Moduli spaces of compact RCD structures

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Abstract

This thesis investigates RCD spaces, which are metric measure spaces with Ricci curvature bounded below and dimension bounded above in a synthetic sense. We introduce moduli spaces of compact RCD structures and study their topology. In particular, we discuss the results obtained in [MN22] (written in collaboration with Andrea Mondino) and [Nav22].

In Chapter 2, we present the primary tools we use in the thesis. We recall Gromov–Hausdorff type topologies and RCD spaces with their covering and moduli spaces. The main contributions of this chapter are the *equivariant measured Gromov–Hausdorff topology* and the *equivariant theorem* (both obtained in [MN22]).

In Chapter 3, we focus on the case of nonnegative curvature. In particular, we obtain topological invariants of $\mathrm{RCD}(0, N)$ spaces using the splitting theorem. In addition, we introduce the *Albanese and soul maps* and prove their continuity. This last result is the most technical part of the chapter and was obtained in [MN22]. Finally, we use the Albanese map to construct examples of moduli spaces with non-trivial higher homotopy groups in every dimension $N \geq 3$.

Chapter 4 is devoted to nonnegative curvature in dimension 2 and discusses the results of [Nav22]. We obtain a classification (up to homeomorphism) of the topological spaces that admit an $\mathrm{RCD}(0, 2)$ structure. For every space appearing in the classification, we compute the homeomorphism type of the moduli space of $\mathrm{RCD}(0, 2)$ structures and show that it is contractible.

Finally, in Chapter 5, we apply Ricci flow techniques to study moduli spaces of $\mathrm{RCD}(-1, 2)$ structures. In particular, we show that if a space has a negative Euler characteristic, then its moduli space of $\mathrm{RCD}(-1, 2)$ structures is homotopy equivalent to its moduli space of hyperbolic metrics.

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Abstract

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Introduction

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This thesis investigates RCD spaces, which are possibly singular metric measure spaces with Ricci curvature bounded below and dimension bounded above. We introduce moduli spaces of compact RCD structures and study their topology. Most of our results appear in [MN22] (written in collaboration with Andrea Mondino) and [Nav22]. Our research first focuses on the topology of $\mathrm{RCD}(0, N)$ spaces, providing examples of moduli spaces with nontrivial topology. Next, we classify topological spaces that admit an $\mathrm{RCD}(0, 2)$ structure and explicitly compute their moduli spaces' homeomorphism type, showing that it is contractible. Lastly, we use the Ricci flow to study the homotopy type of moduli spaces of $\mathrm{RCD}(-1, 2)$ structures.

In this chapter, we outline motivations for studying moduli spaces of metrics that satisfy curvature lower bounds in Section 1.1. Afterwards, in Section 1.2, we present our research problem, which involves constructing and analysing moduli spaces of compact RCD structures. Finally, we introduce the main contributions of this thesis in Section 1.3.

1.1 Motivations

This section presents some motivations for studying curvature lower bounds in geometry. First, we will recall the connection between curvature lower bounds and topology in global Riemannian geometry. Then, we will explain why working with synthetic versions such as Alexandrov and RCD spaces is relevant. Afterwards, we will introduce moduli spaces of metrics satisfying curvature lower bounds. We will survey results about their topology in the smooth setting and the Alexandrov case. Finally, we will move to moduli spaces of RCD structures.

1.1.1 Curvature lower bounds in Geometry

1.1.1.1 Riemannian geometry

Riemannian geometry provides a powerful framework for studying the geometric properties of non-Euclidean spaces. One of its key concepts is curvature, which measures how much a space deviates from Euclidean geometry at an infinitesimal level. There are several ways to define curvature, but three types have been extensively studied: sectional curvature, Ricci curvature, and scalar curvature.

Global Riemannian geometry is an important subfield that seeks to describe the global geometric properties of Riemannian manifolds, including their topology. A fundamental question that has received significant attention is whether we can classify all Riemannian manifolds that satisfy a specific set of geometric constraints up to homotopy, homeomorphism, or diffeomorphism.

Curvature lower bounds are a powerful tool for addressing this question. In particular, while sectional curvature lower bounds are a stronger assumption and lead to stronger results, we will focus on Ricci curvature lower bounds in this thesis.

For example, in dimension 2, the Gauss-Bonnet theorem states that for a closed Riemannian surface (M^2, g), we have $\int_M \sec(g)\,\mathrm{dvol}_g = 2\pi\chi(M)$, where $\sec(g)$ denotes the sectional curvature and $\chi(M)$ is the Euler characteristic of M. Using the uniformisation theorem for surfaces, we can obtain the following result.

Theorem 1.1.1. A closed surface admits a non-negatively Ricci curved metric if and only if it is diffeomorphic to one of the following spaces: $\mathbb{S}^2$, $\mathbb{RP}^2$, $\mathbb{R}^2/\mathbb{Z}^2$, or the Klein bottle $\mathbb{K}^2$.

After introducing the Ricci flow, Hamilton successfully described 3-dimensional closed Riemannian manifolds with nonnegative Ricci curved metric.

Theorem 1.1.2 ([Ham86, Theorem 1.2])**.** A closed 3-dimensional Riemannian manifold with nonnegative Ricci curvature is necessarily diffeomorphic to a quotient of one of the spaces $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{R}^3$ by a group of fixed point free isometries in the standard metrics.

In higher dimensions, lower bounds on the Ricci curvature are not enough to obtain classification results anymore, and one has to add more geometric constraints, such as volume constraints. For instance, while the Bishop–Gromov inequality implies that the volume growth of nonnegatively Ricci curved manifolds is at most Euclidean, one can characterise the Euclidean space by this property.

Proposition 1.1.1. Let (M^n, g, p) be a pointed complete Riemannian manifold such that $\mathrm{Ric}(g) \geq 0$ and $\mathrm{vol}_g(B_R(p)) = V_n(R)$, for every $R > 0$ (where $V_n(R)$ is the volume of a ball of radius R in $\mathbb{R}^n$). Then, (M^n, g) is isometric to $\mathbb{R}^n$.

However, this result is not very flexible as it does not tell us whether a non-negatively Ricci curved manifold with "almost" Euclidean volume growth is "almost" Euclidean. To address this issue, first one needs a way to compare manifolds.

In [GLP98], Gromov introduced a way to compare abstract metric spaces, leading to the Gromov–Hausdorff distance d_{GH} on the set of isometry classes of compact metric spaces. In particular, he observed a crucial property, referred to as Gromov's precompactness theorem.

Theorem 1.1.3. Let $\{(M_i^n, g_i)\}_{i\in\mathbb{N}}$ be a sequence of n-dimensional compact Riemannian manifolds with $\mathrm{Ric} \geq K$ and $\sup_{i\in\mathbb{N}}\{\mathrm{Diam}(M^n, g_i)\} < \infty$ ($n \geq 2$ and $K \in \mathbb{R}$). Then $\{(M_i^n, g_i)\}_{i\in\mathbb{N}}$ admits a subsequence converging to a compact length space, i.e. there exists a compact length space (X, d_X) such that, passing to a subsequence if necessary, we have $\lim_{i\to\infty} \mathrm{d_{GH}}((M_i^n, \mathrm{d}_{g_i}), (X, \mathrm{d}_X)) = 0$.

Limit spaces appearing in Gromov's precompactness theorem are called Ricci-limit spaces and were thoroughly studied by Cheeger and Colding in their seminal papers [CC97], [CC00a], and [CC00b]. In particular, they obtain the following stable version of Proposition 1.1.1 (see [CC97, Theorem 1.1.11]).

Theorem 1.1.4. For every $\epsilon > 0$ and $n \geq 2$, there exists $\delta(n, \epsilon) > 0$ such that, for every pointed complete Riemannian manifold (M^n, g, p) satisfying $\mathrm{Ric}(g) \geq 0$ and $\mathrm{vol}_g(B_R(p)) \geq (1-\delta)V_n(R)$ (for every $R > 0$), we have $\mathrm{d_{GH}}(B^g_{\epsilon^{-1}}(p), B^{\mathbb{R}^n}_{\epsilon^{-1}}(0)) \leq \epsilon$.

The proof of Theorem 1.1.4 proceeds by contradiction. Suppose the result is wrong. Then, there exist $\epsilon > 0$ and a sequence of pointed complete Riemannian manifolds $\{(M_i^n, g_i, p_i)\}_{i\in\mathbb{N}}$ such that $\mathrm{Ric}(g_i) \geq 0$ and for all $i \in \mathbb{N}$ and $R > 0$:

$$\mathrm{vol}_{g_i}(B_R(p_i)) \geq (1-2^{-i})V_n(R) \quad \text{and} \quad \mathrm{d_{GH}}(B^{g_i}_{\epsilon^{-1}}(p_i), B^{\mathbb{R}^n}_{\epsilon^{-1}}(0)) \geq \epsilon. \tag{1.1}$$

By Gromov's precompactness theorem, the sequence subconverges to a complete pointed length space (X, d, p), referred to as a non-collapsed limit space. Cheeger and Colding proved that the Riemannian volume measures converge to the Hausdorff measure $\mathcal{H}^n$ on (X, d). Therefore, passing to the limit in (1.1) yields $\mathcal{H}^n(B_R(p)) = V_n(R)$ for every $R > 0$. In particular, using the "volume cone implies metric cone property" and the generalised splitting theorem for Ricci limits, they show that (X, d) is isometric to $\mathbb{R}^n$. This contradicts $\mathrm{d_{GH}}(B^{\mathrm{d}}_{\epsilon^{-1}}(p), B^{\mathbb{R}^n}_{\epsilon^{-1}}(0)) \geq \epsilon$.

The proof we just described is now classic in global Riemannian geometry and motivates the study of the geometric properties of Ricci limit spaces.

1.1.1.2 Singular geometry

In general, Ricci limit spaces are singular. While they effectively relied on approximating sequences, Cheeger and Colding recognised the need for a more intrinsic definition of Ricci curvature lower bounds, which is referred to as a synthetic definition (see [CC97, Appendix 2]).

Alexandrov spaces. The key to understanding synthetic sectional curvature lower bounds is a result obtained first by A.D. Alexandrov in the case of convex surfaces and later by Toponogov in the general case of Riemannian manifolds. We state the theorem in a simplified form here, but a rigorous statement can be found in Appendix A.2.

Theorem 1.1.5. A complete Riemannian manifold (M^n, g) satisfies sec $\geq K$ if and only if triangles in (M, d_g) are thicker than triangles in the simply connected Riemannian manifold of constant sectional curvature K and dimension 2.

The classical definition of the sectional curvature of a Riemannian manifold relies on its underlying smooth structure. However, thanks to the result above, one can define sectional curvature lower bounds in terms of a Riemannian manifold's geodesic space structure. In particular, it is possible to generalise sectional curvature lower bounds to possibly singular geodesic spaces in the following way.

Definition 1.1.1. A complete geodesic space (X, d) has curvature bounded below by K in the Alexandrov sense if triangles in (X, d) are thicker than triangles in the simply connected Riemannian manifold of constant sectional curvature K and dimension 2.

Burago, Gromov, and Perelman proved in [BGP92] that the definition above is stable under Gromov–Hausdorff convergence. Consequently, any result established for Alexandrov spaces (i.e. geodesic spaces with curvature bounded below in the Alexandrov sense) applies a fortiori to limits of Riemannian manifolds with lower bounds on their sectional curvature. The authors of [BGP92] conducted an extensive investigation into the geometric properties of Alexandrov spaces. This study played a significant role in Perelman's study of the singularities of the Ricci flow in dimension 3, ultimately leading to the proof of the Poincaré conjecture.

RCD spaces. The situation for Ricci curvature lower bounds is different. Indeed, the sectional curvature is related to the distortion of the geodesic distance, which is why sectional curvature lower bounds could be defined in metric terms. However, the Ricci curvature is related to the interplay between the geodesic distance and the Riemannian volume measure (which can be seen from the Bishop–Gromov inequality). Therefore, Ricci curvature lower bounds should be expressed in terms of the metric measure structure of the Riemannian manifold.

The key to generalising Ricci curvature lower bounds comes from Optimal Transport, which we recall briefly. First, let us fix a complete separable metric space (X, d) and probability measures $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ with finite second-order moments. The $\mathcal{L}^2$-Wasserstein distance between μ_0 and μ_1 is defined as:

$$\mathcal{W}_2(\mu_0, \mu_1) := \sqrt{\inf\left\{ \int_{X\times X} \mathrm{d}^2(x,y)\,\mathrm{d}\pi(x,y)\right\}},$$

where the infimum is taken over all probability measures $\pi \in \mathcal{P}(X \times X)$ whose first and second marginal are μ_0 and μ_1, respectively. The Wasserstein distance endows $\mathcal{P}_2(X)$ with a complete separable metric space structure. Now, consider a locally finite measure $\mathfrak{m}$ on (X, d). The triple $(X, \mathrm{d}, \mathfrak{m})$ is called a "metric measure space" (m.m.s. for short). The relative entropy $\mathrm{Ent}(\cdot|\mathfrak{m})$ of $(X, \mathrm{d}, \mathfrak{m})$ is defined as:

$$\mathrm{Ent}(\mu|\mathfrak{m}) := \begin{cases} \int_X \rho \log(\rho)\, \mathrm{d}\,\mathfrak{m}, & \text{if } \mu = \rho\,\mathfrak{m} \ll \mathfrak{m} \text{ and } \rho\log(\rho) \text{ is integrable} \\ \infty, & \text{otherwise} \end{cases},$$

for every $\mu \in \mathcal{P}_2(X, \mathrm{d})$. Finally, $\mathcal{P}_2^*(X, \mathrm{d}, \mathfrak{m}) := \{\mu \in \mathcal{P}_2(X, \mathrm{d}), \mathrm{Ent}(\mu|\mathfrak{m}) < \infty\}$ denotes the domain of the entropy.

Coming back to the Riemannian setting, it was conjectured by Otto and Villani that a lower bound on the Ricci curvature of a Riemannian manifold (M^n, g) can be expressed in terms of the convexity property of the entropy functional $\mathrm{Ent}(\cdot|\mathrm{vol}_g)$ on its domain $\mathcal{P}_2^*(M, \mathrm{d}_g, \mathrm{vol}_g)$. This conjecture was confirmed by Cordero-Erausquin, McCann, and Schmuckenschläger in [CMS01] and by von Renesse and Sturm in [RS05].

Theorem 1.1.6. Let (M^n, g) be a complete Riemannian manifold and let $K \in \mathbb{R}$. Then, $\mathrm{Ric}(g) \geq K$ if and only if the entropy functional $\mathrm{Ent}(\cdot|\mathrm{vol}_g)$ is K-convex on $(\mathcal{P}_2^*(M, \mathrm{d}_g, \mathrm{vol}_g), \mathcal{W}_2)$, i.e., every pair $\mu_0, \mu_1 \in \mathcal{P}_2^*(M, \mathrm{d}_g, \mathrm{vol}_g)$ is joined by a geodesic $\{\mu_t\}_{t\in[0,1]}$ in $(\mathcal{P}_2^*(M, \mathrm{d}_g, \mathrm{vol}_g), \mathcal{W}_2)$ such that:

$$\mathrm{Ent}(\mu_t|\mathfrak{m}) \leq t\mathrm{Ent}(\mu_1|\mathfrak{m}) + (1-t)\mathrm{Ent}(\mu_0|\mathfrak{m}) - \frac{K}{2}t(1-t)\mathcal{W}_2^2(\mu_0, \mu_1),$$

for every $t \in [0, 1]$.

The result above is the key to generalising Ricci curvature lower bounds at the level of metric measure spaces. Indeed, while the inequality $\mathrm{Ric} \geq K$ relies on the underlying smooth structure of the Riemannian manifold, the convexity properties of the $\mathrm{Ent}(\cdot|\mathfrak{m})$ can be defined for a general metric measure space. This observation motivated the following definition, introduced in the seminal papers [LV09] and [Stu06a] by Lott–Villani and Sturm, respectively.

Definition 1.1.2. Let $(X, \mathrm{d}, \mathfrak{m})$ be an m.m.s. and let $K \in \mathbb{R}$. We say that $(X, \mathrm{d}, \mathfrak{m})$ satisfies the $\mathrm{CD}(K, \infty)$ condition if $\mathrm{Ent}(\cdot|\mathfrak{m})$ is K-convex on $(\mathcal{P}_2^*(X, \mathrm{d}, \mathfrak{m}), \mathcal{W}_2)$.

According to the results of [LV09] and [Stu06a], the definition above is stable under measured Gromov–Hausdorff convergence (an analogue of the Gromov–Hausdorff convergence for m.m.s. introduced by Fukaya in [Fuk87]). In particular, any result proven for $\mathrm{CD}(K, \infty)$ spaces holds a fortiori for Ricci limit spaces (endowed with a limit measure as defined in [CC97]).

Since its appearance, the $\mathrm{CD}(K, \infty)$ condition has been refined in various ways. Here, we provide a non-exhaustive list (in chronological order):

- The $\mathrm{CD}(K, N)$ condition was introduced in [LV09] and [Stu06b] to account for upper bounds on dimension, namely $N \in [1, \infty)$.
- The $\mathrm{CD}^*(K, N)$ condition, introduced in [BS10], is implied by $\mathrm{CD}(K, N)$ and satisfies the local to global property (for essentially non-branching spaces). It was proven by Cavalletti–Milman [CM21] that $\mathrm{CD}(K, N)$ and $\mathrm{CD}^*(K, N)$ are equivalent for essentially non-branching m.m.s. of finite total measure.
- The $\mathrm{RCD}(K, \infty)$ condition was introduced in [AGS14] to rule out non-Riemannian Finsler spaces (which satisfy the $\mathrm{CD}^*(K, N)$ condition as observed in [Oht09b]).
- The $\mathrm{RCD}(K, N)$ (and $\mathrm{RCD}^*(K, N)$) condition was proposed in [Gig15] as a finite dimensional counterpart of $\mathrm{RCD}(K, \infty)$, see also [EKS15] and [AMS19].

The classes above generalise Ricci curvature lower bounds and are stable under measured Gromov–Hausdorff convergence. Therefore, they are all considered acceptable synthetic definitions of Ricci curvature lower bounds.

Currently, the $\mathrm{RCD}^*(K, N)$ condition is widely regarded as the most promising synthetic definition of Ricci curvature lower bounds. In the past decade, $\mathrm{RCD}^*(K, N)$ spaces have received significant attention and have been instrumental in resolving longstanding questions related to Ricci limit spaces. For instance, in [CM17a] and [Den20], Cavalletti–Mondino and Deng respectively proved that $\mathrm{RCD}^*(K, N)$ spaces satisfy the Lévy–Gromov isoperimetric inequality and are non-branching.

1.1.2 Moduli spaces

1.1.2.1 The smooth case

So far, we were interested in studying the manifolds that admit a metric satisfying a specific geometric constraint denoted by $\mathcal{C}$. In particular, we introduced various notions of lower bounds on curvature. We highlighted their relevance in the classification of spaces (see, for instance, Theorem 1.1.1 and 1.1.2 in the case where $\mathcal{C} = \mathrm{Ric} \geq 0$). From such results, one hopes to answer the following question: Given a smooth manifold M^n, is there any smooth Riemannian metric g satisfying the constraint $\mathcal{C}$ ($\mathcal{C}$-metric for short)? However, once we know that a manifold M^n admits a $\mathcal{C}$-metric, it is natural to ask: *How many such metrics are there*? This problem might seem vague as there might not be, in general, a finite amount of such metrics. The common way to interpret it is to study the *space and moduli space of* $\mathcal{C}$*-metrics*, which we introduce now.

First of all, the *space of* $\mathcal{C}$*-metrics on* M^n is the set $\mathscr{R}_{\mathcal{C}}(M^n)$ of all smooth Riemannian metrics on M^n that satisfy $\mathcal{C}$. We equip $\mathscr{R}_{\mathcal{C}}(M^n)$ with the topology of smooth convergence on compact subsets of M^n. The topological space $\mathscr{R}_{\mathcal{C}}(M^n)$ is already a good candidate to describe the set of $\mathcal{C}$-metrics. However, depending on our point of view, it may contain too much information. Indeed, in geometry, it is customary to regard Riemannian metrics as equivalent whenever they are isometric. To capture this notion of equivalence, we define the *moduli space of* $\mathcal{C}$*-metrics* as the following quotient:

$$\mathscr{M}_{\mathcal{C}}(M^n) := \mathscr{R}_{\mathcal{C}}(M^n)/\sim,$$

where $\sim$ denotes the isometry equivalence relation. The space $\mathscr{M}_{\mathcal{C}}(M^n)$ is endowed with the quotient topology inherited from $\mathscr{R}_{\mathcal{C}}(M^n)$, and serves to describe the set of $\mathcal{C}$-metrics in a more meaningful way.

A more rigorous approach to study the complexity of the space of $\mathcal{C}$-metrics is to investigate the topological properties of $\mathscr{R}_{\mathcal{C}}(M^n)$ and $\mathscr{M}_{\mathcal{C}}(M^n)$. In general, describing their homeomorphism type can be very difficult. However, one can still try to obtain as much information as possible on their homotopy groups. The cases of positive scalar curvature, nonnegative sectional curvature, and positive Ricci curvature have been thoroughly studied (an extensive survey and new results can be found in [TW15]). Let us present a few striking results about connectedness properties in the compact case.

Positive scalar curvature. In dimension 2, the spaces of positive scalar curvature metrics $\mathscr{R}_{\mathrm{scal}>0}(\mathbb{S}^2)$ and $\mathscr{R}_{\mathrm{scal}>0}(\mathbb{RP}^2)$ are contractible (see [RS01, Theorem 3.4]). In dimension 3, if M^3 is a closed orientable manifold such that $\mathscr{R}_{\mathrm{scal}>0}(M^3) \neq \emptyset$, then $\mathscr{M}_{\mathrm{scal}>0}(M^3)$ is path connected (see [Mar12, Main Theorem]) and $\mathscr{R}_{\mathrm{scal}>0}(M^3)$ is contractible (see [BK19, Theorem 1.1]). In dimension 4, for every $N \in \mathbb{N}$, there exists a closed 4-manifold M^4 such that $\mathscr{M}_{\mathrm{scal}>0}(M^4)$ has at least N path components (see [Rub01, Theorem 6.2]). Furthermore, if M^n is a spherical space form of dimension $n \geq 5$ which is not simply connected, then $\mathscr{M}_{\mathrm{scal}>0}(M^n)$ has infinitely many path compontents (see [BG96, Theorem 0.1]). Finally, in dimension $4n-1$ ($n \geq 2$), the moduli space $\mathscr{M}_{\mathrm{scal}>0}(\mathbb{S}^{4n-1})$ has infinitely many path components (see [TW15, Theorem 6.1.1]).

Positive Ricci or nonnegative sectional curvature. In dimension 2, if M^2 denotes either $\mathbb{S}^2$ or $\mathbb{RP}^2$ and $\lambda \in \mathbb{R}$, then both $\mathscr{R}_{\sec\geq\lambda}(M^2)$ and $\mathscr{R}_{\sec>\lambda}(M^2)$ are homeomorphic to the complete separable hilbert space l^2 (see [BB18, Theorem 1.2], which improves [RS01, Theorem 3.4]). In dimension 5, there are explicit examples of 5-dimensional closed manifolds M^5 such that both $\mathscr{M}_{\sec\geq 0}(M^5)$ and $\mathscr{M}_{\mathrm{Ric}>0}(M^5)$ have infinitely many path components (see [DG21, Theorem A] and [GW22, Theorem A]). Furthermore, in dimension $n = 4k+1 \geq 9$ or $n = 4k+3 \geq 7$, there exists infinitely many pairwise non-homeomorphic closed manifolds M^n of dimension n such that $\mathscr{M}_{\sec\geq 0}(M^n)$ has infinitely many path components (see [Des22, Main Theorem] and [DKT18, Theorem 1.1]). Finally, in dimension $n = 4k+1 \geq 9$ or $n = 4k+3 \geq 7$, there exists n-dimensional closed manifolds M^n of dimension n such that $\mathscr{M}_{\mathrm{Ric}>0}(M^n)$ has infinitely many path components (see [Des22, Main Theorem] and [Wra11, Theorem A]).

In addition to such connectedness results, [CSS18, Corollary 1.9] implies that, given a spin manifold M^n of dimension $n \geq 6$, there exists $g_0 \in \mathscr{R}_{\mathrm{scal}>0}(M^n)$ such that, for infitely many $k \in \mathbb{N}$, $\pi_k(\mathscr{R}_{\mathrm{scal}>0}(M^n), g_0)$ is not trivial (the same statement holds with scal > 0 replaced by Ric > 0 or sec > 0).

While much research has been conducted in the cases above, results about spaces and moduli spaces of nonnegatively Ricci curved metrics remained rare. The first result on the topology of such spaces provides n-dimensional closed manifolds M^n such that $\mathscr{R}_{\mathrm{Ric}\geq 0}$ has infinitely many path components, in each dimension $n = 4k+3 \geq 11$ (see [SW21, Theorem 1.11]). Then, relying on the examples constructed in [DKT18] and applying [SW21, Theorem 1.8], the authors of [TW22] observe that in each dimension $n = 4k + 3 \geq 7$, there exist infinitely many pairwise non-homeomorphic n-dimensional closed manifolds M^n such that $\mathscr{M}_{\mathrm{Ric}\geq 0}$ has infinitely many path components. In particular, by taking products with tori, they close the gap and show that the result mentioned holds for every $n \geq 7$ (see [TW22, Theorem 1.3]). To conclude this survey of results, we state the first result from Tuschmann and Wiemeler concerning higher rational homotopy groups of moduli spaces of nonnegatively Ricci curved metrics together with its corollary.

Theorem 1.1.7 ([TW22, Theorem 1.1])**.** Let M^n be a closed simply connected manifold of dimension $n \geq 2$ and let $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$ the k-dimensional torus, where $k \geq 4$ and $k \neq 8, 9, 10$. Then the moduli space of nonnegative Ricci curved metrics on $M^n \times \mathbb{T}^k$ has non-trivial higher rational homotopy groups. More precisely, we have $\pi_3(\mathscr{M}_{\mathrm{Ric}\geq 0}(M^n \times \mathbb{T}^4)) \otimes \mathbb{Q} \neq 0$ and $\pi_5(\mathscr{M}_{\mathrm{Ric}\geq 0}(M^n \times \mathbb{T}^k)) \otimes \mathbb{Q} \neq 0$, for every $k > 4$.

By taking the product of $\mathbb{T}^4$ with spheres, Theorem 1.1.7 implies the following corollary.

Corollary 1.1.1 ([TW22, Corollary 1.2])**.** In every dimension $n \geq 6$, there exists a closed n-dimensional manifold M^n such that $\pi_1(M^n) \simeq \mathbb{Z}^4$ and such that $\pi_3(\mathscr{M}_{\mathrm{Ric}\geq 0}(M^n) \otimes \mathbb{Q} \neq 0$.

We will be interested in generalising these last two results in a singular setting later on.

1.1.2.2 The singular case

While moduli spaces of smooth metrics satisfying a curvature lower bound have received significant attention, their singular counterparts have been relatively unexplored. In particular, only a few results mention the moduli spaces of Alexandrov metrics. Belegradek [Bel18b] studied moduli spaces of Alexandrov metrics on the 2-sphere. Let us introduce moduli spaces of Alexandrov metrics on topological spaces and present [Bel18b, Theorem 1.1].

Let X be a compact topological space admitting an Alexandrov metric d with nonnegative curvature and finite Hausdorff dimension. We denote by $\mathscr{R}_{\mathrm{curv}\geq 0}(X)$

the set of all nonnegatively curved Alexandrov metrics d on X. The moduli space of nonnegatively curved Alexandrov metrics on X is the quotient space:

$$\mathscr{M}_{\text{curv}\geq 0}(X) := \mathscr{R}_{\text{curv}\geq 0}(X)/\sim,$$

where $\sim$ denotes the isometry equivalence relation. It is natural to equip $\mathscr{M}_{\text{curv}\geq 0}(X)$ with the Gromov–Hausdorff metric. Indeed, Alexandrov spaces are generally non-smooth, and hence, the topology of smooth convergence of metrics is not well-defined.

Before stating Belegradek's theorem, we denote by $\mathcal{K}^s_{2\leq 3}$ the space of convex compact subsets of $\mathbb{R}^3$ with their centre of mass at the origin and dimension in $\{2, 3\}$. We equip $\mathcal{K}^s_{2\leq 3}$ and $\mathcal{K}^s_{2\leq 3}/\operatorname{O}_3(\mathbb{R})$ with the Hausdorff distance $\mathrm{d}^{\mathrm{H}}_{\mathbb{R}^3}$ and the quotient topology, respectively.

Theorem 1.1.8 ([Bel18b, Theorem 1.1])**.** The moduli space $\mathscr{M}_{\text{curv}\geq 0}(\mathbb{S}^2)$ of nonnegatively curved Alexandrov metrics on $\mathbb{S}^2$ is homeomorphic to $\mathcal{K}^s_{2\leq 3}/\operatorname{O}_3(\mathbb{R})$.

To the best of our knowledge, so far, Belegradek's work [Bel18b] is the only paper that has studied moduli spaces of singular metrics with curvature bounded below. In particular, no research has yet been undertaken to study the topological properties of moduli spaces of RCD structures. This leads us to the research problem we will study in this Thesis.

1.2 Research problem

The main goal of this thesis will be the study of moduli spaces of RCD structures. Before we discuss some more specific questions, let us first introduce the notion of an RCD structure on a topological space.

Definition 1.2.1. Let $K \in \mathbb{R}$ and $N \in [1, \infty)$ and let X be a topological space. An $\mathrm{RCD}(K, N)$ structure on X is a triple $(X, \mathrm{d}, \mathfrak{m})$ such that d is a complete separable distance, metrizing the topology of X, $\mathfrak{m}$ is a full support locally finite measure on (X, d), and $(X, \mathrm{d}, \mathfrak{m})$ satisfies the $\mathrm{RCD}(K, N)$ condition.

Remark 1.2.1. A topological space X endowed with an $\mathrm{RCD}(K, N)$ structure $(X, \mathrm{d}, \mathfrak{m})$ is the singular analogue of a smooth manifold M^n of dimension $n \leq N$ endowed with a smooth Riemannian metric g such that $\mathrm{Ric}(g) \geq K$.

In general, there are topological obstructions to the existence of a metric satisfying a specific curvature lower bound (see for instance Theorem 1.1.1 and Theorem 1.1.2). We will study this problem in a singular setting.

Question 1.2.1. What are the topological obstructions to the existence of an $\mathrm{RCD}(K,N)$ structure?

More specifically, in the lower dimensional case, we hope to solve the following classification problem.

Question 1.2.2. Can we classify up to homeomorphism all the compact topological spaces that admit an $\mathrm{RCD}(K,N)$ structure?

The questions above are related to the existence of an $\mathrm{RCD}(K,N)$ structure. However, when a topological space admits such a metric, it is interesting to describe the space of such metrics. To tackle this problem, we first need to introduce the appropriate moduli space.

Definition 1.2.2. Let X be a compact topological space admitting an $\mathrm{RCD}(K,N)$ structure ($K \in \mathbb{R}$ and $N \geq 1$). We denote $\mathfrak{R}_{K,N}(X)$ the set of all $\mathrm{RCD}(K,N)$ structures on X. The *moduli space of* $\mathrm{RCD}(K,N)$ *structures on* X is the following quotient:

$$\mathfrak{M}_{K,N}(X) := \mathfrak{R}_{K,N}(X)/\sim, \tag{1.2}$$

where $\sim$ is the isomorphism of m.m.s. equivalence relation. We equip $\mathfrak{M}_{K,N}(X)$ with the measured Gromov–Hausdorff topology.

Remark 1.2.2. The measured Gromov–Hausdorff topology is well suited to the study of moduli spaces of RCD structures. Indeed, RCD spaces satisfy interesting stability and compactness properties with respect to the aforementioned topology. Moreover, $\mathrm{RCD}(K,N)$ spaces are in general not smooth, and hence, the smooth topology is not well-defined on $\mathfrak{M}_{K,N}(X)$.

The main problem of this thesis will be the following question.

Question 1.2.3. Let X be a compact topological space admitting an $\mathrm{RCD}(K,N)$ structure ($K \in \mathbb{R}$ and $N \geq 1$). What can we say about the topology of the moduli space of $\mathrm{RCD}(K,N)$ structures on X?

We will be particularly interested in obtaining information about the homotopy groups of $\mathfrak{M}_{K,N}(X)$. In particular, in the lower dimensional case, we would like to provide an explicit description of such moduli spaces.

1.3 Thesis structure and main contributions

We conclude this introduction with a presentation of Chapters 2, 3, 4, and 5. In particular, we will highlight the main contributions of each of them.

1.3.1 Chapter 2

Sections 2.1, 2.2, and 2.3 introduce Gromov–Hausdorff type topologies and RCD spaces together with their covering and moduli spaces. The content of Section 2.4 is the main contribution of Chapter 2. There, we introduce the *equivariant pmGH topology* and present the proof of the *equivariant theorem*, which was obtained in collaboration with Andrea Mondino and appears in [MN22]. The rest of this section introduces the equivariant theorem.

1.3.1.1 Equivariant theorem

If X is a compact topological space that admits an $\mathrm{RCD}(K,N)$ structure, then X admits a universal cover $p\colon \tilde{X} \to X$ (see [MW19, Theorem 1.1]). The group of deck transformations of p is also called the revised fundamental group of X and denoted $\overline{\pi}_1(X)$.

Remark 1.3.1. The revised fundamental group of X is isomorphic to $\pi_1(X)$ if and only if X is semi-locally simply connected. In [Wan22], Wang provides promising proof of the semi-local simple connectedness of X. Nevertheless, as this does not alter our result, we will stay on the cautious side and continue writing $\overline{\pi}_1(X)$.

Given an $\mathrm{RCD}(K,N)$ structure $(X,\mathrm{d},\mathfrak{m})$ on X, there exists a unique $\overline{\pi}_1(X)$-equivariant $\mathrm{RCD}^*(K,N)$ structure $p^*(X,\mathrm{d},\mathfrak{m})$ on $\tilde{X}$ (called the *lift of* $(X,\mathrm{d},\mathfrak{m})$) such that $p\colon p^*(X,\mathrm{d},\mathfrak{m}) \to (X,\mathrm{d},\mathfrak{m})$ is a local isomorphism. The equivariant theorem relates the convergence of $\mathrm{RCD}(K,N)$ structures on a compact topological space to the convergence of the associated lifts.

Theorem 1.3.1 (Equivariant theorem)**.** Let X be a compact topological space that admits an $\mathrm{RCD}(K,N)$ structure ($N \in [1,\infty)$ and $K \in \mathbb{R}$) and denote $p\colon \tilde{X} \to X$ the universal cover of X. Moreover, suppose that $\overline{\pi}_1(X)$ is Hopfian (i.e. every surjective homomorphism from $\overline{\pi}_1(X)$ onto itself is an isomorphism). Finally, assume that for every $n \in \mathbb{N} \cup \{\infty\}$:

- $\mathcal{X}_n = (X, \mathrm{d}_n, \mathfrak{m}_n, *_n)$ is a pointed $\mathrm{RCD}(K,N)$ structure on X,

- $\tilde{\mathcal{X}}_n = (\tilde{X}, \tilde{\mathrm{d}}_n, \tilde{\mathfrak{m}}_n, \tilde{*}_n)$ is the associated pointed lift, where $\tilde{*}_n$ is any point in $p^{-1}(*_n)$.

Then, $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ converges to $\mathcal{X}_\infty$ in the pmGH topology (see Definition 2.1.20) if and only if $\{\tilde{\mathcal{X}}_n\}_{n\in\mathbb{N}}$ converges to $\tilde{\mathcal{X}}_\infty$ in the equivariant pmGH topology (see Definition 2.4.8).

Remark 1.3.2. In [MN22], the result was only proven for $\mathrm{RCD}(0, N)$ spaces. For the sake of generality, we present the result in the context of $\mathrm{RCD}(K, N)$ spaces. However, we have to assume that $\overline{\pi}_1(X)$ is Hopfian, i.e. every surjective homomorphism from $\overline{\pi}_1(X)$ onto itself is an isomorphism. Nevertheless, we will see later that the revised fundamental group of a compact $\mathrm{RCD}(0, N)$ space is Hopfian; therefore, we can remove this last assumption in that case.

Remark 1.3.3. Fukaya introduced the equivariant pointed Gromov–Hausdorff topology in [Fuk86]. However, to take the measures into account, we defined the equivariant pmGH topology. We prove in particular that it is metrizable in Appendix A.3.

It is easily seen that isomorphic $\mathrm{RCD}(K, N)$ structures on X have equivariantly isomorphic lifts. Therefore, denoting $\mathfrak{M}^{\mathrm{eq}}_{K,N}(\tilde{X})$ the moduli space of equivariant $\mathrm{RCD}^*(K, N)$ structures on $\tilde{X}$ (see Definition 2.4.3), there is a well-defined map:

$$p^*\colon \mathfrak{M}_{K,N}(X) \to \mathfrak{M}^{\mathrm{eq}}_{K,N}(\tilde{X})$$

called the *lift map*, such that for every $[X, \mathrm{d}, \mathfrak{m}] \in \mathfrak{M}_{K,N}(X)$, we have $p^*([X, \mathrm{d}, \mathfrak{m}]) = [p^*(X, \mathrm{d}, \mathfrak{m})]$ (where brackets denote the equivalence class in the appropriate moduli space). Appealing to a pointed version of the lift map, we can reformulate Theorem 1.3.1 in the following way.

Corollary 1.3.1. Let X be a compact topological space that admits an $\mathrm{RCD}(K, N)$ structure such that $\overline{\pi}_1(X)$ is Hopfian ($N \in [1, \infty)$ and $K \in \mathbb{R}$) and denote $p\colon \tilde{X} \to X$ the universal cover of X. Then, the pointed lift map (introduced in Section 2.4.2):

$$p^*\colon \mathfrak{M}^{\mathrm{p}}_{K,N}(X) \to \mathfrak{M}^{\mathrm{p,eq}}_{K,N}(\tilde{X}),$$

is a homeomorphism, where $\mathfrak{M}^{\mathrm{p,eq}}_{K,N}(\tilde{X})$ and $\mathfrak{M}^{\mathrm{p}}_{K,N}(X)$ are respectively the moduli space of pointed equivariant $\mathrm{RCD}^*(K, N)$ structures on $\tilde{X}$ and the moduli space of pointed $\mathrm{RCD}(K, N)$ structures on X (introduced respectively in Sections 2.2.5 and 2.4.1).

In the case where the universal cover is compact, one can forget about the pointed requirement. More precisely, Corollary 1.3.1 specialises in the following way.

Corollary 1.3.2. Let X be a compact topological space that admits an $\mathrm{RCD}(K,N)$ structure such that $\overline{\pi}_1(X)$ is finite ($N \in [1,\infty)$ and $K \in \mathbb{R}$) and denote $p\colon \tilde{X} \to X$ the universal cover of X. Then, the lift map:

$$p^*\colon \mathfrak{M}_{K,N}(X) \to \mathfrak{M}^{\mathrm{eq}}_{K,N}(\tilde{X}),$$

is a homeomorphism, where $\mathfrak{M}^{\mathrm{eq}}_{K,N}(\tilde{X})$ and $\mathfrak{M}_{K,N}(X)$ are respectively the moduli space of equivariant $\mathrm{RCD}^*(K,N)$ structures on $\tilde{X}$ and the moduli space of $\mathrm{RCD}(K,N)$ structures on X (introduced respectively in Sections 2.2.5 and 2.4.1).

Remark 1.3.4. Corollary 1.3.2 will be especially useful when computing the homeomorphism type of specific examples of moduli spaces (see for example the case of $\mathbb{RP}^2$ in Chapter 4).

1.3.2 Chapter 3

The results presented in Chapter 3 are original and can be found in [MN22], which was written in collaboration with Andrea Mondino. In this chapter, we will analyse the case of nonnegative curvature and general dimension. We will begin by studying the topology of $\mathrm{RCD}(0,N)$ spaces, specifically relying on the splitting theorem to analyse the fundamental group of these spaces. We will then introduce the Albanese and soul maps, which provide insight into how lifts of $\mathrm{RCD}(0,N)$ structures split. The main technical challenge of this chapter will be proving the continuity of these maps. Finally, we will apply this continuity statement to obtain an analogue of Tuschmann and Wiemeler's Theorem 1.1.7.

1.3.2.1 Splittings and topological invariant

Section 3.1 introduces splitting maps and some topological invariants of $\mathrm{RCD}(0,N)$ spaces, hence providing insight into Question 1.2.1. Here, we fix a real number $N \in [1,\infty)$, a compact topological space X that admits an $\mathrm{RCD}(0,N)$ structure, and denote $p\colon \tilde{X} \to X$ the universal cover of X.

A special property enjoyed by $\mathrm{RCD}(0,N)$ spaces is the existence of splittings (see [Gig14, Theorem 1.4]). More precisely, given an $\mathrm{RCD}(0,N)$ structure $(X,\mathrm{d},\mathfrak{m})$ on X and denoting $(\tilde{X},\tilde{\mathrm{d}},\tilde{\mathfrak{m}})$ the associated lift, there exists an isomorphism:

$$\phi\colon (\tilde{X},\tilde{\mathrm{d}},\tilde{\mathfrak{m}}) \to (\overline{X},\overline{\mathrm{d}},\overline{\mathfrak{m}}) \times \mathbb{R}^k, \tag{1.3}$$

where $k \in \mathbb{N} \cap [0, N]$, $\mathbb{R}^k$ is endowed with the Euclidean distance and Lebesgue measure, and $(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}})$ is a compact $\mathrm{RCD}(0, N-k)$ space. The integer k and the space $(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}})$ are called the *degree* and *soul* of ϕ, respectively.

Remark 1.3.5. To our knowledge, the soul first appears in the soul Theorem [CG72, Theorem 2.2]. This result states that given any complete Riemannian manifold with nonnegative sectional curvature (M, g), there exists a closed geodesically convex submanifold $S \subset M$ (called a *soul*) such that M is diffeomorphic to the normal bundle of S. This result is fundamental as it reduces the study of nonnegatively curved complete manifolds to the compact case.

As a consequence of the splitting Theorem [CG71, Theorem 2], a version of the soul Theorem also holds for universal covers of closed Riemannian manifolds with nonnegative Ricci curvature (see [CG71, Theorem 3]). This last fact can be extended to Ricci limit spaces due to the generalised splitting Theorem [CC95, Theorem 2.4]. Finally, the same is true for $\mathrm{RCD}(0, N)$ spaces as a consequence of [Gig14, Theorem 1.4].

The isomorphism ϕ appearing in (1.3) is referred to as a *splitting of* $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ and induces an isomorphism:

$$\phi_* \colon \mathrm{Iso}(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}) \to \mathrm{Iso}(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}}) \times \mathrm{Iso}(\mathbb{R}^k).$$

Moreover, the revised fundamental group $\overline{\pi}_1(X)$ acts by isomorphism on $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$. Therefore, applying ϕ_* and projecting onto $\mathrm{Iso}(\mathbb{R}^k)$, we get a group homomorphism $\rho^{\phi}_{\mathbb{R}} \colon \overline{\pi}_1(X) \to \mathrm{Iso}(\mathbb{R}^k)$. We will prove the following properties (see Corollary 3.1.1 and Propositions 3.1.1 and 3.1.3).

Proposition 1.3.1 (Splitting degree $k(X)$)**.** The degree k does not depend either on the chosen splitting ϕ or the chosen $\mathrm{RCD}(0, N)$ structure $(X, \mathrm{d}, \mathfrak{m})$ on X. In particular, $k(X) \coloneqq k \in [0, N] \cap \mathbb{N}$ is a topological invariant called the *splitting degree of* X.

Proposition 1.3.2 (Crystallographic class $\Gamma(X)$)**.** The image $\Gamma_\phi \coloneqq \mathrm{Im}(\rho^{\phi}_{\mathbb{R}})$ is a crystallographic subgroup of $\mathrm{Iso}(\mathbb{R}^k)$. Moreover, up to conjugation by an affine transformation, it does not depend on the chosen splitting ϕ or the chosen $\mathrm{RCD}(0, N)$ structure $(X, \mathrm{d}, \mathfrak{m})$ on X. In particular, the affine conjugacy class $\Gamma(X) \coloneqq [\Gamma(\phi)]$ is a topological invariant of X called the *crystallographic class of* X.

The revised fundamental group of an $\mathrm{RCD}(0,N)$ space has a special structure. Using the propositions above, we can show the following result (see Corollary 3.1.1 and Proposition 3.1.2).

Proposition 1.3.3. The revised fundamental group $\overline{\pi}_1(X)$ of X is Hopfian (i.e. any surjective homomorphism of $\overline{\pi}_1(X)$ onto itself is an isomorphism). Moreover, $\overline{\pi}_1(X)$ has polynomial growth of order $k(X) \in [0,N] \cap \mathbb{N}$.

1.3.2.2 Albanese and soul maps

In Section 3.2, we apply the results of the previous section to construct the Albanese and soul maps, which reflect how structures on the universal cover split, and we prove their continuity. Here, we fix a real number $N \in [1,\infty)$, a compact topological space X that admits an $\mathrm{RCD}(0,N)$ structure, and denote $p\colon \tilde{X} \to X$ the universal cover of X.

First of all, to any $\mathrm{RCD}(0,N)$ structure $(X,\mathrm{d},\mathfrak{m})$ on X and to any splitting ϕ of its lift, we can associate the compact flat orbifold $(\mathbb{R}^k/\Gamma(\phi), \mathrm{d}_{\Gamma(\phi)})$ (where $\mathrm{d}_{\Gamma(\phi)}$ is the quotient metric). Thanks to Lemma 3.2.1, the isometry class of $(\mathbb{R}^k/\Gamma(\phi), \mathrm{d}_{\Gamma(\phi)})$ and the isomorphism class of the soul $(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}})$ of ϕ only depend on the isomorphism class of $(X,\mathrm{d},\mathfrak{m})$. Hence, we can define:

- the *Albanese variety* $\mathcal{A}([X,\mathrm{d},\mathfrak{m}]) := [\mathbb{R}^k/\Gamma(\phi), \mathrm{d}_{\Gamma(\phi)}]$,
- the *soul* $\mathcal{S}([X,\mathrm{d},\mathfrak{m}]) := [\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}}]$,

associated to $[X,\mathrm{d},\mathfrak{m}]$.

Remark 1.3.6. Let $n \leq N$ and let (M^n, g) be a compact n-dimensional Riemannian manifold with non-negative Ricci curvature, and such that $\pi_1(M) = \mathbb{Z}^k$. In that case, $(M, \mathrm{d}_g, \mathfrak{m}_g)$ is an $\mathrm{RCD}(0,N)$ space, where d_g is the geodesic distance and $\mathfrak{m}_g$ is the Riemannian measure. It is possible to show that the Albanese variety of $(M, \mathrm{d}_g, \mathfrak{m}_g)$ is nothing but the usual Albanese variety of (M,g) (as defined in [Wel86]).

The main contribution of this chapter is the following result.

Theorem 1.3.2. Assume that $\mathcal{X}_n \to \mathcal{X}_\infty$ in $\mathfrak{M}_{0,N}(X)$ in the mGH topology, then $\mathcal{A}(\mathcal{X}_n) \to \mathcal{A}(\mathcal{X}_\infty)$ in the GH topology and $\mathcal{S}(\mathcal{X}_n) \to \mathcal{S}(\mathcal{X}_\infty)$ in the mGH topology.

Remark 1.3.7. Theorem 1.3.2 is essential when computing the homeomorphism type of specific examples of moduli spaces (see for example the case of the Möbius band $\mathbb{M}^2$ and the finite cylinder $\mathbb{S}^1 \times [0,1]$ in Chapter 4). Moreover, the continuity of the Albanese map will be crucial to obtain a singular version of Tuschmann and Wiemeler's Theorem.

1.3.2.3 Applications

The main contribution of Section 3.3 is the proof of a singular version of Theorem 1.1.7. As a first step, we obtain the following corollary of Theorem 1.3.2 and Proposition 1.3.2.

Corollary 1.3.3 (Retract)**.** Let us fix $N \in [1,\infty)$ and $k \geq 2$ and assume that:

- X is a compact topological space that admits an $\mathrm{RCD}(0,N)$ structure such that $\overline{\pi}_1(X) = 0$,
- F is a closed k-dimensional manifold that admits a flat metric.

Then, the moduli space $\mathfrak{M}_{0,N+k}(X \times F)$ retracts onto $\mathscr{M}_{\text{flat}}(F)$, where $\mathscr{M}_{\text{flat}}(F)$ is the *moduli space of flat metrics on* F endowed with the GH topology.

Corollary 1.3.3 implies that the homotopy groups of $\mathscr{M}_{\text{flat}}(F)$ inject in those of $\mathfrak{M}_{0,N+k}(X \times F)$. Thankfully, pieces of information on moduli spaces of flat metrics have been derived in [TW22] (in the case of the torus T^k, with $k \geq 4$ and $k \neq 8,9,10$) and in [Gar20] (in the case of 3 and 4-dimensional closed flat Riemannian manifolds). We are now able to state the singular analogue of Theorem 1.1.7 of Tuschmann and Wiemeler.

Theorem 1.3.3. Let $N \in [1,\infty)$ and let X be a compact topological space that admits an $\mathrm{RCD}(0,N)$ structure such that $\overline{\pi}_1(X) = 0$. In addition, let Y be either $\mathbb{S}^1 \times \mathbb{K}^2$ (where $\mathbb{K}^2$ is the Klein bottle) or a torus of dimension $k \geq 4$ such that $k \neq 8,9,10$. Then, the moduli space $\mathfrak{M}_{0,N+\dim(Y)}(X \times Y)$ has non-trivial higher rational homotopy groups.

As an application of Theorem 1.3.3, we immediately obtain the following corollary, which can be seen as a non-smooth analogue of Corollary 1.1.1.

Corollary 1.3.4. For every $N \geq 3$ (resp. $N \geq 4$ / $N \geq 5$) there exists a compact topological space X such that $\mathfrak{M}_{0,N}(X)$ is not simply connected (resp. has non-trivial third rational homotopy group / non-trivial fifth rational homotopy group).

The results of section 3.3 provide useful insight into Question 1.2.3. In particular, Corollary 1.3.4 should be compared with the main result of Chapter 4, where we show that moduli spaces of RCD$(0,2)$ structures are contractible.

1.3.3 Chapter 4

The results in Chapter 4 are original and appear in [Nav22]. In this chapter, we completely answer Questions 1.2.2 and 1.2.3 in the particular case of RCD$(0,2)$ spaces.

1.3.3.1 Topological obstructions

In Section 4.1, we classify the topological spaces admitting an RCD$(0,2)$ structure.

Notation 1.3.1. Let I and $\mathbb{S}^1$ denote the closed unit interval and the unit circle, respectively. We denote $\mathbb{S}^2$, $\mathbb{RP}^2$, $\mathbb{D}$, $\mathbb{M}^2$, $\mathbb{T}^2$ and $\mathbb{K}^2$ respectively the 2-sphere, the projective plane, the closed 2-disc, the Mobius band, the 2-torus and the Klein bottle.

If $(X, \mathrm{d}, \mathfrak{m})$ is an RCD$(0,2)$ structure on a compact topological space X, then the essential dimension $\dim(X, \mathrm{d}, \mathfrak{m})$ and the splitting degree $k(X)$ can only take the values $\{0,1,2\}$. Proceeding by a case-by-case study and relying on the fundamental result of [LS22], it is possible to prove the following result which answers Question 1.2.2 in the case of RCD$(0,2)$ spaces.

Proposition 1.3.4. Assume that $(X, \mathrm{d}, \mathfrak{m})$ is an RCD$(0,2)$ structure on a compact topological space X. Then we have the following case disjunction (see Proposition 1.3.1 for the definition of $k(X)$):

$\dim(X, \mathrm{d}, \mathfrak{m})$	0	1	1	2	2	2
$k(X)$	0	0	1	0	1	2
X is homeomorphic to	$\{*\}$	I	$\mathbb{S}^1$	$\mathbb{S}^2$, $\mathbb{RP}^2$ or $\mathbb{D}$	$I \times \mathbb{S}^1$ or $\mathbb{M}^2$	$\mathbb{T}^2$ or $\mathbb{K}^2$

Conversely, every topological space appearing in the third row admits an RCD$(0,2)$ structure.

To answer Question 1.2.3 in the case of RCD$(0,2)$ spaces, we will compute the homeomorphism type of $\mathfrak{M}_{0,2}(X)$, where X is any of the topological spaces appearing in Proposition 1.3.4.

1.3.3.2 Helpful results

In Section 4.2, we provide useful results to compute moduli spaces of the form $\mathfrak{M}_{K,N}(X)$, where X is either a closed manifold admitting a flat metric or a topological surface with boundary. This will be relevant in our future computations since most of the spaces appearing in Proposition 1.3.4 are either closed manifolds admitting flat metrics or topological surfaces with boundaries.

Closed flat manifolds. First of all, relying on [BDP18] and [DG18], we introduce a method for computing the moduli space $\mathfrak{M}_{0,N}(X)$, where X is a closed manifold that admits a flat metric. Let us first introduce a few notations before we state the result.

Notation 1.3.2. Let $n \geq 1$ and let Γ be a crystallographic subgroup of $\mathrm{Iso}(\mathbb{R}^n)$. We define:

- $H_\Gamma := \mathfrak{r}(\Gamma) \subset \mathrm{O}_n(\mathbb{R})$ (where $\mathfrak{r}(A, v) := A$, for $A \in \mathrm{GL}_n(\mathbb{R})$ and $v \in \mathbb{R}^n$),
- $\mathfrak{C}_\Gamma := \{A \in \mathrm{GL}_n(\mathbb{R}), AH_\Gamma A^{-1} \subset \mathrm{O}_n(\mathbb{R})\}$,
- $\mathfrak{N}_\Gamma := \mathfrak{r}(\mathrm{N}_{\mathrm{Aff}(\mathbb{R}^n)}(\Gamma))$.

The moduli space of flat metrics on $\mathbb{R}^n/\Gamma$ is the set $\mathscr{M}_{\mathrm{flat}}(\mathbb{R}^n/\Gamma)$ of flat Riemannian metrics on $\mathbb{R}^n/\Gamma$ quotiented by isometries, it is endowed with the GH topology.

Proposition 1.3.5. Let $n \geq 1$, let Γ be a Bieberbach subgroup of $\mathrm{Iso}(\mathbb{R}^n)$, and let $N \in [1, \infty)$. If $N < n$, then there are no $\mathrm{RCD}(0, N)$ structures on $\mathbb{R}^n/\Gamma$. If $N \geq n$, then any $\mathrm{RCD}(0, N)$ structure on $\mathbb{R}^n/\Gamma$ is also an $\mathrm{RCD}(0, n)$ structure. Moreover, there exist homeomorphisms:

$$\mathfrak{M}_{0,n}(\mathbb{R}^n/\Gamma) \simeq \mathbb{R}_{>0} \times \mathscr{M}_{\mathrm{flat}}(\mathbb{R}^n/\Gamma) \simeq \mathbb{R} \times [\mathrm{O}_n(\mathbb{R})\backslash\mathfrak{C}_\Gamma]/\mathfrak{N}_\Gamma,$$

where the left action of $\mathrm{O}_k(\mathbb{R})$ on $\mathfrak{C}_\Gamma$ is given by multiplication on the left, and the right action of $\mathfrak{N}_\Gamma$ on $[\mathrm{O}_n(\mathbb{R})\backslash\mathfrak{C}_\Gamma]$ is defined by $[A]\cdot B := [AB]$ given $[A] \in \mathrm{O}_n(\mathbb{R})\backslash\mathfrak{C}_\Gamma$, and $B \in \mathfrak{N}_\Gamma$.

Surfaces with boundary. Let us first recall that a 2-dimensional $\mathrm{RCD}(K,2)$ space is necessarily an Alexandrov space with curv $\geq K$ thanks to [LS22, Theorem 1.1]. Therefore, relying on results of [DG18] and [Hon20], we can relate the moduli space of $\mathrm{RCD}(K,2)$ structures on a surface with boundary X to the moduli space of Alexandrov metrics with curv $\geq K$ on X.

Proposition 1.3.6. Let X be a topological surface (possibly with boundary) that admits an $\mathrm{RCD}(K,2)$ structure ($K \in \mathbb{R}$) and let $p\colon \tilde{X} \to X$ be its universal cover. Then the map:

$$[X, \mathrm{d}, \mathfrak{m}] \in \mathfrak{M}_{K,2}(X) \to (\mathfrak{m}(X)/\mathcal{H}^2(X), [X, \mathrm{d}]) \in \mathbb{R}_{>0} \times \mathscr{M}_{\mathrm{curv}\geq K}(X)$$

is a homeomorphism, where $\mathscr{M}_{\mathrm{curv}\geq K}(X)$ is the moduli space of Alexandrov metrics with curv $\geq K$ on X and is equipped with the GH topology. Furthermore, the same map induces a homeomorphism:

$$\mathfrak{M}^{\mathrm{eq}}_{K,2}(\tilde{X}) \simeq \mathbb{R}_{>0} \times \mathscr{M}^{\mathrm{eq}}_{\mathrm{curv}\geq K}(\tilde{X}),$$

where $\mathscr{M}^{\mathrm{eq}}_{\mathrm{curv}\geq K}(\tilde{X})$ is the moduli space of equivariant Alexandrov metrics with curv $\geq K$ on $\tilde{X}$ and is equipped with the equivariant GH topology (see Remark 2.4.3).

1.3.3.3 Explicit computations

In Sections 4.3 and 4.4, we compute the homeomorphism type of the moduli spaces of $\mathrm{RCD}(0,2)$ structures on each of the spaces appearing in Proposition 1.3.4. These sections are the main contribution of Chapter 4. In particular, the cases of the projective plane $\mathbb{RP}^2$ and the closed disc $\mathbb{D}$ are the most technical and appeal to convex geometry. The computations of these last two cases are inspired by the proof of Belegradek's Theorem 1.1.8.

Circle, Klein Bottle, and Torus. Using Proposition 1.3.5, we prove that:

- $\mathfrak{M}_{0,2}(\mathbb{S}^1)$ is homeomorphic to $\mathbb{R}^2$ (see Proposition 4.3.1),
- $\mathfrak{M}_{0,2}(\mathbb{K}^2)$ is homeomorphic to $\mathbb{R}^3$ (see Proposition 4.4.2),
- $\mathfrak{M}_{0,2}(\mathbb{T}^2)$ is homeomorphic to $\mathbb{R}^4$ (see Proposition 4.4.1).

The case of the torus utilises the result of [Gar20, Section 2.1].

Interval. The moduli space $\mathfrak{M}_{0,2}(I)$ is closely related to the space of concave functions on I.

Notation 1.3.3 (Space of concave functions)**.** We denote by $\mathcal{C}^*$ the space of concave functions $f\colon I \to \mathbb{R}$ such that f is strictly positive on $\mathrm{int}(I)$. For every $f \in \mathcal{C}^*$, we define $-1 \cdot f(t) := f(1-t)$, which gives rise to an action of $\{\pm 1\}$ on $\mathcal{C}^*$. We equip $\mathcal{C}^*$ with the topology of uniform convergence on compact subsets of $\mathrm{int}(I)$ and $\mathcal{C}^*/\{\pm 1\}$ with the quotient topology.

Relying on results from [CM21], we show that $\mathfrak{M}_{0,2}(I)$ is homeomorphic to $\mathbb{R} \times [\mathcal{C}^*/\{\pm 1\}]$ (see Proposition 4.3.2).

Cylinder and Möbius band. Afterwards, using Theorem 1.3.2, we are able to prove in Propositions 4.4.3 and 4.4.4 that $\mathfrak{M}_{0,2}(\mathbb{M}^2)$ and $\mathfrak{M}_{0,2}(I \times \mathbb{S}^1)$ are homeomorphic to $\mathbb{R}^3$.

Sphere, projective plane, and disc. Now, the only remaining cases are $\mathbb{S}^2$, $\mathbb{RP}^2$ and $\mathbb{D}$. First of all, Corollary 1.3.2 implies that $\mathfrak{M}_{0,2}(\mathbb{RP}^2)$ is homeomorphic to $\mathfrak{M}^{\mathrm{eq}}_{0,2}(\mathbb{S}^2)$ (where the equivariance refers to the natural action of $\{\pm \mathrm{id}_{\mathbb{R}^3}\}$ on $\mathbb{S}^2$). Then, $\mathbb{S}^2$ and $\mathbb{D}$ are topological surfaces. Therefore, thanks to Proposition 1.3.6, we only have to describe the following moduli spaces: $\mathscr{M}_{\mathrm{curv}\geq 0}(\mathbb{S}^2)$, $\mathscr{M}^{\mathrm{eq}}_{\mathrm{curv}\geq 0}(\mathbb{S}^2)$, and $\mathscr{M}_{\mathrm{curv}\geq 0}(\mathbb{D})$. Since [Bel18b, Theorem 1.1] already computes the homeomorphism type of $\mathscr{M}_{\mathrm{curv}\geq 0}(\mathbb{S}^2)$, we only have to describe $\mathscr{M}^{\mathrm{eq}}_{\mathrm{curv}\geq 0}(\mathbb{S}^2)$ and $\mathscr{M}_{\mathrm{curv}\geq 0}(\mathbb{D})$. Both of these moduli spaces are closely related to spaces of convex compacta (we follow the notations of [Bel18b]).

Notation 1.3.4 (Spaces of convex compacta)**.** We denote by $\mathcal{K}$ the set of all compact convex subsets in $\mathbb{R}^3$, and $\mathcal{K}^s$ the subset of $\mathcal{K}$ whose elements have their centre of mass at the origin. Given $0 \leq k \leq l \leq 3$ and $\mathcal{K}' \subset \mathcal{K}$, we denote:

- $\mathcal{K}'_{k \leq l} := \{D \in \mathcal{K}', \dim(D) \in [k, l]\}$
- $\tilde{\mathcal{K}}' := \{D \in \mathcal{K}', D = -D\}$.

We equip every subspace of $\mathcal{K}$ with the Hausdorff distance $\mathrm{d}_{\mathrm{H}}^{\mathbb{R}^3}$.

To help us realise nonnegatively curved metrics on the 2-sphere, let us introduce the following notations from [Bel18b].

Notation 1.3.5. Given $D \in \mathcal{K}^s$, we denote $\Phi_{\mathbb{S}^2}(D) \coloneqq \partial D$ if $\dim(D) = 3$ and $\Phi_{\mathbb{S}^2}(D) \coloneqq \mathcal{D}D$ if $\dim(D) = 2$ (where $\mathcal{D}D$ is the metric double of D).

Our first result establishes the homeomorphism type of $\mathscr{M}^{\mathrm{eq}}_{\mathrm{curv}\geq 0}(\mathbb{S}^2)$ (see Propositions 4.4.5 and 4.4.8).

Theorem 1.3.4. The map:

$$\Psi_{\mathbb{S}^2} \colon [D] \in \tilde{\mathcal{K}}_{2\leq 3} \,/\, \mathrm{O}_3(\mathbb{R}) \to [\Phi_{\mathbb{S}^2}(D)] \in \mathscr{M}^{\mathrm{eq}}_{\mathrm{curv}\geq 0}(\mathbb{S}^2)$$

is a homeomorphism, where $\tilde{\mathcal{K}}_{2\leq 3}$, $\Phi_{\mathbb{S}^2}$, and $\mathscr{M}^{\mathrm{eq}}_{\mathrm{curv}\geq 0}(\mathbb{S}^2)$ are respectively introduced in Notation 1.3.4, Notation 1.3.5, and Proposition 1.3.6, and $\mathrm{O}_3(\mathbb{R})$ acts on the left of $\tilde{\mathcal{K}}_{2\leq 3}$ by translations.

In order to describe nonnegatively curved metrics, let us first introduce spaces of convex compacta with a plane of symmetry.

Notation 1.3.6. Given $\alpha \in \mathbb{S}^2$, we denote $H^-_\alpha \coloneqq \{\langle \alpha, \cdot \rangle \leq 0\}$, $H^+_\alpha \coloneqq \{\langle \alpha, \cdot \rangle \geq 0\}$, and $H_\alpha \coloneqq \{\alpha\}^\perp$ the lower and upper half-planes and the plane induced by α, respectively. We write r_α for the reflection w.r.t. H_α. Then, we denote:

$$\mathcal{K}^\alpha \coloneqq \{D \in \mathcal{K}^s, r_\alpha(D) = D\}$$

and:

$$\mathscr{K}_{2\leq 3} \coloneqq \bigcup_{\alpha \in \mathbb{S}^2} \mathcal{K}^\alpha_{2\leq 3} \times \{\alpha\} \subset \mathcal{K}^s_{2\leq 3} \times \mathbb{S}^2.$$

The following subsets associated with the double of a plane region are essential to obtain nonnegatively curved metrics on $\mathbb{D}$.

Notation 1.3.7. Given $K \in \mathcal{K}^s_{2\leq 3}$ and $\alpha \in \mathrm{Span}(K)$ such that $r_\alpha(K) = K$, we denote $\mathcal{D}K^+_\alpha \subset \mathcal{D}K$ the subspace of $\mathcal{D}K$ obtained by gluing two copies of $K \cap H^+_\alpha$ along ∂K. Observe that $\mathcal{D}K^+_\alpha$ is isometric to a nonnegatively curved metric on $\mathbb{D}$.

Now, we introduce the map that will lead to the correspondence between $\mathscr{M}_{\mathrm{curv}\geq 0}(\mathbb{D})$ and a quotient of $\mathscr{K}_{2\leq 3}$.

Notation 1.3.8. Given $(D, \alpha) \in \mathscr{K}_{2\leq 3}$ (see Notation 1.3.6), we denote:

(i) $\Phi_{\mathbb{D}}(D, \alpha) \coloneqq \partial D \cap H^+_\alpha$ if $\dim(D) = 3$,

(ii) $\Phi_{\mathbb{D}}(D, \alpha) \coloneqq D$ if $\alpha \in \mathrm{Span}(D)^\perp$,

(iii) $\Phi_{\mathbb{D}}(D,\alpha) \coloneqq \mathcal{D}D^+_\alpha$ if $\dim(D) = 2$ and $\alpha \in \operatorname{Span}(D)$ (see Notation 1.3.7).

The following result describes the moduli space of nonnegatively curved metrics on the disc (see Propositions 4.4.7 and 4.4.9).

Theorem 1.3.5. The map:

$$\Psi_{\mathbb{D}}\colon [D,\alpha] \in \mathscr{K}_{2\leq 3}/\operatorname{O}_3(\mathbb{R}) \to [\Phi_{\mathbb{D}}(D,\alpha)] \in \mathscr{M}_{\mathrm{curv}\geq 0}(\mathbb{D})$$

is a homeomorphism, where $\mathscr{K}_{2\leq 3}$ and $\Phi_{\mathbb{D}}$ are respectively introduced in Notations 1.3.6 and 1.3.8, and $\operatorname{O}_3(\mathbb{R})$ acts on the left of $\mathscr{K}_{2\leq 3}$ by translations.

Summary. The following result summarises what has been said above and provides a topological description of moduli spaces of $\operatorname{RCD}(0,2)$ structures.

Theorem 1.3.6. The following table describes moduli spaces of $\operatorname{RCD}(0,2)$ structures on compact topological spaces:

X is homeomorphic to:	$\mathfrak{M}_{0,2}(X)$ is homeomorphic to:
$\{*\}$	$\mathbb{R}$
I	$\mathbb{R} \times [\mathcal{C}^*/\{\pm 1\}]$ (see Notation 1.3.3)
$\mathbb{S}^1$	$\mathbb{R}^2$
$\mathbb{T}^2$	$\mathbb{R}^4$
$\mathbb{S}^1 \times I$, $\mathbb{M}^2$, or $\mathbb{K}^2$	$\mathbb{R}^3$
$\mathbb{S}^2$	$\mathbb{R} \times \mathcal{K}^s_{2\leq 3}/\operatorname{O}_3(\mathbb{R})$ (see Notation 1.3.4)
$\mathbb{RP}^2$	$\mathbb{R} \times \tilde{\mathcal{K}}_{2\leq 3}/\operatorname{O}_3(\mathbb{R})$ (see Notation 1.3.4)
$\mathbb{D}$	$\mathbb{R} \times \mathscr{K}_{2\leq 3}/\operatorname{O}_3(\mathbb{R})$ (see Notation 1.3.6)

In particular, if X is a compact topological space that admits an $\operatorname{RCD}(0,2)$ structure, then the moduli space $\mathfrak{M}_{0,2}(X)$ of $\operatorname{RCD}(0,2)$ structures on X is contractible.

Theorem 1.3.6 should be compared with Corollary 1.3.4. Indeed, Corollary 1.3.4 provides examples of compact topological spaces with non-contractible moduli spaces of $\operatorname{RCD}(0,N)$ structures, for any $N \geq 3$.

1.3.4 Chapter 5

The purpose of Chapter 5 is to investigate the properties of the moduli space of RCD$(-1, 2)$ structures on a connected closed orientable topological surface. This case differs from Chapters 3 and 4 because we cannot rely on splitting maps. Instead, we will use the Ricci flow on Alexandrov spaces, as introduced in [Ric12]. Our main result establishes that the moduli space of RCD$(-1, 2)$ structures on a surface X is homotopy equivalent to the moduli space of hyperbolic metrics (when $\chi(X) < 0$).

1.3.4.1 Ricci flow on smooth manifolds

In Section 5.1, we provide a brief overview of the Ricci flow and obtain uniform convergence estimates for sequences of normalized Ricci flows. The main contribution of this section is the following result, whose proof follows the lines of [CK04, Theorem 5.22] and relies on the convergence properties of curvature potentials.

Proposition 1.3.7. Let X be a connected closed orientable smooth surface such that $\chi(X) < 0$, let $\{h_i\}_{i\in\mathbb{N}\cup\{\infty\}}$ be a family of Riemannian metrics on X, and denote by $\{h_i(t)\}_{i\in\mathbb{N}\cup\{\infty\},t\geq 0}$ the associated family of normalized Ricci flows. If $\{h_i\}_{i\in\mathbb{N}}$ converges smoothly to h_∞, then, there exists a decreasing continuous function $C \in \mathcal{L}^1(\mathbb{R}_{\geq 0})$ such that, for all $i \in \mathbb{N} \cup \{\infty\}$ and $t \geq 0$, we have:

$$|R_i(t) - r_i| \leq C(t),$$

where $R_i(t) = \mathrm{scal}(h_i(t))$ and $r_i \equiv ⨍_X \mathrm{scal}(h_i(t))\,\mathrm{dvol}_{h_i(t))}$.

1.3.4.2 Ricci flow on Alexandrov spaces

In Section 5.2, we recall the definition of the Ricci flow on Alexandrov surfaces, as introduced in [Ric12]. In particular, relying on the short time estimates of the Ricci flow obtained in [Sim12], Proposition 1.3.7, and Hamilton's compactness theorem, we demonstrate the following result.

Theorem 1.3.7. Let X be a connected closed orientable topological surface such that $\chi(X) < 0$ and let $\mathcal{A}$ be a smooth structure on X. Then, the moduli space $\mathscr{M}_{\mathrm{curv}\geq -1}(X)$ of Alexandrov metrics with curv ≥ -1 (equipped with the Gromov–Hausdorff topology) retracts by deformation onto the moduli space $\mathscr{M}_{\mathrm{cste}}(X, \mathcal{A})$ of smooth Riemannian metrics with constant curvature (equipped with the smooth topology).

1.3.4.3 RCD(-1,2) spaces

Finally, in Section 5.3, we apply Theorem 1.3.7 and obtain the following result about the homotopy type of the moduli spaces of RCD$(-1,2)$ structures.

Theorem 1.3.8. Let X be a connected closed orientable topological surface with $\chi(X) < 0$. Then, the moduli space $\mathfrak{M}_{-1,2}(X)$ of RCD$(-1,2)$ structures on X retracts by deformation onto the moduli space $\mathscr{M}_{\mathrm{Hyp}}(X)$ of hyperbolic metrics on X (endowed with the smooth topology).

Remark 1.3.8. The case where $\chi(X) \geq 0$ is more complicated than $\chi(X) < 0$. It is likely that the theorem above will still hold in that case using the results of [AB10] (if $\chi(X) > 0$) and [CK04] (if $\chi(X) = 0$), but one must undertake more research.

Theorem 1.3.8 establishes a bridge between the moduli spaces of RCD$(-1,2)$ structures and the moduli spaces of hyperbolic metrics on surfaces with negative Euler characteristics. The study of moduli spaces of hyperbolic metrics is a well-established research field (see [FM11] for a comprehensive introduction). As a result, in the future, one will be able to draw topological consequences on moduli spaces of RCD$(-1,2)$ structures from this connection.

2
Preliminaries

Contents

This chapter is devoted to introducing the main objects of the thesis. In particular, Sections 2.1, 2.2, and 2.3 introduce Gromov–Hausdorff types topologies and RCD spaces together with their covering and moduli spaces. The content of Section 2.4 is the main contribution of Chapter 2. There, we introduce the *equivariant pmGH topology* and prove the *equivariant theorem*, which was obtained in collaboration with Andrea Mondino and appears in [MN22].

2.1 Gromov–Hausdorff type topologies

Throughout the paper, we will consider sequences of metric spaces and metric measure spaces. However, before discussing converging sequences, it is fundamental to introduce the appropriate topologies. The goal of this section is to present Gromov–Hausdorff type topologies, which are best suited when studying sequences of RCD spaces. We mainly used [BBI22], [GLP98], [ADH13], [Vil09], and [GMS15] as references for this section.

2.1.1 The compact case

We start with a presentation of the Gromov–Hausdorff and measured Gromov–Hausdorff topologies, which are specific to compact metric spaces and metric measure spaces respectively.

2.1.1.1 The GH topology

The *Gromov–Hausdorff distance* metrizes the set of isometry classes of compact metric spaces. Its definition relies on the *Hausdorff distance* between closed subsets of a fixed metric space.

Definition 2.1.1 (Haudorff distance)**.** Let (Z, d_Z) be a metric space and let $A, B \subset Z$ be two closed subsets of (Z, d_Z). We define:

$$\mathrm{d}_{\mathrm{H}}^{(Z,\mathrm{d}_Z)}(A, B) \coloneqq \inf\{\epsilon > 0 \mid A \subset B^\epsilon, B \subset A^\epsilon\},$$

where A^ϵ (resp. B^ϵ) is the ϵ-neighbourhoud of A (resp. B) in (Z, d_Z); more precisely, $A^\epsilon = \{x \in Z, \mathrm{d}_Z(x, A) \leq \epsilon\}$. We will often write d_{H} instead of $\mathrm{d}_{\mathrm{H}}^{(Z,\mathrm{d}_Z)}$, unless it leads to confusion.

Notation 2.1.1. Let $\mathscr{X}$ be the set of isometry classes of compact metric spaces.

Definition 2.1.2 (Gromov–Hausdorff distance)**.** Given $[X_i, \mathrm{d}_i] \in \mathscr{X}$ $(i \in \{1, 2\})$, we define:

$$\mathrm{d}_{\mathrm{GH}}([X_1, \mathrm{d}_1], [X_2, \mathrm{d}_2]) \coloneqq \inf \left\{ \mathrm{d}_{\mathrm{H}} \left(X_1', X_2' \right) \right\},$$

where the infimum is taken over all complete separable metric spaces (Z, d_Z) and isometric embeddings $\iota_i \colon (X_i, \mathrm{d}_i) \hookrightarrow (Z, \mathrm{d}_Z)$, $i \in \{1, 2\}$, with $X_i' := \iota_i(X_i)$, and d_{H} stands for the Hausdorff distance between closed subsets of (Z, d_Z) (see Definition 2.1.1).

The following structure result is a consequence of [BBI22, Proposition 7.3.16] (triangle inequality), [BBI22, Theorem 7.3.30] (separation axiom), [Fuk90, Theorem 1.5] (completeness), and the introduction of Section $3.11\frac{1}{2}{}_{+}$ in [GLP98] (separability).

Theorem 2.1.1. The space $(\mathscr{X}, \mathrm{d}_{\mathrm{GH}})$ is a complete separable metric space, which is contractible but not locally compact.

The topology induced by d_{GH} on $\mathscr{X}$ is called the *Gromov–Hausdorff topology* (GH topology for short). Let us now introduce ϵ-isometries and GH ϵ-approximation, which will be fundamental in the future to characterise converging sequences in $(\mathscr{X}, \mathrm{d}_{\mathrm{GH}})$.

Definition 2.1.3 (ϵ-isometry)**.** Let (X_i, d_i) $(i \in \{1, 2\})$ be compact metric spaces and let $\epsilon > 0$. A Borel measurable map $f \colon (X_1, \mathrm{d}_1) \to (X_2, \mathrm{d}_2)$ is an *ϵ-isometry* from (X_1, d_1) to (X_2, d_2) if it satisfies the following two points:

- for every $x, y \in X_1$, $|\mathrm{d}_2(f(x), f(y)) - \mathrm{d}_1(x, y)| \leq \epsilon$
- for every $y \in X_2$, there exists $x \in X_1$ such that $\mathrm{d}_2(f(x), y) \leq \epsilon$.

The *distorsion* of f is the quantity $\mathrm{Dis}(f) := \sup\{|\mathrm{d}_2(f(x), f(y)) - \mathrm{d}_1(x, y)|\}$,where the supremum is taken over all $x, y \in X_1$.

Observe that if there exists an ϵ-isometry from (X_1, d_1) to (X_2, d_2), there is not necessarily an ϵ-isometry from (X_2, d_2) to (X_1, d_1). In other words, the existence of an ϵ-isometry does not define a symmetric relation. To tackle this issue, we introduce the notion of GH ϵ-approximation.

Definition 2.1.4 (GH ϵ-approximation)**.** Let (X_i, d_i) $(i \in \{1, 2\})$ be compact metric spaces and let $\epsilon > 0$. A *GH ϵ-approximation* between (X_1, d_1) and (X_2, d_2) is a pair (f, g) where $f \colon (X_1, \mathrm{d}_1) \to (X_2, \mathrm{d}_2)$ and $g \colon (X_2, \mathrm{d}_2) \to (X_1, \mathrm{d}_1)$ are Borel maps such that:

- $\max\{\mathrm{Dis}(f), \mathrm{Dis}(g)\} \leq \epsilon$,
- for every $x \in X_1$ and $y \in X_2$, we have $\mathrm{d}_1(g \circ f(x), x) \leq \epsilon$ and $\mathrm{d}_2(f \circ g(y), y) \leq \epsilon$.

The approximate inverse construction detailed on p.751 in [Vil09] and [BBI22, Corollary 7.3.28] implies the following result. It relates converging sequences in $(\mathscr{X}, \mathrm{d}_{\mathrm{GH}})$ to ϵ-isometries and GH ϵ-approximations.

Proposition 2.1.1 (Convergence in $(\mathscr{X}, \mathrm{d}_{\mathrm{GH}})$)**.** Given a family $\{[X_n, \mathrm{d}_n]\}_{n\in\mathbb{N}\cup\{\infty\}}$ in $\mathscr{X}$, the following three points are equivalent:

(i) $\mathrm{d}_{\mathrm{GH}}([X_n, \mathrm{d}_n], [X_\infty, \mathrm{d}_\infty]) \to 0$,

(ii) there exists a sequence of ϵ_n-isometries $f_n\colon (X_n, \mathrm{d}_n) \to (X_\infty, \mathrm{d}_\infty)$ such that $\epsilon_n \to 0$,

(iii) there exists a sequence of GH ϵ_n-approximations (f_n, g_n) between (X_n, d_n) and $(X_\infty, \mathrm{d}_\infty)$ such that $\epsilon_n \to 0$.

In any case, we say that $\{(X_n, \mathrm{d}_n)\}_{n\in\mathbb{N}}$ converges to $(X_\infty, \mathrm{d}_\infty)$ in the GH topology.

2.1.1.2 The mGH topology

Let us now define the measured Gromov–Hausdorff topology, which compares compact metric measure spaces.

Definition 2.1.5 (Metric measure spaces)**.** Let $(X, \mathrm{d}, \mathfrak{m})$ be a triple, where (X, d) is a metric space and $\mathfrak{m}$ is a measure on X. We say that $(X, \mathrm{d}, \mathfrak{m})$ is a metric measure space (m.m.s. for short) when (X, d) is a complete separable metric space and $\mathfrak{m}$ is a locally finite Borel measure on (X, d).

Remark 2.1.1. If $(X, \mathrm{d}, \mathfrak{m})$ is a metric measure space as defined above, then (X, d) is clearly a Polish topological space (i.e. it is homeomorphic to a complete separable metric space). In particular, $\mathfrak{m}$ is necessarily a Radon measure, since it is a locally finite Borel measure on a Polish topological space.

When studying metric measure spaces, there are two different notions of isomorphism (see the discussion in Chapter 27, Section "adding the measure" in [Vil09]). The following definition is best suited to our problem.

Definition 2.1.6 (Isomorphism between m.m.s.)**.** Two m.m.s. $(X, \mathrm{d}_X, \mathfrak{m}_X)$ and $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ are *isomorphic* when there is a bijective isometry $\phi\colon (X, \mathrm{d}_X) \to (Y, \mathrm{d}_Y)$ such that $\phi_*\,\mathfrak{m}_X = \mathfrak{m}_Y$.

Notation 2.1.2. Let $\mathfrak{X}$ be the set of isomorphism classes of compact metric measure spaces. Given $[X, \mathrm{d}, \mathfrak{m}] \in \mathfrak{X}$, we denote $\mathrm{Mass}([X, \mathrm{d}, \mathfrak{m}]) \coloneqq \mathfrak{m}(X)$ and $\mathrm{Diam}([X, \mathrm{d}, \mathfrak{m}]) \coloneqq \mathrm{Diam}(X, \mathrm{d})$.

Before being able to compare compact m.m.s., we first have to compare measures. The following definition introduces the Porkhorov distance between finite measures on a metric space.

Definition 2.1.7 (Prokhorov distance)**.** Let (Z, d_Z) be a complete separable metric space and let μ and ν be finite Borel measures on X. We define:

$$\mathrm{d}_{\mathrm{P}}^{(Z,\mathrm{d}_Z)}(\mu, \nu) \coloneqq \inf \left\{ \epsilon > 0, \forall A \subset X, A \text{ closed implies } \middle| \begin{array}{l} \mu(A) \leq \nu(A^\epsilon) + \epsilon \\ \nu(A) \leq \mu(A^\epsilon) + \epsilon \end{array} \right\}.$$

We will often write d_{P} instead of $\mathrm{d}_{\mathrm{P}}^{(Z,\mathrm{d}_Z)}$, unless it leads to confusion.

