Free field realisation of boundary vertex algebras for Abelian gauge theories in three dimensions

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Abstract

We study the boundary vertex algebras of $\boldsymbol{A}$-twisted $\boldsymbol{\mathcal{N} = 4}$ Abelian gauge theories in three dimensions. These are identified with the BRST quotient (semi-infinite cohomology) of collections of symplectic bosons and free fermions that reflect the matter content of the corresponding gauge theory. We develop various free field realisations for these vertex algebras which we propose to interpret in terms of their localisation on their associated varieties. We derive the free field realisations by bosonising the elementary symplectic bosons and free fermions and then calculating the relevant semi-infinite cohomology, which can be done systematically. An interesting feature of our construction is that for certain preferred free field realisations, the outer automorphism symmetry of the vertex algebras in question (which are identified with the symmetries of the Coulomb branch in the infrared) are made manifest.

Keywords: Supersymmetric Gauge Theories, Vertex Operator Algebras, Free-Field Realisations, Hypertoric Varieties

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1 Introduction

Topologically twisted $\mathcal{N} = 4$ gauge theories in three dimensions admit holomorphic boundary conditions that support boundary vertex operator algebras (VOAs). Constructions of these VOAs were announced in [**?** **?**] and subsequently elaborated upon in [**?** **?**] (see also [**?** **?** **?** **?**] for related developments related to these vertex algebras, and in particular the recent work [**?**] which studies some of the same examples that are discussed here).

Three-dimensional $\mathcal{N} = 4$ theories admit two topological twists, the A- (also H-) and the B- (also C-)twists. In the A-twist, standard (perturbative) boundary conditions that are compatible with the VOA construction are deformations of $(0, 4)$ Neumann boundary conditions. Alternatively in the B-twist there are standard perturbative boundary conditions given by deformations of Dirichlet boundary conditions. The two choices of twist/holomorphic boundary condition lead to very different descriptions for the (distinct) associated VOAs: in the A-twist they are BRST reductions of symplectic bosons and boundary fermions, while in the B-twist they are presented as (essentially) WZW models based on Lie superalgebras. We will refer to the former as "A-twisted VOAs" and to the latter (though we rarely mention them) as "B-twisted VOAs". (For brevity, we sometimes use VOA also in the case of vertex operator superalgebras.)

The main aim of the present paper is to elucidate the A-twisted vertex algebras for a large class of Abelian gauge theories by deriving a variety of free field realisations, and to investigate remarkable geometric interpretations.

The Abelian theories in question can be labelled by a rank-k torus T_G and a representation $V = T^*\mathbb{C}^N$ thereof. The aforementioned BRST reductions that are identified with the boundary VOAs correspond to relative semi-infinite cohomologies of the form

$$\mathcal{V}^A \cong H^{\frac{\infty}{2}+\bullet}(\mathfrak{g}, \mathfrak{g}_0, \mathcal{X}) \ , \tag{1.1}$$

where

$$\mathcal{X} := \bigotimes_{i=1}^{N} (\mathtt{Sb}_i \otimes \mathtt{Ff}_i) \tag{1.2}$$

is a collection of symplectic bosons and free fermions, and $\mathfrak{g}$ is a rank-k level-zero Heisenberg current modelled on a complex moment map $\mu_{\mathbb{C}}$ for the T_G action on V, see (3.7).

We *derive from first principles* the free-field realisations for these vertex algebras by exploiting the relative simplicity of semi-infinite cohomology for Abelian gauge groups. The trick, so to speak, is to bosonise the elementary symplectic bosons and free fermions and embed the entire calculation into the context of a lattice vertex superalgebra. An appropriate change of basis then allows the action of the gauge symmetry to be sequestered from the remaining degrees of freedom, and the corresponding semi-infinite cohomology computation trivialises. The final result is given in terms of the kernel of screening charges acting on some free-field space V_{FFR}, , as we summarise in Proposition 1. Various inequivalent choices of bosonisation of the set of

symplectic bosons exist, and these lead to different free-field realisations that manifest different features of these vertex algebras. For instance, only for certain free-field realisations the outer automorphism symmetry for the vertex algebras (corresponding to the Coulomb branch flavour symmetry of the respective $\mathcal{N} = 4$ theory) lifts to a symmetry of the free field vertex algebra and manifestly acts on the relevant subalgebra. This phenomenon is summarised in Proposition 2.

Although it is in general difficult to identify the kernel of the screening charges, we provide a conjectural collection of (weak) generators for these vertex algebras. We also demonstrate that the seet of vertex algebras that can be obtained in this way contains notable examples, such as $V_1(\mathfrak{psl}(N|N))$ for $N > 1$, for which our technique produce various remarkable free-field realisations.

Geometric interpretation

Our initial approach was inspired by the constructions of free-field realisations for VOAs associated to four-dimensional $\mathcal{N} = 2$ SCFTs in, *e.g.*, [**?** **?** **?**]. Those free field realisations have a distinctly geometric flavour—they exploit the geometry of the Higgs branch of these theories, namely the existence of certain (Zariski) open subsets that can be identified (as holomorphic symplectic varieties) with simple cotangent bundles of the form $(T^*\mathbb{C}^\times)^m \times (T^*\mathbb{C})^n$ for some non-negative integers m and n. A motivation for those constructions is the (speculative) expectation that the vertex algebras $\mathcal{V}$ associated to four-dimensional SCFTs should be realisable as the algebras of global sections of some kind of sheaf of vertex algebras on the *associated variety* $X_\mathcal{V}$. (See, *e.g.*, [**?**] for rigorous work in a similar direction but in a somewhat distinct setting.)

We recall (briefly) the notion of the associated variety of a VOA $\mathcal{V}$ [**?**]. Let $C_2(\mathcal{V})$ denote the vector subspace of $\mathcal{V}$ spanned by all states that can be expressed as a normally ordered product of operators at least one of which has been differentiated. This is an ideal with respect to the normally ordered product, and the quotient,

$$R_\mathcal{V} \coloneqq \mathcal{V}/C_2(\mathcal{V}) \ , \tag{1.3}$$

inherits the structure of a (super)commutative $\mathbb{C}$-algebra dubbed *Zhu's C_2 algebra* [**?**]. (In fact, it further inherits from the simple pole in the OPE for $\mathcal{V}$ a Poisson structure). Discarding nilpotent elements of this algebra, one defines the associated variety,

$$X_\mathcal{V} \coloneqq \text{Spec } (R_\mathcal{V})_{\text{red}} \ , \tag{1.4}$$

which is in general a complex, affine Poisson variety.

In the context of VOAs arising in four dimensions, the *Higgs branch conjecture* of [**?**] postulates the identification of the associated variety with the Higgs branch of vacua of the corresponding SCFT. This in particular would imply that the vertex algebras arising from four-dimensional SCFTs always have associated varieties that are not only Poisson varieties, but symplectic in the sense of having only a finite number of symplectic leaves—such vertex algebras have been designated *quasi-lisse* [**?**].

Though it is surely expected by many experts (the expectation is briefly articulated in [**?**]), to our knowledge the generalisation of this conjecture to three dimensions has not been well advertised. Indeed, in three dimensions there is potentially more variation possible as there is in principle a choice of (deformable) boundary condition involved in the specification of a boundary vertex algebra. Conservatively, we expect that at least in the general class of theories under consideration here an analogous result will hold:[1]

Conjecture 1 (A-twisted Higgs Branch Conjecture)**.** *For the boundary vertex algebras of A-twisted gauge theories with purely free fermion boundary degrees of freedom, the associated variety and the Higgs branch $\mathcal{M}_H$ of the corresponding gauge theory are isomorphic as Poisson varieties.*

For our class of theories, the Higgs branches are hypertoric varieties. These are simple examples of hyperkähler quotients and so, in particular, holomorphic symplectic quotients

$$\mathcal{M}_H \cong \mu_{\mathbb{C}}^{-1}(0) /\!\!/ T_{G,\mathbb{C}} \ , \tag{1.5}$$

which are a indeed a finite-dimensional analogue of the BRST reduction defining the VOAs. If our conjecture is correct, then these constructions in three dimensions will provide another rich source of quasi-lisse vertex algebras to complement those originating in four dimensions. For instance, it would follow from the combination of this conjecture and our study of the vertex algebras that the associated variety of $V_1(\mathfrak{psl}(N|N))$ is an A-type minimal nilpotent orbit, and therefore $V_1(\mathfrak{psl}(N|N))$ is quasi-lisse (we verify this statement for $V_1(\mathfrak{psl}(2|2))$).

Given this expectation, *a posteriori* we can understand our bosonisation approach to constructing these vertex algebras as a chiral version of similar methods for studying Higgs branches/hypertoric varieties, whose geometry has been studied extensively and is well understood (*cf.* [**? ? ?**]). Bosonisation of a symplectic boson vertex algebra amounts to the embedding

$$\mathcal{D}^{ch}(\mathbb{C}) \hookrightarrow \mathcal{D}^{(ch)}(\mathbb{C}^\times) \ , \tag{1.6}$$

where $\mathcal{D}^{(ch)}(X)$ denotes global sections of the sheaf of chiral differential operators on X [**?**], and $\mathcal{D}^{ch}(\mathbb{C})$ can be thought of as the chiralisation of the algebra of functions on $T^*\mathbb{C}$. In this identification the base $\mathbb{C}$ is the space parameterised by one of two scalars in the hypermultiplet. In terms of Higgs branches, the analogous replacement is therefore

$$\mathbb{C}[V] \hookrightarrow \mathbb{C}[V^*] \ , \tag{1.7}$$

where $V \cong T^*\mathbb{C}^N$ is the representation determining the theory, and $V^* \cong (T^*\mathbb{C}^\times)^N$. Performing symplectic reduction on V^* in general produces an open subset of the very simple form,

$$(T^*\mathbb{C}^\times)^{\dim_{\mathbb{H}} \mathcal{M}_H} \hookrightarrow \mathcal{M}_{H,\zeta} \ , \tag{1.8}$$

where $\mathcal{M}_{H,\zeta}$ is in general a *resolution of* the Higgs branch $\mathcal{M}_{H,\zeta}$.

[1] The analogous statement for B-twisted vertex algebras and the Coulomb branch also seems very plausible.

The multiple bosonisations of the free fields which lead to inequivalent free field realisations of the same A-twisted vertex algebras are in correspondence with a multiplicity of inequivalent ways of identifying V as a cotangent bundle of $\mathbb{C}^N$. An interesting phenomenon—which we can already observe in our simplest example—is that choices that manifest the outer automorphism symmetry of the A-twisted vertex algebra correspond to choices of embeddings that do not require to fully resolve the Higgs branch (and in the extreme case are embeddings in the singular (unresolved) Higgs branch), see for instance Lemma 4. This aligns nicely with the physical fact that general Higgs branch resolutions may the Coulomb branch symmetry. The geometric perspective also turns out to be useful to consider "de-bosonisation techniques".

The structure of the remainder of the paper is as follows. In Section 2 we briefly introduce Abelian $\mathcal{N} = 4$ gauge theories in three-dimensions, restricting to theories with unimodular charge matrices. In Section 3 we use this data to introduce the A-twisted boundary VOAs in terms of a semi-infinite cohomology problem applied to a set of symplectic bosons and free fermions. We develop methods to compute this semi-infinite cohomology and to systematically derive various free-field realisations for the VOAs, some of which manifest outer automorphism groups. In Section 4 we apply these methods to the theory $T[\mathrm{SU}(2)]$ and derive two inequivalent free-field realisations for its boundary VOA, which we identify with $V_1(\mathfrak{psl}(2|2))$. Both are given in terms of a half-lattice vertex algebra and two pairs of free fermions, but only one manifest the $\mathrm{SL}(2)_o$ group of outer automorphisms of $V_1(\mathfrak{psl}(2|2))$. We also identify the associated variety with A_1, corroborating Conjecture 1 in this simple example. In Section 5 we develop a finite-dimensional analogue of our free-field realisations, which we interpret in terms of localisations on openly embedded subsets in the respective Higgs branch/associated variety. In particular, we comment on the relations between outer automorphisms of the vertex algebras and resolutions of the associated variety, and on the relations to these localisations to recent work on Coulomb branches. In Section 6 we study these localisations in the examples of the A-type Kleinian singularities and A-type minimal nilpotent orbits, and we interpret *a posteriori* the free-field realisations obtained for $V_1(\mathfrak{psl}(2|2))$ in terms of localisations on the A_1 singularity. In Section 7 we illustrate our general methods to produce free-field realisations for two sets of VOAs: the elements of the first set, which we conjecture are isomorphic to $V_1(\mathfrak{psl}(N|N))$ for $N > 2$, are expected by Conjecture 1 to have A-type minimal nilpotent orbits as associated varieties, whereas elements of the second are expected to have A-type Kleinian singularities. We find that the free-field realisations we derive have neat interpretations in terms of the geometry of these hypertoric varieties.

Remark.

A closely related set of vertex algebras have been considered previously by Kuwabara in [**?**]. In that work, starting with the same set of symplectic bosons, the author includes auxiliary, positive-level Heisenberg current algebras rather than (physically motivated) free fermions to cancel the anomaly in the BRST operator of the naïve gauge theory degrees of freedom. Amongst other things, this has the affect of changing the resulting vertex algebras so they are not quasi-lisse. Furthermore, in that work

the author calculates the relevant semi-infinite cohomology by employing a sophisticated localisation construction involving sheaves of $\hbar$-adic vertex algebras tailored to a particular choice of resolution of the associated hypertoric variety. We expect that a similar approach in the setting involving free fermions would reproduce a subset of our free field realisations, though perhaps not the preferred realisations that manifest the Coulomb branch outer automorphisms.

2 Abelian gauge theories

In this section we introduce 3d $\mathcal{N} = 4$ Abelian gauge theories, to the extent that will be required to discuss the VOAs that emerge at the boundary of their topological A-twist. For a more thorough exposition of some of the aspects discussed here, see for instance [**?**].

An Abelian gauge theory is defined by the data (T_G, V), where

- T_G is the Abelian gauge group, that is a rank-k torus

$$T_G = \prod_{i=1}^{k} \mathrm{U}(1) \ , \tag{2.1}$$

- V is a representation of T_G of the form

$$V = T^*\mathbb{C}^N \ , \tag{2.2}$$

  where $\mathbb{C}^N$ is itself a complex representation.

Physically, V is the representation where the hypermultiplet (matter) fields are valued. We denote by $T_V \cong \mathrm{U}(1)^N$ the maximal torus of the symplectic automorphism group of V. The theory has then a flavour torus that is defined as the coset $T_H = T_V/T_G \cong \mathrm{U}(1)^{N-k}$. More generally, one can define the flavour symmetry group G_H to be the normaliser of T_G in the group of symplectic automorphisms of V, quotiented by T_G.

From now on, we fix bases for the weight lattices $\mathfrak{t}^*_{G,\mathbb{Z}}$, $\mathfrak{t}^*_{V,\mathbb{Z}}$, $\mathfrak{t}^*_{H,\mathbb{Z}}$ in the (dual) Lie algebras $\mathfrak{t}^*_{G,\mathbb{R}}$, $\mathfrak{t}^*_{V,\mathbb{R}}$, $\mathfrak{t}^*_{H,\mathbb{R}}$ of T_G, T_V, T_H. We take T_V to act with weights $+1$ on the base directions of V and with the -1 on the cotangent fibres. The choice of T_G representation can be encoded into a $k \times N$ weight matrix Q (the weights of the actions on the base directions $\mathbb{C}^N$, expressed in the chosen basis), which we call the charge matrix. Representatives for the coset T_H can be taken to act with weights q on the base directions of V, where q is a $(N-k) \times N$ matrix (and correspondingly $-q$ on the fibre directions).

Collectively, the data described above can be encoded in a square $N \times N$ matrix as follows,

$$\mathbf{Q} \coloneqq \begin{pmatrix} Q \\ q \end{pmatrix} \ . \tag{2.3}$$

We will restrict to cases where there is a short exact sequence

$$0 \to \mathfrak{t}_{G,\mathbb{Z}} \cong \mathbb{Z}^k \xrightarrow{Q^T} \mathfrak{t}_{V,\mathbb{Z}} \cong \mathbb{Z}^N \xrightarrow{\widetilde{Q}} \mathfrak{t}_{H,\mathbb{Z}} \cong \mathbb{Z}^{N-k} \to 0 \ , \tag{2.4}$$

where $\mathfrak{t}_{G,\mathbb{Z}}$, $\mathfrak{t}_{V,\mathbb{Z}}$, $\mathfrak{t}_{H,\mathbb{Z}}$ are co-weight lattices of the tori T_G, T_V, T_H. In particular, this assumption requires that the torus T_G has rank $k \leqslant N$, and that the map $\widetilde{Q}$ is surjective. We additionally require $\widetilde{Q}$ to be unimodular, meaning that any minor consisting of $(N-k)$ columns of $\widetilde{Q}$ has determinant ± 1. This implies in particular that $\widetilde{Q}$ can be completed to a matrix $\widetilde{\mathbf{Q}}$ of determinant ± 1. The matrix $\widetilde{\mathbf{Q}}$ is then invertible over the integers, so we can define

$$\widetilde{\mathbf{Q}} = \begin{pmatrix} \tilde{q} \\ \widetilde{Q} \end{pmatrix} \ , \mathbf{Q} = \begin{pmatrix} Q \\ q \end{pmatrix} = (\widetilde{\mathbf{Q}}^{-1})^T \ . \tag{2.5}$$

This provides splitting and projection matrices $q : \mathfrak{t}_{H,\mathbb{Z}} \to \mathfrak{t}_{V,\mathbb{Z}}$ and $\widetilde{q} : \mathfrak{t}_{V,\mathbb{Z}} \to \mathfrak{t}_{G,\mathbb{Z}}$ for the above short exact sequence.

Importantly, the matrices $\widetilde{Q}$, $\widetilde{q}$ also have a clear physical interpretation. In fact, they represent compatible choices of gauge and flavour charge matrices of another 3d $\mathcal{N} = 4$ Abelian gauge theory $(\widetilde{T}_G, \widetilde{V})$ that is *mirror* (in the sense of 3d mirror symmetry) to the one that we started from.

Remark.
Unimodularity is a technical condition that simplifies our analysis in some places, but for many arguments is not required. We expect the approach of this paper to apply, possibly with minor modifications, in more general situations where $k \leqslant N$.

Besides the flavour symmetry T_H (with charge matrix given by q), the theories under consideration enjoy a UV topological symmetry T_C whose charge operators measure the topological type of gauge bundles on a sphere surrounding an operator,

$$T_C \cong \pi_1(T_G)^\vee \cong Z(T_G^\vee) \cong \mathrm{U}(1)^k \ . \tag{2.6}$$

This symmetry can experience a non-Abelian enhancement to a larger Coulomb branch symmetry G_C at the infrared (IR) fixed point. We will therefore occasionally refer to the torus $T_C = U(1)^k$ defined above as the UV topological symmetry.

Under mirror symmetry, the flavour symmetry G_H of a theory defined by $\mathbf{Q}$ is identified with the enhanced Coulomb branch symmetry $\widetilde{G}_C$ of the mirror theory defined by $\widetilde{\mathbf{Q}}$.

2.1 Moduli spaces of vacua

From the data (T_G, V) of a 3d $\mathcal{N} = 4$ Abelian gauge theory one can construct two hypertoric varieties, called the Higgs and Coulomb branch respectively. Physically,

these are branches of the moduli space of vacua of the Abelian gauge theory defined by (T_G, V).

One of the central tenets of this paper is that we expect the VOAs associated to (T_G, V) (which will be defined in Section 3) to have the respective Higgs branch as an associated variety, *cf.* Conjecture 1. We therefore review in some details the construction of the Higgs branches, and briefly discuss how these are related to Coulomb branches by mirror symmetry.

To construct the Higgs branch, notice that T^*V is hyperkähler. The T_G action on V is tri-Hamiltonian, with a triplet of moment maps organised here according to a choice of complex structure,

$$\mu_{\mathbb{R}} : V \to \mathfrak{t}^*_{G,\mathbb{R}} \ , \qquad \mu_{\mathbb{C}} : V \to \mathfrak{t}^*_{G,\mathbb{C}} \ . \tag{2.7}$$

The Higgs branch of vacua is given by the hyperkähler quotient of V by this T_G action,

$$\mathcal{M}_H = V /\!\!/\!\!/ T_G \ , \tag{2.8}$$

which is equivalently the T_G quotient of the zero-locus of the moment maps,

$$\mathcal{M}_H \cong \mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(0)/T_G \ . \tag{2.9}$$

We adopt a conventional physics notation, where we write the basic coordinate functions on $\mathbb{C}^N \subset T^*\mathbb{C}^N$ as X_i, $i = 1, \dots, N$, and the coordinate functions on the cotangent directions as Y^i. Choosing a basis for $\mathfrak{t}^*_G$, we can then write the above moment maps in components,

$$\mu^a_{\mathbb{C}} = \sum_{i=1}^N Q_i^a Y^i X_i \, , \qquad\qquad a \in \{1, \dots, k\} \ , \tag{2.10}$$

$$\mu^a_{\mathbb{R}} = \sum_{i=1}^N \frac{Q_i^a}{2} \left(|X_i|^2 - |Y^i|^2 \right) \ , \quad a \in \{1, \dots, k\} \ . \tag{2.11}$$

As an algebraic symplectic variety, the Higgs branch is given by the holomorphic symplectic quotient,

$$\mathcal{M}_H \cong \mu_{\mathbb{C}}^{-1}(0) /\!\!/ T_{G,\mathbb{C}} \ , \tag{2.12}$$

in terms of the complexified the gauge group $T_{G,\mathbb{C}} = (\mathbb{C}^\times)^k$. The Higgs branch chiral ring/coordinate ring of $\mathcal{M}_H$ is given by

$$R_H = \mathbb{C}\left[\mu_{\mathbb{C}}^{-1}(0)\right]^{G_{\mathbb{C}}} \ . \tag{2.13}$$

By definition, this is a hypertoric variety of quaternionic dimension $(N-k)$ which carries a Hamiltonian action of the (complexified) torus $T_{H,\mathbb{C}} \cong (\mathbb{C}^\times)^{N-k}$.

It will be important to define resolutions of the hypertoric variety $\mathcal{M}_H$. Physically, these are provided by background vector multiplets for T_C (as defined in the UV).

More precisely, the bottom component of a vector multiplet is a real FI parameter $\zeta \in \mathfrak{t}^*_{G,\mathbb{R}}$, and when set to nonzero values one has for the Higgs branch the resolved space

$$\mathcal{M}_{H,\zeta} \cong \mu_{\mathbb{C}}^{-1}(0) /\!\!/_{\zeta} T_{G,\mathbb{C}} \ , \tag{2.14}$$

which is the projective GIT quotient of the zero locus of the complex moment map. This is provides a resolution of singularities

$$\mathcal{M}_{H,\zeta} \twoheadrightarrow \mathcal{M}_H \ . \tag{2.15}$$

The structure of this resolution and the dependence on the FI parameters admits an elegant combinatorial formulation, which will extensively be used in Section 5. With an abuse of notation, let ζ be the image in $\mathfrak{t}^*_{V,\mathbb{R}}$ of the FI parameter under $\widetilde{q}^T$, and define the following hyperplanes in $\mathfrak{t}^*_{H,\mathbb{R}}$,

$$H_i \coloneqq \{x \in \mathfrak{t}^*_{H,\mathbb{R}} \mid x \cdot \widetilde{Q}^i + \zeta^i = 0\} \ . \tag{2.16}$$

The collection of hyperplanes $\mathcal{A}_\zeta = \{H_1, \ldots, H_N\}$ is known as an *arrangement*. An arrangement is called *simple* if every subset of m hyperplanes with nonempty intersection intersects in codimension m, and is called *unimodular* if every collection of $(N-k)$ column vectors $\{\widetilde{Q}^{i_1}, \ldots \widetilde{Q}^{i_{N-k}}\}$ spans $\mathfrak{t}^*_{H,\mathbb{Z}}$ over the integers.

A hyperplane arrangement completely determines the corresponding (resolved) hypertoric variety $\mathcal{M}_{H,\zeta}$. This variety is sometimes written as $\mathcal{M}(\mathcal{A}_\zeta)$ to emphasise this feature. The following are standard results relating properties of the arrangements to the geometry of the hypertoric varieties [**? ?**]:

- The variety $\mathcal{M}(\mathcal{A}_\zeta)$, is a holomorphic symplectic variety with a Hamiltonian T_H action. It is independent of the signs of the row vectors $\widetilde{Q}_i$. (As a consequence, it is also independent of the signs of the column vectors of Q, which are determined by $\widetilde{Q}_i$).
- The resolution $\mathcal{M}(\mathcal{A}_\zeta)$ has at worst orbifold singularities if and only if $\mathcal{A}$ is simple, and is smooth if and only if $\mathcal{A}$ is simple and unimodular.

For the theories we are considering, one can always find choices of resolution parameters so that the resulting arrangements are simple and unimodular. In other words, the Higgs branches in question always admit (physical) smooth resolutions.

2.1.1 Chiral ring generators

It will prove useful in our analysis of the VOAs, and especially in examples, to have a sense for the algebraic generators of the Higgs branch chiral ring R_H introduced in (2.13). We now summarise the characterisation of these chiral ring generators, as explained in detail in, *e.g.*, [**? ?**].

The "mesons" $X_i Y^i$ for $i = 1, \ldots, N$ are certainly in R_H (as they are invariant under the action of T_G), but k linear combinations of these are set to zero by the moment

map condition $\mu_{\mathbb{C}} = 0$. As a basis for linearly independent, non-zero meson operators one can use precisely the $(N-k)$ complex moment maps for the $T_{H,\mathbb{C}}$ action,

$$\mu^a_{H,\mathbb{C}} \coloneqq \sum_{i=1}^{N-k} q^a_i X_i Y^i \ . \tag{2.17}$$

These are all weight-zero with respect to the $T_{H,\mathbb{C}}$ action on $\mathcal{M}_H$.

For the remaining generators one takes gauge-invariant monomials of the elementary hypermultiplet scalars, which will have non-zero $T_{H,\mathbb{C}}$ weights. For any $\mathbb{Z}^N$-valued "weight vector" w, one defines

$$w_{\pm,i} \coloneqq \begin{cases} |w_i| & \text{if} \quad \pm w_i \geqslant 0 \ , \\ 0 & \text{otherwise} \ , \end{cases} \tag{2.18}$$

as well as

$$W^w \coloneqq \prod_{i=1}^{N} (X_i)^{w_{+,i}} (Y^i)^{w_{-,i}} \ . \tag{2.19}$$

Then operators of the form

$$W^A \coloneqq W^{\widetilde{Q}^T \cdot A} \ , \tag{2.20}$$

are gauge-invariant for any vector $A \in \mathbb{Z}^{N-k}$ and descend non-trivially to $R_H \cong \mathbb{C}[\mu_{\mathbb{C}}^{-1}(0)]^{T_{G,\mathbb{C}}}$. One can choose a finite set of these monomials such that together with (2.17) they algebraically generate the Higgs branch chiral ring. (The precise requirement is that the vectors A for the set of monomials must generate the integral lattice inside each cone in so-called "canonical slice" in $\mathfrak{t}^*_{H,\mathbb{R}}$ [**?**].)

2.1.2 Coulomb branch and mirror symmetry

Though we will primarily focus on the A-twist, which foregrounds the Higgs branch from the perspective of the boundary VOA, it will be helpful later to have at hand a concise description of the generators of the Coulomb branch of vacua, $\mathcal{M}_C$. In fact, the Higgs branch of a theory (T_G, V) is isomorphic to the Coulomb branch of the mirror theory $(\widetilde{T}_G, \widetilde{V})$, and it will occasionally be enlightening to study the Higgs branch through the lenses of this isomorphism.

The Coulomb branch chiral ring of a theory (T_G, V) as described above is generated by k complex-valued scalars $\varphi_a = \phi_{1,a} + i\phi_{2,a}$ that arise from the 3d $\mathcal{N} = 4$ vector-multiplet, and by monopole operator $v_{\pm A}$. The latter are labelled by co-characters of the gauge group T_G,

$$A : U(1) \to T_G \cong U(1)^k \ , \tag{2.21}$$

which can be identified with an element $A \in \mathbb{Z}^k$.

From the mirror perspective (that is, under application of the mirror isomorphism between Higgs and Coulomb branch chiral rings), the combinations (5.8) correspond

to the $(N-k)$ complexified vector multiplet scalars $\widetilde{\varphi}_a$ of the mirror theory,

$$\mu^a_{H,\mathbb{C}} \leftrightarrow \widetilde{\varphi}_a \ , \tag{2.22}$$

whereas the monopole operators $\widetilde{v}_A$ of the mirror theory are dual to the operators W^A

$$W^A \leftrightarrow \widetilde{v}_A \ . \tag{2.23}$$

This piece of information is sufficient to determine the Coulomb branch chiral ring. Although we have discuss the mirror isomoprhism at the level of coordinate rings, the two varieties agree as holomorphic symplectic varieties (and in fact as hyperkähler varieities).

