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Calculating the virtual cohomological dimension of the automorphism group of a RAAG

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Abstract

We describe an algorithm to find the virtual cohomological dimension of the automorphism group of a right-angled Artin group. The algorithm works in the relative setting; in particular, it also applies to untwisted automorphism groups and basis-conjugating automorphism groups. The main new tool is the construction of free abelian subgroups of certain Fouxe–Rabinovitch groups of rank equal to their virtual cohomological dimension, generalizing a result of Meucci in the setting of free groups.

1. *Introduction*

Automorphism groups of right-angled Artin groups (or RAAGs) form a diverse and interesting family, encompassing the rich worlds of both integer matrix groups and automorphism groups of free groups. For any right-angled Artin group A_Γ, (determined by a finite graph Γ) Laurence [**20**] gave a generating set for $\mathrm{Aut}(A_\Gamma)$, and since this result authors have worked to understand higher finiteness properties of these groups. In particular, Charney and Vogtmann [**9**] showed that each outer automorphism group $\mathrm{Out}(A_\Gamma)$ has finite *virtual cohomological dimension* (*vcd*). Given recent constructions of classifying spaces for *untwisted subgroups* [**8**] and the analogs of *congruence kernels* for these groups [**13**], it is natural to ask what $\mathrm{vcd}(\mathrm{Out}(A_\Gamma))$ actually is. Indeed, upper and lower bounds for specific examples and interesting subfamilies have been obtained in many cases [**7, 8, 13, 23**], giving the vcd when these bounds coincide.

In this paper, we give an algorithm to compute the virtual cohomological dimension of $\mathrm{Out}(A_\Gamma)$ for an arbitrary graph Γ. More generally, this algorithm gives the virtual cohomological dimension of any outer automorphism group of a right-angled Artin group relative to a collection of special subgroups. This includes the untwisted automorphism groups of [**8**] and partially symmetric (or basis-conjugating) outer automorphism groups of RAAGs.

The relative (outer) automorphism groups mentioned above were studied extensively in [**13**], and are affectionately known as RORGs. Such a group is defined by taking collections $\mathcal{G}$, $\mathcal{H}$ of *special subgroups* (a special subgroup is one of the form A_Δ given by an induced subgraph $\Delta \subset \Gamma$) of a right-angled Artin group A_Γ and looking at the subgroup $\mathrm{Out}(A_\Gamma; \mathcal{G}, \mathcal{H}^t)$ of automorphisms that *preserve* each element of $\mathcal{G}$ and *act trivially* on each element of $\mathcal{H}$ (see Section 2.2). This approach is not an idle exercise in generalization if one wants to understand $\mathrm{Out}(A_\Gamma)$. The main result of [**13**] uses RORGs to construct a subnormal series for $\mathrm{Out}(A_\Gamma)$ (more generally, for an arbitrary RORG) such that the consecutive quotients of this series are either finite, free-abelian groups, copies of $\mathrm{GL}(n,\mathbb{Z})$, or groups known as *Fouxe–Rabinovitch groups*. We call such a normal series a *decomposition series*. In [**13**], decomposition series were used to iteratively construct finite classifying spaces for congruence subgroups of $\mathrm{Out}(A_\Gamma)$. In

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a similar fashion, we will see that the virtual cohomological dimension of $\mathrm{Out}(A_\Gamma)$ is the sum of the vcds of the consecutive quotients appearing in a decomposition series.

To make this process algorithmic, one needs to know how to find the vcd of a Fouxe–Rabinovitch group. Let us first recall the definition of these groups. Let

$$G = G_1 * G_2 * \cdots G_k * F_m$$

be a free factor decomposition of a group G. An element $\Phi \in \mathrm{Out}(G)$ belongs to the *Fouxe–Rabinovitch* group associated to this free factor decomposition if for each G_i there exists a representative $\phi \in \Phi$ restricting to the identity on G_i. For example, the basis-conjugating automorphism group of a free group is the Fouxe–Rabinovitch group given by the free factor decomposition $F_n = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$. Going back to RAAGs, if each $G_i = A_{\Delta_i}$ is a special subgroup, then the Fouxe–Rabinovitch group is the relative automorphism group $\mathrm{Out}(A_\Gamma; \{A_{\Delta_i}\}^t)$.

THEOREM A. *Let $A_\Gamma = A_{\Delta_1} * A_{\Delta_2} * \cdots * A_{\Delta_k} * F_m$ be a free factor decomposition of a right-angled Artin group with $k \geqslant 1$. Let $d(\Delta_i)$ be the size of a maximal clique in each Δ_i, and let $z(\Delta_i)$ be the rank of the center of A_{Δ_i}. Then*

$$\mathrm{vcd}(\mathrm{Out}(A_\Gamma; \{A_{\Delta_i}\}^t)) = (k + 2m - 2) \cdot \max_i \{d(\Delta_i)\} + \sum_{i=1}^{k} (d(\Delta_i) - z(\Delta_i)).$$

There exists a free abelian subgroup of $\mathrm{Out}(A_\Gamma; \{A_{\Delta_i}\}^t)$ of rank equal to the virtual cohomological dimension.

This generalizes a theorem of Meucci [**22**] on relative automorphism groups of free groups and Collins [**11**] on partially symmetric automorphism groups. To prove this theorem, we obtain a lower bound for the virtual cohomological dimension by constructing free-abelian subgroups of the appropriate rank. The upper bound is obtained by a careful analysis of simplex stabilizers for the action of the Fouxe–Rabinovitch group on the spine of Guirardel and Levitt's *relative Outer space* [**19**]. Roughly speaking, we have to make sure that simplices of large dimension have small stabilizers.

In the case where $k = 0$, the virtual cohomological dimension of $\mathrm{Out}(F_m)$ was shown to be $2m - 3$ in Culler and Vogtmann's seminal paper on Outer space [**12**]. There, the lower bound is obtained by finding a copy of $\mathbb{Z}^{2m-3}$ in $\mathrm{Out}(F_m)$ generated by Nielsen automorphisms. The abelian subgroups found in the Fouxe–Rabinovitch case are very similar and made up of transvections and partial conjugations (see Remark 3.6). On the other side of the RAAG spectrum, similar results hold for $\mathrm{GL}(n, \mathbb{Z})$. Here the virtual cohomological dimension is equal to the Hirsch length of the polycyclic subgroup of upper triangular matrices. Given all of this, it is natural to conjecture that for an arbitrary RORG there is also a polycyclic subgroup of rank equal to the virtual cohomological dimension. Indeed, this conjecture holds in all known examples, but we cannot prove it in general. Luckily, we do not need explicit polycyclic subgroups to calculate vcd.

