expected net present value since stabilizing prices requires increasing production when prices are high and decreasing production when prices are low. The implication of the model for such countries is that the valuation of oil fields needs to be adjusted to take into account the fluctuations in production that are due to oil price shocks. The current practice, in many macroeconomic applications (such as fiscal or external sustainability assessment), is to use a deterministic path of production (derived from production capacity) and a deterministic price path (often deduced from future prices) to estimate the value of oil wealth. The model shows that such valuation methods underestimate the value of oil fields

for countries that are able and willing to adjust production with oil prices.

The implications of the model for non-OPEC countries that produce at full capacity because of their urgent need of revenues is that this policy is very costly: inter-temporal revenues could be twice larger with an optimized strategy. We discuss in the last section of the paper how oil producers behaved in the last thirty years and the implications for world oil markets.

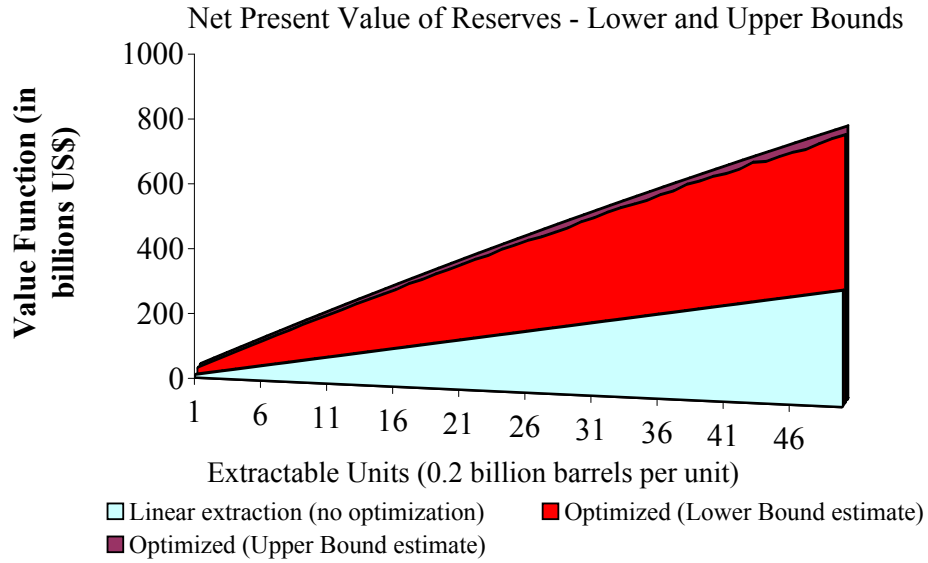


Figure 9. Net present value of reserves as a function of the reserves.

D. U.A.E. - Brazil Comparison

We compare here the optimal extraction policies of Brazil and U.A.E. The two countries have currently similar extraction rates, although their known resources are very different. We show in figure 10 the extraction boundaries for Brazil and the U.A.E. These are functions of the logarithm of the oil price, which we used as a variable in our approximation to the continuation values in the optimal stopping problem. The oil price above which a country should start producing above the minimum requirement is at the intersection of the payoff (which is the oil price, plotted here as a function of the logarithm of the oil price)

and the continuation value, estimated from the regression method as a function of the logarithm of the oil price. The calculations are made under the assumptions that the time horizons are 15 years for Brazil and 100 years for the U.A.E. and the reserves are 12 billion barrels for Brazil and 98 billion barrels for the U.A.E.¹³ If the oil price is higher than the extraction boundary, it is optimal to extract more than the minimum amount; otherwise, the minimum amount should be extracted. Figure 10 shows that the extraction boundary for Brazil is significantly higher for Brazil. According to the model, in 2008, Brazil should only start producing above the minimum when oil prices are above US\$ 73 (at 2000 prices, around 91 US\$ at 2008 prices). The corresponding minimal price for the U.A.E. is US\$ 39 of 2000 (49 US\$ of 2008).

