# Fast Model Predictive Control

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# Abstract

This thesis develops efficient optimization methods for Model Predictive Control (MPC) to enable its application to constrained systems with fast and uncertain dynamics. The key contribution is an active set method which exploits the parametric nature of the sequential optimization problem and is obtained from a dynamic programming (DP) formulation of the MPC problem. This method is first applied to the nominal linear MPC problem and is successively extended to linear systems with additive uncertainty and input constraints or state/input constraints. The thesis discusses both offline (projection-based) and online (active set) methods for the solution of controllability problems for linear systems with additive uncertainty. The active set method uses first-order necessary conditions for optimality to construct parametric programming regions for a particular given active set locally along a line of search in the space of feasible initial conditions. Along this line of search the homotopy of optimal solutions is exploited: a known solution at some given plant state is continuously deformed into the solution at the actual measured current plant state by performing the required active set changes whenever a boundary of a parametric programming region is crossed during the line search operation. The sequence of solutions for the finite horizon optimal control problem is therefore obtained locally for the given plant state. This method overcomes the main limitation of parametric programming methods that have been applied in the MPC context which usually require the offline precomputation of all possible regions. In contrast to this the proposed approach is an online method with very low computational demands which efficiently exploits the parametric nature of the solution and returns exact local DP solutions. The final chapter of this thesis discusses an application of robust tube-based MPC to the nonlinear MPC problem based on successive linearization.

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# Notation

| | |
|---|---|
| $\mathbb{R}$ | The field of reals |
| $\mathbb{R}^n$ | The $n$-dimensional Euclidean vector space |
| $\mathbb{R}^{n\times m}$ | The vector space of real $n \times m$ matrices |
| $\mathcal{S}^m$ | The set of $m \times m$ symmetric matrices |
| $\mathcal{S}^m_{++}$ $(\mathcal{S}^m_{+})$ | The set of $m \times m$ positive (semi-) definite matrices |
| $E(P,\rho)$ for $P \succ 0, \rho > 0$ | The ellipsoidal set $E\left(P,\rho\right) = \left\{x : x^TPx \leq \rho\right\}$ |
| $\partial(\mathcal{X})$ | Boundary of the set $\mathcal{X}$ |
| $relint(\mathcal{X})$ | Relative interior of the set $\mathcal{X}$ |
| $X \succ Y$ $(X \succeq Y)$ | The matrix $X - Y$ is positive (semi-) definite |
| $\mathrm{Tr}(X)$ | Trace of the matrix $X$ |
| $\overline{\lambda}(Y)$, $(\underline{\lambda}(Y))$ | Maximum (minimum) eigenvalue value of the square matrix $Y$ |
| $\overline{\sigma}(X)$, $(\underline{\sigma}(X))$ | Maximum (minimum) singular value of $X$ |
| $\lVert\cdot\rVert$ | 2-norm of a vector, i.e. $\lVert x\rVert = \sqrt{x^Tx}$ |
| $x^+ = f(x,u,w)$ | successor state given by: $x_{k+1} = f(x_k,u_k,w_k)$ |

# Chapter 1

# Introduction

A fundamental problem in control theory is to develop feedback controllers for wide classes of dynamical systems that are optimal with respect to a suitable performance criterion, lead to a stable closed-loop system and satisfy constraints. While the system-theoretical aspects of this problem have been satisfactorily addressed in the Model Predictive Control (MPC) framework even for nonlinear systems [48], there still remains the challenge of how to obtain computationally efficient solutions. Even for linear MPC remarkable computational improvements have been achieved recently by exploiting the underlying structure of the parametric optimization problem; these allow linear MPC to be employed for fast and large systems [18, 21].

Similarly, it is the aim of robust control based on min-max optimality to minimize the worst-case performance of the controlled system over a known set of unknown future disturbances [4, 38, 56] while satisfying constraints robustly. Due to its enormous practical importance, this area has been a major challenge to researchers for decades and the system-theoretical side of this problem is by now well-understood [48], while the computational side still poses considerable challenges. The major challenge lies in the fact that the optimal solutions to min-max control problems have the form of feedback laws [38] and using available dynamic programming methods, the computation needed to optimize over arbitrary feedback laws typically grows rapidly with the system state dimension and horizon length.

For the important class of linear systems with additive, bounded uncertainty there remains the important challenge of obtaining computationally efficient solutions which ideally solve the underlying dynamic programming problem exactly (or at least approximate it to a high degree). In the scenario-based approach of [50] an exact solution is obtained which involves inpracticably large computational loads, since the number of decision variables and constraints grows exponentially with the horizon length. The contributions of [25, 33, 39, 44, 46] select dynamic programming policy parametrizations and optimize over the free parameters. Most nobably in the context of robust MPC these parametrizations consist of: the class of pre-stabilizing policies [39, 44], policies with linear state feedback parameters [33], the more general disturbance-affine policies of [25] and the even more general tube parametrization of [46]. Unfortunately, these methods have in common that they sacrifice optimality in order to gain compuational efficiency.

As an alternative to solving min-max robust control problems online via dynamic programming, approaches based on multiparametric programming have been proposed [1, 19]. These use offline computations to characterize the optimal control law, typically as a piecewise affine state feedback law. However this requires the solution to be determined for all feasible system states, and moreover it relies on being able to efficiently determine which of many polyhedral regions contains the current state. Although efficient point location techniques have been proposed (e.g. [19, 28]), the method is generally applicable only to small problems and short horizons.

It is the goal of this thesis to overcome this limitation by demonstrating that exact and efficient dynamic programming solutions for robust MPC can be obtained if the structure of underlying parametric optimization problems is adequately exploited. As computational power available for controller implementations continues to increase at an astonishing rate, the development of efficient robust MPC algorithms is expected to lead to widespread applications to systems with fast dynamics in the near future (e.g. in aerospace or automotive applictions).

## 1.1 Thesis Outline and Contributions

The greater part of this thesis is concerned with the development of fast algorithms for linear MPC subject to additive, bounded uncertainty. The commom theme is to formulate active set algorithms based on the first-order conditions of optimality and thereby exploiting the underlying parametric nature of the finite-horizon optimal control problem. The main contribution is the introduction of a novel framework for computing exact and efficient dynamic programming solutions to finite horizon robust optimal control problems which form the backbone of any robust MPC algorithm.

We begin by introducing fundamental results from optimization and MPC theory. Below the contents of each chapter are summarized and the contributions that have appeared in the literature or are under review are listed. The necessary background material and relevant literature are presented at the beginning of each chapter.

- In Chapter 2, fundamental results in convex optimization, parametric optimization and dynamic programming are reviewed. The dynamic programming problem is considered in the context of nominal and robust optimal control. The chapter further considers the nominal and robust MPC problem and provides links to the corresponding dynamic programming formulation. Finally, classical stability results for the nominal and robust MPC problem are reviewed.

- In Chapter 3, a dynamic programming based method is proposed which provides an efficient active set solution to the nominal (i.e. uncertainty-free) linear quadratic MPC problem. The proposed algorithm exploits the parametric and sequential nature of the optimization problem and performs a homotopy of solutions. Finally, the issue of degenerate constraints is discussed and a novel solution is provided.

- In Chapter 4, the solution method introduced in Chapter 3 is employed in the context of robust receding horizon control problem for linear systems with additive bounded uncertainty with input-only constraints and quadratic stage costs. The homotopy of solutions can be used to obtain exact feedback saddle point solutions at the given plant state implying that the underlying dynamic program is solved exactly. The method is computationally efficient (similarly to the nominal linear quadratic solution of Chapter 3) and the results provide a major improvement over currently available dynamic programming based robust MPC methods. The results of this chapter have resulted in the contribution [12, 13].

- Chapter 5 addresses state constraints explicitly to overcome the shortcomings of the approach presented in Chapter 4 where a region of attraction has to be determined to guarantee robust stability. The method is again based on the same underlying idea of constructing efficient online active set solutions. A large part of the chapter is devoted to the situation when state and input constraints are degenerate. Finally, the method is compared to state-of-the-art robust MPC methods based on suboptimal feedback policy parametrizations and found to be both computationally and performance-wise superior. The results of this chapter have been accepted for publication in [14]. One limitation of the approach is due to the assumed availability of offline computed robust controllable sets which is the subject of Chapter 6.

- Chapter 6 first discusses the offline solution to the robust controllability problem (i.e. the computation of robust controllability sets) using an exact dynamic programming approach and, as an alternative, a collection of approximation methods. Secondly, this chapter considers a framework for solving the robust controllability problem locally online by considering links to parametric linear programming problems and to the homotopy-based active set method presented in Chapters 3-5.

- Finally Chapter 7 describes how approximate linear robust MPC methods for systems with polytopic uncertainty can be employed for an efficient solution to the nonlinear MPC problem based on successive linearization. The method is based on ellipsoidal tubes which are used to bound the dynamic effects of uncertainty due to the linearization error and are optimized online via second order cone programming. The method is demonstrated to lead to major improvements over classical nonlinear programming based algorithms while only incurring a small degree of suboptimality. The results of this chapter have been published in [16].

# Chapter 2

# Background

This chapter provides an introduction to relevant background material employed throughout this thesis. In particular, Section 2.1 covers the areas of convex optimization and parametric optimization. Then fundemental results in dynamic programming in the context of optimal control are discussed including the important example of linear systems and a quadratic stage cost. In Section 2.2 we consider fundamental results in nominal and robust Model Predictive Control theory. In particular, we provide links to the underlying finite horizon optimal control problems and present fundamental stability concepts for MPC-controlled systems. For basic notation we refer the reader to the Notation Section in the back of this thesis.

## 2.1 Optimization

### 2.1.1 Convex Optimization

Since convex optimization has become a very wide subject, merely a brief overview of the most relevant methods employed in the course of this thesis is provided. For a detailed discussion of the subject the reader may consider [10].

We start with two important definitions in this context: that of a convex set and a convex function.

**Definition 2.1 (Convex Sets)** *The set $\mathcal{C} \subseteq \mathbb{R}^{n_u}$ is convex if $\theta u + (1-\theta)y \in \mathcal{C}$ for any $u, y \in \mathcal{C}$ for any $\theta \in [0,1]$.*

One can understand this definition in the sense that a given set $\mathcal{C} \subseteq \mathbb{R}^{n_u}$ is convex if for any two vectors $u$ and $y$ in $\mathcal{C}$, any point on the connecting line segment between $u$ and $y$ is also in $\mathcal{C}$. One particularly important class of convex sets - frequently used in this thesis - is the polyhedron:

**Definition 2.2 (Polyhedron)** *A polyhedron is defined as the solution set of a finite number of linear equalities and inequalities*

$$\mathcal{P} = \left\{ u \mid a_j^T u \leq b_j, j = 1, \ldots, m, c_j^T u = d_j, j = 1, \ldots, p \right\}$$

and it is therefore the intersection of a finite number of halfspaces and hyperplanes. A bounded polyhedron will be referred to as a polytope.

In Chapter 7 we consider efficient solution methods for nonlinear MPC problems and in this context we will encounter further examples of convex sets: ellipsoids, second order cones and positive semidefinite cones (for precise definitions and illustrations see [10]).

The second important definition relates to the concept of a convex function:

**Definition 2.3 (Convex Functions [10])** *A function $f : \mathcal{C} \rightarrow \mathbb{R}$ is convex if $\mathcal{C} \subseteq \mathbb{R}^{n_u}$ is a convex set and*

$$f(\theta u + (1-\theta)y) \leq \theta f(u) + (1-\theta) f(y)$$

*for all $u, y \in \mathcal{C}$ and $\theta \in [0,1]$.*

Despite our focus on convex optimization methods in this thesis, we first formulate a generic optimization problem:

$$\min_{u} \quad f_0(u) \tag{2.1}$$

$$\begin{aligned} \text{subject to} \quad & f_i(u) \leq 0, \quad \text{for } i = 1, \ldots, m \\ & h_i(u) = 0 \quad \text{for } i = 1, \ldots, p \end{aligned}$$

where the vector $u \in \mathbb{R}^{n_u}$ is referred to as the decision variable and the function $f_0(u) : \mathbb{R}^{n_u} \to \mathbb{R}$ as the objective function. The constraint functions $f_i(u) : \mathbb{R}^{n_u} \to \mathbb{R}$ for $i = 1, \ldots, m$ impose inequality constraints and the functions $h_i(u) : \mathbb{R}^{n_u} \to \mathbb{R}$ for $i = 1, \ldots, p$ impose equality constraints on the decision variable. The optimal decision variable is denoted by $u^*$. The optimization problem defined in (2.1) is very general, but this generality implies that efficient solutions are not always easily obtainable.

In this thesis we mainly consider the important class of convex optimization problems and by this we understand a problem of the form (2.1) where the functions $f_0, \ldots, f_m$ are convex and $h_i$ are affine. The set of all feasible points for Problem (2.1) is referred to as the feasible set $\Omega$. In particular, for convex problems the feasible region $\Omega$ is convex [45]. For optimization-based control synthesis convex optimization bears a number of striking advantages, which is summarized in the following theorem with a proof given in standard textbooks on optimization [45].

**Theorem 2.1** *For a convex optimization problem any locally optimal point is also globally optimal.*

A second related advantage is that it is possible to find computationally efficient algorithms which exploit convexity. This thesis is mainly concerned with solving parametric quadratic programming (pQP) problems in the context of receding horizon control for linear systems based on quadratic stage costs. These problems are, for a given parameter, quadratic programming (QP) problems and this thesis develops efficient structure-exploiting active set methods for their solution based on the first order necessary conditions for optimality and homotopy of solutions. In Chapter 7, in the context of nonlinear MPC, more general methods of convex programming are employed, such as second order cone programmming (SOCP) and semidefinite programming (SDP), for which interior-point methods are very efficient [10]. We will now review the most important classes of convex optimization problems encountered in the context of this thesis. Again, much additional material can be obtained from [10].

#### 2.1.1.1 Linear Programming

A linear program (LP) is a convex optimization problem where a linear objective function is minimized subject to linear constraints:

$$\begin{aligned} \min_{u} \quad & c^T u \\ \text{subject to} \quad & G_i u \leq h_i \quad i = 1, \dots, m \\ & A_i u = b_i \quad i = 1, \dots, p \end{aligned} \tag{2.2}$$

where $u \in \mathbb{R}^{n_u}$ is the decision variable. The feasible set $\Omega$ is given by a polyhedron and the objective function can be visualized as a hyperplane of variable distance from the origin (the distance of which is maximized). This geometric interpretation implies that any local optimum is globally optimal, but it also follows directly from the convexity of objective and constraint functions. We note that the optimizer may be non-unique (when it does not lie on a vertex of the feasible set).

#### 2.1.1.2 Quadratic Programming

If the objective is quadratic and the constraint functions are affine, then we refer to it as a quadratic program (QP):

$$\begin{aligned} \min_{u} \quad & \frac{1}{2} u^T P u + q^T u + r \\ \text{subject to} \quad & G_i u \leq h_i \quad i = 1, \dots, m \\ & A_i u = b_i \quad i = 1, \dots, p \end{aligned} \tag{2.3}$$

where $u \in \mathbb{R}^{n_u}$ is the decision variable. In a QP, a quadratic function is optimized over a polyhedron. If $P \in \mathcal{S}_{+}^{n_u}$, then the objective is convex and a global minimizer to the convex QP exists. If $P \in \mathcal{S}_{++}^{n_u}$, then the objective is stricly convex and a unique global minimizer to the strictly convex QP exists. We note that for $P \notin \mathcal{S}_{+}^{n_u}$ the QP is non-convex and the solution is generally $\mathcal{NP}$-hard.

QPs will be of crucial importance in this thesis, since the receding horizon control problem with nominal or min-max quadratic stage costs, linear constraints and linear dynamics can be related to a sequence of parametric QPs (where the parameters are given by the predicted states).

#### 2.1.1.3 Second Order Cone Programming

The second order cone program (SOCP) has a linear objective and second order cone constraints and is defined by:

$$\begin{aligned} \min_{u} \quad & c^T u \\ \text{subject to} \quad & \|A_i u + b_i\| \leq c_i u + d_i, \quad i = 1, \ldots, m \\ & Fu = g \end{aligned} \tag{2.4}$$

where $u \in \mathbb{R}^{n_u}$ is the decision variable and $A_i \in \mathbb{R}^{n_i \times n_u}$ and $F \in \mathbb{R}^{p \times n_u}$. We note that SOCPs include QPs and LPs as special cases and efficient solution methods are available [10].

#### 2.1.1.4 Semidefinite Programming

A semidefinite program (SDP) is a convex optimization problem with a linear objective function and Linear Matrix Inequality (LMI) constraints. The general form of a primal SDP is given by

$$\begin{aligned} \min_{u} \quad & c^T u \\ \text{subject to} \quad & F(u) \preceq 0 \end{aligned} \tag{2.5}$$

$$Au = b$$

where $u \in \mathbb{R}^{n_u}$ decision variable, $A \in \mathbb{R}^{p \times n_u}$ and $F(u) \preceq 0$ is an LMI in the given matrices $F_0, \ldots, F_{n_u} \in \mathcal{S}^k$ defined by

$$F(u) = F_0 + \sum_{i=1}^{n_u} u_i F_i.$$

We note that semidefinite programming provides a flexible optimization framework which allows for the incorporation of affine constraint sets and multiple LMI constraints. There are many freely available SDP solvers available; unless otherwise stated SeDuMi is used in this thesis [54]. Although efficient solution techniques are available, SDPs are often too computationally demanding to meet the real-time requirements of receding horizon control as demonstrated in Chapter 4 and 7.

### 2.1.2 Parametric Optimization

In the context of MPC the parametric nature of the optimization problem to be solved at each time instant can be exploited to formulate efficient algorithms and this is the subject of Chapters 3-5. The parametric programming problem in its general form may be stated as

$$\begin{aligned} \min_{u} \quad & f_0(u, x) \\ \text{subject to} \quad & f_i(u, x) \leq 0, \quad \text{for } i = 1, \ldots, m \\ & h_i(u, x) = 0 \quad \text{for } i = 1, \ldots, p \end{aligned} \tag{2.6}$$

where $u \in \mathbb{R}^{n_u}$ is the decision variable and $x \in \mathbb{R}^{n_x}$ is the parameter. The solution depends on the value of the parameter $x$, and so the solution is not a point or set, but a function $x \to u^*(x)$ that may be set-valued. Similarly, the optimal objective value is a function $x \to f_0^*(x)$. If the finite horizon optimal control problem encountered in the MPC problem is solved via dynamic programming (DP), then at a given stage of the DP $x$ denotes the predicted state and $u$ the predicted control. The solution to general parametric programming problems is difficult, but fortunately for the class of parametric QPs and LPs the solution is very structured and efficient solutions are obtainable. We will consider parametric QPs in the following, because of their importance in the course of this thesis. A considerable amount of literature on parametric programs has appeared in the context of 'Explicit' MPC. These methods characterize the solution of the receding horizon optimization problem offline, typically as a feedback law that is a piecewise affine function of the model state defined over a polyhedral partition of the feasible set. Secondly, these methods determine which of these polyhedral regions contains the current state $x_0$ and apply the corresponding feedback law. For more details we refer the reader to [2,8].

#### 2.1.2.1 Parametric Quadratic Programming

We first define the following convex parametric QP (pQP) $\mathbb{P}(x)$ as:

$$\begin{array}{rl} \min\limits_{u} & \frac{1}{2}u^T P u + x^T Q^T u + q^T u + r(x) \\ \text{subject to} & G_i u \leq h_i + H_i x \quad i = 1, \ldots, p \end{array} \tag{2.7}$$

where $u \in \mathbb{R}^{n_u}$ is the decision variable, $x \in \mathbb{R}^{n_x}$ is the parameter and $P \in \mathcal{S}^{n_u}_{++}$.

**Definition 2.4 (Feasible Region for pQP)** *$\mathcal{X}_{0,pQP}$ is the region of feasible parameters:*

$$\mathcal{X}_{0,pQP} = \{x \in \mathbb{R}^{n_x} \mid \exists u \in \mathbb{R}^{n_u} \ s.t. \ G_i u \leq h_i + H_i x \quad i = 1, \ldots, p\}$$

Now for some feasible parameter $x \in \mathcal{X}_{0,pQP}$ let the set of active constraints $G_{\mathcal{A}}$, $h_{\mathcal{A}}$ and $H_{\mathcal{A}}$ be defined by the rows corresponding to active constraints as indicated by the 'active' index set $\mathcal{A}$ and let $G_{\mathcal{I}}$,$h_{\mathcal{I}}$ and $H_{\mathcal{I}}$ be defined by the rows corresponding to inactive constraints as indicated by the ‘inactive’ index set $\mathcal{I}$.

Then the following Karush-Kuhn-Tucker (KKT) conditions are necessarily satisfied by a solution to (2.7) [45]:

**Theorem 2.2 (KKT Conditions for pQP)** *If, given a parameter $x \in \mathcal{X}_{0,pQP}$, $u^*(x)$ is the minimizer of Problem $\mathbb{P}(x)$, then there exist Lagrange multipliers $\mu^*(x)$ such that $u^*(x)$ and $\mu^*(x)$ satisfy the following conditions:*

$$\begin{aligned} Pu^* + G_{\mathcal{A}}^T \mu_{\mathcal{A}}^* &= -Qx - q \\ G_{\mathcal{A}} u^* &= h_{\mathcal{A}} + H_{\mathcal{A}} x \\ G_{\mathcal{I}} u^* &\leq h_{\mathcal{I}} + H_{\mathcal{I}} x \\ \mu_{\mathcal{I}}^* &= 0 \\ \mu_{\mathcal{A}}^* &\geq 0 \end{aligned}$$

**Remark 2.1** *Due to the linearity of constraints in Problem $\mathbb{P}(x)$ the set of feasible directions (the tangent cone) is equal to the set of feasible directions if the constraints are linearized and therefore no extra regularity condition is required in Theorem 2.2. If the constraints are nonlinear, then a typical regularity condition is linear independennce of active constraint gradients (at a given feasible point). This implies equality of the tangent cone and the set of feasible directions of the linearized constraints [45].*

Furthermore the optimal primal solution $u^*(x)$ and Lagrange multiplier $\mu_{\mathcal{A}}^*(x)$ can be determined by solving the Karush-Kuhn-Tucker (KKT) system:

$$\begin{bmatrix} P & G_{\mathcal{A}}^T \\ G_{\mathcal{A}} & 0 \end{bmatrix} \begin{bmatrix} u^* \\ \mu_{\mathcal{A}}^* \end{bmatrix} = \begin{bmatrix} -Q \\ H_{\mathcal{A}} \end{bmatrix} x + \begin{bmatrix} -q \\ h_{\mathcal{A}} \end{bmatrix} \tag{2.8}$$

**Definition 2.5 (pQP region)** *For the given strictly convex parametric QP, let $x \in \mathcal{X}_{0,pQP}$. Let $u^*$ denote the unique optimal primal solution and $\mu_{\mathcal{A}}^*$ the optimal*

*Lagrange multiplier corresponding to the active set $\mathcal{A}$. Then the set*

$$\mathcal{X}_{pQP}(\mathcal{A}) = \{x \in \mathcal{X}_{0,pQP} \mid u^*, \mu^*_{\mathcal{A}} \textit{ satisfy the conditions of Theorem 2.2}\}$$

*is called a pQP region of $\mathcal{X}_{0,pQP}$.*

We note that in cases where the matrix on the left hand side of (2.8) is singular, the primal solution $u^* = L^u x + l^u$ is unique, but the optimal Lagrange multipliers will in general be non-unique, i.e. $\mu^*_{\mathcal{A}} = L^\mu x + l^\mu + Z\beta$ where $Z$ corresponds to the null-space of the left hand side matrix of (2.8) and $\beta$ are the available extra degrees of freedom.

**Definition 2.6 (Degenerate QP problem)** *The problem (2.7) is degenerate if the constraints are linearly dependent (for some active set $\mathcal{A}$).*

We note that we can draw one further important conclusion: The optimal solution obtained for a given parameter $x$ is optimal inside a polyhedral region defined by the corresponding KKT conditions as summarized by the following theorem about $\mathcal{X}_{0,pQP}$:

**Theorem 2.3 (Polyhedral partition of $\mathcal{X}_{0,pQP}$)** *If we assume a strictly convex problem $\mathbb{P}(x)$ for $x \in \mathcal{X}_{0,pQP}$ and the existence of a unique Lagrange multipliers. Then the following results hold:*

*(a) $\partial\mathcal{X}_{pQP}(\mathcal{A}_i) \subseteq \mathcal{X}_{pQP}(\mathcal{A}_i)$ and $\mathcal{X}_{pQP}(\mathcal{A}_i) \cap \mathcal{X}_{pQP}(\mathcal{A}_j) = \partial\mathcal{X}_{pQP}(\mathcal{A}_i) \cap \partial\mathcal{X}_{pQP}(\mathcal{A}_j)$ for $i \neq j$*

*(b) $\bigcup_{i \in \mathcal{S}} \mathcal{X}_{pQP}(\mathcal{A}_i) = \mathcal{X}_{0,pQP}$ for finite $\overline{s}$.*

where $\mathcal{S} = \{1, \cdots, \overline{s}\}$ denotes the index set relating to all possible pQP regions.
*Proof:* The pQP regions are closed polyhedra as a result of the linearity of the KKT conditions. Further, due to the convexity of the primal problem for all feasible $x$, $u^*(x)$ is continuous in $x$ and therefore there cannot be a region overlap, i.e. $\mathcal{X}_{pQP}(\mathcal{A}_i) \cap \mathcal{X}_{pQP}(\mathcal{A}_j) = \partial\mathcal{X}_{pQP}(\mathcal{A}_i) \cap \partial\mathcal{X}_{pQP}(\mathcal{A}_j)$ for $i \neq j$ (which

implies that $relint(\mathcal{X}_{pQP}(\mathcal{A}_i)) \cap relint(\mathcal{X}_{pQP}(\mathcal{A}_j)) = \emptyset$ for $i \neq j$) follows from the assumption of uniqueness of the Lagrange multiplier. Statement $(b)$ follows from the assumed feasibility for $x \in \mathcal{X}_{0,pQP}$ and the necessarily finite number of possible active sets. ■

Furthermore, the primal solution and optimal objective on each polyhedral region can be related to the global solution over the entire feasible set $\mathcal{X}_{0,pQP}$:

**Theorem 2.4 (Piecewise Affine solution to $\mathbb{P}(x)$)** *For the strictly convex pQP $\mathbb{P}(x)$ with $x \in \mathcal{X}_{0,pQP}$ the optimal solution is piecewise affine and continuous. Further, on each polyhedron $\mathcal{X}_{pQP}(\mathcal{A}_i)$ it is given by:*

$$u^*_{\mathcal{A}_i}(x) = L^u_{\mathcal{A}_i} x + l^u_{\mathcal{A}_i}$$

*Similarly, the optimal value function is piecewise quadratic and continuous. Further, on each polyhedron $\mathcal{X}_{pQP}(\mathcal{A}_i)$ it is given by:*

$$f^*_{\mathcal{A}_i}(x) = \frac{1}{2} u^*_{\mathcal{A}_i}(x)^T P u^*_{\mathcal{A}_i}(x) + x^T Q^T u^*_{\mathcal{A}_i}(x) + q^T u^*_{\mathcal{A}_i}(x) + r(x)$$

*Proof:* Both results follow from the convexity of the pQP for each given parameter $x$ and the preceding discussion. ■

Both the geometry of the feasible set $\mathcal{X}_{0,pQP}$ as well as the piecewise affine nature of the optimizer are illustrated by a constrained linear-quadratic MPC regulation problem which results in a pQP with the above demonstrated geometric properties [48]. In Figures 2.1 and 2.2 this is illustrated for a double integrator example with $N = 4$, cost matrices $Q = I$, $R = 1$ and $|u| \leq 1$ [37]. These geometric properties will be fundamental in Chapters 3-5.

**Remark 2.2** *For the case of a degenerate problem with active set $\mathcal{A}_i$, in which there exist extra degrees of freedom in the optimal solution of the Lagrange multiplier $\mu^*_{\mathcal{A}}$, the primal solution will remain unique and continuous in $x$. However, there may exist in this case lower dimensional pQP regions $\mathcal{X}_{pQP}(\mathcal{A}_i) =$*

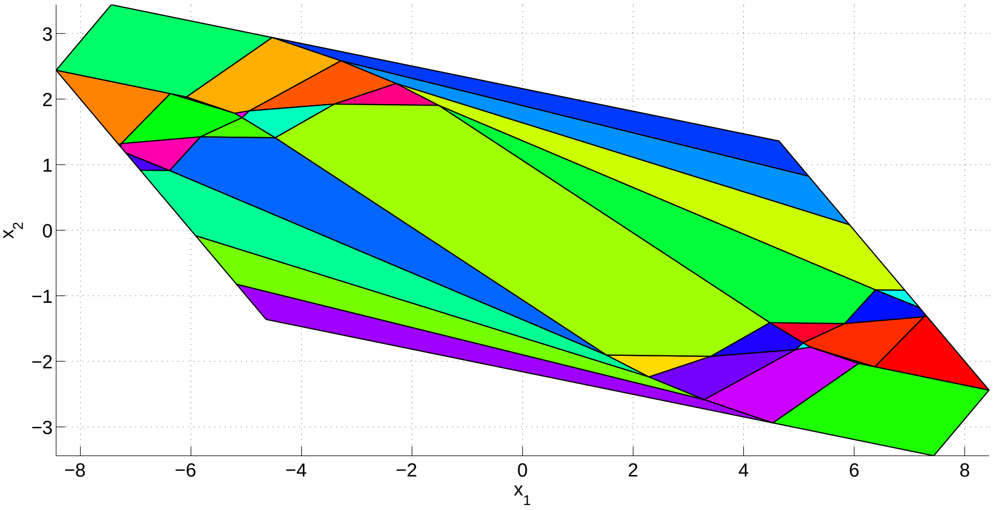

Figure 2.1: Illustration of pQP regions partitioning the feasible set $\mathcal{X}_{0,pQP}$

*$\partial\mathcal{X}_{pQP}(\mathcal{A}_j)\cap\partial\mathcal{X}_{pQP}(\mathcal{A}_k)$ for $i \neq j \neq k$. The intersection $\partial\mathcal{X}_{pQP}(\mathcal{A}_j)\cap\partial\mathcal{X}_{pQP}(\mathcal{A}_k)$ is contained in a subspace of $\mathbb{R}^{n_x}$ defined by the compatibility condition $Z^T H_{\mathcal{A}} x = \mathbf{1}$ associated with the EP solution for the degenerate active set $\mathcal{A}_i$. We note that this situation can only occur on a common boundary of $\mathcal{X}_{pQP}(\mathcal{A}_j)$ and $\mathcal{X}_{pQP}(\mathcal{A}_k)$ due to the uniqueness of the primal solution.*

### 2.1.3 Dynamic Programming and Optimal Control for Nominal Systems

Since the plant state (and the predicted states at each stage) varies as a result of the system dynamics, one part of the natural framework for studying this problem is the area of pQPs. On the other hand, since the successive elements of the sequence of pQPs are dynamically linked, the other part of the natural framework for studying the problem is dynamic programming (DP).

#### 2.1.3.1 Theoretical Background

We illustrate the main features of DP in an optimal control context and to this end we consider time-invariant discrete-time systems with known (nominal) dynamics

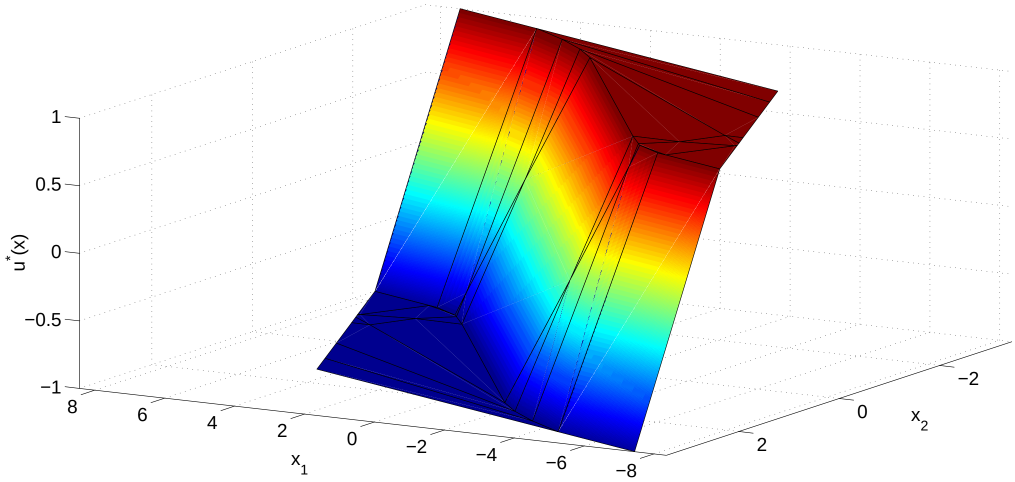

Figure 2.2: Illustration of piecewise-affine nature of optimizer $u^*(x)$ in pQP problems

given by:

$$x^+ = f(x, u)$$

where $f(x, u)$ is a continuous function and the equilibrium is given by $f(0, 0) = 0$. Further, the system is asssumed to be constrained, i.e. $x \in \mathcal{X}$ and $u \in \mathcal{U}$, where $\mathcal{X}$ is closed, $\mathcal{U}$ is compact and both $\mathcal{X}$ and $\mathcal{U}$ contain the origin in their interior. We further define the extended set $\mathcal{Z} = \mathcal{X} \times \mathcal{U}$ and therefore obtain that $(x, u) \in \mathcal{Z}$. We consider now an $N$-stage optimal control problem with a terminal state constraint on $x_N$:

$$x_N \in \mathcal{X}_f$$

where $\mathcal{X}_f$ is closed and contains the origin in its interior and where $k$ denotes the forward-in-time index. The optimal cost at 'starting time' $k$ is denoted as $J_k^*(x)$ with a corresponding domain $\mathcal{X}_k$. If now - for an initial state $x_0 = x$ - the chosen control sequence is taken as $\mathbf{u} = \{u_0, \ldots, u_{N-1}\}$ then the associated cost for the

$N$-stage optimal control problem is given by

$$J_0(x, \mathbf{u}) = \sum_{k=0}^{N-1} l(x_k, u_k) + J_f(x_N)$$

where the stage cost $l(\cdot)$ and terminal cost $J_f(\cdot)$ are assumed to be continuous functions with the properties $l(0,0) = 0$ and $J_f(0) = 0$ and where we make use of the fact that, for each $k$, the predicted state is obtained in the form of $x_k = \Phi_k(x, \mathbf{u})$ (which is obtained if the sequence $\mathbf{u}$ is applied to the system with initial state $x$). The optimal control problem $\mathbb{P}_0(x)$ can therefore be defined in equivalent form with a 'condensed' cost:

$$J_0^*(x) = \min_{\mathbf{u}} J_0(x, \mathbf{u})$$

subject to constraints $(x_k, u_k) \in \mathcal{Z}$, for $k = 0, \ldots, N-1$, and $x_N \in \mathcal{X}_f$ (where $x_k = \Phi_k(x, \mathbf{u})$). This can further be restated by defining the set of admissible inputs as

$$\mathcal{U}_0(x) = \left\{ \mathbf{u} \in \mathbb{R}^{Nn_u} \mid (x_k, u_k) \in \mathcal{Z} \text{ for } k = 0, \ldots, N-1; x_N \in \mathcal{X}_f \right\}$$

where the dynamic constraint $x_k = \Phi_k(x, \mathbf{u})$ is satisfied for each $k$. Therefore $\mathbb{P}_0(x)$ is equivalent to the 'condensed' form:

$$J_0^*(x) = \min_{\mathbf{u}} \{ J_0(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_0(x) \} \tag{2.9}$$

The continuity of $f(\cdot)$ implies that the mappings $\mathbf{u} \to \Phi_k(x, \mathbf{u})$ and $\mathbf{u} \to J_0(x, \mathbf{u})$ are continuous and that the set $\mathcal{U}_0(x)$ is compact. This implies that Weierstrass' theorem applies and the minimum in (2.9) exists at all $x \in \{x \in \mathbb{R}^{n_x} \mid \mathcal{U}_0(x) \neq \emptyset\}$ [48]. For clarity we state the following standing assumptions:

**Assumption 2.1** *The functions $f(\cdot)$, $l(\cdot)$ and $J_f(\cdot)$ are continuous, $f(0,0) = 0$, $l(0,0) = 0$ and $J_f(0) = 0$.*

**Assumption 2.2** *The sets $\mathcal{X}$ and $\mathcal{X}_f$ are closed, $\mathcal{X}_f \subseteq \mathcal{X}$ and $\mathcal{U}$ is compact. Further, each set contains the origin. $\mathcal{X}_f$ (and therefore $\mathcal{X}$) is assumed to contain the origin in its interior.*

The corresponding DP formulation considers a sequence of 'shortened' problems $\mathbb{P}_k(x)$ defined by:

$$J_k^*(x) = \min_{\mathbf{u}_k} \{J_k(x, \mathbf{u}_k) \mid \mathbf{u}_k \in \mathcal{U}_k(x)\}$$

where

$$\mathbf{u}_k = \{u_k, \ldots, u_{N-1}\}$$

$$J_k(x, \mathbf{u}_k) = \sum_{j=k}^{N-1} l(x_j, u_j) + J_f(x_N)$$

$$\mathcal{U}_k(x) = \left\{\mathbf{u}_k \in \mathbb{R}^{(N-k)n_u} \mid (x_j, u_j) \in \mathcal{Z} \text{ for } j = k, k+1, \ldots, N-1; x_N \in \mathcal{X}_f\right\}$$

where $x_j = \Phi_j(x, \mathbf{u}_k)$ (the solution at time $j$ if the state at time $k$ is equal to $x$ and the control sequence is $\mathbf{u}_k$). For each $k$ we define the domain of $J_k^*(x)$ as

$$\mathcal{X}_k = \{x \in \mathbb{R}^{n_x} \mid \mathcal{U}_k(x) \neq \emptyset\}$$

Now we consider the expression for the optimal cost at time $k$:

$$\begin{aligned} J_k^*(x) &= \min_{\mathbf{u}_k} \{J_k(x, \mathbf{u}_k) \mid \mathbf{u}_k \in \mathcal{U}_k(x)\} \\ &= \min_{u} \left\{ l(x, u) + \min_{\mathbf{u}_{k+1}} J_{k+1}(f(x, u), \mathbf{u}_{k+1}) \mid (u, \mathbf{u}_{k+1}) \in \mathcal{U}_k(x) \right\} \end{aligned}$$

Since $x_{k+1} = f(x, u)$, we can employ the equivalence:

$$(u, \mathbf{u}_{k+1}) \in \mathcal{U}_k(x) \Leftrightarrow (x, u) \in \mathcal{Z}, f(x, u) \in \mathcal{X}_{k+1} \text{ and } \mathbf{u}_{k+1} \in \mathcal{U}_{k+1}(f(x, u))$$

and therefore we obtain the DP recursion as

$$J_k^*(x) = \min_{u} \left\{ l(x, u) + J_{k+1}^*(f(x, u)) \mid (x, u) \in \mathcal{Z}, f(x, u) \in \mathcal{X}_{k+1} \right\} \quad \forall x \in \mathcal{X}_k$$

$$\mathcal{X}_k = \{x \in \mathbb{R}^{n_x} \mid \exists u \text{ such that } (x,u) \in \mathcal{Z} \text{ and } f(x,u) \in \mathcal{X}_{k+1}\}$$

with boundary conditions:

$$\mathcal{X}_N = \mathcal{X}_f, \quad J_N^*(x) = J_f(x)$$

Finally, we state two fundamental results of dynamic programming in the optimal control context: firstly, a lemma well-known as the principle of optimality and, secondly, a theorem which links the optimal value function and control law to the DP recursion equations. For proofs of both results, the reader is referred to [48] [3].

**Lemma 2.1 (Principle of Optimality)** *Let* $x \in \mathcal{X}_0$ *and* $\mathbf{u} \in \mathcal{U}_0(x)$ *denote the solution to* $\mathbb{P}_0(x)$ *with a corresponding optimal state trajectory* $\mathbf{x} = \{x, x_1, \dots, x_N\}$, *so that for each* $k$, $x_k = \Phi_k(x, \mathbf{u})$. *Then, for any* $k \in \{0, 1, \dots, N-1\}$, *the control sequence* $\mathbf{u}_k = \{u_k, \dots, u_{N-1}\}$ *is optimal for* $\mathbb{P}_k(x_k)$. *(in words: any forward restriction of an optimal trajectory is optimal)*

**Theorem 2.5 (Optimal value function and control law from DP)** *Suppose that the function* $\Psi_k : \mathbb{R}^{n_x} \to \mathbb{R}$ *satisfies, for all* $k \in \{0, 1, \dots, N-1\}$ *and all* $x \in \mathcal{X}_k$, *the DP recursion:*

$$\Psi_k(x) = \min_u \{l(x,u) + \Psi_{k+1}(f(x,u)) \mid (x,u) \in \mathcal{Z}, f(x,u) \in \mathcal{X}_{k+1}\}$$
$$\mathcal{X}_k = \{x \in \mathbb{R}^{n_x} \mid \exists u \in \mathbb{R}^{n_u} \textit{ such that } (x,u) \in \mathcal{Z}, f(x,u) \in \mathcal{X}_{k+1}\}$$

*with boundary conditions:*

$$\Psi_N(x) = J_f(x) \quad \forall x \in \mathcal{X}_f, \quad \mathcal{X}_N = \mathcal{X}_f$$

*Then* $\Psi_k(x) = J_k^*(x)$ *for* $x \in \mathcal{X}_k$ *for* $k = \{0, 1, \dots, N\}$ *and the DP recursion yields* *the optimal value function* $J_k^*(x)$ *with the corresponding optimal control law:*

$$\kappa_k^*(x) = \arg\min_u \{l(x,u) + \Psi_{k+1}(f(x,u)) \mid (x,u) \in \mathcal{Z}, f(x,u) \in \mathcal{X}_{k+1}\}$$

#### 2.1.3.2 Example: Unconstrained Nominal Linear-Quadratic Control

As an important example we consider the unconstrained linear quadratic regulator as a special case of the general theory presented in the Section 2.1.3.1. The system is therefore defined by:

$$x^+ = Ax + Bu$$

Further, the system is unconstrained, i.e. $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, the stage cost is $l(x,u) = \frac{1}{2}(\|u\|_R^2 + \|x\|_Q^2)$ and the terminal cost is $J_N^*(x) = 0$ for all $x \in \mathbb{R}^{n_x}$. We assume further that $R \in \mathcal{S}_{++}^{n_u}$ and that $Q \in \mathcal{S}_{+}^{n_x}$. The DP recursion then reduces to:

$$J_k^*(x) = \min_u \left\{ \frac{1}{2}(\|u\|_R^2 + \|x\|_Q^2) + J_{k+1}^*(Ax + Bu) \right\} \quad \forall x \in \mathbb{R}^{n_x}$$

with terminal condition

$$J_N^*(x) = 0 \quad \forall x \in \mathbb{R}^{n_x}$$

The derivation uses an induction-like procedure. Therefore we assume that the solution $J_{k+1}^*(\cdot)$ is given by:

$$J_{k+1}^*(x) = \frac{1}{2}\|x\|_{P_{k+1}}^2$$

where $P_{k+1} \in \mathcal{S}_+^{n_x}$. Then the DP recursion can be restated as:

$$J_k^*(x) = \min_u \left\{ \frac{1}{2}(\|u\|_R^2 + \|x\|_Q^2) + \frac{1}{2}\|Ax + Bu\|_{P_{k+1}}^2 \right\} \quad \forall x \in \mathbb{R}^{n_x}$$

Since the matrix $R + B^T P_{k+1} B \in \mathcal{S}_{++}^{n_u}$, the optimization problem has a unique minimizer given by:

$$\kappa_k^*(x) = L_k x \quad \text{with } L_k = -(R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A$$

The corresponding optimal cost is given by

$$\begin{aligned} J_k^*(x) &= \frac{1}{2} \|x\|_{P_k}^2 \\ P_k &= Q + L_k^T R L_k - A^T P_{k+1} B (R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A \end{aligned}$$

The terminal cost is given by $J_N^*(x) = \frac{1}{2} \|x\|_{P_N}^2$ with $P_N = 0 \in \mathcal{S}_{+}^{n_x}$. Therefore we can conclude by induction that the same holds true for $J_k^*(x)$ for all $k = 0, \ldots, N$. The well-known discrete-time algebraic Riccati equation may be obtained by substituting $L_k = -(R + B^T P_{k+1} B)^{-1} B^T P_{k+1}$:

$$P_k = Q + A^T P_{k+1} A - A^T P_{k+1} B (R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A$$

In the limit as $N \to \infty$, we obtain the steady state discrete-time algebraic Riccati equation

$$P = Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A$$

with corresponding solution

$$\kappa_\infty^*(x) = L x \quad \text{with } L = -(R + B^T P B)^{-1} B^T P A$$

**Remark 2.3** *Provided that $(A, B)$ is stabilizable and $(Q^{\frac{1}{2}}, A)$ is detectable, the solution $P \in \mathcal{S}_{++}^{n_x}$ and the closed-loop system under the infinite horizon LQR is exponentially stable. [48]*

### 2.1.4 Dynamic Programming and Min-Max Optimal Control for Systems with Bounded Disturbances

#### 2.1.4.1 Uncertain Systems and Solution Structure

To illustrate the main features of DP in a robust optimal control context, we consider time-invariant discrete-time systems with bounded unknown disturbances given by

$$x^+ = f(x, u, w)$$

where $f(x, u, w)$ is continuous and $x \in \mathcal{X}$, $u \in \mathcal{U}$, $w \in \mathcal{W}$, where $\mathcal{X}$ is closed, $\mathcal{U}, \mathcal{W}$ are compact and $\mathcal{X}, \mathcal{U}, \mathcal{W}$ contain the origin. We note that the uncertain system may alternatively be described by

$$x^+ \in F(x, u) = f(x, u, \mathcal{W}) := \{f(x, u, w) \mid w \in \mathcal{W}\}$$

which may be understood in the following way: if $x$ is the current state, then under the control input $u$, the successor state $x^+$ lies inside the set $F(x, u)$. If a control policy $\boldsymbol{\kappa} = \{\kappa_0(\cdot), \kappa_1(\cdot), \ldots\}$ is used, then the successor state is described by:

$$x^+ \in F(x, \kappa_k(x))$$

For a given policy $\boldsymbol{\kappa}$ this generates a corresponding 'tube' containing all possible trajectories $\mathbf{X}^{tube} = \left\{\mathcal{X}_0^{tube}, \mathcal{X}_1^{tube}, \ldots\right\}$[1] which satisfies the following relationship (which defines a mapping from sets into sets):

$$\mathcal{X}_{k+1}^{tube} = F(\mathcal{X}_k^{tube}, \kappa_k(\cdot)) := \left\{f(x, \kappa_k(x), w) \mid x \in \mathcal{X}_k^{tube}, w \in \mathcal{W}\right\}$$

where $\mathcal{X}_0^{tube}$ is a singleton given by the initial plant state $x_0$. We note that for a given disturbance realization $\mathbf{w} = \{w_0, w_1, \ldots\}$, the tube $\mathbf{X}^{tube}$ collapses to a

---

[1] N.B. $\mathcal{X}_k$ and $\mathcal{X}_k^{tube}$ should not be confused: the first refers to the domain of the optimal cost in an optimal control problem, the second refers to an element of the set sequence generated by all possible disturbance realizations for a given policy $\boldsymbol{\kappa}$

single trajectory.