THEOREM B. *There is an algorithm which, given the input of a finite graph Γ and two collections of special subgroups $\mathcal{G}$ and $\mathcal{H}$ of A_Γ, computes the virtual cohomological dimension of $\mathrm{Out}(A_\Gamma; \mathcal{G}, \mathcal{H}^t)$.*

If one prefers to look at the absolute automorphism group, the vcds of $\mathrm{Aut}(A_\Gamma)$ and $\mathrm{Out}(A_\Gamma)$ differ by the dimension of $A_\Gamma / Z(A_\Gamma)$ (see Remark 4.4).

The main idea behind the proof of Theorem B is as follows. Although virtual cohomological dimension is only subadditive with respect to exact sequences, *rational cohomological* *dimension* is additive (by a theorem of Bieri [**2**]). It is not generally the case that the rational cohomological dimension of a torsion-free group agrees with its (integral) cohomological dimension (an example where they differ is given in the introduction of [**1**]). Luckily for us, the existence of the polycyclic subgroups above imply that for every consecutive quotient in a decomposition series, rational cohomological dimension coincides with vcd. Therefore the vcd of a RORG is the sum of the vcds of the pieces that appear in its decomposition series, which we can find either from previous work in the literature [**4, 12**], or from Theorem A.

Related problems. The groups $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{Out}(F_n)$ are also *virtual duality groups* in the sense of Bieri–Eckmann [**3**]. It is not known if this is the case for $\mathrm{Out}(A_\Gamma)$. As duality groups behave well under exact sequences [**3**, Theorem 3.5], for a positive answer it would be enough to show that every Fouxe–Rabinovitch group associated to a RAAG is a virtual duality group. This may not be true, or at least appears to be a delicate problem.

As mentioned above, we strongly suspect that $\mathrm{Out}(A_\Gamma)$ contains a polycyclic subgroup whose Hirsch length coincides with the vcd (this is most thoroughly discussed in a paper of Millard and Vogtmann [**23**], which contains several positive results in the untwisted setting). It would also be desirable to have a closed formula for the virtual cohomological dimension in terms of (properties of) Γ, which is something we cannot find with the recursive approach given in this paper.

Structure of the paper. We describe the relevant background material on cohomological dimension and automorphism groups in Section 2. In Section 3 we give a proof of Theorem A and in Section 4 we describe how the decomposition series of a RORG can be found algorithmically and complete the proof of Theorem B.

2. *Background*

2.1. *Cohomological dimension*

For a thorough treatment of cohomological dimension, the reader is referred to the books of Bieri [**2**] and Brown [**5**]. Let R be a unital commutative ring. For a group G, the *cohomological dimension of G over R*, denoted $\mathrm{cd}_R(G)$ is given by

$$\mathrm{cd}_R(G) = \max\{n : H^n(G; M) \neq 0 \text{ for some } RG\text{-module } M\}.$$

The cohomological dimension of a group G is given by $\mathrm{cd}(G) = \mathrm{cd}_{\mathbb{Z}}(G)$. The cohomological dimension satisfies $\mathrm{cd}_R(G) \leqslant \mathrm{cd}(G)$ for any ring R. A group G is of *finite type*, or *of type F*, if G is the fundamental group of an aspherical CW-complex with a finite number of cells. If G is of finite type, then to find $\mathrm{cd}_R(G)$ one only needs to look at the cohomology with coefficients in the group ring RG, and

$$\mathrm{cd}_R(G) = \max\{n : H^n(G; RG) \neq 0\}.$$

If $1 \to N \to G \to Q \to 1$ is an exact sequence of groups, then

$$\mathrm{cd}_R(G) \leqslant \mathrm{cd}_R(N) + \mathrm{cd}_R(Q). \tag{1}$$

However, equality does not hold in general. For instance, Dranishnikov [**14**] constructed a family of hyperbolic groups G_p such that $\mathrm{cd}(G_p) = 3$ for all p, but $\mathrm{cd}(G_p \times G_q) = 5$ whenever $p \neq q$. Roughly speaking, the failure of equality in (1) comes from torsion in the top cohomology group (this is explored in detail in [**15**]). Over a field these difficulties disappear, so that one has the following.

THEOREM 2.1 [2, Theorem 5.5]. *If $1 \to N \to G \to Q \to 1$ is an exact sequence of groups of finite type, then*

$$\mathrm{cd}_{\mathbb{Q}}(G) = \mathrm{cd}_{\mathbb{Q}}(N) + \mathrm{cd}_{\mathbb{Q}}(Q).$$

Throughout this paper, we will be working with groups satisfying $\mathrm{cd}(G) = \mathrm{cd}_{\mathbb{Q}}(G)$, and will be able to make use of the following proposition.

PROPOSITION 2.2. *Let*

$$1 \to N \to G \to Q \to 1$$

be an exact sequence of groups. Suppose that N and Q are groups of finite type, with

$$\mathrm{cd}_{\mathbb{Q}}(N) = \mathrm{cd}(N) \text{ and } \mathrm{cd}_{\mathbb{Q}}(Q) = \mathrm{cd}(Q),$$

then

$$\mathrm{cd}_{\mathbb{Q}}(G) = \mathrm{cd}(G) = \mathrm{cd}(N) + \mathrm{cd}(Q).$$

Proof. By applying Theorem 2.1 and equation (1), we have

$$\begin{aligned} \mathrm{cd}(G) &\geqslant \mathrm{cd}_{\mathbb{Q}}(G) \\ &= \mathrm{cd}_{\mathbb{Q}}(N) + \mathrm{cd}_{\mathbb{Q}}(Q) \\ &= \mathrm{cd}(N) + \mathrm{cd}(Q) \\ &\geqslant \mathrm{cd}(G), \end{aligned}$$

so there is equality throughout. □

Any group with torsion has infinite cohomological dimension. However, if G has a finite-index subgroup H with finite cohomological dimension, then a theorem of Serre ([24], or alternatively [5, VIII.3]) asserts that for any other torsion-free finite-index subgroup H' one has $\mathrm{cd}(H) = \mathrm{cd}(H')$. It follows that if G contains a torsion-free subgroup of finite index, then the *virtual cohomological dimension* of G can be defined by

$$\mathrm{vcd}(G) = \{\mathrm{cd}(H) : H \text{ is torsion free and } [G:H] < \infty\}.$$

If P is a torsion-free polycyclic group, then $\mathrm{cd}(P) = \mathrm{cd}_{\mathbb{Q}}(P) = h(P)$, where $h(P)$ is the *Hirsch length* of P — the number of infinite cyclic factors in a normal series for P (this follows from Proposition 2.2). If P is a subgroup of a group G, then $\mathrm{cd}_R(P) \leqslant \mathrm{cd}_R(G)$, so polycyclic groups can be used to find lower bounds for (rational) cohomological dimension. In particular, one has:

