Editors' Suggestion

Multiversality and Unnecessary Criticality in One Dimension

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We present microscopic models of spin ladders which exhibit continuous critical surfaces whose properties and existence, unusually, cannot be inferred from those of the flanking phases. These models exhibit either "multiversality"—the presence of different universality classes over finite regions of a critical surface separating two distinct phases—or its close cousin, "unnecessary criticality"—the presence of a stable critical surface within a single, possibly trivial, phase. We elucidate these properties using Abelian bosonization and density-matrix renormalization-group simulations, and attempt to distill the key ingredients required to generalize these considerations.

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Quantum criticality [1,2] plays a central role in our understanding of zero-temperature phases of matter. The existence of critical points or surfaces can usually be inferred even without probing the transition region, by observing suitably distinct quantum ground states in disjoint parameter regimes. When continuous, their universal scaling properties are likewise assumed to be uniquely determined by the flanking phases, unless fine-tuned. These ideas are thought to hold even when the Landau picture of broken symmetries is modified to include topological distinctions between phases, or in transitions, such as those proposed between distinct broken-symmetry orders, whose fluctuating critical degrees of freedom are not natural excitations of either adjacent phase [3].

Recently, attention has focused on a pair of converse questions: namely (1) whether a phase distinction is necessary for a critical surface to exist and (2) when two distinct phases straddle a critical surface, if this distinction uniquely fixes the universality class of the transition between them. Surprisingly, the answer to both these questions is in the negative. First, it is possible to have a critical surface within the *same* phase, accessed by tuning a single parameter, whose presence is not demanded by phase structure. Second, there exist generic (i.e., non-fine-tuned) transitions between the same pair of phases with distinct universality classes depending on the path in parameter space taken across the critical surface. These phenomena have been identified in a handful of models, usually invoking topology in an essential way. The first class of models with "unnecessary criticality" [4–7] can emerge upon modifying symmetries to remove a topological distinction [8–11] between two phases: a continuous critical surface required by the distinction becomes unnecessary in its absence. An early classical example leveraged topological distinctions within the disordered phase of an XY model in $d = 2$ spatial dimensions augmented with half-vortex defects [12,13]. More recent quantum settings involve Dirac fermions perturbed by topological mass terms and strong interactions in $d = 2$ [14] or coupled to fluctuating non-Abelian gauge fields in $d = 3$ [6]. However, the critical field theories and scaling properties of these examples of multiversality and unnecessary criticality can be challenging to access analytically or even numerically, especially in the $d > 1$ quantum setting. It is thus desirable to identify microscopic models that exhibit both phenomena in an analytically tractable regime, ideally in $d = 1$ where the density-matrix renormalization group (DMRG) allows accurate numerical simulations.

Here, we show that both phenomena arise in $d = 1$ spin ladder models, that can be accessed analytically via (Abelian) bosonization, and numerically via DMRG. We employ both strategies to map out their phase structure, and comment both on their relation to existing work and the possibility of generalizing these ideas to a broader set of models. Our Letter thus provides a basis for deeper investigations of the link between symmetry-protected topological order, criticality, and phase structure, and suggests the ingredients needed to identify further instances of multiversality and unnecessary criticality.

Models.—We begin with a class of two-leg ladder Hamiltonians of the form $H^{M/U} = H_\delta + H_\perp^{M/U}$. Here,

$$H_\delta = \sum_{j=1}^{L} \sum_{\alpha=1,2} [1 + \delta(-1)^j] h_{\alpha j}, \quad \text{with}$$

$$h_{\alpha j} = S^x_{\alpha j} S^x_{\alpha j+1} + S^y_{\alpha j} S^y_{\alpha j+1} + \Delta S^z_{\alpha j} S^z_{\alpha j+1}, \tag{1}$$

where $\vec{S}_{\alpha j} = \frac{1}{2}\vec{\sigma}_{\alpha j}$ are spin-$\frac{1}{2}$ operators written in terms of Pauli matrices $\sigma^\mu_{\alpha j}$. H_δ describes two identical decoupled