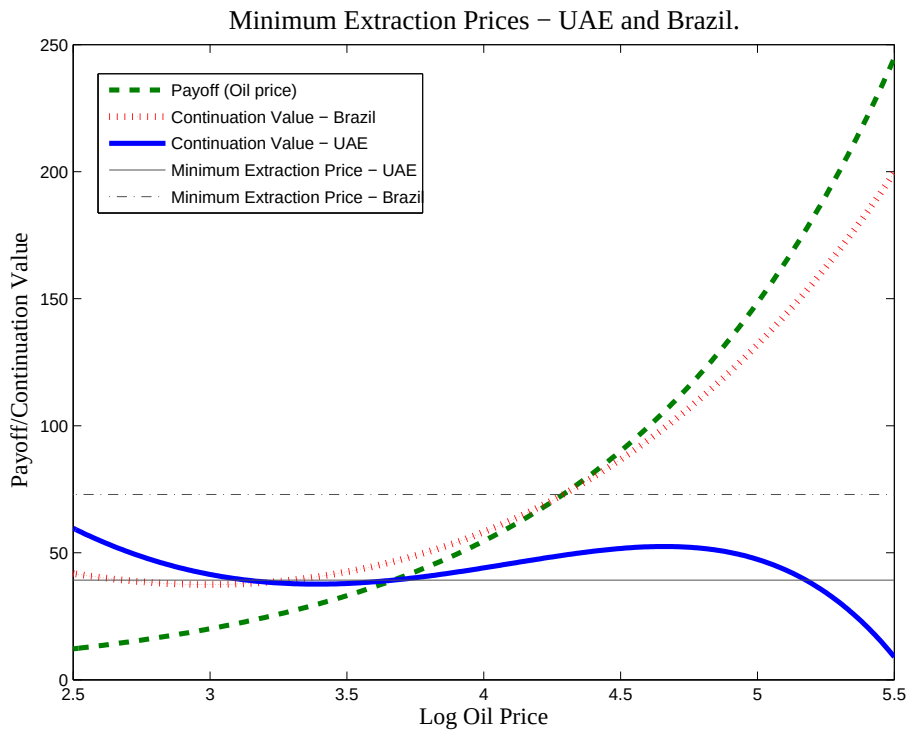


Figure 10. Minimum extraction prices for 2010

¹³We have again assumed minimal extraction of 0.5 billion barrels per year. The oil price parameters are as in table 1 and the real interest rate is assumed to be 2 percent.

E. Investment Decision: Adding New Capacity

In this section, we investigate the benefits of investing in capacity expansion. More precisely, we compute the increase in net present value of the oil fields (the value function) that is obtained when increasing capacity, for a given amount of total reserves. The increase in NPV is our estimate of the shadow price of the yearly production capacity constraint. This number can be used to assess the profitability of a project for which investment costs are known.

We consider two countries with oil reserves estimated at 5 and 10 billion barrels (25 and 50 units respectively).¹⁴ The results are presented in table 5. A capacity expansion of 200 million barrels per year in 2025 (1 additional unit in Year 15) improves the value of the oil reserves by US\$ 2.1 billion, for a country whose reserves are 5 billion barrels. The increase would reach US\$ 6.27 billion for a country that holds 10 billion barrels. Additional capacity has more value for countries with larger reserves, a result that, if it already existed, was not echoed in the oil production literature. The intuition is clear: the additional option provided by higher capacity has more value if it is available for many years and if production capacity was a tighter constraint at times of high prices. Our results therefore suggest that project evaluation needs to be performed in the context of the overall oil strategy of the country, since the viability and profitability (*i.e.* net present value) of projects cannot be assessed project-by-project.

New capacity generally gives access to more reserves in addition to providing more flexibility. We show in table 6 the value of such a project if the additional reserves are worth 1 billion barrels (5 units of 200 million barrels). 200 million barrels of increased capacity in

¹⁴The oil price process parameters are as before. We start with log oil price of 4 ($S = 54.6$). The time horizon is 100 years.

Table 5. Added value by expanded capacity.

Year (unit = 0.2 bn bbl/year)	1-5	5 -10	10-15	15-	25 units	50 units
Current capacity	3	4	5	5	US\$ 325.9bn	US\$ 622.5bn
Increased capacity	3	4	5	6	US\$ 328bn	US\$ 628.8bn
Added value					US\$ 2.1bn	US\$ 6.27bn

2025 and 1 billion more of accessible reserves are worth US\$ 65.6 billion for a country that controls 5 billion barrels of reserves and US\$ 62.7 billion for a country that owns 10 billion barrels. The value of the project is in that case higher for the country with lower reserves because the value of optionality is dwarfed by the value of additional oil reserves, and an additional one billion barrels of oil reserves is worth more for a country that has low reserves than for a country that has high reserves. Indeed, for a similar production capacity, a country with low reserves can ‘distribute’ the extraction of newly-found reserves more optimally than a country that has high reserves.