Notice that under this mirror isomorphism, the flavour torus T_H is mapped to the (UV) topological symmetry $\widetilde{T}_C$ of the mirror theory.

3 *A*-twisted boundary VOAs

We are now in a position to introduce the boundary VOAs. The boundary VOA of an A-twisted Abelian gauge theory of the type reviewed above was identified in [**?**] in terms of a BRST reduction that can be viewed as a chiral analogue of the Higgs branch definition given in Section 2.1.

The input of the VOA definition is the theory data (T_G, V), and in particular the integer $N \times N$ matrix $\mathbf{Q}$ of (2.3). Given this input, we introduce N sets of symplectic bosons $X_a(z), Y^b(w)$ and N sets of complex fermions ξ^α, χ_β [**?**] with standard OPEs,

$$Y^a(z)X_b(w) \sim \frac{\delta^a_b}{z-w} \ , \qquad \chi_\alpha(z)\xi^\beta(w) \sim \frac{\delta^\beta_\alpha}{z-w} \ . \tag{3.1}$$

Here with an abuse of notation we denote the symplectic bosons in the same way as we denoted coordinates on the representation V entering the Higgs branch definition 2.12.

For future convenience, we denote the VOA generated by the a'th symplectic boson pair by $\mathtt{Sb}_a$, and the one generated by α'th complex fermions by $\mathtt{Ff}_\alpha$. We denote the total pre-BRST (henceforth, ungauged) VOA as,

$$\mathcal{X} \coloneqq \bigotimes_{i=1}^{N} (\mathtt{Sb}_i \otimes \mathtt{Ff}_i) \ . \tag{3.2}$$

The VOA $\mathcal{X}$ has a rank-k, level-zero Heisenberg vertex subalgebra $\mathfrak{g} = (\widetilde{\mathfrak{gl}_1})^k$ generated by the currents

$$\mathcal{J}^a = \sum_{i=1}^{N} Q^a_i (X_i Y^i + \chi_i \xi^i) \ , \qquad a = 1, \ldots, k \ . \tag{3.3}$$

where Q_i^a are the entries of the charge matrix Q (2.3), and the VOA is defined as a BRST reduction with respect to this currents.

More precisely, BRST reduction proceeds by introducing k ghost systems (b_a, c^a) (all of weight $(1,0)$),

$$b_a(z)c^b(w) \sim \frac{\delta_a^b}{z-w} \ , \tag{3.4}$$

and forming the Feigin standard complex for the $\mathfrak{g} = (\widetilde{\mathfrak{gl}_1})^k$ module $\mathcal{X}$,

$$\mathcal{X}^\bullet = \mathcal{X} \otimes (b,c)^{\otimes k} \ . \tag{3.5}$$

Cohomological grading is by ghost number (with the c_a assigned ghost number $+1$ and the b^a assigned ghost number -1). The differential is identified with the zero-mode of the BRST current

$$J^{\mathrm{BRST}} = c_a \mathcal{J}^a \ , \qquad d = J_0^{\mathrm{BRST}} \ . \tag{3.6}$$

The boundary VOA is identified with the cohomology of this BRST operator acting on the subcomplex of $\mathcal{X}^\bullet$ that is annihilated by the zero modes of all b-ghosts and gauge currents,

$$\mathcal{V}^A = H_d^\bullet \left(\psi \in \mathcal{X}^\bullet \,|\, \mathcal{J}_0^a \psi = b_0^a \psi = 0\right) \cong H^{\frac{\infty}{2}+\bullet}(\mathfrak{g}, \mathfrak{g}_0, \mathcal{X}) \ . \tag{3.7}$$

Remark.
In Section 5 we will interpret the symplectic bosons in $\mathcal{X}$ as $\mathcal{D}_{\mathrm{ch}}(\mathbb{C}^N)$, a chiralisation of the algebra of functions on V. Following the Higgs branch analogy, the currents (3.3) correspond to the moment maps of the gauge symmetry T_G, *cf.*. (2.10). As anticipated, the definition of the VOA can therefore be seen as a chiral analogue of the symplectic reduction that defines the Higgs branch (2.12).

3.1 A collection of BRST-invariant fields

In the following sections, we will compute the BRST cohomology defined in (3.7) by means of bosonisation techniques. To better characterise the final result, it will prove useful to identify a standard set of BRST-invariant fields in $\mathcal{X}$ (that is, fields written in terms of normal-ordered products of fields in $\mathcal{X}$ that are BRST-closed). We do this first, since it can give a flavour of the structure of the VOAs under consideration.

The easiest way to identify a standard set of BRST-invariant operators is to take a direct analogue of the chiral ring generators of the Higgs branch (the expected underlying associated variety) discussed in Section 2.1. This leads to the following list, whose elements can be easily verified to be BRST-closed:

- Affine currents corresponding to the T_H moment maps μ_H^a for the flavour torus acting on hypermultiplets only,

$$K_a = \sum_{i=1}^{N} \widetilde{Q}_a^i X_i Y^i \ , \qquad a = 1, \ldots, N-k \ . \tag{3.8}$$

- Chiral analogues of the monomial generators W^A of the Higgs branch chiral ring as defined in (2.20). These can always be written unambiguously (no ordering ambiguities arise) as the same monomials as the Higgs branch operators, with X's and Y's reinterpreted as symplectic bosons.
- Fermionic (odd) currents $X_i\xi^i$ and $Y^i\chi_i$ for $i = 1, \ldots, N$.
- Affine currents corresponding to the fermionic versions of the flavour symmetry moment maps,

$$J_a = \sum_{i=1}^{N} \widetilde{Q}_a^i \chi_i \xi^i \ , \qquad a = 1, \ldots, N-k \ . \tag{3.9}$$

- Cousins of the W^A obtained by performing all possible collections of substitutions of the form $X_i \leftrightarrow \chi_i$ or $Y^i \leftrightarrow \xi^i$.

Remark.

Note that in R_H, the moment maps for the action of T_G are set to zero in virtue of the definition of the symplectic reduction (2.12). One can verify that the chiral analogues of these moment maps T_G are not d-closed, and therefore not in the BRST cohomology.

3.2 Semi-infinite cohomology via bosonisation

We now calculate the semi-infinite cohomology (3.7) by means of bosonisation techniques. The solution naturally outputs several free-field realisations of the VOA in question.[2]

Outline of strategy

Before getting started, we outline here the general strategy we will employ. The elementary symplectic bosons and free fermions (henceforth *gauge theory variables*) can be bosonised so as to realise the ungauged vertex algebra $\mathcal{X}$ as a subalgebra of a certain lattice vertex (super)algebra V_L,

$$\mathcal{X} \hookrightarrow V_L^! \hookrightarrow V_L \ . \tag{3.10}$$

Here $V_L^!$ is a subalgebra of V_L and is defined by restricting certain lattice momenta to lie in a degenerate sublattice—this is a generalisation of the half-lattice subalgebras of [**?**]. By performing a change of basis for the lattice and a minor refinement, the BRST differential can be restricted to act only within a rank-k sublattice vertex algebra $V_{\boldsymbol{\eta}}$,

$$V_L^! \hookrightarrow V_{\tilde{L}}^! \cong V_{\boldsymbol{\eta}} \otimes \widetilde{V}_{\mathrm{FFR}} \ , \tag{3.11}$$

where $\widetilde{V}_{\mathrm{FFR}}$ is a VOA generated by free fields.

A relatively straightforward application of a vanishing theorem of Voronov for semi-infinite cohomology [**?**] then shows that the (relative) semi-infinite cohomology of V_η

[2] For readers familiar with the setting of VOAs arising from four-dimensional $\mathcal{N} = 2$ SCFTs, where free-field realisations have been constructed in ad hoc ways [**? ? ?**], notice that the general problem at hand enjoys an advantage owing to the Abelian nature of the semi-infinite cohomology.

is one-dimensional (the trivial vertex algebra), concentrated where the momenta λ in $V_{\boldsymbol{\eta}}$ are set to zero, so we have

$$H^{\frac{\infty}{2}+\bullet}(\mathfrak{g},\mathfrak{g}_0,V_{\tilde{L}}^!)\cong H^{\frac{\infty}{2}+\bullet}(\mathfrak{g},\mathfrak{g}_0,V_{\boldsymbol{\eta}})\otimes\widetilde{V}_{\mathrm{FFR}}\cong\delta_{\lambda,0}\mathbb{C}\otimes\widetilde{V}_{\mathrm{FFR}}\ . \tag{3.12}$$

The image of $\mathcal{X}$ in $V_L^!$ can be characterised as the intersection of kernels of N screening charges. We thus arrive at the result that the solution to the gauge theory BRST cohomology problem can be identified with the intersection of the kernel of N screening charges acting on a VOA V_{FFR}, obtained from $V_L^!$ by changing basis and requiring that all non-zero momenta lie along $\widetilde{V}_{\mathrm{FFR}}$ (and not $V_{\boldsymbol{\eta}}$).

3.2.1 Bosonising elementary fields and relating semi-infinite cohomologies

We first of all bosonise the gauge theory variables (symplectic bosons and fermions) that define $\mathcal{X}$. The details of bosonisations are reviewed in Appendix B. In a nutshell, for $i\in\{1,\dots,N\}$ we introduce Heisenberg vertex algebras V_{ρ_i,σ_i} generated by currents J_{ρ_i} and J_{σ_i} and the Heisenberg vertex algebra V_{γ_i} generated by the current J_{γ_i}, with

$$J_{\rho_i}(z)J_{\rho_i}(w)\sim\frac{1}{(z-w)^2}\ ,\qquad J_{\sigma_i}(z)J_{\sigma_i}(w)\sim\frac{-1}{(z-w)^2}\ ,\qquad J_{\gamma_i}(z)J_{\gamma_i}(w)\sim\frac{1}{(z-w)^2}\ . \tag{3.13}$$

We then take lattice extensions

$$V_{L_{\mathtt{Sb}},i}=\bigoplus_{n,m\in\mathbb{Z}^2}V_{\rho_i,\sigma_i}\cdot e^{n\rho_i}e^{m\sigma_i}\ ,\qquad V^!_{L_{\mathtt{Ff}},i}=\bigoplus_{n\in\mathbb{Z}}V_{\gamma_i}\cdot e^{n\gamma_i}\ . \tag{3.14}$$

as well as

$$V^!_{L_{\mathtt{Sb}},i}=\bigoplus_{n\in\mathbb{Z}}V_{\rho_i,\sigma_i}\cdot e^{n(\rho_i-\sigma_i)}\ , \tag{3.15}$$

Putting these together, we can define

$$V_L^!=\bigotimes_{i=1}^N\left(V^!_{L_{\mathtt{Sb}},i}\otimes V_{L_{\mathtt{Ff}},i}\right)\ . \tag{3.16}$$

and similarly

$$V_L=\bigotimes_{i=1}^N\left(V_{L_{\mathtt{Sb}},i}\otimes V_{L_{\mathtt{Ff}},i}\right)\ . \tag{3.17}$$

Each symplectic boson pair and each free fermion pair enjoys an embedding in one copy of the $V^!_{L_{\mathtt{Sb}},i}$ and $V^!_{L_{\mathtt{Ff}},i}$, respectively. More precisely, for a symplectic boson pair (X,Y) (suppressing indices for convenience) one can define two embeddings

$$\begin{pmatrix}X\\Y\end{pmatrix}\overset{\mathfrak{i}^+}{\longmapsto}\begin{pmatrix}e^{\rho-\sigma}\\J_\rho e^{-\rho+\sigma}\end{pmatrix}\ ,\qquad\qquad\begin{pmatrix}X\\Y\end{pmatrix}\overset{\mathfrak{i}^-}{\longmapsto}\begin{pmatrix}-J_\rho e^{-\rho+\sigma}\\e^{\rho-\sigma}\end{pmatrix}\ , \tag{3.18}$$

see (B12).

When $N = 1$ and there is a single symplectic boson pair in $\mathcal{X}$, the above embeddings are equivalent up to a redefinition of $(X, Y) \to (Y, -X)$. Therefore, choosing an embedding is a matter of convention. For $N > 1$, the different choices of embeddings for the various symplectic bosons may be genuinely inequivalent, due to the relative assignment of weights in Q and their appearence in the BRST differential. (On the other hand, the choice of realisation for the free fermions is always a matter of convention.)

We will therefore label choices of embeddings by a sign vector $\epsilon \in \{\pm\}^N$, which indicates whether we use $\mathfrak{i}^+$ or $\mathfrak{i}^-$ to embed the i'th symplectic boson pair in $V^!_{L_{\mathtt{Sb}}}$. For convenience, we will also correlate the (fully conventional) choice and the i'th free fermion embeddin in $V_{L_{\mathtt{Ff}}}$ to the relevan sign in ϵ, and so realise an ϵ-dependent embedding,

$$\mathcal{X} \overset{\epsilon}{\hookrightarrow} V^!_L \hookrightarrow V_L \ , \tag{3.19}$$

whose image is characterised in terms of the kernel of N screening operators determined by screening charges (see (B16))

$$\mathfrak{s}_i = e^{\rho_i} \ , \quad i \in \{1, \dots N\} \ . \tag{3.20}$$

This implies that contribution of the i'th symplectic boson/free fermion pair to the gauged $\mathfrak{g} = \mathfrak{gl}_1$ current will be given by

$$(X_i Y^i) + (\chi_i \xi^i) = \epsilon_i (J_{\sigma_i} + J_{\gamma_i}) \ , \tag{3.21}$$

where ϵ_i is the sign of the i'th symplectic boson embedding. (Here there is no Einstein summation convention in force.)

The next question is how the above bosonisation of $\mathcal{X}$ in terms free lattice fields interacts with taking relative semi-infinite cohomology. As usual, we have a short exact sequence of $\mathfrak{g}$-modules,

$$0 \to \mathcal{X} \overset{\mathfrak{i}}{\hookrightarrow} V^!_L \twoheadrightarrow \mathcal{Y} \to 0 \ , \tag{3.22}$$

where $\mathcal{Y}$ is defined to complete the sequence. This induces a long-exact sequence in cohomology

$$\cdots \to H^{\frac{\infty}{2}+i-1}(\mathfrak{g}, \mathfrak{g}_0, \mathcal{Y}) \overset{\delta}{\to} H^{\frac{\infty}{2}+i}(\mathfrak{g}, \mathfrak{g}_0, \mathcal{X}) \to H^{\frac{\infty}{2}+i}\left(\mathfrak{g}, \mathfrak{g}_0, V^!_L\right) \to H^{\frac{\infty}{2}+i}(\mathfrak{g}, \mathfrak{g}_0, \mathcal{Y}) \to \cdots \ .$$

We will see presently that the connecting homomorphism δ vanishes (it is the zero map), so we in fact have short exact sequences

$$0 \to H^{\frac{\infty}{2}+i}(\mathfrak{g}, \mathfrak{g}_0, \mathcal{X}) \overset{\mathfrak{i}}{\hookrightarrow} H^{\frac{\infty}{2}+i}\left(\mathfrak{g}, \mathfrak{g}_0, V^!_L\right) \twoheadrightarrow H^{\frac{\infty}{2}+i}(\mathfrak{g}, \mathfrak{g}_0, \mathcal{Y}) \to 0 \ , \tag{3.23}$$

giving rise the inclusion

$$H^{\frac{\infty}{2}+i}(\mathfrak{g}, \mathfrak{g}_0, \mathcal{X}) \hookrightarrow H^{\frac{\infty}{2}+i}(\mathfrak{g}, \mathfrak{g}_0, V^!_L) \ . \tag{3.24}$$

To see that the connecting homomorphism vanishes, it is useful to express $V_L^!$ in terms of Fock modules in a way that separates out the action of the symplectic boson screening currents from the $\mathfrak{g}$ action,

$$V_L^! \cong \bigoplus_{(a,b,\ldots,d)\in\mathbb{Z}^N} (\mathcal{F}_{a\rho_1}\otimes\mathcal{F}_{b\rho_2}\otimes\cdots\otimes\mathcal{F}_{d\rho_n})\otimes \\ \otimes(\mathcal{F}_{-a\sigma_1}\otimes\mathcal{F}_{-b\sigma_2}\otimes\cdots\otimes\mathcal{F}_{-d\sigma_n})\otimes V_{L_{\mathrm{ff}},1}\otimes\cdots\otimes V_{L_{\mathrm{ff}},n}\ . \tag{3.25}$$

For each term in the direct sum, the screening currents e^{ρ_i} act exclusively in the first parenthetical grouping of Fock modules, while the $\mathfrak{g}$ action generated by a linear combination of the σ_i and γ_i plays out exclusively on the second parenthetical grouping. Thus, as one can always find a section of a surjective map of vector spaces, there is a direct sum decomposition *as* $\mathfrak{g}$ *modules*,

$$V_L^! \cong \mathcal{X}\oplus\mathcal{Y}\ , \tag{3.26}$$

i.e., the short exact sequence (3.22) splits in $\mathcal{U}(\mathfrak{g})$-Mod. This, in particular, implies the vanishing of the connecting homomorphism δ.

The gauge theory cohomology can therefore be constructed within the cohomology of the bosonised free field VOA, and what's more, the gauge theory cohomology will be characterised in the free field cohomology by the action of the same screening charges, now treated at the level of cohomology classes. We will see more explicitly how this works after we determine the semi-infinite cohomology of $V_L^!$ below.

3.2.2 A new basis for the free field space

Let us assemble the various elements of our rank $3N$ Heisenberg algebra used to bosonise the ungauged VOA $\mathcal{X}$ into vectors denoted by bold variables, $\boldsymbol{\sigma} = (\sigma_1,\ldots,\sigma_N)^T$, $\boldsymbol{\eta} = (\eta_1,\ldots,\eta_k)^T$, and so on. (Below, we will continue to vectorise various quantities and denote these with boldfaced versions of the same symbols as used to denote the individual components.) Then in order to facilitate the calculation of the BRST cohomology, we consider the following change of basis, defined in terms of a $2N\times 3N$ matrix P_ϵ

$$\begin{pmatrix}\boldsymbol{\eta}\\ \tilde{\boldsymbol{\eta}}\\ \boldsymbol{\phi}\\ \boldsymbol{\delta}\\ \boldsymbol{\omega}\end{pmatrix} = \left(\begin{array}{c|c|c} Q_\epsilon & 0 & Q_\epsilon \\ \hline Q_\epsilon & -Q_\epsilon & 0 \\ \hline \widetilde{Q}_\epsilon & 0 & 0 \\ \hline 0 & -\widetilde{Q}_\epsilon & 0 \\ \hline F_\epsilon^T & -F_\epsilon^T & \mathbf{1}_{N\times N}\end{array}\right)\begin{pmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\rho}\\ \boldsymbol{\gamma}\end{pmatrix}\ , \tag{3.27}$$

where we define the $N\times N$ matrices

$$A_\epsilon \coloneqq \begin{pmatrix}Q_\epsilon\\ \widetilde{Q}_\epsilon\end{pmatrix}\ , \qquad B_\epsilon \coloneqq \begin{pmatrix}Q_\epsilon\\ 0\end{pmatrix}\ , \qquad F_\epsilon = A_\epsilon^{-1}B_\epsilon\ . \tag{3.28}$$

Here Q_ϵ, $\widetilde{Q}_\epsilon$ are defined to be the charge matrices Q with the signs of the columns flipped according to ϵ and its mirror (*cf.* (2.4)).

The details of the choice of basis in (3.27) have been specified in order to simplify the analysis of our BRST problem. First and foremost,

- the BRST operator (3.6) now takes the form

$$J^{\mathrm{BRST}} = \sum_{a=1}^{k} c_a J_{\eta_a} \ , \tag{3.29}$$

  and so the basis vectors $\boldsymbol{\eta}$ are precisely the (null) vector corresponding to the $(\widetilde{\mathfrak{gl}})_1^k$ currents entering the BRST differential.
- There are N basis vectors which pair non-trivially with the original fermionic directions γ_i. Each is a null "bosonic" correction to γ_i, with the correction chosen to enforce orthogonality to the vectors in $\boldsymbol{\eta}$.

Other properties of the change of basis can be inferred from the bilinear form encoding the OPEs of Heisenberg currents, which takes the relatively simple schematic form,

$$\left(\begin{array}{c|c|c|c|c} 0 & \mathrm{OPE}(\boldsymbol{\eta}\times\tilde{\boldsymbol{\eta}}) & 0 & 0 & 0 \\ \hline \mathrm{OPE}(\boldsymbol{\eta}\times\tilde{\boldsymbol{\eta}}) & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathrm{OPE}(\boldsymbol{\phi}) & 0 & 0 \\ \hline 0 & 0 & 0 & \mathrm{OPE}(\boldsymbol{\delta}) & 0 \\ \hline 0 & 0 & 0 & 0 & \mathrm{OPE}(\boldsymbol{\omega}) \end{array}\right) \ , \tag{3.30}$$

where the locations of nontrivial blocks of OPEs are indicated by OPE$(-)$. In particular, notice that

- Only k basis vectors $\widetilde{\boldsymbol{\eta}}$ are non-trivially paired with those entering the BRST differnetial (according the the bilinear form on the lattice L).
- The remaining basis directions ϕ and δ, which are purely bosonic, are mutually orthogonal to all of the above and are of indefinite signature.

Although the above properties do not uniquely select a change of basis (and so other choices would allow us to perform the steps that will follow), they tightly constrain it.

Remark.
The specific form of the change of basis matrix further guarantees that we have

$$\mathrm{OPE}(\boldsymbol{\delta}) = -\mathrm{OPE}(\boldsymbol{\phi}) \ , \qquad \mathrm{OPE}(\boldsymbol{\omega}) = \mathbf{1}_{N\times N} \ . \tag{3.31}$$

In general, OPE$(\boldsymbol{\eta}\times\tilde{\boldsymbol{\eta}})$ and OPE$(\boldsymbol{\phi})$ are not diagonal (though they are always non-degenerate). They instead reflect the matrix of inner products amongst the gauge

charge vectors (rows of Q_ϵ) in the former case, and amongst the flavour charge vectors encoded in the rows of $\widetilde{Q}_\epsilon$ in the latter.

The lattice $L^!$ along which the Heisenberg vertex algebra is extended can be expressed in the basis of these new variables by inspecting the inverse of the change-of-basis matrix in (3.27) (technical details can be found in Appendix C). We ultimately find the expression

$$L^! = \bigoplus_{i=1}^{N} \mathbb{Z}\alpha_i \oplus \mathbb{Z}\beta_i \ , \tag{3.32}$$

where

$$\begin{aligned} \boldsymbol{\alpha} &\coloneqq A_\epsilon^{-1} \begin{pmatrix} \tilde{\boldsymbol{\eta}} \\ \mathbf{0} \end{pmatrix} + A_\epsilon^{-1} \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\phi} + \boldsymbol{\delta} \end{pmatrix} \ , \\ \boldsymbol{\beta} &\coloneqq A_\epsilon^{-1} \begin{pmatrix} \tilde{\boldsymbol{\eta}} \\ \mathbf{0} \end{pmatrix} + \boldsymbol{\omega} \ . \end{aligned} \tag{3.33}$$

Note in particular that $L^!$ (by construction) does include any vectors with nonzero $\boldsymbol{\eta}$ components. We define a lattice refinement, $\tilde{L}^!$ according to[3]

$$L^! \subset \tilde{L}^! = \left(\bigoplus_{i=1}^{N} \mathbb{Z} \left(A_\epsilon^{-1}\right)_i \begin{pmatrix} \tilde{\boldsymbol{\eta}} \\ \mathbf{0} \end{pmatrix} \right) \oplus \left(\bigoplus_{i=1}^{N} \mathbb{Z} \left(A_\epsilon^{-1}\right)_i \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\phi} + \boldsymbol{\delta} \end{pmatrix} \right) \oplus \left(\bigoplus_{i=1}^{N} \mathbb{Z}\, \omega_i \right) \ . \tag{3.34}$$

When the Heisenberg vertex algebra is extended by this refined lattice, then the ω_i can be treated as bosonisations of N complex free fermions,

$$\psi^i = e^{-\omega_i} \ , \qquad \widetilde{\psi}_i = e^{\omega_i} \ , \tag{3.35}$$

with singular OPEs,

$$\psi^i(z)\widetilde{\psi}_j(w) \sim \frac{\delta^i_j}{z-w} \ . \tag{3.36}$$

We then have the decomposition

$$V^!_{\tilde{L}} \cong V_{\boldsymbol{\eta}} \otimes \widetilde{V}_{\mathrm{FFR}} \ , \tag{3.37}$$

where $V_{\boldsymbol{\eta}}$ is the relevant lattice extension (by the first parenthetical grouping in (3.34)) of the Heisenberg vertex algebra generated by the J_{η_a} and $J_{\tilde{\eta}_a}$, and $\widetilde{V}_{\mathrm{FFR}}$ is the free field vertex algebra in which we will ultimately locate our free field realisation, which consists of the lattice extension of the J_{ϕ_i} and J_{δ_i} Heisenbergs by the second parenthetical grouping in (3.34), as well as the N complex free fermions (3.35).

[3]Here for dimensional reasons there is a redundancy amongst the lattice directions appearing in each of the first two direct sums; in examples we will write this extension more compactly by identifying basis vectors over $\mathbb{Z}$ for these vector spaces.

3.2.3 Semi-infinite cohomology of the bosonised algebra

Now let us address the computation of the relative semi-infinite cohomology of $V_L^!$. We can equally consider the semi-infinite cohomology of $V_{\tilde{L}}^!$ and subsequently restrict to the sublattice L. The BRST problem is then isolated $V_{\boldsymbol{\eta}}$, so we have for our semi-infinite cohomology,

$$H^{\frac{\infty}{2}+\bullet}(\mathfrak{g},\mathfrak{g}_0,V_{\tilde{L}}^!) \cong H^{\frac{\infty}{2}+\bullet}(\mathfrak{g},\mathfrak{g}_0,V_{\boldsymbol{\eta}}) \otimes \widetilde{V}_{\mathrm{FFR}} \ . \tag{3.38}$$

Now by a change of basis for the components of $\tilde{\boldsymbol{\eta}}$, we can always diagonalise $\mathrm{OPE}(\boldsymbol{\eta}\times\tilde{\boldsymbol{\eta}})$. Consequently, $V_{\boldsymbol{\eta}}$ is essentially (up to a possible restriction of charge vectors) just the tensor product of k copies of VOAs where various gauged $\widetilde{\mathfrak{gl}}_1$'s act separately in each copy. To ease the notation, without loss of generality we therefore restrict to the case where the lattice extension $V_{\boldsymbol{\eta}} = V_\eta$ has rank-one. More precisely, without loss of generality we will take V_η to be the lattice VOA obtained by extending two Heisenberg currents J_η, $J_{\tilde{\eta}}$ with $(\eta,\tilde{\eta}) = 1$ by the lattice $\mathbb{Z}\tilde{\eta}$.

Remark.

V_η is isomorphic to the algebra of global sections of CDOs on $\mathbb{C}^\times$, *i.e.*, $\mathcal{D}^{(ch)}(\mathbb{C}^\times)$. In general, for CDOs on an algebraic group $\mathcal{G}$ there are two commuting $\mathrm{Lie}(\mathcal{G})$ affine current algebras whose levels add up to $-2h^\vee_{\mathcal{G}}$. In the current identification, we can think of J_η—a level-zero $\widetilde{\mathfrak{gl}}_1$ current—as the generator of the diagonal subalgebra, where in the Abelian case one sets $h^\vee = 0$.

Given the factorisation (3.37) and the fact that the $\mathfrak{g}$ action is generated by η, we have the simplification

$$H^{\frac{\infty}{2}+\bullet}(\mathfrak{g},\mathfrak{g}_0,V_{\tilde{L}}^!) = H^{\frac{\infty}{2}+\bullet}(\mathfrak{g},\mathfrak{g}_0,V_\eta) \otimes \widetilde{V}_{\mathrm{FFR}} \ . \tag{3.39}$$

Thus our only task is to evaluate the relative semi-infinite cohomology of the half-lattice vertex algebra $V_\eta \cong \mathcal{D}^{ch}(\mathbb{C}^\times)$. To perform this evaluation, we observe the following simple structure of V_η as a $\mathfrak{g}$ module,

$$\mathcal{D}^{ch}(\mathbb{C}^\times) \cong \bigoplus_{\lambda\in\mathbb{Z}} \left(\mathcal{U}(\mathfrak{n}_-)\otimes\mathbb{C}_\lambda\otimes\mathcal{U}(\mathfrak{n}_+)^*\right) \ . \tag{3.40}$$

Here we are utilising the semi-infinite structure of $\widetilde{\mathfrak{gl}}_1$ in the sense of Voronov [**?**]—reviewed in Appendix B, *cf.* (B4). In particular, $\mathbb{C}_\lambda$—corresponding to the lattice momentum $e^{-\lambda\tilde{\eta}}$—is the one-dimensional representation of the zero mode of J_η with charge λ. The only slightly nontrivial observation incorporated into (3.40), though it is a straightforward calculation, is that the $J_{\tilde{\eta}}$ Fock space, as an $\mathfrak{n}_+$-module, is isomorphic to the contragredient dual of the free module $\mathcal{U}(\mathfrak{n}_+)$.