XXZ spin chains, whose couplings are staggered when $\delta \neq 0$. We fix the anisotropy $\Delta \in [-(1/\sqrt{2}), 0]$, for reasons discussed below. In this decoupled limit, each leg hosts two gapped phases: a trivial paramagnet for $\delta < 0$ and a symmetry-protected topological (SPT) phase [15,16] with gapless boundary modes—related to the celebrated Haldane phase [17–19]—for $\delta > 0$, separated by a continuous transition at $\delta = 0$. In the fully dimerized, fully decoupled limits $\delta = \pm 1$ the exact ground states of H_δ are

$$|\mathrm{GS}(\delta = \pm 1, J_\perp = 0)\rangle = \prod_{\alpha=1,2} \prod_{j \in \mathcal{J}_\pm} |[\alpha, j; \alpha, j+1]\rangle, \tag{2}$$

where $|[\alpha, i; \beta, j]\rangle$ represents an SU(2) singlet entangling sites (α, i) and (β, j), and $\mathcal{J}_+$ and $\mathcal{J}_-$ denote the set of even and odd sites, respectively. On Abelian bosonization of H_δ [20], keeping only the most relevant terms, we have

$$H_\delta \approx \frac{v}{2\pi} \int dx \sum_{\alpha=1}^{2} \left[\frac{1}{4K} (\partial_x \phi_\alpha)^2 + K (\partial_x \theta_\alpha)^2 \right] + \mathcal{A}^2 \delta \int dx (\cos \phi_1 + \cos \phi_2) \tag{3}$$

where $\phi_\alpha \cong \phi_\alpha + 2\pi$ and $\theta_\alpha \cong \theta_\alpha + 2\pi$ are canonically conjugate compact boson fields satisfying $[\partial_x \phi_\alpha(x), \theta_\beta(y)] = 2\pi i \delta_{\alpha\beta} \delta(x - y)$, $\mathcal{A}$ is a bosonization prefactor whose precise value is unimportant, and the Luttinger parameter $K = (\pi/2)(\pi - \arccos \Delta)^{-1}$ and velocity $v = [K/(2K - 1)] \sin(\pi/2K)$ are determined from the Bethe ansatz solution of the XXZ chain [21]. For $\Delta \in [-(1/\sqrt{2}), 0]$, we have $K \in (1, 2)$, and thus the vertex operators $\mathcal{U}_{1,2} \equiv \cos \phi_{1,2}$, which have scaling dimensions

$$[\mathcal{U}_1] = [\mathcal{U}_2] = K, \tag{4}$$

are relevant [22] and open a gap for any $\delta \neq 0$, pinning the fields at $\langle \phi_{1,2} \rangle = (\pi/2)[1 + \mathrm{sgn}(\delta)]$. Thus, the bosonized description recovers the $J_\perp = 0$ phase structure discussed above, with a critical point at $\delta = 0$. The decoupled model enjoys an $\mathrm{O}(2) \times \mathrm{O}(2)$ symmetry generated by independent U(1) spin rotations $S_{\alpha j}^{\pm} \mapsto e^{\pm i \chi_\alpha} S_{\alpha j}^{\pm}$ and spin reflections $\{S_{\alpha j}^{\pm} \mapsto S_{\alpha j}^{\mp}, S_{\alpha j}^{z} \mapsto -S_{\alpha j}^{z}\}$ on each leg, and $\mathbb{Z}_2$ leg exchange symmetry $\vec{S}_{1j} \leftrightarrow \vec{S}_{2j}$ which enforces the critical points for both legs to coincide.

We now show analytically and verify numerically that introducing two distinct forms of interlayer coupling that preserve different subsets of these symmetries,

$$H_\perp^M = J_\perp^M \sum_j S_{1j}^z S_{2j}^z, \tag{5}$$

$$H_\perp^U = J_\perp^M \sum_j S_{1j}^z S_{2j}^z + J_\perp^U \sum_j (S_{1j}^x S_{2j}^x + S_{1j}^y S_{2j}^y), \tag{6}$$

leads to the phase diagrams in Fig. 1 that respectively exhibit multiversality and unnecessary criticality.