#### 2.1.4.2 Finite Horizon Min-Max Optimal Control Problem

We consider an $N$-stage problem with a constraint on the terminal state $x_N$:

$$x_N \in \mathcal{X}_f$$

where $\mathcal{X}_f$ is closed. To provide worst-case guarantees of performance, we consider, as is common in the literature, a cost which is defined as the maximum over all admissible realizations of the disturbance sequence $\mathbf{w} = \{w_0, \ldots, w_{N-1}\} \in \mathcal{W}^N = \mathcal{W} \times \cdots \times \mathcal{W}$. Due to the uncertain formulation, the optimal solution at a given $x$ is a policy $\boldsymbol{\kappa}^*(x) = \left\{\kappa_0^*(x), \kappa_1^*(\cdot), \ldots, \kappa_{N-1}^*(\cdot)\right\}$, i.e. a sequence of feedback laws. The cost for an initial state $x$ at time $k = 0$, feedback policy $\boldsymbol{\kappa}$ and a particular disturbance realization $\mathbf{w}$ is therefore defined as:

$$J_0(x, \boldsymbol{\kappa}, \mathbf{w}) = J_f(x_N) + \sum_{k=0}^{N-1} l(x_k, u_k, w_k)$$

where the stage cost $l(\cdot)$ and terminal cost $J_f(\cdot)$ are continuous functions and where for the given policy $\boldsymbol{\kappa}$ we obtain, for each $k$, the state $x_k = \Phi_k(x, \boldsymbol{\kappa}, \mathbf{w})$ for the applied input $u_k = \kappa_k(x_k)$. The worst-case cost is therefore defined by:

$$\hat{J}_0(x, \boldsymbol{\kappa}) = \max_{\mathbf{w}} \left\{ J_0(x, \boldsymbol{\kappa}, \mathbf{w}) \mid \mathbf{w} \in \mathcal{W}^N \right\}$$

The robust optimal control problem $\mathbb{P}_0(x)$ is defined by (assuming the minimum exists):

$$J_0^*(x) = \min_{\boldsymbol{\kappa}} \left\{ \hat{J}_0(x, \boldsymbol{\kappa}) \mid \boldsymbol{\kappa} \in \mathcal{M}(x) \right\} \tag{2.10}$$

where $\mathcal{M}(x)$ is defined by

$$\mathcal{M}(x) = \Big\{\boldsymbol{\kappa} \mid \kappa_0(x) = u_0 \in \mathcal{U}, \Phi_k(x, \boldsymbol{\kappa}, \mathbf{w}) \in \mathcal{X}, \kappa_k(\Phi_k(x, \boldsymbol{\kappa}, \mathbf{w})) \in \mathcal{U} \\ \text{for } k = 0, \dots, N-1, \Phi_N(x, \boldsymbol{\kappa}, \mathbf{w}) \in \mathcal{X}_f \quad \forall \mathbf{w} \in \mathcal{W}^N\Big\}$$

with optimal policy $\boldsymbol{\kappa}^*(x) = \big\{\kappa_0^*(x) = u_0^*, \kappa_1^*(\cdot), \dots, \kappa_{N-1}^*(\cdot)\big\}$ and the value function $J_0^*(x) = \hat{J}_0(x, \boldsymbol{\kappa}^*(x))$. As in the nominal case we now consider the corresponding DP formulation. The optimal cost at time $k$ is denoted as $J_k^*(x)$ and its domain as $\mathcal{X}_k$. This is stated as a sequence of problems $\mathbb{P}_k(x)$ each of which is defined by:

$$J_k^*(x) = \min_{u \in \mathcal{U}} \max_{w \in W} \big\{l(x,u,w) + J_{k+1}^*(f(x,u,w)) \mid f(x,u,\mathcal{W}) \in \mathcal{X}_{k+1}\big\} \tag{2.11}$$

$$\kappa_k^*(x) = \arg \min_{u \in \mathcal{U}} \max_{w \in W} \big\{l(x,u,w) + J_{k+1}^*(f(x,u,w)) \mid f(x,u,\mathcal{W}) \in \mathcal{X}_{k+1}\big\} \tag{2.12}$$

$$\mathcal{X}_k = \{x \in \mathcal{X} \mid \exists u \in \mathcal{U} \text{ such that } f(x,u,\mathcal{W}) \in \mathcal{X}_{k+1}\} \tag{2.13}$$

The domain of $J_k^*(x)$, $\mathcal{X}_k$, is the set of states that can be robustly controlled by state feedback to $\mathcal{X}_f$ in $k$ steps while satisfying constraints for all realizations of $\mathbf{w} \in \mathcal{W}^N$. Finally, the boundary conditions are given by:

$$\mathcal{X}_N = \mathcal{X}_f, \quad J_N^*(x) = J_f(x)$$

#### 2.1.4.3 Example: Unconstrained Min-Max Linear-Quadratic Control

Similarly to the LQR solution in the nominal case, we can find the unconstrained $\mathcal{H}_\infty$-controller[1], which is employed in the constrained $\mathcal{H}_\infty$-receding horizon control formulation of Chapter 4 and 5. The system is linear and the uncertainty is

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[1] We use the term $\mathcal{H}_\infty$-control to refer specifically to the feedback laws that solve a particular class of quadratic min-max optimal control problems, rather than the robust control design framework developed by Zames, Doyle, Glover and others [20,57]. The rationale for this terminology is that the solution of these control problems is a feedback law that ensures a given bound on the $l_2$-gain between disturbance and state/input sequences, thus ensuring, for the unconstrained case, a bound on the $\mathcal{H}_\infty$ norm of an associated closed loop transfer function.

additive:

$$x^+ = Ax + Bu + Dw$$

Further, the system is unconstrained, i.e. $x \in \mathbb{R}^{n_x}$,$u \in \mathbb{R}^{n_u}$,$w \in \mathbb{R}^{n_w}$, the stage cost is given as $l(x, u, w) = \frac{1}{2}(\|u\|_R^2 + \|x\|_Q^2 - \gamma^2 \|w\|^2)$ and the terminal cost is $J_N^*(x) = 0$ for all $x \in \mathbb{R}^{n_x}$. We assume further that $R \in \mathcal{S}_{++}^{n_u}$, that $Q \in \mathcal{S}_{+}^{n_x}$ and that $\gamma^2$ is chosen sufficiently large that the optimal control problem is convex (conditions on the value will be provided in the sequel). The DP recursion can now be stated for $k = 0, \ldots, N-1$:

$$J_k^*(x) = \min_u \max_w \left\{ \frac{1}{2}(\|u\|_R^2 + \|x\|_Q^2 - \gamma^2 \|w\|^2) + J_{k+1}^*(Ax + Bu + Dw) \right\}$$

and with terminal condition

$$J_N^*(x) = 0 \quad \forall x \in \mathbb{R}^{n_x}$$

Assuming that $J_{k+1}^*(x) = \frac{1}{2}\|x\|_{P_{k+1}}^2$ the DP recursion can be restated as:

$$J_k^*(x) = \min_u \max_w \left\{ \frac{1}{2}(\|u\|_R^2 + \|x\|_Q^2 - \gamma^2 \|w\|^2) + \frac{1}{2}\|Ax + Bu + Dw\|_{P_{k+1}}^2 \right\}$$

We first consider the 'inner' maximization problem:

$$\hat{J}_k^*(\hat{x}) = \max_w \left\{ -\frac{\gamma^2}{2}\|w\|^2 + \frac{1}{2}\|\hat{x} + Dw\|_{P_{k+1}}^2 \right\}$$

for $\hat{x} = Ax + Bu$. Provided that $-\gamma^2 I + D^T P_{k+1} D \prec 0$, there exists a unique maximizer given by:

$$w_k^*(\hat{x}) = L_k^w \hat{x} \quad \text{with } L_k^w = (\gamma^2 I - D^T P_{k+1} D)^{-1} D^T P_{k+1}$$

Now we define the intermediate cost $\hat{J}_k^*(\hat{x})$ in terms of $\hat{P}_k = P_{k+1}(I + DL_k^w)$ and consider the 'outer' maximization:

$$\begin{aligned}
\hat{J}_k^*(\hat{x}) &= \frac{1}{2}\left\|\hat{x}\right\|_{\hat{P}_k}^2 \\
J_k^*(x) &= \min_u \left\{ \frac{1}{2}(\|u\|_R^2 + \|x\|_Q^2) + \frac{1}{2}\left\|\hat{x}\right\|_{\hat{P}_k}^2 \right\}
\end{aligned}$$

Therefore under the assumptions that $R \in \mathcal{S}_{++}^{n_u}$, $Q \in \mathcal{S}_{+}^{n_x}$, $\gamma^2 I - D^T P_{k+1} D \in \mathcal{S}_{++}^{n_w}$ and the terminal condition $P_N = 0$, the matrix $R + B^T P_{k+1} B \in \mathcal{S}_{++}^{n_u}$ and thus the minimization problem has a unique solution given by:

$$\kappa_k^*(x) = L_k^u x \quad \text{with } L_k^u = -(R + B^T \hat{P}_k B)^{-1} B^T \hat{P}_k$$

The corresponding optimal cost is given by

$$\begin{aligned}
J_k^*(x) &= \frac{1}{2}\left\|x\right\|_{P_k}^2 \\
P_k &= Q + L_k^{uT} R L_k^u - A^T \hat{P}_k B (R + B^T \hat{P}_k B)^{-1} B^T \hat{P}_k A
\end{aligned}$$

which completes the induction proof. With the substitutions $L_k^w = (\gamma^2 I - D^T P_{k+1} D)^{-1} D^T P_{k+1}$ and $L_k^u = -(R + B^T \hat{P}_k B)^{-1} B^T \hat{P}_k$ this results in the generalized discrete-time algebraic Riccati equation (DARE) given by:

$$\begin{aligned}
P_k &= Q + A^T \hat{P}_k A - A^T \hat{P}_k B (R + B^T \hat{P}_k B)^{-1} B^T \hat{P}_k A \\
\hat{P}_k &= P_{k+1}[I + D(\gamma^2 I - D^T P_{k+1} D)^{-1} D^T P_{k+1}]
\end{aligned}$$

**Assumption 2.3** *Let the output be given as* $z = Cx + Du$, *where* $C = [Q^{\frac{1}{2}T} 0^T]^T$ *and* $D = [0^T R^{\frac{1}{2}T}]^T$, *for* $Q \in \mathcal{S}_{+}^{n_x}, R \in \mathcal{S}_{++}^{n_u}$. *Let* $(A, B)$ *be stabilizable and* $(C, A)$ *detectable and let* $(C, A, B)$ *have no zeros on the unit circle.*

In the limit as $N \to \infty$ we obtain the generalized steady-state DARE:

$$\begin{aligned} P &= Q + A^T \hat{P} A - A^T \hat{P} B (R + B^T \hat{P} B)^{-1} B^T \hat{P} A \\ \hat{P} &= P[I + D(\gamma^2 I - D^T P D)^{-1} D^T P] \end{aligned}$$

For the steady-state DARE a positive definite solution $P$ exists provided the following assumption is satisfied [48]:

**Assumption 2.4** *Let $\gamma$ be chosen such that the condition $\gamma^2 I - D^T P D \in \mathcal{S}_{++}^{n_w}$ holds.*

Therefore under Assumptions 2.3 and 2.4 we obtain the steady-state optimal input and worst case disturbance as:

$$\kappa_\infty^*(x) = L^u x \quad \text{with } L^u = -(R + B^T \hat{P} B)^{-1} B^T \hat{P} \tag{2.14}$$

$$w_\infty^*(\hat{x}) = L^w \hat{x} \quad \text{with } L^w = (\gamma^2 I - D^T P D)^{-1} D^T P \tag{2.15}$$

and the closed loop state transition matrices $A + BL^u$ and $(I + DL^w)(A + BL^u)$ are both stable [26].

## 2.2 Model Predictive Control

This section provides a brief overview of control theoretic foundations of MPC relevant to this thesis. We first consider the nominal MPC formulation for non-linear discrete-time systems, and then go on to consider the general robust MPC problem. The fundamental idea of Model Predictive Control is to solve for a given plant state measurement and a given dynamical system model (including constraints on states and inputs) an optimal control problem online. Only the first element of the resulting input solution sequence is taken as an input into the system at a given time instant and then for a new plant state measurement at the next time instant a new optimal control problem is solved. This receding

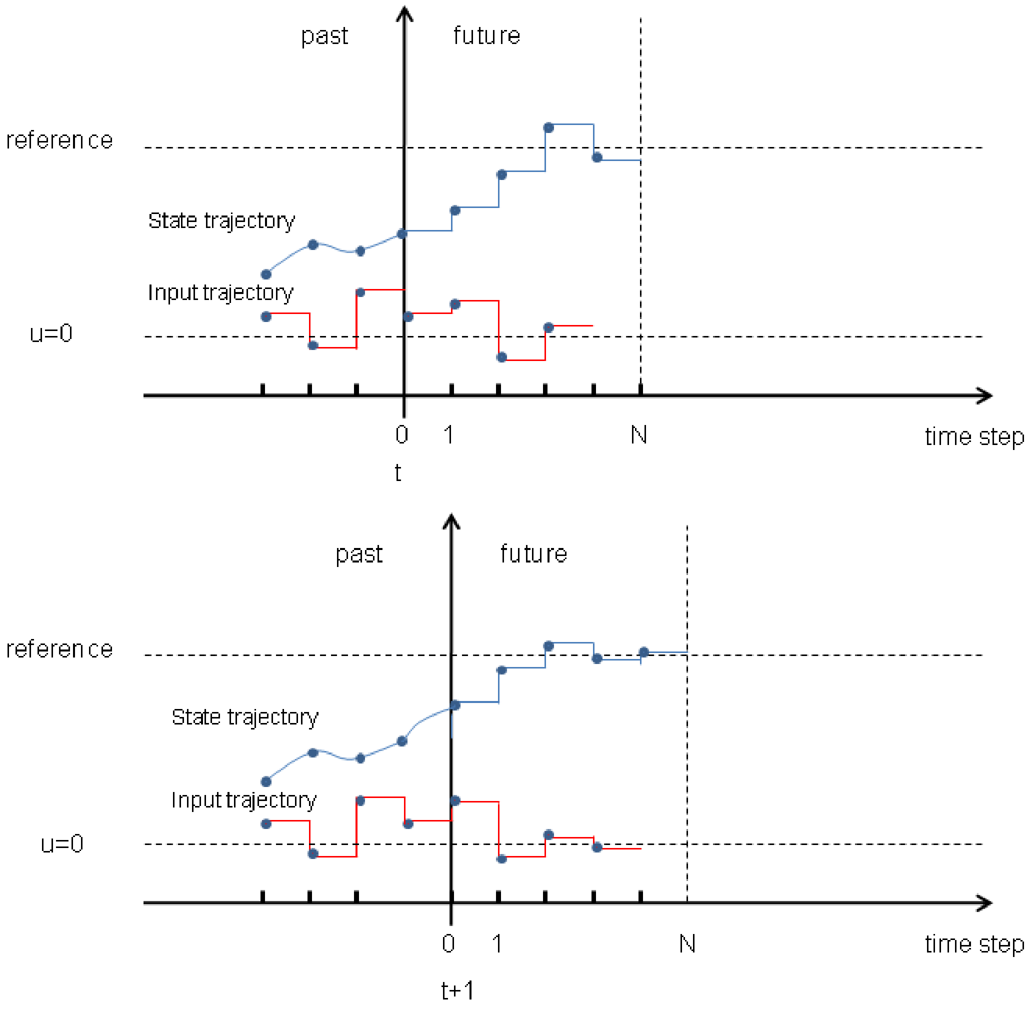

Figure 2.3: Illustration of the receding horizon idea

horizon idea is illustrated in Figure 2.3. Ideally, we would like to perform this optimization over an infinite horizon, but computational limitations necessitate the use of finite horizons (and call for efficient algorithms to minimize computational delays). This fundamental finite horizon requirement has important implications for the stability of the closed loop system under the receding horizon control law. The important link between the finite horizon optimal control problem and DP will play an important role both in the context of stability as well as in the context of computing the MPC control law. For further details, we refer the reader to [48].

### 2.2.1 Nominal MPC

We define the finite horizon optimal control problem $\mathbb{P}_0(x)$ for the class of non-linear discrete time systems in Section 2.1.3 as:

$$J_0^*(x) = \min_{\mathbf{u}} \{J_0(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_0(x)\}$$

and denote its solution as:

$$\begin{aligned}\mathbf{u}^*(x) &= \left\{u_0^*(x), u_1^*(x), \ldots, u_{N-1}^*(x)\right\} \\ &= \left\{\kappa_0^*(x), \kappa_1^*(x_1), \ldots, \kappa_{N-1}^*(x_{N-1})\right\}\end{aligned}$$

In MPC, the first element of this sequence is applied to the plant as $\kappa_0^*(x)$ for any $x \in \mathcal{X}_0$. We note that MPC does not require the determination of the control law $\kappa_0^*(\cdot)$ for all $x \in \mathcal{X}_0$, but only the local determination of the control sequence $\mathbf{u}^*(x)$ for a given $x \in \mathcal{X}_0$ (and this explains why MPC is computationally attractive - it requires only a local solution to a DP). In this section we consider system-theoretical properties of MPC. Therefore we now define positive invariant and control invariant sets which play a key role:

**Definition 2.7** *(a) A set $\mathcal{V} \subseteq \mathbb{R}^{n_x}$ is positive invariant for $x^+ = f(x)$ if $x \in \mathcal{V}$ implies $f(x) \in \mathcal{V}$*

*(b) A set $\mathcal{V} \subseteq \mathbb{R}^{n_x}$ is control invariant for $x^+ = f(x, u)$, $u \in \mathcal{U}$, if , $\forall x \in \mathcal{V} \quad \exists u \in \mathcal{U}$ such that $f(x, u) \in \mathcal{V}$*

Prior to discussing stability of the closed loop system under the MPC control law, we review fundamental concepts of discrete-time Lyapunov theory. We begin by defining $\mathcal{K}_\infty$ and positive-definite functions.

**Definition 2.8** *$\alpha$: $\mathbb{R} \rightarrow \mathbb{R}^+$ is a $\mathcal{K}_\infty$ function if it is continuous, strictly increasing, $\alpha(0) = 0$, and unbounded. A function is positive definite if it is continuous and positive everywhere except at the origin.*

For the system $x^+ = f(x)$, the origin is asymptotically stable with a region of attraction $\mathcal{X}_0$ if there exist a function $V$, a positive invariant set $\mathcal{X}_0$, two $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ and a positive definite function $\alpha_3(\cdot)$ satisfying:

$$\begin{aligned} \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \\ V(f(x)) \leq V(x) - \alpha_3(\|x\|) \end{aligned}$$

for all $x \in \mathcal{X}_0$. If these conditions are satisfied, we refer to $V$ as a 'Lyapunov function'. In MPC stability theory, the standard approach is to employ the optimal finite horizon value function $J_0^*(\cdot)$ as a Lyapunov function.

We now recall Assumptions 2.1 and 2.2 and make one further assumption required for establishing stability under the MPC control law.

**Assumption 2.5** $\min_{u \in \mathcal{U}} \{J_f(f(x,u)) + l(x,u) \mid f(x,u) \in \mathcal{X}_f\} \leq J_f(x) \quad \forall x \in \mathcal{X}_f$

which implies that $\mathcal{X}_f$ is control invariant. This is a crucial step which allows us to consider the objective function $J_0^*(\cdot)$ as a Lyapunov function. We state without proof the following properties of $J_0^*(\cdot)$ which we refer to as descent and monotonicity properties [48].

**Lemma 2.2** *Let Assumptions 2.1, 2.2, 2.5 be satisfied. Then the following descent property holds for $J_0^*(\cdot)$:*

$$J_0^*(f(x, \kappa_0^*(x))) \leq J_0^*(x) - l(x, \kappa_0^*(x)) \quad \forall x \in \mathcal{X}_0$$

**Lemma 2.3** *Under Assumptions 2.1, 2.2 and 2.5 the following monotonicity property holds for $J_0^*(\cdot)$:*

$$\begin{aligned} &J_k^*(x) \leq J_{k+1}^*(x) \quad \forall x \in \mathcal{X}_{k+1} \quad \forall k = 0, \dots, N-1 \\ &J_0^*(x) \leq J_f(x) \quad \forall x \in \mathcal{X}_f \end{aligned}$$

The descent property already hints towards employing $J_0^*(\cdot)$ as a Lyapunov function, but one last assumption is required ensuring that certain bounds on the stage and terminal cost are satisfied.

**Assumption 2.6** *(a) The stage cost $l(\cdot)$ and the terminal cost $J_f(\cdot)$ satisfy*

$$\begin{aligned} l(x,u) &\geq \alpha_1(\|x\|) \quad \forall x \in \mathcal{X}_0, \forall u \in \mathcal{U} \\ J_f(x) &\leq \alpha_2(\|x\|) \quad \forall x \in \mathcal{X}_f \end{aligned}$$

*where $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}_\infty$ functions. or*

*(b) The stage cost $l(\cdot)$ and the terminal cost $J_f(\cdot)$ satisfy*

$$\begin{aligned} l(x,u) &\geq c_1 \|x\|^a \quad \forall x \in \mathcal{X}_0, \forall u \in \mathcal{U} \\ J_f(x) &\leq c_2 \|x\|^a \quad \forall x \in \mathcal{X}_f \end{aligned}$$

*for some $c_1 > 0$ and $c_2 > 0$ and $a > 0$.*

Under the stated assumptions we achieve the following proposition indicating relevant properties of the optimal value function $J_0^*(x)$ for Lyapunov stability of the MPC controlled system [48]:

**Proposition 2.1** *(a) Under Assumptions 2.1, 2.2, 2.5 and 2.6a, there exist $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that $J_0^*(\cdot)$ has the following properties:*

$$\begin{aligned} J_0^*(x) &\geq \alpha_1(\|x\|) \quad \forall x \in \mathcal{X}_0 \\ J_0^*(x) &\leq \alpha_2(\|x\|) \quad \forall x \in \mathcal{X}_f \\ J_0^*(f(x, \kappa_0^*(x))) &\leq J_0^*(x) - \alpha_1(\|x\|) \quad \forall x \in \mathcal{X}_0 \end{aligned}$$

*where $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}_\infty$ functions. or*

*(b) Under Assumptions 2.1, 2.2, 2.5 and 2.6b, there exist positive constants $c_1$,*

*$c_2$ and $a$ such that $J_0^*(\cdot)$ has the following properties:*

$$\begin{aligned} J_0^*(x) &\geq c_1 \|x\|^a \quad \forall x \in \mathcal{X}_0 \\ J_0^*(x) &\leq c_2 \|x\|^a \quad \forall x \in \mathcal{X}_f \\ J_0^*(f(x, \kappa_0^*(x))) &\leq J_0^*(x) - c_1 \|x\|^a \quad \forall x \in \mathcal{X}_0 \end{aligned}$$

We note that these conditions are very similar to the standard discrete-time Lyapunov conditions (apart from the fact that the second condition only holds for all $x \in \mathcal{X}_f$). Nevertheless, the following important theorem providing asymptotic and exponential stability results of the MPC control law holds (where a detailed proof and more general results can be found in [48]):

**Theorem 2.6** *We assume that 2.1, 2.2, 2.5, 2.6a are satisified. Then the origin is asymptotically stable with a region of attraction $\mathcal{X}_0$ for the system $x^+ = f(x, \kappa_0^*(x))$ (N.B. this holds also for unbounded $\mathcal{X}_0$). If, in addition, Assumption 2.6b is satisfied and $\mathcal{X}_0$ is bounded, then the origin is exponentially stable with a region of attraction $\mathcal{X}_0$ for the system $x^+ = f(x, \kappa_0^*(x))$; if $\mathcal{X}_0$ is unbounded, then the origin is exponentially stable with a region of attraction that is any sublevel set of $J_0^*(\cdot)$.*

We now apply Theorem 2.6 to two important examples which will be used in the context of this thesis:

#### 2.2.1.1 Example 1: Nominal Linear MPC with Quadratic Stage Cost

Let the system be linear, i.e. $f(x, u) = Ax + Bu$ and let the stage cost be quadratic and positive definite, i.e. $l(x, u) = \frac{1}{2}(\|x\|_Q^2 + \|u\|_R^2)$, where $R \in \mathcal{S}_{++}^{n_u}$ and $Q \in \mathcal{S}_{++}^{n_x}$ [1]. Finally we assume that $\mathcal{X}$ and $\mathcal{U}$ are polyhedral and satisfy Assumption 2.2. We recall that provided $(A, B)$ is stabilizable the solution to the infinite horizon optimal control problem can be obtained from the solution to the

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[1] If $Q$ is merely positive semidefinite, then an additional assumption is required e.g. the detectability condition of Assumption 2.3

discrete algebraic Riccati equation as discussed in Section 2.1.3.2 and the infinite horizon optimal value function $J_{0,\infty}^{*,uc}(\cdot)$ then satisfies:

$$J_{0,\infty}^{*,uc}(x) = l(x, Kx) + J_{0,\infty}^{*,uc}((A+BK)x) = \frac{1}{2}\left\|x\right\|_P^2$$

for $P \in \mathcal{S}_{++}^{n_x}$ and $K = L$. Therefore if we choose the terminal cost $J_f(x)$ equal to $J_{0,\infty}^{*,uc}(x)$ we obtain:

$$J_f((A+BK)x) + \frac{1}{2}\left\|x\right\|_{Q+K^TPK}^2 - J_f(x) \leq 0 \quad \forall x \in \mathbb{R}^{n_x}$$

Further, we determine the set $\mathcal{X}_f$ such that it is the maximal invariant constraint admissible set for the system $x^+ = (A+BK)x$, i.e. it is the largest set $\mathcal{V}$ such that

(a) $\mathcal{V} \subseteq \{x \in \mathcal{X} \mid Kx \in \mathcal{U}\}$

(b) $x \in \mathcal{V}$ implies that $x_k = (A+BK)^k x \in \mathcal{V} \quad \forall k \geq 0$

**Remark 2.4** *The first requirement means that the linear state feedback $u = Kx$ is chosen such that the closed loop system satisfies both the input and state constraints. The second requirement is an invariance requirement. Both $\mathcal{U}$ and $\mathcal{X}$ contain the origin and therefore $u = Kx$ is feasible at the origin. Since both $\mathcal{U}$ and $\mathcal{X}$ contain the origin in their interior, there also exists a neighbourhood of the origin in which $u = Kx$ is feasible. The existence of an invariant set follows from the fact that the origin is necessarily an invariant set. Now, the hyperplanes relating to those constraints $\mathcal{U}$ and $\mathcal{X}$ which have to be satisfied at each future predicted time instant can be expressed under repeated application of the closed loop state transition matrix $A+BK$. Provided that $A+BK$ is stable, the distance of these hyperplanes from the origin increases for an increasing number of steps over which the closed loop state prediction is performed. This implies a finite 'constraint checking horizon' and, further, that the set $\mathcal{X}_f$ contains the origin* *in its interior. The details of this important result and an efficient numerical procedure for determining* $\mathcal{X}_f$ *based on Linear Programming are discussed in [23].*

The above choices for $J_f(\cdot)$, $\mathcal{X}_f$ and $l(\cdot)$ imply that Assumptions 2.5 and 2.6b and Assumption 2.1 are satisfied. Therefore provided the constraint sets satisfy Assumption 2.2, all the conditions of Theorem 2.6 are satisfied and exponential stability of the origin follows. We note that the region of attraction is $\mathcal{X}_0$, which includes $\mathcal{X}_f$ as a subset and therefore extends the region within which constraints can be handled beyond the maximal constraint admissible set for the system $x^+ = (A + BK)x$.

#### 2.2.1.2 Example 2: Nonlinear MPC with Quadratic Stage Cost

Now, we consider nonlinear systems given by

$$x^+ = f(x, u)$$

where $f(\cdot)$ is twice continuously differentiable. The constraints sets $\mathcal{X}$ and $\mathcal{U}$ are chosen as to satisfy Assumption 2.2 and the stage cost is taken as $l(x, u) = \frac{1}{2}(\|x\|_Q^2 + \|u\|_R^2)$, where $R \in \mathcal{S}_{++}^{n_u}$ and $Q \in \mathcal{S}_{++}^{n_x}$. To demonstrate stability using Theorem 2.6, we need to make choices about $J_f(\cdot)$ and $\mathcal{X}_f$. If Jacobian linearization of the system dynamics at the origin is employed, we obtain

$$x^+ = Ax + Bu$$

where $A = \nabla_x f(0, 0)$ and $B = \nabla_u f(0, 0)$ and $(A, B)$ is assumed stabilizable. We assume that the controller $u = Kx$ is chosen such that $A + BK$ is stable and therefore obtain the stage cost for the controlled linearized system as $l(x, Kx) = \frac{1}{2}\|x\|_{Q+K^T RK}^2$. When $P$ is chosen to satisfy the Lyapunov equation

$$(A + BK)^T P(A + BK) + 2(Q + K^T RK) = P$$

then necessarily $P \in \mathcal{S}^{n_x}_{++}$ holds, since $(A+BK)$ is stable and $Q \in \mathcal{S}^{n_x}_{++}$ [53]. The terminal cost is now chosen as

$$J_f(x) = \frac{1}{2} \|x\|_P^2$$

and it follows from this definition that $J_f(\cdot)$ satisfies:

$$J_f((A+BK)x) + \|x\|^2_{Q+K^TRK} - J_f(x) = 0 \quad \forall x \in \mathbb{R}^{n_x}$$

Instead of the linearized system at the origin we now consider the nonlinear system $x^+ = f(x,u)$ under the linear control $u = Kx$. The quadratic nature of the terminal cost indicates that a reasonable choice for $\mathcal{X}_f$ for the nonlinear system is to define $\mathcal{X}_f$ as a sublevel set of $J_f$. In particular, it is possible to show that there exists a positive scalar $\beta^2$ such that (for the nonlinear system)

$$J_f(f(x,Kx)) + \frac{1}{2}\|x\|^2_{Q+K^TRK} - J_f(x) \leq 0 \quad \forall x \in \left\{x \mid J_f(x) \leq \beta^2\right\} \tag{2.16}$$

holds [48]. Provided $\beta^2$ is chosen so that constraints are also satisfied, i.e. $\{x \mid J_f(x) \leq \beta^2\} \subseteq \mathcal{X}$ and $K\{x \mid J_f(x) \leq \beta^2\} \subseteq \mathcal{U}$, we can define $\mathcal{X}_f$ as:

$$\mathcal{X}_f = \left\{x \mid J_f(x) \leq \beta^2\right\}$$

Invariance of $\mathcal{X}_f$ can be established by considering any $x \in \mathcal{X}_f$. But (2.16) then implies $x^+ \in \mathcal{X}_f$, since $\frac{1}{2}\|x\|^2_{Q+K^TRK}$ is a positive definite function. This implies that Assumption 2.5 is satisfied. We can summarize that provided that $J_f(\cdot)$, $\mathcal{X}_f$ and $l(\cdot)$ are chosen as discussed and that Assumptions 2.1 and 2.2 are satisfied, then Assumptions 2.5 and 2.6b are satisfied and $\mathcal{X}_f$ contains the origin in its interior. Therefore Theorem 2.6 can be applied to establish asymptotic stability of the origin. The region of attraction is given by $\mathcal{X}_0 \supseteq \mathcal{X}_f$.

### 2.2.2 Robust MPC

First let us recall the definition of the finite horizon robust optimal control problem $\mathbb{P}_0(x)$:

$$J_0^*(x) = \min_{\boldsymbol{\kappa}} \left\{ \hat{J}_0(x, \boldsymbol{\kappa}) \mid \boldsymbol{\kappa} \in \mathcal{M}(x) \right\} \tag{2.17}$$

from Section 2.1.4 with the optimal solution obtained as a sequence of *feedback laws* $\boldsymbol{\kappa}^*(x) = \left\{ \kappa_0^*(x), \kappa_1^*(\cdot), \ldots, \kappa_{N-1}^*(\cdot) \right\}$. As in nominal MPC only the first element of this sequence is applied as an input to the plant, i.e. $\kappa_0^*(x)$ for any $x \in \mathcal{X}_0$, and then the optimization process is repeated at the next sampling instant. First, we define what we understand by robust control invariant and robust positive invariant sets.

**Definition 2.9** *A set $\mathcal{V} \subseteq \mathbb{R}^{n_x}$ is robust control invariant for $x^+ = f(x, u, w)$, $w \in \mathcal{W}$ if, for every $x \in \mathcal{V}$, there exists a $u \in \mathcal{U}$ such that $f(x, u, \mathcal{W}) \subseteq \mathcal{V}$.*

**Definition 2.10** *A set $\mathcal{V} \subseteq \mathbb{R}^{n_x}$ is robust positive invariant for $x^+ = f(x, w)$, $w \in \mathcal{W}$ if, for every $x \in \mathcal{V}$, $f(x, \mathcal{W}) \subseteq \mathcal{V}$.*

Similarly to the nominal stability Assumption 2.5 we now assume:

**Assumption 2.7**

$$\min_{u \in \mathcal{U}} \max_{w \in \mathcal{W}} \{ J_f(f(x, u, w)) - J_f(x) + l(x, u, w) \} \leq 0 \quad \forall x \in \mathcal{X}_f \subseteq \mathcal{X}$$

**Remark 2.5** *Assumption 2.7 can be interpreted as the fundamental stability assumption in robust MPC based on a min-max cost (and is in strong analogy to nominal MPC stability assumptions). In particular, it implies robust control invariance of $\mathcal{X}_f$. This implies the existence of some terminal controller $\kappa_f(x)$ such that $\mathcal{X}_f$ is feasible with respect to the constraint sets and robust positive invariant for the resulting closed loop dynamics. This assumption is key in the proofs of the results stated in Theorem 2.7 which involve a backwards induction procedure* *from the stability conditions inside the terminal set $\mathcal{X}_f$ and are recursively propagated to initial time 0 over the $N$-step horizon. The results of Theorem 2.7 are fundamental in obtaining a meaningful $l_2$-stability result of the proposed MPC controller based on an $\mathcal{H}_\infty$-cost.*

Since in this thesis we are mainly interested in the $\mathcal{H}_\infty$-optimal control problem, we make the following assumption on the stage cost and system dynamics and demonstrate that Assumption 2.7 is satisfied:

**Assumption 2.8** *Let $l(x,u,w) = \frac{1}{2}(\|u\|_R^2 + \|x\|_Q^2 - \gamma^2 \|w\|^2)$ and $f(x,u,w) = Ax + Bu + Dw$ and let the assumptions required in Section 2.1.4.3 for a unique min-max-solution for the unconstrained $\mathcal{H}_\infty$-optimal control problem hold.*

$J_f(x)$ is then the infinite horizon optimal cost for the unconstrained $\mathcal{H}_\infty$-optimal control problem of Section 2.1.4.3, i.e. $J_f(x) = \frac{1}{2}\|x\|_P^2$. By optimality we therefore obtain

$$J_f(x) = \max_w \{l(x, L^u x, w) + J_f(f(x, L^u x, w))\}$$

which therefore implies that

$$J_f(f(x, L^u x, w)) - J_f(x) + l(x, L^u x, w) \leq 0 \quad \forall x \in \mathbb{R}^{n_x}, \forall w \in \mathbb{R}^{n_w}$$

Provided that $\mathcal{X}_f$ is chosen to be a robust positive invariant set for $x^+ = (A + BL^u)x + Dw$, $w \in \mathcal{W}$, that satisfies $\mathcal{X}_f \subseteq \mathcal{X}$ and $L^u \mathcal{X}_f \subseteq \mathcal{U}$, it follows that Assumption 2.7 is satisfied.

**Remark 2.6** *Similar to the nominal case there exist numerically efficient ways for computing (in a finite number of steps) the set $\mathcal{X}_f$ for constrained linear systems in the presence of bounded disturbances [32]. We note that an important distinction is that the set $\mathcal{X}_f$ may be empty when the disturbance bounds given by the set $\mathcal{W}$ are too large.*

We can then state the following theorem about recursive feasibility of control policies (a detailed proof is given in [48]):

**Theorem 2.7** *Suppose Assumption 2.7 holds. Then*

*(a)* $\mathcal{X}_0 \supseteq \mathcal{X}_1 \supseteq \cdots \supseteq \mathcal{X}_N = \mathcal{X}_f$

*(b)* $\mathcal{X}_k$ *is robust control invariant for* $x^+ = f(x, u, w) \quad \forall k \in \{0, \cdots, N\}$

*(c)* $\mathcal{X}_k$ *is robust positive invariant for* $x^+ = f(x, \kappa_k^*(x), w) \quad \forall k \in \{0, \cdots, N\}$

*(d)* $J_k^*(x) \leq J_{k+1}^*(x) \quad \forall x \in \mathcal{X}_k \quad \forall k \in \{0, \cdots, N-1\}$

*(e)* $J_0^*(x) \leq J_f(x) \quad \forall x \in \mathcal{X}_f$

*(f)* $J_0^*(f(x, \kappa_0^*(x), w)) - J_0^*(x) + l(x, \kappa_0^*(x), w)$
$\leq J_0^*(f(x, \kappa_0^*(x), w)) - J_1^*(f(x, \kappa_0^*(x), w)) \leq 0 \quad \forall x \in \mathcal{X}_0, \forall w \in \mathcal{W}$

*(g) For any* $x \in \mathcal{X}_0$ $\left\{\kappa_0^*(x), \kappa_1^*(\cdot), \ldots, \kappa_{N-1}^*(\cdot)\right\}$ *is a feasible policy for problem* $\mathbb{P}_0(x)$ *and, for any* $x \in \mathcal{X}_1$, $\left\{\kappa_1^*(x), \ldots, \kappa_{N-1}^*(\cdot)\right\}$ *is a feasible policy for* $\mathbb{P}_1(x)$

In the presence of the disturbance it is not possible to employ classical nominal stability concepts to establish stability or asymptotic stability of the origin, because the optimal finite horizon cost merely satisfies the following conditions:

$$\begin{aligned}
&\alpha_1(\|x\|) \leq J_0^*(x) \leq \alpha_2(\|x\|) \\
&J_0^*(f(x, \kappa_0^*(x), w)) - J_0^*(x) \leq -l(x, \kappa_0^*(x), w) \leq -\alpha_1(\|x\|) + \frac{1}{2}\gamma^2 \|w\|^2
\end{aligned}$$

for all $x \in \mathcal{X}_0$ and $w \in \mathcal{W}$, if $\mathcal{X}_0$ is bounded. The existence of $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ follows from the fact that $\frac{1}{2}\|x\|_Q^2 \leq J_0^*(x) \leq \frac{1}{2}\|x\|_P^2$, where the RHS inequality is a result of the boundedness assumption on $\mathcal{X}_0$ so that the region of validity of this statement can be extended from $\mathcal{X}_f$ to $\mathcal{X}_0$ [48]. Nevertheless, the second inequality does prevent us from establishing asymptotic stability of the origin due to the extra term $\frac{1}{2}\gamma^2 \|w\|^2$. However, it is possible to establish a finite $l_2$-gain property of the closed-loop system as an alternative notion of stability of the closed-loop system. This is the concept of stability that will be employed in the context of this thesis. Theorem 2.7 implies that

$$J_0^*(f(x,\kappa_0^*(x),w)) - J_0^*(x) \leq -l(x,\kappa_0^*(x),w) \quad \forall x \in \mathcal{X}_0, \forall w \in \mathcal{W} \tag{2.18}$$

Now, we can take any $x_0 = x \in \mathcal{X}_0$ and obtain the corresponding infinite time closed loop state trajectory $\mathbf{x} = \{x_0, x_1, \cdots\}$ under robust receding horizon control $u_t = \kappa_0^*(x_t)$ for some realization of the disturbance $\mathbf{w} = \{w_0, w_1, \cdots\}$, where $w_t \in \mathcal{W}$ for all $t = 0, 1, \ldots$. Equation (2.18) implies for each $t$ that

$$J_0^*(f(x_t,\kappa_0^*(x_t),w_t)) - J_0^*(x_t) \leq -l(x_t,\kappa_0^*(x_t),w_t)$$

which can be summed over $t = 0, \cdots M-1$ and then results in:

$$0 \leq J_0^*(x_M) \leq J_0^*(x_0) - \sum_{t=0}^{M-1} l(x_t,\kappa_0^*(x_t),w_t)$$

for any integer $M > 0$. Now substituting $l(x,u,w) = \frac{1}{2}(\|u\|_R^2 + \|x\|_Q^2 - \gamma^2\|w\|^2)$, we obtain:

$$\sum_{t=0}^{M-1}(\|\kappa_0^*(x_t)\|_R^2 + \|x_t\|_Q^2) \leq \gamma^2 \sum_{t=0}^{M-1}\|w_t\|^2 + 2J_0^*(x_0)$$

for any integer $M > 0$. If $\mathbf{w} \in l_2$, i.e. $\sum_{t=0}^{\infty}\|w_t\|^2 < \infty$, then

$$\sum_{t=0}^{\infty}(\|\kappa_0^*(x_t)\|_R^2 + \|x_t\|_Q^2) \leq \gamma^2 \sum_{t=0}^{\infty}\|w_t\|^2 + 2J_0^*(x_0)$$

and the closed loop system has the stated finite $l_2$-gain property from disturbance $w$ to output $z = \begin{bmatrix} Q^{\frac{1}{2}}x \\ R^{\frac{1}{2}}u \end{bmatrix}$.

**Remark 2.7** *The $l_2$-gain bound provided by $2J_0^*(x)$ decreases for increasing horizon length. Although this may seem counterintuitive, this is a result of the monotonicity property of the optimal cost as stated in Theorem 2.7, i.e. $J_k^*(x) \leq J_{k+1}^*(x) \quad \forall x \in \mathcal{X}_k$. $J_k^*(x)$ and $J_{k+1}^*(x)$ correspond to finite horizon problems of* *length* $N-k$ *and* $N-k-1$ *respectively. This implies that a longer horizon length leads to a smaller optimal cost (for the same boundary conditions) and as a result we also obtain a reduced* $l_2$*-gain bound.*

## 2.3 Conclusion

In this chapter we have introduced the mathematical and system-theoretical methods that will be used throughout this thesis. Special focus is placed on embedding the results of the literature into the context of the research reported in the following chapters. In Chapter 3 we start by considering the nominal linear MPC problem and develop a computationally efficient active set solution.

# Chapter 3

# Nominal MPC based on DP

This chapter considers an efficient active set method for solving the nominal linear-quadratic MPC problem. As we noted in Chapter 2, MPC is essentially an efficient way of implementing the solution to a finite horizon DP for a given initial plant state (instead of computing the global DP solution) and it is this idea that is exploited in this chapter. Conventional approaches to linear MPC usually involve a 'condensing' procedure (e.g. [21]) in which the predicted state variables are eliminated from the problem of optimizing predicted performance of the given plant model. The optimization problem is then typically parameterized with the predicted input trajectories as optimization variables. The first step results in:

$$\mathbf{x} = N^x x_0 + N^u \mathbf{u} \tag{3.1}$$

where $\mathbf{x} = \{x_0, \ldots, x_N\}$ and $\mathbf{u} = \{u_0, \ldots, u_{N-1}\}$ are the predicted state and input sequences and $N^x$ and $N^u$ are defined by recursive use of the system model $x^+ = Ax + Bu$. Using Equation (3.1) the cost and constraints can then be expressed in terms of $x_0$ and $\mathbf{u}$. The receding horizon optimization solved online at each $x_0$ is then a quadratic programming problem in a number of variables ($\mathbf{u}$) that depends linearly on the horizon length. Solving this problem (using either active set or interior point methods) requires matrix factorizations with computational complexity that increases cubically with the horizon length. As a result the computational burden of constrained receding horizon control can be prohibitive for systems with fast sampling even when the plant model is linear.

Approaches based on interior point methods have taken advantage of the potential for efficient solutions by incorporating Riccati equation iterations in the solver [47]. This allows the linear equations in each interior point iteration to be solved very efficiently leading to a complexity (per iteration) that depends linearly on the horizon length. However, in contrast to the proposed active set method in this chapter, these methods do not exploit the geometric structure of the multiparametric QP and therefore suffer from the drawback that they cannot be warm-started. Furthermore, in practice, these strategies only provide a reduction in complexity compared to standard QP methods for large-scale problems involving hundreds of inputs and few constraints [18].

Parametric solution methods aim to avoid the online computational load by characterizing the solution of the receding horizon optimization problem offline, typically as a feedback law that is a piecewise affine function of the model state [2] defined over a polyhedral partition of the feasible set $\mathcal{X}_0$. However, whereas MPC typically solves an optimization problem for a given initial condition $x_0 \in \mathcal{X}_0$ at each time-step, this approach requires the solution at all points in $\mathcal{X}_0$. Essentially this 'Explicit' MPC approach solves the entire *global DP* discussed in Chapter 2 offline. The fundamental requirement of having to compute the entire polyhedral partition of $\mathcal{X}_0$ implies that the computational complexity of this offline procedure increases exponentially with the number of constraints (and therefore, for input and/or state constraints, the complexity depends exponentially on the horizon length). Moreover, the method relies on being able to efficiently determine online which of this potentially large number of polyhedral regions contains the current state $x_0$.

Although efficient point location techniques have been proposed for the case of piecewise linear cost functions (e.g. [28]), the method is generally applicable only to small problems and horizon lengths. We consider here an alternative, which avoids the offline global DP solution and instead uses an online active set approach [5, 18, 21]. This avoids the need to compute the solution over the entire state space, and it also forms the basis of an efficient line-search-based point location technique. The crucial idea is that the method only computes the parametric QP regions that are needed to perform a homotopy from some known local DP solution at some initial plant state $x_0$ to the local DP solution at the new measured plant state $x^p$.

Similar to [18] and [51], a Riccati approach is employed to solve the underlying equality constrained optimization problems at each iteration. In contrast to [18] and as in [51] this chapter considers the general setting including input and state constraints. In [51], however, all state constraints are imposed simultaneously implying that computational complexity increases cubically with respect to the total number of active state constraints and may therefore have a cubic complexity increase with horizon length in the worst case. We propose an alternative which uses the inherent structure of the DP and enforces state constraints by considering mixed state-input constraints at the preceding stage using the system model, and the overall complexity therefore grows only linearly with horizon length. Furthermore, this procedure allows to explicitly address the presence of degenerate subproblems at a given stage which can occur frequently e.g. for the class of non-minimum phase systems. As a second distinction we note that the active set solution of [51] requires a *feasible solution at the current (i.e. measured) plant state* $x^p$ and is then followed by a gradient descent, whereas our method is initialized with the *optimal solution at some given plant state* $x_0$ and then uses a homotopy approach to obtain the optimal solution at the current plant state $x^p$ (as in [21], [18]).

## 3.1 Problem Statement

In this chapter we consider linear discrete-time systems with model given by[1] (N.B. in the following development we use $t$ to denote the actual time index, while $k$ is the predicted time index):

$$x_{t+1} = Ax_t + Bu_t, \quad t = 0, 1, \ldots \tag{3.2}$$

with state $x_t \in \mathbb{R}^{n_x}$ and control input $u_t \in \mathbb{R}^{n_u}$. Both state and input are subject to constraints, i.e. $x_t \in \mathcal{X}$, $u_t \in \mathcal{U}$, where $\mathcal{X}$ is assumed to be a convex polyhedral set and $\mathcal{U}$ is assumed to be a convex polytopic set (each containing the origin).

We consider the finite horizon optimization problem encountered in linear MPC based on a quadratic stage cost by considering explicitly its DP formulation for $k = N-1, \ldots, 0$:

$$\kappa_k^*(x) = \arg\min_{u \in \mathcal{U}} J_k(x,u) \tag{3.3a}$$

subject to $Ax + Bu \in \mathcal{X}_{k+1}$ with $J_k$ defined by

$$J_k(x,u) = \frac{1}{2}\left(\|x\|_Q^2 + \|u\|_R^2\right) + J_{k+1}^*\left(x^+\right) \tag{3.3b}$$

where $x^+ = Ax + Bu$ and the $k$th-stage intermediate optimal cost function $J_k^*(x)$ is given by:

$$J_k^*\left(x\right) = J_k\left(x, \kappa_k^*(x)\right) \tag{3.3c}$$

---

[1] The results of the chapter apply to the linear time varying case without major modifications, but the time invariant case has been chosen for notational simplicity

and the controllable set recursion[1] is given by:

$$\mathcal{X}_k = \mathcal{X} \,\cap\, \{x | \, \exists u \in \mathcal{U} \text{ such that } Ax + Bu \in \mathcal{X}_{k+1}\} \tag{3.3d}$$

Furthermore, the terminal conditions are given by:

$$J_N^*(x) = \tfrac{1}{2}\|x\|_P^2, \tag{3.3e}$$

$$\mathcal{X}_N = \mathcal{X}_f \tag{3.3f}$$

where $R \in \mathcal{S}_{++}^{n_u}$ and $Q \in \mathcal{S}_{+}^{n_x}$.

As discussed in Section 2.1.3.2, $P$ is chosen to satisfy the steady-state matrix Riccati equation and $\kappa_\infty^*(x) = Kx$ is the corresponding optimal solution to (3.3a-c) in the limit as $N \to \infty$ in the absence of the constraints $x \in \mathcal{X}$, $u \in \mathcal{U}$ and for terminal condition $\lim_{N\to\infty} J_N^*(x) = 0$. Further, we assume that $\mathcal{X}_f$ is a convex polytopic set (containing the origin in its interior) which ensures feasibility and invariance under the optimal unconstrained feedback law (see Section 2.2.1.1). This implies that all the requirements of Section 2.2.1.1 for exponential stability of the origin are satisfied.

Since we are interested in efficient local solutions to the stated DP, we consider Problem (3.3a-f) for a given $x_0$ and denote the predicted sequences of optimal states and inputs as:

$$\mathbf{x}(x_0) = \{x_0, \ldots, x_N\}, \quad \mathbf{u}(x_0) = \{u_0, \ldots, u_{N-1}\} \tag{3.4}$$

where, for $k = 0, \ldots, N-1$ we define $u_k = \kappa_k^*(x_k)$ and $x_{k+1} = Ax_k + Bu_k$.

---

[1] Equation (3.3d) provides a convenient problem formulation, but for the purposes of implementation it is not necessary to determine $\mathcal{X}_k$ for $k < N$ explicitly since the constraint $Ax + Bu \in \mathcal{X}_{k+1}$ can be equivalently invoked via $Ax + Bu \in \mathcal{X}$ and $x_N \in \mathcal{X}_N$, where $x_N$ is the terminal state along the optimal predicted trajectory starting from state $x$ at prediction time $k$. We note that this simplification does not apply to the uncertain case considered in Chapter 5 (for which it is necessary to determine feasible sets analogous to (3.3d) for $k = N-1, ..., 1$), since in the uncertain case we constrain the one step ahead predicted state to lie in a set that is controllable to $\mathcal{X}_N$ in $N-k-1$ steps under all realizations of future uncertainty.

To differentiate between actual and predicted plant states, we denote the actual plant state at time $t$, which is assumed to be known, as $x_t^p$. A receding horizon control law is defined by implementing, at each time $t$, the input $u_t = \kappa_0^*(x_t^p)$.

## 3.2 Solution Outline and Optimality Conditions

This section outlines a method of solving Problem (3.3a-f) in order to determine $\kappa_0^*(x_0)$ for a given plant state $x_0 = x^p$: For a given active set, we use a Riccati recursion (as in [11]) to determine the solution to the Karush-Kuhn-Tucker (KKT) system for the corresponding equality constrained optimization problem. This provides the optimal sequences of inputs and multipliers as affine functions of $x_k$ which can be related to the solution of Problem (3.3) through the first-order necessary conditions for optimality (see e.g. [45]) and therefore define multiparametric QP (pQP) regions in $x_k$-space (within the feasible region $\mathcal{X}_k$).

For the given active constraint set, our approach determines the optimal state, input and multiplier sequences as functions of the initial state $x_0$ by forward simulation using the system model (3.2). This allows us to express the pQP regions in $x_0$-space (and hence the set of feasible plant states $\mathcal{X}_0$ is partitioned into polyhedral regions). As in [18], we employ a line-search through $x_0$-space, starting from the optimal solution which is assumed to be known at a given initial condition, and successively updating the active set as a function of $x_0$ until $x_0 = x^p$.

Let the polytopic sets $\mathcal{X}_{k+1}$ and $\mathcal{U}$ be defined by:

$$\mathcal{X}_{k+1} = \{x \in \mathbb{R}^{n_x} \mid E_k x \le \mathbf{1}\} \text{ for } k = 0, \ldots, N-1 \tag{3.5}$$

$$\mathcal{U} = \{u \in \mathbb{R}^{n_u} \mid Fu \le \mathbf{1}\} \tag{3.6}$$

for $E_k \in \mathbb{R}^{n_{E_k} \times n_x}$, $F \in \mathbb{R}^{n_F \times n_u}$, $G \in \mathbb{R}^{n_G \times n_w}$, and where $\mathbf{1} = [1 \ \cdots \ 1]^T$ denotes a vector of conformal dimensions. Also let $\lambda_k$ denote the Lagrange multipliers associated with the equality constraints $x_{k+1} = Ax_k + Bu_k$, and let $\nu_k$ and $\mu_k$ denote the Lagrange multipliers for inequality constraints associated with the conditions $Ax_k + Bu_k \in \mathcal{X}_{k+1}$ and $u_k \in \mathcal{U}$ for $k = 0, \ldots, N-1$.