From this, it is immediate to see that we can write

$$\mathcal{D}^{ch}(\mathbb{C}^\times) \cong \mathcal{U}(\mathfrak{n}_+)^* \otimes \left(\bigoplus_{\lambda\in\mathbb{Z}} \mathcal{U}(\mathfrak{n}_-) \otimes \mathbb{C}_\lambda \right) \qquad \text{in} \qquad \mathfrak{n}_+\text{−mod} \ , \tag{3.41}$$

$$\mathcal{D}^{ch}(\mathbb{C}^\times) \cong \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}_+)} \left(\bigoplus_{\lambda\in\mathbb{Z}} \mathbb{C}_\lambda \otimes \mathcal{U}(\mathfrak{n}_+)^* \right) \qquad \text{in} \qquad \mathfrak{g}\text{−mod} \ . \tag{3.42}$$

The first line expressed $\mathcal{D}^{ch}(\mathbb{C}^\times)$ as a direct sum of infinitely many copies of $\mathcal{U}(\mathfrak{n}_+)^*$, which is injective (in fact, co-free) as an $\mathfrak{n}_+$ module. The second line makes it apparent that it is also projective relative to $\mathfrak{b}_+$. The key result is then an unnumbered Lemma of Voronov (specialised here for our purposes):

Theorem 1 (Section 3.11.2 of [**?**]). *Let $\mathfrak{g}$ be a Lie algebra with a semi-infinite structure, and let M be a $\mathfrak{g}$-module that is injective as an $\mathfrak{n}_+$-module and relatively projective over $\mathfrak{b}_+$. Then the relative semi-infinite cohomology of M is given as*

$$H^{\frac{\infty}{2}+i}(\mathfrak{g}, \mathfrak{g}_0, M) = \begin{cases} M^{\mathfrak{n}_+}_{\mathfrak{b}_-} \ , & i = 0 \ , \\ 0 \ , & i \neq 0 \ . \end{cases} \tag{3.43}$$

Here $M^{\mathfrak{n}}_{\mathfrak{b}}$ denotes the semivariants *of M,*

$$M^{\mathfrak{n}}_{\mathfrak{b}} = \frac{\{m \in M \,|\, \mathfrak{n}m = 0\}}{\{m \in M \,|\, \mathfrak{n}m = 0 \text{ and } m = bm' \text{ for } b \in \mathfrak{b}\}} \ . \tag{3.44}$$

Applying the above theorem to the present case, we have seen that the semi-infinite cohomology of V_η will just restrict to semivariants in degree zero (and vanish in other degrees). But it is easy to see that in fact,

$$\left(V_{\eta,\tilde{\eta}} \cdot e^{\lambda\tilde{\eta}} \right)^{\mathfrak{n}_+}_{\mathfrak{b}_-} = \mathbb{C}\delta_{\lambda,0} \ , \tag{3.45}$$

so we have the nice result

$$H^{\frac{\infty}{2}+\bullet}(\mathfrak{g}, \mathfrak{g}_0, V_\eta) = \mathbb{C} \qquad \Longrightarrow \qquad H^{\frac{\infty}{2}+\bullet}(\mathfrak{g}, \mathfrak{g}_0, V^!_L) \cong \mathbb{C}_{\lambda,0} \otimes \widetilde{V}_{\mathrm{FFR}} \ . \tag{3.46}$$

As the cohomology of V_η is supported entirely in the sector with no $\tilde{\eta}$ momentum, the restriction to the sublattice $L^!$ is immediate, and we have

$$H^{\frac{\infty}{2}+\bullet}(\mathfrak{g}, \mathfrak{g}_0, V_{L^!}) \cong V_{\mathrm{FFR}} \ , \tag{3.47}$$

where V_{FFR} is the subalgebra of $V_{L^!}$ that has no $\tilde{\eta}$ momenta, and therefore lies entirely in $\widetilde{V}_{\mathrm{FFR}}$. That is, in terms of modules for the Heisenberg currents J_{δ_i}, J_{ϕ_i},

$i \in \{1, \cdots N-k\}$ J_{ω_i}, $i \in N$

$$V_{\mathrm{FFR}} = \bigoplus_{\substack{m\in\mathbb{Z}^N\, n\in\mathbb{Z}^N \\ m_1=n_1}} e^{m_2\cdot(\boldsymbol{\phi}+\boldsymbol{\delta})} \otimes e^{n\cdot\boldsymbol{\omega}} \tag{3.48}$$

where for $m \in \mathbb{Z}^N$, $m_1 \in \mathbb{Q}^k$ and $m_2 \in \mathbb{Q}^{N-k}$ are such that

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = A_\epsilon^{-1,T} m \ . \tag{3.49}$$

Remark.

By a rational change of variable, the matrix of inner products $\mathrm{OPE}(\boldsymbol{\delta}) = -\mathrm{OPE}(\boldsymbol{\phi})$ can be diagonalised as well. Thus, subject to a refinement of the lattice by which the Heisenberg vertex algebras are extended, we can identify

$$\widetilde{V}_{\mathrm{FFR}} \subseteq V^{\mathrm{ref}}_{\mathrm{FFR}} \cong \mathcal{D}^{(ch)}(\mathbb{C}^\times)^{\otimes(N-k)} \otimes \mathtt{Ff}^{\otimes N} \ . \tag{3.50}$$

Though this is a nice and uniform expression, in our study of examples, we will not always perform such a change of basis and refinement.

3.2.4 *A*-twisted vertex algebra as the kernel of screening operators on a free-field space

Notice that the screening charges (3.20) commute with the BRST differential, as they act in completely different Fock spaces. Thus, one finds that elements of V_{FFR} that lie in the injective image of the BRST cohomology are precisely those that are in the joint kernel of the screening charges $\mathfrak{s}_i = e^{\rho_i}$, restricted to act on V_{FFR}.

These restrictions can be identified using the expression for the ρ_i in terms of the modified Heisenberg basis,

$$\boldsymbol{\rho} = A_\epsilon^{-1} \begin{pmatrix} \boldsymbol{\eta} \\ -\boldsymbol{\delta} \end{pmatrix} - F_\epsilon\, \boldsymbol{\omega} \ . \tag{3.51}$$

We write the restrictions to V_{FFR} as

$$\tilde{\mathfrak{s}}_i = e^{\bar{\rho}_i} \ , \tag{3.52}$$

where

$$\bar{\boldsymbol{\rho}} = -A_\epsilon^{-1} \begin{pmatrix} Q_\epsilon \boldsymbol{\omega} \\ \boldsymbol{\delta} \end{pmatrix} \ . \tag{3.53}$$

We can therefore summarise our results as follows.

Proposition 1. *Let $\mathcal{V}_A$ be the VOA defined from the theory data (T_G, V) as in Definition 3.7. For any choice of sign vector $\epsilon \in \{\pm\}^N$, $\mathcal{V}_A$ is isomorphic to the intersection of the kernel of N screening operators (3.53) acting on V_{FFR} as defined in (3.48).*

In the examples to follow, we will not endeavour to systematically determine the joint kernel of these screening charges for our free field realisations, which may be a difficult problem. However, we can at least provide a set of (weak) generators for a sub-VOA of this kernel by leveraging the BRST-invariant operators identified in Section 3.1. All one needs to do is to compute representatives of these fields in V_{FFR}, which can be obtained in a simple, algorithmic manner:

- We first rewrite the above generators in terms of the bosonised degrees of freedom;
- Performing the relevant change of basis, the closed operators can be rewritten by construction in a form that does not involve the $\tilde{\boldsymbol{\eta}}$ bosons;
- Finally, a choice of BRST representative can then always be made to precisely set to zero all terms involving the $\boldsymbol{\eta}$ bosons, thus projecting into V_{FFR}.

Notice that the last step can be done quite naïvely, because the manifest $\tilde{\boldsymbol{\eta}}$ independence evade normal ordering issues as well as trivialising the problem of finding the BRST-exact correction to eliminate such terms. In particular, application of this algorithm to the currents (3.8) and (3.9) leads to

$$\bar{J}_a = \sum_{i=1}^{N} \widetilde{Q}^i_{\epsilon,a} \psi^i \widetilde{\psi}_i \ , \qquad \bar{K}_a = -J_{\phi_a} \ . \tag{3.54}$$

We do not have a proof that the resulting list forms a set of (weak) generators for the boundary VOA $\mathcal{V}^A$, although this turns out to be true in examples; we therefore turn this into a conjecture.

Conjecture 2. *The images in V_{FFR} of the BRST-closed operators identified in Section 3.1 under the free-field embedding provides a list of weak generators for $\mathcal{V}^A$.*

If this is not the case, then the above operators will generate a sub-VOA and other operators would need to be included. It would be interesting to address this question in general using the screening charge characterisation of these VOAs.

3.3 Outer automorphisms from free fermionic symmetries

We now turn to the interesting question of identifying groups of outer automorphisms in the VOAs constructed above. Physically, it is natural to interpret the outer automorphisms as an avatar of the topological symmetry of the Coulomb branch [**?** **?**].

As discussed in Section 2.1, the UV, this symmetry is $\mathrm{U}(1)^k_C$, but there are possible IR enhancements. In physical language, these enhancements happen precisely when there is a *balanced* subgroup of the gauge group, $\mathrm{U}(1)_S$ subgroup of $T_G \cong \mathrm{U}(1)^k$ where there are exactly two hypermultiplets charged under $\mathrm{U}(1)_S$ (each with charge ± 1), as this is the setting where elementary monopole operators act as additional moment map operators in the Coulomb branch chiral ring.

More mathematically, whenever there is a $U(1)_S \subset\cong T_G$ subgroup that acts on $\mathbb{C}^N \subset V$ with precisely two non-zero weights ± 1, we expect the respective $\mathfrak{u}(1)_C \subset \mathfrak{t}_C$ symmetry group to enhance to $\mathfrak{sl}_2$. We now show that this is enhancement also happens at the level of outer automorphisms of the A-twisted boundary VOAs.

First of all, notice that there are k interesting affine currents, which we can build from the gauge theory fermions as follows,

$$\sum_{j=1}^{N} Q_i^j \chi_j \xi^j \ , \qquad i = 1, \dots, k \ . \tag{3.55}$$

These are *not BRST closed*; one needs to add the analogous currents with symplectic bosons replacing the free fermions to realise the BRST closed (and exact) gauge currents. However, the *zero modes* of these operators commute with the BRST operator when restricted to the relative subcomplex of the Feigin standard complex (where the c-ghost zero modes are absent).[4] These zero modes therefore generate a $\mathfrak{u}(1)^k$ outer automorphism symmetry of the A-twisted VOA.

In the free field realisations described above, these additional currents can be written in terms of the modified (free field) fermions and the components of $\tilde{\boldsymbol{\eta}}$ (*cf.* Appendix C),

$$\sum_{j=1}^{N} Q_{\epsilon,i}^j \widetilde{\psi}_j \psi^j - Q_{\epsilon,i}^a (A_\epsilon^{-1})_a{}^b J_{\tilde{\eta}_b} \ , \qquad i = 1, \dots, k \ . \tag{3.56}$$

In the zero mode of this current, the $J_{\tilde{\eta}_b}$ terms only act to shift an operator by BRST exact terms. Restricting to representatives in $\widetilde{V}_{\mathrm{FFR}}$, we have an action of the zero modes of the operators

$$\sum_{j=1}^{N} Q_{\epsilon,i}^j \widetilde{\psi}_j \psi^j \ . \tag{3.57}$$

Though the currents themselves are not annihilated by the screenings (3.52), their zero modes are diagonalised by the screening currents. Consequently they act within the kernel of the screening charges and so give rise to outer automorphisms of the kernel vertex algebra.

It is then relatively clear what should be the natural mechanism leading to outer automorphism symmetry enhancement. From the complete set of free fermions in $\widetilde{V}_{\mathrm{FFR}}$ we can realise a standard construction of an $V_1(\mathfrak{so}_{2N})$ current algebra [**?** **?**], whose Cartan subalgebra is generated by both the T_C outer automorphism currents above and the moment map currents K_a. Then complementary to the subset of $V_1(\mathfrak{so}_{2N})$ currents that lie in the kernel of the screening charges enhancing the inner affine symmetry of the vertex algebra, there may be further currents whose zero modes *preserve the* $\mathbb{C}$*-span of the screening charges*. These will act as outer automorphisms in the kernel vertex algebra.

[4] We could equivalently consider the affine currents built only from the symplectic bosons with the same charge matrices; the two are related (up to a sign) by the addition of the BRST-exact $\mathcal{J}_a$. We will find it useful to think of these in fermionic terms.

It turns out that indeed, there is an outer automorphism enhancement by zero modes of additional generators of the fermionic $V_1(\mathfrak{so}_{2N})$ precisely when there is such a balanced subgroup *and the corresponding charged hypermultiplets are bosonised appropriately.* We have the following.

Proposition 2. *For a given bosonisation ϵ, we will call a subgroup $\mathrm{U}(1)_S \subset \mathrm{U}(1)^k$* sign balanced *if it acts on $\mathbb{C}^N$ with only two non-zero weights $+1$ and -1 after the sign-flips dictated by ϵ. Then a $\mathfrak{u}(1)_S \subset \mathfrak{u}(1)^k$ outer automorphism will enjoy an $\mathfrak{sl}_2$ enhancement generated by zero modes of free fermionic currents if and only if the corresponding $\mathrm{U}(1)_S$ subgroup of the gauge group is sign balanced.*

Proof. The key to proving this is to rewrite the screening charges in a more convenient form. Recall the expression from (3.53),

$$\tilde{\mathfrak{s}}_i = e^{\bar{\rho}_i} \ , \qquad \bar{\boldsymbol{\rho}} = A_\epsilon^{-1} \begin{pmatrix} Q_\epsilon \boldsymbol{\omega} \\ -\boldsymbol{\delta} \end{pmatrix} \ . \tag{3.58}$$

In light of the definition (3.28) of A_ϵ, this can be rewritten as

$$\bar{\boldsymbol{\rho}} = A_\epsilon^{-1} \begin{pmatrix} \mathbf{0} \\ -\boldsymbol{\delta} + \widetilde{Q}_\epsilon \boldsymbol{\omega} \end{pmatrix} + \boldsymbol{\omega} \ . \tag{3.59}$$

Now in order for any two screening currents—say $\tilde{\mathfrak{s}}_1$ and $\tilde{\mathfrak{s}}_2$—to be related by a free fermionic symmetry of the above type, they must have the same dependence on the components of $\boldsymbol{\delta}$, so in particular we would require

$$\left(A_\epsilon^{-1}\right)_1^j = \left(A_\epsilon^{-1}\right)_2^j \ , \quad j > k \ . \tag{3.60}$$

Thus the only possibility is that the outer automorphism in question is generated by the zero modes of

$$\psi^1 \widetilde{\psi}_2 = e^{-\omega_1 + \omega_2} \ , \qquad \psi^2 \widetilde{\psi}_1 = e^{-\omega_2 + \omega_1} \ , \qquad J_{\omega_1} - J_{\omega_2} \ . \tag{3.61}$$

The two screening currents in question will indeed transform in the standard representation of this $\mathfrak{sl}_2$, and the remaining screening currents will be invariant, if and only if the dependence on ω_1 and ω_2 in $\widetilde{Q}_\epsilon \boldsymbol{\omega}$ is entirely through the combination $\omega_1 + \omega_2$. This in particular means that the vector $(1, -1, 0, \ldots)$ must be orthogonal to the rows of $\widetilde{Q}_\epsilon$, and must therefore be in the span of the rows of Q_ϵ.

To complete the proof, we need only observe that if $(1, -1, 0, \ldots)$ *is* in the gauge charge lattice (so is in the span of the rows of Q_ϵ), then the form of A_ϵ and the basic relation $A_\epsilon A_\epsilon^{-1} = \mathbf{1}_{N \times N}$ implies (3.60). □

Remark.

The analysis here does not preclude the existence of additional outer automorphisms of A-twisted vertex algebras that do not arise in the manner described here for *any* choice of bosonisation/free field realisation. As an example, the A-twisted vertex algebra for SQED[1] is just the symplectic fermion vertex algebra [**?** **?**], which enjoys an $\mathfrak{sl}_2 \cong \mathfrak{sp}_1$ outer automorphism symmetry. However, the free field realisation for this theory produced by our methods here is just the usual one, in which the symplectic fermion is realised as the kernel of a single screening charge in the free fermion VOA. In this case the outer automorphism does not act in the free field space at all.

Physically, this scenario amounts to the presence of an under-balanced U(1), giving an *ugly* theory in the sense of [**?**]. So in this case, the enhanced Coulomb branch symmetry is associated to emergent free fields. Perhaps unsurprisingly, this seems to be a case that requires special treatment.

Remark.

The fact that the outer automorphism symmetry is manifested only with an appropriate choice of signs in the bosonisation has a possible physical and geometrical explanation, that we will comment upon in Section 5.3, see in particular Lemma 4. In a nutshell, we will claim that certain bosonisations are equivalent to interpreting the free fields in terms of CDOs on openly embedded subsets of a *resolution* of the Higgs branch $\mathcal{M}_{H,\zeta}$. Resolving the Higgs branch by turning on an FI parameter ζ in the underlying gauge theory breaks the enhanced $\mathfrak{sl}_2$ symmetry of the Coulomb branch to a Cartan subalgebra, and can therefore obscure the outer automorphisms.

4 $T[\mathrm{SU}(2)]$ and free-field realisations of $V_1(\mathfrak{psl}(2|2))$

We now spell out the arguments and the (occasionally technical) calculations of Section 2 in the simplest non-trivial example, the A-twisted VOA of the theory $T[\mathrm{SU}(2)]$. This is the theory defined by the charge matrices

$$\mathbf{Q} \coloneqq \begin{pmatrix} Q \\ q \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ , \tag{4.1}$$

as well as

$$\widetilde{\mathbf{Q}} \coloneqq \begin{pmatrix} \widetilde{q} \\ \widetilde{Q} \end{pmatrix} = \left(\mathbf{Q}^{-1}\right)^T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \ . \tag{4.2}$$

Notice that by a symplectic automorphism of $T^*\mathbb{C}^2$ and a redefinition of the gauge group (so that the Y^i have charge $+1$), the charge matrix $\tilde{Q} = (-1, 1)$ can be sent to the charge matrix $Q = (1, 1)$. We therefore observe the self-mirror duality of this theory (that is, the theory defined by Q is the same as the theory defined by $\tilde{Q}$).

In Section 4.1 we collect some minimal amount of information about the Higgs branch of this theory, as introduced in Section 2.1. As a holomorphic symplectic variety, it is isomorphic to the A_1 singularity. As explained in Section 3.1, its chiral ring will

determine a selection of BRST-closed operators. In Section 4.2 we explain that based on this selection we expect the A-twisted VOA to be isomorphic to $V_1(\mathfrak{psl}(2|2))$.[5]

The actual calculation of the BRST cohomology presented in Section 3.2 is given in terms of free-field realisations depending upon a choice of sign vector $\epsilon \in \{\pm\}^2$. It can be checked that in this case the choices $(++)$ and $(--)$ are essentially equivalent, as are $(+-)$ and $(-+)$, as they are pairwise related upon redefining the gauge group. We therefore spell out the cases $(+-)$ and $(++)$ in Section 4.3 and Section 4.4, respectively, obtaining two distinct characterisations of the A-twisted boundary VOA in terms of kernels of screening operators on some free-field space. Applied to the choice of generators presented in Section 4.2, our algorithms produce indeed two inequivalent free-field realisations of $V_1(\mathfrak{psl}(2|2))$.

We complement the study of this example by demonstrating in Section 4.5 that the associated variety of $V_1(\mathfrak{psl}(2|2))$ is isomorphic to the A_1 singularity, corroborating Conjecture 1 in this case.

Remark.

The (simple quotient of) $V^1(\mathfrak{psl}(2|2))$ is the same as the one appearing in the description of string theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ with minimal NS flux in the Berkowitz–Vafa–Witten hybrid formalism [**?**], which has recently been investigated in great detail in connection with the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence in [**?**]. In this context, in [**?**] the same vertex algebra was investigated from a similar perspective (in terms of symplectic bosons and free fermions), but without solving the semi-infinite cohomology problem directly and with a slightly different approach to the BRST realisation of the problem. It would be interesting to investigate whether the complete free field realisations presented here have some utility in this string theoretic setting.

4.1 The A_1 singularity as Higgs branch

Recall from (2.12) that the Higgs branch is defined in terms of a holomorphic symplectic quotient detemined by the theory data (T_G, V)

$$\mathcal{M}_H \cong \mu_{\mathbb{C}}^{-1}(0) /\!\!/ T_{G,\mathbb{C}} \ , \tag{4.3}$$

In the present case, $V \cong T^*\mathbb{C}^2$, and the gauge-invariant algebraic functions on V are generated by the entries in the matrix

$$M = \begin{pmatrix} X_1 Y^1 & X_1 Y^2 \\ X_2 Y^1 & X_2 Y^2 \end{pmatrix} \ , \tag{4.4}$$

and the vanishing of the complex moment map is equivalent to requiring $\mathrm{Tr}\, M = 0$. Imposing this tracelessness implies the relation $M^2 = 0$, and indeed the Higgs branch is the just the closure of the minimal (and in this case, unique) nilpotent co-adjoint orbit of $\mathrm{SL}(2,\mathbb{C})$, and the traceless matrix (4.4) is the corresponding $\mathfrak{sl}_2$ moment map.

[5] We adopt the convention that $V^k(\mathfrak{g})$ denotes the universal affine current algebra of type $\mathfrak{g}$ at level k, and $V_k(\mathfrak{g})$ denotes the unique simple quotient thereof.

For later convenience, we can label the matrix entries of M in terms of the Chevalley basis for $\mathfrak{sl}_2$

$$M = \begin{pmatrix} h/2 & e \\ f & -h/2 \end{pmatrix} , \tag{4.5}$$

where $h = X_1Y^1 - X_1Y^2$. The nilpotency condition amounts to the relation $ef = -\frac{1}{4}h^2$, which characterises this nilpotent orbit closure as the simplest Kleinian singularity,

$$\mathcal{M}_H \cong \overline{\mathbb{O}_{\min}(\mathfrak{sl}_2)} \cong A_1 \cong \mathbb{C}^2/\mathbb{Z}_2 . \tag{4.6}$$

4.2 A-twisted vertex algebra and $V_1(\mathfrak{psl}(2|2))$

Recall that the (ungauged) VOA $\mathcal{X}$ entering the definition of the BRST reduction (3.2) is generated by $N = 2$ pairs of symplectic bosons and free fermions. By our standard abuse of notation, the symplectic bosons will be denoted by the same symbols as the chiral ring generators, whereas the free fermions will be denote by (ξ^1, χ_1) and (ξ^2, χ_2).

The BRST current $\mathcal{J}^1 = \mathcal{J}$ of (3.3) in this case reads:

$$\mathcal{J} = X_1Y^1 + X_2Y^2 + \chi_1\xi^1 + \chi_1\xi^2 . \tag{4.7}$$

The BRST-closed operators given in Section 3.1 were modelled on the Higgs branch chiral ring generators, and can be listed in the following suggestive order

$$\begin{aligned} &X_1Y^2 , \quad X_2Y^1 , \quad X_1Y^1 - X_2Y^2 , \\ &\chi_1\xi^2 , \quad \chi_2\xi^1 , \quad \chi_1\xi^1 - \chi_2\xi^2 , \\ &Y^j\chi_\beta , \quad X_i\xi^\alpha , \end{aligned} \tag{4.8}$$

It is straightforward to see that in the ungauged (pre-BRST) VOA $\mathcal{X}$ these generate the VOA $V^1(\mathfrak{sl}(2|2))$. In particular, the first line generates an even $V^{-1}(\mathfrak{sl}_2)$ subalgebra, the second line an even $V^1(\mathfrak{sl}_2)$ subalgebra, and the third line corresponds to the odd generators.

By Theorem 5.5 of [**?**], the $\mathfrak{sl}(2|2)$ current algebra generated by the super-traceless combinations of (4.8) is precisely the Heisenberg coset of $\mathcal{X}$ by the Heisenberg algebra generated by $\mathcal{J}$. Furthermore, $\mathcal{J}$ itself, which is d-exact (it is equal to $d(b)$), is the ordinary trace combination of these currents and this is therefore removed by the cohomological quotient.

We conclude that the image in the BRST cohomology of the VOA generated by our choice of BRST-invariant operators is a level-one affine $\mathfrak{psl}(2|2)$ current algebra. The fact that this results in some quotient of $V^1(\mathfrak{psl}(2|2))$ was already mentioned in [**?**], and we expect it to be precisely the simple quotient vertex algebra $V_1(\mathfrak{psl}(2|2))$.[6]

[6] In vertex algebras arising from four-dimensional superconformal field theories as in [**?**], simplicity is a consequence of four-dimensional unitarity. Though we suspect an analogous statement may hold for boundary vertex algebras arising on $\mathcal{N} = (0, 4)$ boundaries in three-dimensional unitarity theories, we do not pursue such a general result here. Where it is unproven, it will be a standing conjecture that all vertex algebras constructed in this paper by free field methods are, as in the present example, in fact simple.

We therefore conjecture that the A-twisted boundary VOA of $T[\mathrm{SU}(2)]$ is isomorphic to $V_1(\mathfrak{psl}(2|2))$. Concretely, this vertex algebra is generated by:

- Affine currents L_{ab}, which generate a $V_{-1}(\mathfrak{sl}_2)$ subalgebra.
- Affine currents $J^{\alpha\beta}$, which generate a $V_1(\mathfrak{sl}_2)$ subalgebra.
- Odd currents Θ^{α}_{Aa}, which transform in the tri-fundamental representation of the (zero modes algebras of the) previous two $\mathfrak{sl}(2)$ symmetries as well as an $\mathrm{SL}(2)_{\mathrm{o}}$ outer automorphism symmetry (the index A is the outer automorphism fundamental index).

These various currents correspond to the elements ℓ_{ab}, $\mathscr{e}^{\alpha\beta}$, and $(\vartheta_{(A)})^{\alpha}_{a}$ of the $\mathfrak{psl}(2|2)$ superalgebra, the structure of which (along with the precise OPEs for this current algebra) we provide in Appendix A.

In terms of the Chevalley generators $e = {L_1}^2$, $f = {L_2}^2$, and $h = {L_1}^1 - {L_2}^2$, the level $V_{-1}(\mathfrak{sl}(2))$ current subalgebra takes the form

$$\begin{aligned} h(z)h(w) &\sim \frac{-2}{(z-w)^2} \ , \\ h(z)e(w) &\sim \frac{2e(w)}{z-w} \ , \\ h(z)f(w) &\sim \frac{-2f(w)}{z-w} \ , \\ e(z)f(w) &\sim \frac{-1}{(z-w)^2} + \frac{h(w)}{z-w} \ . \end{aligned} \tag{4.9}$$

The OPEs between odd currents Θ^{α}_{Aa} read

$$\begin{aligned} \Theta^{\alpha}_{A1}\Theta^{\beta}_{B1} &\sim \frac{\epsilon_{AB}\epsilon^{\alpha\beta}e(w)}{z-w} \ , \\ \Theta^{\alpha}_{A1}\Theta^{\beta}_{B2} &\sim \frac{\epsilon_{AB}\epsilon^{\alpha\beta}}{(z-w)^2} - \frac{\epsilon_{AB}J^{\alpha\beta}(w) + \epsilon_{AB}\epsilon^{\alpha\beta}h(w)/2}{z-w} \ , \\ \Theta^{\alpha}_{A2}\Theta^{\beta}_{B2} &\sim -\frac{\epsilon_{AB}\epsilon^{\alpha\beta}f(w)}{z-w} \ , \end{aligned} \tag{4.10}$$

4.3 First bosonisation and free-field realisation

Recall from Section 3.2 that in order to provide a solution to the BRST problem in terms of free fields, we need to perform the following steps: first, we need to bosonise the gauge theory variables in $\mathcal{X}$ and change basis; second, we need to identify the screening charges characterising the VOA, and compute the image of the fields described in Section 4.2 in this kernel.

Change of basis.