Multiversality.—$H_\perp^M$ preserves layer exchange and independent spin rotations but only retains simultaneous spin reflections thereby breaking the on site $\mathrm{O}(2) \times \mathrm{O}(2)$ symmetry down to $[\mathrm{U}(1) \times \mathrm{U}(1)] \rtimes \mathbb{Z}_2$. This preserves the $J_\perp^M = 0$ phase structure although it reduces the degeneracy of boundary modes in the nontrivial SPT phase, as the system crosses over from a $\mathrm{O}(2) \times \mathrm{O}(2)$ SPT phase to a $[\mathrm{U}(1) \times \mathrm{U}(1)] \rtimes \mathbb{Z}_2$ SPT phase without any bulk phase transition. To study the effect of $H_\perp^M$ on the $\delta = 0$ critical point, we consider its bosonized form ($\mathcal{B}$ is another bosonization prefactor),

$$H_\perp^M \approx \mathcal{B}^2 J_\perp^M \int dx [\cos(\phi_1 - \phi_2) - \cos(\phi_1 + \phi_2)] + \frac{J_\perp^M}{4\pi^2} \int dx (\partial_x \phi_1 \partial_x \phi_2), \tag{7}$$

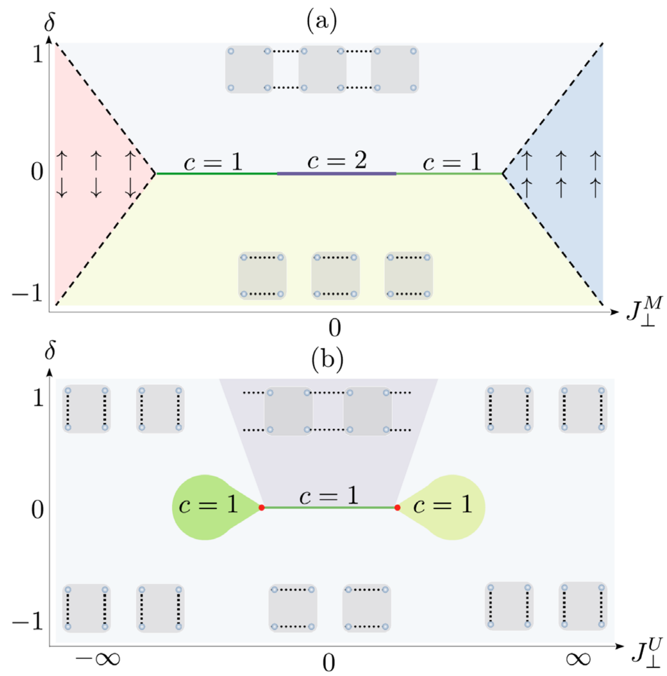

FIG. 1. (a) The Hamiltonian H^M, obtained by perturbing H_δ by $H_\perp^M$, [cf. Eqs. (1) and (5), $\Delta \in [-(1/\sqrt{2}), 0]$], yields a phase diagram exhibiting multiversality. Solid lines along the $\delta = 0$ line denote distinct universality classes separating the same two gapped phases, as reflected by the different central charges c of the respective conformal field theories. Dashed lines denote first-order phase transitions. (b) If instead H_δ is perturbed by $H_\perp^U$ [cf. Eq. (6), with $J_\perp^M < 0$ fixed to a value leading to the $c = 1$ segment in Fig. 1(a)], the phase diagram hosts an unnecessary critical line along $\delta = 0$ with $c = 1$. The critical line terminates into lobes of distinct $c = 1$ critical Luttinger liquid phases through a $c = \frac{3}{2}$ phase transition (red dot). The shaded region hosts stable boundary modes and is separated from the rest of the phase diagram by a boundary transition.

from which we see that it introduces two new vertex operators, $\mathcal{V}_\pm \equiv \cos(\phi_1 \pm \phi_2)$ which involve combinations of the boson fields that are respectively symmetric and antisymmetric under layer exchange. For $J_\perp^M = 0$, both $\mathcal{V}_\pm$ have scaling dimension $2K$, and are hence irrelevant for our choice of $\Delta \in [-(1/\sqrt{2}), 0]$: the critical theory remains a $c = 2$ two-component Luttinger liquid for small $|J_\perp^M|$. As $|J_\perp^M|$ is increased, it changes operator scaling dimensions through its coupling to the exactly marginal operators $\partial_x\phi_1\partial_x\phi_2$: perturbatively in $J_\perp^M$,

$$[\mathcal{V}_\pm] \equiv K_\pm \approx 2K\left(1 \mp \frac{J_\perp^M K}{2\pi v}\right), \qquad (8)$$