Let us define the Gromov–Hausdorff–Prokhorov distance on $\mathfrak{X}$ (introduced in this general setting by Abraham, Delmas, and Hoscheit in [ADH13]).

Definition 2.1.8 (Gromov–Hausdorff–Prokhorov distance)**.** Let $[X_i, \mathrm{d}_i, \mathfrak{m}_i] \in \mathfrak{X}$ and let us denote $\mathcal{X}_i \coloneqq [X_i, \mathrm{d}_i, \mathfrak{m}_i]$ $(i \in \{1, 2\})$. We define:

$$\mathrm{d}_{\mathrm{mGH}}(\mathcal{X}_1, \mathcal{X}_2) \coloneqq \inf \Big\{ \mathrm{d}_{\mathrm{H}} \left(X_1', X_2' \right) + \mathrm{d}_{\mathrm{P}}(\mathfrak{m}_1', \mathfrak{m}_2') \Big\},$$

where:

- the infimum is taken over all complete separable metric spaces (Z, d_Z) and isometric embeddings $\iota_i \colon (X_i, \mathrm{d}_i) \hookrightarrow (Z, \mathrm{d}_Z)$, $i \in \{1, 2\}$, with $X_i' \coloneqq \iota_i(X_i)$ and $\mathfrak{m}_i' \coloneqq \iota_{i*}\,\mathfrak{m}_i$,
- d_{H} stands for the Hausdorff distance between closed subsets of (Z, d_Z),
- d_{P} stands for the Prokhorov distance between finite measures on (Z, d_Z) (see Definition 2.1.7).

We call $\mathrm{d}_{\mathrm{mGH}}$ the *Gromov–Hausdorff–Prokhorov distance.*

We have the following result thanks to [ADH13, Theorem 2.3].

Theorem 2.1.2. The space $(\mathfrak{X}, \mathrm{d}_{\mathrm{mGH}})$ is a complete separable metric space.

The topology induced by $\mathrm{d}_{\mathrm{mGH}}$ on $\mathfrak{X}$ is called the *measured Gromov–Hausdorff topology* (mGH topology for sort). Let us introduce mGH ϵ-approximation, which will help characterise converging sequences in $(\mathfrak{X}, \mathrm{d}_{\mathrm{mGH}})$.

Definition 2.1.9 (mGH ϵ-approximation)**.** Let $(X_i, \mathrm{d}_i, \mathfrak{m}_i)$ $(i \in \{1, 2\})$ be compact m.m.s. and let $\epsilon > 0$. An *mGH ϵ-approximation* between $(X_1, \mathrm{d}_1, \mathfrak{m}_1)$ and $(X_2, \mathrm{d}_2, \mathfrak{m}_2)$ is a GH ϵ-approximation (f, g) between (X_1, d_1) and (X_2, d_2) (see Definition 2.1.4) such that $\max\{\mathrm{d}_{\mathrm{P}}(f_*\mathfrak{m}_1, \mathfrak{m}_2), \mathrm{d}_{\mathrm{P}}(g_*\mathfrak{m}_2, \mathfrak{m}_1)\} \leq \epsilon$.

The notion of convergence in the mGH topology originated from [Fuk87]. However, Fukaya's definition seems to differ from the definition proposed in this section. The following result clarifies the situation and provides a characterisation of converging sequences of metric measure spaces. For the sake of completeness, we prove this result in Appendix A.1.

Theorem 2.1.3 (Convergence in $(\mathfrak{X}, \mathrm{d}_{\mathrm{mGH}})$)**.** Let $\{[X_n, \mathrm{d}_n, \mathfrak{m}_n]\}_{n \in \mathbb{N} \cup \{\infty\}}$ be a family in $\mathfrak{X}$ and let us denote $\mathcal{X}_n := [X_n, \mathrm{d}_n, \mathfrak{m}_n]$ $(n \in \mathbb{N} \cup \{\infty\}\})$. The following three points are equivalent:

(i) $\mathrm{d}_{\mathrm{mGH}}(\mathcal{X}_n, \mathcal{X}_\infty) \to 0$,

(ii) there exists a sequence of ϵ_n-isometries $f_n \colon (X_n, \mathrm{d}_n) \to (X_\infty, \mathrm{d}_\infty)$ such that $\epsilon_n \to 0$ and $f_{n*}\,\mathfrak{m}_n \rightharpoonup \mathfrak{m}_\infty$ in the weak-$*$ topology,

(iii) there exists a sequence of mGH ϵ_n-approximation between $\mathcal{X}_n$ and $\mathcal{X}_\infty$ such that $\epsilon_n \to 0$.

In any case, we say that $\{(X_n, \mathrm{d}_n, \mathfrak{m}_n)\}_{n \in \mathbb{N}}$ converges to $(X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty)$ in the mGH topology.

Now, let us introduce the notion of c-doubling metric measure spaces, which will give rise to a precompactness criterion for subsets of $(\mathfrak{X}, \mathrm{d}_{\mathrm{mGH}})$.

Definition 2.1.10 (c-doubling m.m.s.)**.** Let $c > 0$ and let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space. We say that $(X, \mathrm{d}, \mathfrak{m})$ is c-doubling when $\mathfrak{m}$ is not identically zero and for every $x \in X$ and $R > 0$, we have:

$$\mathfrak{m}(B_{2R}(x)) \leq c\,\mathfrak{m}(B_R(x)).$$

In particular, $\mathfrak{m}$ has full support.

We close this section with the following precompactness result (see [Vil09, Theorem 27.32]).

Theorem 2.1.4 (Precompactness Theorem)**.** Let $c > 0$, $D > 0$ and $0 < m \leq M$ be fixed constants, and let $\mathcal{F}$ be a family of compact m.m.s. such that every element $(X, \mathrm{d}, \mathfrak{m}) \in \mathcal{F}$ satisfies:

$$\begin{cases} \mathrm{Diam}(X, \mathrm{d}) \leq D \\ m \leq \mathfrak{m}(X) \leq M \\ (X, \mathrm{d}, \mathfrak{m}) \text{ is } c\text{-doubling} \end{cases} . \tag{2.1}$$

Then, every sequence in $\mathcal{F}$ admits a subsequence converging in the mGH topology. Moreover, every limit space $(X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty)$ also satisfies (2.1).

2.1.2 The pointed case

We focused on compact metric and metric measure spaces in the previous subsection. However, being able to compare non-compact spaces is fundamental since we will also work with non-compact metric and metric measure spaces in the thesis. Therefore, the role of this subsection is to introduce the pointed Gromov–Hausdorff and pointed measured Gromov–Hausdorff topologies.

2.1.2.1 The pGH topology

Let us first introduce pointed metric spaces and the appropriate notion of isomorphism between them.

Definition 2.1.11 (Pointed metric spaces)**.** A pointed metric space is a triple $(X, \mathrm{d}, *)$, where (X, d) is a metric space and $* \in X$.

Definition 2.1.12 (Isomorphism between pointed metric spaces)**.** Two pointed metric spaces $(X, \mathrm{d}_X, *_X)$ and $(Y, \mathrm{d}_Y, *_Y)$ are *isomorphic* when there is a bijective isometry $\phi\colon (X, \mathrm{d}_X) \to (Y, \mathrm{d}_Y)$ such that $\phi(*_X) = *_Y$.

In [ADH13], results are given in the case of locally compact, geodesic metric spaces; we will therefore restrict our attention to this particular case.

First, we define geodesics and geodesic metric spaces.

Definition 2.1.13. Let (X, d) be a complete separable metric space. A Lipschitz curve $\gamma\colon [0, 1] \to X$ is called a geodesic when it satisfies:

$$\mathrm{d}(\gamma(t), \gamma(s)) = |t - s|\, \mathrm{d}(\gamma(0), \gamma(1)),$$

for every $0 \leq s, t \leq 1$. We say that (X, d) is geodesic if there exists a geodesic between every pair of points. We denote $\mathrm{Geo}(X)$ the space of geodesics in X endowed with the uniform topology.

Remark 2.1.2. Note that our definition of a geodesic corresponds to the Riemannian geometry concept of minimising geodesics parametrised on $[0, 1]$.

Remark 2.1.3. According to [AG13, Section 2.2], if (X, d) is complete and separable, then $\mathrm{Geo}(X)$ is also complete and separable.

Notation 2.1.3. Let $\mathscr{X}_{\mathrm{p}}$ be the set of isomorphism classes of locally compact geodesic pointed metric spaces. Let $\mathscr{X}_{\mathrm{p}}^{\mathrm{c}}$ be the subset of $\mathscr{X}_{\mathrm{p}}$ consisting of equivalence classes of compact spaces.

Before introducing the pointed Gromov–Hausdorff topology, we first restrict our attention to the case of compact pointed metric spaces. The definition below introduces the compact pointed distance (following [ADH13]).

Definition 2.1.14 (Compact pointed distance)**.** Given $[X_i, \mathrm{d}_i, *_i] \in \mathscr{X}_{\mathrm{p}}^{\mathrm{c}}$ $(i \in \{1, 2\})$, we define:

$$\mathcal{D}^{\mathrm{c}}([X_1, \mathrm{d}_1, *_1], [X_2, \mathrm{d}_2, *_2]) \coloneqq \inf \left\{ \mathrm{d}_{\mathrm{H}} \left(X_1', X_2' \right) + \mathrm{d}_Z(*_1', *_2') \right\},$$

where the infimum is taken over all complete separable metric spaces (Z, d_Z) and isometric embeddings $\iota_i \colon (X_i, \mathrm{d}_i) \hookrightarrow (Z, \mathrm{d}_Z)$, $i \in \{1, 2\}$, with $X_i' \coloneqq \iota_i(X_i)$, and $*_i' = \iota_i(*_i)$.

Following [ADH13], but forgetting the measure, we introduce the *pointed Gromov–Hausdorff distance.*

Definition 2.1.15 (Pointed Gromov–Hausdorff distance)**.** Let $[X_i, \mathrm{d}_i, *_i] \in \mathscr{X}_{\mathrm{p}}$ and let us denote $\mathcal{X}_i \coloneqq [X_i, \mathrm{d}_i, *_i]$ $(i \in \{1, 2\})$. We define:

$$\mathrm{d}_{\mathrm{pGH}}(\mathcal{X}_1, \mathcal{X}_2) \coloneqq \int_0^{\infty} \mathrm{e}^{-r} \min \left\{ 1, \mathcal{D}^{\mathrm{c}}\Big(\mathcal{X}_1(r), \mathcal{X}_2(r)\Big) \right\} \mathrm{d}r, \tag{2.2}$$

where for $i \in \{1, 2\}$:

$$\mathcal{X}_i(r) = \left[\overline{B}_i(r), \mathrm{d}_{i|\overline{B}_i(r)}, *_i \right] \in \mathscr{X}_{\mathrm{p}}^{\mathrm{c}},$$

and $\overline{B}_i(r)$ is the closed ball of radius r and center $*_i$ in (X_i, d_i).

Remark 2.1.4. As mentioned in [ADH13, section 2.1], the formula (2.2) is motivated by the definition of the generalised Prokhorov distance.

We have the following result thanks to [ADH13, Theorem 2.7].

Theorem 2.1.5. The space $(\mathscr{X}_{\mathrm{p}}, \mathrm{d}_{\mathrm{pGH}})$ is a complete separable metric space.

The topology induced by $\mathrm{d}_{\mathrm{pGH}}$ on $\mathscr{X}_{\mathrm{p}}$ is called the *pointed Gromov–Hausdorff topology* (pGH topology for short). Let us now introduce pGH ϵ-approximation, which will help characterise converging sequences of $\mathscr{X}_{\mathrm{p}}$.

Definition 2.1.16 (pGH ϵ-approximation)**.** Let $\mathcal{X}_i = (X_i, \mathrm{d}_i, *_i)$ $(i \in \{1,2\})$ be locally compact geodesic pointed metric spaces and let $\epsilon > 0$. A *pGH ϵ-approximation* between $\mathcal{X}_1$ and $\mathcal{X}_2$ is a pair (f,g) where $f\colon X_1 \to X_2$ and $g\colon X_2 \to X_1$ are Borel maps such that:

- $f(*_1) = *_2$ and $g(*_2) = *_1$,
- $\max\{\mathrm{Dis}(f_{|B_1(\epsilon^{-1})}), \mathrm{Dis}(g_{|B_2(\epsilon^{-1})})\} \leq \epsilon$,
- for every $x \in B_1(\epsilon^{-1})$, $\mathrm{d}_1(g \circ f(x), x) \leq \epsilon$, and, for every $x \in B_2(\epsilon^{-1})$, $\mathrm{d}_2(f \circ g(x), x) \leq \epsilon$,

where $B_i(r)$ denotes the ball of radius r and center $*_i$ in (X_i, d_i) $(i \in \{1,2\})$.

We conclude this section with the following result, which characterises converging sequences in $(\mathscr{X}_{\mathrm{p}}, \mathrm{d}_{\mathrm{pGH}})$.

Proposition 2.1.2 (Convergence in $(\mathscr{X}_{\mathrm{p}}, \mathrm{d}_{\mathrm{pGH}})$)**.** Let $\{[X_n, \mathrm{d}_n, *_n]\}_{n \in \mathbb{N} \cup \{\infty\}}$ be a family in $\mathscr{X}_{\mathrm{p}}$. The following three points are equivalent:

(i) $\mathrm{d}_{\mathrm{pGH}}([X_n, \mathrm{d}_n, *_n], [X_\infty, \mathrm{d}_\infty, *_\infty]) \to 0$,

(ii) there exists a sequence of ϵ_n-isometries $f_n\colon (B_n(R_n), \mathrm{d}_n) \to (B_\infty(R_n), \mathrm{d}_\infty)$ such that $f_n(*_n) = *_\infty$, $\epsilon_n \to 0$, and $R_n \to \infty$,

(iii) there exists a sequence of pGH ϵ_n-approximations (f_n, g_n) between $(X_n, \mathrm{d}_n, *_n)$ and $(X_\infty, \mathrm{d}_\infty, *_\infty)$ such that $\epsilon_n \to 0$.

In any case, we say that $\{(X_n, \mathrm{d}_n, *_n)\}_{n \in \mathbb{N}}$ converges to $(X_\infty, \mathrm{d}_\infty, *_\infty)$ in the pGH topology.

The equivalence between points (i) and (ii) is granted by [GMS15, Remark 3.29], whereas the equivalence between (ii) and (iii) can be proven using approximate inverses (see Appendix A.1 and the construction detailed on p.751 in [Vil09]).

2.1.2.2 The pmGH topology

First of all, let us introduce the class of pointed metric measure spaces.

Definition 2.1.17 (Pointed metric measure spaces)**.** A pointed metric measure space (p.m.m.s. for short) is a 4-tuple $(X, \mathrm{d}, \mathfrak{m}, *)$, where $(X, \mathrm{d}, \mathfrak{m})$ is an m.m.s. and $* \in X$.

There are different notions of isomorphisms for pointed metric measure spaces. Nevertheless, we will focus on the following.

Definition 2.1.18 (Isomorphism between p.m.m.s.)**.** Two p.m.m.s. $(X, \mathrm{d}_X, \mathfrak{m}_X, *_X)$ and $(Y, \mathrm{d}_Y, \mathfrak{m}_Y, *_Y)$ are *isomorphic* when there is a bijective isometry $\phi\colon (X, \mathrm{d}_X) \to (Y, \mathrm{d}_Y)$ such that $\phi_* \mathfrak{m}_X = \mathfrak{m}_Y$ and $\phi(*_X) = *_Y$.

Notation 2.1.4. Let $\mathfrak{X}^{\mathrm{p}}$ be the set of isomorphism classes of locally compact geodesic pointed metric measure spaces. Let $\mathfrak{X}^{\mathrm{p}}_c$ be the subset of $\mathfrak{X}^{\mathrm{p}}$ consisting of compact equivalence classes. Given $[X, \mathrm{d}, \mathfrak{m}, *] \in \mathfrak{X}^{\mathrm{p}}$ and $r > 0$, we denote $\mathrm{Mass}_r([X, \mathrm{d}, \mathfrak{m}, *]) := \mathfrak{m}(B_r(*))$.

Following [ADH13], we first introduce the compact pointed distance $\mathfrak{D}^{\mathrm{c}}$ on $\mathfrak{X}^{\mathrm{p}}_c$.

Definition 2.1.19 (Compact pointed distance $\mathfrak{D}^{\mathrm{c}}$)**.** Given two equivalence classes of compact geodesic p.m.m.s. $\mathcal{X}_i = [X_i, \mathrm{d}_i, \mathfrak{m}_i, *_i] \in \mathfrak{X}^{\mathrm{p}}_c$, $i \in \{1, 2\}$, we define:

$$\mathfrak{D}^{\mathrm{c}}(\mathcal{X}_1, \mathcal{X}_2) := \inf \left\{ \mathrm{d}_{\mathrm{H}} \left(X_1', X_2' \right) + \mathrm{d}_{\mathrm{P}}(\mathfrak{m}_1', \mathfrak{m}_2') + \mathrm{d}_Z(*_1', *_2') \right\},$$

where the infimum is taken over all complete separable metric spaces (Z, d_Z) and isometric embeddings $\iota_i\colon (X_i, \mathrm{d}_i) \hookrightarrow (Z, \mathrm{d}_Z)$, $i \in \{1, 2\}$, with $X_i' := \iota_i(X_i)$, $\mathfrak{m}_i' := \iota_{i*} \mathfrak{m}_i$ and $*_i' = \iota_i(*_i)$.

We can now write down the definition of the pointed Gromov–Hausdorff–Prokhorov distance $\mathrm{d}_{\mathrm{pmGH}}$ on $\mathfrak{X}^{\mathrm{p}}$ (introduced in this general setting by Abraham, Delmas and Hoscheit in [ADH13]).

Definition 2.1.20 (Pointed Gromov–Hausdorff–Prokhorov distance)**.** Given two equivalence classes of locally compact geodesic p.m.m.s. $\mathcal{X}_i = [X_i, \mathrm{d}_i, \mathfrak{m}_i, *_i] \in \mathfrak{X}^{\mathrm{p}}$ $(i \in \{1, 2\})$, we define:

$$\mathrm{d}_{\mathrm{pmGH}}(\mathcal{X}_1, \mathcal{X}_2) := \int_0^\infty \mathrm{e}^{-r} \min \left\{ 1, \mathfrak{D}^{\mathrm{c}} \left(\mathcal{X}_1(r), \mathcal{X}_2(r) \right) \right\} \mathrm{d}r,$$

where for $i \in \{1,2\}$:

$$\mathcal{X}_i(r) = \left[\overline{B}_i(r), \mathrm{d}_{i|\overline{B}_i(r)}, \mathfrak{m}_{i|\overline{B}_i(r)}, *_i\right] \in \mathfrak{X}_p^c$$

and $\overline{B}_i(r)$ is the closed ball of radius r and center $*_i$ in (X_i, d_i).

We have the following result thanks to [ADH13, Theorem 2.7].

Theorem 2.1.6. The space $(\mathfrak{X}^{\mathrm{p}}, \mathrm{d}_{\mathrm{pmGH}})$ is a complete separable metric space.

The topology induced by $\mathrm{d}_{\mathrm{pmGH}}$ on $\mathfrak{X}^{\mathrm{p}}$ is called the *pointed measured Gromov–Hausdorff topology* (pmGH topology for short). Before we introduce pmGH ϵ-approximation (which will help to characterise converging sequences of $\mathfrak{X}^{\mathrm{p}}$), we need to introduce a variant of the Prokhorov distance for locally finite measures on a metric space.

Definition 2.1.21. Let $(Z, \mathrm{d}_Z, *_Z)$ be a complete separable pointed metric space, let μ, ν be two locally finite measures on (Z, d_Z), and let $R > 0$. We define:

$$\mathrm{d}_{\mathrm{P}}^{R}(\mu,\nu) := \inf \left\{ \epsilon > 0, \forall A \subset \overline{B}(R), A \text{ closed implies} \;\middle|\; \begin{array}{l} \mu(A) \leq \nu(A^\epsilon) + \epsilon \\ \nu(A) \leq \mu(A^\epsilon) + \epsilon \end{array} \right\},$$

where $\overline{B}(R)$ denote the closed ball of radius R and center $*_Z$.

Definition 2.1.22 (pmGH ϵ-approximation)**.** Let $\mathcal{X}_i = (X_i, \mathrm{d}_i, \mathfrak{m}_i, *_i)$ ($i \in \{1,2\}$) be locally compact geodesic p.m.m.s. and let $\epsilon > 0$. A *pmGH ϵ-approximation* between $\mathcal{X}_1$ and $\mathcal{X}_2$ is a pGH ϵ-approximation (f,g) between $(X_1, \mathrm{d}_1, *_1)$ and $(X_2, \mathrm{d}_2, *_2)$ (see Definition 2.1.16) such that:

$$\max\left\{ \mathrm{d}_{\mathcal{P}}^{\epsilon^{-1}}(f_*\mathfrak{m}_1, \mathfrak{m}_2), \mathrm{d}_{\mathcal{P}}^{\epsilon^{-1}}(g_*\mathfrak{m}_2, \mathfrak{m}_1) \right\} \leq \epsilon,$$

where $\mathrm{d}_{\mathcal{P}}^{\epsilon^{-1}}$ is introduced in Definition 2.1.21.

The following result characterises converging sequences of $\mathfrak{X}^{\mathrm{p}}$.

Proposition 2.1.3 (Convergence in $(\mathfrak{X}^{\mathrm{p}}, \mathrm{d}_{\mathrm{pmGH}})$)**.** Let $\{[X_n, \mathrm{d}_n, \mathfrak{m}_n, *_n]\}_{n \in \mathbb{N} \cup \{\infty\}}$ be a family in $\mathfrak{X}^{\mathrm{p}}$. Then, the following three points are equivalent:

(i) $\mathrm{d}_{\mathrm{pmGH}}([X_n, \mathrm{d}_n, \mathfrak{m}_n, *_n], [X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty, *_\infty]) \to 0$,

(ii) there exists a sequence of ϵ_n-isometries $f_n \colon (B_n(R_n), \mathrm{d}_n) \to (B_\infty(R_n), \mathrm{d}_\infty)$ such that $f_n(*_n) = *_\infty$, $f_{n*}\,\mathfrak{m}_n \rightharpoonup \mathfrak{m}_\infty$ in the weak-$*$ topology, $\epsilon_n \to 0$, and $R_n \to \infty$,

(iii) there are pmGH ϵ_n-approximations (f_n, g_n) between $(X_n, \mathrm{d}_n, \mathfrak{m}_n, *_n)$ and $(X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty, *_\infty)$ such that $\epsilon_n \to 0$.

The equivalence between points (i) and (ii) is granted by [GMS15, Remark 3.29], whereas the equivalence between (ii) and (iii) can be proven using the approximate inverse lemma (see Appendix A.1).

We close this section with the following precompactness result (see [Vil09, Theorem 27.32]).

Theorem 2.1.7 (Precompactness Theorem)**.** Let $r > 0$ and $0 < m \leq M$ be fixed constants, let $c\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a fixed map, and let $\mathcal{F}_p$ be a family of locally compact geodesic p.m.m.s. such that every element $(X, \mathrm{d}, \mathfrak{m}, *) \in \mathcal{F}_p$ satisfies:

(i) for every $R > 0$, $\mathfrak{m}$ is $c(R)$-doubling on $B_X(R)$ in the sense that, for every $x \in B_X(R)$ and $R' > 0$ such that $B_X(x, 2R') \subset B_X(R)$, we have $\mathfrak{m}(B_X(x, 2R')) \leq c(R)\,\mathfrak{m}(B_X(x, R'))$,

(ii) $m \leq \mathfrak{m}(B_X(r)) \leq M$.

Then every sequence in $\mathcal{F}_p$ admits a subsequence converging in the pmGH topology. Moreover, every limit space $(X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty, *_\infty)$ also satisfies condition (i) and (ii) and has full support.

2.2 RCD structures and moduli spaces

The story of RCD spaces has its roots in Gromov's precompactness Theorem (see [Pet06, Corollary 11.1.13]). The result states that sequences of compact Riemannian manifolds with a lower bound on the Ricci curvature, and an upper bound on both the dimension and the diameter are precompact in the Gromov–Hausdorff topology (GH topology for short). Since then, there has been much work to understand the properties of limits of such sequences, called Ricci limit spaces.

In the early '00s, Cheeger and Colding published in [CC97], [CC00a] and [CC00b] an extensive study of the aforementioned spaces. One important observation (already noticed by Fukaya) is that to retain good stability properties in the limit space, it is fundamental to keep track of the Riemannian measures' behaviour associated to the approximating sequence. Since then, it is common to use the measured Gromov–Hausdorff topology (mGH topology for short), endowing Riemannian manifolds with their normalized volume measure.

A related (but slightly different) approach is to introduce a new definition of Ricci curvature lower bounds and dimension upper bound, at the more general level of possibly non-smooth metric measure spaces. Such a definition should generalise the classical notions and be stable when passing to the limit in the mGH topology. The definition of RCD spaces is an example of such a definition. Therefore, any result proven for RCD spaces (with the tools of metric measure theory) would hold a fortiori for Ricci limit spaces. Such a definition is called synthetic, and the situation here is analogous to the case of Alexandrov spaces (see Appendix A.2).

The goal of that part is to introduce $\mathrm{RCD}(K, N)$ spaces (with $K \in \mathbb{R}$ and $N \in [1, \infty)$). In particular, we will state compactness results for $\mathrm{RCD}(K, N)$ spaces and introduce associated moduli spaces.

2.2.1 Wasserstein distance

First of all, we need to introduce some notions of Optimal Transport. More precisely, we introduce some notations and present the Wasserstein distance associated to a complete separable metric space.

Notation 2.2.1 (Subspaces of probability measures)**.** Given a complete separable metric space (X, d), we denote $\mathcal{P}(X, \mathrm{d})$ the space of Borel probability measures on (X, d). For $p \in [1, \infty]$, we denote:

$$\mathcal{P}_p(X, \mathrm{d}) := \{\mu \in \mathcal{P}(X, \mathrm{d}), \exists x_0 \in X, |\mathrm{d}(\cdot, x_0)|_{\mathcal{L}^p(\mu)} < \infty\},$$

the set of Borel probability measures with finite moments of order p. Moreover, if $\mathfrak{m}$ is a measure on (X, d), we denote $\mathcal{P}_p(X, \mathrm{d}, \mathfrak{m})$ the subset of $\mathcal{P}_p(X, \mathrm{d})$ consisting of measures which are absolutely continuous with respect to $\mathfrak{m}$.

Notation 2.2.2 (Transport plans)**.** Let (X, d) be a complete separable metric space and let $\mu, \nu \in \mathcal{P}(X)$ be two probability measures on X. The set of *Transport plans from μ to ν* is the set:

$$\Pi(\mu, \nu) := \{\pi \in \mathcal{P}(X \times X) \mid p_{1*}\pi = \mu, p_{2*}\mu = \nu\},$$

where p_1 and p_2 are respectively the projection of the first and second factors of $X \times X$ onto X. Given a transport plan $\pi \in \Pi(\mu, \nu)$, we denote:

$$\mathcal{C}_2(\pi) := \int_{X \times X} \mathrm{d}^2(x, y) \, \mathrm{d}\, \gamma(x, y),$$

the $\mathcal{L}^2$ cost of π.

Remark 2.2.1. Observe that $\mu \otimes \nu \in \Pi(\mu, \nu)$; therefore, there is always at least one transport plan from μ to ν.

Remark 2.2.2. A transport plan $\pi \in \Pi(\mu, \nu)$ describes a way to transport the mass μ to the mass ν. More precisely, given $A, B \subset X$, $\pi(A \times B)$ is the mass transported by π from the set A to the set B.

Remark 2.2.3. If the transport of a mass unit from x to y costs $\mathrm{d}^2(x, y)$, then the quantity $\mathcal{C}_2(\pi)$ represents the cost necessary to transport μ to ν according to the plan π.

Definition 2.2.1 (Wasserstein distances)**.** Let (X, d) be a complete separable metric space. Given $\mu, \nu \in \mathcal{P}_2(X, \mathrm{d})$ (see Notation 2.2.1), we denote:

$$\mathcal{W}_2(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \sqrt{\mathcal{C}_2(\pi)}.$$

If $\pi \in \Pi(\mu, \nu)$ satisfies $\mathcal{W}_2(\mu, \nu) = \sqrt{\mathcal{C}_2(\pi)}$, then π is called an *optimal transport plan*.

The following result justifies the name of the Wasserstein distance (see [Stu06a, Proposition 2.10]).

Proposition 2.2.1. Let (X, d) be a complete separable metric space. Then $(\mathcal{P}_2(X, \mathrm{d}), \mathcal{W}_2)$ is a complete separable metric space.

When the underlying metric space is geodesic, the Wasserstein distance enjoys additional properties (see [AG13, Theorem 2.10]).

Proposition 2.2.2. Let (X, d) be a complete separable geodesic space. Then $(\mathcal{P}_2(X, \mathrm{d}), \mathcal{W}_2)$ is a geodesic space. Moreover, the following properties are equivalent:

- $t \to \mu_t$ is a geodesic in $(\mathcal{P}_2(X, \mathrm{d}), \mathcal{W}_2)$,
- there exists $\Pi \in \mathcal{P}_2(\mathrm{Geo}(X))$ (see Definition 2.1.13 for the definition $\mathrm{Geo}(X)$) such that $(e_0, e_1)_*\Pi$ is an optimal transport plan from μ_0 to μ_1 and $\mu_t = e_{t*}\Pi$ $(t \in [0, 1])$,

where $e_t \colon \gamma \in \mathrm{Geo}(X) \to \gamma(t) \in X$ is the time t evaluation map.

2.2.2 CD spaces

The first attempts to introduce a synthetic definition of curvature lower bounds for m.m.s. were realised by Lott–Villani and Sturm in the seminal papers [LV09], [Stu06a], and [Stu06b]. The outcome of these papers is the notion of a $\mathrm{CD}(K,N)$ space ($K \in \mathbb{R}$ and $N \in [1,\infty)$), which is roughly a m.m.s. with Ricci curvature bounded below by K and dimension bounded above by N.

Before introducing CD spaces, let us now introduce the distortion coefficients and entropy functionals.

Definition 2.2.2 (Distorsion coefficients)**.** For $\kappa \in \mathbb{R}$ and $\theta \geq 0$, we define:

$$\mathfrak{s}_\kappa(\theta) := \begin{cases} \sqrt{\kappa}^{-1}\sin(\sqrt{\kappa}\theta), & \text{if } \kappa > 0 \\ \theta, & \text{if } \kappa = 0 \\ \sqrt{-\kappa}^{-1}\sinh(\sqrt{-\kappa}\theta), & \text{if } \kappa < 0 \end{cases}. \tag{2.3}$$

Given $K \in \mathbb{R}$, $N \in (0,\infty)$, and $(t,\theta) \in [0,1] \times \mathbb{R}_{\geq 0}$. We denote:

$$\sigma^{(t)}_{K,N}(\theta) := \begin{cases} \infty, & \text{if } K\theta^2 \geq N\pi^2 \\ \frac{\mathfrak{s}_{K/N}(t\theta))}{\mathfrak{s}_{K/N}(\theta)}, & \text{if } K\theta^2 < N\pi^2 \text{ and } K\theta^2 \neq 0 \\ t, & \text{if } K\theta^2 = 0 \end{cases}.$$

If $N = 1$, we define:

$$\tau^{(t)}_{K,1}(\theta) := \begin{cases} \infty, & \text{if } K\theta^2 \geq 0 \\ t, & \text{if } K\theta^2 < 0 \end{cases}.$$

If $N > 1$, we denote $\tau^{(t)}_{K,N}(\theta) := t^{1/N}\{\sigma^{(t)}_{K,N-1}(\theta)\}^{1-1/N}$.

Definition 2.2.3 (Entropy functionals)**.** Let $(X, \mathrm{d}, \mathfrak{m})$ be a m.m.s. equipped with a non-trivial measure, let $N \in [1,\infty)$, and let $\nu = \rho\,\mathfrak{m} \in \mathcal{P}_2(X, \mathrm{d}, \mathfrak{m})$ (see Notation 2.2.1). We define:

$$\mathcal{S}_N(\nu|\mathfrak{m}) := -\int_X \rho^{-\frac{1}{N}}\,\mathrm{d}\nu.$$

The functionnal $\mathcal{S}_N(\cdot|\mathfrak{m})$ is called the *Rényi entropy with parameter N associated to* $\mathfrak{m}$.

While $\mathrm{CD}(K,\infty)$ spaces were introduced simultaneously by Lott–Villani in [LV09] and Sturm in [Stu06a], the more specific $\mathrm{CD}(K,N)$ spaces were introduced by Sturm in [Stu06b]. Such spaces can be defined in various ways. However, in this thesis, we will follow the approach in [BS10], which gives a unified presentation of both CD and CD^* spaces (which will be relevant for us as well).

Definition 2.2.4 (CD(K,N) spaces)**.** Let $N \in [1,\infty)$ and $K \in \mathbb{R}$. A m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is a CD(K,N) *space* if, for every pair $\nu_0, \nu_1 \in \mathcal{P}_\infty(X, \mathrm{d}, \mathfrak{m})$, there exists an optimal transport plan $\pi \in \Pi(\nu_0, \nu_1)$ and a geodesic $\Gamma\colon [0,1] \to (\mathcal{P}_\infty(X, \mathrm{d}, \mathfrak{m}), \mathcal{W}_2)$ from $\nu_0 = \rho_0\, \mathfrak{m}$ to $\nu_1 = \rho_1\, \mathfrak{m}$ such that:

$$\begin{aligned} \mathcal{S}_{N'}(\Gamma(t)|\mathfrak{m}) \leq -\int_{X\times X} &\Big(\tau^{(1-t)}_{K,N'}(\mathrm{d}(x_0,x_1))\rho_0^{-1/N'}(x_0) \\ &+ \tau^{(t)}_{K,N'}(\mathrm{d}(x_0,x_1))\rho_1^{-1/N'}(x_1)\Big)\, \mathrm{d}\pi(x_0,x_1) \end{aligned}.$$

for every $N' \geq N$.

Relying on the fine analysis of the convexity properties of Optimal Transport on Riemannian manifolds realised in [CMS01] (by Cordero-Erausquin, McCann, and Schmuckenschläger), Sturm proves the following result (see [Stu06b, Theorem 1.7]).

Theorem 2.2.1. Let (M^n, g) be a complete Riemannian manifold of dimension $n \geq 1$, let $K \in \mathbb{R}$, and let $N \in [1,\infty)$. Then, $(M^n, \mathrm{d}_g, \mathrm{dvol}_g)$ is a CD(K,N) space if and only if $n \leq N$ and $\mathrm{Ric}(g) \geq K$ (where d_g and dvol_g are respectively the geodesic distance and volume measure of (M^n, g)).

Theorem 2.2.1 shows that CD spaces generalise lower bounds on the Ricci curvature and upper bounds on the dimension. Moreover, CD spaces are stable under mGH convergence as shown by the following result (see [Stu06b, Theorem 3.1]).

Theorem 2.2.2. Let $\{(X_n, \mathrm{d}_n, \mathfrak{m}_n)\}_{n\in\mathbb{N}}$ be a sequence of compact CD(K,N) spaces ($K \in \mathbb{R}$ and $N \in [1,\infty)$) converging to a compact m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ in the mGH topology. Then, $(X, \mathrm{d}, \mathfrak{m})$ is a CD(K,N) spaces.

Let us recall that Ricci limit spaces are mGH limits of Riemannian manifolds with a Ricci curvature lower bound and a dimension upper bound. Theorem 2.2.2 shows that Ricci limit spaces are actually CD spaces. Therefore, as mentioned in this section's introduction, any property proven for CD spaces holds a fortiori for Ricci limit spaces.

In addition to that, Ohta and Petrunin respectively proved in [Oht09b] and [Pet11] that Finsler manifolds and Alexandrov spaces are CD spaces. However, Cheeger and Colding observed that Finsler spaces arise as Ricci-limit spaces if and only if they are actually Riemannian. Since CD spaces' main purpose was to study Ricci limit spaces, this last point is somewhat concerning. In the next section, we will see how to modify the CD definition to rule out Finsler spaces.

2.2.3 RCD spaces

In order to stay as close as possible to the Riemannian situation, Ambrosio, Gigli, and Savaré strengthened the definition of CD spaces by introducing RCD spaces in [AGS14] (ruling out non-Riemannian Finsler spaces). The key is the notion of infinitesimal Hilbertianity.

Let us first introduce the Cheeger energy functional in a metric measure space (we follow the presentation of Section 2.4.1 in [AGS14]).

Definition 2.2.5 (Slope of a Lipschitz function)**.** Let $(X, \mathrm{d}, \mathfrak{m})$ be a m.m.s. and let $f \colon X \to \mathbb{R}$ be a Lipschitz function. We define $\mathrm{Lip}(f) \colon X \to \mathbb{R}_{\geq 0}$ the *slope of* f by:

$$\mathrm{Lip}(f)(x) := \begin{cases} \limsup_{y \to x} \frac{|f(y)-f(x)|}{\mathrm{d}(x,y)}, & \text{if } x \text{ is an accumulation point} \\ 0, & \text{otherwise} \end{cases}.$$

In [AGS14], the Cheeger energy is defined at the level of Borel functions. However, for $\mathcal{L}^2$ functions, it is equal to its relaxation (see [AGS14, equation 2.20)]). Therefore, we define the Cheeger energy in the following way.

Definition 2.2.6 (Cheeger energy)**.** Let $(X, \mathrm{d}, \mathfrak{m})$ be a m.m.s. and let $f \in \mathcal{L}^2(\mathfrak{m})$. The *Cheeger energy* of f is defined as:

$$\mathrm{Ch}(f) := \frac{1}{2} \inf \left\{ \liminf_{n \to \infty} \int \mathrm{Lip}(f_n)^2 \, \mathrm{d}\, \mathfrak{m} \right\},$$

where the infimum is taken over all sequences $\{f_n\}$ of bounded Lipschitz $L^2(\mathfrak{m})$ functions satisfying $f_n \to f$ in $L^2(\mathfrak{m})$.

Using the Cheeger energy, one can define Sobolev functions on general metric measure spaces.

Definition 2.2.7. The space of Sobolev functions on a m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is defined as $W^{1,2}(X, \mathrm{d}, \mathfrak{m}) := \mathrm{D}(\mathrm{Ch}) \cap \mathcal{L}^2(\mathfrak{m})$, where $\mathrm{D}(\mathrm{Ch}) := \{\mathrm{Ch} < \infty\}$ is the domain of the Cheeger energy. $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ is endowed with the following norm:

$$|f|^2_{W^{1,2}} := 2\mathrm{Ch}(f) + |f|^2_{\mathcal{L}^2(\mathfrak{m})},$$

where $f \in H^{1,2}(X, \mathrm{d}, \mathfrak{m})$.

In general, the Sobolev space associated to a m.m.s. is only a separable Banach space. However, when the Cheeger energy is a quadratic form, then the Sobolev space becomes Hilbert (see [AGS14, Proposition 4.11]). In that case, the m.m.s. supports more elaborate calculus tools (such as second-order differential calculus as introduced in [Gig18]). This observation led Ambrosio, Gigli, and Savaré to introduce infinitesimally Hilbertian metric measure spaces.

Definition 2.2.8 (Infinitesimal Hilbertian spaces)**.** A m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is an *infinitesimal Hilbertian* space when the associated Cheeger energy is quadratic.

Riemannian manifolds are infinitesimally Hilbertian (see [LP20, Theorem 4.11] for a proof of this fact in the more general case of weighted Riemannian manifolds). Motivated by this property, the authors of [AGS14] strengthened the CD condition by requiring the spaces to be infinitesimally Hilbertian.

Definition 2.2.9 ($\mathrm{RCD}(K, N)$ spaces)**.** Given $N \in [1, \infty)$ and $K \in \mathbb{R}$, a m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ *space* when it is an infinitesimally Hilbertian $\mathrm{CD}(K, N)$ space.

One of the main results in [AGS14] is that the heat flow (defined as the gradient flow of the Cheeger energy) is linear on RCD spaces. However, it is shown in [OS09] that the heat flow on a Finsler manifold is linear if and only if it is Riemannian. Hence, the RCD condition succeeds at ruling out non-Riemannian Finsler spaces.

Then, even though the infinitesimal Hilbertianity condition is not stable under mGH convergence, it becomes stable when paired with the CD condition (see [AGS14, Theorem 6.11]). Therefore, Ricci limit spaces are also RCD spaces. In addition, based on the description of the infinitesimal structure of Alexandrov spaces realised in [Oht09a], the authors of [AGS14] observe that finite-dimensional Alexandrov spaces are infinitesimally Hilbertian. Thus, they are also RCD spaces.

RCD spaces are a promising candidate for a synthetic definition of Ricci curvature lower bounds. However, establishing that a given space is RCD can be a challenging task, as it necessitates demonstrating that it satisfies the CD property beforehand. This is complicated by the fact that CD spaces do not exhibit a local-to-global property, as pointed out by Rajala in [Raj16]. Consequently, a new class of spaces, called RCD^* spaces, was introduced as a solution to this problem.

2.2.4 RCD* spaces

To address the absence of a local-to-global property for CD spaces, Bacher and Sturm proposed an alternative definition. Specifically, in [BS10], they replaced the original coefficients τ in Definition 2.2.4 by the coefficients σ (see Notation 2.2.2). This approach gave rise to the concept of CD^* spaces, which we introduce below.

Definition 2.2.10 ($\mathrm{CD}^*(K,N)$ spaces)**.** Let $N \in (0,\infty)$ and $K \in \mathbb{R}$. A m.m.s. $(X,\mathrm{d},\mathfrak{m})$ is a $\mathrm{CD}^*(K,N)$ *space* if, for every pair $\nu_0, \nu_1 \in \mathcal{P}_\infty(X,\mathrm{d},\mathfrak{m})$, there exists an optimal transport plan $\pi \in \Pi(\nu_0,\nu_1)$ and a geodesic $\Gamma\colon [0,1] \to (\mathcal{P}_\infty(X,\mathrm{d},\mathfrak{m}), \mathcal{W}_2)$ from $\nu_0 = \rho_0\,\mathfrak{m}$ to $\nu_1 = \rho_1\,\mathfrak{m}$ such that:

$$\begin{aligned}\mathcal{S}_{N'}(\Gamma(t)|\mathfrak{m}) \leq -\int_{X\times X} \Big(&\sigma^{(1-t)}_{K,N'}(\mathrm{d}(x_0,x_1))\rho_0^{-1/N'}(x_0) \\ &+ \sigma^{(t)}_{K,N'}(\mathrm{d}(x_0,x_1))\rho_1^{-1/N'}(x_1)\Big)\,\mathrm{d}\pi(x_0,x_1)\end{aligned}. \tag{2.4}$$

for every $N' \geq N$.

Remark 2.2.4. Observe that in the special case $K = 0$ and $N \in [1,\infty)$, $\mathrm{CD}(0,N)$ spaces and $\mathrm{CD}^*(0,N)$ spaces coincide.

For non-branching m.m.s., the CD^* condition satisfies a local-to-global property, as shown in [BS10, Theorem 5.1]. This property allows us to introduce the most robust definition of synthetic Ricci curvature lower bounds available to date, namely RCD^* spaces.

Definition 2.2.11. A m.m.s. $(X,\mathrm{d},\mathfrak{m})$ is an $\mathrm{RCD}^*(K,N)$ space ($K \in \mathbb{R}$ and $N \in (0,\infty)$) if it is an infinitesimally Hilbertian $\mathrm{CD}^*(K,N)$ space.

The following result, combined with Theorem 2.2.1, implies that RCD^* spaces generalise Ricci curvature lower bounds and dimension upper bounds (see [BS10, Proposition 2.5] for a proof).

Proposition 2.2.3. Let $K \in \mathbb{R}$ and $N \in [1,\infty)$ and let $(X,\mathrm{d},\mathfrak{m})$ be a metric measure space. The following properties hold:

- If $(X,\mathrm{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ space, then it is an $\mathrm{RCD}^*(K,N)$ space.
- If $(X,\mathrm{d},\mathfrak{m})$ is an $\mathrm{RCD}^*(K,N)$ space, then it is an $\mathrm{RCD}(K^*,N)$ space (where $K^* := K\frac{N-1}{N}$).

Proposition 2.2.3 shows how $\mathrm{RCD}^*(K,N)$ and $\mathrm{RCD}(K,N)$ conditions relate to each other. In particular, it shows that both classes of spaces satisfy the same geometric inequalities (with slightly worst constants for the case of $\mathrm{RCD}^*(K,N)$ spaces). Nevertheless, would it be possible for these classes of spaces to coincide under certain conditions?

Thanks to [CM21, Theorem 1.1], $\mathrm{CD}^*(K,N)$ and $\mathrm{CD}(K,N)$ spaces coincide in the case of essentially non-branching metric measure spaces with finite mass. However, as observed in [EKS15, Remark 3.18], $\mathrm{RCD}^*(K,N)$ spaces are also strong $\mathrm{CD}^*(K,N)$ spaces; hence, using [RS14, Theorem 1.1], they are essentially non-branching. Therefore, we have the following fundamental result.

Theorem 2.2.3. *Let $(X,\mathrm{d},\mathfrak{m})$ be a finite mass metric measure space and let $K\in\mathbb{R}$ and $N\in[1,\infty)$. Then, $(X,\mathrm{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ space if and only if it is an $\mathrm{RCD}^*(K,N)$ space.*

Remark 2.2.5. The result above implies that compact $\mathrm{RCD}(K,N)$ spaces and compact $\mathrm{RCD}^*(K,N)$ spaces coincide.

Before we present the stability, compactness, and metric measure properties of RCD^* spaces, we first show how to restrict our attention to locally compact geodesic m.m.s. with full support.

It is clear that a m.m.s. $(X,\mathrm{d},\mathfrak{m})$ is an $\mathrm{RCD}^*(K,N)$ space if and only if $(\mathrm{Spt}(\mathfrak{m}),\mathrm{d},\mathfrak{m})$ is an $\mathrm{RCD}^*(K,N)$ space. Thus, we will always assume that:

$$\mathrm{Spt}(\mathfrak{m}) = X.$$

Then, by definition, $(\mathcal{P}_\infty(X,\mathrm{d},\mathfrak{m}),\mathcal{W}_2)$ is a geodesic metric space. Hence, according to the remark in the first paragraph of the proof of [BS10, Theorem 5.1], $(\mathrm{Spt}(\mathfrak{m}),\mathrm{d})$ is a length space. Furthermore, according to [BS10, Theorem 6.2], $(\mathrm{Spt}(\mathfrak{m}),\mathrm{d})$ is locally compact. In particular, thanks to [BBI22, Theorem 2.5.23], $(\mathrm{Spt}(\mathfrak{m}),\mathrm{d})$ is a geodesic space. Therefore, the following proposition holds.

Proposition 2.2.4. *Let $(X,\mathrm{d},\mathfrak{m})$ be a full support $\mathrm{RCD}^*(K,N)$ space ($K\in\mathbb{R}$ and $N\in[1,\infty)$), then (X,d) is locally compact and geodesic.*

Notation 2.2.3. Given $K\in\mathbb{R}$ and $N\in[1,\infty)$, we denote $\mathfrak{RCD}(K,N)$ (resp. $\mathfrak{RCD}^{*,\mathrm{p}}(K,N)\subset\mathfrak{X}^{\mathrm{p}}$) the set of isomorphism classes of compact $\mathrm{RCD}(K,N)$ spaces with full support (resp. pointed $\mathrm{RCD}^*(K,N)$ spaces with full support).

RCD* spaces satisfy a generalised Bishop–Gromov inequality (see [Stu06b, Theorem 2.3]).

Proposition 2.2.5 (Bishop–Gromov inequality)**.** Let $(X, \mathrm{d}, \mathfrak{m})$ be a full support $\mathrm{RCD}^*(K, N)$ space ($K \in \mathbb{R}$ and $N \in [1, \infty)$). Then, for every $x \in X$ and every $0 < r \leq R \leq \pi\sqrt{N/\max\{0, K\}}$, we have:

$$\frac{\mathfrak{m}(\overline{B}_R(x))}{\mathfrak{m}(\overline{B}_r(x))} \leq \frac{\int_0^R \mathfrak{s}_{K/N}(t)^N \,\mathrm{d}t}{\int_0^r \mathfrak{s}_{K/N}(t)^N \,\mathrm{d}t},$$

where $\mathfrak{s}_{K/N}$ is introduced in Notation 2.2.2.

The Bishop–Gromov inequality implies the following estimates on the doubling constant of an $\mathrm{RCD}^*(K, N)$ space (see [Stu06b, Corollary 2.5]).

Corollary 2.2.1. Let $(X, \mathrm{d}, \mathfrak{m})$ be a full support $\mathrm{RCD}^*(K, N)$ space ($K \in \mathbb{R}$ and $N \in [1, \infty)$). If $K \geq 0$, then $(X, \mathrm{d}, \mathfrak{m})$ is 2^{N+1}-doubling. If $K < 0$, then we have:

$$\frac{\mathfrak{m}(\overline{B}_{2r}(x))}{\mathfrak{m}(\overline{B}_r(x))} \leq 2^{N+1} \cosh^N(r\sqrt{-K/N}),$$

for every $x \in X$ and $r > 0$.

In order to draw conclusions about Ricci limit spaces from the properties of RCD* spaces, it is necessary for the latter to be stable under mGH convergence. Fortunately, the following fundamental compactness result ensures that this condition holds. The arguments to prove it are scattered throughout various references, we will provide a sketch of the proof.

Theorem 2.2.4 (Compact case)**.** Let $N \in [1, \infty)$, $K \in \mathbb{R}$, $0 < m < M$, and $D > 0$ be fixed constants. Then, the subset:

$$\mathfrak{RCD}(K, N) \cap \{\mathrm{Diam} \leq D\} \cap \{m \leq \mathrm{Mass} \leq M\} \subset \mathfrak{X}$$

is compact in the mGH topology ($\mathfrak{RCD}(K, N)$ is defined in Notation 2.2.3).

Sketch of the proof. First of all, Corollary 2.2.1 and Theorem 2.1.4 imply that $\mathfrak{RCD}(K, N) \cap \{\mathrm{Diam} \leq D\} \cap \{m \leq \mathrm{Mass} \leq M\}$ is a precompact subset of $\mathfrak{X}$ (recall that $\mathrm{RCD}(K, N)$ spaces and $\mathrm{RCD}^*(K, N)$ spaces coincide in the compact case). Therefore, we need only prove that it is closed. Let $\{[X_n, \mathrm{d}_n, \mathfrak{m}_n]\}_{n \in \mathbb{N}}$ be a sequence in $\mathfrak{RCD}(K, N) \cap \{\mathrm{Diam} \leq D\} \cap \{m \leq \mathrm{Mass} \leq M\}$ that converges in the mGH topology to $[X, \mathrm{d}, \mathfrak{m}] \in \mathfrak{X}$. Due to Corollary 2.2.1 and Theorem 2.1.4, we can conclude that

$[X, \mathrm{d}, \mathfrak{m}]$ has full support and belongs to $\{\mathrm{Diam} \leq D\} \cap \{m \leq \mathrm{Mass} \leq M\}$. Hence, we only need to demonstrate that $(X, \mathrm{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ space.

First, Theorem 2.2.2 implies that $(X, \mathrm{d}, \mathfrak{m})$ is a $\mathrm{CD}(K, N)$ space. Additionally,[AGS14, Theorem 6.11] implies that $(X, \mathrm{d}, \mathfrak{m})$ is also an $\mathrm{RCD}(K, \infty)$ space (as defined in [AGS14]). Thus, $(X, \mathrm{d}, \mathfrak{m})$ is infinitesimally Hilbertian, which completes the proof. □

In the thesis, we will also need the analogue of Theorem 2.2.4 in the pointed case.

Theorem 2.2.5 (Pointed case)**.** Let $N \in [1, \infty)$, $K \in \mathbb{R}$, $0 < r$, and $0 < m < M$ be fixed constants. Then the subset:

$$\mathfrak{RCD}^{*,\mathrm{p}}(K, N) \cap \{m \leq \mathrm{Mass}_r \leq M\} \subset \mathfrak{X}^{\mathrm{p}}$$

is compact in the pmGH topology (Mass_r and $\mathfrak{RCD}^{*,\mathrm{p}}(K, N)$ are introduced in Notations 2.1.4 and 2.2.3, respectively).

Sketch of the proof. As for the proof of Theorem 2.2.4, Corollary 2.2.1 and Theorem 2.1.7 imply that it is sufficient to prove that accumulation points of $\mathfrak{RCD}^{*,\mathrm{p}}(K, N) \cap \{m \leq \mathrm{Mass}_r \leq M\}$ in the pmGH topology are $\mathrm{RCD}^*(K, N)$ spaces.

Observe that pmGH convergence implies pmG convergence (as defined in [GMS15]). Moreover, $\mathrm{RCD}^*(K, N)$ spaces are in particular $\mathrm{RCD}(K^*, \infty)$ spaces (as defined in [AGS14]). Furthermore, $\mathrm{RCD}(K^*, \infty)$ spaces are stable under pmG convergence (as stated by [GMS15, Theorem 7.2]). Thus, any accumulation point of $\mathfrak{RCD}^{*,\mathrm{p}}(K, N) \cap \{m \leq \mathrm{Mass}_r \leq M\}$ is infinitesimally Hilbertian.

Then, assume that $[X, \mathrm{d}, \mathfrak{m}, *] \in \mathfrak{X}^{\mathrm{p}}$ is an accumulation point of $\mathfrak{RCD}^{*,\mathrm{p}}(K, N) \cap \{m \leq \mathrm{Mass}_r \leq M\}$. Let us denote $\{[X_n, \mathrm{d}_n, \mathfrak{m}_n, *_n]\}_{n \in \mathbb{N}}$ an approximating sequence and let us fix $\mu_0, \mu_1 \in \mathcal{P}_\infty(X, \mathrm{d}, \mathfrak{m})$. Note that there exists $R > 0$ such that $\mathrm{Spt}(\mu_0), \mathrm{Spt}(\mu_1) \subset \overline{B}_R(*)$. In particular, one can approximate μ_i by measures $\mu_i^n \in \mathcal{P}_\infty(X_n, \mathrm{d}_n, \mathfrak{m}_n)$ such that $\mathrm{Spt}(\mu_i^n) \subset \overline{B}_{R+1}(*_n)$ ($i \in \{0, 1\}$). Moreover, since $(X_n, \mathrm{d}_n, \mathfrak{m}_n)$ is a $\mathrm{CD}^*(K, N)$ space, there exists a geodesic $\{\mu_t^n\}_{t \in [0,1]}$ in $\mathcal{P}_\infty(X_n, \mathrm{d}_n, \mathfrak{m}_n)$ and an optimal transport plan π^n from μ_0^n to μ_1^n such that equation (2.4) is satisfied. In addition, observe that, using Proposition 2.2.2, there exists $\Pi^n \in \mathcal{P}(\mathrm{Geo}(X_n, \mathrm{d}_n))$ such that $\mu_t^n = e_{t*}\Pi^n$ for every $0 \leq t \leq 1$. In particular, for every $t \in [0, 1]$, we have:

$$\mathrm{Spt}(\mu_t^n) \subset \{\gamma(t), \gamma \in \mathrm{Geo}(X_n, \mathrm{d}_n), \gamma(0) \in \mathrm{Spt}(\mu_0^n), \gamma(1) \in \mathrm{Spt}(\mu_1^n)\};$$

therefore $\mathrm{Spt}(\mu_t^n) \subset \overline{B}_{2R+3}(*_n)$ (using the triangle inequality and $\mathrm{Spt}(\mu_0^n), \mathrm{Spt}(\mu_1^n) \subset \overline{B}_{R+1}(*_n)$). From there, following the strategy in the proof of [BS10, Theorem 3.1], one can extract a converging subsequence from $\{\mu_t^n\}_{0\leq t\leq 1, n\in\mathbb{N}}$ and $\{\pi^n\}_{n\in\mathbb{N}}$ that will give rise to a geodesic $\{\mu_t\}_{0\leq t\leq 1}$ and an optimal transport plan π from μ_0 to μ_1 satisfying equation (2.4). Therefore, one can conclude that $(X, \mathrm{d}, \mathfrak{m})$ is a $\mathrm{CD}^*(K,N)$ space, which concludes the sketch of the proof. □

To conclude this section, we list several interesting properties of RCD* spaces from the perspective of global Riemannian geometry:

- Brunn–Minkowski inequality (see [BS10, Proposition 6.1]),
- localization Theorem (see [CM17a, Theorem 5.1]),
- p-spectral gap estimate, log-Sobolev inequality, and Sobolev inequality (see Theorem 4.4, Theorem 6.2, and Theorem 7.1 in [CM17b]),
- Obata's Theorem (see [CMS19, Theorem 1.3]),
- Lévy–Gromov–Milman isoperimetric inequality (see [CM17a, Theorem 1.2]),
- splitting theorem, in the case $K = 0$ (see [Gig14, Theorem 1.4]),
- existence of universal covers (see [MW19, Theorem 1.1]).

We will give more details on the existence of universal covers and the splitting theorem in Sections 2.3 and 3.1, respectively.

2.2.5 Moduli spaces of RCD structures

In this thesis, we will mainly focus on RCD structures with a fixed topology. Let us introduce this notion in the following definition.

Definition 2.2.12 (RCD structure on a topological space)**.** Given a topological space X, $N \in [1,\infty)$, and $K \in \mathbb{R}$, an $\mathrm{RCD}(K,N)$ *(resp.* $\mathrm{RCD}^*(K,N)$*) structure on* X is an $\mathrm{RCD}(K,N)$ (resp. $\mathrm{RCD}^*(K,N)$) space $(X, \mathrm{d}, \mathfrak{m})$ such that d metrizes the topology of X and $\mathrm{Spt}(\mathfrak{m}) = X$.

One of our main goals in this thesis is to study the topological properties of the moduli spaces that we introduce in the next two definitions.

Definition 2.2.13. Let $K \in \mathbb{R}$, let $N \in [1, \infty)$, and let X be a compact topological space that admits an $\mathrm{RCD}(K,N)$ structure. The *moduli space of* $\mathrm{RCD}(K,N)$ *structures on* X is the set $\mathfrak{M}_{K,N}(X)$ of $\mathrm{RCD}(K,N)$ structures on X quotiented by isomorphisms, it is endowed with the mGH topology (see Definition 2.1.8).

Definition 2.2.14. Let X be a topological space that admits an $\mathrm{RCD}^*(K,N)$ structure ($K \in \mathbb{R}$ and $N \in [1, \infty)$). The *moduli space of pointed* $\mathrm{RCD}^*(K,N)$ *structures on* X is the set $\mathfrak{M}^{\mathrm{p}}_{K,N}(X)$ of pointed $\mathrm{RCD}^*(K,N)$ structures on X quotiented by isomorphisms, it is endowed with the pmGH topology (see Definition 2.1.20).

2.3 Covering space theory of RCD spaces

In this section, we provide a brief reminder on covering space theory. Then, we will introduce δ-covers and explain how to lift RCD structures to covering spaces. Afterwards, we will explain how universal covers and δ-covers are related to each other. Finally, we will introduce the Dirichlet domain associated to an RCD structure on a compact topological space.

2.3.1 Basic results on covering spaces

This subsection recalls the basic definitions and results about covering spaces (we follow the presentation of [Spa81, Chapter 2] and[Ste99, Chapter 1]). First of all, we define covering triples.

Definition 2.3.1 (Covering triple)**.** Let X and Y be connected, locally path-connected topological spaces and let $p\colon Y \to X$ be a continuous map. The triple $\mathcal{B} := (Y, X, p)$ is called a *covering triple* if every point $x \in X$ admits an open neighbourhood $U_x \subset X$ that is *evenly covered* by p, i.e., such that $p^{-1}(U_x)$ is the disjoint union of open subsets of Y, each of which is mapped homeomorphically onto U_x by p. In that case, we will say that Y, X and p are respectively the *covering space*, the *base space* and the *covering projection* of the covering triple $\mathcal{B}$. If $x \in X$ and $y \in Y$ satisfy $x = p(y)$, then we will call the triple $((Y, y), (X, x), p)$ a *pointed covering triple.*

From now on, when we deal with a covering triple, we will always assume that the covering space and the base space are connected locally path-connected topological spaces (unless otherwise stated). Let us now recall the definition of a morphism between covering triples.

Definition 2.3.2 (Covering morphism)**.** Let $\mathcal{B}_i \coloneqq (Y_i, X_i, p_i)$ be a covering triple ($i \in \{1, 2\}$). We say that $\tilde{\eta}\colon \mathcal{B}_1 \to \mathcal{B}_2$ is a continuous (respectively Borel) *covering morphism* if:

(i) $\tilde{\eta}\colon Y_1 \to Y_2$ is a continuous (respectively Borel) map,

(ii) for every $y_1, y_1' \in Y_1$, we have:

$$p_1(y_1) = p_1(y_1') \implies p_2(\tilde{\eta} \cdot y_1) = p_2(\tilde{\eta} \cdot y_1').$$

In that case, $\tilde{\eta}$ induces a continuous (respectively Borel) map $\eta\colon X_1 \to X_2$, called the *induced base map*, such that the following commutative diagram holds:

$$\begin{array}{ccc} Y_1 & \xrightarrow{\tilde{\eta}} & Y_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ X_1 & \xrightarrow[\eta]{} & X_2 \end{array} .$$

When covering triples share the same base space, we will restrict our attention to a special kind of covering morphism, namely based covering morphism (see [Ste99, Chapter 1], and Lemma 1 in Chapter 2, Section 5 in [Spa81]).

Definition-Proposition 2.3.1 (Based covering morphism)**.** Let $\mathcal{B}_i \coloneqq (Y_i, X, p_i)$ be a covering triple over X ($i \in \{1, 2\}$). A covering morphism $\tilde{\eta}\colon \mathcal{B}_1 \to \mathcal{B}_2$ is a *based covering morphism* if the induced base map $\eta\colon X \to X$ is the identity. In that case, we have the following simplified commutative diagram:

$$\begin{array}{ccc} Y_1 & \xrightarrow{\quad\tilde{\eta}\quad} & Y_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & X & \end{array}$$

and $\tilde{\eta}\colon Y_1 \to Y_2$ is a covering projection. When $\tilde{\eta}$ is a homeomorphism, we say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are *equivalent* as covering triples over X, and $\tilde{\eta}\colon \mathcal{B}_1 \to \mathcal{B}_2$ is called a based covering *isomorphism.*

Let us state the definition of a universal covering space.

Definition 2.3.3 (Universal covering)**.** A covering triple $\tilde{\mathcal{B}} := (\tilde{X}, X, p)$ is a *universal covering triple over* X if, for every other covering triple $\mathcal{B}$ over X, there exists a based covering morphism $\tilde{\eta}\colon \tilde{\mathcal{B}} \to \mathcal{B}$. In that case, $\tilde{X}$ is called a *universal covering space over* X. A universal covering triple over X (if it exists) is unique up to equivalences of based covering triples.

Let us now recall the classification theorem for covering spaces (Chapter 2, Section 5 in [Spa81]).

Theorem 2.3.1 (Classification theorem)**.** Let $\mathcal{B}_i := (Y_i, X, p_i)$ be a covering triple over X ($i \in \{1, 2\}$). The following properties are equivalent:

(i) there exists a based covering morphism $\tilde{\eta}\colon \mathcal{B}_1 \to \mathcal{B}_2$,

(ii) for every $y_1 \in Y_1$ and $y_2 \in Y_1$ such that $p_1(y_1) = p_2(y_2) = x$, $p_{1*}\pi_1(Y_1, y_1)$ is conjugated in $\pi_1(X, x)$ to a subgroup of $p_{2*}\pi_1(Y_2, y_2)$,

(iii) there exists $y_1 \in Y_1$ and $y_2 \in Y_1$ such that $p_1(y_1) = p_2(y_2) = x$, $p_{1*}\pi_1(Y_1, y_1)$ is conjugated in $\pi_1(X, x)$ to a subgroup of $p_{2*}\pi_1(Y_2, y_2)$,

where $p_{1*}\colon \pi_1(Y_1, y_1) \to \pi_1(X, x)$ (respectively $p_{2*}\colon \pi_1(Y_2, y_2) \to \pi_1(X, x)$) is the homomorphism between fundamental groups induced by p_1 (respectively by p_2).

Remark 2.3.1 (Injectivity of homomorphism induced by a covering projection). For any covering triple $\mathcal{B} := (Y, X, p)$ and every $y \in Y$, the induced homomorphism $p_*\colon \pi_1(Y, y) \to \pi_1(X, x)$ (where $x := p(y)$) is always injective.

One of the most important objects associated to a covering triple is its group of deck transformations.

Definition 2.3.4 (Group of deck transformations)**.** Let $\mathcal{B} := (Y, X, p)$ be a covering triple. We denote:

$$G(\mathcal{B}) := \{\tilde{\eta}\colon \mathcal{B} \to \mathcal{B} \mid \tilde{\eta} \text{ based covering isomorphism}\}.$$

$G(\mathcal{B})$ is called the *group of deck transformation* associated to $\mathcal{B}$. For every $x \in X$, there is a natural left action of $G(\mathcal{B})$ on $p^{-1}(x)$ defined by:

$$\forall \tilde{\eta} \in G(\mathcal{B}), \forall x \in X, \forall y \in p^{-1}(x), \tilde{\eta} \cdot y := \tilde{\eta}(y).$$

In the thesis, all of the coverings will be regular. We recall below the definition and basic properties of *regular coverings* (Chapter 2, Section 3 in [Spa81]).

Definition-Proposition 2.3.2 (Regular covering)**.** Let $\mathcal{B} := (Y, X, p)$ be a covering triple. The covering is called *regular* if it satisfies one of the following equivalent properties:

(i) for every $x \in X$, $G(\mathcal{B})$ acts transitively on $p^{-1}(x)$,

(ii) there exists $x \in X$ such that $G(\mathcal{B})$ acts transitively on $p^{-1}(x)$,

(iii) for every $y \in Y$, $p_*\pi_1(Y, y)$ is a normal subgroup of $\pi_1(X, p(y))$ (we will write $p_*\pi_1(Y, y) \trianglelefteq \pi_1(X, p(y)))$,

(iv) there exists $y \in Y$ such that $p_*\pi_1(Y, y) \trianglelefteq \pi_1(X, p(y))$.

Remark 2.3.2 (Continuous discrete action)*.* For any regular covering triple $\mathcal{B} := (Y, X, p)$, it is always possible to see the group of deck transformation $G(\mathcal{B})$ as a discrete group acting continuously on the total space Y. Indeed, any group equipped with its discrete topology is a discrete group. Moreover, since $G(\mathcal{B})$ is a subset of the group of homeomorphism of Y, then it obviously acts continuously on Y.

In the case of a regular covering $\mathcal{B} := (Y, X, p)$, it is possible to identify the base space X with the orbit space $Y/G(\mathcal{B})$ (Chapter 2, Section 6 in [Spa81]).

Proposition 2.3.1 (Quotient of the total space by the group of deck transformations)**.** Given a regular covering triple $\mathcal{B} := (Y, X, p)$, the group of deck transformations $G(\mathcal{B})$ is a discrete group acting continuously on Y. Moreover, the action satisfies the following axioms:

(i) for every $\tilde{\eta} \in G(\mathcal{B})$ and every $y \in Y$ we have:

$$\tilde{\eta} \cdot y = y \implies \tilde{\eta} = \mathrm{id}_y,$$

i.e., the action is free,

(ii) every point $y \in Y$ admits an open neighbourhood $U_y \subset Y$ such that:

$$\forall \tilde{\eta} \in G(\mathcal{B}), \tilde{\eta} \cdot U_y \cap U_y \neq \varnothing \implies \tilde{\eta} = \mathrm{id}_Y .$$

In addition, there exists a homeomorphism $\phi\colon X \to Y/G(\mathcal{B})$ such that the following commutative diagram holds:

$$\begin{array}{ccc} & Y & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle \pi} \\ X & \xrightarrow{\ \phi\ } & Y/G(\mathcal{B}) \end{array} ,$$

where π is the quotient map.

When the base space of a covering triple is Hausdorff, we have the following result (using Proposition 2.3.1, Theorem 1 and Corollary 1 of section 1.5.3 in [Bou16], and Theorem 1 of Section 3.4.5 in [Bou07a]):

Proposition 2.3.2 (Proper action)**.** Let $\mathcal{B} \coloneqq (Y, X, p)$ be a regular covering triple such that the base space X is Hausdorff. Then $G(\mathcal{B})$ acts *properly* on Y, i.e., for every compact subset $K \subset Y$:

$$\left|\left\{\tilde{\eta} \in G(\mathcal{B}) \;\mid\; \tilde{\eta} \cdot K \cap K \neq \varnothing\right\}\right| < +\infty.$$

Let us now introduce a very helpful result (Chapter 2, Section 4 and 5 in [Spa81]) that associates a regular covering triple $\mathcal{B}^{\mathcal{U}}$ to any open covering $\mathcal{U}$ of X.

Proposition 2.3.3 (Regular covering associated to an open covering)**.** Let X be a connected locally path-connected topological space and let $\mathcal{U}$ be an open covering of X. There exists a regular covering triple $\mathcal{B}^{\mathcal{U}} \coloneqq (X^{\mathcal{U}}, X, p^{\mathcal{U}})$ (unique up to equivalence of based covering triples) such that:

$$\forall y \in X^{\mathcal{U}}, p_*^{\mathcal{U}} \pi_1(X^{\mathcal{U}}, y) = \pi_1(\mathcal{U}, p^{\mathcal{U}}(y)),$$

where $\pi_1(\mathcal{U}, p^{\mathcal{U}}(y))$ is composed of homotopy classes of loops of the form $\omega^{-1} * \alpha * \omega$, where α is a loop contained in some $U \in \mathcal{U}$ and ω is a path from $p^{\mathcal{U}}(y)$ to $\alpha(0)$. Moreover, every connected open set $U \in \mathcal{U}$ is evenly covered by $p_{\mathcal{U}}$.

Given a regular covering, let us relate the base space's fundamental group and the group of deck transformation (see Chapter 2, Section 6 in [Spa81]).

Definition-Proposition 2.3.3 (On the relation between the fundamental group and deck transformations)**.** Let $\mathcal{B} \coloneqq (Y, X, p)$ be a regular covering and let $x \in X$ and $y \in p^{-1}(x)$. Since $G(\mathcal{B})$ acts freely and transitively on every fibre, the following map is bijective:

$$\phi_y \colon \tilde{\eta} \in G(\mathcal{B}) \to \tilde{\eta} \cdot y \in p^{-1}(x).$$

For any $[\gamma] \in \pi_1(X, x)$ we denote:

$$\psi_x([\gamma]) \coloneqq \phi_y^{-1}(\tilde{\gamma}(1)) \in G(\mathcal{B}),$$

where $\tilde{\gamma}$ is the unique lift of $\gamma \colon [0,1] \to X$ starting at y (the map ψ_x is denoted this way because it is independent of the chosen point y in the fibre over x). The map ψ_x is a surjective group homomorphism with kernel $p_*(\pi_1(Y, y))$ and thus induces an isomorphism:

$$\pi_1(X, x)/p_*\Big(\pi_1(Y, y)\Big) \xrightarrow[\sim]{\overline{\psi}_x} G(\mathcal{B}) \ .$$

2.3.2 Lift and push forward of RCD structures

In this section, we will introduce specific covering spaces of RCD spaces, namely, δ-covers. In particular, we will see how these covering spaces are related to the universal cover of a space. Finally, we will show how to lift RCD structures from the base space to the universal cover.

Let us start this section with the following fundamental result (see [MW19, Theorem 1.1]).

Theorem 2.3.2. *Let X be a topological space that admits an $\mathrm{RCD}^*(K,N)$ structure ($K \in \mathbb{R}$ and $N \in [1,\infty)$). Then, there exists a universal covering triple $\mathcal{B} = (\tilde{X}, X, p)$ over X. We denote $\overline{\pi}_1(X) := G(\mathcal{B})$ its group of deck transformations and call it the revised fundamental group of X.*

From now on, we fix real numbers $K \in \mathbb{R}$ and $N \in [1,\infty)$, and a compact topological space X admitting an $\mathrm{RCD}(K,N)$ structure. We also denote $p\colon \tilde{X} \to X$ the universal covering projection associated to X. Let us also recall from Theorem 2.2.3 that any $\mathrm{RCD}(K,N)$ structure on X is also an $\mathrm{RCD}^*(K,N)$ structure on X and vice-versa.

The notion of δ-cover was introduced first by Sormani and Wei to prove the existence of a universal cover for Ricci limit spaces (see [SW01, Theorem 1.1]). Later, it has also been used by Mondino and Wei in [MW19] to prove Theorem 2.3.2. These covering spaces will be very important in the proof of the equivariant theorem (see Theorem 1.3.1 in Chapter 1).

Definition 2.3.5. Given $\delta > 0$ and $(X, \mathrm{d}, \mathfrak{m})$ an $\mathrm{RCD}(K,N)$ structure on X, the *δ-cover associated to* $(X, \mathrm{d}, \mathfrak{m})$ is the regular covering $p^\delta_{\mathrm{d}}\colon X^\delta_{\mathrm{d}} \to X$ associated to the open covering $\mathcal{U}(\delta, \mathrm{d})$ consisting of balls of radius δ for the distance d (see Proposition 2.3.3). We write $G(\delta, \mathrm{d})$ the associated group of deck transformations.

In the following result, we introduce the lift of an $\mathrm{RCD}(K,N)$ structure on X to a δ-cover.

Proposition 2.3.4. *Given $\delta > 0$ and $(X, \mathrm{d}, \mathfrak{m})$ an $\mathrm{RCD}(K,N)$ structure on X, there exists a unique $\mathrm{RCD}^*(K,N)$ structure $(X^\delta_{\mathrm{d}}, \mathrm{d}_\delta, \mathfrak{m}_\delta)$ on X^δ_{d} such that $p^\delta_{\mathrm{d}}\colon (X^\delta_{\mathrm{d}}, \mathrm{d}_\delta, \mathfrak{m}_\delta) \to (X, \mathrm{d}, \mathfrak{m})$ is a local isomorphism. Moreover, we have the following properties:*

(i) for every $\tilde{x}, \tilde{y} \in X^\delta_{\mathrm{d}}$, we have $\mathrm{d}_\delta(\tilde{x}, \tilde{y}) = \inf\{\mathcal{L}_{\mathrm{d}}(p^\delta_{\mathrm{d}} \circ \tilde{\gamma})\}$, where the infimum is taken over all continuous path $\tilde{\gamma}\colon [0,1] \to X^\delta_{\mathrm{d}}$ from $\tilde{x}$ to $\tilde{y}$ and $\mathcal{L}_{\mathrm{d}}$ is the length structure induced by d,

(ii) for every Borel subset $\tilde{E} \subset X^\delta_{\mathrm{d}}$ such that $p^\delta_{\mathrm{d}|\tilde{E}}$ is an isometry, we have $\mathfrak{m}_\delta(\tilde{E}) = \mathfrak{m}(E)$,

(iii) the group of deck transformations $G(\delta, \mathrm{d})$ is a subgroup of $\mathrm{Iso}_{\mathrm{m.m.s.}}(X^\delta_{\mathrm{d}}, \mathrm{d}_\delta, \mathfrak{m}_\delta)$,

(iv) for every $\tilde{x} \in X^\delta_{\mathrm{d}}$ and every $r \leq \delta$, the restriction of p^δ_{d} to $B_{\mathrm{d}_\delta}(\tilde{x}, r)$ is a homeomorphism onto $B_{\mathrm{d}}(p^\delta_{\mathrm{d}}(\tilde{x}), r)$,

(v) for every $\tilde{x} \in X^\delta_{\mathrm{d}}$ and every $r \leq \delta/2$, the restriction of p^δ_{d} to $(B_{\mathrm{d}_\delta}(\tilde{x}, r), \mathrm{d}_\delta, \mathfrak{m}_\delta)$ is an isomorphism onto $(B_{\mathrm{d}}(x, r), \mathrm{d}, \mathfrak{m})$.

Proof. First of all, there is obviously at most one $\mathrm{RCD}^*(K,N)$ structure on X^δ_{d} such that p^δ_{d} is a local isomorphism. Then, according to to [MW19, Lemma 2.18], $(X^\delta_{\mathrm{d}}, \mathrm{d}_\delta, \mathfrak{m}_\delta)$ is an $\mathrm{RCD}^*(K,N)$ space (where d_δ and $\mathfrak{m}_\delta$ are defined as in (i) and (ii)). Moreover, it is readily checked that d_δ metrizes the topology of X^δ_{d}, that $\mathrm{Spt}(\mathfrak{m}_\delta) = X^\delta_{\mathrm{d}}$, and that $G(\delta, \mathrm{d})$ acts by isomorphism. Therefore, $(X^\delta_{\mathrm{d}}, \mathrm{d}_\delta, \mathfrak{m}_\delta)$ is an $\mathrm{RCD}^*(K,N)$ structure on X^δ_{d} satisfying point (i) to (iii). Finally, thanks to [Rev08, Proposition 15], and by definition of $\mathfrak{m}_\delta$, points (iv) and (v) are satisfied. □

Now, we put the universal cover of X in relation with δ-covers (see [MW19, Theorem 2.7]).

Theorem 2.3.3. Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ structure on X, and let $\delta(X, \mathrm{d})$ be the supremum of all $\delta > 0$ such that every ball of radius δ in (X, d) is evenly covered by p. Then $\delta(X, \mathrm{d}) > 0$ and, for every $\delta < \delta(X, \mathrm{d})$, p and p^δ_{d} are equivalent and every equivalence map is an isomorphism between $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ and $(X^\delta_{\mathrm{d}}, \mathrm{d}_\delta, \mathfrak{m}_\delta)$.

Relying on Proposition 2.3.4 and Theorem 2.3.3, we can introduce the lift of an $\mathrm{RCD}(K,N)$ structure on X to the universal cover $\tilde{X}$.

Corollary 2.3.1. Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ structure on X. There is a unique $\mathrm{RCD}^*(K,N)$ structure $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ on $\tilde{X}$ (called the *lift of* $(X, \mathrm{d}, \mathfrak{m})$) such that $p\colon (\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}) \to (X, \mathrm{d}, \mathfrak{m})$ is a local isomorphism. Moreover, the revised fundamental group $\overline{\pi}_1(X)$ (introduced in Theorem 2.3.2) acts by isomorphism on $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$.

The following proposition is a sort of converse to Corollary 2.3.1; it introduces the push-forward of an equivariant $\mathrm{RCD}^*(K,N)$ structure on $\tilde{X}$ (see [MW19, Lemma 2.18] and [MMP22, Lemma 2.24]).

Proposition 2.3.5. Let $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ be an $\mathrm{RCD}^*(K,N)$ structure on $\tilde{X}$ such that $\overline{\pi}_1(X)$ acts by isomorphisms on $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$. There is a unique $\mathrm{RCD}(K,N)$ structure $(X, \mathrm{d}, \mathfrak{m})$ on X (called the *push-forward of* $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$) such that $p\colon (\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}) \to (X, \mathrm{d}, \mathfrak{m})$ is a local isomorphism. It satisfies the following properties:

(i) for every $x, y \in X$, we have $\mathrm{d}(x,y) = \inf\{\tilde{\mathrm{d}}(\tilde{x}, \tilde{y})\}$, where the infimum is taken over all $\tilde{x} \in p^{-1}(x)$ and $\tilde{y} \in p^{-1}(y)$,

(ii) for every Borel set $E \in \mathcal{B}(X)$ contained in an open set $U \subset X$ that is evenly covered by p, we have $\mathfrak{m}(E) = \tilde{\mathfrak{m}}(p_{|\tilde{U}}^{-1}(E))$, where $\tilde{U}$ is any open set in $\tilde{X}$ such that $p\colon \tilde{U} \to U$ is a homeomorphism.

Proof. First of all, there is obviously at most one $\mathrm{RCD}(K,N)$ structure on X such that p is a local isomorphism.

Then, let us define d and $\mathfrak{m}$ as in points (i) and (ii). Observe that since $\tilde{X}$ is locally compact and since $\overline{\pi}_1(X)$ acts by isometries, the infimum in the definition of d is achieved and $\mathfrak{m}$ is well-defined. It is then readily checked that d metrizes the topology of X. In particular, (X, d) is separable (since X is compact). Finally, observe that every open set in X is a countable union of evenly covered open sets (using that X is separable). Therefore, one can extend $\mathfrak{m}$ into a pre-measure on the ring of open sets. Furthermore, $\mathfrak{m}$ is finite since it is locally finite (since $\tilde{\mathfrak{m}}$ is locally finite) and X is compact. In particular, Carathéodory's extension theorem implies that $\mathfrak{m}$ can be extended to a finite Borel measure on X, which is unique.

Let us now show that p is a local isomorphism. Let $\tilde{x} \in \tilde{X}$ and define $x \coloneqq p(\tilde{x})$. There exists an open neighborhood $\tilde{U}$ of $\tilde{x}$ such that $p\colon \tilde{U} \to U \coloneqq p(\tilde{U})$ is a homeomorphism. Moreover, there exists $r > 0$ such that $B_{\tilde{\mathrm{d}}}(\tilde{x}, r) \subset \tilde{U}$. Let us show that, for every $0 < r' \leq r$, p is a homeomorphism from $B_{\tilde{\mathrm{d}}}(\tilde{x}, r')$ onto $B_{\mathrm{d}}(x, r')$. First, notice that p is distance decreasing; in particular, we have $p(B_{\tilde{\mathrm{d}}}(\tilde{x}, r')) \subset B_{\mathrm{d}}(x, r')$. Now, let $y \in B_{\mathrm{d}}(x, r')$. Since the infimum in the definition of d is achieved, there exists $\tilde{y} \in p^{-1}(y)$ such that $\tilde{\mathrm{d}}(\tilde{x}, \tilde{y}) = \mathrm{d}(x,y) < r'$. Hence, $p(B_{\tilde{\mathrm{d}}}(\tilde{x}, r')) = B_{\mathrm{d}}(x, r')$. Since $B_{\tilde{\mathrm{d}}}(\tilde{x}, r')$ is a subset of $\tilde{U}$, p is injective on $B_{\tilde{\mathrm{d}}}(\tilde{x}, r')$. Hence, p is a bijective map from $B_{\tilde{\mathrm{d}}}(\tilde{x}, r')$ onto $B_{\mathrm{d}}(x, r')$. However, p is an open map. Hence, it is a homeomorphism from $B_{\tilde{\mathrm{d}}}(\tilde{x}, r')$ onto $B_{\mathrm{d}}(x, r')$.