Table 6. Added value by expanded capacity and increased access to reserves.

Year (unit = 0.2 bn bbl/year)	1-5	5 -10	10-15	15-	25/30	50/55
Current capacity	3	4	5	5	US\$ 325.9bn	US\$ 622.5bn
Increased capacity	3	4	5	6	US\$ 391.4bn	US\$ 685.2bn
Added value					US\$ 65.6bn	US\$ 62.7bn

F. Extraction Costs

The multiple real option approach is flexible enough to allow numerical solutions even when the dimensionality of the problem (the number of state variables) increases. We show briefly in this section how to extend the model to include stochastic extraction (or lifting) costs. The extraction costs are calibrated on data reported by Petrobras (figure 11) assuming the process is a random walk with drift :¹⁵

$$c_t = c + c_{t-1} + \sigma_c W_1^c \quad (16)$$

¹⁵Since Petrobras produces 95% of Brazil’s crude oil, these numbers are accurate estimates for Brazil oil fields.

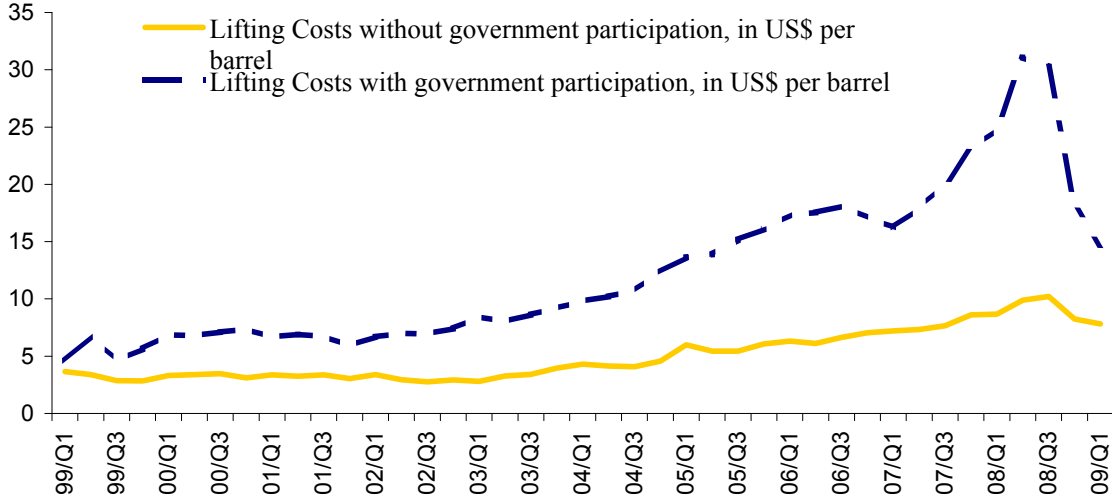


Figure 11. Extraction costs reported by Petrobras of Brazil for the period 1999 - 2009. Source: Petrobras reports.

where W_1^c is a standard normal random variable, c is the drift, and σ_c the volatility (see estimates in Table 7).¹⁶

Table 7. Annualized parameters of the lifting cost process, estimated using quarterly data for the period 1999 - 2009.

Extraction costs	$\hat{c}$	$\hat{\sigma}_c$
Coefficients	0.416	0.602

The payoff from selling a unit of oil at time t is $h_t = S_t - c_t$. In order to account for c_t in the regression algorithm we add it to the set of basis functions ψ . We show the impact of extraction costs on the optimal policy in figure 12. We observe a large difference in optimal policy in year 7: when the extraction cost is US\$ 10.55 per barrel (cost path 1), maximum production is optimal whereas when the cost is US\$ 18.65 (cost path 2), production is reduced to the minimum. For a country like Brazil, for which extraction costs are uncertain but likely to be high, a model that includes stochastic extraction costs suggests that

¹⁶We assume extraction costs and oil prices are independent and estimate on quarterly data $\hat{c} = 0.104$ and $\hat{\sigma}_c = 0.301$. The numbers reported in table 7 are on a yearly basis.

production could be differed on account of even limited increases in extraction costs. In this model, the two state variables are independent, but one can see from figure 12 that optimal policy involves minimum production in year 7 because extraction costs are high, and at the same time oil prices are low.