suggesting that $\mp\cos(\phi_1 \pm \phi_2)$ become relevant for the critical values $\pm J_\perp^{M*} \approx \pm[2\pi v(K-1)/K^2]$ respectively and gap out either the leg-symmetric or leg-antisymmetric components of the Luttinger liquid. The resulting single-component Luttinger liquid corresponds to a critical theory with $c = 1$. In order to show that we have the multiversal line shown in Fig. 1(a), we must verify that the $\delta \neq 0$ gapped phases remain unchanged away from the critical line as we turn on $J_\perp^M$. To do so, we first observe that although the scaling dimensions of $\mathcal{U}_{1,2}$ are modified to

$$[\mathcal{U}_{1,2}] = \begin{cases} \frac{K_+}{4} & \text{for } J_\perp^M < -J_\perp^{M*} \\ \frac{K_+ + K_-}{4} & \text{for } |J_\perp^M| < |J_\perp^{M*}|, \\ \frac{K_-}{4} & \text{for } J_\perp^M > +J_\perp^{M*} \end{cases} \qquad (9)$$

from Eq. (8), they remain relevant as $J_\perp^M$ is tuned through $\pm J_\perp^{M*}$. Now, H_δ pins the fields to the values $\langle\phi_{1,2}\rangle = (\pi/2)[1 + \mathrm{sgn}(\delta)]$ (the minima of $\mathcal{U}_{1,2}$), while $H_\perp^M$ pins $\langle\phi_1 + \mathrm{sgn}(J_\perp^M)\phi_2\rangle = 2\pi\mathbb{Z}$ for $|J_\perp^M| > |J_\perp^{M*}|$ (the minima of $\mathcal{V}_\pm$). Since the minima of $H_\perp^M$ are compatible with those of H_δ (see Fig. 2), there will be no qualitative change in the nature of the $\delta \neq 0$ gapped phases across $J_\perp^{M*}$.

The nature of the ordered phases at large $|J_\perp^M|$ are easily determined from first-order perturbation theory on H_M, as

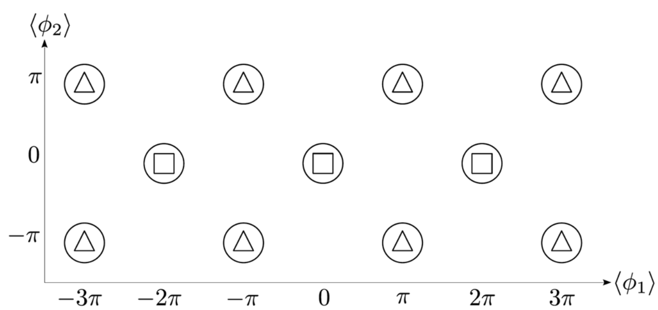

FIG. 2. Values of $\langle\phi_{1,2}\rangle$ pinned by H_δ for $\delta < 0$ (squares), $\delta > 0$ (triangles), and by $H_\perp^M$ (circles). Since the triangles and squares are a subset of the circles, H_δ and $H_\perp^M$ have compatible ground states.

shown in the Supplemental Material [23]. Since the ground states of these ordered phases belong to a different total layer-magnetization sector $S^z_{\mathrm{tot},\alpha} = \sum_j S^z_{\alpha j}$, we expect a first-order transition between them and the original small-$J_\perp^M$ gapped phases.

Combining these results, we obtain the phase diagram in Fig. 1(a) with a multiversal critical line, described by a $c = 2$ or $c = 1$ conformal field theory (CFT) depending on the path taken in $(\delta, J_\perp^M)$ space between the two straddling phases.