Now, let $\tilde{y}, \tilde{z} \in B_{\tilde{\mathrm{d}}}(\tilde{x}, r/3)$. Looking for a contradiction, let us suppose that $\mathrm{d}(y,z) < \tilde{\mathrm{d}}(\tilde{y}, \tilde{z})$, where $y := p(\tilde{y})$ and $z := p(\tilde{z})$. In that case, there exists $\tilde{z}' \in p^{-1}(z)$ such that $\tilde{\mathrm{d}}(\tilde{y}, \tilde{z}') = \mathrm{d}(y,z) < \tilde{\mathrm{d}}(\tilde{y}, \tilde{z}) \le \tilde{\mathrm{d}}(\tilde{x}, \tilde{y}) + \tilde{\mathrm{d}}(\tilde{x}, \tilde{z}) < 2r/3 < r$. However, p is a homeomorphism from $B_{\tilde{\mathrm{d}}}(\tilde{x}, r)$ onto $B_{\mathrm{d}}(x, r)$, so we should have $\tilde{z}' = \tilde{z}$, which is the contradiction we were looking for. Hence, p is an isometry from $B_{\tilde{\mathrm{d}}}(\tilde{x}, r/3)$ onto $B_{\mathrm{d}}(x, r/3)$. Moreover, by definition of $\mathfrak{m}$, this implies that p is an isomorphism of metric measure space from $B_{\tilde{\mathrm{d}}}(\tilde{x}, r/3)$ onto $B_{\mathrm{d}}(x, r/3)$.

Then, it is clear that $\mathrm{Spt}(\mathfrak{m}) = X$. To conclude, we must show that $(X, \mathrm{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ space. However, since p is a local isometry, it preserves the length of curves; therefore, (X, d) is a geodesic space. Summarising: $(X, \mathrm{d}, \mathfrak{m})$ is a compact geodesic space endowed with a finite measure.

Let us prove that $(X, \mathrm{d}, \mathfrak{m})$ is a strong $\mathrm{CD}^{\mathrm{e}}(K,N)$ space (see [EKS15, Definition 3.1]). First, assume that $x \in X$, $R > 0$, and $\mu_i \in \mathcal{P}_2(X, \mathrm{d})$ ($i \in \{0,1\}$) such that $\mathrm{Spt}(\mu_i) \subset B_{R/2}(x)$. Observe that, using Proposition 2.2.2 and proceeding as in the proof of Theorem 2.2.5, we can show that any geodesic $\{\mu_t\}_{t \in [0,1]}$ in $(\mathcal{P}_2(X, \mathrm{d}), \mathcal{W}_2)$ from μ_0 to μ_1 satisfies $\mathrm{Spt}(\mu_t) \subset B_R(x)$, for every $t \in [0,1]$. However, $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ is a strong $\mathrm{CD}^{\mathrm{e}}(K,N)$ space (using [EKS15, Remark 3.18]) and p is a local isomorphism. Therefore, $(X, \mathrm{d}, \mathfrak{m})$ is a strong $\mathrm{CD}^{\mathrm{e}}_{\mathrm{loc}}(K,N)$ space; hence, using [EKS15, Theorem 3.14], $(X, \mathrm{d}, \mathfrak{m})$ is a strong $\mathrm{CD}^{\mathrm{e}}(K,N)$ space.

Let us show that $(X, \mathrm{d}, \mathfrak{m})$ is infinitesimally Hilbertian. First, let us fix a countable dense subset $\mathcal{D} \subset X$ and, for every $x \in \mathcal{D}$, let us fix $y_x \in p^{-1}(x)$. Then, we introduce the countable family $\overline{\mathcal{F}}$ of closed subsets of X of the form $\overline{B}_r(x)$, where $x \in \mathcal{D}$ and $r \in \mathbb{Q}_{>0}$ such that $p\colon \overline{B}_r(y_x) \to \overline{B}_r(x)$ is an isometry. Observe that if $E = \overline{B}_r(x) \in \overline{\mathcal{F}}$ and if we denote $\tilde{E} := \overline{B}_r(y_x)$, then $p\colon (\tilde{E}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}(\tilde{E})^{-1}\tilde{\mathfrak{m}}_{|\tilde{E}}) \to (E, \mathrm{d}, \mathfrak{m}(E)^{-1}\mathfrak{m}_{|E})$ is an isomorphism. In addition, [Vil09, Theorem 30.11] implies $\tilde{\mathfrak{m}}(\partial \tilde{E}) = 0$. Therefore, thanks to [AGS14, Theorem 4.20], $(\tilde{E}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}(\tilde{E})^{-1}\tilde{\mathfrak{m}}_{|\tilde{E}})$ is infinitesimally Hilbertian; hence, $(E, \mathrm{d}, \mathfrak{m}(E)^{-1}\mathfrak{m}_{|E})$ is infinitesimally Hilbertian. Thus, using [EKS15, Theorem 3.25], $(X, \mathrm{d}, \mathfrak{m})$ is infinitesimally Hilbertian.

Summarising, $(X, \mathrm{d}, \mathfrak{m})$ is an infinitesimally Hilbertian strong $\mathrm{CD}^{\mathrm{e}}(K,N)$ space. Therefore, [EKS15, Theorem 3.17] implies that it is an $\mathrm{RCD}^*(K,N)$ space. But X is compact, so $(X, \mathrm{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ space (using Theorem 2.2.3). □

Remark 2.3.3. Observe that if $(X, \mathrm{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ structure on X, then the push-forward of the lift of $(X, \mathrm{d}, \mathfrak{m})$ is equal to $(X, \mathrm{d}, \mathfrak{m})$, thanks to Proposition 2.3.5 and Corollary 2.3.1. The same is true in the other direction; if $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ is an

$\mathrm{RCD}^*(K,N)$ structure on $\tilde{X}$ such that $\overline{\pi}_1(X)$ acts by isomorphisms, then the lift of the push-forward of $(\tilde{X},\tilde{\mathrm{d}},\tilde{\mathfrak{m}})$ is equal to $(\tilde{X},\tilde{\mathrm{d}},\tilde{\mathfrak{m}})$.

We conclude this section with the following result that introduces the Dirichlet domain associated to an $\mathrm{RCD}(K,N)$ structure on X.

Proposition 2.3.6. Let $(X,\mathrm{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ structure on X with lift $(\tilde{X},\tilde{\mathrm{d}},\tilde{\mathfrak{m}})$ and let $\tilde{x}\in\tilde{X}$. We define the *Dirichlet domain with center $\tilde{x}$ associated to* $(X,\mathrm{d},\mathfrak{m})$ by:

$$\mathcal{F}(\tilde{x}) \coloneqq \bigcap_{\eta\in\overline{\pi}_1(X)} \phi_\eta^{-1}(\mathbb{R}_{\geq 0}),$$

where $\phi_\eta(\tilde{y}) \coloneqq \tilde{\mathrm{d}}(\tilde{y},\eta\tilde{x}) - \tilde{\mathrm{d}}(\tilde{y},\tilde{x})$, for $\tilde{y}\in\tilde{X}$. The Dirichlet domain satisfies the following two properties:

(i) for every $\tilde{y}\in\tilde{X}$, there exists $\eta\in\overline{\pi}_1(X)$ such that $\eta\tilde{y}\in\mathcal{F}(\tilde{x})$,

(ii) for every $\tilde{y}\in\mathcal{F}(\tilde{x})$, we have $\tilde{\mathrm{d}}(\tilde{x},\tilde{y}) = \mathrm{d}(x,y)$, where $x\coloneqq p(\tilde{x})$ and $y\coloneqq p(\tilde{y})$.

In particular, $\mathcal{F}(\tilde{x})\subset B_{\tilde{\mathrm{d}}}(\tilde{x},D)$, where $D\coloneqq \mathrm{Diam}(X,\mathrm{d})$.

Proof. We start with the proof of (i). Let $\tilde{y}\in\tilde{X}$ and define $R\coloneqq \tilde{d}(\tilde{x},\tilde{y})$. Then, $p^{-1}(x)\cap\overline{B}_{\tilde{d}}(\tilde{y},R)$ is a compact, discrete, non empty set; hence, it contains finitely many points. In particular, there exists $\eta\in\overline{\pi}_1(X)$ such that $\eta\tilde{x}\in\overline{B}_{\tilde{d}}(\tilde{y},R)$, and such that:

$$\forall\tilde{z}\in p^{-1}(x)\cap\overline{B}_{\tilde{d}}(\tilde{y},R), \tilde{\mathrm{d}}(\tilde{y},\eta\tilde{x})\leq\tilde{\mathrm{d}}(\tilde{y},\tilde{z}). \tag{2.5}$$

Now, assume that $\mu\in\overline{\pi}_1(X)$. If $R\leq\tilde{\mathrm{d}}(\tilde{y},\mu\tilde{x})$, we have $\tilde{\mathrm{d}}(\tilde{y},\eta\tilde{x})\leq\tilde{\mathrm{d}}(\tilde{y},\mu\tilde{x})$ since $\tilde{\mathrm{d}}(\tilde{y},\eta\tilde{x})\leq R$. If $\tilde{\mathrm{d}}(\tilde{y},\mu\tilde{x})<R$, then, equation (2.5) implies $\tilde{\mathrm{d}}(\tilde{y},\eta\tilde{x})\leq\tilde{\mathrm{d}}(\tilde{y},\mu\tilde{x})$. Thus, for every $\mu\in\overline{\pi}_1(X)$, we get $\tilde{\mathrm{d}}(\tilde{y},\eta\tilde{x})\leq\tilde{\mathrm{d}}(\tilde{y},\mu\tilde{x})$. Hence, for every $\mu\in\overline{\pi}_1(X)$, we have $\phi_\mu(\eta^{-1}\tilde{y}) = \tilde{\mathrm{d}}(\eta^{-1}\tilde{y},\mu\tilde{x}) - \tilde{\mathrm{d}}(\eta^{-1}\tilde{y},\tilde{x}) = \tilde{\mathrm{d}}(\tilde{y},\eta\mu\tilde{x}) - \tilde{\mathrm{d}}(\tilde{y},\eta\tilde{x})\geq 0$. In conclusion, $\eta^{-1}\tilde{y}\in\mathcal{F}(\tilde{x})$.

Now we prove (ii). Assume that $\tilde{y}\in\mathcal{F}(\tilde{x})$ and let $\tilde{\gamma}\colon[0,1]\to\tilde{X}$ be a minimizing geodesic from $\tilde{x}$ to $\tilde{y}$. Then, we define $y\coloneqq p(\tilde{y})$ and we assume that $\beta\colon[0,1]\to X$ is a minimizing geodesic from x to y. Let $\tilde{\beta}$ be the lift of β starting at $\tilde{x}$ and let $\eta\in\overline{\pi}_1(X)$ such that $\tilde{\beta}(1)=\eta\tilde{y}$. Looking for a contradiction, let us suppose that $\mathrm{d}(x,y)<\tilde{\mathrm{d}}(\tilde{x},\tilde{y})$. Then, observe that $\tilde{\mathrm{d}}(\tilde{x},\eta\tilde{y})\leq\mathcal{L}(\beta)=\mathrm{d}(x,y)$; in particular, $\tilde{\mathrm{d}}(\tilde{x},\eta\tilde{y})=\mathrm{d}(x,y)$, since p contracts distances. Hence, we have $\phi_{\eta^{-1}}(\tilde{y}) = \tilde{\mathrm{d}}(\tilde{y},\eta^{-1}\tilde{x}) - \tilde{\mathrm{d}}(\tilde{x},\tilde{y}) = \tilde{\mathrm{d}}(\eta\tilde{y},\tilde{x}) - \tilde{\mathrm{d}}(\tilde{x},\tilde{y}) = \mathrm{d}(x,y) - \tilde{\mathrm{d}}(\tilde{x},\tilde{y}) < 0$. In particular, $\tilde{y}\notin\mathcal{F}(\tilde{x})$, which is the contradiction we were looking for. Thus, $\mathrm{d}(x,y)\geq\tilde{\mathrm{d}}(\tilde{x},\tilde{y})$ and, since p contracts distances, we have $\mathrm{d}(x,y)=\tilde{\mathrm{d}}(\tilde{x},\tilde{y})$. This concludes the proof. $\square$

2.4 Equivariance

This section is the main contribution of Chapter 2. Here, we are going to introduce the equivariant mGH topology and prove the equivariant theorem (see Theorem 1.3.1 introduced in Chapter 1).

Throughout this section, $N \in [1, \infty)$ and $K \in \mathbb{R}$ are fixed real numbers, X is a compact topological space that admits an $\mathrm{RCD}(K, N)$ structure, $p\colon \tilde{X} \to X$ is the universal cover of X, and $\overline{\pi}_1(X)$ is the revised fundamental group of X (see Theorem 2.3.2).

2.4.1 Equivariant topology

As we saw in Corollary 2.3.1 and Proposition 2.3.5, $\mathrm{RCD}(K, N)$ structures on X are closely related to $\mathrm{RCD}^*(K, N)$ structures on $\tilde{X}$ that admit $\overline{\pi}_1(X)$ as a subgroup of isomorphisms. The goal of this section is to introduce equivariant $\mathrm{RCD}^*(K, N)$ structures on $\tilde{X}$. We will then define the moduli space of equivariant $\mathrm{RCD}^*(K, N)$ structures on $\tilde{X}$ together with its topology.

2.4.1.1 The compact case

In this subsection, we assume that $\tilde{X}$ is compact.

Definition 2.4.1 (Equivariant $\mathrm{RCD}^*(K, N)$ structures)**.** An $\mathrm{RCD}^*(K, N)$ structure $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ on $\tilde{X}$ is called equivariant if $\overline{\pi}_1(X)$ acts by isomorphisms on $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$.

Definition 2.4.2 (Equivariant isomorphism)**.** Let $\tilde{\mathfrak{X}}_i = (\tilde{X}, \tilde{\mathrm{d}}_i, \tilde{\mathfrak{m}}_i)$ $(i \in \{1, 2\})$ be an equivariant $\mathrm{RCD}^*(K, N)$ structure on $\tilde{X}$. We say that $\tilde{\mathfrak{X}}_1$ and $\tilde{\mathfrak{X}}_2$ are equivariantly isomorphic when there is an isomorphism ϕ of $\overline{\pi}_1(X)$ and an isomorphism $f\colon \tilde{\mathfrak{X}}_1 \to \tilde{\mathfrak{X}}_2$ of m.m.s. such that $f(\gamma x) = \phi(\gamma) f(x)$, for every $\gamma \in \overline{\pi}_1(X)$, and every $x \in \tilde{X}$.

Definition 2.4.3. The *moduli space of equivariant* $\mathrm{RCD}^*(K, N)$ *structures on* $\tilde{X}$ is the set $\mathfrak{M}^{\mathrm{eq}}_{K,N}(\tilde{X})$ of equivariant $\mathrm{RCD}^*(K, N)$ structures on $\tilde{X}$ quotiented by equivariant isomorphisms.

We introduce the equivariant pseudo-distance $\mathfrak{D}^{\mathrm{eq}}$ to compare equivariant $\mathrm{RCD}^*(K, N)$ structures on $\tilde{X}$.

Definition 2.4.4. Let $\tilde{\mathfrak{X}}_i = (\tilde{X}, \tilde{\mathrm{d}}_i, \tilde{\mathfrak{m}}_i)$ $(i \in \{1, 2\})$ be an equivariant $\mathrm{RCD}^*(K, N)$ structure on $\tilde{X}$ and let $\epsilon > 0$. An *equivariant mGH ϵ-approximation* between $\tilde{\mathfrak{X}}_1$ and $\tilde{\mathfrak{X}}_2$ is a triple (f, g, ϕ) where $f \colon \tilde{X} \to \tilde{X}$ and $g \colon \tilde{X} \to \tilde{X}$ are Borel maps and ϕ is an isomorphism of $\overline{\pi}_1(X)$ such that:

(i) for every $x, y \in \tilde{X}$, $|\mathrm{d}_2(f(x), f(y)) - \mathrm{d}_1(x, y)| \leq \epsilon$ and $|\mathrm{d}_1(g(x), g(y)) - \mathrm{d}_2(x, y)| \leq \epsilon$,

(ii) for every $x \in \tilde{X}$, $\mathrm{d}_1(g \circ f(x), x) \leq \epsilon$ and $\mathrm{d}_2(f \circ g(x), x) \leq \epsilon$,

(iii) for every $\gamma \in \overline{\pi}_1(X)$ and $x \in \tilde{X}$, $f(\gamma x) = \phi(\gamma) f(x)$ and $g(\gamma x) = \phi^{-1}(\gamma) g(x)$,

(iv) $\max\{\mathrm{d}_{\mathcal{P}}(f_* \tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2), \mathrm{d}_{\mathcal{P}}(g_* \tilde{\mathfrak{m}}_2, \tilde{\mathfrak{m}}_1)\} \leq \epsilon$.

We define $\mathfrak{D}^{\mathrm{eq}}(\tilde{\mathfrak{X}}_1, \tilde{\mathfrak{X}}_2)$ the *equivariant pseudo-distance between $\tilde{\mathfrak{X}}_1$ and $\tilde{\mathfrak{X}}_2$* as the minimum between $1/24$ and the infimum of all $\epsilon > 0$ such that there exists an equivariant ϵ-isometry between $\tilde{\mathfrak{X}}_1$ and $\tilde{\mathfrak{X}}_2$.

Remark 2.4.1. In the process of writing [MN22], we first constructed the pointed version $\mathfrak{D}^{\mathrm{eq}}_{\mathrm{p}}$ of the equivariant distance (see Section 2.4.1.2 below). The (non-pointed) equivariant distance $\mathfrak{D}^{\mathrm{eq}}$ is then obtained from $\mathfrak{D}^{\mathrm{eq}}_{\mathrm{p}}$ by forgetting about the pointed requirements. In particular, all the properties proven for the pointed distance hold a fortiori for the (non-pointed) one.

Remark 2.4.2. The choice of the constant $1/24$ in the definition above can seem surprising and will be explained later for the pointed version (see Remark 2.4.4 below).

The equivariant pseudo-distance satisfies all the axioms of a distance apart from the triangle inequality. Indeed, given equivariant $\mathrm{RCD}^*(K, N)$ structures $\tilde{\mathfrak{X}}_i = (\tilde{X}, \tilde{\mathrm{d}}_i, \tilde{\mathfrak{m}}_i)$ $(i \in \{1, 2, 3\})$, we only have the following inequality:

$$\mathfrak{D}^{\mathrm{eq}}(\tilde{\mathfrak{X}}_1, \tilde{\mathfrak{X}}_3) \leq 4(\mathfrak{D}^{\mathrm{eq}}(\tilde{\mathfrak{X}}_1, \tilde{\mathfrak{X}}_2) + \mathfrak{D}^{\mathrm{eq}}(\tilde{\mathfrak{X}}_2, \tilde{\mathfrak{X}}_3)), \tag{2.6}$$

which is proven in the pointed case in Appendix A.3. Even though the equivariant pseudo-distance $\mathfrak{D}^{\mathrm{eq}}$ is a priori not a distance, it induces a metrizable topology as shown by the following proposition (see Appendix A.3 for a proof in the pointed case).

Proposition 2.4.1. The equivariant pseudo-distance $\mathfrak{D}^{\mathrm{eq}}$ induces a metrizable topology on $\mathfrak{M}^{\mathrm{eq}}_{K,N}(\tilde{X})$, which we call the *equivariant mGH topology.*

Remark 2.4.3. Later, we will sometimes forget about point (iv) in Definition 2.4.4, leading to the notion of *equivariant GH ϵ-approximation* and *equivariant GH distance* $\mathcal{D}^{\mathrm{eq}}$. Moduli spaces of equivariant metrics will be endowed with $\mathcal{D}^{\mathrm{eq}}$ (equivariant GH topology).

2.4.1.2 The pointed case

In this subsection, $\tilde{X}$ is not necessarily compact.

Definition 2.4.5. A pointed $\mathrm{RCD}^*(K,N)$ structure $(\tilde{X},\tilde{\mathrm{d}},\tilde{\mathfrak{m}},\tilde{*})$ on $\tilde{X}$ is called equivariant if $\overline{\pi}_1(X)$ acts by isomorphisms on $(\tilde{X},\tilde{\mathrm{d}},\tilde{\mathfrak{m}})$.

Definition 2.4.6. For $i\in\{1,2\}$, let $\tilde{\mathfrak{X}}_i=(\tilde{X},\tilde{\mathrm{d}}_i,\tilde{\mathfrak{m}}_i,\tilde{*}_i)$ be a pointed equivariant $\mathrm{RCD}^*(K,N)$ structure on $\tilde{X}$. We say that $\tilde{\mathfrak{X}}_1$ and $\tilde{\mathfrak{X}}_2$ are equivariantly isomorphic when there is an isomorphism ϕ of $\overline{\pi}_1(X)$ and an isomorphism $f\colon\tilde{\mathfrak{X}}_1\to\tilde{\mathfrak{X}}_2$ of p.m.m.s. such that $f(\gamma x)=\phi(\gamma)f(x)$, for every $\gamma\in\overline{\pi}_1(X)$, and every $x\in\tilde{X}$.

Definition 2.4.7. The *moduli space of pointed equivariant* $\mathrm{RCD}^*(K,N)$ *structures on* $\tilde{X}$ is the set $\mathfrak{M}^{\mathrm{p,eq}}_{K,N}(\tilde{X})$ of pointed equivariant $\mathrm{RCD}^*(K,N)$ structures on $\tilde{X}$ quotiented by equivariant isomorphisms.

To define a topological structure on $\mathfrak{M}^{\mathrm{p,eq}}_{K,N}(\tilde{X})$, we introduce the pointed equivariant distance.

Definition 2.4.8. Let $\epsilon>0$, and, for $i\in\{1,2\}$, let $\tilde{\mathfrak{X}}_i=(\tilde{X},\tilde{\mathrm{d}}_i,\tilde{\mathfrak{m}}_i,\tilde{*}_i)$ be a pointed equivariant $\mathrm{RCD}^*(K,N)$ structure on $\tilde{X}$. An *equivariant pmGH ϵ-approximation* between $\tilde{\mathfrak{X}}_1$ and $\tilde{\mathfrak{X}}_2$ is a triple (f,g,ϕ) where $f\colon\tilde{X}\to\tilde{X}$ and $g\colon\tilde{X}\to\tilde{X}$ are Borel maps and ϕ is an isomorphism of $\overline{\pi}_1(X)$ such that:

(i) $f(*_1)=*_2$ and $g(*_2)=*_1$,

(ii) for every $\gamma\in\overline{\pi}_1(X)$ and $x\in\tilde{X}$, $f(\gamma x)=\phi(\gamma)f(x)$ and $g(\gamma x)=\phi^{-1}(\gamma)g(x)$,

(iii) for every $x,y\in\tilde{X}$, $\mathrm{d}_1(x,y)\le\epsilon^{-1}\implies|\mathrm{d}_2(f(x),f(y))-\mathrm{d}_1(x,y)|\le\epsilon$, and $\mathrm{d}_2(x,y)\le\epsilon^{-1}\implies|\mathrm{d}_1(g(x),g(y))-\mathrm{d}_2(x,y)|\le\epsilon$,

(iv) for every $x\in\tilde{X}$, $\mathrm{d}_1(g\circ f(x),x)\le\epsilon$ and $\mathrm{d}_2(f\circ g(x),x)\le\epsilon$,

(v) $\max\{\mathrm{d}_{\mathcal{P}}^{\epsilon^{-1}}(f_*\tilde{\mathfrak{m}}_1,\tilde{\mathfrak{m}}_2),\mathrm{d}_{\mathcal{P}}^{\epsilon^{-1}}(g_*\tilde{\mathfrak{m}}_2,\tilde{\mathfrak{m}}_1)\}\le\epsilon$.

We define $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}(\tilde{\mathfrak{X}}_1, \tilde{\mathfrak{X}}_2)$ the *pointed equivariant distance between* $\tilde{\mathfrak{X}}_1$ *and* $\tilde{\mathfrak{X}}_2$ as the minimum between $1/24$ and the infimum of all $\epsilon > 0$ such that there exists an equivariant pmGH ϵ-approximation between $\tilde{\mathfrak{X}}_1$ and $\tilde{\mathfrak{X}}_2$.

Remark 2.4.4. The choice of the constant $1/24$ in the definition of $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}$ may be surprising. First, the role of this constant is to obtain a finite distance on the moduli space of equivariant $\mathrm{RCD}^*(K,N)$ structures. Then, proving that $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}$ satisfies a modified version of the triangle inequality is somewhat technical. This difficulty is due to the fact that, given $\epsilon > 0$ and locally finite measures $\mathfrak{m}_i$ ($i \in \{1,2,3\}$), we are not able to obtain $\mathrm{d}_{\mathcal{P}}^{\epsilon^{-1}}(\mathfrak{m}_1, \mathfrak{m}_3) \leq \mathrm{d}_{\mathcal{P}}^{\epsilon^{-1}}(\mathfrak{m}_1, \mathfrak{m}_2) + \mathrm{d}_{\mathcal{P}}^{\epsilon^{-1}}(\mathfrak{m}_2, \mathfrak{m}_3)$ (as shown in Remark A.3.1). The choice of the constant $1/24$ becomes necessary to overcome this difficulty and we use it explicitly in Appendix A.3 to prove Proposition 2.4.2 below.

The following result shows that we can endow $\mathfrak{M}_{K,N}^{\mathrm{p,eq}}(\tilde{X})$ with a metrizable topology (see Appendix A.3 for a proof).

Proposition 2.4.2. $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}$ induces a metrizable uniform structure on $\mathfrak{M}_{K,N}^{\mathrm{p,eq}}(\tilde{X})$.

From now on, we endow $\mathfrak{M}_{K,N}^{\mathrm{p,eq}}(\tilde{X})$ with the topology induced by $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}$, which we call the *equivariant pmGH topology.*

2.4.2 Lift and push-forward maps

We observed in Section 2.3.2 that $\mathrm{RCD}(K,N)$ structures on X are in 1-1 correspondence with equivariant $\mathrm{RCD}^*(K,N)$ structures on $\tilde{X}$. This section presents the *lift and push-forwards maps*, which will express the correspondence at the level of moduli spaces.

First of all, relying on Corollary 2.3.1, we can define the lift of a pointed $\mathrm{RCD}(K,N)$ structure .

Definition 2.4.9. Let $(X, \mathrm{d}, \mathfrak{m}, x)$ be a pointed $\mathrm{RCD}(K,N)$ structure on X and let $\tilde{x} \in p^{-1}(x)$. We define $p^*_{\tilde{x}}(X, \mathrm{d}, \mathfrak{m}, x) := (\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}, \tilde{x})$, where $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ is the lift of $(X, \mathrm{d}, \mathfrak{m})$.

Remark 2.4.5. Let $(X, \mathrm{d}_i, \mathfrak{m}_i, x_i)$ be a pointed $\mathrm{RCD}(K,N)$ structure on X and let $\tilde{x}_i \in p^{-1}(x_i)$ ($i \in \{1,2\}$). If $(X, \mathrm{d}_1, \mathfrak{m}_1, x_1)$ and $(X, \mathrm{d}_2, \mathfrak{m}_2, x_2)$ are isomorphic, then $p^*_{\tilde{x}_1}(X_1, \mathrm{d}_1, \mathfrak{m}_1, x_1)$ is equivariantly isomorphic to $p^*_{\tilde{x}_1}(X_2, \mathrm{d}_2, \mathfrak{m}_2, x_2)$.

Thanks to Remark 2.4.5, we can define the pointed lift map associated to X.

Definition 2.4.10 (Pointed lift map)**.** The *pointed lift map associated to* X is the unique map $p^*\colon \mathfrak{M}^{\mathrm{p}}_{K,N}(X) \to \mathfrak{M}^{\mathrm{p,eq}}_{K,N}(\tilde{X})$ that satisfies $p^*[X, \mathrm{d}, \mathfrak{m}, x] = [p^*_{\tilde{x}}(X, \mathrm{d}, \mathfrak{m}, x)]$ for every pointed $\mathrm{RCD}(K, N)$ structure $(X, \mathrm{d}, \mathfrak{m}, x)$ on X and $\tilde{x} \in p^{-1}(x)$.

As an application of Proposition 2.3.5, we can define the push-forward of a pointed equivariant $\mathrm{RCD}^*(K, N)$ structure.

Definition 2.4.11. Let $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}, \tilde{x})$ be a pointed equivariant $\mathrm{RCD}^*(K, N)$ structures on $\tilde{X}$. We define $p_*(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}, \tilde{x})$ as the unique pointed $\mathrm{RCD}(K, N)$ structure on X such that $p\colon (\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}, \tilde{x}) \to p_*(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}, \tilde{x})$ is a pointed local isomorphism.

Remark 2.4.6. Let $(\tilde{X}, \tilde{\mathrm{d}}_i, \tilde{\mathfrak{m}}_i, \tilde{x}_i)$ be a pointed equivariant $\mathrm{RCD}^*(K, N)$ structure on $\tilde{X}$ $(i \in \{1, 2\})$. If $(\tilde{X}, \tilde{\mathrm{d}}_1, \tilde{\mathfrak{m}}_1, \tilde{x}_1) \sim (\tilde{X}, \tilde{\mathrm{d}}_2, \tilde{\mathfrak{m}}_2, \tilde{x}_2)$, then $p_*(\tilde{X}, \tilde{\mathrm{d}}_1, \tilde{\mathfrak{m}}_1, \tilde{x}_1)$ is isomorphic to $p_*(\tilde{X}, \tilde{\mathrm{d}}_2, \tilde{\mathfrak{m}}_2, \tilde{x}_2)$.

Thanks to Remark 2.4.6, we can define the push-forward map associated to X.

Definition 2.4.12 (Push-forward map)**.** The *push-forward map associated to* X is the unique map

$$p_*\colon \mathfrak{M}^{\mathrm{p,eq}}_{K,N}(\tilde{X}) \to \mathfrak{M}^{\mathrm{p}}_{K,N}(X)$$

satisfying $p_*[\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}, \tilde{x}] = [p_*(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}, \tilde{x})]$ for every pointed equivariant $\mathrm{RCD}^*(K, N)$ structure $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}, \tilde{x})$ on $\tilde{X}$.

Remark 2.3.3 implies the following proposition.

Proposition 2.4.3. The pointed lift map $p^*\colon \mathfrak{M}^{\mathrm{p}}_{K,N}(X) \to \mathfrak{M}^{\mathrm{p,eq}}_{K,N}(\tilde{X})$ and the push-forward map $p_*\colon \mathfrak{M}^{\mathrm{p,eq}}_{K,N}(\tilde{X}) \to \mathfrak{M}^{\mathrm{p}}_{K,N}(X)$ are respectively inverse to each other.

Remark 2.4.7. It is of course possible to define non-pointed versions of the pointed lift and push-forward maps. In particular, these non-pointed versions also satisfy Proposition 2.4.3. This will be interesting when the universal cover is compact.

2.4.3 Equivariant theorem

In this subsection, we will prove Theorem 1.3.1 presented in Chapter 1. The equivariant theorem is one of the main results of [MN22], joint work with Andrea Mondino. It relates the convergence of RCD structures on a compact topological space to the convergence of their lifts. In contrast with [MN22], the version of this thesis treats the more general case of $\mathrm{RCD}(K, N)$ spaces. However, we have to assume that the revised fundamental group of the base space is Hopfian.

Definition 2.4.13. A group G is called *Hopfian* when every surjective group homomorphism $\phi\colon G \to G$ is necessarily an isomorphism.

Remark 2.4.8. As we will se in Chapter 3, the revised fundamental group of a compact $\mathrm{RCD}(0, N)$ space is Hopfian.

For the reader's convenience, we restate the equivariant theorem below.

Theorem 2.4.1 (Equivariant theorem)**.** Let X be a compact topological space that admits an $\mathrm{RCD}(K, N)$ structure such that $\overline{\pi}_1(X)$ is Hopfian ($N \in [1, \infty)$ and $K \in \mathbb{R}$) and denote $p\colon \tilde{X} \to X$ the universal cover of X. Assume that for every $n \in \mathbb{N} \cup \{\infty\}$:

- $\mathcal{X}_n = (X, \mathrm{d}_n, \mathfrak{m}_n, *_n)$ is a pointed $\mathrm{RCD}(K, N)$ structure on X,
- $\tilde{\mathcal{X}}_n = (\tilde{X}, \tilde{\mathrm{d}}_n, \tilde{\mathfrak{m}}_n, \tilde{*}_n)$ is the associated pointed lift, where $\tilde{*}_n$ is any point in $p^{-1}(*_n)$.

Then, $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ converges to $\mathcal{X}_\infty$ in the pmGH topology (see Definition 2.1.20) if and only if $\{\tilde{\mathcal{X}}_n\}_{n\in\mathbb{N}}$ converges to $\tilde{\mathcal{X}}_\infty$ in the equivariant pmGH topology (see Definition 2.4.8).

Remark 2.4.9. Note that, since X is compact, it is also possible to formulate Theorem 2.4.1 as follows (forgetting about the reference points in the base space): Assume that for every $n \in \mathbb{N} \cup \{\infty\}$, $\mathcal{X}^*_n = (X, \mathrm{d}_n, \mathfrak{m}_n)$ is an $\mathrm{RCD}(K, N)$ structure on X with lift $\tilde{\mathcal{X}}^*_n = (\tilde{X}, \tilde{\mathrm{d}}_n, \tilde{\mathfrak{m}}_n)$. Then $\{\mathcal{X}^*_n\}_{n\in\mathbb{N}}$ converges to $\mathcal{X}^*_\infty$ in the mGH topology if and only if for every $n \in \mathbb{N} \cup \{\infty\}$, there exist $\tilde{*}_n \in \tilde{X}$ such that $\{(\tilde{\mathcal{X}}^*_n, \tilde{*}_n)\}_{n\in\mathbb{N}}$ converges to $(\tilde{\mathcal{X}}^*_\infty, \tilde{*}_\infty)$ in the equivariant pmGH topology.

Theorem 2.4.1 and Proposition 2.4.3 imply the following corollary.

Corollary 2.4.1. Let $K \in \mathbb{R}$ and $N \in [1, \infty)$ be fixed real numbers. Let X be a compact topological space that admits an $\mathrm{RCD}(K, N)$ structure such that $\overline{\pi}_1(X)$ is Hopfian and denote $p\colon \tilde{X} \to X$ the universal cover of X. Then the pointed lift map:

$$p^*\colon \mathfrak{M}^{\mathrm{p}}_{K,N}(X) \to \mathfrak{M}^{\mathrm{p,eq}}_{K,N}(\tilde{X}),$$

is a homeomorphism (introduced in Section 2.4.2), where $\mathfrak{M}^{\mathrm{p,eq}}_{K,N}(\tilde{X})$ and $\mathfrak{M}^{\mathrm{p}}_{K,N}(X)$ are respectively the moduli space of equivariant pointed $\mathrm{RCD}^*(K, N)$ structures on $\tilde{X}$ and the moduli space of pointed $\mathrm{RCD}(K, N)$ structures on X (introduced respectively in Sections 2.4.1 and 2.2.5).

In particular, when the universal cover is compact, one can forget about the pointed requirement and reformulate Corollary 2.4.1 in the following simpler form (see Remark 2.4.7).

Corollary 2.4.2. Let X be a compact topological space that admits an $\mathrm{RCD}(K,N)$ structure such that $\overline{\pi}_1(X)$ is finite ($N \in [1,\infty)$ and $K \in \mathbb{R}$) and denote $p\colon \tilde{X} \to X$ the universal cover of X. Then, the lift map:

$$p^*\colon \mathfrak{M}_{K,N}(X) \to \mathfrak{M}^{\mathrm{eq}}_{K,N}(\tilde{X}),$$

is a homeomorphism, where $\mathfrak{M}^{\mathrm{eq}}_{K,N}(\tilde{X})$ and $\mathfrak{M}_{K,N}(X)$ are respectively the moduli space of equivariant $\mathrm{RCD}^*(K,N)$ structures on $\tilde{X}$ and the moduli space of $\mathrm{RCD}(K,N)$ structures on X (introduced respectively in Sections 2.2.5 and 2.4.1).

Remark 2.4.10. Corollary 2.4.2 will be particularly useful when computing the homeomorphism type of specific examples of moduli spaces (see, for example, the case of $\mathbb{RP}^2$ in Chapter 4).

Remark 2.4.11. Observe that it is more straightforward to obtain Corollary 2.4.1 by using Theorem 2.4.1 than its equivalent version given in Remark 2.4.9.

Before we prove Theorem 2.4.1, let us first introduce the systole associated to an $\mathrm{RCD}(K,N)$ structure on X. Finding a uniform lower bound on the systoles associated to a converging sequence will be crucial.

Definition 2.4.14 (Systole of an $\mathrm{RCD}(K,N)$ structure)**.** The systole associated to an $\mathrm{RCD}(K,N)$ structure $(X,\mathrm{d},\mathfrak{m})$ on X is the quantity $\mathrm{sys}(X,\mathrm{d}) \coloneqq \inf\{\tilde{\mathrm{d}}(\eta\cdot\tilde{x},\tilde{x})\}$, where the infimum is taken over all point $\tilde{x} \in \tilde{X}$ and $\eta \in \overline{\pi}_1(X)\backslash\{\mathrm{id}\}$. Whenever $\overline{\pi}_1(X)$ is trivial, we define $\mathrm{sys}(X,\mathrm{d}) \coloneqq \infty$.

The following proposition relates the systole of an $\mathrm{RCD}(K,N)$ structure $(X,\mathrm{d},\mathfrak{m})$ on X and the quantity $\delta(X,\mathrm{d})$ introduced in Theorem 2.3.3.

Proposition 2.4.4. Let $(X,\mathrm{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ structure on X. Then, $\mathrm{sys}(X,\mathrm{d}) = 2\delta(X,\mathrm{d})$, where $\delta(X,\mathrm{d})$ is defined in Theorem 2.3.3.

Proof. Let $\delta < \delta(X,\mathrm{d})$, let $\eta \in \overline{\pi}_1(X)\backslash\{\mathrm{id}\}$, and let $\tilde{x} \in \tilde{X}$. Then, by Proposition 2.3.4 and Theorem 2.3.3, p induces a homeomorphism from $B_{\tilde{\mathrm{d}}}(\tilde{x},\delta)$ and $B_{\tilde{\mathrm{d}}}(\eta\cdot\tilde{x},\delta)$ onto $B_{\mathrm{d}}(x,\delta)$, where $x \coloneqq p(\tilde{x})$. Seeking for a contradiction, assume that there exists $\tilde{y} \in B_{\tilde{\mathrm{d}}}(\tilde{x},\delta) \cap B_{\tilde{\mathrm{d}}}(\eta\cdot\tilde{x},\delta)$. Then, $\mathrm{d}(\eta\cdot\tilde{y},\eta\cdot\tilde{x}) = \mathrm{d}(\tilde{y},\tilde{x}) < \delta$. In particular, $\tilde{y}$ and

$\eta \cdot \tilde{y}$ are two distinct elements of $B_{\tilde{\mathrm{d}}}(\eta \cdot \tilde{x}, \delta)$ which have the same image under p, which is the contradiction we were looking for. Hence, $B_{\tilde{\mathrm{d}}}(\tilde{x}, \delta) \cap B_{\tilde{\mathrm{d}}}(\eta \cdot \tilde{x}, \delta) = \varnothing$. In particular, $\tilde{\mathrm{d}}(\eta \cdot \tilde{x}, \tilde{x}) \geq 2\delta$; thus, $2\delta \leq \mathrm{sys}(X, \mathrm{d})$. Since that holds for every $\delta < \delta(X, \mathrm{d})$, we have $2\delta(X, \mathrm{d}) \leq \mathrm{sys}(X, \mathrm{d})$.

Now assume that $\delta(X, \mathrm{d}) < \delta$. Then, there is some $x \in X$ such that $B_{\mathrm{d}}(x, \delta)$ is not evenly covered by p. Therefore, given any $\tilde{x} \in p^{-1}(x)$, there exists $\tilde{y}_i \in B_{\tilde{\mathrm{d}}}(\tilde{x}, \delta)$ ($i \in \{1, 2\}$) such that $p\tilde{y}_1 = p\tilde{y}_2$, $\tilde{y}_1 \neq \tilde{y}_2$. Hence, there exists $\gamma \in \overline{\pi}_1(X) \backslash \{\mathrm{id}\}$ such that $\tilde{y}_2 = \gamma \tilde{y}_1$; thus, $\tilde{\mathrm{d}}(\tilde{y}_1, \tilde{y}_2) = \tilde{\mathrm{d}}(\gamma \tilde{y}_1, \tilde{y}_1) \leq 2\delta$. Therefore, we have $\mathrm{sys}(X, \mathrm{d}) \leq 2\delta$. Thus, letting δ go to $\delta(X, \mathrm{d})$, we finally obtain $\mathrm{sys}(X, \mathrm{d}) \leq 2\delta(X, \mathrm{d})$. □

The next result shows that we can find a positive uniform lower bound on the systoles associated to a converging sequence of $\mathrm{RCD}(K, N)$ structures on X.

Proposition 2.4.5. Assume that $\{(X, \mathrm{d}_n, \mathfrak{m}_n)\}_{n \in \mathbb{N}}$ converges to $(X, \mathrm{d}_\infty, \mathfrak{m}_\infty)$ in the mGH-topology, where, for every $n \in \mathbb{N} \cup \{\infty\}$, $(X, \mathrm{d}_n, \mathfrak{m}_n)$ is an $\mathrm{RCD}(K, N)$ structure on X. If $\overline{\pi}_1(X)$ is Hopfian, then $0 < \inf_{n \in \mathbb{N}} \{\delta(X, \mathrm{d}_n)\}$.

Proof. First of all, observe that Theorem 2.3.3 implies $\delta(X, \mathrm{d}_n) > 0$ for every $n \in \mathbb{N}$. In particular, it is sufficient to prove that there exists a constant $\delta > 0$ such that $\delta(X, \mathrm{d}_n) \geq \delta$ whenever n is large enough.

We define $\epsilon_n := \mathrm{d}_{\mathrm{GH}}((X, \mathrm{d}_n), (X, \mathrm{d}_\infty)) \to 0$, $\delta_2 := \delta(X, \mathrm{d}_\infty)/2$, and $\delta_1 := \delta(X, \mathrm{d}_\infty)/3$. Observe that thanks to Theorem 2.3.3, $\delta_1 > 0$. Thus, whenever n is large enough, we have $\delta_1 > 20\epsilon_n$ and $\delta_2 > \delta_1 + 10\epsilon_n$. Hence, by [SW01, Theorem 3.4], there is a surjective group homomorphism $\psi_n \colon G(\delta_1, \mathrm{d}_n) \to G(\delta_2, \mathrm{d}_\infty)$. Moreover, since $\delta_2 < \delta(X, \mathrm{d}_\infty)$, then $G(\delta_2, \mathrm{d}_\infty)$ is isomorphic to $\overline{\pi}_1(X)$ by Proposition 2.3.3. Now, fixing $\tilde{x} \in \tilde{X}$, and $x_1 \in X^{\delta_1}_{\mathrm{d}_n}$, such that $x := p(\tilde{x}) = p^{\delta_1}_{\mathrm{d}_n}(x_1)$, we have a surjective homomorphism:

$$q \colon \pi_1(X, x)/p_*\pi_1(\tilde{X}, \tilde{x}) \to (\pi_1(X, x)/p_*\pi_1(\tilde{X}, \tilde{x}))/(p^{\delta_1}_{\mathrm{d}_n *}\pi_1(X^{\delta_1}_{\mathrm{d}_n}, x_1)/p_*\pi_1(\tilde{X}, \tilde{x})).$$

However, the domain of q is isomorphic to $\overline{\pi}_1(X)$, whereas its codomain is isomorphic to $G(\delta_1, \mathrm{d}_n)$ (see Definition-Proposition 2.3.3). Therefore, q gives rise to a surjective homomorphism ν_n from $\overline{\pi}_1(X)$ onto $G(\delta_1, \mathrm{d}_n)$. Hence, we have a surjective group homomorphism:

$$\overline{\pi}_1(X) \xrightarrow{\nu_n} G(\delta_1, \mathrm{d}_n) \xrightarrow{\psi_n} G(\delta_2, \mathrm{d}_\infty) \xrightarrow{\sim} \overline{\pi}_1(X). \tag{2.7}$$

However, $\overline{\pi}_1(X)$ is a Hopfian group; thus, the surjective group homomorphism in equation (2.7) is an isomorphism. In particular, $\nu_n \colon \overline{\pi}_1(X) \to G(\delta_1, \mathrm{d}_n)$ is injective; hence, it is an isomorphism and it implies that q is also an isomorphism. In particular, we necessarily have $p^{\delta_1}_{\mathrm{d}_n *}\pi_1(X^{\delta_1}_{\mathrm{d}_n}, x_1) = p_*\pi_1(\tilde{X}, \tilde{x})$; hence, by the classification Theorem 2.3.1, $(X^{\delta_1}_{\mathrm{d}_n}, X, p^{\delta_1}_{\mathrm{d}_n})$ is equivalent to $(\tilde{X}, X, p)$. In particular, every ball of radius δ_1 in (X, d_n) is evenly covered by p; thus, $\delta(X, \mathrm{d}_n) \geq \delta_1$, which concludes the proof. □

The following proposition is a converse to Proposition 2.4.5; it will be essential to prove the converse implication of Theorem 1.3.1.

Proposition 2.4.6. Assume that $\{(\tilde{X}, \tilde{\mathrm{d}}_n, \tilde{\mathfrak{m}}_n, \tilde{*}_n)\}$ converges to $(\tilde{X}, \tilde{\mathrm{d}}_\infty, \tilde{\mathfrak{m}}_\infty, \tilde{*}_\infty)$ in the equivariant pmGH-topology, where, for every $n \in \mathbb{N} \cup \{\infty\}$, $(\tilde{X}, \tilde{\mathrm{d}}_n, \tilde{\mathfrak{m}}_n, \tilde{*}_n)$ is an equivariant pointed $\mathrm{RCD}^*(K, N)$ structure on $\tilde{X}$. Then $0 < \inf_{n\in\mathbb{N}}\{\delta(X, \mathrm{d}_n)\}$, where $(X, \mathrm{d}_n, \mathfrak{m}_n)$ is the push-forward of $(\tilde{X}, \tilde{\mathrm{d}}_n, \tilde{\mathfrak{m}}_n)$.

Proof. We fix a sequence $\{(\tilde{f}_n, \tilde{g}_n, \phi_n, \epsilon_n)\}$ realising the equivariant pointed convergence. Looking for a contradiction, assume that $\inf_{n\in\mathbb{N}}\{\mathrm{sys}(X, \mathrm{d}_n)\} = 0$. Without loss of generality, we can assume (passing to a subsequence if necessary) that there exist sequences $\{\tilde{x}_n\}$ in $\tilde{X}$ and $\{\gamma_n\}$ in $\overline{\pi}_1(X)\backslash\{\mathrm{id}\}$ such that $\tilde{\mathrm{d}}_n(\gamma_n\tilde{x}_n, \tilde{x}_n) \to 0$. However, when n is large enough so that $\tilde{\mathrm{d}}_n(\gamma_n\tilde{x}_n, \tilde{x}_n) \leq \epsilon_n^{-1}$, we have $\mathrm{sys}(X, \mathrm{d}_\infty) \leq \tilde{\mathrm{d}}_\infty(\tilde{f}_n(\tilde{x}_n), \phi_n(\gamma_n)\tilde{f}_n(\tilde{x}_n)) = \tilde{\mathrm{d}}_\infty(\tilde{f}_n(\tilde{x}_n), \tilde{f}_n(\gamma_n\tilde{x}_n)) \leq \tilde{\mathrm{d}}_n(\gamma_n\tilde{x}_n, \tilde{x}_n) + \epsilon_n \to 0$. Therefore, $\mathrm{sys}(X, \mathrm{d}_\infty) = 0 = \delta(X, \mathrm{d}_\infty)$ (using Proposition 2.4.4), which is the contradiction we were looking for. Hence $0 < \inf_{n\in\mathbb{N}}\{\mathrm{sys}(X, \mathrm{d}_n)\}$; therefore, we have $0 < \inf_{n\in\mathbb{N}}\{\delta(X, \mathrm{d}_n)\}$ (using Proposition 2.4.4). □

We can now prove Theorem 2.4.1.

Proof of Theorem 2.4.1, direct implication. To avoid trivialities, we will consider that $\overline{\pi}_1(X)$ is not trivial. Assume that $\{\mathcal{X}_n = (X, \mathrm{d}_n, \mathfrak{m}_n, *_n)\}$ converges in the pmGH-topology to $\mathcal{X}_\infty = (X, \mathrm{d}_\infty, \mathfrak{m}_\infty, *_\infty)$. Let us prove that $\{\tilde{\mathcal{X}}_n\}$ converges in the equivariant pmGH-topology to $\tilde{\mathcal{X}}_\infty$.

Part I: Construction of the realising sequence $\{\tilde{f}_n, \tilde{g}_n, \psi_n, \epsilon_n\}$

First of all, we fix a sequence $\{f_n, g_n, \epsilon'_n\}$ realising the convergence of $\{\mathcal{X}_n\}$ to $\mathcal{X}_\infty$ in the pmGH-topology. Then, we define $\delta \coloneqq \inf_{n\in\mathbb{N}\cup\{\infty\}}\{\delta(X, \mathrm{d}_n)\}$, which satisfies $\delta > 0$ (thanks to Proposition 2.4.5) and $\delta \leq \mathrm{Diam}(X, \mathrm{d}_n)$ (since $\overline{\pi}_1(X)$ is not trivial). By Proposition 2.4.4, we have $\mu_0 \coloneqq \inf_{n\in\mathbb{N}\cup\{\infty\}}\{\mathrm{sys}(X, \mathrm{d}_n)\} = 2\delta > 0$. We define $r_0 \coloneqq \delta/2$, and we assume that n is large enough so that:

$$5\epsilon'_n < r_0 < \mu_0/2 - 3\epsilon'_n/2. \tag{2.8}$$

Theorem 2.3.3 implies that $(\tilde{X}, \tilde{\mathrm{d}}_\infty, \tilde{\mathfrak{m}}_\infty)$ is isomorphic to $(X^{r_0}_{\mathrm{d}_\infty}, \mathrm{d}_{\infty,r_0}, \mathfrak{m}_{\infty,r_0})$, since $r_0 < \delta(X, \mathrm{d}_\infty)$. Now, according to [Rev08, Theorem 16] (and the construction in its proof), there exists a triple $(\tilde{f}_n, \tilde{g}_n, \psi_n)$ such that:

- $\tilde{f}_n : (\tilde{X}, \tilde{*}_n) \to (\tilde{X}, \tilde{*}_\infty)$ (resp. $\tilde{g}_n : (\tilde{X}, \tilde{*}_\infty) \to (\tilde{X}, \tilde{*}_n)$) satisfy $p \circ \tilde{f}_n = f_n \circ p$ (resp. $p \circ \tilde{g}_n = g_n \circ p$),
- for every $\tilde{x} \in \tilde{X}$, we have $\tilde{\mathrm{d}}_n(\tilde{g}_n \circ \tilde{f}_n(\tilde{x}), \tilde{x}) \leq \epsilon'_n$ and $\tilde{\mathrm{d}}_\infty(\tilde{f}_n \circ \tilde{g}_n(\tilde{x}), \tilde{x}) \leq \epsilon'_n$,
- for every $\tilde{x} \in \tilde{X}$ and $\eta \in \overline{\pi}_1(X)$, we have $\tilde{f}_n(\eta \cdot \tilde{x}) = \psi_n(\eta) \cdot \tilde{f}_n(\tilde{x})$ and $\tilde{g}_n(\eta \cdot \tilde{x}) = \psi_n^{-1}(\eta) \cdot \tilde{g}_n(\tilde{x})$.

Moreover, thanks to inequality 2.8 and [Rev08, Theorem 16], the following two inequalities hold for every $\tilde{x}, \tilde{y} \in \tilde{X}$:

$$|\tilde{\mathrm{d}}_\infty(\tilde{f}_n(\tilde{x}), \tilde{f}_n(\tilde{y})) - \tilde{\mathrm{d}}_n(\tilde{x}, \tilde{y})| \leq 3\epsilon'_n(\tilde{\mathrm{d}}_n(\tilde{x}, \tilde{y})/r_0 + 1),$$
$$|\tilde{\mathrm{d}}_n(\tilde{g}_n(\tilde{x}), \tilde{g}_n(\tilde{y})) - \tilde{\mathrm{d}}_\infty(\tilde{x}, \tilde{y})| \leq 3\epsilon'_n(\tilde{\mathrm{d}}_\infty(\tilde{x}, \tilde{y})/r_0 + 1).$$

We fix $C > 0$ such that $C + 3/r_0 \leq C^2$ and we define $\epsilon_n \coloneqq C\sqrt{\epsilon'_n}$. When n is large enough so that $\epsilon'_n \leq \epsilon_n$, we have:

$$(\tilde{f}_n, \tilde{g}_n, \psi_n, \epsilon_n) \text{ satisfies point (i) to (iv) of Definition 2.4.8 w.r.t. } \tilde{\mathfrak{X}}_n \text{ and } \tilde{\mathfrak{X}}_\infty. \tag{2.9}$$

Let us prove that when n is large enough, $\tilde{f}_n$ and $\tilde{g}_n$ are Borel maps. Let $\tilde{x} \in \tilde{X}$, and let $r < \delta/3$. Proposition 2.3.4 and property (2.9) imply the following:

$$\tilde{f}_n^{-1}(\tilde{B}_\infty(\tilde{x}, r)) = (f_n \circ p)^{-1}(B_\infty(x, r)) \cap \tilde{B}_n(\tilde{g}_n(\tilde{x}), r + 2\epsilon_n),$$

when n is large enough so that $\delta/3 + 4\epsilon_n < \delta/2 < \epsilon_n^{-1}$ and where $x \coloneqq p(\tilde{x})$. However, f_n is a Borel map and p is continuous; therefore $\tilde{f}_n^{-1}(\tilde{B}_\infty(\tilde{x}, r))$ is a Borel subset of $\tilde{X}$. We have shown that when n is large enough, the pre-image by $\tilde{f}_n$ of balls of radius $r < \delta/3$ are Borel subsets of $\tilde{X}$. Therefore, for n large enough, $\tilde{f}_n$ is a Borel map and the same is true for $\tilde{g}_n$ with the same procedure.

Part II: Measured convergence

Making ϵ_n larger if necessary (but keeping $\epsilon_n \to 0$), we want to prove the following inequality:

$$\max\left\{ \mathrm{d}_{\mathcal{P}}^{\{\epsilon_n{}^{-1}\}}(\tilde{f}_{n*}\tilde{\mathfrak{m}}_n, \tilde{\mathfrak{m}}_\infty), \mathrm{d}_{\mathcal{P}}^{\{\epsilon_n{}^{-1}\}}(\tilde{g}_{n*}\tilde{\mathfrak{m}}_\infty, \tilde{\mathfrak{m}}_n) \right\} \leq \epsilon_n.$$

It is sufficient to show that $\{\mathrm{d}_{\mathcal{P}}^{\{R\}}(\tilde{f}_{n*}\tilde{\mathfrak{m}}_n, \tilde{\mathfrak{m}}_\infty)\}$ and $\{\mathrm{d}_{\mathcal{P}}^{\{R\}}(\tilde{g}_{n*}\tilde{\mathfrak{m}}_\infty, \tilde{\mathfrak{m}}_n)\}$ converge to 0 as n goes to infinity, for every $R > 0$.

First of all, observe that $\lim_{n\to\infty} \mathrm{d}_{\mathcal{P}}^{\{R\}}(\tilde{f}_{n*}\tilde{\mathfrak{m}}_n, \tilde{\mathfrak{m}}_\infty) = 0$ for every $R > 0$ if and only if $\{\tilde{f}_{n*}\tilde{\mathfrak{m}}_n\}$ converge to $\tilde{\mathfrak{m}}_\infty$ in the weak-$*$ topology. Then, note that the space $\mathcal{M}_{\mathrm{loc}}(\tilde{X}, \tilde{\mathrm{d}}_\infty)$ of Radon measures on $(\tilde{X}, \tilde{\mathrm{d}}_\infty)$ endowed with the weak-$*$ topology is metrizable (see [DV03, Theorem A2.6.III]). Hence, it is sufficient to show that any subsequence of $\{\tilde{f}_{n*}\tilde{\mathfrak{m}}_n\}$ admits a subsequence converging to $\tilde{\mathfrak{m}}_\infty$. Without loss of generality (reindexing the sequence if necessary), let us just show that $\{\tilde{f}_{n*}\tilde{\mathfrak{m}}_n\}$ admits a subsequence converging to $\tilde{\mathfrak{m}}_\infty$.

First, let us show that $\{\tilde{f}_{n*}\tilde{\mathfrak{m}}_n\}$ is precompact. It is sufficient to show that, for every $R > 0$, the sequence $\{\tilde{f}_{n*}\tilde{\mathfrak{m}}_n(B_{\tilde{\mathrm{d}}_\infty}(R))\}_{n\in\mathbb{N}}$ is uniformly bounded (see Theorem A2.6.IV and Theorem A2.4.I in [DV03]). Let us fix $R > 0$. We recall the definition of r_0 and define M in the following way:

$$r_0 = \inf_{n\in\mathbb{N}\cup\{\infty\}} \{\delta(X, \mathrm{d}_n)\}/2 > 0 \quad \text{and} \quad M \coloneqq \sup_{n\in\mathbb{N}\cup\{\infty\}} \{\mathfrak{m}_n(X)\}.$$

Observe that M is finite since $\{\mathfrak{m}_n(X)\}_{n\in\mathbb{N}}$ converges to $\mathfrak{m}_\infty(X) < \infty$. Then, thanks to point (v) of Proposition 2.3.4, we have $\tilde{\mathfrak{m}}_n(B_{\tilde{\mathrm{d}}_n}(r_0)) = \mathfrak{m}_n(B_{\mathrm{d}_n}(r_0)) \leq M$, for every $n \in \mathbb{N}$. Moreover, we fix n large enough so that $\tilde{f}_n^{-1}(B_{\tilde{\mathrm{d}}_\infty}(R)) \subset B_{\tilde{\mathrm{d}}_n}(2R)$ (which is possible thanks to property (2.9)). Afterwards, consider the following cases:

- If $K > 0$, then Bonnet–Myers diameter estimate implies $\mathrm{Diam}(\tilde{X}, \tilde{\mathrm{d}}_n) \leq \pi\sqrt{N/K} =: D_{K,N}$ (see [Stu06b, Corollary 2.6]). Thanks to the Bishop–Gromov inequality (see Proposition 2.2.5), we have the following inequalities:

$$\begin{aligned}
\tilde{f}_{n*}\tilde{\mathfrak{m}}_n(B_{\tilde{\mathrm{d}}_\infty}(R)) &\leq \tilde{\mathfrak{m}}_n(B_{\tilde{\mathrm{d}}_n}(2R)), \\
&\leq \tilde{\mathfrak{m}}_n(\tilde{X}), \\
&\leq \left[\frac{\int_0^{D_{K,N}} \mathfrak{s}_{K/N}(t)^N \,\mathrm{d}t}{\int_0^{r_0} \mathfrak{s}_{K/N}(t)^N \,\mathrm{d}t}\right] \tilde{\mathfrak{m}}_n(B_{\tilde{\mathrm{d}}_n}(r_0)), \\
&\leq \left[\frac{\int_0^{D_{K,N}} \mathfrak{s}_{K/N}(t)^N \,\mathrm{d}t}{\int_0^{r_0} \mathfrak{s}_{K/N}(t)^N \,\mathrm{d}t}\right] M.
\end{aligned}$$

- If $K \leq 0$ and $R \leq r_0/2$, we have $\tilde{f}_{n*}\tilde{\mathfrak{m}}_n(B_{\tilde{\mathrm{d}}_\infty}(R)) \leq \tilde{\mathfrak{m}}_n(B_{\tilde{\mathrm{d}}_n}(2R)) \leq \tilde{\mathfrak{m}}_n(B_{\tilde{\mathrm{d}}_n}(r_0)) \leq M$.

- If $K \leq 0$ and $r_0/2 < R$, then, proceeding as for the first case and using Bishop–Gromov inequality (see Proposition 2.2.5), we obtain:

$$\tilde{f}_{n*}\tilde{\mathfrak{m}}_n(B_{\tilde{\mathrm{d}}_\infty}(R)) \leq \left[\frac{\int_0^R \mathfrak{s}_{K/N}(t)^N \,\mathrm{d}t}{\int_0^{r_0} \mathfrak{s}_{K/N}(t)^N \,\mathrm{d}t}\right] M.$$

In particular, for every $R > 0$, the sequence $\{\tilde{f}_{n*}\tilde{\mathfrak{m}}_n(B_{\tilde{\mathrm{d}}_\infty}(R))\}$ is uniformly bounded; hence, $\{\tilde{f}_{n*}\tilde{\mathfrak{m}}_n\}$ is precompact.

Now, passing to a subsequence if necessary, we can assume that $\{\tilde{f}_{n*}\tilde{\mathfrak{m}}_n\}_{n\in\mathbb{N}}$ converges to some $\mathfrak{m} \in \mathcal{M}_{\mathrm{loc}}(\tilde{X}, \tilde{\mathrm{d}}_\infty)$. Let us show that $\mathfrak{m} = \tilde{\mathfrak{m}}_\infty$. It is sufficient to prove that, for every $\tilde{x} \in \tilde{X}$ and $0 < r < r_0$, we have $\mathfrak{m}(B_{\tilde{\mathrm{d}}_\infty}(\tilde{x}, r)) = \tilde{\mathfrak{m}}_\infty(B_{\tilde{\mathrm{d}}_\infty}(\tilde{x}, r))$; since small balls generate the Borel σ-algebra of $\tilde{X}$.

First, observe that, for every $n \in \mathbb{N}$, we have $\tilde{\mathfrak{m}}_n(B_{\tilde{\mathrm{d}}_n}(r_0)) = \mathfrak{m}_n(B_{\mathrm{d}_n}(r_0)) \geq m$, where

$$m := \inf_{n\in\mathbb{N}\cup\{\infty\}} \{\mathfrak{m}_n(B_{\mathrm{d}_n}(r_0))\}.$$

In addition, m is positive since $\{\mathfrak{m}_n(B_{\mathrm{d}_n}(r_0))\}_{n\in\mathbb{N}}$ is a sequence of positive numbers converging to $\mathfrak{m}_\infty(B_{\mathrm{d}_\infty}(r_0)) > 0$. Therefore, $\{\tilde{\mathfrak{X}}_n\}_{n\in\mathbb{N}}$ is a sequence of pointed $\mathrm{RCD}^*(K, N)$ spaces with measures uniformly bounded from below; hence, thanks to Theorem 2.2.5, any limit point in the pmGH-topology is an $\mathrm{RCD}^*(K, N)$ space with full support. However, the sequence converges in the pmGH-topology to $(\tilde{X}, \tilde{\mathrm{d}}_\infty, \mathfrak{m}, \tilde{*}_\infty)$. Thus, $(\tilde{X}, \tilde{\mathrm{d}}_\infty, \mathfrak{m}, \tilde{*}_\infty)$ is a full support $\mathrm{RCD}^*(K, N)$ space. In particular, [Vil09, Theorem 30.11] implies $\mathfrak{m}(\partial B_{\tilde{\mathrm{d}}_\infty}(\tilde{x}, R)) = 0$ for every $R > 0$ and $\tilde{x} \in \tilde{X}$. Hence, thanks to [DV03, Proposition A2.6.II], for every $R > 0$ and $\tilde{x} \in \tilde{X}$ we have:

$$\mathfrak{m}(B_{\tilde{\mathrm{d}}_\infty}(\tilde{x}, R)) = \lim_{n\to\infty} \tilde{f}_{n*}\tilde{\mathfrak{m}}_n(B_{\tilde{\mathrm{d}}_\infty}(\tilde{x}, R)). \tag{2.10}$$

Now, let $\tilde{x} \in \tilde{X}$ and $0 < r < r_0$, and let us show that we have $\mathfrak{m}(B_{\tilde{\mathrm{d}}_\infty}(\tilde{x}, r)) = \tilde{\mathfrak{m}}_\infty(B_{\tilde{\mathrm{d}}_\infty}(\tilde{x}, r))$. First, when n is large enough so that $r \leq \epsilon_n^{-1}$, we can use property (2.9) to get:

$$B_{\tilde{\mathrm{d}}_n}(\tilde{g}_n(\tilde{x}), r - 2\epsilon_n) \subset \tilde{f}_n^{-1}(B_{\tilde{\mathrm{d}}_\infty}(\tilde{x}, r)) \subset B_{\tilde{\mathrm{d}}_n}(\tilde{g}_n(\tilde{x}), r + 2\epsilon_n)).$$

In particular, defining $A := \mathfrak{m}(B_{\tilde{\mathrm{d}}_\infty}(\tilde{x}, r))$ and using equation (2.10), we have:

$$\limsup_{n\to\infty} \tilde{\mathfrak{m}}_n(B_{\tilde{\mathrm{d}}_n}(\tilde{g}_n(\tilde{x}), r - 2\epsilon_n)) \leq A \leq \liminf_{n\to\infty} \tilde{\mathfrak{m}}_n(B_{\tilde{\mathrm{d}}_n}(\tilde{g}_n(\tilde{x}), r + 2\epsilon_n)).$$

Moreover, when n is large enough, we have $r + 2\epsilon_n < r_0 < \delta(X, \mathrm{d}_n)$; hence, point (v) of Proposition 2.3.4 implies:

$$\limsup_{n\to\infty} \mathfrak{m}_n(B_{\mathrm{d}_n}(g_n(x), r - 2\epsilon_n)) \leq A \leq \liminf_{n\to\infty} \mathfrak{m}_n(B_{\mathrm{d}_n}(g_n(x), r + 2\epsilon_n)),$$

where $x := p(\tilde{x})$. Now, observe that when n is large enough so that $r + 4\epsilon_n \leq \epsilon_n^{-1}$, we can use property (2.9) to get:

$$B_{\mathrm{d}_n}(g_n(x), r + 2\epsilon_n) \subset f_n^{-1}(B_{\mathrm{d}_\infty}(x, r + 4\epsilon_n)),$$
$$f_n^{-1}(B_{\mathrm{d}_\infty}(x, r - 4\epsilon_n)) \subset B_{\mathrm{d}_n}(g_n(x), r - 2\epsilon_n).$$

In particular, for every $\eta > 0$, we have:

$$\limsup_{n\to\infty} f_{n_*}\, \mathfrak{m}_n(B_{\mathrm{d}_\infty}(x, r - \eta)) \leq A \leq \liminf_{n\to\infty} f_{n_*}\, \mathfrak{m}_n(B_{\mathrm{d}_\infty}(x, r + \eta)).$$

However, $\{f_{n_*}\, \mathfrak{m}_n\}$ converges to $\mathfrak{m}_\infty$ and $\mathcal{X}_\infty$ is a full support $\mathrm{RCD}(K, N)$ space. Thus, we can apply [Vil09, Theorem 30.11] and [DV03, Theorem A2.3.II] to get:

$$\limsup_{n\to\infty} f_{n_*}\, \mathfrak{m}_n(B_{\mathrm{d}_\infty}(x, r - \eta)) = \mathfrak{m}_\infty(B_{\mathrm{d}_\infty}(x, r - \eta)),$$
$$\liminf_{n\to\infty} f_{n_*}\, \mathfrak{m}_n(B_{\mathrm{d}_\infty}(x, r + \eta)) = \mathfrak{m}_\infty(B_{\mathrm{d}_\infty}(x, r + \eta)).$$

Hence, for every $\eta > 0$, we have:

$$\mathfrak{m}_\infty(B_{\mathrm{d}_\infty}(x, r - \eta)) \leq \mathfrak{m}(B_{\tilde{\mathrm{d}}_\infty}(\tilde{x}, r)) \leq \mathfrak{m}_\infty(B_{\mathrm{d}_\infty}(x, r + \eta)).$$

In particular, letting η go to 0, we have the following:

$$\mathfrak{m}(B_{\tilde{\mathrm{d}}_\infty}(\tilde{x}, r)) = \mathfrak{m}_\infty(B_{\mathrm{d}_\infty}(x, r)) = \tilde{\mathfrak{m}}_\infty(B_{\tilde{\mathrm{d}}_\infty}(\tilde{x}, r)),$$

using $r < r_0 < \delta/2$ for the last equality. Therefore $\{\tilde{f}_{n_*}\tilde{\mathfrak{m}}_n\}$ converges to $\tilde{\mathfrak{m}}_\infty$.

For every $R' > 0$, we define $\epsilon(n, R') := \mathrm{d}_{\mathcal{P}}^{\{R'\}}(\tilde{f}_{n_*}\tilde{\mathfrak{m}}_n, \tilde{\mathfrak{m}}_\infty)$. Thanks to the discussion above, we have $\lim_{n\to\infty} \epsilon(n, R') \to 0$, for every $R' > 0$. Let $R > 0$, and let us show that $\lim_{n\to\infty} \mathrm{d}_{\mathcal{P}}^{\{R\}}(\tilde{g}_{n_*}\tilde{\mathfrak{m}}_\infty, \tilde{\mathfrak{m}}_n) = 0$.

Let $A \subset \overline{B}_{\tilde{\mathrm{d}}_n}(R)$, and observe that we have $\tilde{\mathfrak{m}}_n(A) \leq \tilde{f}_{n_*}\tilde{\mathfrak{m}}_n(\tilde{f}_n(A))$. Also, when n is large enough, we can use property (2.9) to get $\tilde{f}_n(A) \subset \overline{B}_{\tilde{\mathrm{d}}_\infty}(2R)$. Therefore, we have $\tilde{\mathfrak{m}}_n(A) \leq \tilde{\mathfrak{m}}_\infty(\{\tilde{f}_n(A)\}^{\epsilon(n,2R)}) + \epsilon(n, 2R)$. Then, when n is large enough, we can use property (2.9) to obtain $\{\tilde{f}_n(A)\}^{\epsilon(n,2R)} \subset \tilde{g}_n^{-1}(\{A\}^{2\epsilon_n+\epsilon(n,2R)})$. Thus, we have $\tilde{\mathfrak{m}}_n(A) \leq \tilde{g}_{n_*}\tilde{\mathfrak{m}}_\infty(\{A\}^{2\epsilon_n+\epsilon(n,2R)}) + \epsilon(n, 2R)$. Applying the same arguments, we also have $\tilde{g}_{n_*}\tilde{\mathfrak{m}}_\infty(A) \leq \tilde{\mathfrak{m}}_n(\{A\}^{2\epsilon_n+\epsilon(n,2R)}) + \epsilon(n, 2R)$. Therefore, $\mathrm{d}_{\mathcal{P}}^{\{R\}}(\tilde{g}_{n_*}\tilde{\mathfrak{m}}_\infty, \tilde{\mathfrak{m}}_n) \leq \epsilon(n, 2R) + 2\epsilon_n$; in particular, $\lim_{n\to\infty} \mathrm{d}_{\mathcal{P}}^{\{R\}}(\tilde{g}_{n_*}\tilde{\mathfrak{m}}_\infty, \tilde{\mathfrak{m}}_n) = 0$. This concludes the proof. □

Proof of Theorem 2.4.1, converse implication. Assume that $\{\tilde{\mathfrak{X}}_n = (\tilde{X}, \tilde{\mathrm{d}}_n, \tilde{\mathfrak{m}}_n, \tilde{*}_n)\}$ converges in the equivariant pmGH-topology to $\tilde{\mathfrak{X}}_\infty = (\tilde{X}, \tilde{\mathrm{d}}_\infty, \tilde{\mathfrak{m}}_\infty, \tilde{*}_\infty)$. Let us prove that $\{\mathfrak{X}_n = (X, \mathrm{d}_n, \mathfrak{m}_n, *_n)\}$ converges in the pmGH-topology to $\mathfrak{X}_\infty = (X, \mathrm{d}_\infty, \mathfrak{m}_\infty, *_\infty)$.

Let $\{\tilde{f}_n, \tilde{g}_n, \phi_n, \epsilon_n\}$ be a sequence realising the convergence of $\{\tilde{\mathfrak{X}}_n\}$ to $\tilde{\mathfrak{X}}_\infty$ in the equivariant pmGH-topology. Thanks to the equivariant requirement, there exists pointed Borel maps $f_n\colon (X, *_n) \to (X, *_\infty)$ and $g_n\colon (X, *_\infty) \to (X, *_n)$ such that $p \circ \tilde{f}_n = f_n \circ p$ and $p \circ \tilde{g}_n = g_n \circ p$.

Let us fix $x \in X$ and $\tilde{x} \in p^{-1}(x)$. Observe that $\mathrm{d}_n(g_n(f_n(x)), x) = \inf\{\tilde{\mathrm{d}}_n(\tilde{y}, \tilde{x})\}$, where the infimum is taken over all $\tilde{y} \in \tilde{X}$ such that $p(\tilde{y}) = g_n(f_n(x))$. However, we have $p(\tilde{g}_n(\tilde{f}_n(\tilde{x}))) = g_n(f_n(x))$. Therefore, we have $\mathrm{d}_n(g_n(f_n(x)), x) \leq \tilde{\mathrm{d}}_n(\tilde{g}_n(\tilde{f}_n(\tilde{x})), \tilde{x}) \leq \epsilon_n$. The same argument shows that $\mathrm{d}_\infty(f_n(g_n(x)), x) \leq \epsilon_n$.

Let $y_i \in X$ ($i \in \{1, 2\}$) and let $\tilde{y}_i$ such that $p(\tilde{y}_i) = y_i$ and $\tilde{\mathrm{d}}_\infty(\tilde{y}_1, \tilde{y}_2) = \mathrm{d}_\infty(y_1, y_2)$. Assume that $D_\infty \coloneqq \mathrm{Diam}(X, \mathrm{d}_\infty) \leq \epsilon_n^{-1}$ and observe that, since $p(\tilde{g}_n(\tilde{y}_i)) = g_n(y_i)$, we have:

$$\begin{aligned} \mathrm{d}_n(g_n(y_1), g_n(y_2)) - \mathrm{d}_\infty(y_1, y_2) &\leq \tilde{\mathrm{d}}_n(\tilde{g}_n(\tilde{y}_1), \tilde{g}_n(\tilde{y}_2)) - \tilde{\mathrm{d}}_\infty(\tilde{y}_1, \tilde{y}_2) \\ &\leq \epsilon_n. \end{aligned} \tag{2.11}$$

Then, let $x_i \in X$ ($i \in \{1, 2\}$) such that $\mathrm{d}_n(x_1, x_2) \leq \epsilon_n^{-1}$, and fix $\tilde{x}_i \in \tilde{X}$ such that $p(\tilde{x}_i) = x_i$ and $\tilde{\mathrm{d}}_n(\tilde{x}_1, \tilde{x}_2) = \mathrm{d}_n(x_1, x_2)$. Observe that we have $p(\tilde{f}_n(\tilde{x}_i)) = f_n(x_i)$, therefore:

$$\begin{aligned} \mathrm{d}_\infty(f_n(x_1), f_n(x_2)) - \mathrm{d}_n(x_1, x_2) &\leq \tilde{\mathrm{d}}_\infty(\tilde{f}_n(\tilde{x}_1), \tilde{f}_n(\tilde{x}_2)) - \tilde{\mathrm{d}}_n(\tilde{x}_1, \tilde{x}_2) \\ &\leq \epsilon_n. \end{aligned} \tag{2.12}$$

Let us show that $\{D_n \coloneqq \mathrm{Diam}(X, \mathrm{d}_n)\}$ is a bounded sequence. Let $x_i \in X$ ($i \in \{1, 2\}$) and observe that $\mathrm{d}_n(g_n(f_n(x_1)), g_n(f_n(x_2))) \leq \epsilon_n + D_\infty$ as a consequence of inequality (2.11) (when $D_\infty \leq \epsilon_n^{-1}$). However, we have $|\mathrm{d}_n(x_1, x_2) - \mathrm{d}_n(g_n(f_n(x_1)), g_n(f_n(x_2)))| \leq \mathrm{d}_n(g_n(f_n(x_1)), x_1) + \mathrm{d}_n(g_n(f_n(x_2)), x_2) \leq 2\epsilon_n$. Therefore, $\mathrm{d}_n(x_1, x_2) \leq D_\infty + 3\epsilon_n$. We can conclude that $\{D_n\}$ is bounded.

Since $\{D_n\}$ is bounded, we have (thanks to inequality (2.12)):

$$\forall x_1, x_2 \in X, \mathrm{d}_\infty(f_n(x_1), f_n(x_2)) - \mathrm{d}_n(x_1, x_2) \leq \epsilon_n,$$

when n is large enough. Also, we have $\mathrm{d}_n(x_1, x_2) \leq \mathrm{d}_n(g_n(f_n(x_1)), g_n(f_n(x_2))) + 2\epsilon_n$. Therefore, using inequality 2.11 we obtain, $\mathrm{d}_n(x_1, x_2) - \mathrm{d}_\infty(f_n(x_1), f_n(x_2)) \leq 3\epsilon_n$. Hence, we can conclude that $\mathrm{Dis}(f_n) \leq 3\epsilon_n$. The same argument also gives $\mathrm{Dis}(g_n) \leq 3\epsilon_n$, which concludes the proof of the second metric requirement.

Finally, using Lemma 2.4.6, and applying the same procedure as in Part II of the direct implication, we can prove that (making ϵ_n smaller if necessary but keeping $\epsilon_n \to 0$) we have:

$$\max\{\mathrm{d}_{\mathcal{P}}(f_{n_*}\,\mathfrak{m}_n, \mathfrak{m}_\infty), \mathrm{d}_{\mathcal{P}}(g_{n_*}\,\mathfrak{m}_\infty, \mathfrak{m}_n)\} \leq \epsilon_n.$$

Hence, $\{f_n, g_n, \epsilon_n\}$ is a sequence realising the convergence of $\{\mathcal{X}_n\}$ to $\mathcal{X}_\infty$ in the pmGH-topology, which concludes the proof. □

3

Structure theory in nonnegative curvature

Contents

The results presented in this Chapter are original and appear in [MN22], written in collaboration with Andrea Mondino. In this chapter, we will analyse the case of nonnegative curvature and general dimension. We will begin by studying the topology of $\mathrm{RCD}(0, N)$ spaces in Section 3.1, specifically relying on the splitting theorem to analyse the fundamental group of these spaces. We will then introduce the Albanese and soul maps in Section 3.2, which provide insight into how lifts of $\mathrm{RCD}(0, N)$ structures split. The main technical challenge of this chapter will be proving the continuity of these maps (see Theorem 1.3.2 presented in Chapter 1). Finally, we will apply this continuity statement to obtain an analogue of Tuschmann and Wiemeler's Theorem 1.1.7 in Section 3.3.

3.1 Splittings and topological invariants

In this section, we are going to introduce splitting maps and obtain some topological invariants of compact $\mathrm{RCD}(0, N)$ spaces, namely the splitting degree and the crystallographic class. From now on, we fix a compact topological space X admitting an $\mathrm{RCD}(0,N)$ structure and denote $p\colon \tilde{X} \to X$ its universal cover.

Definition 3.1.1. Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0,N)$ structure on X and denote $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ its lift. A *splitting of* $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ is an isomorphism:

$$\phi\colon (\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}) \to (\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}}) \times \mathbb{R}^k,$$

where $\mathbb{R}^k$ is endowed with the Euclidean distance and Lebesgue measure, $k \in \mathbb{N} \cap [0,N]$ is called the *degree of* ϕ, and $(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}})$ is a compact $\mathrm{RCD}(0, N-k)$ space with trivial revised fundamental group called the *soul of* ϕ (if $N - k < 1$, then $\overline{X}$ is a singleton).

Thanks to [MW19, Theorem 1.3], which in turn built on top of the splitting theorem for $\mathrm{RCD}(0,N)$ spaces [Gig14, Theorem 1.4], we have the following existence result.

Theorem 3.1.1. Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0,N)$ structure on X. Then, its lift $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ admits a splitting. Moreover, for every splitting ϕ of $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$, the group of isomorphisms of $(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}}) \times \mathbb{R}^k$ splits, i.e., we have:

$$\mathrm{Iso}_{\mathrm{m.m.s.}}((\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}}) \times \mathbb{R}^k) = \mathrm{Iso}_{\mathrm{m.m.s.}}(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}}) \times \mathrm{Iso}(\mathbb{R}^k),$$

where $(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}})$ is the soul of ϕ and k is the degree of ϕ.

As a consequence of Theorem 3.1.1, Theorem 2.3.3, and Proposition 2.3.4, we can introduce the following notations.

Notation 3.1.1. Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0,N)$ structure on X and let ϕ be a splitting of $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ with degree k and soul $(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}})$. We write:

(i) p^{ϕ}_{S*} (resp. $p^{\phi}_{\mathbb{R}*}$) the projection of $\mathrm{Iso}_{\mathrm{m.m.s.}}((\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}}) \times \mathbb{R}^k)$ onto $\mathrm{Iso}_{\mathrm{m.m.s.}}(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}})$ (resp. $\mathrm{Iso}(\mathbb{R}^k)$),

(ii) ι the inclusion of $\overline{\pi}_1(X)$ into $\mathrm{Iso}_{\mathrm{m.m.s.}}(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$,

(iii) ϕ_* the isomorphism from $\mathrm{Iso}_{\mathrm{m.m.s.}}(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ onto $\mathrm{Iso}_{\mathrm{m.m.s.}}((\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}}) \times \mathbb{R}^k)$ defined by $\phi_*(\eta) \coloneqq \phi \circ \eta \circ \phi^{-1}$ for every $\eta \in \mathrm{Iso}_{\mathrm{m.m.s.}}(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$.