PROPOSITION 2.3. *Suppose that G acts properly and cocompactly on a contractible complex of dimension d and contains a polycyclic subgroup P with Hirsch length $h(P) = d$. Then for any finite-index, torsion-free subgroup H of G, one has*

$$cd_{\mathbb{Q}}(H) = \mathrm{cd}(H) = d.$$

In particular, if G has a finite-index torsion-free subgroup, then $\mathrm{vcd}(G) = d$. □

Culler and Vogtmann [12] use the spine of Outer space and the existence of a free abelian subgroup of rank $2n-3$ to show that for any torsion-free, finite-index subgroup H of $\mathrm{Out}(F_n)$, one has

$$\mathrm{cd}_{\mathbb{Q}}(H) = \mathrm{cd}(H) = 2n - 3.$$

Similarly, by combining Borel and Serre's calculation of the vcd [4] with the upper-triangular matrices in $\mathrm{GL}(n,\mathbb{Z})$, we see that

$$\mathrm{cd}_{\mathbb{Q}}(H) = \mathrm{cd}(H) = \frac{n(n-1)}{2},$$

for any torsion-free, finite index subgroup H of $\mathrm{GL}(n,\mathbb{Z})$.

2.2. *RAAGs and RORGs*

2.2.1. *Automorphism groups of RAAGs.* Let A_Γ be the right-angled Artin group determined by a finite graph Γ. Let us first fix names and notation for some common automorphisms:

- *Graph symmetries.* Any automorphism of the graph induces an automorphism of the group via the corresponding permutation of the generating set. These elements of $\mathrm{Aut}(A_\Gamma)$ are called *graph symmetries.*
- *Inversions.* If v is a vertex of Γ, then there is an *inversion* ι_v that sends v to v^{-1} and fixes all other generators of A_Γ.
- *Transvections.* Suppose v and w are distinct vertices of Γ with $\mathrm{lk}(v) \subset \mathrm{st}(w)$. There is a *right transvection* ρ_v^w which takes v to vw and fixes all other generators of A_Γ. There is also a *left transvection* λ_v^w taking v to wv and fixing all other generators.
- *Extended partial conjugations.* Let v be a vertex of Γ and let K be a union of connected components of $\Gamma - \mathrm{st}(v)$. There is an *extended partial conjugation* π_K^v which sends w to vwv^{-1} if w is a vertex of K, and fixes each generator which is not a vertex of K.

By a theorem of Laurence [20], the above automorphisms generate the whole automorphism group $\mathrm{Aut}(A_\Gamma)$. Given $\phi \in \mathrm{Aut}(A_\Gamma)$, we use $[\phi]$ or Φ to denote the outer automorphism represented by ϕ. We will use the names of the automorphisms above to also describe their images in $\mathrm{Out}(A_\Gamma)$. This should be clear based on context; we will mostly be working in $\mathrm{Out}(A_\Gamma)$ below. Furthermore, we will often pass to the finite index subgroup $\mathrm{Out}^0(A_\Gamma)$ of $\mathrm{Out}(A_\Gamma)$ generated by inversions, transvections, and extended partial conjugations.

2.2.2. *Relative outer automorphism groups of RAAGs.* If Δ is a full subgraph of Γ, we use A_Δ to denote the *special subgroup* generated by the vertices contained in Δ. An outer automorphism Φ of A_Γ *preserves* A_Δ if there exists a representative $\phi \in \Phi$ that restricts to an automorphism of A_Δ. An outer automorphism Φ *acts trivially on* A_Δ if there exists a representative $\phi \in \Phi$ acting as the identity on A_Δ.

Definition 2.4 (RORGs). If $\mathcal{G}$, $\mathcal{H}$ are collections of special subgroups of A_Γ, then the *relative outer automorphism group* (or *RORG*), $\mathrm{Out}(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ consists of automorphisms that preserve each $A_\Delta \in \mathcal{G}$ and act trivially on each $A_\Delta \in \mathcal{H}$.

Similarly to the absolute case, we can define the finite-index subgroup

$$\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t) := \mathrm{Out}(A_\Gamma;\mathcal{G},\mathcal{H}^t) \cap \mathrm{Out}^0(A_\Gamma)$$

Laurence's theorem also extends to the relative setting:

Theorem 2.5 [13, Theorem D]. *The group* $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ *is generated by inversions, transvections, and extended partial conjugations.*

For working with examples and, importantly for this paper, making a process algorithmic it is important for us to be able to answer the following questions.

(1) Which inversions, transvections, and partial conjugations are contained in $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$?
(2) If A_Δ is a special subgroup of A_Γ, which inversions, transvections, and partial conjugations preserve A_Δ? Which of these fix A_Δ?
(3) Given a special subgroup A_Δ (not necessarily in $\mathcal{G}$ or $\mathcal{H}$), is A_Δ preserved by $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$? If so, is A_Δ invariant under $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$?

Note that if we can check (2) for a generator Φ, then we can also check (1) by checking if Φ preserves every element of $\mathcal{G}$ and fixes every element of $\mathcal{H}$. Similarly, once we have a generating set for $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ we can check if another subgroup A_Δ is invariant or fixed under this group by running the check generator-by-generator. The conditions for (2) are covered by [**13**, Lemma 2.2]:

LEMMA 2.6 [**13**, Lemma 2.2]. *Let A_Δ be a special subgroup of A_Γ and let $[\iota_v]$, $[\rho_v^w]$, and $[\pi_K^v]$ be an inversion, transvection, and extended partial conjugation, respectively.*

- *The inversion $[\iota_v]$ acts trivially on A_Δ if and only if $v \notin \Delta$. It always preserves A_Δ.*
- *The transvection $[\rho_v^w]$ acts trivially on A_Δ if and only if $v \notin \Delta$. It preserves A_Δ if and only if $v \notin \Delta$ or both $v, w \in \Delta$.*
- *The extended partial conjugation $[\pi_K^v]$ acts trivially on Δ if and only if*

$$K \cap \Delta = \emptyset \text{ or } \Delta - \mathrm{st}(v) \subset K. \qquad (*)$$

The subgroup A_Δ is preserved if and only if Δ satisfies $(*)$ or $v \in \Delta$.

Lemma 2.6 is often enough for dealing with small examples in practice. We will finish this section by introducing some slightly more sophisticated language that is useful for applying Lemma 2.6 (and hence answering questions (1)–(3)) when working with the theory in general or dealing with larger examples.