This phase diagram can be numerically verified via infinite-size DMRG (IDMRG) [39], most efficiently by restricting attention to the $\delta = 0$ line, as shown in the Supplemental Material [23] and $J_\perp^M > 0$ (the latter since $J_\perp^M \to -J_\perp^M$ is a unitary transformation). We extract $K_\pm$, the scaling dimensions of $\mathcal{V}_\pm$, via the correlation functions of two independent scaling operators $S^+_{1j}S^\pm_{2j} \sim e^{i(\theta_1 \pm \theta_2)}$ [23] and the central charge c through finite-entanglement scaling [40,41]. We also use the string order parameter

$$\mathcal{O}_{\mathrm{SC}} = \lim_{r\to\infty}\left\langle \prod_{l=j}^{j+r} \sigma^z_{1l}\sigma^z_{2l} \right\rangle, \qquad (10)$$

which picks up an expectation value when $\langle\phi_1 \pm \phi_2\rangle = 0$ [23,42] and the central charge changes to $c = 1$. By tracking the evolution of $K_\pm$, $\mathcal{O}_{\mathrm{SC}}$, and c along the $\delta = 0$ line (Fig. 3), we see that the central charge drops from $c = 2$ to $c = 1$ as $J_\perp^M$ is tuned through $\pm J_\perp^{M*}$ while H_δ remains relevant [cf. Eq. (9)].

Unnecessary criticality.—We now consider $H_\perp^U$ in Eq. (6), with a fixed value of $J_\perp^M < -J_\perp^{M*}$ [such that $c = 1$ when $J_\perp^U = 0$, cf. Fig. 1(a)]. This preserves the leg-exchange symmetry of $H_\perp^M$ but breaks the on site symmetry down to the O(2) generated by simultaneous spin rotations and reflections in both legs. $H_\perp^U$ eliminates the distinction between the gapped regions of $H_\perp^M$ [23] for different signs of δ. The easiest way to see this is by observing that both exact ground states in Eq. (2) evolve to the *same* product state as $J_\perp^U \to \infty$, without a bulk phase transition:

$$|GS(\delta = \pm 1, J_\perp^U \to \infty)\rangle = \prod_{j=1}^{L} |[1, j; 2, j]\rangle, \qquad (11)$$

where $|[1, j; 2, j]\rangle$ denotes a singlet along the jth rungs of the ladder [Fig. 1(b)]. A similar result obtains for $J_\perp^U \to -\infty$ but with the singlet replaced by a different entangled Bell pair. This implies that there is a single gapped phase in the periphery of the entire $(\delta, J_\perp^U)$ region. To determine the fate of the system closer to the origin $\delta = J_\perp^U = 0$, we use the bosonized version of $H_\perp^U$,

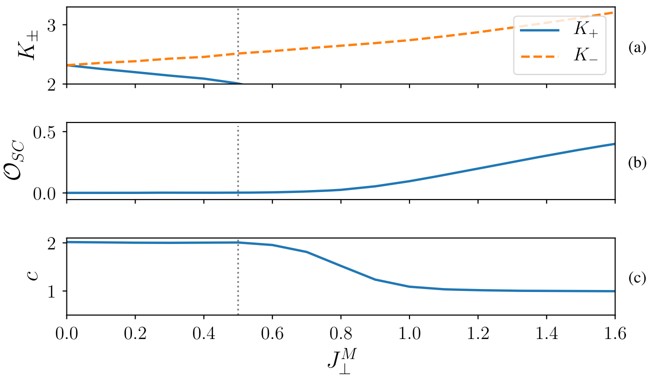

FIG. 3. Multiversality. From top to bottom: (a) Luttinger parameters $K_\pm$, (b) string order parameter $\mathcal{O}_{\rm SC}$, and (c) central charge c computed along the $\delta = 0$ line of H^M for various $J_\perp^M > 0$ with fixed $\Delta = -0.25$ using IDMRG, as shown in the Supplemental Material [23]. The vertical dotted line ($J_\perp^{M*} \approx 0.5$) denotes where $K_+ = 2$ and the critical CFT changes from $c = 2$ to $c = 1$.

