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LOWER SEMICONTINUITY, STOILOW FACTORIZATION AND PRINCIPAL MAPS

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Abstract. We consider a refinement of the usual quasiconvexity condition of Morrey in two dimensions that allows us to prove lower semicontinuity and existence of minimizers for a class of functionals which are unbounded as the determinant vanishes and are non-polyconvex in general. This notion, that we call *principal quasiconvexity*, arose from the planar theory of quasiconformal mappings and mappings of finite distortion. We compare it with other quasiconvexity conditions that have appeared in the literature and provide a number of concrete examples of principally quasiconvex functionals that are not polyconvex. We also describe local conditions which combined with quasiconvexity yield principal quasiconvexity. The Stoilow factorization, that in the context of maps of integrable distortion was developed by Iwaniec and Šverák, plays a prominent role in our approach.

Dedicated to Vladimir Šverák on the occasion of his 65th birthday.

1. **Introduction.**

Given a real $n \times n$ matrix A, its (outer) distortion is defined by

$$K_A \equiv \begin{cases} |A|^n/\det A & \text{if } \det A > 0, \\ 1 & \text{if } A = 0, \\ +\infty & \text{if } A \neq 0 \text{ and } \det A \leqslant 0, \end{cases}$$

where $|A| \equiv \max_{x\in\mathbb{S}^{n-1}} |Ax|$ is the operator norm and $|Ax|$ is the usual euclidean norm of $Ax \in \mathbb{R}^n$. When Ω is an open, connected, non-empty subset of $\mathbb{R}^n$ (henceforth called a domain) and $u\colon \Omega \to \mathbb{R}^n$ is locally of Sobolev class $\mathrm{W}^{1,1}$ in Ω, denoted $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, its *distortion function* $K_u(x) \equiv K_{\mathrm{D}u(x)}$ is an $\mathscr{L}^n$-almost everywhere defined extended real-valued function on Ω. If the distortion $K_u < +\infty$ almost everywhere and the *Jacobian* $J_u \equiv J_{\mathrm{D}u} \equiv \det \mathrm{D}u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, then u is called a *mapping of finite distortion*, and if furthermore $K_u \in \mathrm{L}^\infty(\Omega)$, then u is called a *weakly quasiregular map*. A striking result of Reshetnyak states that a *quasiregular map*, so a map $u \in \mathrm{W}^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ with $K_u \in \mathrm{L}^\infty(\Omega)$, is either constant or admits a representative that is continuous, open and discrete. In the two-dimensional case,

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$n = 2$, that is our focus here, this result is a consequence of the so-called *Stoilow factorization*. When Ω is a domain in $\mathbb{C}$ it asserts that a non-constant map $f\colon \Omega \to \mathbb{C}$ is continuous, open and discrete if and only if it admits the factorization $f = H \circ F$, where $F\colon \Omega \to \mathbb{C}$ is a homeomorphism and $H\colon F(\Omega) \to \mathbb{C}$ is holomorphic (see [77], and [58] for a modern exposition).

In [54] Tadeusz Iwaniec and Vladimir Šverák brought together ideas from Geometric Function Theory and Nonlinear Elasticity proving, again in two dimensions, that if $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega) \equiv \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{C})$ is not a.e. constant and has distortion function $K_u \in \mathrm{L}^1(\Omega)$, then (the precise representative of) u is continuous, open and discrete (see also [7] for various refinements). The aim of this note is to give a systematic presentation of the notion of principal quasiconvexity considered first in [4]. In particular we aim to explain how it, in conjunction with Stoilow factorization and its extension by Iwaniec and Šverák, allows to prove lower semicontinuity results in situations where the energy functionals are not polyconvex and do not satisfy the standard growth assumptions. Relevant examples include the typical variational models for isotropic planar hyperelasticity and the variational problems arising in Geometric Function Theory.

We briefly review some background results from the Calculus of Variations before elaborating further on the Geometric Function Theory concepts, such as Stoilow factorization, the distortion function and the principal maps.

First recall [63, 18] that when Ω is a bounded domain in $\mathbb{R}^n$ and the exponent $p \geqslant 1$, then the $\mathrm{W}^{1,p}$-quasiconvexity of a functional $\mathbf{E}\colon \mathbb{R}^{n\times n} \to \mathbb{R}$, meaning that

$$\mathbf{E}(A) \leqslant \fint_{\Omega} \mathbf{E}(\mathrm{D}u(x))\,\mathrm{d}x \qquad \text{for all } u \in A + \mathrm{W}^{1,p}_0(\Omega, \mathbb{R}^n) \text{ and } A \in \mathbb{R}^{2\times 2},$$

is equivalent to the sequential weak lower semicontinuity on $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^n)$ (briefly $\mathrm{W}^{1,p}$-swlsc) of the corresponding variational integral

$$\mathscr{E}[u] \equiv \int_{\Omega} \mathbf{E}(\mathrm{D}u)\,\mathrm{d}x, \tag{1}$$

provided that $\mathbf{E}$ is non-negative and in addition has standard p-growth:

$$|\mathbf{E}(A)| \leqslant c\big(|A|^p + 1\big) \quad \forall\, A \in \mathbb{R}^{n\times n}. \tag{2}$$

It is easy to see that this equivalence persists if the functional $\mathbf{E}$ satisfies (2) but, instead of being non-negative, satisfies the weaker condition

$$\liminf_{|A|\to+\infty} \frac{\mathbf{E}(A)}{|A|^p} \geqslant 0. \tag{3}$$

However, when a $\mathrm{W}^{1,p}$ quasiconvex functional $\mathbf{E}$ has only the standard p-growth (2) for an exponent $p > 1$, then the variational integral (1) can fail to be $\mathrm{W}^{1,p}$-swlsc, due to possible concentration effects at the boundary [63, 18, 59, 1, 38]. For instance, when $n = p = 2$ an example of this with $\mathbf{E}(A) = \det A$, $A \in \mathbb{R}^{2\times 2}$, can be found in [32]. On the other hand, as observed by Meyers [62], the $\mathrm{W}^{1,p}$-swslc result persists provided the variational integral $\mathscr{E}$ is restricted to Dirichlet classes: if $g \in \mathrm{W}^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$, then for a functional $\mathbf{E}$ satisfying (2), the variational integral $\mathscr{E}$ is $\mathrm{W}^{1,p}$-swlsc on $g + \mathrm{W}^{1,p}_0(\Omega, \mathbb{R}^n)$ if and only if $\mathbf{E}$ is $\mathrm{W}^{1,p}$-quasiconvex. We refer to [22] for further discussions and to [30] for some related results also linking quasiconvexity to coercivity of $\mathscr{E}$ in Dirichlet classes.

In the absence of the standard p-growth condition (2) the problem of $\mathrm{W}^{1,p}$-swlsc is much harder, little can be said in the general case and most results in the literature require some additional assumptions for their validity. In fact already the definition of the variational integral $\mathscr{E}[u]$ is unclear at this level of generality. When $\mathbf{E}$ is allowed to be valued in $\mathbb{R} \cup \{+\infty\}$, or perhaps even in $\mathbb{R} \cup \{\pm\infty\}$, then we have issues already for smooth and compactly supported maps u. The approach we shall follow here is in a sense the naive one, namely to use (1), but with the integral interpreted as an upper Lebesgue integral. We often refer to this as the pointwise definition of the variational integral. The other approach to the definition of the variational integral is to define it via a relaxation process, which in many situations is more natural and often means that the ensuing variational integral has better properties. This approach is often called the Lebesgue-Serrin-Marcellini definition and in the present context goes back to [60]. If we use the pointwise definition of the variational integral (1), then both the $\mathrm{W}^{1,p}$-quasiconvexity and the rank-one convexity conditions remain necessary for $\mathrm{W}^{1,p}$-swlsc, see [18]. However, neither is sufficient. The rank-one convexity does not even imply $\mathrm{W}^{1,\infty}$-quasiconvexity, regardless of growth conditions, by Šverák's counter example [79, 80] except if the mapping is into the target $\mathbb{R}^2$, where the problem remains open, see also [42]. In this connection we refer to [36] and [68] for the particularity of $\mathbb{R}^{2\times 2}$ matrices for this problem and also some partial results. The sufficiency of $\mathrm{W}^{1,p}$-quasiconvexity for $\mathrm{W}^{1,p}$-swlsc of $\mathscr{E}$ on the Dirichlet class $g + \mathrm{W}_0^{1,p}(\Omega, \mathbb{R}^n)$ is an open problem when the functional $\mathbf{E}$ is assumed continuous. Easy counterexamples exist if the functional $\mathbf{E}$ is allowed to be discontinuous and extended real-valued, see [18]. In this situation one introduces a strengthening of the $\mathrm{W}^{1,p}$-quasiconvexity condition, the so-called *closed* $\mathrm{W}^{1,p}$*-quasiconvexity* [73]. It is a sufficient condition for $\mathrm{W}^{1,p}$-swlsc of $\mathscr{E}$ on $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^n)$, provided the functional $\mathbf{E}$ satisfies also (3). However, the necessity becomes unclear, and in fact, when $\mathbf{E}$ is allowed to be lower semicontinuous and extended real-valued, the necessity of closed $\mathrm{W}^{1,p}$-quasiconvexity fails, see [57].

These issues become particularly relevant in dealing with variational problems where the effective domain of the energy functional is contained in the set of matrices with finite distortion $\{K_A < +\infty\}$. We denote

$$\mathbb{R}_+^{n\times n} \equiv \mathbb{R}^{n\times n} \cap \{\det > 0\}, \tag{4}$$

and record that $\{K_A < +\infty\} = \mathbb{R}_+^{n\times n} \cup \{0\}$. Such functionals arise naturally in, for instance, elasticity [14, 15, 16] and in geometric function theory [7], where a common example is the classical second invariant

$$\mathbf{I_2}(A) = K_A + \frac{1}{K_A}, \qquad A \in \mathbb{R}_+^{2\times 2} \cup \{0\}, \tag{5}$$

which is discussed at length in [7, Chapt. 21] and in [8, 61]. However, for applications to elasticity in particular, two central aspects of the theory in this context remain somewhat obscure. Firstly, it is not clear how $\mathbf{E}(A)$ should diverge when $\det(A)$ tends to 0, in order to have a feasible theory for the corresponding variational integral $\mathscr{E}$. Secondly, it seems desirable to rule out pathological deformations (for example homeomorphisms whose determinant is zero a.e or sequences of maps which converge to such maps [48, 35]). Thus it is not obvious what are the appropriate function spaces in which minimisers should be sought, and unfortunately it appears that experiments do not help to clarify these issues partly because we are close to a regime where the elasticity theory breaks down. Let us emphasize that variational problems on the classes of $\mathrm{W}^{1,p}_{\mathrm{loc}}$ homeomorphisms, lead for $p \geqslant 2$ naturally to the so-called topologically monotone maps introduced by Morrey, see [7] and [52, 53], whereas for $p < 2$ a more complicated condition, the no crossing condition, introduced by De Philippis and Pratelli in [33] becomes relevant (see also [28] for the case $p = 1$). For these reasons, and since the general structure of quasiconvex functionals is far from clear, the functionals defined on $\mathbb{R}_+^{n\times n}$ that have been successfully used in the elasticity literature, have mostly been polyconvex. See for example, [16, 17, 31, 41] and particularly [14, 15, 65, 47] for the setting and success of polyconvexity.

Similar obstacles were faced in our recent study [4] of the local Burkholder functional $\mathbf{B}_K^{\mathrm{loc}}$, see (13) and (14) below for the definitions. This functional is real-valued on the K-quasiconformal well

$$Q_2(K) \equiv \{A \in \mathbb{R}^{2\times 2} : K_A \leqslant K\}, \tag{6}$$

while outside the well it is $+\infty$, $\mathbf{B}_K^{\mathrm{loc}}(A) = +\infty$ when $A \notin Q_2(K)$. Because $\mathbf{B}_K^{\mathrm{loc}}$ is non-positive on $Q_2(K)$ it is easy to see that the closed $\mathrm{W}^{1,p}$-quasiconvexity of $\mathbf{B}_K^{\mathrm{loc}}$ is a necessary condition for $\mathrm{W}^{1,p}$-swlsc of the associated variational integral. On the other hand, the sufficiency is now non-obvious. This is exactly the opposite situation of what happens for non-negative extended real-valued functionals.

For later reference we finish this subsection by recalling a rather pleasant property of $\mathrm{W}^{1,1}$-homeomorphisms with integrable distortion: If $f\colon \Omega \to \mathbb{C}$ is a homeomorphism (onto $f(\Omega)$) of class $\mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$ with $K_f \in \mathrm{L}^1(\Omega)$, then its inverse $f^{-1}\colon f(\Omega) \to \Omega$ is of class $\mathrm{W}^{1,2}(f(\Omega))$ and

$$\int_\Omega K_f(z)\,\mathrm{d}m(z) = \int_{f(\Omega)} |\mathrm{D}f^{-1}(w)|^2\,\mathrm{d}m(w). \tag{7}$$

See [7], [48].

1.1. **Principal maps.**

In order to address some of the issues described above we next turn to the role of Stoilow factorization in the setting of lower semicontinuity. Here the key notion is that of a principal map, which has for a long time been implicitly used in geometric complex analysis, for instance in connection with planar quasiconformal mappings and the Beltrami equations; explicitly the term is used in [7]. We require a slightly more flexible definition:

Definition 1.1. A *principal map* is an orientation preserving homeomorphism $f\colon \mathbb{C} \to \mathbb{C}$ of class $\mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{C})$, with integrable distortion K_f on the open unit disk $\mathbb{D}$, which is conformal outside the closed unit disc $\overline{\mathbb{D}}$ with the Laurent expansion

$$f(z) = b_0 z + \sum_{j=1}^{\infty} \frac{b_j}{z^j} \text{ for } |z| > 1,$$

where $|b_1| < |b_0|$.

In connection with the principal map f we often write

$$A_f(z) \equiv b_0 z + b_1 \bar{z} \text{ for } z \in \mathbb{C}, \text{ and } \phi_f(z) \equiv \sum_{j=2}^{\infty} \frac{b_j}{z^j} \text{ for } |z| \geqslant 1.$$

Here the linear map $A_f \in \mathbb{R}^{2\times 2}$ (that we identify with its matrix) has in particular positive determinant $|b_0|^2 - |b_1|^2$, and the function ϕ_f extends continuously to all of $\overline{\mathbb{C}}$, defining a holomorphic map on $\mathbb{C}\backslash\overline{\mathbb{D}}$. Since $1/z = \bar{z}$ for $|z| = 1$, we hereby have $f(z) = A_f(z) + \phi_f(z)$ for $|z| = 1$. We also record that

$$\fint_{\mathbb{D}} \mathrm{D}f(z)\,\mathrm{d}m(z) = A_f. \tag{8}$$

Notice that it is standard to normalize principal maps to have $b_0 = 1$. Here we shall refer to such maps as *normalized principal maps.* It is useful to recall that the class of all normalized $\mathrm{W}^{1,2}$-principal maps forms a normal family (see [7, Theorems 20.1.6 and 20.2.3]); in fact, any $\mathrm{W}^{1,2}_{\mathrm{loc}}$-bounded sequence of such normalized principal maps admits a subsequence that converges uniformly on $\mathbb{C}$ to a normalized principal map. On the other hand, normalized principal maps with the mere $\mathrm{W}^{1,1}_{\mathrm{loc}}$-regularity *do not* form a normal family; a sequence of such Sobolev homeomorphisms can have a BV-limit (see [7, Chapt. 21]).

We also recall that via (7) the inverse of a principal map lies in $\mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{C})$. If the principal map itself lies in $\mathrm{W}^{1,2}_{\mathrm{loc}}$, then the inverse has likewise an integrable distortion.

Our next goal is the definition of $\mathrm{W}^{1,p}$-principal quasiconvexity, which is instrumental for the lower semicontinuity results in this paper. It can be viewed as a refinement, adapted to the setting described above, of the notions of $\mathrm{W}^{1,p}$-quasiconvexity that was introduced by Ball and Murat in [18] and of closed $\mathrm{W}^{1,p}$-quasiconvexity that was introduced by Pedregal in [72]. The condition is motivated by the Stoilow factorization mentioned above.

For one of our key examples it is important that we allow the considered integrands to be signed, so before giving our definition of principal quasiconvexity let us for comparison and precision also review the corresponding definitions of $\mathrm{W}^{1,p}$-quasiconvexity and closed $\mathrm{W}^{1,p}$-quasiconvexity in this context.

A lower semicontinuous functional $\mathbf{E}\colon \mathbb{R}^{2\times 2}_+ \to \mathbb{R}\cup\{+\infty\}$ is said to be $\mathrm{W}^{1,p}$-quasiconvex at $A \in \mathbb{R}^{2\times 2}_+$ provided

$$\mathbf{E}(A) \leqslant {\int\!\!\!\!\!\!-}^{*}_{\mathbb{D}} \mathbf{E}(\mathrm{D}f(z))\,\mathrm{d}m(z) \tag{9}$$

holds for all $f \in A + \mathrm{W}^{1,p}_0(\mathbb{D})$[1]. If this Jensen inequality holds for all $A \in \mathbb{R}^{2\times 2}_+$, then $\mathbf{E}$ is said to be $\mathrm{W}^{1,p}$-quasiconvex.

As observed in [18] this condition depends on the exponent p and it refines Morrey's original definition of quasiconvexity in [63] that in this terminology corresponds to $\mathrm{W}^{1,\infty}$-quasiconvexity. The definition is arguably natural, but it still has some deficiencies. For instance, as observed in [18], an extended real-valued integrand need not be rank-one convex. The next definition strengthens the notion of $\mathrm{W}^{1,p}$-quasiconvexity and does always imply rank-one convexity.