DEFINITION 2.7 ($\mathcal{J}^v$, $\mathcal{J}$-paths, and $\mathcal{J}$-components). Let $\mathcal{J}$ be a collection of special subgroups of A_Γ. Given a vertex $v \in \Gamma$ and a collection of special subgroups $\mathcal{J}$, we define $\mathcal{J}^v$ to be the subset of $\mathcal{J}$ consisting of special subgroups that *do not* contain v, so that:

$$\mathcal{J}^v = \{A_\Delta \in \mathcal{J} : v \notin \Delta\}.$$

A *$\mathcal{J}$-path* in Γ is a sequence of vertices $v_1, \ldots, v_k$ of Γ such that each pair (v_i, v_{i+1}) either span an edge of Γ or are contained in some common element of $\mathcal{J}$. A *$\mathcal{J}$-component* of a subgraph $\Delta \subset \Gamma$ is a maximal subgraph $C \subset \Delta$ with the property that any two vertices in C are connected by a $\mathcal{J}$-path in Λ. Equivalently, we can obtain $\mathcal{J}$-components of Δ by gluing up connected components of Δ when they both intersect the same element of $\mathcal{J}$.

We define the partial pre-order $\leqslant_{(\mathcal{G},\mathcal{H})}$ on $V(\Gamma)$ by saying that $v \leqslant_{(\mathcal{G},\mathcal{H})} w$ if and only if $\mathrm{lk}(v) \subset \mathrm{st}(w)$ and $v \notin \mathcal{G}^w \cup \mathcal{H}$ (equivalently $[\rho_v^w] \in \mathrm{Out}(A_\Gamma;\mathcal{G},\mathcal{H}^t)$). Given a subgraph $\Delta \subset \Gamma$ we say that Δ is *upwards closed* under $\leqslant_{(\mathcal{G},\mathcal{H})}$ if $v \in \Delta$ and $v \leqslant_{(\mathcal{G},\mathcal{H})} w$ implies that $w \in \Delta$. For partial conjugations, Lemma 2.6 implies that an extended partial conjugation $[\pi_K^v]$ is an element of $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ if and only if K is a union of $\mathcal{G}^v \cup \mathcal{H}$ components of $\Gamma - \mathrm{st}(v)$. We say that Δ is $(\mathcal{G},\mathcal{H})$*–star-separated* by a vertex v if Δ intersects more than one $(\mathcal{G}^v \cup \mathcal{H})$–component of $\Gamma - \mathrm{st}(v)$. This is equivalent to the existence of an extended partial conjugation $[\pi_K^v] \in \mathrm{Out}(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ which acts on A_Δ as a non-inner automorphism. Putting all of this together with the fact that A_Δ is invariant (respectively, fixed) by every element of $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ if and only if A_Δ is invariant (respectively, fixed) by all of its generators, we have the following:

Proposition 2.8. *Let A_Δ be a special subgroup of A_Γ.*

- *A_Δ is invariant under $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ if and only if Δ is upwards closed under $\leqslant_{(\mathcal{G},\mathcal{H})}$ and Δ is not $(\mathcal{G},\mathcal{H})$–star-separated by a vertex $v\in\Gamma-\Delta$.*
- *The group $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ acts trivially on A_Δ if and only if $v\leqslant_{(\mathcal{G},\mathcal{H})} w$ implies that $v=w$ for every $v\in\Delta$, the graph Δ is not $(\mathcal{G},\mathcal{H})$–star-separated by* any *vertex of Γ, and every element of Δ is contained in an element of $\mathcal{H}$.*

Proof. We have $[\rho_v^w]\in\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ if and only if $v\neq w$ and $v\leqslant_{(\mathcal{G},\mathcal{H})} w$. From Lemma 2.6 it follows that Δ is upwards closed under $\leqslant_{(\mathcal{G},\mathcal{H})}$ if and only if A_Δ is invariant under every transvection in $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$. Similarly, A_Δ is invariant under every extended partial conjugation in $\mathrm{Out}(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ if and only if Δ is not $(\mathcal{G},\mathcal{H})$–star-separated by a vertex $v\in\Gamma-\Delta$. The argument for when A_Δ is fixed runs along similar lines, with the final condition ensuring that all of the inversions in $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ also fix A_Δ. □

3. *The virtual cohomological dimension of a Fouxe–Rabinovitch group*

In this section, we use the relative outer space of Guirardel and Levitt to find the virtual cohomological dimension of a Fouxe–Rabinovitch group associated to a free factor decomposition of a right-angled Artin group.

3.1. *Fouxe–Rabinovitch groups and congruence subgroups*

Let $G=G_1*G_2*\cdots*G_k*F_m$ be a free factor decomposition of a group. We let $\mathcal{G}=\{G_i\}$ and define the *Fouxe–Rabinovitch group* associated to this free factor decomposition to be

$$\mathrm{FR}_\mathcal{G}=\mathrm{Out}(G;\mathcal{G}^t).$$

This is the subgroup of $\mathrm{Out}(G)$ acting trivially on each G_i in the decomposition. We do not assume the free factor decomposition is maximal: it need not be the Grushko decomposition of G. We do, however, require that this free factor decomposition is non-trivial in the sense that $k\geqslant 1$ and $k+m\geqslant 2$.

The level 3 congruence subgroup of $\mathrm{FR}_\mathcal{G}$ is defined in the same way as the subgroups of $\mathrm{GL}(n,\mathbb{Z})$ of the same name. It is the finite-index subgroup $\mathrm{FR}_\mathcal{G}^{[3]}$ acting trivially on $H_1(G;\mathbb{Z}/3\mathbb{Z})$. As the action of $\mathrm{FR}_\mathcal{G}$ on each G_i is trivial, this is the same as the subgroup acting trivially on $H_1(F_m;\mathbb{Z}/3\mathbb{Z})$ via the quotient map $\mathrm{FR}_\mathcal{G}\to\mathrm{Out}(F_m)$. If each G_i and each $G_i/Z(G_i)$ is torsion-free, then so is each level 3 congruence subgroup of $\mathrm{FR}_\mathcal{G}$ (see [**19**, Theorem 5.2]).

3.2. *Relative outer space*

Following the work of Guirardel and Levitt in [**17, 18**], we recall the definition of the spine of the relative outer space given by a free factor decomposition of a group.

Definition 3.1. A *Grushko $\mathcal{G}$-tree* is a minimal action of G on a simplicial tree T with trivial edge stabilizers such that each element of $\mathcal{G}$ is elliptic in T and each vertex stabilizer is either trivial or conjugate to an element of $\mathcal{G}$. Two Grushko $\mathcal{G}$-trees T_1 and T_2 are equivalent if there is a G-equivariant homeomorphism $f:T_1\to T_2$.