We call $\rho^{\phi}_{S} \coloneqq p^{\phi}_{S*} \circ \phi_* \circ \iota$ (resp. $\rho^{\phi}_{\mathbb{R}} \coloneqq p^{\phi}_{\mathbb{R}*} \circ \phi_* \circ \iota$) the *soul homomorphism associated to* ϕ (resp. the *Euclidean homomorphism associated to* ϕ) and we write $K(\phi) \coloneqq \mathrm{Ker}(\rho^{\phi}_{\mathbb{R}})$ and $\Gamma(\phi) \coloneqq \mathrm{Im}(\rho^{\phi}_{\mathbb{R}})$.

The next result shows that the kernel and the image of the Euclidean homomorphism associated to a splitting enjoy particular group structures.

Proposition 3.1.1. Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0, N)$ structure on X and let ϕ be a splitting of $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ with degree k and soul $(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}})$. Then, $K(\phi)$ is a finite normal subgroup of $\overline{\pi}_1(X)$ and $\Gamma(\phi)$ is a crystallographic subgroup of $\mathrm{Iso}(\mathbb{R}^k)$ (i.e. it acts cocompactly and discretely on $\mathbb{R}^k$).

Proof. First, let us show that $K(\phi)$ is finite. Observe that every element $\eta \in K(\phi)$ satisfies $\eta(\overline{X} \times \{0\}) \cap (\overline{X} \times \{0\}) \neq \varnothing$. However, $\overline{X} \times \{0\}$ is a compact subset of $\overline{X} \times \mathbb{R}^k$ and $\overline{\pi}_1(X)$ acts properly on $\overline{X} \times \mathbb{R}^k$; thus, $K(\phi)$ is finite.

Now, let us show that $\Gamma(\phi)$ acts cocompactly on $\mathbb{R}^k$. Applying the first isomorphism theorem for topological spaces, there is a continuous map μ such that the following diagram is commutative:

$$\begin{array}{ccc} \overline{X} \times \mathbb{R}^k & \xrightarrow{\quad p_{\mathbb{R}^k} \quad} & \mathbb{R}^k \\ \Big\downarrow q_1 & & \Big\downarrow q_2 \\ (\overline{X} \times \mathbb{R}^k)/\overline{\pi}_1(X) & \xrightarrow{\quad \mu \quad} & \mathbb{R}^k/\Gamma(\phi) \end{array}$$

where q_i $(i \in \{1, 2\})$ are the quotient maps. Moreover, μ is surjective since $q_2 \circ p_{\mathbb{R}^k}$ is surjective. Finally, X is homeomorphic to $(\overline{X} \times \mathbb{R}^k)/\overline{\pi}_1(X)$; in particular, $(\overline{X} \times \mathbb{R}^k)/\overline{\pi}_1(X)$ is compact and $\mathbb{R}^k/\Gamma(\phi)$ is compact, being the image of a compact topological space by a continuous surjective map. In conclusion, $\Gamma(\phi)$ acts cocompactly on $\mathbb{R}^k$.

Let us prove that $\Gamma(\phi)$ acts discretely on $\mathbb{R}^k$ (i.e. its orbits are discrete subsets of $\mathbb{R}^k$). First, observe that it is sufficient to prove that $\Gamma(\phi)$ acts properly on $\mathbb{R}^k$. To prove this, let K be a compact subset of $\mathbb{R}^k$ and let us show that there are only finitely many elements $g \in \Gamma(\phi)$ such that $gK \cap K \neq \varnothing$. By definition of $\Gamma(\phi)$, we have:

$$\{g \in \Gamma(\phi), gK \cap K \neq \varnothing\} = \rho^{\phi}_{\mathbb{R}}(\{\eta \in \overline{\pi}_1(X), \eta(\overline{X} \times K) \cap (\overline{X} \times K) \neq \varnothing\}).$$

However, $\overline{\pi}_1(X)$ acts properly on $\overline{X} \times \mathbb{R}^k$ and $\overline{X} \times K$ is compact; hence:

$$\#\{\eta \in \overline{\pi}_1(X), \eta(\overline{X} \times K) \cap (\overline{X} \times K) \neq \varnothing\} < \infty.$$

Thus, $\{g \in \Gamma(\phi), gK \cap K \neq \varnothing\}$ is finite, being the image of a finite set. $\square$

The following corollary of Proposition 3.1.1 defines the splitting degree of X (see also [MMP22, Proposition 2.25]).

Corollary 3.1.1 (Splitting degree $k(X)$)**.** The revised fundamental group $\overline{\pi}_1(X)$ is a finitely generated group which has polynomial growth of order $k(X) \in \mathbb{N} \cap [0, N]$. Moreover, given any $\mathrm{RCD}(0, N)$ structure $(X, \mathrm{d}, \mathfrak{m})$ on X with lift $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$, the degree of any splitting ϕ of $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ is equal to $k(X)$. We call $k(X)$ the *splitting degree of* X.

Proof. Due to Proposition 3.1.1, $\Gamma(\phi)$ is a crystallographic subgroup of $\mathrm{Iso}(\mathbb{R}^k)$, where $k \in [0, N] \cap \mathbb{N}$ is the degree of ϕ. We need to prove that $\overline{\pi}_1(X)$ has polynomial growth of order k.

By Bieberbach's first Theorem (see [Cha86, Theorem 3.1]), $\Gamma(\phi)$ admits a normal subgroup $\Gamma(\phi) \cap \mathbb{R}^k$ such that $\Gamma(\phi) \cap \mathbb{R}^k$ is isomorphic to $\mathbb{Z}^k$ and $\Gamma(\phi) \cap \mathbb{R}^k$ has finite index in $\Gamma(\phi)$. In particular, $\Gamma(\phi) \cap \mathbb{R}^k$ is finitely generated, has polynomial growth of order k, and is a normal subgroup of $\Gamma(\phi)$ with finite index; thus, $\Gamma(\phi)$ is also finitely generated and has polynomial growth of order k. Now, $\overline{\pi}_1(X)/K(\phi)$ is isomorphic to $\Gamma(\phi)$; hence it is finitely generated with polynomial growth of order k. However, $K(\phi)$ is finite and is a normal subgroup of $\overline{\pi}_1(X)$; thus, $\overline{\pi}_1(X)$ is also finitely generated and has polynomial growth of order k. □

The revised fundamental group satisfies the following additional group property.

Proposition 3.1.2. The revised fundamental group $\overline{\pi}_1(X)$ is a Hopfian group, i.e., every surjective group homomorphism from $\overline{\pi}_1(X)$ onto itself is an isomorphism.

Proof. First of all, let us recall some results about Group theory:

(i) Noetherian groups (every subgroup is finitely generated) are Hopfian groups.

(ii) If H is a normal subgroup of G such that both H and G/H are Noetherian, then G is Noetherian.

(iii) Finite groups are Noetherian.

(iv) Finitely generated abelian groups are Noetherian.

Let us fix an $\mathrm{RCD}(0,N)$ structure $(X,\mathrm{d},\mathfrak{m})$ on X and let ϕ be a splitting of its lift $(\tilde{X},\tilde{\mathrm{d}},\tilde{\mathfrak{m}})$. By Proposition 3.1.1 and Corollary 3.1.1, $\Gamma(\phi)$ is a crystallographic subgroup of $\mathrm{Iso}(\mathbb{R}^{k(X)})$. Hence, by Bieberbach's 1st Theorem ([Cha86, Theorem 3.1]), $\Gamma(\phi)\cap\mathbb{R}^{k(X)}$ is isomorphic to $\mathbb{Z}^{k(X)}$. In particular, $\Gamma(\phi)\cap\mathbb{R}^{k(X)}$ is Noetherian thanks to (iv). Moreover, $\Gamma(\phi)\cap\mathbb{R}^k$ is normal in $\Gamma(\phi)$ and the quotient $\Gamma(\phi)/\Gamma(\phi)\cap\mathbb{R}^k$ is finite. As a consequence of (iii), $\Gamma(\phi)/\Gamma(\phi)\cap\mathbb{R}^k$ is Noetherian, and, using (ii), $\Gamma(\phi)$ is Noetherian. In addition, $K(\phi)$ is finite by Proposition 3.1.1, so it is Noetherian by (iii). Finally, $\overline{\pi}_1(X)/K(\phi)$ is isomorphic to $\Gamma(\phi)$ so it is Noetherian. In conclusion, because of (ii), $\overline{\pi}_1(X)$ is Noetherian; hence, it is Hopfian using (i). □

Remark 3.1.1. The result above will be fundamental to prove Theorem 1.3.2 introduced in Chapter 1.

Given $k\in\mathbb{N}$, two crystallographic subgroups of $\mathrm{Iso}(\mathbb{R}^k)$ are called equivalent if they are conjugated by an affine transformation. The set $\mathrm{Crys}(k)$ of equivalence classes of crystallographic subgroups of $\mathrm{Iso}(\mathbb{R}^k)$ is a finite set according to Bieberbach's third Theorem (see [Cha86, Theorem 7.1]). The following result defines the crystallographic class of X.

Proposition 3.1.3 (Crystallographic class $\Gamma(X)$)**.** For $i\in\{1,2\}$, let $(X,\mathrm{d}_i,\mathfrak{m}_i)$ be an $\mathrm{RCD}(0,N)$ structure on X, and let ϕ_i be a splitting of $(\tilde{X},\tilde{\mathrm{d}}_i,\tilde{\mathfrak{m}}_i)$. Then $\Gamma(\phi_1)$ and $\Gamma(\phi_2)$ are equivalent as crystallographic subgroups of $\mathrm{Iso}(\mathbb{R}^{k(X)})$. We denote by $\Gamma(X)$ the common equivalence class and call it the *crystallographic class of* X.

Proof. By Bieberbach's second Theorem (see [Cha86, Theorem 4.1]), two crystallographic subgroups of $\mathrm{Iso}(\mathbb{R}^k)$ are conjugated by an affine transformation if and only if they are isomorphic (we let $k\coloneqq k(X)$). We need to show that $\Gamma(\phi_1)$ and $\Gamma(\phi_2)$ are isomorphic. Observe that, for $i\in\{1,2\}$, we have the following exact sequence of groups:

$$\{1\}\longrightarrow K(\phi_i)\xrightarrow{\ \iota\ }\overline{\pi}_1(X)\xrightarrow{\ \rho_{\mathbb{R}}^{\phi_i}\ }\Gamma(\phi_i)\longrightarrow\{1\}\ ,$$

where ι is just the inclusion, $\Gamma(\phi_i)$ is a crystallographic subgroup of $\mathrm{Iso}(\mathbb{R}^k)$, and $K(\phi_i)$ is finite. Thanks to [Wil00, Remark 2.5], $K(\phi_i)=\iota(K(\phi_i))$ is uniquely characterized as the maximal finite normal subgroup of $\overline{\pi}_1(X)$. In particular, we necessarily have $K(\phi_1)=K(\phi_2)$. In conclusion, $\Gamma(\phi_1)\simeq\overline{\pi}_1(X)/K(\phi_1)=\overline{\pi}_1(X)/K(\phi_2)\simeq\Gamma(\phi_2)$; thus, $\Gamma(\phi_1)$ is isomorphic to $\Gamma(\phi_2)$. □

Given $k \in \mathbb{N}$, and Γ a crystallographic subgroup of $\mathrm{Iso}(\mathbb{R}^k)$, the quotient space $\mathbb{R}^k/\Gamma$ has the structure of a compact flat orbifold of dimension k, whose orbifold metric d_Γ satisfies:

$$\mathrm{d}_\Gamma([x],[y]) = \inf\{|x'-y'|\}, \tag{3.1}$$

where $x, y \in \mathbb{R}^k$, $[x]$ and $[y]$ are their equivalence class in $\mathbb{R}^k/\Gamma$, and the infimum is taken over all $x' \in [x]$ and $y' \in [y]$. Moreover, equivalent crystallographic groups give rise to orbifolds that are affinely equivalent.

Conversely, given a compact flat orbifold (X, d) of dimension k, the orbifold fundamental group $\pi_1^{\mathrm{orb}}(X)$ acts by isometries on the orbifold universal cover, which is $\mathbb{R}^k$ (the action being discrete and cocompact). Hence, one can associate a crystallographic group to (X, d). Finally, two affinely equivalent flat orbifolds of dimension k have isomorphic orbifold fundamental groups. Hence, by Bieberbach's second Theorem (see [Cha86, Theorem 4.1]), they give rise to equivalent crystallographic groups (see the introduction of Section 2.1 in [BDP18] for more details and some references).

Therefore, there is a one-to-one correspondence between equivalence classes of crystallographic subgroups of $\mathrm{Iso}(\mathbb{R}^k)$ and affine equivalence classes of compact flat orbifolds of dimension k. This leads us to the definition of the Albanese class of X.

Definition 3.1.2 (Albanese class $A(X)$)**.** We write $A(X)$ the affine equivalence class of compact flat orbifold determined by $\Gamma(X)$, and call it the *Albanese class of X*. More explicitly, $A(X)$ is the set of all flat orbifolds $(\mathbb{R}^{k(X)}/\Gamma, \mathrm{d}_\Gamma)$, where $\Gamma \in \Gamma(X)$ and d_Γ is defined in equation (3.1).

3.2 Albanese and soul maps

In this section, we fix a compact topological space X admitting an $\mathrm{RCD}(0, N)$ structure and denote $p\colon \tilde{X} \to X$ its universal cover. We will introduce the *Albanese and soul maps* associated to X, reflecting how the lift of $\mathrm{RCD}(0, N)$ structures on X split. The main challenge of this section will be the proof of Theorem 1.3.2 (introduced in Chapter 1), which states the continuity of these maps.

3.2.1 Definition

First of all, we introduce the moduli space of flat metrics on the Albanese class $A(X)$ (introduced in Definition 3.1.2). This moduli space will act as the codomain of the Albanese map.

Definition 3.2.1 ($\mathscr{M}_{\text{flat}}(A(X))$)**.** The *moduli space of flat metrics on* $A(X)$ is the quotient of $A(X)$ by isometry equivalence, endowed with the Gromov–Hausdorff distance d_{GH} (see [BBI22, Definition 7.3.10]).

The following remark will be helpful in the proof of Theorem 1.3.3. It is also interesting on its own as it gives a more explicit way to see the moduli space $\mathscr{M}_{\text{flat}}(A(X))$.

Remark 3.2.1. Given any element $\Gamma \in \Gamma(X)$, $\mathscr{M}_{\text{flat}}(A(X))$ is isometric to the moduli space $\mathscr{M}_{\text{flat}}(\mathbb{R}^k/\Gamma)$ of flat metrics on the compact orbifold $\mathbb{R}^k/\Gamma$ equipped with the GH topology (see [BDP18, Section 4.2] for more details on $\mathscr{M}_{\text{flat}}(\mathbb{R}^k/\Gamma)$).

The next lemma is fundamental to introduce the Albanese and soul maps associated to X.

Lemma 3.2.1. For $i \in \{1,2\}$, let $(X, \mathrm{d}_i, \mathfrak{m}_i)$ be an $\mathrm{RCD}(0,N)$ structure on X, and let ϕ_i be a splitting of $(\tilde{X}, \tilde{\mathrm{d}}_i, \tilde{\mathfrak{m}}_i)$ with soul $(\overline{X}_i, \overline{\mathrm{d}}_i, \overline{\mathfrak{m}}_i)$. If $(X, \mathrm{d}_1, \mathfrak{m}_1)$ and $(X, \mathrm{d}_2, \mathfrak{m}_2)$ are isomorphic, then $(\overline{X}_1, \overline{\mathrm{d}}_1, \overline{\mathfrak{m}}_1)$ is isomorphic to $(\overline{X}_2, \overline{\mathrm{d}}_2, \overline{\mathfrak{m}}_2)$, and $(\mathbb{R}^k/\Gamma(\phi_1), \mathrm{d}_{\Gamma(\phi_1)})$ is isometric to $(\mathbb{R}^k/\Gamma(\phi_2), \mathrm{d}_{\Gamma(\phi_2)})$.

Proof. Let us fix an isomorphism $\phi\colon (X, \mathrm{d}_1, \mathfrak{m}_1) \to (X, \mathrm{d}_2, \mathfrak{m}_2)$. We can lift ϕ to the universal covers to get an isomorphism $\tilde{\phi}\colon (\tilde{X}, \tilde{\mathrm{d}}_1, \tilde{\mathfrak{m}}_1) \to (\tilde{X}, \tilde{\mathrm{d}}_2, \tilde{\mathfrak{m}}_2)$ such that $p \circ \tilde{\phi} = \phi \circ p$. Now, let $\mu := \phi_2 \circ \tilde{\phi} \circ \phi_1^{-1}$. Since, $\overline{X}_1$ and $\overline{X}_2$ are compact, μ is of the form $\mu = (\mu_S, \mu_{\mathbb{R}})$, where $\mu_S\colon (\overline{X}, \overline{\mathrm{d}}_1, \overline{\mathfrak{m}}_1) \to (\overline{X}, \overline{\mathrm{d}}_2, \overline{\mathfrak{m}}_2)$ is an isomorphism, and $\mu_{\mathbb{R}} \in \mathrm{Iso}(\mathbb{R}^k)$ (where $k := k(X)$). In particular $(\overline{X}_1, \overline{\mathrm{d}}_1, \overline{\mathfrak{m}}_1)$ is isomorphic to $(\overline{X}_2, \overline{\mathrm{d}}_2, \overline{\mathfrak{m}}_2)$.

We are going to show that $\Gamma(\phi_2) = \mu_{\mathbb{R}} \Gamma(\phi_1) \mu_{\mathbb{R}}^{-1}$. Let $\bar{x}_1 \in \overline{X}_1$, let $t \in \mathbb{R}^k$, let $\alpha \in \overline{\pi}_1(X)$ and define $\tilde{z} := \phi_1^{-1}(\bar{x}_1, t)$. By definition of the soul and Euclidean homomorphisms associated to ϕ_1 and ϕ_2, we have:

$$\begin{aligned}\mu(\rho_S^{\phi_1}(\alpha) \cdot \bar{x}_1, \rho_{\mathbb{R}}^{\phi_1}(\alpha) \cdot t) &= (\rho_S^{\phi_2}(\eta) \cdot \mu_S(\bar{x}_1), \rho_{\mathbb{R}}^{\phi_2}(\eta) \cdot \mu_{\mathbb{R}}(t)) \\ &= (\mu_S(\rho_S^{\phi_1}(\alpha) \cdot \bar{x}_1), \mu_{\mathbb{R}}(\rho_{\mathbb{R}}^{\phi_1}(\alpha) \cdot t)),\end{aligned}$$

where $\eta \coloneqq \tilde{\phi}_*(\alpha)$ and $\tilde{\phi}_*$ is the automorphism of $\overline{\pi}_1(X)$ defined by $\tilde{\phi}_*(\alpha) \coloneqq \tilde{\phi}\circ\alpha\circ\tilde{\phi}^{-1}$. In particular, for every $t \in \mathbb{R}^k$ and $\alpha \in \overline{\pi}_1(X)$, we have $\mu_{\mathbb{R}}(\rho_{\mathbb{R}}^{\phi_1}(\alpha) \cdot t) = \rho_{\mathbb{R}}^{\phi_2} \circ \tilde{\phi}_*(\alpha) \cdot \mu_{\mathbb{R}}(t)$. Thus, for every $\alpha \in \overline{\pi}_1(X)$, we have $\mu_{\mathbb{R}} \circ \rho_{\mathbb{R}}^{\phi_1}(\alpha) \circ \mu_{\mathbb{R}}^{-1} = \rho_{\mathbb{R}}^{\phi_2} \circ \tilde{\phi}_*(\alpha)$. In particular, by definition of $\Gamma(\phi_1)$ and $\Gamma(\phi_2)$, and since $\tilde{\phi}_*(\overline{\pi}_1(X)) = \overline{\pi}_1(X)$, we have $\Gamma(\phi_2) = \mu_{\mathbb{R}}\Gamma(\phi_1)\mu_{\mathbb{R}}^{-1}$. In conclusion, using [BDP18, Lemma 4.1], $(\mathbb{R}^k/\Gamma(\phi_1), \mathrm{d}_{\Gamma(\phi_1)})$ is isometric to $(\mathbb{R}^k/\Gamma(\phi_2), \mathrm{d}_{\Gamma(\phi_2)})$, which concludes the proof. □

We can define the Albanese and soul maps thanks to Lemma 3.2.1.

Definition 3.2.2 (Albanese and soul maps)**.** Given an $\mathrm{RCD}(0, N)$ structure $(X, \mathrm{d}, \mathfrak{m})$ on X, and given a splitting ϕ of $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ with soul $(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}})$, we define:

- $\mathcal{A}([X, \mathrm{d}, \mathfrak{m}]) \coloneqq [\mathbb{R}^k/\Gamma(\phi), \mathrm{d}_{\Gamma(\phi)}] \in \mathscr{M}_{\mathrm{flat}}(A(X))$,
- $\mathcal{S}([X, \mathrm{d}, \mathfrak{m}]) \coloneqq [\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}}] \in \mathfrak{RCD}(0, N - k(X))$.

The maps $\mathcal{A}\colon \mathfrak{M}_{0,N}(X) \to \mathscr{M}_{\mathrm{flat}}(A(X))$ and $\mathcal{S}\colon \mathfrak{M}_{0,N}(X) \to \mathfrak{RCD}(0, N - k(X))$ are called the *Albanese and soul maps associated to* X.

We end this section with the following surjectivity result.

Proposition 3.2.1. The Albanese map associated to X is surjective from $\mathfrak{M}_{0,N}(X)$ onto $\mathscr{M}_{\mathrm{flat}}(A(X))$.

Proof. First of all, let $(X, \mathrm{d}_0, \mathfrak{m}_0)$ be a reference $\mathrm{RCD}(0, N)$ structure on X, and let ϕ_0 be a splitting of its lift $(\tilde{X}, \tilde{\mathrm{d}}_0, \tilde{\mathfrak{m}}_0)$ with soul $(\overline{X}_0, \overline{\mathrm{d}}_0, \overline{\mathfrak{m}}_0)$. Now, let $\Gamma \in \Gamma(X)$ and let us show that there is some $(X, \mathrm{d}, \mathfrak{m}) \in \mathfrak{R}_{0,N}(X)$ such that $\mathcal{A}([X, \mathrm{d}, \mathfrak{m}]) = [\mathbb{R}^k/\Gamma, \mathrm{d}_\Gamma]$.

Since $\Gamma(\phi_0) \in \Gamma(X)$, there is $\alpha \in \mathrm{Aff}(\mathbb{R}^k)$ such that $\Gamma = \alpha\Gamma(\phi_0)\alpha^{-1}$. Now, let $\psi \coloneqq (\mathrm{id}_{\overline{X}_0}, \alpha) \circ \phi_0$, and consider the metric measure structure $(\tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ defined as the pull back by ψ of $(\overline{\mathrm{d}}_0 \times \mathrm{d}_{\mathrm{eucli}}, \overline{\mathfrak{m}}_0 \otimes \mathcal{L}_k)$. Note that ψ is a homeomorphism, and $(\overline{X}_0 \times \mathbb{R}^k, \overline{\mathrm{d}}_0 \times \mathrm{d}_{\mathrm{eucli}}, \overline{\mathfrak{m}}_0 \otimes \mathcal{L}_k)$ is an $\mathrm{RCD}(0, N)$ space; hence, $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ is an $\mathrm{RCD}(0, N)$ structure on $\tilde{X}$.

Now, we are going to show that $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ is the lift of some $(X, \mathrm{d}, \mathfrak{m})$. Due to Remark 2.3.3, it is equivalent to show that $\overline{\pi}_1(X) \subset \mathrm{Iso}_{\mathrm{m.m.s.}}(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$, which is itself equivalent to $\psi_*(\overline{\pi}_1(X)) \subset \mathrm{Iso}_{\mathrm{m.m.s.}}(\overline{X}_0 \times \mathbb{R}^k, \overline{\mathrm{d}}_0 \times \mathrm{d}_{\mathrm{eucli}}, \overline{\mathfrak{m}}_0 \otimes \mathcal{L}_k)$. Let $\eta \in \overline{\pi}_1(X)$, then:

$$\psi_*(\eta) = \psi \circ \eta \circ \psi^{-1} = (\mathrm{id}_{\overline{X}_0}, \alpha) \circ \phi_{0_*}(\eta) \circ (\mathrm{id}_{\overline{X}_0}, \alpha^{-1}) = (\rho_S^{\phi_0}(\eta), \alpha \circ \rho_{\mathbb{R}}^{\phi_0}(\eta) \circ \alpha^{-1}).$$

Note that $\rho_S^{\phi_0}(\eta) \in \mathrm{Iso}_{\mathrm{m.m.s.}}(\overline{X}_0, \overline{\mathrm{d}}_0, \overline{\mathfrak{m}}_0)$ and $\alpha \circ \rho_{\mathbb{R}}^{\phi_0}(\eta) \circ \alpha^{-1} \in \alpha\Gamma(\phi_0)\alpha^{-1} = \Gamma \subset \mathrm{Iso}(\mathbb{R}^k)$; hence, $\psi_*(\eta) \in \mathrm{Iso}_{\mathrm{m.m.s.}}(\overline{X}_0 \times \mathbb{R}^k, \overline{\mathrm{d}}_0 \times \mathrm{d}_{\mathrm{eucli}}, \overline{\mathfrak{m}}_0 \otimes \mathcal{L}_k)$. In conclusion, there is an $\mathrm{RCD}(0,N)$ structure $(X, \mathrm{d}, \mathfrak{m}) \in \mathfrak{R}_{0,N}(X)$ whose lift is $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$. By construction, ψ is a splitting of $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ with soul $(\overline{X}_0, \overline{\mathrm{d}}_0, \overline{\mathfrak{m}}_0)$. Moreover, we have seen above that, for every $\eta \in \overline{\pi}_1(X)$, we have $\rho_{\mathbb{R}}^{\psi}(\eta) = p_{\mathbb{R}*}^{\psi} \circ \psi_*(\eta) = \alpha \circ \rho_{\mathbb{R}}^{\phi_0}(\eta) \circ \alpha^{-1}$. Hence, $\Gamma(\psi) = \alpha\Gamma(\phi_0)\alpha^{-1} = \Gamma$, and we get $\mathcal{A}([X, \mathrm{d}, \mathfrak{m}]) = [\mathbb{R}^k/\Gamma, \mathrm{d}_\Gamma]$. □

3.2.2 Continuity statement

In this section, we prove Theorem 1.3.2 (introduced in Chapter 1). This result is one of the main contributions of [MN22], written in collaboration with Andrea Mondino. For the reader's convenience, we restate the theorem below.

Theorem 3.2.1. Assume that $\mathcal{X}_n \to \mathcal{X}_\infty$ in $\mathfrak{M}_{0,N}(X)$ in the mGH topology, then $\mathcal{A}(\mathcal{X}_n) \to \mathcal{A}(\mathcal{X}_\infty)$ in the GH topology and $\mathcal{S}(\mathcal{X}_n) \to \mathcal{S}(\mathcal{X}_\infty)$ in the mGH topology.

Strategy of the proof Assume that $\{(X, \mathrm{d}_n, \mathfrak{m}_n)\}_{n\in\mathbb{N}}$ converges to $(X, \mathrm{d}_\infty, \mathfrak{m}_\infty)$ in the mGH topology, where, for every $n \in \mathbb{N} \cup \{\infty\}$, $(X, \mathrm{d}_n, \mathfrak{m}_n)$ is an $\mathrm{RCD}(0,N)$ structure on X with lift $(\tilde{X}, \tilde{\mathrm{d}}_n, \tilde{\mathfrak{m}}_n)$. For every $n \in \mathbb{N} \cup \{\infty\}$, we fix ϕ_n a splitting of $(\tilde{X}, \tilde{\mathrm{d}}_n, \tilde{\mathfrak{m}}_n)$ with soul $(\overline{X}_n, \overline{\mathrm{d}}_n, \overline{\mathfrak{m}}_n)$ and we denote $k := k(X)$ the splitting degree of X (see Corollary 3.1.1 for the definition of the splitting degree). In order to prove Theorem 3.2.1, we are going to prove that:

$$\{\overline{X}_n, \overline{\mathrm{d}}_n, \overline{\mathfrak{m}}_n\} \text{ converges to } (\overline{X}_\infty, \overline{\mathrm{d}}_\infty, \overline{\mathfrak{m}}_\infty) \text{ in the mGH topology} \tag{3.2}$$

and:

$$\{\mathbb{R}^k/\Gamma(\phi_n), \mathrm{d}_{\Gamma(\phi_n)}\} \text{ converges to } (\mathbb{R}^k/\Gamma(\phi_\infty), \mathrm{d}_{\Gamma(\phi_\infty)}) \text{ in the GH-topology.} \tag{3.3}$$

Observe that, since X is compact, we can find a family $\{*_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ of points in X such that $\{(X, \mathrm{d}_n, \mathfrak{m}_n, *_n)\}_{n\in\mathbb{N}}$ converges to $(X, \mathrm{d}_\infty, \mathfrak{m}_\infty, *_\infty)$ in the pmGH topology. Then, for every $n \in \mathbb{N} \cup \{\infty\}$, let us fix $\tilde{*}_n \in p^{-1}(*_n)$. Moreover, we can assume without loss of generality that, for every $n \in \mathbb{N} \cup \{\infty\}$, we have $p_{\mathbb{R}^k}(\phi_n(\tilde{*}_n)) = 0$. For every $n \in \mathbb{N} \cup \{\infty\}$, we denote:

- $\mathcal{X}_n := (X, \mathrm{d}_n, \mathfrak{m}_n, *_n)$,
- $\tilde{\mathcal{X}}_n := (\tilde{X}, \tilde{\mathrm{d}}_n, \tilde{\mathfrak{m}}_n, \tilde{*}_n) = p_{\tilde{*}_n}^*(\mathcal{X}_n)$ ($p_{\tilde{*}_n}^*$ is defined in Definition 2.4.9),
- $\overline{\mathcal{X}}_n := (\overline{X}_n, \overline{\mathrm{d}}_n, \overline{\mathfrak{m}}_n, \overline{*}_n)$, where $\overline{*}_n := p_{\overline{X}_n}(\phi_n(\tilde{*}_n))$.

As a consequence of the equivariant theorem (see Theorem 2.4.1) and since $\overline{\pi}_1(X)$ is Hopfian (see Proposition 3.1.2), $\{\tilde{\mathfrak{X}}_n\}_{n\in\mathbb{N}}$ converges to $\tilde{\mathfrak{X}}_\infty$ in the equivariant pmGH topology. Therefore, there exists a sequence $\{f_n, g_n, \epsilon_n\}_{n\in\mathbb{N}}$ (resp. $\{\tilde{f}_n, \tilde{g}_n, \phi_n, \epsilon_n\}_{n\in\mathbb{N}}$) realising the convergence of $\{\mathfrak{X}_n\}_{n\in\mathbb{N}}$ to $\mathfrak{X}_\infty$ (resp. of $\{\tilde{\mathfrak{X}}_n\}_{n\in\mathbb{N}}$ to $\tilde{\mathfrak{X}}_\infty$) in the pmGH topology (resp. in the equivariant pmGH topology), such that $p \circ \tilde{f}_n = f_n \circ p$ and $p \circ \tilde{g}_n = g_n \circ p$. Finally, for every $n \in \mathbb{N}$, we define:

- $k_n := \phi_\infty \circ \tilde{f}_n \circ \phi_n^{-1}$, $k_n^{\mathbb{R}} := p_{\mathbb{R}^k} \circ k_n(\overline{*}_n, \cdot)$, and $k_n^S := p_{\overline{X}_\infty} \circ k_n(\cdot, 0)$,
- $l_n := \phi_n \circ \tilde{g}_n \circ \phi_\infty^{-1}$, $l_n^{\mathbb{R}} := p_{\mathbb{R}^k} \circ l_n(\overline{*}_\infty, \cdot)$, and $l_n^S := p_{\overline{X}_n} \circ l_n(\cdot, 0)$.

The main difficulty of the proof will be to prove that k_n and l_n almost split. More precisely, we will show that $k_n \simeq (k_n^S, k_n^{\mathbb{R}})$ and $l_n \simeq (l_n^S, l_n^{\mathbb{R}})$ whenever n is large enough (we will give a precise meaning to $\simeq$). Then, we will deduce property (3.2) and property (3.3) from that.

First of all, we prove that $\{\mathrm{Diam}(\overline{X}, \overline{\mathrm{d}}_n)\}$ is bounded.

Proposition 3.2.2. The sequence $\{\mathrm{Diam}(\overline{X}, \overline{\mathrm{d}}_n)\}$ is bounded.

Proof. Looking for a contradiction, let us suppose that $\limsup_{n\to\infty} \mathrm{Diam}(\overline{X}_n, \overline{\mathrm{d}}_n) = \infty$. Passing to a subsequence if necessary, we can assume that $\mathrm{Diam}(\overline{X}_n, \overline{\mathrm{d}}_n) > 2^{n+1}$, for every $n \in \mathbb{N}$. Hence, there are sequences $\{\overline{x}_n\}_{n\in\mathbb{N}}$ and $\{\overline{z}_n\}_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, we have $\overline{x}_n, \overline{z}_n \in \overline{X}_n$, and $\overline{d}_n(\overline{x}_n, \overline{z}_n) = 2^{n+1}$.

For every $n \in \mathbb{N}$, let $\overline{\gamma}_n\colon [-2^n, 2^n] \to \overline{X}_n$ be a minimizing geodesic parametrized by arc length from $\overline{x}_n$ to $\overline{z}_n$, and let us denote $\tilde{\gamma}_n := (\overline{\gamma}_n, 0)$. Thanks to Proposition 2.3.6, there exists $\eta \in \overline{\pi}_1(X)$ such that $\eta\tilde{\gamma}_n(0) \in B_{\overline{X}_n\times\mathbb{R}^k}((\overline{*}_n, 0), D)$, where $D := \sup_{n\in\mathbb{N}\cup\{\infty\}}\{\mathrm{Diam}(X, \mathrm{d}_n)\} < \infty$ (D being finite because (X, d_n) converges to (X, d_∞) in the GH-topology).

Then, let us define $\tilde{\beta}_n := \eta\tilde{\gamma}_n$, and denote $\overline{\beta}_n := p_{\overline{X}_n}(\tilde{\beta}_n)$, and $v_n := p_{\mathbb{R}^k}(\tilde{\beta}_n) = \rho_{\mathbb{R}}^{\phi_n}(\eta)(0)$. Observe that the sequence $\{\tilde{\beta}_n\}_{n\in\mathbb{N}}$ consists of isometric embeddings such that $\tilde{\beta}_n(0) \in B_{\overline{X}_n\times\mathbb{R}^k}((\overline{*}_n, 0), D)$. Moreover, $\{k_n, l_n\}_{n\in\mathbb{N}}$ realises the convergence of $\{\overline{\mathfrak{X}}_n \times (\mathbb{R}^k, 0)\}_{n\in\mathbb{N}}$ to $(\overline{\mathfrak{X}}_\infty \times (\mathbb{R}^k, 0))$ in the pmGH topology. Therefore, as a consequence of Arzelà–Ascoli Theorem (see [Vil09, Proposition 27.20]), we can assume (passing to a subsequence if necessary) that $\{k_n \circ \tilde{\beta}_n\}$ converges locally uniformly to an isometric embedding $\tilde{\beta}\colon \mathbb{R} \to \overline{X}_\infty \times \mathbb{R}^k$. However, $(\overline{X}_\infty, \overline{\mathrm{d}}_\infty)$ is compact; thus, applying [SW91, Lemma 1], there exist $a, b \in \mathbb{R}^k$ and $\overline{y}_\infty \in \overline{X}_\infty$ such that, for every $t \in \mathbb{R}$, $\tilde{\beta}(t) = (\overline{y}_\infty, at + b)$, and $\|a\| = 1$.

Now, we define $\overline{y}_n := \overline{\beta}_n(0)$ and, for $u \in \mathbb{R}^k$, $\Phi_n(u) := (\overline{y}_n, u) \in \overline{X}_n \times \mathbb{R}^k$. Observe that $\{\Phi_n\}_{n\in\mathbb{N}}$ is a sequence of isometric embeddings such that, for every $n \in \mathbb{N}$, we have $\Phi_n(0) \in B_{\overline{X}_n\times\mathbb{R}^k}((\overline{*}_n, 0), D)$. Therefore, thanks to Arzelà–Ascoli Theorem (see [Vil09, Proposition 27.20]), we can assume (passing to a subsequence if necessary) that $\{k_n \circ \Phi_n\}_{n\in\mathbb{N}}$ converges locally uniformly to an isometric embedding $\Phi\colon \mathbb{R}^k \to \overline{X}_\infty \times \mathbb{R}^k$. Moreover, since $\overline{X}_\infty$ is compact, we can easily deduce from [SW91, Lemma 1] that there exist $\phi \in \mathrm{Iso}(\mathbb{R}^k)$ and $\overline{z}_\infty \in \overline{X}_\infty$ such that $\Phi(t) = (\overline{z}_\infty, \phi(t))$, for every $t \in \mathbb{R}^k$.

Notice that $\tilde{\beta}_n(0) = (\overline{y}_n, v_n) = \Phi_n(v_n)$. Moreover, observe that $|v_n| \leq D$; hence, passing to a subsequence if necessary, we can assume that $v_n \to v \in \mathbb{R}^k$. Thus, we have $\mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}(\Phi(v), \tilde{\beta}(0)) = \lim_{n\to\infty} \mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}(k_n \circ \Phi_n(v_n), k_n \circ \tilde{\beta}_n(0)) = 0$. In particular, $\overline{y}_\infty = \overline{z}_\infty$, and $\phi(v) = b$. Now, let $c \in \mathbb{R}^k$ such that $[\phi - \phi(0)](c) = a$; thus, we have $\Phi(ct + v) = at + b = \tilde{\beta}(t)$, for every $t \in \mathbb{R}$. Now, observe that we have $0 = \mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}(\Phi(c + v), \tilde{\beta}(1))$. Therefore:

$$\mathrm{d}_{\overline{X}_n\times\mathbb{R}^k}(\Phi_n(c + v_n), \tilde{\beta}_n(1)) = (1 + \|c\|^2)^{1/2} \to 0.$$

Therefore, we have $0 = (1 + \|c\|^2)^{1/2} > 0$, which is the contradiction we were looking for. ☐

As a result of Proposition 3.2.2, we can introduce the following notations.

Notation 3.2.1. We denote $D := \sup_{n\in\mathbb{N}\cup\infty}\{\mathrm{Diam}(X, \mathrm{d}_n)\} < \infty$ (finiteness being granted by the convergence of $\{X, \mathrm{d}_n\}_{n\in\mathbb{N}}$ to (X, d_∞) in the GH-topology). We also denote $\overline{D} := \sup_{n\in\mathbb{N}\cup\{\infty\}}\{\mathrm{Diam}(\overline{X}_n, \overline{\mathrm{d}}_n)\} < \infty$.

Our first goal will be to obtain a convergence result on the following “splitting quantities”.

Notation 3.2.2 (Splitting quantities)**.** Given $n \in \mathbb{N}$ and $R > 0$, we define:

(i) $\alpha(n, R) := \sup\{\mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}(k_n(\overline{y}_n, t), (k_n^S(\overline{y}_n), k_n^{\mathbb{R}}(t)))\}$, the supremum being taken over $\overline{y}_n \in \overline{X}_n$ and $|t| \leq R$,

(ii) $\beta(n, R) := \sup\{\mathrm{d}_{\overline{X}_n\times\mathbb{R}^k}(l_n(\overline{y}_\infty, t), (l_n^S(\overline{y}_\infty), l_n^{\mathbb{R}}(t)))\}$, the supremum being taken over $\overline{y}_\infty \in \overline{X}_\infty$ and $|t| \leq R$.

The next two technical lemmas will be our main ingredients in proving the convergence result of the splitting quantities.

Lemma 3.2.2. Let $\{\overline{y}_n\}_{n\in\mathbb{N}}$ be a sequence such that, for every $n \in \mathbb{N}$, $\overline{y}_n \in \overline{X}_n$. For every $n \in \mathbb{N}$, and $t \in \mathbb{R}^k$, we define $\Phi_n\colon t \in \mathbb{R}^k \to (\overline{y}_n, t) \in \overline{X}_n \times \mathbb{R}^k$. Then, the sequence of maps $\{k_n \circ \Phi_n\colon \mathbb{R}^k \to \overline{X}_\infty \times \mathbb{R}^k\}_{n\in\mathbb{N}}$ admits a subsequence converging locally uniformly to a map $\Phi\colon \mathbb{R}^k \to \overline{X}_\infty \times \mathbb{R}^k$. Moreover, for any such limit Φ, there exists $\overline{y}_\infty$, and $\phi \in \mathrm{O}_k(\mathbb{R})$ such that $\forall t \in \mathbb{R}^k, \Phi(t) = (\overline{y}_\infty, \phi(t))$.

Proof. Observe that, for every $n \in \mathbb{N}$, Φ_n is an isometric embedding that satisfies

$$\Phi_n(0) \in B_{\overline{X}_n\times\mathbb{R}^k}((\overline{*}_n, 0), \overline{D})\,.$$

Therefore, applying Arzelà–Ascoli Theorem (see [Vil09, Proposition 27.20]) as in the proof of Proposition 3.2.2, we can assume without loss of generality that $\{k_n\circ\Phi_n\}_{n\in\mathbb{N}}$ converges locally uniformly to an isometric embedding $\Phi\colon \mathbb{R}^k \to \overline{X}_\infty\times\mathbb{R}^k$. Moreover, using [SW91, Lemma 1], there exist $\phi \in \mathrm{Iso}(\mathbb{R}^k)$ and $\overline{y}_\infty \in \overline{X}_\infty$ such that $\Phi(t) = (\overline{y}_\infty, \phi(t))$, for every $t \in \mathbb{R}^k$. To conclude, we must show that $\phi(0) = 0$. First, observe that:

$$\overline{\mathrm{d}}_\infty(\overline{*}_\infty, \overline{y}_\infty) \le \mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}((\overline{*}_\infty, 0), \Phi(0)) \le \overline{\mathrm{d}}_n(\overline{*}_n, \overline{y}_n) + u_n,$$

whenever n is large enough (so that $\overline{D} \le \epsilon_n^{-1}$), and where $u_n = \epsilon_n + \mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}(\Phi(0), k_n \circ \Phi_n(0)) \to 0$. Now, let $t \in \mathbb{R}^k$ such that $\phi(t) = 0$, and observe that:

$$\begin{aligned}\overline{\mathrm{d}}_n(\overline{*}_n, \overline{y}_n) &\le \mathrm{d}_{\overline{X}_n\times\mathbb{R}^k}((\overline{*}_n, 0), (\overline{y}_n, t))\\ &\le \mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}((\overline{*}_\infty, 0), \Phi(t)) + v_n = \overline{\mathrm{d}}_\infty(\overline{*}_\infty, \overline{y}_\infty) + v_n,\end{aligned}$$

when n is large enough (so that $(\overline{D}^2 + |t|^2)^{1/2} \le \epsilon_n^{-1}$), and where $v_n \coloneqq \epsilon_n + \mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}(k_n \circ \Phi_n(t), \Phi(t)) \to 0$. Hence, by combining the two inequalities above, we obtain the following:

$$|\mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}((\overline{*}_\infty, 0), \Phi(0)) - \overline{\mathrm{d}}_\infty(\overline{*}_\infty, \overline{y}_\infty)| \le u_n + v_n \to 0.$$

In particular, $\overline{\mathrm{d}}^2_\infty(\overline{*}_\infty, \overline{y}_\infty) = \overline{\mathrm{d}}^2_\infty(\overline{*}_\infty, \overline{y}_\infty) + |\phi(0)|^2$. In conclusion, $\phi(0) = 0$. □

Lemma 3.2.3. Let $\{\overline{y}_n\}_{n\in\mathbb{N}}$ and $\{\overline{z}_n\}_{n\in\mathbb{N}}$ be sequences such that, for every $n \in \mathbb{N}$, $\overline{y}_n, \overline{z}_n \in \overline{X}_n$. For every $n \in \mathbb{N}$ and $t \in \mathbb{R}^k$, we define $\Phi_n(t) \coloneqq (\overline{y}_n, t)$ and $\Psi_n(t) \coloneqq (\overline{z}_n, t)$. Assume that (passing to a subsequence if necessary) the sequences of maps $\{k_n \circ \Phi_n\}_{n\in\mathbb{N}}$ and $\{k_n \circ \Psi_n\}_{n\in\mathbb{N}}$ converge locally uniformly, respectively to $\Phi = (\overline{y}_\infty, \phi)$ and $\Psi = (\overline{z}_\infty, \psi)$, where $\overline{y}_\infty, \overline{z}_\infty \in \overline{X}_\infty$ and $\phi, \psi \in \mathrm{O}_k(\mathbb{R})$. Then, we necessarily have $\phi = \psi$.

Proof. Looking for a contradiction, suppose that $\phi \neq \psi$. In that case, there exists $t \in \mathbb{R}^k \backslash \{0\}$ such that $\phi(t) \neq \psi(t)$, which implies $\lim_{s\to\infty} \mathrm{d}_{\overline{X}_\infty \times \mathbb{R}^k}(\Phi(st), \Psi(st)) = \infty$. In particular, there exists $s \in \mathbb{R}$ such that $\mathrm{d}_{\overline{X}_\infty \times \mathbb{R}^k}(\Phi(st), \Psi(st)) \geq \overline{D} + 1$. However, using $\overline{\mathrm{d}}_n(\overline{y}_n, \overline{z}_n) = \mathrm{d}_{\overline{X}_n \times \mathbb{R}^k}(\Phi_n(st), \Psi_n(st))$, we have:

$$\mathrm{d}_{\overline{X}_\infty \times \mathbb{R}^k}(\Phi(st), \Psi(st)) \leq \overline{\mathrm{d}}_n(\overline{y}_n, \overline{z}_n) + u_n \leq \overline{D} + u_n, \tag{3.4}$$

where $u_n = \epsilon_n + \mathrm{d}_{\overline{X}_\infty \times \mathbb{R}^k}(\Phi(st), k_n \circ \Phi_n(st)) + \mathrm{d}_{\overline{X}_\infty \times \mathbb{R}^k}(\Psi(st), k_n \circ \Psi_n(st))$, and when n is large enough (so that $\overline{D} \leq \epsilon_n^{-1}$). Now, observe that $\lim_{n\to\infty} u_n = 0$; therefore, passing to the limit in inequality (3.4), we have $\mathrm{d}_{\overline{X}_\infty \times \mathbb{R}^k}(\Phi(st), \Psi(st)) \leq \overline{D}$, which contradicts $\mathrm{d}_{\overline{X}_\infty \times \mathbb{R}^k}(\Phi(st), \Psi(st)) \geq \overline{D} + 1$. □

We can now state the convergence result on the splitting quantities.

Lemma 3.2.4. For every $R > 0$, we have $\lim_{n\to\infty} \alpha(n, R) = \lim_{n\to\infty} \beta(n, R) = 0$.

Proof. Part I: $\lim_{n\to\infty} \alpha(n, R) = 0$

Looking for a contradiction, we assume that $\lim_{n\to\infty} \alpha(n, R) \neq 0$. Passing to a subsequence if necessary, there exist $\epsilon > 0$, and sequences $\{\overline{y}_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ such that:

$$\epsilon \leq \mathrm{d}_{\overline{X}_\infty \times \mathbb{R}^k}(k_n(\overline{y}_n, t_n), (k_n^S(\overline{y}_n), k_n^{\mathbb{R}}(t_n))), \tag{3.5}$$

$\overline{y}_n \in \overline{X}_n$, and $|t_n| \leq R$. Moreover, since $\{t_n\}_{n\in\mathbb{N}}$ is bounded, we can assume that $t_n \to t$. Now, applying Lemma 3.2.2, and passing to a subsequence if necessary, we can assume that $\{k_n \circ \Phi_n\}_{n\in\mathbb{N}}$ converges locally uniformly to $\Phi = (\overline{y}_\infty, \phi)$, where $\Phi_n(s) := (\overline{y}_n, s)$, $\phi \in \mathrm{O}_k(\mathbb{R})$ and $\overline{y}_\infty \in \overline{X}_\infty$. In particular, we have:

$$\lim_{n\to\infty} \overline{\mathrm{d}}_\infty(p_{\overline{X}_\infty} \circ k_n(\overline{y}_n, t_n), k_n^S(\overline{y}_n)) = \overline{\mathrm{d}}_\infty(p_{\overline{X}_\infty} \circ \Phi(t), p_{\overline{X}_\infty} \circ \Phi(0)) = 0. \tag{3.6}$$

Now, applying Lemma 3.2.2 and Lemma 3.2.3, we can assume that $\{k_n \circ \Psi_n\}_{n\in\mathbb{N}}$ converges locally uniformly to $\Psi = (\overline{z}_\infty, \phi)$, where $\Psi_n(s) := (\overline{*}_n, s)$, and $\overline{z}_\infty \in \overline{X}_\infty$. Thus, we have:

$$\lim_{n\to\infty} \mathrm{d}_{\mathrm{eucli}}(p_{\mathbb{R}^k} \circ k_n(\overline{y}_n, t_n), k_n^{\mathbb{R}}(t_n)) = \mathrm{d}_{\mathrm{eucli}}(p_{\mathbb{R}^k} \circ \Phi(t), p_{\mathbb{R}^k} \circ \Psi(t)) = 0. \tag{3.7}$$

Hence, using equations (3.6) and (3.7), we have:

$$\lim_{n\to\infty} \mathrm{d}_{\overline{X}_\infty \times \mathbb{R}^k}(k_n(\overline{y}_n, t_n), (k_n^S(\overline{y}_n), k_n^{\mathbb{R}}(t_n))) = 0,$$

which contradicts inequality (3.5).

Part II: $\lim_{n\to\infty} \beta(n,R) = 0$

Looking for a contradiction, we assume that $\lim_{n\to\infty} \beta(n,R) \neq 0$. Passing to a subsequence if necessary, there exist $\epsilon > 0$, and sequences $\{\overline{y}_\infty^{(n)}\}_{n\in\mathbb{N}} \in \overline{X}_\infty^{\mathbb{N}}$ and $|t_n| \leq R$ such that:

$$\epsilon \leq \mathrm{d}_{\overline{X}_n\times\mathbb{R}^k}(l_n(\overline{y}_\infty^{(n)}, t_n), (l_n^S(\overline{y}_\infty^{(n)}), l_n^{\mathbb{R}}(t_n))). \tag{3.8}$$

Observe that $\{t_n\}_{n\in\mathbb{N}}$ is bounded and $\overline{X}_\infty$ is compact; therefore, we can assume that $t_n \to t$ and $\overline{y}_\infty^{(n)} \to \overline{y}_\infty$.

Now, let us define $(\overline{y}_n, s_n) := l_n(\overline{y}_\infty^{(n)}, t_n)$, and $\Phi_n(u) := (\overline{y}_n, u)$, $u \in \mathbb{R}^k$. Applying Lemma 3.2.2, we can assume (passing to a subsequence if necessary) that $\{k_n \circ \Phi_n\}_{n\in\mathbb{N}}$ converges locally uniformly to $(\overline{z}_\infty, \phi)$, where $\overline{z}_\infty \in \overline{X}_\infty$ and $\phi \in \mathrm{O}_k(\mathbb{R})$. Observe that $\{s_n\}_{n\in\mathbb{N}}$ is bounded since $\{(\overline{y}_\infty^{(n)}, t_n)\}_{n\in\mathbb{N}}$ converges. Hence, we can assume that $s_n \to s$. However, we have:

$$\lim_{n\to\infty} k_n \circ \Phi_n(s_n) = \Phi(s) = \lim_{n\to\infty} k_n \circ l_n(\overline{y}_\infty^{(n)}, t_n) = (\overline{y}_\infty, t).$$

Therefore, $\overline{z}_\infty = \overline{y}_\infty$, and $\phi(s) = t$. Now, observe that:

$$\begin{aligned}\overline{\mathrm{d}}_n(p_{\overline{X}_n} \circ l_n(\overline{y}_\infty^{(n)}, t_n), l_n^S(\overline{y}_\infty^{(n)})) \leq &\mathrm{d}_{\overline{X}_n\times\mathbb{R}^k}(\Phi_n(0), l_n(\Phi(0))) \\ &+ \mathrm{d}_{\overline{X}_n\times\mathbb{R}^k}(l_n(\overline{y}_\infty, 0), l_n(\overline{y}_\infty^{(n)}, 0)).\end{aligned}$$

Thus:

$$\lim_{n\to\infty} \overline{\mathrm{d}}_n(p_{\overline{X}_n} \circ l_n(\overline{y}_\infty^{(n)}, t_n), l_n^S(\overline{y}_\infty^{(n)})) = 0. \tag{3.9}$$

Then, applying Lemma 3.2.2 and Lemma 3.2.3, we can assume that $\{k_n \circ \Psi_n\}_{n\in\mathbb{N}}$ converges locally uniformly to $\Psi = (\overline{z}'_\infty, \phi)$, where $\Psi_n(u) := (\overline{*}_n, u)$, and $\overline{z}'_\infty \in \overline{X}_\infty$. Moreover, $\Psi(0) = \lim_{n\to\infty} k_n \circ \Psi_n(0) = (\overline{*}_\infty, 0)$; therefore, $\overline{z}'_\infty = \overline{*}_\infty$. Now, using $p_{\mathbb{R}^k} \circ l_n(\overline{y}_\infty^{(n)}, t_n) = s_n$, and $l_n^{\mathbb{R}}(t_n) = p_{\mathbb{R}^k} \circ l_n(\Psi(\phi^{-1}(t_n)))$, observe that:

$$\begin{aligned}\mathrm{d}_{\mathrm{eucli}}(p_{\mathbb{R}^k} \circ l_n(\overline{y}_\infty^{(n)}, t_n), l_n^{\mathbb{R}}(t_n)) \leq &\mathrm{d}_{\overline{X}_n\times\mathbb{R}^k}(\Psi_n(\phi^{-1}(t_n)), l_n(\Psi(\phi^{-1}(t_n)))) \\ &+ \mathrm{d}_{\mathrm{eucli}}(s_n, \phi^{-1}(t_n)).\end{aligned}$$

Thus, using $\lim_{n\to\infty} s_n = s = \phi^{-1}(t) = \lim_{n\to\infty} \phi^{-1}(t_n)$, we obtain:

$$\lim_{n\to\infty} \mathrm{d}_{\mathrm{eucli}}(p_{\mathbb{R}^k} \circ l_n(\overline{y}_\infty^{(n)}, t_n), l_n^{\mathbb{R}}(t_n)) = 0. \tag{3.10}$$

Finally, observe that equations (3.9) and (3.10) contradict inequality (3.8), which concludes the proof. □

The continuity of the soul map is a consequence of the following proposition, which gives us property (3.2) as a corollary.

Proposition 3.2.3. The sequence $\{k_n^S, l_n^S\}_{n\in\mathbb{N}}$ (resp. $\{k_n^{\mathbb{R}}, l_n^{\mathbb{R}}\}_{n\in\mathbb{N}}$) realises the convergence of $\{\overline{X}_n, \overline{\mathrm{d}}_n, \overline{\mathfrak{m}}_n\}_{n\in\mathbb{N}}$ (resp. $\{\mathbb{R}^k, |\cdot|, \mathcal{L}_k, 0\}_{n\in\mathbb{N}}$) to $(\overline{X}_\infty, \overline{\mathrm{d}}_\infty, \overline{\mathfrak{m}}_\infty)$ (resp. $(\mathbb{R}^k, |\cdot|, \mathcal{L}_k, 0)$) in the mGH topology (resp. pmGH topology).

Proof. Part I: $\{k_n^{\mathbb{R}}, l_n^{\mathbb{R}}\}$ realises the convergence of $\{\mathbb{R}^k, |\cdot|, \mathcal{L}_k, 0\}$ to $(\mathbb{R}^k, |\cdot|, \mathcal{L}_k, 0)$

We are going to show that there exists a map $\epsilon^{\mathbb{R}} \colon \mathbb{N} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for every $R > 0$:

(i) $\lim_{n\to\infty} \epsilon^{\mathbb{R}}(n, R) = 0$,

(ii) for every $|t| \leq R$, we have $\mathrm{d}_{\mathrm{eucli}}(k_n^{\mathbb{R}} \circ l_n^{\mathbb{R}}(t), t) \leq \epsilon^{\mathbb{R}}(n, R)$, and $\mathrm{d}_{\mathrm{eucli}}(l_n^{\mathbb{R}} \circ k_n^{\mathbb{R}}(t), t) \leq \epsilon^{\mathbb{R}}(n, R)$ (when n is large enough),

(iii) $\max\{\mathrm{Dis}(k^{\mathbb{R}}_{n\,|B_{\mathbb{R}^k}(0,R)}), \mathrm{Dis}(l^{\mathbb{R}}_{n\,|B_{\mathbb{R}^k}(0,R)})\} \leq \epsilon^{\mathbb{R}}(n, R)$ (when n is large enough).

Then we will prove that $\{k^{\mathbb{R}}_{n*}\mathcal{L}_k\}_{n\in\mathbb{N}}$ converges to $\mathcal{L}_k$ for the weak-$*$ topology.

Let $R > 0$ and let $t \in \mathbb{R}^k$ such that $|t| \leq R$. Observe that:

$$\mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}(k_n(\overline{*}_n, t), (\overline{*}_\infty, k_n^{\mathbb{R}}(t))) \leq \alpha(n, R).$$

Hence, thanks to Lemma 3.2.4, we get:

$$\mathrm{d}_{\overline{X}_n\times\mathbb{R}^k}(l_n \circ k_n(\overline{*}_n, t), l_n(\overline{*}_\infty, k_n^{\mathbb{R}}(t))) \leq \alpha(n, R) + \epsilon_n \leq R,$$

when n is large enough. In addition, we have $\mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}((\overline{*}_\infty, 0), k_n(\overline{*}_n, t)) \leq R + \epsilon_n$, when n is large enough. In particular, this implies $|k_n^{\mathbb{R}}(t)| \leq R + \alpha(n, R) + \epsilon_n \leq 2R$. Therefore, we get $\mathrm{d}_{\overline{X}_n\times\mathbb{R}^k}(l_n(\overline{*}_\infty, k_n^{\mathbb{R}}(t)), (\overline{*}_n, l_n^{\mathbb{R}} \circ k_n^{\mathbb{R}}(t))) \leq \beta(n, 2R)$. In conclusion, when n is large enough, we obtain $\mathrm{d}_{\mathrm{eucli}}(t, l_n^{\mathbb{R}} \circ k_n^{\mathbb{R}}(t)) \leq 2\epsilon_n + \alpha(n, R) + \beta(n, 2R)$. The same strategy leads to $\mathrm{d}_{\mathrm{eucli}}(t, k_n^{\mathbb{R}} \circ l_n^{\mathbb{R}}(t)) \leq 2\epsilon_n + \beta(n, R) + \alpha(n, 2R)$, for n large enough. Therefore, we obtain point (ii) if we set $\epsilon^{\mathbb{R}}(n, R) \coloneqq 2(\epsilon_n + \alpha(n, 2R) + \beta(n, 2R))$. Moreover, Lemma 3.2.4 implies $\lim_{n\to\infty} \epsilon^{\mathbb{R}}(n, R) = 0$, for every $R > 0$.

Now, given $t_1, t_2 \in \mathbb{R}^k$ such that $|t_1| \leq R$ and $|t_2| \leq R$, we define:

$$\begin{aligned} A \coloneqq{} & ||\mathrm{d}_{\mathrm{eucli}}(k_n^{\mathbb{R}}(t_1), k_n^{\mathbb{R}}(t_2)) - \mathrm{d}_{\mathrm{eucli}}(t_1, t_2)| \\ & - |\mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}(k_n(\overline{*}_n, t_1), k_n(\overline{*}_n, t_2)) - \mathrm{d}_{\mathrm{eucli}}(t_1, t_2)|| \end{aligned}$$

Using $k_n^S(\overline{*}_n) = \overline{*}_\infty$, we get:

$$\begin{aligned}A &\leq |\mathrm{d}_{\mathrm{eucli}}(k_n^{\mathbb{R}}(t_1), k_n^{\mathbb{R}}(t_2)) - \mathrm{d}_{\overline{X}_\infty \times \mathbb{R}^k}(k_n(\overline{*}_n, t_1), k_n(\overline{*}_n, t_2))| \\ &\leq \mathrm{d}_{\overline{X}_\infty \times \mathbb{R}^k}(k_n(\overline{*}_n, t_1), (\overline{*}_\infty, k_n^{\mathbb{R}}(t_1))) + \mathrm{d}_{\overline{X}_\infty \times \mathbb{R}^k}(k_n(\overline{*}_n, t_2), (\overline{*}_\infty, k_n^{\mathbb{R}}(t_2))) \\ &\leq 2\alpha(n, R).\end{aligned}$$

Hence, for n large enough, we have:

$$\begin{aligned}|\mathrm{d}_{\mathrm{eucli}}(k_n^{\mathbb{R}}(t_1), k_n^{\mathbb{R}}(t_2)) - \mathrm{d}_{\mathrm{eucli}}(t_1, t_2)| &\leq |\mathrm{d}_{\overline{X}_\infty \times \mathbb{R}^k}(k_n(\overline{*}_n, t_1), k_n(\overline{*}_n, t_2)) - \mathrm{d}_{\mathrm{eucli}}(t_1, t_2)| \\ &\quad + A \\ &\leq 2\alpha(n, R) + \epsilon_n.\end{aligned}$$

In particular, this implies $\mathrm{Dis}(k^{\mathbb{R}}_{n\,|B_{\mathbb{R}^k}(0,R)}) \leq 2\alpha(n, R) + \epsilon_n$. Moreover, k_n and l_n playing symmetric roles, we also have $\mathrm{Dis}(l^{\mathbb{R}}_{n\,|B_{\mathbb{R}^k}(0,R)}) \leq 2\beta(n, R) + \epsilon_n$. We replace $\epsilon^{\mathbb{R}}(n, R)$ by $\epsilon^{\mathbb{R}}(n, R) + 2\alpha(n, R) + 2\beta(n, R) + \epsilon_n$. This concludes the proof of (i), (ii), and (iii) (thanks to Lemma 3.2.4).

Let us prove that $\{k^{\mathbb{R}}_{n_*}\mathcal{L}_k\}_{n\in\mathbb{N}}$ converges to $\mathcal{L}_k$ in the weak-$*$ topology. Here, the strategy will be the same as in the proof of Theorem 1.3.1. More precisely, the weak-$*$ topology on $\mathcal{M}_{\mathrm{loc}}(\mathbb{R}^k)$ is metrizable; therefore, it is equivalent to prove that every subsequence of $\{k^{\mathbb{R}}_{n_*}\mathcal{L}_k\}_{n\in\mathbb{N}}$ admits a subsequence converging to $\mathcal{L}_k$ in the weak-$*$ topology. Let us just prove that $\{k^{\mathbb{R}}_{n_*}\mathcal{L}_k\}_{n\in\mathbb{N}}$ admits a subsequence converging to $\mathcal{L}_k$ in the weak-$*$ topology (the proof for a subsequence being the same). First of all, as a result of Lemma 3.2.2, we can assume (passing to a subsequence if necessary) that $\{t \in \mathbb{R}^k \to k_n(\overline{*}_n, t) \in \overline{X}_\infty \times \mathbb{R}^k\}_{n\in\mathbb{N}}$ converges locally uniformly to $(\overline{y}_\infty, \phi)$ for some $\overline{y}_\infty \in \overline{X}_\infty$ and $\phi \in \mathrm{O}_k(\mathbb{R})$. In particular, $\{k_n^{\mathbb{R}}\}_{n\in\mathbb{N}}$ converges locally uniformly to ϕ. Then, notice that $\overline{y}_\infty = \lim_{n\to\infty} p_{\overline{X}_\infty} \circ k_n(\overline{*}_n, 0) = \overline{*}_\infty$. Now let $R > 0$, and let $f \in \mathcal{C}_c(\mathbb{R}^k)$ be a continuous function such that $\mathrm{Spt}(f) \subset B_{\mathbb{R}^k}(0, R)$, and let us show that $\lim_{n\to\infty} \int_{\mathbb{R}^k} f \,\mathrm{d}\, k^{\mathbb{R}}_{n_*}\mathcal{L}_k = \int_{\mathbb{R}^k} f \,\mathrm{d}\,\mathcal{L}_k$.

First, observe that if $f \circ k_n^{\mathbb{R}}(t) \neq 0$, then $|k_n^{\mathbb{R}}(t)| \leq R$. Therefore:

$$\mathrm{d}_{\mathrm{eucli}}(k_n^{\mathbb{R}}(t), k_n^{\mathbb{R}}(0)) \leq R + \mathrm{d}_{\mathrm{eucli}}(k_n^{\mathbb{R}}(0), 0).$$

In particular, we have:

$$\mathrm{d}_{\overline{X}_\infty \times \mathbb{R}^k}(k_n(\overline{*}_n, t), k_n(\overline{*}_n, 0)) \leq \tilde{R}_n\,,$$

where $\tilde{R}_n \coloneqq (\overline{D}^2 + (R + \mathrm{d}_{\mathrm{eucli}}(k_n^{\mathbb{R}}(0), 0))^2)^{1/2}$, and $\overline{D}$ is defined in Notation 3.2.1. Hence, if $f \circ k_n^{\mathbb{R}}(t) \neq 0$, then we have $|t| = \mathrm{d}_{\overline{X}_n \times \mathbb{R}^k}((\overline{*}_n, t), (\overline{*}_n, 0)) \leq 3\epsilon_n + \tilde{R}_n \leq$

$2\tilde{R} \coloneqq 2(\overline{D}^2 + R^2)^{1/2}$ (then n is sufficiently large). Hence, whenever n is large enough, we have:

$$\begin{aligned}|\int_{\mathbb{R}^k} f \,\mathrm{d}\, k^{\mathbb{R}}_{n_*}\mathcal{L}_k - \int_{\mathbb{R}^k} f \,\mathrm{d}\,\phi_*\mathcal{L}_k| &\leq \int_{B(2\tilde{R})} |f \circ k^{\mathbb{R}}_n(t) - f \circ \phi(t)| \,\mathrm{d}\,\mathcal{L}_k(t) \\ &\leq \mathcal{L}_k(B(2\tilde{R}))\omega_n,\end{aligned}$$

where $\omega_n \coloneqq \sup_{|x-y|\leq \nu_n}\{|f(x)-f(y)|\}$ and $\nu_n \coloneqq \sup_{t\in B(2\tilde{R})}\{|k^{\mathbb{R}}_n(t)-\phi(t)|\}$. Observe that $\{k^{\mathbb{R}}_n\}_{n\in\mathbb{N}}$ converges locally uniformly to ϕ; thus $\nu_n \to 0$. In particular, since f has compact support, $\omega_n \to 0$; hence $\lim_{n\to\infty}\int_{\mathbb{R}^k} f \,\mathrm{d}\, k^{\mathbb{R}}_{n_*}\mathcal{L}_k = \int_{\mathbb{R}^k} f \,\mathrm{d}\,\phi_*\mathcal{L}_k$. In conclusion, passing to a subsequence if necessary, $\{k^{\mathbb{R}}_{n_*}\mathcal{L}_k\}_{n\in\mathbb{N}}$ converges in the weak-$*$ topology to $\phi_*\mathcal{L}_k = \mathcal{L}_k$ (using $\phi \in \mathrm{O}_k(\mathbb{R})$).

Part II: $\{k^S_n, l^S_n\}$ realises the convergence of $\{\overline{X}_n, \overline{\mathrm{d}}_n, \overline{\mathfrak{m}}_n\}$ to $(\overline{X}_\infty, \overline{\mathrm{d}}_\infty, \overline{\mathfrak{m}}_\infty)$

Let $\overline{y}_n \in \overline{X}_n$, and observe that $\mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}(k_n(\overline{y}_n, 0), (k^S_n(\overline{y}_n), 0)) \leq \alpha(n, 0)$. Due to Lemma 3.2.4, we have $\mathrm{d}_{\overline{X}_n\times\mathbb{R}^k}(l_n \circ k_n(\overline{y}_n, 0), l_n(k^S_n(\overline{y}_n), 0)) \leq \alpha(n, 0) + \epsilon_n$ (when n is large enough). However, we have $\mathrm{d}_{\overline{X}_n\times\mathbb{R}^k}(l_n(k^S_n(\overline{y}_n), 0), (l^S_n \circ k^S_n(\overline{y}_n), 0)) \leq \beta(n, 0)$. In conclusion, we have:

$$\overline{\mathrm{d}}_n(\overline{y}_n, l^S_n \circ k^S_n(\overline{y}_n)) = \mathrm{d}_{\overline{X}_n\times\mathbb{R}^k}((\overline{y}_n, 0), (l^S_n \circ k^S_n(\overline{y}_n), 0)) \leq \alpha(n, 0) + \beta(n, 0) + 2\epsilon_n \eqqcolon \epsilon^S_n. \tag{3.11}$$

Since k_n and l_n play symmetric roles, we also have:

$$\overline{\mathrm{d}}_\infty(\overline{y}_\infty, k^S_n \circ l^S_n(\overline{y}_\infty)) \leq \epsilon^S_n, \tag{3.12}$$

for every $\overline{y}_\infty \in \overline{X}_\infty$. Observe that, thanks to Lemma 3.2.4, we have $\lim_{n\to\infty}\epsilon^S_n = 0$.

Now, let $y_1, y_2 \in \overline{X}_n$, and define:

$$A \coloneqq ||\overline{\mathrm{d}}_\infty(k^S_n(y_1), k^S_n(y_2)) - \overline{\mathrm{d}}_n(y_1, y_2)| - |\mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}(k_n(y_1, 0), k_n(y_2, 0)) - \overline{\mathrm{d}}_n(y_1, y_2)||.$$

Then, we have:

$$\begin{aligned}A &\leq |\overline{\mathrm{d}}_\infty(k^S_n(y_1), k^S_n(y_2)) - \mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}(k_n(y_1, 0), k_n(y_2, 0))| \\ &\leq \mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}(k_n(y_1, 0), (k^S_n(y_1), 0)) + \mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}(k_n(y_2, 0), (k^S_n(y_2), 0)) \\ &\leq 2\alpha(n, 0).\end{aligned}$$

Hence, we finally get (for n large enough):

$$\begin{aligned}|\overline{\mathrm{d}}_\infty(k^S_n(y_1), k^S_n(y_2)) - \overline{\mathrm{d}}_n(y_1, y_2)| &\leq A + |\mathrm{d}_{\overline{X}_\infty\times\mathbb{R}^k}(k_n(y_1, 0), k_n(y_2, 0)) - \overline{\mathrm{d}}_n(y_1, y_2)|| \\ &\leq 2\alpha(n, 0) + \epsilon_n.\end{aligned}$$

In particular, we have $\mathrm{Dis}(k_n^S) \leq 2\alpha(n,0) + \epsilon_n$. Then, k_n and l_n playing symmetric roles, we also have $\mathrm{Dis}(l_n^S) \leq 2\beta(n,0) + \epsilon_n$. Finally, replacing ϵ_n^S by $\epsilon_n^S + \epsilon_n + 2\max\{\alpha(n,0), \beta(n,0)\}$, and applying Lemma 3.2.4, we have $\epsilon_n^S \to 0$; therefore, using inequalities (3.11) and (3.12), we can conclude that $\{\overline{X}_n, \overline{\mathrm{d}}_n\}_{n\in\mathbb{N}}$ converges to $(\overline{X}_\infty, \overline{\mathrm{d}}_\infty)$ in the GH-topology.

Now let us prove that $\{k^S_{n*}\overline{\mathfrak{m}}_n\}_{n\in\mathbb{N}}$ converges to $\overline{\mathfrak{m}}_\infty$ in the weak-$*$ topology. As we saw in Part I, it is sufficient to show that $\{k^S_{n*}\overline{\mathfrak{m}}_n\}_{n\in\mathbb{N}}$ admits a subsequence converging to $\overline{\mathfrak{m}}_\infty$ in the weak-$*$ topology (the weak-$*$ topology being metrizable).

First, let us show that $\{k^S_{n*}\overline{\mathfrak{m}}_n\}_{n\in\mathbb{N}}$ is precompact in the space $\mathcal{M}(\overline{X}_\infty)$ of Radon measure on $\overline{X}_\infty$, which is implied by the uniform boundedness of the sequence $\{\overline{\mathfrak{m}}_n(\overline{X}_n)\}_{n\in\mathbb{N}}$. Let us fix $r_0 \in (0, \delta/2)$, where $\delta := \inf_{n\in\mathbb{N}\cup\{\infty\}}\{\delta(X, \mathrm{d}_n)\}$ (δ being positive as stated by Proposition 2.4.5). Then, using point (v) of Proposition 2.3.4 and Theorem 2.3.3, observe that $\overline{\mathfrak{m}}_n \otimes \mathcal{L}_k(B_{\overline{X}_n\times\mathbb{R}^k}((\overline{*}_n, 0), r_0)) = \mathfrak{m}_n(B_{\mathrm{d}_n}(*_n, r_0)) \leq M$, where $M := \sup_{n\in\mathbb{N}\cup\{\infty\}}\{\mathfrak{m}_n(X)\}$ is finite. Moreover, notice that $B_{\overline{\mathrm{d}}_n}(\overline{*}_n, r_0/\sqrt{2}) \times B_{\mathbb{R}^k}(0, r_0/\sqrt{2}) \subset B_{\overline{X}_n\times\mathbb{R}^k}((\overline{*}_n, 0), r_0)$. Hence, we have $\overline{\mathfrak{m}}_n(B_{\overline{\mathrm{d}}_n}(\overline{*}_n, r_0/\sqrt{2})) \leq (\sqrt{2}/r_0)^k M/\omega_k$, where $\omega_k := \mathcal{L}_k(B_{\mathbb{R}^k}(0,1))$. In particular, for every $n \in \mathbb{N}$, we can apply Bishop–Gromov inequality (see Proposition 2.2.5), and get:

$$\overline{\mathfrak{m}}_n(\overline{X}_n) \leq (\sqrt{2}/r_0)^N \overline{D}^{N-k} M/\omega_k =: \overline{M},$$

where $\overline{D}$ is defined in Noation 3.2.1. In conclusion, $\{\overline{\mathfrak{m}}_n(\overline{X}_n)\}_{n\in\mathbb{N}}$ is uniformly bounded; thus $\{k^S_{n*}\overline{\mathfrak{m}}_n\}_{n\in\mathbb{N}}$ is precompact in the weak-$*$ topology.

Now, passing to a subsequence if necessary, we can assume that $\{k^S_{n*}\overline{\mathfrak{m}}_n\}_{n\in\mathbb{N}}$ converges to some Radon measure $\overline{\mathfrak{m}}$ on $\overline{X}_\infty$. We need to prove that $\overline{\mathfrak{m}} = \overline{\mathfrak{m}}_\infty$. Observe that it is equivalent to prove that $\overline{\mathfrak{m}} \otimes \mathcal{L}_k = \overline{\mathfrak{m}}_\infty \otimes \mathcal{L}_k$. First, thanks to the first part of the proof, $\{k^{\mathbb{R}}_{n*}\mathcal{L}_k\}_{n\in\mathbb{N}}$ converges to $\mathcal{L}_k$ in the weak-$*$ topology; hence $\{(k_n^S, k_n^{\mathbb{R}})_*[\overline{\mathfrak{m}}_n \otimes \mathcal{L}_k]\}_{n\in\mathbb{N}}$ converges to $\overline{\mathfrak{m}} \otimes \mathcal{L}_k$ in the weak-$*$ topology. In addition, as a result of Theorem 1.3.1, $\{k_{n*}[\overline{\mathfrak{m}}_n \otimes \mathcal{L}_k]\}_{n\in\mathbb{N}}$ converges to $\overline{\mathfrak{m}}_\infty \otimes \mathcal{L}_k$. Now, let $\phi \in \mathcal{C}_c(\overline{X}_\infty \times \mathbb{R}^k)$ and $R > 0$ such that $\mathrm{Spt}(\phi) \subset B_{\overline{X}_\infty\times\mathbb{R}^k}(R)$. Then, proceeding as in Part I of the proof, we obtain:

$$\mathrm{Spt}(\phi \circ (k_n^S, k_n^{\mathbb{R}})) \cup \mathrm{Spt}(\phi \circ k_n) \subset \overline{X}_\infty \times B_{\mathbb{R}^k}(0, 2\tilde{R}),$$

when n is sufficiently large and where $\tilde{R} := (\overline{D}^2 + R^2)^{1/2}$. In particular, we have:

$$\left| \int_{\overline{X}_n\times\mathbb{R}^k} \phi(k_n^S(\overline{x}), k_n^{\mathbb{R}}(t)) - \phi(k_n(\overline{x}, t)) \,\mathrm{d}\, \overline{\mathfrak{m}}_n \otimes \mathcal{L}_k(\overline{x}, t) \right| \leq (2\tilde{R})^k \omega_k \overline{M} \omega_\phi(\alpha(n, 2\tilde{R})),$$

where ω_ϕ is a modulus of uniform continuity associated to ϕ. Then, thanks to Lemma 3.2.4, we have $\lim_{n\to\infty} \omega_\phi(\alpha(n, 2\tilde{R})) = 0$. Thus, for every $\phi \in \mathcal{C}_c(\overline{X}_\infty \times \mathbb{R}^k)$, we have:

$$\lim_{n\to\infty} \int_{\overline{X}_n\times\mathbb{R}^k} \phi \,\mathrm{d}(k_n^S, k_n^{\mathbb{R}})_*[\overline{\mathfrak{m}}_n \otimes \mathcal{L}_k] = \lim_{n\to\infty} \int_{\overline{X}_n\times\mathbb{R}^k} \phi \,\mathrm{d}\, k_{n_*}[\overline{\mathfrak{m}}_n \otimes \mathcal{L}_k].$$

In particular, $\{(k_n^S, k_n^{\mathbb{R}})_*[\overline{\mathfrak{m}}_n \otimes \mathcal{L}_k]\}_{n\in\mathbb{N}}$ and $\{k_{n_*}[\overline{\mathfrak{m}}_n \otimes \mathcal{L}_k]\}_{n\in\mathbb{N}}$ have the same limit, i.e. $\overline{\mathfrak{m}} \otimes \mathcal{L}_k = \overline{\mathfrak{m}}_\infty \otimes \mathcal{L}_k$. This concludes the proof. □

Inspired by the proof of [TW22, Theorem 5.4], we introduce the following "shrunk" metrics.

Definition 3.2.3. Given $n \in \mathbb{N} \cup \{\infty\}$, and $m \in \mathbb{N}$, let $\tilde{\mathrm{d}}_{n,m} \coloneqq \phi_n^*(2^{-m}\overline{\mathrm{d}}_n \times \mathrm{d}_{\mathrm{eucli}})$, and let $(X, \mathrm{d}_{n,m}, \mathfrak{m}_n)$ be the push-forward of $(\tilde{X}, \tilde{\mathrm{d}}_{m,n}, \tilde{m}_n)$ (see Proposition 2.3.5).

Remark 3.2.2. Given $n \in \mathbb{N} \cup \{\infty\}$, $m \in \mathbb{N}$, and $\tilde{x}, \tilde{y} \in \tilde{X}$, we have $\tilde{\mathrm{d}}_{n,m}(\tilde{x}, \tilde{y}) \le \tilde{\mathrm{d}}_n(\tilde{x}, \tilde{y})$. Indeed, defining $(\overline{x}, t_x) \coloneqq \phi_n(x)$ and $(\overline{y}, t_y) \coloneqq \phi_n(y)$, we have:

$$\tilde{\mathrm{d}}_{n,m}(\tilde{x}, \tilde{y})^2 = 2^{-2m}\overline{\mathrm{d}}_n^2(\overline{x}, \overline{y}) + \mathrm{d}^2_{\mathrm{eucli}}(t_x, t_y) \le \overline{\mathrm{d}}_n^2(\overline{x}, \overline{y}) + \mathrm{d}^2_{\mathrm{eucli}}(t_x, t_y) = \tilde{\mathrm{d}}_n(\tilde{x}, \tilde{y})^2.$$

In particular, this implies $\mathrm{d}_{n,m} \le \mathrm{d}_n$.

The following lemma shows that the "shrunk" metrics associated to the sequence $\{X, \mathrm{d}_n, \mathfrak{m}_n\}_{n\in\mathbb{N}}$ are close to the corresponding Albanese varieties.

Lemma 3.2.5. We have $\mathrm{d}_{\mathrm{GH}}([X, \mathrm{d}_{n,m}], \mathcal{A}([X, \mathrm{d}_n, \mathfrak{m}_n])) \le 2^{-m+1}\overline{D}$, for every $m \in \mathbb{N}$, and $n \in \mathbb{N} \cup \{\infty\}$ (where $\overline{D}$ is defined in Notation 3.2.1).

Proof. First of all, there exists a continuous map $a\colon X \to (\mathbb{R}^k/\Gamma(\phi_n), \mathrm{d}_{\Gamma(\phi_n)})$ such that $a \circ p = q \circ p_{\mathbb{R}^k} \circ \phi_n$, where $q\colon \mathbb{R}^k \to \mathbb{R}^k/\Gamma(\phi_n)$ is the quotient map. Notice that, $q \circ p_{\mathbb{R}^k} \circ \phi_n$ is surjective; hence, a is also surjective. Now, let $y, z \in X$ and let us show that $|\mathrm{d}_{n,m}(y, z) - \mathrm{d}_{\Gamma(\phi_n)}(a(y), a(z))| \le 2^{-m}\overline{D}$.