A lower semicontinuous integrand $\mathbf{E}\colon \mathbb{R}^{2\times 2}_+ \to \mathbb{R}\cup\{+\infty\}$ is closed $\mathrm{W}^{1,p}$-quasiconvex at $A \in \mathbb{R}^{2\times 2}_+$ provided

$$\mathbf{E}(A) \leqslant \int^{*}_{\mathbb{R}^{2\times 2}_+} \mathbf{E}\,\mathrm{d}\nu \tag{10}$$

holds for all homogeneous $\mathrm{W}^{1,p}$ gradient Young measures ν with centre of mass A. If it holds for all $A \in \mathbb{R}^{2\times 2}_+$, then $\mathbf{E}$ is said to be closed $\mathrm{W}^{1,p}$-quasiconvex. We refer for the definitions and basic properties of Young measures to the monographs [64, 73, 74]. In particular, we will use the notation and terminology of gradient Young measures as described in our paper [4]. In fact, closed quasiconvexity provides an abstract solution to a closely related lower semicontinuity problem when the

[1] The asterix on the integral signifies that the integral is intended as an upper Lebesgue integral, meaning that when both $\mathbf{E}(\mathrm{D}f)^+$ and $\mathbf{E}(\mathrm{D}f)^-$ integrate to $+\infty$ over $\mathbb{D}$, then the integral is taken to be $+\infty$. The interested reader can verify that this is equivalent to requiring, for each $k \in \mathbb{N}$, the validity of the inequality (9) when $\mathbf{E}$ is replaced by $\max\{\mathbf{E}, -k\}$.

considered integrands are assumed bounded from below, see [57]. We refer to Section 2 and also [4] for further discussion of these quasiconvexity notions. The main drawback is that it is very difficult to verify whether a $W^{1,p}$-quasiconvex integrand is closed quasiconvex unless we have control on the generating sequence of the gradient young measures.

Now we are ready for our definition of principal quasiconvexity.

Definition 1.2. A lower semicontinuous integrand $\mathbf{E}\colon \mathbb{R}^{2\times 2}_+ \to \mathbb{R} \cup \{+\infty\}$ is $W^{1,p}$-*principal quasiconvex* at $A \in \mathbb{R}^{2\times 2}_+$ if the Jensen inequality

$$\mathbf{E}(A) \leqslant \fint^*_{\mathbb{D}} \mathbf{E}(\mathrm{D}f(z))\,\mathrm{d}m(z). \tag{11}$$

holds for all principal maps $f\colon \mathbb{C} \to \mathbb{C}$ of class $W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ with $\fint_{\mathbb{D}} \mathrm{D}f(z)\,\mathrm{d}m(z) = A$. We say the $\mathbf{E}$ is $W^{1,p}$-principal quasiconvex if the Jensen inequality (11) holds for all $A \in \mathbb{R}^{2\times 2}_+$ and for the corresponding $W^{1,p}_{\mathrm{loc}}$-principal maps f.

We have stated an abstract definition depending on p parallel to that of $W^{1,p}$-quasiconvexity and closed $W^{1,p}$-quasiconvexity, however our examples all are $W^{1,1}$-principal quasiconvex (Notice that we are testing the Jensen inequality with gradients of homeomorphism). We also remark that allowing for extended real-valued integrands is a matter of convenience, but it also means that it is easy to construct pathological examples. For example, a $W^{1,1}$-principal quasiconvex integrand need not be rank-one convex: take the integrand that is zero at the two matrices $\mathrm{diag}(1,1)$, $\mathrm{diag}(1,2)$ and $+\infty$ elsewhere. It is unclear if similar examples exist when the functional is required to be continuous.

Often in our lower semicontinuity theorems it is convenient that the integrands $\mathbf{E}$ are defined on the full matrix space $\mathbb{R}^{2\times 2}$ and not merely on $\mathbb{R}^{2\times 2}_+$. We achieve this by simply declaring

$$\mathbf{E}(A) = +\infty \ \text{ on } \ \mathbb{R}^{2\times 2}\backslash\big(\mathbb{R}^{2\times 2}_+ \cup \{0\}\big) \tag{12}$$

and throughout the entire paper this will be our standing assumption for the considered integrands that might not be explicitly defined beyond $\mathbb{R}^{2\times 2}_+$. The reader will notice that we did not specify any value for $\mathbf{E}$ at $0 \in \mathbb{R}^{2\times 2}$. This is on purpose because the 0 matrix is in this context special and it is convenient to have an ad hoc approach here. Sometimes we extend $\mathbf{E}$ to 0 by lower semicontinuity and sometimes we declare that it is $+\infty$ or some other value there.

Next, we compare principal quasiconvexity with the classical notions of semiconvexity in the vectorial calculus of variations. Firstly, it is immediate from Jensen's inequality for convex functions and (8) that any convex functional on $\mathbb{R}^{2\times 2}$ restricts to a $W^{1,1}$-principal quasiconvex functional on $\mathbb{R}^{2\times 2}_+$. Next, notice that if $f \in C^1(\mathbb{D}) \cap W^{1,\infty}(\mathbb{D})$ with $\det \mathrm{D}f > 0$, and for some linear $A \in \mathbb{R}^{2\times 2}_+$ we have $f \in A + W^{1,\infty}_0(\mathbb{D})$, so that f is a classical test function for the $W^{1,\infty}$-quasiconvexity inequality, then f can be extended as principal map off the unit disc. Indeed, if $A(z) = b_0 z + b_1 \bar{z}$, then the map $f^{\mathrm{prin}}(z) \equiv f(z)\mathbf{1}_{\mathbb{D}} + \big(b_0 z + b_1/z\big)\mathbf{1}_{\mathbb{C}\backslash\mathbb{D}}$ is principal in the sense of Definition 1.1. Therefore $W^{1,\infty}$-principal quasiconvexity implies the standard quasiconvexity inequality for sufficiently smooth maps. A standard argument then yields that a $W^{1,\infty}$-principal quasiconvex functional is locally rank-one convex on the interior of its effective domain and therefore that it is locally Lipschitz there (see [32]).

On the other hand, as we shall see in Section 2 below, the classical area formula implies that minus the determinant, $-\det(A)$, is $\mathrm{W}^{1,1}$-principal quasiconvex (although not $\mathrm{W}^{1,p}$-quasiconvex for $1 \leqslant p < 2$) , while the determinant itself, $\det(A)$, is not. This simple example shows, in particular, that polyconvexity, and hence also quasiconvexity, does not imply principal quasiconvexity in general. For general functionals on $\mathbb{R}^{2\times 2}_{+}$ we present an additional local condition, see (31), which together with standard quasiconvexity implies principal quasiconvexity, see Proposition 2.6.

In addition, the class of boundary values of all principal maps, i.e. of all the maps f for which (11) applies, is much larger than just the linear ones. In fact, if Ω is a Jordan domain in $\overline{\mathbb{C}}$ with $\infty \in \Omega$, take a conformal map $\Psi\colon \mathbb{C}\backslash\overline{\mathbb{D}} \to \Omega$ with $\Psi(\infty) = \infty$. Then if the boundary $\partial\Omega$ is sufficiently regular, we can extend Ψ to a principal map of $\overline{\mathbb{C}}$. In particular, we see that the set of boundary values of principal maps on $\partial\mathbb{D}$ coincides with the set of boundary values on $\partial\mathbb{D}$ of all such Riemann maps.

Let us then informally explain why principal quasiconvexity is relevant for establishing lower semicontinuity of variational integrals. The proof of Morrey [63] for the lower semicontinuity of functionals with standard quasiconvex integrands goes basically as follows. Firstly, given a weakly converging sequence (ψ_j) of Sobolev functions, one localises the sequence to achieve convergence to a linear map. Second, one replaces this localized sequence with one of affine boundary values, $u_j - Az \in \mathrm{W}^{1,2}_0(\Omega)$, via a carefully chosen cutoff. And third, one applies directly the definition of quasiconvexity to each such u_j. However, such a strategy breaks down for functionals that are defined and finite only on $\mathbb{R}^{2\times 2}_{+}$, simply because the cut-off modification might give negative determinants on sets of positive measure.

We must therefore approach the lower semicontinuity problem via other methods, for example gradient Young measures and the related closed quasiconvexity. However, to deal with gradient Young measures, and investigate whether quasiconvex functionals are closed quasiconvex in practise one needs information of the generating sequence of the Young measure. It is here that the principal quasiconvexity shows its usefulness. For instance, in [3] one first shows that weakly converging sequences of quasiregular mappings generate gradient Young measures that are supported on the quasiconformal wells $Q_2(K)$, as defined in (6). And second, thanks to Stoilow factorization, the properties of those Young measures can be analysed with the help of principal maps, via the next result from [4]. After these observations, the definition of principal quasiconvexity seems the natural one.

Proposition 1.3. *Consider a homogeneous* $\mathrm{W}^{1,s}$ *gradient Young measure* ν *supported on the* K*-quasiconformal well* $Q_2(K)$*, where the exponent* $2K/(K+1) < s < 2K/(K-1)$*. Then there is a sequence of* K*-quasiconformal principal maps which generates in* $\mathbb{D}$ *the Young measure* ν*.*

For mappings of bounded distortion, aka the quasiconformal maps, the classical Stoilow factorization is among the main tools in the above result. However, looking for similar methods for mappings of finite or integrable distortion, it is the Iwaniec–Šverák version of Stoilow factorization that provides us the following important improvement, see [4, Theorem 4.4].

Proposition 1.4. *Let* ν *be a homogeneous Young measure generated by* $(\mathrm{D}\psi_j)$*, where* (ψ_j) *is a sequence of homeomorphisms that is bounded in* $\mathrm{W}^{1,2}(\mathbb{D})$ *and such that* $\sup_j \int_{\mathbb{D}} K^q_{\psi_j}\, \mathrm{d}m(z) < +\infty$ *for some* $q > 1$*. Then there is a sequence of homeomorphisms* $f_j\colon \mathbb{C} \to \mathbb{C}$ *such that:*

1. *f_j are principal maps;*
2. *(f_j) is bounded in $\mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{D})$ and $(\mathrm{D}f_j)$ generates ν;*
3. *$\psi_j = h_j \circ f_j$ for some conformal maps $h_j\colon f_j(\mathbb{D}) \to \psi_j(\mathbb{D})$.*

We recently proved [5] a more general version of this theorem that applies to sequences of maps (ψ_j) with distortions bounded merely in L^1. In particular we emphasize that the maps ψ_j there need not necessarily be homeomorphisms. The proof has some new ideas compared to [4] and will appear elsewhere.

In order to apply Theorem 1.4 for a functional $\mathbf{E}$, we need that the sequence $\big(\mathbf{E}(\mathrm{D}f_j)\big)$ is equi-integrable. Thus the proofs of results concerning such general functionals, considered in Theorems 1.6 and 1.9 with Corollary 1.7, split naturally into two propositions. The first states that the growth conditions, such as given in (17) below, together with a higher integrability of the distortion functions, guarantee that the sequence $\big(\mathbf{E}(\mathrm{D}f_j)\big)$ is equi-integrable. This rules out concentration effects.

The second proposition, needed in Theorem 1.6, takes care of the oscillation effects and yields lower semicontinuity once equi-integrability is available. As a matter of fact, in [4] a similar issue arose for the important Burkholder functional itself. Fortunately the scope of principal quasiconvexity and Stoilow factorization goes beyond the basic oscillation effects. Indeed, our third result, Theorem 1.5 shows that in the case of the Burkholder functional, principal quasiconvexity allows us to use a blow-up technique to prove lower semicontinuity in the borderline case, where equi-integrability cannot be assumed.

We next describe the results.

1.2. Main results.

To illustrate the uses of principal quasiconvexity we next present three lower semicontinuity theorems, and start with the classical Burkholder functional $\mathbf{B}_p\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$, which for exponents $p \geqslant 2$ is given by

$$\mathbf{B}_p(A) \equiv \left((\frac{p}{2} - 1)|A|^2 - \frac{p}{2}\det A \right)|A|^{p-2}, \qquad A \in \mathbb{R}^{2\times 2}. \tag{13}$$

We note that $\mathbf{B}_2(A) = -\det(A)$, while for $p > 2$, in terms of the quasiconformal well (6), we have $\mathbf{B}_p(A) \leqslant 0$ if and only if $A \in Q_2\big(p/(p-2)\big)$. Thus, fixing $K \equiv p/(p-2)$, or equivalently, $p = 2K/(K-1)$, one is led [4] to consider the *local Burkholder functional* defined as

$$\mathbf{B}^{\mathrm{loc}}_K(A) \equiv \begin{cases} \mathbf{B}_p(A) & \text{if } A \in Q_2(K) \\ +\infty & \text{if } A \in \mathbb{R}^{2\times 2}\backslash Q_2(K). \end{cases} \tag{14}$$

For the origins of the Burkholder functional we refer to [25, 26]. Its relation to Morrey's problem about rank-one and quasiconvexity was first observed by T. Iwaniec and it has since been the subject of intense investigation by a number of authors, including [9, 10, 12, 44, 45, 78, 51]. See also [7, 20, 21, 24, 67, 69] for results about the norm of the Beurling-Ahlfors operator and the Iwaniec conjecture. We emphasize the rather unusual feature of $\mathbf{B}^{\mathrm{loc}}_K$ that it is nonpositive and p-homogeneous on its effective domain $Q_2(K)$.

As is well known, a planar K-quasiregular map lies in $\mathrm{W}^{1,p}_{\mathrm{loc}}$ for every $p < p_K \equiv 2K/(K-1)$, while in general the integrability of the derivative of such a map fails at the borderline exponent p_K. Instead, in general its derivative is locally in the Marcinkiewicz space $\mathrm{L}^{p_K,\infty}$ (see [2]). On the other hand, an interesting result of [9] states that for K-quasiregular maps $f\colon \Omega \to \mathbb{C}$ the function $\mathbf{B}_{p_K}(\mathrm{D}f)$ is locally integrable on Ω, that is,

$$0 \geqslant \int_C \mathbf{B}_{p_K}(\mathrm{D}f)\,\mathrm{d}m(z) > -\infty$$

holds for compact subsets C of Ω. However, even when K-quasiregular maps satisfy $f_j \rightharpoonup f$ weakly in W^{1,p_K} it still is not clear whether $\big(\mathbf{B}_{p_K}(\mathrm{D}f_j)\big)$ is locally equi-integrable. Thus the closed quasiconvexity of $\mathbf{B}_K^{\mathrm{loc}}$, proven in [4, Theorem 1.4], led there to results that correspond to the $\mathrm{W}^{1,q}$-swlsc of $\mathbf{B}_K^{\mathrm{loc}}$ only for $q > p_K$.

In this note we in particular extend, using principal quasiconvexity and Stoilow factorization, the $\mathrm{W}^{1,q}$-swlsc of the local Burkholder functional up to the borderline case $q = p_K$. This is a consequence of a stronger and in a sense more natural result. To explain why our result is natural we fix a bounded open subset Ω of $\mathbb{C}$ and a K-quasiregular map $g\colon \mathbb{C} \to \mathbb{C}$, where $K > 1$. Let $p = p_K$ be the corresponding borderline exponent and consider the variational problem of minimizing the Burkholder energy

$$\mathcal{B}[v] \equiv \int_\Omega \mathbf{B}_p(\mathrm{D}v)\,\mathrm{d}m(z)$$

over all K-quasiregular maps $v \in g + \mathrm{W}_0^{1,2}(\Omega)$. First one notes that from [9] (see also Proposition 3.6 below for a more precise lower bound) it follows that the Burkholder energy $\mathcal{B}[v]$ is bounded below over $v \in g + \mathrm{W}_0^{1,2}(\Omega)$. Next, let $(u_j) \subset g + \mathrm{W}_0^{1,2}(\Omega)$ be a minimizing sequence, so that in particular each $u_j : \Omega \to \mathbb{C}$ is K-quasiregular and, in view of the Dirichlet condition, the sequence (u_j) is bounded in $\mathrm{W}^{1,2}(\Omega)$. By a normality result for K-quasiregular maps [7], (u_j) admits a subsequence (u_{j_k}) that converges uniformly to a K-quasiregular map $u \in g + \mathrm{W}_0^{1,2}(\Omega)$. In order to conclude that u is a minimizer one therefore needs lower semicontinuity of $\mathcal{B}[v]$ along the uniformly convergent subsequence (u_{j_k}). Apart from the *technical assumption* on the limit map u the following result yields exactly that.

Theorem 1.5. *Let $p > 2$, $K \equiv p/(p-2)$, Ω be a domain in $\mathbb{C}$ and suppose u_j, $u\colon \Omega \to \mathbb{C}$ are K-quasiregular maps such that $u_j \to u$ locally uniformly on Ω. If $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$, then*

$$\int_{\Omega'} \mathbf{B}_p(\mathrm{D}u)\,\mathrm{d}m(z) \leqslant \liminf_{j\to\infty} \int_{\Omega'} \mathbf{B}_p(\mathrm{D}u_j)\,\mathrm{d}m(z) \tag{15}$$

holds for each subset $\Omega' \Subset \Omega$ with $\mathscr{L}^2(\partial\Omega') = 0$.

Since any locally uniform limit u of a sequence (u_j) of K-quasiregular maps is itself K-quasiregular, we see from [2] that the limit function u in Theorem 1.5 automatically has the Marcinkiewicz-regularity $\mathrm{D}u \in \mathrm{L}^{p,\infty}_{\mathrm{loc}}$. It remains open if merely this suffices in Theorem 1.5.

We refer to Corollary 3.7 for a global version of the Burkholder lower semicontinuity result on Dirichlet classes. Since we do not know if the sequence $\big(\mathbf{B}_p(\mathrm{D}u_j)\big)$ in Theorem 1.5 is locally equi-integrable, we need to handle the concentration and oscillation phenomena simultaneously. In this setting we will actually prove that the singular part of the reduced defect measure for the sequence $\big(\mathbf{B}_p(\mathrm{D}u_j)\big)$ is zero.