In the case $(+-)$, the change of basis (3.27) reads

$$\begin{pmatrix} \eta \\ \widetilde{\eta} \\ \phi \\ \delta \\ \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \rho_1 \\ \rho_2 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} . \tag{4.11}$$

In this new basis, the bilinear form (3.30) encoding the norms of the chiral bosons is transformed to

$$P_{(+-)} B P^T_{(+-)} = \begin{pmatrix} 0 & -2 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} , \tag{4.12}$$

In particular, we notice that we have two bosons δ and ϕ norm

$$(\phi, \phi) = -(\delta, \delta) = 2 . \tag{4.13}$$

Inspecting the inverse change-of-basis matrix,

$$P^{-1}_{(+-)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \end{pmatrix} , \tag{4.14}$$

we see that our original (sub)lattice of charges $L^!$ takes the form

$$L^! = \mathbb{Z}\left(-\tfrac{\widetilde{\eta}}{2} + \tfrac{\phi+\delta}{2}\right) \oplus \mathbb{Z}\left(\tfrac{\widetilde{\eta}}{2} + \tfrac{\phi+\delta}{2}\right) \oplus \mathbb{Z}\left(\tfrac{\widetilde{\eta}}{2} - \omega_1\right) \oplus \mathbb{Z}\left(\tfrac{\widetilde{\eta}}{2} + \omega_2\right) . \tag{4.15}$$

The (integral) refinement of the lattice (3.34) reads,

$$L^! \subset \tilde{L}^! = \mathbb{Z}\left(\tfrac{\widetilde{\eta}}{2}\right) \oplus \mathbb{Z}\left(\tfrac{\phi+\delta}{2}\right) \oplus \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 . \tag{4.16}$$

Thus, we see that we can identify the bosonised VOA according to

$$V^!_{\tilde{L}} \cong V_\eta \otimes \widetilde{V}_{\mathrm{FFR}} , \tag{4.17}$$

where V_η is the lattice extension of the rank-two Heisenberg algebra generated by J_η and $J_{\widetilde{\eta}}$ along the lattice $\mathbb{Z}(\frac{\widetilde{\eta}}{2})$. Notice that the $\omega_{1,2}$ directions can be re-interpreted as

bosonisations of free fermions ψ_A^α,

$$\psi_1^1 \coloneqq -e^{\omega_2} \ , \qquad \psi_2^2 \coloneqq e^{-\omega_2} \ , \tag{4.18}$$

$$\psi_1^2 \coloneqq e^{-\omega_1} \ , \qquad \psi_2^1 \coloneqq e^{\omega_1} \ . \tag{4.19}$$

The OPEs for these are,

$$\psi_A^\alpha(z)\psi_B^\beta \sim \frac{\epsilon_{AB}\epsilon^{\alpha\beta}}{z-w} \ . \tag{4.20}$$

The free-field VOA V_{FFR} that is obtained after performing the BRST cohomology is then identified with the lattice VOA defined as extension of the Heisenberg currents J_δ, J_ϕ, J_{ω_1} and J_{ω_2} along all lattice directions in $L^!$ that do not contain $\tilde{\eta}$ momenta

$$\mathbb{Z}\left(\phi+\delta\right) \oplus \mathbb{Z}\left(\tfrac{\phi+\delta}{2}+\omega_1\right) \oplus \mathbb{Z}\left(\tfrac{\phi+\delta}{2}+\omega_2\right) \ . \tag{4.21}$$

Embedding of the VOA in free-field space.

The VOA corresponds to the image of $\mathcal{X}$ in V_{FFR}. Observe that the screening operators characterising this image in the new variables are as follows,

$$\begin{aligned} \mathfrak{s}_1 &= \mathrm{Res}\, e^{\rho_1} = \mathrm{Res}\, e^{\frac{1}{2}\eta+\frac{1}{2}\delta-\frac{1}{2}\omega_1+\frac{1}{2}\omega_2} \ , \\ \mathfrak{s}_2 &= \mathrm{Res}\, e^{\rho_2} = \mathrm{Res}\, e^{-\frac{1}{2}\eta+\frac{1}{2}\delta+\frac{1}{2}\omega_1-\frac{1}{2}\omega_2} \ . \end{aligned} \tag{4.22}$$

These restrict to V_{FFR},

$$\begin{aligned} \tilde{\mathfrak{s}}_1 &= \mathrm{Res}\, e^{\frac{1}{2}\delta-\frac{1}{2}\omega_1+\frac{1}{2}\omega_2} \ , \\ \tilde{\mathfrak{s}}_2 &= \mathrm{Res}\, e^{\frac{1}{2}\delta+\frac{1}{2}\omega_1-\frac{1}{2}\omega_2} \ . \end{aligned} \tag{4.23}$$

In Section 4.2 we identified a set of fields in $\mathcal{X}$, whose image in V_{FFR} belongs by construction to this kernel. As discussed therein, in this case we know that expect that the kernel should correspond to $V_1(\mathfrak{psl}(2|2))$, and that the images of these fields should provide generators upon application of the algorithm outlined in Section 3.2.4.

The details of all the steps of the algorithm are shown in Table 1, The current J_ψ appearing in the table is the Cartan generator of the additional $V_1(\mathfrak{sl}_2)$ current algebra in the free fermion system,

$$J_\psi = -\frac{\epsilon_{\alpha\beta}\psi_1^\alpha\psi_2^\beta}{2} \ . \tag{4.24}$$

We note that as an expected consequence of BRST-invariance, after changing variables, all of the fields of Section 4.2 are independent of $\widetilde{\eta}$. (It is also clear that they are in the kernel of the screening charges (4.23).) In the final step, which amounts to finding good BRST representatives, all terms involving J_η are set to zero.

It is straightforward to check that the free-field realisation provided in Table 1 reproduces the $V_1(\mathfrak{psl}(2|2))$ OPEs, and in particular (4.9) and (4.10).

	Gauge theory	Bosonisation	Change of Variables	FFR Representative
e	X_1Y^2	$e^{-\sigma_1-\sigma_2+\rho_1+\rho_2}$	$e^{\phi+\delta}$	$e^{\phi+\delta}$
f	X_2Y^1	$-J_{\rho_1}J_{\rho_2}e^{\sigma_1+\sigma_2-\rho_1-\rho_2}$	$\left((J_\psi+\frac{1}{2}J_\eta)^2-\frac{1}{4}J_\delta^2\right)e^{-\phi-\delta}$	$\left(J_\psi^2-\frac{1}{4}J_\delta^2\right)e^{-\phi-\delta}$
h	$X_1Y^1-X_2Y^2$	$J_{\sigma_1}+J_{\sigma_2}$	$-J_\phi$	$-J_\phi$
Θ^1_{11}	$-Y^2\chi_2$	$-e^{-\sigma_2+\rho_2+\gamma_2}$	$\psi_1^1 e^{\frac{1}{2}(\phi+\delta)}$	$\psi_1^1 e^{\frac{1}{2}(\phi+\delta)}$
Θ^2_{11}	$Y^2\chi_1$	$e^{-\sigma_2+\rho_2-\gamma_1}$	$\psi_1^2 e^{\frac{1}{2}(\phi+\delta)}$	$\psi_1^2 e^{\frac{1}{2}(\phi+\delta)}$
Θ^1_{21}	$X_1\xi^1$	$e^{-\sigma_1+\rho_1+\gamma_1}$	$\psi_2^1 e^{\frac{1}{2}(\phi+\delta)}$	$\psi_2^1 e^{\frac{1}{2}(\phi+\delta)}$
Θ^2_{21}	$X_1\xi^2$	$e^{-\sigma_1+\rho_1-\gamma_2}$	$\psi_2^2 e^{\frac{1}{2}(\phi+\delta)}$	$\psi_2^2 e^{\frac{1}{2}(\phi+\delta)}$
Θ^1_{12}	$Y^1\chi_2$	$J_{\rho_1}e^{\sigma_1-\rho_1+\gamma_2}$	$-\left(\frac{1}{2}(J_\delta+J_\eta)+J_\psi\right)\psi_1^1 e^{-\frac{1}{2}(\phi+\delta)}$	$-\left(\frac{1}{2}J_\delta+J_\psi\right)\psi_1^1 e^{-\frac{1}{2}(\phi+\delta)}$
Θ^2_{12}	$-Y^1\chi_1$	$-J_{\rho_1}e^{\sigma_1-\rho_1-\gamma_1}$	$-\left(\frac{1}{2}(J_\delta+J_\eta)+J_\psi\right)\psi_1^2 e^{-\frac{1}{2}(\phi+\delta)}$	$-\left(\frac{1}{2}J_\delta+J_\psi\right)\psi_1^2 e^{-\frac{1}{2}(\phi+\delta)}$
Θ^1_{22}	$X_2\xi^1$	$-J_{\rho_2}e^{\sigma_2-\rho_2+\gamma_1}$	$-\left(\frac{1}{2}(J_\delta-J_\eta)-J_\psi\right)\psi_2^1 e^{-\frac{1}{2}(\phi+\delta)}$	$-\left(\frac{1}{2}J_\delta-J_\psi\right)\psi_2^1 e^{-\frac{1}{2}(\phi+\delta)}$
Θ^2_{22}	$X_2\xi^2$	$-J_{\rho_2}e^{\sigma_2-\rho_2-\gamma_2}$	$-\left(\frac{1}{2}(J_\delta-J_\eta)-J_\psi\right)\psi_2^2 e^{-\frac{1}{2}(\phi+\delta)}$	$-\left(\frac{1}{2}J_\delta-J_\psi\right)\psi_2^2 e^{-\frac{1}{2}(\phi+\delta)}$
J_1^1	$\frac{1}{2}(\chi_1\xi^1-\chi_2\xi^2)$	$\frac{1}{2}(e^{\gamma_1}e^{-\gamma_1}-e^{-\gamma_2}e^{\gamma_2})$	$\frac{1}{2}(\psi_1^1\psi_2^2+\psi_1^2\psi_2^1)$	$\frac{1}{2}(\psi_1^1\psi_2^2+\psi_1^2\psi_2^1)$
J_2^2	$-\frac{1}{2}(\chi_1\xi^1-\chi_2\xi^2)$	$-\frac{1}{2}(e^{\gamma_1}e^{-\gamma_1}-e^{-\gamma_2}e^{\gamma_2})$	$-\frac{1}{2}(\psi_1^1\psi_2^2+\psi_1^2\psi_2^1)$	$-\frac{1}{2}(\psi_1^1\psi_2^2+\psi_1^2\psi_2^1)$
J_1^2	$\chi_1\xi^2$	$e^{\gamma_1}e^{\gamma_2}$	$\psi_1^2\psi_2^2$	$\psi_1^2\psi_2^2$
J_2^1	$\chi_2\xi^1$	$e^{-\gamma_2}e^{-\gamma_1}$	$\psi_2^1\psi_1^1$	$\psi_2^1\psi_1^1$

Table 1 Outline of derivation of the free-field realisation of $V_1(\mathfrak{psl}(2|2))$ of type $(+-)$.

Relation to known free-field realisations of the subalgebras.

It is worth dwelling upon the relations of our free-field realisations to previous results in the literature, in particular results on free-field realisations of the even subalgebra $V_{-1}(\mathfrak{sl}(2))\otimes V_1(\mathfrak{sl}(2))$ of $V_1(\mathfrak{psl}(2|2))$.

First of all, the free-field realisation of $V_1(\mathfrak{sl}(2))$ we obtain in Table 1 is given in terms of complex fermions ψ_A^α. This is a standard free-field realisation, going back to Kac-Frenkel and Segal [**?** **?**].

The free-field realisation of $V_{-1}(\mathfrak{sl}(2))$ we obtain in Table 1 is explicitly given by

$$e = e^{\phi+\delta}\ , \qquad h = -J_\phi \equiv -\partial\phi\ . \tag{4.25}$$

and

$$f = \left(-\frac{1}{4}(J_\delta)^2+T\right)e^{-\phi-\delta}\ . \tag{4.26}$$

where $T := J_\psi^2$ can be identified with the Virasoro stress tensor of $V_1(\mathfrak{sl}(2))$. It has central charge $c_T = 1$, and so obeys

$$T(z)T(w) \sim \frac{1/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w}\ . \tag{4.27}$$

Thus, the resulting realisation is given in terms of a lattice extension of a rank-two Heisenberg vertex algebra (by vertex operators of the form $e^{n(\phi+\delta)}$) of the Heisenberg algebra generated by J_ϕ and J_δ, and an auxiliary Virasoro stress tensor, which arises from the free fields realising $V_1(\mathfrak{sl}(2))$. This free-field realisation coincides with the one obtained in [**?**] (see [**?**] for the first appearance in the context of VOAs for higher dimensional supersymmetric theories).

Notice that it is precisely because the identification of the stress tensor T in (4.26) with the stress tensor of the subalgebra $V_1(\mathfrak{sl}(2))$ that is responsible for the reduction to $\mathfrak{psl}(2|2)$ from $\mathfrak{sl}(2|2)$. If this had not been the case, and an abstract stress tensor had

been used instead to free-field realise $V_{-1}(\mathfrak{sl}(2))$, there would have been a J_ψ arising from the odd currents OPEs, playing the role of the ordinary trace generator in $\mathfrak{sl}(2|2)$, and that would have needed to be removed to pass to $\mathfrak{psl}(2|2)$. So it is precisely the identification of the stres tensor that leads to the reduced current algebra.

Remark.

We notice that the lattice extensions that enters this realisation is a half-lattice vertex algebra the in the sense of [**?**], and can also be identified with the algebra of global sections of the sheaf of chiral differential operators (CDOs) [**?**] on the rank-one algebraic torus, $\mathcal{D}^{\mathrm{ch}}(\mathbb{C}^\times)$. This is the vertex algebra obtained from the symplectic boson $\beta\gamma$ VOA by formally inverting (say) γ. It was suggested in [**?**] and subsequent work that this sort of a realisation should be thought of as the starting point for writing a localisation of the vertex algebra at hand on the (Zariski) open subset of its associated variety where $e \neq 0$. In this simple example, one could indeed have guessed such a free-field realisation based on this geometric intuition. We will elaborate further on this finite-dimensional analogy in Section 6.1.1.

Conformal vectors.

The direct solution for the semi-infinite cohomology presented above allows us to express the stress tensor/conformal vector of $V_1(\mathfrak{psl}(2|2))$ in the free-field variables easily. The starting point is the stress tensor for the gauge theory degrees of freedom, including the (b, c) ghosts associated to the bulk vector multiplet,

$$T = \frac{1}{2}\left(\partial X_a Y^a - \partial Y^a X_a\right) + \frac{1}{2}(\partial\chi^a\xi_a + \partial\xi_a\chi^a) + (\partial c)b \ . \tag{4.28}$$

This operator is always BRST closed. The realisation of the same operator in $V_{L^!}$ (in terms of the redefined basis) is as follows,

$$T = T_\phi + T_\delta + T_{\omega_1} + T_{\omega_2} - \frac{1}{2}J_\eta J_{\tilde{\eta}} + (\partial c)b \ , \tag{4.29}$$

where we have the following expressions for stress tensors of the various chiral bosons appearing in the free field realisation,

$$T_\phi \ = -\frac{1}{4}J_\phi J_\phi \ , \qquad T_\delta \ = \frac{1}{4}J_\delta J_\delta - \frac{1}{2}\partial J_\delta \ , \tag{4.30}$$

$$T_{\omega_1} = \frac{1}{2}J_{\omega_1}J_{\omega_1} \ , \qquad T_{\omega_2} = \frac{1}{2}J_{\omega_2}J_{\omega_2} \ . \tag{4.31}$$

On the other hand, the stress tensor for V_η and that for the (b, c) ghosts combine to be BRST-exact,

$$-\frac{1}{2}J_\eta J_{\tilde{\eta}} + (\partial c)b = -\frac{1}{2}d\left(J_{\tilde{\eta}}b\right) \ , \tag{4.32}$$

so indeed, in V_{FFR} we have

$$T = T_\phi + T_\delta + T_{\omega_1} + T_{\omega_2} \ . \tag{4.33}$$

This assigns conformal weights $\pm\frac{1}{2}$ to $e^{\pm\frac{1}{2}(\phi+\delta)}$, while the fermions ψ^α_A are all symmetrically weighted (with conformal weight $\frac{1}{2}$). It can be confirmed by direct calculation that in our free field realisation, this is identically equal to the Sugawara stress tensor for $V_1(\mathfrak{psl}(2|2))$ as well as the sum of Sugawara stress tensors for $V_1(\mathfrak{sl}_2)$ and $V_{-1}(\mathfrak{sl}_2)$ subalgebras. Thus the null vector relating these Sugawara vectors is indeed set to zero in this vertex algebra—this reinforces the expectation that the VOA is the simple quotient.

Outer automorphisms.

Notice that the gauge group for the theory $T[\mathrm{SU}(2)]$ is signed-balanced in the sense of Proposition 2, as its charge matrix is $Q_{(+-)} = (1, -1)$. Thus, Proposition 2 implies that the free-field realisations should manifest the $\mathrm{SL}(2)_\mathrm{o}$ group of outer automorphisms of $V_1(\mathfrak{psl}(2|2))$. In fact, it can be checked explicitly that the screening charges (4.23) transform in representations of an $\mathfrak{sl}_2$ outer automorphism symmetry.

More concretely, however, observe that the free field expressions for the last four odd currents Θ^α_{A2} in Table 1 are the only one that are not written in a manner that is manifestly covariant with respect to $\mathrm{SL}(2)_o$. However, they are in fact identically equal to the covariant expressions

$$\Theta^\alpha_{A2} = \frac{1}{2}\left(\partial\psi^\alpha_A - \psi^\alpha_A\partial\delta + \frac{1}{3}\epsilon_{\beta\gamma}\epsilon^{BC}\psi^\alpha_B\psi^\beta_A\psi^\gamma_C\right)e^{-\frac{1}{2}(\phi+\delta)} \ . \tag{4.34}$$

Thus, all generators can be written in a $\mathrm{SL}(2)_\mathrm{o}$ covariant way, and the outer automorphism symmetry is manifest.

Remark.

We observe that at the level of groups, the symmetry acting on the free fermions is in fact the full $\mathrm{SU}(2)_C$, rather than the $\mathrm{SO}(3)_C$ Coulomb branch symmetry that acts faithfully in the bulk [**?** **?**]. Incidentally, the same holds for the (inner) Higgs branch flavour symmetry: the odd generators Θ^α_{Aa} transform in representations of $\mathrm{SU}(2)_H$ instead of $\mathrm{SO}(3)_H$ (the symmetry acting faithfully on bulk Higgs branch operators). It would be interesting to explore consequences of these phenomena in the context of the generalised symmetries of the bulk theory, which were recently explored in [**?**].

4.4 Second bosonisation and free-field realisation

We now briefly describe the free-field realisation of $V_1(\mathfrak{psl}(2|2))$ obtained by implementing the $(++)$ type bosonisation. The steps to obtain the free-field realisation are as in 4.3 *mutatis mutandis*, and for the sake of notational simplicity we will recycle the same symbols for the free field bosons here. We make the slightly different identification for the free fermions in terms of their lattice bosons (the difference with respect to (4.18) arises from our different conventions for bosonisation),

$$\begin{aligned} \psi^1_1 &\coloneqq -e^{-\omega_2} \ , & \psi^2_2 &\coloneqq e^{\omega_2} \ , \\ \psi^2_1 &\coloneqq \ \ e^{-\omega_1} \ , & \psi^1_2 &\coloneqq e^{\omega_1} \ , \end{aligned} \tag{4.35}$$

	Gauge theory	Bosonisation	Change of Variables	FFR Representative
e	X_1Y^2	$J_{\rho_2}e^{-(\sigma_1-\sigma_2)+(\rho_1-\rho_2)}$	$\left(\frac{1}{2}(J_\delta+J_\eta)+J_\psi\right)e^{-\phi-\delta}$	$\left(J_\psi+\frac{1}{2}J_\delta\right)e^{-\phi-\delta}$
f	X_2Y^1	$J_{\rho_1}e^{(\sigma_1-\sigma_2)-(\rho_1-\rho_2)}$	$\left(-\frac{1}{2}(J_\delta-J_\eta)+J_\psi\right)e^{\phi+\delta}$	$\left(J_\psi-\frac{1}{2}J_\delta\right)e^{\phi+\delta}$
h	$X_1Y^1-X_2Y^2$	$J_{\sigma_1}-J_{\sigma_2}$	J_ϕ	J_ϕ
Θ^1_{11}	$-Y^2\chi_2$	$-J_{\rho_2}e^{\sigma_2-\rho_2-\gamma_2}$	$\left(\frac{1}{2}(J_\delta+J_\eta)+J_\psi\right)\psi_1^1e^{-\frac{1}{2}(\phi+\delta)}$	$\left(\frac{1}{2}J_\delta+J_\psi\right)\psi_1^1e^{-\frac{1}{2}(\phi+\delta)}$
Θ^2_{11}	$Y^2\chi_1$	$J_{\rho_2}e^{\sigma_2-\rho_2-\gamma_1}$	$\left(\frac{1}{2}(J_\delta+J_\eta)+J_\psi\right)\psi_1^2e^{-\frac{1}{2}(\phi+\delta)}$	$\left(\frac{1}{2}J_\delta+J_\psi\right)\psi_1^2e^{-\frac{1}{2}(\phi+\delta)}$
Θ^1_{21}	$X_1\xi^1$	$e^{-\sigma_1+\rho_1+\gamma_1}$	$\psi_2^1e^{\frac{1}{2}(-\phi+\delta)}$	$\psi_2^1e^{-\frac{1}{2}(\phi+\delta)}$
Θ^2_{21}	$X_1\xi^2$	$e^{-\sigma_1+\rho_1+\gamma_2}$	$\psi_2^2e^{\frac{1}{2}(-\phi+\delta)}$	$\psi_2^2e^{-\frac{1}{2}(\phi+\delta)}$
Θ^1_{12}	$Y^1\chi_2$	$J_{\rho_1}e^{\sigma_1-\rho_1-\gamma_2}$	$\left(\frac{1}{2}(J_\delta-J_\eta)-J_\psi\right)\psi_1^1e^{\frac{1}{2}(\phi+\delta)}$	$\left(\frac{1}{2}J_\delta-J_\psi\right)\psi_1^1e^{\frac{1}{2}(\phi+\delta)}$
Θ^2_{12}	$-Y^1\chi_1$	$-J_{\rho_1}e^{\sigma_1-\rho_1-\gamma_1}$	$\left(\frac{1}{2}(J_\delta-J_\eta)-J_\psi\right)\psi_1^2e^{\frac{1}{2}(\phi+\delta)}$	$\left(\frac{1}{2}J_\delta-J_\psi\right)\psi_1^2e^{\frac{1}{2}(\phi+\delta)}$
Θ^1_{22}	$X_2\xi^1$	$e^{-\sigma_2+\rho_2+\gamma_1}$	$\psi_2^1e^{\frac{1}{2}(\phi+\delta)}$	$\psi_2^1e^{\frac{1}{2}(\phi+\delta)}$
Θ^2_{22}	$X_2\xi^2$	$e^{-\sigma_2+\rho_2+\gamma_2}$	$\psi_2^2e^{\frac{1}{2}(\phi+\delta)}$	$\psi_2^2e^{\frac{1}{2}(\phi+\delta)}$
J_1^1	$\frac{1}{2}(\chi_1\xi^1-\chi_2\xi^2)$	$\frac{1}{2}(e^{\gamma_1}e^{-\gamma_1}-e^{\gamma_2}e^{-\gamma_2})$	$\frac{1}{2}(\psi_1^1\psi_2^2+\psi_1^2\psi_2^1)$	$\frac{1}{2}(\psi_1^1\psi_2^2+\psi_1^2\psi_2^1)$
J_2^2	$-\frac{1}{2}(\chi_1\xi^1-\chi_2\xi^2)$	$-\frac{1}{2}(e^{\gamma_1}e^{-\gamma_1}-e^{\gamma_2}e^{-\gamma_2})$	$-\frac{1}{2}(\psi_1^1\psi_2^2+\psi_1^2\psi_2^1)$	$-\frac{1}{2}(\psi_1^1\psi_2^2+\psi_1^2\psi_2^1)$
J_1^2	$\chi_1\xi^2$	$e^{\gamma_1}e^{-\gamma_2}$	$\psi_1^2\psi_2^2$	$\psi_1^2\psi_2^2$
J_2^1	$\chi_2\xi^1$	$e^{\gamma_2}e^{-\gamma_1}$	$\psi_2^1\psi_1^1$	$\psi_2^1\psi_1^1$

Table 2 Outline of derivation of the free-field realisation of $V_1(\mathfrak{psl}(2|2))$ of type $(++)$.

and the current J_ψ is again given by the expression (4.24). We display the relevant expressions for the BRST derivation of a free field realisation in Table 2. The expressions in the final column give rise to the same $V_1(\mathfrak{psl}(2|2))$ OPEs as before, but there are some notable differences to observe here compared to the $(+-)$ case.

First, observe that the conformal weights of the free fields are much different here. This can be seen empirically from the free field realisation itself, but can be derived from first principles by analysing stress tensors. As before, the stress tensor of the gauge theory degrees of freedom can be rewritten so that up to the BRST-exact contribution $-d(J_{\tilde{\eta}}b+\partial b)$ (which is set to zero in V_{FFR}) it takes the form

$$T = T_\phi + T_\delta + T_{\omega_1} + T_{\omega_2} \tag{4.36}$$

where now there are different background charges for the various chiral bosons compared to (4.30),

$$T_\phi = -\frac{1}{4}J_\phi J_\phi\ , \qquad T_{\omega_1} = \frac{1}{2}J_{\omega_1}J_{\omega_1} + \frac{1}{2}\partial J_{\omega_1}\ , \tag{4.37}$$

$$T_\delta = +\frac{1}{4}J_\delta J_\delta\ , \qquad T_{\omega_2} = \frac{1}{2}J_{\omega_2}J_{\omega_2} + \frac{1}{2}\partial J_{\omega_2}\ . \tag{4.38}$$

The fermions are assigned asymmetric conformal weights, with ψ_1^α being of dimension zero and ψ_2^α being of dimension one. Moreover, $e^{\pm\frac{1}{2}(\phi+\delta)}$ carries conformal weight 0. This is in alignment with the finite dimensional localisation picture described above, where $e^{\phi+\delta}$ should be thought of as the chiral version of the function that parameterises a patch on the exceptional divisor, and so is weight zero under the conformal $\mathbb{C}^\times$ action on the Higgs branch (which fixes the exceptional divisor pointwise).

Finally, it is impossible to write the generators in Table 2 in a manifestly $\mathrm{SL}(2)_o$-covariant way. In fact, in this case $Q_{(++)} = (1,1)$ is not signed-balanced in the sense

of Proposition 2, and so the screening charges

$$\begin{aligned}\tilde{\mathfrak{s}}_1 &= e^{\rho_1} = e^{-\frac{1}{2}\delta-\frac{1}{2}\omega_1-\frac{1}{2}\omega_2} \ , \\ \tilde{\mathfrak{s}}_2 &= e^{\rho_2} = e^{\frac{1}{2}\delta-\frac{1}{2}\omega_1-\frac{1}{2}\omega_2} \ .\end{aligned} \tag{4.39}$$

do not transform in a representation of the outer automorphism group.

4.5 The A_1 singularity as associated variety

We end the treatment of this simple example by determining the associated variety of $V_1(\mathfrak{psl}(2|2))$ and prove that it is isomorphic to the Higgs branch $\mathcal{M}_H$ of the theory, that is the A_1 singularity, corroborating Conjecture 1. This provides a motivation for the finite-dimensional analogy of our bosonisation techniques that we will develop in Section 5, and apply to this case in particular in Section 6.1.