The set of Grushko $\mathcal{G}$-trees forms a poset, where $T_1<T_2$ if there is a (G-equivariant) subforest in T_2 which collapses to give the action of G on T_1. The geometric realization of this poset is called the *spine of relative Outer space* and we will denote it by $X_\mathcal{G}$.

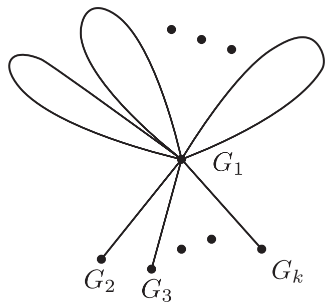

FIGURE 1. *A rose with m petals and $k-1$ leaves, whose Bass–Serre tree gives a vertex in the spine of relative Outer space.*

By a theorem of Guirardel and Levitt [**18**], the spine $X_\mathcal{G}$ is contractible. The spine admits an action of $\mathrm{Out}(G;\mathcal{G})$ by precomposing the action of G on a tree T with the automorphism. We will only need the restriction of this action to $\mathrm{FR}_\mathcal{G} = \mathrm{Out}(G;\mathcal{G}^t)$.

Each n-simplex corresponds to a chain $T_0 < T_1 < \cdots < T_n$ of Grushko $\mathcal{G}$-trees. As the action of $\mathrm{FR}_\mathcal{G}$ preserves the number of edge orbits in a Grushko $\mathcal{G}$-tree, the action of $\mathrm{FR}_\mathcal{G}^{[3]}$ on $X_\mathcal{G}$ is *rigid* (any automorphism preserving a simplex fixes it pointwise). The lemma below uses ideas from the proof of [**19**, Proposition 3.7] and gives a description of simplex stabilizers for the action.

LEMMA 3.2. *Let σ be a simplex in $X_\mathcal{G}$ corresponding to the chain $T_0 < T_1 < \cdots < T_n$ of Grushko $\mathcal{G}$-trees. Then the stabilizer of σ in $\mathrm{FR}_\mathcal{G}^{[3]}$ is a finite-index subgroup of*

$$\oplus_{i=1}^k G_i^{v_i}/Z(G_i),$$

where v_i is the number of G_i-orbits of edges at the vertex fixed by G_i in T_n and $Z(G_i)$ is embedded in $G_i^{v_i}$ diagonally. Furthermore,

$$\sum_{i=1}^k v_i \leqslant 2(m+k-1) - n.$$

Proof. First, we show that automorphisms in $\mathrm{FR}_\mathcal{G}^{[3]}$ preserving a Grushko $\mathcal{G}$-tree T act trivially on the quotient graph T/G, and therefore preserve all collapses of T. This can be seen since the leaf vertices in T/G must have a non-trivial stabilizer in G, so must be in $\mathcal{G}$. Hence the action fixes these vertices. Any such action induces a finite-order element of $\mathrm{Out}(\pi_1(T/G)) \cong \mathrm{Out}(F_m)$. However, the image of $\mathrm{FR}_\mathcal{G}^{[3]}$ in $\mathrm{Out}(F_m)$ is also a level 3 congruence subgroup and is torsion-free. Hence the action on T/G must be trivial (note that if T/G is a circle, all vertices are fixed). As each T_j is a collapse of T_n, any automorphism in $\mathrm{FR}_\mathcal{G}^{[3]}$ that fixes T_n in $\mathrm{FR}_\mathcal{G}^{[3]}$ also fixes each T_j with $j < n$, so that:

$$\mathrm{Stab}_{\mathrm{FR}_\mathcal{G}^{[3]}}(\sigma) = \mathrm{Stab}^0_{\mathrm{FR}_\mathcal{G}}(T_n) \cap \mathrm{FR}_\mathcal{G}^{[3]},$$

where $\mathrm{Stab}^0_{\mathrm{FR}_\mathcal{G}}(T_n)$ denotes the stabilizer of T_n in $\mathrm{FR}_\mathcal{G}$ that acts trivially on T_n/G. This is the *group of twists* of the splitting [**21**, Section 2.4] and satisfies

$$\mathrm{Stab}^0_{\mathrm{FR}_\mathcal{G}}(T_n) \cong \oplus_{i=1}^k G_i^{v_i}/Z(G_i),$$

where, as in the hypothesis, each v_i is the number of G_i-orbits of edges in T_n at the vertex fixed by G_i.

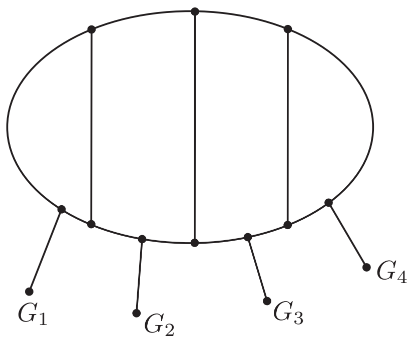

FIGURE 2. *A graph of groups decomposition of G with a maximal number of edge orbits in the case that $k = m = 4$.*

It remains to justify the final inequality. Note that we have to collapse at least n orbits of edges in T_n to obtain T_0, so we may assume T_n has $N \geqslant n$ orbits of vertices with trivial stabilizer. In total, the quotient graph T_n/G has $N + k$ vertices and $N + k + m - 1$ edges (as the fundamental group is F_m). There are at least three half-edges adjacent to each of the N vertices with trivial stabilizer. Subtracting these 3N half-edges from the total half-edge count gives:

$$\sum_{i=1}^{k} v_i \leqslant 2(N + k + m - 1) - 3N$$
$$= 2(m + k - 1) - N$$
$$\leqslant 2(m + k - 1) - n,$$

as required. □

As each $v_i \geqslant 1$, the inequality in Lemma 3.2 shows that $n \leqslant 2m + k - 2$. It is not hard to check that the dimension of the spine is equal to $2m + k - 2$ by exhibiting a graph of groups decomposition of G with $2m + k - 2$ trivalent vertices with trivial stabilizers, trivial edge groups, and each non-trivial vertex group corresponding to a G_i (see Figure 2).

The above work allows one to bound the geometric dimension of a Fouxe–Rabinovitch group via the following theorem:

THEOREM 3.3 [**16**, Theorem 7.3.3]. *Let X be a contractible, rigid G-CW complex with* $\dim(X) \leqslant N$. *For each n, suppose that the stabilizer of each n-cell in X has geometric dimension at most d_n. Then G has geometric dimension*

$$\mathrm{gd}(G) \leqslant \max\{d_n + n : 0 \leqslant n \leqslant N\}.$$

We will apply this to the specific case of right-angled Artin groups below.