The minimization subproblems (3.3a) at time-step $k$ may be degenerate due to linear dependence in the active constraint set, and for $k > 0$ this results in compatibility conditions consisting of equality constraints in the preceding stage of the optimization (i.e. at prediction time-step $k-1$). Accordingly, we introduce equality constraints into the minimization subproblems at time-step $k$ as $C_k x_{k+1} = \mathbf{1}$ ($C_k \in \mathbb{R}^{n_{d_{k+1}} \times n_x}$) in order to account for possible degeneracy of the minimization subproblem at stage $k+1$ and we denote the associated Lagrange multipliers as $\zeta_k$. This leads to the following theorem:

**Theorem 3.1** *The KKT conditions defining first-order necessary conditions for the optimal solution of Problem (3.3) with equality constraints $C_k x_{k+1} = \mathbf{1}$ for $k = 0, ..., N-2$ can be expressed as follows:*
*for $k = 0, \ldots, N-1$:*

$$x_{k+1} = Ax_k + Bu_k \tag{3.7}$$

*for $k = 0, \ldots, N-2$*

$$\lambda_k = A^T \lambda_{k+1} + Qx_{k+1} + C_k^T \zeta_k + E_k^T \nu_k \tag{3.8}$$

$$C_k x_{k+1} = \mathbf{1} \tag{3.9}$$

*for $k = 0, \ldots, N-1$:*

$$Ru_k + F^T \mu_k + B^T \lambda_k = 0 \tag{3.10a}$$

$$\nu_k \geq 0, \quad \nu_k^T (\mathbf{1} - E_k x_{k+1}) = 0, \quad \mathbf{1} - E_k x_{k+1} \geq 0 \tag{3.10b}$$

$$\mu_k \geq 0, \quad \mu_k^T (\mathbf{1} - Fu_k) = 0, \quad \mathbf{1} - Fu_k \geq 0 \tag{3.10c}$$

*with initial and terminal conditions:*

$$x_0 = x^p \tag{3.11}$$

$$\lambda_{N-1} = Px_N + E_{N-1}^T \nu_{N-1}. \tag{3.12}$$

*Proof:* We first define the Lagrangian for stage $k$ of Problem (3.3) subject to $C_{k+1}x_{k+1} = \mathbf{1}$ (and drop the index $k$ for notational simplicity):

$$\begin{aligned} L(x, u, \lambda, \mu, \nu, \zeta, x^+) = \frac{1}{2}\|x\|_Q^2 + \frac{1}{2}\|u\|_R^2 - \lambda^T(x^+ - Ax - Bu) \\ - \mu^T(\mathbf{1} - Fu) - \nu^T(\mathbf{1} - Ex^+) - \zeta^T(\mathbf{1} - Cx^+) + J^{*+}(x^+) \end{aligned}$$

The first order optimality conditions are:

$$\begin{aligned} \nabla_u L &= Ru + F^T\mu + B^T\lambda = 0 \\ \nabla_{x^+} L &= -\lambda + E^T\nu + C^T\zeta + \nabla_{x^+} J^{*+}(x^+) = 0 \\ \nabla_\lambda L &= x^+ - Ax - Bu = 0 \\ \nabla_\zeta L &= \mathbf{1} - Cx^+ = 0 \end{aligned}$$

These imply (3.7), (3.9) and (3.10a). Further, we obtain the following complementarity, primal feasibility and dual feasibility conditions on the inequality constraints from standard results in constrained optimization [10]:

$$\begin{aligned} &\nu^T(\mathbf{1} - Ex^+) = 0 \quad && \mathbf{1} - Ex^+ \geq 0 \quad && \mu \geq 0 \\ &\mu^T(\mathbf{1} - Fu) = 0 \quad && \mathbf{1} - Fu \geq 0 \quad && \nu \geq 0 \end{aligned}$$

These imply conditions (3.10b),(3.10c).

From the definition of $L$, and (3.7), (3.9), and the complementarity conditions above, we have $\nabla_x J^*(x) = \nabla_x L(x, u, \lambda, \mu, \nu, \zeta, x^+)$, and since the gradients of $L$ with respect to $u$, $\lambda$, $\zeta$ and $x^+$ are each zero by optimality, this gives $\nabla_x J^*(x) =$

$Qx + A^T\lambda$. At any stage $k < N-1$, this argument applied to the successor stage gives $\nabla_{x^+} J^{*+}(x^+) = Qx^+ + A^T\lambda^+$, and hence

$$\nabla_{x^+} L = -\lambda + E^T\nu + C^T\zeta + Qx^+ + A^T\lambda^+ = 0$$

which implies (3.8).

At stage $k = N-1$ we have $\nabla_{x^+} J^{*+}(x^+) = Px_N$, which directly implies (3.12). ■

## 3.3 Solution to Equality Constrained Problem via Riccati Recursion

An active set approach solves the optimization Problem (3.3) by solving a sequence of equality constrained problems. Let $\mathcal{A} = \{\mathcal{A}_0, \ldots, \mathcal{A}_{N-1}\}$ define a set of active constraints in (3.3), namely a set of constraints that are satisfied with equality at a solution of (3.3) for some initial state $x_0$.

For convenience we redefine $E_k$ and define $F_k$ such that they denote the matrices consisting of the rows of $E_k$, $F$ corresponding to active constraints at time $k$, and let $\nu_{a,k}$, $\mu_{a,k}$ denote the vectors of active constraint multipliers so that

$$Ru_k + F_k^T\mu_{a,k} + B^T\lambda_k = 0 \tag{3.13a}$$

$$E_k x_{k+1} = \mathbf{1} \tag{3.13b}$$

$$F_k u_k = \mathbf{1}. \tag{3.13c}$$

Consider first the equality constrained maximization subproblem at stage $k$ for the case that the equality constrained minimization subproblem at $k+1$ is not degenerate. In order to solve this EP for given $\mathcal{A}$ using a Riccati recursion, we first express the costate variables as

$$\lambda_k = P_k x_{k+1} + q_k + E_k^T \nu_{a,k}. \tag{3.14}$$

From (3.7) and (3.13) we therefore obtain

$$\begin{bmatrix} R + B^T P_k B & B^T E_k^T & F_k^T \\ E_k B & 0 & 0 \\ F_k & 0 & 0 \end{bmatrix} \begin{bmatrix} u_k \\ \nu_{a,k} \\ \mu_{a,k} \end{bmatrix} = - \begin{bmatrix} B^T P_k \\ E_k \\ 0 \end{bmatrix} A x_k + \begin{bmatrix} -B^T q_k \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}. \tag{3.15}$$

Suppose now that the matrix $[B^T E_k^T \quad F_k^T]$ is rank-deficient of order $d_k$ (so that $rank[B^T E_k^T \quad F_k^T] = n_u - d_k$). Then the general solution of (3.15) is given by

$$\begin{bmatrix} u_k \\ \nu_{a,k} \\ \mu_{a,k} \end{bmatrix} = \begin{bmatrix} L_k^u \\ L_k^\nu \\ L_k^\mu \end{bmatrix} x_k + \begin{bmatrix} l_k^u \\ l_k^\nu \\ l_k^\mu \end{bmatrix} + \begin{bmatrix} 0 \\ Z_{1,k} \\ Z_{2,k} \end{bmatrix} \beta_k \tag{3.16}$$

where $\beta_k \in \mathbb{R}^{d_k}$ contains the free variables in the solution of (3.15) and $Z_k = [Z_{1,k}^T \quad Z_{2,k}^T]^T$ is the full-rank matrix satisfying

$$Z_k^T \begin{bmatrix} E_k B \\ F_k \end{bmatrix} = 0, \quad Z_k^T \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \mathbf{1}.$$

In this degenerate case, (3.15) admits solutions if and only if $x_k$ satisfies the compatibility condition

$$C_{k-1} x_k = \mathbf{1}, \text{ where } C_{k-1} = Z_{1,k}^T E_k A,$$

implying an equality constraint in the minimization at time-step $k-1$ with associated multiplier $\zeta_{k-1} \in \mathbb{R}^{d_k}$.[1] From (3.8) therefore, $\lambda_{k-1} = P_{k-1} x_k + q_{k-1} +$

---

[1] One may see this by performing Gaussian elimination steps on the linear system. In the degenerate case, this will lead to zero rows on the left hand side. The corresponding expressions

$E_{k-1}^T\nu_{a,k-1} + C_{k-1}^T(\beta_k + \zeta_{k-1})$.

Using (3.7) and (3.13) once more, we obtain

$$\begin{bmatrix} R+B^TP_{k-1}B & B^TC_{k-1}^T & B^TE_{k-1}^T & F_{k-1}^T \\ C_{k-1}B & 0 & 0 & 0 \\ E_{k-1}B & 0 & 0 & 0 \\ F_{k-1} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{k-1} \\ \beta_k+\zeta_{k-1} \\ \mu_{a,k-1} \\ \nu_{a,k-1} \end{bmatrix} = - \begin{bmatrix} B^TP_{k-1} \\ C_{k-1} \\ E_{k-1} \\ 0 \end{bmatrix} Ax_{k-1} + \begin{bmatrix} -B^Tq_{k-1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} \tag{3.17}$$

with general solution (assuming $[B^TC_{k-1}^T \; B^TE_{k-1}^T \; F_{k-1}^T]$ is rank-deficient of order $d_{k-1}$):

$$\begin{bmatrix} u_{k-1} \\ \beta_k+\zeta_{k-1} \\ \nu_{a,k-1} \\ \mu_{a,k-1} \end{bmatrix} = \begin{bmatrix} L_{k-1}^u \\ L_{k-1}^{\beta+\zeta} \\ L_{k-1}^{\nu} \\ L_{k-1}^{\mu} \end{bmatrix} x_{k-1} + \begin{bmatrix} l_{k-1}^u \\ l_{k-1}^{\beta+\zeta} \\ l_{k-1}^{\nu} \\ l_{k-1}^{\mu} \end{bmatrix} + \begin{bmatrix} 0 \\ Z_{1,k-1} \\ Z_{2,k-1} \\ Z_{3,k-1} \end{bmatrix} \beta_{k-1}. \tag{3.18}$$

where $\beta_{k-1} \in \mathbb{R}^{d_{k-1}}$, and $Z_{k-1} = [Z_{1,k-1}^T \; Z_{2,k-1}^T \; Z_{3,k-1}^T]^T$ is the full-rank matrix satisfying

$$Z_{k-1}^T \begin{bmatrix} C_{k-1}B \\ E_{k-1}B \\ F_{k-1} \end{bmatrix} = 0, \qquad Z_{k-1}^T \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \mathbf{1},$$

and compatibility of (3.17) requires that

$$C_{k-2}x_{k-1} = \mathbf{1}, \quad C_{k-2} = \begin{bmatrix} Z_{1,k-1}^T & Z_{2,k-2}^T \end{bmatrix} \begin{bmatrix} C_{k-1} \\ E_{k-1} \end{bmatrix} A.$$

---

on the right hand side (which are affine in the state $x_k$) have to equal zero and are considered as compatibility conditions.

We now employ equation (3.8) which, for the given active set, yields

$$\lambda_{k-2} = P_{k-2}x_{k-1} + q_{k-2} + E_{k-1}^T \nu_{a,k-1} + C_{k-2}^T(\zeta_{k-2} + \beta_{k-1}) \tag{3.19}$$

and therefore the cost recursion:

$$\begin{aligned}\begin{bmatrix} P_{k-2} & q_{k-2} \end{bmatrix} &= \begin{bmatrix} Q + A^T P_{k-1} A & A^T q_{k-1} \end{bmatrix} \\ &\quad + A^T \begin{bmatrix} P_{k-1}B & C_{k-1}^T & E_{k-1}^T \end{bmatrix} \begin{bmatrix} L_{k-1}^u & l_{k-1}^u \\ L_{k-1}^{\beta+\zeta} & l_{k-1}^{\beta+\zeta} \\ L_{k-1}^{\nu} & l_{k-1}^{\nu} \end{bmatrix}.\end{aligned} \tag{3.20}$$

Finally, since (3.12) has the form of (3.14), with $P_{N-1} = P$, $q_{N-1} = 0$, it follows by induction that the general solution to the KKT conditions (3.7-3.12) is given by (3.18) for $k = 0, \ldots, N-1$.

## 3.4 Active Set Method

The following lemma gives conditions for optimality of the Riccati recursion (3.18, 3.20). We note that due to the strict convexity of Problem (3.3) the KKT conditions admit a unique solution for a given $x_0$.

**Remark 3.1** *We note that the choice of $\zeta$ does not influence the EP primal solution. We obtain $\zeta + \gamma$ as a state feedback law, which can be employed to eliminate the degrees of freedom that result from degenerate constraints at later stages on the horizon (where $\zeta$ appears as an intermediate variable). Finally, we note the possibility of degrees of freedom in the solution at stage 0 and may be interpreted as true degrees of freedom in the dual solution of Problem (3.3). The primal solution, on the other hand, is always unique due to the strict convexity of Problem (3.3). We crucially note that the choice $\zeta = 0$ ensures continuity of the dual solution at QP region boundaries (while the primal solution is always* *continuous). Any element of the vector $\beta$ is equal to zero for a multiplier of any new degenerate constraint that has been added to the constraint set at the given region boundary. This fact allows us to formulate an active set algorithm based on homotopy of solutions.*

**Lemma 3.1** *The optimal solution to Problem (3.3a-f) is given by*

$$u_k^*(x_k) = \kappa_k^*(x_k) = L_k^u x_k + l_k^u \tag{3.21}$$

*if and only if the first-order necessary (KKT) conditions*

$$F(L_k^u x_k + l_k^u) \leq 1, \quad E_k x_{k+1} \leq 1 \tag{3.22a}$$

$$L_k^\nu x_k + l_k^\nu + Z_{2,k}\beta_k \geq 0, \quad L_k^\mu x_k + l_k^\mu + Z_{3,k}\beta_k \geq 0 \tag{3.22b}$$

*and the compatibility conditions (satisfied with $\zeta_k = 0$)*

$$\beta_{k+1} = L_k^{\beta+\zeta} x_k + l_k^{\beta+\zeta} + Z_{1,k}\beta_k \tag{3.23}$$

*and the compatibility condition (for a degenerate subproblem at stage $k = 0$)*

$$Z_{1,0}^T E_0 A x_0 = \mathbf{1} \tag{3.24}$$

*hold for a given $x_0 = x^p$.*

*Proof:* This follows from Theorem 3.1 and the discussion of Section 3.3 which yielded explicit feedback solutions for a given active set at a given state $x_0$. ■

The extra compatibility conditions (3.23) and (3.24) are required so that the feedback solutions given by (3.21) are consistent solutions to the sequential linear KKT systems. This demonstrates the fact that in the MPC problem formulation we solve the underlying finite horizon DP locally and obtain the expressions for the optimal feedback solution sequence when these are evaluated locally along the predicted state sequence.

By using the feedback law (3.21) in conjunction with (3.7) to simulate forward over the prediction horizon, we obtain:

$$x_k = \Phi_k x_0 + \phi_k \quad \text{for } k = 1, \dots, N \tag{3.25}$$

where $\Phi_k \in \mathbb{R}^{n_x \times n_x}$ and $\phi_k \in \mathbb{R}^{n_x}$ are defined by

$$\Phi_{k+1} = (A + BL_k^u)\Phi_k \tag{3.26a}$$

$$\phi_{k+1} = (A + BL_k^u)\phi_k + Bl_k^u \tag{3.26b}$$

with initial values $\Phi_0 = I$ and $\phi_0 = 0$.

Therefore the input sequence $\mathbf{u}(x_0)$ can be determined as an affine function of $x_0$ by substituting (3.25) into (3.18). The multiplier sequences $\boldsymbol{\mu}(x_0, \beta_0) = \{\mu_0, \dots, \mu_{N-1}\}$, $\boldsymbol{\nu}(x_0, \beta_0) = \{\nu_0, \dots, \nu_{N-1}\}$ can be obtained as affine functions of $x_0$ and $\beta_0$ by removing the degrees of freedom $\beta_k$ at stages $k > 0$ via compatibility conditions (3.23).

**Remark 3.2** *We note that an active set containing linearly dependent constraints at any stage $k$ may still lead to a full-dimensional pQP region in $x_0$-space. However, if the subproblem at $k = 0$ is degenerate the required compatibility conditions on $x_0$ define a lower dimensional subspace of $x_0$-space and we therefore obtain lower dimensional pQP regions. We note that Problem (3.3) can be reformulated as a single convex pQP with parameter $x_0$, the primal solution of which is a continuous, piecewise affine function of $x_0$. The possibility of non-concavity of the dual problem is reflected by the non-uniqueness of the Lagrange multipliers in lower dimensional pQP regions [2].*

Hence, for a given active set $\mathcal{A}$, we can define a region of the space $\mathcal{X}_{pQP}(\mathcal{A}) \subset \mathbb{R}^{n_x}$ in which the KKT conditions hold for given $\beta_0$:

$$\mathcal{X}_{pQP}(\mathcal{A}) = \left\{ x_0 \mid \mathbf{x}(x_0) \text{ satisfies } (3.22 \text{ a,b}, 3.23, 3.24) \text{ for given } \beta_0 \right\}$$

**Lemma 3.2** *The sets $\mathcal{X}_{pQP}(\mathcal{A})$ are convex polyhedra in $\mathbb{R}^{n_x - d_0}$ where $d_0$ is the degree of freedom in the subproblem at $k = 0$. The union $\bigcup_{\mathcal{A} \in \Sigma} \mathcal{X}_{pQP}(\mathcal{A})$ of all admissible active sets forms a partition of $\mathcal{X}_0$ (i.e. the set of feasible initial conditions for Problem (3.3)).*

*Proof:* This follows from the fact that Problem (3.3) is a convex pQP. ■

The contribution of [34] considers the nominal linear quadratic MPC problem in the explicit/multiparametric context and applies dynamic programming to a sequence of smaller (essentially condensed) subproblems, in which each subproblem is solved via convex multi-parametric QP methods. The paper claims that this avoids overlapping of pQP regions which may arise as a result of the piecewise quadratic nature of the cost-to-go in the dynamic programming formulation of Problem (3.3). We remark that the possibility of condensing each $k^{th}$-stage DP subproblem (with $x_k$ as the parameter) shows that each of these subproblems is a convex pQP and the possibility of overlapping can therefore be excluded. This argument similarly applies to our proposed method, but we note that in both cases one may have the occurence of lower dimensional regions contained within hyperplanes at boundaries of pQP regions in $x_k$ space, as discussed in Remark 3.2.

**Definition 3.1** *Suppose two feedback solutions to the dynamic programming formulation of the finite horizon optimal control Problem (3.3) are evaluated at initial states $x_0^a$ and $x_0^b$ as $\boldsymbol{\kappa}^*(x_0^a) \in \mathcal{U}^N$ and $\boldsymbol{\kappa}^*(x_0^b) \in \mathcal{U}^N$ and that a new initial state is defined by $x = x_0^a + \alpha(x_0^b - x_0^a)$ for $\alpha \in [0,1]$. Then a homotopy is defined as the continuous mapping $H : \mathcal{U}^N \times [0,1] \to \mathcal{U}^N$ such that $H(x_0^a, 0) = \boldsymbol{\kappa}^*(x_0^a)$ and $H(x_0^a, 1) = \boldsymbol{\kappa}^*(x_0^b)$ for $\alpha \in [0,1]$.*

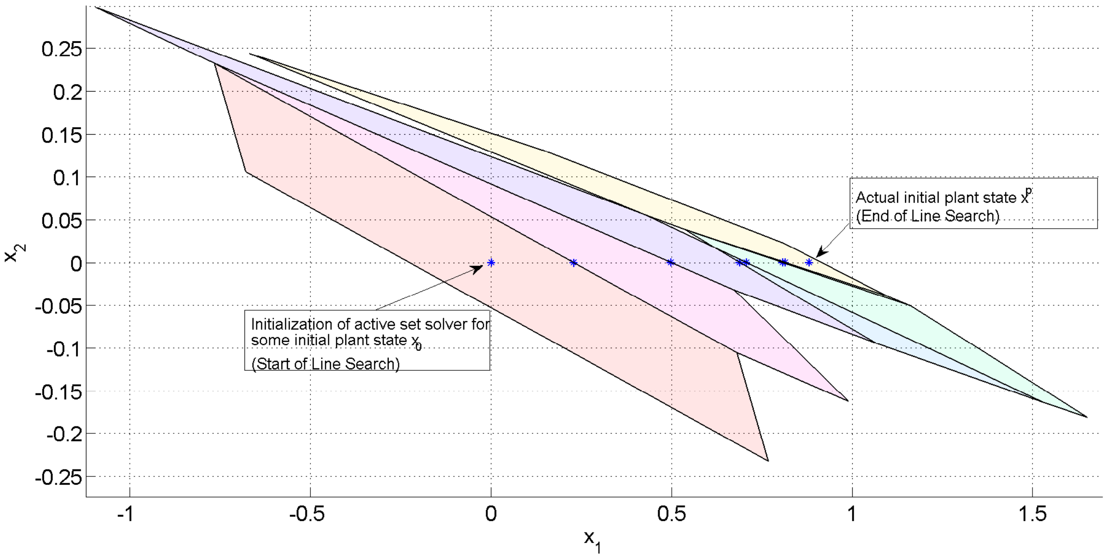

Figure 3.1: Illustration of active set method based on homotopy of optimal solutions (between the unconstrained solution at the origin and a constrained solution at the actual plant state $x^p$)

We note that if both $x_0^a$ and $x_0^b$ are inside the same the QP region, the same sequence of feedback functions is optimal and is linearly blended according to the choice of $\alpha$. If $x_0^a$ and $x_0^b$ are in different QP regions, the active set is updated along the line of search. We crucially employ continuity results (of inputs and states) of the solution to Problem (3.3) in the update procedure of the proposed active set algorithm below.

The algorithm we propose solves Problem (3.3) by solving the equality constrained problem for an estimate of the optimal active set, and then updates this active set at successive iterations. At each iteration $i = 0, 1, \ldots$ the algorithm determines $\mathcal{A}^{(i+1)}$ from $\mathcal{A}^{(i)}$ by performing a line search over $x_0 \in \mathcal{X}_{pQP}(\mathcal{A}^{(i)})$ in the direction of the current plant state $x^p$ for the case that there is no degenerate subproblem at stage $k = 0$. *In each of these steps constraints are added to the active constraint set if primal conditions (3.22a) become satisfied with equality and constraints are removed from the active constraint set if Lagrange multipliers (3.22b) for previously active constraints become zero.*

For degenerate subproblems at stage $k = 0$ Remark 3.2 implies that degenerate pQP regions are embedded in hyperplanes in $\mathbb{R}^{n_x}$ (i.e. $\beta_0 \in \mathbb{R}$). Then a search in both directions along the kernel of the KKT matrix returns either the previous active set or the next active set corresponding to neighbouring full dimensional regions. We make the following assumption on the homotopy path defined by the algorithm:

**Assumption 3.1** *For each $\mathcal{A} \in \Sigma$, the line segment $\{x_0 \mid x_0 = x_0^{(0)} + \alpha(x^p - x_0^{(0)}), \alpha \in [0,1]\}$ intersects the boundary $\partial\mathcal{X}_{pQP}(\mathcal{A})$ at most at two discrete points.*

This assumption excludes the possibility that more than one constraint can become active or inactive simulaneously, and is a common assumption for active set solvers (e.g. [22]). We note that if Assumption 3.1 is violated as a result of the path for $x_0$ intersecting the boundary of a pQP region along a line segment of non-zero length, we can perform an infinitesimal perturbation of $x_0^{(0)}$ and detect active set changes as before. Further, Assumption 3.1 clearly does not imply that the active sets of neighbouring pQP regions differ by only a single constraint - this is expected to occur as indicated by Remark 3.2.

Overall, the algorithm results in a sequence of dual-feasible iterates $x_0^{(i)}$ that generate trajectories satisfying (3.22, 3.23, 3.24) but not necessarily (3.11). This is illustrated in Figure 3.1, which shows a line search that is initialized at the origin of state space, where the unconstrained optimal solution is feasible. Then the search takes place in the direction of the plant state and whenever the boundary of a given pQP region is reached, the active set is updated with a new EP solution and a new pQP region. This procedure continues until the current plant state is located, since then the homotopy is completed and the optimal constrained solution has been found. We note that homotopies can be performed starting from any known local DP solution for some given corresponding plant state and this can be exploited to warm start the algorithm.

**Algorithm 3.1** *We set $i = 0$ and initialize the algorithm with $x_0^{(0)}$ and a corresponding optimal active set $\mathcal{A}^{(0)}$. This means, we require that $x_0^{(0)} \in \mathcal{X}_{pQP}(\mathcal{A}^{(0)})$. (we consider case (a): no degenerate constraints and case (b): degenerate constraints))*

*At iteration $i = 0, 1, \dots$:*

*(i) Compute $\{P_k, q_k\}$ for $k = N-1, \dots, 0$, and $\{\Phi_k, \phi_k\}$ for $k = 0, \dots, N-1$, and hence $\mathcal{X}_{pQP}(\mathcal{A}^{(i)})$.*

*(ii) (a) Perform the line search:*
$\alpha^{(i)} = \max_{\alpha \in (0,1]} \left\{ \alpha \mid x_0^{(i)} + \alpha(x^p - x_0^{(i)}) \in \mathcal{X}_{pQP}(\mathcal{A}^{(i)}) \right\}$.
*(b) If the EP subproblem (3.13) at stage $k = 0$ is degenerate, then the multipliers are determined by the scalar $\beta_0$ and the line search is performed along the kernel direction:*

$$\beta_{0,+}^{(i)} = \max_{\beta_0 \in \mathbb{R}} \{\beta_0 \mid \beta_0 Z_0 \in \mathcal{X}_{pQP}(\mathcal{A}^{(i)})\}.$$
$$\beta_{0,-}^{(i)} = \max_{\beta_0 \in \mathbb{R}} \{\beta_0 \mid -\beta_0 Z_0 \in \mathcal{X}_{pQP}(\mathcal{A}^{(i)})\}.$$

*$\beta_0^{(i)}$ is selected corresponding to the active set which is different from the previous active set.*

*(iii) (a) If $\alpha^{(i)} < 1$, then set $x_0^{(i+1)} := x_0^{(i)} + \alpha^{(i)}(x^p - x_0^{(i)})$, $i := i + 1$, and update $\mathcal{A}^{(i)}$ on the basis of the new set of active constraints. Return to step (i).*
*(b) Set $x_0^{(i+1)} := x_0^{(i)}$, $i := i + 1$, and update $\mathcal{A}^{(i)}$ on the basis of the new set of active constraints. Return to step (i).*
*If $\alpha^{(i)} = 1$ and $x_0^{(i)}$ lies in $\mathcal{X}_{pQP}(\mathcal{A}^{(i)})$ set $\mathcal{A}^* := \mathcal{A}^{(i)}$, compute $\kappa_0^*(x^p)$ and stop.*

**Remark 3.3** *We note that after backwards sweep and forward simulation, all KKT conditions are parametrized in terms of the homotopy parameter $\alpha$ and it is merely necessary to check a list of scalar inequalites to update the active set accordingly with negligible computational effort as compared to the linear algebra operations required in the EP solution*

**Theorem 3.2** *Algorithm 3.1 converges to* $\mathcal{A}^*$ *such that Problem (3.3) is minimized by the trajectories for* $\mathbf{x}$, $\mathbf{u}$ *that are generated by (3.7) and (3.21) with* $\mathcal{A} = \mathcal{A}^*$.

*Proof:* Lemma 3.2 implies that each iterate $x_0^{(i)}$ lies on the line segment $x_0^{(i+1)} := x_0^{(i)} + \alpha^{(i)}(x^p - x_0^{(i)})$, either at an intersection of the line with the boundary $\partial\mathcal{X}_{pQP}(\mathcal{A}^{(i)})$ or at $x_0^{(i)} = x^p$. For a non-degenerate subproblem at $k = 0$ for a given plant state $x_0 = x^p$, primal and dual variables are continuous in $x_0$ and the active set change at region boundaries is unique. For degenerate subproblems at $k = 0$ with $\beta_0 \in \mathbb{R}$ two entries of the active set can change at a given $x_0$ with a unique active set change as determined by step (ii)b in Algorithm 3.1. Therefore all non-degenerate steps are of finite length (with the possibility of an intermediate degenerate step at region boundaries) and the sequence $\{\alpha^{(i)},\ i = 0, 1, \ldots\}$ must converge to 1 after a finite number of iterations (due to the finite number of admissible active sets $\mathcal{A} \in \Sigma$). ■

**Remark 3.4** *A trivial initialization for Algorithm 1 is the choice* $x_0^{(0)} = 0$ *and* $\mathcal{A}^{(0)} = \{0, \ldots, 0\}$. *On the other hand, if* $x_t^p$, $x_{t+1}^p$ *are assumed to be feasible initial conditions for (3.3) at time* $t$ *and* $t+1$ *respectively, then (in the context of MPC) further computational savings can be achieved by warm-starting Algorithm 3.1. This can be done by choosing* $x_0^{(0)}$ *at time* $t+1$ *equal to* $x_t^p$ *and choosing the initial active set at* $t+1$ *as the optimal active set obtained at time* $t$, *i.e.* $\mathcal{A}^{(0)}(t+1) = \{\mathcal{A}_0^*(t), \ldots, \mathcal{A}_{N-1}^*(t)\}$.

**Remark 3.5** *Since the terminal cost is defined as the infinite horizon cost-to-go for the optimal unconstrained feedback law, the solution of each underlying EP is unchanged if* $N$ *is taken to be the minimum horizon over which input and state constraints are active (for the case that terminal constraints are inactive). To reduce computation and improve numerical robustness, it is therefore possible to use, similarly to [18], a reduced horizon* $\hat{N}$ *in (3.3), which is adjusted at each* *iteration of Algorithm 3.1 via* $\hat{N} = min(N, max(k : \mathcal{A}_k \neq \emptyset))$ *so that each active input and state constraint corresponds to a prediction time* $k \leq \hat{N}$.

**Remark 3.6** *From the definition of* $\mathcal{X}_{pQP}(\mathcal{A})$ *it follows that* $\mathcal{A}$ *is an optimal active set for (3.3) if and only if* $x^p \in \mathcal{X}_{pQP}(\mathcal{A})$*: Furthermore, the boundaries* $\partial\mathcal{X}_{pQP}(\mathcal{A})$ *are independent of initial conditions, and the linear inequalities in* $x_0$ *corresponding to constraints could in principle be computed offline. In common with pQP solution methods [2], this is clearly likely to result in very large storage requirements for the online algorithm. However, it would avoid the online computation in step* $(i)$ *of Algorithm 3.1. The contribution of [55] has demonstrated that this is likely to provide an improvement in online complexity over classical pQP methods, which typically require the online evaluation of at least one function in each region* $\mathcal{X}_{pQP}(\mathcal{A})$ *to solve (3.3), where the number of regions is exponential in* $N$.

## 3.5 Computation

We consider the computational effort for solving a one-stage degenerate subproblem (in order to get conservative overall estimations): We make the assumption that (3.17) is solved using the Schur complement method (see e.g. [45]). This approach, applied to (3.17), involves computing the QR decomposition of $\begin{bmatrix} B^T C_k^T & B^T E_k^T & F_k^T \end{bmatrix}^T$ which requires $O(2(n_F + n_E + d)n_u^2)$ floating point operations but can be further reduced using incremental rank-1 updates (where $n_G$ and $n_E$ are the number of rows of $G_k$, $E_k$ and $d$ is the order of degeneracy in the minimization at time $k+1$) . Furthermore, the required matrix inverses and multiplications require $O\big(a_1 n_u^3 + a_2 n_u^2(n_F + n_E + d) + a_3 n_u (n_F + d)^2\big)$ operations. The constants $a_1, a_2, a_3$ depend on the implementation of the Schur complement method and the underlying functions used for the Cholesky and QR decompositions.

The other significant contribution in computing (3.18-3.20) is due to the matrix multiplications in (3.20), requiring $O\big(3n_x^2 + 2n_x^2(n_u + n_E + d) + 2n_x^2(n_E + n_u)\big)$ operations. Combining the above estimates for $N$ stages gives an $O(N)$ dependence of computational complexity.

Noting that the computation required for the forward simulation is $O(n_x^2 N)$ (since only the projection, $\Phi_k(x^p - x_0^{(i)})$, of $\Phi_k$ in (3.26a,b) is needed), and also that the computation involved in the line search in step (ii) is comparatively insignificant, we estimate the computation per iteration of Algorithm 1 to grow as $O(N)$.

Thus the dependence of computation per iteration on the horizon length, $N$ is linear. This is in contrast to similar approaches such as [51] with worst case $O(N^3)$ complexity in case of many active state constraints, and [21] where despite the use of efficient matrix factorization the complexity per iteration is of order $O(N^2)$, since the problem is condensed prior to the solution of the linear KKT system.

The required number of iterations of the active set method is problem-dependent, but can be minimized using warm-starts (as described in Remark 3.4).

**Remark 3.7** *Limitations of the proposed active set method are mainly common to standard QP methods in the literature: from a theoretical point of view it may only be possible to obtain large upper bounds on the required maximum number of iterations, but it remains an open problem to quantify the dependence of the number of iterations on relevant parameters such as horizon length and the system state and input dimension. However, the numerical examples of Chapters 4 and 5 in this thesis indicate that the practical number of iterations encountered even in the more difficult min-max setting are reasonably small (and in many instances seem to increase roughly linearly with horizon length). Also, there has not been found to be a significant dependence of the number of iterations on the system state and input dimension in these examples.*

## 3.6 Numerical Example

To demonstrate the performance of the active set algorithm consider the linear continuous-time model of an aircraft's pitch dynamics as in [52]:

$$J\ddot{\alpha} + b\dot{\alpha} + (C_{ZE}l + C_{CW}d)\alpha = C_{ZE}lE$$

$$m\ddot{h} = (C_{ZE} + C_{CW})\alpha - C_{ZE}E$$

where $m$ and $J$ denote the mass and moment or inertia about the pitch axis, $C_{ZE}, C_{CW}$ are the elevator lift and wing lift coefficients respectively and $b$ is a friction coefficient. $d, l$ are the distances between center of gravity and center of lift and between center of gravity and the point where the elevator lift force is applied. We denote the state by $x = [\alpha, \dot{\alpha}, h, \dot{h}]^T$, where $\alpha, h$ denote the pitch angle (deg.) and vertical height (m) of the aircraft. The input is the elevator angle, i.e. $u = E$. We take the parameters as in [52] to be $J = 1, m = 1, b = 4, C_{ZE} = 1, C_{ZW} = 5, l = 3, d = 0.2$ resulting in a four dimensional state space model in continuous time which is discretized with a sampling time of $0.2s$:
The resulting linear discrete-time dynamics are described by:

$$A = \begin{bmatrix} 0.9384 & 0.1341 & 0 & 0 \\ -0.5363 & 0.4022 & 0 & 0 \\ 0.1186 & 0.0066 & 1 & 0.2 \\ 1.1737 & 0.0923 & 0 & 1 \end{bmatrix} B = \begin{bmatrix} 0.0462 \\ 0.4022 \\ -0.0190 \\ -0.1803 \end{bmatrix}$$

The constraint sets are $\mathcal{U} = \{u \in \mathbb{R} \mid -25 \leq u \leq 25\}$ and cost weights are $Q = \text{diag}\{0, 0, 0.04, 0\}$, $R = 2.5 \times 10^{-4}$. Terminal constraints are employed as explained in Section 3.1.

In Figure 3.2 we illustrate the computational complexity increase per iteration as a function of horizon length for the Riccati approach by comparison to

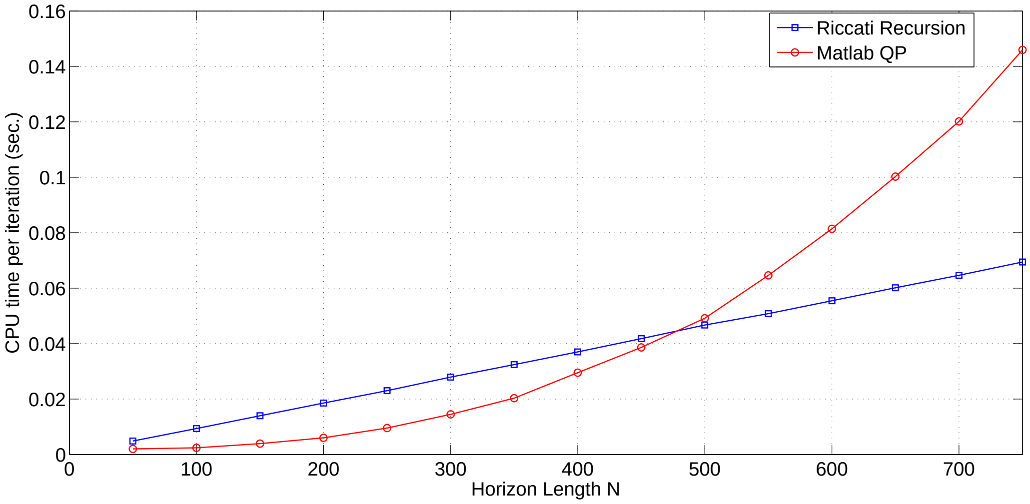

Figure 3.2: Comparison of CPU time per iteration as function of $N$ of Riccati approach and MATLAB QP solution

MATLAB's standard QP solver (quadprog) [1]. One can clearly see, in agreement with the bounds given in Section 3.5, the linear vs. cubic complexity increase of the Riccati approach and the standard QP solver, respectively. Therefore there exists a horizon length $N^* \geq 0$ such that for $N > N^*$ the required computations per iteration are smaller for the Riccati approach as compared to MATLAB's QP solver. The results shown in Figure 3.2 require a large horizon length before an improvement over MATLAB's 'quadprog' solver is seen. One important reason for the results are that sparsity in the linear equations at each iteration is also adequately addressed using 'quadprog'. Nevertheless, significant improvements are expected if computationally intensive parts of the active set algorithm (in particular the matrix multiplications involved in the backward sweep and forwards simulation) are implemented e.g. using Blas/Lapack subroutines and compiled as MATLAB MEX-Files (we note that such significant computational savings have been demonstrated in [18]).

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[1] All simulations in this thesis have been performed using [42]. The computing hardware was equipped with an Intel(R) Core(TM) i5 M450 processor with 2.4GHz and 4GB RAM

## 3.7 Conclusion

An efficient optimization method for the linear model predictive control problem is proposed which is based on the underyling dynamic programming structure, both for the solution of the EP problems (based on Riccati recursions) and during the active set update which is performed using a line search method through pQP regions. Since the solution holds for linear time-varying systems, the solution method may prove to be very efficient in tackling the Nonlinear MPC problem via sequential QP methods. However, if one chooses successive linearization methods, it seems difficult to establish recursive feasibility if one does not explicitly account for the uncertainty in the prediction dynamics due the linearization error. This and many other sources of uncertainty encountered in practice call for extensions of efficient MPC methods to the robust case and this is the subject of the following chapters.

# Chapter 4

# Input-Constrained Robust MPC based on DP

This chapter extends the ideas employed for the nominal linear-quadratic MPC problem discussed in Chapter 3 to the robust linear MPC problem based on min-max costs for systems with additive bounded disturbances and input constraints. In contrast to most of the existing robust MPC literature [25,39,44,46] we obtain exact solutions to the underlying finite horizon min-max optimal control problem based on dynamic programming. The exact method of [50] solves individual optimization problems for each extreme realization of the disturbance sequence, however the price paid for this approach is an enormous computational load for all but the simplest examples which is compounded by the exponential growth of the number of optimization variables with horizon length [46]. The homotopy approach of this chapter is based on the observation that a sequence of pQPs is solved for a given initial plant state in order to solve the given finite horizon min-max optimal control problem. Riccati recursions allow us to obtain efficient and exact DP solutions provided that the active set is correctly traced from some initial plant state to the actual measured plant state. Similary to Chapter 3 the solution is deeply rooted in the fact that MPC seeks local solutions to finite horizon DP problems. A strong advantage of the proposed method is that it scales favourably with respect to the dimension of state, input and disturbance and horizon length. However, since state constraints are not explicitly included, the receding horizon application of the input-constrained algorithm does not have the guarantee of stability that is provided by conventional receding horizon control laws (for which initial feasibility implies stability). Despite this fact it is possible to compute - either exactly or approximately - a region of attraction of a robust control invariant set for the closed loop system in which a guaranteed $l_2$-gain bound obtains. This chapter finally gives theoretical comparisons with max-min and open loop strategies, as well as numerical comparisons with the suboptimal min-max strategy [24] and a 'saturated' unconstrained $\mathcal{H}_\infty$-controller.

## 4.1 Problem Statement

We consider linear discrete time systems with model

$$x_{t+1} = Ax_t + Bu_t + Dw_t, \quad t = 0, 1, \ldots \tag{4.1}$$

with state $x_t \in \mathbb{R}^{n_x}$, control input $u_t \in \mathbb{R}^{n_u}$ and disturbance input $w_t \in \mathbb{R}^{n_w}$ at time $t$. Here $u_t$ and $w_t$ are subject to constraints: $u_t \in \mathcal{U}$, $w_t \in \mathcal{W}$, and $\mathcal{U}$ and $\mathcal{W}$ are assumed to be convex polytopic sets containing the origin defined by

$$\mathcal{U} = \{u \in \mathbb{R}^{n_u} \mid Fu \leq \mathbf{1}\}, \quad \mathcal{W} = \{w \in \mathbb{R}^{n_w} \mid Gw \leq \mathbf{1}\}$$

for $F \in \mathbb{R}^{n_F \times n_u}$, $G \in \mathbb{R}^{n_G \times n_w}$, where $\mathbf{1} = [1 \ \cdots \ 1]^T$ denotes a vector of conformal dimensions.

We define the feedback law $\kappa_0^*(x)$ as the solution to the following closed loop robust optimal control problem [1, 43] over a finite horizon of $N$ time-steps by using the robust DP recursion as established in Equation (2.11). For $k = N -$

$1, \dots 0$ we therefore consider: [1]:

$$\big(\kappa_k^*(x), w_k^*(x,u)\big) = \arg\min_{u\in\mathcal{U}}\max_{w\in\mathcal{W}} J_k(x,u,w) \tag{4.2a}$$

with $J_k$ defined by

$$J_k(x,u,w) = \tfrac{1}{2}(\|x\|_Q^2 + \|u\|_R^2 - \gamma^2\|w\|^2) + J_{k+1}^*(x^+) \tag{4.2b}$$

where $x^+ = Ax + Bu + Dw$ and the $k$th-stage intermediate optimal cost function $J_k^*(x)$ is given by

$$J_k^*(x) = J_k\big(x, \kappa_k^*(x), w_k^*(x, \kappa_k^*(x))\big) \tag{4.2c}$$

Furthermore, the terminal cost is given by:

$$J_N^*(x) = \tfrac{1}{2}\|x\|_P^2. \tag{4.2d}$$

Here $R \in \mathcal{S}_{++}^{n_u}$, $Q \in \mathcal{S}_{+}^{n_x}$ and the scalar $\gamma$ is chosen (as stated in Proposition 4.1) to be sufficiently large that (4.2a) defines a strictly convex-concave min-max problem (more details are provided in Section 4.4). We assume that $P$ is chosen so that $\|x_0\|_P^2 = \sum_{t=0}^{\infty}(\|x_t\|_Q^2 + \|u_t\|_R^2 - \gamma^2\|w_t\|^2)$ with $u_t = \kappa_\infty^*(x_t) = L^u x_t$ and $w_t = w_\infty^*(x_t, u_t) = L^w(Ax_t + Bu_t)$, where $\kappa_\infty^*(\cdot), w_\infty^*(\cdot)$ are the optimal solutions of (4.2a-c) in the limit as $N \to \infty$ and in the absence of constraints $u \in \mathcal{U}$, $w \in \mathcal{W}$. Therefore Assumptions 2.3 and 2.4 are made to ensure the existence of an unconstrained $\mathcal{H}_\infty$ solution as discussed in Section 2.1.4.3.

The problem defined in (4.2a-d) is formulated under the assumption that the disturbance $w_t$ is unknown when the control input $u_t$ is chosen at time $t$. Since the solutions $\kappa_k^*(\cdot)$ and $w_k^*(\cdot)$ depend on $x$ and on $(x,u)$ respectively, (4.2a-d) defines a closed loop optimal control problem (see e.g. [38]). The sequential nature of this

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[1] We note that, in the case that only input constraints are present, stability has to be ensured in an alternative fashion (i.e. not by direct enforement of the state constraints defined in Equation (2.11)) and will be the subject of Section 4.3

min-max problem and the fact that the optimization is performed over arbitrary feedback laws $\{\kappa_k^*(x), w_k^*(x,u),\ k = N-1,\ldots,0\}$ imply that, unlike open-loop formulations of robust MPC (e.g. [15]), (4.2a-d) cannot be solved exactly by a single quadratic program.

For given $x_0$, we denote the optimal predicted state, input and disturbance sequences as

$$\mathbf{x}(x_0) = \{x_0,\ldots,x_N\}, \quad \mathbf{u}(x_0) = \{u_0,\ldots,u_{N-1}\}, \quad \mathbf{w}(x_0) = \{w_0,\ldots,w_{N-1}\}$$

where, for $k = 0,\ldots,N-1$ we define $u_k = \kappa_k^*(x_k)$, $w_k = w_k^*(x_k, u_k)$ and $x_{k+1} = Ax_k + Bu_k + Dw_k$. As in Chapter 3, we differentiate between actual and predicted plant states and we denote the actual plant state at time $t$, which is assumed to be known at $t$, as $x_t^p$. A receding horizon control law is defined by implementing, at each time $t$, the input $u_t = \kappa_0^*(x_t^p)$.

## 4.2 Active Set Method via Riccati Recursion

This section describes a method of solving (4.2a-d) in order to determine $\kappa_0^*(x)$ for a given plant state, $x = x^p$. We use a Riccati recursion to solve the Karush-Kuhn-Tucker (KKT) conditions [36, 45] that provide first-order necessary optimality conditions for Problem (4.2). Using a backwards sweep approach [11], both the optimal control and the worst-case disturbance inputs for an equality constrained problem corresponding to a given active constraint set are obtained as a sequence of affine state feedback functions. We give necessary and sufficient conditions for optimality of this policy with respect to Problem (4.2). For the given active constraint set, our approach then determines state, control, disturbance and multiplier sequences as functions of the initial state $x_0$ using the system model (4.1). As in Chapter 3, we use a line-search through $x_0$-space to update the active set, and the process is repeated until $x_0 = x^p$. Finally we discuss how the computation required by this approach depends on the problem size.

### 4.2.1 Solution to Equality Constrained Problem via Riccati Recursion

**Theorem 4.1** *Let* $\lambda_k$ *denote a Lagrange multiplier associated with the constraint* $x_{k+1} = Ax_k + Bu_k + Dw_k$ *and let* $\mu_k$ *and* $\eta_k$ *denote the Lagrange multipliers of constraints* $u_k \in \mathcal{U}$ *and* $w_k \in \mathcal{W}$ *respectively, for* $k = 0, \dots, N-1$. *Then the KKT conditions defining first order necessary conditions for the optimal solution of Problem (4.2) can be expressed as:*

$$x_{k+1} = Ax_k + Bu_k + Dw_k, \qquad k = 0, \dots, N-1 \tag{4.3}$$

$$\lambda_{k-1} = A^T\lambda_k + Qx_k, \qquad k = 1, \dots, N-1 \tag{4.4}$$

*and, for* $k = 0, \dots, N-1$*:*

$$Ru_k = -B^T\lambda_k - F^T\mu_k \tag{4.5a}$$

$$\mu_k \geq 0, \quad \mu_k^T(\mathbf{1} - Fu_k) = 0, \quad \mathbf{1} - Fu_k \geq 0 \tag{4.5b}$$

$$\gamma^2 w_k = D^T\lambda_k - G^T\eta_k \tag{4.6a}$$

$$\eta_k \geq 0, \quad \eta_k^T(\mathbf{1} - Gw_k) = 0, \quad \mathbf{1} - Gw_k \geq 0 \tag{4.6b}$$

*with the terminal and initial conditions:*

$$\lambda_{N-1} = Px_N. \tag{4.7}$$

$$x_0 = x^p \tag{4.8}$$

*Proof:* Using results from classical optimization theory (e.g. [45]), the KKT conditions defining first order necessary conditions can be derived. Section 5.3 provides a detailed derivation for the more general case including state constraints and therefore the KKT conditions stated here follow as a special case (with $\lambda_k =$

$\hat{\lambda}_k$). ■

An active set approach solves the optimization Problem (4.2) by solving a sequence of problems involving only equality constraints. Let $\mathcal{A} = (\mathcal{A}^u, \mathcal{A}^w)$ define a set of active constraints in (4.2), namely a set of constraints that are satisfied with equality at a solution of (4.2) for some initial state $x_0$. Specifically, let $\mathcal{A}^u = \{A^u_{0,i}, \ldots, A^u_{N-1,i},\ i = 1, \ldots, n_F\}$ and $\mathcal{A}^w = \{A^w_{0,i}, \ldots, A^w_{N-1,i},\ i = 1, \ldots, n_G\}$, where $A^u_{i,k}, A^w_{i,k}$ can take values of 0 or 1, and rewrite (4.5b) and (4.6b) as

$$\left.\begin{aligned} e_i^T F u_k &= 1 \\ e_i^T \mu_k &\geq 0 \end{aligned}\right\} \text{ if } A^u_{k,i} = 1, \quad \left.\begin{aligned} e_i^T F u_k &\leq 1 \\ e_i^T \mu_k &= 0 \end{aligned}\right\} \text{ if } A^u_{k,i} = 0 \tag{4.9}$$

$$\left.\begin{aligned} e_i^T G w_k &= 1 \\ e_i^T \eta_k &\geq 0 \end{aligned}\right\} \text{ if } A^w_{k,i} = 1, \quad \left.\begin{aligned} e_i^T G w_k &\leq 1 \\ e_i^T \eta_k &= 0 \end{aligned}\right\} \text{ if } A^w_{k,i} = 0 \tag{4.10}$$

where $e_i$ denotes the $i$th column of an identity matrix of conformal dimensions. Also let $F_k$, $G_k$ denote the matrices that consist of the rows of $F$, $G$ corresponding to the active sets indicated by $A^u_{k,i} = 1$, $i = 1, \ldots, n_u$, and $A^w_{k,i} = 1$, $i = 1, \ldots, n_w$ respectively, and denote the multipliers of these active constraints as $\mu_{a,k}$ and $\eta_{a,k}$. Then the equality constraints in (4.5a),(4.9) and (4.6a),(4.10) are equivalent to

$$R u_k = -B^T \lambda_k - F_k^T \mu_{a,k} \qquad \text{and} \qquad F_k u_k = \mathbf{1}, \tag{4.11}$$

$$\gamma^2 w_k = D^T \lambda_k - G_k^T \eta_{a,k} \qquad \text{and} \qquad G_k w_k = \mathbf{1}. \tag{4.12}$$

Let $\Sigma$ denote the set of all $\mathcal{A}$ such that (4.3),(4.4),(4.11),(4.12) are feasible for some $x_0$. Then, for given $\mathcal{A} \in \Sigma$, these constraints and (4.7),(4.8) define a two-point boundary value problem. To prove that this equality constrained problem can be solved using a Riccati recursion, we perform a backwards induction proof:

We assume that $\lambda_k$ can be expressed as

$$\lambda_k = P_k x_{k+1} + q_k \tag{4.13}$$

which clearly holds for the boundary condition (4.7), i.e. $\lambda_{N-1} = P x_N$ with $P_{N-1} = P$ and $q_{N-1} = 0$. For the induction step we assume that $\lambda_k = P_k x_{k+1} + q_k$ holds for some $k$. Then, using (4.3), (4.12) gives

$$\begin{bmatrix} \gamma^2 I - D^T P_k D & G_k^T \\ G_k & 0 \end{bmatrix} \begin{bmatrix} w_k \\ \eta_{a,k} \end{bmatrix} = \begin{bmatrix} D^T P_k (A x_k + B u_k) + D^T q_k \\ \mathbf{1} \end{bmatrix}. \tag{4.14}$$

Therefore, if (4.14) has the unique solution:

$$\begin{bmatrix} w_k \\ \eta_{a,k} \end{bmatrix} = \begin{bmatrix} M_k^w \\ M_k^\eta \end{bmatrix} (A x_k + B u_k) + \begin{bmatrix} m_k^w \\ m_k^\eta \end{bmatrix}, \tag{4.15}$$

then (4.13) gives $\lambda_k = \hat{P}_k (A x_k + B u_k) + \hat{q}_k$ with 'intermediate' cost terms $\hat{P}_k, \hat{q}_k$ defined by:

$$\hat{P}_k = P_k + P_k D M_k^w \tag{4.16a}$$

$$\hat{q}_k = q_k + P_k D m_k^w \tag{4.16b}$$

and hence (4.11) gives

$$\begin{bmatrix} R + B^T \hat{P}_k B & F_k^T \\ F_k & 0 \end{bmatrix} \begin{bmatrix} u_k \\ \mu_{a,k} \end{bmatrix} = \begin{bmatrix} -B^T \hat{P}_k A x_k - B^T \hat{q}_k \\ \mathbf{1} \end{bmatrix}. \tag{4.17}$$

Furthermore, assuming that (4.17) has a unique solution:

$$\begin{bmatrix} u_k \\ \mu_{a,k} \end{bmatrix} = \begin{bmatrix} L_k^u \\ L_k^\mu \end{bmatrix} x_k + \begin{bmatrix} l_k^u \\ l_k^\mu \end{bmatrix}, \tag{4.18}$$

equation (4.4) yields $\lambda_{k-1} = P_{k-1}x_k + q_{k-1}$, where

$$P_{k-1} = Q + A^T\hat{P}_k(A + BL_k^u) \tag{4.19a}$$

$$q_{k-1} = \hat{q}_k + A^T\hat{P}_kBl_k^u \tag{4.19b}$$

Therefore, it follows by induction that (4.13) holds for $k = N-1, \dots, 0$.