It follows from completely general considerations (due to the nilpotency of Grassmann-odd Θ generators at the level of Zhu's C_2 algebra) that we will have

$$X_{V_1(\mathfrak{psl}(2|2))} \subseteq X_{V_1(\mathfrak{sl}_2)} \times X_{V_{-1}(\mathfrak{sl}_2)} \cong \{pt.\} \times \mathfrak{sl}_2^* \ . \tag{4.40}$$

To proceed further requires some information about the null states of this VOA that do not lie in the $V_1(\mathfrak{sl}_2)$ subalgebra. To find the state we need, it is sufficient to observe the existence of a conformal embedding [**?** **?**],

$$V_1(\mathfrak{sl}_2) \otimes V_{-1}(\mathfrak{sl}_2) \hookrightarrow V_1(\mathfrak{psl}(2|2)) \ . \tag{4.41}$$

This is equivalent to the identity (in the simple quotient) between various Sugawara stress tensors,

$$T_{\text{sug}}^{\mathfrak{psl}(2|2)_1} = T_{\text{sug}}^{\mathfrak{sl}(2)_1} + T_{\text{sug}}^{\mathfrak{sl}(2)_{-1}} \ . \tag{4.42}$$

Let us denote the *un-normalised* Segal–Sugawara vectors for the current algebra of type $\hat{\mathfrak{g}}_k$ by

$$S_{\mathfrak{g}_k} = \kappa_{AB}(J^A J^B) \ , \tag{4.43}$$

where κ_{AB} is the Killing form on $\mathfrak{g}$. The Sugawara stress tensor is given by

$$T_{\mathfrak{g}_k} = \frac{k \dim \mathfrak{g}}{k + h^\vee_{\mathfrak{g}}} S_{\mathfrak{g}} \ . \tag{4.44}$$

Then the null relation (4.42) becomes (substituting in the expression for the $\mathfrak{psl}(2|2)$ Segal–Sugawara vector) the following relation,

$$S_{\mathfrak{psl}(2|2)} = -2\left(S_{\mathfrak{sl}(2)_1} + \Theta^\alpha_{Aa}\Theta^{Aa}_\alpha - S_{\mathfrak{sl}(2)_{-1}}\right) = S_{\mathfrak{sl}(2)_1} - 3S_{\mathfrak{sl}(2)_{-1}} \ , \tag{4.45}$$

which we rewrite as

$$S_{\mathfrak{sl}(2)_{-1}} = \frac{3}{5} S_{\mathfrak{sl}(2)_1} + \frac{2}{5}\Theta^\alpha_{Aa}\Theta^{Aa}_\alpha \ . \tag{4.46}$$

The right hand side is nilpotent in the Zhu C_2 algebra, so we see that the Segal–Sugawara vector of the $\mathfrak{sl}(2)_{-1}$ subalgebra also vanishes at the level of the associated variety. In particular, this implies the inclusion

$$X_{V_1(\mathfrak{psl}(2|2))} \subseteq \overline{\mathbb{O}_{\min}(\mathfrak{sl}_2)} \subset \mathfrak{sl}_2^* \ , \tag{4.47}$$

As the associated variety must admit an SL(2) action descending from the zero modes of the level -1 affine currents, it must be either the entirety of the nilpotent orbit closure above or a point.

One can see easily that the associated variety cannot be zero-dimensional. The state $(L_{11})^n$ is non-null for all n; moreover is the unique state of conformal dimension n and charge n with respect to L_{12} so cannot participate in a null relation with any other states in the VOA. Thus we have the identification

$$X_{V_1(\mathfrak{psl}(2|2))} \cong \overline{\mathbb{O}_{\min}(\mathfrak{sl}_2)} \cong \mathcal{M}_H \ . \tag{4.48}$$

5 Finite-dimensional considerations

We now turn to the development of an instructive *finite-dimensional* (non-chiral) analogy of the solution to the BRST reduction presented above. The guiding principle is that according to Conjecture 1, the associated variety of a boundary VOA of the type considered here is expected to be isomorphic as a Poisson variety to the Higgs branch $\mathcal{M}_H$ of the respective theory. We then expect that a solution to the BRST problem given in terms of free fields obtained from a certain bosonisation prescription corresponds to a localisation on a particular Zariski open subset of the associated variety.

In VOAs arising in connection with four-dimensional SCFTs, these open subsets have been found to often take the form $T^*\left(\mathbb{C}^\times\right)^m \times T^*\left(\mathbb{C}\right)^n$, endowed with a standard symplectic structure. The finite-dimensional procedure in the present examples will lead to just such a result. As a byproduct, this analogy will provide insights on the possible origin of the manifestation of outer automorphisms in the free-field realisations observed in Section 3.3, and on possible "de-bosonisation" techniques that we will further explore in the examples of Section 7.

This section is structured as follows. In Section 5.1, we review how in general a given bosonisation prescription corresponds to restriction to a subspace $T^*\left(\mathbb{C}^\times\right)^N \subset T^*\mathbb{C}^N$ of the free hypermultiplet Higgs branch prior to performing symplectic reduction, which then results in the cotangent bundle $T^*\left(\mathbb{C}^\times\right)^{N-k}$ (we recall here that N and k are the number of hypermultiplets and the rank of the gauge group, respectively). In Section 5.2, working in terms of hyperplane arrangements, we show that for each bosonisation, for some choice of resolution parameter $\zeta \in \mathfrak{t}_G^*$ the "reduced torus" $T^*\left(\mathbb{C}^\times\right)^{N-k}$ can be identified with an open subset of the resolved Higgs branch $\mathcal{M}_{H,\zeta}$. We also provide a possible finite-dimenensional explanation of the manifestation of

outer automorphisms observed in Section 3.3. In Section 5.3, we show that for particular choices of bosonisation, the reduced torus enjoys an open immersion into the (unresolved) singular affine variety $\mathcal{M}_H$ itself. In Section 5.4, we take a mirror perspective and we relate these embeddings to those arising from restriction functors for the (mirror) Coulomb branch chiral rings discussed in [**?**]. (This perspective has the additional benefit of predicting when some of the $T^*\mathbb{C}^\times$ components can be extended across the origin, providing open subsets of the form $T^*\left(\mathbb{C}^\times\right)^m \times T^*\left(\mathbb{C}\right)^n$. This may suggest a potential for *de-bosonisation* of some free field realisations in terms of symplectic bosons, *cf.* 7.1.4.)

Throughout this section, all groups T_G, T_H, T_C, T_V will be complexified unless otherwise stated and we omit the subscript $\mathbb{C}$. The corresponding complex Lie algebras will similarly be denoted by $\mathfrak{t}_G$, $\mathfrak{t}_H$, $\mathfrak{t}_C$, $\mathfrak{t}_V$.

Remark.

It would be interesting to incorporate the free fermion degrees of freedom into the finite-dimensional analogue discussed here by considering the symplectic reduction of $V \oplus \Pi V$, where ΠV is the Grassmann-odd vector space where the free fermions live. The physical interpretation of such an analogy is somewhat unclear to us, but mathematically it may be relevant for understanding the interplay between boundary fermionic degrees of freedom and localisation on the associated variety.

5.1 Hypertoric symplectic reduction

To a bosonisation of the symplectic bosons (X_i, Y^i) we associate a finite-dimensional (non-chiral) analogue. Namely, considering the corresponding functions (Y^i, X_i) on $T^*\mathbb{C}$, we can choose to identify two different subspaces $T^*\mathbb{C}^\times \subset T^*\mathbb{C}$ by inverting either X_i or Y^i. Given N symplectic boson pairs, a choice of bosonisation is encoded in a sign vector ϵ, and we identify this with a choice of subset $T^*\left(\mathbb{C}^\times\right)^N \subset T^*V \cong T^*\left(\mathbb{C}\right)^N$, where either X_i or Y^i is inverted according to the sign ϵ_i (X_i if $+$, Y^i if $-$). As a matter of convenience, throughout this section we will utilise an equivalent way of encoding these choices in terms of subsets $U \subset \{1, \ldots, N\}$, indicating that we invert Y^i if $i \in U$ and X_i if $i \notin U$. Thus, for a given choice of bosonisation ϵ

$$U \coloneqq \left\{ i \in \{1, \ldots, N\} \mid \epsilon_i = - \right\} . \tag{5.1}$$

We denote the corresponding subset $T^*\left(\mathbb{C}^\times\right)^N \subset T^*\mathbb{C}^N$ by T^*V_U. This is a holomorphic symplectic variety with symplectic form pulled back from T^*V. It carries an action of T_G and we can consider the corresponding holomorphic symplectic quotient.

Lemma 1. *The symplectic reduction of T^*V_U by T_G is isomorphic as a holomorphic symplectic variety to $T^*T_H \cong T^*\left(\mathbb{C}^\times\right)^{N-k}$.*

Proof. We proceed by defining appropriate generators for functions on the torus T^*V_U. In particular, we will see that

$$\mathbb{C}\left[T^*V_U\right] \cong \mathbb{C}\left[T^*T_G\right] \otimes \mathbb{C}\left[T^*T_H\right] \ , \tag{5.2}$$

and the result follows immediately.

We include amongst our generators $\mu^a_{\mathbb{C}}$, $a = 1, \ldots, k$, and $\mu^b_{H,\mathbb{C}}$, $b = 1, \ldots N-k$. Then for a $\mathbb{Z}^N$-vector w with components w_i, we define

$$W_U^w := \prod_{i \in U} (X_i)^{w_i} \prod_{i \notin U} Y^{i\,-w_i} \ . \tag{5.3}$$

For any w, W_U^w is invertible by construction. We define the additional functions

$$W^a_{U,H} := W^{\widetilde{Q}_a} \ , \qquad W^b_U := W^{\widetilde{q}_b} \ , \tag{5.4}$$

for $a = 1, \ldots, k$, $b = 1, \ldots, N-k$. Then one has for Poisson brackets,[7]

$$\{\mu^a_{\mathbb{C}}, {W_U}^a\} = {W_U}^a \ , \qquad \{\mu^a_{H,\mathbb{C}}, W^b_{U,H}\} = W^b_{U,H} \ , \tag{5.6}$$

with all other brackets vanishing identically. Thus we have the following identifications of Poisson algebras,

$$\begin{aligned}
&\mathbb{C}\left[\mu^1_{\mathbb{C}}, \ldots, \mu^k_{\mathbb{C}}, W^1_U, \ldots W^k_U, \left(W^1_U\right)^{-1}, \ldots, \left(W^k_U\right)^{-1}\right] \cong \mathbb{C}\left[T^*T_G\right] \ , \\
&\mathbb{C}\left[\mu^1_{H,\mathbb{C}}, \ldots, \mu^{N-k}_{H,\mathbb{C}}, W^1_{U,H}, \ldots, W^{N-k}_{U,H}, \left(W^1_{U,H}\right)^{-1}, \ldots, \left(W^{N-k}_{U,H}\right)^{-1}\right] \cong \mathbb{C}\left[T^*T_H\right] \ .
\end{aligned} \tag{5.7}$$

Then (5.2) follows immediately upon noting that:

- As $\widetilde{Q}$ is unimodular by assumption, a monomial built from the W^a_U and $W^b_{U,H}$ will recover Y^i, $(Y^i)^{-1}$ for $i \in U$ and X_i, X_i^{-1} for $i \notin U$;
- Every monomial $Y^i X_i$ can be written as a linear combination of the $\mu^a_{\mathbb{C}}$ and $\mu^a_{H,\mathbb{C}}$.

□

5.2 Embedding localised reductions

We have seen that for each choice of localisation before symplectic reduction, the resulting space can be identified with T^*T_H. Depending on the localisation, however, these reductions will come with different maps into $\mathcal{M}_H$ and its resolutions. In this section we will show that in general, the localised reductions can be understood as open subsets of the resolutions $\mathcal{M}_{H,\zeta}$ for appropriate choices of resolution ζ. We will also see that for certain choices of localisation, the localised reduction is an open subset of (the smooth locus) of the singular Higgs branch $\mathcal{M}_H$.

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[7] Here we use the following elementary relations amongst (mirror) charge vectors,

$$\begin{aligned}
\widetilde{Q}^T Q &= \mathbf{0}_{(N-k)\times(N-k)} \ , & \widetilde{Q}^T q &= \mathbf{1}_{(N-k)\times(N-k)} \ , \\
\widetilde{q}^T q &= \mathbf{0}_{k\times k} \ , & \widetilde{q}^T Q &= \mathbf{1}_{k\times k} \ .
\end{aligned} \tag{5.5}$$

In the first instance, we will demonstrate that for each U there exists a resolution parameter ζ such that the symplectic reduction T^*T_H of T^*T_U enjoys an open immersion into $\mathcal{M}_{H,\zeta}$.

Let us fix a resolution parameter ζ so that $\mathcal{M}_{H,\zeta}$ is fully resolved, and pick as in (2.16) a lift via $\widetilde{q}$ that in an abuse of notation, we will also denote this lift by ζ. The real and complex moment maps for the flavour action T_H on $\mathcal{M}_{H,\zeta}$ are given by

$$\begin{aligned} \mu^a_{H,\mathbb{C}} &= \sum_{i=1}^N q_i^a Y^i X_i \ , & a \in \{1,\dots,N-k\} \ , \\ \mu^a_{H,\mathbb{R}} &= \sum_{i=1}^N \frac{q_i^a}{2}\left(|X_i|^2 - |Y^i|^2 - \zeta^i\right) \ , & a \in \{1,\dots,N-k\} \ . \end{aligned} \tag{5.8}$$

These lead to a fibration,

$$(\pi_{\mathbb{C}}, \pi_{\mathbb{R}}) : \mathcal{M}_{H,\zeta} \to \mathfrak{t}^*_{H,\mathbb{C}} \oplus \mathfrak{t}^*_{H,\mathbb{R}} \ , \tag{5.9}$$

with generic fibres $\left(S^1\right)^{N-k}$ that degenerate along hyperplanes where $Y^i = X_i = 0$ (*cf.* [**?**]). If we impose the constraints $Y^i \neq 0$ or $X_i \neq 0$ for every i before symplectic reduction, then we will have removed all of the degeneration loci of this fibration (and more). What remains will be open subsets that (if non-empty) are isomorphic to $\left(S^1 \times \mathbb{R}^3\right)^{N-k} \cong T^*\left(\mathbb{C}^\times\right)^{N-k}$. Whether these subsets are actually non-empty or not can depend on the value of ζ.

We can treat this in greater detail using a description of hypertoric varieties in terms of cotangent bundles to toric varieties, glued together in a combinatorial fashion. The relation to toric geometry arises by first restricting to the zero set of the complex moment map, also known as the extended core

Definition 5.1. The *extended core* $\mathcal{E}$ is the subset $\pi_{\mathbb{C}}^{-1}(0) \subset \mathcal{M}_{H,\zeta}$.

Equivalently, $\mathcal{E}$ consists of those points $p \in \mathcal{M}_{H,\zeta}$ such that for each i, $X_i Y^i = 0$ at p. Now let $A \subset \{1,\dots N\}$ and define

$$E_A \coloneqq \{(Y^1, X_1, \dots, Y^N, X_N) \in T^*V \mid X_i = 0 \text{ if } i \in A \text{ and } Y^i = 0 \text{ if } i \notin A\} \tag{5.10}$$

(note that here we are requiring Y^i or X_i to be zero, rather than non-vanishing). Then we have that

$$X_A \coloneqq E_A /\!\!/_\zeta T_G \ , \tag{5.11}$$

are $(N-k)$-dimensional Kähler, Lagrangian subvarieties of $\mathcal{M}_{H,\zeta} = \mathcal{M}(\mathcal{A}_\zeta)$. For example, if $A = \emptyset$, then

$$X \coloneqq X_\emptyset = \mathbb{C}^N /\!\!/_\zeta T_G \ , \tag{5.12}$$

can be interpreted as the fixed-point set of the natural $\mathbb{C}^\times$-action on V that rescales the cotangent fibres

$$X_\emptyset = \mathcal{M}(\mathcal{A}_\zeta)^{\mathbb{C}^\times} \ . \tag{5.13}$$

The collection of these Kähler Lagrangian subvarieties cover $\mathcal{E}$. Furthermore, their normal bundle in M_{H_ζ} can be identified with their cotangent bundle (as the subvarieties are Lagrangian), and the collection of T^*X_A covers the hypertoric variety $\mathcal{M}_{H,\zeta}$ itself. Additionally, the X_A themselves are actually toric varieties, and their gluing can is described in a completely combinatorial fashion.

Key to these facts is the identification of the codomain of the fibration (5.9), restricted to $\mathfrak{t}^*_{H,\mathbb{R}}$, with the ambient space of the hyperplane arrangement $\mathcal{A}_\zeta$. Upon this identification, one has that

$$\pi_{\mathbb{R}}^{-1}(H_i) = \{p \in \mathcal{M}_{H,\zeta} \mid |X_i|^2 - |Y^i|^2 = 0 \} \, . \tag{5.14}$$

This implies that X_A can be identified with the pre-image of the polyhedron

$$P_A \coloneqq \{x \in \mathfrak{t}^*_{H,\mathbb{R}} \mid x \cdot \widetilde{Q}_i + \zeta^i \leqslant 0 \text{ if } i \notin A \;\text{ and }\; x \cdot \widetilde{Q}_i + \zeta^i \geqslant 0 \text{ if } i \in A\} \, . \tag{5.15}$$

It follows that the Kähler variety X_A is a toric variety classified by the respective polyhedron P_A (as essentially follows from the fibration structure). Some of the polyhedra P_A could be empty, but their union covers $\mathfrak{t}^*_{H,\mathbb{R}}$. Moreover, the gluing of the Kähler varieties X_A is dictated by the intersections of the respective polyhedra. The above observations are summarised in the following:

Lemma 2 (Remark 2.1.6 of [**?**])**.** *Let $\mathcal{A}_\zeta$ be a hyperplane arrangement determining a hypertoric variety $\mathcal{M}(\mathcal{A}_\zeta)$. Let $A \subset \{1, \dots N\}$ and P_A be the polyhedra defined in (5.15). Then $\mathcal{M}(\mathcal{A}_\zeta)$ can be described in terms of a union of cotangent bundles T^*X_A to toric varieties X_A classified by P_A, glued together equivariantly and symplectically based on their intersections in $\mathfrak{t}^*_{H,\mathbb{R}}$.*

This description of hypertoric varieties is a useful starting point for our analysis of the localisations determined by U. Suppose that we remove the coordinate hyperplanes $X_i = 0$ or $Y^i = 0$. If we impose $Y^i \neq 0$, on the extended core we necessarily have $X_i = 0$, and *vice-versa*. This means that by imposing a choice of localisation U, in $\mathfrak{t}^*_{H,\mathbb{R}}$ we are left with the interior a single polyhedron,

$$O_U \coloneqq \{x \in \mathfrak{t}^*_{H,\mathbb{R}} \mid x \cdot \widetilde{Q}^i + \zeta^i > 0 \text{ if } i \notin U \;\text{ and }\; x \cdot \widetilde{Q}^i + \zeta^i < 0 \text{ if } i \in U\} \, . \tag{5.16}$$

Now for a given U, it remains to be shown that there exists a choice of ζ so that O_U is the toric diagram for $(\mathbb{C}^\times)^{N-k}$. In other words, we want to prove the following:

Lemma 3. *For each U, there exists a (not necessarily unique) ζ such that O_U is non-empty and classifies a toric variety $T_{U,\zeta} \cong (\mathbb{C}^\times)^{N-k} \cong T_H$.*

Notice that it then follows that there is an open embedding $T^*T_{U,\zeta} \hookrightarrow \mathcal{M}_{H,\zeta}$.

Proof. Since O_U has no faces, it suffices to prove that there is a lift of $\zeta = (\zeta^1, \dots, \zeta^k)$ such that O_U is non-empty. The existence of such a ζ can for instance be argued in terms of bounded linear functions on Gale dual arrangements [**?**].

Alternatively, chasing through the definitions, this is equivalent to showing that for a fixed ϵ there are at least a $\zeta = (\zeta^1, \ldots, \zeta^k)$ and a point $(x^1, \ldots, x^{N-k}) \in \mathfrak{t}^*_{H,\mathbb{R}} \cong \mathbb{R}^{N-k}$ such that the N inequalities

$$\left(\tilde{q}_\epsilon^T \ \tilde{Q}_\epsilon^T\right) \left(\zeta^1, \ \ldots, \zeta^k, \ x^1, \ \ldots, x^{N-k}\right)^T > 0 \tag{5.17}$$

are satisfied. But since $\left(\tilde{q}_\epsilon^T \ \tilde{Q}_\epsilon^T\right)$ has full rank by assumption, this is guaranteed. □

5.3 Embeddings into singular varieties

We next want to ask whether a resolution is at all necessary. In the language of the previous section, this is equivalent to asking whether setting some of the components of ζ to zero (thus taking some hyperplanes in $\mathfrak{t}^*_{H,\mathbb{R}}$ to pass through the origin) ruins the identification of O_U as classifying the toric variety T_H.

To put this question into context, it is useful to introduce the notion of the core. Suppose that for a given U we can find a ζ satisfying the requirements of the previous section. That is, we have made a choice of localisation U and found an open immersion $T^*T_H \hookrightarrow \mathcal{M}_{H,\zeta}$. Now there is always a surjective map,

$$\mathcal{M}_{H,\zeta} \twoheadrightarrow \mathcal{M}_H \ , \tag{5.18}$$

given by affinisation. This map is in fact an equivariant orbifold resolution. One then defines the *core* as follows,

Definition 5.2. The core $L(\mathcal{A}_\zeta) \subset \mathcal{M}_{H,\zeta} \cong \mathcal{M}(\mathcal{A}_\zeta)$ is the pre-image of the most singular point $0 \in \mathcal{M}_H$ under the surjective map (5.18).

For our purposes, it is best to express the core in terms of the polyhedra P_A. In fact, the following simple characterisation holds [**?**],

$$L(\mathcal{A}_\zeta) = \bigcup_{P_A \text{ bounded}} X_A \ , \tag{5.19}$$

where the bounded polyhedra are by definition the polyhedra of finite volume. Now it is clear that whenever we set some of the components of ζ to zero, some of the bounded polyhedra will degenerate. The question of whether the open subsets $T^*T_{U,\zeta} \cong T^*T_H$ remain openly embedded in $\mathcal{M}_H$ corresponds to the question of whether the respective polyhedra degenerate or not.

This question is relevant for the manifestation (at the level of free-field realisations) of the enhanced outer automorphism group of the A-twisted boundary VOAs observed in Proposition 2 of Section 3.3. We noted therein that for a theory with a balanced gauge group (a $U(1)_S$ subgroup of the gauge symmetry T_G whose action on $\mathbb{C}^N \subset V$ has only two non-zero weights ± 1), the enhanced outer automorphism group is manifest only if the $U(1)_S$ subgroup is *sign*-balanced with respect to a choice of localisation U (or equivalently bosonisation ϵ). Translated into the language of this section, we

speculated that this might be due to the necessity of resolving $\mathcal{M}_H$ by a resolution parameter ζ associated to $U(1)_S$ in order of O_U to be non-empty. We now show that this is the case.

Lemma 4. *Suppose that in (T_G, V) there is a balanced gauge group $U(1)_S \subset T_G$, so that without loss of generality Q has the first row equal to $(1, 1, 0, \dots 0)$. Then there exists a resolution parameter ζ satisfying $\zeta^1 = 0$ so that O_U is non-empty only if the respective sign-vector ϵ is of the form $(+ - \dots)$ or $(- + \dots)$, and in particular $U(1)_S$ is sign-balanced with respect to ϵ in the sense of Proposition 2.*

Proof. We now from the proof of Lemma 3 that O_U is non-empty if and only if

$$\left(\tilde{q}_\epsilon^T \;\; \tilde{Q}_\epsilon^T\right) \left(\zeta^1, \dots, \zeta^k, \, x^1, \dots, x^{N-k}\right)^T > 0 \tag{5.20}$$

for some $\zeta = (\zeta^1, \dots, \zeta^k)$ and a point $(x^1, \dots, x^{N-k}) \in \mathfrak{t}^*_{H,\mathbb{R}} \cong \mathbb{R}^{N-k}$. Now notice that $\left(\tilde{q}_\epsilon^T \;\; \tilde{Q}_\epsilon^T\right) = \tilde{\mathbf{Q}}_\epsilon^T$, and by (2.5)

$$\mathbf{Q}_\epsilon \tilde{\mathbf{Q}}_\epsilon^T = \mathbf{1}_{N\times N} \; . \tag{5.21}$$

Thus

$$\mathbf{Q}_\epsilon \tilde{\mathbf{Q}}_\epsilon^T \left(\zeta^1, \dots, \zeta^k, \, x^1, \dots, x^{N-k}\right)^T = \left(\zeta^1, \dots, \zeta^k, \, x^1, \dots, x^{N-k}\right)^T \; . \tag{5.22}$$

From this we can derive $\zeta^1 = \epsilon_1 a + \epsilon_2 b$ for some $a, b > 0$, and this can be satisfied for $\zeta^1 = 0$ only if $\epsilon = (+ - \cdots)$ or $\epsilon = (- + \dots)$. □

Finally, it is interesting to ask under which conditions U determines a non-degenerate chamber O_U even if *all* components of ζ are set to zero. Suppose for simplicity that $U = \{1, \dots N\}$ and consider the hyperplane arrangement $\mathcal{A}_{\zeta=0}$. If there is $x \in \mathbb{R}^{N-k} \cong \mathfrak{t}^*_{H,\mathbb{R}}$ such that $x \cdot \widetilde{Q}^i > 0$ for all $i \in \{1, \dots, N\}$, then by the proof of Lemma 3 indeed O_U will be non-empty. For other choices of U, one simply has to adjust the signs in the previous inequalities. The goal of our next subsection will be to understand this condition from the perspective of restriction functors on (mirror) Coulomb branch chiral rings, as described in [**?**].

5.4 The mirror Coulomb branch perspective

Still in the unresolved case, we can adopt a mirror perspective and make contact with some general results of [**?**] related to restriction functors on Coulomb branches. More precisely, we can view $\mathcal{M}_H$ as the Coulomb branch of the mirror theory defined by the charge matrix $\widetilde{Q}$ (we denote this by $\mathcal{M}_C[\widetilde{Q}]$). Then the following lemma, which is a simple consequence of Theorem 2.9 of [**?**], applies.

Lemma 5. *Let $\mathbb{C}[\mathcal{M}_C[\widetilde{Q}]]$ be the Coulomb branch chiral ring associated to a an Abelian theory with charge matrix $\widetilde{Q}$. Let A be a co-character (defining a monopole operator v_A) that pairs non-negatively with the weights encoded in $\widetilde{Q}$,*

$$\langle A, \widetilde{Q}^i \rangle \geqslant 0 \; . \tag{5.23}$$

Then there is an isomorphism

$$\mathbb{C}\left[\mathcal{M}_C[\widetilde{Q}]\right][v_A^{-1}] \cong \mathbb{C}\left[\mathcal{M}_C[\widetilde{Q}^{(0)}]\right] \ , \tag{5.24}$$

where $\widetilde{Q}^{(0)}$ is the representation obtained by restricting to those weights whose pairing with A is zero.

In general, there is an explicit homomorphism from $\mathbb{C}[\mathcal{M}_C[\widetilde{Q}]]$ to $\mathbb{C}[\mathcal{M}_C[\widetilde{Q}^{(0)}]]$ (see [**?**]),

$$v_A \mapsto \prod_{\langle \widetilde{Q}^i, A\rangle < 0} \prod_{j=\langle \widetilde{Q}^i, A\rangle}^{-1} \varphi^{\widetilde{Q}^i} w_A \ , \tag{5.25}$$

where the operators w_A satisfy the monopole operator algebra of the new theory. The above Lemma from [**?**] tells us that under the given assumptions, upon inverting the monopole operator v_A this map becomes an isomorphism.

Now suppose that $\langle A, \widetilde{Q}^i\rangle > 0$ for all i. Then the above lemma tells us that there is an open subset of $\mathcal{M}_C[\widetilde{Q}] \cong \mathcal{M}_{H,\zeta}$ that can be identified with the Coulomb branch of a rank-$(N-k)$ pure gauge theory. This Coulomb branch is isomorphic as a holomorphic symplectic variety to $T^*\left(\mathbb{C}^\times\right)^{N-k}$, and so we see some alignment with our results from the previous section; indeed, the existence of such a co-character A is equivalent to the statement that the interior of the chamber $O_\emptyset$ is non-degenerate.

If for some i $\langle A, \widetilde{Q}^i\rangle < 0$ instead, then we can consider the equivalent theory defined by a charge matrix with $\widetilde{Q}^i$ replaced by $-\widetilde{Q}^i$. If $\langle A, \widetilde{Q}^i\rangle = 0$, however, then we may not be able to find such an openly embedded $T^*(\mathbb{C}^\times)^{N-k}$ in $\mathcal{M}_C[\widetilde{Q}]$. Indeed, based on our results in the previous sections, we expect that generically we will be forced to introduce deformation parameters in order to find such patches.

This mirror perspective highlights an interesting additional feature compared to what followed from our previous considerations. Namely, there is a special case in which, for a given co-character A, $\widetilde{Q}^{(0)}$ turns out to define $m \leqslant N-k$ copies of a U(1) gauge theory a single hypermultiplet of charge one (SQED[1]). This theory mirror to a free hypermultiplet, so the Coulomb is just $T^*\mathbb{C}$. Thus, by inverting the monopole operator defined by A one finds the ring of functions on $T^*\mathbb{C}^m \times T^*(\mathbb{C}^\times)^{N-k-m}$. While we will not explore this feature in full generality, we will see some examples of this phenomenon in relation to our free-field construction of the boundary VOA of SQED[N].

6 Examples of openly embedded subsets

In this section we apply the general considerations of Section 5 to an interesting set of examples. In Section 6.1 we study the A_1 singularity and relate its geometry to our findings of Section 4. We then generalise this example for $N > 1$ to the A_{N-1} singularities (Section 6.2) and minimal nilpotent orbit closures in $\mathfrak{sl}(N)^*$ (Section 6.3), which will be relevant for the VOAs studied in Section 7.