We next consider lower semicontinuity results for general principal quasiconvex functionals, which are consistent with the blow up condition

$$\mathbf{E}(A) \text{ diverges as } \det A \to 0^+. \tag{16}$$

For simplicity we restrict attention here to energy densities satisfying the following growth conditions: There exist $C > 0$ and $p \in [1, 2)$ such that

$$\begin{cases} \big|\mathbf{E}(A)\big| \leqslant C \max\big\{|A|^p, -\log(\det A), K_A\big\} + C \ \text{ on } \ \mathbb{R}^{2\times 2}_+, \\ \\ \mathbf{E}(A) = +\infty \ \text{ on } \ \mathbb{R}^{2\times 2}\backslash\big(\mathbb{R}^{2\times 2}_+ \cup \{0\}\big). \end{cases} \tag{17}$$

Note we do not impose any condition for the value of $\mathbf{E}$ at 0 and that the functional $\mathbf{W}$ defined at (27) satisfies (17) with $p = 1$. With these bounds we have:

Theorem 1.6. *Let $\mathbf{E}\colon \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{+\infty\}$ be Borel measurable and satisfy (17). Assume that $\mathbf{E}$ is $\mathrm{W}^{1,1}$ principal quasiconvex. Fix an exponent $q > \frac{p}{2-p}$ and let $g\colon \mathbb{C} \to \mathbb{C}$ be a homeomorphism of class $\mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{C})$ with distortion $K_g \in \mathrm{L}^q_{\mathrm{loc}}(\mathbb{C})$.*

Let (u_j) be a sequence of homeomorphisms in $g + \mathrm{W}^{1,2}_0(\Omega)$ with $\sup_j \|K_{u_j}\|_{\mathrm{L}^q(\Omega)} < \infty$ and suppose that $u_j \rightharpoonup u$ in $\mathrm{W}^{1,2}(\Omega)$. Then $u \in g + \mathrm{W}^{1,2}_0(\Omega)$ is a homeomorphism with $K_u \in \mathrm{L}^q(\Omega)$ and

$$\liminf_{j\to\infty} \int_\Omega \mathbf{E}(\mathrm{D}u_j(z))\,\mathrm{d}m(z) \geqslant \int_\Omega \mathbf{E}(\mathrm{D}u(z))\,\mathrm{d}m(z).$$

This result yields in a standard way an existence result for a variational problem related to hyperelasticity:

Corollary 1.7. *Let $\mathbf{E}\colon \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{+\infty\}$ be Borel measurable, satisfy (17) and be $\mathrm{W}^{1,1}$-principal quasiconvex. Suppose $\mathbf{P}\colon \mathbb{R}^{2\times 2} \to \mathbb{R}\cup\{+\infty\}$ is lower semicontinuous, polyconvex and satisfies the coercivity condition*

$$\mathbf{P}(A) \geqslant c\big(|A|^2 + K_A^q\big) \tag{18}$$

for all $A \in \mathbb{R}^{2\times 2}$, where $c > 0$, $q > \frac{p}{2-p}$ are constants. For a homeomorphism $g\colon \mathbb{C} \to \mathbb{C}$ of class $\mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{C})$ and with $\mathbf{E}(\mathrm{D}g) + \mathbf{P}(\mathrm{D}g) \in \mathrm{L}^1(\Omega)$ we put

$$\mathcal{A}_g \equiv \Big\{ u \in g + \mathrm{W}^{1,2}_0(\Omega) : u\colon \Omega \to g(\Omega) \ \text{homeomorphism} \Big\}.$$

Then, for each $F \in \mathrm{W}^{-1,2}(\Omega) \equiv \mathrm{W}^{1,2}_0(\Omega)^*$, the variational problem

$$\inf_{u\in\mathcal{A}_g} \left(\int_\Omega \big(\mathbf{E}(\mathrm{D}u) + \mathbf{P}(\mathrm{D}u)\big)\,\mathrm{d}m(z) + \big\langle F, u \big\rangle \right)$$

admits a minimizer $u \in \mathcal{A}_g$ with $K_u \in \mathrm{L}^q(\Omega)$.

Our last result provides lower semicontinuity for principal quasiconvex functionals along sequences of maps that have asymptotically finite distortion in the following sense.

Definition 1.8. Let (u_j) be a sequence in $\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega)$ and $\mathbb{A}$ be a measurable subset of Ω. We say that the maps have *asymptotically finite distortion* on $\mathbb{A}$ if

$$\limsup_{j\to\infty} \mathbf{K}(\mathrm{D}u_j(x)) < +\infty \quad \text{ for a.e. } x \in \mathbb{A}. \tag{19}$$

We then present a general lower semicontinuity result for sequences with asymptotically finite distortion. We emphasize that the mappings do not necessarily have finite distortion everywhere, but that the distortion condition is merely required on the measurable set $\mathbb{A}$.

Theorem 1.9. *Let* $\mathbf{E}\colon \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{+\infty\}$ *be lower semicontinuous and* $\mathrm{W}^{1,2}$*-principal quasiconvex, with* $\mathbf{E} = +\infty$ *on* $\mathbb{R}^{2\times 2}\backslash\big(\mathbb{R}^{2\times 2}_+ \cup \{0\}\big)$*, and assume that*

$$\big|\mathbf{E}(A)\big| \leqslant \mathbf{C}(K_A)\big(|A|^2 + 1\big), \quad \forall A \in \mathbb{R}^{2\times 2}_+, \tag{20}$$

where $\mathbf{C}\colon [1,\infty) \to [1,\infty)$ *is an increasing function. Let* $\mathbb{A}$ *be a measurable subset of* Ω*.*

Then, if $(u_j) \subset \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega)$ *has asymptotically finite distortion on* $\mathbb{A}$*, if the sequence converges weakly to* u *in* $\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega)$ *and if*

$$\big(\mathbf{E}(\mathrm{D}u_j)^-\big) \text{ is equi-integrable on } \mathbb{A}, \tag{21}$$

then we have

$$\liminf_{j\to\infty} \int_{\mathbb{A}} \mathbf{E}(\mathrm{D}u_j)\,\mathrm{d}m(z) \geqslant \int_{\mathbb{A}} \mathbf{E}(\mathrm{D}u)\,\mathrm{d}m(z).$$

Remark 1.10. Notice that we have not imposed any condition on the value of $\mathbf{E}$ at 0, so any value for $\mathbf{E}(0)$ that is compatible with lower semicontinuity of $\mathbf{E}$ is permitted in Theorem 1.9.

In this introduction we have stated the homogeneous versions of the lower semicontinuity estimates, but in the bulk of the text we will deal with general non-autonomous z-dependent versions of the integrands. Although we have stated the results for planar maps, the reader interested in applications could think of higher dimensional plane strains, which often can be modeled by two-dimensional deformations [19, 70, 71]. For example this could be relevant for isochoric Mooney-Rivlin materials altered in a quasiconvex but non-polyconvex way; see [23, 76] and the references therein. We remark that our results in particular yield semicontinuity results for strain energies that decompose as a sum of a polyconvex energy density and a principally quasiconvex energy density. This is particularly relevant for a class of rank-one convex functionals with an additive isochoric-volumetric split, see [81, 82] and [4, Sect. 11]. We also remark that under natural coercivity conditions in terms of either the distortion or the second invariant one obtains existence of minimizers subject to, for instance, Dirichlet boundary conditions. See [4, Sect. 12].

Before we embark on the proofs, we discuss and present examples of principal quasiconvex functionals in section 2. We start by recalling our earlier work on the Burkholder functional and isotropic functionals that admit a volumetric isochoric split. Additionally, we show that for any functional depending only on distortion, principal quasiconvexity is equivalent to rank-one convexity and to polyconvexity. We then investigate the natural question of which additional conditions, on top of quasiconvexity, imply principal quasiconvexity. Here, Proposition 2.6 is the main technical result. The proof originates from [4], utilizing the theory of quadrature domains, but in this work we deal more efficiently with concentration. This part combines the approximation result [49, 27] together with some careful manipulations with Orlicz functions in order to transfer equiintegrability from K_f to $\log(1+\frac{1}{J_f})$, in analogy with the L^p estimates of Koskela and Onninen. In addition to natural L^1 type growth conditions (30), condition (31) upgrades $\mathrm{W}^{1,\infty}$-quasiconvexity to $\mathrm{W}^{1,1}$-principal quasiconvexity. The condition (31) is not difficult to verify but requires to test on all conformal maps. Therefore we present stronger local conditions easier to verify. Namely under growth condition (30), any $\mathrm{W}^{1,\infty}$-quasiconvex functional $\mathbf{E}$, such that

$$\mathbb{C}\backslash\{0\} \ni w \mapsto \mathbf{E}(wA) \text{ is locally concave}, \tag{22}$$

is $\mathrm{W}^{1,1}$-principal quasiconvex. For an isotropic functional we give a more precise condition in Lemma 2.7. We end the examples section with a brief discussion of the so-called complex Burkholder functionals introduced in [11]. It is interesting to note that these functionals have a different symmetry group, but still the arguments persist in this wider context.

In section 3 we describe the proofs for the semicontinuity and existence results. The proofs of Theorems 1.6 and 1.9 follow a well-known strategy. First we use Young measures for the localization, we then deal with the homogeneous gradient Young measures via Stoilow factorization, and finally we apply principal quasiconvexity. The localization for the case of asymptotic finite distortion is however rather delicate, and involves a number of technical steps and results, where for instance the 0-1 law from [3] is used. The proof of Theorem 1.5 is more subtle as we cannot rely on gradient Young measure theory. Instead we analyze the limit measure of the L^1-bounded sequence $\mathbf{B}_p(\mathrm{D}u_j)$ and localize differently at points where the limit measure is absolutely continuous with respect to Lebesgue measure, and at points where it is singular. This blow up technique adjusts very well to principal quasiconvexity, though to treat the singular part we need to assume critical integrability for the limit map (not for the sequence though). Finally, we gather all results and prove in this section the results of existence of minimizers. The reader not interested in proofs can directly check our local conditions and apply the existence theorem.

2. **Examples of principal quasiconvex functionals.** In this section we discuss and give examples of functionals satisfying the principal quasiconvexity condition of Definition 1.2. Some of the examples are better understood using complex notation for matrices $A \in \mathbb{R}^{2\times 2}$. Recall that this amounts to identifying A with $(a_+, a_-) \in \mathbb{C}^2$, where, in terms of the usual identification of $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$ with $\omega = \omega_1 + \mathrm{i}\omega_2 \in \mathbb{C}$,

$$A\omega = a_+\omega + a_-\overline{\omega}. \tag{23}$$

Accordingly we get with $A^+ \equiv (A + \mathrm{cof}A)/2$, $A^- \equiv (A - \mathrm{cof}A)/2$ that $A^+ = (a_+, 0)$, $A^- = (0, a_-)$ and this is the reason for calling (a_+, a_-) the conformal-anticonformal coordinates of A. In this connection, we also record that $(\rho \mathrm{e}^{\mathrm{i}\theta}, 0) = \rho R_\theta$, where $R_\theta \in \mathrm{SO}(2)$ represents the (anti-clockwise) θ-angle rotation about the origin. Similarly, the complex conjugate mapping, $\omega \mapsto \overline{\omega}$, is identified with the reflection $\mathrm{diag}(1, -1) = (0, 1)$.

For later use, we also recall that for $A \in \mathbb{R}^{2\times 2}$ we have the identities: $|A| = |a_+| + |a_-|$, $J_A = |a_+|^2 - |a_-|^2$. When $A \in \mathbb{R}^{2\times 2}_+$ we have $K_A = (|a_+| + |a_-|)/(|a_+| - |a_-|) = (1 + |\mu_A|)/(1 - |\mu_A|)$, where the ratio $\mu_A \equiv a_-/a_+$ is called the complex dilation of A.

A functional $\mathbf{E}\colon \mathbb{R}^{2\times 2}_+ \to \mathbb{R} \cup \{+\infty\}$ is isotropic provided $\mathbf{E}(QAR) = \mathbf{E}(A)$ holds for all $A \in \mathbb{R}^{2\times 2}_+$ and Q, $R \in \mathrm{SO}(2)$. There are a number of equivalent ways to characterize isotropy and here we emphasize that both $A \mapsto J_A$ and $A \mapsto K_A$ are isotropic. These two functionals can be considered the building blocks for all isotropic functionals in the sense that one can show $\mathbf{E}$ to be isotropic precisely when $\mathbf{E}(A) = E(K_A, J_A)$ holds for all $A \in \mathbb{R}^{2\times 2}_+$ for some extended real-valued function $E\colon (0, \infty)^2 \to \mathbb{R} \cup \{+\infty\}$. Below we shall be considering subclasses of the isotropic functionals that are naturally expressed in such terms. Finally, it will be convenient on a few occasions to use the short-hand zA for the matrix $\rho R_\theta A = (z, 0)(a_+, a_-) = (za_+, za_-)$, where $z = \rho\mathrm{e}^{\mathrm{i}\theta}$. For later reference we also record the general multiplication rule for matrices $A = (a_+, a_-)$ and $B = (b_+, b_-)$

in complex notation

$$AB = (a_+, a_-)(b_+, b_-) = \big(a_+b_+ + a_-\overline{b_-}, a_+b_- + a_-\overline{b_+}\big). \tag{24}$$

A first simple class of examples is furnished by the following result:

Proposition 2.1. *Assume $P\colon \mathbb{R}^{2\times 2}\times\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ is an extended real-valued, lower semicontinuous and convex function for which the partial function $t\mapsto P(A,t)$ is non-increasing on $[0,|A|^2]$ for each fixed $A\in\mathbb{R}^{2\times 2}$. Then the functional $\mathbf{P}(A)\equiv P(A,J_A)$, $A\in\mathbb{R}^{2\times 2}_+$, is $\mathrm{W}^{1,1}$-principal quasiconvex.*

Proof. Let $f\colon\mathbb{C}\to\mathbb{C}$ be an orientation-preserving homeomorphism of class $\mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{C})$, which is conformal on $\mathbb{C}\backslash\overline{\mathbb{D}}$ with the Laurent expansion

$$f(z) = b_0 z + \frac{b_1}{z} + \sum_{n=2}^{\infty}\frac{b_n}{z^n} \text{ for } |z|>1. \tag{25}$$

Note that a principal map in the sense of our Definition 1.1 in particular will satisfy these conditions. According to the area formula (see [7, Th. 2.10.1 & Cor. 2.10.2]) we have, in terms of the notation introduced after Definition 1.1, that

$$\fint_{\mathbb{D}} J_f\,\mathrm{d}m(z) \leqslant \det A_f - \sum_{n=2}^{\infty} n|b_n|^2, \tag{26}$$

and therefore in particular, $0\leqslant \fint_{\mathbb{D}} J_f\,\mathrm{d}m(z)\leqslant \det A_f$. We employ this inequality in conjunction with Jensen's inequality whereby

$$\fint_{\mathbb{D}} \mathbf{P}(\mathrm{D}f)\,\mathrm{d}m(z) \geqslant P\left(A_f, \fint_{\mathbb{D}} J_f\,\mathrm{d}m(z)\right).$$

Here we have $0\leqslant \fint_{\mathbb{D}} J_u\,\mathrm{d}m(z)\leqslant \det A_f\leqslant |A_f|^2$, where the last inequality is Hadamard's. We may now conclude with the desired inequality if we use that $t\mapsto P(A_f,t)$ is non-increasing on $[0,|A_f|^2]$. □

We record that the above argument also illustrates why $A\mapsto J_A$ is not $\mathrm{W}^{1,\infty}$-principal quasiconvex. Indeed the area inequality (26) immediately implies that $\fint_{\mathbb{D}} J_f\,\mathrm{d}m(z) < \det A_f$ whenever f is a $\mathrm{W}^{1,\infty}_{\mathrm{loc}}$ principal map where at least one of the coefficients $b_n\neq 0$ for $n\geqslant 2$ in its Laurent expansion (25). Prime examples of polyconvex functionals that are also $\mathrm{W}^{1,1}$-principal quasiconvex are, minus the determinant, $-J_A$, and the distortion functional $\mathbf{K}(A)\equiv K_A$. It is well-known from [18] that $A\mapsto -J_A$ is $\mathrm{W}^{1,p}$-quasiconvex if and only if $p\geqslant 2$, so the fact that we in the definition of $\mathrm{W}^{1,1}$-principal quasiconvexity only test with special $\mathrm{W}^{1,1}$ maps is essential. See also [66] for further discussion of the $\mathrm{W}^{1,p}$-quasiconvexity condition that involves the distributional Jacobian.

The second invariant $\mathbf{I_2}(A) = K_A + 1/K_A$ is also simultaneously polyconvex and $\mathrm{W}^{1,1}$-principal quasiconvex. This is a consequence of the following result concerning so-called isochoric, or conformally invariant, free energy functionals, meaning those isotropic functionals that are also homogeneous of degree 0. The canonical example of an isochoric functional is the distortion $A\mapsto K_A$ and it can be shown that $\mathbf{H}\colon\mathbb{R}^{2\times 2}_+\to\mathbb{R}\cup\{+\infty\}$ is isochoric precisely when $\mathbf{H}(A) = H(K_A)$ for all $A\in\mathbb{R}^{2\times 2}_+$, where $H\colon[1,\infty)\to\mathbb{R}\cup\{+\infty\}$. For instance, see [81] and the references therein. This class of energy functionals and the associated variational problems have been discussed by a number of authors, including [7, Chap. 21], [8], [61] and [81, 82, 83].