First, thanks to Proposition 2.3.5, there exists $\tilde{y} \in p^{-1}(y)$ and $\tilde{z} \in p^{-1}(z)$ such that $\mathrm{d}_{n,m}(y, z) = \tilde{\mathrm{d}}_{m,n}(\tilde{y}, \tilde{z})$. Let $(\overline{y}, t_y) \coloneqq \phi_n(\tilde{y})$ and $(\overline{z}, t_z) \coloneqq \phi_n(\tilde{z})$. Observe that $a(y) = q(t_y)$ and $a(z) = q(t_z)$, and $\mathrm{d}^2_{n,m}(y, z) = 2^{-2m}\overline{\mathrm{d}}_n^2(\overline{y}, \overline{z}) + \mathrm{d}^2_{\mathrm{eucli}}(t_y, t_z)$. Now, note that, by definition of $\mathrm{d}_{\Gamma(\phi_n)}$, we have $\mathrm{d}_{\Gamma(\phi_n)}(a(y), a(z)) \le \mathrm{d}_{\mathrm{eucli}}(t_y, t_z)$; in particular:

$$0 \le \mathrm{d}_{n,m}(y, z) - \mathrm{d}_{\Gamma(\phi_n)}(a(y), a(z)). \tag{3.13}$$

Then, by definition of $\mathrm{d}_{\Gamma(\phi_n)}$, there exists $\eta \in \overline{\pi}_1(X)$ such that $\mathrm{d}_{\Gamma(\phi_n)}(a(y), a(z)) = \mathrm{d}_{\mathrm{eucli}}(t_y, t_2)$, where $t_2 \coloneqq \rho_{\mathbb{R}}^{\phi_n}(\eta) \cdot t_z$. Observe that:

$$\tilde{\mathrm{d}}_{n,m}(\tilde{y}, \tilde{z}) = \mathrm{d}_{n,m}(y, z) \leq \tilde{\mathrm{d}}_{n,m}(\tilde{y}, \eta \cdot \tilde{z});$$

hence:

$$\begin{aligned}
\mathrm{d}_{n,m}(y,z) - \mathrm{d}_{\Gamma(\phi_n)}(a(y), a(z)) &\leq (2^{-2m}\overline{\mathrm{d}}_n^2(\overline{y}, \rho_S^{\phi_n}(\eta) \cdot \overline{z}) + \mathrm{d}_{\mathrm{eucli}}(t_y, t_2))^{1/2} \\
&\qquad - \mathrm{d}_{\mathrm{eucli}}(t_y, t_2) \\
&\leq 2^{-m}\overline{\mathrm{d}}_n(\overline{y}, \rho_S^{\phi_n}(\eta) \cdot \overline{z}) + \mathrm{d}_{\mathrm{eucli}}(t_y, t_2) - \mathrm{d}_{\mathrm{eucli}}(t_y, t_2) \\
&\leq 2^{-m}\overline{D}.
\end{aligned}$$

In particular, we have $|\mathrm{d}_{n,m}(y,z) - \mathrm{d}_{\Gamma(\phi_n)}(a(y), a(z))| \leq 2^{-m}\overline{D}$ using inequality (3.13). Therefore, recalling that a is surjective, and using [BBI22, Corollary 7.3.28], we get $\mathrm{d}_{\mathrm{GH}}([X, \mathrm{d}_{n,m}], \mathcal{A}([X, \mathrm{d}_n, \mathfrak{m}_n])) \leq 2^{-m+1}\overline{D}$. □

To prove property (3.3), we must obtain a convergence result for the following quantities.

Notation 3.2.3. Given $R > 0$, $n, m \in \mathbb{N}$, we denote:

(i) $\epsilon(n,m,R) \coloneqq \sup\{|\tilde{\mathrm{d}}_{\infty,m}(\tilde{f}_n(\tilde{y}_1), \tilde{f}_n(\tilde{y}_2)) - \tilde{\mathrm{d}}_{n,m}(\tilde{y}_1, \tilde{y}_2)|\}$, the supremum being taken over $\tilde{y}_i \in \tilde{B}_n(R)$,

(ii) $\epsilon'(n,m,R) \coloneqq \sup\{|\tilde{\mathrm{d}}_{n,m}(\tilde{g}_n(\tilde{y}_1), \tilde{g}_n(\tilde{y}_2)) - \tilde{\mathrm{d}}_{\infty,m}(\tilde{y}_1, \tilde{y}_2)|\}$, the supremum being taken over $\tilde{y}_i \in \tilde{B}_\infty(R)$.

Lemma 3.2.6. For every $R > 0$, $\lim_{n,m\to\infty} \epsilon(n,m,R) = \lim_{n,m\to\infty} \epsilon'(n,m,R) = 0$.

Proof. Let us only prove that $\lim_{n,m\to\infty} \epsilon(n,m,R) = 0$, the proof for $\epsilon'(n,m,R)$ being exactly the same. Let $\tilde{y}_i \in \tilde{B}_n(R)$, $i \in \{1,2\}$. For $i \in \{1,2\}$, we denote $(\overline{y}_i, t_i) \coloneqq \phi_n(\tilde{y}_i)$, $\tilde{y}_i^\infty \coloneqq \tilde{f}_n(\tilde{y}_i)$, $(\overline{y}_i^\infty, t_i^\infty) \coloneqq \phi_\infty(\tilde{y}_i^\infty)$, and $A \coloneqq |\tilde{\mathrm{d}}_{\infty,m}(\tilde{f}_n(\tilde{y}_1), \tilde{f}_n(\tilde{y}_2)) - \tilde{\mathrm{d}}_{n,m}(\tilde{y}_1, \tilde{y}_2)|$. Using the fact that, for every $x, y \in \mathbb{R}_{\geq 0}$, we have $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}$, we get:

$$\begin{aligned}
A &\leq (|2^{-2m}(\overline{\mathrm{d}}_\infty^2(\overline{y}_1^\infty, , \overline{y}_2^\infty) - \overline{\mathrm{d}}_n^2(\overline{y}_1, , \overline{y}_2)) + (\mathrm{d}_{\mathrm{eucli}}^2(t_1^\infty, t_2^\infty) - \mathrm{d}_{\mathrm{eucli}}^2(t_1, t_2))|)^{1/2} \\
&\leq 2^{-m}(|(\overline{\mathrm{d}}_\infty^2(\overline{y}_1^\infty, , \overline{y}_2^\infty) - \overline{\mathrm{d}}_n^2(\overline{y}_1, , \overline{y}_2))|)^{1/2} + (|\mathrm{d}_{\mathrm{eucli}}^2(t_1^\infty, t_2^\infty) - \mathrm{d}_{\mathrm{eucli}}^2(t_1, t_2)|)^{1/2}.
\end{aligned}$$

However, note that $|(\overline{\mathrm{d}}_\infty^2(\overline{y}_1^\infty, \overline{y}_2^\infty) - \overline{\mathrm{d}}_n^2(\overline{y}_1, \overline{y}_2))| \leq 2\overline{D}^2$, where $\overline{D}$ is introduced in Notation 3.2.1. Then, observe that $\mathrm{d}_{\mathrm{eucli}}(t_1^\infty, t_2^\infty) \leq \tilde{\mathrm{d}}_\infty(\tilde{y}_1^\infty, \tilde{y}_2^\infty) \leq \tilde{\mathrm{d}}_n(\tilde{y}_1, \tilde{y}_2) + \epsilon_n \leq$

$2R+\epsilon_n$, when n is large enough. Therefore, $\mathrm{d}_{\mathrm{eucli}}(t_1^\infty,t_2^\infty)+\mathrm{d}_{\mathrm{eucli}}(t_1,t_2)\leq 4R+\epsilon_n$. Then, using $|t_i|\leq R$, and denoting $B:=|\mathrm{d}_{\mathrm{eucli}}(t_1^\infty,t_2^\infty)-\mathrm{d}_{\mathrm{eucli}}(t_1,t_2)|$, we have:

$$\begin{aligned} B &\leq |\mathrm{d}_{\mathrm{eucli}}(t_1^\infty,t_2^\infty)-\mathrm{d}_{\mathrm{eucli}}(k_n^{\mathbb{R}}(t_1),k_n^{\mathbb{R}}(t_2))|+|\mathrm{d}_{\mathrm{eucli}}(k_n^{\mathbb{R}}(t_1),k_n^{\mathbb{R}}(t_2))-\mathrm{d}_{\mathrm{eucli}}(t_1,t_2)| \\ &\leq \mathrm{Dis}(k^{\mathbb{R}}_{n|B_{\mathbb{R}^k}(0,R)})+\mathrm{d}_{\mathrm{eucli}}(k_n^{\mathbb{R}}(t_1),t_1^\infty)+\mathrm{d}_{\mathrm{eucli}}(k_n^{\mathbb{R}}(t_2),t_2^\infty) \\ &\leq \mathrm{Dis}(k^{\mathbb{R}}_{n|B_{\mathbb{R}^k}(0,R)})+2\alpha(n,R). \end{aligned}$$

In conclusion, we obtain the following:

$$A\leq 2^{-m+1/2}\overline{D}+(4R+\epsilon_n)^{1/2}(\mathrm{Dis}(k^{\mathbb{R}}_{n|B_{\mathbb{R}^k}(0,R)})+2\alpha(n,R))^{1/2}=:\tilde{\epsilon}(n,m,R).$$

Therefore, passing to the supremum as $\tilde{y}_i\in\tilde{B}_n(R)$ ($i\in\{1,2\}$), we obtain $\epsilon(n,m,R)\leq\tilde{\epsilon}(n,m,R)$. Thanks to Lemma 3.2.4 and Proposition 3.2.3, we have $\lim_{n,m\to\infty}\tilde{\epsilon}(n,m,R)=0$. Thus, $\lim_{n,m\to\infty}\epsilon(n,m,R)=0$, which concludes the proof. □

We conclude this section with the following proposition, which states the continuity of the Albanese map by proving property (3.3).

Proposition 3.2.4. The sequence $\{[\mathbb{R}^k/\Gamma(\phi_n),\mathrm{d}_{\Gamma(\phi_n)}]=\mathcal{A}(X,\mathrm{d}_n,\mathfrak{m}_n)\}$ converges in the GH-topology to $[\mathbb{R}^k/\Gamma(\phi_\infty),\mathrm{d}_{\Gamma(\phi_\infty)}]=\mathcal{A}(X,\mathrm{d}_\infty,\mathfrak{m}_\infty)$.

Proof. Lemma 3.2.5 implies that $\mathrm{d}_{\mathrm{GH}}([X,\mathrm{d}_{n,n}],\mathcal{A}(X,\mathrm{d}_n,\mathfrak{m}_n))\leq 2^{-n+1}\overline{D}$, for every $n\in\mathbb{N}\cup\{\infty\}$ (where $\mathrm{d}_{n,n}$ is defined in Definition 3.2.3 and $\overline{D}$ is introduced in Notation 3.2.1). In particular, using the triangle inequality for the Gromov–Hausdorff distance d_{GH}, we obtain:

$$\mathrm{d}_{\mathrm{GH}}(\mathcal{A}(X,\mathrm{d}_n,\mathfrak{m}_n),\mathcal{A}(X,\mathrm{d}_\infty,\mathfrak{m}_\infty))\leq 2^{-n+2}\overline{D}+\mathrm{d}_{\mathrm{GH}}([X,\mathrm{d}_{n,n}],[X,\mathrm{d}_{\infty,n}]).$$

Therefore, to conclude, it is sufficient to prove that:

$$\lim_{n,m\to\infty}\mathrm{d}_{\mathrm{GH}}([X,\mathrm{d}_{n,m}],[X,\mathrm{d}_{\infty,m}])=0,$$

which is what we are going to prove.

Let $n,m\in\mathbb{N}$, and $y_1,y_2\in X$. There exists $\tilde{y}_1\in p^{-1}(y_1)$, and $\tilde{y}_2\in p^{-1}(y_2)$, such that $\tilde{\mathrm{d}}_n(\tilde{*}_n,\tilde{y}_1)=\mathrm{d}_n(*_n,y_1)$, and $\tilde{\mathrm{d}}_{n,m}(\tilde{y}_1,\tilde{y}_2)=\mathrm{d}_{n,m}(y_1,y_2)$. Then, for $i\in\{1,2\}$, we denote $(\overline{y}_i,t_i):=\phi_n(\tilde{y}_i)$. Observe that $\tilde{\mathrm{d}}_n(\tilde{y}_1,\tilde{y}_2)=(\overline{\mathrm{d}}_n^2(\overline{y}_1,\overline{y}_2)+\mathrm{d}^2_{\mathrm{eucli}}(t_1,t_2))^{1/2}$, where $\overline{\mathrm{d}}_n(\overline{y}_1,\overline{y}_2)\leq\overline{D}$, and $\mathrm{d}_{\mathrm{eucli}}(t_1,t_2)\leq\mathrm{d}_{n,m}(y_1,y_2)\leq\mathrm{d}_n(y_1,y_2)\leq D$ (using Remark 3.2.2). Therefore, we get:

$$\tilde{\mathrm{d}}_n(\tilde{*}_n,\tilde{y}_2)\leq D+(\overline{D}^2+D^2)^{1/2}=:\tilde{D}. \tag{3.14}$$

Now, using $\mathrm{d}_{n,m}(y_1, y_2) = \tilde{\mathrm{d}}_{n,m}(\tilde{y}_1, \tilde{y}_2)$, and $\tilde{f}_n(\tilde{y}_i) \in p^{-1}(f_n(y_i))$, we have:

$$\begin{aligned} \mathrm{d}_{\infty,m}(f_n(y_1), f_n(y_2)) - \mathrm{d}_{n,m}(y_1, y_2) &\leq \tilde{\mathrm{d}}_{\infty,m}(\tilde{f}_n(\tilde{y}_1), \tilde{f}_n(\tilde{y}_2)) - \tilde{\mathrm{d}}_{n,m}(\tilde{y}_1, \tilde{y}_2) \\ &\leq \epsilon(n, m, \tilde{D}), \end{aligned}$$

where $\epsilon(n, m, R)$ is introduced in Notation 3.2.3. Since $\tilde{f}_n$ and $\tilde{g}_n$ play symmetric roles, we also have:

$$\forall y_1', y_2' \in X, \mathrm{d}_{n,m}(g_n(y_1'), g_n(y_2')) - \mathrm{d}_{\infty,m}(y_1, y_2) \leq \epsilon'(n, m, \tilde{D}),$$

where $\epsilon'(n, m, \tilde{D})$ is also introduced in Notation 3.2.3. In particular, this implies:

$$\forall y_1, y_2 \in X, \mathrm{d}_{n,m}(g_n \circ f_n(y_1), g_n \circ f_n(y_2)) - \mathrm{d}_{\infty,m}(f_n(y_1), f_n(y_2)) \leq \epsilon'(n, m, \tilde{D}).$$

Hence, defining $A \coloneqq \mathrm{d}_{n,m}(y_1, y_2) - \mathrm{d}_{\infty,m}(f_n(y_1), f_n(y_2))$, we have:

$$\begin{aligned} A &\leq \mathrm{d}_{n,m}(y_1, y_2) - \mathrm{d}_{n,m}(g_n \circ f_n(y_1), g_n \circ f_n(y_2)) + \epsilon'(n, m, \tilde{D}) \\ &\leq \mathrm{d}_{n,m}(g_n \circ f_n(y_1), y_1) + \mathrm{d}_{n,m}(g_n \circ f_n(y_2), y_2) + \epsilon'(n, m, \tilde{D}) \\ &\leq \mathrm{d}_n(g_n \circ f_n(y_1), y_1) + \mathrm{d}_n(g_n \circ f_n(y_2), y_2) + \epsilon'(n, m, \tilde{D}) \\ &\leq 2\epsilon_n + \epsilon'(n, m, \tilde{D}). \end{aligned}$$

In conclusion, we have:

$$|\mathrm{d}_{\infty,m}(f_n(y_1), f_n(y_2)) - \mathrm{d}_{n,m}(y_1, y_2)| \leq 2\epsilon_n + \epsilon(n, m, \tilde{D}) + \epsilon'(n, m, \tilde{D}). \tag{3.15}$$

Moreover, since $\mathrm{d}_{\infty,m} \leq \mathrm{d}_\infty$, and since f_n is an ϵ_n-isometry from (X, d_n) onto (X, d_∞), we have:

$$\forall x \in X, \exists y \in X, \mathrm{d}_{\infty,m}(x, f_n(y)) \leq \epsilon_n. \tag{3.16}$$

Hence, thanks to inequalities (3.15) and (3.16), f_n is a $2\epsilon_n + \epsilon(n, m, \tilde{D}) + \epsilon'(n, m, \tilde{D})$-isometry from $(X, \mathrm{d}_{n,m})$ to $(X, \mathrm{d}_{\infty,m})$. Therefore, using [BBI22, Corollary 7.3.28], we have:

$$\mathrm{d}_{\mathrm{GH}}([X, \mathrm{d}_{n,m}], [X, \mathrm{d}_{\infty,m}]) \leq 2(2\epsilon_n + \epsilon(n, m, \tilde{D}) + \epsilon'(n, m, \tilde{D})).$$

However, thanks to Lemma 3.2.6, we have $\lim_{n,m\to\infty} 2\epsilon_n + \epsilon(n, m, \tilde{D}) + \epsilon'(n, m, \tilde{D}) = 0$, which concludes the proof. □

3.3 Applications

In this last section of Chapter 3, we apply Theorem 3.2.1 to obtain some examples of compact topological spaces whose moduli spaces of $\mathrm{RCD}(0, N)$ structures have a non-trivial topology.

First of all, using Theorem 3.2.1, we are going to prove the following result.

Corollary 3.3.1. Let $N \in [1, \infty)$, let X be a compact topological space that admits an $\mathrm{RCD}(0, N)$ structure such that $\overline{\pi}_1(X) = 0$ (see Theorem 2.3.2 for the definition of $\overline{\pi}_1(X)$), and let Γ be a Bieberbach subgroup of $\mathbb{R}^k$ ($k \geq 2$). Then, the moduli space $\mathfrak{M}_{0,N+k}(X \times \mathbb{R}^k/\Gamma)$ retracts onto $\mathscr{M}_{\text{flat}}(\mathbb{R}^k/\Gamma)$.

Proof. Let us describe the crystallographic class $\Gamma(X \times \mathbb{R}^k/\Gamma)$ (introduced in Proposition 3.1.3). First, observe that, since $\overline{\pi}_1(X) = 0$, then the universal cover of $X \times \mathbb{R}^k/\Gamma$ is $X \times \mathbb{R}^k$, and the covering map is just $\mathrm{id}_X \times q$, where $q\colon \mathbb{R}^k \to \mathbb{R}^k/\Gamma$ is the usual quotient map. Now, let g be the flat Riemannian metric on $\mathbb{R}^k/\Gamma$ such that q is a local isometry, and fix an $\mathrm{RCD}(0, N)$ structure $(X, \mathrm{d}_0, \mathfrak{m}_0)$ on X. Observe that $(X, \mathrm{d}_0, \mathfrak{m}_0) \times (\mathbb{R}^k/\Gamma, \mathrm{d}_g, \mathfrak{m}_g)$ is an $\mathrm{RCD}(0, N + k)$ structure on $X \times \mathbb{R}^k/\Gamma$, where d_g and $\mathfrak{m}_g$ are respectively the Riemannian distance and measure associated to g. Moreover, the lifted $\mathrm{RCD}(0, N + k)$ structure on $X \times \mathbb{R}^k$ is equal to $(X, \mathrm{d}_0, \mathfrak{m}_0) \times (\mathbb{R}^k, \mathrm{d}_{\text{eucli}}, \mathcal{L}_k)$. In particular, the identity map $\mathrm{id}_{X\times\mathbb{R}^k}$ is a splitting of $(X, \mathrm{d}_0, \mathfrak{m}_0) \times \mathbb{R}^k$. Moreover, since $\overline{\pi}_1(X \times \mathbb{R}^k/\Gamma)$ acts trivially on X, we have $\Gamma(\mathrm{id}_{X\times\mathbb{R}^k}) = \Gamma$. Hence, the crystallographic class $\Gamma(X \times \mathbb{R}^k/\Gamma)$ equals the set of crystallographic subgroups of $\mathrm{Iso}(\mathbb{R}^k)$ isomorphic to Γ. Thanks to Remark 3.2.1, it implies that $\mathscr{M}_{\text{flat}}(A(X \times \mathbb{R}^k/\Gamma))$ is isometric to $\mathscr{M}_{\text{flat}}(\mathbb{R}^k/\Gamma)$.

Now, according to Theorem 3.2.1, the Albanese map associated to $X \times \mathbb{R}^k/\Gamma$ is continuous from $\mathfrak{M}_{0,N+k}(X \times \mathbb{R}^k/\Gamma)$ onto $\mathscr{M}_{\text{flat}}(A(X \times \mathbb{R}^k/\Gamma))$. Hence, it gives rise to a continuous surjective map ϕ from $\mathfrak{M}_{0,N+k}(X \times \mathbb{R}^k/\Gamma)$ onto $\mathscr{M}_{\text{flat}}(\mathbb{R}^k/\Gamma)$. Given, $[\mathbb{R}^k/\Gamma, \mathrm{d}] \in \mathscr{M}_{\text{flat}}(\mathbb{R}^k/\Gamma)$, we define:

$$s([\mathbb{R}^k/\Gamma, \mathrm{d}]) := [(X, \mathrm{d}_0, \mathfrak{m}_0) \times (\mathbb{R}^k/\Gamma, \mathrm{d}, \mathcal{H}_{\mathrm{d}})] \in \mathfrak{M}_{0,N+k}(X \times \mathbb{R}^k/\Gamma),$$

where $\mathcal{H}_{\mathrm{d}}$ is the Hausdorff measure associated to $(\mathbb{R}^k/\Gamma, \mathrm{d})$. Observe that s is a section of ϕ; therefore, we only have to show that s is continuous to conclude the proof.

Let us show that $s\colon \mathscr{M}_{\text{flat}}(\mathbb{R}^k/\Gamma) \to \mathfrak{M}_{0,N+k}(X \times \mathbb{R}^k/\Gamma)$ is continuous. To do so, let $\{(\mathbb{R}^k/\Gamma, \mathrm{d}_n)\}$ converge in the Gromov–Hausdorff sense to $(\mathbb{R}^k/\Gamma, \mathrm{d}_\infty)$, where, for every $n \in \mathbb{N} \cup \{\infty\}$, d_n is a flat metric on $\mathbb{R}^k/\Gamma$. Let us prove that $\{(X, \mathrm{d}_0, \mathfrak{m}_0) \times (\mathbb{R}^k/\Gamma, \mathrm{d}_n, \mathcal{H}_{\mathrm{d}_n})\}$ converges in the measured Gromov–Hausdorff sense to $(X, \mathrm{d}_0, \mathfrak{m}_0) \times (\mathbb{R}^k/\Gamma, \mathrm{d}_\infty, \mathcal{H}_{\mathrm{d}_\infty})$. Observe that it is sufficient to prove that $\{(\mathbb{R}^k/\Gamma, \mathrm{d}_n, \mathcal{H}_{\mathrm{d}_n})\}$ converges in the mGH sense to $(\mathbb{R}^k/\Gamma, \mathrm{d}_\infty, \mathcal{H}_{\mathrm{d}_\infty})$. However, since d_∞ is a flat metric on $\mathbb{R}^k/\Gamma$, the Hausdorff dimension of $(\mathbb{R}^k/\Gamma, \mathrm{d}_\infty)$ is equal to k. In particular, by [DG18, Theorem 1.2], $\{(\mathbb{R}^k/\Gamma, \mathrm{d}_n, \mathcal{H}_{\mathrm{d}_n})\}$ converges in the mGH sense to $(\mathbb{R}^k/\Gamma, \mathrm{d}_\infty, \mathcal{H}_{\mathrm{d}_\infty})$. In conclusion, s is continuous. □

Proposition 3.3.1 implies that homotopy groups of $\mathscr{M}_{\text{flat}}(\mathbb{R}^k/\Gamma)$ injects in those of $\mathfrak{M}_{0,N+k}(X \times \mathbb{R}^k/\Gamma)$. Thankfully, information on moduli spaces of flat metrics have been derived in [TW22] (in the case of the torus T^k, with $k \geq 4$ and $k \neq 8, 9, 10$) and in [Gar20] (in the case of 3 and 4-dimensional closed flat Riemannian manifolds). We can now prove Theorem 1.3.3, a singular analogue of Theorem 1.1.7 of Tuschmann and Wiemeler. We restate Theorem 1.3.3 below for the reader's convenience.

Theorem 3.3.1. Let $N \in [1, \infty)$ and let X be a compact topological space that admits an $\mathrm{RCD}(0, N)$ structure such that $\overline{\pi}_1(X) = 0$. In addition, let Y be either $\mathbb{S}^1 \times \mathbb{K}^2$ (where $\mathbb{K}^2$ is the Klein bottle) or a torus of dimension $k \geq 4$ such that $k \neq 8, 9, 10$. Then, the moduli space $\mathfrak{M}_{0,N+\dim(Y)}(X \times Y)$ has non-trivial higher rational homotopy groups.

Proof. Observe that [Gar20, Theorem 3.4.3] and [TW22, Proposition 5.5] imply that the moduli space $\mathscr{M}_{\text{flat}}(N)$ has non-trivial higher rational homotopy groups. Therefore, Proposition 3.3.1 concludes the proof. □

The following corollary can be seen as a singular analogue of Corollary 1.1.1 of Tuschmann and Wiemeler.

Corollary 3.3.2. For every $N \geq 3$ (resp. $N \geq 4$ / $N \geq 5$), there exists a compact topological space X such that $\mathfrak{M}_{0,N}(X)$ is not simply connected (resp. has non-trivial third rational homotopy group / non-trivial fifth rational homotopy group).

Proof. First of all, observe that thanks to [Gar20, Theorem 3.4.3], the moduli space of flat metrics on $X_3 := \mathbb{S}^1 \times \mathbb{K}^2$ is homotopy equivalent to a circle (where $\mathbb{K}^2$ is the Klein bottle). Then, let us define $X_4 := [0, 1] \times X_3$, and $X_N := \mathbb{S}^{N-3} \times X_3$ ($N \geq 5$). Proposition 3.3.1 implies that, for every $N \geq 3$, $\mathfrak{M}_{0,N}(X_N)$ retracts onto $\mathscr{M}_{\text{flat}}(X_3)$. In particular, for every $N \geq 3$, $\mathfrak{M}_{0,N}(X_N)$ has non trivial fundamental group.

Then, to conclude the proof, we apply the same idea, using the fact that $\pi_3(\mathscr{M}_{\text{flat}}(\mathbb{T}^4)) \otimes \mathbb{Q} \simeq \mathbb{Q}$, and $\pi_5(\mathscr{M}_{\text{flat}}(\mathbb{T}^5)) \otimes \mathbb{Q} \simeq \mathbb{Q}$ (see [TW22, Proposition 5.5]). $\square$

Figure 3.1: Statement of authorship for "A. Mondino and D. Navarro. Moduli spaces of compact RCD(0,N)-structures. Mathematische Annalen, 2022."

Statement of Authorship for joint/multi-authored papers for PGR thesis

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Student Confirmation

Student Name:	Dimitri Navarro		
Contribution to the Paper	The authors contributed in equal terms to the mathematics of the paper, mostly in discussions at the blackboard. Navarro worked out most of the details of the proofs and wrote a first draft of the paper.		
Signature		Date	20/02/2023

Supervisor Confirmation

By signing the Statement of Authorship, you are certifying that the candidate made a substantial contribution to the publication, and that the description described above is accurate.

Supervisor name and title: Professor Andrea Mondino			
Supervisor comments. I confirm that Navarro made a substantial contribution to the paper and that the description described above is accurate.			
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This completed form should be included in the thesis, at the end of the relevant chapter.

4
RCD(0,2) spaces and convex geometry

Contents

In this Chapter, we are going to completely answer Questions 1.2.2 and 1.2.3 (introduced in Section 1.2) in the particular case of RCD$(0,2)$ spaces. In Section 4.1, we will classify up to homeomorphisms the topological spaces admitting an RCD$(0,2)$ structure. In Section 4.2, we will investigate the particular cases of flat manifolds and topological surfaces. Sections 4.3 and 4.4 are the core of this chapter. There, we will show that moduli spaces of compact $\mathrm{RCD}(0,2)$ structures are contractible. The results of this chapter are original and appear in [Nav22].

4.1 Essential dimension and topological obstructions

In this section, we classify (up to homeomorphism) the compact topological spaces that admit an $\mathrm{RCD}(0,2)$ structure. This classification relies on the notion of essential dimension, which we introduce below.

Given an $\mathrm{RCD}(0,N)$ structure $(X,\mathrm{d},\mathfrak{m})$ on a topological space X, there exists a unique $k \in \mathbb{N} \cap [0,N]$ such that the k-dimensional regular set $\mathcal{R}_k$ associated to $(X,\mathrm{d},\mathfrak{m})$ has positive $\mathfrak{m}$-measure (see [BS20, Theorem 0.1], after [MN19]). This integer k is called the *dimension of* $(X,\mathrm{d},\mathfrak{m})$ and denoted:

$$\dim(X,\mathrm{d},\mathfrak{m}) := k. \tag{4.1}$$

Moreover, thanks to [KM18] (see also the independent proofs in [PMR17] and [GP21]), $\mathfrak{m}$ is absolutely continuous with respect to the k-dimensional Hausdorff measure $\mathcal{H}^k$ of (X,d). Finally, if $k = N$, then there exists $a > 0$ such that $\mathfrak{m} = a\mathcal{H}^N$ (as a result of [Hon20, Corollary 1.3]). We summarise the discussion in the following proposition.

Proposition 4.1.1. Let $N \in [1,\infty)$ and let $(X,\mathrm{d},\mathfrak{m})$ be an $\mathrm{RCD}(0,N)$ structure on a compact topological space X. Then $\mathfrak{m}$ is absolutely continuous with respect to $\mathcal{H}^k$, where $k = \dim(X,\mathrm{d},\mathfrak{m})$. Moreover, if $k = N$, then there exists $a > 0$ such that $\mathfrak{m} = a\mathcal{H}^N$.

We can now classify the compact topological spaces admitting an $\mathrm{RCD}(0,2)$ structure.

Notation 4.1.1. Let I and $\mathbb{S}^1$ denote the closed unit interval and the unit circle, respectively. We denote $\mathbb{S}^2$, $\mathbb{RP}^2$, $\mathbb{D}$, $\mathbb{M}^2$, $\mathbb{T}^2$, and $\mathbb{K}^2$ the 2-sphere, the projective plane, the closed 2-disc, the Möbius band, the 2-torus, and the Klein bottle, respectively.

Proposition 4.1.2. Assume that $(X, \mathrm{d}, \mathfrak{m})$ is an $\mathrm{RCD}(0,2)$ structure on a compact topological space X. Then we have the following case disjunction (see Corollary 3.1.1 for the definition of $k(X)$):

$\dim(X, \mathrm{d}, \mathfrak{m})$	0	1	1	2	2	2
$k(X)$	0	0	1	0	1	2
X is homeomorphic to	$\{*\}$	I	$\mathbb{S}^1$	$\mathbb{S}^2$, $\mathbb{RP}^2$ or $\mathbb{D}$	$I \times \mathbb{S}^1$ or $\mathbb{M}^2$	$\mathbb{T}^2$ or $\mathbb{K}^2$

Conversely, the topological spaces in the third row admit an $\mathrm{RCD}(0,2)$ structure.

Proof. The converse part is straightforward, so we will focus on the direct part. First, we assume that $\dim(X, \mathrm{d}, \mathfrak{m}) = 0$. According to [BS20, Theorem 1.12], there is a measurable subset $\mathcal{R}_0^* \subset \mathcal{R}_0 \subset X$ such that $\mathfrak{m}$ is concentrated on $\mathcal{R}_0^*$ and such that $\mathfrak{m}$ and $\mathcal{H}^0$ are absolutely continuous with respect to each other on $\mathcal{R}_0^*$. In particular, $\mathcal{R}_0^* \neq \varnothing$. Moreover, picking $x \in \mathcal{R}_0^*$, we have $\mathcal{H}^0(\{x\}) = 1$, hence $\mathfrak{m}(\{x\}) \neq 0$. Thus, thanks to [Vil09, Corollary 30.9], $\mathfrak{m}$ is a Dirac mass. In particular, since $\mathfrak{m}$ has full support, X is a singleton.

If $\dim(X, \mathrm{d}, \mathfrak{m}) = 1$ then $\mathcal{R}_1 \neq \varnothing$. Therefore, [KL16, Theorem 1.1] implies that X is homeomorphic to either $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, I or $\mathbb{S}^1$. However, since X is compact, it is either homeomorphic to I (if $k(X) = 0$) or $\mathbb{S}^1$ (if $k(X) = 1$).

From now on, we assume that $\dim(X, \mathrm{d}, \mathfrak{m}) = 2$. Let $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ be the lift of $(X, \mathrm{d}, \mathfrak{m})$ and $\phi\colon (\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}) \to (\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}}) \times \mathbb{R}^k$ be a splitting of $(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ (see Definition 3.1.1). Also, let us recall that $k = k(X)$ (see Corollary 3.1.1) and that $\dim(\tilde{X}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}) = 2 = k + \dim(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}})$ (p being a local isomorphism).

If $k(X) = 2$, then $\dim(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}}) = 0$; thus $\overline{X}$ is a singleton $\{*\}$. In particular, $\rho_{\mathbb{R}}^{\phi}$ coincides with ϕ_* (see Notation 3.1.1). Hence, ϕ induces a homeomorphism $X = \tilde{X}/\overline{\pi}_1(X) \simeq \mathbb{R}^2/\Gamma(\phi)$ (where $\Gamma(\phi)$ is introduced in Notation 3.1.1). Moreover, $\overline{\pi}_1(X)$ acts freely on X; hence $\Gamma(\phi)$ acts freely on $\mathbb{R}^2$. Therefore, being a crystallographic subgroup of $\mathrm{Iso}(\mathbb{R}^2)$ (see Proposition 3.1.1), $\Gamma(\phi)$ is a Bieberbach subgroup of $\mathrm{Iso}(\mathbb{R}^2)$ (see [Cha86, Proposition 1.1]). However, there are only two Bieberbach subgroups of $\mathrm{Iso}(\mathbb{R}^2)$ (up to isomorphism), leading respectively to X being homeomorphic to $\mathbb{T}^2$ or $\mathbb{K}^2$.

If $k(X) = 1$, then $\dim(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}}) = 1$. Moreover, $(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}})$ is a compact $\mathrm{RCD}(0,1)$-space. Therefore, according to Proposition 4.1.1, there exists $a > 0$ such that $\overline{\mathfrak{m}} = a\mathcal{H}^1$. Also, $\overline{X}$ has a trivial revised fundamental group. Hence, as a result of [KL16, Theorem 1.1], there exists $r > 0$ such that $(\overline{X}, \overline{\mathrm{d}})$ is isometric to $([0,r], \mathrm{d_E})$.

Therefore, we can assume that $(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}}) = ([0,1], r\,\mathrm{d_E}, a\mathcal{H}^1)$. Now, observe that $\mathrm{Iso}([0,1], r\,\mathrm{d_E}, a\mathcal{H}^1) \simeq \mathbb{Z}/2\mathbb{Z}$, where a generator is given by $s\colon t \to 1-t$. In particular, there are two cases, either $\mathrm{Im}(\rho_S^\phi) = 0$ or $\mathrm{Im}(\rho_S^\phi) \simeq \mathbb{Z}/2\mathbb{Z}$ (where ρ_S^ϕ is introduced in Notation 3.1.1).

First, let us suppose that $\mathrm{Im}(\rho_S^\phi) = 0$. Then $\phi_*(\gamma) = (\mathrm{id}, \rho_{\mathbb{R}}^\phi(\gamma))$ for every $\gamma \in \overline{\pi}_1(X)$. Hence, X is homeomorphic to $[0,1] \times \{\mathbb{R}/\Gamma(\phi)\}$. Moreover, $\Gamma(\phi)$ acts freely on $\mathbb{R}$, i.e. $\Gamma(\phi)$ is a Bieberbach subgroup of $\mathrm{Iso}(\mathbb{R})$ (see [Cha86, Proposition 1.1]). However, $\mathbb{Z}$ is the only Bieberbach subgroup of $\mathrm{Iso}(\mathbb{R})$ (up to isomorphism), which leads to X homeomorphic to $[0,1] \times \mathbb{S}^1$.

Now, we assume that $\mathrm{Im}(\rho_S^\phi) \simeq \mathbb{Z}/2\mathbb{Z}$. In particular, for every $\gamma \in \overline{\pi}_1(X)$, we have $\rho_S^\phi(\gamma)(1/2) = 1/2$. Hence, $\Gamma(\phi)$ acts freely on $\mathbb{R}$. In particular, $\Gamma(\phi)$ is a Bieberbach subgroup of $\mathrm{Iso}(\mathbb{R})$ (see [Cha86, Proposition 1.1]), i.e. it is conjugated to $\mathbb{Z}$ by an affine transformation. Then, note that $\rho_{\mathbb{R}}^\phi$ is injective. Indeed, let us assume that $\rho_{\mathbb{R}}^\phi(\gamma) = 0$ and, looking for a contradiction, assume that $\rho_S^\phi(\gamma) = s$. In that case, we have $\phi_*(\gamma)(1/2, t) = (1/2, t)$ for any fixed $t \in \mathbb{R}$, which is not possible as $\overline{\pi}_1(X)$ acts freely on $\tilde{X}$. Therefore, $\rho_{\mathbb{R}}^\phi$ is injective and $\overline{\pi}_1(X)$ is isomorphic to $\mathbb{Z}$. In conclusion, there is a unique generator γ of $\overline{\pi}_1(X)$ such that $\phi_*(\gamma)(\overline{x}, t) = (1-\overline{x}, t+a)$ for some $a > 0$ and every $(\overline{x}, t) \in [0,1] \times \mathbb{R}$; therefore, X is homeomorphic to $\mathbb{M}^2$.

If $k(X) = 0$, then $\overline{\pi}_1(X)$ is finite. Moreover, due to Proposition 4.1.1, there exists $a > 0$ such that $\mathfrak{m} = a\mathcal{H}^2$. In particular, $(X, \mathrm{d}, \mathcal{H}^2)$ is an $\mathrm{RCD}(0,2)$-space. Thanks to [LS22, Theorem 1.1], X is a topological surface with boundary, which implies $\pi_1(X) \simeq \overline{\pi}_1(X)$. However, up to homeomorphism, there are only three compact topological surfaces with a finite fundamental group, namely $\mathbb{S}^2$, $\mathbb{RP}^2$ and $\mathbb{D}$. This concludes the proof. □

4.2 Helpful results

Our goal in Chapter 4 is to compute the homeomorphism type of moduli spaces of $\mathrm{RCD}(0,2)$ structures. Note that the topological spaces in Proposition 4.1.2 are either closed manifolds admitting flat metrics or topological surfaces with boundaries. In this section, we give helpful results to simplify the computation of the homeomorphism type of $\mathfrak{M}_{K,N}(X)$ when X is a closed flat manifold or a topological surface with boundary.

4.2.1 The case of compact flat manifolds

Using [BDP18], we are going to present a way to compute $\mathfrak{M}_{0,N}(X)$ in the case where X is homeomorphic to a compact flat manifold.

Definition 4.2.1. Let $n \geq 1$ and let Γ be a crystallographic subgroup of $\mathrm{Iso}(\mathbb{R}^n)$. We define:

(i) $H_\Gamma \coloneqq \mathfrak{r}(\Gamma) \subset \mathrm{O}_n(\mathbb{R})$ (where $\mathfrak{r}(A, v) \coloneqq A$, for $A \in \mathrm{GL}_n(\mathbb{R})$ and $v \in \mathbb{R}^n$),

(ii) $\mathcal{C}_\Gamma \coloneqq \{A \in \mathrm{GL}_n(\mathbb{R}), AH_\Gamma A^{-1} \subset \mathrm{O}_n(\mathbb{R})\}$,

(iii) $\mathcal{N}_\Gamma \coloneqq \mathfrak{r}(\mathrm{N}_{\mathrm{Aff}(\mathbb{R}^n)}(\Gamma))$.

The moduli space of flat metrics on $\mathbb{R}^n/\Gamma$ is the set $\mathscr{M}_{\mathrm{flat}}(\mathbb{R}^n/\Gamma)$ of flat Riemannian metrics on $\mathbb{R}^n/\Gamma$ quotiented by isometries, equipped with the GH topology.

Proposition 4.2.1. Let $n \geq 1$, let Γ be a Bieberbach subgroup of $\mathrm{Iso}(\mathbb{R}^n)$, and let $N \in [1, \infty)$. If $N < n$ there are no $\mathrm{RCD}(0, N)$ structures on $\mathbb{R}^n/\Gamma$. If $N \geq n$, any $\mathrm{RCD}(0, N)$ structure on $\mathbb{R}^n/\Gamma$ is also an $\mathrm{RCD}(0, n)$ structure. Moreover, there exist homeomorphisms:

$$\mathfrak{M}_{0,n}(\mathbb{R}^n/\Gamma) \simeq \mathbb{R}_{>0} \times \mathscr{M}_{\mathrm{flat}}(\mathbb{R}^n/\Gamma) \simeq \mathbb{R} \times [\mathrm{O}_n(\mathbb{R})\backslash\mathcal{C}_\Gamma]/\mathcal{N}_\Gamma,$$

where the left action of $\mathrm{O}_n(\mathbb{R})$ on $\mathcal{C}_\Gamma$ is given by multiplication on the left, and the right action of $\mathcal{N}_\Gamma$ on $[\mathrm{O}_n(\mathbb{R})\backslash\mathcal{C}_\Gamma]$ is defined by $[A]\cdot B \coloneqq [AB]$ given $[A] \in \mathrm{O}_n(\mathbb{R})\backslash\mathcal{C}_\Gamma$, and $B \in \mathcal{N}_\Gamma$.

Proof. We denote $X \coloneqq \mathbb{R}^n/\Gamma$. Let us show that, for $N < n$, there are no $\mathrm{RCD}(0, N)$ structures on X. Indeed, Γ is a Bieberbach subgroup of $\mathrm{Iso}(\mathbb{R}^n)$; thus, X is a topological manifold and $\overline{\pi}_1(X) \simeq \pi_1(\mathbb{R}^n/\Gamma) \simeq \Gamma$. Hence, as a consequence of Bieberbach's first Theorem (see [Cha86, Theorem 3.1]), we have $k(X) = n$ (see Corollary 3.1.1). If $(X, \mathrm{d}, \mathfrak{m})$ is an $\mathrm{RCD}(0, N)$ structure on X, then Corollary 3.1.1 implies that the degree of any splitting is equal to n and belong to $[0, N]$; hence, $n \leq N$.

Now, assume that $n \leq N$. Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0, N)$ structure on X, and let us prove that it is $\mathrm{RCD}(0, n)$. We denote $(\mathbb{R}^n, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}) \coloneqq p^*(X, \mathrm{d}, \mathfrak{m})$ the associated lift (where $p\colon \mathbb{R}^n \to \mathbb{R}^n/\Gamma = X$ is the quotient map) and we fix a splitting ϕ of $(\mathbb{R}^n, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}})$ with soul $(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}})$. Note that ϕ has degree n. Let us show that $\overline{X}$ is a singleton. Seeking for a contradiction, we assume that there exists $\overline{x}, \overline{y} \in \overline{X}$

such that $\overline{x} \neq \overline{y}$. Let $\overline{\gamma}\colon [0,L] \to \overline{X}$ be a minimizing geodesic from $\overline{x}$ to $\overline{y}$, which is parametrized by arclength. Observe that ϕ induces an isometric embedding $\phi^{-1}\colon \overline{\gamma}(]0,L[) \times (\mathbb{R}^n, \mathrm{d_E}) \to (\mathbb{R}^n, \tilde{\mathrm{d}})$. However, $\overline{\gamma}(]0,L[) \times \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^{n+1}$. Hence, ϕ gives rise to a continuous injective map $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$; but no such map exists (see [HPM02, Corollary 2B.7]). In conclusion, $\overline{X}$ is a singleton $\{*\}$. Now, since $\overline{\mathfrak{m}}$ has full support, there exists $a > 0$ such that $\overline{\mathfrak{m}} = a\delta_*$. Hence, $(\overline{X}, \overline{\mathrm{d}}, \overline{\mathfrak{m}}) \times \mathbb{R}^n$ is isomorphic to $(\mathbb{R}^n, \mathrm{d_E}, a\mathcal{H}^n)$. Moreover, since $\overline{X}$ is a singleton, $\rho^\phi_{\mathbb{R}}$ is injective and coincides with ϕ_* (see Notation 3.1.1). Thus, $\Gamma(\phi) \simeq \Gamma$ and ϕ induces an isomorphism $(X, \mathrm{d}, \mathfrak{m}) \simeq (\mathbb{R}^n/\Gamma(\phi), \mathrm{d}_{\Gamma(\phi)}, a\mathcal{H}^n)$. Now, observe that $\mathrm{d}_{\Gamma(\phi)}$ and $\mathcal{H}^n$ are respectively the Riemannian distance and measure associated to $\mathbb{R}^n/\Gamma(\phi)$, which is flat of dimension n. Hence, $(\mathbb{R}^n/\Gamma(\phi), \mathrm{d}_{\Gamma(\phi)}, a\mathcal{H}^n_{\mathrm{d}_{\Gamma(\phi)}})$ is an $\mathrm{RCD}(0,n)$ space and $(X, \mathrm{d}, \mathfrak{m})$ as well a fortiori.

Now, let us prove that $\mathfrak{M}_{0,n}(\mathbb{R}^n/\Gamma) \simeq \mathbb{R} \times [\mathrm{O}_n(\mathbb{R})\backslash\mathcal{C}_\Gamma]/\mathcal{N}_\Gamma$. We have shown above that if $(X, \mathrm{d}, \mathfrak{m})$ is an $\mathrm{RCD}(0,n)$ structure on X, then $[X, \mathrm{d}] \in \mathscr{M}_{\mathrm{flat}}(X)$ and there exists $a > 0$ such that $\mathfrak{m} = a\mathcal{H}^n$. In particular, the map $\Phi\colon \mathfrak{M}_{0,n}(X) \to \mathscr{M}_{\mathrm{flat}}(X) \times \mathbb{R}_{>0}$ defined by $\Phi([X, \mathrm{d}, \mathfrak{m}]) \coloneqq ([X, \mathrm{d}], \mathfrak{m}(X)/\mathcal{H}^n(X))$ is well-defined. Then, the map $\Psi\colon \mathscr{M}_{\mathrm{flat}}(X) \times \mathbb{R}_{>0} \to \mathfrak{M}_{0,n}(X)$ defined by $\Psi([X, \mathrm{d}], a) \coloneqq [X, \mathrm{d}, a\mathcal{H}^n]$ is also well-defined, and it is clear that Ψ and Φ are respectively inverse to each other.

Let us show that Φ is continuous. Assume that $[X, \mathrm{d}_k, \mathfrak{m}_k] \to [X, \mathrm{d}_\infty, \mathfrak{m}_\infty]$ in $\mathfrak{M}_{0,n}(X)$ and, for $k \in \mathbb{N} \cup \{\infty\}$, let us denote $a_k \coloneqq \mathfrak{m}_k(X)/\mathcal{H}^n(X)$. Observe that we necessarily have $[X, \mathrm{d}_k] \to [X, \mathrm{d}_\infty]$ in the GH topology. Then, notice that (X, d_∞) has Hausdorff dimension n; hence, [DG18, Theorem 1.2] implies that $[X, \mathrm{d}_k, \mathcal{H}^n] \to [X, \mathrm{d}_\infty, \mathcal{H}^n]$ in the mGH topology. Therefore, $a_k \to a_\infty$ and Φ is continuous.

Conversely, assume that $[X, \mathrm{d}_k] \to [X, \mathrm{d}_\infty]$ in $\mathscr{M}_{\mathrm{flat}}(X)$, and let $a_k \to a_\infty$ in $\mathbb{R}_{>0}$. Observe that, thanks to [DG18, Theorem 1.2], $[X, \mathrm{d}_k, \mathcal{H}^n] \to (X, \mathrm{d}_\infty, \mathcal{H}^n)$ in the mGH topology. Hence, $[X, \mathrm{d}_k, a_k\mathcal{H}^n] \to [X, \mathrm{d}_\infty, a_\infty\mathcal{H}^n]$ in the mGH topology and Ψ is continuous.

Now, we have shown that $\mathfrak{M}_{0,n}(\mathbb{R}^n/\Gamma)$ is homeomorphic to $\mathbb{R}_{>0} \times \mathscr{M}_{\mathrm{flat}}(\mathbb{R}^n/\Gamma)$. In order to conclude, notice that, as a consequence of [BDP18, Proposition 4.3], $\mathscr{M}_{\mathrm{flat}}(\mathbb{R}^n/\Gamma)$ is homeomorphic to $[\mathrm{O}_n(\mathbb{R})\backslash\mathcal{C}_\Gamma]/\mathcal{N}_\Gamma$. □

Remark 4.2.1. Given $n \geq 1$ and Γ a Bieberbach subgroup of $\mathrm{Iso}(\mathbb{R}^n)$, $\mathrm{O}_n(\mathbb{R})\backslash\mathcal{C}_\Gamma$ is homeomorphic to $\mathbb{R}^d$ for some $d \in \mathbb{N}$ (see [BDP18, Theorem B]). In particular, $\mathfrak{M}_{0,N}(\mathbb{R}^n/\Gamma)$ is connected for every $N \geq n$.

4.2.2 The case of surfaces

We are going to relate $\mathrm{RCD}(K,2)$-structures on X to Alexandrov metrics on X with $\mathrm{curv} \geq K$, which we introduce below (see Appendix A.2 for the definition of Alexandrov spaces).

Definition 4.2.2. Let X be a compact topological manifold with boundary (possibly empty) and let $K \in \mathbb{R}$. We say that d is an *Alexandrov metric on X with* $\mathrm{curv} \geq K$ if:

(i) d metrizes the topology of X,

(ii) (X,d) has curvature bounded from below by K in the Alexandrov sense.

We denote $\mathscr{R}_{\mathrm{curv}\geq K}(X)$ the set of all metric spaces (X,d), where d is an Alexandrov metric on X with $\mathrm{curv} \geq K$.

We also need to introduce the moduli space of Alexandrov metrics.

Definition 4.2.3. Let X be a compact topological manifold with boundary (possibly empty) and let $K \in \mathbb{R}$. The *moduli space $\mathscr{M}_{\mathrm{curv}\geq K}(X)$ of Alexandrov metrics on X with* $\mathrm{curv} \geq K$ is the quotient of $\mathscr{R}_{\mathrm{curv}\geq K}(X)$ by isometry equivalence, equipped with the GH topology.

Let X be a topological surface with boundary (possibly empty) that admits an $\mathrm{RCD}(K,2)$ structure and let $(X,\mathrm{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,2)$ structure on X ($K \in \mathbb{R}$). Proceeding as in the proof of Proposition 4.1.2, it is clear that if $\dim(X,\mathrm{d},\mathfrak{m}) \in \{0,1\}$, then X is homeomorphic to either a singleton, a circle, or a compact interval. Therefore, we necessarily have $\dim(X,\mathrm{d},\mathfrak{m}) = 2$. Hence, applying Proposition 4.1.1, there exists $a > 0$ such that $\mathfrak{m} = a\mathcal{H}^2$. In particular, due to [LS22, Theorem 1.1], (X,d) is an Alexandrov space with $\mathrm{curv} \geq K$.

Thus, we obtain the following result, proceeding exactly as in the last part of the proof of Proposition 4.2.1.

Lemma 4.2.1. Let X be a topological surface with boundary (possibly empty) that admits an $\mathrm{RCD}(K,2)$ structure and let $p\colon \tilde{X} \to X$ be its universal cover. Then the map:

$$[X,\mathrm{d},\mathfrak{m}] \in \mathfrak{M}_{K,2}(X) \to (\mathfrak{m}(X)/\mathcal{H}^2(X), [X,\mathrm{d}]) \in \mathbb{R}_{>0} \times \mathscr{M}_{\mathrm{curv}\geq K}(X)$$

is a homeomorphism, where $\mathscr{M}_{\mathrm{curv}\geq K}(X)$ is the moduli space of Alexandrov metrics with curv $\geq K$ on X, endowed with the GH topology. In addition, the same map induces a homeomorphism:

$$\mathfrak{M}^{\mathrm{eq}}_{K,2}(\tilde{X}) \simeq \mathbb{R}_{>0} \times \mathscr{M}^{\mathrm{eq}}_{\mathrm{curv}\geq K}(\tilde{X}),$$

where $\mathscr{M}^{\mathrm{eq}}_{\mathrm{curv}\geq K}(\tilde{X})$ is the moduli space of equivariant Alexandrov metrics with curv $\geq K$ on $\tilde{X}$, equipped with the equivariant GH topology (see Remark 2.4.3).

4.3 The 1-dimensional case

In this section, we compute the homeomorphism type of $\mathfrak{M}_{0,N}(X)$ when X is either the circle or the unit interval. In particular, we will rely on Proposition 4.2.1 (proven in the previous section) and results of [CM21].

4.3.1 The circle

Proposition 4.3.1. For every $N \geq 1$, the moduli space $\mathfrak{M}_{0,N}(\mathbb{S}^1)$ of $\mathrm{RCD}(0,N)$ structures on $\mathbb{S}^1$ is homeomorphic to $\mathbb{R}^2$; in particular, it is contractible.

Proof. Given $n \geq 1$ and $N \geq n$, Proposition 4.2.1 implies that we have a homeomorphism:

$$\mathfrak{M}_{0,N}(\mathbb{R}^n/\mathbb{Z}^n) \simeq \mathbb{R} \times [\mathrm{O}_n(\mathbb{R})\backslash \mathrm{GL}_n(\mathbb{R})]/\mathrm{GL}_n(\mathbb{Z}).$$

When $n = 1$, we have $\mathrm{O}_1(\mathbb{R}) = \mathrm{GL}_1(\mathbb{Z}) = \{\pm 1\}$. In addition, $\mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^*$ is commutative; hence, $\mathrm{GL}_1(\mathbb{Z})$ acts trivially on $\mathrm{O}_1(\mathbb{R})\backslash \mathrm{GL}_1(\mathbb{R})$. Therefore, $[\mathrm{O}_1(\mathbb{R})\backslash \mathrm{GL}_1(\mathbb{R})]/\mathrm{GL}_1(\mathbb{Z})$ is homeomorphic to $\{\pm 1\}\backslash\mathbb{R}^*$, which is homeomorphic to $\mathbb{R}_{>0}$, which is itself homeomorphic to $\mathbb{R}$. In conclusion, for every $N \geq 1$, $\mathfrak{M}_{0,N}(\mathbb{S}^1)$ is homeomorphic to $\mathbb{R}^2$. □

Remark 4.3.1. It is readily checked that:

$$\Psi\colon [\mathbb{S}^1,\mathrm{d},\mathfrak{m}] \in \mathfrak{M}_{0,N}(\mathbb{S}^1) \to (\mathrm{Diam}(\mathbb{S}^1,\mathrm{d}), \mathfrak{m}(\mathbb{S}^1)) \in \mathbb{R}^2_{>0}$$

is an explicit homeomorphism.

4.3.2 The interval

The moduli space of $\mathrm{RCD}(0,N)$ structures on I is closely related to concave functions.

Notation 4.3.1 (Space of concave functions)**.** We denote $\mathcal{C}^*$ the space of concave functions $f\colon I \to \mathbb{R}$ such that f is strictly positive on $\mathrm{int}(I)$, equipped with the topology of uniform convergence on compact subsets of $\mathrm{int}(I)$. The aforementioned topology is metrizable with the following distance:

$$\mathrm{d}_{\mathcal{C}^*}(f,g) := \sum_{k=0}^{\infty} 2^{-k} \min\{1, \mathrm{d}_k(f,g)\},$$

where $f,g \in \mathcal{C}^*$, and $\mathrm{d}_k(f,g) := \sup_{t\in[2^{-k},1-2^{-k}]}\{|f(t)-g(t)|\}$. For every $f \in \mathcal{C}^*$, we define $-1 \cdot f(t) := f(1-t)$, which gives rise to an action of $\{\pm 1\}$ on $\mathcal{C}^*$. We denote $\mathcal{C}^*/\{\pm1\}$ the quotient of $\mathcal{C}^*$ by the action of $\{\pm1\}$, endowed with the quotient topology.

Remark 4.3.2. Observe that $\{\pm1\}$ acts by isometries on $(\mathcal{C}^*, \mathrm{d}_{\mathcal{C}^*})$. Therefore, the distance $\mathrm{d}_{\mathcal{C}^*/\{\pm1\}}([f],[g]) := \min\{\mathrm{d}_{\mathcal{C}^*}(f,g), \mathrm{d}_{\mathcal{C}^*}(f,-1\cdot g)\}$ metrizes the topology of $\mathcal{C}^*/\{\pm1\}$.

Proposition 4.3.2. The moduli space $\mathfrak{M}_{0,1}(I)$ is homeomorphic to $\mathbb{R}^2$. Moreover, for every $N \in (1,\infty)$, the moduli space $\mathfrak{M}_{0,N}(I)$ is homeomorphic to $\mathbb{R} \times [\mathcal{C}^*/\{\pm1\}]$ (which is contractible).

Proof. Part I: The case $N = 1$

Let us assume that $(I, \mathrm{d}, \mathfrak{m})$ is an $\mathrm{RCD}(0,1)$ structure on I. Thanks to Proposition 4.1.2, we necessarily have $\dim(X, \mathrm{d}, \mathfrak{m}) = 1$. Therefore, Proposition 4.1.1 implies that there exists $a > 0$ such that $\mathfrak{m} = a\mathcal{H}^1$. Moreover, as a result of [KL16, Theorem 1.1], (I, d) is isometric to $(I, L\,\mathrm{d}_{\mathrm{E}})$, where $L := \mathrm{Diam}(I, \mathrm{d})$. Hence, the map $\Psi\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathfrak{M}_{0,1}(I)$ defined by $\Psi(L,a) := [I, L\,\mathrm{d}_{\mathrm{E}}, a\mathcal{H}^1]$ is surjective. It is then readily checked that $\Phi\colon \mathfrak{M}_{0,1}(I) \to \mathbb{R}^+ \times \mathbb{R}^+$ defined by $\Phi([I,\mathrm{d},\mathfrak{m}]) := (\mathrm{Diam}(I,\mathrm{d}), \mathfrak{m}(I)/\mathcal{H}^1(I))$ is an inverse. Moreover, relying on [DG18, Theorem 1.2], we can prove that Φ is continuous (proceeding in the same way as in the last part of the proof of Proposition 4.2.1). Then, the proof of Ψ's continuity is trivial. Thus, $\mathfrak{M}_{0,1}(I)$ is homeomorphic to $\mathbb{R}^2$.

Part II: The case $1 < N$

If $(I, \mathrm{d}, \mathfrak{m})$ is an $\mathrm{RCD}(0, N)$ structure on I, then $(I, \mathrm{d}, \mathfrak{m})$ is isomorphic to $(I, L\,\mathrm{d}_{\mathrm{E}}, \mathfrak{m}')$ (as an application of [KL16, Theorem 1.1]), where $L \coloneqq \mathrm{Diam}(I, \mathrm{d})$ and $\mathfrak{m}'$ is a finite Radon measure on I. Due to [CM21, Theorem A.2], there exists $g \in \mathcal{C}^*$ (see Notation 4.3.1) such that $\mathfrak{m}' = g^{N-1}\mathcal{L}^1$. Conversely, [CM21, Theorem A.2]implies that, for every $g \in \mathcal{C}^*$ and $L > 0$, $(I, L\,\mathrm{d}_{\mathrm{E}}, g^{N-1}\mathcal{L}^1)$ is an $\mathrm{RCD}(0, N)$ structure on I. Now, if $g_1, g_2 \in \mathcal{C}^*$ satisfy $[I, \mathrm{d}_{\mathrm{E}}, g_1^{N-1}\mathcal{L}^1] = [I, \mathrm{d}_{\mathrm{E}}, g_2^{N-1}\mathcal{L}^1]$, then, there exists $\phi \in \mathrm{Iso}(I, \mathrm{d}_{\mathrm{E}})$ such that $g_2^{N-1}\mathcal{L}^1 = \phi_*(g_1^{N-1}\mathcal{L}^1) = (g_1 \circ \phi^{-1})^{N-1}\mathcal{L}^1$; hence, $g_1 = g_2 \circ \phi$. However, $\mathrm{Iso}(I, \mathrm{d}_{\mathrm{E}})$ consists of 2 elements, the identity id_I and the symmetry $t \to 1 - t$. Thus, $[g_1] = [g_2] \in \mathcal{C}^*/\{\pm 1\}$. Conversely, if $g_1, g_2 \in \mathcal{C}^*$ satisfy $[g_1] = [g_2] \in \mathcal{C}^*/\{\pm 1\}$, then $(I, \mathrm{d}_{\mathrm{E}}, g_1^{N-1})$ is isomorphic to $(I, \mathrm{d}_{\mathrm{E}}, g_2^{N-1})$. Therefore, the following two maps are well-defined and respectively inverse to each other:

- $\Phi\colon \mathfrak{M}_{0,N}(I) \to \mathbb{R} \times \mathcal{C}^*/\{\pm 1\}$ defined by $\Phi([I, \mathrm{d}, \mathfrak{m}]) \coloneqq (L, [g])$, where $L \coloneqq \mathrm{Diam}(I, \mathrm{d})$ and $g \in \mathcal{C}^*$ satisfies $[I, \mathrm{d}, \mathfrak{m}] = [I, L\,\mathrm{d}_{\mathrm{E}}, g^{N-1}\mathcal{L}^1]$,
- $\Psi\colon \mathbb{R} \times \mathcal{C}^*/\{\pm 1\} \to \mathfrak{M}_{0,N}(I)$ defined by $\Psi(L, [g]) \coloneqq [I, L\,\mathrm{d}_{\mathrm{E}}, g^{N-1}\mathcal{L}^1]$.

Part III: Φ is continuous

Assume that $\{[I, \mathrm{d}_n, \mathfrak{m}_n]\}_{n\in\mathbb{N}}$ is converging to $[I, \mathrm{d}_\infty, \mathfrak{m}_\infty]$ in $\mathfrak{M}_{0,N}(I)$ and, for every $n \in \mathbb{N} \cup \{\infty\}$, denote $(L_n, [g_n]) \coloneqq \Phi([I, \mathrm{d}_n, \mathfrak{m}_n])$. Observe that $\{(I, \mathrm{d}_n)\}_{n\in\mathbb{N}}$ converges to (I, d_∞) in the GH topology; in particular, $L_n \to L_\infty$. Now, let us show that $[g_n] \to [g_\infty]$ in $\mathcal{C}^*/\{\pm 1\}$. Note that it is sufficient to prove that every subsequence of $\{g_n\}_{n\in\mathbb{N}}$ admits a subsequence converging to $\nu \cdot g_\infty$ for some $\nu \in \{\pm 1\}$. We'll just show that $\{g_n\}_{n\in\mathbb{N}}$ admits a subsequence converging to $\nu \cdot g_\infty$ for some $\nu \in \{\pm 1\}$ (the proof for a subsequence of $\{g_n\}_{n\in\mathbb{N}}$ being exactly the same).

Observe first that $\{g_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(I)$. Indeed, $f_n \coloneqq g_n^{N-1}$ is a $\mathrm{CD}(0, N)$-density on $\mathrm{int}(I)$ (see [CM21, Definition A.1]). Hence, for every $n \in \mathbb{N}$, we have $|f_n|_{L^\infty} \leq N|f_n|_{L^1}$ (see [CM21, Lemma A.8]). Moreover, observe that $|f_n|_{L^1} = \mathfrak{m}_n(I) \leq M$, where $M \coloneqq \sup_{n\in\mathbb{N}}\{\mathfrak{m}_n(I)\}$. Hence, for every $n \in \mathbb{N}$, we obtain $|g_n|_{L^\infty} \leq (NM)^{1/N-1} \eqqcolon \overline{M}$.

Now, observe that for every $\epsilon > 0$, $\{g_n\}_{n\in\mathbb{N}}$ is equicontinuous on $I_\epsilon \coloneqq [\epsilon, 1-\epsilon]$. Indeed, let $n \in \mathbb{N}$ and observe that, since g_n is concave and positive on $\mathrm{int}(I)$, then we have:

$$-\overline{M}/1 - x \leq -g_n(x)/1 - x \leq g'_{n,r}(x) \leq g'_{n,l}(x) \leq g_n(x)/x \leq \overline{M}/x,$$

for every $x \in I_\epsilon$ and every $t \in [0,1]$. Hence, for every $x \in I_\epsilon$, we have $\mathrm{Lip}_x(g_n) \leq \overline{M}/\epsilon$. Therefore, $\{g_n\}_{n\in\mathbb{N}}$ is equicontinuous on I_ϵ.

Now, passing to a subsequence if necessary, we can assume that $\{g_n\}_{n\in\mathbb{N}}$ is converging uniformly to some continuous function $g\colon (0,1) \to \mathbb{R}$ on every compact subset $K \subset (0,1)$ (applying Arzelà–Ascoli Theorem and a diagonal argument). Observe that g is nonnegative and concave on $(0,1)$; hence, we can assume that g is continuous on I. A fortiori, note that $\{g_n^{N-1}\}_{n\in\mathbb{N}}$ converges uniformly to g^{N-1} on compact subsets of $(0,1)$. Let us prove that there exists $\nu \in \{\pm 1\}$ such that $g = \nu \cdot g_\infty$.

First, observe that $g_n^{N-1}\mathcal{L}^1 \to g^{N-1}\mathcal{L}^1$ in the weak-$*$ topology. Indeed, let us fix $f \in \mathcal{C}^0(I)$ and let $\epsilon > 0$. Denoting $\mathrm{d}_\epsilon(g_n^{N-1}, g^{N-1}) \coloneqq \sup_{I_\epsilon}(|g_n^{N-1} - g^{N-1}|)$, and splitting the integral into three part, we easily obtain:

$$\left| \int_I f(g_n^{N-1} - g^{N-1}) \right| \leq 4\epsilon \overline{M}^{N-1} |f|_{L^\infty} + |f|_{L^\infty} \mathrm{d}_\epsilon(g_n^{N-1}, g^{N-1}).$$

In particular, for every $\epsilon > 0$, we have $\limsup_{n\to\infty} |\int_I f(g_n^{N-1} - g^{N-1})| \leq \epsilon$. Hence, for every continuous function $f \in \mathcal{C}^0(I)$, we obtain $\lim_{n\to\infty} \int_I f g_n^{N-1} = \int_I f g^{N-1}$. This implies that $\{(I, L_n \,\mathrm{d_E}, g_n^{N-1}\mathcal{L}^1)\}_{n\in\mathbb{N}}$ converges in the mGH topology to $(I, L_\infty \,\mathrm{d_E}, g^{N-1}\mathcal{L}^1)$. However, $\{(I, L_n \,\mathrm{d_E}, g_n^{N-1}\mathcal{L}^1)\}_{n\in\mathbb{N}}$ also converges to $(I, L_\infty \,\mathrm{d_E}, g_\infty^{N-1}\mathcal{L}^1)$ in the mGH topology. Thus, there exists an isometry:

$$\phi\colon (I, L_\infty \,\mathrm{d_E}) \to (I, L_\infty \,\mathrm{d_E})$$

such that $\phi_*(g_\infty^{N-1}\mathcal{L}^1) = g^{N-1}\mathcal{L}^1$, i.e. $g = g_\infty \circ \phi^{-1}$. However, $\mathrm{Iso}(I, L_\infty \,\mathrm{d_E})$ consists of two elements, the identity id_I and the symmetry with center $1/2$. Thus, $g = \nu \cdot g_\infty$ for some $\nu \in \{\pm 1\}$, which concludes the proof of Φ's continuity.

Part IV: $\Psi = \Phi^{-1}$ is continuous

Let $\{L_n, [g_n]\}_{n\in\mathbb{N}}$ converge to $(L_\infty, [g_\infty])$ in $\mathbb{R}^+ \times [\mathcal{C}^*/\{\pm 1\}]$. Then, there exists a sequence $\{\nu_n\}_{n\in\mathbb{N}}$ in $\{\pm 1\}$ such that $\{\nu_n \cdot g_n\}_{n\in\mathbb{N}}$ converges to g_∞ uniformly on compact subsets of $(0,1)$. Let us denote $\tilde{g}_n \coloneqq \nu_n \cdot g_n$ and observe that, for every $n \in \mathbb{N}$, $(I, L_n, g_n^{N-1}\mathcal{L}^1)$ and $(I, L_n, \tilde{g}_n^{N-1}\mathcal{L}^1)$ are isomorphic. We need to show that $\{(I, L_n, \tilde{g}_n^{N-1}\mathcal{L}^1)\}_{n\in\mathbb{N}}$ converges to $(I, L_\infty, g_\infty^{N-1}\mathcal{L}^1)$ in the mGH topology. Since $|L_n - L_\infty| \to 0$, it is sufficient to show that $\{\tilde{g}_n^{N-1}\mathcal{L}^1\}_{n\in\mathbb{N}}$ converges to $g_\infty^{N-1}\mathcal{L}^1$ in the weak-$*$ topology. Moreover, proceeding exactly as in the last paragraph, it is enough to prove that $\{\tilde{g}_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(I)$. Let $n \in \mathbb{N}$ and observe that, thanks to the nonnegativity and concavity of $\tilde{g}_n$, we have the following three cases:

- if $x < 1/4$, we have $\tilde{g}_n(x) \leq \tilde{g}_n(1/2) + 2(1-2x)(\tilde{g}_n(1/4) - \tilde{g}_n(1/2))$, which implies that $\tilde{g}_n(x) \leq 3\max_{[1/4,3/4]}\{\tilde{g}_n\}$,
- if $x > 3/4$, we have $\tilde{g}_n(x) \leq \tilde{g}_n(1/2) + 2(2x-1)(\tilde{g}_n(3/4) - \tilde{g}_n(1/2))$, which implies that $\tilde{g}_n(x) \leq 3\max_{[1/4,3/4]}\{\tilde{g}_n\}$,
- if $x \in [1/4, 3/4]$, then $\tilde{g}_n(x) \leq \max_{[1/4,3/4]}\{\tilde{g}_n\}$.

Hence, for every $n \in \mathbb{N}$, we have $|\tilde{g}_n|_{L^\infty} \leq 3\max_{[1/4,3/4]}\{\tilde{g}_n\}$. However, $\{\tilde{g}_n\}_{n\in\mathbb{N}}$ converges uniformly to g_∞ on $[1/4, 3/4]$, hence $\sup_{n\in\mathbb{N}}\{\max_{[1/4,3/4]}\{\tilde{g}_n\}\} < \infty$, which concludes the proof of Ψ's continuity. Therefore, $\mathfrak{M}_{0,N}(I)$ is homeomorphic to $I \times [\mathcal{C}^*/\{\pm 1\}]$.

Part V: $\mathcal{C}^*/\{\pm 1\}$ is contractible

Indeed, observe that the map $H\colon I \times [\mathcal{C}^*/\{\pm 1\}] \to \mathcal{C}^*/\{\pm 1\}$ defined by $H(t, [f]) \coloneqq [t\tilde{1} + (1-t)f]$ is a retract by deformation of $\mathcal{C}^*/\pm 1$ onto $\{[\tilde{1}]\}$ (where $\tilde{1}$ is the function constant equal to 1). This concludes the proof of Proposition 4.3.2. □

4.4 The 2-dimensional case

This section will describe the moduli spaces $\mathfrak{M}_{0,2}(X)$, where X has dimension 2. We will start with the spaces whose splitting degree $k(X)$ equals 2, namely the 2-torus and the Klein bottle $\mathbb{K}^2$. Then, we will proceed with the spaces satisfying $k(X) = 1$; namely, the cylinder $\mathbb{S}^1 \times I$ and the Möbius band $\mathbb{M}^2$. Finally, we will study the case where $k(X) = 0$, which corresponds to the 2-sphere $\mathbb{S}^2$, the projective plane $\mathbb{RP}^2$, and the closed 2-disc $\mathbb{D}$.

4.4.1 The torus and the Klein bottle

The torus and the Klein bottle are closed manifolds that admit flat metrics; therefore, we will use Proposition 4.2.1.

4.4.1.1 The torus

Proposition 4.4.1. The moduli space $\mathfrak{M}_{0,2}(\mathbb{T}^2)$ is homeomorphic to $\mathbb{R}^4$; in particular, it is contractible.

Proof. According to Proposition 4.2.1, there is a homeomorphism:

$$\mathfrak{M}_{0,2}(\mathbb{T}^2) \simeq \mathbb{R} \times \mathscr{M}_{\text{flat}}(\mathbb{T}^2). \tag{4.2}$$

Moreover, as a result of [Gar20, Section 2.1], $\mathscr{M}_{\text{flat}}(\mathbb{T}^2)$ is homeomorphic to $\mathbb{R}^3$. Therefore, $\mathfrak{M}_{0,2}(\mathbb{T}^2)$ is homeomorphic to $\mathbb{R}^3$. □

4.4.1.2 The Klein bottle

Proposition 4.4.2. The moduli space $\mathfrak{M}_{0,2}(\mathbb{K}^2)$ of $\mathrm{RCD}(0,2)$ structures on $\mathbb{K}^2$ is homeomorphic to $\mathbb{R}^3$; in particular, it is contractible.

Proof. We denote Γ the Bieberbach subgroup of $\mathrm{Iso}(\mathbb{R}^2)$ generated by $a := (I_2, e_1)$ and $b := (\sigma, e_2)$, where (e_1, e_2) is the canonical basis of $\mathbb{R}^2$ and $\sigma := \mathrm{diag}(-1, 1)$. Let us recall that by definition, $\mathbb{K}^2 = \mathbb{R}^2/\Gamma$. Therefore, thanks to Proposition 4.2.1, there is a homeomorphism:

$$\mathfrak{M}_{0,2}(\mathbb{K}^2) \simeq \mathbb{R} \times [\mathrm{O}_2(\mathbb{R})\backslash \mathcal{C}_\Gamma]/\mathcal{N}_\Gamma. \tag{4.3}$$

It is then readily checked that $\mathrm{N}_{\mathrm{Aff}(\mathbb{R}^2)}(\Gamma) = \{(\mathrm{diag}(\epsilon_1, \epsilon_2), v), \epsilon_i \in \{\pm 1\}, 2\langle v, e_1\rangle \in \mathbb{Z}\}$. Therefore, we have $\mathcal{N}_\Gamma = \{\mathrm{diag}(\epsilon_1, \epsilon_2), \epsilon_i \in \{\pm 1\}\}$ (see Definition 4.2.1). Then, [BDP18, Proposition 4.8] implies that we have $\mathcal{C}_\Gamma = \mathrm{O}_2(\mathbb{R}) \cdot \mathcal{Z}$, where $\mathcal{Z}$ is the centralizer of H_Γ in $\mathrm{GL}_2(\mathbb{R})$. In addition, H_Γ is the subgroup of $\mathrm{O}_2(\mathbb{R})$ generated by σ; hence, it is easy to see that $\mathcal{Z} = \{\mathrm{diag}(a_1, a_2), a_i \in \mathbb{R}^*\}$. Now, observe that $\mathcal{N}_\Gamma$ acts trivially on $\mathrm{O}_2(\mathbb{R})\backslash \mathcal{C}_\Gamma$. Indeed, given $A \in \mathcal{C}_\Gamma$ and $B \in \mathcal{N}_\Gamma$, there exists $C \in \mathrm{O}_2(\mathbb{R})$ and $D \in \mathcal{Z}$ such that $A = CD$. In particular, we have $AB = CDB = CBD$ (since both B and D are diagonal matrices). Therefore, using the fact that $B \in \mathcal{N}_\Gamma \subset \mathrm{O}_2(\mathbb{R})$ and $C \in \mathrm{O}_2(\mathbb{R})$,we obtain $[D] = [CD] = [CBD]$ (where $[\cdot]$ denotes the class of a matrix in $\mathrm{O}_2(\mathbb{R})\backslash \mathcal{C}_\Gamma$). However, $[A] = [CD]$ and $[A] \cdot B = [CDB] = [CBD]$. Hence, $[A] = [A] \cdot B$, i.e. $\mathcal{N}_\Gamma$ acts trivially on $\mathrm{O}_2(\mathbb{R})\backslash \mathcal{C}_\Gamma$. Thus, as a result of (4.3), we have a homeomorphism:

$$\mathfrak{M}_{0,2}(\mathbb{K}^2) \simeq \mathbb{R} \times \mathrm{O}_2(\mathbb{R})\backslash \mathcal{C}_\Gamma. \tag{4.4}$$

Now, according to [BDP18, Corollary 4.9], $\mathrm{O}_2(\mathbb{R})\backslash \mathcal{C}_\Gamma$ is homeomorphic to the quotient space $\mathrm{O}_2(\mathbb{R}) \cap \mathcal{Z}\backslash \mathcal{Z}$. Then, observe that the map $\mathrm{diag}(a, b) \in \mathcal{Z} \to (|a|, |b|) \in \mathbb{R}^+ \times \mathbb{R}^+$ passes to the quotient, giving rise to a homeomorphism $[\mathrm{diag}(a, b)] \in \mathrm{O}_2(\mathbb{R}) \cap \mathcal{Z}\backslash \mathcal{Z} \to (|a|, |b|) \in \mathbb{R}^+ \times \mathbb{R}^+$. Hence, using (4.4), we finally obtain $\mathfrak{M}_{0,2}(\mathbb{K}^2) \simeq \mathbb{R}^3$. □

4.4.2 The Möbius band and the cylinder

Theorem 3.2.1 will be crucial to compute the homeomorphism type of the moduli spaces of RCD(0, 2) structures on the Möbius band $\mathbb{M}^2$ and the cylinder $\mathbb{S}^1 \times I$.

4.4.2.1 The Möbius band

Proposition 4.4.3. The moduli space $\mathfrak{M}_{0,2}(\mathbb{M}^2)$ of RCD(0, 2) structures on $\mathbb{M}^2$ is homeomorphic to $\mathbb{R}^3$; in particular, it is contractible.

Proof. Assume that $(\mathbb{M}^2, \mathrm{d}, \mathfrak{m})$ is an RCD(0, 2) structure on $\mathbb{M}^2$ and let $(I \times \mathbb{R}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}) := p^*(\mathbb{M}^2, \mathrm{d}, \mathfrak{m})$ be the associated lift (where $p\colon I \times \mathbb{R} \to \mathbb{M}^2$ is the quotient map). Observe that, following the proof of Proposition 4.1.2 (case $k(X) = 1$), there exists a splitting $\phi\colon (I \times \mathbb{R}, \tilde{\mathrm{d}}, \tilde{\mathfrak{m}}) \to ([0, r], \mathrm{d_E}, a\mathcal{L}^1) \times (\mathbb{R}, \mathrm{d_E}, \mathcal{L}^1)$, where $a, r \in \mathbb{R}_{>0}$. Moreover, following the same proof, there exists a generator γ of $\overline{\pi}_1(X)$ and $b > 0$ such that, for every $(\overline{x}, t) \in [0, r] \times \mathbb{R}$, we have $\phi_*(\gamma)(\overline{x}, t) = (r - \overline{x}, t + b)$. In particular, we have $[\mathbb{M}^2, \mathrm{d}, \mathfrak{m}] = [\mathbb{M}^2, \mathrm{d}_{r,b}, a\mathcal{H}^2]$, where $(\mathbb{M}^2, \mathrm{d}_{r,b})$ is the metric quotient of $(I \times \mathbb{R}, r\,\mathrm{d_E} \times b\,\mathrm{d_E})$ by the action of $\mathbb{Z}$ on $I \times \mathbb{R}$ defined by $1 \cdot (\overline{x}, t) := (1 - \overline{x}, t + 1)$. Moreover, it is easily seen from the definitions that $\mathcal{A}([\mathbb{M}^2, \mathrm{d}, \mathfrak{m}]) = (\mathbb{R}, \mathrm{d_E})/b\mathbb{Z}$ and $\mathcal{S}([\mathbb{M}^2, \mathrm{d}, \mathfrak{m}]) = [I, r\,\mathrm{d_E}, a\mathcal{H}^1]$ (where the Albanese and soul maps were defined respectively in Section 3.2.1). In particular, we have $a = \mathrm{Mass}(\mathcal{S}([\mathbb{M}^2, \mathrm{d}, \mathfrak{m}]))/\,\mathrm{Diam}(\mathcal{S}([\mathbb{M}^2, \mathrm{d}, \mathfrak{m}]))$, $r = \mathrm{Diam}(\mathcal{S}([\mathbb{M}^2, \mathrm{d}, \mathfrak{m}]))$, and $b = 2\,\mathrm{Diam}(\mathcal{A}([\mathbb{M}^2, \mathrm{d}, \mathfrak{m}]))$. Hence, $\Phi\colon a, b, r \in (\mathbb{R}_{>0})^3 \to [\mathbb{M}^2, \mathrm{d}_{r,b}, a\mathcal{H}^2] \in \mathfrak{M}_{0,2}(\mathbb{M}^2)$ is invertible, and its inverse satisfies $\Phi^{-1} = (\psi_1, \psi_2, \psi_3)$ where:

$$\psi_1([\mathbb{M}^2, \mathrm{d}, \mathfrak{m}]) = \frac{\mathrm{Mass}(\mathcal{S}([\mathbb{M}^2, \mathrm{d}, \mathfrak{m}]))}{\mathrm{Diam}(\mathcal{S}([\mathbb{M}^2, \mathrm{d}, \mathfrak{m}]))},$$

$$\psi_2([\mathbb{M}^2, \mathrm{d}, \mathfrak{m}]) = 2\,\mathrm{Diam}(\mathcal{A}([\mathbb{M}^2, \mathrm{d}, \mathfrak{m}])),$$
$$\psi_3([\mathbb{M}^2, \mathrm{d}, \mathfrak{m}]) = \mathrm{Diam}(\mathcal{S}([\mathbb{M}^2, \mathrm{d}, \mathfrak{m}])).$$

Now, observe that Φ and Φ^{-1} are continuous. Indeed, notice that, thanks to Theorem 3.2.1, Φ^{-1} is continuous. Then, assume that $(a_k, b_k, r_k) \to (a_\infty, b_\infty, r_\infty)$. It is clear that $\{(I \times \mathbb{R}, r_k\,\mathrm{d_E} \times b_k\,\mathrm{d_E}, a_k\mathcal{H}^2, 0)\}_{k\in\mathbb{N}}$ converges to $(I \times \mathbb{R}, r_\infty\,\mathrm{d_E} \times b_\infty\,\mathrm{d_E}, a_\infty\mathcal{H}^2, 0)$ in the equivariant pmGH topology (see Definition 2.4.8) w.r.t. the action of $\mathbb{Z}$ on $I \times \mathbb{R}$ that we introduced above. Hence, applying the equivariant theorem (see Theorem 2.4.1), the sequence of associated quotients $\{(\mathbb{M}^2, \mathrm{d}_{r_k,b_k}, a_k\mathcal{H}^2)\}$ converges to $(\mathbb{M}^2, \mathrm{d}_{r_\infty,b_\infty}, a_\infty\mathcal{H}^2)$ in the mGH topology. Thus, Φ is continuous, which concludes the proof. □

4.4.2.2 The cylinder

Proposition 4.4.4. The moduli space $\mathfrak{M}_{0,2}(\mathbb{S}^1 \times I)$ of $\mathrm{RCD}(0,2)$ structures on $\mathbb{S}^1 \times I$ is homeomorphic to $\mathbb{R}^3$; in particular, it is contractible.