3.3. *Free product decompositions of RAAGs*

For a finite graph Γ, we define $d(\Gamma)$ to be the size of the largest clique in Γ. This is the same as the dimension of the Salvetti complex of A_Γ. We define $z(\Gamma)$ to be the number of vertices in Γ that are adjacent to every other vertex. This is the same as the rank of the center of A_Γ, which is a finitely generated free-abelian group. As $\mathrm{FR}_{\mathcal{G}}^{[3]}$ is finite-index in $\mathrm{FR}_{\mathcal{G}}$, the next theorem and its corollary imply Theorem A from the introduction.

Theorem 3.4. *Let* $\mathrm{FR}_{\mathcal{G}}$ *be the Fouxe–Rabinovitch group associated to a non-trivial free factor decomposition*

$$A_\Gamma = A_{\Delta_1} * A_{\Delta_2} * \cdots * A_{\Delta_k} * F_m$$

of a right-angled Artin group, and let $\mathrm{FR}_{\mathcal{G}}^{[3]}$ *be its level* 3 *congruence subgroup. Then*

$$\mathrm{gd}(\mathrm{FR}_{\mathcal{G}}^{[3]}) = (k + 2m - 2) \cdot \max_i\{d(\Delta_i)\} + \sum_{i=1}^{k}(d(\Delta_i) - z(\Delta_i)).$$

Proof. As above, let $X_{\mathcal{G}}$ be the spine of relative Outer space and let σ be a simplex of dimension n given by the chain of trees $T_0 < T_1 \cdots < T_n$. By Lemma 3.2, the stabilizer $\mathrm{Stab}_{\mathrm{FR}_{\mathcal{G}}^{[3]}}(\sigma)$ of σ is a finite index subgroup of

$$\oplus_{i=1}^{k} A_{\Delta_i}^{v_i}/Z(A_{\Delta_i}) \cong \oplus_{i=1}^{k}(A_{\Delta_i}^{v_i-1} \oplus A_{\Delta_i}/Z(A_{\Delta_i})),$$

and $\sum_{i=1}^{k}(v_i - 1) \leqslant 2m + k - 2 - n$. As the geometric dimension of A_{Δ_i} is $d(\Delta_i)$ and the geometric dimension of $A_{\Delta_i}/Z(A_{\Delta_i})$ is $d(\Delta_i) - z(\Delta_i)$, it follows that

$$\begin{aligned}\mathrm{gd}(\mathrm{Stab}_{\mathrm{FR}_{\mathcal{G}}^{[3]}}(\sigma)) &\leqslant (2m + k - 2 - n) \cdot \max_i\{d(\Delta_i)\} + \sum_{i=1}^{k}(d(\Delta_i) - z(\Delta_i)) \\ &\leqslant [(2m + k - 2) \cdot \max_i\{d(\Delta_i)\} + \sum_{i=1}^{k}(d(\Delta_i) - z(\Delta_i))] - n.\end{aligned}$$

Therefore, Theorem 3.3 implies

$$\mathrm{gd}(\mathrm{FR}_{\mathcal{G}}^{[3]}) \leqslant (2m + k - 2) \cdot \max_i\{d(\Delta_i)\} + \sum_{i=1}^{k}(d(\Delta_i) - z(\Delta_i)).$$

To establish equality, it is enough to find a free abelian subgroup of $\mathrm{FR}_{\mathcal{G}}$ of rank equal to the right-hand side of this equation. If we reorder the vertices so that A_{Δ_1} has maximal dimension, we can find such a group inside the stabilizer of (the tree corresponding to) the rose given in Figure 1. In this case, Lemma 3.2 tells us that the stabilizer of the 0-cell given by the rose in $X_{\mathcal{G}}$ under $\mathrm{FR}_{\mathcal{G}}^{[3]}$ is a finite index subgroup of

$$A_{\Delta_1}^{2m+k-1}/Z(A_{\Delta_1}) \oplus_{i=2}^{k} A_{\Delta_i}/Z(A_{\Delta_i}) \cong A_{\Delta_1}^{2m+k-2} \oplus_{i=1}^{k} A_{\Delta_i}/Z(A_{\Delta_i}),$$

which contains a free-abelian subgroup of the desired rank. □

The following corollary is immediate from the proof:

Corollary 3.5. *There exists a free abelian subgroup of rank equal to* $\mathrm{gd}(\mathrm{FR}_{\mathcal{G}}^{[3]})$*, so that*

$$\mathrm{gd}(\mathrm{FR}_{\mathcal{G}}^{[3]}) = \mathrm{cd}(\mathrm{FR}_{\mathcal{G}}^{[3]}) = \mathrm{cd}_{\mathbb{Q}}(\mathrm{FR}_{\mathcal{G}}^{[3]}).$$

Remark 3.6. The free abelian subgroup used in the proof of Theorem 3.4 can be given quite explicitly. First, order the factors of the decomposition so that A_{Δ_1} has maximal dimension and let A_i be the vertices of a maximal clique in Δ_i. Let X be the set of vertices generating the free factor F_m. Then, take all left and right transvections ρ_x^a and λ_x^a for $x \in X$ and $a \in A_1$. Adding the partial conjugations of the subgroups A_{Δ_j}, for $j > 1$, by elements of A_1 gives a free abelian group of rank $(2m + k - 1) \cdot d(\Delta_1)$ in $\mathrm{Aut}(A_\Gamma)$.

For each $i = 1, \ldots, k$, and each vertex $v \in A_i$, add the partial conjugation $\pi_{\Delta_i}^v$. Such partial conjugations are trivial if v is in the center of A_{Δ_i}, and since a maximal clique in Δ_i must contain all vertices in the center, this gives us $d(\Delta_i) - z(\Delta_i)$ partial conjugations for each i. One can check that all of automorphisms above generate a free abelian subgroup of $\mathrm{Aut}(A_\Gamma)$ of rank

$$(2m + k - 1) \cdot d(\Delta_1) + \sum_{i=1}^{k} (d(\Delta_i) - z(\Delta_i)).$$

The only inner automorphisms that appear in the above group come from products of generators with acting letter $a \in A_1$, so that the intersection of this subgroup with the inner automorphisms has rank $d(\Delta_1)$. Subtracting this gives the rank in $\mathrm{Out}(A_\Gamma)$.

4. *Calculating the vcd*

We now give the details of the algorithm to compute the vcd of a RORG. As a first step, we explain how the decomposition procedure for a RORG given in [**13**] is algorithmic.