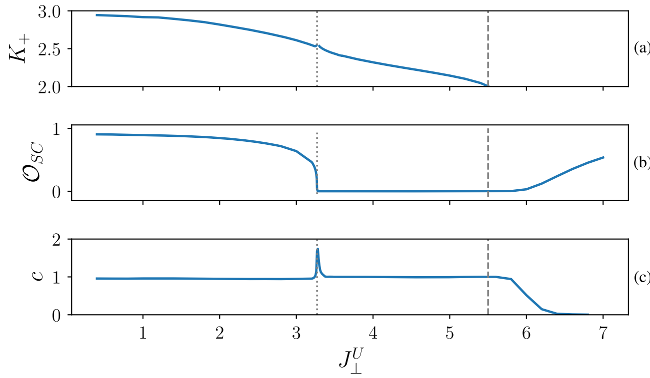

FIG. 4. Unnecessary criticality. From top to bottom: (a) Luttinger parameter K_+, (b) string order parameter $\mathcal{O}_{\rm SC}$, and (c) central charge c along the $\delta = 0$ line of H^U for various $J_\perp^U > 0$ with fixed $\Delta = -0.05$ and $J_\perp^M = -5.2$ using IDMRG, as shown in the Supplemental Material [23]. The dotted line ($J_\perp^U \approx 3.27$) denotes the $c = \frac{3}{2}$ point when the XY_2 critical line transitions to XY_1 lobes. The dashed line ($J_\perp^U \approx 5.5$) denotes the point where $K_+ = 2$ when the system transitions to a trivial gapped phase.

$$H_\perp^U \approx H_\perp^M + J_\perp^U \mathcal{C}^2 \int dx \, \cos(\theta_1 - \theta_2), \tag{12}$$

with $H_\perp^M$ as in Eq. (7) and $\mathcal{C}$ again an unimportant prefactor. For nonzero $J_\perp^M$ and $J_\perp^U$, at least one of $\mathcal{W}_- \equiv \cos(\theta_1 - \theta_2)$ or $\mathcal{V}_-$ is always relevant; if $\mathcal{V}_+$ is irrelevant the system flows to a gapless $c = 1$ theory. However, the nature of this theory depends on which of the two operators dominates at large distances. When $\mathcal{W}_-$ dominates, $\theta_1 - \theta_2$ is pinned, while $\phi_1 - \phi_2$ fluctuates. Instead, when $\mathcal{V}_-$ dominates, $\theta_1 - \theta_2$ fluctuates while $\phi_1 - \phi_2$ is pinned. Following the terminology of Ref. [43], we refer to these Luttinger liquids as XY_1 and XY_2 respectively. We find that there exists a range of values of fixed $\Delta \in (-1, 0)$ and $J_\perp^M < -J_\perp^{M*}$ such that we get a stable extended XY_2 unnecessary critical line extending from the origin along $\delta = 0$. For $\delta \neq 0$ away from this line, $\mathcal{U}_{1,2}$ are relevant and drive the system to a gapped phase. While there are other possible ways for this line to terminate [23], for the chosen parameters, the XY_2 line first transitions to XY_1 on each end and then terminates. In the XY_1 regions, $\mathcal{U}_{1,2}$ decay exponentially and H_δ cannot gap out the system. As a result, the XY_1 line opens up into small islands of gapless *phases*, that persist until $\mathcal{V}_+$ becomes relevant and drives the system to a fully gapped trivial phase. The two XY_1 lobes on each end of the critical line, in fact, correspond to distinct phases, i.e. cannot be connected without going through a phase transition. This is due to the gapless degrees of freedom in the two lobes carrying different symmetry charges as explained in Ref. [23] and the Supplemental Material [23].

A schematic of the phase diagram is shown in Fig. 1(b). We can once again numerically verify all aspects of the phase diagram via DMRG at $\delta = 0$ (Fig. 4). $\mathcal{O}_{\rm SC}$ [the same string operator defined in Eq. (10)] now picks up an expectation value in the XY_2 critical region and in the trivial gapped phase, but not the XY_1 lobes. Since $J_\perp^U \mapsto -J_\perp^U$ is a unitary transformation, we restrict our attention to $J_\perp^U > 0$. We see that indeed a stable XY_2 line with $c = 1$ persists until it transitions to XY_1. In the numerics this is marked by a jump in the central charge: indeed the transition from XY_1 to XY_2 is known to happen through a $c = \frac{3}{2}$ critical point corresponding to the gapped sector undergoing an Ising transition [45]. Finally, at larger values of $J_\perp^U$, K_+ dips below 2, and the system gaps out.

We have thus embedded a $c = 1$ CFT as an unnecessary critical line not demanded by the phase structure (as there is a unique gapped phase in the phase diagram) that can be accessed by tuning a single parameter.