The necessary and sufficient conditions for optimality of the Riccati recursion in (4.15),(4.16) and (4.18),(4.19) are as follows.

**Proposition 4.1** *The optimal solution of (4.2a-d) is given by*

$$w_k^*(x_k, u_k) = M_k^w(Ax_k + Bu_k) + m_k^w \tag{4.20a}$$

$$u_k^*(x_k) = \kappa_k^*(x_k) = L_k^u x_k + l_k^u \tag{4.20b}$$

*if and only if*

$$G_{k,\perp}^T(\gamma^2 I - D^T P_k D)G_{k,\perp} \succ 0 \tag{4.21a}$$

$$F_{k,\perp}^T(R + B^T\hat{P}_kB)F_{k,\perp} \succ 0 \tag{4.21b}$$

*(where the columns of $F_{k,\perp}$ and $G_{k,\perp}$ form bases for the kernels of $F_k$ and $G_k$ respectively), and*

$$G\big(M_k^w(Ax_k+Bu_k)+m_k^w\big) \le 1, \qquad F(L_k^u x_k+l_k^u) \le 1 \tag{4.22a}$$

$$M_k^\eta(Ax_k+Bu_k)+m_k^\eta \ge 0, \qquad L_k^\mu x_k+l_k^\mu \ge 0. \tag{4.22b}$$

*Proof:* Conditions (4.22a,b) ensure that a solution of the equality constraint problem (4.3),(4.4),(4.11),(4.12), coincides with that of the KKT conditions (4.3)-(4.8) for the given active set $\mathcal{A}$. Furthermore Problem (4.2) is strictly concave in $w_k$ and strictly convex in $u_k$ if and only if conditions (4.21a) and (4.21b) hold. Therefore, under (4.21a,b), the KKT conditions (4.3)-(4.8) admit a unique solution and are sufficient as well as necessary for optimality [45]. ■

**Remark 4.1** *Condition (4.21b) is necessarily satisfied since $R \succ 0$ by assumption and $Q \succeq 0$ implies $P_k, \hat{P}_k \succ 0$ for all $k$. However (4.21a) is very difficult to verify in practice, since this would require checking all active sets $\mathcal{A} \in \Sigma$. In this chapter we therefore assume that $\gamma$ is sufficiently large to satisfy (4.21a) for all active sets likely to be encountered. We note that the terms $P_k$ and $q_k$ define the optimal DP cost at stage $k$ in the solution to Problem 4.2 (omitting the constant term for simplicity, since it is irrelevant in the decision making).*

### 4.2.2 Active Set Method

Using the feedback law (4.20) in conjunction with (4.3) to simulate forward over the $N$-step horizon, we obtain

$$x_k = \Phi_k x_0 + \phi_k \quad \text{for } k = 1, \dots, N \tag{4.23}$$

where $\Phi_k \in \mathbb{R}^{n_x \times n_x}$ and $\phi_k \in \mathbb{R}^{n_x}$ are defined by

$$\Phi_{k+1} = (I + DM_k^w)(A + BL_k^u)\Phi_k \tag{4.24a}$$

$$\phi_{k+1} = (I + DM_k^w)\big((A + BL_k^u)\phi_k + Bl_k^u\big) + Dm_k^w \tag{4.24b}$$

with inital conditions $\Phi_0 = I$ and $\phi_0 = 0$. Therefore the input, disturbance and costate sequences $\mathbf{u}(x_0)$, $\mathbf{w}(x_0)$ and $\boldsymbol{\lambda}(x_0) = \{\lambda_0, \dots, \lambda_{N-1}\}$, as well as the corresponding multiplier sequences $\boldsymbol{\mu}(x_0) = \{\mu_0, \dots, \mu_{N-1}\}$ and $\boldsymbol{\eta}(x_0) = \{\eta_0, \dots, \eta_{N-1}\}$ can be determined as affine functions of $x_0$ by substituting (4.23) into (4.15), (4.18) and (4.13). Hence, for a given active set $\mathcal{A}$, we can define a region of state space $\mathcal{X}_{pQP}(\mathcal{A}) \subset \mathbb{R}^{n_x}$ in which the KKT conditions hold:

$$\mathcal{X}_{pQP}(\mathcal{A}) = \big\{x_0 \mid \mathbf{x}(x_0) \text{ satisfies } (4.22\text{a,b})\big\}. \tag{4.25}$$

**Lemma 4.1** *The sets $\mathcal{X}_{pQP}(\mathcal{A})$ defined by (4.25) are convex polyhedra. Furthermore the union $\cup_{\mathcal{A}\in\Sigma}\mathcal{X}_{pQP}(\mathcal{A})$ of all admissible active sets defines a partition of the set of feasible initial conditions for (4.2) with $\mathcal{X}_{pQP}(\mathcal{A})$ as its elements. (see e.g. [27]) In particular, this implies the following properties:*

$$\partial\mathcal{X}_{pQP}(\mathcal{A}) \subseteq \mathcal{X}_{pQP}(\mathcal{A}) \tag{4.26a}$$

$$\mathcal{X}_{pQP}(\mathcal{A}_1) \cap \mathcal{X}_{pQP}(\mathcal{A}_2) = \partial\mathcal{X}_{pQP}(\mathcal{A}_1) \cap \partial\mathcal{X}_{pQP}(\mathcal{A}_2) \tag{4.26b}$$

*for any two distinct active sets $\mathcal{A}_1, \mathcal{A}_2 \in \Sigma$ (where $\partial\mathcal{X}_{pQP}(\mathcal{A})$ denotes the boundary of $\mathcal{X}_{pQP}$).*

*Proof:* A solution of (4.2) exists for all feasible $x^p$, so $\cup_{\mathcal{A}\in\Sigma}\mathcal{X}_{pQP}(\mathcal{A})$ necessarily covers the set of feasible initial conditions $x^p$. The convexity property and (4.26a) follow from the linear inequality constraints (4.22a,b) that define the boundary of $\mathcal{X}_{pQP}(\mathcal{A})$. Property (4.26b) results from the piecewise continuity of the trajectories $\mathbf{x}(x_0), \boldsymbol{\lambda}(x_0), \mathbf{u}(x_0), \mathbf{w}(x_0), \boldsymbol{\mu}(x_0)$ and $\boldsymbol{\eta}(x_0)$ which follows from (4.14),(4.16),(4.17),(4.19) (and implies the fact that $relint(\mathcal{X}_{pQP}(\mathcal{A}_1)) \cap relint(\mathcal{X}_{pQP}(\mathcal{A}_2)) = \emptyset$). ■

The algorithm we propose solves (4.2) by solving the equality constrained problem for an estimate of the optimal active set, and then updates this active set at successive iterations. At each iteration $i$ the algorithm determines $\mathcal{A}^{(i+1)}$ from $\mathcal{A}^{(i)}$ by performing a line search over $x_0 \in \mathcal{X}_{pQP}(\mathcal{A}^{(i)})$ in the direction of the current plant state $x^p$. This results in a sequence of dual-feasible iterates $x_0^{(i)}$ that generate trajectories satisfying (4.22a,b) but not necessarily (4.8). In the following we make the assumption, similary to Chapter 3:

**Assumption 4.1** *For each $\mathcal{A} \in \Sigma$, the line segment $\{x_0 \mid x_0 = x_0^{(0)} + \alpha(x^p - x_0^{(0)}), \alpha \in [0,1]\}$ intersects the boundary $\partial\mathcal{X}_{pQP}(\mathcal{A})$ at most at two discrete points.*

**Algorithm 4.1** *Initialize with $x_0^{(0)}$ and an active set $\mathcal{A}^{(0)}$ such that $x_0^{(0)} \in \mathcal{X}_{pQP}(\mathcal{A}^{(0)})$, and set $i = 0$. At iteration $i = 0, 1, \ldots$:*

*(i).* *Compute* $\{P_k, q_k\}$ *for* $k = N-1, \dots, 0$, *and* $\{\Phi_k, \phi_k\}$ *for* $k = 0, \dots, N-1$, *and hence* $\mathcal{X}_{pQP}(\mathcal{A}^{(i)})$.

*(ii).* *Perform the line search:*

$$\alpha^{(i)} = \max_{\alpha \in (0,1]} \{\alpha \mid x_0^{(i)} + \alpha(x^p - x_0^{(i)}) \in \mathcal{X}_{pQP}(\mathcal{A}^{(i)})\}.$$

*(iii).* *If* $\alpha^{(i)} < 1$, *then set* $x_0^{(i+1)} := x_0^{(i)} + \alpha^{(i)}(x^p - x_0^{(i)})$, $i := i+1$, *and update* $\mathcal{A}^{(i)}$ *on the basis of the new set of active constraints. Return to step (i).*

*(iv).* *Otherwise set* $\mathcal{A}^* := \mathcal{A}^{(i)}$, *compute* $\kappa_0^*(x^p)$ *and stop.*

**Theorem 4.2** *Algorithm 4.1 converges after a finite number of iterations to* $\mathcal{A}^*$ *such that the trajectories for* $\mathbf{x}$, $\mathbf{u}$ *and* $\mathbf{w}$ *generated by (4.1) and (4.20) with* $\mathcal{A} = \mathcal{A}^*$ *are optimal for (4.2).*

*Proof:* The line search in step (ii) of Algorithm 4.1 implies that each iterate $x_0^{(i)}$ lies on the line segment defined by $x_0^{(0)} + \beta^{(i)}(x^p - x_0^{(0)})$ with $\beta^{(i)} \in [0,1]$. Since the sequence $\{\beta^{(i)},\ i = 0, 1, \dots\}$ is non-decreasing and each iterate $x^{(i)}$ lies either at an intersection of the line with the boundary $\partial\mathcal{X}_{pQP}(\mathcal{A}^{(i)})$ or at $x_0^{(i)} = x^p$, the sequence $\{\beta^{(i)},\ i = 0, 1, \dots\}$ must converge to 1 after a finite number of iterations due to the finite number of admissible active sets $\mathcal{A} \in \Sigma$ and the results of Lemma 4.1 (which implies that there is no region overlap). It follows that Algorithm 4.1 terminates with $x_0^{(i)} = x^p$ after a finite number of iterations. ■

**Remark 4.2** *If (4.2) is strictly convex-concave in the absence of constraints (i.e. if (4.21a,b) hold with* $F_{k,\perp} = I$ *and* $G_{k,\perp} = I$ *for* $k = 1, \dots, N$*), then* $\kappa_k^*(0) = 0$ *and* $w_k^*(0,0) = 0$ *for* $k = 1, \dots, N$, *and hence a possible initialization of Algorithm 4.1 is* $x_0^{(0)} = 0$ *and* $\mathcal{A}^{(0)} = \{0, \dots, 0\}$.

**Remark 4.3** *In the context of MPC, the computation required by Algorithm 4.1 may be reduced through warm-starting. For example, Algorithm 4.1 can be initialized at time* $t+1$ *using the time-shifted optimal sequence computed at time* $k$ *by setting* $x_0^{(0)} := Ax_t + B\kappa_0^*(x_t) + Dw_0^*(x_t, \kappa_0^*(x_t))$ *and* $\mathcal{A}^{(0)} := \{A_1^*(t), \dots, A_N^*(t), 0\}$

*at $t+1$, where $\mathcal{A}^*(t)$ is the optimal sequence of active sets at time $t$. The choice of the $N^{th}$ element of $\mathcal{A}^{(0)}$ to be equal to zero corresponds to an implicit assumption (discussed in Section 4.3) that the state of (4.3) enters a terminal set after $N$ time-steps within which the unconstrained optimal control law and worst-case disturbances satisfy the constraints $u \in \mathcal{U}$ and $w \in \mathcal{W}$.*

### 4.2.3 Computation

In order to estimate how the computational complexity of Algorithm 4.1 depends on the problem size, we make the assumption that (4.14) and (4.17) are solved using the Schur complement method commonly employed by QP active set solvers (see e.g. [22]). This approach, applied to (4.14), involves computing the QR decomposition of $F_k$, which requires $O(n_w^2)$ floating point operations (assuming that incremental rank-1 updates are employed), as well as calculating the inverse of the matrix on the LHS of (4.21a), which requires $O\big((n_w - n_F)^3\big)$ operations (assuming Cholesky decomposition is used), where $n_F \leq n_w$ is the number of rows of $F_k$. Applying the same approach to the solution of (4.17) requires $O(n_u^2)$ operations for the QR decomposition of $G_k$ plus $O\big((n_u - n_G)^3\big)$ operations for the Cholesky decomposition of the LHS of (4.21b), where $n_G \leq n_u$ is the number of rows of $G_k$. The other significant contribution to the computation in (4.15)-(4.19) is due to the matrix multiplications in (4.16) and (4.19), which require $O\big((2n_x^3 + (3n_u + 2n_w)n_x^2 + n_u^2 n_x)\big)$ operations.

Combining these estimates, and noting that the computation required for the forward simulation is $O(n_x^2 N)$ (since only the projection, $\Phi_k(x^p - x_0^{(i)})$, of $\Phi_k$ in (4.24a,b) is needed), and also that the computation involved in the line search in step (ii) is comparatively insignificant, we estimate the computation per iteration of Algorithm 4.1 to grow with the problem size as

$$O\Big(\big(2n_x^3 + n_x^2(3n_u + 2n_w) + c_1(n_w^3 + n_u^3) + c_2(n_w^2 + n_u^2)\big)N\Big).$$

Here $c_1$, $c_2$ are constants that depend on the implementation of Cholesky and QR decompositions, and we have used conservative approximations: $n_u - n_G \approx n_u$, $n_w - n_F \approx n_w$.

Thus the dependence of computation per iteration on the horizon length $N$ is linear. The required number of iterations is problem-dependent, but empirical evidence (see e.g. the example of Section 4.6) suggests that this also grows approximately linearly with $N$. Furthermore the number of iterations can be minimized using warm-starting, as described in Remark 4.3. This is in stark contrast to existing schemes for min-max receding horizon control, which, for the case of optimal approaches that are based on dynamic programming, have computational loads that depend exponentially on $N$ (see e.g. [31, 50]). Likewise, approximate approaches such as [24] based on suboptimal controller parameterizations require the solution of a convex optimization in a number of variables that grows quadratically with the horizon length, which (as demonstrated in Section 4.6) leads to much higher computational loads than Algorithm 4.1.

## 4.3 Closed Loop Stability and $l_2$-Gain Bound

This section discusses the stability and disturbance attenuation properties of the receding horizon controller defined by the solution of Problem (4.2). The problem description (4.2a-d) does not include inequality state constraints, and hence it does not allow terminal state constraints to be included in the definition of the receding horizon policy. Nevertheless, the associated receding horizon control law $\kappa_0^*(\cdot)$ ensures robust stability and induces a specified disturbance $l_2$-gain bound for a particular set of initial conditions.

To demonstrate this we define a robust, controlled, positively invariant set $\mathcal{X}_f$ under the infinite horizon unconstrained optimal solutions, $\kappa_\infty^*(\cdot), w_\infty^*(\cdot)$, of Problem (4.2). The control law $\kappa_0^*(\cdot)$ ensures that the specified $l_2$-gain bound holds for $x_0 \in \mathcal{X}_0$, where $\mathcal{X}_0$ is a set of initial conditions from which $\mathcal{X}_f$ is reached in $N$ steps under $u = \kappa_0^*(x)$. This section concludes with a discussion of the connections between min-max, max-min and open-loop problem formulations.

**Definition 4.1** *A set $\mathcal{X}_f \subseteq \mathbb{R}^{n_x}$ is robust positively invariant for (4.1) under $u_t = \kappa_\infty^*(x_t)$ and feasible for $w_t = w_\infty^*(x_t, \kappa_\infty^*(x_t))$ if: (i). $\kappa_\infty^*(x) \in \mathcal{U}$ for all $x \in \mathcal{X}_f$; (ii). $Ax + B\kappa_\infty^*(x) + Dw \in \mathcal{X}_f$ for all $x \in \mathcal{X}_f$, $w \in \mathcal{W}$; and (iii). $w_\infty^*(x, \kappa_\infty^*(x)) \in \mathcal{W}$ for all $x \in \mathcal{X}_f$.*

**Definition 4.2** *For $t = N, \ldots, 1$, the set $\mathcal{X}_{t-1}$ is the preimage of $\mathcal{X}_t$ for (4.1) under $u_t = \kappa_0^*(x_t)$ if $\mathcal{X}_N = \mathcal{X}_f$ and $\mathcal{X}_{t-1} = \{x \in \mathbb{R}^{n_x} \mid Ax + B\kappa_0^*(x_t) + Dw \in \mathcal{X}_t, \forall w \in \mathcal{W}\}$.*

**Remark 4.4** *Definition 4.1 and the terminal cost (4.2d) imply that $\kappa_0^*(x) = \kappa_\infty^*(x)$, for all $x \in \mathcal{X}_f$. From Definition 4.2 it follows that $\mathcal{X}_f \subseteq \mathcal{X}_{N-1} \subseteq \cdots \subseteq \mathcal{X}_0$, and therefore $\mathcal{X}_0$ is robust positive invariant and a region of attraction of $\mathcal{X}_f$. We note that due to the nonlinear (piecewise affine) nature of the receding horizon control law, the pre-image sets are not necessarily convex[1].*

The main result of this section, concerning stability of the uncertain system $x_{t+1} = Ax_t + Bu_t + Dw_t$, $u_t \in \mathcal{U}$, $w_t \in \mathcal{W}$, under the receding horizon control law $u_t = \kappa_0^*(x_t)$, is stated next.

**Proposition 4.2** *If $x_0 \in \mathcal{X}_0$, then for any admissible disturbance sequence $\{w_t \in \mathcal{W}, t = 0, 1 \ldots\}$, the state of (4.1) with $u_t = \kappa_0^*(x_t)$ satisfies $x_t \in \mathcal{X}_f$ for all $t \geq N$, and*

$$\sum_{t=0}^{\infty}(\|x_t\|_Q^2 + \|u_t\|_R^2) \leq \gamma^2 \sum_{t=0}^{\infty} \|w_t\|^2 + 2J_0^*(x_0). \tag{4.27}$$

*Proof:* Definitions 4.1, 4.2 and Remark 4.4 imply that $x_t \in \mathcal{X}_f$ for all $t \geq N$ if $x_0 \in \mathcal{X}_0$. To demonstrate the bound on $l_2$-gain, consider the evolution of the optimal cost, $J_0^*(x_t)$. Due to the optimality of $w_0^*(x_t, u_t)$ we have

$$J_0^*(x_t) \geq \tfrac{1}{2}\big(\|x_t\|_Q^2 + \|u_t\|_R^2 - \gamma^2\|w_t\|^2\big) + J_1^*(x_{t+1}).$$

---

[1] Note however that we don't need to know these sets in order to determine the control law

Also $J_1^*(x_{t+1}) \geq J_0^*(x_{t+1})$ since $x_{t+1} \in \mathcal{X}_1$, so

$$J_0^*(x_t) \geq \tfrac{1}{2}\big(\|x_t\|_Q^2 + \|u_t\|_R^2 - \gamma^2\|w_t\|^2\big) + J_0^*(x_{t+1}).$$

Summing this inequality over $t = 0, 1\ldots$ and using the fact that $J_0^*(x) = \frac{1}{2}\|x\|_P^2$ for all $x \in \mathcal{X}_f$ yields the $l_2$-gain bound (4.27) (see Section 2.2.2 for more details).

■

## 4.4 Min-Max and Max-Min Optimal Control

The formulation of Problem (4.2) assumes that $w_t$ is unknown when $u_t$ is determined. If, on the other hand, $u_t$ is allowed to depend explicitly on $w_t$ (e.g. corresponding to the situation in practice that a measurement of the disturbance at the current time is available, but all other future disturbances are unknown), then it is straightforward to adapt Algorithm 4.1 to solve the max-min problem defined by:

$$\big(\tilde{w}_k^*(x), \tilde{\kappa}_k^*(x,w)\big) = \arg\max_{w\in\mathcal{W}}\min_{u\in\mathcal{U}} \tilde{J}_k(x,u,w), \tag{4.28}$$

where

$$\tilde{J}_k(x,u,w) = \tfrac{1}{2}(\|x\|_Q^2 + \|u\|_R^2 - \gamma^2\|w\|^2) + \tilde{J}_{k+1}^*(x^+)$$

and

$$\tilde{J}_k^*(x) = \tilde{J}_k\big(x, \tilde{\kappa}_k^*(x, \tilde{w}_k^*(x)), \tilde{w}_k^*(x)\big)$$

with terminal cost $\tilde{J}_N^*(x) = \frac{1}{2}\|x\|_P^2$. It is easy to show that the receding horizon max-min control law $u_t = \tilde{\kappa}_0^*(x_t, w_t)$ ensures the performance bound (4.27) with $J_0^*(x_0)$ replaced by $\tilde{J}_0^*(x_0)$. The additional information that is available when computing $\tilde{\kappa}^*(x,w)$ would be expected to lead to a reduced $l_2$-gain bound, and this is indeed confirmed by the following result.

**Theorem 4.3** *For a given active set $\mathcal{A}$, Problem (4.2) is strictly convex-concave for $\gamma \geq \underline{\gamma}(\mathcal{A})$ and Problem (4.28) is strictly concave-convex for $\gamma \geq \underline{\tilde{\gamma}}(\mathcal{A})$, where*

$\underline{\tilde{\gamma}}(\mathcal{A}) \leq \underline{\gamma}(\mathcal{A})$.

*Proof:* Problem (4.28) is strictly concave-convex for a given active set $\mathcal{A}$ if and only if:

$$F_{k,\perp}^T(R + B^T \tilde{P}_k B)F_{k,\perp} \succ 0 \tag{4.29a}$$

$$G_{k,\perp}^T(\gamma^2 I - D^T \hat{\tilde{P}}_k D)G_{k,\perp} \succ 0 \tag{4.29b}$$

where $\hat{\tilde{P}}_k = \tilde{P}_k - \tilde{P}_k B \tilde{L}_k^u$, $\tilde{P}_{k-1} = Q + A^T \hat{\tilde{P}}_k(A + D\tilde{M}_k^w)$ and $\tilde{\kappa}_k^*(x, w) = \tilde{L}_k^u(Ax + Dw) + \tilde{l}_k^w$, $\tilde{w}_k^*(x) = \tilde{M}_k^w x + \tilde{m}_k^w$ for $k = 0, \ldots, N-1$, with $\tilde{P}_{N-1} = P$. Since $P_{N-1} = P$ and since $\kappa_k^*(x)$ and $\tilde{w}_k^*(x)$ are suboptimal solutions for $\tilde{\kappa}_k^*(x, w)$ and $w_k^*(x, u)$ respectively, it follows by induction that $\tilde{P}_k \preceq P_k$ for $k = 0, \ldots, N-1$. Therefore $\underline{\lambda}(D^T \hat{\tilde{P}}_k D) \leq \underline{\lambda}(D^T P_k D)$ and hence, for all $k = 0, \ldots, N-1$, the lower bound on $\gamma$ imposed by (4.21a) exceeds that imposed by (4.29b). ■

The max-min formulation (4.28) cannot however achieve better performance than the min-max formulation (4.2) whenever $\gamma$ is large enough to ensure that the min-max problem is strictly convex-concave, as shown by the following result.

**Theorem 4.4** *For given $\mathcal{A}$, $J_0^*(x) = \tilde{J}_0^*(x)$ if $\gamma \geq \underline{\gamma}(\mathcal{A})$.*

*Proof:* Problem (4.2) is strictly convex-concave if $\gamma \geq \underline{\gamma}(\mathcal{A})$, and hence the Wolfe dual [22] can be used to reformulate each stage of Problem (4.2) as a single minimization (for clarity, we use the notation used in Chapter 5, i.e. $\hat{x} = Ax + Bu$ and indices $k$ are dropped. The reader may find it helpful to read Section 5.3):

$$J^*(x) = \min_{u,\,\hat{x}} \; \min_{w,\eta,x^+,\lambda} \left\{ \hat{L}(\hat{x}, w, \eta, x^+, \lambda) \right\} \quad \text{subject to}$$

$$\begin{cases} \nabla_w \hat{L} = 0 \quad \nabla_{x^+} \hat{L} = 0 \quad \hat{x} = Ax + Bu \\ \qquad\qquad\qquad\qquad\qquad Fu \leq \mathbf{1} \quad \eta \geq 0 \end{cases}$$

where the Lagrangian for the maximization subproblem is defined as:

$$\hat{L}(\hat{x}, w, \eta, x^+, \lambda) = -\frac{\gamma^2}{2}\|w\|^2 + J^{+*}(x^+) + \eta^T(\mathbf{1} - Gw) - \lambda^T\big(x^+ - (\hat{x} + Dw)\big).$$

The order in which $u_k^*$ and $w_k^*$ are determined can then be interchanged, and subsequent re-application of the Wolfe dual, which is applicable since $\gamma \geq \underline{\tilde{\gamma}}(\mathcal{A})$ by Theorem 4.3, yields Problem (4.28). ■

Finally we note that, using similar arguments, it can be shown that the open loop min-max problem defined by

$$\big(\mathbf{u}_{ol}^*(x), \mathbf{w}_{ol}^*(x, \mathbf{u}_{ol}^*(x))\big) =$$
$$\arg \min_{\substack{\{u_0,\ldots u_{N-1}\} \\ u \in \mathcal{U}}} \max_{\substack{\{w_0,\ldots w_{N-1}\} \\ w_k \in \mathcal{W}}} \sum_{k=0}^{N-1} \tfrac{1}{2}(\|x_k\|_Q^2 + \|u_k\|_R^2 - \gamma^2\|w_k\|^2) + \tfrac{1}{2}\|x_N\|_P^2$$

is strictly convex-concave for $\gamma \geq \underline{\gamma}_{ol}(\mathbf{s}) \geq \underline{\gamma}(\mathbf{s}) \geq \underline{\tilde{\gamma}}(\mathbf{s})$, and the optimal value of its objective function is equal to $\tilde{J}_0^*(x)$ if $\gamma \geq \underline{\gamma}_{ol}(\mathbf{s})$.

**Remark 4.5** *We note that the KKT conditions are identical for min-max, max-min and open-loop formulations of Problem (4.2). The crucial difference is the required value of $\gamma$ to ensure convex-concavity of each problem formulation. This level of $\gamma$ therefore results in different disturbance rejection capabilities of each controller.*

## 4.5 Numerical Computation of Region of Attraction

As noted in Remark 4.4, the exact region of attraction under the receding horizon control law is non-convex in general, due to the nonlinear nature of the closed loop system and this fact is illustrated for a one-step problem in Figure 4.1. Two heuristic approaches can be employed to calculate this set: one tackles the problem sequentially, the other tackles the global $N$-step problem directly. One straightforward sequential approach is to first take samples of states on a hypersphere for a given radius. Then, for each fixed direction, the radius is increased step-wise until the successor state is not mapped robustly into the successor set $\mathcal{X}_{t+1}$ or the radius reaches a maximal value (N.B. only the vertices of the disturbance set $\mathcal{W}$ have to be tested). The previous state iterate is then guaranteed to lie within $\mathcal{X}_t$. The accuracy of the approximation of $\mathcal{X}_t$ depends of how fine the gridding in the radial and angular directions are chosen. The difficulty is that for given convex $\mathcal{X}_{t+1}$, the pre-image set $\mathcal{X}_t$ will generally be non-convex and therefore the membership testing becomes more difficult. One might first consider obtaining a convex polytopic inner approximation of the resulting non-convex pre-image, but this may be a difficult task in high dimensional state spaces. Alternatively one can avoid this problem by computing the region of attraction $\mathcal{X}_0$ for the entire $N$-step problem directly. This requires again a gridding of state-space e.g. on a hypersphere, however one now has to test whether a given sampled state is mapped robustly into the terminal set $\mathcal{X}_f$ under the receding horizon control law. This requires to test all extreme realizations of the disturbance sequence and therefore (offline) computation grows exponentially with horizon length $N$. For both approaches the sampling approach leads to $rp^{n_x}$ points in state-space, where $p$ is the number of angular gridding points and $r$ the number of radial gridding points.

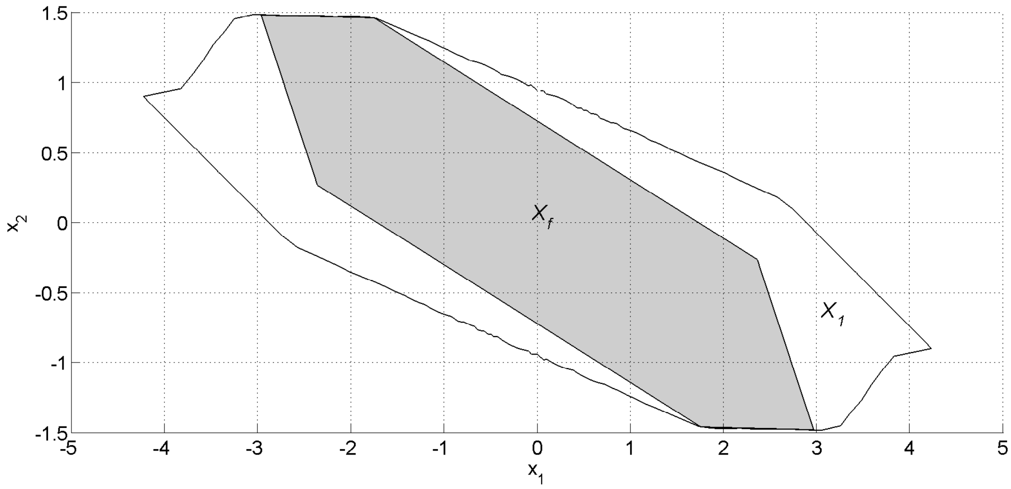

Figure 4.1: Region of attraction illustration for one-step problem for double integrator system for Example A

## 4.6 Numerical Examples

### 4.6.1 Example A

The proposed robust MPC algorithm was applied to (4.1) with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and with constraint sets $\mathcal{U} = \{u \in \mathbb{R} \mid -1 \leq u \leq 1\}$ and $\mathcal{W} = \{w \in \mathbb{R}^2 \mid -0.3 \leq w_i \leq 0.3 \text{ for } i = 1, 2\}$, and cost weights $Q = [1 \quad 0]^T[1 \quad 0]$, $R = 0.8$.

The unconstrained optimal solution is obtained as in Section 2.1.4.3 with $P$, $L^u$ and $L^w$ computed for $\gamma^2 = 20$ as:

$$P = \begin{bmatrix} 2.7829 & 2.3435 \\ 2.3435 & 3.7400 \end{bmatrix}, \quad L^u = \begin{bmatrix} -0.5606 & -1.3965 \end{bmatrix} \quad L^w = \begin{bmatrix} 0.1849 & 0.1708 \\ 0.1708 & 0.2546 \end{bmatrix}.$$

Fig. 4.2 compares the computational load of the exact dynamic programming (DP) active set algorithm of Alg. 4.1 with that of a Disturbance Affine (DA) feedback policy [24] applied to Problem (4.2) for 90 initial conditions equispaced

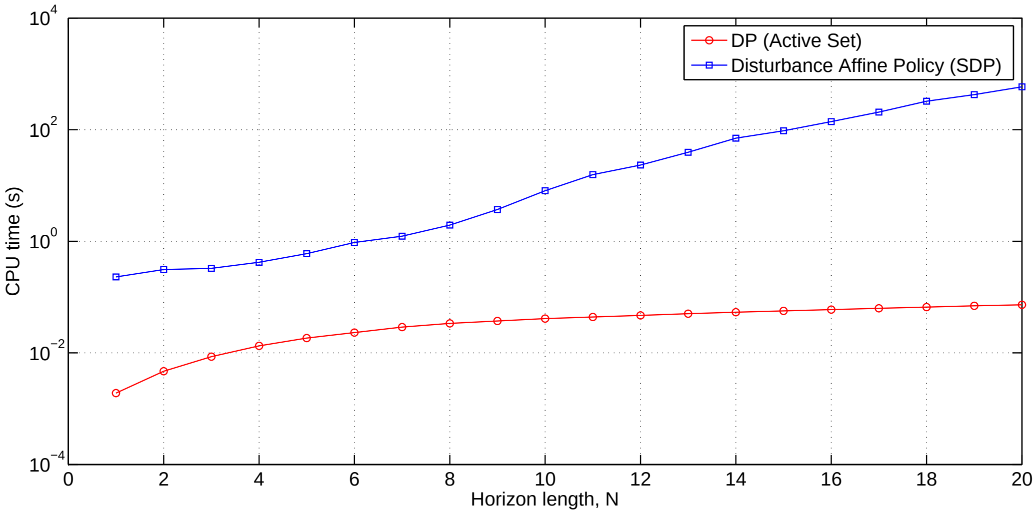

Figure 4.2: CPU time of Algorithm 4.1 and the Disturbance Affine policy of [24] averaged over 90 plant states with $||x_0|| = 2.5$ for Example A

around a circle of radius 2.5 (which is contained within $\mathcal{X}_0$, where $\mathcal{X}_9 = \mathcal{X}_f$). The DA policy was formulated as a semidefinite program (as in [24]), and CPU times are SeDuMi [54] execution times. This comparison shows that Alg. 4.1 is several orders of magnitude faster than the DA approach where Alg. 4.1 was initialized with $x_0^{(0)} = 0$. We also note that the DA policy can be significantly suboptimal; for $N = 12$, $\gamma^2 = 20$ and $x_0 = (18, -5.4)$, the predicted costs for the DA policy and Alg.4.1 are 1014.2 and 851.7 respectively, so in this case the DA policy is 19% suboptimal. Finally, the closed loop performance for the input-constrained $\mathcal{H}_\infty$ controller is compared to a saturated unconstrained $\mathcal{H}_\infty$-controller. For the initial condition $x_0 = (18, -5.4)$, the closed loop performance in both cases is determined under the assumption that the first element $w_0$ of the worst-case predicted sequence of disturbances $\mathbf{w}$ of the constrained $\mathcal{H}_\infty$ controller is realized (both for the constrained and saturated $\mathcal{H}_\infty$ controller). The closed loop costs are given by 191.19 and 273.43 respectively for the input-constrained and a saturated unconstrained $\mathcal{H}_\infty$ controller (where the unconstrained solution is projected onto the set $\mathcal{U}$), indicating the practical benefits of the Alg. 4.1.

### 4.6.2 Example B

We consider a numerical example describing airplane pitch dynamics given by:

$$A = \begin{bmatrix} 0.9384 & 0.1341 & 0 & 0 \\ -0.5363 & 0.4022 & 0 & 0 \\ 0.1186 & 0.0066 & 1 & 0.2 \\ 1.1737 & 0.0923 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0.0462 \\ 0.4022 \\ -0.0190 \\ -0.1803 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The constraint sets are given by $\mathcal{U} = \{u \in \mathbb{R} \mid -25 \leq u \leq 25\}$ and $\mathcal{W} = \{w \in \mathbb{R}^2 \mid -0.25 \leq w_i \leq 0.25 \text{ for } i = 1, 2\}$, and cost weights by $Q = \operatorname{diag}[0, 0, 0.04, 0]$, $R = 2.5 \cdot 10^{-4}$ and $\gamma^2 = 5$ to meet the relevant conditions for Alg. 4.1. Figure 4.3 demonstrates that the computational complexity per iteration of Algorithm 4.1 scales linearly with horizon length $N$, in agreement with Section 4.2.3. Average computation times for an implementation in Matlab are given for 100 initial states, $x_0$, equispaced around a circle of radius 50 and Alg. 4.1 was initialized with $x_0^{(0)} = 0$. For this example, the number of iterations required to reach convergence initially increases linearly with $N$, but reaches a maximum when $N$ is sufficiently large that an increase in $N$ causes no more constraints to become active. Thus, for example, the circle of radius 50 lies entirely within $\mathcal{X}_0$ (for $\mathcal{X}_{12} = \mathcal{X}_f$), so the number of iterations is constant for $N \geq 12$ for the case of $\|x_0\| = 50$, and hence the computation time of Alg. 4.1 increases quadratically for $N < 12$ and linearly for $N > 12$. This example shows that the method is applicable for higher dimensional systems with computational complexity of similar order to that of active set methods for nominal MPC [18].

## 4.7 Conclusion

This chapter considers a robust min-max MPC problem for input constrained linear systems with bounded disturbances. We give necessary and sufficient con-

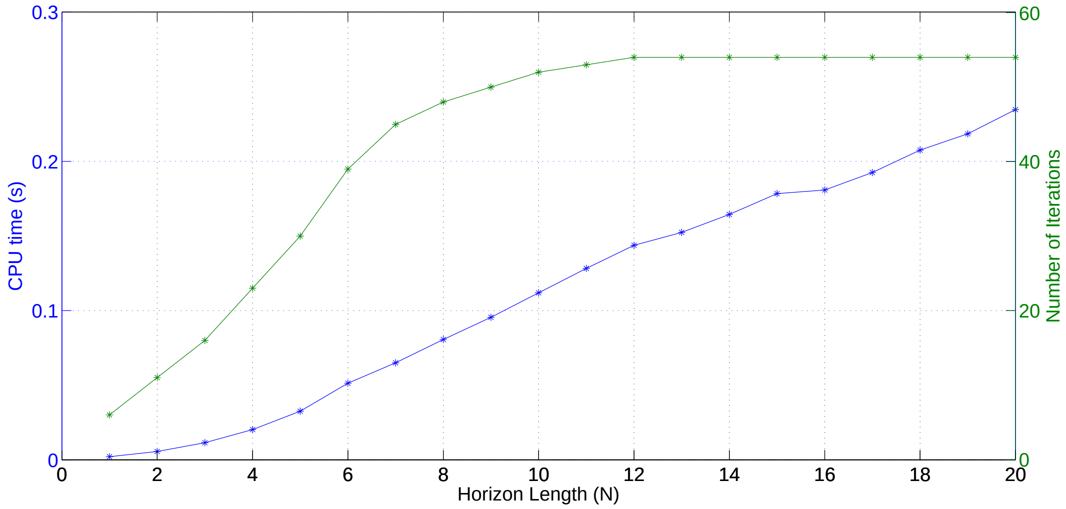

Figure 4.3: Average CPU time (blue) and number of iterations (green) for convergence of Alg. 4.1, for 100 initial states with $||x_0|| = 50$ for Example B

ditions for optimality and propose an algorithm based on Riccati recursions which can be warm started. In addition, a guaranteed $l_2$-gain bound is derived under the condition that the initial system state is inside the region of attraction of the closed loop system. Clearly, the lack of a clean theoretical framework for the computation of a region of attraction poses a limitation of the input-constrained approach and therefore Chapter 5 modifies the framework to include input and state constraints.

# Chapter 5

# Input and State Constrained Min-Max MPC based on DP

This chapter extends the ideas developed in Chapters 3 and 4 to the input- and state-constrained case and shows that optimal solutions can be determined efficiently by exploiting the linearity and continuity properties of the corresponding dynamic programming solutions. This avoids suboptimal formulations such as considered in the contributions of [25,39,44,46]. Moreover, we demonstrate that the computation required by each iteration of the proposed algorithm grows only linearly with the horizon length of the control problem. As in Chapters 3 and 4 we formulate an active set algorithm based on homotopy of solutions, but now the proposed approach makes use of robust controllable sets computed offline [6] and solves a multi-stage min-max robust optimal control problem locally online. We propose a method of handling linearly dependent active constraints which circumvents the possibility of discontinuous optimal control laws (see e.g. [43]) by selecting those active set changes which lead to continuity of the primal and dual variables along the line-of-search and therefore convergence of the algorithm to a point satisfying the first order necessary conditions for optimality of a related problem can be ensured. An $\mathcal{H}_\infty$ performance index is employed and hence a convex-concave min-max problem is obtained in the case that the initial plant state is inside a non-overlapping pQP region and the solution to the KKT conditions is therefore unique. We show that the resulting receding horizon control law ensures closed loop stability and a specified disturbance $l_2$-gain bound for the convex-concave case as well as for the situtations involving degenerate and non-concave subproblems. Numerical examples demonstrate significant advantages in terms of computation and performance over the Disturbance-Affine feedback strategy of [24]. Further, this chapter provides detailed derivations of the first order optimality conditions and discussions of the active set method including degenerate and non-concave subproblems.

## 5.1 Problem Statement

This chapter is concerned with linear discrete time systems (where $t$ denotes the real time index and $k$ denotes the predicted time index):

$$x_{t+1} = Ax_t + Bu_t + Dw_t, \quad t = 0, 1, \ldots \tag{5.1}$$

with state $x_t \in \mathbb{R}^{n_x}$, control input $u_t \in \mathbb{R}^{n_u}$ and disturbance input $w_t \in \mathbb{R}^{n_w}$. The matrices $A$, $B$, $D$ are assumed to be constant and known, and variables $x_t$, $u_t$, $w_t$ are subject to constraints: $x_t \in \mathcal{X}$, $u_t \in \mathcal{U}$, $w_t \in \mathcal{W}$, where $\mathcal{X}$ is a polyhedral set and $\mathcal{U}$, $\mathcal{W}$ are polytopic sets that contain the origin.

We consider the following closed loop robust optimal control problem based on the dynamic programming problem defined in Equations (2.11) and (2.13) for $k = N-1, \ldots, 0$:

$$\big(\kappa_k^*(x), w_k^*(x,u)\big) = \arg\min_{u\in\mathcal{U}} \max_{w\in\mathcal{W}} J_k(x,u,w) \tag{5.2a}$$

subject to $Ax + Bu \in \mathcal{X}_{k+1} \ominus D\mathcal{W}$[1] (), with $J_k$ defined by

$$J_k(x,u,w) = \frac{1}{2}\big(\|x\|_Q^2 + \|u\|_R^2 - \gamma^2\|w\|^2\big) + J_{k+1}^*(x^+) \tag{5.2b}$$

where $x^+ = Ax + Bu + Dw$ and the $k$th-stage intermediate optimal cost function $J_k^*(x)$ is given by

$$J_k^*(x) = J_k\big(x, \kappa_k^*(x), w_k^*(x, \kappa_k^*(x))\big) \tag{5.2c}$$

with the robust controllable set recursion given by:

$$\mathcal{X}_k = \mathcal{X} \cap \{x \in \mathbb{R}^{n_x} \mid \exists u \in \mathcal{U},\ Ax + Bu \in \mathcal{X}_{k+1} \ominus D\mathcal{W}\} \tag{5.2d}$$

Furthermore, the terminal conditions are given by:

$$J_N^*(x) = \tfrac{1}{2}\|x\|_P^2, \tag{5.2e}$$

$$\mathcal{X}_N = \mathcal{X}_f. \tag{5.2f}$$

In (5.2), $R \in \mathcal{S}_{++}^{n_u}$, $Q \in \mathcal{S}_{+}^{n_x}$ and the scalar $\gamma$ is generally assumed to be chosen sufficiently large that the min-max problem in (5.2a) is strictly convex-concave in $(u,w)$. Conditions on $\gamma$ that ensure this are derived in Section 5.3. We assume that $P$ in (5.2f) is chosen so that $\|x_0\|_P^2 = \sum_{k=0}^{\infty}(\|x_k\|_Q^2 + \|u_k\|_R^2 - \gamma^2\|w_k\|^2)$ with $u_k = \kappa_\infty^*(x_k)$ and $w_k = w_\infty^*(x_k, u_k)$, where $\kappa_\infty^*(\cdot), w_\infty^*(\cdot)$ are the optimal solutions of (5.2a-d) in the limit as $N \to \infty$ in the absence of constraints $x \in \mathcal{X}$, $u \in \mathcal{U}$, $w \in \mathcal{W}$. Therefore Assumptions 2.3 and 2.4 are made to ensure the existence of an unconstrained $\mathcal{H}_\infty$ solution as discussed in Section 2.1.4.3. Furthermore $\mathcal{X}_f$ satisfies Definition 4.1 and can be obtained through the procedure of [32] (provided that $\mathcal{W}$ is small enough to ensure the existence of a non-empty terminal set $\mathcal{X}_f$).

We denote predicted sequences of minimizing control inputs and maximiz-

---

[1] The Pontryagin difference of set $\mathcal{X}$ with respect to set $\mathcal{W}$ is given by $\mathcal{X} \ominus \mathcal{W} = \{x \mid\ x + w \in \mathcal{X}\ \forall w \in \mathcal{W}\}$ [6]

ing disturbance inputs in (5.2) as $\{u_0, \dots, u_{N-1}\}$ and $\{w_0, \dots, w_{N-1}\}$, and the corresponding predicted state sequence as $\{x_0, \dots, x_{N-1}\}$, so that $u_k = \kappa_k^*(x_k)$, $w_k = w_k^*(x_k, u_k)$ and $x_{k+1} = Ax_k + Bu_k + Dw_k$ for $k = 0, \dots, N-1$. As in Chapters 3 and 4 we assume the initial plant state to be known at time $t$, as $x_t^p$, and define the applied control input as $u_t = \kappa_0^*(x_t^p)$.

## 5.2 Solution Outline

Firstly, in Section 5.3 the Karush-Kuhn-Tucker (KKT) conditions providing first-order necessary conditions for optimality (see e.g. [45]) are derived for Problem (5.2). In Section 5.4 we derive Riccati recursions to determine the solution of an associated problem involving only equality constraints, using a sweep method (as in [11]). Thus we obtain the sequence of optimal control and worst-case disturbance functions (for the given cost) for an equality constrained problem (EP) corresponding to a given active constraint set (resulting in a sequence of affine state feedback laws). Finally, in Section 5.5 necessary conditions are derived for this control policy to be optimal with respect to the inequality constrained Problem (5.2).

We consider: (a) the special case that the state and input constraints are linearly independent and necessary conditions are also sufficient (leading to unique solutions for given $x_0$); (b) the general case of degenerate and non-concave subproblems in which the necessary conditions can be used to obtain feasible solutions for the given plant state. A consistent treatment of cases (a) and (b) is obtained in Section 5.6 by formulation of a line search algorithm based on homotopy. To this end, after the backward sweep, we determine optimal state, input, disturbance and multiplier sequences for (5.2a-f) as functions of the initial state $x_0$ by forward simulation using the system model (5.1). We then devise a line-search algorithm through polyhedral regions which cover the set of feasible plant states, starting from the optimal solution at a given initial condition (e.g. the unconstrained optimal solution at the origin, $x_0 = 0$), and successively updating the active set as a function of $x_0$ until $x_0 = x^p$, as in Chapters 3 and 4.