4.1. *Dismantling RORGs*

The finite-index subgroup $\mathrm{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H}^t)$ of a RORG can be broken up in the following way:

THEOREM 4.1 [**13**, Theorem A]. *The group* $\mathrm{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H}^t)$ *admits a subnormal series*

$$1 = H_0 < H_1 < H_2 < \cdots < H_K = \mathrm{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H}^t)$$

such that each quotient $Q_i = H_i/H_{i-1}$ *is either:*

- ($\mathcal{D}1$) *a finitely generated free abelian group;*
- ($\mathcal{D}2$) *isomorphic to* $\mathrm{GL}(m, \mathbb{Z})$*; or*
- ($\mathcal{D}3$) *a Fouxe–Rabinovitch group given by a free factor decomposition of a special subgroup of* A_Γ.

Note that $\mathrm{Out}(F_m)$ may arise as a quotient via case ($\mathcal{D}3$). As in the introduction, we call such a subnormal series a *decomposition series* for the group.

The most natural way to find the consecutive quotients in a decomposition series is to first build a *decomposition tree* for $\mathrm{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H}^t)$. This is a rooted tree where every internal vertex is labeled by a group of the form $\mathrm{Out}^0(\Gamma_v; \mathcal{G}_v, \mathcal{H}_v)$, with Γ_v a subgraph of Γ and $\mathcal{G}_v, \mathcal{H}_v$ sets of special subgroups of A_{Γ_v}. Our initial group is at the root. Each internal vertex G_v has two descendants K_v and I_v forming an exact sequence

$$1 \to K_v \to G_v \to I_v \to 1.$$

Every leaf of this tree is labeled by a group of the form ($\mathcal{D}1$), ($\mathcal{D}2$), or ($\mathcal{D}3$) and one can show (for example, using induction on the size of the tree) that the leaves of the tree give consecutive quotients in a subnormal series for the root. An example of such a tree is given in [**13**, Figure 6].

PROPOSITION 4.2. *There is an algorithm that produces a decomposition tree for* $\mathrm{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H}^t)$.

Proof. The process for obtaining a tree is iterative. Given a vertex v in the tree, labeled by $\mathrm{Out}^0(\Gamma_v; \mathcal{G}_v, \mathcal{H}_v^t)$, we describe below how to either:

1. recognize $\mathrm{Out}^0(\Gamma_v; \mathcal{G}_v, \mathcal{H}_v^t)$ as a group of type ($\mathcal{D}1$), ($\mathcal{D}2$), or ($\mathcal{D}3$); or
2. extend the tree by adding two edges and two new descendants v_1 and v_2 of v, such that the group labelling v_1 and v_2 is either a RORG of lower complexity (see [**13**, Theorem 5.9] for details on the complexity), or a group of type ($\mathcal{D}1$), ($\mathcal{D}2$), or ($\mathcal{D}3$).

If a new vertex has not been recognized as ($\mathcal{D}1$), ($\mathcal{D}2$), or ($\mathcal{D}3$), then we repeat the process on this vertex. Because the complexity of RORGs decreases as we get further from the root, this algorithm will terminate.

Given $\mathrm{Out}^0(\Gamma_v;\mathcal{G}_v,\mathcal{H}_v^t)$, the first step is to extend $\mathcal{G}_v$ to its *saturation* $\mathcal{G}'_v$ relative to $(\mathcal{G}_v,\mathcal{H}_v)$, which is the collection $\mathcal{G}'_v$ of all special subgroups that are invariant under $\mathrm{Out}^0(\Gamma_v;\mathcal{G}_v,\mathcal{H}_v^t)$. The invariant special subgroups can be determined from the input using Proposition 2.8.

Now assume that $\mathcal{G}_v$ is saturated with respect to $(\mathcal{G}_v,\mathcal{H}_v)$. By [**13**, Theorem E], for each special subgroup A_Δ, the image of $\mathrm{Out}^0(\Gamma_v;\mathcal{G}_v,\mathcal{H}_v^t)$ under the restriction map R_Δ is equal to $\mathrm{Out}^0(A_\Delta;(\mathcal{G}_v)_\Delta,(\mathcal{H}_v)_\Delta^t)$, where

$$(\mathcal{G}_v)_\Delta = \{A_{\Delta\cap\Theta} \mid A_\Theta \in \mathcal{G}_v\} - \{A_\Delta\},$$

and $(\mathcal{H}_v)_\Delta$ is defined similarly. This image is non-trivial if and only if there is an inversion, extended partial conjugation, or transvection with non-trivial image under R_Δ. This is a finite list of elements, and checking if each one has non-trivial image is a simple process.

We now divide into cases according to the nature of the images of restriction maps.

Case 1. There is a restriction map R_Δ with non-trivial image.

In this case, we use the exact sequence

$$1 \to \mathrm{Out}^0(A_{\Gamma_v};\mathcal{G}_v,(\mathcal{H}_v\cup\{A_\Delta\})^t) \to \mathrm{Out}^0(A_{\Gamma_v};\mathcal{G}_v,\mathcal{H}_v^t) \xrightarrow{R_\Delta} \mathrm{Out}^0(A_\Delta;(\mathcal{G}_v)_\Delta,(\mathcal{H}_v)_\Delta^t) \to 1.$$

given by [**13**, Theorem E]. As per the proof of [**13**, Theorem 5.9], the complexity of the RORG in the kernel and quotient is strictly lower than that of $\mathrm{Out}^0(\Gamma_v;\mathcal{G}_v,\mathcal{H}_v^t)$. In this case, we add two new descendants below v in the tree, with vertices labeled by the kernel and image above.

Case 2. All restriction maps have trivial image.

As in [**13**, Section 5], we can break into five subcases.

Case 2a. Γ_v is disconnected and Γ_v is $\mathcal{G}_v$-disconnected.

Here $\mathrm{Out}^0(\Gamma_v;\mathcal{G}_v,\mathcal{H}_v^t)$ is a Fouxe–Rabinovitch group where F_m is a free group on m isolated vertices not contained in any element of $\mathcal{G}_v$ and the A_{Δ_i} are the remaining $\mathcal{G}_v$–connected components ([**13**, Proposition 5.2]).

Case 2b. Γ_v is disconnected and Γ_v is $\mathcal{G}_v$-connected.

The vertices which $(\mathcal{G}_v,\mathcal{H}_v)$-star-separate form a complete graph Θ and, as per the proof of [**13**, Proposition 5.2], $\mathrm{Out}^0(\Gamma_v;\mathcal{G}_v,\mathcal{H}_v^t)$ is a free-abelian group of rank equal to $|\Theta|$.