Proposition 2.2. *Let* $H\colon [1,\infty) \to \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous and extended real-valued function. Define* $\mathbf{H}(A) \equiv H(K_A)$*,* $A \in \mathbb{R}^{2\times 2}_+$*. Then the following are equivalent:*

(i) $\mathbf{H}$ *is rank-one convex;*
(ii) $\mathbf{H}$ *is* $\mathrm{W}^{1,1}$*-principal quasiconvex;*
(iii) $\mathbf{H}$ *is polyconvex;*
(iv) H *is convex and non-decreasing.*

Proof. The pairwise equivalences of (i), (iii) and (iv) follow from [81]. To see the equivalence to (ii) we first observe that an argument similar to the one used in the proof of Proposition 2.1 yields (ii) from (iv). We therefore conclude the proof if we can show that (i) follows from (ii). Assume that (ii) holds and fix matrices A_0, $A_1 \in \mathbb{R}^{2\times 2}_+$ that are rank-one connected. Put for a weight $\lambda \in (0,1)$, $A_\lambda = (1-\lambda)A_0 + \lambda A_1$. We can without loss in generality assume that $\mathbf{H}(A_0)$, $\mathbf{H}(A_1) < +\infty$. Note that $A_\lambda \in \mathbb{R}^{2\times 2}_+$ and $1 \leqslant K_{A_i} < +\infty$, hence by rank-one convexity of the distortion, $K_{A_\lambda} \leqslant (1-\lambda)K_{A_0} + \lambda K_{A_1} \leqslant K \equiv \max\{K_{A_0}, K_{A_1}\} < +\infty$. Write $A_1 - A_0 = a \otimes n$ with $|n| = 1$, take $\theta \in \mathrm{SO}(2)$ with $\theta e_1 = n$, denote $Q = (0,1)^2$ and $R = \theta\big(Q\big)$. Define, in terms of real notation,

$$v(z) \equiv \begin{cases} (A_0 - A_\lambda)z + (1-\lambda)a & \text{in } R_0, \\ (A_1 - A_\lambda)z & \text{in } R_1, \end{cases}$$

where $R_i = \theta\big(Q_i\big)$, $Q_0 = (0, 1-\lambda) \times (0,1)$, $Q_1 = (1-\lambda, 1) \times (0,1)$. Observe that hereby $v\colon R \to \mathbb{R}^2$ is a Lipschitz map that is constant in the direction perpendicular to n and $v\big(\theta(te_2)\big) = v\big(n + \theta(te_2)\big)$ for $t \in (0,1)$. If therefore we extend v to $\mathbb{R}^2$ by R-periodicity, then we obtain a Lipschitz and R-periodic map that we again denote $v\colon \mathbb{R}^2 \to \mathbb{R}^2$. By inspection

$$\mathrm{D}v = \begin{cases} A_0 - A_\lambda & \text{in } R_0, \\ A_1 - A_\lambda & \text{in } R_1, \end{cases}$$

hence $\int_R \mathrm{D}v \,\mathrm{d}m(z) = 0$. Put $u_j(z) \equiv A_\lambda z + \frac{1}{j}v(jz)$, $z \in \mathbb{D}$, and record that u_j is Lipschitz with

$$\mathrm{D}u_j = \begin{cases} A_0 & \text{in } R_{0,j}, \\ A_1 & \text{in } R_{1,j}, \end{cases}$$

where the disjoint sets $R_{0,j}$, $R_{1,j} \subset \mathbb{D}$ satisfy $\mathscr{L}^2(R_{0,j}) = (1-\lambda)\pi + \mathrm{o}(1)$, $\mathscr{L}^2(R_{1,j}) = \lambda\pi + \mathrm{o}(1)$. In particular, $K_{u_j} \leqslant K$, so u_j is K-quasiregular and we may Stoilow factorize it: $u_j = h_j \circ f_j$ on $\mathbb{D}$, where $f_j\colon \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal normalized principal map with $K_{f_j} = K_{u_j}$ and $h_j\colon f_j(\mathbb{D}) \to \mathbb{C}$ is holomorphic. It follows from the Riemann-Lebesgue lemma that $u_j \overset{*}{\rightharpoonup} A_\lambda$ in $\mathrm{W}^{1,\infty}(\mathbb{D})$ and so by uniqueness of the Stoilow factorization we find that $f_j \to f$ uniformly on $\mathbb{C}$ and weakly in $\mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{C})$ and $h_j \to h$ locally uniformly on $f(\mathbb{D})$. Here we have in terms of $A_\lambda = (a_{\lambda,+}, a_{\lambda,-})$ and in complex notation,

$$f(z) = \begin{cases} z + \frac{a_{\lambda,-}}{a_{\lambda,+}}\bar{z} & \text{if } |z| \leqslant 1, \\ z + \frac{a_{\lambda,-}}{a_{\lambda,+}}\frac{1}{z} & \text{if } |z| > 1, \end{cases} \qquad \text{and} \qquad h(z) = a_{\lambda,+}z.$$

Because

$$K_{f_j} = K_{u_j} = \begin{cases} K_{A_0} \text{ on } R_{0,j}, \\ K_{A_1} \text{ on } R_{1,j}, \end{cases}$$

we get

$$\fint_{\mathbb{D}} \mathbf{H}(\mathrm{D}f_j)\,\mathrm{d}m(z) = (1-\lambda)\mathbf{H}(A_0) + \lambda\mathbf{H}(A_1) + \mathrm{o}(1)$$

and here the left-hand side is estimated using $\mathrm{W}^{1,1}$-principal quasiconvexity,

$$\mathbf{H}\left(\fint_{\mathbb{D}} \mathrm{D}f_j \,\mathrm{d}m(z)\right) \leqslant \fint_{\mathbb{D}} \mathbf{H}(\mathrm{D}f_j)\,\mathrm{d}m(z).$$

Because $\fint_{\mathbb{D}} \mathrm{D}f_j \,\mathrm{d}m(z) \to A_f = A_\lambda / a_{\lambda,+}$ and since $K_{A_\lambda/a_{\lambda,+}} = K_{A_\lambda}$ we may use the lower semicontinuity of H to conclude. □

Next we turn to the class of free energy functionals that are isotropic and admit an additive isochoric-volumetric split, meaning they must have the general form

$$\mathbf{E}(A) = H(K_A) + G(J_A), \quad A \in \mathbb{R}^{2\times 2}_+,$$

where H and G are extended real-valued functions. Such free energy functionals have been considered by many authors and are often used to model slightly compressible materials, see for instance [37, 29, 46, 70]. Whereas the case of rank-one convexity for the individual terms $H(K_A)$ and $G(J_A)$ is characterized by Propositions 2.2 and 2.1, respectively, the rank-one convexity of $\mathbf{E}(A)$ is more involved and does not necessarily mean that both $H(K_A)$ and $G(J_A)$ are rank-one convex. The precise conditions have been worked out in the interesting paper [81, Th. 2.6] and following [82] we shall here focus on the special case when the isochoric part is rank-one convex. In that situation it was shown in [82] (see also [4, Sect. 11]) that $\mathbf{E}$ is rank-one convex if and only if there exists an isotropic polyconvex functional $\mathbf{P}$ and a non-negative constant $c \geqslant 0$ such that $\mathbf{E} = \mathbf{P} + c\mathbf{W}$, where

$$\mathbf{W}(A) \equiv K_A - \log K_A + \log J_A \ \text{ for } A \in \mathbb{R}^{2\times 2}_+, \tag{27}$$

as observed in [82], is an isotropic rank-one convex, non-polyconvex functional. Indeed it cannot be polyconvex because $\mathbf{W}(tA) \to -\infty$ as $t \to 0$. If we follow the convention (12) of this paper and extend $\mathbf{W}$ to full matrix space $\mathbb{R}^{2\times 2}$ and at the same time insist that the extension be lower semicontinuous, then we must define

$$\mathbf{W}(A) = \begin{cases} K_A - \log K_A + \log J_A & \text{if } A \in \mathbb{R}^{2\times 2}_+, \\ -\infty & \text{if } A = 0, \\ +\infty & \text{if } A \in \mathbb{R}^{2\times 2} \setminus \big(\mathbb{R}^{2\times 2}_+ \cup \{0\}\big). \end{cases}$$

We are therefore forced to consider extended real-valued integrands that assume both $\pm\infty$. This example, together with the local Burkholder functional (14), are our reasons for adopting an ad hoc approach to the value at 0 when extending functionals from $\mathbb{R}^{2\times 2}_+$ to $\mathbb{R}^{2\times 2}$. We remark that this extension of $\mathbf{W}$ remains isotropic and rank-one convex (with the natural definitions, see for instance [4]). The following result is proved in [4, Cor. 1.7, Th. 1.8 & Th. 1.9]. We emphasize that it in particular yields a positive solution to Morrey's problem within the considered class of functionals.

Theorem 2.3. *Let $\mathbf{E}(A) = H(K_A) + G(J_A)$, $A \in \mathbb{R}^{2\times 2}_+$, where $H\colon [1,\infty) \to \mathbb{R} \cup \{+\infty\}$ and $G\colon (0,\infty) \to \mathbb{R} \cup \{+\infty\}$ are lower semicontinuous extended real-valued functions. Assume H is convex. Then $\mathbf{E}$ is rank-one convex if and only if $\mathbf{E}$ is $\mathrm{W}^{1,2}$-quasiconvex.*

Furthermore, the functional $\mathbf{W}$ from (27) is $\mathrm{W}^{1,1}$-principal quasiconvex.

We return to the Burkholder functional $\mathbf{B}_p\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ that was defined for exponents $p \geqslant 2$ at (13) in the introduction. In complex notation it is given by

$$\mathbf{B}_p(A) = \big((p-1)|a_-| - |a_+|\big)\big(|a_+| + |a_-|\big)^{p-1} \tag{28}$$

for a matrix $A \in \mathbb{R}^{2\times 2}$ that has conformal-anticonformal coordinates $(a_+, a_-) \in \mathbb{C}^2$. It is isotropic, p-homogeneous and rank-one convex, but it is an open question whether it is also $\mathrm{W}^{1,\infty}$-quasiconvex. Its quasiconvexity would among other things provide an alternative approach to the optimal higher integrability properties of quasiconformal maps [2], see [51] where the reader can also find interesting extensions to higher dimensions $n > 2$. In the recent work, [4], building on [9, 10], it was shown that the Burkholder functional is quasiconvex when restricted to quasiconformal test functions with appropriate distortion. The precise result is conveniently stated in terms of the local Burkholder functional.

Theorem 2.4. *Let $p > 2$, $p' = p/(p-1)$ and $K = p/(p-2)$. The local Burkholder functional $\mathbf{B}_K^{\mathrm{loc}}$ defined at (14) is simultaneously $\mathrm{W}^{1,1}$-principal quasiconvex, $\mathrm{W}^{1,p'}$-quasiconvex and closed $\mathrm{W}^{1,q}$-quasiconvex for each $q > p'$.*

We refer to [4, Th. 1.4 & Th. 1.6] for the proofs. Indeed, as quasiconformal homeomorphisms of Sobolev class $\mathrm{W}^{1,1}_{\mathrm{loc}}$ are automatically [2] in $\mathrm{W}^{1,2}_{\mathrm{loc}}$ we see that the local Bukholder functional is actually $\mathrm{W}^{1,1}$-principal quasiconvex. However, if we test with the staircase laminate introduced in [34, Example 5.4] we see that it is not closed $\mathrm{W}^{1,q}$-quasiconvex for $q < \frac{2K}{K+1}$. Moreover if we test with the map obtained in [6, Theorem 3.2] we see that the local Burkholder functional is not even $\mathrm{W}^{1,q}$-quasiconvex for $q < \frac{2K}{K+1}$.

We have already noted that $\mathbf{B}_K^{\mathrm{loc}}$ is non-positive on its effective domain which is the K-quasiconformal well $Q_2(K)$. Indeed, it is 0 on its boundary and strictly negative in its interior. If we combine Theorem 2.4 with Jensen's inequality for convex functions we deduce the following corollary.

Corollary 2.5. *Let $p > 2$, $p' = p/(p-1)$ and $K = p/(p-2)$. Assume $\theta\colon (-\infty, 0) \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous convex and non-decreasing function satisfying $\theta(t) \to +\infty$ as $t \to 0-$. Then the functional $\mathbf{B}_p^\theta\colon \mathbb{R}^{2\times 2} \to \mathbb{R}\cup\{+\infty\}$ defined by*

$$\mathbf{B}_p^\theta(A) \equiv \begin{cases} \theta\big(\mathbf{B}_p(A)\big) & \text{if } A \in \mathbb{R}^{2\times 2}_+ \text{ and } K_A < K, \\ +\infty & \text{if } A \in \mathbb{R}^{2\times 2} \text{ and } A = 0 \text{ or } K_A \geqslant K, \end{cases}$$

is $\mathrm{W}^{1,p'}$-principal quasiconvex, $\mathrm{W}^{1,p'}$-quasiconvex and closed $\mathrm{W}^{1,q}$-quasiconvex for each $q > p'$.

We leave the elementary verification to the interested reader. We shall exhibit one particular example from this and start by recalling that $\mathbf{B}_p$ is isotropic. If therefore we restrict $\mathbf{B}_p(A)$ to $A \in \mathbb{R}^{2\times 2}_+$ it can be expressed in terms of the distortion and the Jacobian of A, where, if we assume $p > 2$ and keep writing $K = p/(p-2)$, it becomes

$$\mathbf{B}_p(A) = \frac{p-2}{2}\big(K_A - K\big)K_A^{\frac{p-2}{2}} J_A^{\frac{p}{2}}, \quad A \in \mathbb{R}^{2\times 2}_+.$$

It clearly does not admit an additive isochoric-volumetric split. However, it does admit a multiplicative isochoric-volumetric split as

$$\mathbf{B}_p(A) = \Phi(K_A)\Psi(J_A),$$

where $\Phi(s) = \frac{p-2}{2}(s-K)s^{\frac{p-2}{2}}$, $s \geqslant 1$, and $\Psi(t) = t^{\frac{p}{2}}$, $t > 0$. It is interesting to note that both Φ and Ψ are increasing and convex on their respective domains. We

[2]This is classical. A proof follows by applying, for example [7, Corollary 3.3.6], to get that the Jacobian is in L^1 and deduce the square integrability of the full derivative by the distortion inequality

exploit this as follows and note first that $\Phi(s) < 0$ for $s \in [1, K)$ corresponding to $\mathbf{B}_p(A) < 0$ on the interior of the K-quasiconformal well. We may therefore define the functional $\mathbf{B}_p^\theta$ corresponding to $\theta(t) = -\log(-t)$, $t < 0$, whereby the functional

$$\mathbf{L}_p(A) \equiv \mathbf{B}_p^\theta(A) = \begin{cases} -\log\big(-\Phi(K_A)\big) - \log\big(\Psi(J_A)\big) & \text{if } A \in \mathbb{R}_+^{2\times 2} \text{ and } K_A < K, \\ +\infty & \text{otherwise,} \end{cases} \tag{29}$$

results and it clearly has the additive isochoric-volumetric split form. By virtue of Propositions 2.1 and 2.2 both the isochoric and volumetric parts are polyconvex, so $\mathbf{L}_p$ is in particular polyconvex too.

The following proposition gives, modulo a one-sided growth condition, a local condition for a quasiconvex function to be automatically principal quasiconvex. The interested reader will see that the statement in particular applies to the functional $\mathbf{W}$ defined at (27).

Proposition 2.6. *Let $\mathbf{E}\colon \mathbb{R}_+^{2\times 2} \to \mathbb{R}$ be $\mathrm{W}^{1,\infty}$-quasiconvex. Suppose that*

$$\mathbf{E}(A) \leqslant c\left(1 + K_A + \log\left(1 + \frac{1}{J_A}\right) + J_A^\alpha\right) \tag{30}$$

and that the composite function

$$\Omega \ni z \mapsto \mathbf{E}(h'(z)A) \ \textit{ is superharmonic} \tag{31}$$

for all $A \in \mathbb{R}_+^{2\times 2}$ and all conformal maps $h\colon \Omega \subset \mathbb{C} \to \mathbb{C}$, where $c > 1$ and $\alpha \in (0,1)$ are constants. Then $\mathbf{E}$ is $\mathrm{W}^{1,1}$-principal quasiconvex.

One may give versions of this result also for functionals defined on quasiconformal wells and that are modelled on the local Burkholder functionals; we intend to report on that elsewhere. It is possible to simplify the proof of Proposition 2.6 below if we replace the condition (31) by the stronger condition that the function

$$\mathbb{C}\backslash\{0\} \ni w \mapsto \mathbf{E}(wA) \ \text{ is locally concave.} \tag{32}$$

Local concavity amounts to concavity of the function near each $w_0 \neq 0$ and is used here because the set $\mathbb{C}\backslash\{0\}$ clearly is not convex. It suffices for (31) since it allows us, via Jensen's inequality, to check that the composite function $z \mapsto \mathbf{E}(h'(z)A)$ locally satisfies the mean-value inequality that is known to be equivalent to superharmonicity. However, as $\mathbf{E}$ is real-valued and quasiconvex it is also rank-one convex and so locally Lipschitz. It is then not difficult to see that the local concavity (32) implies concavity in the sense that $\mathbf{E}(wA)$ lies below its supporting hyperplanes everywhere on $\mathbb{C}\backslash\{0\}$. The interested reader will see that this observation allows a simplification of the proof below.

The following lemma characterizes the condition (31) in the case the functional $\mathbf{E}$ is isotropic. Its proof is a straightforward calculation that is left to the interested reader.

Lemma 2.7. *Suppose $\mathbf{E}\colon \mathbb{R}_+^{2\times 2} \to \mathbb{R}$ is locally Lipschitz and isotropic. Then the composite function $z \mapsto \mathbf{E}(h'(z)A)$ is superharmonic for each conformal map $h\colon \Omega \subset \mathbb{C} \to \mathbb{C}$ and each $A \in \mathbb{R}_+^{2\times 2}$ if and only if the function*

$$(0,\infty) \ni t \mapsto t\left\langle \mathbf{E}'(tA), A\right\rangle$$

for each $A \in \mathbb{R}_+^{2\times 2}$ is non-increasing.

The proof of Proposition 2.6 below requires a number of further results that we will quote in the course of the proof and it also uses elements from [4]. The most important is [4, Lemma 7.2] that we for convenience of the reader state here in a form adapted to our setting.