## 5.3 First-Order Necessary Conditions

Let the the polyhedral sets $\mathcal{X}_{k+1} \ominus D\mathcal{W}$ and the polytopic sets $\mathcal{U}$ and $\mathcal{W}$ be given in their irreducible form by:

$$\mathcal{X}_{k+1} \ominus D\mathcal{W} = \{x \in \mathbb{R}^{n_x} \mid E_k x \leq \mathbf{1}\} \tag{5.3}$$

$$\mathcal{U} = \{u \in \mathbb{R}^{n_u} \mid Fu \leq \mathbf{1}\} \tag{5.4}$$

$$\mathcal{W} = \{w \in \mathbb{R}^{n_w} \mid Gw \leq \mathbf{1}\} \tag{5.5}$$

for $E_k \in \mathbb{R}^{n_E \times n_x}$, $F \in \mathbb{R}^{n_F \times n_u}$, $G \in \mathbb{R}^{n_G \times n_w}$, and where $\mathbf{1} = [1 \ \cdots \ 1]^T$ denotes a vector of conformal dimensions. Also let $\lambda$ and $\hat{\lambda}$ denote the Lagrange multipliers associated with constraints $x^+ = \hat{x} + Dw$ and $\hat{x} = Ax + Bu$, and let $\nu$, $\mu$ and $\eta$ denote the Lagrange multipliers for inequality constraints $\hat{x} \in \mathcal{X}_{k+1} \ominus D\mathcal{W}$, $u \in \mathcal{U}$ and $w \in \mathcal{W}$ respectively.[1] Since the min-max subproblem (5.2a) at each stage may be degenerate due to linear dependence in the active constraint set, we include equality state constraints $C_k\hat{x} = \mathbf{1}$ and $\hat{C}_k x^+ = -\mathbf{1}$ in the minimization and maximization ($\hat{C}_k \in \mathbb{R}^{n_{d_{k+1}} \times n_x}$ and $C_m \in \mathbb{R}^{n_{\hat{d}_k} \times n_x}$) with associated multipliers $\zeta$ and $\hat{\zeta}$.

**Definition 5.1** *The kth-stage min-max optimization problem is defined as the following sequence of sub-problems*

$$J_k^*(x) = \min_{u,\hat{x}} \left\{ \tfrac{1}{2}\|x\|_Q^2 + \tfrac{1}{2}\|u\|_R^2 + \hat{J}_k^*(\hat{x}) \right\} \quad \textit{subject to} \begin{cases} E_k\hat{x} \leq \mathbf{1}, Fu \leq \mathbf{1}, C_k\hat{x} = \mathbf{1}, \\ \hat{x} = Ax + Bu, \end{cases} \tag{5.6a}$$

---

[1] Note the dropping of the time indices on the optimization variables for readibility

*and*

$$\hat{J}_k^*(\hat{x}) \;=\; \max_{w,\,x^+}\Big\{-\frac{\gamma^2}{2}\|w\|^2 \;+\; J_{k+1}^*(x^+)\Big\} \quad \textit{subject to} \quad \begin{cases} Gw \leq \mathbf{1},\;\; \hat{C}_k x^+ = -\mathbf{1}, \\ x^+ = \hat{x} + Dw, \end{cases} \tag{5.6b}$$

*with* $J_N^*(x) = \frac{1}{2}\|x\|_P^2$.

*And correspondingly we define the associated Lagrangian functions:*

$$\begin{aligned} L_k(x,u,\mu,\nu,\hat{\zeta},\hat{\lambda},\hat{x}) = \frac{1}{2}\|x\|_Q^2 + \frac{1}{2}\|u\|_R^2 + \hat{J}_k^*(\hat{x}) - \mu^T(\mathbf{1} - Fu) - \nu^T(\mathbf{1} - E_k\hat{x}) \\ - \hat{\zeta}^T(\mathbf{1} - C_k\hat{x}) - \hat{\lambda}^T\big(\hat{x} - (Ax + Bu)\big) \end{aligned} \tag{5.7a}$$

*and*

$$\begin{aligned} \hat{L}_k(\hat{x},w,\eta,\zeta,\lambda,x^+) = -\frac{\gamma^2}{2}\|w\|^2 + J_{k+1}^*(x^+) + \eta^T(\mathbf{1} - Gw) + \zeta^T(\mathbf{1} + \hat{C}_k x^+) \\ - \lambda^T\big(x^+ - (\hat{x} + Dw)\big). \end{aligned} \tag{5.7b}$$

Then the first order necessary conditions for optimality of Problem (5.6b) are given by the KKT conditions (e.g. [45]):

$$\nabla_w \hat{L}_k = 0 \quad \Longrightarrow \quad \gamma^2 w + G^T\eta - D^T\lambda = 0 \tag{5.8a}$$

$$\nabla_\zeta \hat{L}_k = 0 \quad \Longrightarrow \quad \hat{C}_k x^+ = -\mathbf{1} \tag{5.8b}$$

$$\nabla_\lambda \hat{L}_k = 0 \quad \Longrightarrow \quad x^+ = \hat{x} + Dw \tag{5.8c}$$

$$\nabla_{x^+} \hat{L}_k = 0 \quad \Longrightarrow \quad \lambda = \nabla_{x^+} J_{k+1}^*(x^+) + \hat{C}_k^T\zeta \tag{5.8d}$$

and

$$\eta^T(\mathbf{1} - Gw) = 0, \quad \eta \geq 0, \quad Gw \leq \mathbf{1}. \tag{5.8e}$$

The first order necessary conditions for optimality of Problem (5.6a) are given similarly by

$$\nabla_u L_k = 0 \quad \Longrightarrow \quad Ru + F^T\mu + B^T\hat{\lambda} = 0 \tag{5.9a}$$

$$\nabla_{\hat{\zeta}} L_k = 0 \quad \Longrightarrow \quad C_k\hat{x} = \mathbf{1} \tag{5.9b}$$

$$\nabla_{\hat{\lambda}} L_k = 0 \quad \Longrightarrow \quad \hat{x} = Ax + Bu \tag{5.9c}$$

$$\nabla_{\hat{x}} L_k = 0 \quad \Longrightarrow \quad \hat{\lambda} = \nabla_{\hat{x}}\hat{J}_k^*(\hat{x}) + E_k^T\nu + C_k^T\hat{\zeta} \tag{5.9d}$$

and

$$\mu^T(\mathbf{1} - Fu) = 0, \quad \mu \geq 0, \quad Fu \leq \mathbf{1}, \tag{5.9e}$$

$$\nu^T(\mathbf{1} - E_k\hat{x}) = 0, \quad \nu \geq 0, \quad E_k\hat{x} \leq \mathbf{1}. \tag{5.9f}$$

**Lemma 5.1** *Equations (5.8d) and (5.9d) are respectively equivalent to*

$$\lambda = \begin{cases} A^T\hat{\lambda}^+ + Qx^+ + \hat{C}_k^T\zeta, & \text{if } k < N-1 \\ Px^+ + \hat{C}_k^T\zeta, & \text{if } k = N-1 \end{cases} \tag{5.10a}$$

$$\hat{\lambda} = \lambda + E_k^T\nu + C_k^T\hat{\zeta}, \tag{5.10b}$$

*where $\hat{\lambda}^+$ denotes the multiplier associated with the constraint $\hat{x} = Ax + Bu$ in the $(k+1)$-stage maximization problem.*

*Proof:* Let $w^*(\hat{x})$, $\eta^*(\hat{x})$, $\zeta^*(\hat{x})$, $\lambda^*(\hat{x})$, $x^{+*}(\hat{x})$, denote the optimal values of primal variables and multipliers for Problem (5.6b) for given $\hat{x}$. Then, using (5.8a-d), we have

$$\begin{aligned} \nabla_{\hat{x}}\hat{J}_k^*(\hat{x}) &= \nabla_{\hat{x}}\hat{L}_k\big(\hat{x}, w^*(\hat{x}), \eta^*(\hat{x}), \zeta^*(\hat{x}), \lambda^*(\hat{x}), x^{+*}(\hat{x})\big) \\ &= \frac{\partial \hat{L}_k}{\partial \hat{x}} + \frac{\partial w^*}{\partial \hat{x}}\nabla_w\hat{L}_k + \cdots + \frac{\partial x^{+*}}{\partial \hat{x}}\nabla_{x^+}\hat{L}_k \\ &= \lambda^*(\hat{x}). \end{aligned}$$

Similarly denoting the optimal primal variables and multipliers for Problem (5.6a) for given $x$ as $u^*(x)$, $\mu^*(x)$, $\nu^*(x)$, $\hat{\zeta}^*(x)$, $\hat{\lambda}^*(x)$, $\hat{x}^*(x)$, and using (5.9a-d), we also have

$$\begin{aligned}\nabla_x J_k^*(x) &= \nabla_x L_k\big(x, u^*(x), \mu^*(x), \nu^*(x), \hat{\zeta}^*(x), \hat{\lambda}^*(x), \hat{x}^*(x)\big)\\ &= \frac{\partial L_k}{\partial x} + \frac{\partial u^*}{\partial x}\nabla_u L_k + \cdots + \frac{\partial \hat{x}^*}{\partial x}\nabla_{\hat{x}} L_k\\ &= A^T\hat{\lambda}^*(x) + Qx.\end{aligned}$$

so that $\nabla_{x^+} J_{k+1}^*(x^+) = A^T\hat{\lambda}^+ + Qx^+$ if $k < N-1$, and $\nabla_{x^+} J_{k+1}^*(x^+) = Px^+$ if $k = N-1$, which implies (5.10a,b). ■

**Remark 5.1** *The optimal values of the decision variables* $(u^*(x), \hat{x}^*(x))$ *and* $(w^*(\hat{x}), x^{+*}(\hat{x}))$ *are in general multivalued functions of their arguments,* $x$ *and* $\hat{x}$*, respectively. Hence* $J_k^*(x)$ *and* $\hat{J}_k^*(\hat{x})$ *are also multivalued in general. This situation occurs as a result of linear dependence in the active constraint set at a given minimization stage. This has the effect that the gradient of the value function at the given stage is discontinuous in the given parameter and causes the optimization problems at prior stages to be non-convex-concave, in general. However, the proposed active set method uses a selection that enables the gradients introduced in the proof of Lemma 5.1 to be defined uniquely along the line of search and ensures convergence to the optimal solution of Problem (5.2) in the convex-concave case and to a feasible solution of Problem (5.2) otherwise.*

## 5.4 The Active Set Equality Constraint Problem

Let $\mathcal{A}_k = \{\mathcal{A}_k^u, \mathcal{A}_k^{\hat{x}}, \mathcal{A}_k^w\}$ denote the indices of the active constraints in the $k$th-stage minimization and maximization problems in (5.6a,b), so that the solution for primal variables $u^*, w^*, \hat{x}^*$ and dual variables $\mu^*, \nu^*, \eta^*$, satisfies

$$e_i^T F u^* = 1, \quad i \in \mathcal{A}_k^u, \qquad e_i^T \mu^* = 0, \quad i \notin \mathcal{A}_k^u \tag{5.11a}$$

$$e_i^T E_k \hat{x}^* = 1, \quad i \in \mathcal{A}_k^{\hat{x}}, \qquad e_i^T \nu^* = 0, \quad i \notin \mathcal{A}_k^{\hat{x}} \tag{5.11b}$$

$$e_i^T G w^* = 1, \quad i \in \mathcal{A}_k^w, \qquad e_i^T \eta^* = 0, \quad i \notin \mathcal{A}_k^w \tag{5.11c}$$

where $e_i$ is the $i$th column of an identity matrix of conformal dimensions. To simplify notation, for a given active set $\mathcal{A}_k$ we define $F_k$, $G_k$, and redefine $E_k$ so that the active constraints of (5.11a-c) are equivalent to:

$$F_k u = \mathbf{1}, \quad E_k \hat{x} = \mathbf{1}, \quad G_k w = \mathbf{1}, \tag{5.12}$$

and similarly we redefine $\mu$, $\nu$ and $\eta$ as the vectors of multipliers of the active constraints indexed by $\mathcal{A}_k^u$, $\mathcal{A}_k^{\hat{x}}$ and $\mathcal{A}_k^w$ respectively. To ensure full generality we assume that the active constraints at each stage $k \leq N-1$ may be linearly dependent. In addition we assume for notational convenience that the equality constraints $\hat{C}_k x^+ = -\mathbf{1}$ and $C_k \hat{x} = \mathbf{1}$ appearing in (5.6a,b) are the compatibility conditions associated with linearly dependent constraints.

### 5.4.1 Single-Stage Problem

We note that the maximization subproblem at stage $N-1$ never contains any compatibility conditions since there is no subsequent subproblem. The preceding minimization at stage $N-1$ does not contain compatibility conditions since the linear constraint sets are assumed to be given in irreducible form. For the equality constraint problem (EP) associated with active set $\mathcal{A}_{N-1}^w$ in the maximization (5.6b) at $k = N-1$ we have, from (5.10a)

$$\lambda = P x^+. \tag{5.13}$$

Eliminating $\lambda$ and $x^+$ from (5.8a-c) and invoking $G_{N-1}w = \mathbf{1}$ therefore gives

$$\begin{bmatrix} \gamma^2 I - D^T P D & G_{N-1}^T \\ G_{N-1} & 0 \end{bmatrix} \begin{bmatrix} w \\ \eta \end{bmatrix} = \begin{bmatrix} D^T P \\ 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix}. \tag{5.14}$$

The solution of (5.14) can be expressed as

$$\begin{bmatrix} w \\ \eta \end{bmatrix} = \Psi_{N-1} \begin{bmatrix} D^T P \\ 0 \end{bmatrix} \hat{x} + \Psi_{N-1} \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix} \tag{5.15}$$

where it has been assumed that $G_{N-1,\perp}^T(\gamma^2 I - D^T P D)G_{N-1,\perp}$ is non-singular, (see e.g. [45] for details of how to compute $\Psi_{N-1}$).

For the EP with active set $\{\mathcal{A}_{N-1}^{\hat{x}}, \mathcal{A}_{N-1}^{u}\}$ in the minimization (5.6a) at $k = N-1$, (5.13), (5.10b) and (5.15) imply

$$\begin{gathered} \hat{\lambda} = \hat{P}_{N-1}\hat{x} + \hat{q}_{N-1} + E_{N-1}^T \nu, \\ \begin{bmatrix} \hat{P}_{N-1} & \hat{q}_{N-1} \end{bmatrix} = \begin{bmatrix} P & 0 \end{bmatrix} + \begin{bmatrix} PD & 0 \end{bmatrix} \Psi_{N-1} \begin{bmatrix} D^T P & 0 \\ 0 & \mathbf{1} \end{bmatrix}. \end{gathered} \tag{5.16}$$

By eliminating $\hat{\lambda}$ and $\hat{x}$ from (5.9a-c) and invoking $E_{N-1}\hat{x} = \mathbf{1}$ and $F_{N-1}u = \mathbf{1}$, we therefore obtain

$$\begin{bmatrix} R + B^T \hat{P}_{N-1} B & B^T E_{N-1}^T & F_{N-1}^T \\ E_{N-1} B & 0 & 0 \\ F_{N-1} & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ \nu \\ \mu \end{bmatrix} = - \begin{bmatrix} B^T \hat{P}_{N-1} \\ E_{N-1} \\ 0 \end{bmatrix} Ax + \begin{bmatrix} -B^T \hat{q}_{N-1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}. \tag{5.17}$$

Assuming that $R + B^T \hat{P}_{N-1} B$ is non-singular, the general solution of (5.17) (which includes the degenerate case in which the columns of $[B^T E_{N-1}^T \; F_{N-1}^T]$ are linearly dependent) can be expressed as

$$\begin{bmatrix} u \\ \nu \\ \mu \end{bmatrix} = -\Theta_{N-1} \begin{bmatrix} B^T \hat{P}_{N-1} \\ E_{N-1} \\ 0 \end{bmatrix} Ax + \Theta_{N-1} \begin{bmatrix} -B^T \hat{q}_{N-1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} - \begin{bmatrix} 0 \\ Z_{1,N-1} \\ Z_{2,N-1} \end{bmatrix} \beta \tag{5.18}$$

(again, see e.g. [45] for details of how to compute $\Theta_{N-1}$). In the degenerate case, $\beta$ contains the free variables in the solution of (5.17), and $Z_{N-1} = [Z_{1,N-1}^T \ Z_{2,N-1}^T]^T$ is the full-rank matrix satisfying

$$Z_{N-1}^T \begin{bmatrix} E_{N-1} B \\ F_{N-1} \end{bmatrix} = 0, \qquad Z_{N-1}^T \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \mathbf{1}.$$

For the non-degenerate case that the columns of $[B^T E_{N-1}^T \ F_{N-1}^T]$ are linearly independent, we set $Z_{N-1} = 0$, and $\beta = 0$. In the degenerate case, (5.17) admits solutions if and only if $x$ satisfies the compatibility condition

$$\hat{C}_{N-2} x = -\mathbf{1}, \text{ where } \hat{C}_{N-2} = -Z_{N-1}^T E_{N-1} A. \tag{5.19}$$

Thus (5.19) constitutes an equality constraint in the maximization problem (5.6b) at $k = N-2$, and (5.16), (5.18) and (5.19) imply

$$A^T \hat{\lambda} + Qx = P_{N-1} x + q_{N-1} + \hat{C}_{N-2}^T \beta, \tag{5.20a}$$

$$\begin{aligned} \begin{bmatrix} P_{N-1} & q_{N-1} \end{bmatrix} &= \begin{bmatrix} Q + A^T \hat{P}_{N-1} A & A^T \hat{q}_{N-1} \end{bmatrix} \\ &\quad + A^T \begin{bmatrix} \hat{P}_{N-1} B & E_{N-1}^T & 0 \end{bmatrix} \Theta_{N-1} \begin{bmatrix} -B^T \hat{P}_{N-1} A & -B^T \hat{q}_{N-1} \\ -E_{N-1} A & \mathbf{1} \\ 0 & \mathbf{1} \end{bmatrix}. \end{aligned} \tag{5.20b}$$

### 5.4.2 Multi-Stage Problem

Consider now the EP at stage $k < N-1$ with active set $\mathcal{A}_k$. If the minimization problem (5.6a) at stage $k+1$ is degenerate due to linearly dependent constraints, then its solution contains free variables, denoted $\beta^+$ in (5.21), which can be determined by solving the maximization problem (5.6b) at stage $k$, as we now discuss. From (5.10a) and (5.20a) we have

$$\lambda = P_{k+1}x^+ + q_{k+1} + \hat{C}_k^T(\zeta + \beta^+). \tag{5.21}$$

By eliminating $\lambda$ and $x^+$ from (5.8a-c) and using $G_k w = \mathbf{1}$ we obtain

$$\begin{bmatrix} \gamma^2 I - D^T P_{k+1} D & -D^T \hat{C}_k^T & G_k^T \\ -\hat{C}_k D & 0 & 0 \\ G_k & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ \zeta + \beta^+ \\ \eta \end{bmatrix} = \begin{bmatrix} D^T P_{k+1} \\ \hat{C}_k \\ 0 \end{bmatrix} \hat{x} + \begin{bmatrix} D^T q_{k+1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}. \tag{5.22}$$

The general solution of (5.22) can be expressed as

$$\begin{aligned} \begin{bmatrix} w \\ \zeta + \beta^+ \\ \eta \end{bmatrix} &= \Psi_k \begin{bmatrix} D^T P_{k+1} \\ \hat{C}_k \\ 0 \end{bmatrix} \hat{x} + \Psi_k \begin{bmatrix} D^T q_{k+1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} - \begin{bmatrix} 0 \\ \hat{Z}_{1,k} \\ \hat{Z}_{2,k} \end{bmatrix} \hat{\beta} \\ &= \begin{bmatrix} M_k^w \\ M_k^{\zeta+\beta^+} \\ M_k^\eta \end{bmatrix} \hat{x} + \begin{bmatrix} m_k^w \\ m_k^{\zeta+\beta^+} \\ m_k^\eta \end{bmatrix} - \begin{bmatrix} 0 \\ \hat{Z}_{1,k} \\ \hat{Z}_{2,k} \end{bmatrix} \hat{\beta} \end{aligned} \tag{5.23}$$

For the case that (5.22) is degenerate (i.e. the columns of $[-D^T \hat{C}_k^T \; G_k^T]$ are linearly dependent), $\hat{\beta}$ contains the free variables in the solution of (5.22) and $\hat{Z}_k = [\hat{Z}_{1,k}^T \; \hat{Z}_{2,k}^T]^T$ is the full-rank matrix satisfying

$$\hat{Z}_k^T \begin{bmatrix} -\hat{C}_k D \\ G_k \end{bmatrix} = 0, \qquad \hat{Z}_k^T \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \mathbf{1},$$

whereas $\hat{Z}_k = 0$, $\hat{\beta} = 0$ if (5.22) is non-degenerate. In the degenerate case, $\hat{x}$ must satisfy the compatibility condition given by the following constraint in the minimization (5.6a) at stage $k$:

$$C_k \hat{x} = \mathbf{1}, \text{ where } C_k = -\hat{Z}_{1,k}^T \hat{C}_k. \tag{5.24}$$

Therefore (5.21), (5.10b) and (5.23) imply

$$\hat{\lambda} = \hat{P}_k \hat{x} + \hat{q}_k + E_k^T \nu + C_k^T (\hat{\zeta} + \hat{\beta}) \tag{5.25}$$

where

$$\begin{bmatrix} \hat{P}_k & \hat{q}_k \end{bmatrix} = \begin{bmatrix} P_{k+1} & q_{k+1} \end{bmatrix} + \begin{bmatrix} P_{k+1} D & \hat{C}_k^T & 0 \end{bmatrix} \Psi_k \begin{bmatrix} D^T P_{k+1} & D^T q_{k+1} \\ \hat{C}_k & \mathbf{1} \\ 0 & \mathbf{1} \end{bmatrix}. \tag{5.26}$$

Eliminating $\hat{\lambda}$ and $\hat{x}$ from (5.9a-c) and invoking $E_k \hat{x} = \mathbf{1}$ and $F_k u = \mathbf{1}$ gives

$$\begin{bmatrix} R + B^T \hat{P}_k B & B^T C_k^T & B^T E_k^T & F_k^T \\ C_k B & 0 & 0 & 0 \\ E_k B & 0 & 0 & 0 \\ F_k & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ \hat{\zeta} + \hat{\beta} \\ \nu \\ \mu \end{bmatrix} = - \begin{bmatrix} B^T \hat{P}_k \\ C_k \\ E_k \\ 0 \end{bmatrix} Ax + \begin{bmatrix} -B^T \hat{q}_k \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}. \tag{5.27}$$

The general solution of (5.27) can be expressed as

$$\begin{bmatrix} u \\ \hat{\zeta} + \hat{\beta} \\ \nu \\ \mu \end{bmatrix} = -\Theta_k \begin{bmatrix} B^T \hat{P}_k \\ C_k \\ E_k \\ 0 \end{bmatrix} Ax + \Theta_k \begin{bmatrix} -B^T \hat{q}_k \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} - \begin{bmatrix} 0 \\ Z_{1,k} \\ Z_{2,k} \\ Z_{3,k} \end{bmatrix} \beta \tag{5.28}$$

$$= \begin{bmatrix} L_k^u \\ L_k^{\hat{\zeta}+\hat{\beta}} \\ L_k^\nu \\ L_k^\mu \end{bmatrix} x + \begin{bmatrix} l_k^u \\ l_k^{\hat{\zeta}+\hat{\beta}} \\ l_k^\nu \\ l_k^\mu \end{bmatrix} - \begin{bmatrix} 0 \\ Z_{1,k} \\ Z_{2,k} \\ Z_{3,k} \end{bmatrix} \beta$$

For the degenerate case, in which the columns of the matrix $[B^T C_k^T \;\; B^T E_k^T \;\; F_k^T]$ are linearly dependent, $\beta$ contains the free variables in the solution of (5.27) and $Z_k = [Z_{1,k}^T \; Z_{2,k}^T \; Z_{3,k}^T]^T$ is the full-rank matrix satisfying

$$Z_k^T \begin{bmatrix} C_k B \\ E_k B \\ F_k \end{bmatrix} = 0, \qquad Z_k^T \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \mathbf{1},$$

whereas $Z_k = 0$ and $\beta = 0$ if (5.27) is non-degenerate. In the degenerate case, (5.27) admits solutions if and only if $x$ satisfies the compatibility condition

$$\hat{C}_{k-1} x = -\mathbf{1}, \text{ where } \hat{C}_{k-1} = -\begin{bmatrix} Z_{1,k}^T & Z_{2,k}^T \end{bmatrix} \begin{bmatrix} C_k \\ E_k \end{bmatrix} A. \tag{5.29}$$

From (5.25), (5.28) and making use of (5.29) we therefore have

$$A^T \hat{\lambda} + Qx = P_k x + q_k + \hat{C}_{k-1}^T \beta \tag{5.30a}$$

$$\begin{bmatrix} P_k & q_k \end{bmatrix} = \begin{bmatrix} Q + A^T \hat{P}_k A & A^T \hat{q}_k \end{bmatrix} + A^T \begin{bmatrix} \hat{P}_k B & C_k^T & E_k^T & 0 \end{bmatrix} \Theta_k \begin{bmatrix} -B^T \hat{P}_k A & -B^T \hat{q}_k \\ -C_k A & \mathbf{1} \\ -E_k A & \mathbf{1} \\ 0 & \mathbf{1} \end{bmatrix} \tag{5.30b}$$

The following theorem summarises the Riccati recursion defined by the procedure described above for solving the min-max problems (5.6a,b) corresponding to a given sequence of active sets $\mathcal{A} = \{\mathcal{A}_0, \dots, \mathcal{A}_{N-1}\}$.

**Theorem 5.1** *The solutions of the EPs corresponding to the min-max problems (5.6a,b) with active set* $\mathcal{A}_k = \{\mathcal{A}_k^u, \mathcal{A}_k^{\hat{x}}, \mathcal{A}_k^w\}$, $u_k = u_k^*(x_k)$, $w_k = w_k^*(x_k, u_k)$, *are given by (5.23) and (5.28), where* $\hat{P}_k$, $\hat{q}_k$, $P_k$ *and* $q_k$ *are defined recursively by (5.26) and (5.30b) for* $k = N-1, \ldots, 0$.

*Proof:* This can be shown by induction on $\lambda$ in (5.21). Thus, for $k = N-1$ (5.21) reduces to (5.13), while for $k < N-1$, (5.10a) and (5.30b) imply that the multiplier $\lambda$ in (5.7b), with $k$ replaced by $k-1$, is given by (5.21) with $k$ replaced by $k-1$. ■

## 5.5 Optimality of Inequality Constrained Problem

The following lemma gives conditions for optimality of the Riccati recursion (5.23), (5.26) and (5.28), (5.30b). We note that in general the KKT conditions admit non-unique solutions in the case of degenerate constraints, since the corresponding Lagrange multipliers are non-unique. In the following we state the conditions under which the underlying equality constrained problem is convex-concave. This solution corresponds to a local solution to Problem (5.2). For a degenerate active set, the EP solution provides a solution to Problem (5.2) if this is modified to include the required compatibility constraints at preceding stages in its formulation. Therefore we make the following assumption prior to stating the lemma. We note that modifiying Problem (5.2) has implications on system theoretical closed-loop properties and requires further assumptions (see Section 5.7).

**Assumption 5.1** *If* $\mathcal{A}$ *is degenerate, we assume that Problem (5.2) is modified such that it includes compatibility constraints required for the equality constraint solution explicitly in its formulation.*

**Theorem 5.2** *$w_k^*(\hat{x}_k), u_k^*(x_k)$[1] is a feedback solution to Problem (5.2) defined by*

$$w_k^*(\hat{x}_k) = M_k^w \hat{x}_k + m_k^w \tag{5.31a}$$

$$u_k^*(x_k) = \kappa_k^*(x_k) = L_k^u x_k + l_k^u \tag{5.31b}$$

*if and only if for some active set $\mathcal{A}_k$ the first-order necessary conditions:*

$$\begin{gathered} G(M_k^w \hat{x}_k + m_k^w) \leq 1, \\ F(L_k^u x_k + l^u) \leq 1, \quad E_k \hat{x}_k \leq 1 \end{gathered} \tag{5.32a}$$

$$\begin{gathered} M_k^\eta \hat{x}_k + m_k^\eta - \hat{Z}_{2,k}\hat{\beta}_k \geq 0, \\ L_k^\nu x_k + l_k^\nu - Z_{2,k}\beta_k \geq 0, \quad L_k^\mu x_k + l_k^\mu - Z_{3,k}\beta_k \geq 0 \end{gathered} \tag{5.32b}$$

*and the second-order sufficient conditions:*

$$\begin{bmatrix} -D^T \hat{C}_k^T & G_k^T \end{bmatrix}_\perp^T (-\gamma^2 I + D^T P_k D) \begin{bmatrix} -\hat{C}_k D \\ G_k \end{bmatrix}_\perp \prec 0 \tag{5.33a}$$

$$\begin{bmatrix} B^T C_k^T & B^T E_k^T & F_k^T \end{bmatrix}_\perp (R + B^T \hat{P}_k B) \begin{bmatrix} C_k B \\ E_k B \\ F_k \end{bmatrix}_\perp \succ 0 \tag{5.33b}$$

*hold for each stage $k = N-1, \dots, 0$ and the compatibility conditions with $\zeta_k = 0$ and $\hat{\zeta}_k = 0$:*

$$\beta_{k-1} = M_k^{\zeta+\beta} \hat{x}_k + m_k^{\zeta+\beta} - \hat{Z}_{1,k}\hat{\beta}_k \tag{5.34a}$$

$$\hat{\beta}_k = L_k^{\hat{\zeta}+\hat{\beta}} x_k + l_k^{\hat{\zeta}+\hat{\beta}} - Z_{1,k}\beta_k \tag{5.34b}$$

*hold for $k = N-2, \dots, 0$ and, in the case that $\mathcal{A}_0$ is degenerate, the compatibility condition $\hat{C}_{-1} x_0 = -1$ holds.*

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[1] We re-instate the time index $k$ to indicate the $k$th stage of the DP for the remainder of this chapter

*Proof:* The first-order conditions follow from the KKT conditions (5.8), (5.9) and Theorem 5.1. If their solution is unique, then for a given EP the second-order sufficient conditions as given in ( [45], Theorem 12.6) lead to conditions (5.33) and ensure that $w_k^*, u_k^*$ are the optimal solutions to Problem (5.2). In the case of degenerate constraints, the dual variables satisfying the KKT conditions are non-unique due to the existence of the free variables $\beta$ and $\hat{\beta}$ in (5.28) and (5.23). However the inclusion of compatibility constraints in Problem (5.2) and the choices $\zeta = 0$ and $\hat{\zeta} = 0$ in the EP define a unique solution to the KKT conditions which satisfies the second order sufficient conditions for optimality with respect to the modified problem formulation. This solution provides a feasible solution to the original formulation of Problem (5.2), but it may be suboptimal as a result of the inclusion of compatibility constraints as additional state constraints.

■

**Remark 5.2** *Although the choice of $\zeta$ and $\hat{\zeta}$ does not affect the EP primal solution, it does have an influence on the dual solution. The choices $\zeta = 0$ and $\hat{\zeta} = 0$ ensure continuity of the dual solution at QP region boundaries (while the primal solution is always continuous) - with elements of $\beta$ and $\hat{\beta}$ equal to zero for multipliers referring to new degenerate constraints at the particular region boundary. This is crucial, since this allows to formulate an active set algorithm based on homotopy of solutions.*

**Remark 5.3** *For the special case in which $\mathcal{A}$ contains no degenerate constraints $\beta = 0$ and $\hat{\beta} = 0$. In this case the KKT conditions (5.8), (5.9) admit unique solutions and therefore the EP solution corresponds to the unique solution of Problem (5.2).*

In the case that at a particular stage $k$ the reduced Hessian in (5.33$a$) is indefinite, there exists a unique subspace $\mathcal{Q}^+ \subseteq \mathcal{Q} \subseteq \mathbb{R}^{n_w}$ (where $\mathcal{Q} = \text{span} \begin{bmatrix} -\hat{C}_k D \\ G_k \end{bmatrix}_\perp$) in which (5.33$a$) is positive definite. Therefore the solution obtained from the first order conditions corresponds to a saddle point (instead of a maximizing) solution to the $w$-optimization subproblem at the particular stage. The backwards sweep and line search can be performed as in the convex-concave case, because there is no effect on the continuity of the solution at active set changes. Despite the existence of degenerate and non-concave subproblems it can therefore be shown (as explained in Section 5.6) that there exists a homotopy path between any two states $x_0$ in the feasible set $\mathcal{X}_0$.

We note that demonstrating the existence of an $l_2$-gain bound in closed loop requires special consideration (as discussed in Section 5.7). In both cases this is due to the fact that, in general, there exist realizations of the disturbance $w$, such that the actual cost may be greater than the predicted worst case cost. In the degenerate case this is due to the restriction of the maximization subproblem to a subspace of the space of $w$ for the given parameter $\hat{x}$ due to extra compatibility constraints required for the solution to the minimization subproblem, whereas in the non-concave case this is due to the fact that the KKT conditions correspond to a saddle point solution in the space of $w$ obtained for the given parameter $\hat{x}$ (and not a maximizer).

## 5.6 Active Set Method

This section proposes a method of solving (5.2) for a given plant state $x^p$ using the Riccati recursion of Section 5.3. Starting from an initial state $x_0 \neq x^p$ for which the optimal active set is known (e.g. $x_0 = 0$, $\mathcal{A} = \emptyset$), and using (5.23), (5.28) to determine optimal control, disturbance and multiplier sequences for (5.2) as functions of $x_0$, the approach uses a line-search in the space of $x_0$ to update the active set until $x_0 = x^p$. We discuss continuity properties of solutions before describing the algorithm, its convergence and computation. As in Chapters 3 and 4 we make the following assumption, which is common to active set solvers that employ floating point arithmetic:

**Assumption 5.2** *Only a single constraint can become active or inactive at each iteration.*

Using (5.1) and (5.31), for given $x_0$ and active set $\mathcal{A}$ we obtain:

$$x_k = \Phi_k x_0 + \phi_k, \quad k = 1, \dots, N \tag{5.35}$$

where $\Phi_k$, $\phi_k$ are defined by $\Phi_0 = I$, $\phi_0 = 0$ and

$$\Phi_{k+1} = (I + DM_k^w)(A + BL_k^u)\Phi_k \tag{5.36a}$$

$$\phi_{k+1} = (I + DM_k^w)\big((A + BL_k^u)\phi_k + Bl_k^u\big) + Dm_k^w. \tag{5.36b}$$

Therefore $\{u_0, \dots, u_{N-1}\}$ and $\{w_0, \dots, w_{N-1}\}$ can be determined as affine functions of $x_0$, which we denote as $\mathbf{u}(x_0, \mathcal{A})$ and $\mathbf{w}(x_0, \mathcal{A})$, respectively. Similarly, from (5.23), (5.28) and (5.34a,5.34b), the multiplier sequences $\{\mu_0, \dots, \mu_{N-1}\}$, $\{\nu_0, \dots, \nu_{N-1}\}$ and $\{\eta_0, \dots, \eta_{N-1}\}$ are obtained as affine functions of $x_0$, and (in the case that $\mathcal{A}$ contains a degenerate subproblem at $k = 0$) $\beta_0$, which we denote as $\boldsymbol{\mu}(x_0, \beta_0, \mathcal{A})$, $\boldsymbol{\nu}(x_0, \beta_0, \mathcal{A})$ and $\boldsymbol{\eta}(x_0, \beta_0, \mathcal{A})$. Hence the collection, $\mathbf{p}$, of predicted sequences for given $x_0, \beta_0$ and $\mathcal{A}$ can be expressed

$$\mathbf{p}(x_0, \beta_0, \mathcal{A}) = \big\{\mathbf{u}(x_0, \mathcal{A}), \mathbf{w}(x_0, \mathcal{A}), \boldsymbol{\mu}(x_0, \beta_0, \mathcal{A}), \boldsymbol{\nu}(x_0, \beta_0, \mathcal{A}), \boldsymbol{\eta}(x_0, \beta_0, \mathcal{A})\big\}. \tag{5.37}$$

Let $\mathcal{X}_{pQP}(\mathcal{A})$ denote the subset of the feasible initial conditions for (5.2) within which the EP solution $\mathbf{p}(x_0, \beta_0, \mathcal{A})$ satisfies the optimality conditions of Theorem 5.2, namely

$$\mathcal{X}_{pQP}(\mathcal{A}) = \{x_0 \mid \mathbf{p}(x_0, \beta_0, \mathcal{A}) \text{ satisfies } (5.32a), (5.32b), (5.34a), (5.34b) \text{ for some } \beta_0\}. \tag{5.38}$$

Then clearly $\bigcup_{\mathcal{A} \in \Sigma} \mathcal{X}_{pQP}(\mathcal{A})$, where $\Sigma$ is the set of all possible active sets, must cover the feasible set of plant states $\mathcal{X}_0$. Also by linearity of (5.32a,b), (5.34a,5.34b) and (5.35), $\mathcal{X}_{pQP}(\mathcal{A})$ is a convex polyhedral set.

**Lemma 5.2** *If $\mathcal{A}'$ is an active set that is not degenerate at $k = 0$, so that $\mathbf{p} = \mathbf{p}(x_0, \mathcal{A}')$ for all $x_0 \in \mathcal{X}_{pQP}(\mathcal{A}')$, then:*

*(i). For each $x_0 \in \partial\mathcal{X}_{pQP}(\mathcal{A}')$ (where $\partial\mathcal{X}_{pQP}(\mathcal{A})$ is the boundary of $\mathcal{X}_{pQP}(\mathcal{A})$), $x_0 \in \partial\mathcal{X}_{pQP}(\mathcal{A}'')$ for a unique active set $\mathcal{A}''$.*

*(ii). If $x_0 \in \partial\mathcal{X}_{pQP}(\mathcal{A}') \cap \partial\mathcal{X}_{pQP}(\mathcal{A}'')$, then $\mathbf{p}(x_0, \mathcal{A}') = \mathbf{p}(x_0, \mathcal{A}'')$ if $\mathcal{A}''$ is not degenerate at $k = 0$, and $\mathbf{p}(x_0, \mathcal{A}') = \mathbf{p}(x_0, \beta_0, \mathcal{A}'')$ for some $\beta_0$ otherwise.*

*If $\mathcal{A}'$ is degenerate at $k = 0$, then:*

*(iii). $\mathcal{X}_{pQP}(\mathcal{A}')$ is contained in a hyperplane in $\mathbb{R}^{n_x}$.*

*(iv). If $x_0 \in \mathcal{X}_{pQP}(\mathcal{A}')$, then either $x_0 \in \partial\mathcal{X}_0$, or $x_0 \in \partial\mathcal{X}_{pQP}(\mathcal{A}'')$ and $x_0 \in \partial\mathcal{X}_{pQP}(\mathcal{A}''')$ for some active sets $\mathcal{A}'', \mathcal{A}'''$ that are non-degenerate at $k = 0$.*

*(v). For each $x_0 \in \mathcal{X}_{pQP}(\mathcal{A}')$, $\mathbf{p}(x_0, \beta_0, \mathcal{A}')$ satisfies the optimality conditions of Theorem 5.2 for all $\beta_0 \in [\beta_{0,-}, \beta_{0,+}]$, where $\mathbf{p}(x_0, \beta_{0,-}, \mathcal{A}') = \mathbf{p}(x_0, \mathcal{A}'')$ and $\mathbf{p}(x_0, \beta_{0,+}, \mathcal{A}') = \mathbf{p}(x_0, \mathcal{A}''')$.*

*Proof:* Properties (i), (ii), (iv) and (v) follow from the complementarity conditions of Theorem 5.2 and the linearity and uniqueness of $\mathbf{p}$ as a function of $x_0$ and $\beta_0$ for given $\mathcal{A}$, whereas (iii) is due to the compatibility condition (5.34b), which applies to $x_0$ if $\mathcal{A}'$ is degenerate at $k = 0$. ■

Lemma 5.2 implies that, if $\mathcal{A}$ is degenerate at stage $k = 0$, then $\mathbf{u}, \mathbf{w}$ are continuous in a neighbourhood of $x_0 \in \mathcal{X}_{pQP}(\mathcal{A})$, whereas the multiplier sequences $\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\eta}$ are discontinuous at $x_0 \in \mathcal{X}_{pQP}(\mathcal{A})$. As a result, although the optimal cost $J_0^*(x_0)$ is continuous in a neighbourhood of $x_0 \in \mathcal{X}_{pQP}(\mathcal{A})$, the gradient of $J_0^*(x_0)$ is not uniquely defined. Due to the sequential min-max nature of (5.2a), this causes non-uniqueness of the active set satisfying the optimality conditions of Theorem 5.2 at $x_0 \in \mathcal{X}_{pQP}(\mathcal{A}')$, whenever $\mathcal{A}'$ contains a degenerate subproblem

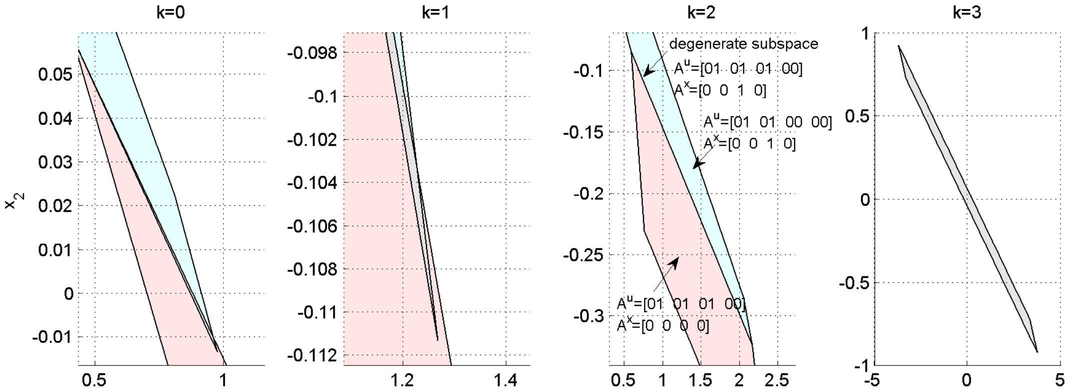

Figure 5.1: Neighbouring (overlapping) pQP regions for sequential minimization subproblems (i.e. after the maximization has been performed) for $N = 4$

at some stage $k > 0$, i.e. there exists at least one other active set $\mathcal{A}''$ such that $\mathcal{X}_{pQP}(\mathcal{A}') \subseteq \mathcal{X}_{pQP}(\mathcal{A}'')$. This non-uniqueness of active sets indicates that the globally optimal solution of (5.2) can be discontinuous at $x_0 \in \mathcal{X}_{pQP}(\mathcal{A}')$ if $\mathcal{A}'$ is degenerate at $k > 0$.

A geometric illustration is provided in Figure 5.1 for a two-dimensional SISO example for $N = 4$, where a state and input constraint are degenerate at $k = 2$. At $k = 2$, the common boundary of the two neighbouring regions is equal to the compatibility condition which is enforced for $k = 1$. As can been see in the figure, the state-spaces for $x_0$ and $x_1$ contain overlapping pQP regions.

The active set algorithm described below avoids the discontinuous solutions that could occur if the line-search always moved in the forward direction towards the current plant state by considering active set changes that ensure the continuity of optimal solution trajectories. This idea is illustrated in Figure 5.2, which shows the sequential min-max pQP regions in the space of $x_0$. An overlap occurs between points $A$ and $B$. The proposed homotopy approach moves to point $A$, detects the active set change which allows to deform the solution trajectories continuously resulting in a change of direction of the line-search (the search moves backwards inside this degenerate pQP region) until point $B$ is reached. Here the active set changes again and allows to move to point $C$. We show below that the required changes in direction of the line search are implemented simply by invoking conditions (5.32a), (5.32b), (5.34a), (5.34b) at the pQP region

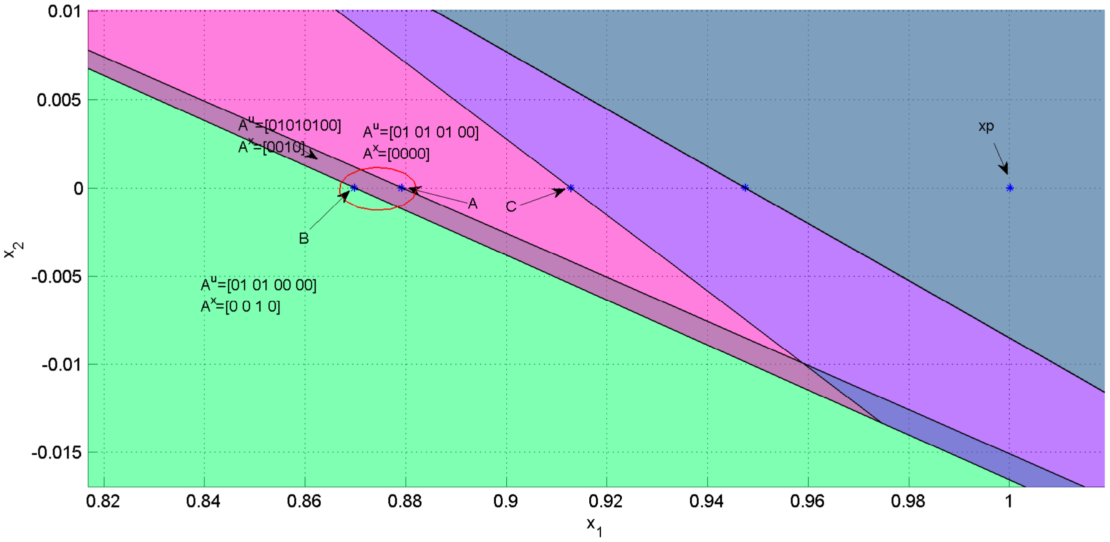

Figure 5.2: pQP regions for all sequential problems as expressed in $x_0$-space for $N = 4$

boundaries.

The general formulation of the algorithm is therefore as follows: At each iteration $i = 0, 1, \dots$ the solver determines $x_0^{(i+1)}$ and $\mathcal{A}^{(i+1)}$ from $x_0^{(i)}$ and $\mathcal{A}^{(i)}$ via a search over $x_0 \in \mathcal{X}_{pQP}(\mathcal{A}^{(i)})$ along the line $x_0 = x_0^{(i)} + \alpha(x^p - x_0^{(i)}), \alpha \in \mathbb{R}$. Since $\mathcal{X}_{pQP}(\mathcal{A}^{(i)})$ may overlap $\mathcal{X}_{pQP}(\mathcal{A}^{(i-1)})$, the search direction is determined by the condition $x_0 \in \mathcal{X}_{pQP}(\mathcal{A}^{(i)})$. For the special case of a degenerate subproblem in $\mathcal{A}^{(i)}$ at stage $k = 0$, the search is performed over $\beta_0$ with $x_0$ fixed, where under Assumption 5.2, $\beta_0$ is scalar. The algorithm results in a sequence of iterates satisfying the conditions stated in Theorem 5.2 for some $x_0$, but not necessarily for the initial condition $x_0 = x^p$.

**Algorithm 5.1** *Initialize with* $x_0^{(0)}$ *and an active set* $\mathcal{A}^{(0)}$ *such that* $x_0^{(0)} \in \mathcal{X}_{pQP}(\mathcal{A}^{(0)})$, *and set* $i := 0$. *At iteration* $i = 0, 1, \dots$*:*

1. *Compute* $\{P_k, q_k\}$ *for* $k = N-1, \dots, 0$, *and* $\{\Phi_k, \phi_k\}$ *for* $k = 0, \dots, N-1$, *and hence* $\mathcal{X}_{pQP}(\mathcal{A}^{(i)})$ *using (5.32a), (5.32b), (5.34a), (5.34b).*

2. *If* $i = 0$*:*

   $$\alpha^{(i)} := \max_{\alpha \le 1} \{\alpha \mid x_0^{(i)} + \alpha(x^p - x_0^{(i)}) \in \mathcal{X}_{pQP}(\mathcal{A}^{(i)})\}.$$

   *Else if* $\mathcal{A}^{(i)}$ *is non-degenerate at* $k = 0$*:*

$$\alpha^{(i)} := \max_{\alpha\leq 1}\big\{|\alpha| \mid x_0^{(i)} + \alpha(x^p - x_0^{(i)}) \in \mathcal{X}_{pQP}(\mathcal{A}^{(i)})\big\}.$$

*Else:*

$$\beta_{0,-}^{(i)} := \min_{\beta_0\in\mathbb{R}}\big\{\beta_0 \mid \mathbf{p}(x_0^{(i)}, \beta_0, \mathcal{A}^{(i)}) \textit{ satisfies } (5.32a), (5.32b), (5.34a), (5.34b)\big\}$$

$$\beta_{0,+}^{(i)} := \max_{\beta_0\in\mathbb{R}}\big\{\beta_0 \mid \mathbf{p}(x_0^{(i)}, \beta_0, \mathcal{A}^{(i)}) \textit{ satisfies } (5.32a), (5.32b), (5.34a), (5.34b)\big\}$$

*and select $\beta_0^{(i)}$ so that $\mathcal{A}^{(i+1)}$ is not equal to $\mathcal{A}^{(i)}$.*

*If $\mathcal{A}^{(i)}$ is degenerate at $k = 0$:*

*set $x_0^{(i+1)} := x_0^{(i)}$, $i := i + 1$ and update $\mathcal{A}^{(i)}$ using the new active constraints.*

*Else:*

*if $\alpha^{(i)} = 1$, set $\mathcal{A}^* := \mathcal{A}^{(i)}$, compute $\kappa_0^*(x^p)$ and stop.*

*if $\alpha^{(i)} < 1$, set $x_0^{(i+1)} := x_0^{(i)} + \alpha^{(i)}(x^p - x_0^{(i)})$, $i := i + 1$,*

*and update $\mathcal{A}^{(i)}$ using the new active constraints.*

*Return to step 1.*

**Remark 5.4** *If $x^p$ is infeasible, Algorithm 5.1 necessarily terminates with an infeasible line-search over $\beta_0$ in step 2.*

**Remark 5.5** *$x^p$ is assumed to be a feasible initial condition for (5.2), therefore a trivial initialization for Algorithm 5.1 is the choice $x_0^{(0)} = 0$ and $\mathcal{A}^{(0)} = \{\emptyset, \dots, \emptyset\}$. In the context of MPC, further computational savings can be achieved by warm-starting Algorithm 5.1. This can be done by choosing $x_0^{(0)}$ at time $t + 1$ equal to the plant state $x^p$ at time $t$. The active set can then be initially chosen as $\mathcal{A}^{(0)}(t + 1) = \mathcal{A}^*(t)$*

**Theorem 5.3** *For $x^p \in \mathcal{X}_0$, Algorithm 5.1 terminates after a finite number of iterations at $\mathcal{A}^*$ such that $x^p \in \mathcal{X}_{pQP}(A^*)$.*

*Proof:* Each iteration involves a line-search over one of the finite number of line segments that are defined by the intersections of the line containing $x^{(0)}$ and $x^p$ with the regions $\mathcal{X}_{pQP}(\mathcal{A})$ for all $\mathcal{A} \in \Sigma$. The line-search is constructed so that $x_0^{(i+1)} \in \partial\mathcal{X}_{pQP}(\mathcal{A}^{(i)}) \cap \partial\mathcal{X}_{pQP}(\mathcal{A}^{(i+1)})$ for all $i \geq 0$, and it follows from Lemma 5.2 that the active set change at each iteration is uniquely defined and since the EP solution for each active set is uniquely defined in terms of $x_0$ and $\beta_0$, the sequence $\{\mathcal{A}^{(i)}, i = 0, 1, \ldots\}$ is therefore uniquely defined by $x_0^{(0)}$ and $x^p$. Furthermore, each active set can either appear at most once in the sequence $\{\mathcal{A}^{(i)}, i = 0, 1, \ldots\}$ or else the sequence is necessarily periodic. We now show that the cycling between active sets that is implied by the latter case cannot occur by considering separately the two methods described in Remark 5.5 for initializing the algorithm.