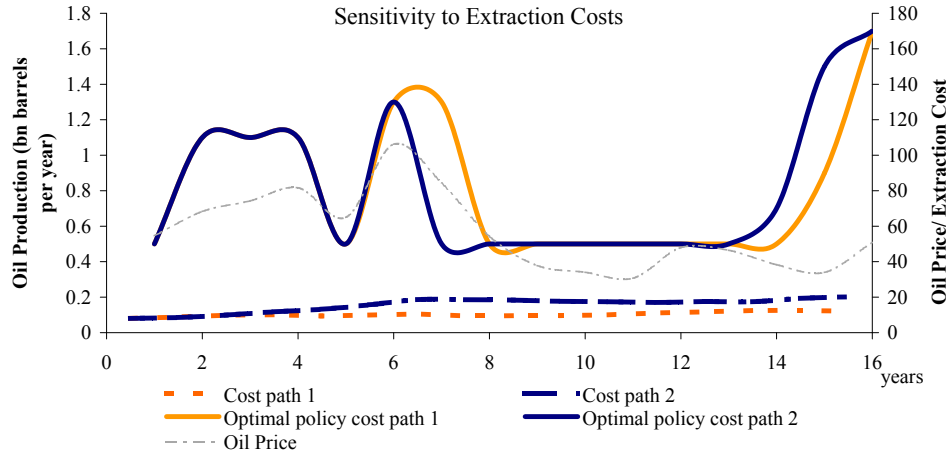


Figure 12. Optimal extraction policy - dependence on extraction costs

VIII. THE GLOBAL SUPPLY CURVE

The model gives us two points in the theoretical world market supply curve (Brazil and the U.A.E.), and the aim of this section is to derive a rough approximation of the rest of the supply curve. We ranked countries with capacity higher than 500 thousand bb/d by their reserves and cumulated their production (see table 8).

Kuwait expected production profile and reserves are very similar to those of the U.A.E. and therefore its decision rule should be the same. Algeria, Mexico and Brazil should also have comparable policies (threshold around US\$90) while Venezuela's threshold price would lie in between. Norway and Egypt policies should be to extract at prices higher than US\$90 only. Roughly speaking, the theoretical supply curve could be represented as in figure 13.

Country	Reserves	Production	Cumulated production
Australia	1.5	586.4	79345.8
Colombia	1.5	600.6	78759.3
Argentina	2.6	792.3	78158.7
U.K.	3.6	1583.9	77366.4
Egypt	3.7	630.6	75782.6
Malaysia	4.0	727.2	75152.0
Indonesia	4.4	1051.0	74424.8
Ecuador	4.5	505.1	73373.8
Oman	5.5	761.0	72868.7
India	5.6	883.5	72107.7
Norway	6.9	2466.0	71224.2
Azerbaijan	7.0	875.2	68758.3
Angola	9.0	2014.5	67883.1
Mexico	11.7	3185.6	65868.6
Brazil	12.2	2422.0	62682.9
Algeria	12.2	2179.8	60260.9
Qatar	15.2	1207.6	58081.1
China	16.0	3973.1	56873.6
U.S.	21.3	8514.2	52900.4
Kazakhstan	30.0	1429.3	44386.3
Nigeria	36.2	2168.9	42957.0
Libya	41.5	1875.2	40788.1
Russia	60.0	9789.8	38912.9
Venezuela	87.0	2642.9	29123.1
U.A.E.	97.8	3046.5	26480.2
Kuwait	104.0	2741.4	23433.7
Iraq	115.0	2385.4	20692.4
Iran	138.4	4174.4	18306.9
Canada	178.6	3350.4	14132.5
Saudi Arabia	266.8	10782.1	10782.1

source: IEA

Table 8. Reserves (in billion barrels) and production (in thousand bb/d) in 2008

In such a world, the supply elasticity would be very large and oil price volatility would never reach the levels attained in 2008-2009. Indeed, demand would need to fall as low as 30 million bb/d to see prices declining to US\$ 40. On average, production would be lower (as several countries would find the oil price to be below their threshold price). As a result, prices would be higher but less volatile.