6.1 The A_1 singularity

We start with a thorough investigation of the example relevant for $T[\mathrm{SU}(2)]$. We recall the computation of the Higgs branch/Higgs branch chiral ring via holomorphic symplectic reduction (see (4.3)),

$$\mathbb{C}[\mathcal{M}_H] = \left(\mathbb{C}[T^*\mathbb{C}^2]/\langle X_1Y^1 + X_2Y^2\rangle\right)^{T_{G,\mathbb{C}}} \cong \mathbb{C}[A_1] \ . \tag{6.1}$$

In Sections 4.3 and 4.4 we identified two possible bosonisations (up to equivalence), labelled by the sign vectors $(+-)$ and $(++)$. According to the finite-dimensional analogy of Section 5, these correspond to two openly embedded subsets in resolutions of the A_1 singularity, which we now discuss. Since this example is sufficiently simple, we do this very concretely at the level of rings.

6.1.1 Localisation of type $(+-)$ and symplectic reduction

In the finite-dimensional analogy we have been developing in this section, the bosonisation $(+-)$ utilised in Section 4.3 corresponds to removing the hyperplanes $X_1 = 0$ and $Y^2 = 0$ from $T^*\mathbb{C}^2$ before performing symplectic reduction. At the level of rings of functions, this leaves us with the localisation

$$\mathbb{C}\left[T^*(\mathbb{C}^\times)^2\right] = \mathbb{C}\left[\left(Y^2\right)^{\pm 1}, (X_1)^{\pm 1}, Y^1, X_2\right] \ . \tag{6.2}$$

Thus, Lemma 1 in this case reduces to the statement that

$$\begin{aligned}\mathbb{C}\left[T^*(\mathbb{C}^\times)^2\right] &\cong \mathbb{C}\left[X_1Y^1 + X_2Y^2, \left(Y^2\right)^{\pm 1}\right] \otimes \mathbb{C}\left[X_1Y^1, \left(X_1Y^2\right)^{\pm 1}\right] \\ &\cong \mathbb{C}\left[T^*T_{G,\mathbb{C}}\right] \otimes \mathbb{C}\left[T^*\mathbb{C}^\times\right] \ .\end{aligned} \tag{6.3}$$

Here we have identified $X_1Y^1 + X_2Y^2$ with the cotangent directions to $T_{G,\mathbb{C}}$ via the complex moment map, (Y^2) with the standard coordinate on $T_{G,\mathbb{C}}$ (as it has weight $+1$), and the Poisson brackets follow from (4.2). On the other hand, the coordinates on the complementary $T^*\mathbb{C}^\times$ are gauge invariant, so symplectic reduction just gives

$$R_{(+-)} \coloneqq \mathbb{C}\left[T^*\mathbb{C}^\times\right] \ , \tag{6.4}$$

with coordinate functions

$$\widetilde{e}^{\pm 1} \coloneqq (X_1Y^2)^{\pm 1} \ , \qquad \widetilde{h} \coloneqq X_1Y^1 = \frac{1}{2}(X_1Y^1 - X_2Y^2) \ . \tag{6.5}$$

There is a Poisson algebra homomorphism

$$\begin{aligned} R_H &\hookrightarrow R_{(+-)} \ , \\ (e,\, h,\, f) &\mapsto \left(\widetilde{e},\, \widetilde{h},\, -\frac{\widetilde{h}^2}{4\widetilde{e}}\right) \ , \end{aligned} \tag{6.6}$$

and from this we see that there is an isomorphism of Poisson algebras,

$$R_{(+-)} \cong R_H[e^{-1}] \ . \tag{6.7}$$

This identifies the spectrum of $R_{(+-)}$ with an open subvariety of (the smooth locus of) the Higgs branch,

$$T^*\mathbb{C}^\times \hookrightarrow \mathcal{M}_H \ . \tag{6.8}$$

This is precisely the finite-dimensional geometric construction that has been proposed to underlie many of the four-dimensional free field realisations discussed elsewhere by one of the authors.

6.1.2 The mirror Coulomb branch perspective

In light of the results of Section 5.4, the above localisation can be interpreted from a mirror viewpoint. Recall from Section 4 that $T[\mathrm{SU}(2)]$ is self-mirror. It follows that we can identify $\mathcal{M}_H$ with the Coulomb branch $\mathcal{M}_C$ of the same theory. In this case (up to some choices of normalisation), the isomorphism outlined in (2.22) and (2.23) reads

$$\begin{pmatrix} \frac{1}{2}h \\ e \\ f \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} \varphi \\ v_+ \\ v_- \end{pmatrix} \ , \tag{6.9}$$

where $v_\pm$ are monopole operators defined by co-characters $A = (\pm 1)$. In particular, e is identified the monopole operator v_1 associated to the co-character $A = (1)$.

As in Section (6.1.2), let us denote by $w_{\pm 1}$ the elementary monopole operators of a pure U(1) gauge theory. These satisfy the simple relation

$$w_1 w_{-1} = 1 \ , \tag{6.10}$$

and along with the vector multiplet scalar, φ, can be identified with coordinate functions on $T^*\mathbb{C}^\times$. The homomorphism (6.6) can be interpreted as a homomorphism between Coulomb branch chiral rings

$$v_- \mapsto \varphi^2 w_- \ , \qquad v_+ \mapsto w_+ \ , \qquad \varphi \mapsto \varphi \ . \tag{6.11}$$

Since the pairing between the co-character $A = (1)$ and the characters $\widetilde{Q}$ are positive— $\langle A, \widetilde{Q}^j_{(+-)} \rangle = 1$ for $j = 1, 2$— Lemma 5 tells us that this homomorphism becomes an isomorphism upon inverting e in the Coulomb branch chiral ring of $T[\mathrm{SU}(2)]$, which is precisely the statement in (6.7).

6.1.3 An alternative localisation: (++)

Consider now the inequivalent (++) bosonisation. In the finite-dimensional analogy, we are localising with respect to X_1 and X_2 prior to symplectic reduction.

After localising, we again have the cotangent to an algebraic torus $T^*(\mathbb{C}^\times)^2$, this time with coordinate ring

$$\mathbb{C}\left[T^*(\mathbb{C}^\times)^2\right] = \mathbb{C}\left[X_1^{\pm 1}, X_2^{\pm 1}, Y^1, Y^2\right] \ . \tag{6.12}$$

This time Lemma 1 reduces to the following statement

$$\begin{aligned}\mathbb{C}\left[T^*(\mathbb{C}^\times)^2\right] &\cong \mathbb{C}\left[Y^1X_1 + Y^2X_2, (X_2)^{\pm 1}\right] \otimes \mathbb{C}\left[Y^1X_1, \left(X_1X_2^{-1}\right)^{\pm 1}\right] \\ &\cong \mathbb{C}\left[T^*T_{G,\mathbb{C}}\right] \otimes \mathbb{C}\left[T^*\mathbb{C}^\times\right] \ ,\end{aligned} \tag{6.13}$$

where all isomorphisms are as Poisson algebras. Symplectic reduction then kills off $T^*T_{G,\mathbb{C}}$ and we have (at the level of functions),

$$R_{(++)} = \mathbb{C}[T^*\mathbb{C}^\times] \cong \mathbb{C}\left[\widetilde{h}, \widetilde{e}^{\pm 1}\right] \tag{6.14}$$

where

$$\widetilde{e} \coloneqq X_1X_2^{-1} \ , \qquad \widetilde{h} \coloneqq Y^1X_1 \ . \tag{6.15}$$

Now for our Poisson algebra homomorphism we have

$$\begin{aligned} R_H &\hookrightarrow R_{(++)} \ , \\ (e,\, h,\, f) &\mapsto \left(-\widetilde{h}\widetilde{e},\, \widetilde{h},\, \widetilde{h}\widetilde{e}^{-1}\right) \ . \end{aligned} \tag{6.16}$$

We can see that this does not determine an open immersion of $R_{(++)}$ into $\mathcal{M}_H$ (let alone into the smooth locus). Indeed, all of $\{\widetilde{h} = 0\}$ maps to the origin of $\mathcal{M}_H$, so the fibre over zero is a full copy of $\mathbb{C}^\times$.

However, by Lemma 3 there must exist an open immersion into some resolution of the Higgs branch. It is easily seen that this is the case. For a given choice of non-zero FI parameter $\zeta \in \mathfrak{t}_G^* \cong \mathbb{R}$ we have

$$\widetilde{\mathcal{M}}_H \cong \mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(\zeta)/T_G \ . \tag{6.17}$$

This resolution just gives the cotangent bundle of $\mathbb{CP}^1$,

$$T^*\mathbb{CP}^1 \twoheadrightarrow A_1 \ , \tag{6.18}$$

with the zero section being the exceptional divisor.

Consider the choice $\zeta < 0$. Then the $\mathbb{C}^\times$ fibre of our map over the origin of $\mathcal{M}_H$ is sent to the $\mathbb{C}^\times \subset \mathbb{CP}^1$ subset of the exceptional divisor where the north and south poles are removed.

The fact that for this choice of localisation we must pick $\zeta < 0$ to find this open immersion is compatible with Lemmas 2 and 4, as well our findings of Section 4.4: the outer automorphisms of $V_1(\mathfrak{psl}(2|2))$ are not manifest in the free-field realisation

obtained from the bosonisation (++), because the only gauge group of the theory is not sign-balanced with respect to (++).

Remark.

From this geometric viewpoint, it should be possible to rewrite the free field expressions of Section 4.4 in terms of a symplectic boson rather than a half-lattice vertex algebra by appropriate field redefinitions. This would in fact amount to "adding the north pole back in" to the exceptional divisor. These de-bosonisation techniques will be explored in Section 7.1.

6.2 A_{N-1} singularities

Consider the rank $k = (N-1)$ theory defined with N hypermultiplets defined by the following charge matrices

$$\mathbf{Q} = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ -1 & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \end{pmatrix} , \qquad \widetilde{\mathbf{Q}} = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & -1 \\ -1 & -1 & -1 & \cdots & -1 \end{pmatrix} . \tag{6.19}$$

The charge matrix Q encodes the A_{N-1} quiver. The chiral ring is generated by the operators

$$Z = \prod_{i=1}^{N} Y^i , \quad W = \prod_{i=1}^{N} X_i , \tag{6.20}$$

as well as

$$h = \frac{X_1 Y^1}{2} , \tag{6.21}$$

with

$$WZ = \frac{(-1)^N}{2^N} h^N , \tag{6.22}$$

which are the equations determining the A_{N-1} singularity

$$A_{N-1} \cong \mathbb{C}/\mathbb{Z}_N . \tag{6.23}$$

The singularity can be resolved by introducing real FI parameters $\zeta \in \mathfrak{t}^*_G \cong \mathbb{R}^{N-1}$.

6.2.1 Open patches

We spell out in details the A_2 case, which is sufficiently generic so that the other A_{N-1} cases can be obtained with relatively straightforward modifications. We will look at our hyperplane arrangement in $\mathfrak{t}^*_{H,\mathbb{R}} \cong \mathbb{R}$, which is identified with the values of the moment map $\mu_{H,\mathbb{R}}$. There are three hyperplanes,

$$\begin{aligned} H_1 &: x = 0 , \\ H_2 &: x = \zeta^1 , \\ H_3 &: x = \zeta^2 , \end{aligned} \tag{6.24}$$

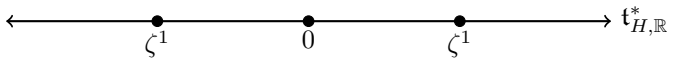
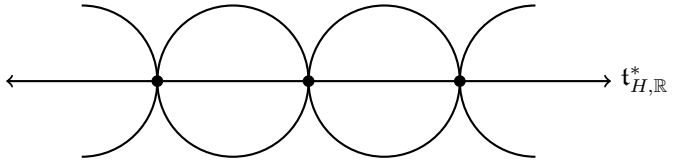


Fig. 1 Hyperplane arrangement for the A_2 theory

where which in this case are just points whose positions depend on the values of the two FI parameters. Such an arrangement is displayed in Fig. 1. with the normal vectors all pointing to the left. Generically (meaning, away from the hyperplanes) above each point we have a copy of $\mathbb{R}^2 \times S^1$ in $\mathcal{M}_{H,\zeta}$. $\mathbb{R}^2$ corresponds to the values of the complex moment map, whereas S^1 is the generic fibre of the fibration (5.9). We set the complex moment map to zero to inspect the extended core $\mathcal{E}$ and the toric varieties associated to the various polyhedra (intervals) into which $\mathfrak{t}^*_{H,\mathbb{R}}$ is divided.

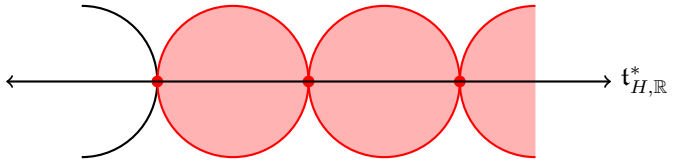
Let us fix $\zeta^1 < 0 < \zeta^2$. There are then four non-degenerate polyhedra P_{A_i} associated to the choices $A_1 = \{1,2,3\}$, $A_2 = \{1,3\}$, $A_3 = \{3\}$, $A_4 = \emptyset$. Up to rescaling, these polyhedra are recognised to be $\mathbb{R}^{\geqslant 0}$, $[0,1]$, $[0,1]$, $\mathbb{R}^{\geqslant 0}$ respectively, and subsequent polyhedra intersect at a point. Thus, the extended core is build out of Kähler varieties $\mathbb{C}$, $\mathbb{P}^1$, $\mathbb{P}^1$, $\mathbb{C}$. glued together at points. In this case, we can easily visualise this by depicting the radius of the S^1 fibre, which provides the classical picture of the resolution of the A_2 singularity



Let us now consider the localisation choice $U = \emptyset$. This amounts to inverting the function $e \coloneqq W$. In algebraic terms, we are left with a ring of functions

$$\mathbb{C}[p, e, e^{-1}] \tag{6.25}$$

where $p = h/2$. This is clearly the ring of functions on a copy of $T^*\mathbb{C}^\times$. In fact, we can see this very clearly in the above picture. Recall that picking the inversion choice $U = \emptyset$ corresponds to removing all chambers (that is, polyhedra) but the interior of $P_{\{1,2,3\}}$. If we colour in red the chambers that we lose by inverting X_i, then



so that we are indeed left with a $\mathbb{C}^\times$ in the extended core, or a $T^*\mathbb{C}^\times$ in $\mathcal{M}_{H,\zeta}$. Moreover, notice that setting $\zeta = 0$ (that is, collapsing the hyperplanes) the left-over chamber interior would remain untouched. It is an interior of an unbounded chamber, which is not part of the core. In other words, after setting the FI parameters to zero this $T^*\mathbb{C}^\times$ embeds openly into the A_2 singularity itself.

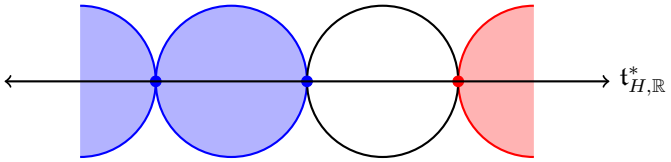
Let us now choose instead $U = \{1, 2\}$. Algebraically, we are inverting the function

$$e' = (Y^1)^{-1}(Y^2)^{-1}X_3 \ , \tag{6.26}$$

and we are left with a ring of functions

$$\mathbb{C}[p, e', e'^{-1}] \ . \tag{6.27}$$

To see which $T^*\mathbb{C}^\times$ this corresponds to, let us depict in red the chambers lost from inverting X_3 and in blue the chamber lost from inverting Y^1, Y^2.



We clearly see that are left with one copy of $\mathbb{C}^\times$, but this time it is part of the core. If we set $\zeta = 0$, it collapses with the core itself.

Finally, let us consider the inversion choice $U = \{1, 3\}$. With this choice of FI parameter we would remove the entire (resolved) affine singularity. However, we know based on our previous abstract arguments that there must be a value of the FI parameter ζ^a that would make $O_{\{1,3\}}$ isomorphic to $\mathbb{C}^\times$. It can be verified that one such choice is $\zeta^2 < 0 < \zeta^1$.

6.2.2 Mirror perspective

Let us now take a mirror perspective and consider the charge matrix

$$\widetilde{Q} = (-1, -1, \cdots, -1) \ . \tag{6.28}$$

If we invert the monopole operator defined by the co-character $A = (-1)$ (corresponding to the operator W in the Higgs branch of the original theory) then clearly $\langle \widetilde{Q}^i, A\rangle = 1$ for all i and so $\widetilde{Q}^{(0)}$ is empty. Therefore we are left with the Coulomb branch of a U(1) pure gauge theory, which is a copy of $T^*\mathbb{C}^\times$. This is the open patch that we discussed above for the choice $U = \emptyset$.

We can alternatively flip the signs of all the hypermultiplets, and invert the mirror co-character $A = (-1)$ instead to find another openly embedded patch $T^*\mathbb{C}^\times$. This choice corresponds to the inversion of the operator Z in the Higgs branch of the original theory.

Notice that no further sign flip is compatible with the condition $\langle \widetilde{Q}, A\rangle \geqslant 0$, which is a necessary assumption in Lemma 5.

6.3 Minimal nilpotent orbit closures

Now we consider SQED[N]. The charge matrix $\mathbf{Q}$ and its mirror charge matrix $\widetilde{\mathbf{Q}}$ read

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} , \qquad \widetilde{\mathbf{Q}} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \\ -1 & 0 & 0 & \cdots & 1 \end{pmatrix} . \tag{6.29}$$

Famously, the Higgs branch of SQED[N] is the minimal nilpotent orbit closure in $\mathfrak{sl}_N^*$. As in the case $N = 2$, this can be seen by putting the collection of gauge invariant operators $Y^i X_j$ into a $N \times N$ charge matrix M. The complex moment map reads

$$\mu_{\mathbb{C}} = \sum_{i=1}^{N} Y^i X_i \ . \tag{6.30}$$

Setting this to zero results in the conditions $M^N = \mathrm{Tr}(M) = 0$, which define the minimal nilpotent orbit closure. If we introduce a non-zero real FI parameter $\zeta \in \mathbb{R}$, then it is easy to see that

$$\mathcal{M}_{H,\zeta} \cong T^*\mathbb{CP}^{N-1} \ , \tag{6.31}$$

with core $\mathbb{CP}^{N-1}$. This is the Springer resolution. There are two chambers for the resolutions, $\zeta > 0$ and $\zeta < 0$, corresponding to whether we are requiring at least one of the Y^is to be invertible, or one of the X_is.

6.3.1 Open patches in SQED[3]

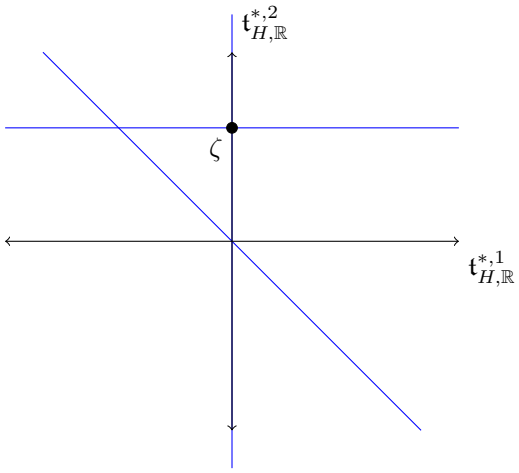
We now discuss the open patches in these minimal nilpotent orbit closures that are realised in our approach for $N = 3$. Recall that the mirror matrix reads

$$\widetilde{Q} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} , \tag{6.32}$$

so we have three hyperplanes

$$\begin{aligned} H_1 &: \ -x_1 - x_2 = 0 \ , \\ H_2 &: x_1 = \zeta \ , \\ H_3 &: x_2 = 0 \ . \end{aligned} \tag{6.33}$$

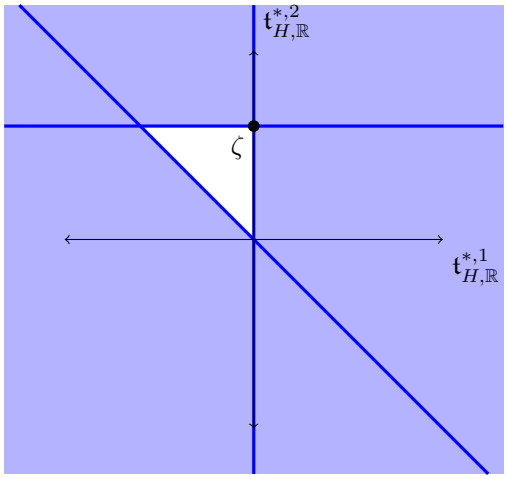
We chose $\zeta > 0$, so that we can represent the hyperplane arrangement as in the Figure below.



We consider two choices of inversions U that are compatible with $\zeta > 0$: $U = \{2, 3\}$, which descends to a viable choice on the nilpotent orbit closure, and $U = \{1, 2, 3\}$, which does not. Following the above convention, we colour in blue the chambers that we lose by inverting X_is, and in red the chambers that we lose by inverting Y^is.

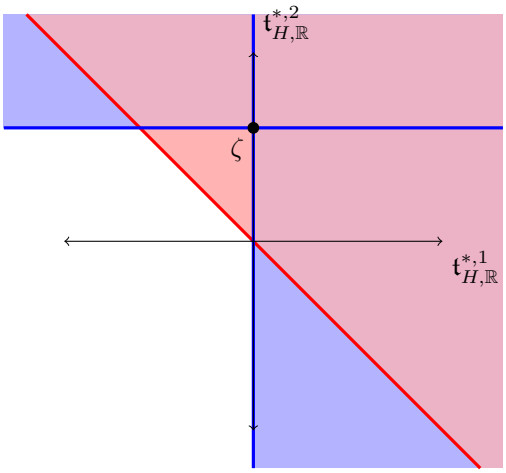
Choice $U = \{1, 2, 3\}$

With this choice we get a $T^* \left(\mathbb{C}^\times\right)^2$ where the that has components in the core (the area that remains white).



Choice $U = \{2, 3\}$

In this case, the remaining $T^*(\mathbb{C}^\times)^2$ does not degenerate when $\zeta = 0$. In fact, we are inverting the operators $X_1 Y^2$ and $e_2 = X_1 Y^3$, which are both in the chiral ring of the minimal nilpotent orbit closure.



6.3.2 Mirror perspective

We now take a mirror perspective and discuss open patches that openly embeds in the (unresolved) minimal nilpotent orbit closures for general N.

To this end, consider the charge matrix $\widetilde{Q}$. We can flip the signs of the charges of the last $N-1$ hypermultiplets to obtain the matrix

$$\widetilde{Q}_U = \begin{pmatrix} -1 & -1 & 0 & \cdots & 0 \\ -1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ -1 & 0 & 0 & \cdots & -1 \end{pmatrix} . \tag{6.34}$$

Then we can invert the mirror monopole operator defined by the co-character $(-1, 0, \cdots, 0)$ (corresponding to the operator $X_1 Y^2$ of the original theory) to obtain

$$\widetilde{Q}^{(0)}_U = \begin{pmatrix} 0 & \cdots & 0 \\ -1 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{pmatrix} . \tag{6.35}$$

This is the matrix defining a U(1) pure gauge theory and $N-2$ copies of SQED[2], we have therefore identified a patch $T^*\mathbb{C}^\times \times T^*\mathbb{C}^{N-2} \subset \mathcal{M}_C[\widetilde{Q}] \cong \mathcal{M}_H$.

This patch should in principle be good enough to construct a free-field realisation, and in fact we are going to discuss such a construction in Section 7.1. However, we could further invert the mirror monopoles associated to the various mirror co-characters $(\ldots, 0, -1, 0 \ldots)$ to obtain an openly embedded patch $T^*(\mathbb{C}^\times)^{N-2} \subset \mathcal{M}_H$. This are precisely the operators that we invert with the $U = \{2, \cdots, N\}$ choice.

7 Further examples of free-field realisation

In this final section, we apply our general free field analysis to SQED[N] and A_{N-1} quiver gauge theories. The (singular) Higgs branches for these theories are minimal nilpotent orbit closures in $\mathfrak{sl}_N^*$ and A_{N-1} Kleinian singularities, respectively (see Section 6.2 and 6.3).

By exactly the same arguments presented in Section 4.2, we conjecture that the A-twisted boundary VOA of SQED[N] to be isomorphic to $V_1(\mathfrak{psl}(N|N))$.

7.1 SQED[N] and free-field realisations of $V_1(\mathfrak{psl}(N|N))$

Our first set of theories are U(1) gauge theories with N hypermultiplets of weight 1. The charge matrix $\mathbf{Q}$ and its mirror charge matrix $\widetilde{\mathbf{Q}}$ were already reported in (6.29)

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} , \qquad \widetilde{\mathbf{Q}} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \\ -1 & 0 & 0 & \cdots & 1 \end{pmatrix} . \tag{7.1}$$

The A-twisted VOA is the BRST reduction of N symplectic boson pairs (X_i, Y^i) and N complex fermion pairs $(\chi_\alpha, \xi^\alpha)$ with respect to the zero modes of the BRST current

$$J^{\text{BRST}} = c\,(X_a Y^a + \chi_\alpha \xi^\alpha) \ , \tag{7.2}$$

where (b, c) is a ghost system of weight $(1, 0)$.

The standard list of BRST-invariant operators (*cf.* 3.1) reduces to the following. For $i \neq j$ and $k \neq 1$, we have BRST-closed bi-linears composed of symplectic bosons,

$$X_i Y^j \ , \qquad -X_1 Y^1 + X_k Y^k \ , \tag{7.3}$$

which generate a $V_{-1}(\mathfrak{sl}_N)$ VOA. Similarly (for the same conditions $i \neq j$, $k \neq 1$), we have BRST-closed bi-linears composed of the complex fermions

$$\xi^i \chi_j \ , \qquad -\xi^1 \chi_1 + \xi^k \chi_k \ , \tag{7.4}$$

which generate a $V_1(\mathfrak{sl}(N))$ VOA. Finally, we have odd, fermionic current generators,

$$Y^i \chi_j \ , \qquad X_i \xi^j \ . \tag{7.5}$$

Together, these generate a quotient of the level one $V^1(\mathfrak{sl}(N|N))$, which after taking cohomology with respect to the BRST operator gets reduced to a quotient (we expect the simple quotient) of $V^1(\mathfrak{psl}(N|N)))$.

We can now apply our general procedure to these examples. After briefly describing the change of lattice coordinates in Section 7.1.1, in Section 7.1.2 we give a short exposition

of the free-field realisation that arises by applying the methods of Section 2. We also compare our result with the free-field realisation of $V_{-\frac{3}{2}}(\mathfrak{sl}(3))$ proposed in [**?**], as well as an extensions of that result to $V_{-1}(\mathfrak{sl}_N)$. We comment on the geometric aspects of this comparison in Section 7.1.3, and see a nice application of our Coulomb branch analysis above. The actual free-field constructions are compared in Section 7.1.4, where inspired by the finite-dimensional geometry we de-bosonise some combinations of our free-fields.

7.1.1 Change of coordinates

We will work with the $\epsilon = (+--\cdots-)$ choice of bosonisation, which generalises the one used in our first analysis of SQED[2] in Section 4. From the point of view of the Higgs branch, this corresponds to localising on a $T^*\left(\mathbb{C}^\times\right)^N \subset \mathcal{M}_H$ subset of the singular Higgs branch, as we described in Section 6.3. The modified charge matrices now take the form

$$Q_\epsilon = \begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \end{pmatrix} \ , \qquad \widetilde{Q}_\epsilon = \begin{pmatrix} -1 & -1 & 0 & \cdots & 0 \\ -1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ -1 & 0 & 0 & \cdots & -1 \end{pmatrix} \ , \tag{7.6}$$

from which we compute the remaining players in the change of basis matrix (3.27),

$$A_\epsilon^{-1} = \frac{1}{N}\begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \\ -1 & -N+1 & 1 & \cdots & 1 \\ \vdots & & \ddots & & \vdots \\ -1 & 1 & 1 & -N+1 & 1 \\ -1 & 1 & 1 & 1 & -N+1 \end{pmatrix} \ , \quad F_\epsilon^T = \frac{1}{N}\begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \\ -1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -1 & 1 & 1 & \cdots & 1 \end{pmatrix} \ . \tag{7.7}$$

For our free fields, we therefore take the following basis,

$$\begin{aligned} \phi_i &= -\sigma_1 - \sigma_{i+1} \ , & & i = 1, \dots, N-1 \ , \\ \delta_i &= \rho_1 + \rho_{i+1} \ , & & i = 1, \dots, N-1 \ , \\ \omega_i &= \gamma_i + \frac{(-1)^{\delta_{1i}}}{N}\left(\sum_{j=1}^N (\sigma_j - \rho_j) - 2(\sigma_1 - \rho_1)\right) \ , & & i = 1, \dots, N \ . \end{aligned} \tag{7.8}$$

The two point functions in this basis are not diagonal, but rather we have pairings

$$\begin{aligned} (\delta_i, \delta_j) &= -(\phi_i, \phi_j) = 1 + \delta_{ij} \ , \\ (\omega_i, \omega_j) &= \delta_{ij} \ . \end{aligned} \tag{7.9}$$

The lattice extension of this Heisenberg which defines $\widetilde{V}_{\mathrm{FFR}}$ is given by

$$\bigoplus_{i=1}^{N-1}\mathbb{Z}\left(\delta_i+\phi_i-\sum_{j=1}^{N-1}\frac{\delta_j+\phi_j}{N}\right)\oplus\bigoplus_{i=1}^{N}\mathbb{Z}\left(\omega_i\right)\ . \tag{7.10}$$

As we described below (3.50), free fields with mutually vanishing OPEs could be obtained by a rational change of variables. Though such a refinement could be useful in interpreting the free fields in terms of CDOs on $T^*(\mathbb{C}^\times)^{N-1}$, for our purposes it will be simpler to continue with the present basis.[8].