Proof. Proceeding precisely as in section 4.4.2.1, it is readily checked that the map $\Phi\colon a, b, r \in (\mathbb{R}_{>0})^3 \to [\mathbb{S}^1 \times I, \mathrm{d}_{r,b}, a\mathcal{H}^2] \in \mathfrak{M}_{0,2}(\mathbb{S}^1 \times I)$ is a homeomorphism, where $d_{r,b} = b\,\mathrm{d}_{\mathbb{S}^1} \times r\,\mathrm{d}_{\mathrm{E}}$, and $\mathrm{d}_{\mathbb{S}^1}$ is the length metric on the circle with perimeter 1. Therefore, the result follows. □

4.4.3 The sphere, the projective plane, and the closed disc

In this section, we will compute the moduli spaces of $\mathrm{RCD}(0,2)$ structures on the 2-sphere $\mathbb{S}^2$, the projective plane $\mathbb{RP}^2$, and the closed disc $\mathbb{D} \subset \mathbb{R}^2$. As we will see later, these moduli spaces are all homeomorphic to specific spaces of convex compacta. We will start by introducing some notations of convex geometry. We will then prove realisation results for nonnegatively curved metrics on $\mathbb{RP}^2$ and $\mathbb{D}$. Then, we will introduce convergence lemmas that will be fundamental to proving continuity statements. Finally, we will compute the aforementioned moduli spaces.

4.4.3.1 Notations

We start by introducing some notations from [Bel18b].

Notation 4.4.1 (Spaces of convex compacta)**.** We denote $\mathcal{K}$ the set of all compact convex subsets in $\mathbb{R}^3$, and $\mathcal{K}^s$ the subset of $\mathcal{K}$ whose elements have their Steiner point at the origin (see [Bel18a, section 4] for some properties of the Steiner point). Given $0 \leq k \leq l \leq 3$ and $\mathcal{K}' \subset \mathcal{K}$, we denote:

- $\mathcal{K}'_{k\leq l} \coloneqq \{D \in \mathcal{K}', \dim(D) \in [k, l]\}$
- $\tilde{\mathcal{K}}' \coloneqq \{D \in \mathcal{K}', D = -D\}$.

Every subspace of $\mathcal{K}$ will be endowed with the Hausdorff distance $\mathrm{d}_{\mathrm{H}}^{\mathbb{R}^3}$. Also, we will generically denote B, K, L and $\{*\}$ elements of $\mathcal{K}$ with dimension 3, 2, 1 and 0.

Remark 4.4.1. Observe that if $D \in \tilde{\mathcal{K}}$, then $s(D) = s(-D) = -s(D)$ (where $s(D)$ is the Steiner point of D), i.e. $s(D) = 0$. In particular, we have $\tilde{\mathcal{K}} \subset \mathcal{K}^s$.

The following maps will be important later when comparing the boundaries of two different convex bodies in $\mathcal{K}^s$.

Notation 4.4.2 (Central projection)**.** Let $n \geq 1$ and assume that D is an n-dimensional compact convex subset of $\mathbb{R}^n$ whose Steiner point is at the origin. Given $x \in \mathbb{R}^n \backslash \{0\}$, the open half line $\mathbb{R}_{>0} \cdot x$ intersects ∂D in a single point, which we denote $p^{\mathrm{c}}_{\partial D}(x)$. The map $p^{\mathrm{c}}_{\partial D} \colon \mathbb{R}^n \backslash \{0\} \to \partial D$ is called the *central projection on* ∂D.

A classical result is that the orthogonal projection on a closed convex subspace of a Hilbert space is well-defined.

Notation 4.4.3 (Orthogonal projection)**.** Let $n \geq 1$ and assume that D is a closed convex subset of $\mathbb{R}^n$. Given $x \in \mathbb{R}^n$, there exists a unique point $p_D(x) \in D$ such that $\mathrm{d_E}(x, p_D(x)) = \mathrm{d_E}(x, D)$. The map $p_D \colon \mathbb{R}^n \to D$ is called the *orthogonal projection on* D. Given $\alpha \in \mathbb{R}^n \backslash \{0\}$, we denote $p_\alpha \coloneqq p_{\mathbb{R}\alpha}$, and $p_\alpha^\perp \coloneqq p_{\{\alpha\}^\perp}$.

Let us now clarify what we mean when we speak about a set's boundary and interior.

Notation 4.4.4 (Boundary and interior)**.** Let $M \subset \mathbb{R}^n$ $(n \geq 1)$ be a topological submanifold with boundary. We denote ∂M the boundary of M, and $\mathring{M} \coloneqq M \backslash \partial M$ the interior of M (we will also write $\mathrm{int}(M)$).

Every Lipschitz submanifold of $\mathbb{R}^n$ $(n \geq 1)$ admits two canonical metrics: extrinsic and intrinsic.

Notation 4.4.5 (Intrinsic and extrinsic metrics)**.** Given $n \in \mathbb{N}$, and given a connected Lipschtiz submanifold $X \subset \mathbb{R}^n$ (possibly with boundary), we denote d_X the *intrinsic metric of* X. More precisely, given $x, y \in X$, we have $\mathrm{d}_X(x,y) = \inf\{\mathcal{L}(\gamma)\}$, where the infimum is computed over the set of rectifiable curves in X joining x to y. The *extrinsic metric* on X is simply the restriction of the Euclidean distance $\mathrm{d_E}$ to $X \subset \mathbb{R}^n$. We will usually only write X to speak about the metric space (X, d_X), and we will specify when we endow X with its extrinsic distance $\mathrm{d_E}$. For example, given $B \in \mathcal{K}^s$ of dimension 3, will usually write ∂B to speak about the metric space $(\partial B, \mathrm{d}_{\partial B})$.

Let us introduce the double of a metric space. This notion will be crucial to realise metrics on the 2-sphere, the projective plane and the disc.

Notation 4.4.6 (Double of a metric space)**.** Given a topological manifold X with boundary, we denote:

$$\mathcal{D}X \coloneqq \sqcup_{\mathbf{i}=\mathbf{1},\mathbf{2}}\{\mathbf{i}\} \times X/\sim,$$

where $(\mathbf{1}, x) \sim (\mathbf{2}, x)$, whenever $x \in \partial X$. We call $\mathcal{D}X$ the *double of* X and let q be the quotient map.

Given $\mathbf{i} \in \{\mathbf{1}, \mathbf{2}\}$, we denote $X^{\mathbf{i}}$ (resp. $\mathring{X}^{\mathbf{i}}$) the image of $\{\mathbf{i}\} \times X$ (resp. $\{\mathbf{i}\} \times \mathring{X}$) under q. Since $\{\mathbf{1}\} \times \partial X$ and $\{\mathbf{2}\} \times \partial X$ are identified when passing to the quotient, we also denote ∂X, the image of these by q. Then, given $x \in X$ we will write $x^{\mathbf{i}} \coloneqq q(\mathbf{i}, x) \in X^{\mathbf{i}}$.

Given a length metric d on X, there exists a unique length metric $\mathcal{D}\,\mathrm{d}$ on $\mathcal{D}X$ whose restriction to $X^{\mathbf{1}}$ and $X^{\mathbf{2}}$ coincides with d. More precisely, given $x, y \in X$,we define $\mathcal{D}\,\mathrm{d}(x^{\mathbf{1}}, y^{\mathbf{1}}) \coloneqq \mathrm{d}(x, y) \eqqcolon \mathcal{D}\,\mathrm{d}(x^{\mathbf{2}}, y^{\mathbf{2}})$, and:

$$\mathcal{D}\,\mathrm{d}(x^{\mathbf{1}}, y^{\mathbf{2}}) \coloneqq \inf\{\mathrm{d}(x, z) + \mathrm{d}(z, y), z \in \partial X\}.$$

We denote $\mathcal{D}(X, \mathrm{d}) \coloneqq (\mathcal{D}X, \mathcal{D}\,\mathrm{d})$.

Very often, there will be no confusion about the metric X is endowed with; therefore, we usually only write $\mathcal{D}X$ instead of $\mathcal{D}(X, \mathrm{d})$. For example, given $K \in \mathcal{K}$ of dimension 2, we will write $\mathcal{D}K$ to speak about the metric space $\mathcal{D}(K, \mathrm{d_E})$.

The following maps will be relevant when studying doubles of metric spaces.

Notation 4.4.7. Assume that K is a 2-dimensional compact convex subset of $\mathbb{R}^2$. We denote $s_K \colon \mathcal{D}K \to \mathcal{D}K$ the isometry defined by $s_K(x^{\mathbf{1}}) \coloneqq x^{\mathbf{2}}$ and $s_K(x^{\mathbf{2}}) \coloneqq x^{\mathbf{1}}$, for $x \in K$.

Notation 4.4.8. Assume that (X, d_X) and (Y, d_Y) are metric spaces homeomorphic to topological manifolds with boundary, and assume that $\phi \colon (X, \mathrm{d}_X) \to (Y, \mathrm{d}_Y)$ is an isometry (in particular $\phi(\partial X) = \partial Y$). We denote $\phi_{\mathcal{D}} \colon \mathcal{D}(X, \mathrm{d}_X) \to \mathcal{D}(Y, \mathrm{d}_Y)$ the isometry defined by $\phi_{\mathcal{D}}(x^{\mathbf{i}}) \coloneqq (\phi(x))^{\mathbf{i}}$, for $x \in X$ and $\mathbf{i} \in \{\mathbf{1}, \mathbf{2}\}$.

4.4.3.2 Realisation of nonnegatively curved metrics

In [Bel18b], Belegradek introduces a homeomorphism between the quotient space $\mathcal{K}^s_{2\leq 3}/\,\mathrm{O}_3(\mathbb{R})$ and the moduli space $\mathscr{M}_{\mathrm{curv}\geq 0}(\mathbb{S}^2)$ of nonnegatively curved metrics on $\mathbb{S}^2$. Let us first introduce the correspondence.

Notation 4.4.9. Given $D \in \mathcal{K}^s$ we denote $\Phi_{\mathbb{S}^2}(D) \coloneqq \partial D$ if $\dim(D) = 3$, $\Phi_{\mathbb{S}^2}(D) \coloneqq \mathcal{D}D$ if $\dim(D) = 2$, and $\Phi_{\mathbb{S}^2}(D) \coloneqq D$ if $\dim(D) \in \{0, 1\}$.

Notice that given $D \in \mathcal{K}^s_{2\leq 3}$, the isometry class $[\Phi_{\mathbb{S}^2}(D)]$ belongs to $\mathscr{M}_{\text{curv}\geq 0}(\mathbb{S}^2)$. Moreover, given $\phi \in \mathrm{O}_3(\mathbb{R})$, we have $[\Phi_{\mathbb{S}^2}(D)] = [\Phi_{\mathbb{S}^2}(\phi(D))]$. Therefore, there exists a unique map:

$$\Psi_{\mathbb{S}^2} \colon \mathcal{K}^s_{2\leq 3}/\,\mathrm{O}_3(\mathbb{R}) \to \mathscr{M}_{\text{curv}\geq 0}(\mathbb{S}^2) \tag{4.5}$$

such that, for every $D \in \mathcal{K}^s_{2\leq 3}$, we have $\Psi_{\mathbb{S}^2}([D]) = [\Phi_{\mathbb{S}^2}(D)]$.

The following result is inspired by the realisation Theorem of Alexandrov (see Theorem 1 page 237 in [Kut05]) and is proven by Belegradek in [Bel18b, section 2].

Theorem 4.4.1. The map $\Psi_{\mathbb{S}^2} \colon \mathcal{K}^s_{2\leq 3}/\,\mathrm{O}_3(\mathbb{R}) \to \mathscr{M}_{\text{curv}\geq 0}(\mathbb{S}^2)$ introduced in (4.5) is a homeomorphism.

We will show that the equivariant nonnegatively curved metrics on $\mathbb{S}^2$ correspond with symmetric convex compacta with dimensions between 2 and 3. In what follows, by equivariant, we always mean equivariant with respect to the action of $\{\pm 1\}$ on the relevant spaces.

Remark 4.4.2. Note that there might be various actions of $\{\pm 1\}$ on our spaces. Given $B \in \tilde{\mathcal{K}}_{2\leq 3}$ of dimension 3, we let -1 act as $-\mathrm{id}_{\mathbb{R}^3}$ on ∂B. Given $K \in \tilde{\mathcal{K}}_{2\leq 3}$ of dimension 2, we let -1 act on $\mathcal{D}K$ in the following way: for $x \in K$, $-1 \cdot x^{\mathbf{1}} := (-x)^{\mathbf{2}}$ and $-1 \cdot x^{\mathbf{2}} := (-x)^{\mathbf{1}}$.

First, note that if $D \in \tilde{\mathcal{K}}_{2\leq 3}$, it is clear that there exists an equivariant nonnegatively curved metric d on $\mathbb{S}^2$ such that $(\mathbb{S}^2, \mathrm{d})$ is equivariantly isometric to $\Phi_{\mathbb{S}^2}(D)$; hence, $[\Phi_{\mathbb{S}^2}(D)] \in \mathscr{M}^{\text{eq}}_{\text{curv}\geq 0}(\mathbb{S}^2)$. Then, note that if $\phi \in \mathrm{O}_3(\mathbb{R})$, then $\Phi_{\mathbb{S}^2}(D)$ and $\Phi_{\mathbb{S}^2}(\phi(D))$ are equivariantly isometric. Therefore, there exists a unique map:

$$\Psi^{\text{eq}}_{\mathbb{S}^2} \colon\ \tilde{\mathcal{K}}_{2\leq 3}\,/\,\mathrm{O}_3(\mathbb{R}) \to \mathscr{M}^{\text{eq}}_{\text{curv}\geq 0}(\mathbb{S}^2) \tag{4.6}$$

such that, for every $D \in \tilde{\mathcal{K}}_{2\leq 3}$, we have $\Psi^{\text{eq}}_{\mathbb{S}^2}([D]) = [\Phi_{\mathbb{S}^2}(D)]$. We will prove the following realisation theorem for equivariant metrics on the 2-sphere.

Proposition 4.4.5. The map $\Psi^{\text{eq}}_{\mathbb{S}^2} \colon\ \tilde{\mathcal{K}}_{2\leq 3}\,/\,\mathrm{O}_3(\mathbb{R}) \to \mathscr{M}^{\text{eq}}_{\text{curv}\geq 0}(\mathbb{S}^2)$ introduced in (4.6) is a 1-1 correspondence.

We will need the following Lemma to prove Proposition 4.4.5.

Lemma 4.4.1. Let $K \subset \mathbb{R}^2$ and $\Sigma \subset \mathbb{R}^2$ be 2-dimensional compact convex subsets of $\mathbb{R}^2$. If $\phi \colon \mathcal{D}K \to \mathcal{D}\Sigma$ is an isometry, then $\phi(\partial K) = \partial\Sigma$.

Proof. Looking for a contradiction, let us assume that $\partial K \cap \phi^{-1}(\mathring{\Sigma}^{\mathbf{2}}) \neq \varnothing$.

First of all, observe that if $x, y \in \partial K \cap \phi^{-1}(\mathring{\Sigma}^{\mathbf{2}})$, then there exists a unique shortest path $[\phi(x)\phi(y)]$ between $\phi(x)$ and $\phi(y)$. In particular, there exists a unique shortest path $[xy]$ between x and y. However, as $x, y \in \partial K$, then we necessarily have $[xy] \subset \partial K$. Moreover, observe that $[\phi(x)\phi(y)] \subset \mathring{\Sigma}^{\mathbf{2}}$ and that $\phi([xy]) = [\phi(x)\phi(y)]$; therefore, $[xy] \subset \partial K \cap \phi^{-1}(\mathring{\Sigma}^{\mathbf{2}})$.

Then, observe that if $x, y, z \in \partial K \cap \phi^{-1}(\mathring{\Sigma}^{\mathbf{2}})$, then x, y, z are necessarily aligned. Indeed, if not, then the triangle $\Delta(x, y, z)$ is equal to ∂K (since it is homeomorphic to $\mathbb{S}^1$) and is a subset of $\phi^{-1}(\mathring{\Sigma}^{\mathbf{2}})$. In particular, thanks to the first observation, every pair of points on ∂K can be joined by a unique shortest path, which is a contradiction. Using the two observations above, there exists $x \neq y \in \partial K$ such that $(xy) = \partial K \cap \phi^{-1}(\mathring{\Sigma}^{\mathbf{2}})$, where (xy) is the interior of $[xy]$ seen as a subspace of $\partial K \subset \mathbb{R}^2$. Note that we necessarily have:

(i) $[xy] \subset \partial K$,

(ii) $\phi(x), \phi(y) \in \partial\Sigma$.

In particular, there exists a unique shortest path from x to y in $\mathcal{D}K$ (using (i)). Therefore, there is a unique shortest path from $\phi(x)$ to $\phi(y)$, which implies that $\phi([xy]) = [\phi(x)\phi(y)] \subset \partial\Sigma$ (using (ii)). However, let us recall that by assumption $(xy) = \partial K \cap \phi^{-1}(\mathring{\Sigma}^{\mathbf{2}})$. Hence, $\phi((xy)) \subset \partial\Sigma \cap \mathring{\Sigma}^{\mathbf{2}} = \varnothing$, which is the contradiction we were seeking. Furthermore, the same argument leads to $\partial K \cap \phi^{-1}(\mathring{\Sigma}^{\mathbf{1}}) = \varnothing$.

We have shown that $\phi(\partial K) \subset \partial\Sigma$. However, the same argument applied to ϕ^{-1} leads to $\phi^{-1}(\partial\Sigma) \subset \partial K$; therefore, $\partial\Sigma \subset \phi(\partial K)$, which concludes the proof. □

Lemma 4.4.1 implies the following Proposition.

Proposition 4.4.6. Let $K \subset \mathbb{R}^2$ and $\Sigma \subset \mathbb{R}^2$ be 2-dimensional compact convex subsets of $\mathbb{R}^2$ with Steiner point at the origin, and let $\phi\colon \mathcal{D}K \to \mathcal{D}\Sigma$ be an isometry. Then, there exists $\mu \in \mathrm{O}_2(\mathbb{R})$ such that $\mu(K) = \Sigma$, and $\phi = \mu_{\mathcal{D}}$ if $\phi(K^{\mathbf{2}}) = \Sigma^{\mathbf{2}}$ (resp. $s_\Sigma \circ \phi = \phi \circ s_K = \mu_{\mathcal{D}}$ if $\phi(K^{\mathbf{2}}) = \Sigma^{\mathbf{1}}$), where we introduced $\mu_{\mathcal{D}}$ (resp. s_Σ and s_K) in Notation 4.4.8 (resp. Notation 4.4.7).

Proof. First of all, Lemma 4.4.1 implies $\phi(\partial K) = \partial\Sigma$. In particular, ϕ maps the connected components $\mathring{K}^{\mathbf{1}}$ and $\mathring{K}^{\mathbf{2}}$ to $\mathring{\Sigma}^{\mathbf{1}} \sqcup \mathring{\Sigma}^{\mathbf{2}}$. Hence, composing ϕ with s_Σ if necessary, we can assume that $\phi(K^{\mathbf{i}}) = \Sigma^{\mathbf{i}}$ ($\mathbf{i} \in \{\mathbf{1}, \mathbf{2}\}$). In particular, there exists isometries $\mu\colon K \to \Sigma$ and $\nu\colon K \to \Sigma$ such that $\mu_{|\partial K} = \nu_{|\partial K}$, and such that, for every

$x \in K$, we have $\phi(x^{\mathbf{2}}) = \mu(x)^{\mathbf{2}}$ and $\phi(x^{\mathbf{1}}) = \nu(x)^{\mathbf{1}}$. According to [ATV01, Theorem 2.2], we can assume that μ and ν are isometries of $\mathbb{R}^2$. However $\mathrm{Span}(\partial K) = \mathbb{R}^2$; therefore, since μ and ν coincide on ∂K, we necessarily have $\mu = \nu$. Then, K and Σ have their Steiner point at the origin and $\mu(K) = \Sigma$; thus, we necessarily have $\mu(0) = 0$, i.e. $\mu \in \mathrm{O}_2(\mathbb{R})$. Hence, we can conclude that $\phi = \mu_{\mathcal{D}}$. □

We can now prove Proposition 4.4.5.

Proof of Proposition 4.4.5. First of all, assume that $\Psi^{\mathrm{eq}}_{\mathbb{S}^2}([D_1]) = \Psi^{\mathrm{eq}}_{\mathbb{S}^2}([D_2])$ for some $D_1, D_2 \in \tilde{\mathcal{K}}_{2\leq 3}$. Then $\Phi_{\mathbb{S}^2}(D_1)$ is equivariantly isometric to $\Phi_{\mathbb{S}^2}(D_1)$. In particular $\Psi_{\mathbb{S}^2}([D_1]) = \Psi_{\mathbb{S}^2}([D_2])$. Hence, as an application of Theorem 4.4.1, we have $[D_1] = [D_2]$. Thus, $\Psi^{\mathrm{eq}}_{\mathbb{S}^2}$ is injective.

Let us now show that $\Psi^{\mathrm{eq}}_{\mathbb{S}^2}$ is surjective. Let d be an equivariant nonnegatively curved metric on $\mathbb{S}^2$ and let us show that there exists $D \in \tilde{\mathcal{K}}_{2\leq 3}$ such that $(\mathbb{S}^2, \mathrm{d})$ is equivariantly isometric to $\Phi_{\mathbb{S}^2}(D)$. Thanks to Theorem 4.4.1, either there exists $B \in \mathcal{K}^{\mathrm{s}}_{2\leq 3}$ of dimension 3 such that $(\mathbb{S}^2, \mathrm{d})$ is isometric to ∂B, or there exists $K \in \mathcal{K}^{\mathrm{s}}_{2\leq 3}$ of dimension 2 such that $(\mathbb{S}^2, \mathrm{d})$ is isometric to $\mathcal{D}K$.

Let us first assume that we have an isometry $\phi\colon (\mathbb{S}^2, \mathrm{d}) \to \partial B$. We define $f\colon x \in \partial B \to \phi(-\phi^{-1}(x)) \in \partial B$. It is sufficient to prove that f coincides with $-\mathrm{id}_{\mathbb{R}^3}$. To do so, note that f is a self-isometry of ∂B. Therefore, [Bur+92, Theorem 5.2.1] implies that f is also a self-isometry of $(\partial B, \mathrm{d_E})$. Thus, as a result of [ATV01, Theorem 2.2], we can extend f into an isometry of $(\mathbb{R}^3, \mathrm{d_E})$ (which we also denote f). Observe that f is in particular an affine transformation; thus $f(B) = f(\{\mathrm{Conv}(\partial B)\}) = \mathrm{Conv}\{f(\partial B)\} = \mathrm{Conv}\{\partial B\} = B$. Therefore, we have $f(0) = f(s(B)) = s(f(B)) = s(B) = 0$ (where $s(B)$ is the Steiner point of B), i.e. $f \in \mathrm{O}_3(\mathbb{R})$. Now, note that by definition, f is involutive on ∂B. Moreover, ∂B spans $\mathbb{R}^3$. Hence, f is an orthogonal involution of $\mathbb{R}^3$ and can be diagonalized with eigenvalues ± 1. However, by definition, f has no fixed point on ∂B, so 1 cannot be an eigenvalue of f. Hence, $f = -\mathrm{id}_{\mathbb{R}^3}$.

Now, let us assume that we have an isometry $\phi\colon (\mathbb{S}^2, \mathrm{d}) \to \mathcal{D}K$. Without loss of generality, we may assume that $K \subset \mathbb{R}^2$. As above, we introduce $f\colon x \in \mathcal{D}K \to \phi(-\phi^{-1}(x)) \in \mathcal{D}K$, which is an involutive isometry without any fixed points. Let us prove that f coincides with the action of -1 on $\mathcal{D}K$. Applying Lemma 4.4.1, we have either $f(K^{\mathbf{2}}) \subset K^{\mathbf{2}}$ or $f(K^{\mathbf{2}}) \subset K^{\mathbf{1}}$. If $f(K^{\mathbf{2}}) \subset K^{\mathbf{2}}$, then Brower's fix point Theorem implies that f has a fix point on $K^{\mathbf{2}}$ (which is not possible by definition of f). Hence, we necessarily have $f(K^{\mathbf{2}}) \subset K^{\mathbf{1}}$. Note that, thanks to Proposition

4.4.6, there exists an isometry $\mu \in \mathrm{O}_2(\mathbb{R})$ such that $\mu(K) = K$ and $s_K \circ f = \mu_{\mathcal{D}}$ (see notations 4.4.7 and 4.4.8). In addition, let us recall that $f^2 = \mathrm{id}_{\mathcal{D}K}$. Thus, using $\mu_{\mathcal{D}} \circ s_K = s_K \circ \mu_{\mathcal{D}}$ and $s_K^2 = \mathrm{id}_{\mathcal{D}K}$, we obtain that μ is involutive on K. However, $\mathrm{Span}(K) = \mathbb{R}^2$; hence, $\mu^2 = \mathrm{id}_{\mathbb{R}^2}$. In particular, μ is diagonalizable on $\mathbb{R}^2$ with eigenvalues ± 1. However, if 1 is an eigenvalue, then f admits a fixed point on the boundary of K, which can't happen. Therefore, $\mu = -\mathrm{id}_{\mathbb{R}^2}$; which implies $K = -K$ and that f coincides with the action of -1 on $\mathcal{D}K$. □

We will focus on how to realise nonnegatively curved metrics on $\mathbb{D}$ using convex compacta. But first, we need to introduce some notations.

Notation 4.4.10. Given $\alpha \in \mathbb{S}^2$, we denote $H_\alpha^- := \{\langle \alpha, \cdot \rangle \leq 0\}$ and $H_\alpha^+ := \{\langle \alpha, \cdot \rangle \geq 0\}$ respectively the lower half-plane and upper half-plane induced by α, and $H_\alpha := \{\alpha\}^\perp$. We write r_α for the reflection w.r.t. H_α. Then, we denote $\mathcal{K}^\alpha := \{D \in \mathcal{K}^s, r_\alpha(D) = D\}$, and:

$$\mathscr{K} := \bigcup_{\alpha \in \mathbb{S}^2} \mathcal{K}^\alpha \times \{\alpha\} \subset \mathcal{K}^s \times \mathbb{S}^2,$$

and $\mathscr{K}_{2\leq 3} := \mathscr{K} \cap \mathcal{K}^s_{2\leq 3} \times \mathbb{S}^2$. The last two spaces introduced are endowed with the direct product topology.

Remark 4.4.3. Assume that $(K, \alpha) \in \mathscr{K}$ such that $\dim(K) = 2$, then we have either $\alpha \in \mathrm{Span}(K)$ or $\alpha \in \mathrm{Span}(K)^\perp$. Indeed, let us assume that $\alpha \notin \mathrm{Span}(K)$. Looking for a contradiction, assume that r_α does not coincide with id on $\mathrm{Span}(K)$. Then, since $r_\alpha(K) = K$, there exists $x \in \mathrm{Span}(K)\backslash\{0\}$ such that $r_\alpha(x) = -x$. Moreover, since $\alpha \notin \mathrm{Span}(K)$, then $\mathrm{Span}(x, \alpha)$ has dimension 2. However, r_α coincides with $-\mathrm{id}$ on $\mathrm{Span}(x, \alpha)$, which contradicts the fact that $\dim(\mathrm{Ker}(r_\alpha + \mathrm{id})) = 1$. Therefore, r_α necessarily coincides with id on $\mathrm{Span}(K)$; hence, $\mathrm{Ker}(r_\alpha - \mathrm{id}) = \mathrm{Span}(K) = \{\alpha\}^\perp$, i.e. $\alpha \in \mathrm{Span}(K)^\perp$.

Proceeding with the same idea, we can show that if $(L, \alpha) \in \mathscr{K}$ satisfies $\dim(L) = 1$, then either $\alpha \in \mathrm{Span}(L)$ or $\alpha \perp \mathrm{Span}(L)$.

The following subsets associated with the double of a plane region will be crucial to obtain nonnegatively curved metrics on $\mathbb{D}$.

Notation 4.4.11. Assume that $(K, \alpha) \in \mathscr{K}$ such that $\dim(K) = 2$ and $\alpha \in \mathrm{Span}(K)$. We denote $\mathcal{D}K_\alpha^\pm := \cup_{\mathbf{i}=\mathbf{1}}^{\mathbf{2}} (K \cap H_\alpha^\pm)^{\mathbf{i}} \subset \mathcal{D}K$, $\mathcal{D}\mathring{K}_\alpha^\pm := \cup_{\mathbf{i}=\mathbf{1}}^{\mathbf{2}} (K \cap \mathring{H}_\alpha^\pm)^{\mathbf{i}} \subset \mathcal{D}K$, and $\mathcal{D}K_\alpha := \cup_{\mathbf{i}=\mathbf{1}}^{\mathbf{2}} (K \cap H_\alpha)^{\mathbf{i}} = \mathcal{D}K_\alpha^+ \cap \mathcal{D}K_\alpha^- \subset \mathcal{D}K$. Observe that all sets above are convex subsets of $\mathcal{D}K$. In particular, $\mathcal{D}K_\alpha^+$ is isometric to a nonnegatively curved metric on $\mathbb{D}$.

It is always possible to symmetrize a convex compactum w.r.t. a specific direction.

Notation 4.4.12. Given $\alpha \in \mathbb{S}^2$ and $D \in \mathcal{K}^s$, we denote $D^\alpha := (D + r_\alpha(D))/2$, where $+$ is the Minkowski sum here.

We now introduce the map that will lead to the correspondence between $\mathscr{M}_{\mathrm{curv}\geq 0}(\mathbb{D})$ and a particular space of convex compacta.

Notation 4.4.13. Given $(D, \alpha) \in \mathscr{K}$ (see Notation 4.4.10), we denote:

(i) $\Phi_{\mathbb{D}}(D, \alpha) := \partial D \cap H_\alpha^+$ if $\dim(D) = 3$,

(ii) $\Phi_{\mathbb{D}}(D, \alpha) := D$ if $\alpha \in \mathrm{Span}(D)^\perp$,

(iii) $\Phi_{\mathbb{D}}(D, \alpha) := \{\mathcal{D}D\}_\alpha^+$ if $\dim(D) = 2$ and $\alpha \in \mathrm{Span}(D)$ (see Notation 4.4.11),

(iv) $\Phi_{\mathbb{D}}(D, \alpha) := D \cap H_\alpha^+$ if $\dim(D) = 1$ and $\alpha \in \mathrm{Span}(D)$.

Note that in any case we have $\Phi_{\mathbb{D}}(D, \alpha) \subset \Phi_{\mathbb{S}^2}(D)$ (see Notation 4.4.9).

First, observe that if $(D, \alpha) \in \mathscr{K}_{2\leq 3}$ (see Notation 4.4.10), it is clear that there exists a nonnegatively curved metric d on $\mathbb{D}$ such that $(\mathbb{D}, \mathrm{d})$ is isometric to $\Phi_{\mathbb{D}}(D, \alpha)$; hence, $[\Phi_{\mathbb{D}}(D, \alpha)] \in \mathscr{M}_{\mathrm{curv}\geq 0}(\mathbb{D})$. Then, note that if $\phi \in \mathrm{O}_3(\mathbb{R})$, then $\Phi_{\mathbb{D}}(D, \alpha)$ and $\Phi_{\mathbb{D}}(\phi(D), \phi(\alpha))$ are isometric. Therefore, there exists a unique map:

$$\Psi_{\mathbb{D}} \colon \mathscr{K}_{2\leq 3}/\,\mathrm{O}_3(\mathbb{R}) \to \mathscr{M}_{\mathrm{curv}\geq 0}(\mathbb{D}) \tag{4.7}$$

such that, for every $(D, \alpha) \in \mathscr{K}_{2\leq 3}$, we have $\Psi_{\mathbb{D}}([D, \alpha]) = [\Phi_{\mathbb{D}}(D, \alpha)]$. Our next goal is to prove the following proposition.

Proposition 4.4.7. The map $\Psi_{\mathbb{D}} \colon \mathscr{K}_{2\leq 3}/\,\mathrm{O}_3(\mathbb{R}) \to \mathscr{M}_{\mathrm{curv}\geq 0}(\mathbb{D})$ introduced in (4.7) is a 1-1 correspondence.

Proof. Let us first prove that $\psi_{\mathbb{D}}$ is surjective. Assume that d is a nonnegatively curved metric on $\mathbb{D}$. Then, thanks to Perelman Doubling Theorem (see [BGP92, section 13.3]), the double metric space $\mathcal{D}(\mathbb{D}, \mathrm{d})$ (see Notation 4.4.6) is also an Alexandrov space with nonnegative curvature. In addition, $\mathcal{D}\mathbb{D}$ is homeomorphic to $\mathbb{S}^2$. Therefore, according to [Bel18b, Theorem 1.1], either there exists $B \in \mathcal{K}^{\mathrm{s}}_{2\leq 3}$

of dimension 3 such that $\mathcal{D}(\mathbb{D},\mathrm{d})$ is isometric to ∂B, or, there exists $K \in \mathcal{K}^{\mathrm{s}}_{2\leq 3}$ of dimension 2 such that $\mathcal{D}(\mathbb{D},\mathrm{d})$ is isometric to $\mathcal{D}K$. Before considering each case, let us denote $\phi\colon \mathcal{D}\mathbb{D} \to \mathcal{D}\mathbb{D}$ the map defined by $\phi(x^{\mathbf{1}}) := x^{\mathbf{2}}$ and $\phi(x^{\mathbf{2}}) := x^{\mathbf{1}}$, for every $x \in \mathbb{D}$. Observe that ϕ is an involutive isometry of $\mathcal{D}(\mathbb{D},\mathrm{d})$, whose set of fix points $\mathrm{Fix}(\phi)$ is equal to $\partial\mathbb{D} = \mathbb{S}^1$.

Let us consider the case where there exists an isometry $f\colon \mathcal{D}(\mathbb{D},\mathrm{d}) \to \partial B$. Note that the map $\psi := f \circ \phi \circ f^{-1}$ is an involutive isometry of ∂B whose set of fix points $\mathrm{Fix}(\psi)$ is homeomorphic to $\mathbb{S}^1$. Arguing as for the 3-dimensional case in the proof of Proposition 4.4.5, we can assume that ψ belong to $\mathrm{O}_3(\mathbb{R})$, satisfies $\psi^2 = \mathrm{id}_{\mathbb{R}^3}$, and $\psi(B) = B$. In particular, ψ is diagonalizable with eigenvalues ± 1. Observe that 1 necessarily has multiplicity 2 in order for $\mathrm{Fix}(\psi)$ to be homeomorphic to $\mathbb{S}^1$; hence, there exists $\alpha \in \mathbb{S}^2$ such that $\psi = r_\alpha$. In particular, $r_\alpha(B) = B$, i.e. $(B,\alpha) \in \mathscr{K}_{2\leq 3}$ (see Notation 4.4.10). Now, notice that since $f(\mathrm{Fix}(\phi)) = \mathrm{Fix}(\psi)$, i.e. $f(\partial\mathbb{D}) = \partial B \cap H_\alpha$. Hence, f maps the disjoint union $\mathring{\mathbb{D}}^{\mathbf{1}} \sqcup \mathring{\mathbb{D}}^{\mathbf{2}}$ onto $\{\partial B \cap \mathring{H}^-_\alpha\} \sqcup \{\partial B \cap \mathring{H}^+_\alpha\}$. In particular, changing α into $-\alpha$ if necessary, we have $f(\mathring{\mathbb{D}}^{\mathbf{2}}) = \partial B \cap \mathring{H}^+_\alpha$. Therefore, f is an isometry from $(\mathbb{D},\mathrm{d})$ to $\Phi_{\mathbb{D}}(B,\alpha)$.

Now, we assume that there exists an isometry $f\colon \mathcal{D}(\mathbb{D},\mathrm{d}) \to \mathcal{D}K$, and, without loss of generality, that $K \subset \mathbb{R}^2$. As before, $\psi := f \circ \phi \circ f^{-1}$ is an involutive isometry of $\mathcal{D}K$ such that $\mathrm{Fix}(\psi)$ is homeomorphic to $\mathbb{S}^1$. As stated by Lemma 4.4.1, we either have $\psi(K^{\mathbf{2}}) \subset K^{\mathbf{2}}$ or $\psi(K^{\mathbf{2}}) \subset K^{\mathbf{1}}$.

If $\psi(K^{\mathbf{2}}) \subset K^{\mathbf{2}}$, then by Proposition 4.4.6 there exists $\mu \in \mathrm{O}_2(\mathbb{R})$ such that $\mu(K) = K$ and $\psi = \mu_{\mathcal{D}}$ (see Notation 4.4.8). Since ψ is involutive and $\mathrm{Span}(K) = \mathbb{R}^2$, we have $\mu^2 = \mathrm{id}_{\mathbb{R}^2}$; which implies that μ is diagonalizable with eigenvalues ± 1. Then, note that 1 necessarily has multiplicity 1, otherwise $\mathrm{Fix}(\psi)$ is not homeomorphic to $\mathbb{S}^1$. Hence, there exists $\alpha \in \mathrm{Span}(K)$ such that $\mu = r_\alpha$; in particular, $(K,\alpha) \in \mathscr{K}_{2\leq 3}$. Observe that $f(\mathrm{Fix}(\phi)) = f(\partial\mathbb{D}) = \mathrm{Fix}(\psi) = \mathcal{D}K_\alpha$ (see Notation 4.4.11). Therefore, f maps the disjoint union $\mathring{\mathbb{D}}^{\mathbf{1}} \sqcup \mathring{\mathbb{D}}^{\mathbf{2}}$ onto $\mathcal{D}\mathring{K}^-_\alpha \sqcup \mathcal{D}\mathring{K}^+_\alpha$. In particular, changing α into $-\alpha$ if necessary, we may assume that $f(\mathring{D}^{\mathbf{2}}) = \mathcal{D}\mathring{K}^+_\alpha$. Therefore, f is an isometry from $(\mathbb{D},\mathrm{d})$ to $\mathcal{D}K^+_\alpha = \Phi_{\mathbb{D}}(K,\alpha)$.

If $\psi(K^{\mathbf{2}}) \subset K^{\mathbf{1}}$, then we can use Proposition 4.4.6 and proceed as above to get $\mu \in \mathrm{O}_2(\mathbb{R})$, such that $\mu(K) = K$, $\mu^2 = \mathrm{id}_{\mathbb{R}^2}$, and $s_K \circ \psi = \mu_{\mathcal{D}}$ (see notations 4.4.7 and 4.4.8). As before, μ is diagonalizable with eigenvalues ± 1. Observe that 1 necessarily has multiplicity 2 for $\mathrm{Fix}(\psi)$ to be homeomorphic to $\mathbb{S}^1$; therefore, $\mu = \mathrm{id}_{\mathbb{R}^2}$. Then, note that f maps $\mathrm{Fix}(\phi) = \partial\mathbb{D}$ onto $\mathrm{Fix}(\psi) = \partial K$. Hence, composing f with s_K if necessary, we can assume that $f(\mathring{D}^{\mathbf{2}}) = \mathring{K}^{\mathbf{2}}$. Therefore, fixing any $\alpha \in \mathrm{Span}(K)^{\perp}$, f induces an isometry between $(\mathbb{D}, \mathrm{d})$ and $K = \Phi_{\mathbb{D}}(K, \alpha)$.

Now let us prove that $\Psi_{\mathbb{D}}$ is injective. Let $(D_i, \alpha_i) \in \mathscr{K}_{2\leq 3}$ $(i \in \{1,2\})$ and assume that there exists an isometry $\phi\colon \Phi_{\mathbb{D}}(D_1, \alpha_1) \to \Phi_{\mathbb{D}}(D_2, \alpha_2)$. We need to prove that there exists $\psi \in \mathrm{O}_3(\mathbb{R})$ such that $D_2 = \psi(D_1)$ and $\alpha_2 = \psi(\alpha_1)$. First of all, observe that ϕ induces an isometry $\phi_{\mathcal{D}}\colon \mathcal{D}\{\Phi_{\mathbb{D}}(D_1, \alpha_1)\} \to \mathcal{D}\{\Phi_{\mathbb{D}}(D_2, \alpha_2)\}$. Moreover, there exists an isometry $\nu_i\colon \mathcal{D}\{\Phi_{\mathbb{D}}(D_i, \alpha_i)\} \to \Phi_{\mathbb{S}^2}(D_i)$ (this is readily checked via a case by case study). In particular, Theorem 4.4.1 implies $\dim(D_1) = \dim(D_2)$.

If $\dim(D_1) = \dim(D_2) = 3$, then $\psi \coloneqq \nu_2 \circ \phi_{\mathcal{D}} \circ \nu_1^{-1}\colon \partial D_1 \to \partial D_2$ is an isometry. Arguing as for the 3-dimensional case in the proof of Proposition 4.4.5, we can assume that $\psi \in \mathrm{O}_3(\mathbb{R})$ satisfies $\psi(D_1) = D_2$. Moreover, we can easily chose ν_i $(i \in \{1,2\})$ so that $\psi(\partial D_1 \cap H_{\alpha_1}) = \partial D_2 \cap H_{\alpha_2}$, which implies that $\psi(H_{\alpha_1}) = H_{\alpha_2}$. Hence, composing ψ with r_{α_2} if necessary, we can conclude that $\psi(\alpha_1) = \alpha_2$.

Let us assume that $\dim(D_1) = \dim(D_2) = 2$. If $\alpha_i \in \mathrm{Span}(D_i)^{\perp}$ $(i \in \{1,2\})$, then $\phi\colon D_1 \to D_2$ is an isometry. Thanks to [ATV01, Theorem 2.2], and since $s(D_1) = s(D_2) = 0$, we can assume that $\phi \in \mathrm{O}_3(\mathbb{R})$. In particular, $\phi(\mathrm{Span}(D_1)^{\perp}) = \mathrm{Span}(D_2)^{\perp}$. Therefore, composing ϕ with r_{α_2} if necessary, we obtain $\phi(D_1) = D_2$ and $\phi(\alpha_1) = \alpha_2$. If $\alpha_1 \in \mathrm{Span}(D_1)^{\perp}$ and $\alpha_2 \in \mathrm{Span}(D_2)$, then D_1 is isometric to $\{\mathcal{D}D_2\}^{+}_{\alpha_2}$. In particular, we necessarily have $\phi(\partial D_1) = \partial\{\mathcal{D}D_2\}^{+}_{\alpha_2} = \{\mathcal{D}D_2\}_{\alpha_2}$. However, $\{\mathcal{D}D_2\}_{\alpha_2}$ is a convex subset of $\{\mathcal{D}D_2\}_{\alpha_2}$ and ∂D_1 is not a convex subset of D_1; thus, that case cannot happen. The case where $\alpha_i \in \mathrm{Span}(D_i)$ $(i \in \{1,2\})$ can be treated in the same way as the case $\dim(D_1) = \dim(D_2) = 3$. □

4.4.3.3 Approximation Lemmas

In this section, we introduce approximation lemmas. The goal here is to define ϵ-isometries between spaces with various dimensions and find an upper bound for the distortion of these maps. This will be crucial later when we will prove that $\Psi^{\mathrm{eq}}_{\mathbb{S}^2}$ (see (4.6)) and $\Psi_{\mathbb{D}}$ (see (4.7)) are homeomorphisms.

First of all, let us recall the following result (see the proof of [BBI22, Lemma 10.2.7]).

Lemma 4.4.2 (3 to 3)**.** Let $B, B' \in \mathcal{K}^s$ such that $\dim(B) = \dim(B') = 3$, and assume that there exists $\epsilon \in (0,1)$ such that $(1-\epsilon)B \subset B' \subset (1+\epsilon)B$ and $(1-\epsilon)B' \subset B \subset (1+\epsilon)B'$. We denote f and g respectively the restriction of $p^{\mathrm{c}}_{\partial B}$ to $\partial B'$ and the restriction of $p^{\mathrm{c}}_{\partial B'}$ to ∂B (see Notation 4.4.2). Then, (f, g) is a GH ν-approximation (see Definition 2.1.4) between $\partial B'$ and ∂B, where $\nu := 6(\mathrm{Diam}(B) + \mathrm{Diam}(B'))\epsilon$. Moreover, if $B, B' \in \tilde{\mathcal{K}}$ (see Notation 4.4.1), then (f, g, id) is an equivariant GH ν-approximation between $(\partial B', \{\pm 1\})$ and $(\partial B, \{\pm 1\})$ (see Remark 2.4.3).

Remark 4.4.4. The result is a bit different than what appears in the proof of [BBI22, Lemma 10.2.7]; we just used the fact that $\mathrm{Diam}(\partial B) \leq \pi \,\mathrm{Diam}(B)$ (which also appears in the proof mentioned above).

Let us introduce the following notation, which relates 3-dimensional and 2-dimensional convex spaces.

Notation 4.4.14. Let $B \in \mathcal{K}^s$ such that $\dim(B) = 3$, let $v \in \mathbb{S}^2$, and denote $K := p_v^{\perp}(B)$ (see Notation 4.4.3). Then, there exist two functions $\phi^{\mathbf{1}} \colon K \to \mathbb{R}$ and $\phi^{\mathbf{2}} \colon K \to \mathbb{R}$ that are respectively convex and concave such that, for every $x \in K$, we have:

$$(x + \mathbb{R}v) \cap B = [x + \phi^{\mathbf{1}}(x)v, x + \phi^{\mathbf{2}}(x)v].$$

Moreover, ∂B can be partitioned as $\partial B = \partial B^{\mathbf{1}} \sqcup \partial B^{\mathbf{2}} \sqcup \partial B^{\mathbf{L}}$, where $\partial B^{\mathbf{i}} := \{x + \phi^{\mathbf{i}}(x)v, x \in \mathring{K}\}$ ($\mathbf{i} \in \{\mathbf{1}, \mathbf{2}\}$), and $\partial B^{\mathbf{L}} := \{y, y \in [x + \phi^{\mathbf{1}}(x)v, x + \phi^{\mathbf{2}}(x)v], x \in \partial K\}$. That allows us to introduce:

$$f_{B,v} \colon \partial B \to \mathcal{D}K,$$

the map defined by $f_{B,v}(x + \phi^{\mathbf{i}}(x)v) := x^{\mathbf{i}}$ (for $x \in \mathring{K}$ and $\mathbf{i} \in \{\mathbf{1}, \mathbf{2}\}$), and $f_{B,v}(y) := x$ (for every $x \in \partial K$ and $y \in [x + \phi^{\mathbf{1}}(x)v, x + \phi^{\mathbf{2}}(x)v]$). Finally, we introduce:

$$g_{B,v} \colon \mathcal{D}K \to \partial B,$$

the map defined by $g_{B,v}(x^{\mathbf{i}}) := x + \phi^{\mathbf{i}}(x)v$ (for $x \in \mathring{K}$ and $\mathbf{i} \in \{\mathbf{1}, \mathbf{2}\}$), and $g_{B,v}(x) := x + [(\phi^{\mathbf{1}}(x) + \phi^{\mathbf{2}}(x))/2]v$ (for $x \in \partial K$).

The next lemma introduces approximations between a 3-dimensional and a 2-dimensional convex space.

Lemma 4.4.3 (3 to 2)**.** Let $B \in \mathcal{K}^s_{2\leq 3}$ such that $\dim(B) = 3$, let $v \in \mathbb{S}^2$, and denote $K := p_v^{\perp}(B)$. We denote $f := f_{B,v}$, $g := g_{B,v}$ (see Notation 4.4.14), and $\epsilon := \sup_{x \in B}\{\mathrm{d_E}(x, p_v^{\perp}(x))\}$. Then, (f, g) is a GH 10ϵ-approximation between ∂B and $\mathcal{D}K$. Moreover, if $B = -B$, then (f, g, id) is an equivariant GH 10ϵ-approximation between $(\partial B, \{\pm 1\})$ and $(\mathcal{D}K, \{\pm 1\})$.

Proof. First of all, assume that $x, y \in \mathring{K}$. Given $t \in [0,1]$, let us denote $\alpha(t) \coloneqq tx+(1-t)y$, and $\gamma(t) \coloneqq \alpha(t)+\phi_{\mathbf{2}}(\alpha(t))v$. Observe that γ is a curve with values in ∂B, such that $\gamma(0) = g(x^{\mathbf{2}})$, and $\gamma(1) = g(y^{\mathbf{2}})$. In particular, $\mathrm{d}_{\partial B}(g(x^{\mathbf{2}}), g(y^{\mathbf{2}})) \leq \mathcal{L}(\gamma) \leq \mathcal{L}(\alpha) + \mathcal{L}(\beta)$, where $\beta(t) \coloneqq \phi_{\mathbf{2}}(\alpha(t))$. However, $\mathcal{L}(\alpha) = \mathrm{d_E}(x, y) = \mathrm{d}_{\mathcal{D}K}(x^{\mathbf{2}}, y^{\mathbf{2}})$. In addition, β is a concave function from $[0, 1]$ to $[-\epsilon, \epsilon]$ (using the fact that $\phi_{\mathbf{2}}$ is concave and $B \subset K + [-\epsilon, \epsilon]v$). Hence, there exists $t_0 \in [0, 1]$ such that β is increasing on $[0, t_0]$ and non-increasing on $[t_0, 1]$. Therefore, $\mathcal{L}(\beta) = \int_0^1 |\beta'| = \int_0^{t_0} \beta' - \int_{t_0}^1 \beta' = 2\beta(t_0) - \beta(1) - \beta(0)$. In particular, we have $\mathcal{L}(\beta) \leq 4\epsilon$; which implies that:

$$\mathrm{d}_{\partial B}(g(x^{\mathbf{2}}), (g(y^{\mathbf{2}})) - \mathrm{d}_{\mathcal{D}K}(x^{\mathbf{2}}, y^{\mathbf{2}}) \leq 4\epsilon. \tag{4.8}$$

Conversely, let us assume that γ is a geodesic between $g(x^{\mathbf{2}})$ and $g(y^{\mathbf{2}})$ on ∂B. Then, note that $p_v^{\perp}(\gamma)$ has shorter length and joins x to y, thus:

$$\mathrm{d}_{\mathcal{D}K}(x^{\mathbf{2}}, y^{\mathbf{2}}) = \mathrm{d_E}(x, y) \leq \mathcal{L}(p_v^{\perp}(\gamma)) \leq \mathcal{L}(\gamma) = \mathrm{d}_{\partial B}(g(x^{\mathbf{2}}), g(y^{\mathbf{2}})). \tag{4.9}$$

Hence, thanks to (4.8) and (4.9), we obtain:

$$|\mathrm{d}_{\partial B}(g(x^{\mathbf{2}}), g(y^{\mathbf{2}}) - \mathrm{d}_{\mathcal{D}K}(x^{\mathbf{2}}, y^{\mathbf{2}})| \leq 4\epsilon.$$

Applying this argument, it is then readily checked that $\mathrm{Dis}(g) \leq 10\epsilon$.

It is then easy to see that, for every $x \in \partial B$, we have $\mathrm{d}_{\partial B}(g \circ f(x), x) \leq 2\epsilon$, and that $f \circ g = \mathrm{id}_{\mathcal{D}K}$. Finally, the argument used to estimate the distortion of g can be used to show that $\mathrm{Dis}(f) \leq 10\epsilon$.

Now, notice that if $B = -B$, then, for every $x \in K$, we have $\phi_{\mathbf{1}}(-x) = -\phi_{\mathbf{2}}(x)$ and $\phi_{\mathbf{2}}(-x) = -\phi_{\mathbf{1}}(x)$. Therefore, both f and g are equivariant. □

The next lemma compares a 3-dimensional convex space with its projection on a line.

Lemma 4.4.4 (3 to 1)**.** Let $B \in \mathcal{K}^s$ such that $\dim(B) = 3$, let $v \in \mathbb{S}^2$, and denote $L \coloneqq p_v(B)$ (see Notation 4.4.3). Then, denoting $f \colon \partial B \to L$ the restriction of p_v to ∂B, and $\epsilon \coloneqq \sup_{x \in B}\{\mathrm{d_E}(x, p_v(x))\}$, we have $\mathrm{Dis}(f) \leq (8 + \pi)\epsilon$.

Proof. First, denoting $\mathcal{B} \coloneqq B_\epsilon(0) \cap \{v\}^\perp$, observe that $B \subset L + \mathcal{B} \eqqcolon C$. Let $x, y \in \partial B$ and observe that $\mathrm{d}_L(f(x), f(y)) = \mathrm{d_E}(p_v(x), p_v(y)) \leq \mathrm{d_E}(x, y) \leq \mathrm{d}_{\partial B}(x, y)$. Hence:

$$\forall x, y \in \partial B, 0 \leq \mathrm{d}_{\partial B}(x, y) - \mathrm{d}_L(f(x), f(y)). \tag{4.10}$$

Now, let us prove that $p_B\colon \partial C \to \partial B$ is surjective (see Notation 4.4.3). Let $x \in \partial B$ and observe that, since B is convex, there exists $w \in \mathbb{S}^2$ such that:

$$B \subset \{y \in \mathbb{R}^3, \langle y - x, w\rangle \leq 0\}.$$

Moreover, since $B \subset C$, there exists $y \in \partial C \cap \{x + \mathbb{R}^+ w\}$; in particular, $x = p_B(y)$. Therefore, $p_B\colon \partial C \to \partial B$ is surjective.

Let $x_i \in \partial B$ and let $x_i' \in \partial C$ such that $x_i = p_B(x_i')$ $(i \in \{1,2\})$. Observe that there exists $x_i'' \in L + \partial\mathcal{B} \subset \partial C$ such that $\mathrm{d}_{\partial C}(x_i', x_i'') = \mathrm{d}_{\mathrm{E}}(x_i', x_i'') \leq \epsilon$ $(i \in \{1,2\})$. Let γ be a geodesic from x_1'' to x_2'' in ∂C. Note that there exists a geodesic γ_L in L, and a geodesic $\gamma_{\partial\mathcal{B}}$ in $\partial\mathcal{B}$, such that $\gamma = \gamma_L + \gamma_{\partial\mathcal{B}}$. Therefore, we have:

$$\begin{aligned}
\mathrm{d}_{\partial B}(x_1, y_2) - \mathrm{d}_L(f(x_1), f(x_2)) &\leq \mathrm{d}_{\partial C}(x_1', x_2') - \mathrm{d}_{\mathrm{E}}(p_v(x_1), p_v(x_2))\\
&\leq 2\epsilon + \mathrm{d}_{\partial C}(x_1'', x_2'') - \mathrm{d}_{\mathrm{E}}(p_v(x_1), p_v(x_2))\\
&\leq 2\epsilon + \mathcal{L}(\gamma_L) + \mathcal{L}(\gamma_{\partial\mathcal{B}}) - \mathrm{d}_{\mathrm{E}}(p_v(x_1), p_v(x_2))\\
&\leq (2+\pi)\epsilon + \mathrm{d}_{\mathrm{E}}(p_v(x_1''), p_v(x_2'')) - \mathrm{d}_{\mathrm{E}}(p_v(x_1), p_v(x_2))\\
&\leq (2+\pi)\epsilon + \sum_{i=1,2} \mathrm{d}_{\mathrm{E}}(x_i, x_i') + \mathrm{d}_{\mathrm{E}}(x_i', x_i'')\\
&\leq (4+\pi)\epsilon + \sum_{i=1,2} \mathrm{d}_{\mathrm{E}}(x_i, x_i'),
\end{aligned}$$

where we used the Busemann–Feller Lemma for the first inequality (see the proof of [BBI22, Lemma 10.2.7]), the fact that $\mathrm{d}_{\partial C}(x_1'', x_2'') = \mathcal{L}(\gamma) \leq \mathcal{L}(\gamma_L) + \mathcal{L}(\gamma_{\partial\mathcal{B}})$ for the third inequality, the fact that γ_L and $\gamma_{\partial\mathcal{B}}$ are geodesics, and that $\mathrm{Diam}(\partial\mathcal{B}) = \pi\epsilon$ for the fourth inequality, and the fact that p_v is 1-Lipschitz for the fifth inequality. Moreover, note that $\mathrm{d}_{\mathrm{E}}(x_i, x_i') = \mathrm{d}_{\mathrm{E}}(x_i', B)$ $(i \in \{1,2\})$. In addition, writing $x_i' = t_i + v_i$ for some $t_i \in L$ and $v_i \in \mathcal{B}$ $(i \in \{1,2\})$, it is clear that $\{t_i + \mathcal{B}\} \cap B \neq \varnothing$. Therefore, $\mathrm{d}_{\mathrm{E}}(x_i', B) \leq \mathrm{Diam}(\mathcal{B}) = 2\epsilon$. Hence, we also have shown that:

$$\forall x, y \in \partial B, \mathrm{d}_{\partial B}(x, y) - \mathrm{d}_L(f(x), f(y)) \leq (8+\pi)\epsilon \tag{4.11}$$

Thanks to (4.10) and (4.11), we can conclude that $\mathrm{Dis}(f) \leq (8+\pi)\epsilon$. □

We now introduce a way to compare doubles of plane convex regions.

Lemma 4.4.5 (2 to 2)**.** Let $K, K' \in \mathcal{K}^s$ such that $\dim(K) = \dim(K') = 2$, and assume that there exists $\epsilon \in (0,1)$ such that $(1-\epsilon)K \subset K' \subset (1+\epsilon)K$ and $(1-\epsilon)K' \subset K \subset (1+\epsilon)K'$. We denote $f\colon \mathcal{D}K \to \mathcal{D}K'$ the map defined by $f(x^{\mathbf{i}}) := [(1-\epsilon)x]^{\mathbf{i}}$ (for $x \in \mathring{K}$ and $\mathbf{i} \in \{\mathbf{1},\mathbf{2}\}$) and $f(x) := p^{\mathrm{c}}_{\partial K'}(x)$ (for

$x \in \partial K$). We define $g\colon \mathcal{D}K' \to \mathcal{D}K$ similarly by exchanging the roles of K and K'. Then, (f, g) is a GH ν-approximation between $\mathcal{D}K$ and $\mathcal{D}K'$, where $\nu := 4(\mathrm{Diam}(K) + \mathrm{Diam}(K'))\epsilon$. Moreover, if $K, K' \in \tilde{\mathcal{K}}$, then (f, g, id) is an equivariant GH ν-approximation between $(\mathcal{D}K, \{\pm 1\})$ and $(\mathcal{D}K', \{\pm 1\})$.

Proof. First, assume that $x, y \in \mathrm{int}(K)$, and observe that:

$$|\mathrm{d}_{\mathcal{D}K'}(f(x^{\mathbf{2}}), f(y^{\mathbf{2}})) - \mathrm{d}_{\mathcal{D}K}(x^{\mathbf{2}}, y^{\mathbf{2}})| = \epsilon\, \mathrm{d}_{\mathrm{E}}(x, y).$$

Then, assume that $y \in \partial K$, and observe that $p^{\mathrm{c}}_{\partial K'}(y) = \lambda y$ for some $\lambda > 0$. However, $(1-\epsilon)y \in K'$; hence, $\lambda \geq (1-\epsilon)$. Then, note that $(1+\epsilon)^{-1} p^{\mathrm{c}}_{\partial K'}(y) \in K$. In particular, we have $q_K((1+\epsilon)^{-1} p^{\mathrm{c}}_{\partial K'}(y)) \leq 1$ (where q_K is the Minkowski gauge associated to K). However, since $y \in \partial K$, then $q_K(y) = 1$. Hence, using $q_K((1+\epsilon)^{-1} p^{\mathrm{c}}_{\partial K'}(y)) = (1+\epsilon)^{-1}\lambda q_K(y) = (1+\epsilon)^{-1}\lambda \leq 1$, we have $\lambda \leq 1+\epsilon$. In particular, we have $|1-\lambda| \leq \epsilon$. Therefore, given $x \in \mathrm{int}(K)$ we have:

$$\begin{aligned}
|\mathrm{d}_{\mathcal{D}K'}(f(x^{\mathbf{2}}), f(y)) - \mathrm{d}_{\mathcal{D}K}(x^{\mathbf{2}}, y)| &= |\mathrm{d}_{\mathrm{E}}((1-\epsilon)x, p^{\mathrm{c}}_{\partial K'}(y)) - \mathrm{d}_{\mathrm{E}}(x, y)| \\
&\leq \mathrm{d}_{\mathrm{E}}((1-\epsilon)x, x) + \mathrm{d}_{\mathrm{E}}(\lambda y, y) \\
&\leq |1-\lambda||y| + \epsilon\, \mathrm{Diam}(K) \\
&\leq 2\epsilon\, \mathrm{Diam}(K).
\end{aligned}$$

Proceeding the same way, we also have $|\mathrm{d}_{\mathcal{D}K'}(f(x^{\mathbf{1}}), f(y)) - \mathrm{d}_{\mathcal{D}K}(x^{\mathbf{1}}, y)| \leq 2\epsilon\, \mathrm{Diam}(K)$. Finally, given $x, y \in \mathrm{int}(K)$, there exists $z \in \partial K$ such that $\mathrm{d}_{\mathcal{D}K}(x^{\mathbf{2}}, y^{\mathbf{1}}) = \mathrm{d}_{\mathrm{E}}(x, z) + \mathrm{d}_{\mathrm{E}}(y, z)$. Hence, we have the following:

$$\begin{aligned}
\mathrm{d}_{\mathcal{D}K'}(f(x^{\mathbf{2}}), f(y^{\mathbf{1}})) &\leq \mathrm{d}_{\mathcal{D}K'}(f(x^{\mathbf{2}}), f(z)) + \mathrm{d}_{\mathcal{D}K'}(f(z), f(y^{\mathbf{1}})) \\
&\leq 4\epsilon\, \mathrm{Diam}(K) + \mathrm{d}_{\mathrm{E}}(x, z) + \mathrm{d}_{\mathrm{E}}(z, y) \\
&\leq 4\epsilon\, \mathrm{Diam}(K) + \mathrm{d}_{\mathcal{D}K}(x^{\mathbf{2}}, y^{\mathbf{1}}).
\end{aligned}$$

Moreover, there exists $z' \in \partial K'$ such that $\mathrm{d}_{\mathcal{D}K'}(f(x^{\mathbf{2}}), f(y^{\mathbf{1}})) = \mathrm{d}_{\mathcal{D}K'}(f(x^{\mathbf{2}}), z') + \mathrm{d}_{\mathcal{D}K'}(z', f(y^{\mathbf{1}}))$. In addition, denoting $z'' := p^{\mathrm{c}}_{\partial K}(z')$, we have $z' = f(z'')$. Therefore, we obtain the following:

$$\begin{aligned}
\mathrm{d}_{\mathcal{D}K'}(f(x^{\mathbf{2}}), f(y^{\mathbf{1}})) &= \mathrm{d}_{\mathcal{D}K'}(f(x^{\mathbf{2}}), f(z'')) + \mathrm{d}_{\mathcal{D}K'}(f(z''), f(y^{\mathbf{1}})) \\
&\geq \mathrm{d}_{\mathcal{D}K}(x^{\mathbf{2}}, z'') + \mathrm{d}_{\mathcal{D}K}(z'', y^{\mathbf{1}}) - 4\epsilon\, \mathrm{Diam}(K) \\
&\geq \mathrm{d}_{\mathcal{D}K}(x^{\mathbf{2}}, y^{\mathbf{1}}) - 4\epsilon\, \mathrm{Diam}(K).
\end{aligned}$$

Hence, we have shown that $\mathrm{Dis}(f) \leq 4\epsilon\,\mathrm{Diam}(K)$. We show the same way that $\mathrm{Dis}(g) \leq 4\,\mathrm{Diam}(K')\epsilon$.

Now, observe that given $x \in K$ and $\mathbf{i} \in \{\mathbf{1}, \mathbf{2}\}$, we have:

$$\mathrm{d}(x^{\mathbf{i}}, g \circ f(x^{\mathbf{i}})) \leq (1 - (1-\epsilon)^2)|x| \leq 2\,\mathrm{Diam}(K)\epsilon.$$

The same holds, replacing f by g and exchanging the roles of K and K'. Hence, (f, g) is a GH ν-approximation between $\mathcal{D}K$ and $\mathcal{D}K'$.

Finally, f and g are equivariant whenever $K, K' \in \tilde{\mathcal{K}}_{2\leq 3}$. □

The following result shows how to compare the double of a plane convex region with its projection on a line.

Lemma 4.4.6 (2 to 1)**.** Let $K \in \mathcal{K}^s$ such that $\dim(K) = 2$, let $v \in \mathrm{Span}(K)\backslash\{0\}$, and let $L \coloneqq p_v(K)$. Let us denote $f\colon \mathcal{D}K \to L$ the map defined by $f(x^{\mathbf{i}}) \coloneqq p_v(x)$ (for $x \in K$ and $\mathbf{i} \in \{\mathbf{1}, \mathbf{2}\}$). Then, we have $\mathrm{Dis}(f) \leq 4\epsilon$, where $\epsilon \coloneqq \sup_{x\in K}\{\mathrm{d_E}(x, p_v(x))\}$.

Proof. Note first that if $x, y \in K$, and if $\mathbf{i} \in \{\mathbf{1}, \mathbf{2}\}$, then:

$$\begin{aligned}
|\mathrm{d}_L(f(x^{\mathbf{i}}), f(y^{\mathbf{i}})) - \mathrm{d}_{\mathcal{D}K}(x^{\mathbf{i}}, y^{\mathbf{i}})| &= |\mathrm{d_E}(p_v(x), p_v(y)) - \mathrm{d_E}(x, y)| \\
&\leq \mathrm{d_E}(x, p_v(x)) + \mathrm{d_E}(y, p_v(y)) \\
&\leq 2\epsilon.
\end{aligned}$$

Now, assume that $x, y \in K$ and let $z \in \partial K$ such that $\mathrm{d}_{\mathcal{D}K}(x^{\mathbf{2}}, y^{\mathbf{1}}) = \mathrm{d_E}(x, z) + \mathrm{d_E}(y, z)$. Note that:

$$\begin{aligned}
\mathrm{d}_L(f(x^{\mathbf{2}}), f(y^{\mathbf{1}})) &= \mathrm{d_E}(p_v(x), p_v(y)) \\
&\leq \mathrm{d_E}(x, y) \\
&\leq \mathrm{d_E}(x, z) + \mathrm{d_E}(z, y) = \mathrm{d}_{\mathcal{D}K}(x^{\mathbf{2}}, y^{\mathbf{1}}).
\end{aligned}$$

Then, let $\beta \in \mathrm{Span}(K)$ such that $\{v, \beta\}$ is an orthonormal basis of $\mathrm{Span}(K)$. Let $s \in [-\epsilon, \epsilon]$ be chosen such that $z' \coloneqq (p_v(x) + p_v(y))/2 + s\beta \in \partial K$. Note that we have the following:

$$\begin{aligned}
\mathrm{d_E}(x, z') &\leq \mathrm{d_E}(x, p_v(x)) + \mathrm{d_E}(z', p_v(z')) + \mathrm{d_E}(p_v(x), p_v(z')) \\
&\leq 2\epsilon + \mathrm{d_E}(p_v(x), p_v(z')),
\end{aligned}$$

and, proceeding the same way, we have $\mathrm{d_E}(y,z') \le 2\epsilon + \mathrm{d_E}(p_v(z'), p_v(y))$. Hence, we have the following:

$$\begin{aligned} \mathrm{d}_{\mathcal{D}K}(x^{\mathbf{2}}, y^{\mathbf{1}}) &\le \mathrm{d_E}(x,z') + \mathrm{d_E}(y,z') \\ &\le 4\epsilon + \mathrm{d_E}(p_v(x), p_v(z')) + \mathrm{d_E}(p_v(z'), p_v(y)) \\ &\le 4\epsilon + \mathrm{d_E}(p_v(x), p_v(y)) = 4\epsilon + \mathrm{d}_L(f(x^{\mathbf{2}}), f(y^{\mathbf{1}})), \end{aligned}$$

where we used the fact that $p_v(z') \in [p_v(x), p_v(y)]$. Therefore, we can conclude that $\mathrm{Dis}(f) \le 4\epsilon$. □

This next lemma is trivial but will be needed for completeness.

Lemma 4.4.7 (1 to 1)**.** Let $L, L' \in \mathcal{K}^s$ such that $\dim(L) = \dim(L') = 1$, and assume that there exists $\epsilon \in (0,1)$ such that $(1-\epsilon)L \subset L' \subset (1+\epsilon)L$ and $(1-\epsilon)L' \subset L \subset (1+\epsilon)L'$. We denote $f\colon L \to L'$ the map defined by $f(x) := (1-\epsilon)x$. We define $g\colon L' \to L$ similarly by exchanging the roles of L and L'. Then, (f,g) is a GH ν-approximation between L and L', where $\nu := 2(\mathrm{Diam}(L) + \mathrm{Diam}(L'))\epsilon$.

The following result treats the case where convex compacta collapse to a point.

Lemma 4.4.8 (Collapsing case)**.** If $D \in \mathcal{K}^s$, then $\mathrm{Diam}(\Phi_{\mathbb{S}^2}(D)) \le \pi\,\mathrm{Diam}(D)$ (see Notation 4.4.9).

Proof. First of all, note that the result is trivial if $\dim(D) \in \{0,1\}$. Then, if $\dim(D) = 2$, it is clear from the definition of the double of a metric space that $\mathrm{Diam}(\mathcal{D}D) \le 2\,\mathrm{Diam}(D)$. Finally, if $\dim(D) = 3$, then we have $\mathrm{Diam}(\partial D) \le \pi\,\mathrm{Diam}(D)$ (see the proof of [BBI22, Lemma 10.2.7]). □

Given a sequence $D_n \to D_\infty$ in $\mathcal{K}^s$ with $\dim(D_n) = 2$ (for $n \in \mathbb{N} \cup \{\infty\}$), there is not necessarily any $\epsilon_n \to 0$ such that $(1-\epsilon_n)D_\infty \subset D_n \subset (1+\epsilon_n)D_\infty$. Indeed, this would hold only if we had $D_n \subset \mathrm{Span}(D_\infty)$ when n is large enough. The following lemma is going to help us fix this issue.

Lemma 4.4.9. Assume that $D_n \to D_\infty$ in $\mathcal{K}^s$ satisfy $\dim(D_\infty) \le \dim(D_n)$ (for every $n \in \mathbb{N}$). Then, there exists $\phi_n \to \mathrm{id}_{\mathbb{R}^3}$ in $\mathrm{O}_3(\mathbb{R})$ such that $\phi_n^{-1}(\mathrm{Span}(D_\infty)) \subset \mathrm{Span}(D_n)$. In addition, if $(D_n, \alpha_n) \to (D_\infty, \alpha_\infty)$ in $\mathscr{K}$ (see Notation 4.4.10), we can also ask $\{\phi_n\}$ to satisfy $\phi_n(\alpha_n) = \alpha_\infty$.

Proof. First of all, we assume that $1 \leq k \coloneqq \dim(D_\infty)$, the case $k = 0$ being trivial. Then let $\{w_i\}_{i=1}^3$ be an oriented orthonormal basis of $\mathbb{R}^3$ such that $\mathrm{Span}(D_\infty) = \mathrm{Span}(\{w_i\}_{i=1}^k)$. Let $r > 0$ such that $B_r(0) \cap \mathrm{Span}(D_\infty) \subset D_\infty$ (such an $r > 0$ always exists since the Steiner point of D_∞ is at the origin and belongs to the relative interior of D_∞). Then, for every $n \in \mathbb{N}$ and $i \in \mathbb{N} \cap [1, k]$, there exists $u_i^n \in D_n$ such that $\mathrm{d_E}(u_i^n, rw_i) \leq \epsilon_n \coloneqq \mathrm{d}_{\mathrm{H}}^{\mathbb{R}^3}(D_n, D_\infty) \to 0$. We can then apply the Gram–Schmidt orthonormalization process to the family $\{u_i^n\}_{i=1}^k$ and get $\{v_i^n\}_{i=1}^k$ such that $\mathrm{Span}(\{v_i^n\}_{i=1}^k) \subset \mathrm{Span}(D_n)$ and, for $i \in \mathbb{N} \cap [1, k]$, $v_i^n \to w_i$.

Let us construct $\{v_i^n\}_{k<i}$ such that $\{v_i^n\}_{i=1}^3$ is an orthonormal basis of $\mathbb{R}^3$ and for $k < i$ we have $v_i^n \to w_i$. If $k = 3$, we are done already. If $k = 2$, we can define $v_3^n \coloneqq v_1^n \wedge v_2^n$. Let us now assume that $k = 1$. In that case, whenever n is large enough, $p_{v_1^n}^\perp \colon \{v_1\}^\perp \to \{v_1^n\}^\perp$ is an isomorphism. We can then define $u_i^n \coloneqq p_{v_1^n}^\perp(w_i)$ for $i \in \{2, 3\}$. Applying the Gram–Schmidt orthonormalization process to the family $\{u_2^n, u_3^n\}$ gives rise to a family $\{v_2^n, v_3^n\}$ satisfying the desired properties.

To conclude the first part of the proof, let $\phi_n \in \mathrm{O}_3(\mathbb{R})$ such that $\phi_n(v_i^n) = w_i$ ($i \in \{1, 2, 3\}$), and observe that ϕ_n satisfies the desired properties by construction.

Now, let us assume that $(D_n, \alpha_n) \to (D_\infty, \alpha_\infty)$ in $\mathscr{K}$. It is readily checked, proceeding case by case (and remembering that $(D_n, \alpha_n) \in \mathscr{K}$ implies either $\alpha_n \in \mathrm{Span}(D_n)$ or $\alpha_n \perp \mathrm{Span}(D_n)$), that we can construct $\{v_i^n\}$ and $\{w_i\}$ so that $\alpha_\infty = w_i$ for some $i \in \{1, 2, 3\}$, and $v_i^n = \alpha_n$ for every $n \in \mathbb{N}$. This concludes the proof. □

4.4.3.4 Moduli spaces of nonnegatively curved metrics

We have seen in Theorem 4.4.1 that $\mathscr{M}_{\mathrm{curv}\geq 0}(\mathbb{S}^2)$ is homeomorphic to $\mathcal{K}^{\mathrm{s}}_{2\leq 3} / \mathrm{O}_3(\mathbb{R})$. We are now going to prove results in the same spirit for $\mathscr{M}^{\mathrm{eq}}_{\mathrm{curv}\geq 0}(\mathbb{S}^2)$ and $\mathscr{M}_{\mathrm{curv}\geq 0}(\mathbb{D})$.

Proposition 4.4.8. The map $\Psi^{\mathrm{eq}}_{\mathbb{S}^2} \colon \tilde{\mathcal{K}}_{2\leq 3} / \mathrm{O}_3(\mathbb{R}) \to \mathscr{M}^{\mathrm{eq}}_{\mathrm{curv}\geq 0}(\mathbb{S}^2)$ introduced in (4.7) is a homeomorphism.

Proof. We have already seen in Proposition 4.4.5 that $\Psi^{\mathrm{eq}}_{\mathbb{S}^2}$ is bijective; therefore, we just need to prove that $\Psi^{\mathrm{eq}}_{\mathbb{S}^2}$ and $\{\Psi^{\mathrm{eq}}_{\mathbb{S}^2}\}^{-1}$ are continuous. However, observe that convergence in the equivariant GH topology implies convergence in the GH topology. Thus, thanks to Theorem 4.4.1, $\{\Psi^{\mathrm{eq}}_{\mathbb{S}^2}\}^{-1}$ is continuous.

Let $D_n \to D_\infty$ in $\tilde{\mathcal{K}}_{2\leq 3}$ and let us prove that $\mathcal{D}^{\mathrm{eq}}(\Phi_{\mathbb{S}^2}(D_n), \Phi_{\mathbb{S}^2}(D_\infty)) \to 0$ (see Remark 2.4.3 for the definition of $\mathcal{D}^{\mathrm{eq}}$), which will imply $\Psi^{\mathrm{eq}}_{\mathbb{S}^2}$'s continuity. Since the dimension of compact convex sets is lower semi-continuous with respect to the Hausdorff distance, we can assume that for every $n \in \mathbb{N}$, we have $\dim(D_\infty) \leq \dim(D_n)$ (forgetting the first terms of the sequence if necessary). According to Lemma 4.4.9, there exists $\phi_n \to \mathrm{id}_{\mathbb{R}^3}$ in $\mathrm{O}_3(\mathbb{R})$ such that $\phi_n^{-1}(\mathrm{Span}(D_\infty)) \subset \mathrm{Span}(D_n)$. Let us introduce $H_n \coloneqq \phi_n^{-1}(\mathrm{Span}(D_\infty))$, $p_n \coloneqq p_{H_n}$ (see Notation 4.4.3), $D'_n \coloneqq p_n(D_n)$, and $\epsilon_n \coloneqq \sup_{x\in D_n}\{\mathrm{d_E}(x, p_n(x))\}$. Observe that $\epsilon_n \to 0$. Indeed, given $x \in D_n$, we have $\mathrm{d_E}(x, p_n(x)) \leq \mathrm{d_E}(x, p_\infty(x)) + \mathrm{d_E}(p_n(x), p_\infty(x))$, where $p_\infty = p_{\mathrm{Span}(D_\infty)}$. In particular, we have $\mathrm{d_E}(x, p_n(x)) \leq |p_n - p_\infty| \,\mathrm{Diam}(D_n) + \mathrm{d_H}(D_n, D_\infty)$. However, since $\phi_n \to \mathrm{id}_{\mathbb{R}^3}$, it is readily checked that $|p_n - p_\infty| \to 0$; hence, since $\{\mathrm{Diam}(D_n)\}$ is bounded, we have $\epsilon_n \to 0$ (note that the proof also works if $\dim(D_\infty) = 1$).

Now, let us prove that $\mathcal{D}^{\mathrm{eq}}(\Phi_{\mathbb{S}^2}(D'_n), \Phi_{\mathbb{S}^2}(D_\infty)) \to 0$. First of all, note that $\Phi_{\mathbb{S}^2}(D'_n)$ is equivariantly isometric to $\Phi_{\mathbb{S}^2}(D''_n)$, where $D''_n \coloneqq \phi_n(D'_n)$; therefore, we only have to show $\mathcal{D}^{\mathrm{eq}}(\Phi_{\mathbb{S}^2}(D''_n), \Phi_{\mathbb{S}^2}(D_\infty)) \to 0$. Note that $\mathrm{d_H}(D'_n, D''_n) \leq \mathrm{Diam}(D_n)|\phi_n - \mathrm{id}|$, $\mathrm{d_H}(D_n, D'_n) \leq \epsilon_n$; hence, applying the triangle inequality, we have $\mathrm{d_H}(D''_n, D_\infty) \leq \mathrm{d_H}(D_n, D_\infty) + \mathrm{Diam}(D_n)|\phi_n - \mathrm{id}| + \epsilon_n \to 0$. Moreover, observe that $D''_n \subset \mathrm{Span}(D_\infty)$ (due to the properties of ϕ_n). Thus, there exists $\mu_n \to 0$ such that $(1-\mu_n)D_\infty \subset D''_n \subset (1+\mu_n)D_\infty$ and $(1-\mu_n)D''_n \subset D_\infty \subset (1+\mu_n)D''_n$. In particular, applying Lemma 4.4.2 or 4.4.5 (depending on $\dim(D_\infty)$ being 2 or 3), we obtain $\mathcal{D}^{\mathrm{eq}}(\Phi_{\mathbb{S}^2}(D''_n), \Phi_{\mathbb{S}^2}(D_\infty)) \to 0$.

Observe now that if $\dim(D_\infty) = \dim(D_n)$ then $D_n = D'_n$ by assumption on ϕ_n. We will therefore assume that $\dim(D'_n) = \dim(D_\infty) < \dim(D_n)$ to avoid trivialities. Thanks to Lemma 4.4.3, and using $\epsilon_n \to 0$, we obtain $\mathcal{D}^{\mathrm{eq}}(\Phi_{\mathbb{S}^2}(D_n), \Phi_{\mathbb{S}^2}(D'_n)) \to 0$.

We proved $\mathcal{D}^{\mathrm{eq}}(\Phi_{\mathbb{S}^2}(D_n), \Phi_{\mathbb{S}^2}(D'_n)) \to 0$ and $\mathcal{D}^{\mathrm{eq}}(\Phi_{\mathbb{S}^2}(D'_n), \Phi_{\mathbb{S}^2}(D_\infty)) \to 0$, which implies $\mathcal{D}^{\mathrm{eq}}(\Phi_{\mathbb{S}^2}(D_n), \Phi_{\mathbb{S}^2}(D_\infty)) \to 0$ as a consequence of the modified triangle inequality satisfied by $\mathcal{D}^{\mathrm{eq}}$ (see (2.6)). $\square$

Proposition 4.4.9. The map $\Psi_{\mathbb{D}} \colon \mathscr{K}_{2\leq 3}/\,\mathrm{O}_3(\mathbb{R}) \to \mathscr{M}_{\mathrm{curv}\geq 0}(\mathbb{D})$ introduced in (4.7) is a homeomorphism.

Proof. Let us recall that $\Psi_{\mathbb{D}}$ is a 1-1 correspondence thanks to Proposition 4.4.7. We will start by proving the continuity of $\Psi_{\mathbb{D}}$. Let us assume that $(D_n, \alpha_n) \to (D_\infty, \alpha_\infty)$ in $\mathscr{K}$ and let us prove that $\{\Phi_{\mathbb{D}}(D_n, \alpha_n)\}$ converges to $\Phi_{\mathbb{D}}(D_\infty, \alpha_\infty)$ in the GH topology (note that this is stronger than proving $\Psi_{\mathbb{D}}$'s continuity since here we do not ask $\dim(D_n) \in \{2, 3\}$). Without loss of generality, we will assume that for every $n \in \mathbb{N}$, we have $\dim(D_\infty) \leq \dim(D_n)$. Note that if $\dim(D_\infty) = 0$, then Lemma 4.4.8 implies $\mathrm{Diam}(\Phi_{\mathbb{D}}(D_n, \alpha_n)) \leq \mathrm{Diam}(\Phi_{\mathbb{S}^2}(D_n)) \leq \pi\,\mathrm{Diam}(D_n) \to 0$. In particular, $\Phi_{\mathbb{D}}(D_n, \alpha_n) \to 0 = \Phi_{\mathbb{D}}(D_\infty, \alpha_\infty)$. Now we assume that $1 \leq \dim(D_\infty)$. Let $\phi_n \to \mathrm{id}$, $p_n \to p_\infty$, D'_n, D''_n, and $\epsilon_n \to 0$ be defined exactly as in the proof of Proposition 4.4.8, asking that $\phi_n(\alpha_n) = \alpha_\infty$ (which is possible due to Lemma 4.4.9).