Lemma 2.8. *Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is a principal map of class $\mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{C})$ with Laurent expansion (25) in the complement of the unit disk. Put $R_f(z) \equiv b_0 z + b_1/z$ and $h \equiv f \circ R_f^{-1}$ on $\mathbb{C}\backslash A_f(\overline{\mathbb{D}})$. Then $h\colon \mathbb{C}\backslash A_f(\overline{\mathbb{D}}) \to f(\mathbb{C}\backslash\overline{\mathbb{D}})$ is a conformal homeomorphism and*

$$g \equiv \begin{cases} f & \text{in } \overline{\mathbb{D}}, \\ h \circ A_f & \text{in } \mathbb{C}\backslash\overline{\mathbb{D}}, \end{cases}$$

is a homeomorphism of class $\mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{C})$. Moreover, $h(z) = z + \mathrm{O}(z^{-2})$ as $|z| \to \infty$.

Proof of Proposition 2.6. Fix a principal map $f\colon \mathbb{C} \to \mathbb{C}$ of class $\mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{C})$ and use the notation of Lemma 2.8. For $r > 1$ we consider the dilated map $g_r(z) \equiv g(rz)$, $z \in \mathbb{C}$. It has the same properties as g, but is additionally conformal for all $|z| > 1/r$. To avoid complicated notation we shall in the following denote this dilated version simply as g again and likewise we write h for h_r. Also denote $A = A_g = (rb_0, b_1/r)$. We now employ [49, 27] to find a sequence (g_n) of C^1-diffeomerphisms $g_n\colon \mathbb{D} \to g(\mathbb{D})$ with $g_n = g$ on $\partial\mathbb{D}$, $g_n \to g$ uniformly on $\mathbb{D}$ and strongly in $\mathrm{W}^{1,1}(\mathbb{D})$, $g_n^{-1} \to g^{-1}$ uniformly on $g(\mathbb{D})$ and strongly in $\mathrm{W}^{1,2}(g(\mathbb{D}))$. Put $\mathcal{E} \equiv A(\overline{\mathbb{D}})$ and note it is an ellipse. If we extend each g_n to $\mathbb{C}$ by $h \circ A_f$ we obtain Lipschitz maps with the property that $g_n \circ A_f^{-1} = h$ is conformal on $\mathbb{C}\backslash\mathcal{E}$ with Lipschitz derivative h' and so $h'(z) = 1 + \mathrm{O}(z^{-3})$ as $|z| \to \infty$. Because $\mathbf{E}$ as mentioned before must be locally Lipschitz we may use a simple change of variables to see, using the asymptotics of h, that $\mathbf{E}(\mathrm{D}g_n) - \mathbf{E}(A)\big)$ is integrable over $\mathbb{C}$ and proceed similarly to [4, Lemma 7.2] to obtain, for each fixed $n \in \mathbb{N}$, from the assumed $\mathrm{W}^{1,\infty}$-quasiconvexity at $A = A_{g_n}$ that

$$\int_{\mathbb{C}} \big(\mathbf{E}(\mathrm{D}g_n) - \mathbf{E}(A)\big)\, \mathrm{d}m(z) \geqslant 0.$$

We continue with

$$\begin{aligned} \int_{\mathbb{D}} \big(\mathbf{E}(\mathrm{D}g_n) - \mathbf{E}(A)\big)\, \mathrm{d}m(z) &\geqslant -\int_{\mathbb{C}\backslash\mathbb{D}} \big(\mathbf{E}(h'(Az)A) - \mathbf{E}(A)\big)\, \mathrm{d}m(z) \\ &= -\frac{1}{J_A} \int_{\mathbb{C}\backslash\mathcal{E}} \big(\mathbf{E}(h'(z)A) - \mathbf{E}(A)\big)\, \mathrm{d}m(z), \end{aligned}$$

where we note that the last integrand is integrable and, by assumption (31) is also superharmonic over the complement $\mathbb{C}\backslash\mathcal{E}$. But the complement of an ellipse is by a result of Sakai [75] a null quadrature domain, so the integral of an integrable superharmonic function there is non-positive, hence we arrive at

$$\int_{\mathbb{D}} \big(\mathbf{E}(\mathrm{D}g_n) - \mathbf{E}(A)\big)\, \mathrm{d}m(z) \geqslant 0.$$

In order to pass to the limit $n \to \infty$ here we must invoke the growth condition (30). First recall (7) and using that $g_n(\mathbb{D}) = g(\mathbb{D})$ we get by the convergence properties of g_n:

$$\int_{\mathbb{D}} K_{g_n}\, \mathrm{d}m(z) = \int_{g(\mathbb{D})} |\mathrm{D}(g_n^{-1})|^2\, \mathrm{d}m(z) \to \int_{g(\mathbb{D})} |\mathrm{D}(g^{-1})|^2\, \mathrm{d}m(z) = \int_{\mathbb{D}} K_g\, \mathrm{d}m(z).$$

Since also $K_{g_n} \to K_g$ in $\mathscr{L}^2$ measure it follows that $K_{g_n} \to K_g$ in $\mathrm{L}^1(\mathbb{D})$. In particular, the sequence (K_{g_n}) is equi-integrable on $\mathbb{D}$. For the Jacobians J_{g_n} we note that $g_n(\mathbb{D}_r) = g(\mathbb{D}_r)$ for all $r \geqslant 1$, hence the area formula gives $\int_{\mathbb{D}_r} J_{g_n}\,\mathrm{d}m(z) = \mathscr{L}^2(g(\mathbb{D}_r))$, so (J_{g_n}) is bounded in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{C})$. But then $(J^{\alpha}_{g_n})$ is equi-integrable on $\mathbb{D}$ by de la Vallée Poussin's criterion.

We turn to the term $\log\bigl(1+1/J_{g_n}\bigr)$ and remark that as a $\mathrm{W}^{1,1}$-homeomorphism of integrable distortion the map g has $J_g > 0$ a.e. (see [48]), so clearly $\log\bigl(1+1/J_{g_n}\bigr) \to \log\bigl(1+1/J_g\bigr)$ in $\mathscr{L}^2$ measure on $\mathbb{D}$. Even though $g_n = g$ on $\partial\mathbb{D}$ so that the local bound from [55, 56] can be improved to a global bound [4, Prop. 12.2], and in fact that we have a very precise bound resulting from $\mathrm{W}^{1,1}$-principal quasiconvexity of the functional $\mathbf{W}$ from Theorem 2.3, we only get that the sequence $\bigl(\log\bigl(1+1/J_{g_n}\bigr)\bigr)$ is bounded in $\mathrm{L}^1(\mathbb{D})$. We must however exclude concentration in order to proceed with the limit, so the bounds must be upgraded.

To that end note that if $\Psi\colon [0,\infty) \to [0,\infty)$ is an increasing and continuous function, then we get by elementary calculus since $g_n(\mathbb{D}) = g(\mathbb{D})$ that

$$\int_{\mathbb{D}} \Psi\left(\frac{1}{J_{g_n}}\right) J_{g_n}\,\mathrm{d}m(z) = \int_{g(\mathbb{D})} \Psi\bigl(J_{g_n^{-1}}\bigr)\,\mathrm{d}m(z).$$

If $\Theta\colon [0,\infty) \to [0,\infty)$ is an increasing, continuous function with $\Theta(t)/t \to 1$ as $t \to 0^+$ and $\Theta(t) \to \infty$ as $t \to \infty$, and we put $\Psi(t) = \Theta(t)t$ above then we find

$$\int_{\mathbb{D}} \Theta\left(\frac{1}{J_{g_n}}\right)\,\mathrm{d}m(z) = \int_{g(\mathbb{D})} \Theta\left(J_{g_n^{-1}}\right) J_{g_n^{-1}}\,\mathrm{d}m(z). \tag{33}$$

Recall that g_n equals g outside $\mathbb{D}$, that g is smooth away from $\mathbb{D}$ and that $g(\mathbb{C}) = \mathbb{C}$. Take a cube Q centered at 0 and with axes parallel to the coordinate axes such that $g(\mathbb{D}) \subset Q$. We now employ [43, Th. 2]. Accordingly, if $\theta\colon [0,\infty) \to [0,\infty)$ is increasing, continuous $\theta(t)/t \to 1$ as $t \to 0^+$ and $\theta(t) \to \infty$ as $t \to \infty$, then there exists a constant $C > 0$ such that with

$$\Psi(t) = \theta(t)t + t\int_0^t \frac{\theta(s)}{s}\,\mathrm{d}s,\ t \geqslant 0,$$

we have

$$\int_Q \Psi\bigl(J_{g_n^{-1}}\bigr)\,\mathrm{d}m(z) \leqslant C\int_{2Q} \theta\bigl(|\mathrm{D}(g_n^{-1})|^2\bigr)|\mathrm{D}(g_n^{-1})|^2\,\mathrm{d}m(z) \tag{34}$$

for all $n \in \mathbb{N}$. Because $\bigl(\mathrm{D}(g_n^{-1})\bigr)$ is 2-equi-integrable it follows from the de la Vallée-Poussin criterion that we can chose a function θ as above and such that the right-hand side of (34) is bounded for $n \in \mathbb{N}$. Put

$$\Theta(t) \equiv \int_0^t \frac{\theta(s)}{s}\,\mathrm{d}s = \int_0^{\log(1+t)} \theta\bigl(\mathrm{e}^r - 1\bigr)\frac{\mathrm{e}^r}{\mathrm{e}^r - 1}\,\mathrm{d}r$$

and note that Θ hereby is a continuous, increasing function with $\Theta(0) = 0$ and

$$\frac{\Theta(t)}{\log(1+t)} \to \infty \ \text{ as } \ t \to \infty.$$

Consequently, if we define

$$\psi(s) \equiv \int_0^s \theta\bigl(\mathrm{e}^r - 1\bigr)\frac{\mathrm{e}^r}{\mathrm{e}^r - 1}\,\mathrm{d}r,\ s \geqslant 0,$$

then $\psi\colon [0,\infty) \to [0,\infty)$ is continuous, increasing, $\psi(0) = 0$ and $\psi(s)/s \to \infty$ as $s \to \infty$. Since $\psi\big(\log(1+t)\big)t = \Theta(t)t \leqslant \Psi(t)$ and we use Ψ in (33) we arrive at

$$\int_{\mathbb{D}} \psi\left(\log\left(1+\frac{1}{J_{g_n}}\right)\right) \mathrm{d}m(z) \leqslant C \int_{2Q} \Theta\big(|\mathrm{D}(g_n^{-1})|^2\big)\big|\mathrm{D}(g_n^{-1})\big|^2 \,\mathrm{d}m(z),$$

where the right-hand side as observed in (34) is bounded for $n \in \mathbb{N}$. De la Vallée-Poussin yields equi-integrability of the sequence $\big(\log\big(1+1/J_{g_n}\big)\big)$ on $\mathbb{D}$. Taken together with the previous observations we conclude that the sequence of positive parts $\big(\mathbf{E}(\mathrm{D}g_n)^+\big)$ is equi-integrable on $\mathbb{D}$ and since it also converges in measure to $\mathbf{E}(\mathrm{D}g)^+$ it follows from Vitali's convergence theorem that

$$\int_{\mathbb{D}} \mathbf{E}(\mathrm{D}g_n)^+ \,\mathrm{d}m(z) \to \int_{\mathbb{D}} \mathbf{E}(\mathrm{D}g)^+ \,\mathrm{d}m(z).$$

For the sequence $\big(\mathbf{E}(\mathrm{D}g_n)^-\big)$ of negative parts we can estimate using Fatou's lemma, and consequently we have shown that

$$⨍_{\mathbb{D}} \mathbf{E}(\mathrm{D}g)\,\mathrm{d}m(z) \geqslant \mathbf{E}(A).$$

Because $\mathbf{E}$ is locally Lipschitz we can take $r \to 1^+$ to conclude the proof. □

Remark 2.9. The interested reader will see that the proof of Proposition 2.6 can give slightly more general results. We also note that the argument leading to the equi-integrability of the sequence $\big(\log(1+1/J_{g_n})\big)$ can be upgraded to give an Orlicz space version of the estimates in [55] and we will present that elsewhere [5].

We now give a larger, more general family of examples, with different symmetries. In order to motivate these examples we recall that the profound implications of the restricted quasiconvexity of the Burkholder functional on the higher integrability theory of quasiconformal maps led the authors of [11] to define and investigate a more general family of functionals, so-called complex Burkholder functionals. These functionals describe not only the integrability properties of quasiconformal mappings, loosely speaking corresponding to the stretch of such maps, but also the rotation of such maps. While its definition is a bit cumbersome it is very natural from the quasiconformal view point, and we now describe it in some detail.

It was discovered in [9, 11] that a fruitful way to understand the Burkholder functional is to see $\int_{\mathbb{D}} \mathbf{B}_p(\mathrm{D}f)\,\mathrm{d}m(z)$ as a weighted L^p norm of $\mathrm{D}f$, where the weight depends on p and f: indeed, first recall that any K-quasiconformal map is a $\mathrm{W}^{1,2}_{\mathrm{loc}}$-solution to the Beltrami equation

$$f_{\overline{z}} = \mu(z) f_z,$$

where the complex dilatation $\mu = \mu_f$ satisfies $|\mu_f(z)| \leqslant \frac{K-1}{K+1} < 1$ for a.e. z. Further, note that for such a map we may write

$$\mathbf{B}_p(\mathrm{D}f) = \Big(\frac{p|\mu_f|}{1+|\mu_f|} - 1\Big)\Big|(1+|\mu_f|)f_z\Big|^p.$$

In order to generalize this definition to describe the optimal complex powers for quasiconformal maps we take the exponent $\mathtt{p}$ to be a complex number which satisfies

$$1 \leqslant |\mathtt{p}-1|, \qquad \mathtt{p} \neq 0, \tag{35}$$

and we define $\beta = \beta(\mathtt{p})$ as the complex number determined uniquely through the equations

$$|\beta| + |\beta - 2| = 2|\mathtt{p}-1| \quad \text{and} \quad \mathrm{Re}(\beta/\mathtt{p}) = 1.$$

Then we set

$$\mathbf{B}_p(A) = \Big(\frac{\mathrm{p}\eta|\mu_A|}{1+\eta|\mu_A|} - 1\Big)|(a_+ + \eta|\mu_A|a_+)^\beta|.$$

Here η depends on $|\mu_A|$ and $\mathtt{p}$ through the implicit relations

$$|\eta(z)| = 1 \quad \text{and} \quad \arg(p\eta) = \arg(1+\eta|\mu_A|).$$

We readily see that the structure of the complex Burkholder functional is similar to that of $\mathbf{B}_p$ but that it contains complex powers of f_z. Thus, as mentioned above, its quasiconvexity relates not only to integrabilty issues but also to how fast quasiconformal maps can wind a line into a spiral. Concerning the structure of the functional we note that, for a complex parameter β, the functional is $(\operatorname{Re}\beta)$-homogeneous and the group of symmetries is described by the logarithmic spiral $S_\beta = \{z : |z^\beta| = 1\}$. We invite the reader to find a function $b\colon \mathbb{R} \to \mathbb{C}$ such that $\mathbf{B}_p(A) = |a_+^\beta| b(|\mu_A|)$ and thus will meet the requirements of Proposition 2.6. In a companion paper [5] we prove that the full family of local complex Burkholder functionals is principal quasiconvex:

Theorem 2.10. *Let $\mathbf{B}_\mathrm{p}$ be the $\mathtt{p}$-Burkholder functional with the complex exponent $\mathtt{p}$ satisfying condition (35). Then $\{\mathbf{B}_\mathrm{p} \leqslant 0\} = Q_2\left(\frac{|\mathrm{p}-1|+1}{|\mathrm{p}-1|-1}\right)$ and*

$$\mathcal{B}(A) \equiv \begin{cases} \mathbf{B}_\mathrm{p}(A) & \text{if } \mathbf{B}_\mathrm{p}(A) \leqslant 0, \\ +\infty & \text{if } \mathbf{B}_\mathrm{p}(A) > 0, \end{cases}$$

is $\mathrm{W}^{1,1}$-principal quasiconvex.

3. Proofs of semicontinuity and existence results.

3.1. Lower semicontinuity along sequences with asymptotically finite distortion.

We begin by giving the proof of a more general version of Theorem 1.9. The strategy is as follows: first we apply a general semicontinuity result for Young measures [13]. Then we localize and show that our assumed growth conditions and principal quasiconvexity imply that the integrands satisfy the Jensen inequality for the corresponding homogeneous gradient Young measures.

Theorem 3.1. *Let $\mathbf{E}\colon \Omega \times \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{+\infty\}$ be a normal integrand. Assume that*

$$\big|\mathbf{E}(z,A)\big| \leqslant \mathbf{C}(z,K_A)\big(|A|^2+1\big) \quad \forall (z,A) \in \Omega \times \mathbb{R}^{2\times 2}_+, \tag{36}$$

where $\mathbf{C}\colon \Omega \times [1,+\infty) \to [1,+\infty)$ is Borel measurable and $\mathbf{C}(z,\cdot)\colon [1,\infty) \to [1,\infty)$ is an increasing function for each $z \in \Omega$, and that for each $z \in \Omega$, $\mathbf{E}(z,\cdot)$ is $\mathrm{W}^{1,2}$-principal quasiconvex.