We will need a few notational ingredients to produce our free field realisation. First, we define our free fermions as follows (for later convenience, we are introducing a relative minus sign for ω_1 compared to (3.35))

$$J_{\omega_i}=(-1)^{\delta_{i1}}\psi^i\widetilde{\psi}_i\ . \tag{7.12}$$

We then introduce the following current, whose zero mode generates the $\mathrm{U}(1)_C$ outer automorphism,

$$J_\psi\coloneqq\frac{1}{N}\left(\sum_{i=1}^{N}\psi^i\widetilde{\psi}_i\right)\ . \tag{7.13}$$

The BRST representatives in $\widetilde{V}_{\mathrm{FFR}}$ of the currents J_{ρ_i} can now be written as[9]

$$\bar{J}_{\rho_i}=(-1)^{\delta_{i1}}\left(-\frac{1}{N}\sum_{j=1}^{N-1}J_{\delta_j}-J_\psi\right)+J_{\delta_{i-1}}\ . \tag{7.14}$$

7.1.2 Free-field realisation

Given the ingredients above, we can efficiently produce a free-field realisation of the $\mathfrak{psl}(N|N)$ current algebra. Indeed, the computations are directly parallel to the $N=2$ case (*cf.* Tables 1 and 2). First, provided $i\neq j$, we have replacements,

$$X_iY^j\quad\mapsto\quad\begin{cases}e^{(\phi_{j-1}+\delta_{j-1})} & i=1\ ,\\ -\bar{J}_{\rho_i}e^{-(\phi_{i-1}+\delta_{i-1})+(\phi_{j-1}+\delta_{j-1})} & i\neq1,\ j\neq1\ ,\\ -\bar{J}_{\rho_1}\bar{J}_{\rho_i}e^{-(\phi_{i-1}+\delta_{i-1})} & j=1\ .\end{cases} \tag{7.15}$$

The currents in (7.3) are then simply identified with the J_{ϕ_i}. One can check directly that these fields generate indeed a level -1 $\mathfrak{sl}_N$ current algebra.

[8]The relevant lattice refinement would correspond to an orthogonalisation of $\widetilde{Q}_\epsilon$, which reads,

$$\widetilde{Q}_\epsilon^{\mathrm{orth}}=\begin{pmatrix}-1 & -1 & 0 & \cdots & 0\\ -\frac{1}{2} & \frac{1}{2} & 1 & \cdots & 0\\ \vdots & \vdots & & \ddots & \\ -\frac{1}{N-1} & \frac{1}{N-1} & \frac{1}{N-1} & \cdots & 1\end{pmatrix}\ . \tag{7.11}$$

[9]Here and below, we use the shorthand notation that fields with index $\leqslant 0$ vanish.

The $V_1(\mathfrak{sl}_N)$ generators are given by the usual construction in terms of the free fermions, with

$$\xi^i\chi_j \;\mapsto\; -\psi^i\widetilde{\psi}_j \;, \qquad i\neq j \;, \tag{7.16}$$

and a basis for the Cartan subalgebra given by

$$\psi^1\widetilde{\psi}_1 - \psi^j\widetilde{\psi}_j \;, \qquad j=2,\dots,N \;. \tag{7.17}$$

Finally, the odd generators read

$$\begin{aligned} Y^i\chi_j \;&\mapsto\; \begin{cases} \bar{J}_{\rho_1} e^{-\frac{1}{N}\sum_{l=1}^{N-1}(\phi_l+\delta_l)}\psi^j \;, & i=1 \;, \\ e^{-\frac{1}{N}\sum_{l=1}^{N-1}(\phi_l+\delta_l)+(\phi_{i-1}+\delta_{i-1})}\psi^j \;, & \text{else} \;, \end{cases} \\ X_i\xi^j \;&\mapsto\; \begin{cases} e^{\frac{1}{N}\sum_{l=1}^{N-1}(\phi_l+\delta_l)}\widetilde{\psi}_j & i=1 \;, \\ -\bar{J}_{\rho_i} e^{\frac{1}{N}\sum_{l=1}^{N-1}(\phi_l+\delta_l)-(\phi_{i-1}+\delta_{i-1})}\widetilde{\psi}_j & \text{else} \;. \end{cases} \end{aligned} \tag{7.18}$$

The full set of $\mathfrak{psl}(N|N)$ OPEs can be verified for all of these operators by direct computation.

As we mentioned above, the zero-mode of the trace current J_ψ (which is not included in $V_{-1}(\mathfrak{sl}_N)$) generates a $\mathfrak{u}(1)$ outer automorphism symmetry. This corresponds to the $\mathrm{U}(1)_C$ topological symmetry of SQED[N], which for $N>2$ is not enhanced. Indeed, from (7.14) one can see that the screening currents $e_i^{\bar{\rho}}$ will not enjoy any non-Abelian automorphism enhancement for $N=2$, as they each depend differently on the δ bosons. This is as it should be given the lack of a balanced subgroup of the gauge symmetry in this example. From the point of view of the outer automorphism symmetry, then, our choice of bosonisation is in this case was not particularly important.

7.1.3 More on finite-dimensional geometry

In order to understand a little more concretely the inner workings of our free-field realisation, it is instructive to compare our result here with a related construction in four dimensions [**?**], where free-field realisations of the simple affine Kac-Moody algebra $V_\kappa(\mathfrak{sl}_3)$ at level $\kappa=-\frac{3}{2}$ was produced. This VOA has been proven to have the minimal nilpotent orbit in $\mathfrak{sl}_3^*$ as associated variety, and the construction of [**?**]—much in the spirit of the *ad hoc* free-field realisation of $V_{-1}(\mathfrak{psl}(2|2))$ presented in Section 4—exploits the geometry of the minimal nilpotent orbit to produce a free-field realisation. Furthermore, it was announced in [**?**] that similar constructions can be performed for $V_1(\mathfrak{sl}_N)$, but at the cost of introducing an additional auxiliary $\mathfrak{gl}_1$ Heisenberg current. Intuitively, this is because the associated varieties to $V_{-1}(\mathfrak{sl}_N)$ *are not* the respective minimal nilpotent orbits [**?**], but include some additional direction in $\mathfrak{sl}_N^*$ related to the extra current.

Per Conjecture 1, we expect the associated variety for $V_1(\mathfrak{psl}(N|N))$ to be the minimal nilpotent orbit, so we may expect some connection between the geometric construction of [**?**]—including the extra Heisenberg current—and our free-field realisation, but the extra current should be accounted for by extra nilpotent degrees of freedom. We will indeed demonstrate that this is the case.

The geometric aspects of $\mathfrak{sl}_N^*$ minimal nilpotent orbit closures used in [**?**] are based on the following decomposition of $\mathfrak{sl}_N$,

$$\mathfrak{sl}_N \cong (\mathfrak{sl}_{N-2} \oplus \mathbb{C}h_\perp \oplus (\mathfrak{sl}_2)_\theta) \oplus (\mathfrak{R}, \mathbf{2}) \ , \tag{7.19}$$

where $(\mathfrak{sl}_2)_\theta$ is an $\mathfrak{sl}_2$-triple associated to a choice of highest root, and $\mathfrak{R}$ transforms in the representation $(\mathbf{N-2})_+ \oplus \overline{(\mathbf{N-2})}_-$ of $\mathfrak{sl}_{N-2} \oplus \mathbb{C}$ (here $+$ and $-$ correspond to positive and negative weights with respect to the extra Cartan generator $h_\perp$). We will pick a basis $(e_\theta, f_\theta, h_\theta)$ for $(\mathfrak{sl}_2)_\theta$ and decompose the $(\mathfrak{R}, 2)$ generators in terms of their h_θ weight,

$$(\mathfrak{R}, 2) = \mathfrak{R}^+ \oplus \mathfrak{R}^- \ . \tag{7.20}$$

We write e_a, $\tilde{e}_a$ $a \in \{1, \cdots, N-2\}$ as bases for the two $\mathfrak{sl}_{N-2}$ representations in $\mathfrak{R}^+$. A key observation in [**?**] is that upon inversion of e_θ, the generators of the minimal nilpotent orbit can be expressed in terms of the functions $(e_\theta, e_\theta^{-1}, h_\theta)$ and e^a, $\widetilde{e}_a$ which have non-vanishing Poisson brackets

$$\{h_\theta, e_\theta\} = 2e_\theta \ , \quad \{e^a, \widetilde{e}_b\} = \delta^a_b e_\theta \ , \quad \{h_\theta, e^a\} = e^a \quad \{h_\theta, \widetilde{e}_a\} = e_a \ . \tag{7.21}$$

Moreover, if one introduces $(\boldsymbol{d}_\theta, \boldsymbol{\beta}_a, \boldsymbol{\gamma}_a)$ for $a \in \{1, \cdots, N-2\}$ according to,

$$\boldsymbol{\beta}^a \coloneqq e_\theta^{-\frac{1}{2}} e_a \ , \qquad \boldsymbol{\gamma}_a = e_\theta^{-\frac{1}{2}} \widetilde{e}_a \ , \qquad \boldsymbol{d}_\theta = e_\theta^{\frac{1}{2}} \ , \tag{7.22}$$

then one can realise all of the coordinate functions in terms of $\mathfrak{h}_\theta$, $\boldsymbol{d}_\theta$, $\boldsymbol{\beta}^a$, $\boldsymbol{\gamma}_a$ with non-vanishing Poisson brackets given by

$$\{h_\theta, \boldsymbol{d}_\theta\} = \boldsymbol{d}_\theta \ , \qquad \{\boldsymbol{\beta}^a, \boldsymbol{\gamma}_b\} = \delta^a_b \ . \tag{7.23}$$

These are interpreted as coordinate functions on

$$T^*\mathbb{C}^\times \times T^*\mathbb{C}^{N-2} \ , \tag{7.24}$$

and so h_θ, e_θ, e^a, $\widetilde{e}_a$ can be interpreted as functions on

$$\left(T^*\mathbb{C}^\times \times T^*\mathbb{C}^{N-2}\right)/\mathbb{Z}_2 \ , \tag{7.25}$$

with $\mathbb{Z}_2$ acting on $\mathbb{C}^\times$ by negation. The free fields introduced in [**?**] correspond to CDOs on the space (7.24).

Let us compare this picture with the open patches we discussed in Section 6.3, and in particular the restriction functor on Coulomb branches as presented in Section 6.3.2. We saw there that inversion of $e = e_\theta$ indeed leads to an openly embedded patch of the form

$$\left(T^*\mathbb{C}^\times \times T^*\mathbb{C}^{N-2}\right) \ , \tag{7.26}$$

where $T^*\mathbb{C}^{N-2}$ can be identified with $N-2$ copies of the Coulomb branch of SQED[1]. The way in which this patch is obtained, however, is slightly different from the one

presented in [**?**]. In fact, instead of the redefinition (7.22), the isomorphism (5.25) is such that factors of e_θ are mapped into the new variables in an asymmetric fashion, namely

$$w_+^a \coloneqq e_\theta e_a \ , \qquad w_-^a = \widetilde{e}_a \ . \tag{7.27}$$

The relation between the two open patches can be understood in terms of an isomorphism

$$\left(T^*\mathbb{C}^\times \times T^*\mathbb{C}^{N-2}\right)/\mathbb{Z}_2 \cong \left(T^*\mathbb{C}^\times \times T^*\mathbb{C}^{N-2}\right) \ , \tag{7.28}$$

where the two sides are subject to different $\mathbb{C}^\times$ scaling. Finally, notice that the open patch $T^*\left(\mathbb{C}^\times\right)^N$ that underlies the free-field realisation of Section 7.1.2 can be obtained by further inverting the functions e_a.

7.1.4 More on free-field realisations and de-bosonisation

The discussion above raises the question of whether there is a chiral analogue of these relations. More precisely, starting from the free-field realisation presented above, perhaps one can *de-bosonise* some auxiliary $\beta\gamma$-systems to obtain the generators proposed in [**?**] for the $V_{-1}(\mathfrak{sl}_N)$ component of our current algebra, with the extra Heisenberg current provided by the free fermions. In this section, we will show that this is precisely the case.

By an abuse of notation, we use the same letters for the functions we introduce above and their chiral analogues. According to the discussion around (7.15), we have for the relevant generators in our free field realisation

$$\begin{aligned} h_\theta &= -J_{\phi_1} \ , \qquad & e^a &= e^{\phi_{a+1}+\delta_{a+1}} \ , \\ e_\theta &= e^{\phi_1+\delta_1} \ , \qquad & \widetilde{e}_a &= -\bar{J}_{a+2} e^{-(\phi_{a+1}+\delta_{a1+})+(\phi_1+\delta_1)} \ , \end{aligned} \tag{7.29}$$

with OPEs in accordance with (7.21)

$$\begin{aligned} h_\theta(z)e_\theta(w) &\sim \frac{2e_\theta(w)}{z-w} \ , \\ e^a(z)\widetilde{e}_b(w) &\sim \frac{\delta^a_b e_\theta(w)}{z-w} \ , \\ h_\theta(z)e_a(w) &\sim \frac{e_a(w)}{z-w} \ . \end{aligned} \tag{7.30}$$

If we now define (for $a \in \{1, \cdots, N-2\}$),

$$\begin{aligned} \boldsymbol{d}_\theta &\coloneqq e^{\frac{1}{2}(\phi_1+\delta_1)} \ , \\ \boldsymbol{\beta}^a &\coloneqq e^{(\phi_{a+1}+\delta_{a+1})-\frac{1}{2}(\phi_1+\delta_1)} \ , \\ \boldsymbol{\gamma}_a &\coloneqq -\bar{J}_{a+2} e^{-(\phi_{a+1}+\delta_{a+1})+\frac{1}{2}(\phi_1+\delta_1)} \ , \end{aligned} \tag{7.31}$$

	Gauge theory	FFR Representative	De-bosonisation
e_θ	X_1Y^2	$e^{\phi_1+\delta_1}$	$e^{\phi_1+\delta_1}$
f_θ	X_2Y^1	$\left(-\frac{1}{4}J_{\delta_1}^2+T\right)e^{-(\delta_1+\phi_1)}$	$\left(-\frac{1}{4}J_{\delta_1}^2+T\right)e^{-(\delta_1+\phi_1)}$
h_θ	$(Y^1X_1-Y^2X_2)$	$-J_{\phi_1}$	$-J_{\phi_1}$
e^a	X_1Y^{a+2}	$e^{\phi_{a+1}+\delta_{a+1}}$	$\boldsymbol{\beta}^a\boldsymbol{d}_\theta$
$\widetilde{e}_a$	$X_{a+2}Y^2$	$\left(-J_{\delta_{a+1}}+J_\nu\right)e^{-(\phi_{a+1}+\delta_{a+1})+(\phi_1+\delta_1)}$	$\boldsymbol{\gamma}_a\boldsymbol{d}_\theta$
f^a	X_2Y^{a+2}	$\left(-J_{\delta_1}+J_\nu\right)e^{(\phi_{a+1}+\delta_{a+1})-(\phi_1+\delta_1)}$	$\boldsymbol{\beta}^a\boldsymbol{d}_\theta^{-1}$
$\widetilde{f}_a$	$X_{a+2}Y^1$	$J_\nu\left(-J_{\delta_{a+1}}+J_\nu\right)e^{-\phi_{a+1}-\delta_{a+1}}$	$J_\nu\boldsymbol{\gamma}_a\boldsymbol{d}_\theta^{-1}$
c_a^b	$X_{a+2}Y^{b+2}$	$\left(-J_{\delta_{a+1}}+J_\nu\right)e^{-(\phi_{a+1}+\delta_{a+1})+(\phi_{b+1}+\delta_{b+1})}$	$\boldsymbol{\gamma}_a\boldsymbol{\beta}^b$
t_a	$(-\frac{1}{2}Y^1X_1-\frac{1}{2}Y^2X_2+Y^aX_a)$	$-J_{\phi_a}-\frac{1}{2}J_{\phi_1}$	$J_\nu+\boldsymbol{\beta}^a\boldsymbol{\gamma}_a+\frac{1}{2}J_{\delta_1}$
	χ_jY^1	$J_\nu e^{-\frac{1}{N}\sum_{l=1}^{N-1}(\phi_l+\delta_l)}\psi^j$	$J_\nu\boldsymbol{d}_\theta^{-1}\Psi^j$
	χ_jY^2	$e^{-\frac{1}{N}\sum_{l=1}^{N-1}(\phi_l+\delta_l)+(\phi_1+\delta_1)}\psi^j$	$\boldsymbol{d}_\theta\Psi^j$
	χ_jY^{2+a}	$e^{-\frac{1}{N}\sum_{l=1}^{N-1}(\phi_l+\delta_l)+(\phi_{a+1}+\delta_{a+1})}\psi^j$	$\boldsymbol{\beta}_a\Psi^j$
	$X_1\xi^j$	$e^{\frac{1}{N}\sum_{l=1}^{N-1}(\phi_l+\delta_l)}\widetilde{\psi}_j$	$\boldsymbol{d_\theta}\widetilde{\Psi}_j$
	$X_2\xi^j$	$\left(-J_{\delta_1}+J_\nu\right)e^{\frac{1}{N}\sum_{l=1}^{N-1}(\phi_l+\delta_l)-(\phi_1+\delta_1)}\widetilde{\psi}_j$	$\left(-J_{\delta_1}+J_\nu\right)\boldsymbol{d}_\theta^{-1}\widetilde{\Psi}_j$
	$X_{2+a}\xi^j$	$\left(-J_{\delta_{a+1}}+J_\nu\right)e^{\frac{1}{N}\sum_{l=1}^{N-1}(\phi_l+\delta_l)-(\phi_a+\delta_a)}\widetilde{\psi}_j$	$\boldsymbol{\gamma}_a\widetilde{\Psi}_j$

Table 3 Relation between free field realisations of $V_1(\mathfrak{psl}(N|N))$. Here $a,b\in\{1,\cdots N-2\}$, $i,j\in\{1,\cdots N\}$ and we are assuming $a\neq b$. The top block corresponds to $V_{-1}(\mathfrak{sl}_N)$. f^a, $\widetilde{f}_a$ are the generators corresponding to a basis of $\mathfrak{R}^-$, c_a^b, t_a be generators of $\mathfrak{sl}_{(N-2)}\oplus\mathbb{C}h_\perp$. The second block corresponds to the odd generators. We omit the copy of $V_1(\mathfrak{sl}_N)$ generated by free fermions, which are standard.

then these fields enjoy the OPEs of CDOs on $T^*\mathbb{C}^\times\times T^*\mathbb{C}^{N-2}$,

$$\begin{aligned}\boldsymbol{\beta}^a(z)\boldsymbol{\gamma}_b(w) &\sim \frac{\delta_a^b}{z-w}\ ,\\ h_\theta(z)\boldsymbol{d}_\theta(w) &\sim \frac{\boldsymbol{d}_\theta}{z-w}\ .\end{aligned}\tag{7.32}$$

Indeed, we can see that the pairs $(\boldsymbol{\beta}^a,\boldsymbol{\gamma}_a)$ in (7.31) are in fact $(N-2)$ bosonised $\beta\gamma$-systems by making the following identifications,

$$\boldsymbol{\beta}^a\coloneqq e^{\upsilon_a-\tau_a}\ ,\qquad \boldsymbol{\gamma}_a\coloneqq J_{\upsilon_a}e^{-\upsilon_a+\tau_a}\ ,\tag{7.33}$$

where we define yet another basis of free bosons,

$$\begin{aligned}\upsilon_a&\coloneqq\delta_{a+1}-\nu\ ,\\ \tau_a&\coloneqq-\phi_{a+1}-\frac{1}{2}(\phi_1+\delta_1)-\nu\ .\end{aligned}\tag{7.34}$$

For convenience, we have introduced

$$\nu\coloneqq\frac{1}{N}\left(\sum_{j=1}^{N-1}\delta_j+\sum_{i=1}^{N}(-1)^{\delta_{1i}}\omega_i\right)\ .\tag{7.35}$$

By further redefining the free fermions ψ^i, $\widetilde{\psi}_j$, we can absorb much of the remaining dependence on the lattice bosons. Indeed, if we set

$$\begin{aligned} \Psi^j &:= e^{-\frac{1}{N}\sum_{l=1}^{N-1}(\phi_l+\delta_l)+\frac{1}{2}(\delta_1+\phi_1)}\psi^j \ , \\ \widetilde{\Psi}_j &:= e^{\frac{1}{N}\sum_{l=1}^{N-1}(\phi_l+\delta_l)-\frac{1}{2}(\delta_1+\phi_1)}\widetilde{\psi}_j \ , \end{aligned} \tag{7.36}$$

then these still enjoy free fermion OPEs but also commute with $\boldsymbol{\beta}^a$, $\boldsymbol{\gamma}_a$, $\boldsymbol{d}_\theta$, h_θ. Moreover, we now have

$$J_\nu = -\frac{N}{2}J_\Psi + \frac{1}{2}\boldsymbol{\beta}^a\boldsymbol{\gamma}_a + \frac{1}{2}J_{\delta_1} \ . \tag{7.37}$$

where we have defined

$$J_\Psi := \frac{1}{N}\sum_{i=1}^{N}\Psi^i\widetilde{\Psi}_i \ . \tag{7.38}$$

Now let us consider the form of the lowering operator in $\mathfrak{sl}(2)_\theta$,

$$f_\theta = \left(-\frac{1}{4}J_{\delta_1}^2 + T\right)e^{-(\delta_1+\phi_1)} \tag{7.39}$$

where

$$T = \left(J_\nu - \frac{1}{2}J_{\delta_1}\right)\left(J_\nu - \frac{1}{2}J_{\delta_1}\right) \ , \tag{7.40}$$

which we can now rewrite in our new variables as

$$T = \left(-\frac{N}{2}J_\Psi + \frac{1}{2}\boldsymbol{\beta}^a\boldsymbol{\gamma}_a\right)^2 \ . \tag{7.41}$$

So indeed, we have J_Ψ acting as the additional Heisenberg current compared to the free field realisation of [**?**].

More generally it can be verified that all generators of $V_1(\mathfrak{psl}(N|N))$ can in fact be expressed in terms of these de-bosonised free fields. We therefore obtain a free-field realisation of $V_1(\mathfrak{psl}(N|N)$ modelled on a $T^*\mathbb{C}^\times \times T^*\mathbb{C}^{N-2}$ patch on the associated variety. Details of this and the comparison between the different free-field realisations is provided in Table 3.

7.2 A_{N-1} quiver example

We now consider the mirror of SQED[N], for which we take the charge matrices defined in (6.19)

$$\mathbf{Q} = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ -1 & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \end{pmatrix} \ , \qquad \widetilde{\mathbf{Q}} = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & -1 \\ -1 & -1 & -1 & \cdots & -1 \end{pmatrix} \ . \tag{7.42}$$

The charge matrix Q encodes the A_{N-1} quiver. The VOA for $N = 3$ has recently been studied in [**?**] where the OPEs were constructed directly.

The theory has a $\mathfrak{su}(N)_C$ Coulomb branch symmetry (the symmetry of the $\mathfrak{sl}_N$ minimal nilpotent orbit) which enhances the $\mathfrak{u}(1)_C^{N-2}$ topological symmetry visible in the UV. Per our discussion in Section 3.3, we expect two choices of bosonisation to make this symmetry manifest as outer automorphism symmetry of the free field realisation: $(+ + \cdots +)$ and $(- - \cdots -)$. In terms of charts on the Higgs branch, as described in Section 6.3, these choices indeed correspond to localisation on an open $T^*\mathbb{C}^\times \subset \mathcal{M}_H$ of the singular Higgs branch.

Below, after reviewing the prescribed change of basis, we list the expected generators in terms of free fields in the $\epsilon = (+ + \cdots +)$ bosonisation and observe the manifest $\mathrm{SL}(N)_o$ outer automorphism symmetry. Finally, specialise to $N = 3$ case and study the VOA in more detail in Section 7.2.3.

7.2.1 Change of coordinates

We compute the relevant matrices for the change of basis (3.27):

$$A_\epsilon^{-1} = \frac{1}{N}\begin{pmatrix} -1 & -1 & \cdots & -1 & 1 \\ N-1 & -1 & \cdots & -1 & 1 \\ \vdots & & \ddots & \vdots & \vdots \\ -1 & & N-1 & -1 & 1 \\ -1 & -1 & \cdots & N-1 & 1 \end{pmatrix} , \; F_\epsilon = \frac{1}{N}\begin{pmatrix} N-1 & -1 & \cdots & -1 & -1 \\ -1 & N-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & -1 \\ -1 & -1 & -1 & \cdots & N-1 \end{pmatrix} . \tag{7.43}$$

Thus, the new basis for the lattice vertex algebra will be

$$\begin{aligned} \phi &= -\sum_{i=1}^{N} \sigma_i \ , \\ \delta &= \sum_{i=1}^{N} \rho_i \ , \\ \omega_i &= \gamma_i + (\sigma_i - \rho_i) + \frac{1}{N}\sum_{j=1}^{N} (\rho_j - \sigma_j) \ . \end{aligned} \tag{7.44}$$

The fields have the normalisation

$$(\delta, \delta) = -(\phi, \phi) = N \ , \qquad (\omega_i, \omega_j) = \delta_{ij} \ , \tag{7.45}$$

and the lattice extension of this algebra which defines $\widetilde{V}_{\mathrm{FFR}}$ is given by

$$\mathbb{Z}\left(\frac{\delta + \phi}{N}\right) \oplus \bigoplus_{i=1}^{N} \mathbb{Z}\,\omega_i \ . \tag{7.46}$$

We define free fermions

$$\psi^i := e^{-\omega_i} \ , \qquad \widetilde{\psi}_i := e^{\omega_i} \ , \tag{7.47}$$

so the free field space is the tensor product of a half lattice $\mathcal{D}^{(ch)}(\mathbb{C}^\times)$ with N free fermions as expected. With an eye towards outer automorphisms, we adopt the notation ψ^A, $\widetilde{\psi}_A$ where $A, B, C \ldots = 1, \cdots N$ are now indices fundamental/anti-fundamental representations of SL(N). We introduce the SL(N) invariant (trace) current,

$$J_\psi = \psi^A \widetilde{\psi}_A \ . \tag{7.48}$$

The BRST representatives $\bar{J}_{\rho_i}$ of the currents J_{ρ_i} read as follows

$$\bar{J}_{\rho_i} = \tfrac{1}{N} \left(J_\delta - N \psi^i \widetilde{\psi}_i + J_\psi \right) \ . \tag{7.49}$$

The corresponding screening currents are given by

$$\tilde{\mathfrak{s}}_i = \exp \left(\tfrac{1}{N} \delta - \tfrac{1}{N} \sum_{j=1}^{N} \omega_j + \omega_i \right) \ , \tag{7.50}$$

and indeed these transform in the standard representation of horizontal $\mathfrak{sl}_N$ subalgebra of the free fermionic $V_1(\mathfrak{sl}_N)$ that commutes with J_ψ.