Boundary transitions meet the bulk.—A curious feature of the phase diagram of $H_\perp^U$ is the presence of stable boundary modes above the unnecessary critical line. This can be seen in the limiting case $\delta = 1$, where the effective boundary Hamiltonian on each end acts on two spins and has the form (suppressing site labels for brevity)

$$H_\partial = J_\perp^M S_1^z S_2^z + J_\perp^U (S_1^x S_2^x + S_1^y S_2^y). \tag{13}$$

This has a twofold degenerate ground state for $|J_\perp^U| < -J_\perp^M$ so that H^U has boundary modes, and a unique ground state for $|J_\perp^U| > -J_\perp^M$ so that H^U has no boundary modes. We can numerically verify [23] that the boundary modes are stable even as we reduce δ [shaded region in Fig. 1(b)]. Remarkably, the boundary transition (at $|J_\perp^U| = -J_\perp^M$ for $\delta = 1$) terminates at the $c = \frac{3}{2}$ point, the same as the unnecessary critical line. If the bulk and boundary transitions are treated on equal footing, the unnecessary critical line becomes part of a phase boundary separating "boundary-obstructed" topological phases [46], leading to a more conventional-looking phase diagram.

We conjecture that unnecessary critical lines in the bulk generically terminate by turning into boundary critical lines and enclose regions with stable boundary modes. While this is true in all known one-dimensional examples [4,7], it would be interesting to verify in higher-dimensional examples too [6,47], as it suggests an intriguing universal connection between unnecessary criticality, boundary criticality, and stable gapless modes.

Stability of phase diagrams.—The field theories shown in Eqs. (3), (7), and (12) already contain the most relevant symmetry-allowed scaling operators. As a result, the phase diagrams shown in Fig. 1 are stable to arbitrary (but small) symmetry-allowed perturbations and small variations of existing parameters. These can only introduce corrections to parameters of the field theory which in turn only quantitatively change (Fig. 1). In particular, both the critical lines hosting unnecessary criticality and multiversality can be reached by tuning a single parameter with no additional fine-tuning.

Discussion.—We conclude by sketching conditions to generate models with multiversality and unnecessary criticality. It is illuminating to anchor the discussion to the region on the critical surface where the universality class is about to change or the surface is about terminate. Broadly, we need two ingredients: (i) a single parameter δ that couples to all relevant operators that lead to the gapped phase(s) and (ii) a marginal operator $\mathcal{O}_M$ whose energy can be minimized simultaneously with that of the operators coupled to δ. The change along the critical surface occurs when $\mathcal{O}_M$ changes from marginally irrelevant to marginally relevant. For the examples in this Letter, δ couples to $\mathcal{U}_{1,2}$ and $\mathcal{O}_M \propto \mathcal{V}_\pm$ in the bosonized language. These conditions are neither necessary nor sufficient, but are useful guides. Two additional ingredients also serve to simplify our analysis. The first is the existence of an exactly marginal operator $\partial_x\phi_1\partial_x\phi_2$ that can tune the scaling dimension of $\mathcal{O}_M$ along the critical surface. The second is the "failed SPT" premise [4,6,7,47,48] which provides a template to construct phase diagrams using results from the classification of SPT phases [23,49]. Using similar ingredients, it is likely that one can engineer examples of both phenomena in higher dimensions.

Finally, we flag some possible extensions of this Letter. First, note that our two-leg models can be straightforwardly generalized to $2N$ legs where the possible critical phenomena are richer, as shown in the Supplemental Material [23]. Second, note that we restricted our focus to multiversality on the critical surface of a phase transition separating a trivial from a nontrivial SPT phase, which lies outside the Landau paradigm of symmetry-breaking orders. It would be equally interesting to find examples where the transition is *not* Landau forbidden, but one of the multiversality classes is [23,50]. Third, we conjectured that unnecessary criticality, boundary criticality, and stable boundary modes are intimately connected. It would be useful to make this more concrete, e.g., via a field-theoretic formulation, particularly in higher dimensions. A fourth open question is whether phase diagrams analogous to those studied in this Letter can be obtained in models with quenched randomness. Finally, it would be particularly exciting to find experimental examples of either of these phenomena. Given the simplicity of the models presented here, we are optimistic that this is a question that can be answered positively in the not-too-distant future.

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Note added.—In the final stages of the preparation of this manuscript, we became aware of an upcoming independent work [52] which also studies unnecessary criticality in spin chains. We thank the authors for alerting us about their results.

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