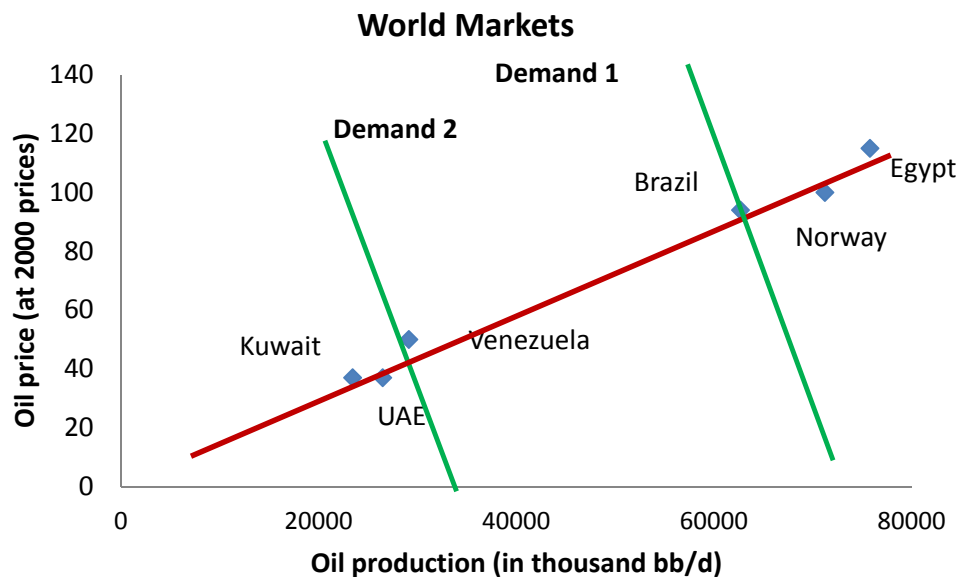


Figure 13. World oil market

There are several potential explanations to why the supply curve is much less elastic in reality than the model would predict (see for instance Krichene, 2005, who shows that short-term supply is insensitive to price) and we discuss below the two that we think are most relevant.

- The NPV model is not appropriate if a country's discount rate is contingent on oil production. This will be the case for oil producers whose budget and external

accounts are highly dependent on oil revenues. For such countries, cutting production at the time prices are low will push those accounts in deficit and trigger borrowing at increased rates, which governments and oil companies will be unwilling to do.

However, given the significant losses in NPV (100 percent in the extreme case), strategies that lead to more volatile production should be considered, especially for producers with larger reserves and for those who already accumulated large financial reserves. Casual data investigation did not support the hypothesis that countries with higher reserves produce at more volatile rates, although OPEC production is indeed more variable than non-OPEC production. For most countries, production has been at full capacity in the last 30 years.

- There may be technical limitations as well that reduce the benefits of choosing variable production. For instance, there is a minimum turn-down that is necessary to have the plant on, to avoid rusting, losing staff, etc. Similarly, if extraction costs are a function of oil production, the solution of the model will differ somewhat and optimal oil production will be less volatile. This is why we also computed the NPV gains for optimizing over part of the capacity (and we found it to be in excess of 20 percent). Depending on the specificity of these technical costs, countries could either optimize the functioning of each extraction units over half of the capacity or take decisions every year to shut down or run at full capacity half of the production units.

IX. CONCLUSION

We proposed a Monte Carlo real option approach as a solution to the optimization problem of a price-taker oil producer. This approach allows us to replicate the basic results of the ‘waiting to invest’ literature in a multiple-period setting. The Monte Carlo numerical solution is accurate and flexible enough to solve the problem with increasing capacity and with several state variables. We also investigated the value of increasing capacity and

showed that the marginal value of additional capacity is higher for countries with higher resources, because additional capacity generates new options to extract oil, and such options have more value for countries with larger resources and longer horizons of production.