Let (u_j) be a sequence in $\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega)$ and suppose that the maps have asymptotically finite distortion on a measurable subset $\mathbb{A}$ of Ω:

$$K \equiv \limsup_{j\to\infty} \mathbf{K}(\mathrm{D}u_j) < +\infty \quad \mathscr{L}^2 \text{ a.e. in } \mathbb{A}. \tag{37}$$

Then, if $u_j \rightharpoonup u$ in $\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega)$ and

$$\big(\mathbf{E}(\cdot,\mathrm{D}u_j)^-\big) \text{ is equi-integrable on } \mathbb{A}, \tag{38}$$

we have

$$\liminf_{j\to\infty} \int_{\mathbb{A}} \mathbf{E}(\cdot,\mathrm{D}u_j)\,\mathrm{d}m(z) \geqslant \int_{\mathbb{A}} \mathbf{E}(\cdot,\mathrm{D}u)\,\mathrm{d}m(z).$$

Remark 3.2. The conclusion of Theorem 3.1 fails in general without the assumption (38), but can be reinstated if formulated in terms of biting convergence as in [50] and [40]. Under the assumptions of Theorem 3.1 except (38) we have that each subsequence of (u_j) admits a further subsequence (u_{j_k}) such that

$$b*\lim_{k\to\infty}\mathbf{E}(\cdot,\mathrm{D}u_{j_k})\geqslant\mathbf{E}(\cdot,\mathrm{D}u)\quad\mathscr{L}^2\text{ a.e. in }\mathbb{A}.$$

In fact, this is a consequence of Jensen's inequality for $\mathbf{E}$ and the Young measures generated by subsequences of $(\mathrm{D}u_{j_k})$. That they hold is part of our proof for Theorem 3.1 and we invite the reader to check that their validity do not hinge on the equi-integrability assumption (38). Indeed, if $(\nu_z)_{z\in\Omega}$ is a Young measures generated by a subsequence of $(\mathrm{D}u_{j_k})$, then it is not difficult to see that

$$b*\lim_{k\to\infty}\mathbf{E}(\cdot,\mathrm{D}u_{j_k})\geqslant\int_{\mathbb{R}^{2\times2}}\mathbf{E}(\cdot,A)\,\mathrm{d}\nu_{\cdot}(A)\geqslant\mathbf{E}(\cdot,\mathrm{D}u)\quad\mathscr{L}^2\text{ a.e. in }\mathbb{A}.$$

The key to the proof is the following localization principle for the Young measures that are generated by the derivatives of maps in a weakly converging sequence in $\mathrm{W}^{1,2}_{\mathrm{loc}}$ satisfying the asymptotic distortion condition (37). We emphasize that in the subsequent statements and proofs we consider all functions and maps in terms of their precise representatives.

Proposition 3.3. *Let (u_j) be a bounded sequence in $\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega)$ of maps satisfying (37). If $(\mathrm{D}u_j)$ generates the Young measure $(\nu_z)_{z\in\Omega}$, then for $\mathscr{L}^2$ almost all $z\in\mathbb{A}$ the measure ν_z is a homogeneous $\mathrm{W}^{1,2}$ gradient Young measure with support in $Q_2\big(K(z)\big)$.*

Proof. Put $M_j(z)\equiv\sup_{s\geqslant j}\mathbf{K}(\mathrm{D}u_s)$, $z\in\Omega$. Then $M_j\colon\Omega\to[1,+\infty]$ is measurable and

$$M_j(z)\geqslant M_{j+1}(z)\searrow K(z)\ \text{ as }\ j\nearrow\infty\ \text{ pointwise in }\ z\in\Omega. \tag{39}$$

We recall that $K(z)$ is defined at (37), where it is also assumed that $K(z)<+\infty$ holds $\mathscr{L}^2$ almost everywhere on $\mathbb{A}$. Because K is also clearly measurable it follows from Lusin's theorem that it is $\mathscr{L}^2$ quasi-continuous on $\mathbb{A}$: for each $\varepsilon>0$ there exists a measurable subset $G=G_\varepsilon\subset\mathbb{A}$ such that K is finite and continuous relative to G and $\mathscr{L}^2(\mathbb{A}\backslash G)<\varepsilon$. Fix an $\varepsilon>0$ and corresponding set $G=G_\varepsilon$. Fix $a>1$ and put $G_j=G^a_j\equiv\{z\in G:\ M_j(z)<aK(z)\}$ and record that G_j are measurable sets that by virtue of (39) form an ascending sequence that in the limit $j\to\infty$ exhaust G:

$$G_j\subset G_{j+1}\ \text{ and }\ \bigcup_{j\in\mathbb{N}}G_j=G. \tag{40}$$

We next note that the space $\mathrm{C}_0(\mathbb{D}\times\mathbb{R}^{2\times2})$ with the supremum norm is separable and that we may find countable families $\mathcal{F}_1$ in $\mathrm{C}^1_c(\mathbb{D})$ and $\mathcal{F}_2$ in $\mathrm{C}^1_c(\mathbb{R}^{2\times2})$ such that the countable family $\mathcal{F}\equiv\big\{\eta\otimes\Phi:\ \eta\in\mathcal{F}_1,\ \Phi\in\mathcal{F}_2\big\}$ is total in $\mathrm{C}_0(\mathbb{D}\times\mathbb{R}^{2\times2})$. Consequently, if $(\kappa^i_z)_{z\in\mathbb{D}}$, $i=1,2$, are two Young measures on $\mathbb{D}$ and

$$\int_{\mathbb{D}}\langle\kappa^1_z,\Phi\rangle\eta(z)\,\mathrm{d}m(z)=\int_{\mathbb{D}}\langle\kappa^2_z,\Phi\rangle\eta(z)\,\mathrm{d}m(z)$$

holds for all $\eta\otimes\Phi\in\mathcal{F}$, then the Young measures are the same: $\kappa^1_z=\kappa^2_z$ for $\mathscr{L}^2$ almost all $z\in\mathbb{D}$.

Because the sequence (u_j) is bounded in $\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega)$ the sequence of measures $(|\mathrm{D}u_j|^2\mathscr{L}^2\llcorner\Omega)$ is bounded in $\mathrm{C}_c(\Omega)^*$ and hence by Banach-Alaoglu's compactness theorem admits a weak* converging subsequence there. Extracting such a subsequence (that we for notational convenience do not relabel) we can assume that

$$|\mathrm{D}u_j|^2 \mathscr{L}^2 \llcorner \Omega \overset{*}{\rightharpoonup} \lambda \text{ in } \mathrm{C}_c(\Omega)^*, \tag{41}$$

where λ is a positive Radon measure on Ω. In particular we then have for each disk $\mathbb{D}_r(z_0)$ that is compactly contained in Ω that

$$\limsup_{j\to\infty} ⨍_{\mathbb{D}_r(z_0)} |\mathrm{D}u_j|^2 \,\mathrm{d}m(z) \leqslant \frac{\lambda\big(\overline{\mathbb{D}_r(z_0)}\big)}{\mathscr{L}^2\big(\mathbb{D}_r(z_0)\big)},$$

and hence by Lebesgue's differentiation theorem that

$$\limsup_{r\searrow 0} \limsup_{j\to\infty} ⨍_{\mathbb{D}_r(z_0)} |\mathrm{D}u_j|^2 \,\mathrm{d}m(z) \leqslant \frac{\mathrm{d}\lambda}{\mathrm{d}\mathscr{L}^2}(z_0) < +\infty \tag{42}$$

holds for $\mathscr{L}^2$ almost all $z_0 \in \Omega$. We now identify a set of *bad* points in Ω that we would like to avoid in the subsequent argument. First recall the countable family $\mathcal{F}_2$, and note that for each $\Phi \in \mathcal{F}_2$ the function $z \mapsto \langle \nu_z, \Phi \rangle$ is an $\mathrm{L}^\infty(\Omega)$ function, so that $\mathscr{L}^2$ almost all $z_0 \in \Omega$ are Lebesgue points:

$$\lim_{r\searrow 0} ⨍_{\mathbb{D}_r(z_0)} \langle \nu_z, \Phi \rangle \,\mathrm{d}m(z) = \langle \nu_{z_0}, \Phi \rangle \in \mathbb{R}.$$

Let N_Φ denote the complement in Ω of this set of Lebesgue points. Next, let N denote the set of points in Ω, where (42) fails. Define $\mathcal{N} \equiv N \cup \bigcup_{\Phi\in\mathcal{F}_2} N_\Phi$ and record that $\mathscr{L}^2(\mathcal{N}) = 0$.

Define for $\mathbb{D}_r(z_0) \Subset \Omega$ the maps

$$v_{j,r}(z) \equiv \frac{u_j(z_0 + rz) - (u_j)_{z_0,r}}{r}, \quad z \in \mathbb{D},$$

where $(u_j)_{z_0,r}$ is the integral mean of u_j over $\mathbb{D}_r(z_0)$. Hereby $v_{j,r} \in \mathrm{W}^{1,2}(\mathbb{D})$ and $(v_{j,r})_{0,1} = 0$. Before the next display we introduce some convenient notation for functions $\theta\colon \mathbb{C} \to \mathbb{R}$: the reflected function, $\tilde{\theta}$, is defined as $\tilde{\theta}(z) \equiv \theta(-z)$, and the L^1-dilated function with factor $r > 0$, θ_r, is defined as $\theta_r(z) \equiv \theta\big(z/r\big)/r^2$. In these terms we note that for each $z_0 \in \Omega\backslash\mathcal{N}$ and $\eta \otimes \Phi \in \mathcal{F}$ the above Lebesgue point property and a standard result about convolution yield (here η is extended to $\mathbb{C}\backslash\mathbb{D}$ by 0):

$$\begin{aligned}
\lim_{r\searrow 0}\lim_{j\to\infty} \int_{\mathbb{D}} \eta\Phi(\mathrm{D}v_{j,r}) \,\mathrm{d}m(z) &= \lim_{r\searrow 0}\lim_{j\to\infty} \int_{\Omega} \tilde{\eta}_r(z_0 - z)\Phi(\mathrm{D}u_j) \,\mathrm{d}m(z) \\
&= \lim_{r\searrow 0} \int_{\Omega} \tilde{\eta}_r(z_0 - z)\langle \nu_z, \Phi \rangle \,\mathrm{d}m(z) \\
&= \langle \nu_{z_0}, \Phi \rangle \int_{\mathbb{D}} \eta \,\mathrm{d}m(z).
\end{aligned}$$

The bound (42) translates to

$$\limsup_{r\searrow 0} \limsup_{j\to\infty} \int_{\mathbb{D}} |\mathrm{D}v_{j,r}|^2 \,\mathrm{d}m(z) \leqslant \pi \frac{\mathrm{d}\lambda}{\mathrm{d}\mathscr{L}^2}(z_0) < +\infty.$$

Fix a point $z_0 \in G\backslash\mathcal{N}$ that is also a density point for G. In addition to the above properties we now also have the following double limit:

$$\lim_{r\searrow 0}\lim_{j\to\infty} \frac{\mathscr{L}^2\big(\mathbb{D}_r(z_0) \cap G_j\big)}{\mathscr{L}^2\big(\mathbb{D}_r(z_0)\big)} = 1.$$

In order to choose a good null sequence $r_j \searrow 0$ we employ a diagonalization argument in connection with the above double limits: Hereby we find for $v_j \equiv v_{j,r_j}$ that

$$\lim_{j\to\infty}\int_{\mathbb{D}}\eta\Phi(\mathrm{D}v_j)\,\mathrm{d}m(z)=\int_{\mathbb{D}}\eta\,\mathrm{d}m(z)\langle\nu_{z_0},\Phi\rangle\quad\forall\,\eta\otimes\Phi\in\mathcal{F},\tag{43}$$

$$\sup_{j\in\mathbb{N}}\int_{\mathbb{D}}|\mathrm{D}v_j|^2\,\mathrm{d}m(z)<+\infty\tag{44}$$

and

$$\lim_{j\to\infty}\frac{\mathscr{L}^2\big(\mathbb{D}_{r_j}(z_0)\cap G_j\big)}{\mathscr{L}^2\big(\mathbb{D}_{r_j}(z_0)\big)}=1.\tag{45}$$

Because $(v_j)_{0,1}=0$ we deduce using Poincaré's inequality and (44) that the sequence (v_j) is bounded in $\mathrm{W}^{1,2}(\mathbb{D})$. Therefore any subsequence of $(\mathrm{D}v_j)$ admits a further subsequence that generates a Young measure. By (43) this Young measure is the homogeneous Young measure $(\nu_{z_0})_{z\in\mathbb{D}}$. A routine argument then shows that the full sequence $(\mathrm{D}v_j)$ generates the homogeneous Young measure $(\nu_{z_0})_{z\in\mathbb{D}}$. It is then easy to check that (v_j) converges weakly in $\mathrm{W}^{1,2}(\mathbb{D})$ to the linear map $\langle\nu_{z_0},\mathrm{Id}\rangle$. In particular it follows that $(\nu_{z_0})_{z\in\mathbb{D}}$ is a homogeneous $\mathrm{W}^{1,2}$ gradient Young measure. We use (45) to get information about the support of ν_{z_0}. Recall the parameter $a>1$ used in the definition of G_j and put

$$\mathcal{E}=\mathcal{E}_a\equiv\big\{A\in\mathbb{R}^{2\times2}:\,|A|^2>a^2K(z_0)\det A\big\}.$$

This is clearly an open set in $\mathbb{R}^{2\times2}$ and we note that for $z\in\mathbb{D}$ we have $\mathrm{D}v_j(z)\in\mathcal{E}$ precisely when $|\mathrm{D}u_j(z_0+r_jz)|^2>a^2K(z_0)\det\mathrm{D}u_j(z_0+r_jz)$. Now recall that K is continuous and finite relative to G and so for some $j_a\in\mathbb{N}$ we have $K(z_0+r_jz)<aK(z_0)$ for all $z\in\mathbb{D}\cap\big((G-z_0)/r_j\big)$ for $j\geqslant j_a$. In view of the definition of G_j we therefore have when $j\geqslant j_a$ that

$$\begin{aligned}|\mathrm{D}u_j(z_0+r_jz)|^2&\leqslant aK(z_0+r_jz)\det\mathrm{D}u_j(z_0+r_jz)\\&\leqslant a^2K(z_0)\det\mathrm{D}u_j(z_0+r_jz)\end{aligned}\tag{46}$$

holds for all $z\in\mathbb{D}\cap\big((G_j-z_0)/r_j\big)$. Because $\mathcal{E}$ is an open set its indicator function $\mathbf{1}_{\mathcal{E}}$ is lower semicontinuous, so we can find functions $\Phi_k\in\mathrm{C}_c(\mathbb{R}^{2\times2})$ such that $0\leqslant\Phi_k\leqslant\Phi_{k+1}\nearrow\mathbf{1}_{\mathcal{E}}$ as $k\nearrow+\infty$. Now for each $k\in\mathbb{N}$,

$$\liminf_{j\to\infty}\fint_{\mathbb{D}}\mathbf{1}_{\mathcal{E}}(\mathrm{D}v_j)\,\mathrm{d}m(z)\geqslant\liminf_{j\to\infty}\fint_{\mathbb{D}}\Phi_k(\mathrm{D}v_j)\,\mathrm{d}m(z)=\langle\nu_{z_0},\Phi_k\rangle$$

and so taking $k\nearrow+\infty$ we get by virtue of Lebesgue's monotone convergence theorem

$$\liminf_{j\to\infty}\fint_{\mathbb{D}}\mathbf{1}_{\mathcal{E}}(\mathrm{D}v_j)\,\mathrm{d}m(z)\geqslant\nu_{z_0}\big(\mathcal{E}\big).$$

To conclude the proof we estimate the left-hand side using (46) for $j\geqslant j_a$:

$$\begin{aligned}\fint_{\mathbb{D}}\mathbf{1}_{\mathcal{E}}(\mathrm{D}v_j)\,\mathrm{d}m(z)&=\frac{1}{\pi}\left(\int_{\mathbb{D}\cap(G_j-z_0)/r_j}+\int_{\mathbb{D}\setminus(G_j-z_0)/r_j)}\right)\mathbf{1}_{\mathcal{E}}(\mathrm{D}v_j)\,\mathrm{d}m(z)\\&\leqslant\frac{\mathscr{L}^2\big(\mathbb{D}_{r_j}(z_0)\setminus G_j\big)}{\mathscr{L}^2\big(\mathbb{D}_{r_j}(z_0)\big)}\end{aligned}$$

and the latter tends by (45) to 0 as $j\to\infty$. Consequently, $\nu_{z_0}\big(\mathcal{E}\big)=0$. Recall that $\mathcal{E}=\mathcal{E}_a$ and that $a>1$ is arbitrary, so by monotone convergence we deduce that $\nu_{z_0}\big(\bigcup_{a>1}\mathcal{E}_a\big)=0$. But then ν_{z_0} is carried by $Q_2(K(z_0))$, and since $Q_2(K(z_0))$ is a closed set it must contain the support of ν_{z_0}, as required. The above holds for all

$z_0 \in G\backslash\mathcal{N}$ that are density points of G, where we recall that G was a measurable subset of $\mathbb{A}$ such that $\mathscr{L}^2(\mathbb{A}\backslash G) < \varepsilon$. Here $\varepsilon > 0$ was arbitrary and it is then routine to conclude the proof. □

Proof of Theorem 3.1. Because of (38) we can without loss of generality assume, considering a suitable subsequence that we for notational convenience do not relabel, that

$$\lim_{j\to\infty}\int_{\mathbb{A}} \mathbf{E}(\cdot, \mathrm{D}u_j)\,\mathrm{d}m(z) = \liminf_{j\to\infty}\int_{\mathbb{A}} \mathbf{E}(\cdot, \mathrm{D}u_j)\,\mathrm{d}m(z) \in \mathbb{R}$$

and that the sequence $(\mathrm{D}u_j)$ generates the Young measure $(\nu_z)_{z\in\Omega}$. Here, for each $z \in \Omega$, the partial functional $\mathbf{E}(z,\cdot)$ is lower semicontinuous on $\mathbb{R}^{2\times 2}_+$, and so it coincides with its lower semicontinuous envelope

$$\mathbf{E}_*(z, A) \equiv \liminf_{A'\to A} \mathbf{E}(z, A')$$

there. Note that $\mathbf{E}_* \colon \Omega \times \mathbb{R}^{2\times 2} \to [-\infty, +\infty]$ is Borel measurable and that

$$\lim_{j\to\infty}\int_{\Omega} \mathbf{E}(\cdot, \mathrm{D}u_j)\,\mathrm{d}m(z) = \lim_{j\to\infty}\int_{\Omega} \mathbf{E}_*(\cdot, \mathrm{D}u_j)\,\mathrm{d}m(z) \in \mathbb{R},$$

where again the negative parts $\big(\mathbf{E}_*(\cdot, \mathrm{D}u_j)^-\big)$ form an equi-integrable sequence on $\mathbb{A}$. We may then appeal to a general lower semicontinuity result from Young measure theory [13] whereby

$$\lim_{j\to\infty}\int_{\mathbb{A}} \mathbf{E}_*(\cdot, \mathrm{D}u_j)\,\mathrm{d}m(z) \geqslant \int_{\mathbb{A}}^{*} \langle \nu_z, \mathbf{E}_*(z,\cdot)\rangle\,\mathrm{d}m(z).$$

It is at this stage we employ the localization principle from Proposition 3.3. Accordingly ν_z is, for $\mathscr{L}^2$ almost all $z \in \mathbb{A}$, a homogeneous $\mathrm{W}^{1,2}$ gradient Young measure supported on $Q_2\big(K(z)\big)$. Because $\mathbf{E}(z,\cdot)$ in particular is lower semicontinuous there we infer that

$$\langle \nu_z, \mathbf{E}_*(z,\cdot)\rangle = \langle \nu_z, \mathbf{E}(z,\cdot)\rangle \quad \text{for } \mathscr{L}^2 \text{ a.e. } z \in \mathbb{A}.$$

Moreover, $\mathbf{E}(z,\cdot)$ is principal quasiconvex and real-valued on $\mathbb{R}^{2\times 2}_+$, so must be rank-one convex and thus locally Lipschitz on $\mathbb{R}^{2\times 2}_+\backslash\{0\}$. It need not be continuous at 0 relative to $\mathbb{R}^{2\times 2}_+$, but is lower semicontinuous there by assumption.