Case 2c. Γ_v is connected and the center $Z(A_{\Gamma_v})$ of A_{Γ_v} is trivial.

In this case, $\mathrm{Out}^0(\Gamma_v;\mathcal{G}_v,\mathcal{H}_v^t)$ is generated by commuting partial conjugations with acting letters v that have N_v $(\mathcal{G},\mathcal{H})$-connected components. It is not hard to check (for example, using the first Johnson homomorphism [**25**]) that these elements form a free-abelian group of rank $\sum(N_v-1)$.

Case 2d. Γ_v is connected and $Z(A_{\Gamma_v})$ is a proper, non-trivial subgroup.

If $\Delta = \Gamma_v - Z(\Gamma_v)$, we apply [**13**, Proposition 5.6]. There is a projection homomorphism P_Δ with image $\mathrm{Out}^0(A_\Delta;(\mathcal{G}_v)_\Delta^t)$ (with $(\mathcal{G}_v)_\Delta$ as defined above) whose kernel is a free abelian group with basis given by the *leaf transvections* in $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$. These are transvections ρ_w^u with $w \in Z(\Gamma)$ and $u \notin Z(\Gamma)$, see [**10**]. We therefore add two descendants below v, one labeled by a free abelian group of the appropriate rank, and the other labeled by $\mathrm{Out}^0(A_\Delta;(\mathcal{G}_v)_\Delta^t)$.

Case 2e. Γ_v is complete and $A_{\Gamma_v} = \mathbb{Z}^n$ for some n.

It is described in [**13**, Proposition 5.8] how the group fits in the exact sequence

$$1 \to A \to \mathrm{Out}^0(\Gamma_v;\mathcal{G}_v,\mathcal{H}_v^t) \to \mathrm{GL}(m,\mathbb{Z}) \to 1,$$

where A is a finitely generated free abelian group of matrices, so that the rank is easy to compute. We thus add two descendants below v, one labeled by A and the other by $\mathrm{GL}(m,\mathbb{Z})$. □

Note that the construction of a decomposition tree involves many choices, as at each step we only pick *some* invariant special subgroup A_Δ for which there is a restriction map.

QUESTION 4.3. Does the set of consecutive quotients in a decomposition series depend on the set of choices made to dismantle $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$?

In this direction, Brück [**6**, Section 7] uses careful choices of restriction maps to construct a decomposition tree for $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ where the leaves can be described quite explicitly. As a trade-off, the leaves that appear in the decomposition tree of Brück are slightly more general (there are groups generated by partial conjugations that are not necessarily of type $(\mathcal{D}1)$, $(\mathcal{D}2)$, or $(\mathcal{D}3)$).

4.2. *Completing the proof of Theorem B*

To complete the proof of Theorem B, we describe how to compute the vcd of a RORG step-by-step:

Step 1: Build a decomposition tree for $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$. This is detailed in Proposition 4.2.

Step 2: Find the vcd of each leaf. Each leaf is free-abelian, a copy of $\mathrm{GL}(n,\mathbb{Z})$, or a Fouxe–Rabinovitch group, so this can be read off via the calculations of Borel–Serre [**4**] and Culler–Vogtmann [**12**] discussed in Section 2.1 and Theorem A.

Step 3: Add the vcds of the leaves to find the vcd of the root. We do not need to explain how to carry out this step, but we should justify why it works. This is where the discussion of rational cohomological dimension given in Section 2.1 comes into play. The key point here is that we can restrict to the congruence subgroup $\mathrm{Out}^{[3]}(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ of $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$. This is the torsion-free, finite-index subgroup given by the elements acting trivially on $H_1(A_\Gamma,\mathbb{Z}/3\mathbb{Z})$. By [**13**, Theorem 4.8], the short exact sequence

$$1 \to \mathrm{Out}^0(A_{\Gamma_v};\mathcal{G}_v,(\mathcal{H}_v\cup\{A_\Delta\})^t) \to \mathrm{Out}^0(A_{\Gamma_v};\mathcal{G}_v,\mathcal{H}_v^t) \xrightarrow{R_\Delta} \mathrm{Out}^0(A_\Delta;(\mathcal{G}_v)_\Delta,(\mathcal{H}_v)^t_\Delta) \to 1,$$

coming from each projection map restricts to a short exact sequence

$$1 \to \mathrm{Out}^{[3]}(A_{\Gamma_v};\mathcal{G}_v,(\mathcal{H}_v\cup\{A_\Delta\})^t) \to \mathrm{Out}^{[3]}(A_{\Gamma_v};\mathcal{G}_v,\mathcal{H}_v^t) \xrightarrow{R_\Delta} \mathrm{Out}^{[3]}(A_\Delta;(\mathcal{G}_v)_\Delta,(\mathcal{H}_v)^t_\Delta) \to 1.$$

for congruence subgroups. Similar behavior happens with the projection maps that appear in Case 2d and Case 2e during the construction of the decomposition tree (one can see this as both of the projection maps split). As a result, one obtains an analogous decomposition tree for $\mathrm{Out}^{[3]}(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ where each vertex is a level 3 congruence subgroup of the corresponding vertex in the decomposition tree for $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$. This gives a subnormal series

$$1 = H_0 < H_1 < H_2 < \cdots < H_K = \mathrm{Out}^{[3]}(A_\Gamma;\mathcal{G},\mathcal{H}^t)$$

of $\mathrm{Out}^{[3]}(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ where the consecutive quotients are congruence subgroups of the leaves of the decomposition tree for $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ (and the leaves given by free-abelian groups are still free-abelian of the same rank). Some leaves, in particular those isomorphic to $\mathrm{GL}(1,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, will now be trivial. All of these groups are of finite type, and have rational cohomological dimension equal to their cohomological dimension (using either the discussion in Section 2.1 or Theorem A). By Bieri's theorem (Theorem 2.1) and Proposition 2.2, the sum of the (rational) cohomological dimensions of the leaves is equal to the cohomological dimension of $\mathrm{Out}^{[3]}(A_\Gamma;\mathcal{G},\mathcal{H}^t)$, justifying the calculation of the vcd of $\mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ given above.

Remark 4.4. The above work shows that the rational cohomological dimension of a (relative) outer automorphism group is the same as its cohomological dimension. As the inner automorphisms are isomorphic to $A_\Gamma/Z(A_\Gamma)$, the same is true for $\mathrm{Inn}(A_\Gamma)$. Bieri's theorem implies that the vcds of $\mathrm{Out}(A_\Gamma)$ and $\mathrm{Aut}(A_\Gamma)$ differ by the dimension of $A_\Gamma/Z(A_\Gamma)$.

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