If Algorithm 5.1 is initialized with $x_0^{(0)} = 0$, then the line segment joining $x_0^{(0)}$ and $x^p$ necessarily intersects $\mathcal{X}_f$ and hence contains points at which Problem (5.2) is known to have a unique solution, namely the unconstrained optimal control law (since this satisfies the conditions of Theorem 5.2 within $\mathcal{X}_f$ by construction). It follows that $\mathcal{A}^{(0)}$ appears only once in the sequence $\{\mathcal{A}^{(i)}, i = 0, 1, \ldots\}$, and, because of the uniqueness of the active set change at each iteration, cycling is therefore not possible. Given that $x^p \in \mathcal{X}_0$ Lemma 5.2 implies that $\{\mathcal{A}^{(i)}, i = 0, 1, \ldots\}$ must in this case terminate at an active set $\mathcal{A}^*$ such that $x^p \in \mathcal{X}_{pQP}(\mathcal{A}^*)$. Finally this must occur after a finite number of iterations since the total number of possible active sets is finite.

Consider next the case that Algorithm 5.1 is initialized with arbitrary $x_0^{(0)} \in \mathcal{X}_0$ by warm-starting. There must exist a homotopy between the EP solutions in any two contiguous regions $\mathcal{X}_{pQP}(\mathcal{A})$ and $\mathcal{X}_{pQP}(\mathcal{A}')$ that intersect the line segment joining $x_0^{(0)}$ and $x^p$ since the preceeding argument (and the fact that $\mathcal{X}_f$ contains an open neighbourhood of the origin) implies that, starting from some point in $\mathcal{X}_f$, it is possible to construct by homotopy the EP solution along a line that intersects the boundary $\partial\mathcal{X}_{pQP}(\mathcal{A}) \cap \partial\mathcal{X}_{pQP}(\mathcal{A}')$ and to place this intersection point arbitrarily close to the intersection point of the line segment joining $x_0^{(0)}$ and $x^p$ with the same boundary. The existence of such a homotopy follows by a construction of successive partial homotopy paths which result from warm starting for subsequent time instants $t$. Therefore the sequence $\{\mathcal{A}^{(i)}, i = 0, 1, \ldots\}$ necessarily terminates at an active set $\mathcal{A}^*$ such that $x^p \in \mathcal{X}_{pQP}(\mathcal{A}^*)$, and this must be achieved in a finite number of iterations since $\Sigma$ contains a finite number of active sets. ■

The possibility of overlapping regions in $x_0$-space has the implication that the line-search path changes direction whenever a new degenerate subproblem is encountered.

**Proposition 5.1** *If $\mathcal{A}$ is a degenerate active set then every point in $\mathcal{X}_{pQP}(\mathcal{A})$ also lies in $\mathcal{X}_{pQP}(\mathcal{A}')$ for some non-degenerate active set $\mathcal{A}'$.*

*Proof:* Suppose that at stage $k$ and iteration $i$ constraint $C1$ is active and that at iteration $i+1$ constraint $C2$ becomes active such that $C1$ and $C2$ are linearly dependent. The solution to the corresponding minimization subproblem therefore requires that a compatibility constraint $C3$ is satisfied and enforced in the maximization at stage $k-1$. Suppose we have moved an $O(\epsilon)$-distance (for $\epsilon \to 0$) along the line-of-search for the degenerate subproblem.

First, we make the linearly dependent constraint $C2$ at $k$ weakly active by choosing $\beta^+$ so that the multiplier of $C2$ is zero (step a). Secondly, we remove constraint $C2$ from the minimization at $k$ (step b). And finally, we remove the compatibility equality constraint $C3$ from the maximization at $k-1$ (step c). Moving $O(\epsilon)$ along the line of search implies that the multiplier of $C2$ is initially of $O(\epsilon)$. Therefore $\zeta$ is $O(\epsilon)$ in step b, so the removal of $C3$ from the maximization in step c causes a change in all primal variables and multipliers of $O(\epsilon)$, and hence the only constraint that can become active or inactive during step c is $C2$.

But $\zeta > 0$ after step a, implying that $C3$ is equivalent to an inequality constraint such that the optimal maximization cost will increase if it is relaxed. But $C2$ and $C3$ are equivalent since $C3$ enforces $C2$ in the minimization at stage $k$, so relaxing $C3$ must be equivalent to tightening $C2$ since this also causes an increase in the optimal maximization cost. Therefore $C2$ cannot become active in step c, and hence the KKT conditions are satisfied for the active set with both $C2$ and $C3$ removed from it. This is the same as the active set before the linearly dependent constraint $C2$ became active, which implies that the initial state $x_0$ must have moved back into the associated region where only $C1$ is active. Generalizations to higher-order degenerate subproblems and degenerate subproblems in maximization stages follow along similar lines. ■

**Remark 5.6** *Overlapping regions arise as a result of the combination of degenerate state and input constraints and the presence of maximization subproblems at prediction times* $k > 0$*. Therefore region overlaps do not occur in the nominal problem formulation (which can be condensed) and input-constrained problem formulation (which has no degeneracy) considered in Chapters 3 and 4.*

### 5.6.1 Computation

We consider the computational effort for solving a one-stage degenerate subproblem (in order to get conservative overall estimations). We make the assumption that (5.22) and (5.27) are solved using the Schur complement method (see e.g. [45]). This approach, applied to (5.22), involves computing the QR decomposition of $\begin{bmatrix} -D^T\hat{C}_k^T & G_k^T \end{bmatrix}^T$, which requires $O(2(n_G + d_{u+})n_w^2)$ floating point operations but can be further reduced using incremental rank-1 updates ($n_G$ is the number of rows of $G_k$ and $d_{u+}$ is the order of degeneracy in the minimization at stage $k+1$). Further the required matrix inversions and matrix multiplications require $O\big(a_1 n_w^3 + a_2 n_w^2(n_G + d_{u+}) + a_3 n_w(n_G + d_{u+})^2\big)$ operations. Applying the same approach to the solution of (5.27) requires

$$O\big(b_1 n_u^3 + b_2 n_u^2(n_F + n_E + \hat{d}_w) + b_3 n_u(n_F + n_E + \hat{d}_w)^2\big)$$

(where $n_G$ and $n_E$ are the number of rows of $G_k$, $E_k$ and $\hat{d}_w$ is the order of degeneracy in the maximization at stage $k$). The constants $a_1, a_2, a_3$ and $b_1, b_2, b_3$ depend on the implementation of the Schur complement method and the underlying functions used for the Cholesky and QR decompositions.

The other significant contribution is due to the matrix multiplications in (5.26) and (5.30b), which require $O\big(6n_x^3 + 2n_x^2(n_w + n_w + n_E + \hat{d}_w + d_u) + 2n_x[\hat{d}_w d_u + (n_E + \hat{d}_w)(n_u + d_{u+})]\big)$ operations. Combining the above estimates for $N$ stages gives an $O(N)$ dependence of computational complexity.

Noting that the computation required for the forward simulation is $O(n_x^2 N)$ (since only the projection, $\Phi_t(x^p - x_0^{(i)})$, of $\Phi_t$ in (5.36a,b) is needed), and also that the computation involved in the line search in step (ii) is comparatively insignificant, we estimate the computation per iteration of Algorithm 5.1 to grow as $O(N)$.

Thus the dependence of computation per iteration on the horizon length $N$ is linear. The required number of iterations is problem-dependent and depends on whether exact representations of the robust controllability sets are used as discussed in Section 5.9. Furthermore, the number of iterations can be minimized using warm-starts (as described in Remark 5.5 and demonstrated in Section 5.9). The computational load is in stark contrast to existing schemes for min-max receding horizon control, which, for the case of optimal approaches that are based on dynamic programming, have computational loads that depend exponentially on $N$ (see e.g. [31, 50]). Likewise, the approach of [24] based on a suboptimal (affine) controller parameterization requires the solution of a semidefinite program (SDP) in a number of optimization variables that grows quadratically with the horizon length and, as illustrated in Section 5.9, this typically results in a much higher computational load than Algorithm 5.1.

## 5.7 Closed Loop Stability and $l_2$-Gain Bound

First, assuming dual mode predictions, we define the unconstrained optimal feedback law as the terminal control law (for the given $\mathcal{H}_\infty$-type stage cost) and determine a robust, controlled, positive invariant set $\mathcal{X}_f$ as the corresponding terminal set as given by Definition 4.1. We then ensure recursive feasibility through the use of the robust controllable sets $\mathcal{X}_k$ in the dynamic programming recursion as defined in Problem (5.2). The proposed receding horizon application of Algorithm 5.1 therefore ensures a specified $l_2$-gain bound for all $x_p \in \mathcal{X}_0$, for a given initial horizon length $N_0$. Since the following results make use of a possibly time-varying horizon length (in constrast to a fixed horizon $N$ for all $t$), we make the following definition:

**Definition 5.2** *Let $\mathcal{X}_0^t$ be the feasible set $\mathcal{X}_0$ for Problem (5.2) with horizon length $N = N_t$*

**Definition 5.3** *At a given time $t$, let $R_{N_t} \subseteq \mathcal{X}_0^t$ be the region in which the subproblems in the solution to Problem (5.2) with horizon length $N = N_t$ are non-degenerate and convex-concave.*

**Definition 5.4** *At a given time $t$, let $J_0^*(x_t^p)$ denote the predicted cost corresponding to the current plant state $x_t^p$ and the selected active set $\mathcal{A}^*(t)$*

**Assumption 5.3** *Let initially $N_t = N_0$ for $t \geq 0$. If at time step $t = T+1$ we find that $\frac{1}{2}(\|x_T^p\|_Q^2 + \|\kappa_0^*(x_T^p)\|_R^2 - \gamma^2\|w_T\|^2) + J_0^*(x_{T+1}^p) > J_0^*(x_T^p)$, then let $N_{T+1} := N_T - 1$ and define the corresponding shortened finite horizon optimal control problem (5.2) with $N_t$ redefined as $N_t = N_{T+1}$ for $t > T$.*

**Theorem 5.4** *If $x_0^p \in \mathcal{X}_0^0$, then for any admissible disturbance sequence $\{w_t \in \mathcal{W},\ t = 0, 1, \ldots\}$, the state of (5.1) with $u_t = \kappa_0^*(x_t^p)$ satisfies $x_t^p \in \mathcal{X}_0^0$ for all $t \geq 0$ and*

*(i)*

$$\sum_{t=0}^{\infty}(\|x_t^p\|_Q^2 + \|u_t\|_R^2) \leq \gamma^2 \sum_{t=0}^{\infty}\|w_t\|^2 + 2J_0^*(x_0^p) \text{ if } x_t^p \in R_{N_0}\ \forall t \geq 0 \tag{5.39}$$

*(ii)*

$$\sum_{t=0}^{\infty}(\|x_t^p\|_Q^2 + \|u_t\|_R^2) \leq \gamma^2 \sum_{t=0}^{\infty}\|w_t\|^2 + 2J_0^*(x_0^p) + \Delta \quad \text{for some finite } \Delta \tag{5.40}$$

$$\text{if } \exists t \geq 0 \text{ s.t. } x_t^p \notin R_{N_0}$$

*Proof:* The robust positive invariance of $\mathcal{X}_0^0$ follows from its definition. The $l_2$-gain bound (5.39) follows from the monotonicity property $J^*_{0,N_0}(x_t^p) \leq J^*_{0,N_0-1}(x_t^p)$ (where we make the shortened horizon length explicit through the index notation) for the optimal value function and the optimality of the value function $J^*_{0,N_0}(x_t^p)$. This gives the expression:

$$\begin{aligned}&\tfrac{1}{2}(\|x_t^p\|_Q^2 + \|\kappa^*_{0,N_0}(x_t^p)\|_R^2 - \gamma^2\|w_t\|^2) + \\ &J^*_{0,N_0}(Ax_t^p + B\kappa^*_{0,N_0}(x_t^p) + Dw_t) \leq J^*_{0,N_0}(x_t^p) \quad x_t^p \in R_{N_0} \quad \forall w_t \in \mathcal{W}\end{aligned} \tag{5.41}$$

In the presence of degenerate or non-concave subproblems, there is no guarantee that the solution obtained using Algorithm 5.1 is globally optimal for given $x_t^p$. Therefore to demonstrate the bound stated in (5.40), we consider the following modified statement, which introduces an additional term $\delta_t(w, x_t^p)$ to account for the possible non-monotonicity of the predicted cost that may result:

$$\begin{aligned}&\tfrac{1}{2}(\|x_t^p\|_Q^2 + \|\kappa^*_{0,N_t}(x_t^p)\|_R^2 - \gamma^2\|w_t\|^2) + J^*_{0,N_t}(Ax_t^p + B\kappa^*_{0,N_t}(x_t^p) + Dw_t) \\ &\leq J^*_{0,N_t}(x_t^p) + \delta_t(w_t, x_t^p) \leq J^*_{0,N_t}(x_t^p) + \tilde{\delta}_t \quad x_t^p \notin R_{N_t} \quad \forall w_t \in \mathcal{W}\end{aligned} \tag{5.42}$$

where $\tilde{\delta}_t$ is an upper bound on $\delta(w_t, x_t^p)$ which is obtained by maximizing over all admissible $x_t^p$ and $w_t$ for a given horizon $N_t$.

Further, $J^*_{0,N_t}(x_t^p)$ is bounded for all $t$ and suboptimility gaps $\tilde{\delta}_t$ are incurred at a finite number of instances due to Assumption 5.3. In the worst case the horizon can shrink $N_0$ times, after which $x_t^p$ will remain in $\mathcal{X}_f$ for all $t$, so we necessarily obtain for a finite $\Delta_1 = 2\sum_{t=0}^{N-1} \tilde{\delta}_t$ and $\Delta_2 = 2\sum_{t=0}^{N-1} J^*_{0,N_{t+1}}(x_{t+1}^p) - J^*_{0,N_t}(x_{t+1}^p)$:

$$\sum_{t=0}^{\infty}(\|x_t^p\|_Q^2 + \left\|\kappa^*_{0,N_t}(x_t^p)\right\|_R^2) \leq \gamma^2 \sum_{t=0}^{\infty}(\|w_t\|^2) + 2J^*_{0,N_0}(x_0^p) + \Delta_1 + \Delta_2 \tag{5.43}$$

whenever $x_t^p \notin R_{N_0}$ implying the $l_2$-bound (5.40) with $\Delta = \Delta_1 + \Delta_2$. ■

**Remark 5.7** *We note that in numerical examples performed so far, the line* *search resulted in a non-degenerate solution at the current plant state* $x_t^p$ *in almost all practical situations. Therefore, provided* $\gamma$ *is chosen reasonably large to ensure convex-concavity, Assumption 5.3 does result in a receding horizon control law with fixed horizon length and* $l_2$*-bound (5.39) in practice. We also note that when Condition (5.40) applies,* $\Delta$ *may be still be zero, because the horizon is not necessarily shortened every time* $x_t^p \notin R_{N_0}$ *as this condition is necessary but not sufficient for the horizon to be shortened on the basis of the criterion in Assumption 5.3.*

## 5.8 Approximate DP via Policy Choice

In the robust MPC literature the major focus has been to solve the underlying DP approximately via choice of a suboptimal feedback policy. The application for two of such policies with major impact in the literature shall be discussed in this section, the pre-stabilizing feedback policy [35, 39, 44] and the disturbance-affine policy [24, 25]. For the pre-stabilizing feedback policy, we will consider a homotopy-based approach and we also note that is possible to solve other robust MPC methods based on QP via a homotopy-based active set method (such as the tube-based approach of [44]).

### 5.8.1 Pre-stabilizing DP Policy

The $\mathcal{H}_\infty$ Receding Horizon problem was tackled via pre-stabilizing policies in [40], however this paper does not consider the state-constrained case and does not exploit the underlying structure of the pQPs in order to improve computational performance. This section provides the main ideas to overcome these limitations. The problem to be considered is given by:

$$\min_{\boldsymbol{\kappa} \in \mathcal{M}(x)} \left\{ \max_{\mathbf{w} \in \mathcal{W}^N} J_0(x, \boldsymbol{\kappa}, \mathbf{w}) \right\} \tag{5.44}$$

where $\boldsymbol{\kappa}$ is the feedback policy that defines $u_k$ at each stage $k$ and $\mathbf{w}$ the sequence of disturbances. It is assumed that the cost due to an individual realization of the disturbance $\mathbf{w}$ is given by

$$J_0(x, \boldsymbol{\kappa}, \mathbf{w}) = \sum_{k=0}^{N-1} \frac{1}{2}(\|x_k\|_Q^2 + \|u_k\|_R^2 - \gamma^2 \|w_k\|^2) + J_N^*(x_N) \tag{5.45}$$

Now we assume that the class of pre-stabilizing feedback policies is employed:

$$\boldsymbol{\kappa} = \{\kappa_0, \ldots, \kappa_{N-1}\} \tag{5.46}$$

with $\kappa_k = Kx_k + c_k$. This class is characterized by an offline choice of a 'prestabilizing' feedback matrix $K$ and an online perturbation sequence $\mathbf{c} = \{c_0, \ldots, c_{N-1}\}$. For a given realization of the disturbance sequence $\mathbf{w}$ the state prediction sequence $\mathbf{x}$ can be expressed as:

$$\mathbf{x} = N^x x + N^c \mathbf{c} + N^w \mathbf{w} \tag{5.47}$$

and the set of admissible policies for general mixed state and input constraints $\overline{E}\mathbf{x} + \overline{F}\mathbf{u} \leq \mathbf{1}$ is given by:

$$\mathcal{M}^{PS}(x) = \left\{\mathbf{c} \mid \mathbf{E}\mathbf{c} \leq \mathbf{1} - \mathbf{F}x - \mathbf{G}\mathbf{w} \quad \forall \mathbf{w} \in \mathcal{W}^N\right\} \tag{5.48}$$

where $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$ are determined by a condensing procedure using the chosen policy and the system dynamics. This is equivalent to the following formulation:

$$\mathcal{M}^{PS}(x) = \{\mathbf{c} \mid \mathbf{E}\mathbf{c} \leq \mathbf{1} - \mathbf{F}x - \mathbf{g}\} \tag{5.49}$$

where we assume that the Minkowski set difference operation has been performed (offline) by solving a number of LPs to yield

$$\mathbf{g} = \max_{\mathbf{w} \in \mathcal{W}^N} \mathbf{G}\mathbf{w} \tag{5.50}$$

Essentially this policy enables removal of the 'prestabilized' effects of uncertainty from the feasibility side of the DP problem by performing offline computations. Furthermore, the prediction structure can be used to express the cost as:

$$\begin{aligned} &J_0(x, \boldsymbol{\mu}, \mathbf{w}) = J_0(x, \mathbf{c}, \mathbf{w}) = \\ &\frac{1}{2}(x^T A_{xx} x + \mathbf{c}^T A_{cc}\mathbf{c} + \mathbf{w}^T A_{ww}(\gamma)\mathbf{w} + 2x^T A_{xc}\mathbf{c} + 2x^T A_{xw}\mathbf{w} + 2\mathbf{w}^T A_{wc}\mathbf{c}) \end{aligned} \tag{5.51}$$

where $A_{xx}$, $A_{cc}$, $A_{ww}(\gamma)$,$A_{xc}$,$A_{xw}$ and $A_{wc}$ are obtained as a result of the condensing process. We note that $A_{cc} \in \mathcal{S}_{++}^{Nn_u}$and assume that the maximization problem is concave for all active sets likely to be encountered. This requires $\gamma$ to be chosen large enough so that the reduced Hessian based on $-A_{ww}(\gamma)$ is a positive definite matrix. Overall the problem is therefore convex-concave in $\mathbf{c}$ and $\mathbf{w}$. Assuming $\mathcal{W}^N$ is defined by:

$$\mathcal{W}^N = \{\mathbf{w} \mid Z\mathbf{w} \leq \mathbf{1}\} \tag{5.52}$$

we can now formulate the Lagrangian function in order to obtain necessary conditions for a saddle point solution:

$$L(x, \mathbf{c}, \mathbf{w}, \boldsymbol{\eta}, \boldsymbol{\mu}) = J_0(x, \mathbf{c}, \mathbf{w}) + \boldsymbol{\eta}^T(\mathbf{1} - Z\mathbf{w}) - \boldsymbol{\mu}^T(\mathbf{1} - \mathbf{E}\mathbf{c} - \mathbf{F}x - \mathbf{g}) \tag{5.53}$$

This leads to the first order necessary conditions of optimality (KKT) expressed as the following block matrix equation (where $\mathcal{A}/\mathcal{I}$ indicate the active/inactive rows of $\mathbf{E}, \mathbf{F}, \mathbf{g}$ and $Z$:

$$\begin{bmatrix} A_{cc} & A_{wc}^T & \mathbf{E}_{\mathcal{A}}^T & 0 \\ A_{wc} & A_{ww} & 0 & Z_{\mathcal{A}}^T \\ \mathbf{E}_{\mathcal{A}} & 0 & 0 & 0 \\ 0 & Z_{\mathcal{A}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{w} \\ \boldsymbol{\mu}_{\mathcal{A}} \\ \boldsymbol{\eta}_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} A_{xc}^T \\ A_{xw}^T \\ -\mathbf{F}_{\mathcal{A}} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \mathbf{1} - \mathbf{g}_{\mathcal{A}} \\ \mathbf{1} \end{bmatrix}$$

which defines the equality constrained solution, while the multipliers and inactive primal constraints satisfy

$$\begin{aligned} \boldsymbol{\mu}_{\mathcal{I}} = 0 \quad \mathbf{1} - \mathbf{E}_{\mathcal{I}}\mathbf{c} - \mathbf{F}_{\mathcal{I}}x - \mathbf{g}_{\mathcal{I}} \leq 0 \quad \boldsymbol{\mu}_{\mathcal{A}} \geq 0 \\ \boldsymbol{\eta}_{\mathcal{I}} = 0 \quad \mathbf{1} - Z_{\mathcal{I}}\mathbf{w} \leq 0 \quad \boldsymbol{\eta}_{\mathcal{A}} \geq 0 \end{aligned}$$

for the inequality constrained problem. We note that this result can be used to formulate an active set method based on homotopy. Further, $l_2$-stability results carry over provided terminal constraints are satisfied as discussed in Section 5.7. An advantage of this approach is that the possibility of overlapping regions can be excluded, since the problem can be condensed and be reformulated as a single parametric, quadratic min-max problem. In this sense the solution structure is similar to the nominal MPC problem discussed in Chapter 3. In addition it appears possible to exploit more structure via a Riccati recursion approach to solve the underlying EP problems and would lead to linear complexity increase per iteration. We will not follow up on this method due to the performance advantages of exact DP methods in general. However, when the system dimension is too high to employ exact DP methods, suboptimal approaches which scale well with the system dimension offer clear advantages and therefore the described approach would appear very attractive in fast sampling and high-dimensional applications.

### 5.8.2 Disturbance Affine DP Policy

The disturbance affine policy assumes each input is parametrized as an affine function of the sequence of past disturbances. The disturbance affine policy includes the pre-stabilizing policy as a special case (see [25] for details) and, since it offers a widely accepted compromise between computational efficiency and degree of suboptimality as compared to DP, we choose it as a benchmark for

Algorithm 5.1:

$$\mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{c} \tag{5.54}$$

where $\mathbf{M}$ is given by:

$$\mathbf{M} = \begin{bmatrix} 0 & \dots & \dots & 0 \\ M_{1,0} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \dots & M_{N-1,N-2} & 0 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_0 \\ \vdots \\ \vdots \\ c_{N-1} \end{bmatrix} \tag{5.55}$$

This allows to define set of feasible parameters for this policy as:

$$\mathcal{M}^{DA}(x) = \left\{ (\mathbf{M}, \mathbf{c}) \mid (\mathbf{M}, \mathbf{c}) \text{ s.t. } (5.55),\ \tilde{\mathbf{E}}\mathbf{c} + (\tilde{\mathbf{E}}\mathbf{M} + \tilde{\mathbf{F}})\mathbf{w} \leq \mathbf{1} - \tilde{\mathbf{G}}x\ \forall \mathbf{w} \in \mathcal{W}^N \right\} \tag{5.56}$$

where $\tilde{\mathbf{E}}$, $\tilde{\mathbf{F}}$, $\tilde{\mathbf{G}}$ are determined by a condensing procedure using the chosen policy and the system dynamics (for details see [25]). The crucial insight is that the set $\mathcal{M}^{DA}(x)$ can be shown to be convex [25] and therefore methods from convex optimization can be employed (see Chapter 2). In particular the $\mathcal{H}_\infty$ Receding Horizon problem can be tackled using disturbance affine feedback policies and as discussed in [24] this requires the solution of a semidefinite program at each sampling instant. Further, due to the triangular structure of the feedback matrix, computation grows polynomially with horizon length. Both of these issues imply that in general this method is intractable for large/fast systems and long horizon lengths.

## 5.9 Numerical Examples

### 5.9.1 Example A

We consider the same example as in Section 3.6 (considering airplane pitch dynamics) with linear discrete-time dynamics described by:

$$A = \begin{bmatrix} 0.9384 & 0.1341 & 0 & 0 \\ -0.5363 & 0.4022 & 0 & 0 \\ 0.1186 & 0.0066 & 1 & 0.2 \\ 1.1737 & 0.0923 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0.0462 \\ 0.4022 \\ -0.0190 \\ -0.1803 \end{bmatrix}$$

The constraint sets are given by $\mathcal{U} = \{u \in \mathbb{R} \mid -25 \leq u \leq 25\}$ and $\mathcal{W} = \{w \in \mathbb{R}^2 \mid -0.25 \leq w_i \leq 0.25 \text{ for } i = 1, 2\}$, and cost weights by $Q = \text{diag}[0, 0, 0.04, 0]$, $R = 2.5 \cdot 10^{-4}$. As discussed in Section 2.1.4.3, by solving the generalized Riccati equation, $P$, $L^u$, and $L^w$ are computed with $\gamma$ chosen sufficiently large to satisfy conditions (5.33a,b) of Theorem 5.2. For $N = 15$ this gives

$$P = \begin{bmatrix} 0.7171 & 0.1188 & 0.3680 & 0.2953 \\ 0.1188 & 0.0198 & 0.0599 & 0.0486 \\ 0.3680 & 0.0599 & 0.2762 & 0.1609 \\ 0.2953 & 0.0486 & 0.1609 & 0.1238 \end{bmatrix}, \quad L^u = \begin{bmatrix} -25.6258 \\ -4.6188 \\ -6.4620 \\ -8.5796 \end{bmatrix}^T$$

$$L^w = \begin{bmatrix} 0.0244 & 0.0041 & 0.0124 & 0.0100 \\ 0.0608 & 0.0100 & 0.0331 & 0.0255 \end{bmatrix}, \quad \gamma^2 = 5.$$

A robust positive invariant set $\mathcal{X}_f$ is obtained using the procedure of [32] and the robust controllability sets $\mathcal{X}_k$ for $k = 14, \ldots, 0$ were computed offline as in [6].

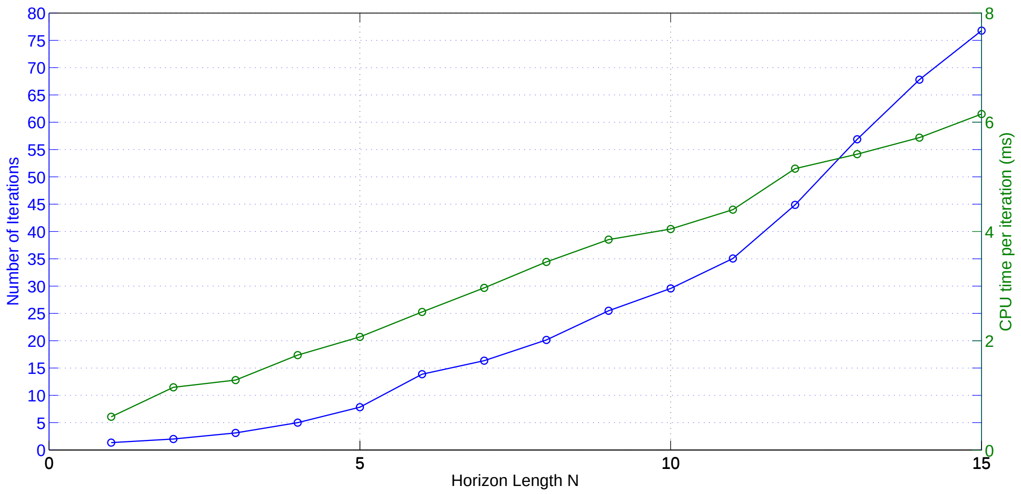

Figure 5.3: Number of iterations and CPU time/iteration vs. horizon length

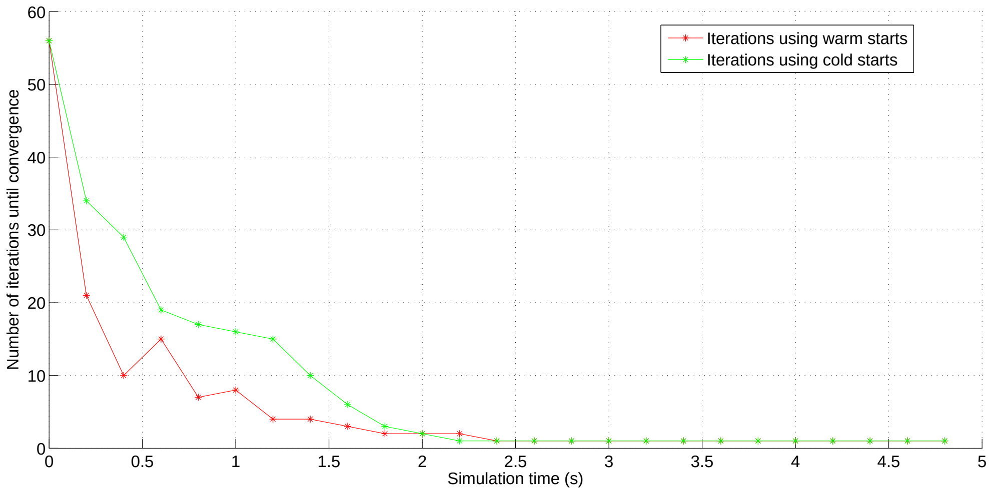

Figure 5.4: Closed loop simulation - benefits of warm-starting ($N = 10$)

#### 5.9.1.1 Dependence of CPU Time per Iteration and Number of Iterations on Horizon Length $N$

To illustrate how the number of iterations and CPU time per iteration varies with horizon length 400 plant states were sampled inside $\mathcal{X}_{N-1} \setminus \mathcal{X}_N$ for $N = 14, ..., 0$. The CPU time per iteration grows linearly with horizon length in agreement with the theoretical results (see Figure 5.3). The growth of the number of iterations with horizon length depends on the example chosen, but may be explained to some degree by the increasing complexity of the robust controllability sets ($\mathcal{X}_0$,

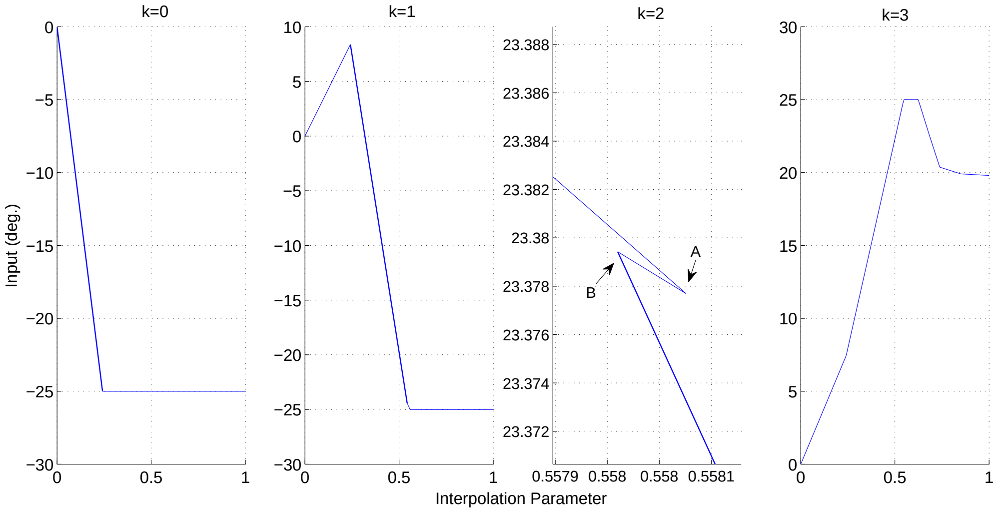

Figure 5.5: Input $u = E$ as function of interpolation parameter ($N = 4$)

for example, is described by 231 inequalities). For $N = 15$ the average CPU time per iteration is 6.24 ms and the average number of iterations 77.31 (total time 0.48 s).

### 5.9.1.2 Number of Iterations in Closed Loop Operation: Cold vs. Warm Starting

The number of iterations can be significantly reduced in closed loop operation by using warm starts. This is illustrated for $N = 10$ for $x_p = \begin{bmatrix} 0 & 0 & 60 & 0 \end{bmatrix}^T$ (which is inside $\mathcal{X}^9 \setminus \mathcal{X}^{10}$) for a simulation time of 5 s (see Figure 5.4).

### 5.9.1.3 Handling of Degenerate Constraints

Since the aircraft pitch dynamics are non-minimum phase (as the closed loop response confirms in Figure 5.6), the example is adequate to demonstrate the situation of degenerate input and state constraints. For $N = 4$ the input profiles are depicted as functions of the interpolation parameter (which lies in [0,1] and the algorithm is cold started). Figure 5.5 shows that the input is a piecewise affine function along this parameter, which allows for overlapping regions due to non-unique Lagrange multipliers. This example illustrates the continuity of the input

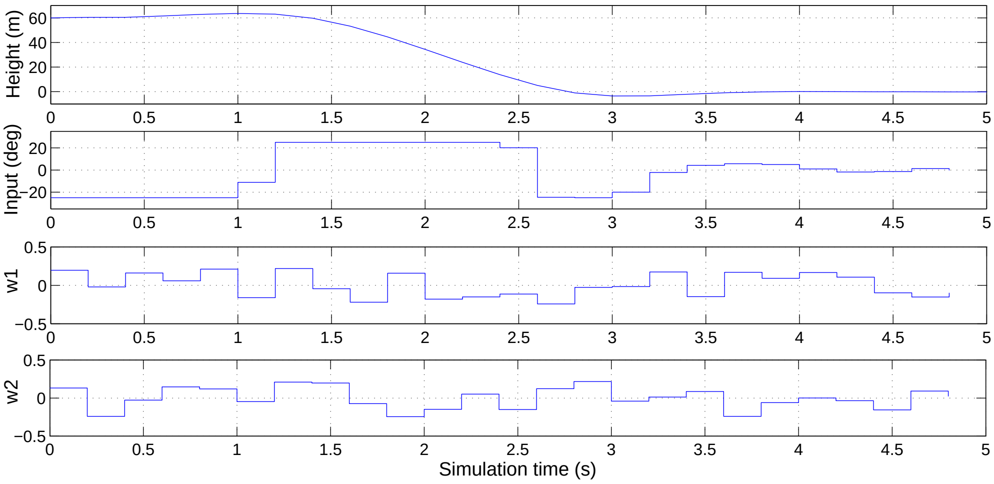

Figure 5.6: Closed loop response of aircraft height ($x_3$) for $N = 10$

at the $k = 2$ stages-to-go problem. Here a forward step (inside a non-degenerate pQP region) is followed by a backward step (inside a degenerate pQP region) and then another forward step (again inside a non-degenerate pQP region).

### 5.9.2 Example B

The proposed robust MPC algorithm was applied to (5.1) with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and with constraint sets $\mathcal{U} = \{u \in \mathbb{R} \mid -1 \leq u \leq 1\}$ and $\mathcal{W} = \{w \in \mathbb{R}^2 \mid -0.3 \leq w_i \leq 0.3 \text{ for } i = 1, 2\}$, and cost weights $Q = [1 \quad 0]^T[1 \quad 0]$, $R = 0.8$. This example represents a double-integrator with bounded control and disturbance inputs, which is commonly used as a benchmark problem in robust MPC [46]. $P$, $L^u$, and $L^w$ were computed for $\gamma^2 = 20$ giving:

$$P = \begin{bmatrix} 2.7829 & 2.3435 \\ 2.3435 & 3.7400 \end{bmatrix}, \quad L^u = \begin{bmatrix} -0.5606 & -1.3965 \end{bmatrix} \quad L^w = \begin{bmatrix} 0.1849 & 0.1708 \\ 0.1708 & 0.2546 \end{bmatrix}.$$

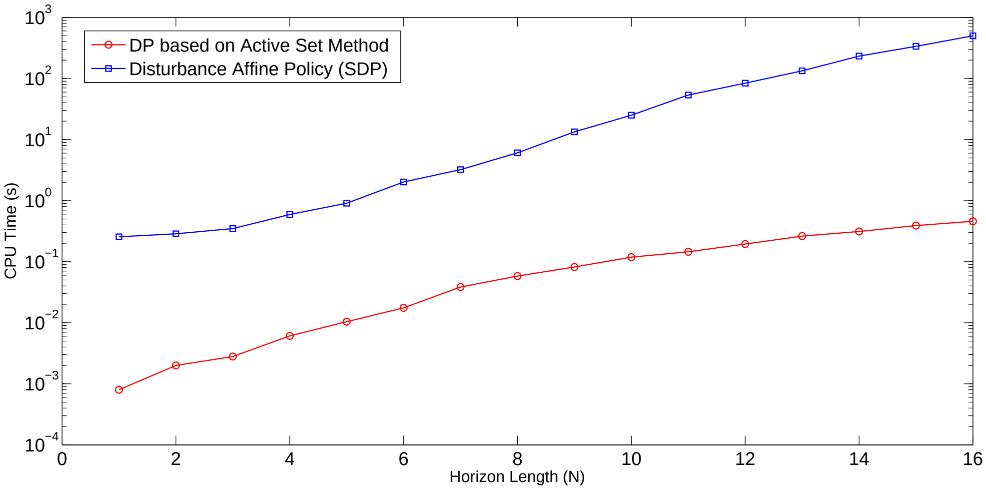

Figure 5.7: CPU time of DP active set algorithm and Disturbance Affine policy for Example B

Figure 5.7 compares the computational load of the exact active set Algorithm 5.1 (implemented in Matlab) with that required by a disturbance affine policy [25] applied to problem (5.2). The DA policy is formulated as a semidefinite program (as in [24]), and CPU times are SeDuMi [54] solution times. The results were obtained for a set of 50 plant states $x^p$ sampled inside the set $\mathcal{X}_0^{DA} \setminus \mathcal{X}_1^{DA}$, for $N = 15, \dots, 0$ and in each case Algorithm 5.1 was initialized with $x_0^{(0)} = 0$. This illustrates that the active set algorithm is several orders of magnitude faster than the DA approach. An approximation of the feasible set $\mathcal{X}_0^{DA}$ is obtained by considering 50 plant state equally spaced along the boundary of the set $\mathcal{X}_0^{DP}$ and decreasing them successively by 2% until the affine policy becomes feasible (N.B. to demonstrate that this procedure leads to a reasonable approximation the set $0.98\mathcal{X}_0^{DP}$ is also given in Figure 5.8). The predicted performance for these plant states using the DA policy was found to be 4.23% suboptimal (where in order to provide a meaningful comparison plant states were selected for which no degenerate or non-concave subproblems occured). We note that although the DA policy can lead to near-optimal results, it can also be significantly suboptimal: We consider the above example for $N = 12$ (using again $\gamma^2 = 20$). For the plant state $x_p = [16.8756 \quad -5.0627]^T$ we obtain predicted costs $J_{DA} = 743.4$

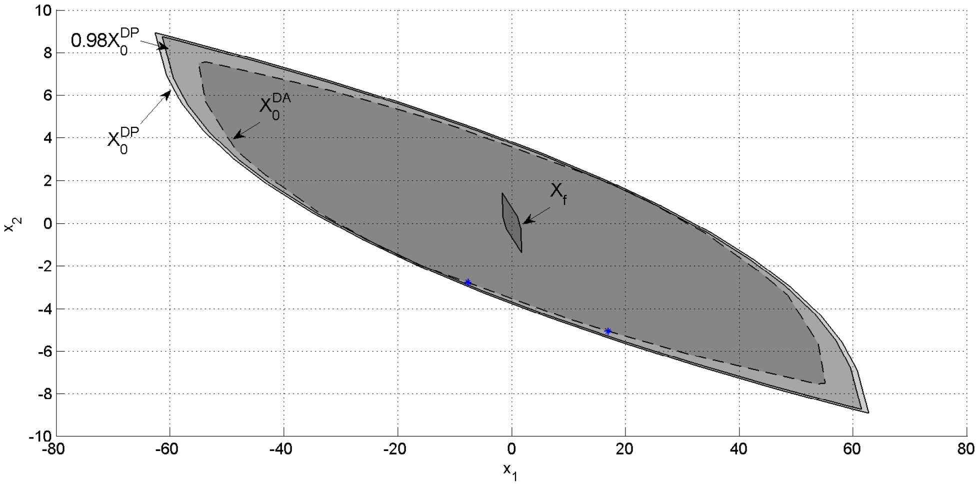

Figure 5.8: Feasible set for DP and DA-Policy for Example B with $N = 16$

and $J_{DP} = 637.1$ for the DA and DP approach respectively, indicating 16.7% suboptimality of the DA policy.

## 5.10 Conclusions

This chapter has discussed an efficient active set algorithm for the robust min-max MPC problem for input and state constrained linear systems with bounded disturbances based on the first order necessary conditions for optimality. The method results in remarkable improvements both in terms of computational efficiency and performance over established solution methods based on disturbance-affine parametrizations. A interesting extension is to compute robust controllable sets locally online using a homotopy approach, which would provide a completion of the contribution presented in this chapter, which requires the offline, global computation of robust controllable sets which can lead to computationally intensive offline computations. This topic is considered in Chapter 6.

# Chapter 6

# Robust Controllability for Linear Discrete-Time Systems

The active set solution to the dynamic programming formulation of the min-max receding horizon control problem presented in Chapter 5 requires the computation of robust controllable sets to ensure recursive feasibility. This chapter discusses a variety of ways of obtaining exact and approximate representations of robust controllable sets for linear discrete time systems with additive bounded disturbances (where a general discussion of this problem is available in [4, 6]). As noted earlier, many suboptimal formulations to the robust MPC problem have been presented, in particular in the 'tube' MPC context, which avoid the necessity to compute controllability sets and instead ensure recursive feasibility through a tube construction [39, 44, 46]. If exact dynamic programming based robust MPC is employed, the availability of such sets is generally assumed in the literature [1, 43]. This chapter first considers offline methods to obtain exact and approximate representations of global robust controllable sets. We then discuss a parametric linear programming approach to solve via homotopy - analogously to the active set approach presented in Chapters 3-5 - the robust controllability problem locally which would make it an attractive online method. The discussion of this approach starts with a derivation of the corresponding KKT conditions for a

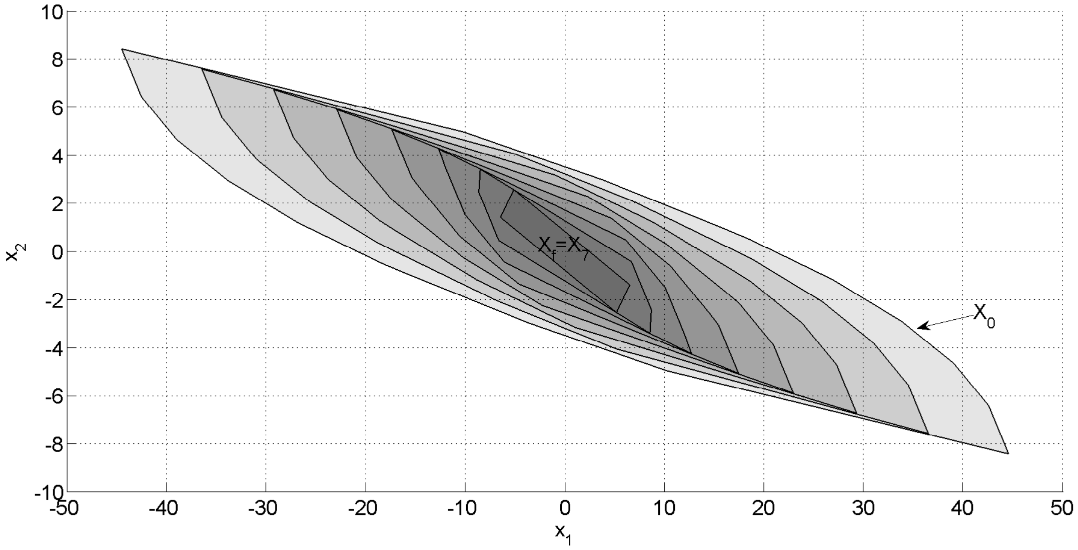

Figure 6.1: Exact robust controllable sets for $N = 7$ for a double integrator example

single-step controllability problem, and for this single-step problem the proposed active set approach is formulated and finally verified by a numerical instance. Finally some remarks on possible generalizations to the solution of the multiple step problem are provided.[1]

## 6.1 Offline Computation of Robust Controllable Sets

### 6.1.1 Exact Methods based on Projection

In this section we determine exact representations of $k$-step robust controllable sets to a convex polyhedral target set $\mathcal{X}_f$ for linear systems with additive bounded disturbances and input constraints:

$$x^+ = Ax + Bu + Dw \tag{6.1}$$

---

[1] The derivation presented in this chapter is limited to robust controllability to a target set for simplicity, but this may be generalized to the inclusion of state path constraints [6]

where $u \in \mathcal{U}$, $w \in \mathcal{W}$ and both $\mathcal{U}$ and $\mathcal{W}$ are convex polytopic sets given by:

$$\mathcal{U} = \{u \in \mathbb{R}^{n_u} \mid Fu \leq \mathbf{1}\}, \quad \mathcal{W} = \{w \in \mathbb{R}^{n_w} \mid Gw \leq \mathbf{1}\}$$

for $F \in \mathbb{R}^{n_F \times n_u}$, $G \in \mathbb{R}^{n_G \times n_w}$ and the target set $\mathcal{X}_f$ is given by:

$$\mathcal{X}_f = \left\{x \in \mathbb{R}^{n_x} \mid E^N x \leq \mathbf{1}\right\} \tag{6.2}$$

where the matrix $E^N$ has $n_{E^N}$ rows.

We first recall what is understood by robust controllable sets. Then we perform the following set recursion:

$$\mathcal{X}_k = \{x \in \mathbb{R}^{n_x} \mid \exists u \in \mathcal{U},\ Ax + Bu + Dw \in \mathcal{X}_{k+1} \quad \forall w \in \mathcal{W}\} \tag{6.3}$$

Performing this set recursion $N$ times, for $k = N-1, \ldots, 0$, with initial condition $\mathcal{X}_N = \mathcal{X}_f$ defines the $N$-step controllable set to $\mathcal{X}_0$, which we denote as $\mathcal{X}_0^{DP}$ in the sequel. For notational convenience, we denote $\mathcal{X}_k$ as $\mathcal{X}$ and $\mathcal{X}_{k+1}$ as $\mathcal{X}^+$. Since $\mathcal{X}^+$ is given as a convex polyhedral set we obtain

$$\mathcal{X}^+ = \left\{x \in \mathbb{R}^{n_x} \mid E^+ x \leq \mathbf{1}\right\} \tag{6.4}$$

and we therefore restate the set recursion as:

$$\mathcal{X} = \left\{x \in \mathbb{R}^{n_x} \mid \exists u \in \mathcal{U},\ E^+(Ax + Bu + Dw) \leq \mathbf{1} \quad \forall w \in \mathcal{W}\right\} \tag{6.5}$$

As a first step we can consider an equivalent problem by considering the worst case realization of the disturbance for each row of $E^+$, denoted by $E_j^+$ with $j = 1, \ldots, n_{E^+}$ (i.e. we perform a Pontryagin Set Difference):

$$\mathcal{X} = \{x \in \mathbb{R}^{n_x} \mid \exists u \in \mathcal{U}, E^+(Ax + Bu) \leq \mathbf{1} - e^+\} \tag{6.6}$$

where $e^+$ consists of the elements $e_j^+ = \max_{w \in \mathcal{W}} E_j^+ Dw$ with $j = 1, \ldots, n_{E^+}$. Thus we can define a polyhedron in the extended space of states and controls:

$$\mathcal{Z} = \left\{ (x, u) \in \mathbb{R}^{n_x + n_u} \mid \; Fu \leq \mathbf{1} \text{ and } E^+(Ax + Bu) \leq \mathbf{1} - e^+ \right\} \tag{6.7}$$

There may exist redundant inequalities in this description, and it is important for the subsequent steps in the procedure that these are removed, e.g. by solvng a sequence of linear programming problems. The next step involves the solution to the existence problem in the above definition via projection of the polyhedron $\mathcal{Z}$ from $(x, u)$-space onto $x$-space:

$$\mathcal{X} = Proj_x(\mathcal{Z}) \tag{6.8}$$

which may e.g. be done via Fourier-Motzkin elimination (see [58]). There are two important facts to be noted: Since $\mathcal{X}^+$ is polyhedral and $\mathcal{U}$ and $\mathcal{W}$ are polytopic, $\mathcal{X}$ is also polyhedral and the controllability sets have the property: $\mathcal{X} \supseteq \mathcal{X}^+$ (as illustrated in Figure 6.1 for a polytopic terminal set $\mathcal{X}_f$).

The second fact is that the required set operations lead to polyhedra with increasing complexity in general. We have found in examples that performing these operations exactly is computationally viable for small/medium-scale systems with at most about five states, two inputs, five disturbance inputs for horizon lengths in the order of ten to twenty. The calculations can take several hours to complete (using a typical desktop PC) and this may pose a limitation on the current approach. We refer the interested reader to the PhD thesis [30] for details on polyhedral projection algorithms and their limitations. The advantage of using exact controllable sets is that the overall DP optimal solution can be obtained in the receding horizon algorithm.