We first prove that $\mathrm{d_{GH}}(\Phi_{\mathbb{D}}(D'_n, \alpha_n), \Phi_{\mathbb{D}}(D_\infty, \alpha_\infty)) \to 0$. Observe that $\Phi_{\mathbb{D}}(D'_n, \alpha_n)$ is isometric to $\Phi_{\mathbb{D}}(D''_n, \alpha_\infty)$. Moreover, proceeding exactly as in the proof of Proposition 4.4.8, we can show that there exists $\mu_n \to 0$ such that $(1-\mu_n)D_\infty \subset D''_n \subset (1+\mu_n)D_\infty$ and $(1-\mu_n)D''_n \subset D_\infty \subset (1+\mu_n)D''_n$. Let $f_n\colon \Phi_{\mathbb{S}^2}(D''_n) \to \Phi_{\mathbb{S}^2}(D_\infty)$ and $g_n\colon \Phi_{\mathbb{S}^2}(D_\infty) \to \Phi_{\mathbb{S}^2}(D''_n)$ be defined as in Lemma 4.4.2, 4.4.5, or 4.4.7 (depending on $\dim(D_\infty)$ being 3, 2, or 1). Observe that in every case, we have $f_n(\Phi_{\mathbb{D}}(D''_n, \alpha_\infty)) \subset \Phi_{\mathbb{D}}(D_\infty, \alpha_\infty)$ and $g_n(\Phi_{\mathbb{D}}(D_\infty, \alpha_\infty)) \subset \Phi_{\mathbb{D}}(D''_n, \alpha_\infty)$; in particular, $\mathrm{d_{GH}}(\Phi_{\mathbb{D}}(D_\infty, \alpha_\infty)), \Phi_{\mathbb{D}}(D''_n, \alpha_\infty)) \leq \mathrm{d_{GH}}(\Phi_{\mathbb{S}^2}(D_\infty), \Phi_{\mathbb{S}^2}(D''_n)) \to 0$ (using the estimates of the lemmas mentioned before depending on the dimension of $\dim(D_\infty)$).

Now, let us prove that $\mathrm{d_{GH}}(\Phi_{\mathbb{D}}(D_n, \alpha_n), \Phi_{\mathbb{D}}(D'_n, \alpha_n)) \to 0$. Observe that if $\dim(D_n) = \dim(D_\infty)$, then $\phi_n^{-1}(\mathrm{Span}(D_\infty)) \subset \mathrm{Span}(D_n)$ implies that $D_n = D'_n$. Hence, we will assume that $\dim(D_\infty) = \dim(D'_n) < \dim(D_n)$ to avoid trivialities. Let $f_n\colon \Phi_{\mathbb{S}^2}(D_n) \to \Phi_{\mathbb{S}^2}(D'_n)$ be defined as in Lemma 4.4.3, 4.4.4, or 4.4.6 (depending on $(\dim(D_n), \dim(D_\infty))$ being equal to $(3,2)$, $(3,1)$, or $(2,1)$). It is readily checked that in every case, we have $f_n(\Phi_{\mathbb{D}}(D_n, \alpha_n)) = \Phi_{\mathbb{D}}(D'_n, \alpha_n)$; therefore, applying the previous lemmas' estimates, and using $\epsilon_n \to$, we obtain $\mathrm{d_{GH}}(\Phi_{\mathbb{D}}(D_n, \alpha_n), \Phi_{\mathbb{D}}(D'_n, \alpha_n)) \to 0$. This concludes the proof of $\Psi_{\mathbb{D}}$'s continuity.

Now, we are going to prove that $\Psi_{\mathbb{D}}^{-1}$ is continuous. Let us assume that $[\mathbb{D}, \mathrm{d}_n] \to [\mathbb{D}, \mathrm{d}_\infty]$ w.r.t. the GH topology on $\mathscr{M}_{\mathrm{curv}\geq 0}(\mathbb{D})$. According to Proposition 4.4.7, for every $n \in \mathbb{N} \cup \{\infty\}$ there exists $(D_n, \alpha_n) \in \mathscr{K}_{2\leq 3}$ such that $(\mathbb{D}, \mathrm{d}_n)$ is isometric to $\Phi(D_n, \alpha_n)$. We need to show that $[D_n, \alpha_n] \to [D_\infty, \alpha_\infty]$ in $\mathscr{K}_{2\leq 3}/\mathrm{O}_3(\mathbb{R})$. Note that, since $\mathscr{K}_{2\leq 3}/\mathrm{O}_3(\mathbb{R})$ is a metric space, it is sufficient to prove that every subsequence of $\{[D_n, \alpha_n]\}$ admits a subsequence converging to $[D_\infty, \alpha_\infty]$. Reindexing the sequence if necessary, let us just prove that $\{[D_n, \alpha_n]\}_{n\in\mathbb{N}}$ admits a subsequence converging to $[D_\infty, \alpha_\infty]$.

First of all, observe that since $\mathbb{S}^2$ is compact, we can assume that $\alpha_n \to \alpha \in \mathbb{S}^2$ (reindexing the sequence if necessary). Then, observe that if $\dim(D_n) = 3$ then $\mathrm{Diam}(D_n) \leq \mathrm{Diam}(\partial D_n) \leq 2\,\mathrm{Diam}(\partial D_n \cap H^+_{\alpha_n}) = 2\,\mathrm{Diam}(\mathbb{D}, \mathrm{d}_n)$. Moreover, if $\dim(D_n) = 2$, then:

- either $\alpha_n \perp \mathrm{Span}(D_n)$, which implies $\mathrm{Diam}(D_n) = \mathrm{Diam}(\mathbb{D}, \mathrm{d}_n)$,

- or $\alpha_n \in \mathrm{Span}(D_n)$; in which case we have the following inequality $\mathrm{Diam}(D_n) \leq \mathrm{Diam}(\mathcal{D}D_n) \leq 2\,\mathrm{Diam}(\{\mathcal{D}D_n\}^+_{\alpha_n}) = 2\,\mathrm{Diam}(\mathbb{D}, \mathrm{d}_n)$.

Since $\{\mathrm{Diam}(\mathbb{D}, \mathrm{d}_n)\}_{n\in\mathbb{N}}$ is bounded, we can conclude that there exists $r \in (0, \infty)$ such that, for every $n \in \mathbb{N}$, we have $\mathrm{Diam}(D_n) \leq r$. Then, observe that, for every $n \in \mathbb{N}$, we have $0 = s(D_n) \in D_n$. Hence $\{D_n\}$ is a sequence of convex compact subsets of $\overline{B}_r(0)$. Thanks to Blaschke Theorem (see [BBI22, Theorem 7.3.8]), we can assume (passing to a subsequence if necessary) that $D_n \to D$ w.r.t. $\mathrm{d}_{\mathrm{H}}^{\mathbb{R}^3}$, where D is a compact convex subset if $\overline{B}_r(0)$. Note that since $\alpha_n \to \alpha$, it is readily checked that $D_n^{\alpha_n} \to D^\alpha$ (see Notation 4.4.12). However, $D_n^{\alpha_n} = D_n \to D$; therefore $(D, \alpha) \in \mathscr{K}$. To conclude, we just need to prove that there exists $\phi \in \mathrm{O}_3(\mathbb{R})$ such that $\phi(D) = D_\infty$ and $\phi(\alpha) = \alpha_\infty$.

Thanks to the first part of the proof, $(D_n, \alpha_n) \to (D, \alpha)$ implies $\Phi_{\mathbb{D}}(D_n, \alpha_n) \to \Phi_{\mathbb{D}}(D, \alpha)$. However, $\Phi_{\mathbb{D}}(D_n, \alpha_n) \to \Phi_{\mathbb{D}}(D_\infty, \alpha_\infty)$ by assumption. Therefore, there exists an isometry between $\Phi_{\mathbb{D}}(D_\infty, \alpha_\infty)$ and $\Phi_{\mathbb{D}}(D, \alpha)$. However, $\Phi_{\mathbb{D}}(D_\infty, \alpha_\infty)$ is homeomorphic to $\mathbb{D}$; therefore, we necessarily have $2 \leq \dim(D)$, i.e. $(D, \alpha) \in \mathscr{K}_{2\leq 3}$. Thus, since $\Psi_{\mathbb{D}} \colon \mathscr{K}_{2\leq 3}/\,\mathrm{O}_3(\mathbb{R}) \to \mathscr{M}_{\mathrm{curv}\geq 0}(\mathbb{D})$ is a 1-1 correspondence, we have $[D, \alpha] = [D_\infty, \alpha_\infty]$, which concludes the proof. □

4.4.3.5 Moduli spaces of RCD(0, 2) structures

We finally show that the moduli spaces of RCD(0, 2) structures $\mathfrak{M}_{0,2}(\mathbb{S}^2)$, $\mathfrak{M}_{0,2}(\mathbb{RP}^2)$, and $\mathfrak{M}_{0,2}(\mathbb{D})$ are contractible.

Proposition 4.4.10. The moduli space $\mathfrak{M}_{0,2}(\mathbb{S}^2)$ of RCD(0, 2) structures on $\mathbb{S}^2$ is homeomorphic to $\mathbb{R} \times \{\mathcal{K}^{\mathrm{s}}_{2\leq 3} \,/\, \mathrm{O}_3(\mathbb{R})\}$. In particular, $\mathfrak{M}_{0,2}(\mathbb{S}^2)$ is contractible.

Proof. Lemma 4.2.1 and Theorem 4.4.1 implie that $\mathfrak{M}_{0,2}(\mathbb{S}^2)$ is homeomorphic to $\mathbb{R} \times \{\mathcal{K}^{\mathrm{s}}_{2\leq 3} \,/\, \mathrm{O}_3(\mathbb{R})\}$. To conclude, we only need to show that $\mathcal{K}^{\mathrm{s}}_{2\leq 3} \,/\, \mathrm{O}_3(\mathbb{R})$ is contractible. To do so, we define $H \colon I \times \mathcal{K}^{\mathrm{s}}_{2\leq 3} \to \mathcal{K}^{\mathrm{s}}_{2\leq 3}$ by $H(t, D) \coloneqq t\mathbb{B} + (1-t)D$ for every $(t, D) \in I \times \mathcal{K}^{\mathrm{s}}_{2\leq 3}$ (where $+$ is the Minkowski sum and $\mathbb{B}$ is the unit ball in $\mathbb{R}^3$). Observe that H is $\mathrm{O}_3(\mathbb{R})$-equivariant, satisfies $H(0, \cdot) = \mathrm{id}_{\mathcal{K}^{\mathrm{s}}_{2\leq 3}}$, and $H(1, \cdot)$ is the constant function equal to $\mathbb{B}$. Hence, we just need to show that H is continuous. Let $D_1, D_2 \in \mathcal{K}^{\mathrm{s}}_{2\leq 3}$ and let $t, s \in I$. Assume that $y \in \mathbb{B}$ and $z \in D_1$, and let $x_t \coloneqq ty + (1-t)z \in H(t, D_1)$ and $x_s \coloneqq sy + (1-s)z \in H(s, D_1)$. Observe that $\mathrm{d}_{\mathrm{E}}(x_t, x_s) = |t-s|\,\mathrm{d}_{\mathrm{E}}(y, z) \leq |t-s|(1 + \mathrm{Diam}(D_1))$. Hence, we have the following:

$$\mathrm{d}_{\mathrm{H}}^{\mathbb{R}^3}(H(D_1, t), H(D_1, s)) \leq |t-s|(1 + \mathrm{Diam}(D_1)). \tag{4.12}$$

Then, observe that there exists $z' \in D_2$ such that $\mathrm{d_E}(z, z') \leq \epsilon$ (where $\epsilon > 0$ is any positive number such that $\mathrm{d}_{\mathrm{H}}^{\mathbb{R}^3}(D_1, D_2) < \epsilon$). Denoting $x'_s := sy + (1-s)z' \in H(s, D_2)$, we have $\mathrm{d_E}(x_s, x'_s) = (1-s)\,\mathrm{d_E}(z, z') \leq (1-s)\epsilon$. Therefore, we have $\mathrm{d}_{\mathrm{H}}^{\mathbb{R}^3}(H(D_1, s), H(D_2, s)) \leq (1-s)\epsilon$, and, letting ϵ go to $\mathrm{d}_{\mathrm{H}}^{\mathbb{R}^3}(D_1, D_2)$, we obtain:

$$\mathrm{d}_{\mathrm{H}}^{\mathbb{R}^3}(H(s, D_1), H(s, D_2)) \leq (1-s)\,\mathrm{d}_{\mathrm{H}}^{\mathbb{R}^3}(D_1, D_2). \tag{4.13}$$

Applying the triangle inequality to $\mathrm{d}_{\mathrm{H}}^{\mathbb{R}^3}$, and using (4.12) and (4.13) we finally get:

$$\mathrm{d}_{\mathrm{H}}^{\mathbb{R}^3}(H(t, D_1), H(s, D_2)) \leq |t-s|(2+\mathrm{Diam}(D_1)+\mathrm{Diam}(D_2))+(1-s)\,\mathrm{d}_{\mathrm{H}}^{\mathbb{R}^3}(D_1, D_2).$$

In particular, H is continuous. □

Proposition 4.4.11. The moduli space $\mathfrak{M}_{0,2}(\mathbb{RP}^2)$ of $\mathrm{RCD}(0,2)$ structures on $\mathbb{RP}^2$ is homeomorphic to $\mathbb{R} \times \{\tilde{\mathfrak{K}}_{2\leq 3} \,/\, \mathrm{O}_3(\mathbb{R})\}$; in particular, it is contractible.

Proof. First of all, note that thanks to Corollary 2.4.2, the lift map $p^* : \mathfrak{M}_{0,2}(\mathbb{RP}^2) \to \mathfrak{M}^{\mathrm{eq}}_{0,2}(\mathbb{S}^2)$ is a homeomorphism (where $p\colon \mathbb{S}^2 \to \mathbb{S}^2/\{\pm 1\} = \mathbb{RP}^2$ is the quotient map). Moreover, applying Lemma 4.2.1, $\mathfrak{M}^{\mathrm{eq}}_{0,2}(\mathbb{S}^2)$ is homeomorphic to $\mathbb{R} \times \mathscr{M}^{\mathrm{eq}}_{\mathrm{curv}\geq 0}(\mathbb{S}^2)$. Thus, Proposition 4.4.8 implies that $\mathscr{M}^{\mathrm{eq}}_{\mathrm{curv}\geq 0}(\mathbb{S}^2)$ is homeomorphic to $\tilde{\mathfrak{K}}_{2\leq 3} \,/\, \mathrm{O}_3(\mathbb{R})$. Now, observe that the map $H\colon I \times \mathfrak{K}^{\mathrm{s}}_{2\leq 3} \to \mathfrak{K}^{\mathrm{s}}_{2\leq 3}$ introduced in the proof of Proposition 4.4.10 satisfies $H(I \times \tilde{K}_{2\leq 3}) \subset \tilde{K}_{2\leq 3}$. Hence, $\tilde{\mathfrak{K}}_{2\leq 3} \,/\, \mathrm{O}_3(\mathbb{R})$ is contractible, and, a fortiori, $\mathfrak{M}_{0,2}(\mathbb{RP}^2)$ is contractible. □

Proposition 4.4.12. The moduli space $\mathfrak{M}_{0,2}(\mathbb{D})$ of $\mathrm{RCD}(0,2)$ structures on $\mathbb{D}$ is homeomorphic to $\mathbb{R} \times (\mathscr{K}_{2\leq 3}/\, \mathrm{O}_3(\mathbb{R}))$ (where $\mathscr{K}_{2\leq 3}$ is introduced in Notation 4.4.10); in particular, it is contractible.

Proof. Observe that, according to Lemma 4.2.1, $\mathfrak{M}_{0,2}(\mathbb{D})$ is homeomorphic to $\mathbb{R} \times \mathscr{M}_{\mathrm{curv}\geq 0}(\mathbb{D})$. Thus, thanks to Proposition 4.4.9, $\mathfrak{M}_{0,2}(\mathbb{D})$ is homeomorphic to $\mathbb{R} \times \mathscr{K}_{2\leq 3}/\, \mathrm{O}_3(\mathbb{R})$. Now let us consider the map $H_{\mathbb{D}}\colon I \times \mathscr{K}_{2\leq 3} \to \mathscr{K}_{2\leq 3}$ defined by $H_{\mathbb{D}}(t, D, \alpha) := (H(t, D), \alpha)$, where $H\colon I \times \mathfrak{K}^{\mathrm{s}}_{2\leq 3} \to \mathfrak{K}^{\mathrm{s}}_{2\leq 3}$ is introduced in the proof of Proposition 4.4.10. Observe that $H_{\mathbb{D}}$ is continuous since H is continuous. Moreover, note that $H_{\mathbb{D}}$ is equivariant w.r.t. the action of $\mathrm{O}_3(\mathbb{R})$ on $\mathscr{K}_{2\leq 3}$. To conclude, note that since $\mathrm{O}_3(\mathbb{R})$ acts transitively on $\mathbb{S}^2$, we have $[H_{\mathbb{D}}(1, \cdot)] \equiv [\mathbb{B}, (0,0,1)] \in \mathscr{K}_{2\leq 3}/\, \mathrm{O}_3(\mathbb{R})$; hence $\mathscr{K}_{2\leq 3}/\, \mathrm{O}_3(\mathbb{R})$ is contractible. □

5

RCD(-1,2) spaces and Ricci flow

Contents

The purpose of this chapter is to investigate the properties of the moduli space of $\mathrm{RCD}(-1,2)$ structures on a connected closed orientable topological surface. This case differs from Chapters 3 and 4 because we cannot rely on splitting maps. Instead, we will use the Ricci flow on Alexandrov spaces, as introduced in [Ric12], to establish that the moduli space of $\mathrm{RCD}(-1,2)$ structures on a surface with negative Euler characteristic is homotopy equivalent to the moduli space of hyperbolic metrics.

In Section 5.1, we provide a brief overview of Ricci flow and establish uniform convergence estimates for sequences of normalized Ricci flows. In Section 5.2, we recall the definition of the Ricci flow on Alexandrov surfaces, as introduced in [Ric12]. In particular, we demonstrate that the moduli space of Alexandrov metrics with curv ≥ -1 retracts by deformation onto the moduli space of hyperbolic metrics (in the case of negative Euler characteristic). Finally, in Section 5.3, we apply these results to the moduli spaces of $\mathrm{RCD}(-1,2)$ structures.

5.1 Smooth Ricci flow

First of all, in Section 5.1.1, we introduce Hamilton's Ricci flow, present some compactness theorem, and introduce the normalized Ricci flow on surfaces. Then, in Section 5.1.2, we introduce curvature potentials and some technical lemmas. Finally, in Section 5.1.3, we obtain uniform convergence estimates for sequences of normalized Ricci flows.

5.1.1 Basics of Ricci flow

5.1.1.1 Hamilton's Ricci flow

The Ricci flow is a geometric evolution equation introduced by Richard Hamilton in [Ham82] as a tool to study the geometry of 3-manifolds. In particular, it played a crucial role in proving Poincaré's conjecture. First of all, we introduce the long-time existence theorem for the Ricci flow on a closed manifold (see [Ham82, Theorem 14.1]).

Theorem 5.1.1 (Short-time existence)**.** Let (X^n, g_0) be a closed Riemannian manifold of dimension $n \geq 2$. The evolution equation:

$$\begin{cases} \partial_t g(t) = -2\,\mathrm{Ric}(g(t)) \\ g(0) = g_0 \end{cases},$$

admits a unique solution $\{g(t)\}_{t\in[0,T)}$ on a maximal time interval $[0,T)$, where $0 < T \leq \infty$. The maximal solution $\{g(t)\}_{t\in[0,T)}$ is called the *(maximal) Ricci flow starting from* g_0.

It is in general possible to compute the evolution equation of most of the geometric quantities associated to a Ricci flow. The following result provides the volume evolution equation (see [Ham82, Section 3]).

Proposition 5.1.1 (Evolution of the volume)**.** Let (X^n, g_0) be a connected closed orientable Riemannian manifold of dimension $n \geq 2$ and let $\{g(t)\}_{t\in[0,T)}$ be the maximal Ricci flow starting from g_0. The volume satisfies the following evolution equation:

$$\partial_v \operatorname{vol}(g(t)) = -\int_X \operatorname{scal}(g(t)) \operatorname{dvol}_{g(t)} . \tag{5.1}$$

In particular, if $n = 2$, then $\partial_t \operatorname{vol}(g(t)) = -4\pi\chi(X)$ (applying Gauss–Bonnet theorem).

5.1.1.2 Compactness results

In this section we fix $-\infty \leq a < 0 < b \leq \infty$ and $n \geq 2$. First, we recall the notion of Ricci flows' convergence in the Hamilton–Cheeger–Gromov sense introduced by Hamilton in [Ham95] (stated in the more recent formalism of [Top14]).

Definition 5.1.1. A sequence $\{(M_i^n, g_i(t), p_i)\}_{i\in\mathbb{N},t\in(a,b)}$ of pointed complete Ricci flows converges to a pointed complete Ricci flow $\{(M^n, g(t), p)\}_{t\in(a,b)}$ *in the Hamilton–Cheeger–Gromov sense* if there exists:

(i) a sequence of compact sets $\Omega_i \subset M^n$ exhausting M^n such that, for all $i \in \mathbb{N}$, $p_i \in \operatorname{int}(\Omega_i)$,

(ii) a sequence of smooth maps $\phi_i \colon \Omega_i \to M_i^n$, diffeomorphic onto their images, and with $\phi_i(p) = p_i$, such that $\phi_i^* g_i(t)$ converges smoothly and locally to $g(t)$ on $M^n \times (a, b)$ as $i \to \infty$.

Remark 5.1.1. Using the notations of Definition 5.1.1, assume that M is compact. Then, for i large enough, we have $M = \Omega_i$. In particular, $\operatorname{Im}(\phi)$ is an open-closed subset of M_i; hence, ϕ_i is a diffeomorphism.

In [Ham95], Hamilton proved a fundamental compactness theorem for sequences of Ricci flows (we state it in the more recent formalism of [Top14]).

Theorem 5.1.2. Let $\{(M_i^n, g_i(t))\}_{i\in\mathbb{N},t\in(a,b)}$ be a sequence of complete Ricci flows such that:

$$\sup_{i\in\mathbb{N},t\in(a,b),x\in M_i} \{|\operatorname{Riem}_x(g_i(t))|\} < \infty,$$

and assume that there exists a sequence of points $p_i \in M_i$ such that:

$$\inf_{i\in\mathbb{N}} \{\operatorname{inj}(g_i(0), p_i)\} > 0.$$

Then (passing to a subsequence if necessary), there exists a pointed complete Ricci flow $\{(M, g(t), p)\}_{t\in(a,b)}$ such that:

$$\lim_{i\to\infty}\{(M_i^n, g_i(t), p_i)\}_{i\in\mathbb{N},t\in(a,b)} = \{(M, g(t), p)\}_{t\in(a,b)},$$

in the Hamilton–Cheeger–Gromov sense.

Uniform lower bounds on injectivity radii are important in order to apply Hamilton's compactness Theorem. Combining [Pet06, Lemma 6.4.7] (proven by Klingenberg) and [Che70, Theorem 2.1], we have the following estimate on the injectivity radii of compact manifolds with two-sided sectional curvature bounds.

Theorem 5.1.3. Let us fix $n \geq 2$, $K, V, D > 0$, and $H \leq K$. Then, there exists a constant $c_n(H, K, V, D) > 0$ such that, for every closed manifold (M^n, g) satisfying:

$$\begin{cases} H \leq \sec(g) \leq K \\ V \leq \operatorname{vol}(M^n, g) \\ \operatorname{Diam}(M^n, g) \leq D \end{cases},$$

we have $\operatorname{inj}(M^n, g) \geq c_n(H, K, V, D)$.

5.1.1.3 Normalized Ricci flow

In this section, we will be focusing on surfaces.

Notation 5.1.1. Given a connected closed orientable Riemannian surface (X, g), we use the notations of [CK04] and denote by $K(g) = \sec(g)$, $R(g) = \operatorname{scal}(g)$, and $r(g) := \fint_X R(g)\,\mathrm{dvol}_g$ the Gauss, scalar, and average scalar curvature of g, respectively. We have the following relations $K(g) = R(g)/2$, $\operatorname{Ric}(g) = K(g)g$, and $r(g) = 4\pi\chi(X)/\operatorname{vol}_g(X)$ (using Gauss–Bonnet theorem).

The normalized Ricci flow is a modification of the classical Ricci flow which was introduced to avoid the collapse of the manifold and improve the behaviour of the flow near singularities. The following result introduces the normalized Ricci flow on surfaces and states its long-time existence (see [CK04, Proposition 5.19]).

Proposition 5.1.2 (normalized Ricci flow on surfaces)**.** Let (X, g_0) be a connected closed orientable Riemannian surface. Then the evolution equation:

$$\begin{cases} \partial_t g(t) = -(R(g(t)) - r(g_0))g \\ g(0) = g_0 \end{cases},$$

admits a unique solution $\{g(t)\}_{t\geq 0}$ defined on $\mathbb{R}_{\geq 0}$ (where R and r are both introduced in Notation 5.1.1). We call $\{g(t)\}_{t\geq 0}$ the *normalized Ricci flow starting from* g_0.

The normalized Ricci flow preserves the volume as stated by the following result (see [Ham82, Section 3]).

Lemma 5.1.1. Let (X, g_0) be a connected closed orientable Riemannian surface and denote by $\{g(t)\}_{t\geq 0}$ the normalized Ricci flow starting from g_0. Then we have the following:

$$\operatorname{vol}(g(t)) \equiv \operatorname{vol}(g(0)) \quad \text{and} \quad r(g(t)) \equiv r(g_0) = 4\pi\chi(X)/\operatorname{vol}(g_0),$$

for all $t \geq 0$.

One can easily transform a Ricci flow into a normalized Ricci flow (and vice versa) by rescaling the flow and reparametrising the time (see [Ham82, Section 3]).

Lemma 5.1.2. Let (X, g_0) be a connected closed orientable surface and denote by $\{g(t)\}_{t\in[0,T)}$ the maximal Ricci flow starting from g_0. We define:

$$\psi(t) := \frac{\operatorname{vol}(g_0)}{\operatorname{vol}(g(t))}, \quad \phi(t) := \int_0^t \psi(s)\,\mathrm{d}s, \quad \text{and} \quad \tilde{g}(\phi(t)) := \psi(t)g(t), \tag{5.2}$$

for $t \in [0, T)$. Then $\phi\colon [0, T) \to [0, \infty)$ is an increasing diffeomorphism and $\{\tilde{g}(\tau)\}_{\tau\geq 0}$ is the normalized Ricci flow starting from g_0.

Remark 5.1.2. Using the notations of Lemma 5.1.2 and applying Lemma 5.1.1, we easily compute:

$$\psi(t) = \frac{1}{1 - r(g_0)t} \quad \text{and} \quad \phi(t) = \begin{cases} t, & \text{if } \chi(X) = 0 \\ r(g_0)^{-1}\log(\psi(t)), & \text{otherwise} \end{cases},$$

for all $t \geq 0$. Moreover, we also have the following:

$$\phi^{-1}(\tau) = \begin{cases} t, & \text{if } \chi(X) = 0 \\ r(g_0)^{-1}(1 - \mathrm{e}^{-r(g_0)t}), & \text{otherwise} \end{cases},$$

for all $\tau \geq 0$.

Lemma 5.1.2 and Remark 5.1.2 give rise to the following explicit formula for the maximal existence time of the Ricci flow on surfaces.

Lemma 5.1.3. Let (X, g_0) be a connected closed orientable surface and denote by $\{g(t)\}_{t\in[0,T)}$ the maximal Ricci flow starting from g_0. Then, we have the following:

$$T = \begin{cases} r(g_0)^{-1} = \mathrm{vol}(g_0)/4\pi\chi(X), & \text{if } \chi(X) > 0 \\ \infty, & \text{otherwise} \end{cases}. \tag{5.3}$$

Proof. According to Lemma 5.1.2, we have $T = \lim_{\tau\to\infty} \phi^{-1}(\tau)$. The result follows from Remark 5.1.2. □

We state below the most fundamental result about the normalized Ricci flow on surfaces (see Theorems 5.22, 5.28, and 5.77 in [CK04]).

Theorem 5.1.4. Let (X, g_0) be a connected closed orientable surface and let $\{(X, h(t))\}_{t\geq 0}$ be a normalized Ricci flow. Then $\{h(t)\}_{t\geq 0}$ converges smoothly to a smooth constant curvature Riemannian metric $h(\infty)$.

We will see later that the Ricci flow on Alexandrov surfaces is well-defined thanks to [Ric12]. In particular, Theorem 5.1.4 is a first step towards proving that the moduli space of Alexandrov metrics with curv ≥ -1 on a surface X retracts by deformation onto the moduli space of metrics with constant curvature on X (when $\chi(X) < 0$). However, in order to prove such a result, we first have to obtain uniform convergence estimates for sequences of normalized Ricci flows. Our goal until the end of Section 5.1 will be proving the following result.

Proposition 5.1.3. Let X be a connected closed orientable surface satisfying $\chi(X) < 0$, let $\{h_i\}_{i\in\mathbb{N}\cup\{\infty\}}$ be a family of Riemannian metrics on X, and let us denote by $\{h_i(t)\}_{i\in\mathbb{N}\cup\{\infty\},t\geq 0}$ the associated family of normalized Ricci flows. If $\{h_i\}_{i\in\mathbb{N}}$ converges smoothly to h_∞, then, there exists a decreasing continuous function $C \in \mathcal{L}^1(\mathbb{R}_{\geq 0})$ such that, for all $i \in \mathbb{N}\cup\{\infty\}$, we have:

$$|R_i(t) - r_i| \leq C(t),$$

where $R_i(t) = R(h_i(t))$ and $r_i = r(h_i(0))$ (see Notation 5.1.1).

The following Lemma motivates the proof of Proposition 5.1.3.

Lemma 5.1.4. Let X be a connected closed orientable surface, let $C \in \mathcal{L}^1(\mathbb{R}_{\geq 0})$ be a decreasing continuous function, and let $\{(X, h(t))\}_{t\geq 0}$ be a normalized Ricci flow such that, for all $t \geq 0$, we have $|R(t) - r| \leq C(t)$, where $R(t) \coloneqq R(h(t))$ and $r \coloneqq r(h(0))$ (see Notation 5.1.1). Then, for every $0 \leq s \leq t \leq \infty$, we have:

$$\mathrm{d}_{\mathrm{GH}}((X, \mathrm{d}_{h(s)}), (X, \mathrm{d}_{h(t)})) \leq F(0, \infty) \operatorname{Diam}(h(0)) F(s, t),$$

where $F(s,t) \coloneqq \exp(\int_s^t C(\tau)\, \mathrm{d}\tau)^{1/2}$ and $h(\infty) \coloneqq \lim_{t\to\infty} h(t)$ (whose existence is granted by Theorem 5.1.4).

Proof. First of all, for all $t \geq 0$ and $X \in \mathfrak{X}(X)$, we have the following:

$$|\partial_t(\log(h_t(X, X))| = |R(t) - r| \leq C(t).$$

In particular, for $0 \leq s \leq t \leq \infty$, we have the inequality:

$$\left| \log\left(\frac{h_t(X, X)}{h_s(X, X)} \right) \right| \leq \int_s^\infty C(\tau) d\tau.$$

Hence, using the notations of the lemma, we obtain the following:

$$F(s,t)^{-1}\, \mathrm{d}_{h(s)} \leq \mathrm{d}_{h(t)} \leq F(s,t)\, \mathrm{d}_{h(s)},$$

for every $0 \leq s \leq t \leq \infty$, which implies $|\mathrm{d}_{h(t)} - \mathrm{d}_{h(s)}| \leq F(t,s)\, \mathrm{d}_s$ (using $F(t,s) \geq 1$). However, observe that $\mathrm{d}_s \leq F(0,s)\, \mathrm{d}_{h(0)} \leq F(0,\infty) \operatorname{Diam}(h(0))$, which concludes the proof. □

5.1.2 Curvature potentials

The key towards proving Proposition 5.1.3 is to use curvature potentials (as inspired by the proof of [CK04, Theorems 5.22]). First, let us state the existence theorem for the Poisson problem $\Delta f = \phi$ (see [Aub98, Theorem 4.7]).

Theorem 5.1.5. Let (X^n, g) be a connected closed orientable Riemannian manifold and let $\phi \in \mathcal{C}^\infty(X)$ such that $\int_X \phi\, \mathrm{dvol}_g = 0$. Then, there exists $f \in \mathcal{C}^\infty(X)$ (unique up to the addition of a constant) such that $\Delta_g f = \phi$.

The following lemma states the continuity of the solution to the Poisson problem with respect to the metric g and the function ϕ. It is well-known in the community but it seems difficult to find a proof in the literature. We prove the Lemma in Appendix A.4.

Lemma 5.1.5. Let X^n be a connected closed orientable manifold and let us fix families of Riemannian metrics $\{h_i\}_{i\in\mathbb{N}\cup\{\infty\}}$, smooth functions $\{\phi_i\}_{i\in\mathbb{N}\cup\{\infty\}}$, and real numbers $\{a_i\}_{i\in\mathbb{N}\cup\{\infty\}}$. Assume that for every $i \in \mathbb{N}\cup\{\infty\}$, we have $\int_X \phi_i \,\mathrm{dvol}_{h_i} = 0$, and let us denote by f_i the unique solution to the Poisson problem:

$$\begin{cases} \Delta_{h_i} f_i & = \phi_i \\ \fint_X f_i \,\mathrm{dvol}_{h_i} & = a_i \end{cases}.$$

If $\{h_i\}_{i\in\mathbb{N}}$ and $\{\phi_i\}_{i\in\mathbb{N}}$ converge smoothly respectively to h_∞ and ϕ_∞, and if $a_i \to a_\infty$, then $\{f_i\}_{i\in\mathbb{N}}$ converges smoothly to f_∞.

The next proposition is a slightly modified version of [CK04, Lemma 5.12]. For completeness, we recall the proof of Chow and Knopf and explain why the curvature potential is smooth in Appendix A.4.

Proposition 5.1.4. Let X be a connected closed orientable surface and assume that $\{h(t)\}_{t\geq 0}$ is a normalized Ricci flow on X. Then, there exists $f \in \mathcal{C}^\infty(X \times \mathbb{R}_{\geq 0})$ such that:

$$\begin{cases} \Delta_{h(t)} f(\cdot, t) = R(t) - r \\ \partial_t f(\cdot, t) = \Delta_{h(t)} f(\cdot, t) + r f(\cdot, t) \\ \fint_X f(\cdot, 0) \,\mathrm{dvol}_{h(0)} = 0 \end{cases},$$

where $R(t) := R(h(t))$ and $r = r(h(0))$ (see Notation 5.1.1). Such a function f is called a *curvature potential.*

5.1.3 Uniform convergence estimates

Our goal in this section is to prove Proposition 5.1.3. We restate the result below for the reader's convenience.

Proposition 5.1.5. Let X be a connected closed orientable surface such that $\chi(X) < 0$, let $\{h_i\}_{i\in\mathbb{N}\cup\{\infty\}}$ be a family of Riemannian metrics on X, and denote by $\{h_i(t)\}_{i\in\mathbb{N}\cup\{\infty\}, t\geq 0}$ the associated family of normalized Ricci flows. If $\{h_i\}_{i\in\mathbb{N}}$ converges smoothly to h_∞, then, there exists a decreasing continuous function $C \in \mathcal{L}^1(\mathbb{R}_{\geq 0})$ such that, for all $i \in \mathbb{N}\cup\{\infty\}$, we have:

$$|R_i(t) - r_i| \leq C(t),$$

where $R_i(t) = R(h_i(t))$ and $r_i = r(h_i(0))$ (see Notation 5.1.1).

The following result implies Proposition 5.1.5. Its proof follows the lines [CK04, Corollary 5.17] and utilises Lemma 5.1.5.

Proposition 5.1.6. Let X be a connected closed orientable surface, let $\{h_i\}_{i\in\mathbb{N}\cup\{\infty\}}$ be a family of Riemannian metrics on X, and denote by $\{h_i(t)\}_{i\in\mathbb{N}\cup\{\infty\},t\geq 0}$ the associated family of normalized Ricci flows. If $\{h_i\}_{i\in\mathbb{N}}$ converges smoothly to h_∞, then, there exists $C>0$ such that, for all $i\in\mathbb{N}\cup\{\infty\}$ and $t\in[0,\infty)$, the following holds:

$$|R_i(t)-r_i|\leq C\mathrm{e}^{r_i t}, \tag{5.4}$$

where $R_i(t)=R(h_i(t))$ and $r_i=r(h_i(0))$ (see Notation 5.1.1).

Proof. Part I: Lower bound

Let $\Phi_i(t) \coloneqq R_i(t)-r_i$ ($i\in\mathbb{N}\cup\{\infty\}$ and $t\geq 0$) and note that, thanks to [CK04, Corollary 5.8], we have the following:

$$\begin{aligned}\partial_t\Phi_i &= \partial_t R_i,\\ &= \Delta_{h_i(t)}R_i + R_i(R_i-r_i),\\ &= \Delta_{h_i(t)}\Phi_i + \Phi_i^2 + r_i\Phi_i,\\ &\geq \Delta_{h_i(t)}\Phi_i + r_i\Phi_i.\end{aligned}$$

In particular, if we denote by ϕ_i the unique solution to the ODE $\partial_t\phi_i = r_i\phi_i$ such that $\phi_i(0)=\min_X\{\Phi_i(0)\}$, then $\Phi_i(t)\geq s_i(t)$ for all $t\geq 0$ (according to the maximum principle [CK04, Theorem 4.4]). Therefore, for all $t\geq 0$, we have the following:

$$\Phi_i(t)\geq \min_X\{\Phi_i(0)\}\mathrm{e}^{r_i t}.$$

However, observe that $r_i=\frac{4\pi\chi(X)}{\mathrm{vol}(h_i(0))}$ converges to $r_\infty=\frac{4\pi\chi(X)}{\mathrm{vol}(h_\infty(0))}$ and $\{R_i(0)\}_{i\in\mathbb{N}}$ converges smoothly to $R_\infty(0)$, since $\{h_i\}_{i\in\mathbb{N}}$ converges smoothly to h_∞. Thus, $\{\min_X\{\Phi_i(0)\}\}_{i\in\mathbb{N}\cup\{\infty\}}$ converges to $\min_X\{\Phi_\infty(0)\}$. In particular, there exists $C\in\mathbb{R}$ such that $\Phi_i(t)\geq C\mathrm{e}^{r_i t}$, for all $i\in\mathbb{N}\cup\{\infty\}$ and $t\geq 0$.

Part II: Upper bound

For every $i\in\mathbb{N}\cup\{\infty\}$, let $f_i(x,t)\in\mathcal{C}^\infty(X\times\mathbb{R}_{\geq 0})$ be a curvature potential associated to $\{(X,h_i(t))\}_{t\geq 0}$ (as defined in Proposition 5.1.4). Following [CK04], we introduce the following quantity:

$$H_i(\cdot,t)\coloneqq R_i(t)-r_i+|\nabla_{h_i(t)}f_i(\cdot,t)|_{h_i(t)},$$

where $i \in \mathbb{N}\{\infty\}$ and $t \geq 0$. Observe that $H_i \in \mathcal{C}^\infty(X \times \mathbb{R}_{\geq 0})$ as a result of Proposition 5.1.4. Then, [CK04, Proposition 5.16] implies the following:

$$\begin{aligned}\partial_t H_i &= \Delta_{h_i(t)} H_i - 2|M_i|^2_{h_i(t)} + r_i H_i, \\ &\leq \Delta_{h_i(t)} H_i + r_i H_i,\end{aligned}$$

where M_i is the trace-free part of the Hessian of f_i. In particular, as a result of the maximum principle [CK04, Theorem 4.4], we have the following:

$$H_i(t) \leq \max_X \{H_i(0)\} \mathrm{e}^{r_i t}.$$

Now, note that $H_i(0) = R_i(0) - r_i + |\nabla_{h_i(0)} f_i(\cdot, 0)|_{h_i(0)}$. However, $f_i^0 := f_i(\cdot, 0)$ is the unique solution to the Poisson problem:

$$\begin{cases} \Delta_{h_i(0)} f_i^0 = R_i(0) - r_i \\ \fint_X f_i^0 \,\mathrm{dvol}_{h_i(0)} = 0 \end{cases}.$$

In addition, as observed in the first part of the proof, $\{h_i\}_{i\in\mathbb{N}}$ converges smoothly to h_∞ and $\{R_i(0) - r_i\}_{i\in\mathbb{N}}$ converges smoothly to $R_\infty(0) - r_\infty$. Therefore, as a consequence of Lemma 5.1.5, $\{f_i^0\}_{i\in\mathbb{N}}$ converges smoothly to f_∞^0. Hence, $\{\max_X\{H_i(0)\}\}_{i\in\mathbb{N}}$ converges to $\max_X\{H_\infty(0)\}$. In particular, there exists $C' \in \mathbb{R}$ such that we have the following:

$$R_i(t) - r_i \leq H_i(t) \leq C' \mathrm{e}^{r_i t},$$

which concludes the proof. □

5.2 Ricci flow on Alexandrov spaces

This section focuses on Alexandrov surfaces. First, in Section 5.2.1, we introduce moduli spaces of metrics with constant curvature. Then, in Section 5.2.2, we present the Ricci flow on Alexandrov spaces, as introduced in [Ric12]. In particular, we show that a sequence of Ricci flows on Alexandrov spaces whose initial conditions converge in the Gromov–Hausdorff topology necessarily converges in the Hamilton–Cheeger–Gromov sense. Finally, in Section 5.2.3, we show that the moduli space of constant curvature metrics on a surface with negative Euler characteristic is a deformation retract of the moduli space of Alexandrov metrics with curv ≥ -1 on the surface.

5.2.1 Moduli spaces of constant curvature metrics

In this section, we fix a connected closed orientable topological surface X. We will introduce various moduli spaces of constant curvature metrics on X and show that they coincide. First of all, we recall the definition of smooth structures.

Definition 5.2.1 ([Lee13, p.13])**.** A *smooth structure* on X is a maximal atlas $\mathcal{A}$ of smooth charts on X.

Remark 5.2.1. As observed in [Lee13, p.40], a consequence of [Mun60] and [Moi77] is that, given two smooth structures $\mathcal{A}_1$ and $\mathcal{A}_2$ on X, there exists a diffeomorphism between $(X, \mathcal{A}_1)$ and $(X, \mathcal{A}_2)$.

Definition 5.2.2. Given $K \geq -1$, a metric d metrizing the topology of X *has constant curvature* K if there exists a smooth structure $\mathcal{A}$ on X and a Riemannian metric g on $(X, \mathcal{A})$ such that $K(g) \equiv K$ and $\mathrm{d} = \mathrm{d}_g$.

Remark 5.2.2. Assume that g_1 and g_2 are Riemannian metrics on X with respect to smooth structures $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Moreover, suppose that $\mathrm{d}_{g_1} = \mathrm{d}_{g_2}$. Then, $\mathcal{A}_1 = \mathcal{A}_2$ and $g_1 = g_2$. Indeed, the identity is an isometry between (X, d_{g_1}) and (X, d_{g_2}). Hence, the identity is a smooth diffeomorphism between $(X, \mathcal{A}_1)$ and $(X, \mathcal{A}_2)$ and $\mathrm{id}^* g_2 = g_1$. Therefore, $\mathcal{A}_1 = \mathcal{A}_2$ and $g_1 = g_2$. As a consequence, the smooth structure and Riemannian metric associated to a metric with constant curvature are unique (see Definition 5.2.2).

Notation 5.2.1 (Spaces of constant curvature metrics)**.** Given $K \geq -1$, we denote by $\mathscr{R}_K(X)$ the set of metric spaces (X, d), where d metrizes the topology of X and has constant curvature K. We define $\mathscr{R}_{\mathrm{cste}}(X) := \bigcup_{K \geq -1} \mathscr{R}_K(X)$ the space of metrics with constant curvature on X.

Notation 5.2.2 (Moduli spaces of constant curvature metrics)**.** We denote by $\mathscr{M}_K(X)$ (resp. $\mathscr{M}_{\mathrm{cste}}(X)$) the quotient of $\mathscr{R}_K(X)$ (resp. $\mathscr{R}_{\mathrm{cste}}(X)$) by isometry equivalence, equipped with the GH topology.

Remark 5.2.3. It is clear that, for all $K \geq -1$, we have the following inclusion:

$$\mathscr{R}_K(X) \subset \mathscr{R}_{\mathrm{cste}}(X) \subset \mathscr{R}_{\mathrm{curv} \geq -1}(X),$$

where $\mathscr{R}_{\mathrm{curv}\geq -1}(X)$ is defined in Definition 4.2.3. In particular, there exist natural topological embeddings:

$$\iota_K\colon \mathscr{M}_K(X) \to \mathscr{M}_{\mathrm{cste}}(X) \quad \text{and} \quad \iota\colon \mathscr{M}_{\mathrm{cste}}(X) \to \mathscr{M}_{\mathrm{curv}\geq -1}(X),$$

where $\mathscr{M}_{\mathrm{curv}\geq -1}(X)$ is defined in Definition 4.2.3. Therefore, we will consider $\mathscr{M}_K(X)$ and $\mathscr{M}_{\mathrm{cste}}(X)$ as subsets of $\mathscr{M}_{\mathrm{curv}\geq -1}(X)$.

The following easy lemma implies that $\mathscr{M}_K(X)$ is homotopy equivalent to $\mathscr{M}_{\mathrm{cste}}(X)$ for some $K \in \{-1, 0, 1\}$.

Lemma 5.2.1. Let us defined $K(X) \coloneqq \mathrm{sign}(\chi(X)) \in \{-1, 0, 1\}$ (where $\mathrm{sign}(0) = 0$ by convention). Then $\mathscr{M}_{K(X)}(X)$ is a deformation retract of $\mathscr{M}_{\mathrm{cste}}(X)$.

Proof. Assume that $[X, \mathrm{d}] \in \mathscr{M}_{\mathrm{cste}}(X)$ and let $\mathcal{A}$ be a smooth structure on X and g be a Riemannian metric on $(X, \mathcal{A})$ such that $\mathrm{d} = \mathrm{d}_g$. By definition, there exists $K \geq -1$ such that $K(g) \equiv K$. In particular, thanks to Gauss–Bonnet theorem, we have $K = 2\pi\chi(X)/\mathcal{H}^2(X, \mathrm{d})$, which implies that the case $\chi(X) = 0$ is trivial. Then, if $\chi(X) \neq 0$, it is clear that the map $H\colon \mathscr{M}_{\mathrm{cste}}(X) \times [0, 1] \to \mathscr{M}_{\mathrm{cste}}(X)$ defined by:

$$H([X, \mathrm{d}], t) \coloneqq \left[X, \left((1-t) + \frac{2\pi\chi(X)}{\mathcal{H}^2(X, \mathrm{d})K(X)}t\right)^{1/2} \mathrm{d}\right] \tag{5.5}$$

is a deformation retract of $\mathscr{M}_{\mathrm{cste}}(X)$ onto $\mathscr{M}_{K(X)}(X)$. □

Notation 5.2.2 seems to differ from the following more standard definition.

Definition 5.2.3. Let $K \geq -1$, let $\mathcal{A}$ be a smooth structure on X and denote by $\mathscr{R}_K(X, \mathcal{A})$ the set of Riemannian metrics g which are smooth on $(X, \mathcal{A})$ such that $K(g) \equiv K$. The *moduli space* $\mathscr{M}_K(X, \mathcal{A})$ is the quotient of $\mathscr{R}_K(X, \mathcal{A})$ by the action of the diffeomorphism group of $(X, \mathcal{A})$. We equip $\mathscr{R}_K(X, \mathcal{A})$ with the topology of smooth convergence and $\mathscr{M}_K(X, \mathcal{A})$ with the quotient topology (referred to as the smooth topology).

The moduli space $\mathscr{M}_K(X, \mathcal{A})$ can also be endowed with the Gromov–Hausdorff topology by setting $\mathrm{d}_{\mathrm{GH}}([g_1], [g_2]) \coloneqq \mathrm{d}_{\mathrm{GH}}([X, \mathrm{d}_{g_1}], [X, \mathrm{d}_{g_2}])$. In particular, one may wonder whether the Gromov–Hausdorff topology coincides with the smooth topology on $\mathscr{M}_K(X, \mathcal{A})$. The following result answer that question.

Proposition 5.2.1. The smooth topology coincides with the Gromov–Hausdorff on $\mathscr{M}_K(X, \mathcal{A})$.

Proof. Assume that $d_{GH}((X, g_i), (X, g_\infty)) \to 0$, where, for all $i \in \mathbb{N} \cup \{\infty\}$, we have $g_i \in \mathscr{R}_K(X, \mathcal{A})$. In particular, there exists $V > 0$ and $D > 0$ such that, for all $i \in \mathbb{N}$, we have the following:

$$\begin{cases} \sec(g_i) \equiv K \\ \mathrm{vol}(g_i) \geq V \\ \mathrm{Diam}(g_i) \leq D \end{cases}.$$

Moreover, $\nabla^{g_i} R(g_i) = 0$ for all $i \in \mathbb{N}$. Therefore, thanks to [Fuk90, Proposition 5.10], there exists a sequence $\{\phi_i\}_{i\in\mathbb{N}}$ of diffeomorphisms of X such that $\{\phi_i^* g_i\}_{i\in\mathbb{N}}$ converges smoothly to g_∞. In particular, $[X, g_i] \to [X, g_\infty]$ in the smooth topology. Therefore, the GH topology is stronger than the smooth topology on $\mathscr{M}_K(X, \mathcal{A})$. Conversely, the smooth topology is stronger than the GH topology on $\mathscr{M}_K(X, \mathcal{A})$, which concludes the proof. □

The following lemma shows that $\mathscr{M}_K(X, \mathcal{A})$ (introduced in Definition 5.2.3) and $\mathscr{M}_K(X)$ (introduced in Notation 5.2.2) are essentially the same.

Lemma 5.2.2. Let $\mathcal{A}$ be a smooth structure on X and let $K \geq -1$. Then $\mathscr{M}_K(X)$ is isometric to $\mathscr{M}_K(X, \mathcal{A})$ equipped with the Gromov–Hausdorff metric.

Proof. First of all, assume that $(X, \mathrm{d}) \in \mathscr{R}_K(X)$. In particular, following Remark 5.2.2, there exists a unique smooth structure $\mathcal{B}$ on X and a unique smooth Riemannian metric g on $(X, \mathcal{B})$ such that $\mathrm{d} = \mathrm{d}_g$ and $K(g) \equiv K$. According to Remark 5.2.1, there exists a diffeomorphism $\phi\colon (X, \mathcal{A}) \to (X, \mathcal{B})$. Therefore, $\phi^* g \in \mathscr{R}_K(X, \mathcal{A})$. In addition, if $\psi\colon (X, \mathcal{A}) \to (X, \mathcal{B})$ is another diffeomorphism, then $\psi^* g = (\phi^{-1} \circ \psi)^* \phi^* g$. In particular, $[\phi^* g] = [\psi^* g] \in \mathscr{M}_K(X, \mathcal{A})$. Thus, we have a well-defined map $\Phi\colon \mathscr{R}_K(X) \to \mathscr{M}_K(X, \mathcal{A})$.

Assume that $[X, \mathrm{d}_1] = [X, \mathrm{d}_2] \in \mathscr{M}_K(X)$. Then, by definition, there exists an isometry $\phi\colon (X, \mathrm{d}_1) \to (X, \mathrm{d}_2)$. Moreover, by definition and using the discussion above, given $i \in \{1, 2\}$, there exists a smooth structure $\mathcal{A}_i$ on X and a smooth Riemannian metric g_i on $(X, \mathcal{A}_i)$ such that $\mathrm{d}_i = \mathrm{d}_{g_i}$. Now, since ϕ is an isometry between (X, d_{g_1}) and (X, d_{g_2}), it is necessarily a Riemannian isometry, i.e. $\phi\colon (X, \mathcal{A}_1) \to (X, \mathcal{A}_2)$ is a diffeomorphism and $g_1 = \phi^* g_2$. Hence, if we fix a diffeomorphism $\psi_1\colon (X, \mathcal{A}) \to (X, \mathcal{A}_1)$, then $\psi_2 \coloneqq \phi \circ \psi_1\colon (X, \mathcal{A}) \to (X, \mathcal{A}_2)$ is a diffeomorphism. In particular, by definition of Φ, we have $\Phi(X, \mathrm{d}_2) = [\psi_2^* g_2] = [\psi_1^* \phi^* g_2] = [\psi_1^* g_1] = \Phi(X, \mathrm{d}_1)$. Therefore, Φ gives rise to a well-defined map $\Phi\colon \mathscr{M}_K(X) \to \mathscr{M}_K(X, \mathcal{A})$.

Then, the map $\Psi\colon [g] \in \mathscr{M}_K(X,\mathcal{A}) \to [X, \mathrm{d}_g] \in \mathscr{M}_K(X)$ is clearly well-defined and it is readily checked that Ψ and Φ are respectively inverse to each other. Moreover, it is also clear from their definitions that both Φ and Ψ conserve Gromov–Hausdorff distances. □

5.2.2 Convergence of the flows

In his PhD thesis, Thomas Richard proved that one can start a Ricci flow from an Alexandrov surface. The following formulation of his main result will be best suited for our goals (see [Ric12, Proposition 3.1.4] and [Ric18, Proposition 0.6]).

Proposition 5.2.2. Let X be a connected closed orientable topological surface. Assume that d is an Alexandrov metric on X with $\mathrm{curv} \geq -1$. Then, there exists a smooth structure on X and a Ricci flow $\{(X, g(t))\}_{t\in(0,T)}$ that is smooth for that structure such that:

(i) $\mathrm{d}_{g(t)}$ converges uniformly to d when t goes to 0,

(ii) $K(g(t)) \geq -1$, for all $t \in (0,T)$.

Moreover, assume that $\{(X, g_i(t))\}_{t\in(0,T_i)}$ $(i \in \{1,2\})$ are two smooth Ricci flows (for a priori different smooth structures on X) satisfying (i) and (ii). Then, the smooth structures agree and $g_1(t) = g_2(t)$ for all $t \in (0, \min(T_1, T_2))$.

Thanks to Proposition 5.2.2, we can now introduce the following notation without any confusion.

Notation 5.2.3. Given $(X, \mathrm{d}) \in \mathscr{R}_{\mathrm{curv}\geq -1}(X)$, we denote by $T_{\max}(X, \mathrm{d})$ the existence time of the maximal Ricci flow starting from (X, d) and we define $r(X, \mathrm{d}) := 4\pi\chi(X)/\mathcal{H}^2(X, \mathrm{d})$.

We can generalise Lemma 5.1.3 to Alexandrov spaces.

Lemma 5.2.3. Let X be a connected closed orientable topological surface and assume that $(X, \mathrm{d}) \in \mathscr{R}_{\mathrm{curv}\geq -1}(X)$. Then $T_{\max}(X, \mathrm{d})$ satisfies the following:

$$T_{\max}(X,\mathrm{d}) = \begin{cases} \infty, & \text{if } \chi(X) \leq 0 \\ r(X,\mathrm{d})^{-1}, & \text{otherwise} \end{cases},$$

where $r(X, \mathrm{d}) = 4\pi\chi(X)/\mathcal{H}^2(X, \mathrm{d})$ is introduced in Notation 5.2.3.

Proof. Observe that $T := T_{\max}(X, \mathrm{d}) > 0$ due to Proposition 5.2.2. In particular, fixing any $0 < t_0 < T$, $\{(X, g(t + t_0))\}_{t\in[0,T-t_0)}$ is the maximal smooth Ricci flow starting from $(X, g(t_0))$. In particular, Lemma 5.1.3 implies the following:

$$T - t_0 = \begin{cases} \mathrm{vol}(g(t_0))/4\pi\chi(X), & \text{if } \chi(X) > 0 \\ \infty, & \text{otherwise} \end{cases}. \tag{5.6}$$

Observe that $\lim_{t_0\to 0} \mathrm{vol}(g(t_0)) = \mathcal{H}^2(X, \mathrm{d})$; therefore, passing to the limit as t_0 goes to 0 in (5.6) concludes the proof. □

The following result provides uniform short-time estimates of the geometric quantities associated to a Ricci flow starting from an Alexandrov surface. It is a slight generalisation of Theorem 1.1 of [Ric18], which is itself a slight modification of Theorem 7.1 of [Sim12]. For the sake of completeness, we provide an extended proof, regrouping the arguments of [Ric12] and [Ric18].

Proposition 5.2.3 (Uniform short-time estimates)**.** For every $V > 0$ and $D > 0$, there exists $T > 0$ and $\kappa > 0$ such that, if d is an Alexandrov metric on a compact topological surface X without boundary satisfying:

$$\begin{cases} \mathrm{curv}(X, \mathrm{d}) \geq -1 \\ \mathrm{Diam}(X, \mathrm{d}) \leq D \\ V/2 \leq \mathcal{H}^2(X, \mathrm{d}) \leq V \end{cases}, \tag{5.7}$$

then, the maximal Ricci flow $\{(X, g(t))\}_{t\in(0,T_{\max}(X,\mathrm{d}))}$ starting from (X, d) satisfies $T \leq T_{\max}(X, \mathrm{d})$ and the following properties:

$$\begin{cases} -1 \leq K(g(t)) \leq \kappa/t \\ \mathrm{Diam}(X, g(t)) \leq 4D \\ V/8 \leq \mathrm{vol}(X, g(t)) \leq 4V \\ d_{g(s)} - \kappa(\sqrt{t} - \sqrt{s}) \leq d_{g(t)} \leq \mathrm{e}^{\kappa(t-s)} d_{g(s)} \end{cases},$$

for all $0 \leq s < t < T$ (where, by abuse of notation, we define $\mathrm{d}_{g(0)} := \mathrm{d}$).

Proof. Let (X, d) be a compact Alexandrov surface without boundary satisfying (5.7). Thanks to Lemma 2.4 of [IRV15], there exists a sequence of closed Riemannian surfaces $\{(M_i, g_i)\}_{i\in\mathbb{N}}$ converging to (X, d) in the GH topology such that, for all $i \in \mathbb{N}$, $-1 \leq K(g_i)$. In particular, since $\mathrm{vol}(M_i, g_i) \to \mathcal{H}^2(X, \mathrm{d})$ and $\mathrm{Diam}(M_i, g_i) \to$

$\mathrm{Diam}(X, \mathrm{d})$, we can assume without loss of generality that, for all $i \in \mathbb{N}$, we have the following:

$$\begin{cases} \mathrm{Diam}(M_i, g_i) \leq 2D \\ V/4 \leq \mathrm{vol}(M_i, g_i) \leq 2V \end{cases}.$$

Using the notations of Theorem 1.1 of [Ric18], we define:

$$T \coloneqq \min(T(V/2, 2D), T(V, 2D), T(2V, 2D)),$$
$$\kappa \coloneqq \kappa(V/2, 2D) + \kappa(V, 2D) + \kappa(2V, 2D)),$$

and $\{(M_i, g_i(t))\}_{t\in[0,T^i)}$ the maximal Ricci flow starting from (M_i, g_i). For all $i \in \mathbb{N}$, $\{(M_i, g_i(t))\}_{t\in[0,T^i)}$ satisfies $T \leq T^i_{\max}$ and the following properties:

$$\begin{cases} -1 \leq K(g_i(t)) \leq \kappa/t \\ \mathrm{Diam}(M_i, g_i(t)) \leq 4D \\ V/8 \leq \mathrm{vol}(M_i, g_i(t)) \leq 4V \\ d_{g_i(s)} - \kappa(\sqrt{t} - \sqrt{s}) \leq d_{g_i(t)} \leq \mathrm{e}^{\kappa(t-s)} d_{g_i(s)} \end{cases}, \tag{5.8}$$

for all $0 \leq s < t < T$. Now, Theorem 5.1.3 implies the following:

$$\mathrm{inj}(M_i, g_i(T/2)) \geq c_2(-1, 2\kappa/T, V/8, 4D) > 0.$$

Moreover, there is a non-increasing continuous function $C(t) = \max\{1, \kappa/t\}$ such that, for all $t \in (0, T)$, we have the following:

$$\sup_{i\in\mathbb{N}, x\in M_i} \{|K_{g_i(t)}(x)|\} \leq C(t).$$

Hence, using Theorem 5.1.2 and a diagonal argument and passing to a subsequence if necessary, we can assume that $\{(M_i, g_i(t))\}_{i\in\mathbb{N}, t\in(0,T)}$ converges to a closed Ricci flow $\{(\tilde{M}, \tilde{g}(t))\}_{t\in(0,T)}$ satisfying (5.8) ($\tilde{M}$ is closed thanks to Remark 5.1.1 and because $\{\mathrm{Diam}(g_i(t))\}_{i\in\mathbb{N}, t\in[0,T)}$ is uniformly bounded by $4D$). As a consequence of Proposition 5.2.2, we just need to show that $(\tilde{M}, d_{\tilde{g}(t)})$ converges uniformly to a metric space $(\tilde{M}, \tilde{\mathrm{d}})$ isometric to (X, d) as t goes to 0.

First, observe that (5.8) implies the following:

$$|\mathrm{d}_{\tilde{g}(t)} - \mathrm{d}_{\tilde{g}(s)}| \leq \max\left\{\kappa(\sqrt{t} - \sqrt{s}), 4D(\mathrm{e}^{\kappa(t-s)} - 1)\right\},$$

for all $0 < s \leq t < T$. In particular, $\{\mathrm{d}_{\tilde{g}(t)}\}_{t\in(0,T)}$ is Cauchy as t goes to 0; therefore, it converges to a pseudo-metric $\tilde{\mathrm{d}}$ on $\tilde{M}$. Moreover, observe that passing to the limit as s goes to 0 in the last property of (5.8), we obtain $\tilde{\mathrm{d}} \geq \mathrm{e}^{-\kappa t}\mathrm{d}_{\tilde{g}(t)}$, for all

$t \in (0,T)$. In particular, $(\tilde{M}, \tilde{\mathrm{d}})$ is a metric space. Now, let $\epsilon > 0$ and observe that (5.8) implies the following:

$$\mathrm{d}_{\mathrm{GH}}((\tilde{M}, \tilde{\mathrm{d}}), (\tilde{M}, \tilde{g}(\epsilon))) \leq \max\left\{\kappa\sqrt{\epsilon}, 4D(\mathrm{e}^{\kappa\epsilon} - 1)\right\},$$
$$\mathrm{d}_{\mathrm{GH}}((M_i, g_i), (M_i, g_i(\epsilon))) \leq \max\left\{\kappa\sqrt{\epsilon}, 4D(\mathrm{e}^{\kappa\epsilon} - 1)\right\}.$$

Moreover, for i large enough, we have:

$$\mathrm{d}_{\mathrm{GH}}((\tilde{M}, \tilde{g}(\epsilon)), (M_i, g_i(\epsilon))) \leq \epsilon,$$
$$\mathrm{d}_{\mathrm{GH}}((M_i, g_i), (X, \mathrm{d})) \leq \epsilon.$$

In particular, fixing i large enough, and applying the triangle inequality in the order $(X, \mathrm{d}) \to (M_i, g_i) \to (M_i, g_i(\epsilon)) \to (\tilde{M}, \tilde{g}(\epsilon)) \to (\tilde{M}, \tilde{\mathrm{d}})$, we have:

$$\mathrm{d}_{\mathrm{GH}}((X, \mathrm{d}), (\tilde{M}, \tilde{\mathrm{d}})) \leq 2\max\left\{\kappa\sqrt{\epsilon}, 4D(\mathrm{e}^{\kappa\epsilon} - 1)\right\} + 2\epsilon.$$

Therefore, letting ϵ go to 0, we obtain $\mathrm{d}_{\mathrm{GH}}((X, \mathrm{d}), (\tilde{M}, \tilde{\mathrm{d}})) = 0$, which concludes the proof. □

We conclude this section with the following convergence result, which will be crucial to construct a homotopy between $\mathscr{M}_{\mathrm{curv}\geq -1}(X)$ and $\mathscr{M}_{\mathrm{cste}}(X)$ (when $\chi(X) < 0$).

Proposition 5.2.4. Let X be a connected closed orientable topological surface, let $[X, \mathrm{d}_i] \to [X, \mathrm{d}_\infty]$ in $\mathscr{M}_{\mathrm{curv}\geq -1}(X)$ and, for every $i \in \mathbb{N} \cup \{\infty\}$, denote by $\{(X, g_i(t))\}_{t\in(0,T_i)}$ the maximal Ricci flow starting from (X_i, d_i). Then, the following properties hold:

(i) $\lim_{i\to\infty} T_i = T_\infty$,

(ii) for every $0 < T < T_\infty$, every subsequence of $\{(X, g_i(t))\}_{t\in(0,T)}$ admits a subsequence converging to $\{(X, g_\infty(t))\}_{t\in(0,T)}$ in the Hamilton–Cheeger–Gromov sense,

(ii) for every $t_i \to 0$, $\{(X_i, g_i(t_i))\}_{i\in\mathbb{N}}$ converges to (X, d_∞) in the Gromov–Hausdorff topology.

Proof. Part I: Proof of (i)

If $\chi(X) \leq 0$, then we trivially have $\lim_{i\to\infty} T_i = T_\infty = \infty$ thanks to Lemma 5.2.3. If $\chi(X) > 0$, then we have $T_i = \mathcal{H}^2(X, \mathrm{d}_i)/4\pi\chi(X)$, for all $i \in \mathbb{N} \cup \{\infty\}$ (due to Lemma 5.2.3). However, $\mathcal{H}^2(X, \mathrm{d}_i) \to \mathcal{H}^2(X, \mathrm{d}_\infty)$ since $[X, \mathrm{d}_i] \to [X, \mathrm{d}_\infty]$ in $\mathscr{M}_{\mathrm{curv} \geq -1}(X)$ (see [DG18, Theorem 1.2] for example); therefore, $T_i \to T_\infty$.

Part II: Proof of (ii)

Let $0 < T < T_\infty$ and let us show that $\{(X, g_i(t))\}_{t \in (0,T)}$ admits a subsequence converging to $\{(X, g_\infty(t))\}_{t \in (0,T)}$ in the Hamilton–Cheeger–Gromov sense (the proof for a subsequence being the same).

First of all, observe that there exists $V, D > 0$ such that, for all $i \in \mathbb{N} \cup \{\infty\}$, we have the following:

$$\begin{cases} \mathrm{curv}(X, \mathrm{d}_i) \geq -1 \\ \mathrm{Diam}(X, \mathrm{d}_i) \leq D \\ V/2 \leq \mathcal{H}^2(X, \mathrm{d}_i) \leq V \end{cases}.$$

Therefore, as a result of Theorem 5.2.3, there exists $T_0, \kappa \in \mathbb{R}_{>0}$ such that, for all $i \in \mathbb{N} \cup \{\infty\}$, we have $T_0 \leq T_i$ and the following properties:

$$\begin{cases} -1 \leq K(g_i(t)) \leq \kappa/t \\ \mathrm{Diam}(X, g_i(t)) \leq 4D \\ V/8 \leq \mathrm{vol}(X, g_i(t)) \leq 4V \\ d_{g_i(s)} - \kappa(\sqrt{t} - \sqrt{s}) \leq d_{g_i(t)} \leq \mathrm{e}^{\kappa(t-s)} d_{g_i(s)} \end{cases}, \tag{5.9}$$

for all $0 \leq s < t < T_0$ (where, by abuse of notation, we define $\mathrm{d}_{g_i(0)} \coloneqq \mathrm{d}_i$). Proceeding as in the proof of Theorem 5.2.3 and passing to a subsequence if necessary, we can assume that $\{(X, g_i(t))\}_{i \in \mathbb{N}, t \in (0,T_0)}$ is converging to $\{(X, g_\infty(t))\}_{t \in (0,T_0)}$ in the Hamilton–Cheeger–Gromov sense. Replacing $(X, g_i(t))$ by $(X, \phi_i^* g_i(t))$ for some diffeomorphism ϕ_i if necessary (see Definition 5.1.1), we can assume that $\{g_i(t)\}_{i \in \mathbb{N}, t \in (0,T_0)}$ converges smoothly to $\{g_\infty(t)\}_{t \in (0,T_0)}$ on compact subsets of $X \times (0, T_0)$.

Now, given $i \in \mathbb{N} \cup \{\infty\}$, let $\{(X, h_i(t))\}_{t \geq 0}$ be the normalized Ricci flow starting from $(X, g_i(T_0/2))$. Observe that $\{h_i(0)\}_{i \in \mathbb{N}}$ converges smoothly to $h_\infty(0)$. In particular, as a consequence of Proposition 5.1.6, there exists $C > 0$ such that, for all $i \in \mathbb{N} \cup \{\infty\}$ and $t \in [0, \infty)$, the following holds:

$$|R(h_i(t)) - r_i| \leq C\mathrm{e}^{r_i t}, \tag{5.10}$$

where $r_i \coloneqq r(h_i(0)))$ (using Notation 5.1.1). However, Lemma 5.1.2 and Remark 5.1.2 imply that, for all $t \in [0, T_i - T_0/2)$, we have the following:

$$h_i(\phi_i(t)) = \psi_i(t) g_i(t + T_0/2), \tag{5.11}$$

where $\psi_i(t) = (1 - r_i t)^{-1}$ and $\phi_i(t) = \int_0^t \psi_i(\tau)\,\mathrm{d}\tau$. In particular, for all $t \in [0, T_i - T_0/2)$, we have the following inequality:

$$|R(g_i(t + T_0/2))| \leq \psi_i(t)(C\mathrm{e}^{r_i\phi_i(t)} + |r_i|) = \psi_i(t)(C\psi_i(t) + |r_i|), \tag{5.12}$$

due to (5.10), (5.11), and Remark 5.1.2. For all $t \in [0, T - T_0/2]$ and i large enough so that $T < T_i$ (say when $i \geq i_0$), we have the following:

$$\psi_i(t) = (1 - r_i t)^{-1} \leq \max\left\{1, (1 - r_i(T - T_0/2))^{-1}\right\}.$$

Since $\{r_i\}_{i\in\mathbb{N}}$ converges, there exists $C' > 0$ such that, for all $t \in [0, T - T_0/2]$ and $i \geq i_0$, we have the following $\psi_i(t) \leq C'$. Therefore, (5.12) implies the following:

$$\sup_{i\geq i_0, t\in[T_0/2,T], x\in X} |R_x(g_i(t))| < C'(CC' + \sup_{i\geq i_0}\{|r_i|\}) < \infty. \tag{5.13}$$

Finally, using (5.13) and applying the strategy of the proof of Proposition 5.1.6, $\{(X, g_i(t)\}_{i\in\mathbb{N}, t\in(0,T)}$ admits a subsequence converging to a $\{(X, g_\infty(t)\}_{t\in(0,T)}$ in the Hamilton–Cheeger–Gromov sense.

Part III: Proof of (iii)

We are keeping the notations from the previous part. Proceeding as in the proof of Proposition 5.1.6 and using (5.9), we easily obtain:

$$\mathrm{d}_{\mathrm{GH}}((X, \mathrm{d}_i), (X, g_i(t)) \leq \max\{\kappa\sqrt{t}, 4D(\mathrm{e}^{\kappa t} - 1)\},$$

for all $t \in (0, T_0]$. In particular, if $t_i \to 0$, then we have the following:

$$\mathrm{d}_{\mathrm{GH}}((X, g_i(t_i)), (X, \mathrm{d}_\infty)) \leq \max\{\kappa\sqrt{t_i}, 4D(\mathrm{e}^{\kappa t_i} - 1)\} + \mathrm{d}_{\mathrm{GH}}((X, \mathrm{d}_i), (X, \mathrm{d}_\infty)) \to 0,$$

which concludes the proof. □

5.2.3 Homotopy

In this section, we fix a connected closed orientable topological surface X.

Definition 5.2.4. Let d be an Alexandrov metric d with curv ≥ -1 on X. Let $\{(X, g(t))\}_{t\in(0,T)}$ be the maximal Ricci flow starting from (X, d) and, for $t \in [0, T)$, define $\psi(t) := \mathcal{H}^2(X, \mathrm{d})\,\mathrm{vol}(g(t))^{-1}$ and $\phi(t) := \int_0^t \psi(\tau)\,\mathrm{d}\tau$. The *normalized Ricci flow* starting from (X, d) is defined as $\{(X, \tilde{g}(t))\}_{t\in(0,\infty)}$, where $\tilde{g}(\phi(t)) := \psi(t)g(t)$ for every $t \in (0, T)$.

Remark 5.2.4. The normalized Ricci flow satisfies the following equation for all $t > 0$:

$$\partial_t \tilde{g}(t) = (\tilde{r} - \tilde{R}(t))\tilde{g}(t),$$

where $\tilde{R}(t) \coloneqq R(\tilde{g}(t))$ and $\tilde{r} = r(X, \mathrm{d})$ (see Notation 5.2.3).

Proposition 5.2.5. There is a well-defined map:

$$H\colon \mathscr{M}_{\mathrm{curv}\geq -1}(X) \times [0,\infty] \to \mathscr{M}_{\mathrm{curv}\geq -1}(X) \tag{5.14}$$

satisfying $H([X,\mathrm{d}],t) \coloneqq [X,\tilde{g}(t)] \in \mathscr{M}_{\mathrm{curv}\geq -1}(X)$ for every $[X,\mathrm{d}] \in \mathscr{M}_{\mathrm{curv}\geq -1}(X)$ and $t \in [0,\infty]$ (where $\{(X,\tilde{g}(t))\}_{t\in[0,\infty)}$ is the normalized Ricci flow starting from (X,d) introduced in Definition 5.2.4, $\tilde{g}(\infty) = \lim_{t\to\infty}\tilde{g}(t)$ is well-defined thanks to Theorem 5.1.4, and by abuse of notations, we denote $[X,\tilde{g}(0)] \coloneqq [X,\mathrm{d}]$).

Proof. Part I: $K(\tilde{g}(t)) \geq -1$

Assume that $(X,\mathrm{d}) \in \mathscr{R}_{\mathrm{curv}\geq -1}(X)$ and let $\{(X,g(t))\}_{t\in(0,T_{\max})}$ and $\{(X,\tilde{g}(t)\}_{t\in(0,\infty)}$ be the maximal Ricci flow and normalized Ricci flow starting from (X,d), respectively. We need to show that $K(\tilde{g}(t)) \geq -1$ for all $t \in (0,\infty)$.

First of all, let us recall that, according to Proposition 5.2.2, there exists $T \in (0, T_{\max})$ such that $K(g(t)) \geq -1$, for all $t \in (0,T)$. In particular, using the notations of Definition 5.2.4, we have the following:

$$\tilde{R}(\phi(t)) = \psi(t)^{-1}R(t) \geq -2\psi(t)^{-1} \tag{5.15}$$

for every $t \in (0,T)$, where $R(t) \coloneqq R(g(t))$ and $\tilde{R}(t) \coloneqq R(\tilde{g}(t))$. In particular, for every $t \in (0,T)$, we also have the following:

$$\tilde{r} \equiv \frac{\int_X \tilde{R}(\phi(t))\,\mathrm{dvol}_{\tilde{g}(\phi(t))}}{\mathrm{vol}(\tilde{g}(\phi(t)))} \geq -2\psi(t)^{-1}. \tag{5.16}$$

Then, Remark 5.2.4 and [CK04, Corollary 5.8] imply that the scalar curvature satisfies the following:

$$\partial_t \tilde{R} = \Delta_{\tilde{g}(t)}\tilde{R} + \tilde{R}(\tilde{R} - \tilde{r}), \tag{5.17}$$

for all $t \in (0,\infty)$. In particular, for every $t_0 \in (0,T)$ and $t \geq \phi(t_0)$, the following three cases hold (due to the maximum principle [CK04, Lemma 5.9] and (5.15)):

- if $\tilde{r} < 0$, then $\tilde{R}(t) \geq \tilde{r} + (-2\psi(t_0)^{-1} - \tilde{r})\mathrm{e}^{\tilde{r}t} \geq \tilde{r} + (-2\psi(t_0)^{-1} - \tilde{r})\mathrm{e}^{\tilde{r}\phi(t_0)}$ (using (5.16) for the second inequality),

- if $\tilde{r} = 0$, then $\tilde{R}(t) \geq \frac{-2\psi(t_0)^{-1}}{1+2\psi(t_0)^{-1}t} \geq \frac{-2\psi(t_0)^{-1}}{1+2\psi(t_0)^{-1}\phi(t_0)}$,
- if $\tilde{r} > 0$, then $\tilde{R}(t) \geq -2\psi(t_0)^{-1}\mathrm{e}^{-\tilde{r}t} \geq -2\psi(t_0)^{-1}\mathrm{e}^{-\tilde{r}\phi(t_0)}$.

In particular, denoting:

$$C_{\tilde{r}}(t_0) := \begin{cases} \tilde{r} + (-2\psi(t_0)^{-1} - \tilde{r})\mathrm{e}^{\tilde{r}\phi(t_0)}, & \text{if } \tilde{r} < 0 \\ \frac{-2\psi(t_0)^{-1}}{1+2\psi(t_0)^{-1}\phi(t_0)}, & \text{if } \tilde{r} = 0 \\ -2\psi(t_0)^{-1}\mathrm{e}^{-\tilde{r}\phi(t_0)}, & \text{if } \tilde{r} > 0 \end{cases}$$

we have the following:

$$\forall t \in (0,\infty), \tilde{R}(t) \geq \liminf_{t_0 \to 0} C_{\tilde{r}}(t_0) = -2,$$

using $\lim_{t_0\to 0}\phi(t_0) = 0$ and $\lim_{t_0\to 0}\psi(t_0) = 1$. Therefore, for all $t \in (0,\infty)$, $K(\tilde{g}(t)) = \tilde{R}/2 \geq -1$, which concludes the first part of the proof.

Part II: Passing to the quotient

Assume that $[X, \mathrm{d}_1] = [X, \mathrm{d}_2] \in \mathscr{M}_{\mathrm{curv}\geq -1}(X)$ and let $\phi\colon (X, \mathrm{d}_1) \to (X, \mathrm{d}_2)$ be an isometry. Given $i \in \{1,2\}$, let us denote by $\{(X, g_i(t))\}_{t\in(0,T_i)}$ the maximal Ricci flow starting from (X, d_i), which is smooth with respect to a smooth structure $\mathcal{A}_i$ on X. Note that $\phi^*\mathcal{A}_2$ is a smooth structure on X and $\{(X, \phi^* g_2(t))\}_{t\in(0,T_2)}$ is a Ricci flow starting from (X, d_1), which is smooth with respect to $\phi^*\mathcal{A}_2$. In particular, as a result of Proposition 5.2.2, $\phi^*\mathcal{A}_2 = \mathcal{A}_1$, $T_2 \leq T_1$, and for every $t \in (0, T_2)$, we have $g_1(t) = \phi^* g_2(t)$. Proceeding in the same way with ϕ^{-1}, we conclude that $T_1 = T_2 =: T$, that $\phi\colon (X, \mathcal{A}_1) \to (X, \mathcal{A}_2)$ is a diffeomorphism, and that for all $t \in (0,T)$, $\phi^* g_2(t) = g_1(t)$. In particular, denoting $\{(X, \tilde{g}_i(t))\}_{t\in(0,\infty)}$ the normalized Ricci flow starting from (X, d_i) ($i \in \{1,2\}$), we have $\phi^*\tilde{g}_2(t) = \tilde{g}_1(t)$ for all $t \in (0,\infty)$. Therefore, for all $t \in [0,\infty]$, we have $H([X, \mathrm{d}_1], t) = H[X, \mathrm{d}_2], t)$, which concludes the proof. □

The following Theorem is the key to studying the homotopy type of moduli spaces of $\mathrm{RCD}(-1,2)$ structures.

Theorem 5.2.1. If $\chi(X) < 0$, then the map $H\colon \mathscr{M}_{\mathrm{curv}\geq -1}(X) \times [0,\infty] \to \mathscr{M}_{\mathrm{curv}\geq -1}(X)$ introduced in Proposition 5.2.5 is continuous and satisfies the following properties:

(i) $H(\cdot, 0) = \mathrm{id}_{\mathscr{M}_{\mathrm{curv}\geq -1}(X)}$,

(ii) $H([X,\mathrm{d}], \infty) \in \mathscr{M}_{\mathrm{cste}}(X)$ for all $[X, \mathrm{d}] \in \mathscr{M}_{\mathrm{curv}\geq -1}(X)$,

(iii) $H(\cdot,t)_{|\mathscr{M}_{\mathrm{cste}}(X)} = \mathrm{id}_{\mathscr{M}_{\mathrm{cste}}(X)}$,

where $[0,\infty]$ is the compactification of $[0,\infty)$. In particular, $\mathscr{M}_{\mathrm{cste}}(X)$ is a deformation retract of $\mathscr{M}_{\mathrm{curv}\geq -1}(X)$ (see Notation 5.2.2).

Proof. Part I: Introducing the notations

Let $[X,\mathrm{d}_i] \to [X,\mathrm{d}_\infty]$ in $\mathscr{M}_{\mathrm{curv}\geq -1}(X)$ and, for every $i \in \mathbb{N}\cup\{\infty\}$, we denote by $\{(X,g_i(t))\}_{t\in(0,T_i)}$ the maximal Ricci flow starting from (X_i,d_i). Then, for every $i \in \mathbb{N}\cup\{\infty\}$, we denote by $\{(X,\tilde{g}_i(t))\}_{t\in(0,\infty)}$ the normalized Ricci flow starting from (X,d_i). By definition and thanks to Remark 5.1.2, for all $t\in(0,T_i)$, we have $\tilde{g}_i(\phi_i(t)) = \psi_i(t)g_i(t)$, where:

$$\psi_i(t) = \frac{1}{1-r_i t} \quad \text{and} \quad \phi_i(t) = \begin{cases} t, & \text{if } \chi(X) = 0 \\ r_i^{-1}\log(\psi_i(t)), & \text{otherwise} \end{cases}, \tag{5.18}$$

and $r_i = 4\pi\chi(X)/\mathcal{H}^2(X,\mathrm{d}_i)(= T_i^{-1}$ if $\chi(X)>0)$. We also have:

$$\mathrm{d}_{\tilde{g}_i(\phi_i(t))} = \sqrt{\psi_i(t)}\,\mathrm{d}_{g_i(t)}, \tag{5.19}$$

for all $t\in[0,T_i)$ (where, by abuse of notation, we define $\mathrm{d}_{\tilde{g}_i(0)} = \mathrm{d}_{g_i(0)} = \mathrm{d}_i$). Finally, let us define $\tilde{g}_i(\infty) \coloneqq \lim_{\tau\to\infty}\tilde{g}_i(\tau)$, which is well-defined because of Theorem 5.1.4.