We showed that, in theory, the net present value of these countries' oil reserves are grossly underestimated (at around half of their true value, in the most extreme case) if one does not take into account optimizing strategies. We also showed that the optimal policies imply production profiles that are much more volatiles than the observed ones - even when one assumes that some minimum extraction is needed to finance operations or government budgets. As a result, the theoretical supply is much more elastic than the actual one, and the equilibrium consequences of that result is that oil prices are lower, but much more volatile than what would be the case if oil producers were optimizing by maximizing the NPV of oil fields. The puzzle could be partially resolved by modeling extraction costs as a function of oil production, but in practice, costs tend to depend on capacity rather than on production itself. Technical costs may also limit the scope for varying production, but even optimizing over half of capacity would yield substantial benefits to oil producers, while reducing oil price volatility. The most convincing explanation seems to be that countries and companies' borrowing conditions would be worsened by such pro-cyclical extraction policies, and this is why such extraction policies are not in place. Finding ways to reduce the pro-cyclicality of borrowing conditions could therefore yield substantive benefits at the individual country level at the same time as stabilizing the oil market.

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APPENDIX

To simplify notations the discount factors are kept implicit in the following expressions.

The optimal payoff at time $t < T$ is the maximum of:

1. the payoff obtained by producing at full capacity k_{t-1} and leaving the remaining barrels $m - k_t$ for later (with expected value of the remaining barrels $\mathbb{E}_t[V_{t+1}^{*,m-k_t,\mathbf{k}}]$);
1. the payoff obtained by producing one barrel less than full capacity $k_{t-1} - 1$ and leaving the remaining barrels $m - (k_t - 1)$ for later; etc.

For $0 \leq i \leq k_t$ the quantity

$$(k_t - i)h_t + \mathbb{E}_t[V_{t+1}^{*,m-k_t+i,\mathbf{k}}]$$

is the payoff under the extracting of the m -th, $m - 1$ -th, ..., $m - k_t + i + 1$ -th units of oil at time t plus the expected future payoff under the remaining $m - k_t + i$ units. At time T (when the model ends) the only thing that can be done is to extract all barrels allowed by production capacity and the value of the field is $k_T h_T$.

The dynamic programming can therefore be formulated as:

Lemma 1. (Optimal extraction value function - Dynamic programming formulation). The value function $V_t^{*,m,\mathbf{k}}$ at time t is given by

$$\begin{aligned} \text{for } t = T, \quad V_T^{*,m,\mathbf{k}} &= k_T h_T, \\ \forall t < T, \quad V_t^{*,m,\mathbf{k}} &= \max\{k_t h_t + \mathbb{E}_t[V_{t+1}^{*,m-k_t,\mathbf{k}}], (k_t - 1)h_t + \mathbb{E}_t[V_{t+1}^{*,m-(k_t-1),\mathbf{k}}], \\ &\quad \dots, h_t + \mathbb{E}_t[V_{t+1}^{*,m-1,\mathbf{k}}], \mathbb{E}_t[V_{t+1}^{*,m,\mathbf{k}}]\}. \end{aligned}$$

For $m \leq k_t$ we have

$$\begin{aligned} \text{for } t = T, \quad V_T^{*,m,\mathbf{k}} &= m h_T, \\ \forall t < T, \quad V_t^{*,m,\mathbf{k}} &= \max\{m h_t, (m - 1)h_t + \mathbb{E}_t[V_{t+1}^{*,1,\mathbf{k}}], \dots, \mathbb{E}_t[V_{t+1}^{*,m,\mathbf{k}}]\}. \end{aligned}$$

The problem is also equivalent to the following optimal stopping problem formulation:

Lemma 2. (*Optimal extraction value function - Optimal stopping problem formulation*).
The value function $V_t^{*,m,\mathbf{k}}$ is given by

$$V_t^{*,m,\mathbf{k}} = \max_{t \leq \tau \leq T} \mathbb{E}_t \left[\max \{ k_\tau h_\tau + \mathbb{E}_\tau [V_{\tau+1}^{*,m-k_\tau,\mathbf{k}}], (k_\tau - 1)h_\tau + \mathbb{E}_\tau [V_{\tau+1}^{*,m-(k_\tau-1),\mathbf{k}}], \right. \\ \left. \dots, h_\tau + \mathbb{E}_\tau [V_{\tau+1}^{*,m-1,\mathbf{k}}] \} \right]$$

(In the max bracket only those terms that exist are taken)