### 6.1.2 Approximation Methods for Robust Controllability

Two alternatives have been evaluated in the context of approximating the controllable set: the first is a numerical approach based on ‘sampling and scaling’ so as to satisfy the definition of the robust controllable sets with any degree of accuracy, while the second method is based on suboptimal policy parametrizations and polyhedral projection.

The first method involves a sampling based approach and is performed sequentially (as in exact projection based approaches). We further assume that the target set is convex polytopic. The extended feasible set in $(x, u)$-space is given as before:

$$\mathcal{Z} = \left\{(x, u) \in \mathbb{R}^{n_x+n_u} \mid Fu \leq \mathbf{1} \text{ and } E^+(Ax + Bu) \leq \mathbf{1} - e^+\right\} \tag{6.9}$$

Suppose a set of $d$ points is known on the boundary of a convex polytopic set $\mathcal{X}^+$, given by $\{q_1, \ldots, q_d\}$, then feasible points in $\mathcal{X}$ can be found by solution of an LP for a given point $q_i$ and a corresponding scaling parameter $\delta_i \in \mathbb{R}$:

$$(\delta_i^*, u^*) = \arg\min_{\delta_i,u} -\delta_i \tag{6.10}$$

$$s.t. \quad Fu \leq \mathbf{1} \tag{6.11}$$

$$E^+(A\delta_i q_i + Bu) \leq \mathbf{1} - e^+ \tag{6.12}$$

The convex hull of the pre-image points $\delta_i^* q_i$ for all $i = 1, \ldots, d$ is then an inner approximation of the set $\mathcal{X}$, since the exact representation of $\mathcal{X}$ is known to be convex polytopic. While the set $\mathcal{X}$ can be approximated arbitrarily close via sampling of sufficiently many points and for each sampling point only an LP has to be solved, the method suffers from two drawbacks: First the method may result in conservative approximations of the robust controllable sets and there is no clear procedure for selecting the sampling points (e.g. one may choose the vertices of $\mathcal{X}^+$). Secondly, one has to perform a convex hull operation explicity at each stage along the horizon as well as converting between vertex and hyperplane representations of $\mathcal{X}$, which may be computationally intensive.

The second method considers the feasible sets of parameters for class of pre-stabilizing (PS) and disturbance affine (DA) DP policies denoted as $\mathcal{M}^{DA}(x)$ and $\mathcal{M}^{PS}(x)$ (see Section 5.8). The exact feasible region (with respect to this parametrization) can be found by projecting these sets from the space $(\mathbf{M}, \mathbf{c}, x)$ and $(\mathbf{c}, x)$ respectively onto $x$-space:

$$\mathcal{X}_0^{DA} = Proj_x(\mathcal{M}^{DA}(x)) \tag{6.13}$$

$$\mathcal{X}_0^{PS} = Proj_x(\mathcal{M}^{PS}(x)) \tag{6.14}$$

We note that these sets have the following property:

$$\mathcal{X}_0^{PS} \subseteq \mathcal{X}_0^{DA} \subseteq \mathcal{X}_0^{DP} \tag{6.15}$$

where the LHS follows from the fact that the disturbance-affine policy contains the pre-stabilizing policy as a special case. The RHS follows from the fact that both policies are suboptimal in general with respect to exact DP methods (where no structure is pre-imposed). Unfortunately, this method - although conceptually attractive - involves computationally intensive operations, since projections from high dimensional spaces onto $x$-space are required this is in particularly difficult for the disturbance affine policy with additional parameters in the feedback matrix $\mathbf{M}$. Due to the limitations posed by currently available projection algorithms [30], this method can only handle very small problems with short horizon lengths and is consequently found to be inferior to exact DP methods in general. One advantage of the method is that the required $N$ sequential set operations in the direct DP approach are replaced by a single projection operation for the $N$-step problem when parametrizations are employed.

A numerical instance is given in Figure 6.2 for a double integrator system:

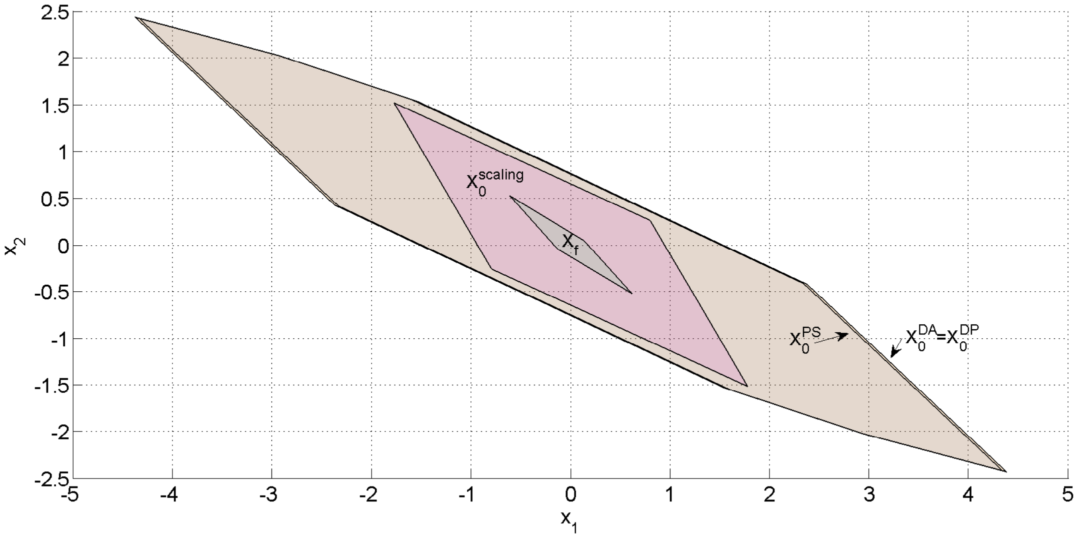

Figure 6.2: Approximations of 2-step robust controllable set (i.e. the feasible set $\mathcal{X}_0$ for $N = 2$) for a double integrator example

one can see that the DA- and PS-policy parametrizations result in very good approximations of the exact DP controllable set. In fact, the DA-policy feasible set is identical to the DP feasible set for this simple example. We emphasize the fact that this is not the case for more realistic examples as discussed in Section 5.9.2. The accuracy of the sampling based approach strongly depends on the number of scaling points chosen (which in Figure 6.2 is done via scaling the vertices of the terminal set $\mathcal{X}_f$). We note that the sampling based approach may be improved by adding additional 'scaling'-points and it may be an alternative for high dimensional problems where other methods are computationally intractable.

## 6.2 An Active Set Solver for Robust Controllability

In this section we illustrate the potential of using parametric linear programming (pLP) methods for a local active set solution to a one-step feasibility problem before considering generalizations to multiple step problems. We begin by giving some general results on pLPs, we then state the particular pLP relating to the one-step feasibility problem and finally we derive the necessary conditions of optimality for this problem. The proposed active set method is then illustrated on a numerical example.

### 6.2.1 Single-Stage Robust Controllability and Multiparametric Linear Programming

We first state what we understand by a parametric linear program (pLP) [1]:

$$\begin{aligned} J^*_{pLP}(x) = \min_u \quad & c^T u \\ \text{s.t.} \quad & Cu \leq \mathbf{1} - Zx \end{aligned} \tag{6.16}$$

where $C \in \mathbb{R}^{nC \times n_u}$ and $Z \in \mathbb{R}^{nC \times n_x}$, $u$ is the optimization vector and $x$ the parameter. We define $\mathcal{X}_{pLP}$ as the set of parameters for which a solution to pLP (6.16) exists. The optimizing function $J^*_{pLP}(x)$ is a scalar-valued function defining a mapping $\mathcal{X}_{pLP} \rightarrow \mathbb{R}$ and $u^*(x)$ is a vector-valued function definining a mapping $\mathcal{X}_{pLP} \rightarrow \mathbb{R}^{n_u}$. We now state two important results on pLPs without proof (which may be obtained from [1]):

**Theorem 6.1** *For pLP (6.16), the set $\mathcal{X}_{pLP}$ is a convex polyhedral set, the optimizer $u^*(x)$ is a continuous and piecewise affine function of $x$, and the optimizing function $J^*_{pLP}(x)$ is a convex and continuous piecewise affine function of $x$.*

We note that the optimizer $u^*(x)$ may be a non-unique piecewise affine function, but despite this multivalued nature a continuous optimizer can always be selected [7]. In this section we make the simplifying assumption that the optimizer $u^*(x)$ is unique for a given state $x$.

**Lemma 6.1** *Let $J_{pLP}(u, x)$ be a convex piecewise affine function. Then, the multiparametric optimization problem*

$$J^*_{pLP}(x) = \min_u \quad J_{pLP}(x, u) \tag{6.17}$$

$$\text{s.t.} \quad Cu \le \mathbf{1} - Zx$$

*is a pLP.*

The proof of this lemma given in [1] makes use of the important equivalence of the maximum representation and the piecewise representation of polyhedral functions [49]:

$$J_{pLP}(x,u) = \max_{j=1,\ldots,m} \left\{ A_j^x x + A_j^u u + a_j \right\} \text{ for } (x,u) \in \mathcal{Z}_{pLP} \quad \Leftrightarrow$$

$$J_{pLP}(x,u) = A_j^x x + A_j^u u + a_j \text{ for } (x,u) \in \mathcal{Z}_{pLP}(\mathcal{A}_j) \text{ and } \cap_{j=1}^m \mathcal{Z}_{pLP}(\mathcal{A}_j) = \mathcal{Z}_{pLP}$$

where $\mathcal{Z}_{pLP} = \{(x,u) \mid Cu \le \mathbf{1} - Zx\}$ and $\mathcal{A}_j$ is a chosen active set.

Next, we reconsider the one-step feasibility problem stated in Equation (6.5) by considering the following equivalent formulation:

$$J_{pLP}^*(x) = \min_u \max_{j=1,\ldots,nE^+} \left\{ E_j^+ Ax + E_j^+ Bu + e_j^+ \right\} \qquad (6.18)$$
$$\text{s.t.} \quad Fu \le \mathbf{1}$$

with $\mathcal{X} = \mathcal{X}_{pLP}$. Further using Lemma 6.1 the min-max optimization in (6.18) can alternatively be solved by considering the following pLP:

$$\begin{aligned} J_{pLP}^*(x) = \min_{u,\alpha} \quad & \alpha \\ \text{s.t.} \quad & E^+ Ax + E^+ Bu + e^+ \le \alpha \mathbf{1} \\ & Fu \le \mathbf{1} \end{aligned} \qquad (6.19)$$

where $x$ is the given parameter and $E^+$ and $e^+$ consist of the rows $E_j^+$ and $e_j^+$ for $j = 1, \ldots, nE^+$. Methods for obtaining solutions of pLPs of the form (6.16) have been explored in particular in the 'Explicit' MPC literature [2], but this typically requires the offline computation of all pLP regions (a global DP solution) and an online location of the solution for a given parameter $x$.

Since our goal is to consider active set solutions to pLPs in the robust controllability context, we now consider in more detail how the above pLP (6.19) can be solved using a primal-dual approach. Theorem 6.1 implies that the solution of pLP (6.19) leads to a continuous, piecewise affine optimal objective function $\alpha^*(x)$ and optimizer $u^*(x)$[1], both of which are continuous, piecewise affine functions. We first consider the Lagrangian for this optimization problem and introduce multipliers $\mu$ and $\nu$ for corresponding input and state constraints:

$$L(u, x, \alpha) = \alpha - \nu^T(\alpha\mathbf{1} - E^+Bu - E^+Ax - e^+) - \mu^T(\mathbf{1} - Fu) \tag{6.20}$$

which leads to the following KKT conditions:

$$\nabla_\alpha L = \mathbf{1}^T\nu - 1 = 0 \tag{6.21}$$

$$\nabla_u L = B^T E^{+T}\nu + F^T\mu = 0 \tag{6.22}$$

$$\alpha\mathbf{1} - E^+Bu - E^+Ax - e^+ \geq 0 \tag{6.23}$$

$$\nu^T(\alpha\mathbf{1} - E^+Bu - E^+Ax - e^+) = 0 \quad \nu \geq 0$$

$$\mathbf{1} - Fu \geq 0, \quad \mu^T(\mathbf{1} - Fu) = 0, \quad \mu \geq 0 \tag{6.24}$$

Now we assume that a given active set of constraints $\mathcal{A}$ for this pLP is selected. Therefore we can solve the resulting set of linear equations and obtain the optimizing function $\alpha^*(x)$ and the optimizing state feedback law $u^*(x)$. For a given active set the KKT conditions for this pLP become:

$$\alpha\mathbf{1} - E^+_{\mathcal{A}}Bu = E^+_{\mathcal{A}}Ax + e^+_{\mathcal{A}} \tag{6.25}$$

$$\alpha\mathbf{1} - E^+_{\mathcal{I}}Bu \geq E^+_{\mathcal{I}}Ax + e^+_{\mathcal{I}} \tag{6.26}$$

$$F_{\mathcal{A}}u = \mathbf{1} \tag{6.27}$$

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[1] The solution $u^*(x)$ is the feedback solution to the feasibility problem and may not be confused with the feedback solution to the MPC problem obtained in Chapter 5

$$F_{\mathcal{I}} u \leq \mathbf{1} \tag{6.28}$$

$$B^T E_{\mathcal{A}}^{+T} \nu_{\mathcal{A}} + F_{\mathcal{A}}^T \mu_{\mathcal{A}} = 0, \quad \nu_{\mathcal{A}} \geq 0 \quad \nu_{\mathcal{I}} = 0 \tag{6.29}$$

$$\mathbf{1}^T \nu_{\mathcal{A}} = 1, \quad \mu_{\mathcal{A}} \geq 0 \quad \mu_{\mathcal{I}} = 0 \tag{6.30}$$

where the subscripts $\mathcal{A}$ and $\mathcal{I}$ denote the rows of matrices and elements of vectors that correspond to active and inactive constraints respectively.

The primal KKT system is given by:

$$\begin{bmatrix} \mathbf{1} & -E_{\mathcal{A}}^{+} B \\ 0 & F_{\mathcal{A}} \end{bmatrix} \begin{bmatrix} \alpha \\ u \end{bmatrix} = \begin{bmatrix} E_{\mathcal{A}}^{+} A \\ 0 \end{bmatrix} x + \begin{bmatrix} e_{\mathcal{A}}^{+} \\ \mathbf{1} \end{bmatrix} \tag{6.31}$$

and its solution (under the assumption that the primal solution is unique) is given by:

$$\begin{bmatrix} \alpha \\ u \end{bmatrix} = \begin{bmatrix} E^{\alpha} \\ L^{u} \end{bmatrix} x + \begin{bmatrix} e^{\alpha} \\ l^{u} \end{bmatrix} \tag{6.32}$$

Further, the dual KKT system is given by:

$$\begin{bmatrix} B^T E_{\mathcal{A}}^{+T} & F_{\mathcal{A}}^T \\ \mathbf{1}^T & 0 \end{bmatrix} \begin{bmatrix} \nu_{\mathcal{A}} \\ \mu_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{6.33}$$

While the assumption on the uniqueness of the primal solution for given $x$ excludes the possibility of non-uniqueness of the dual solution in the interior of a pLP region[1], it is necessarily non-unique on pLP region boundaries. For parameters $x \in \partial \mathcal{X}_{pLP}(\mathcal{A})$, we assume that at most one degree of freedom occurs. In geometric terms this means that optimal solution is assumed to change over edges of the feasible set, i.e. from one vertex to another as the parameter of the

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[1] Note that the pLP region $\mathcal{X}_{pLP}(\mathcal{A})$ is defined by evaluating the primal KKT conditions in (6.25-6.30) using the equality constrained solution for the given corresponding active set $\mathcal{A}$

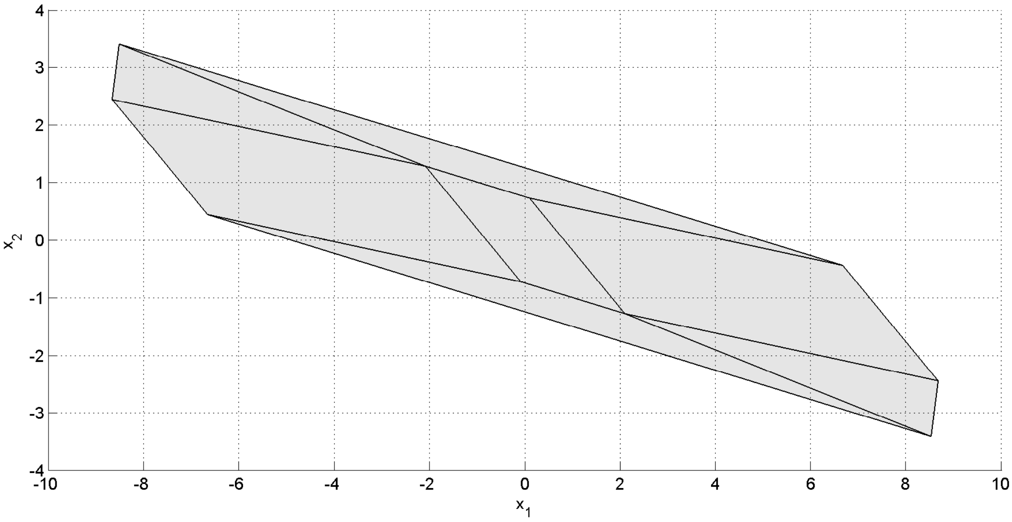

Figure 6.3: Illustration of the solution to a one-step robust controllability problem with respect to $\mathcal{X}_f$ via parametric linear programming for a double integrator example

pLP varies and this is a reasonable[1] assumption made in the context of active set solutions to general linear programming problems [22]:

$$\begin{bmatrix} \nu_{\mathcal{A}} \\ \mu_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} l^{\nu} \\ l^{\mu} \end{bmatrix} + Z\beta \tag{6.34}$$

where $\beta \in [\underline{\beta}, \overline{\beta}] \in \mathbb{R}$ and $Z$ is the kernel of the LHS matrix in Equation (6.33). We note that a parameter $x$ is feasible provided that it satisfies $\alpha^*(x) \leq 1$.

This method has been applied to a double integrator example defined in Section 5.9.2. Figure 6.3 illustrates that the one-step robust controllabilty set with respect to the terminal set $\mathcal{X}_f$ can be obtained by taking the union of pLP regions corresponding to active sets for which the KKT conditions are satisfied (since $\mathcal{X}_{pLP} = \cap_{j=1}^{m} \mathcal{X}_{pLP}(\mathcal{A}_j)$). Therefore this approach can also be used as an alternative global method to solving the controllability problem, although expensive set operation methods are required (in particular the set union operation). However, it is example dependent whether gains are achieved as compared to

---

[1] This is a reasonable assumption because it excludes the possibility that the pLP region is not uniquely defined on the other side of the pLP region boundary (as could be the case, for example, at a vertex of $\mathcal{X}_{pLP}(\mathcal{A})$)

direct polyhedral projection methods.

Alternatively, one can see the potential at this point for considering a homotopy based active set approach: provided a solution for some parameter $x$ in a corresponding pLP region $\mathcal{X}_{pLP}(\mathcal{A})$ is known, we can move along a line to a new local pLP solution for some other parameter $x''$ say inside a directly neighbouring region $\mathcal{X}_{pLP}(\mathcal{A}'')$ by updating the active set on the common region boundary $\partial\mathcal{X}_{pLP}(\mathcal{A}) \cap \partial\mathcal{X}_{pLP}(\mathcal{A}'')$ at point $x'$ on the common boundary(where $x' = x + a(x'' - x)$ for some $a \in \mathbb{R}$). Under our assumptions of exactly one degree of freedom for a problem at $x' \in \partial\mathcal{X}_{pLP}(\mathcal{A}) \cap \partial\mathcal{X}_{pLP}(\mathcal{A}'')$ after addition of the corresponding extra constraint, the active set change at this common region boundary is unique. This is illustrated in Figure 6.4, where a homotopy of solutions[1] is performed from the solution at point $x_a = [-15, 2]$ to $x_b = [15, 4.5]$ (where the one-step robust controllable set is calculated with respect to $\mathcal{X}_4$ as the target set for a double integrator example). Although we have not encountered degeneracy problems (i.e. multiple optimizers) in single step pLP calculations for the double integrator example, degeneracy problems have been reported in the LP based MPC context (with possible remedies offered by using lexicographic perturbation methods [29]).

### 6.2.2 Multi-Stage Robust Controllability and Multiparametric Linear Programming

In order to replace the global robust controllable set in multi-stage problems as encountered in MPC, we need a computationally convenient and verifiable condition under which the terminal set $\mathcal{X}_f$ can be robustly entered under application of some feasible state feedback law sequence, from any predicted state in the finite horizon optimal control problem of Chapter 5. This section aims at characterizing

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[1] N.B. In contrast to the pQP problem, we understand the homotopy of solutions is performed in two steps: variations of $x$ along a line inside pLP regions lead to continuous variations of the primal solution and variations along the kernel direction lead to continuous variations of the dual solution.

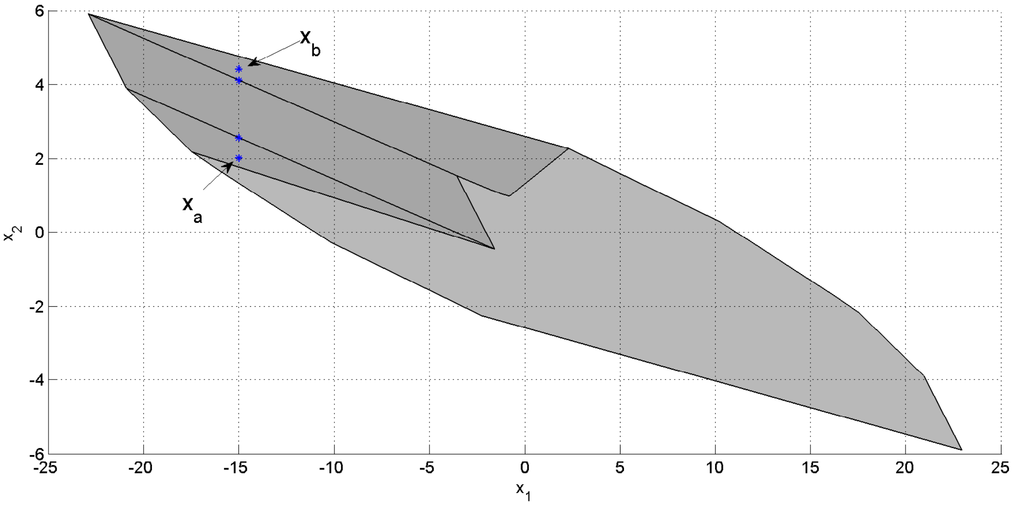

Figure 6.4: Illustration of a homotopy between solutions of the one-step robust controllability problem with respect to the target set $\mathcal{X}_4$ for a double integrator example

properties of the multiple-step robust controllability problem. We first summarize briefly the results of Section 6.2.1 which considers the single-step problem: we first fundamentally assume the availability of the successor set $\mathcal{X}^+$. Given this it is then relatively straightforward to perform a Pontryagin difference (by solving a number of LPs or alternatively performing the maximization directly over the vertices of the uncertainty set $\mathcal{W}$) to remove the effects of uncertainty in the single stage problem. Finally we obtain a pLP, which can be solved globally to obtain the exact representation of $\mathcal{X}$. This global existence solution answers the posed controllability question by returning a set $\mathcal{X}$ of all such robustly controllable states $x$ and therefore the question whether a specific $x$ is controllable can be addressed by checking that $x \in \mathcal{X}$. A local solution to the same pLP then essentially means that we have a verifiable condition of whether a given state $x$ can be robustly controlled via state feedback to the set $\mathcal{X}^+$. Therefore the fundamental goal is to obtain such a checkable condition that the current state is robustly controllable to $\mathcal{X}^+$ for multiple-step applications of some feasible state feedback without reverting to the solution of the global existence problem (which involves sequential set computations with increasing complexity).

The particular challenge in this respect can be illustrated by considering a local uncertainty 'tube' originating from a specific initial plant state $x_0$ as shown in Figure 6.5. For the purposes of this illustration we assume that the global pLP solution is known for multiple steps (i.e. the single step method has been applied sequentially). Of course, the ultimate goal is to avoid such computations. In Figure 6.5 we first apply the feedback law $u_0^l(x_0)$ to the inital state $x_0$, but due to the presence of the unknown disturbance (for which the extreme realizations are assumed) the predicted state lies inside some set at step 1 and this set may intersect with different pLP regions. This means that different feedback laws $u_1^j(x_1^1)$ and $u_1^k(x_1^2)$ will have to be applied to generate the optimal tube. We note that as a consequence of the continuous, piecewise affine nature of the feedback laws and the convexity of the disturbance set $\mathcal{W}$ it follows that the tube cross sections are convex. We can see that for a given state $x_0$ the worst case trajectory corresponds to an active vertex of the terminal tube cross section with respect to some scaled set $\alpha\mathcal{X}_f$. As the parameter $x_0$ varies it would then appear possible to track active set changes so as to determine how the associated worst case trajectory changes.

There are several challenges to achieve such a method: firstly, it is unclear which objective to select in the multi-step optimization, since global $k$ steps-ahead sets are unknown (apart from the terminal set $\mathcal{X}_f$). Considering the solution to the single step problem for some given active set $\mathcal{A}$ at a given parameter $x$, we have local information of the optimizing function $\alpha^*(x)$, the optimizer $u^*(x)$ and a hyperplane representation of the pLP region $\mathcal{X}_{pLP}(\mathcal{A})$. It is therefore reasonable to employ this local information in the preceding stage optimization problem (e.g. one may be able to (a) define a suitable cost-to-go by using the optimizing function $\alpha^*(x)$ from the successor stage optimization (b) update the active set when the state $x$ reaches the boundary of the current pLP region defined by the KKT conditions of the one-step-ahead feasibility problem).

A second challenge is to suitably define an active set update procedure using the feedback laws obtained in the backwards solution. This would enable

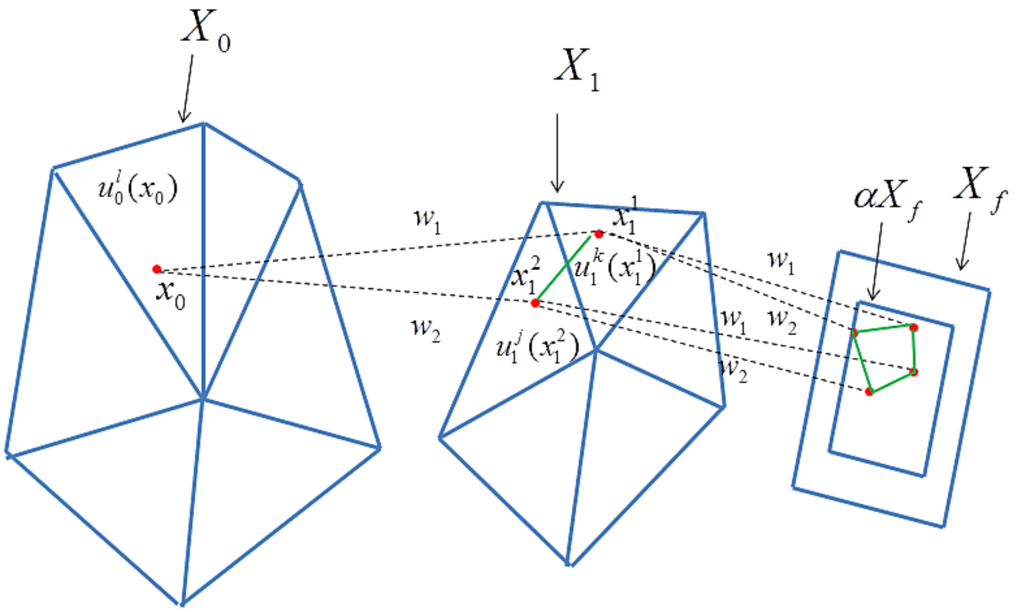

Figure 6.5: Illustration of a forward simulation in multi-stage robust controllability problems

tracking of the worst case state sequence that determines active terminal constraints with respect to a suitably rescaled terminal constraint set $\alpha^*(x_N)\mathcal{X}_f$ (a role similar to the objective function $\alpha^*(x)$ in the one step problem in Section 6.2.1). We note that the actual terminal constraint would then become active for $\alpha^*(x_N) = 1$. Since the predicted state lies within a tube dictated by the polytopic disturbance set $\mathcal{W}$, we might expect the computational load of updating the active set to increase exponentially with the horizon length $N$. However the active set method outlined here would provide computational advantages over dynamic programming approaches (such as [50]) provided that each iteration requires the computation of a number of feedback laws $u_k^j(x)$ that depends only linearly on $N$.

## 6.3 Conclusion

This chapter first discusses offline methods for the computation of exact and approximate robust controllable sets. Secondly a detailed solution to the one-step controllability problem based on parametric linear programming is discussed and a local solution method is considered. Finally some remarks on the multi-step robust controllability problem are made with the hope that some of the presented ideas may make it possible in near future to rigorously address this important open problem with one significant application in the min-max robust control problem presented in Chapter 5.

# Chapter 7

# Nonlinear MPC based on Robust Ellipsoidal Tubes

An important situation involving model uncertainty arises when nonlinear dynamics are linearized either about an operating point or a system trajectory. On the other hand, a very promising approach for a broad class of nonlinear MPC problems is based on successive linearization [16]. If a linearization is performed along a given feasible trajectory, then a first order dynamic model approximation typically leads to a linear time varying system with parametric uncertainty or additive bounded uncertainty (if a more conservative approximation of the errors is employed). The material presented in this chapter demonstates the potential of linear robust MPC methods for solving nonlinear MPC problems and thereby demonstrates a very interesting future extension of the active set approach presented in Chapters 3-5.

In this chapter, instead of an exact dynamic programming approach to solve the uncertain linearized MPC subproblem, we parametrize the uncertain subproblems via the class of pre-stabilizing DP policies and use robust tubes to bound the effects of uncertainty over several time steps by using a sequence of single step conditions [17, 44]. An advantage of this approach is that it is possible to terminate the NMPC optimization early, but to still retain closed loop stability. To ensure closed loop stability given that predicted trajectories are only feasible, the error bounds have to be non-conservative for the case of zero linearization error, which requires the cross sections to vary along predicted trajectories. Variable polytopic tubes for bounding the effects of linearization errors have been used in the successive linearization NMPC setting of [41]. However, this approach involved large numbers of optimization variables and this limitation is overcome with the ellipsoidal approach presented in this chapter.

## 7.1 Problem Formulation

The system of interest is a nonlinear discrete time model of the form:

$$x_{t+1} = f(x_t, u_t) \tag{7.1}$$

where $x_t \in \mathbb{R}^{n_x}$ and $u_t \in \mathbb{R}^{n_u}$ are the state and input of the system and are constrained, i.e. $(x_t, u_t) \in \mathcal{Z}$ with

$$\mathcal{Z} = \left\{ (x, u) \in \mathbb{R}^{n_x + n_u} \mid Fx + Gu \leq h \right\} \tag{7.2}$$

where $F \in \mathbb{R}^{n_c \times n_x}$, $G \in \mathbb{R}^{n_c \times n_u}$. Also $f$ is assumed to be twice continuously differentiable for all $(x, u)$ in some operating region and an equilibrium exists at $x = 0$, $u = 0$, i.e. $f(0, 0) = 0$. We seek to solve (an approximation) to a finite horizon optimal control problem (in 'non-condensed' form) defined by :

$$J_0^*(\mathbf{x}^*, \mathbf{u}^*) = \min_{\mathbf{u},\mathbf{x}} J_0(\mathbf{x}, \mathbf{u}) \tag{7.3}$$

where the cost is defined by

$$J_0(\mathbf{x}, \mathbf{u}) = \sum_{k=0}^{N-1} \frac{1}{2}(\|x_k\|_Q^2 + \|u_k\|_R^2) + \frac{1}{2}\|x_N\|_P^2 \tag{7.4}$$

and constraints are given by:

$$x_{k+1} = f(x_k, u_k) \quad k = 0, 1, \dots, N-1 \tag{7.5}$$

$$(x_k, u_k) \in \mathcal{Z} \quad k = 0, 1, \dots, N-1 \tag{7.6}$$

$$x_N \in \mathcal{X}_N \tag{7.7}$$

where $Q \in \mathcal{S}_+^{n_x}, R \in \mathcal{S}_{++}^{n_u}$ and the terminal set $\mathcal{X}_N = \mathcal{X}_f$ and terminal weight $P \in \mathcal{S}_+^{n_x}$ are defined in the sequel (see Sections 7.2 and 7.3). $\mathbf{u} = \{u_0, \dots, u_{N-1}\}$, $\mathbf{x} = \{x_0, \dots, x_N\}$ denote the input and state sequence respectively (and $*$ is used to denote optimal values).

As usual in receding horizon control, the implemented input $u_t$ at time $t$ is equal to the first predicted input element $u_0^*$. This is determined for a current plant state measurement $x_t$ which is equal to the first element of the predicted state sequence $x_0^*$[1].

The idea of the NMPC strategy is to take feasible (but suboptimal) input and corresponding state trajectories as initial future predicted state and input trajectories known as ‘seed’ trajectories. To improve from this initial solution, successive linearization is used about the predicted ‘seed’ trajectories. The goal is to compute an optimal sequence of perturbations on the feasible trajectories by solving a receding horizon optimization problem.

Suppose now that an initial seed trajectory $(x_k^0, u_k^0)$ for $k = 0, \dots, N-1$ is available which satisfies the system model and constraints over an $N$-step horizon: i.e. $x_{k+1}^0 = f(x_k^0, u_k^0)$ and $Fx_k^0 + Gu_k^0 \le h$ for $k = 0, ..., N-1$ and $x_N^0 \in \mathcal{X}_N$. We denote $x_k^\delta, u_k^\delta$ as perturbations on the seed trajectories which are to be determined in an optimal way, i.e. $x_k = x_k^0 + x_k^\delta$, $u_k = u_k^0 + u_k^\delta$, where these sums satisfy the system equation in the following way: $x_{k+1}^0 + x_{k+1}^\delta = f(x_k^0 + x_k^\delta, u_k^0 + u_k^\delta)$ for

---

[1] N.B. The reader should note that this approach aims at identifying the optimal input and state sequence for the given state $x_t$ which is implicitly defined by sequence of feedback laws as discussed in Section 2.1.3.1. However, here we do not obtain the state feedback solution explicitly which would require solution to a dynamic program

$k = 0, ..., N-1$ with initial $x_0^\delta = 0$.

The remainder of this section demonstrates that this optimal perturbation problem leads to a linear robust MPC subproblem with polytopic uncertainty. Although these problems could be solved exactly by DP, we seek a computationally and conceptually convenient approach and parametrize the robust MPC subproblems using the class of pre-stabilizing DP policies, i.e. $u_k^\delta$ is taken as the sum of a linear feedback law and feedforward term:

$$u_k^\delta = K_k x_k^\delta + v_k \tag{7.8}$$

At each time $k$ an upper bound on the cost (7.4) is minimized over the trajectory of perturbations $\{v_k,\ k = 0, \dots, N-1\}$ subject to the constraints (7.6) and a tube approach is used to bound the effects of uncertainty due to the linearization. Employing (7.8) to linearize (7.5) around the seed $(x_k^0, u_k^0)$ we obtain the linear time-varying model with parametric uncertainty:

$$x_{k+1}^\delta = \Phi_k x_k^\delta + B_k v_k + w_k \tag{7.9}$$

where $\Phi_k = A_k + B_k K_k$ and $A_k = \frac{\partial f}{\partial x}|_{(x_k^0, u_k^0)}$,$B_k = \frac{\partial f}{\partial u}|_{(x_k^0, u_k^0)}$. Here $w_k$ denotes the linearization error and using the mean value theorem [9] it is possible to bound the linearization error within a polytopic set defined by the convex hull of the vertices $[C_j \quad D_j][{x^\delta_k}^T \quad {u_k^\delta}^T]^T$ for $j = 1, \dots, n_w$, i.e.:

$$w_k \in Co \left\{ C_j x_k^\delta + D_j u_k^\delta, j = 1, \dots, n_w \right\} \tag{7.10}$$

The predictions for $k = N, N+1, \dots$ steps ahead are determined by linearizing (7.5) about the set point $(x, u) = (0, 0)$ and using a pre-determined linear feedback law $\hat{K}x$ :

$$x_{k+1} = \hat{\Phi} x_k + \hat{w}_k \tag{7.11}$$

$$u_k = \hat{K} x_k \tag{7.12}$$

where $\hat{\Phi} = \hat{A} + \hat{B}\hat{K}$ and $\hat{A} = \frac{\partial f}{\partial x}|_{(0,0)}, \hat{B} = \frac{\partial f}{\partial u}|_{(0,0)}$. Similarly to the above discussion for $i \geq N$ the errors $\hat{w}_k$ of the approximate linear dynamics (7.11) are assumed to lie in the polytopic set:

$$\hat{w}_k \in Co\Big\{\hat{C}_j x_k + \hat{D}_j u_k, j = 1, \ldots, n_w\Big\} \tag{7.13}$$

**Remark 7.1** *We note that if $w_k$ satisfies a Lipschitz condition of the form $|w_k| \leq \Gamma_x |x_k^\delta| + \Gamma_u |u_k^\delta|$, then the matrices $[C_j \quad D_j]$ can be chosen equal to $[\Gamma_x \quad \Gamma_u] S_j$ where $\{S_j, j = 1, \ldots, 2^{n_x + n_u}\}$ denotes the collection of diagonal matrices with elements equal to $\pm 1$.*

## 7.2 Construction of Tubes for Linearization Errors

In this section we discuss how to bound the effects of linearization errors on predicted trajectories by constructing tubes which contain that component of the predicted perturbation state $x^\delta$ that results from the linearization error $w$. This is achieved by using one-step-ahead predictions. For notational convenience the prediction of $x_k^\delta$ is split into a nominal component $z_k$ and an 'error' component $e_k$ which depends only on the linearization error $w_k$:

$$x_k^\delta = z_k + e_k \tag{7.14}$$

$$z_{k+1} = \Phi_k z_k + B_k v_k \tag{7.15}$$

$$e_{k+1} = \Phi_k e_k + w_k \tag{7.16}$$

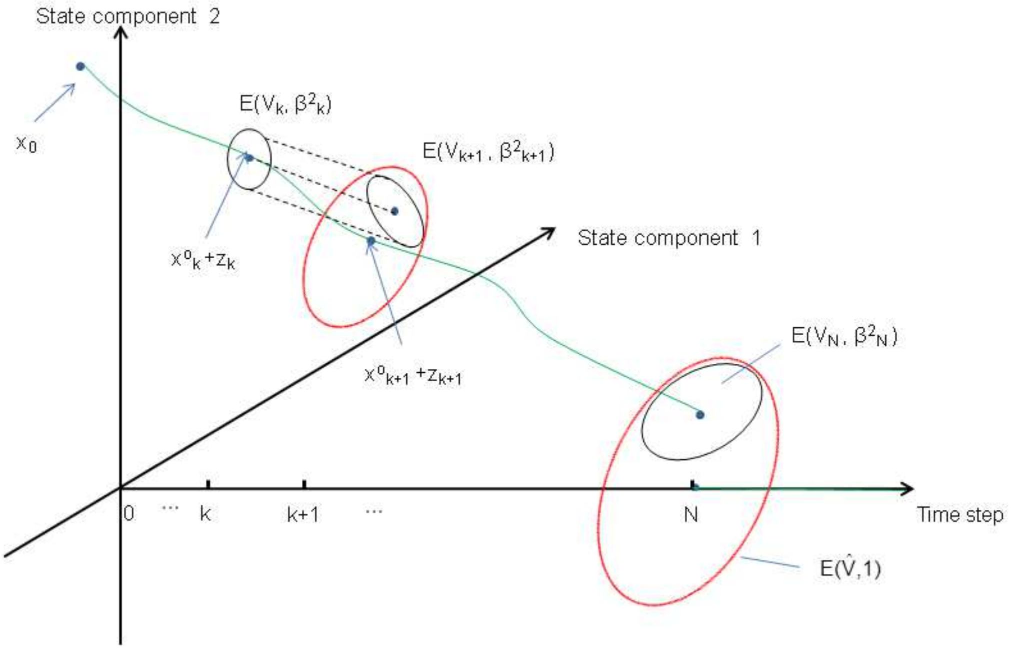

Figure 7.1: Illustration of the tube containing predicted state sequences

with $z_0 = e_0 = 0$. Our proposed method determines tubes with time-varying ellipsoidal cross-sections which are satisfied in mode 1 (i.e. for $k = 1, \ldots, N$):

$$e_k \in E\left(V_k, \beta_k^2\right), \tag{7.17}$$

and ensures satisfaction of an ellipsoidal terminal condition in mode 2 (i.e. for $i = N, N+1, \ldots$):

$$x_k \in E\left(\hat{V}, 1\right). \tag{7.18}$$

The approach of using recursive tube membership conditions and an ellipsoidal terminal condition is illustrated in Figure 7.1: At a point $x_k^0 + z_k$ on the nominal trajectory, the uncertainty due to the linearization errors at earlier prediction times is accounted for by the set $E(V_k, \beta_k^2)$ (which is centered at the point $x_k^0 + z_k$) and in which the predicted state $x_k$ is guaranteed to be contained. Now, we aim to find the set $E(V_{k+1}, \beta_{k+1}^2)$ in which the predicted state $x_{k+1}$ is guaranteed to be contained.

The advantage of ellipsoidal sets is that the scalings $\{\beta_k, i = 1, ..., N\}$ can be incorporated into an optimization expressed as a second order cone program (see Section 2.1). We note that from Condition (7.18) it follows that the ellipsoid $E\left(\hat{V}, 1\right)$ defines a terminal constraint set. Therefore this set must be invariant under the system dynamics in mode 2, i.e. under Conditions (7.11) and (7.12) and feasible with respect to the system constraints (7.6), so that constraints are satisfied over an infinite prediction horizon. These two conditions require that:

$$\hat{\Phi}x + \hat{w} \in E\left(\hat{V}, 1\right), \forall \hat{w} \in Co\left\{(\hat{C}_j + \hat{D}_j\hat{K})x\right\} \forall j = 1, \ldots, n_w, \forall x \in E\left(\hat{V}, 1\right) \tag{7.19}$$

is satisfied for robust invariance and

$$\left(F + G\hat{K}\right)x \leq h, \forall x \in E\left(\hat{V}, 1\right) \tag{7.20}$$

is satisfied for robust feasibility. A fundamental indicator of control performance is the size of the region of attraction of the closed loop system and it is clearly desirable that this region should be as large as possible. In a dual mode MPC scheme such as the one proposed in this chapter, one crucial ingredient in achieving this is to maximize offline the terminal constraint set $\mathcal{X}_f = E\left(\hat{V}, 1\right)$ over $\hat{K}$ and $\hat{V}$ subject to the invariance and feasibility conditions (7.19), (7.20). In fact one can solve the following equivalent determinant maximization problem, which can be formulated as an SDP (see Section 2.1) as stated in the following lemma:

**Lemma 7.1** *Solving the following determinant maximization problem*[1] *is equivalent to maximizing the volume of the ellipsoidal set* $E\left(\hat{V}, 1\right)$ *over* $\hat{K}$ *and* $\hat{V}$ *subject to the invariance and feasibility conditions (7.19), (7.20):*

$$\max_{\hat{S}, \hat{Y}} \left[logdet(\hat{S})\right] \tag{7.21}$$

---

[1] A determinant maximization problem of the given form can be converted to an SDP by maximizing the geometric mean of the eigenvalues of $\hat{S}$ [54]

$$s.t. \begin{bmatrix} \hat{S} & \left[(\hat{A}+\hat{C}_j)\hat{S}+(\hat{B}+\hat{D}_j)\hat{Y}\right]^T \\ * & \hat{S} \end{bmatrix} \succeq 0 \quad j=1,\dots,n_w \tag{7.22}$$

$$\begin{bmatrix} h_q^2 & F_q\hat{S}+G_q\hat{Y} \\ * & \hat{S} \end{bmatrix} \succeq 0, \quad q=1,\dots,n_c$$

*where the volume of* $E\left(\hat{V},1\right)$ *is maximized with* $\hat{V}=\hat{S}^{-1}$ *and* $\hat{K}=\hat{Y}\hat{S}^{-1}$.

*Proof:* (7.19) is equivalent to requiring that:

$$\begin{aligned} & x^T\left[\left(\hat{A}+\hat{C}_j\right)+\left(\hat{B}+\hat{D}_j\right)\hat{K}\right]^T\hat{V}\left[\left(\hat{A}+\hat{C}_j\right)+\left(\hat{B}+\hat{D}_j\right)\hat{K}\right]x\leq 1 \\ & \forall x \text{ s.t} \quad 1-x^T\hat{V}x\geq 0 \end{aligned}$$

which is equivalent by the S-Procedure [9] to condition:

$$\left[\left(\hat{A}+\hat{C}_j\right)+\left(\hat{B}+\hat{D}_j\right)\hat{K}\right]^T\hat{V}\left[\left(\hat{A}+\hat{C}_j\right)+\left(\hat{B}+\hat{D}_j\right)\hat{K}\right]x\preceq\hat{V}$$

If one pre- and postmultiplies with $\hat{V}^{-1}$ and sets $\hat{S}=\hat{V}^{-1}, \hat{Y}=\hat{K}\hat{S}$, then one can write:

$$\left[\left(\hat{A}+\hat{C}_j\right)\hat{S}+\left(\hat{B}+\hat{D}_j\right)\hat{Y}\right]^T\hat{S}^{-1}\left[\left(\hat{A}+\hat{C}_j\right)\hat{S}+\left(\hat{B}+\hat{D}_j\right)\hat{Y}\right]\preceq\hat{S}$$

Using Schur complements (noting that $\hat{S}=\hat{S}^T\succeq 0$ for an ellipsoid) this can be brought equivalently into LMI form:

$$\begin{bmatrix} \hat{S} & \left[\left(\hat{A}+\hat{C}_j\right)\hat{S}+\left(\hat{B}+\hat{D}_j\right)\hat{Y}\right]^T \\ * & \hat{S} \end{bmatrix} \succeq 0, j=1,...,n_w \tag{7.23}$$

The feasibility condition leads to the following set of scalar inequalities:

$$\left(F_q+G_q\hat{K}\right)\hat{V}^{-1}\left(F_q+G_q\hat{K}\right)^T\leq {h_q}^2$$

which, using $\hat{S} = \hat{V}^{-1}, \hat{Y} = \hat{K}\hat{S}$ can be written as:

$$\left(F_q\hat{S} + G_q\hat{Y}\right)\hat{S}^{-1}\left(F_q\hat{S} + G_q\hat{Y}\right)^T \leq {h_q}^2$$

Using Schur Complements this results in the LMI:

$$\begin{bmatrix} {h^2}_q & F_q\hat{S} + G_q\hat{Y} \\ * & \hat{S} \end{bmatrix} \succeq 0, q = 1, ..., n_c \tag{7.24}$$

($[\cdot]_q$ denoting the $q^{th}$ row of $[\cdot]$). Therefore the LMI conditions in the optimization problem are equivalent to invariance and feasibility conditions (7.19), (7.20). Maximizing the given objective is therefore equivalent to maximizing the volume of the set $E(\hat{V}, 1)$. Together this implies the statement. ■

In mode 1, the tube membership condition (7.17) has the analogous form:

$$\begin{aligned} &\Phi_k e + w \in E\left(V_{k+1}, \beta_{k+1}^2\right) \\ &\qquad \forall w \in Co\left\{C_j x_k^\delta + D_j u_k^\delta, j = 1, ..., n_w\right\} \forall e \in E\left(V_k, \beta_k^2\right) \end{aligned} \tag{7.25}$$

Two approaches to ensure this condition will be described in the following. In the first approach the feedback gains $K_k$ and the matrices $V_k$ defining the shape of the tube cross sections are recomputed online via a sequence of SDPs each time the linearization trajectory $(x_k^0, u_k^0), k = 0, ..., N-1$ is updated. The second approach is to reduce the online computational load by fixing $K_k$ and $V_k$ offline to the terminal values $\hat{K}$ and $\hat{V}$.

The terminal matching condition (i.e. that $x_N \in E\left(\hat{V}, 1\right)$) is similarly ensured by the condition:

$$x_N^0 + z_N + e \in E\left(\hat{V}, 1\right) \quad \forall e \in E\left(V_N, \beta_N^2\right) \tag{7.26}$$

### 7.2.1 Varying Feedback Gains and Tube Cross-Sections

We now use conditions (7.9) and (7.14) to show that the recursive membership condition (7.25) becomes:

$$\beta^+ \geq \|(\Phi + C_j + D_jK)\, e + (C_j + D_jK)\, z + D_jv\|_{V^+} \; \forall e \in E(V, \beta^2) \tag{7.27}$$

where $j = 1, .., n_w$ and $(V^+, \beta^+) = (V_{k+1}, \beta_{k+1})$

This is equivalent to taking the maximum over $e$ of the RHS of Equation (7.27) over the region $E(V, \beta^2)$:

$$\beta^+ \geq \max_{e \in E(V,\beta^2)} \|(\Phi + C_j + D_jK)\, e + (C_j + D_jK)\, z + D_jv\|_{V^+} \tag{7.28}$$

We can now use the triangle inequality to obtain a sufficient condition:

$$\beta^+ \geq \|(C_j + D_jK)z + D_jv\|_{V^+} + \max_{e \in E(V,\beta^2)} \|(\Phi + C_j + D_jK)e\|_{V^+} \tag{7.29}$$

**Lemma 7.2** *Condition (7.29) can be stated equivalently in terms of two conditions on $V$ and $\beta$:*

$$V \succeq (\Phi + C_j + D_jK)^T V^+ (\Phi + C_j + D_jK) \tag{7.30}$$

$$\beta^+ \geq \|(C_j + D_jK)z + D_jv\|_{V^+} + \beta, j = 1, ..., n_w \tag{7.31}$$

*Proof:* Let $\gamma_j = \beta^+ - \|((C_j + D_jK)z + D_jv)\|_{V^+}$ and $\Gamma_j = (\Phi + C_j + D_jK)^T V^+ (\Phi + C_j + D_jK)$. Now we seek to show that $\gamma_j^2 - e^T\Gamma_j e \geq 0$ holds for all $e$ such that $\beta^2 - e^T V e \geq 0$. The S-Procedure implies now that this is true iff [9]:

$$\begin{bmatrix} -\Gamma_j & 0 \\ 0 & \gamma_j^2 \end{bmatrix} \succeq \lambda_j \begin{bmatrix} -V & 0 \\ 0 & \beta^2 \end{bmatrix}, j = 1, ..., n_w$$

After multiplication of both sides with $-1$ this yields Conditions (7.30) and (7.31), where without loss of generality all $\lambda_j$ can be set to 1, because $\lambda_j$ multiplies both $V$ and $\beta$. ■

Using the triangle inequality leads to some conservativeness in the approach: we can see this by setting $\tilde{\Phi}_j = \Phi + C_j + D_j K$ and $\tilde{z}_j = (C_j + D_j K)z + D_j v$. For this illustration, we now assume that we only look at one particular realization vertex $j$, so that we drop the $j$-index, and we assume that $\tilde{\Phi}$ is invertible. Defining $V'$ as $\tilde{\Phi}^T V^+ \tilde{\Phi}$ the right hand side (RHS) of (7.28) is equal to $\max\limits_{\left\|V^{1/2}e\right\| \leq \beta} \left\| e + \tilde{\Phi}^{-1}\tilde{z} \right\|_{V'} = \max\limits_{\left\|V^{1/2}e\right\| \leq \beta} \left\| V'^{1/2}e + V'^{1/2}\tilde{\Phi}^{-1}\tilde{z} \right\|$. We can now define $x_1 = V'^{1/2}\tilde{\Phi}^{-1}\tilde{z}$ and $x_2 = V'^{1/2}e$, so that the RHS of (7.28) becomes equal to: $\max\limits_{\left\|V^{1/2}V'^{-1/2}x_2\right\| \leq \beta} \|x_1 + x_2\|$. Now we can consider the two cases where $V = V'$ and $V > V'$: In the first case we obtain $\max\limits_{\|x_2\| \leq \beta} \|x_1 + x_2\| = \beta + \|x_1\|$ and the triangle inequality in fact reduces to equality, which can be seen in Figure 7.2. For the second case (i.e. when setting $\overline{V}^{1/2} = V^{1/2}V'^{-1/2}$) we obtain $\max\limits_{\left\|\overline{V}^{1/2}x_2\right\| \leq \beta} \|x_1 + x_2\| \leq \max\limits_{\left\|\overline{V}^{1/2}x_2\right\| \leq \beta} \|x_2\| + \|x_1\|$.