Next, for $\mathscr{L}^2$ a.e. $z \in \mathbb{A}$, from [3] we find a sequence of principal $K(z)$-quasiconformal maps $f_j \colon \mathbb{C} \to \mathbb{C}$ such that $f_j \rightharpoonup \langle \nu_z, \mathrm{Id}\rangle$ in $\mathrm{W}^{1,p}(\mathbb{D})$ for each $p < 2K(z)/(K(z)-1)$ and $(\mathrm{D}f_j)$ generates ν_z. Recall that according to the 0-1 Law for homogeneous gradient Young measures from [3] we have that either $\nu_z(\{0\}) = 0$ or $\nu_z = \delta_0$. For measures ν_z of the former type, $\mathbf{E}(z,\cdot)$ is therefore continuous ν_z almost everywhere and so as the quadratic growth (36) ensures that the sequence $\big(\mathbf{E}(\mathrm{D}f_j)\big)$ is equiintegrable on $\mathbb{D}$, we arrive at

$$\fint_{\mathbb{D}} \mathbf{E}(z, \mathrm{D}f_j(w))\,\mathrm{d}m(w) \to \langle \nu_z, \mathbf{E}(z,\cdot)\rangle.$$

Since we also have that $f_j \rightharpoonup A_f$ in $\mathrm{W}^{1,2}(\mathbb{D})$, where $A_f = \langle \nu_z, \mathrm{Id}\rangle$, we get that $A_{f_j} \to A_f$. It now follows directly from principal quasiconvexity that $\langle \nu_z, \mathbf{E}(z,\cdot)\rangle \geqslant \mathbf{E}\big(z, \langle \nu_z, \mathrm{Id}\rangle\big)$ and this concludes the proof for measures with $\nu_z(\{0\}) = 0$.

For the other measures $\nu_z = \delta_0$ we simply use that $\mathbf{E}(z,\cdot)$ is lower semicontinuous at 0, that the quadratic growth (36) still ensures that the sequence $\big(\mathbf{E}(\mathrm{D}f_j)\big)$ is equiintegrable on $\mathbb{D}$ (in particular the negative parts) and then the general lower semicontinuity result for Young measures [13]. This concludes the proof. □

3.2. **Lower semicontinuity with integrable distortion.** In this subsection we give the proof of a more general version of Theorem 1.6, which follows the proof in [4, Section 12] concerning the lower semicontinuity of the non polyconvex functional $\mathbf{W}$ whose definition was recalled at (27). This functional is the Shield transform of a functional introduced in [51], [9] and it was also studied in the recent works [82, 83]. As the result below shows, the crucial feature of $\mathbf{W}$ is its principal quasiconvexity.

Theorem 3.4. *Let $\mathbf{E}\colon \Omega\times\mathbb{R}^{2\times 2}\to\mathbb{R}\cup\{+\infty\}$ be a Borel measurable functional such that $\mathbf{E}(z,\cdot)$ is $\mathrm{W}^{1,1}$ principal quasiconvex for each $z\in\Omega$. Assume that there exist constants $C>0$ and $p\in[1,2)$ such that*

$$\big|\mathbf{E}(z,A)\big|\leqslant C\max\big\{|A|^p,-\log(\det A),K_A\big\}+C \tag{47}$$

holds for all $(z,A)\in\Omega\times\mathbb{R}^{2\times 2}_+$, while $\mathbf{E}(z,A)=+\infty$ for $(z,A)\in\Omega\times\big(\mathbb{R}^{2\times 2}\backslash\big(\mathbb{R}^{2\times 2}_+\cup\{0\}\big)\big)$. Let $q>\frac{p}{2-p}$ be an exponent and $g\in\mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{C})$ a homeomorphism with $K_g\in\mathrm{L}^q_{\mathrm{loc}}(\mathbb{C})$. If (u_j) is a sequence of homeomorphisms, $u_j\colon\Omega\to u_j(\Omega)$, $u_j\in g+\mathrm{W}^{1,2}_0(\Omega)$ with $\sup_j\|K_{u_j}\|_{\mathrm{L}^q(\Omega)}<+\infty$ and $u_j\rightharpoonup u$ in $\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega)$, then $u\in g+\mathrm{W}^{1,2}_0(\Omega)$ is a homeomorphism with $K_u\in\mathrm{L}^q(\Omega)$ and

$$\liminf_{j\to\infty}\int_\Omega\mathbf{E}(\cdot,\mathrm{D}u_j)\,\mathrm{d}m(z)\geqslant\int_\Omega\mathbf{E}(\cdot,\mathrm{D}u)\,\mathrm{d}m(z).$$

Apart from the use of principal quasiconvexity the overall proof strategy is standard. The implementation is however not and it might have other applications and so be of wider interest. First we show that the assumed growth conditions suffice to ensure equiintegrability of the energy densities. This relies on [56, 55] as presented in [4, Proposition 12.8]. Next, by Young measure theory the proof then reduces to obtaining Jensen's inequality for a homogeneous gradient Young measure, which is the content of Proposition 1.4.

Proposition 3.5. *Let $\mathbf{E}$ satisfy (47) and let (v_j) be a bounded sequence in $\mathrm{W}^{1,2}(\Omega)$ with $\sup_j\|K_{u_j}\|_{\mathrm{L}^q(\Omega)}<\infty$ for some $q>\frac{p}{2-p}$. Then the sequence $\big(\mathbf{E}(\cdot,\mathrm{D}v_j)\big)$ is equiintegrable in Ω.*

Proof. The assumptions imply that both $|\mathrm{D}v_j|^p$ and K_{v_j} are equiintegrable. In order to see that $\log J_{v_j}$ is equiintegrable too we invoke the pointwise estimate:

$$|\log(J_{v_j})|\leqslant\log\left(e+\frac{1}{J_{v_j}}\right)+(J_{v_j})^{\frac{1}{2}}. \tag{48}$$

Here the first term on the right-hand side is equiintegrable by virtue of [4, Proposition 12.8], while equiintegrability of the second follows from the assumed $\mathrm{W}^{1,2}$ bound. □

Proof of Theorem 3.4. It is well-known that the limit $u\in g+\mathrm{W}^{1,2}_0(\Omega)$ is a homeomorphism with distortion $K_u\in\mathrm{L}^q(\Omega)$, see [7, Ch. 21]. In view of Proposition 3.5 the sequence $\big(\mathbf{E}(\cdot,\mathrm{D}u_j)\big)$ is equiintegrable on Ω, and for a subsequence that we for convenience do not relabel we have

$$\int_\Omega\mathbf{E}(\cdot,\mathrm{D}u_j)\,\mathrm{d}m(z)\to\ell\in\mathbb{R}.$$

Extracting a further subsequence, again not relabelled, we may assume that $(\mathrm{D}u_j)$ generates the $\mathrm{W}^{1,2}$-gradient Young measure $(\nu_z)_{z\in\Omega}$. First, note that the L^q bound entails

$$\int_\Omega\langle\nu_z,\mathbf{K}\rangle\,\mathrm{d}m(z)=\lim_{j\to\infty}\int_\Omega K_{u_j}\,\mathrm{d}m(z)<+\infty,$$

so that, in particular, $\langle \nu_z, \mathbf{K} \rangle < +\infty$ holds for a.e. $z \in \Omega$. If therefore $\mathbf{E}_*(z,\cdot)$ denotes the lower semicontinuous envelope of $\mathbf{E}(z,\cdot)$, then $\mathbf{E}_*(z,\cdot) = \mathbf{E}(z,\cdot)$ ν_z-a.e. for $\mathscr{L}^2$ a.e. $z \in \Omega$. Hence we get by general Young measure theory [13],

$$\lim_{j\to\infty} \int_\Omega \mathbf{E}(z, \mathrm{D}u_j)\,\mathrm{d}m(z) = \ell \geqslant \int_\Omega \int_{\mathbb{R}^{2\times 2}} \mathbf{E}(z,A)\,\mathrm{d}\nu_z(A)\,\mathrm{d}m(z) > -\infty. \tag{49}$$

For a.e. $z \in \Omega$ it is routine to check that we have

$$A_z \equiv \langle \nu_z, \mathrm{Id} \rangle \in \mathbb{R}^{2\times 2}_+ \ \text{ and } \ \nu_z \text{ is a homogeneous } \mathrm{W}^{1,2} \text{ gradient Young measure.} \tag{50}$$

Let us fix a point $z \in \Omega$ for which (50) holds. The measure ν_z is generated, as a homogeneous Young measure over $\mathbb{D}$, by taking a suitable diagonal subsequence (ψ_{j,λ_j}) of $\psi_{j,\lambda}(w) = \lambda^{-1}\big(u_j(z_0 + \lambda w) - u_j(z_0)\big)$, where $j \to \infty$ and $\lambda_j \to 0$, see [4, Theorem 2.8]. In particular, $\psi_{j,\lambda_j} \colon \mathbb{D} \to \mathbb{C}$ is a sequence of homeomorphisms such that $\psi_{j,\lambda_j} \to A_z$ weakly in $\mathrm{W}^{1,2}(\mathbb{D})$ and $\sup_j \|K_{\psi_{j,\lambda_j}}\|_{\mathrm{L}^q(\mathbb{D})} < \infty$. Then we are precisely in position to apply Proposition 1.4, whereby we find a sequence (f_j) of principal homeomorphisms of class $\mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{C})$ satisfying $f_j \to A_z$ locally uniformly in $\mathbb{C}$, $f_j \to A_z$ weakly in $\mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{D})$, $A_{f_j} = ⨍_{\mathbb{D}} \mathrm{D}f_j\,\mathrm{d}m(z) \to A_z$, $K_{f_j} = K_{\psi_{j,\lambda_j}}$ a.e. in $\mathbb{D}$ and $(\mathrm{D}f_j|_{\mathbb{D}})$ generates ν_z. Now as f_j are principal maps it follows that (J_{f_j}) is bounded in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{C})$, and therefore we have for $s \in (\frac{p}{2-p}, q)$ that

$$|\mathrm{D}f_j|^p = J_{f_j}^{\frac{p}{2}} K_{f_j}^{\frac{p}{2}} \leqslant J_{f_j}^{\frac{ps}{2s-p}} + K_{f_j}^s$$

is equiintegrable on $\mathbb{D}$. In view of Proposition 3.5 the sequence $\big(\mathbf{E}(z, \mathrm{D}f_j(w))\big)$ is equiintegrable over $\mathbb{D}$ and since $\mathbf{E}(z,\cdot)$ is principal quasiconvex, it is in particular rank-one convex on $\mathbb{R}^{2\times 2}_+$, and so continuous there, hence

$$\int_{\mathbb{R}^{2\times 2}} \mathbf{E}(z,A)\,\mathrm{d}\nu_z(A) = \lim_{j\to\infty} ⨍_{\mathbb{D}} \mathbf{E}(z, \mathrm{D}f_j(w))\,\mathrm{d}m(w) \geqslant \lim_{j\to\infty} \mathbf{E}(z, A_{f_j}) = \mathbf{E}(z, A_z),$$

where the last inequality follows by the assumed principal $\mathrm{W}^{1,1}$ quasiconvexity. Inserting the Jensen inequality for each ν_z in (49) yields the required lower semicontinuity. □

Proof of Corollary 1.7. We start by establishing a coercivity inequality and note that the growth condition (17) implies that $\mathbf{E}(A) \geqslant -C\big(1 + |A|^p + K_A + |\log J_A|\big)$ holds for all $A \in \mathbb{R}^{2\times 2}_+$. Consequently, invoking Young's inequality in a routine manner we find positive constants c_1, $c_2 > 0$ such that $\mathbf{E}(A) + \mathbf{P}(A) \geqslant c_1|A|^2 + c_1 K_A^q - c_2 - C|\log J_A|$ holds for all $A \in \mathbb{R}^{2\times 2}_+$. Next, combining the bound (48) with [4, Proposition 12.8] as in the proof of Proposition 3.5 above we arrive at

$$\int_\Omega \big(\mathbf{E}(\mathrm{D}u) + \mathbf{P}(\mathrm{D}u)\big)\,\mathrm{d}m(z) \geqslant c_0 \int_\Omega \big(|\mathrm{D}u|^2 + K_u^q\big)\,\mathrm{d}m(z) - C_0$$

for all $u \in \mathcal{A}_g$, where c_0, $C_0 > 0$ are positive constants that in particular are independent of u. Again using Young's inequality in a standard manner we infer that the energy including the forcing term,

$$\int_\Omega \big(\mathbf{E}(\mathrm{D}u) + \mathbf{P}(\mathrm{D}u)\big)\,\mathrm{d}m(z) + \langle F, u \rangle$$

is bounded below on $\mathcal{A}_g$ and that, since $\mathbf{E}(\mathrm{D}g) + \mathbf{P}(\mathrm{D}g)$ is integrable over Ω by assumption, any minimizing sequence $(u_j) \subset \mathcal{A}_g$ is bounded in $\mathrm{W}^{1,2}(\Omega)$ and has distortions (K_{u_j}) bounded in $\mathrm{L}^q(\Omega)$. By general principles we may then extract a subsequence (for convenience not relabelled) such that $u_j \to u$ weakly in $\mathrm{W}^{1,2}(\Omega)$.

It then follows that $u \in g + \mathrm{W}_0^{1,2}(\Omega)$ and, see [7, Ch. 21], that $u\colon \Omega \to g(\Omega)$ is a homeomorphism with $K_u \in \mathrm{L}^q(\Omega)$. In particular, $u \in \mathcal{A}_g$ and by [14] and Theorem 3.4 u is therefore a minimizer, as required. □

3.3. Lower Semicontinuity of the critical Burkholder energy.

3.3.1. *An integrability property of the Burkholder functional.* Before starting the proof we record the following integrability property of the Burkholder functional. That a result of this type holds is well-known [7], but our statement here is more explicit.

Proposition 3.6 (L^1 boundedness)**.** *Let $p > 2$ and put $K \equiv p/(p-2)$. There exists a constant $C = C(p)$ with the following property. If $u\colon \Omega \to \mathbb{C}$ is a K-quasiregular map and $\mathbb{D}_R(z_0) \Subset \Omega$, then for all $\delta \in (0,1)$ we have*

$$\fint_{\mathbb{D}_{\delta R}(z_0)} \mathbf{B}_p(\mathrm{D}u)\,\mathrm{d}m(z) \geqslant -\frac{C}{R^{pK}(1-\delta)^{pK}} \sup_{\mathbb{D}_R(z_0)} |u|^p. \tag{51}$$

The point here is, as was mentioned in the Introduction, that a K-quasiregular map need not be of class $\mathrm{W}^{1,p}_{\mathrm{loc}}$. That the Burkholder functional is locally integrable despite being nonpositive and p-homogeneous on the derivative of a K-quasiregular map comes down to it vanishing on the boundary of the K-quasiconformal well, where the large values of the derivative must concentrate.

Proof. Changing variables we may without loss of generality assume that $z_0 = 0$ and $R = 1$. We then Stoilow factorize u on $\mathbb{D}$, whereby $u = h \circ f$ for a K-quasiconformal normalized principal map $f\colon \mathbb{C} \to \mathbb{C}$ and a holomorphic map $h\colon f(\mathbb{D}) \to \mathbb{C}$. If $m \equiv \sup_{\mathbb{D}} |u|$, then also $m = \sup_{f(\mathbb{D})} |h|$ and so by Cauchy's integral formula and Hölder properties of K-quasiconformal principal maps [7] we estimate

$$\sup_{\delta\mathbb{D}} |h' \circ f| \leqslant \frac{m}{\mathrm{dist}(f(\delta\mathbb{D}), \partial f(\mathbb{D}))} \leqslant c_p \frac{m}{(1-\delta)^K}.$$

Next, using that $\mathbf{B}_p$ is nonpositive and the Burkholder area inequality [4]:

$$\begin{aligned}
\int_{\delta\mathbb{D}} \mathbf{B}_p(\mathrm{D}u)\,\mathrm{d}m(z) &= \int_{\delta\mathbb{D}} |h' \circ f|^p \mathbf{B}_p(\mathrm{D}f)\,\mathrm{d}m(z) \\
&\geqslant \sup_{\delta\mathbb{D}} |h' \circ f|^p \int_{\delta\mathbb{D}} \mathbf{B}_p(\mathrm{D}f)\,\mathrm{d}m(z) \\
&\geqslant c_p^p \frac{m^p}{(1-\delta)^{pK}} \int_{\mathbb{D}} \mathbf{B}_p(\mathrm{D}f)\,\mathrm{d}m(z) \\
&\geqslant -c_p^p \frac{m^p}{(1-\delta)^{pK}} \pi,
\end{aligned}$$

as required. □

3.4. Proof of Theorem 1.5.

Proof. We split the proof into two steps and remark that the assumption $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$ only is used in the second step.