Part II: Goal of the proof

First of all, it is clear that H satisfies points (i), (ii), and (iii). Therefore, we only need to show that H is continuous. Let $\tau_i \to \tau_\infty$ in $[0,\infty]$ and define $t_i \coloneqq \phi_i^{-1}(\tau_i)$ $(i\in\mathbb{N}\cup\{\infty\})$. From the expressions of ψ_i and ϕ_i in (5.18), it is clear that $t_i \to t_\infty$, and that $\psi_i(t_i) \to \psi_\infty(t_\infty)$ (if $t_\infty < T_\infty$). Our goal is to prove that:

$$\mathrm{d}_{\mathrm{GH}}(H([X,\mathrm{d}_i],\tau_i), H([X,\mathrm{d}_\infty],\tau_\infty)) \to 0. \tag{5.20}$$

Part III: The case $\tau_\infty = 0$

If $\tau_\infty = 0$, then $t_i \to \phi_\infty^{-1}(0) = 0$. We will assume $\tau_i > 0$ for all $i\in\mathbb{N}$ since we already know that $\mathrm{d}_{\mathrm{GH}}((X,\mathrm{d}_i),(X,\mathrm{d}_\infty)) \to 0$. Now, thanks to Proposition 5.2.4, we have $\mathrm{d}_{\mathrm{GH}}((X,g_i(t_i)),(X,\mathrm{d}_\infty)) \to 0$. Moreover, (5.19) implies $\mathrm{d}_{\mathrm{GH}}((X,\tilde{g}_i(\tau_i),(X,g_i(t_i))) \leq |1-\sqrt{\psi_i(t_i)}|\,\mathrm{Diam}(g_i(t_i))$. However, $\mathrm{Diam}(g_i(t_i)) \to \mathrm{Diam}(X,\mathrm{d})$ (due to Proposition 5.2.4) and $\psi_i(t_i) \to \psi_\infty(0) = 1$. Therefore, $\mathrm{d}_{\mathrm{GH}}((X,\tilde{g}_i(\tau_i)),(X,\mathrm{d}_\infty)) \to 0$, i.e. (5.20) holds in the case where $\tau_\infty = 0$.

Part IV: The case $\tau_\infty \in \mathbb{R}_{>0}$

If $\tau_\infty \in \mathbb{R}_{>0}$, then $t_i \to t_\infty \in \mathbb{R}_{>0}$. Let us show that $\{(X, \tilde{g}_i(\tau_i))\}_{i\in\mathbb{N}}$ admits a subsequence converging to $(X, \tilde{g}_\infty(\tau_\infty))$ (the proof for a subsequence being the same).

Forgetting the first terms of the sequence if necessary, there exists $0 < T < T_\infty$ such that, for all $i \in \mathbb{N} \cup \{\infty\}$, we have $0 < t_i < T$. In particular, thanks to Proposition 5.2.4, $\{(X, g_i(t))\}_{i\in\mathbb{N}, t\in(0,T)}$ admits a subsequence converging to $\{(X, g_\infty(t))\}_{t\in(0,T)}$ in the Hamilton–Cheeger–Gromov sense. In particular, we have $\mathrm{d_{GH}}((X, g_i(t_i)), (X, g_\infty(t_\infty))) \to 0$. Furthermore, using (5.19) and applying the triangle inequality in the order $(X, \tilde{g}_i(\tau_i)) \to (X, g_i(t_i)) \to (X, g_\infty(t_\infty)) \to (X, \tilde{g}_\infty(\tau_\infty))$, we have the following:

$$\begin{aligned}\mathrm{d_{GH}}((X, \tilde{g}_i(\tau_i)), (X, \tilde{g}_\infty(\tau_\infty))) \leq{} & \mathrm{d_{GH}}((X, g_i(t_i)), (X, g_\infty(t_\infty))) \\ & + |1 - \sqrt{\psi_\infty(t_\infty)}| \operatorname{Diam}(g_\infty(t_\infty)). \\ & + |1 - \sqrt{\psi_i(t_i)}| \operatorname{Diam}(X, g_i(t_i))\end{aligned}$$

Therefore, $\mathrm{d_{GH}}((X, \tilde{g}_i(\tau_i)), (X, \tilde{g}_\infty(\tau_\infty))) \to 0$, i.e. (5.20) holds in the case where $\tau_\infty \in \mathbb{R}_{>0}$.

Part V: The case $\tau_\infty = \infty$

Let us show that $\{(X, \tilde{g}_i(\tau_i))\}_{i\in\mathbb{N}}$ admits a subsequence converging to $(X, \tilde{g}_\infty(\infty))$ (the proof for a subsequence being the same).

First of all, proceeding as in the previous case, we can fix $0 < T < T_\infty$ and assume that $\{(X, g_i(t))\}_{i\in\mathbb{N}, t\in(0,T)}$ admits a subsequence converging to $\{(X, g_\infty(t))\}_{t\in(0,T)}$ in the Hamilton–Cheeger–Gromov sense. Replacing $(X, g_i(t))$ by $(X, \phi_i^* g_i(t))$ for some diffeomorphism ϕ_i if necessary (see Definition 5.1.1), we can assume that $\{g_i(t)\}_{i\in\mathbb{N}, t\in(0,T)}$ converges smoothly to $\{g_\infty(t)\}_{t\in(0,T)}$ on compact subsets of $X \times (0, T)$.

In particular, denoting $\tau \coloneqq \phi_\infty(T/2)$, we have $\tilde{g}_i(\tau) = \psi_i(\phi_i^{-1}(\tau)) g_i(\phi_i^{-1}(\tau))$. Moreover, $\phi_i^{-1}(\tau) \to \phi_\infty^{-1}(\tau) = T/2$; therefore, $\{\tilde{g}_i(\tau))\}_{i\in\mathbb{N}}$ converges smoothly to $\tilde{g}_\infty(\tau))$. Now, applying Proposition 5.1.5 and Lemma 5.1.4 and reusing their notations, we obtain:

$$\mathrm{d_{GH}}((X, \tilde{g}_i(t)), (X, \tilde{g}_i(s)) \leq F(0, \infty) \operatorname{Diam}(\tilde{g}_i(\tau)) F(s - \tau, t - \tau)$$

for all $\tau \leq s \leq t \leq \infty$. In particular, denoting $D \coloneqq \sup_{i\in\mathbb{N}\cup\{\infty\}}\{\operatorname{Diam}(\tilde{g}_i(\tau))\} < \infty$ and $G(s, t) \coloneqq F(0, \infty) F(s - \tau, t - \tau) D$, we have the following:

$$\mathrm{d_{GH}}((X, \tilde{g}_i(t)), (X, \tilde{g}_i(s)) \leq G(s, t), \tag{5.21}$$

for all $\tau \leq s \leq t \leq \infty$, where $G(s,t)$ is independent of i and satisfies $G(s,t) \leq G(s,\infty)$ and $\lim_{s\to\infty} G(s,\infty) = 0$.

Now let $\epsilon > 0$ and let $A > 0$ large enough such that $G(A,\infty) \leq \epsilon$. Then, proceeding as above, we can assume that $\{\tilde{g}_i(A)\}_{i\in\mathbb{N}}$ converges smoothly to $\tilde{g}_\infty(A)$. Therefore, applying the triangle inequality on the order $(X, \tilde{g}_i(\tau_i)) \to (X, \tilde{g}_i(A)) \to (X, \tilde{g}_\infty(A)) \to (X, \tilde{g}_\infty(\infty))$ and using (5.21), we have the following:

$$\mathrm{d}_{\mathrm{GH}}((X, \tilde{g}_i(\tau_i))(X, \tilde{g}_\infty(\infty))) \leq 2G(A,\infty) + \mathrm{d}_{\mathrm{GH}}((X, \tilde{g}_i(A)), (X, \tilde{g}_\infty(A))). \quad (5.22)$$

Therefore, $\limsup_{i\in\mathbb{N}} \mathrm{d}_{\mathrm{GH}}((X, \tilde{g}_i(\tau_i))(X, \tilde{g}_\infty(\infty))) \leq 2\epsilon$; hence, letting ϵ go to 0 implies $\mathrm{d}_{\mathrm{GH}}((X, \tilde{g}_i(\tau_i))(X, \tilde{g}_\infty(\infty))) \to 0$, which concludes the proof. □

5.3 RCD(-1,2) spaces

In this section, we apply the results of Section 5.2 to investigate the case of RCD$(-1,2)$ spaces. The result we present provides a homotopy between moduli spaces of RCD$(-1,2)$ structures and smooth moduli spaces of hyperbolic metrics.

Theorem 5.3.1. Let X be a connected closed orientable topological surface with $\chi(X) < 0$ and let $\mathcal{A}$ be a smooth structure on X. Then, the moduli space $\mathfrak{M}_{-1,2}(X)$ of RCD$(-1,2)$ structures on X retracts by deformation onto $\mathscr{M}_{-1}(X,\mathcal{A})$ (endowed with the smooth topology).

Proof. First of all, Lemma 4.2.1 implies that $\mathfrak{M}_{-1,2}(X)$ is homeomorphic to $\mathbb{R} \times \mathscr{M}_{\mathrm{curv}\geq -1}(X)$; in particular, it retracts by deformation on to $\mathscr{M}_{\mathrm{curv}\geq -1}(X)$. However, as stated by Theorem 5.2.1, $\mathscr{M}_{\mathrm{curv}\geq -1}(X)$ retracts by deformation onto $\mathscr{M}_{\mathrm{cste}}(X)$. Furthermore, thanks to Lemma 5.2.1, $\mathscr{M}_{\mathrm{cste}}(X)$ retracts by deformation onto $\mathscr{M}_{\mathrm{sign}(\chi(X))}(X)$ (endowed with the GH topology). Finally, Proposition 5.2.1 and Lemma 5.2.2 imply that $\mathscr{M}_{\mathrm{sign}(\chi(X))}(X)$ is homeomorphic to $\mathscr{M}_{\mathrm{sign}(\chi(X))}(X,\mathcal{A})$ (equipped with the smooth topology), which concludes the proof. □

Appendices

A
Technical proofs

Contents

A.1 Convergence in the mGH topology

Our goal in this section is to prove Theorem 2.1.3. First of all, observe that the Prokhorov distance $\mathrm{d_P}$ metrizes weak-$*$ convergence of measures. Therefore, points (ii) and (iii) of Theorem 2.1.3 are equivalent thanks to the following lemma (which I have not been able to find in the literature).

Lemma A.1.1 (Approximate inverse)**.** *Let $(X, \mathrm{d}_X, \mathfrak{m}_X)$ and $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ be compact metric measure spaces and let $\epsilon \geq 0$. Assume that f is an ϵ-isometry from (X, d_X) to (Y, d_Y) such that $\mathrm{d_P}(f_* \mathfrak{m}_X, \mathfrak{m}_Y) \leq \epsilon$. Then, setting $\tilde{\epsilon} := 10\epsilon$, there is an $\tilde{\epsilon}$-isometry g from (Y, d_Y) to (X, d_X) such that $\mathrm{d_P}(\mathfrak{m}_X, g_* \mathfrak{m}_Y) \leq \tilde{\epsilon}$ and $g \circ f$ (resp. $f \circ g$) is $\tilde{\epsilon}$-close to id_X (resp. to id_Y). Such a map g will be called an approximate inverse associated to f with parameter ϵ.*

Proof. Let us first construct the map g. Since (Y, d_Y) is compact, there exists a partition of Y into finitely many Borel subsets $\{B_i\}_{1 \leq i \leq n}$ with diameter less that ϵ.

Then, for every $1 \leq i \leq n$, pick a reference point $*_i \in B_i$ and let $x_i \in X$ such that $\mathrm{d}_Y(*_i, f(x_i)) \leq \epsilon$. Now, given $y \in Y$, there is a unique $1 \leq i \leq n$ such that $y \in B_i$; we define $g(y) \coloneqq x_i$.

The map g is Borel measurable because for any Borel subset B of X, $g^{-1}(B) = \cup_{i \in J} B_i$ is a finite union of Borel subset of Y, where J is the set of indexes $1 \leq i \leq n$ such that $x_i \in B$.

Now, let us show that $X = \{g(Y)\}^{3\epsilon}$. Given $x \in X$, there is $1 \leq i \leq n$ such that $f(x) \in B_i$. However, B_i having diameter less than ϵ, we have $\mathrm{d}_Y(*_i, f(x)) \leq \epsilon$. Moreover, f being an ϵ-isometry, we have $\mathrm{d}_X(x, x_i) \leq \epsilon + \mathrm{d}_Y(f(x_i), f(x))$. Hence, using the triangle inequality we obtain $\mathrm{d}_X(x, x_i) \leq 3\epsilon$, where $x_i = g(*_i)$. In conclusion, we have shown that $X = \{g(Y)\}^{3\epsilon}$.

Let us show that $\mathrm{Dis}(g) \leq 5\epsilon$. It is enough to prove that, for every $y_i \in B_i$ and $y_j \in B_j$ (where $1 \leq i, j \leq n$), we have $|\mathrm{d}_X(x_i, x_j) - \mathrm{d}_Y(y_i, y_j)| \leq 5\epsilon$. Indeed, we have:

$$\begin{aligned}
|\mathrm{d}_X(x_i, x_j) - \mathrm{d}_Y(y_i, y_j)| &\leq |\mathrm{d}_X(x_i, x_j) - \mathrm{d}_Y(f(x_i), f(x_j))| \\
&\quad + |\mathrm{d}_Y(f(x_i), f(x_j)) - \mathrm{d}_Y(y_i, y_j)| \\
&\leq \epsilon + \mathrm{d}_Y(f(x_i), y_i) + \mathrm{d}_Y(f(x_j), y_j) \\
&\leq \epsilon + \mathrm{d}_Y(f(x_i), *_i) + \mathrm{d}_Y(f(x_j), *_j) + \mathrm{d}_Y(*_i, y_i) + \mathrm{d}_Y(*_j, y_j) \\
&\leq 5\epsilon,
\end{aligned}$$

using the fact that f is an ϵ-isometry for the second inequality and the fact that B_i and B_j have diameter less than ϵ for the last inequality.

Now we show that $g \circ f$ is 3ϵ-close to id_X. Let $x \in X$ and let $1 \leq i \leq n$ such that $f(x) \in B_i$, so that $g \circ f(x) = x_i$. We have:

$$\begin{aligned}
\mathrm{d}_X(x, g \circ f(x)) &\leq \epsilon + \mathrm{d}_Y(f(x), f(x_i)) \\
&\leq \epsilon + \mathrm{d}_Y(f(x_i), *_i) + \mathrm{d}_Y(*_i, f(x)) \\
&\leq 3\epsilon.
\end{aligned}$$

Then, we show that $f \circ g$ is 2ϵ-close to id_Y. Let $y \in Y$ and let $1 \leq i \leq n$ such that $y \in B_i$, so that $f \circ g(y) = f(x_i)$. We have:

$$\begin{aligned}
\mathrm{d}_X(y, f \circ g(y)) &\leq \mathrm{d}_Y(y, *_i) + \mathrm{d}_Y(*_i, f(x_i)) \\
&\leq 2\epsilon.
\end{aligned}$$

Finally, we show that $\mathrm{d_P}(g_* \mathfrak{m}_Y, \mathfrak{m}_X) \le 10\epsilon$. Let A be a closed subset of X and let us first prove that $\mathfrak{m}_X(A) \le 10\epsilon + g_* \mathfrak{m}_Y(\{A\}^{10\epsilon})$. Indeed, we have:

$$\begin{aligned}\mathfrak{m}_X(A) &\le \mathfrak{m}_X(f^{-1}(g^{-1}(\{A\}^{3\epsilon}))) = f_* \mathfrak{m}_X(g^{-1}(\{A\}^{3\epsilon}))\\ &\le f_* \mathfrak{m}_X\left(\overline{g^{-1}(\{A\}^{3\epsilon})}\right)\\ &\le \epsilon + \mathfrak{m}_Y\left(\left\{\overline{g^{-1}(\{A\}^{3\epsilon})}\right\}^{\epsilon}\right),\end{aligned}$$

using the fact that $g \circ f$ is 3ϵ-close to id_X in order to get $A \subset f^{-1}(g^{-1}(\{A\}^{3\epsilon}))$ at the first line and the fact that $\mathrm{d_P}(f_* \mathfrak{m}_X, \mathfrak{m}_Y) < \epsilon$ in order to get the last inequality. Now, let us show that $\left\{\overline{g^{-1}(\{A\}^{3\epsilon})}\right\}^{\epsilon} \subset g^{-1}(\{A\}^{10\epsilon})$. Given $y \in \left\{\overline{g^{-1}(\{A\}^{3\epsilon})}\right\}^{\epsilon}$, there is $y' \in g^{-1}(\{A\}^{3\epsilon})$ such that $\mathrm{d}_Y(y, y') \le 2\epsilon$. Now, using $\mathrm{Dis}(g) \le 5\epsilon$, we have $\mathrm{d}_X(g(y), g(y')) \le 7\epsilon$, where $g(y') \in \{A\}^{3\epsilon}$. Hence, there is $x \in A$ such that $\mathrm{d}_X(x, g(y')) \le 3\epsilon$, so that $\mathrm{d}_X(g(y), x) \le 10\epsilon$. In conclusion, we have $\left\{\overline{g^{-1}(\{A\}^{3\epsilon})}\right\}^{\epsilon} \subset g^{-1}(\{A\}^{10\epsilon})$; thus:

$$\begin{aligned}\mathfrak{m}_X(A) &\le \epsilon + \mathfrak{m}_Y(g^{-1}(\{A\}^{10\epsilon})) = \epsilon + g_* \mathfrak{m}_Y(\{A\}^{10\epsilon})\\ &\le 10\epsilon + g_* \mathfrak{m}_Y(\{A\}^{10\epsilon}).\end{aligned}$$

Now, we show that $g_* \mathfrak{m}_Y(A) \le 10\epsilon + \mathfrak{m}_X(\{A\}^{10\epsilon})$. Indeed, we have:

$$\begin{aligned}g_* \mathfrak{m}_Y(A) &\le \mathfrak{m}_Y\left(\overline{g^{-1}(A)}\right)\\ &\le \epsilon + f_* \mathfrak{m}_X\left(\left\{\overline{g^{-1}(A)}\right\}^{\epsilon}\right)\\ &\le 10\epsilon + \mathfrak{m}_X(\{A\}^{10\epsilon}),\end{aligned}$$

where we used the fact that $f^{-1}\left(\left\{\overline{g^{-1}(A)}\right\}^{\epsilon}\right) \subset \{A\}^{10\epsilon}$ for the last inequality, which concludes the proof. □

We can now prove the rest of Theorem 2.1.3, i.e. show that points (i) and (ii) are equivalent. The proof relies on [Mie09, Proposition 7], which shows the result in the case where all measures are probability measures.

Proof of Theorem 2.1.3. We will only prove the result in the case where all m.m.s. are endowed with non-trivial measures, the other case being easier to prove.

Let us first assume that point (ii) of Theorem 2.1.3 is satisfied and let us prove (i), i.e. let us show that $\mathfrak{D}([X_n, \mathrm{d}_n, \mathfrak{m}_n], [X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty]) \to 0$. We start by proving that:

$$\mathfrak{D}([X_n, \mathrm{d}_n, \tilde{\mathfrak{m}}_n], [X_\infty, \mathrm{d}_\infty, \tilde{\mathfrak{m}}_\infty]) \to 0,$$

where, for every $n \in \mathbb{N} \cup \{\infty\}$, $\tilde{\mathfrak{m}}_n := \frac{\mathfrak{m}_n}{\mathfrak{m}_n(X_n)}$ is the normalization of $\mathfrak{m}_n$. By [Mie09, Proposition 7], it is equivalent to show that there are ϵ_n-isometries f_n from (X_n, d_n) to $(X_\infty, \mathrm{d}_\infty)$ and g_n from $(X_\infty, \mathrm{d}_\infty)$ to (X_n, d_n) such that $\epsilon_n \to 0$ and:

$$\max\{\mathrm{d_P}(f_{n*}\tilde{\mathfrak{m}}_n, \tilde{\mathfrak{m}}_\infty), \mathrm{d_P}(g_{n*}\tilde{\mathfrak{m}}_\infty, \tilde{\mathfrak{m}}_n)\} \to 0.$$

By assumption, there exists a sequence of ϵ_n-isometries $f_n \colon (X_n, \mathrm{d}_n) \to (X_\infty, \mathrm{d}_\infty)$ such that $\epsilon_n \to 0$ and $f_{n*}\,\mathfrak{m}_n \rightharpoonup \mathfrak{m}_\infty$. By Lemma A.1.1, there are maps g_n such that for every $n \in \mathbb{N}$, g_n is an approximate inverse associated to f_n with parameter δ_n, where:

$$\delta_n := \max(\epsilon_n, \mathrm{d_P}(f_{n*}\,\mathfrak{m}_n, \mathfrak{m}_\infty)).$$

However, $\mathrm{d_P}$ induces weak-$*$ convergence of finite measures on $(X_\infty, \mathrm{d}_\infty)$; in particular $\delta_n \to 0$. Moreover, $\mathrm{d_P}(g_{n*}\,\mathfrak{m}_\infty, \mathfrak{m}_n) \leq 10\delta_n$ and we get $\mathrm{d_P}(g_{n*}\,\mathfrak{m}_\infty, \mathfrak{m}_n) \to 0$. Now, observe that $\{f_{n*}\,\mathfrak{m}_n\}_{n\in\mathbb{N}}$ converges to $\mathfrak{m}_\infty$ in the weak-$*$ topology. Hence, we have $\lim_{n\to\infty} f_{n*}\,\mathfrak{m}_n(X_\infty) = \mathfrak{m}_\infty(X_\infty) > 0$. In particular:

$$m := \inf_{n\in\mathbb{N}\cup\{\infty\}} \{\mathfrak{m}_n(X_n)\} > 0.$$

Then, for every $n \in \mathbb{N}$, we can show that:

$$\mathrm{d_P}(f_{n*}\tilde{\mathfrak{m}}_n, \tilde{\mathfrak{m}}_\infty) \leq \mathrm{d_P}(f_{n*}\,\mathfrak{m}_n, \mathfrak{m}_\infty) \max\left\{1, \frac{2}{m}\right\} \to 0 \tag{A.1}$$

$$\mathrm{d_P}(g_{n*}\tilde{\mathfrak{m}}_\infty, \tilde{\mathfrak{m}}_n) \leq \mathrm{d_P}(g_{n*}\,\mathfrak{m}_\infty, \mathfrak{m}_n) \max\left\{1, \frac{2}{m}\right\} \to 0. \tag{A.2}$$

Indeed, let us prove inequality A.1. Assume, $\mathrm{d_P}(f_{n*}\,\mathfrak{m}_n, \mathfrak{m}_\infty) < \epsilon$. Then, for every closed subset A of X_∞ we have, by definition of the Prokhorov distance:

$$\begin{aligned} f_{n*}\tilde{\mathfrak{m}}_n(A) &\leq \frac{\mathfrak{m}_\infty(\{A\}^\epsilon) + \epsilon}{\mathfrak{m}_n(X_n)} = \frac{\mathfrak{m}_\infty(\{A\}^\epsilon)}{\mathfrak{m}_\infty(X_\infty)} + \mathfrak{m}_\infty(\{A\}^\epsilon)\frac{\mathfrak{m}_\infty(X_\infty) - \mathfrak{m}_n(X_n)}{\mathfrak{m}_\infty(X_\infty)\,\mathfrak{m}_n(X_n)} + \frac{\epsilon}{\mathfrak{m}_n(X_n)} \\ &\leq \tilde{\mathfrak{m}}_\infty(\{A\}^\epsilon) + \frac{|\mathfrak{m}_\infty(X_\infty) - \mathfrak{m}_n(X_n)|}{\mathfrak{m}_n(X_n)} + \frac{\epsilon}{m} \\ &\leq \tilde{\mathfrak{m}}_\infty(\{A\}^\epsilon) + \frac{2\epsilon}{m}, \end{aligned}$$

where we used $|\mathfrak{m}_\infty(X_\infty) - \mathfrak{m}_n(X_n)| \leq \epsilon$ and $\mathfrak{m}_n(X_n) \geq m$. Using exactly the same technique, we also show that $\tilde{\mathfrak{m}}_\infty(A) \leq f_{n*}\tilde{\mathfrak{m}}_n(\{A\}^\epsilon) + \frac{2\epsilon}{m}$. Thus, we have $\mathrm{d_P}(f_{n*}\tilde{\mathfrak{m}}_n, \tilde{\mathfrak{m}}_\infty) \leq \epsilon \max\{1, \frac{2}{m}\}$. Letting ϵ go to $\mathrm{d_P}(f_{n*}\,\mathfrak{m}_n, \mathfrak{m}_\infty)$, we finally get inequality A.1. The same strategy gives inequality A.2. Since we have inequalities A.1 and A.2, we can apply [Mie09, Proposition 7] and get:

$$\mathfrak{D}([X_n, \mathrm{d}_n, \tilde{\mathfrak{m}}_n], [X_\infty, \mathrm{d}_\infty, \tilde{\mathfrak{m}}_\infty]) \to 0.$$

Now, let us show that:

$$\mathfrak{D}([X_n, \mathrm{d}_n, \mathfrak{m}_n], [X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty]) \to 0.$$

First, observe that $\mathfrak{D}([X_n, \mathrm{d}_n, \tilde{\mathfrak{m}}_n], [X_\infty, \mathrm{d}_\infty\, \tilde{\mathfrak{m}}_\infty]) \to 0$; hence, there exist isometric embeddings $\iota_n \colon (X_n, \mathrm{d}_n) \hookrightarrow (Z_n, \mathrm{d}_{Z_n})$ and $j_n \colon (X_\infty, \mathrm{d}_\infty) \hookrightarrow (Z_n, \mathrm{d}_{Z_n})$ such that:

- $\mathrm{d}_\mathrm{H}(X'_n, X'_\infty) \leq \alpha_n$,
- $\mathrm{d}_\mathrm{P}(\tilde{\mathfrak{m}}'_n, \tilde{\mathfrak{m}}'_\infty) \leq \alpha_n$,
- $\alpha_n \to 0$,

where $X'_n = \iota_n(X_n)$, $X'_\infty = j_n(X_\infty)$, $\tilde{\mathfrak{m}}'_n = \iota_{n*}\tilde{\mathfrak{m}}_n$, and $\tilde{\mathfrak{m}}'_\infty = j_{n*}\tilde{\mathfrak{m}}_\infty$. However, it is possible to show that:

$$\mathrm{d}_\mathrm{P}(\mathfrak{m}'_n, \mathfrak{m}'_\infty) \leq \max\{\alpha_n, \alpha_n M + \delta_n\}, \tag{A.3}$$

where $M \coloneqq \sup_{n \in \mathbb{N}\cup\{\infty\}}\{\mathfrak{m}_n(X_n)\} < \infty$, $\mathfrak{m}'_n = \iota_{n*}\, \mathfrak{m}_n$, and $\mathfrak{m}'_\infty = j_{n*}\, \mathfrak{m}_\infty$. Indeed, for every $\mathrm{d}_\mathrm{P}(\tilde{\mathfrak{m}}'_n, \tilde{\mathfrak{m}}'_\infty) < \epsilon_1$ and $\mathrm{d}_\mathrm{P}(f_{n*}\, \mathfrak{m}_n, \mathfrak{m}_\infty) < \epsilon_2$ and every closed subset A of Z_n, we have:

$$\begin{aligned}
\mathfrak{m}'_\infty(A) = \mathfrak{m}_\infty(X_\infty)\tilde{\mathfrak{m}}'_\infty(A) &\leq \mathfrak{m}_\infty(X_\infty)(\tilde{\mathfrak{m}}_n(\{A\}^{\epsilon_1}) + \epsilon_1) \\
&\leq \frac{\mathfrak{m}_\infty(X_\infty)}{\mathfrak{m}_n(X_n)}\, \mathfrak{m}'_n(\{A\}^{\epsilon_1}) + \mathfrak{m}_\infty(X_\infty)\epsilon_1 \\
&\leq \mathfrak{m}'_n(\{A\}^{\epsilon_1})\Big(1 + \frac{|\mathfrak{m}_\infty(X_\infty) - \mathfrak{m}_n(X_n)|}{\mathfrak{m}_n(X_n)}\Big) + M\epsilon_1 \\
&\leq \mathfrak{m}'_n(\{A\}^{\epsilon_1}) + \frac{\mathfrak{m}'_n(\{A\}^{\epsilon_1})}{\mathfrak{m}_n(X_n)}\epsilon_2 + M\epsilon_1 \\
&\leq \mathfrak{m}'_n(\{A\}^{\epsilon_1}) + M\epsilon_1 + \epsilon_2,
\end{aligned}$$

where we used $|\mathfrak{m}_\infty(X_\infty) - \mathfrak{m}_n(X_n)| \leq \epsilon_2$. Proceeding the same way, we show $\mathfrak{m}'_n(A) \leq \mathfrak{m}'_\infty(\{A\}^{\epsilon_1}) + M\epsilon_1 + \epsilon_2$. Hence, $\mathrm{d}_\mathrm{P}(\mathfrak{m}'_n, \mathfrak{m}'_\infty) \leq \max\{\epsilon_1, \epsilon_2 + M\epsilon_1\}$. Letting ϵ_1 go to α_n and ϵ_2 go to δ_n, we finally get A.3. In particular, this implies that:

$$\mathfrak{D}([X_n, \mathrm{d}_n, \mathfrak{m}_n], [X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty]) \leq \alpha_n + \max\{\alpha_n, \alpha_n M + \delta_n\} \to 0.$$

Hence, the first point of Theorem 2.1.3 is proven.

Now, let us assume point (i) of Theorem 2.1.3, and let us prove point (ii). Let $\iota_n \colon (X_n, \mathrm{d}_n) \hookrightarrow (Z_n, \mathrm{d}_{Z_n})$ and $j_n \colon (X_\infty, \mathrm{d}_\infty) \hookrightarrow (Z_n, \mathrm{d}_{Z_n})$ be isometric embeddings such that $d_H(X'_n, X'_\infty) \le \alpha_n$, $d_P(\mathfrak{m}'_n, \mathfrak{m}'_\infty) \le \alpha_n$, and $\alpha_n \to 0$ (where $X'_n = \iota_n(X_n)$, $X'_\infty = j_n(X_\infty)$, $\mathfrak{m}'_n = \iota_{n*}\,\mathfrak{m}_n$, and $\mathfrak{m}'_\infty = j_{n*}\,\mathfrak{m}_\infty$). As before:

$$|\mathfrak{m}_n(X_n) - \mathfrak{m}_\infty(X_\infty)| \le d_P(\mathfrak{m}'_n, \mathfrak{m}'_\infty) \le \alpha_n \to 0;$$

hence, $m := \inf_{n\in\mathbb{N}\cup\{\infty\}}\{\mathfrak{m}_n(X_n)\} > 0$ and $M := \sup_{n\in\mathbb{N}\cup\{\infty\}} \mathfrak{m}_n(X_n) < \infty$. Proceeding as for the proof of A.1 we have:

$$d_P(\tilde{\mathfrak{m}}'_n, \tilde{\mathfrak{m}}'_\infty) \le \alpha_n \max\left\{1, \frac{2}{m}\right\} \to 0.$$

In particular, $\mathfrak{D}([X_n, d_n, \tilde{\mathfrak{m}}_n], [X_\infty, d_\infty, \tilde{\mathfrak{m}}_\infty]) \to 0$. Thus, applying [Mie09, Proposition 7], there are ϵ_n-isometries $f_n\colon (X_n, d_n) \to (X_\infty, d_\infty)$ such that $d_P(\tilde{\mathfrak{m}}_\infty, f_{n*}\tilde{\mathfrak{m}}_n) \le \epsilon_n$ and $\epsilon_n \to 0$. However, proceeding as for the proof of A.3 we have:

$$d_P(\mathfrak{m}_\infty, f_{n*}\,\mathfrak{m}_n) \le \max\{\epsilon_n, M\epsilon_n + \alpha_n\} \to 0,$$

which concludes the proof. □

A.2 Alexandrov spaces

In this section, we briefly introduce compact Alexandrov spaces.

Notation A.2.1. Let $\mathrm{Sec}(K, N)$ be the set of isometry classes of compact Riemannian manifolds with sectional curvature bounded below by $K \in \mathbb{R}$ and dimension bounded above by $N \in \mathbb{N}$.

Using [Pet06, Corollary 11.1.13], we have the following precompactness result.

Theorem A.2.1 (Gromov precompactness Theorem)**.** Given $K \in \mathbb{R}$, $D > 0$ and $N \in \mathbb{N}$, the set $\mathrm{Sec}(K, N) \cap \{\mathrm{Diam} \le D\}$ is precompact for the GH topology.

Following Theorem A.2.1, special attention has been given to limits of sequences in $\mathrm{Sec}(K, N) \cap \{\mathrm{Diam} \le D\}$, i.e., points of the boundary set $\partial\mathrm{Sec}(K, N) \subset \mathscr{X}$ (see Notation 2.1.1). However, such limits are not necessarily smooth and it is in general not possible to apply the tools of Riemannian geometry to study them. Hence, Alexandrov introduced a new definition of sectional curvature lower bounds at the level of geodesic spaces.

Definition A.2.1 (Alexandrov spaces)**.** Let (X, d) be a compact geodesic metric space, let $K \in \mathbb{R}$, and let M_K be the simply connected Riemannian manifold of constant sectional curvature K and dimension 2. A triangle Δabc in X is K-*admissible* whenever there exists a comparison triangle in M_K uniquely defined up to isometry. We say that (X, d) has *curvature bounded below by* $K \in \mathbb{R}$ (also denote $\mathrm{curv} \geq K$) if, for every K-admissible triangle Δabc in X, we have the following inequality:

$$\forall d \in [a, c], |bd| \geq |\overline{bd}|,$$

where $\Delta\bar{a}\bar{b}\bar{c}$ is a comparison triangle, and $\bar{d} \in [\bar{a}, \bar{c}]$ is the unique point satisfying $|ad| = |\bar{a}\bar{d}|$. Compact geodesic metric spaces with curvature bounded below are called *Alexandrov spaces.*

Notation A.2.1. *Let* $\mathrm{Alex}(K, N)$ *be the set of isometry classes of compact Alexandrov spaces with* $\mathrm{curv} \geq K \in \mathbb{R}$*, and Hausdorff dimension bounded above by* $N \in \mathbb{N}$*. Let* Riem *be the set of isometry classes of compact Riemannian manifolds.*

Alexandrov's definition satisfies the following properties:

(a) it does not rely on any underlying smooth structure,

(b) it generalizes the classical definition of lower bounds for the sectional curvature (see the Cartan–Alexandrov–Topogonov comparison Theorem [BBI22, Theorem 6.5.6]),

(c) it is stable under Gromov–Hausdorff limits (since, by Theorem 8.5 of [BGP92], the set $\mathrm{Alex}(K, N) \cap \{\mathrm{Diam} \leq D\}$ is compact w.r.t. the Gromov–Hausdorff distance $\mathcal{D}$ for every $D > 0$).

A definition satisfying points (a), (b) and (c) is called *synthetic*. Such a definition is particularly interesting in order to study the limits of sequences in $\mathrm{Sec}(K, N) \subset \mathscr{X}$. Indeed, if (X, d_X) is the GH limit of a sequence $\{M_k, g_k\}_{k\in\mathbb{N}}$ in $\mathrm{Sec}(K, N)$, then (X, d_X) is a fortiori an Alexandrov space with $\mathrm{curv} \geq K$. Therefore, any result proven for Alexandrov spaces holds in particular for limits of sequences in $\mathrm{Sec}(K, N)$. See [BBI22] for a comprehensive introduction to Alexandrov spaces.

A.3 Equivariant mGH topology is metrizable

Before proving Proposition 2.4.2, let us point out the following technicality on the Prokhorov distance (see Definition 2.1.21).

Remark A.3.1. There are various notions of distances between restricted measures. Indeed, given a pointed complete separable metric space $(Y, \mathrm{d}, *)$ endowed with two boundedly finite measures $\mathfrak{m}_1$ and $\mathfrak{m}_2$, and $R > 0$, we can define:

$$\mathrm{d}_{\mathcal{P}}^R(\mathfrak{m}_1, \mathfrak{m}_2) := \inf \left\{ \epsilon > 0, \forall A \subset \overline{B}_R, A \text{ closed implies } \middle| \begin{array}{l} \mathfrak{m}_1(A) \leq \mathfrak{m}_2(A^\epsilon) + \epsilon \\ \mathfrak{m}_2(A) \leq \mathfrak{m}_1(A^\epsilon) + \epsilon \end{array} \right\},$$

or:

$$\mathrm{d}_{\mathcal{P}}(\mathfrak{m}_1^R, \mathfrak{m}_2^R) := \inf \left\{ \epsilon > 0, \forall A \subset X, A \text{ closed implies } \middle| \begin{array}{l} \mathfrak{m}_1^R(A) \leq \mathfrak{m}_2^R(A^\epsilon) + \epsilon \\ \mathfrak{m}_2^R(A) \leq \mathfrak{m}_1^R(A^\epsilon) + \epsilon \end{array} \right\},$$

where $\mathfrak{m}_1^R$ (resp. $\mathfrak{m}_2^R$) is the restriction of $\mathfrak{m}_1$ (resp. $\mathfrak{m}_2$) to $\overline{B}_R$. Let us see how these two notions differ.

First of all, we easily obtain:

$$\mathrm{d}_{\mathcal{P}}^R(\mathfrak{m}_1, \mathfrak{m}_2) \leq \mathrm{d}_{\mathcal{P}}(\mathfrak{m}_1^R, \mathfrak{m}_2^R). \tag{A.4}$$

Then, if $\mathrm{d}_{\mathcal{P}}^R(\mathfrak{m}_1, \mathfrak{m}_2) \leq \epsilon$, we have:

$$\mathrm{d}_{\mathcal{P}}(\mathfrak{m}_1^R, \mathfrak{m}_2^R) \leq \epsilon + \mathfrak{m}_1(\overline{B}_{R+\epsilon} \backslash \overline{B}_R) + \mathfrak{m}_2(\overline{B}_{R+\epsilon} \backslash \overline{B}_R). \tag{A.5}$$

Therefore, both notions lead to the same convergence. Indeed, given a boundedly finite measure $\mathfrak{m}_\infty$ and a sequence $\{\mathfrak{m}_k\}$ of boundedly finite measures, we have the following equivalence thanks to inequalities (A.4) and (A.5):

$$\mathfrak{m}_k \overset{*}{\rightharpoonup} \mathfrak{m}_\infty \iff \forall R > 0, \mathrm{d}_{\mathcal{P}}(\mathfrak{m}_k^R, \mathfrak{m}_\infty^R) \to 0 \iff \forall R > 0, \mathrm{d}_{\mathcal{P}}^R(\mathfrak{m}_k, \mathfrak{m}_\infty) \to 0.$$

In Definition 2.4.8, we are using $\mathrm{d}_{\mathcal{P}}^R(\mathfrak{m}_1, \mathfrak{m}_2)$ instead of $\mathrm{d}_{\mathcal{P}}(\mathfrak{m}_1^R, \mathfrak{m}_2^R)$ because the first one is a non-decreasing function of R whereas the other one is a priori not. In doing so, we are losing the triangle inequality. In fact, given three boundedly finite measure $\mathfrak{m}_i$ ($i \in \{1, 2, 3\}$) such that $\mathrm{d}_{\mathcal{P}}^R(\mathfrak{m}_1, \mathfrak{m}_2) \leq \delta$ and $\mathrm{d}_{\mathcal{P}}^R(\mathfrak{m}_2, \mathfrak{m}_3) \leq \eta$, we only have:

$$\mathrm{d}_{\mathcal{P}}^{R-(\delta+\eta)}(\mathfrak{m}_1, \mathfrak{m}_3) \leq \delta + \eta. \tag{A.6}$$

This will be sufficient to investigate the properties of $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}$.

Proof of Proposition 2.4.2. Part I: Properties of $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}$

Observe that $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}$ is symmetric, nonnegative and invariant under equivariant isomorphisms. However, $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}$ does not satisfy the triangle inequality a priori. Nevertheless, it will be sufficient for our purposes to show that $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}$ satisfies the following modified triangle inequality:

$$\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}(\tilde{\mathfrak{X}}_1, \tilde{\mathfrak{X}}_3) \leq 4(\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}(\tilde{\mathfrak{X}}_1, \tilde{\mathfrak{X}}_2) + \mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}(\tilde{\mathfrak{X}}_2, \tilde{\mathfrak{X}}_3)) \tag{A.7}$$

where $\tilde{\mathfrak{X}}_i = (\tilde{X}, \tilde{\mathrm{d}}_i, \tilde{\mathfrak{m}}_i, \tilde{*}_i)$ is any pointed equivariant $\mathrm{RCD}^*(K,N)$ structure on $\tilde{X}$ $(i \in \{1,2,3\})$.

Observe that the inequality is trivially true whenever $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}(\tilde{\mathfrak{X}}_1, \tilde{\mathfrak{X}}_2)$ or $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}(\tilde{\mathfrak{X}}_2, \tilde{\mathfrak{X}}_3)$ is equal to $1/24$.

Now, assume that for $i \in \{1,2\}$, we have $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}(\tilde{\mathfrak{X}}_i, \tilde{\mathfrak{X}}_{i+1}) \leq \epsilon_{i,i+1}$, where $\epsilon_{i,i+1} \in (0, 1/24)$, and let $(f_{i,i+1}, g_{i,i+1}, \phi_{i,i+1})$ be an associated equivariant pointed $\epsilon_{i,i+1}$-isometry. We define $f := f_{23} \circ f_{12}$, $g := g_{12} \circ g_{23}$, and $\phi := \phi_{23} \circ \phi_{12}$. We want to show that (f, g, ϕ) is an equivariant pointed $4(\epsilon_{12} + \epsilon_{23})$-isometry between $\tilde{\mathfrak{X}}_1$ and $\tilde{\mathfrak{X}}_3$.

First, point (i) and point (ii) of Definition 2.4.8 are trivially satisfied.

Then, let $x, y \in \tilde{X}$ such that $\mathrm{d}_1(x,y) \leq [4(\epsilon_{12} + \epsilon_{23})]^{-1}$. Notice that $[4(\epsilon_{12} + \epsilon_{23})]^{-1} \leq \epsilon_{12}^{-1}$. Therefore $\tilde{\mathrm{d}}_2(f_{12}(x), f_{12}(y)) \leq \epsilon_{12} + [4(\epsilon_{12} + \epsilon_{23})]^{-1} \leq \epsilon_{23}^{-1}$ (where we used $\epsilon_{i,i+1} < 1/24$). Hence, we have $|\tilde{\mathrm{d}}_3(f(x), f(y)) - \tilde{\mathrm{d}}_2(f_{12}(x), f_{12}(y))| \leq \epsilon_{23}$ and $|\tilde{\mathrm{d}}_2(f_{12}(x), f_{12}(y)) - \tilde{\mathrm{d}}_1(x,y)| \leq \epsilon_{12}$. Thus, we get:

$$\begin{aligned} |\tilde{\mathrm{d}}_3(f(x), f(y)) - \tilde{\mathrm{d}}_1(x,y)| &\leq \epsilon_{12} + \epsilon_{23} \\ &\leq 4(\epsilon_{12} + \epsilon_{23}). \end{aligned}$$

Using the same argument for g, we can conclude that point (iii) of Definition 2.4.8 is satisfied.

Now, let $x \in \tilde{X}$ and observe that $\tilde{\mathrm{d}}_2(g_{23} f_{23} f_{12}(x), f_{12}(x)) \leq \epsilon_{23} \leq \epsilon_{12}^{-1}$. Hence, we have:

$$\begin{aligned} \tilde{\mathrm{d}}_1(gf(x), x) &\leq \tilde{\mathrm{d}}_1(x, g_{12} f_{12}(x)) + \tilde{\mathrm{d}}_1(g_{12} g_{23} f_{23} f_{12}(x), g_{12} f_{12}(x)) \\ &\leq \epsilon_{12} + \epsilon_{23} + \epsilon_{12} \\ &\leq 4(\epsilon_{12} + \epsilon_{23}). \end{aligned}$$

Arguing the same way for $f \circ g$, we can conclude that point (iv) of Definition 2.4.8 is satisfied.

Now, let us show that point (v) of Definition 2.4.8 is also satisfied. First of all, let us find an upper bound on $\mathrm{d}_P^{[2(\epsilon_{12}+\epsilon_{23})]^{-1}}(f_*\tilde{\mathfrak{m}}_1, f_{23_*}\tilde{\mathfrak{m}}_2)$. Let A be a closed subset of $\overline{B}_3([2(\epsilon_{12}+\epsilon_{23})]^{-1})$. We easily get that $f_{23}^{-1}(A) \subset \overline{B}_2(\epsilon_{12}^{-1})$, which implies:

$$f_*\tilde{\mathfrak{m}}_1(A) = f_{23_*}(f_{12_*}\tilde{\mathfrak{m}}_1)(A) \leq \tilde{\mathfrak{m}}_2((f_{23}^{-1}(A))^{\epsilon_{12}}) + \epsilon_{12}.$$

Then, note that we have $(f_{23}^{-1}(A))^{\epsilon_{12}} \subset f_{23}^{-1}((A)^{\epsilon_{12}+\epsilon_{23}})$. Therefore, we have $f_*\tilde{\mathfrak{m}}_1(A) \leq f_{23_*}\tilde{\mathfrak{m}}_2(A^{\epsilon_{12}+\epsilon_{23}}) + \epsilon_{12}$. Doing the same in the opposite direction gives us:

$$\mathrm{d}_P^{[2(\epsilon_{12}+\epsilon_{23})]^{-1}}(f_*\tilde{\mathfrak{m}}_1, f_{23_*}\tilde{\mathfrak{m}}_2) \leq \epsilon_{12} + \epsilon_{23}.$$

Finally, observe that $[4(\epsilon_{12}+\epsilon_{23})]^{-1} \leq [2(\epsilon_{12}+\epsilon_{23})]^{-1} - (\epsilon_{12}+2\epsilon_{23})$. Thus, applying inequality (A.6) of Remark A.3.1, we have:

$$\begin{aligned}
\mathrm{d}_P^{[4(\epsilon_{12}+\epsilon_{23})]^{-1}}(f_*\tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_3) &\leq \mathrm{d}_P^{[2(\epsilon_{12}+\epsilon_{23})]^{-1}-(\epsilon_{12}+2\epsilon_{23})}(f_*\tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_3) \\
&\leq \mathrm{d}_P^{[2(\epsilon_{12}+\epsilon_{23})]^{-1}}(f_{23_*}\tilde{\mathfrak{m}}_2, \tilde{\mathfrak{m}}_3) + \mathrm{d}_P^{[2(\epsilon_{12}+\epsilon_{23})]^{-1}}(f_{23_*}\tilde{\mathfrak{m}}_2, f_*\tilde{\mathfrak{m}}_1) \\
&\leq \epsilon_{12} + 2\epsilon_{23} \\
&\leq 4(\epsilon_{12}+\epsilon_{23}).
\end{aligned}$$

Applying the same argument, we also get $\mathrm{d}_P^{[4(\epsilon_{12}+\epsilon_{23})]^{-1}}(g_*\tilde{\mathfrak{m}}_3, \tilde{\mathfrak{m}}_1) \leq 4(\epsilon_{12}+\epsilon_{23})$. Therefore, point (v) of Definition 2.4.8 is satisfied.

This concludes the proof of the modified triangle inequality (A.7).

Part II: Hausdorff uniform structure

Let $\mathcal{B} := \{\{\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}} \leq 2^{-n}\}, n \in \mathbb{N}\}$ be a family of subsets of $\mathfrak{M}_{K,N}^{\mathrm{p,eq}}(\tilde{X}) \times \mathfrak{M}_{K,N}^{\mathrm{p,eq}}(\tilde{X})$. Thanks to the fact that $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}$ is well-defined on $\mathfrak{M}_{K,N}^{\mathrm{p,eq}}(\tilde{X})$, symmetric, nonnegative, and satisfies the modified triangle inequality (A.7), we can easily check the axioms introduced p.141 of [Bou07a]. Therefore, $\mathcal{B}$ is a fundamental system of neighbourhoods of a uniform structure on $\mathfrak{M}_{K,N}^{\mathrm{p,eq}}(\tilde{X})$.

Let us now show that the uniform structure is Hausdorff. Let $\tilde{\mathfrak{X}}_i = (\tilde{X}, \tilde{\mathrm{d}}_i, \tilde{\mathfrak{m}}_i, \tilde{*}_i)$ be a pointed equivariant $\mathrm{RCD}^*(K,N)$ structure on $\tilde{X}$ $(i \in \{1,2\})$. Let us assume that $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}(\tilde{\mathfrak{X}}_1, \tilde{\mathfrak{X}}_2) = 0$. We need to prove that $\tilde{\mathfrak{X}}_1$ and $\tilde{\mathfrak{X}}_2$ are equivariantly isomorphic.

First, since $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}(\tilde{\mathfrak{X}}_1, \tilde{\mathfrak{X}}_2) = 0$, there is a sequence of equivariant pointed ϵ_n-isometries (f_n, g_n, ϕ_n) such that $\epsilon_n \to 0$. Let us fix a countable dense subset $\mathcal{D}_1$ in $(\tilde{X}, \tilde{\mathrm{d}}_1)$ and observe that for every $x \in \mathcal{D}_1$, the sequence $\{f_n(x)\}$ is bounded. Therefore, applying Cantor's diagonal argument, we can assume that there is a map $f\colon \mathcal{D}_1 \to (\tilde{X}, \tilde{\mathrm{d}}_2)$ such that, for every $x \in \mathcal{D}_1$, we have $f_n(x) \to f(x)$. Then, observe that given $x, y \in \mathcal{D}_1$, and n large enough, we have $|\tilde{\mathrm{d}}_2(f_n(x), f_n(y)) - \tilde{\mathrm{d}}_1(x, y)| \leq \epsilon_n \to 0$. Therefore, f is an isometric embedding; hence, it can be extended uniquely into an isometric embedding $f\colon (\tilde{X}, \tilde{\mathrm{d}}_1) \to (\tilde{X}, \tilde{\mathrm{d}}_2)$. Then, it is not hard to prove that, given any sequence $\{x_n\}$ in $\tilde{X}$ converging to $x \in \tilde{X}$, we have $f_n(x_n) \to f(x)$. Now, we can apply the same procedure to the sequence $\{g_n\}$, and get an isometric embedding $g\colon (\tilde{X}, \tilde{\mathrm{d}}_2) \to (\tilde{X}, \tilde{\mathrm{d}}_1)$ such that, given any sequence $\{x_n\}$ in $\tilde{X}$ converging to $x \in \tilde{X}$, we have $g_n(x_n) \to g(x)$. In particular, given $x \in \tilde{X}$, we have $\lim_{n\to\infty} \tilde{\mathrm{d}}_1(g_n(f_n(x)), x) = 0 = \tilde{\mathrm{d}}_1(g(f(x)), x)$. Therefore, f and g are respectively inverse to each other. Moreover, we have $f(\tilde{*}_1) = \lim_{n\to\infty} f_n(\tilde{*}_1) = \tilde{*}_2$. The same argument gives us $g(\tilde{*}_2) = \tilde{*}_1$. Hence, $f\colon (\tilde{X}, \tilde{\mathrm{d}}_1, \tilde{*}_1) \to (\tilde{X}, \tilde{\mathrm{d}}_2, \tilde{*}_2)$ is an isomorphism of pointed metric space. Now, observe that $\mathrm{d}_{\mathcal{P}}^{\epsilon_n^{-1}}(f_{n*}\tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2) \leq \epsilon_n \to 0$. Hence, Remark A.3.1 implies that $\{f_{n*}\tilde{\mathfrak{m}}_1\}$ converges to $\tilde{\mathfrak{m}}_2$ in the weak-$*$ topology. Let us show that $f_*\tilde{\mathfrak{m}}_1 = \tilde{\mathfrak{m}}_2$. We fix $R > 0$ and $h \in \mathcal{C}^0(\tilde{X})$ such that $\mathrm{Spt}(h) \subset \overline{B}_2(R)$. Observe that we have $\int_{\tilde{X}} h \,\mathrm{d}\,\tilde{\mathfrak{m}}_2 = \lim_{n\to\infty} \int_{\tilde{X}} h \,\mathrm{d}\, f_{n*}\tilde{\mathfrak{m}}_1$. However, $h \circ f_n$ is point-wise converging to $h \circ f$. Also, whenever n is large enough, we have $\mathrm{Spt}(h \circ f_n) \subset \overline{B}_1(2R)$. Hence, applying the dominated convergence theorem, we obtain:

$$\int_{\tilde{X}} h \,\mathrm{d}\,\tilde{\mathfrak{m}}_2 = \lim_{n\to\infty} \int_{\tilde{X}} h \,\mathrm{d}\, f_{n*}\tilde{\mathfrak{m}}_1 = \int_{\tilde{X}} h \,\mathrm{d}\, f_*\tilde{\mathfrak{m}}_1.$$

Therefore, since $\tilde{\mathfrak{m}}_2$ and $f_*\tilde{\mathfrak{m}}_1$ are Radon measures, we necessarily have $\tilde{\mathfrak{m}}_2 = f_*\tilde{\mathfrak{m}}_1$. Finally, given $\gamma \in \overline{\pi}_1(X)$ and $x \in \tilde{X}$, we have:

$$p(f(x)) = \lim_{n\to\infty} p(f_n(x)) = \lim_{n\to\infty} p(f_n(\gamma x)) = p(f(\gamma x)).$$

Thus, $p(f\gamma f^{-1}) = p$. We define $\phi \in \mathrm{Iso}(\overline{\pi}_1(X))$ by $\phi(\gamma) := f\gamma f^{-1}$, which satisfies $f(\gamma x) = \phi(\gamma) f(x)$.

Hence, we can conclude that $\tilde{\mathcal{X}}_1$ is equivariantly isomorphic to $\tilde{\mathcal{X}}_2$.

Part III: Metrizable uniform structure

We have seen that $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}$ induces a Hausdorff uniform structure on $\mathfrak{M}_{K,N}^{\mathrm{p,eq}}(\tilde{X})$. Moreover, $\mathcal{B}$ is a countable system of fundamental neighbourhoods for this uniform structure. Therefore, according to Proposition 2 p.126 in [Bou07b], there exists a distance $\mathrm{d}\colon \mathfrak{M}_{K,N}^{\mathrm{p,eq}}(\tilde{X}) \times \mathfrak{M}_{K,N}^{\mathrm{p,eq}}(\tilde{X}) \to [0, +\infty]$ such that d induces the same uniform structure as $\mathfrak{D}_{\mathrm{p}}^{\mathrm{eq}}$. Observe that we can assume, without loss of generality, that d is finite (replacing d by $\min\{1, \mathrm{d}\}$ if necessary), which concludes the proof. □

A.4 Curvature potentials: technical proofs

This appendix is devoted to the proof of Lemma 5.1.5 and Proposition 5.1.4. First of all, we prove Lemma 5.1.5.

Proof of Lemma 5.1.5. We only need to prove that, for all $m \in \mathbb{N}$, every subsequence of $\{f_i\}_{i\in\mathbb{N}}$ sub-converges to f_∞ in the $\mathcal{C}^m$ topology. Let $m \in \mathbb{N}$ ($m \geq 2$) and, to avoid cumbersome notations, let us only prove that $\{f_i\}_{i\in\mathbb{N}}$ sub-converges to f_∞ in the $\mathcal{C}^m$ topology. First, let us fix $\alpha \in (0,1)$ and cover X by finitely many charts $\{(U_k, \psi_k)\}_{1\leq k\leq N}$ such that $\overline{B}_4(0) \subset V_k := \psi_k(U_k)$, and $\{W_k := \psi_k^{-1}(B_1(0))\}_{1\leq k\leq N}$ covers X.

Part I: $\{f_i \circ \psi_k^{-1}\}_{i\in\mathbb{N}}$ is bounded in $\mathcal{C}^{m,\alpha}(\overline{B}_1(0))$, for all $1 \leq k \leq N$

Let $1 \leq k \leq N$ and let us show that $\{f_i \circ \psi_k^{-1}\}_{i\in\mathbb{N}}$ is bounded in $\mathcal{C}^{m,\alpha}(\overline{B}_1(0))$. To avoid cumbersome notations, we will write f_i instead of $f_i \circ \psi_k^{-1}$. In the chart (U_k, ψ_k), for all functions $u \in \mathcal{C}^2(V_k)$ (we recall $V_k = \psi_k(U_k)$), we have:

$$\begin{cases} \Delta_{h_i} u = h_i^{r,s}(\partial_r\partial_s - \Gamma^t_{i;r,s}\partial_t)u \\ \int_{V_k} u \,\mathrm{dvol}_{h_i} = \int_{V_k} u(x) J_i(x) dx \end{cases},$$

where $J_i = \sqrt{\det[h_{i;r,s}]_{1\leq r,s\leq n}}$. Since $\{h_i\}_{i\in\mathbb{N}}$ converges smoothly, there exists $\lambda > 0$ such that, for all $i \in \mathbb{N}$, we have:

- $\lambda|\xi|^2 \leq h_i^{r,s}(x)\xi_r\xi_s \leq \lambda^{-1}|\xi|^2$, for all $x \in \overline{B}_4(0)$ and $\xi \in \mathbb{R}^n$,
- $|h_i^{r,s}|_{m,\alpha;\overline{B}_4(0)}, |\Gamma^t_{i;r,s}|_{m,\alpha;\overline{B}_4(0)} \leq \lambda^{-1}$, for all $1 \leq r,s,t \leq n$.

Therefore, since $\mathrm{d}(B_1(0), \partial B_2(0)) = 1 > 0$ and $\mathrm{Diam}(\overline{B}_2(0)) = 4 < \infty$, we can use Schauder's interior estimates (see Theorem 6.2 and exercise 6.1 of [GT01]) to get $C_1 = C_1(n,m,\alpha,\lambda) \in \mathbb{R}_{>0}$ such that, for all $i \in \mathbb{N}$, we have:

$$|f_i|_{m,\alpha;B_1(0)} \leq C_1(|f_i|_{0;B_2(0)} + |\phi_i|_{m-2,\alpha;B_2(0)}).$$

Note that $\{|\phi_i|_{m-2,\alpha;B_2(0)}\}_{i\in\mathbb{N}}$ is bounded, since $\{\phi_i\}_{i\in\mathbb{N}}$ converges smoothly. Hence, we only have to show that $\{|f_i|_{0;B_2(0)}\}$ is bounded. However, due to Corollary 9.21 of [GT01], there exists $C_2 = C_2(n,\lambda) > 0$ such that, for all $i \in \mathbb{N}$, we have:

$$|f_i|_{0;B_2(0)} \leq C_2\Big(|B_4(0)|^{-1/2}|f_i|_{\mathcal{L}^2(B_4(0))} + 2\lambda^{-1}|\phi_i|_{\mathcal{L}^n(B_4(0))}\Big).$$

Moreover, $\{|\phi_i|_{\mathcal{L}^n(B_4(0))}\}_{i\in\mathbb{N}}$ is bounded, since $\{\phi_i\}_{i\in\mathbb{N}}$ converges smoothly. Thus, we only need to bound $\{|f_i|_{\mathcal{L}^2(B_4(0))}\}_{i\in\mathbb{N}}$. However, observe that $\{h_i\}_{i\in\mathbb{N}}$ converges smoothy; therefore, there exists $C_3, \kappa, D > 0$ such that, for all $i \in \mathbb{N}$, we have:

$$\begin{cases} |f_i|^2_{\mathcal{L}^2(B_4(0))} \leq C_3 \int_{U_k} f_i^2 \,\mathrm{dvol}_{h_i} = C_3 |f_i|^2_{\mathcal{L}^2(h_i)} \\ \mathrm{Ric}(h_i) \geq -(n-1)\kappa h_i \\ \mathrm{Diam}(X, h_i) \leq D \end{cases}.$$

In particular, we just need to prove that $\{|f_i|^2_{\mathcal{L}^2(h_i)}\}_{i\in\mathbb{N}}$ is bounded. Moreover, thanks to the local Poincaré inequality (see for example Theorem 1.14 of [BCH18]), there exists $C_4 = C_4(n, \kappa, D) > 0$ such that, for all $i \in \mathbb{N}$, we have:

$$\int_X |f_i - a_i|^2 \,\mathrm{dvol}_{h_i} \leq C_4 \int_X |\nabla f_i|^2 \,\mathrm{dvol}_{h_i}.$$

where we recall $a_i = ⨍_X f_i \,\mathrm{dvol}(h_i)$. In addition, observe that:

$$\begin{aligned} \int_X |f_i - a_i|^2 \,\mathrm{dvol}_{h_i} &= |f_i|^2_{\mathcal{L}^2(h_i)} + |a_i|^2_{\mathcal{L}^2(h_i)} - 2\int_X f_i a_i \,\mathrm{dvol}_{h_i}, \\ &\geq |f_i|^2_{\mathcal{L}^2(h_i)} + |a_i|^2_{\mathcal{L}^2(h_i)} - \left(\frac{1}{2}|f_i|^2_{\mathcal{L}^2(h_i)} + 2|a_i|^2_{\mathcal{L}^2(h_i)}\right), \\ &\geq \frac{1}{2}|f_i|^2_{\mathcal{L}^2(h_i)} - |a_i|^2_{\mathcal{L}^2(h_i)}. \end{aligned}$$

Thus, we have:

$$\frac{1}{2}|f_i|^2_{\mathcal{L}^2(h_i)} \leq C_4 \int_X |\nabla f_i|^2 \,\mathrm{dvol}_{h_i} + |a_i|^2_{\mathcal{L}^2(h_i)}.$$

But, for every $\epsilon > 0$, we have:

$$\begin{aligned} \int_X |\nabla f_i|^2 \,\mathrm{dvol}_{h_i} &\leq \int_X |\Delta_{h_i} f_i||f_i| \,\mathrm{dvol}_{h_i}, \\ &\leq \epsilon^{-1}|\Delta_{h_i} f_i|^2_{\mathcal{L}^2(h_i)} + \epsilon |f_i|^2_{\mathcal{L}^2(h_i)}, \\ &\leq \epsilon^{-1}|\phi_i|^2_{\mathcal{L}^2(h_i)} + \epsilon |f_i|^2_{\mathcal{L}^2(h_i)}. \end{aligned}$$

Therefore, fixing $\epsilon = \frac{1}{4C_4}$, we obtain:

$$|f_i|^2_{\mathcal{L}^2(h_i)} \leq 8C_4^2 |\phi_i|^2_{\mathcal{L}^2(h_i)} + 4|a_i|^2_{\mathcal{L}^2(h_i)}.$$

However, under the assumption of the lemma, both $\{|a_i|^2_{\mathcal{L}^2(h_i)}\}_{i\in\mathbb{N}}$ and $\{|\phi_i|^2_{\mathcal{L}^2(h_i)}\}_{i\in\mathbb{N}}$ converge. Therefore, $\{|f_i|^2_{\mathcal{L}^2(h_i)}\}_{i\in\mathbb{N}}$ is bounded, which concludes the first part of the proof.

Part II: Conclusion

The inclusion $\mathcal{C}^{m,\alpha}(\overline{B}_1(0)) \subset \mathcal{C}^m(\overline{B}_1(0))$ is compact since $B_1(0)$ is a smooth domain of $\mathbb{R}^n$ (see Lemma 6.36 of [GT01]). Therefore, passing to a subsequence if necessary, we can assume that, for every $1 \leq k \leq N$, $\{f_{i|W_k}\}_{i\in\mathbb{N}}$ converges to some $f^{(k)} \in \mathcal{C}^m(\overline{W}_k)$ in the $\mathcal{C}^m(\overline{W}_k)$ topology (we recall $W_k = \psi_k^{-1}(B_1(0))$). In addition, for any $1 \leq k, l \leq N$, we necessarily have $f^{(k)}_{|W_k\cap W_l} = f^{(l)}_{|W_k\cap W_l}$. In particular, since $\{W_k\}_{1\leq k\leq N}$ covers X, there exists a uniquely defined function $f \in \mathcal{C}^m(X)$ such that, for every $1 \leq k \leq N$, we have $f_{|W_k} = f^{(k)}$. In particular, $\{f_i\}_{i\in\mathbb{N}}$ necessarily converges to f in the $\mathcal{C}^m$ topology (where we assumed $m \geq 2$). Now, since $h_i \xrightarrow{\mathcal{C}^\infty} h_\infty$, $\phi_i \xrightarrow{\mathcal{C}^\infty} \phi_\infty$, $f_i \xrightarrow{\mathcal{C}^2} f$, and $a_i \to a_\infty$, then f is necessarily the solution to the following Poisson problem:

$$\begin{cases} \Delta_{h_\infty} f & = \phi_\infty \\ \fint_X f \,\mathrm{dvol}_{h_\infty} & = a_\infty \end{cases}.$$

Therefore, by the uniqueness of the solution, $f = f_\infty$. □

The following remark will be useful in the proof of the next lemma.

Remark A.4.1. Assume that $t \in \mathbb{R}_{\geq 0} \to f(\cdot, t) \in \mathcal{C}^\infty(\mathbb{R})$ belongs to $\mathcal{C}^k(\mathbb{R}_{\geq 0}, \mathcal{C}^\infty(\mathbb{R}))$. Then, thanks to Schwartz's symmetry Theorem (see for instance Theorem 4.3 of [HW96]), we have the following property:

$$\forall l \leq k, \forall m \in \mathbb{N}, \partial_t^l \partial_x^m f = \partial_x^m \partial_t^l f.$$

In particular, for every $l \leq k$ and $m \in \mathbb{N}$, the map $t \in \mathbb{R}_{\geq 0} \to \partial_t^l \partial_x^m f(\cdot, t) \in \mathcal{C}^\infty(\mathbb{R})$ belongs to $\mathcal{C}^{k-l}(\mathbb{R}_{\geq 0}, \mathcal{C}^\infty(\mathbb{R}))$.

The first step towards proving Proposition 5.1.4 is the following lemma.

Lemma A.4.1. Let X^n be a connected closed orientable manifold and let $\{h(t)\}_{t\geq 0}$ be a smooth family of Riemannian metrics on X. In addition, let $l \geq 0$ and let us fix functions $\phi \in \mathcal{C}^l(\mathbb{R}_{\geq 0}, \mathcal{C}^\infty(X))$ and $\gamma \in \mathcal{C}^l(\mathbb{R}_{\geq 0})$ such that, for every $t \geq 0$, we have $\int_X \phi(\cdot, t) \,\mathrm{dvol}_{h(t)} = 0$. Finally, let us denote $f(\cdot, t)$ the unique solution to the Poisson problem:

$$\begin{cases} \Delta_{h(t)} f(\cdot, t) & = \phi(\cdot, t) \\ \int_X f(\cdot, t) \,\mathrm{dvol}_{h(t)} & = \gamma(t) \end{cases}.$$

Then, we necessarily have $f \in \mathcal{C}^l(\mathbb{R}_{\geq 0}, \mathcal{C}^\infty(X))$.

Proof. Given $k \in \mathbb{N}$, let us introduce our induction hypothesis:

$$\mathbf{H}_k := \text{The lemma holds for every } 0 \leq l \leq k.$$

We will prove that $\mathbf{H}_k$ holds for every $k \in \mathbb{N}$.

Part I: k=0

Let f satisfy the assumptions of the lemma for $l = 0$, i.e.:

$$\begin{cases} \Delta_{h(t)} f(\cdot, t) & = \phi(\cdot, t) \\ \int_X f(\cdot, t)\,\mathrm{dvol}_{h(t)} & = \gamma(t) \end{cases},$$

where $\phi \in \mathcal{C}^0(\mathbb{R}_{\geq 0}, \mathcal{C}^\infty(X))$ and $\gamma \in \mathcal{C}^0(\mathbb{R}_{\geq 0})$.

Let $t_n \to t_\infty$ in $\mathbb{R}_{\geq 0}$, we need to prove that $\{f(\cdot, t_n)\}_{n\in\mathbb{N}}$ converges smoothly to $f(\cdot, t_\infty)$.

First, given $n \in \mathbb{N} \cup \{\infty\}$, let us denote $h_n := h(t_n)$, $\phi_n := \phi(\cdot, t_n)$, $a_n := \mathrm{vol}(X, h_n)^{-1}\gamma(t_n)$, and $f_n := f(\cdot, t_n)$. Observe that, by assumption, $\{h_n\}_{n\in\mathbb{N}}$ and $\{\phi_n\}_{n\in\mathbb{N}}$ converge smoothly respectively to h_∞ and ϕ_∞, and $a_n \to a_\infty$. Therefore, as a result of Lemma 5.1.5, $\{f_i\}_{i\in\mathbb{N}}$ converges smoothly to f_∞, which concludes the first part of the proof.

Part II: Induction

We assume that $\mathbf{H}_k$ is true for some $k \in \mathbb{N}$. Assume that f satisfies the assumption of the lemma for $l = k + 1$, i.e.:

$$\begin{cases} \Delta_{h(t)} f(\cdot, t) & = \phi(\cdot, t) \\ \int_X f(\cdot, t)\,\mathrm{dvol}_{h(t)} & = \gamma(t) \end{cases},$$

where $\phi \in \mathcal{C}^{k+1}(\mathbb{R}_{\geq 0}, \mathcal{C}^\infty(X))$ and $\gamma \in \mathcal{C}^{k+1}(\mathbb{R}_{\geq 0})$. In particular, thanks to $\mathbf{H}_k$, $f \in \mathcal{C}^k(\mathbb{R}_{\geq 0}, \mathcal{C}^\infty(X))$. We need to show that $f \in \mathcal{C}^{k+1}(\mathbb{R}_{\geq 0}, \mathcal{C}^\infty(X))$.

Given $\psi \in \mathcal{C}^\infty(X)$, and $t, t_0 \geq 0$, we denote

$$\mathcal{C}^\infty(X) \ni L_t^{t_0}\psi := \begin{cases} \frac{\Delta_{h(t)}\psi - \Delta_{h(t_0)}\psi}{t - t_0} & \text{if } t \neq t_0 \\ \frac{\partial \Delta_{h(t)}\psi}{\partial_t}_{|t=t_0} & \text{otherwise} \end{cases},$$

and:

$$\mathcal{C}^\infty(X) \ni \phi_1^{t_0}(\cdot, t) := \begin{cases} \frac{\phi(\cdot,t) - \phi(\cdot,t_0)}{t - t_0} & \text{if } t \neq t_0 \\ \frac{\partial\phi}{\partial_t}(\cdot, t_0) & \text{otherwise} \end{cases}.$$

Observe that, $(x, t, t_0) \in X \times \mathbb{R}^2_{\geq 0} \to L_t^{t_0}\psi(x)$ is smooth. Moreover, if $t_0 \geq 0$ is fixed, then $\phi_1^{t_0} \in \mathcal{C}^k(\mathbb{R}_{\geq 0}, \mathcal{C}^\infty(X))$. In addition, for every $t, t_0 \geq 0$, we have $\Delta_{h(t)}\psi = \Delta_{h(t_0)}\psi + (t - t_0)L_t^{t_0}\psi$, and $\phi(\cdot, t) = \phi(\cdot, t_0) + (t - t_0)\phi_1^{t_0}(\cdot, t)$.

Now, we fix $t_0 \geq 0$, and observe that, for every $t \geq 0$, we have:

$$\int_X \phi_1^{t_0}(\cdot, t) - L_t^{t_0} f(\cdot, t_0)\,\mathrm{dvol}_{h(t)} = 0.$$

Therefore, for $t \geq 0$, we can introduce $f_1^{t_0}(\cdot, t)$ the unique solution to the Poisson problem:

$$\begin{cases} \Delta_{h(t)} f_1^{t_0}(\cdot, t) = \phi_1^{t_0}(\cdot, t) - L_t^{t_0} f(\cdot, t_0) \\ \int_X f_1^{t_0}(\cdot, t) \,\mathrm{dvol}_{h(t)} = \gamma_1^{t_0}(t) \end{cases}, \tag{A.8}$$

where we define $\gamma_1^{t_0}(t)$ in the following way:

$$\gamma_1^{t_0}(t) := \begin{cases} \int_X \frac{f(\cdot,t)-f(\cdot,t_0)}{t-t_0} \,\mathrm{dvol}_{h(t)} & \text{if } t \neq t_0 \\ \lim_{t \to t_0} \int_X \frac{f(\cdot,t)-f(\cdot,t_0)}{t-t_0} \,\mathrm{dvol}_{h(t)} & \text{otherwise} \end{cases}.$$

Observe that $\gamma_1^{t_0}$ is $\mathcal{C}^k$ since, for $t \neq t_0$, we have:

$$\gamma_1^{t_0}(t) = \frac{\gamma(t) - \gamma(t_0)}{t - t_0} + \int_X f(\cdot, t_0) \frac{\mathrm{dvol}_{h(t)} - \mathrm{dvol}_{h(t_0)}}{t - t_0},$$

and:

$$\lim_{t \to t_0} \gamma_1^{t_0}(t) = \dot{\gamma}(t_0) + \int_X f(\cdot, t_0) \frac{\partial \,\mathrm{dvol}_{h(t)}}{\partial_t}_{|t=t_0}.$$

Observe that, for every $t \geq 0$, our construction implies:

$$\begin{cases} \Delta_{h(t)}[f(\cdot, t_0) + (t - t_0) f_1^{t_0}(\cdot, t)] = \phi(\cdot, t) \\ \int_X [f(\cdot, t_0) + (t - t_0) f_1^{t_0}(\cdot, t)] \,\mathrm{dvol}_{h(t)} = \gamma(t) \end{cases}.$$

Hence, by the uniqueness of the solution to such a Poisson problem, for all $t \geq 0$, we have:

$$f(\cdot, t) = f(\cdot, t_0) + (t - t_0) f_1^{t_0}(\cdot, t).$$

Now, observe that $t \to \phi_1^{t_0}(\cdot, t) - L_t^{t_0} f(\cdot, t_0)$ is in $\mathcal{C}^k(\mathbb{R}_{\geq 0}, \mathcal{C}^\infty(X))$ and $\gamma_1^{t_0} \in \mathcal{C}^k(\mathbb{R}_{\geq 0})$. Therefore, since $\mathbf{H}_0$ holds, and since $f_1^{t_0}$ satisfies A.8, then $f_1^{t_0} \in \mathcal{C}^0(\mathbb{R}_{\geq 0}, \mathcal{C}^\infty(X))$. In particular, f is differentiable with respect to time.

More specifically, for every $t_0 \geq 0$, we have:

$$\frac{\partial f}{\partial t}(\cdot, t_0) = f_1^{t_0}(\cdot, t_0) \in \mathcal{C}^\infty(X).$$

Even better, A.8 implies the following:

$$\begin{cases} \Delta_{h(t_0)} \frac{\partial f}{\partial t}(\cdot, t_0) = \phi_1^{t_0}(\cdot, t_0) - L_{t_0}^{t_0} f(\cdot, t_0) = \frac{\partial \phi}{\partial t}(\cdot, t_0) + \frac{\partial \Delta_{h(t)} f(\cdot, t_0)}{\partial_t}_{|t=t_0} \\ \int_X \frac{\partial f}{\partial t}(\cdot, t_0) \,\mathrm{dvol}_{h(t_0)} = \gamma_1^{t_0}(t_0) = \dot{\gamma}(t_0) + \int_X f(\cdot, t_0) \frac{\partial \,\mathrm{dvol}_{h(t)}}{\partial_t}_{|t=t_0} \end{cases}.$$

However, observe that:

$$\left[t_0 \to \frac{\partial \phi}{\partial t}(\cdot, t_0) + \frac{\partial \Delta_{h(t)} f(\cdot, t_0)}{\partial_t}_{|t=t_0} \right] \in \mathcal{C}^k(\mathbb{R}_{\geq 0}, \mathcal{C}^\infty(X)),$$

$$\left[t_0 \to \dot{\gamma}(t_0) + \int_X f(\cdot, t_0) \frac{\partial \,\mathrm{dvol}_{h(t)}}{\partial_t}_{|t=t_0} \right] \in \mathcal{C}^k(\mathbb{R}),$$

where we used Remark A.4.1 for the first inclusion. In particular, $\mathbf{H}_k$ implies $\partial_t f \in \mathcal{C}^k(\mathbb{R}_{\geq 0}, \mathcal{C}^\infty(X))$; which concludes the induction. $\square$

Lemma A.4.1 implies the following corollary.

Corollary A.4.1. Let X^n be a connected closed orientable manifold and let $\{h(t)\}_{t\geq 0}$ be a smooth family of Riemannian metrics on X. In addition, let us fix smooth functions $\phi \in \mathcal{C}^\infty(X \times \mathbb{R}_{\geq 0})$ and $\gamma \in \mathcal{C}^\infty(\mathbb{R}_{\geq 0})$ such that, for every $t \geq 0$, we have $\int_X \phi(\cdot, t)\,\mathrm{dvol}_{h(t)} = 0$. Finally, let us denote $f(\cdot, t)$ the unique solution to the Poisson problem:

$$\begin{cases} \Delta_{h(t)} f(\cdot, t) & = \phi(\cdot, t) \\ \int_X f(\cdot, t)\,\mathrm{dvol}_{h(t)} & = \gamma(t) \end{cases}.$$

Then, f is necessarily smooth on $X \times \mathbb{R}_{\geq 0}$.

Proof. Observe first that Lemma A.4.1 implies $f \in \mathcal{C}^\infty(\mathbb{R}_{\geq 0}, \mathcal{C}^\infty(X))$. Let $\psi\colon U \subset X^n \to V \subset \mathbb{R}^n$ be a chart and, to avoid cumbersome notations, let us write $f(x,t)$ instead of $f \circ \psi^{-1}(x,t)$. Note that, according to Schwartz's symmetry Theorem (see for instance Theorem 4.3 of [HW96]), $f \in \mathcal{C}^\infty(V \times \mathbb{R}_{\geq 0})$ if and only if, for every $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}$, $\partial_x^\alpha \partial_t^\beta f \in \mathcal{C}^0(V \times \mathbb{R}_{\geq 0})$. However, observe that given $(x_n, t_n) \to (x_\infty, t_\infty)$ in $V \times \mathbb{R}_{\geq 0}$, we have:

$$\begin{aligned} |\partial_x^\alpha \partial_t^\beta f(x_n, t_n) - \partial_x^\alpha \partial_t^\beta f(x_\infty, t_\infty)| \leq & |\partial_x^\alpha \partial_t^\beta f(x_n, t_n) - \partial_x^\alpha \partial_t^\beta f(x_n, t_\infty)| \\ & + |\partial_x^\alpha \partial_t^\beta f(x_n, t_\infty) - \partial_x^\alpha \partial_t^\beta f(x_\infty, t_\infty)| \end{aligned}.$$

Note that, because of Lemma A.4.1, we have $\partial_t^\beta f(\cdot, t_\infty) \in \mathcal{C}^\infty(X)$. In particular:

$$\lim_{n\to\infty} |\partial_x^\alpha \partial_t^\beta f(x_n, t_\infty) - \partial_x^\alpha \partial_t^\beta f(x_\infty, t_\infty)| = 0.$$

Finally, thanks to Lemma A.4.1, $\{\partial_t^\beta f(\cdot, t_n)\}_{n\in\mathbb{N}}$ converges smoothly to $\partial_t^\beta f(\cdot, t_\infty)$, which implies:

$$\lim_{n\to\infty} \sup_{z\in V} \{|\partial_x^\alpha \partial_t^\beta f(z, t_n) - \partial_x^\alpha \partial_t^\beta f(z, t_\infty)|\} = 0.$$

Hence, $|\partial_x^\alpha \partial_t^\beta f(x_n, t_n) - \partial_x^\alpha \partial_t^\beta f(x_n, t_\infty)| \to 0$, which concludes the proof. $\square$

We can now prove Proposition 5.1.4.

Proof of Proposition 5.1.4. First of all, for $t \geq 0$, let $f_0(\cdot, t)$ be the unique solution to the Poisson problem:

$$\begin{cases} \Delta_{h(t)} f_0(\cdot, t) = R(t) - r \\ ⨍_X f_0(\cdot, t)\,\mathrm{dvol}_{h(t)} = 0 \end{cases}. \tag{A.9}$$

As a result of Corollary A.4.1, f_0 is smooth on $X \times \mathbb{R}_{\geq 0}$. Moreover, observe that thanks to Corollary 5.5 of [CK04], we have:

$$\partial_t \Delta_{h(t)} f_0 = \Delta_{h(t)} \partial_t f_0 + (R - r) \Delta_{h(t)} f_0. \tag{A.10}$$

However, due to Corollary 5.8 of [CK04], we have $\partial_t R = \Delta_{h(t)} R + R(R - r)$; hence, using A.9, we have $\partial_t \Delta_{h(t)} f_0 = \Delta_{h(t)} R + R(R - r)$. Reorganising A.10, we have:

$$\Delta_{h(t)} \partial_t f_0 = \Delta_{h(t)} R + R(R - r) - (R - r)^2 = \Delta_{h(t)} R + r(R - r) = \Delta_{h(t)} [R + r f_0].$$

In particular, since X is closed, then $\gamma \coloneqq \partial_t f_0 - R - r f_0$ is independent of x, i.e. $\gamma \in \mathcal{C}^\infty(\mathbb{R}_{\geq 0})$. It is then easy to prove that $f(\cdot, t) \coloneqq f_0(\cdot, t) + c(t)$ is a curvature potential, where $c(t) \coloneqq -\mathrm{e}^{rt} \int_0^t \mathrm{e}^{-r\tau} \gamma(\tau) d\tau$. □

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