Figure 7.2 shows the resulting conservativeness if $x_2$ lies within an ellipsoidal set instead of being inside a spherical set. The gap indicated shows that using the triangle inequality results in the one-step-ahead ellipsoid containing the image of the current ellipsoid, however in general this image will not touch the one-step-ahead ellipsoid. This gap reduces as the set in which $x_2$ lies becomes more spherical, which is the crucial idea in the following.

If the uncertainty polytope had only one vertex (instead of $n_w$), it would be possible to set $V = (\Phi + C_j + D_j K)^T V^+ (\Phi + C_j + D_j K)$ and thus the ellipsoid at the step-ahead contains the ellipsoid at the current step non-conservatively (as in the first case of the above discussion). Since, in general, the uncertainty polytope has $n_w > 1$ vertices, we will have to accept some conservativeness, since in general then $V \succeq (\Phi + C_j + D_j K)^T V^+ (\Phi + C_j + D_j K)$ (see discussion for the second case). However, one can formulate an SDP, which ensures that (7.30) is tight in some sense. Therefore for a given scaling the maximal eigenvalue of

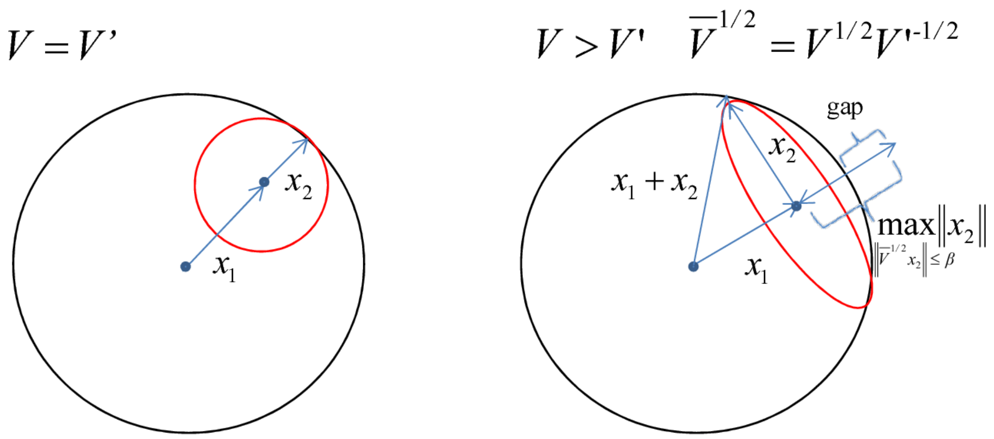

Figure 7.2: Illustration of conservativeness of tube approach

the ellipsoidal shape matrix $V$ at current time is minimized under the constraint that the corresponding ellipsoid encloses the union of the $n_w$ possible step-ahead ellipsoids. The minimization of the maximal eigenvalue can be formulated by ensuring that shrinking the ball outside the current ellipsoid shrinks it along the axis of its largest eigenvalue. This and condition (7.30) can be formulated thus as follows:

$$\max_{S,Y,\gamma} \quad \gamma \tag{7.32}$$

$$\text{s.t. } S \succeq \gamma I,$$

$$\begin{bmatrix} S & [(A_k + C_j)S + (B_k + D_j)Y]^T \\ * & V_{k+1}^{-1} \end{bmatrix} \succeq 0, \quad j = 1, \ldots, n_w$$

where $V_k = S^{-1}$, $K_k = YS^{-1}$. The proposed method is to start with $V_N = \hat{V}$ and then (in order to minimize the degree of conservativeness of Condition (7.30)) to calculate recursively values for $V_k$ and $K_k$ for $k = N-1, \ldots, 1$ by solving the above SDP for $k = N-1, \ldots, 1$. Therefore we can employ Condition (7.31) with variable tube shapes over the horizon. For the terminal constraint we set $V_N = \hat{V}$ and therefore the terminal matching Condition (7.26) can be ensured by the following condition:

$$1 \geq \left\| x_N^0 + z_N \right\|_{\hat{V}} + \beta_N \tag{7.33}$$

### 7.2.2 Fixed Feedback Gains and Tube Cross-Sections

In order to optimize $V_k$ and $K_k$ for $k = 1, \dots, N-1$, the sequence of SDPs described in the last section has to be solved after each update of the trajectory $(x_k^0, u_k^0)$ about which the successive linearization is performed. Minimizing the degree of conservativeness when invoking the recursive membership condition (7.28) has the advantage of increasing the feasible set of plant states and the degree of optimality with respect to the optimal NLP cost. The disadvantage of the approach is that the SDPs have to be solved online. Therefore for large systems and systems requiring fast sampling this could be computationally intractable (although the above approach leads to a sequence of $N$ SDPs, each of which has $O(n_x)$ variables, so that the computation scales only linearly with horizon length).

An alternative approach is to avoid as much of the online computations as possible but to still optimize the tube volume online by retaining the tube scalings as optimization variables. The idea is therefore to fix the shape of the tubes $V_k$ and the feedback gains $K_k$ to the terminal values, but to keep $\beta$ as a variable in the online optimization. Thus we set:

$$V_k = \hat{V}, K_k = \hat{K}, k = 1, \dots, N-1. \tag{7.34}$$

Before stating the corresponding tube-membership conditions, we first recall the submultiplicative property of induced norms:

$$\|Ax\| \leq \overline{\sigma}(A) \|x\| \tag{7.35}$$

where $x \in \mathbb{R}^{n_x}$. This expression will also hold if we take the maximum of both sides with $x \in \left\{x \mid \|x\|^2 \leq \alpha^2\right\}$:

$$\max_{x \in \left\{x \mid \|x\|^2 \leq \alpha^2\right\}} \|Ax\| \leq \overline{\sigma}(A)\alpha \tag{7.36}$$

Hence we can reformulate the first term of the RHS of the recursive membership condition (7.28) for fixed shapes and fixed feedback gains as:

$$\begin{aligned}
&\max_{e \in E(\hat{V}, \beta^2)} \left\|\hat{V}^{1/2}(\Phi + C_j + D_j\hat{K})\hat{V}^{-1/2}\hat{V}^{1/2}e\right\| && (7.37)\\
&\leq \overline{\sigma}[\hat{V}^{1/2}(\Phi + C_j + D_j\hat{K})\hat{V}^{-1/2}] \max_{e \in E(\hat{V}, \beta^2)} \left\|\hat{V}^{1/2}e\right\| && (7.38)\\
&= \beta\overline{\sigma}[|\hat{V}^{1/2}(\Phi + C_j + D_j\hat{K})\hat{V}^{-1/2}] && (7.39)
\end{aligned}$$

Applying the triangle inequality allows to write

$$\overline{\sigma}[\hat{V}^{1/2}(\Phi + C_j + D_j\hat{K})\hat{V}]^{-1/2}) \leq \overline{\sigma}(\hat{V}^{1/2}\Phi\hat{V}^{-1/2}) + \overline{\sigma}[\hat{V}^{1/2}(C_j + D_j\hat{K})\hat{V}^{-1/2}] \tag{7.40}$$

which leads to the following lemma:

**Lemma 7.3** *A sufficient condition for membership condition (7.28) is:*

$$\begin{aligned}
\beta^+ \geq& \left\|(C_j + D_j\hat{K})z + D_j v\right\|_{\hat{V}} + && (7.41)\\
&\beta\left\{\overline{\sigma}(\hat{V}^{1/2}\Phi\hat{V}^{-1/2}) + \overline{\sigma}[\hat{V}^{1/2}(C_j + D_j\hat{K})\hat{V}^{-1/2})]\right\}, j = 1, ..., n_w
\end{aligned}$$

*Proof:* This follows from the above discussion ■

This allows the $n_w$ singular values of the matrices $\hat{V}^{1/2}(C_j + D_j\hat{K})\hat{V}^{-1/2}$ to be computed offline and only leaves the determination of the singular values of $\hat{V}^{1/2}\Phi_k\hat{V}^{-1/2}$ online for $k = 1, ..., N-1$.

## 7.3 Cost and Constraint Bounds

The tubes constructed in the last section to bound the linearization errors can now be employed to determine an upper bound on the predicted cost-to-go function and to construct robust constraints that ensure that predicted states and inputs satisfy (7.6) over an infinite prediction horizon. We restate the finite horizon objective to be solved optimized:

$$J_0(\mathbf{x},\mathbf{u}) = \sum_{k=0}^{N-1} \frac{1}{2}(\|x_k\|_Q^2 + \|u_k\|_R^2) + \frac{1}{2}\|x_N\|_P^2 \tag{7.42}$$

The weight $P$ on the terminal state is computed so as to upper bound the cost-to-go over prediction times $k \geq N$: $\sum_{k=N}^{\infty} \frac{1}{2}(\|x_k\|_Q^2 + \|u_k\|_R^2) \leq \frac{1}{2}\|x_N\|_P^2$. We note that the nonlinear system is embedded in the convex hull of $n_w$ linear systems. Due to this fact the Lyapunov equation discussed in Section 2.2.1.2 cannot in general hold with equality for all $i = 1, ..., n_w$. We therefore seek a non-conservative upper bound on the cost over the remaining horizon and to this end we minimize the trace of $P$. We therefore solve the following SDP to obtain $P$:

$$\begin{aligned} &\min_P \; tr(P) \\ &\text{s.t.} \quad P - (\hat{\Phi} + \hat{C}_j + \hat{D}_j\hat{K})^T P(\hat{\Phi} + \hat{C}_j + \hat{D}_j\hat{K}) \succeq Q + \hat{K}^T R\hat{K}, \quad j = 1, \ldots, n_w \end{aligned} \tag{7.43}$$

which can be performed offline.

In order to formulate the optimal perturbation problem as a convex optimization problem we consider the minimization of a bound on $J(\mathbf{x},\mathbf{u})$ which is derived from bounds on the individual terms in (7.42) (which in turn is derived from the bounds on the linearization errors). We note that the state decomposition (7.14) and membership conditions (7.25) imply that $x = x^0 + z + e$ and $e \in E(V, \beta^2)$ (where the subscripts $k$ have been omitted for simplicity). Application of the triangle inequality gives

$$\begin{aligned}\|x\|_Q &\leq \left\|x^0+z\right\|_Q + \|e\|_Q \leq \left\|x^0+z\right\|_Q + \overline{\sigma}(Q^{1/2}V^{-1/2}) \max_{e\in E(V,\beta^2)} \left\|V^{1/2}e\right\| \quad (7.44)\\ &= \left\|x^0+z\right\|_Q + \beta\overline{\sigma}(Q^{1/2}V^{-1/2})\end{aligned}$$

Similarly from (7.14), (7.25) and the predicted feedback law (7.9) we obtain $u = u^0 + K(z+e) + v$, and then applying the triangle inequality as before implies $\|u\|_R \leq \|u^0 + Kz + v\|_R + \beta\overline{\sigma}(R^{1/2}KV^{-1/2})$. These upper bounds on the individual terms can be employed in the definition of the upper bound on the predicted cost. We define $l_{x,k}$ and $l_{u,k}$ for $k = 0, ..., N-1$ by

$$l_{x,k} = \left\|x_k^0 + z_k\right\|_Q + \beta_k\overline{\sigma}(Q^{1/2}V_k^{-1/2}) \quad (7.45)$$

$$l_{u,k} = \left\|u_k^0 + K_k z_k + v_k\right\|_R + \beta_k\overline{\sigma}(R^{1/2}K_kV_k^{-1/2}) \quad (7.46)$$

$$l_{x,N} = \left\|x_N^0 + z_N\right\|_P + \beta_N\overline{\sigma}(P^{1/2}\hat{V}^{-1/2}) \quad (7.47)$$

We define the upper bound cost $\overline{J_0}(\mathbf{v}, \boldsymbol{\beta}, \mathbf{x}^0, \mathbf{u}^0)$ for given sequences $\mathbf{v} = \{v_0, \ldots, v_{N-1}\}$ and $\boldsymbol{\beta} = \{\beta_0, \ldots, \beta_{N-1}\}$ as

$$\overline{J_0}(\mathbf{v}, \boldsymbol{\beta}, \mathbf{x}^0, \mathbf{u}^0) = \sum_{k=0}^{N-1} \frac{1}{2}(l_{x,k}^2 + l_{u,k}^2) + \frac{1}{2}l_{x,N}^2 \quad (7.48)$$

We note that from the above definition of upper bounds on $\|x_k\|_Q$ and $\|u_k\|_R$ for $k = 0, ..., N-1$ and $\|x_N\|_P$ the following result can be obtained:

**Lemma 7.4** *Under feedback law (7.9) and state decomposition (7.14) we obtain:*

$$J_0(\mathbf{x}, \mathbf{u}) \leq \overline{J_0}(\mathbf{v}, \boldsymbol{\beta}, \mathbf{x}^0, \mathbf{u}^0) \quad (7.49)$$

*for any* $\mathbf{x}, \boldsymbol{\beta}$ *satisfying the membership conditions (7.25) and (7.26).*

We impose the system constraints (7.6) on the predicted state and input trajectories by imposing constraints on the nominal component of the predicted state $z_k$, the nominal component of the input $v_k$ and the scaling of the tube $\beta_k$.

**Lemma 7.5** *Given feedback law (7.9) and state decomposition (7.14), sufficient conditions for satisfaction of the constraints for the predicted states and inputs* $Fx_k + Gu_k \leq h$ *are:*

$$(F_q + G_qK_k)z_k + G_qv_k + \beta_k\overline{\sigma}[(F_q + G_qK_k)V_k^{-1/2}] \leq h_q - (F_qx_k^0 + G_qu_k^0) \quad (7.50)$$

*for* $q = 1, ..., n_c$*, for any* $\beta_k$ *such that the membership condition (7.25) holds.*

*Proof:* Using (7.9) and (7.14) a sufficient condition is given by:

$$\begin{aligned}&F_qx_k + G_qu_k = F_qx_k^0 + G_qu_k^0 + (F_q + G_qK_k)z_k + G_qv_k + (F_q + G_qK_k)e_k \leq\\&F_qx_k^0 + G_qu_k^0 + (F_q + G_qK_k)z_k + G_qv_k + \max_{e\in E(V_k,\beta_k^2)} (F_q + G_qK_k)e =\\&F_qx_k^0 + G_qu_k^0 + (F_q + G_qK_k)z_k + G_qv_k + \beta_k\overline{\sigma}[(F_q + G_qK_k)V_k^{-1/2}] \leq h_q\end{aligned}$$

where $h_q$ are the elements of $h$ and $F_q, G_q$ denoting the rows of $F, G$ for $q = 1, ..., n_c$. This follows by a variable change $e = V_k^{-1/2}\beta_k\tilde{e}$, which transforms the feasible ellipsoidal set into a spherical set of unit radius. This transforms the objective into $(F_q + G_qK_k)V_k^{-1/2}\beta_k\tilde{e}$ and this implies that the maximum value over the spherical set is $\beta_k\overline{\sigma}[(F_q + G_qK_k)V_k^{-1/2}]$. ■

In the next section a procedure will be outlined which splits the online MPC optimization into a sequence of iterations, each of which consists of minimizing the objective $\overline{J_0}(\mathbf{v}, \boldsymbol{\beta}, \mathbf{x}, \mathbf{u})$ over $\mathbf{v}, \boldsymbol{\beta}$ subject to the 'robustified' constraints (7.50) combined with the conditions defining the ellipsoidal tubes in Section 7.2.

The optimization to be performed at each iteration can be expressed as:

$$(\mathbf{v}^*, \boldsymbol{\beta}^*) = \arg\min_{\mathbf{v},\boldsymbol{\beta}} \sum_{k=0}^{N-1} \frac{1}{2}(l_{x,k}^2 + l_{u,k}^2) + \frac{1}{2}l_{x,N}^2 \quad (7.51)$$

subject to

$$z_{k+1} = \Phi_k z_k + B_k v_k \tag{7.52}$$

$$\beta_{k+1} \geq \lambda_{k,j}\beta_k + \|(C_j + D_j K_k)z_k + D_j v_k\|_{V_{k+1}} \tag{7.53}$$

$$l_{x,k} \geq \left\|x_k^0 + z_k\right\|_Q + \beta_k \overline{\sigma}(Q^{1/2}V_k^{-1/2}) \tag{7.54}$$

$$l_{u,k} \geq \left\|u_k^0 + K_k z_k + v_k\right\|_R + \beta_k \overline{\sigma}(R^{1/2}K_k V_k^{-1/2}) \tag{7.55}$$

$$h_q - (F_q x_k^0 + G_q u_k^0) \geq F_q + G_q K_k z_k + G_q v_k \tag{7.56}$$

$$+ \beta_k \overline{\sigma}[(F_q + G_q K_k)V_k^{-1/2}]$$

for $k = 0, ..., N-1$, and

$$z_0 = 0 \tag{7.57}$$

$$\beta_0 = 0 \tag{7.58}$$

$$1 \geq \left\|x_N^0 + z_N\right\|_{\hat{V}} + \beta_N \tag{7.59}$$

$$l_{x,N} \geq \left\|x_N^0 + z_N\right\|_P + \beta_N \overline{\sigma}(P^{1/2}\hat{V}^{-1/2}) \tag{7.60}$$

The scalars $\lambda_{k,j}$ are chosen as $\lambda_{k,j} = 1$ for time-varying tubes and feedback gains. For fixed tubes and feedback gains (i.e. $V_k = \hat{V}$, $K_k = \hat{K}$) the scalars $\lambda_{k,j}$ are chosen as $\lambda_{k,j} = \overline{\sigma}(\hat{V}^{1/2}\Phi_k \hat{V}^{-1/2}) + \overline{\sigma}[\hat{V}^{1/2}(C_j + D_j\hat{K})\hat{V}^{-1/2}]$.

**Remark 7.2** *We note that the above optimization can be expressed as a second order cone program (SOCP) (see Section 2.1 and note that* $\hat{J}_0^* = \sqrt{\overline{J_0}^*}$*):*

$$(\hat{J}_0{}^*, \mathbf{v}^*, \boldsymbol{\beta}^*, l_x^*, l_u^*) = \min_{\hat{J}_0, \mathbf{v}, \boldsymbol{\beta}, l_x, l_u} \hat{J}_0$$

*subject to*

$$\hat{J}_0 \geq \left\| \begin{matrix} \frac{1}{\sqrt{2}} l_x & \frac{1}{\sqrt{2}} l_u \end{matrix} \right\|$$

$$(7.52) - (7.60)$$

*where* $l_x = \{l_{x,0}, \ldots, l_{x,N}\}$ *and* $l_u = \{l_{u,0}, \ldots, l_{u,N-1}\}$.

In order to demonstrate that the cost and constraint bounds derived in this section are not overly conservative, we note that in the special case of zero perturbation, i.e. $\mathbf{v}^* = 0$, the bounds are in fact non-conservative. This fact is demonstrated in the next lemma:

**Lemma 7.6** *For any* $\mathbf{x}^0, \mathbf{u}^0$ *satisfying (7.5) we have:*

$$\overline{J_0}(0, 0, \mathbf{x}^0, \mathbf{u}^0) = J(\mathbf{x}^0, \mathbf{u}^0) \tag{7.61}$$

*Furthermore, if* $\mathbf{v}^* = 0$*, then* $\boldsymbol{\beta}^* = 0$*, and the constraints of (7.51) are equivalent to:*

$$h \leq (Fx_k^0 + Gu_k^0) \tag{7.62}$$

$$x_N^0 \in E(\hat{V}, 1). \tag{7.63}$$

*Proof:* If $\boldsymbol{\beta} = 0, \mathbf{v} = 0$ then (7.54), (7.55) and (7.60) imply that $\overline{J_0}(0, 0, \mathbf{x}^0, \mathbf{u}^0) = J_0(\mathbf{x}^0, \mathbf{u}^0)$ (N.B. this simply follows from the definition of the cost and does not require the optimization problem to be solved). Furthermore, if $\mathbf{v}^* = 0$ (i.e. zero is a solution to the optimization problem), then (7.52),(7.57) imply that $z_k = 0$ for all $k$ and hence according to (7.53),(7.58) and due to the fact that the objective is to minimize the upper bound cost (7.51) it follows that $\boldsymbol{\beta}^* = 0$. This implies that the RHS of (7.56) is zero and that condition (7.59) becomes equivalent to $x_N^0 \in E(\hat{V}, 1)$. ■

## 7.4 Receding Horizon Control Law

This section provides a description of the proposed MPC optimization procedure and discusses the system theoretic properties of the associated receding horizon control law. The optimization procedure computes an optimal sequence of perturbations on a feasible predicted trajectory by solving the SOCP (7.51), which is based on the Jacobian linearization about this trajectory. Then it updates the feasible trajectory using the optimal perturbations and repeats the process. Since a feasible predicted trajectory is available at each step, the procedure can be terminated after any chosen maximum number of iterations ($maxiters$), or when the input perturbation vector falls below any given tolerance ($solutiontolerance$). This can be summarized in the following algorithm:

**Algorithm 7.1** *$Offline$: Compute $\hat{V}, \hat{K}$ defining the terminal set and feedback law by solving (7.21) and the terminal cost weight $P$ by solving (7.43). Find an initial input trajectory $\mathbf{u}^0$ such that $\mathbf{u}^0, \mathbf{x}^0$ satisfy equations (7.62) and (7.63).*

*Online: At times $t = 0, 1, ...$ (N.B. the time index $t$ is dropped for simplicity):*

1. *Set iter $= 1$. Given $\mathbf{u}^0$, compute $\mathbf{x}^0$ satisfying the model (7.5) with $x_0^0 = x_t$*
2. *Linearize model (7.5) about $\mathbf{u}^0, \mathbf{x}^0$ to determine $A_k, B_k$ for $k = 0, ..., N-1$.*
3. *If time-varying feedback gains and tube shapes are used, compute $V_k$ and $K_k$ by solving (7.32) for $k = N-1, ..., 1$ with $V_N = \hat{V}$.*
4. *Solve (7.51) to determine $\mathbf{v}^*$.*
5. *Compute $\mathbf{x}$ and $\mathbf{u}$ satisfying (7.5) and (7.8) with $\mathbf{v} = \mathbf{v}^*$.*
6. *If $iter < maxiters$ and $\|\mathbf{v}^*\| \geq solutiontolerance$, set $\mathbf{x}^0 = \mathbf{x}$, $\mathbf{u}^0 = \mathbf{u}$, $iter := iter + 1$ and return to step 2.*
7. *Otherwise set*

$$\mathbf{u}^{0+} = \left\{u_1, ..., u_{N-1}, \hat{K}x_N\right\} \text{ (i.e. the tail of the optimal input sequence)} \tag{7.64}$$

*and implement $u_t = u_0^0 + v_0^*$.*

**Lemma 7.7** *If $\mathbf{x}^0, \mathbf{u}^0$ is feasible with respect to constraints (7.62),(7.63) at time $t = 0$, then the SOCP (7.51) in step 4 of Algorithm 7.1 is feasible at each iteration and for all $t \geq 0$.*

*Proof:* By Lemma 7.6, if $\mathbf{x}^0, \mathbf{u}^0$ is feasible for (7.62),(7.63), then $(\mathbf{v}, \boldsymbol{\beta}) = (0, 0)$ is feasible for the optimization (7.51). Lemma 7.2, 7.3 and 7.5, which are ensured by corresponding constraints in the optimization (7.51), imply that the updated trajectory $\mathbf{u}^0, \mathbf{x}^0$ is feasible with respect to (7.62) and (7.63). Similarly, the robust invariance and feasibility conditions (7.19), (7.20) ensure feasibility of the tail at the subsequent instant of time. ■

The following results establish that employing Algorithmn 7.1 for more iterations is sensible, since the corresponding iteration sequence of the cost is monotonically non-increasing. This is expressed by the following result:

**Theorem 7.1** *Let $\overline{J_0}^j$ denote the optimal value of the objective of (7.51) in step 4 of Algorithm 7.1 after $j$ iterations at time $t$. Then for all $j \geq 1$ we have*

$$\overline{J_0}^{j+1} \leq \overline{J_0}^j. \tag{7.65}$$

*Proof:* Lemma 7.4 implies that the trajectory $\mathbf{x}^0, \mathbf{u}^0$ generated in step 4 of the $j$th iteration of Algorithm 7.1 necessarily satisfies $J_0(\mathbf{x}^0, \mathbf{u}^0) \leq \overline{J_0}^j$. However $\overline{J_0}^{j+1}$ is the solution to the optimization problem (7.51), so that $\overline{J_0}^{j+1} \leq \overline{J_0}(0, 0, \mathbf{x}^0, \mathbf{u}^0)$. Lemma 7.6 implies that $\overline{J_0}(0, 0, \mathbf{x}^0, \mathbf{u}^0) = J(\mathbf{x}^0, \mathbf{u}^0)$ and this implies that $\overline{J_0}^{j+1} \leq \overline{J_0}(0, 0, \mathbf{x}^0, \mathbf{u}^0) = J_0(\mathbf{x}^0, \mathbf{u}^0) \leq \overline{J_0}^j$. ■

**Remark 7.3** *We note that the upper bound cost $\overline{J_0}$ is bounded from below and holds with equality only for zero perturbation $\mathbf{v}^* = 0$ at iteration $j+1$. So if $\overline{J_0}^{j+1} = \overline{J_0}^j$ we have $\overline{J_0}^j = \overline{J_0}^{j+1} \leq \overline{J_0}(0, 0, \mathbf{x}^0, \mathbf{u}^0) \leq \overline{J_0}^j = \overline{J_0}^{j+1}$ which implies $\overline{J_0}^j = \overline{J_0}^{j+1} = \overline{J_0}(0, 0, \mathbf{x}^0, \mathbf{u}^0)$. So assuming $\overline{J_0}^j \neq \overline{J_0}(0, 0, \mathbf{x}^0, \mathbf{u}^0)$ implies $\overline{J_0}^{j+1} < \overline{J_0}^j$ according to Theorem 7.1. This implies asymptotic convergence to $(\mathbf{v}^*, \boldsymbol{\beta}^*) = 0$ as $j \to \infty$ and therefore we have convergence to a (possibly locally) optimal point for the problem of minimizing $\overline{J_0}(0, 0, \mathbf{x}^0, \mathbf{u}^0)$ over $\mathbf{u}^0$ subject to constraints (7.62), (7.63). We note that a minimum point of $J_0(\mathbf{x}^0, \mathbf{u}^0)$ is necessarily also a minimum of $\overline{J_0}(0, 0, \mathbf{x}^0, \mathbf{u}^0)$ with respect to $\mathbf{u}^0$. To see this* *we assume that* $\mathbf{u}^0$ *is locally optimal for* $J_0(\mathbf{x}^0, \mathbf{u}^0)$. *Then (7.61) implies that* $J_0(\mathbf{x}^0, \mathbf{u}^0) = \overline{J_0}(0, 0, \mathbf{x}^0, \mathbf{u}^0)$ *and (7.49) implies that* $J_0(\mathbf{x}^0, \mathbf{u}^0) \le \overline{J_0}(\mathbf{v}, \boldsymbol{\beta}, \mathbf{x}^0, \mathbf{u}^0)$. *But this implies that* $\overline{J_0}(0, 0, \mathbf{x}^0, \mathbf{u}^0) \le \overline{J_0}(\mathbf{v}, \boldsymbol{\beta}, \mathbf{x}^0, \mathbf{u}^0)$. *Thus since the optimal perturbation is zero, we have demonstrated that* $\overline{J_0}(0, 0, \mathbf{x}^0, \mathbf{u}^0)$ *is optimal w.r.t the same* $\mathbf{u}^0$ *that is optimal for* $J_0$. *However the converse does not necessarily hold, i.e. a local minimum of* $\overline{J_0}(0, 0, \mathbf{x}^0, \mathbf{u}^0)$ *may not be optimal for* $J_0(\mathbf{x}^0, \mathbf{u}^0)$.

**Remark 7.4** *We note further that in the discussion of this chapter assumes that reasonable bounds on* $u^\delta$, $x^\delta$ *have been selected offline for which the successive linearization method is performed. Selecting such bounds involves a tradeoff: the larger the bounds are chosen, the larger the uncertainty incurred over the horizon implying larger tubes and therefore a smaller region of attraction and higher degree of suboptimality with respect to the optimal NLP cost. On the other hand, larger bounds also allow larger allowable step-sizes for each iteration of the algorithm and therefore convergence is generally faster. It would therefore be a sensible future extension to allow for online adjustments of the bounds to find the best tradeoff between optimality and convergence rate.*

Unfortunately, we cannot prove stability directly by employing the standard results established in Section 2.2.1.2 for the nonlinear MPC problem. These depend on global optimality of the value function and this assumption is not satisfied in the suboptimal approach of this chapter. Therefore we seek an alternative approach by using the upper bound cost as a Lyapunov function and demonstrating asymptotic stability. This results in the following theorem:

**Theorem 7.2** $x = 0$ *is an asymptotically stable equilibrium of (7.5) under the MPC law of Algorithm 7.1 with a region of attraction equal to the set of feasible initial conditions for (7.62), (7.63).*

*Proof:* The LMI constraint on P in (7.43) and (7.13) imply that

$$\|x\|_P^2 - \left\|\hat{\Phi}x + (\hat{C}_j + \hat{D}_j\hat{K})x\right\|_P^2 \ge \|x\|_Q^2 + \left\|\hat{K}x\right\|_R^2 \tag{7.66}$$

$$\left\| f(x,\hat{K}x) \right\|_P^2 \leq \left\| \hat{\Phi}x + (\hat{C}_j + \hat{D}\hat{K})x \right\|_P^2 \tag{7.67}$$

for $x \in \mathcal{X}_f$. This implies that

$$\|x\|_P^2 - \left\| f(x,\hat{K}x) \right\|_P^2 \geq \|x\|_Q^2 + \left\| \hat{K}x \right\|_R^2 \tag{7.68}$$

for $x \in \mathcal{X}_f$. We can use this result and the definition of $\mathbf{u}^{0+}$ (+ denotes the next instant in time $t+1$) in (7.64) to show that the trajectory $\mathbf{x}^{0+}$,$\mathbf{u}^{0+}$ satisfies:

$$J_0(\mathbf{x}^{0+},\mathbf{u}^{0+}) = \sum_{k=1}^{N-1} \frac{1}{2}(\|x_k\|_Q^2 + \|u_k\|_R^2) \tag{7.69}$$

$$+ \frac{1}{2}\|x_N\|_Q^2 + \frac{1}{2}\left\| \hat{K}x_N \right\|_R^2 + \frac{1}{2}\left\| f(x_N,\hat{K}x_N) \right\|_P^2 \tag{7.70}$$

$$\leq \sum_{k=0}^{N-1} \frac{1}{2}(\|x_k\|_Q^2 + \|u_k\|_R^2) + \frac{1}{2}\|x_N\|_P^2 - \frac{1}{2}(\|x_0\|_Q^2 + \frac{1}{2}\|u_0\|_R^2) \tag{7.71}$$

$$= J_0(\mathbf{x}^0,\mathbf{u}^0) - \frac{1}{2}(\|x_0\|_Q^2 + \|u_0\|_R^2) \tag{7.72}$$

However, we know from Lemma 7.4 that this implies

$$J_0(\mathbf{x}_+^0,\mathbf{u}_+^0) \leq \overline{J_0}(\mathbf{v}^*,\boldsymbol{\beta}^*,\mathbf{x}^0,\mathbf{u}^0) - \frac{1}{2}(\|x_0\|_Q^2 + \|u_0\|_R^2) \tag{7.73}$$

Furthermore, the optimization is feasible at the time instant $t+1$ for no perturbation, i.e.$(\mathbf{v}^{*+},\boldsymbol{\beta}^{*+}) = (0,0)$, and application of Lemma 7.6 then gives

$$\overline{J_0}(\mathbf{v}^{*+},\boldsymbol{\beta}^{*+},\mathbf{x}^{0+},\mathbf{u}^{0+}) \leq \overline{J_0}(0,0,\mathbf{x}^{0+},\mathbf{u}^{0+}) = J_0(\mathbf{x}^{0+},\mathbf{u}^{0+}) \tag{7.74}$$

$$\leq \overline{J_0}(\mathbf{v}^*,\boldsymbol{\beta}^*,\mathbf{x}^0,\mathbf{u}^0) - \frac{1}{2}(\|x_0\|_Q^2 + \|u_0\|_R^2) \tag{7.75}$$

$$= \overline{J_0}(\mathbf{v}^*,\boldsymbol{\beta}^*,\mathbf{x}^0,\mathbf{u}^0) - \frac{1}{2}(\|x_{0,t}\|_Q^2 + \|u_{0,t}\|_R^2) \tag{7.76}$$

For a Lyapunov stability argument we take $\overline{J_0}(\mathbf{v}^*, \boldsymbol{\beta}^*, \mathbf{x}^0, \mathbf{u}^0)$ as a Lyapunov function. $\overline{J_0}$ is a positive definite function of $x_0$ and we obtain

$$\overline{J_0}(\mathbf{v}^{*+}, \boldsymbol{\beta}^{*+}, \mathbf{x}^{0+}, \mathbf{u}^{0+}) - \overline{J_0}(\mathbf{v}^*, \boldsymbol{\beta}^*, \mathbf{x}^0, \mathbf{u}^0) \leq -\frac{1}{2}(\|x_t\|_Q^2 + \|u_t\|_R^2) \leq 0 \tag{7.77}$$

which implies stability (note that we made explicit that the first predicted element is equal to the plant measurement/actual applied input respectively, i.e. $x_0 = x_{0,t} = x_t$ and $u_0 = u_{0,t} = u_t$). Further we can establish attractiveness of the origin:

$$\sum_{t=0}^{\infty} \frac{1}{2}(\|x_t\|_Q^2 + \|u_t\|_R^2) \leq \overline{J_0}(\mathbf{v}_0^*, \boldsymbol{\beta}_0^*, \mathbf{x}_0^0, \mathbf{u}_0^0) \tag{7.78}$$

The finiteness of the RHS implies asymptotic stability of $x = 0$. ■

**Remark 7.5** *It has to be stressed that a key assumption of the proposed algorithm is the availability of an initial trajectory that is feasible w.r.t. constraints (7.62), (7.63). In practice one could find a trajectory which satisfies the system model 7.5 without these constraints and then perturb the input and state trajectory so that the maximum violation of Constraints (7.56), (7.59) is minimized. This could be formulated as an SOCP problem, which analogously to Lemma 7.7 and Theorem 7.1 is guaranteed to give a non-increasing bound on the maximum constraint violation.*

## 7.5 Simulation Example

The control law of Algorithm 7.1 is applied to a planar model of a fixed-rotor helicopter:

$$\ddot{y} = (u^1 + g)\sin\alpha \tag{7.79}$$

$$\ddot{z} = (u^1 + g)\cos\alpha - g \tag{7.80}$$

$$\ddot{\alpha} = u^2 \tag{7.81}$$

Here $y, z, \alpha$ represent horizontal, vertical and angular displacement, $g$ is acceleration due to gravity, and the inputs $u^1, u^2$ are proportional to the net thrust and torque acting on the aircraft. The MPC cost is defined with $Q = I$, $R = 10^{-3}I$ and the system is subject to input constraints ($t$ is the real time of the closed loop system, $k$ the predicted time):

$$\left|u_t^1\right| \leq 10, \left|u_t^2\right| \leq 10. \tag{7.82}$$

A discrete time model with state $x_t = (y(tT), z(tT), \dot{y}(tT), \dot{z}(tT), \alpha(tT), \dot{\alpha}(tT))$ and sample period $T = 0.1$s is computed offline by numerical integration. The linearization error bounds (7.13) are computed offline by using the constraints (7.82) and bounds on the state and input perturbations: $\left|x_t^\delta\right| \leq \overline{x}_t^\delta$ $\left|u_t^\delta\right| \leq \overline{u}_t^\delta$ with $\overline{x}^\delta = (0.5, 1, 0.5, 1, 0.05, 1)$ and $\overline{u}^\delta = (1, 1)$. The offline computation of $\hat{K}, \hat{V}$ was performed by modifying (7.21) to include the bound $\max\limits_{x \in E(\hat{V},1)} \|x\|_P^2 \leq 10$ on the mode 2 cost. For simplicity fixed tubes were employed for bounding the effects of linearization errors.

Figures 7.3 and 7.4 show the predicted input and state trajectories for horizon length $N = 10$, an initial condition $x_0 = (0, -1, 0, 0, -0.5, 0)$ and an initial trajectory $\mathbf{u}^0, \mathbf{x}^0$. Algorithm 7.1 has not converged (with $solutiontolerance = 10^{-3}$) to the optimal solution of the underlying NLP. The costs shown in Table 7.1 indicate that the predicted costs at $t = 0$ are 13% and 7% suboptimal for $Maxiters = 1$ and $Maxiters = 3$ respectively, however in the first case Algorithm 7.1 only requires 5% of the CPU time of the NLP-solver. As can also be noted, the degree of suboptimality is significantly reduced by the receding horizon implementation of Algorithm 7.1 which is only 5% suboptimal in closed loop operation.

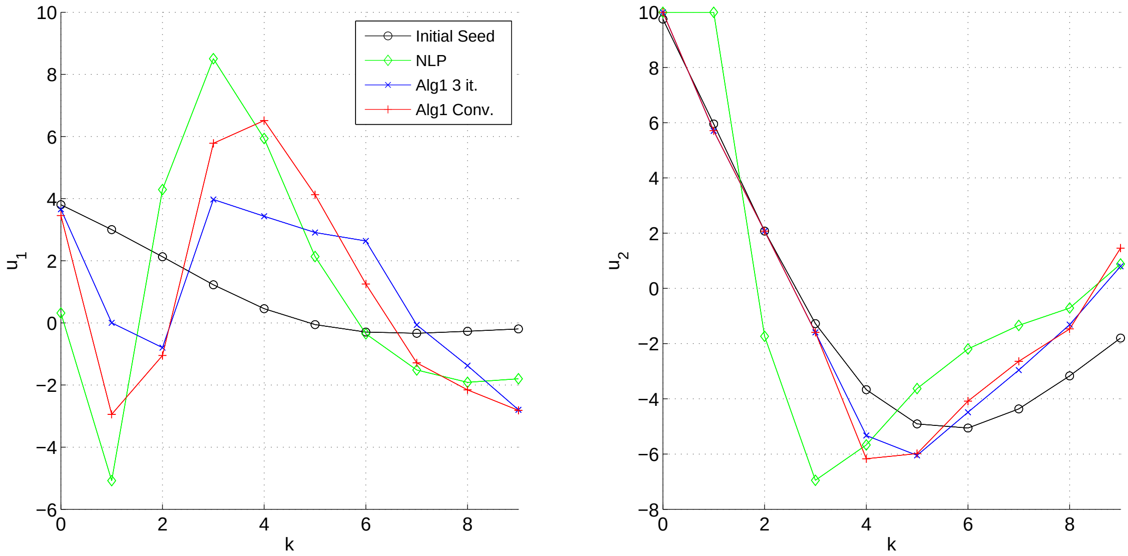

Figure 7.3: Input sequences predicted at $t = 0$

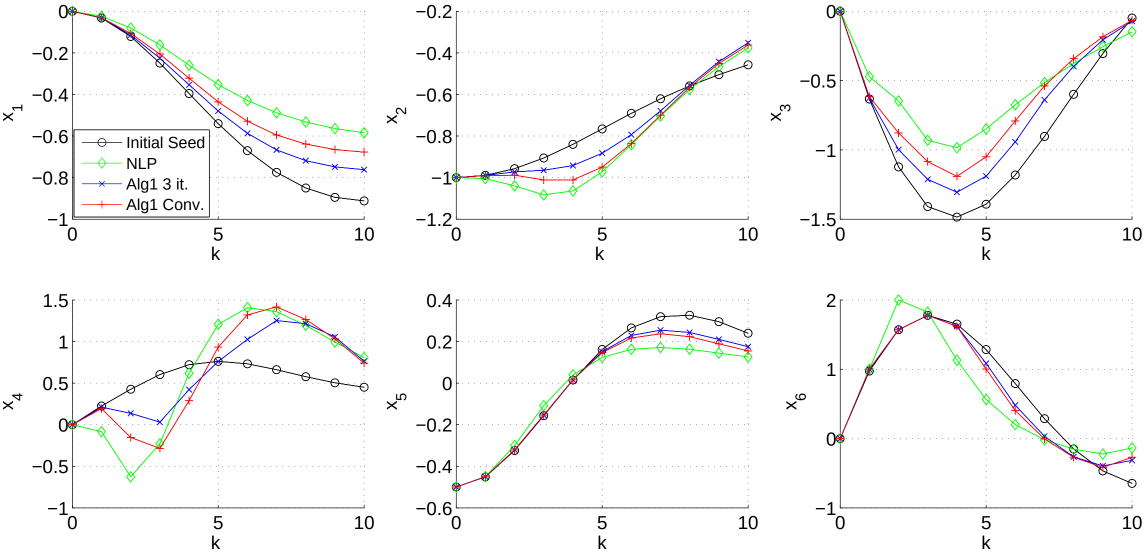

Figure 7.4: State sequences predicted at $t = 0$

## 7.6 Conclusions

The chapter discusses a NMPC algorithm based on successive approximization of the underlying NLP. The effects of the approximation errors on predicted trajectories are bounded using tubes with ellipsoidal cross sections which are optimized online simultaneously with the MPC cost by solving an SOCP. The approach is

| | $J_0(\mathbf{x}_0, \mathbf{u}_0)$ | CPU time | $J(closedloop)$ |
|---|---|---|---|
| Alg.1($Maxiters$=1) | 22.89 | 0.44 | 21.14 |
| Alg.1($Maxiters$=3) | 21.70 | 1.28 | 20.95 |
| NLP(fmincon) | 20.26 | 9.37 | 19.97 |

Table 7.1: Predicted and closed loop cost, CPU times

recursively feasible and successive iterations converge to an optimum of the upper bound cost. The approach further has asymptotic stability and can be terminated early for computational convenience but at the price of suboptimality. However, the approach does not necessarily guarantee convergence to a local optimum of the underlying NLP and the linearization error grows quadratically whereas the assumed bounds on the error are taken to be linear. Some promising extensions of the discussed approach are discussed in the concluding remarks of this thesis (see Chapter 8).

# Chapter 8

# Conclusions

In this thesis we have introduced a framework for the efficient solution of finite horizon optimal control problems both in the linear nominal and linear uncertain situation. These methods have been demonstrated to be computationally very efficient, while at the same time providing exact solutions to the underlying dynamic program.

To conclude, we summarize the main contributions of each chapter and outline some promising future research directions.

## 8.1 Summary

The following results have been presented in this thesis:

- In Chapter 3, an efficient DP based method is proposed which solves linear-quadratic MPC problems. This chapter further introduces a general framework for exploiting the parametric and sequential nature of the optimization problem in order to derive an efficient line-search based point location method based on homotopy of solutions, where Riccati recursions are used to solve the underlying equality constrained DPs efficiently whenever the active set has changed. Further this chapter discusses the issue and resolution of degenerate subproblems in the nominal MPC context.

- In Chapter 4, the ideas of Chapter 3 are extended and shown to be applicable in the context of min-max robust control for linear uncertain systems with input-only constraints. The solution leads to feedback saddle point solutions which solve the underlying DP exactly. The method is computationally of similar efficiency as the nominal linear quadratic solution and therefore is a major improvement over current DP based robust MPC methods.

- Chapter 5 extends the approach of Chapter 4 to include problems with explicit state constraints. As well as allowing for a larger class of control problems this extension also addresses the shortcoming of Chapter 4 that, due to the lack of state constraints, a (possibly nonconvex) region of attraction has to be determined to guarantee closed loop stability. Furthermore, true system state constraints could not be considered in that framework. Both issues are overcome in this chapter, again based on the same underlying idea of constructing an efficient online active set solver. The chapter also addresses the important issue of degenerate subproblems, which can lead - in the uncertain case - to overlapping pQP regions and require a modification of the active set method. The method is compared to state-of-the-art robust MPC methods based on suboptimal feedback policies and found to be both computationally and performancewise superior. A possible limitation of the approach is that it requires the offline computation of the sequence of $k$-step robust controllable sets to the terminal set.

- Chapter 6 discusses methods for computing robust controllable sets offline by considering a collection of exact and approximate solutions. Secondly, the possibility of solving the controllability problem locally is discussed by considering pLPs and considering active set solutions based on homotopy. This method is discussed in detail for the single-step situation, for which theoretical and numerical results have been obtained. The multiple-step controllability problem is discussed and a possible solution approach is described in outline. Given the promising results obtained for the single-step case, it is expected that the multiple-step problem will be addressed in future research.

- Finally Chapter 7 describes how techniques for robust MPC can be efficiently applied to the Nonlinear MPC problem based on Successive Linearization. This based is based on ellipsoidal tubes, which are optimized online and are used to bound the dynamic effects of uncertainty due to the linearization error. The method is demonstrated to lead to major improvements over classical nonlinear programming based algorithms while only incurring a small degree of suboptimality.

## 8.2 Future research directions

- The local online solution to the robust controllability problem remains a very challenging and important open problem. This problem is especially challenging due to the exponential increase in complexity with horizon length in offline DP methods which do not exploit the local nature of the DP solution involved in the Model Predictive Control problem. It is unclear at present, if a method based on pLPs as described in Chapter 6 overcomes this combinatorial complexity increase, but it would certainy seem preferrable to obtain local verifiable conditions of whether a state is robustly controllable to the target set by state feedback without having to compute the solution of a sequence of global existence problems. If this were indeed possible, in combination with the method of Chapter 5, it would constitute a significant advance in the field of parametric robust MPC methods.
- A very interesting future path is to tackle the finite horizon stochastic optimal control problem with linear constraints using the proposed homotopy

active set approach based on Riccati recursions at each iteration. This would involve replacement of the Max-operator (of the sequential quadratic min-max problems considered in Chapters 4 and 5 with the Expectation operator. However, it does appear straightforward to obtain the governing KKT conditions and follow a similar approach as considered in this thesis. Similarly, it will be necessary to solve a stochastic equivalent of the robust controllabilty problem, either globally offline as controllability sets, or locally online.

- It would further seem possible to tackle the output robust MPC problem via two approaches. One way would be to select an off-the-shelf estimation method (e.g. a Luenberger observer) and solve the corresponding modified robust MPC problem with bounded disturbances (including the extra uncertainty arising from the estimation). However, the receding horizon estimation problem may also be tackled directly via a homotopy approach to solve the corresponding forward dynamic programming problem locally (i.e. in contrast to the backward dynamic programming used throughout this thesis).

- Another very challenging open problem in the context of efficient parametric methods for robust MPC is the DP based min-max problem for the class of linear systems with polytopic uncertainty and state and input constraints for the case of piecewise-affine performance objective. The solution to the corresponding dynamic programming problem can be shown to be equivalent to a sequence of multi-parametric linear programming problems and thus the space of initial conditions can be partitioned into polytopic regions [1]. Therefore the possibility exists for the formulation of an efficient line-search point location technique for solving the DP subproblems. The challenges are how to select the cost function appropriately (e.g. minimizing the distance from the minimal robust positive invariant set) and how to overcome non-uniqueness (degeneracy) issues of the dual problem. Furthermore the issue of robust controllability would seem as a major challenge in this context although the exact robust controllable sets can be shown to be convex [6].

- Chapter 7 discusses a successive linearization NMPC method which uses ellipsoidal bounds to bound the effects of uncertainty that arise as a result of the linearization error. Similarly, it is possible to bound the dynamic effects of the linearization error using polytopic bounds (e.g. using Lipschitz bounds as in [41]). This would enable one to use linear robust MPC methods based on polytopic disturbance bounds such as the one proposed in Chapter 5. One main challenge is how to set up the robust MPC subproblems and how to treat the effects of the linearization error as additive disturbances or multiplicative uncertainty. It is expected that solving subproblems with additive uncertainty is computationally more efficient but may lead to a higher degree of suboptimality as compared to using multiplicative uncertainty descriptions. Another main challenge is the online computation of robust controllability sets for the linear subproblems which has to be performed online due to the successive nature of the problem. This leads to time varying linear robust subproblems and efficient ways of tackling these problems (in the nominal context) have been suggested in Chapter 3.

- We note that under the presented results of this thesis, robust MPC based on exact DP formulations can now be applied to fast sampling applications (even of high dimensionality) if only input constraints are present. If state constraints are present and if the system is of small/medium scale such that the required offline set calculations can be performed, then the approach is similarly promising for applications.

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