By Proposition 3.6, the sequence $\big(\mathbf{B}_p(\mathrm{D}u_j)\big)$ is bounded in $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$. Because $\mathrm{L}^1_{\mathrm{loc}}(\Omega) \subset \mathrm{C}_c(\Omega)^*$ continuously, the Banach-Alaoglu compactness theorem implies that any subsequence of $\big(\mathbf{B}_p(\mathrm{D}u_j)\big)$ admits a further subsequence that converges weakly* to a Radon measure on Ω. Any such limit must necessarily be a nonpositive Radon measure. Assume that $-\lambda$ is such a weak* limit and consider the Lebesgue-Radon-Nikodym decomposition

$$\lambda = \frac{\mathrm{d}\lambda}{\mathrm{d}\mathscr{L}^2}\mathscr{L}^2 + \lambda^s, \text{ where } \lambda^s \perp \mathscr{L}^2.$$

We next estimate each term in this decomposition.

Step 1. $-\frac{\mathrm{d}\lambda}{\mathrm{d}\mathscr{L}^2} \geqslant \mathbf{B}_p(\mathrm{D}u)$ holds $\mathscr{L}^2$ almost everywhere in Ω. The proof consist of a rather general localization argument at a point of differentiabilty of u. Arguments of this sort are known in the literature as blow-up arguments and in the present context go back to I. Fonseca and S. Müller [39]. The fact that the Burkholder functional is nonpositive on its effective domain means that one must be more careful than in the classical set-up, where the considered functionals are bounded below and therefore concentration effects are not so harmful.

Let $z_0 \in \Omega$ be a point satisfying

$$u \text{ is differentiable at } z_0 \text{ with Jacobi matrix } A_0 \equiv \mathrm{D}u(z_0) \tag{52}$$

and

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\mathscr{L}^2}(z_0) = \lim_{r\searrow 0}\frac{\lambda(\mathbb{D}_r(z_0))}{\mathscr{L}^2(\mathbb{D}_r(z_0))}. \tag{53}$$

From the local uniform convergence of u_j to u and (52) follows that

$$\lim_{r\searrow 0}\lim_{j\to\infty}\frac{1}{r}\sup_{z\in\mathbb{D}_r(z_0)}\left|u_j(z)-u_j(z_0)-A_0(z-z_0)\right| = 0.$$

Next, for $r \in (0, \operatorname{dist}(z_0, \partial\Omega))$ with $\lambda(\partial\mathbb{D}_r(z_0)) = 0$ we have

$$\lim_{j\to\infty}\int_{\mathbb{D}_r(z_0)}\mathbf{B}_p(\mathrm{D}u_j)\,\mathrm{d}m(z) = -\lambda(\mathbb{D}_r(z_0)),$$

and so by (53) we get in particular

$$\lim_{E\not\ni r\searrow 0}\lim_{j\to\infty}⨍_{\mathbb{D}_r(z_0)}\mathbf{B}_p(\mathrm{D}u_j)\,\mathrm{d}m(z) = -\frac{\mathrm{d}\lambda}{\mathrm{d}\mathscr{L}^2}(z_0),$$

where $E \equiv \{r \in (0, \operatorname{dist}(z_0, \partial\Omega)) : \lambda(\partial\mathbb{D}_r(z_0)) > 0\}$ is an at most countable set. Fix $s \in (0,1)$. In view of the above we may choose a null sequence $r_j \searrow 0$ such that in terms of

$$v_j(z) \equiv \frac{u_j(z_0 + r_j z) - u_j(z_0)}{r_j},\ z \in \mathbb{D},$$

we have

$$\lim_{j\to\infty}\int_{\mathbb{D}_s(0)}\mathbf{B}_p(\mathrm{D}v_j)\,\mathrm{d}m(z) = -\pi s^2\frac{\mathrm{d}\lambda}{\mathrm{d}\mathscr{L}^2}(z_0).$$

Here it is clear that v_j are K-quasiregular and $v_j \to A_0$ uniformly on $\mathbb{D}$. The matrix A_0 is K-quasiconformal, so if its conformal-anticonformal coordinates are (A_0^+, A_0^-), then $|A_0^-| \leqslant |A_0^+|/(p-1)$. Its Stoilow factorization is therefore (when $A_0 \neq 0$) $A_0 = h \circ f$ with $h(z) \equiv A_0^+ z$ and

$$f(z) \equiv \begin{cases} z + \frac{A_0^-}{A_0^+}\bar{z} & \text{if } |z| \leqslant 1, \\ z + \frac{A_0^-}{A_0^+}\frac{1}{z} & \text{if } |z| > 1. \end{cases}$$

If $v_j = h_j \circ f_j$ is the Stoilow factorization of v_j, then using uniqueness of the factorization it is not difficult to see that $f_j \to f$ uniformly on $\mathbb{C}$ (if $A_0 = 0$ we use normality to extract a convergent subsequence and this then defines f) and $h_j \to h$

locally uniformly on $f(\mathbb{D})$. Consequently we have that $h_j'(f_j(z)) \to A_0^+$ uniformly in $z \in \mathbb{D}_s(0)$, hence using the properties of $\mathbf{B}_p$ and in particular the Burkholder area inequality (see [4]) and that $A_{f_j} \to (1, \frac{A_0^-}{A_0^+})$:

$$\begin{aligned} -\pi s^2 \frac{\mathrm{d}\lambda}{\mathrm{d}\mathscr{L}^2}(z_0) &= \lim_{j\to\infty} \int_{\mathbb{D}_s(0)} \mathbf{B}_p(\mathrm{D}v_j)\,\mathrm{d}m(z) \\ &= \lim_{j\to\infty} \int_{\mathbb{D}_s(0)} \left|h_j' \circ f_j\right|^p \mathbf{B}_p(\mathrm{D}f_j)\,\mathrm{d}m(z) \\ &= \lim_{j\to\infty} \int_{\mathbb{D}_s(0)} |A_0^+|^p \mathbf{B}_p(\mathrm{D}f_j)\,\mathrm{d}m(z) \\ &\geqslant |A_0^+|^p \limsup_{j\to\infty} \int_{\mathbb{D}} \mathbf{B}_p(\mathrm{D}f_j)\,\mathrm{d}m(z) \\ &\geqslant |A_0^+|^p \limsup_{j\to\infty} \big(\pi \mathbf{B}_p(A_{f_j})\big) = \pi \mathbf{B}_p(A_0). \end{aligned}$$

Finally, taking $s \nearrow 1$ we conclude that $-\frac{\mathrm{d}\lambda}{\mathrm{d}\mathscr{L}^2}(z_0) \geqslant \mathbf{B}_p(\mathrm{D}u(z_0))$ holds at all points $z_0 \in \Omega$ satisfying (52) and (53). Because this includes $\mathscr{L}^2$ almost all points in Ω the proof of Step 1 is finished.

Step 2. $\lambda^s = 0$ when $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$.

It suffices to show that $\lambda^s(B) = 0$ for each disk $B = \mathbb{D}_R(w_0) \Subset \Omega$. Fix such a disk B. Let $z_0 \in B$ satisfy

$$\lim_{r\searrow 0} \frac{\lambda(\mathbb{D}_r(z_0))}{\mathscr{L}^2(\mathbb{D}_r(z_0))} = +\infty \tag{54}$$

and

$$\lim_{r\searrow 0} \frac{1}{\lambda(\mathbb{D}_r(z_0))} \int_{\mathbb{D}_r(z_0)} |\mathrm{D}u|^p\,\mathrm{d}m(z) = 0. \tag{55}$$

Put

$$v_{j,r}(z) \equiv \frac{u_j(z_0 + rz) - u_j(z_0)}{r\rho_r}, \quad z \in \mathbb{D}, \text{ where } \rho_r \equiv \left(\frac{\lambda(\mathbb{D}_r(z_0))}{\mathscr{L}^2(\mathbb{D}_r(z_0))}\right)^{\frac{1}{p}}.$$

We invoke (55) as follows. Recall that u_j are K-quasiregular and that $u_j \to u$ locally uniformly on Ω and boundedly in $\mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega)$ for each $q < p$, hence we get as $j \to \infty$:

$$\begin{aligned} \int_{\mathbb{D}} |\mathrm{D}v_{j,r}|^2\,\mathrm{d}m(z) &= \frac{\pi}{\rho_r^2} ⨍_{\mathbb{D}_r(z_0)} |\mathrm{D}u_j|^2\,\mathrm{d}m(z) \leqslant \frac{\pi K}{\rho_r^2} ⨍_{\mathbb{D}_r(z_0)} \det \mathrm{D}u_j\,\mathrm{d}m(z) \\ &\to \frac{\pi K}{\rho_r^2} ⨍_{\mathbb{D}_r(z_0)} \det \mathrm{D}u\,\mathrm{d}m(z) \leqslant \frac{\pi K}{\rho_r^2} ⨍_{\mathbb{D}_r(z_0)} |\mathrm{D}u|^2\,\mathrm{d}m(z) \\ &\leqslant \pi K \left(\rho_r^{-p} ⨍_{\mathbb{D}_r(z_0)} |\mathrm{D}u|^p\,\mathrm{d}m(z)\right)^{\frac{2}{p}} \\ &= \pi K \left(\frac{1}{\lambda\big(\mathbb{D}_r(z_0)\big)} \int_{\mathbb{D}_r(z_0)} |\mathrm{D}u|^p\,\mathrm{d}m(z)\right)^{\frac{2}{p}}. \end{aligned}$$

Consequently,

$$\lim_{r\searrow 0} \lim_{j\to\infty} \int_{\mathbb{D}} |\mathrm{D}v_{j,r}|^2\,\mathrm{d}m(z) = 0. \tag{56}$$

Next, we get as in Step 2,

$$\lim_{j\to\infty}\int_{\mathbb{D}_r(z_0)}\mathbf{B}_p(\mathrm{D}u_j)\,\mathrm{d}m(z) = -\lambda(\mathbb{D}_r(z_0))$$

when $\lambda(\partial\mathbb{D}_r(z_0)) = 0$, hence

$$\lim_{E\not\ni r\searrow 0}\lim_{j\to\infty}\int_{\mathbb{D}}\mathbf{B}_p(\mathrm{D}v_{j,r})\,\mathrm{d}m(z) = -1,$$

where $E \equiv \{r \in (0, \mathrm{dist}(z_0, \partial\Omega)) : \lambda(\partial\mathbb{D}_r(z_0)) > 0\}$ is an at most countable set. In order to combine this with (54) we require some routine arguments from measure theory that we briefly recall here for convenience of the reader. Define

$$\Lambda(t) \equiv \limsup_{E\not\ni r\searrow 0}\frac{\lambda\big(\mathbb{D}_{tr}(z_0)\big)}{\lambda\big(\mathbb{D}_r(z_0)\big)},\; t \in (0,1].$$

We assert that (54) implies that $\Lambda(t) \geqslant t^2$ for all $t \in (0,1]$. Indeed suppose for a contradiction that it fails at some $s \in (0,1)$, so that $\Lambda(s) < s^2$. First note that since the function $r \mapsto \lambda\big(\mathbb{D}_r(z_0)\big)$ is left-continuous and the set E is at most countable we actually have that

$$\Lambda(t) = \limsup_{r\searrow 0}\frac{\lambda\big(\mathbb{D}_{tr}(z_0)\big)}{\lambda\big(\mathbb{D}_r(z_0)\big)},\; t \in (0,1],$$

and so choosing $t \in (0,s)$ with $\Lambda(s) < t^2$ we get by iteration that for some $\delta > 0$,

$$\lambda\big(\mathbb{D}_{s^j\delta}(z_0)\big) < t^{2j}\lambda\big(\mathbb{D}_\delta(z_0)\big)$$

holds for all $j \in \mathbb{N}$. But this contradicts (54) since then, as $j \to \infty$,

$$\frac{\lambda\big(\mathbb{D}_{s^j\delta}(z_0)\big)}{s^{2j}\delta^2} < \left(\frac{t}{s}\right)^{2j}\frac{\lambda\big(\mathbb{D}_\delta(z_0)\big)}{\delta^2} \to 0.$$

In terms of the maps $v_{j,r}$ this amounts to that the bound

$$\limsup_{E\not\ni r\searrow 0}\lim_{j\to\infty}\int_{\mathbb{D}_s(0)}\mathbf{B}_p(\mathrm{D}v_{j,r})\,\mathrm{d}m(z) \leqslant -\pi s^2 \tag{57}$$

holds for each $s \in (0,1]$. In order to conclude the proof we fix $s \in (0,1)$, use (56) and (57) to select a null sequence $r_j \searrow 0$ such that $v_j \equiv v_{j,r_j} \to 0$ strongly in $\mathrm{W}^{1,2}(\mathbb{D})$ and

$$\lim_{j\to\infty}\int_{\mathbb{D}_s(0)}\mathbf{B}_p(\mathrm{D}v_j)\,\mathrm{d}m(z) \leqslant -\pi s^2.$$

Now $v_j\colon \mathbb{D} \to \mathbb{C}$ are K-quasiregular maps and so we may Stoilow factorize as $v_j = h_j \circ f_j$, where $f_j\colon \mathbb{C} \to \mathbb{C}$ are K-quasiconformal normalized principal and $h_j\colon f_j(\mathbb{D}) \to \mathbb{C}$ are holomorphic. Taking a further subsequence if necessary (not relabelled) we may assume that $f_j \to f$ uniformly on $\mathbb{C}$, where $f\colon \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal normalized principal map. We then also have that $h_j \to 0$ locally uniformly on $f(\mathbb{D})$. It follows that $h_j' \circ f_j \to 0$ uniformly on $\mathbb{D}_s(0)$ and therefore

$$\begin{aligned}
-\pi s^2 &\geqslant \lim_{j\to\infty}\int_{\mathbb{D}_s(0)}\big|h_j' \circ f_j\big|^p\mathbf{B}_p(\mathrm{D}f_j)\,\mathrm{d}m(z)\\
&\geqslant \sup_{\mathbb{D}_s(0)}\big|h_j' \circ f_j\big|^p\int_{\mathbb{D}}\mathbf{B}_p(\mathrm{D}f_j)\,\mathrm{d}m(z)\\
&\geqslant -\pi\sup_{\mathbb{D}_s(0)}\big|h_j' \circ f_j\big|^p \to 0
\end{aligned}$$

as $j \to \infty$, which is impossible. The set of points $z_0 \in B$ satisfying (54) and (55) is therefore empty, and since λ^s almost all points in B have those properties we infer that $\lambda^s(B) = 0$ as required. □

For the Dirichlet classes we have the following almost immediate corollary.

Corollary 3.7. *Let $p > 2$ and put $K \equiv p/(p-2)$. Let Ω be a bounded open subset of $\mathbb{C}$ and $g\colon \mathbb{C} \to \mathbb{C}$ be a K-quasiregular map of class $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{C})$ If $u_j \in g + \mathrm{W}^{1,p}_0(\Omega)$ are K-quasiregular and $u_j \to u$ weakly in $\mathrm{W}^{1,p}(\Omega)$, then $u \in g + \mathrm{W}^{1,p}_0(\Omega)$ is K-quasiregular and*

$$\liminf_{j \to \infty} \int_\Omega \mathbf{B}_p(\mathrm{D}u_j)\,\mathrm{d}m(z) \geqslant \int_\Omega \mathbf{B}_p(\mathrm{D}u)\,\mathrm{d}m(z).$$

We emphasize that we assume $u_j \to u$ weakly in the critical Sobolev space $\mathrm{W}^{1,p}(\Omega)$ and that this of course automatically entails that $u \in \mathrm{W}^{1,p}(\Omega)$. If we only knew that u_j, u were K-quasiregular and $u_j \to u$ uniformly on Ω, then we would merely have that $\mathrm{D}u$ belonged to the local Marcinkiewicz space $\mathrm{L}^{p,\infty}_{\mathrm{loc}}(\Omega)$. Theorem 1.5 does not cover this situation.

Proof. Consider the metric neighbourhood $\Omega_r \equiv \mathbb{D}_r(\Omega) \equiv \{z \in \mathbb{C} :\, \mathrm{dist}(z,\Omega) < r\}$ for $r > 0$. It is clearly precompact and $\Omega \Subset \Omega_r$. Since the function $\mathrm{dist}(\cdot,\Omega)$ is Lipschitz we may use the coarea formula to see that also $\mathscr{L}^2(\partial\Omega_r) = 0$ holds for $\mathscr{L}^1$ almost all $r > 0$, and we fix such an r. Now extend the maps u_j, u to $\mathbb{C}\backslash\Omega$ by g, and record that hereby u_j, u are K-quasiregular maps in the critical Sobolev space $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{C})$ and that $u_j \to u$ weakly in $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{C})$. The latter in particular implies that also $u_j \to u$ uniformly on $\mathbb{C}$, and so according to Theorem 1.5

$$\liminf_{j \to \infty} \int_{\Omega_r} \mathbf{B}_p(\mathrm{D}u_j)\,\mathrm{d}m(z) \geqslant \int_{\Omega_r} \mathbf{B}_p(\mathrm{D}u)\,\mathrm{d}m(z)$$

holds. Note that here we have $u_j = u = g$ on $\Omega_r\backslash\Omega$ and so by standard properties of Sobolev functions and approximate derivatives we get in particular that $\mathrm{D}u_j = \mathrm{D}u$ a.e. on $\Omega_r\backslash\Omega$ (also in the pathological case where $\mathscr{L}^2(\partial\Omega) > 0$ that has not been excluded). Because $\mathbf{B}_p(\mathrm{D}u) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{C})$ the required conclusion follows from this. □

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