7.2.2 A_{N-1} generators and outer automorphism symmetry

In this short section, we briefly outline the list of generators we expect in the A_{N-1} case. From the flavour moment map and its fermion counterpart, we will have two Heisenberg currents,

$$\begin{aligned} \sum_{i=1}^{N} X_i Y^i \quad &\mapsto \quad - J_\phi \ , \\ \sum_{i=1}^{N} \chi_i \xi^i \quad &\mapsto \quad J_\psi \ . \end{aligned} \tag{7.51}$$

The N Grassmann-odd currents take the form

$$\begin{aligned} X_i \xi^i \quad &\mapsto \quad J^+_A \quad := \quad e^{\frac{1}{N}(\phi+\delta)} \widetilde{\psi}_A \ , \\ \chi_i Y^i \quad &\mapsto \quad J^{-,A} \quad := \quad \left(\tfrac{1}{N} \left(-J_\delta - J_\psi \right) \psi^A + \partial \psi^A \right) e^{-\frac{1}{N}(\phi+\delta)} \ . \end{aligned} \tag{7.52}$$

Finally, the additional generators are realised by starting with $X_1 X_2 \cdots X_N$ and $Y^1 Y^2 \cdots Y^N$ and performing the substitutions $X_i \leftrightarrow \chi_i$, $Y^i \leftrightarrow \xi^i$. We obtain operators

of the form,[10]

$$\begin{aligned} W &:= e^{\phi+\delta} \,, \\ W^A &:= \psi^A e^{\frac{N-1}{N}(\phi+\delta)} \,, \\ W^{AB} &:= \psi^A \psi^B e^{\frac{N-2}{N}(\phi+\delta)} \,, \\ \cdots &\quad \cdots \,. \end{aligned} \tag{7.53}$$

as well as

$$\begin{aligned} Z &:= e^{-\phi-\delta} \,, \\ Z_A &:= \prod_{i\neq A} \left(-\frac{1}{N}(J_\delta + J_\psi) + \psi^i \widetilde{\psi}_i \right) \widetilde{\psi}_A e^{-\frac{N-1}{N}(\phi+\delta)} \,, \\ Z_{AB} &:= \prod_{i\notin\{A,B\}} \left(-\frac{1}{N}(J_\delta + J_\psi) + \psi^i \widetilde{\psi}_i \right) \widetilde{\psi}_A \widetilde{\psi}_B e^{-\frac{N-2}{N}(\phi+\delta)} \,, \\ \cdots &\quad \cdots \,. \end{aligned} \tag{7.54}$$

To see that these operators enjoy an $\mathrm{SL}(N)_o$ outer automorphism symmetry, it is sufficient to note that the affine currents (7.52) have been written in a manifestly covariant form, and the monomial operators given here appear in first-order poles when taking repeated OPEs of the singlets W and Z with $J^{-,A}$ and J^+_A, respectively. This means, in particular, that the expressions in (7.54) that are not transparently covariant nevertheless always can be massaged into a covariant form.

7.2.3 Free-field realisation for the A_2 quiver gauge theory

Finally, we include more detail in the special case $N = 3$. In this case, we can complete the list of Z and W-type generators and put them in covariant form as expected. In particular, we have the very simple expressions for the W's,

$$\begin{aligned} W &:= e^{\phi+\delta} \,, \\ W^A &:= \psi^A e^{\frac{2}{3}(\phi+\delta)} \,, \\ W^{AB} &:= \psi^A \psi^B e^{\frac{1}{3}(\phi+\delta)} \,, \\ W^{ABC} &:= \psi^A \psi^B \psi^C \,. \end{aligned} \tag{7.55}$$

[10] To avoid confusion, we remind the reader that here $A, B, \cdots$ denote fundamental indices of the $\mathrm{SL}(N)_o$ outer automorphism symmetry.

Moreover, we have the following for the Z's,

$$\begin{aligned}
Z &\coloneqq e^{-\phi-\delta} \ , \\
Z_A &\coloneqq \left(\tfrac{1}{9}(\partial\delta + J_\psi)(J_\delta - 2J_\psi)\widetilde{\psi}_A - \tfrac{1}{3}(J_\delta + J_\psi)\widetilde{\psi}'_A - \tfrac{1}{12}\epsilon_{ABC}\epsilon^{DEF}\psi^B\psi^C\widetilde{\psi}_D\widetilde{\psi}_E\widetilde{\psi}_F\right) e^{-\frac{2}{3}(\phi+\delta)} \ , \\
Z_{AB} &\coloneqq \left(-\tfrac{1}{3}\left(\partial\delta + J_\psi\right)\widetilde{\psi}_A\widetilde{\psi}_B + \tfrac{1}{6}\epsilon_{ABC}\epsilon^{DEF}\psi^C\widetilde{\psi}_D\widetilde{\psi}_E\widetilde{\psi}_F\right) e^{-\frac{1}{3}(\phi+\delta)} \ , \\
Z_{ABC} &\coloneqq \widetilde{\psi}_A\widetilde{\psi}_B\widetilde{\psi}_C \ .
\end{aligned} \tag{7.56}$$

The $\mathrm{SL}(3)_o$ outer automorphism symmetry allows us to express the OPEs amongst these somewhat compactly. We report here some of the non-vanishing ones

$$\begin{aligned}
J_\phi(z)W^{AB\cdots}(w) &\sim \frac{-(3-|AB\cdots|)W^{AB\cdots}}{z-w} \ , \\
J_\phi(z)Z_{AB\ldots}(w) &\sim \frac{+(3-|AB\cdots|)Z_{AB\cdots}}{z-w} \ .
\end{aligned} \tag{7.57}$$

Here $|AB\cdots|$ simply denotes the number of indices. Then, we have

$$\begin{aligned}
J^{+,A}(z)W^{BC\cdots}(w) &\sim \frac{-W^{ABC\cdots}}{z-w} \ , \\
J^-_A(z)W^{BC\cdots}(w) &\sim \frac{\delta^B_A W^{C\cdots} + \delta^C_A W^{B\cdots} + \cdots}{z-w} \ ,
\end{aligned} \tag{7.58}$$

as well as

$$\begin{aligned}
J^-_A(z)Z_{BC\ldots}(w) &\sim \frac{Z_{ABC\cdots}}{z-w} \ , \\
J^{+,A}(z)Z_{BC\ldots}(w) &\sim \frac{\delta^A_B Z_{C\ldots} + \delta^A_C Z_{B\ldots} + \cdots}{z-w} \ .
\end{aligned} \tag{7.59}$$

The OPEs amongst cubic operators are slightly more complicated. For example, we have

$$\begin{aligned}
W(z)Z(w) \sim\ & \frac{1}{(z-w)^3} + \frac{J_\phi}{(z-w)^2} + \frac{1}{(z-w)}\left(\frac{1}{2}\partial J_\phi + \frac{1}{3}J_\phi J_\phi + \frac{1}{6}J_\phi J_\psi + \frac{1}{6}J_\psi J_\phi\right) \\
&+ \frac{1}{(z-w)}\left(-\frac{1}{6}J_\psi J_\psi + \frac{1}{2}\sum_{A=1}^{3}\left(J^{+,A}J^-_A - J^-_A J^{A,+}\right)\right) \ .
\end{aligned} \tag{7.60}$$

One may checked that the OPEs then close amongst the selected generators.

7.2.4 Stress tensors

In particular, the above VOA does not require a separate stress tensor as a generator. Indeed, there is a Sugawara-type expression for the stress tensor in terms of the various

currents. First, we have the stress tensors associated to the free fields,

$$\begin{aligned} T_\phi &:= -\frac{1}{6} J_\phi J_\phi \ , \\ T_\delta &:= \frac{1}{6}\left(J_\delta J_\delta - 3 J_\delta\right) \ , \\ T_{\psi^i} &:= \frac{1}{2}\left(\partial\psi^i \widetilde{\psi}_i - \psi^i \partial\widetilde{\psi}_i\right) \ , \end{aligned} \tag{7.61}$$

where the background charge for J_δ ensures that the vertex operators have the correct dimension. It can then be checked explicitly that in terms of the free fields, we have the identity

$$T_\phi + T_\delta + \sum_{i=1}^{3} T_{\psi^i} = T_{\mathrm{Sug}} \ , \tag{7.62}$$

where

$$T_{\mathrm{Sug}} := -\frac{1}{6}\left(J_\phi J_\psi + J_\psi J_\phi\right) + \frac{1}{3} J_\psi J_\psi - \frac{1}{2}\sum_{i=A}^{3}\left(J^{+,A} J_A^- - J_A^- J^{+,A}\right) \ . \tag{7.63}$$

This provides the required Sugawara-type construction in the A_2 case.

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Declarations

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Appendix A Conventions for the structure of (affine) $\mathfrak{psl}(2|2)$

In this appendix we establish our notation for the Lie superalgebra $\mathfrak{psl}(2|2)$ and its $\mathrm{SL}(2)_o$ group of outer automorphisms.

A.1 $\mathfrak{psl}(2|2)$

Starting with the larger superalgebra $\mathfrak{gl}(2|2)$, we denote by $\ell_a{}^b$ and $\mathscr{r}_\alpha{}^\beta$ a basis for the $\mathfrak{gl}(2) \times \mathfrak{gl}(2)$ (Grassmann-)even subalgebra. In reference to the realisation of these subalgebras as symmetries in the $T[\mathrm{SU}(2)]$ theory, we will refer to the subalgebra spanned by the $\ell_a{}^b$ as the *bosonic subalgebra*, and that spanned by $\mathscr{r}_\alpha{}^\beta$ as the *fermionic subalgebra*. As a basis for the (Grassmann-)odd part of the algebra we take $\vartheta_a{}^\alpha$, $(\widetilde{\vartheta})^b{}_\beta$. In matrix form, these are assembled as follows,

$$M = \left(\begin{array}{cc|cc} \ell_1{}^1 & \ell_1{}^2 & \vartheta_1{}^1 & \vartheta_1{}^2 \\ \ell_2{}^1 & \ell_2{}^2 & \vartheta_2{}^1 & \vartheta_2{}^2 \\ \hline \tilde{\vartheta}^1{}_1 & \tilde{\vartheta}^2{}_1 & \mathscr{r}_1{}^1 & \mathscr{r}_1{}^2 \\ \tilde{\vartheta}^1{}_2 & \tilde{\vartheta}^2{}_2 & \mathscr{r}_2{}^1 & \mathscr{r}_2{}^2 \end{array}\right) . \tag{A1}$$

The subalgebra which includes only super-traceless combinations of diagonal elements is identified with the simple Lie superalgebra $\mathfrak{sl}(2|2)$, and one can define the super-traceless version of the matrix elements of (A1) according to

$$\widehat{M}_a{}^b = M_a{}^b - \frac{1}{4}\delta_a{}^b \,\mathrm{Str}\, M \ . \tag{A2}$$

Further taking the quotient by the ideal generated by the *ordinary* trace defines $\mathfrak{psl}(2|2)$. In this quotient, one has that the $\ell_a{}^b$ and $\mathscr{r}_\alpha{}^\beta$ generate $\mathfrak{sl}_2$ even subalgebras (in particular, $\ell_1{}^1 + \ell_2{}^2 = \mathscr{r}_1{}^1 + \mathscr{r}_2{}^2 = 0$). Adopting identical notation as before for

our matrix elements, but *now denoting their avatars in* $\mathfrak{psl}(2|2)$, we have the following non-vanishing brackets,

$$\begin{aligned}
[\ell_a{}^b, X_c] &= +\delta_c^b X_a - \frac{1}{2}\delta_a^b X_c \ , \\
[\ell_a{}^b, Y^c] &= -\delta_a^c Y^b + \frac{1}{2}\delta_a^b Y^c \ , \\
[\mathscr{k}_\alpha{}^\beta, X_\gamma] &= +\delta_\gamma^\beta Y^\alpha - \frac{1}{2}\delta_\alpha^\beta X_\gamma \ , \\
[\mathscr{k}_\alpha{}^\beta, Y^\gamma] &= -\delta_\alpha^\gamma Y^\beta + \frac{1}{2}\delta_\alpha^\beta X_\gamma \ , \\
\{\vartheta_a{}^\alpha, \tilde{\vartheta}^b{}_\beta\} &= +\delta_a^b \mathscr{k}_\alpha{}^\beta + \delta_\alpha^\beta \ell_a{}^b \ ,
\end{aligned} \tag{A3}$$

with the other anti-commutators involving ϑ_α^a and $\tilde{\vartheta}_b^\beta$ being zero. Here X is a placeholder for any elements of the algebra with the appropriate index structure.

This superalgebra enjoys the action of an $\mathrm{SL}(2)_o$ outer automorphism group. This acts only on the Grassmann odd part of the algebra, and mixes the upper-right-hand and lower-left-hand blocks of (A1). In particular, if we define the following alternative notation for the odd entries in the lower-left-hand block of the matrix in A1,

$$(\vartheta^{(1)})_a{}^\alpha = \vartheta_a{}^\alpha \ , \qquad (\vartheta^{(2)})_a{}^\alpha = \epsilon_{ab}(\tilde{\vartheta})^b{}_\beta \epsilon^{\beta\alpha} \ , \tag{A4}$$

then the $(\vartheta^{(A)})_a{}^\alpha$ transform in the standard representation of $\mathrm{SL}(2)_o$ with A being a fundamental index. The last (anti)commutator in (A3) then becomes

$$\{(\vartheta^{(A)})_a{}^\alpha, (\vartheta^{(B)})_b{}^\beta\} = \epsilon^{AB}\left(\epsilon^{\alpha\beta}\ell_{ab} + \epsilon_{ab}\,\mathscr{k}^{\alpha\beta}\right) \ . \tag{A5}$$

It is a simple exercise to check that the defining commutation relations are invariant under these outer automorphism rotations.

A.2 $V^k(\mathfrak{psl}(2|2))$

For the current algebra corresponding to the $\mathfrak{psl}(2|2)$ Lie superalgebra, we promote ℓ, $\mathscr{k}$, and ϑ to affine currents L, J, and Θ, with non-vanishing singular OPEs (at level

k) given by

$$\begin{aligned}
L_{ab}(z)L_{cd}(w) &\sim \frac{\frac{1}{2}k(\epsilon_{bc}\epsilon_{ad}+\epsilon_{ac}\epsilon_{bd})}{(z-w)^2} - \frac{\frac{1}{2}\left(\epsilon_{ac}L_{bd}(w)+\epsilon_{bc}L_{ad}(w)+\epsilon_{ad}L_{bc}(w)+\epsilon_{bd}L_{ac}(w)\right)}{z-w} \,, \\
J^{\alpha\beta}(z)J^{\gamma\delta}(w) &\sim \frac{-\frac{1}{2}k(\epsilon^{\beta\gamma}\epsilon^{\alpha\delta}+\epsilon^{\alpha\gamma}\epsilon^{\beta\delta})}{(z-w)^2} - \frac{\frac{1}{2}\left(\epsilon^{\alpha\gamma}J^{\beta\delta}(w)+\epsilon^{\beta\gamma}J^{\alpha\delta}(w)+\epsilon^{\alpha\delta}J^{\beta\gamma}(w)+\epsilon^{\beta\delta}J^{\alpha\gamma}(w)\right)}{z-w} \,, \\
\Theta^{\alpha}_{Aa}\Theta^{\beta}_{Bb} &\sim -\frac{k\epsilon_{AB}\epsilon^{\alpha\beta}\epsilon_{ab}}{(z-w)^2} + \frac{\epsilon_{AB}\left(\epsilon_{ab}J^{\alpha\beta}(w)+\epsilon^{\alpha\beta}L_{ab}(w)\right)}{z-w} \,, \\
L_{ab}(z)\Theta^{\gamma}_{Ac} &\sim -\frac{\epsilon_{bc}\Theta^{\gamma}_{Aa}(w)+\epsilon_{ac}\Theta^{\gamma}_{Ab}(w)}{z-w} \,, \\
J^{\alpha\beta}(z)\Theta^{\gamma}_{Aa} &\sim -\frac{\epsilon^{\beta\gamma}\Theta^{\alpha}_{Aa}(w)+\epsilon^{\alpha\gamma}\Theta^{\beta}_{Aa}(w)}{z-w} \,.
\end{aligned} \tag{A6}$$

Here the trace parts of the two $\mathfrak{sl}_2$ subalgebras are absent, so $L_{12} = L_{21}$ and $J^{12} = J^{21}$, and we recall our conventions that $\epsilon^{12} = \epsilon_{21} = 1$.

Appendix B Lattice vertex algebras and bosonisations of elementary fields

In this appendix, we establish notation and recall standard results used in the discussion of bosonisation and lattice vertex algebras used in the analysis of VOAs for Abelian gauge theories in three dimensions.

B.1 Lattice vertex algebras

Given an Abelian Lie algebra $\mathfrak{h}$ equipped with a bilinear form $(,)$, let $\widetilde{\mathfrak{h}} \cong \mathfrak{h}[t,t^{-1}]$ denote the loop algebra for $\mathfrak{h}$, and let $\widehat{\mathfrak{h}}$ denote its central extension according to the given bilinear form, with underlying vector space $\widetilde{\mathfrak{h}} \oplus \mathbb{C}K$ and Lie bracket

$$[at^m, bt^n] = mK\delta_{m+n,0}(a,b) \,. \tag{B1}$$

We denote the graded components (with loop grading) of $\widehat{\mathfrak{h}}$ by $\mathfrak{h}_n$, $n \in \mathbb{Z}$,

$$\mathfrak{h}_{n\neq 0} = \mathrm{span}\{at^n \mid a \in \mathfrak{h}\} \,, \qquad \mathfrak{h}_0 = \mathrm{span}\{K; a \mid a \in \mathfrak{h}\} \,. \tag{B2}$$

and we have the grade decomposition

$$\widehat{\mathfrak{h}} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{h}_n \,. \tag{B3}$$

It proves useful to introduce various (Abelian) subalgebras associated with the semi-infinite structure on $\widehat{\mathfrak{h}}$ (*cf.* [**?**]),

$$\begin{aligned} \mathfrak{b}_- &= \bigoplus_{i\leqslant 0} \mathfrak{h}_i \ , \qquad \mathfrak{b}_+ = \bigoplus_{i\geqslant 0} \mathfrak{h}_i \ , \\ \mathfrak{n}_- &= \bigoplus_{i< 0} \mathfrak{h}_i \ , \qquad \mathfrak{n}_+ = \bigoplus_{i> 0} \mathfrak{h}_i \ . \end{aligned} \tag{B4}$$

From $\widehat{\mathfrak{h}}$ one constructs the corresponding Heisenberg vertex operator algebra/Abelian affine current algebra with underlying vector space (vacuum module)

$$V_{\mathfrak{h}} = \mathcal{U}(\mathfrak{h}) \otimes_{\mathcal{U}(\mathfrak{b}_+)} \mathbb{C}_{1,0} \ , \tag{B5}$$

where $\mathbb{C}_{1,0}$ is the one-dimensional $\mathfrak{b}_+$-module on which K acts as the identity and all other basis elements act as zero. This vertex algebra is strongly generated by affine currents $\{J_{a_i}(z)\}$, where the a_i form a basis for $\mathfrak{h}$, with singular OPEs taking the form

$$J_a(z)J_b(w) \sim \frac{(a,b)}{(z-w)^2} \ . \tag{B6}$$

This vacuum module is the simplest example of a Fock module for $\widehat{\mathfrak{h}}$.

Definition B.1 (Fock module)**.** For any $\lambda \in \mathfrak{h}^*$, we denote by $\mathcal{F}_\lambda$ the *Fock module of charge* λ, which is the induced module

$$\mathcal{F}_\lambda = \mathcal{U}(\mathfrak{h}) \otimes_{\mathcal{U}(\mathfrak{b}_+)} \mathbb{C}_{1,\lambda} \ , \tag{B7}$$

where the action of $\mathfrak{h}_0$ in the one-dimensional representation $\mathbb{C}_{1,\lambda}$ is given by λ. We will usually abuse notation and write Fock modules with $\lambda \in \mathfrak{h}$ with the corresponding element of $\mathfrak{h}^*$ arising via the bilinear form on $\mathfrak{h}$.

Certain collections of Fock modules can be given the structure of a vertex operator algebra or a vertex operator superalgebra, extending the vertex algebra structure on $V_{\mathfrak{h}}$.[11]

Definition B.2 (Lattice vertex (super)algebra)**.** For any integral lattice (with respect to the bilinear form $(,)$) $L \subset V$, define the $V_{\mathfrak{h}}$ module

$$M_L = \bigoplus_{\lambda\in L} \mathcal{F}_\lambda = \bigoplus_{\lambda\in L} V_{\mathfrak{h}} \cdot e^\lambda \ . \tag{B8}$$

Then M_L can be given the structure of a vertex operator superalgebra, the *lattice vertex (super)algebra* V_L. In V_L, vectors lying in Fock modules corresponding to points in L with even/odd squared length are assigned even/odd Grassmann parity.

[11] This sketch of a definition is primarily intended to establish notation; for complete details on the construction of the vertex algebra structure on V_L, see, *e.g.*, [**?**].

B.2 Bosonisation of the symplectic boson

The symplectic bosons and free fermions appearing in the gauge theory description of A-twisted boundary vertex algebras can both be realised as (subalgebras of) lattice vertex algebras of the type just reviewed. We start with the case of the symplectic bosons vertex algebra following [**?**] (see also the recent comprehensive discussion in [**?**]).

Consider the integral lattice $L_{\mathtt{Sb}} = (\mathbb{Z}\rho \oplus \mathbb{Z}\sigma)$, where $(\rho,\rho) = 1$ and $(\sigma,\sigma) = -1$. Let $\mathfrak{h}_{\mathtt{Sb}} = L_{\mathtt{Sb}} \otimes_{\mathbb{Z}} \mathbb{C}$; the corresponding Heisenberg vertex algebra $V_{\mathfrak{h}_{\mathtt{Sb}}}$ is generated by currents $\{J_\rho, J_\sigma\}$ obeying

$$J_\rho(z)J_\rho(w) \sim \frac{1}{(z-w)^2} \ , \qquad J_\sigma(z)J_\sigma(w) \sim \frac{-1}{(z-w)^2} \ . \tag{B9}$$

The corresponding lattice vertex superalgebra has underlying vector space

$$V_{L_{\mathtt{Sb}}} \coloneqq \bigoplus_{\lambda \in L_{\mathtt{Sb}}} \mathcal{F}_\lambda \ , \tag{B10}$$

where the vacuum state of a given Fock module is represented by the vertex operator e^λ.

Next consider the isotropic sublattice of $L_{\mathtt{Sb}}$ given by $L^!_{\mathtt{Sb}} = \mathbb{Z}(\rho_i - \sigma_i)$. Restricting the Fock modules in (B10) to lie in $L^!_{\mathtt{Sb}}$ then gives the canonical example of a *half-lattice vertex algebra* [**?**], which we denote by

$$V^!_{L_{\mathtt{Sb}}} = \bigoplus_{\lambda \in L^!_{\mathtt{Sb}}} V_{\mathfrak{h}} \cdot e^\lambda \ . \tag{B11}$$

The symplectic boson VOA can be realised within $V^!_{L_{\mathtt{Sb}}}$ according to either of two embeddings (*cf.* [**?**]),

$$\begin{pmatrix} X \\ Y \end{pmatrix} \underset{\mathtt{i}^+}{\longmapsto} \begin{pmatrix} e^{\rho-\sigma} \\ J_\rho e^{-\rho+\sigma} \end{pmatrix} \ , \qquad \begin{pmatrix} X \\ Y \end{pmatrix} \underset{\mathtt{i}^-}{\longmapsto} \begin{pmatrix} -J_\rho e^{-\rho+\sigma} \\ e^{\rho-\sigma} \end{pmatrix} \ , \tag{B12}$$

with the internal Heisenberg subalgebra generated by the current

$$(XY) = \pm J_\sigma \ . \tag{B13}$$

Here the sign on the right hand side is correlated with the embedding choice.

It is instructive to note that this half-lattice vertex algebra is isomorphic to the algebra of global sections of the sheaf of chiral differential operators [**?**] on $\mathbb{C}^\times$,

$$V_{L^!_{\mathtt{Sb}}} \cong \mathcal{D}^{ch}(\mathbb{C}^\times) \ . \tag{B14}$$

On the other hand, symplectic bosons themselves are the global sections of CDOs on the affine line,

$$\mathtt{Sb} \cong \mathcal{D}^{ch}(\mathbb{C}) \ . \tag{B15}$$

The half lattice vertex algebra can thus be thought of as arising by (chiral) localisation from $\mathbb{C}$ to $\mathbb{C}^\times$, and the free field realisation/bosonisation of the symplectic bosons amounts to corresponding restriction of sections.

For both of the embeddings in (B12), we can define the *screening charge* $\mathfrak{s}_\pm$ acting on $V_{L^!_{\mathtt{Sb}}}$,

$$\begin{aligned} \mathfrak{s}_\pm : V_{L^!_{\mathtt{Sb}}} &\longrightarrow \bigoplus_{\lambda \in \rho + L^!_{\mathtt{Sb}}} \mathcal{F}_\lambda \\ v &\longmapsto \operatorname{Res} e^{\rho(z)} v \ . \end{aligned} \tag{B16}$$

Then the image of the inclusions $\mathfrak{i}^\pm$ can be characterised as precisely the kernel of the screening charge [**?**],

$$\mathtt{Sb} \cong \operatorname{Im} \mathfrak{i}_\pm = \operatorname{Ker} \mathfrak{s}_\pm \ . \tag{B17}$$

B.3 Bosonisation of the free fermion

Complex free fermions can also famously be bosonised. Take the unit lattice $L_{\mathtt{Ff}} = \mathbb{Z}\gamma$ with $(\gamma, \gamma) = 1$, the corresponding Heisenberg current J_γ obeys

$$J_\gamma(z) J_\gamma(w) \sim \frac{1}{(z-w)^2} \ . \tag{B18}$$

Then the lattice vertex superalgebra $V_{L_{\mathtt{Ff}}}$ is isomorphic to the free fermion vertex algebra $\mathtt{Ff}$,

$$V_{L_{\mathtt{Ff}}} \cong \mathtt{Ff} \ , \tag{B19}$$

with the identification given by determined by either of the following (equivalent) identifications of strong generators,

$$\begin{pmatrix} \chi \\ \xi \end{pmatrix} \overset{\longleftrightarrow}{+} \begin{pmatrix} e^{\gamma} \\ e^{-\gamma} \end{pmatrix} \ , \qquad \text{or} \qquad \begin{pmatrix} \chi \\ \xi \end{pmatrix} \overset{\longleftrightarrow}{-} \begin{pmatrix} e^{-\gamma} \\ e^{\gamma} \end{pmatrix} \ . \tag{B20}$$

As in the symplectic boson case, the internal Heisenberg current of the free fermion vertex algebra is realised with a relative sign depending on the choice of identification convention,

$$(\xi\chi) = \pm J_\gamma \ . \tag{B21}$$

Appendix C Change of basis and its inverse

We derive the inverse of the change of basis first introduced in (**??**) in the special case of $T[\mathrm{SU}(2)]$ and then more generally in (3.27). This is given by a matrix

$$P = \left(\begin{array}{c|c|c} Q & 0 & Q \\ \hline Q & -Q & 0 \\ \hline \widetilde{Q} & 0 & 0 \\ \hline 0 & -\widetilde{Q} & 0 \\ \hline F & -F & \mathbf{1}_{N\times N} \end{array}\right) \tag{C1}$$

where

$$F^T = \begin{pmatrix} Q \\ \widetilde{Q} \end{pmatrix}^{-1} \begin{pmatrix} Q \\ 0 \end{pmatrix} \ . \tag{C2}$$

Q and $\widetilde{Q}$ are orthogonal. Let us introduce

$$A = \begin{pmatrix} Q \\ \widetilde{Q} \end{pmatrix} \ , \ B = \begin{pmatrix} Q \\ 0 \end{pmatrix} \ , \ C = \begin{pmatrix} 0 \\ \widetilde{Q} \end{pmatrix} \ , \tag{C3}$$

so that

$$F = A^{-1}B \ . \tag{C4}$$

Using the identities

$$BC^T = 0 \ , \ A^{-1}B = \mathbf{1}_{N\times N} - A^{-1}C \tag{C5}$$

it is easy to verify that F is idempotent (in fact, F^T is by definition a projector onto the span of Q) and that moreover

$$FF^T = F \ . \tag{C6}$$

Now swapping the second and third row-blocks in P leads to the matrix

$$\widetilde{P} = \left(\begin{array}{c|c|c} A & 0 & B \\ \hline B & -A & 0 \\ \hline (A^{-1}B)^T & -(A^{-1}B)^T & \mathbf{1}_{N\times N} \end{array}\right) \ . \tag{C7}$$

This can be thought of as a 2×2 block matrix with the bottom-right block being the identity, and the upper-left block (which is itself a 2×2 block) being invertible. The inverse can then be computed by repeated use of standard formulæfor 2×2 block-matrices. The result simplifies thanks to the identities

$$F^T A^{-1} = F A^{-1} = \left(A^{-1}\right)^{(k)} \tag{C8}$$

where $\left(A^{-1}\right)^{(k)}$ is a matrix obtained from A^{-1} by setting to zero the last $(N-k)$ columns. We find

$$\widetilde{P}^{-1} = \left(\begin{array}{c:c:c} A^{-1} & \left(A^{-1}\right)^{(k)} & -F \\ \hdashline \left(A^{-1}\right)^{(k)} & -A^{-1}+\left(A^{-1}\right)^{(k)} & -F \\ \hdashline 0 & -\left(A^{-1}\right)^{(k)} & \mathbf{1}_{N\times N} \end{array}\right) . \tag{C9}$$

From here, the inverse P^{-1} is recovered by exchanging columns $\{k, k+1, \ldots, N\}$ with columns $\{N+1, N+2, \ldots, 2N-k\}$.