Information-Theoretical Witnesses of Non-Classicality in Hybrid Systems



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A thesis submitted for the degree of

Doctor of Philosophy

Hilary 2026

Acknowledgements

I would like to express my deepest gratitude to my supervisors and collaborators, Chiara Marletto and Vlatko Vedral, for guiding me throughout this incredible journey. Their deep and profound knowledge of quantum physics has always inspired and influenced me, fuelling the curiosity that motivated me to pursue this DPhil. They encouraged me to always aim high and involved me in any activity that could benefit my DPhil in any way. Above all, however, I will always be grateful to them for giving me the most precious resource in academia: time to think about what truly interests me.

Allow me to spend a few more words thanking Chiara. Qualifying her simply as a supervisor or a collaborator would be reductive. She is the person who helped me flourish academically and made me the researcher I am today. Our constant conversations have provided me with unique and enlightening perspectives, both academically and beyond. She has never failed to offer her support and encouragement, and what I appreciated about her right from the start is her ability to turn every situation, even the most annoying, into an opportunity for growth and improvement. She is an invaluable mentor on my journey, and I will always be grateful to her.

I am grateful for the opportunity to have worked with amazing collaborators: Chiara Marletto, Vlatko Vedral, Simone Rijavec, Salvatore Raia, Fabrizio Piacentini, Matteo Bordin, Ettore Bernardi, Ekaterina Moreva, Carmine Napoli, Ivo Pietro Degiovanni, Marco Genovese, Gaurav Bhole, James Eaton, Andrew J. Baldwin, Jonathan A. Jones, Aditya Iyer, Samuel Hagh Shenas, Erik Rydow, Yao-Chih Kuo, Antonia Weber, Eduardo Dias and Mia Stamatova. Each and every one of them has taught me something new that has contributed not only to the development of the projects I present in this Thesis and those I am actively working on, but also to my growth as a researcher. A special thanks goes to Aditya, Samuel and Antonia for taking the time to read parts of this thesis and provide precious comments that helped improve its final version.

In a field that can be very isolating such as theoretical physics, I have been fortunate enough to find a group of people who became my scientific family: the past and present members of the "Frontiers of Quantum Information" research group. A heartfelt thanks therefore goes to: Abdulla Alhajri, Aditya Iyer, Amy

Searle, Antonia Weber, Aroosa Ijaz, Charles Bédard, Cosmin Andrei, Eduardo Dias, Jiaxuan Zhang, Jinzhao Sun, Jerry Qu, Lodovico Scarpa, Lucia Vilchez-Estevez, Maria Violaris, Mattheus Burkard, Mia Stamatova, Michele Minervini, Nicetu Tibau Vidal, Salvatore Raia, Samuel Hagh Shenas, Samuel Kuypers, Simone Rijavec, and Tom O'Leary. Each and every one of you has made a unique contribution to making this an unforgettable experience, not only in terms of the science but also, and above all, outside of work. I believe there are very few groups that are as fun and sociable as this one, where you can learn so much between meals and memes. Among the members of the group, I would like to say a special thank you to Tristan Farrow. He has always been there for me whenever I needed, teaching me the experimental side of things and providing precious advice that I will always carry with me.

Oxford is a really special place, where you make lifelong friends that already are, or will soon be, spread all over the world. Among these friends, a special thank you goes to: Mehtab Hayre, Xiao Sun, Erik Rydow, Eryk Ratajczyk, Harrison Nicholls, Hao Zha, Shreya Jain, Katerina Szyllo, Ziyun Liang, Gladson Vaghela, Jairus Patock, Giorgio Canalella, Lorenzo Versini, Yao-Chih Kuo, Rayanika Dutta, Sofia Marchetti and Kasturi Pindar.

I would also like to thank Fabio Anzà and Tad Komacek, whom I was privileged to have as mentors. They guided me through some of the most important decisions of my career with clear-sighted and expert insight.

I would not be writing this acknowledgements section without a group of people that, on the personal side, have been walking alongside me since long before the beginning of this journey. The rest of this section is for them, and please bear with me if I write the following in Italian.

È difficile condensare in poche righe il ringraziamento che andrebbe fatto alla persona più importante della mia vita, a Roberta. Tra tutti, lei c'è sempre stata, quotidianamente, instancabilmente, costantemente. Anche nei momenti più impegnativi di questo percorso e non solo, lei non ha mai smesso di credere in me, dandomi l'energia necessaria per superarli e diventare la versione migliore di me stesso. Mi ritengo davvero fortunato a poter percorrere il cammino della vita al suo fianco: avere un faro ad illuminarlo è una vera benedizione, e io ne sono profondamente grato. Spero di essere anche solo un decimo per lei di quanto lei è per me. Quindi grazie per aver scelto di condividere tutto questo con me, per essere semplicemente te stessa, per avere reso la mia vita bella da morire.

A Miriam e Alfonso, amici fraterni di una vita, persone che sanno rendere speciale qualunque momento e stare insieme alle quali è sempre una festa. Anche se ormai sparsi per l'Italia, è come se in realtà non ci fossimo mai separati. La sensazione è così forte che mi viene quasi da pensare che MiriammeFofò si scriva tutto unito. Grazie per la vostra leggerezza.

A Martina, sorella per scelta. Ne abbiamo fatta di strada da quel banco di scuola, ma anche oggi, a migliaia di chilometri di distanza, so che come allora lei rappresenta un punto di riferimento imprescindibile. E no, non soltanto perché ormai ho una dottoressa personale pronta a salvarmi la vita in qualsiasi momento. Grazie per esserci.

A Paola, la mia cogni, che ha sempre reso ogni ritorno a casa divertente ed indimenticabile grazie alle sue continue organizzazioni. Ci ha portati a conoscere posti in Sicilia ormai leggendari, quelli di cui si racconta intorno al fuoco di una sera d'estate, sul quale lei mette anche ad arrostire della salsiccia. Grazie per questi momenti.

A Simone, persona così speciale nella mia vita da spacciarlo per mio cugino giusto per aggiungere quel legame in più che viene dalla famiglia. Principale mia fonte di gossip paesano, non posso che ringraziarlo per avere fin da sempre creduto in me.

A Samuel e Adi, per le nostre infinite chiacchierate che vanno dalla fisica quantistica fino a come curare uno strappo al gluteo. L'uno ha portato nuova linfa nel gruppo, l'altro ne è il cuore pulsante, entrambi mi hanno insegnato tanto e ho ancora molto da imparare da loro. Grazie per la quotidianità recente.

A Paola, Marco, Stefano, e Giulia, i miei fisichi fantastichi, che ho avuto la fortuna di incontrare già dai primi passi in questo campo. Loro sono la mia fonte di aggiornamento su tutte quelle tematiche di cui non capisco niente ma, soprattutto, di risate assicurate.

Questi ringraziamenti non sarebbero completi se non dedicassi qualche parola alle persone che mi hanno accompagnato alla scoperta del meraviglioso mondo della fisica. Il primo grazie va certamente alla mia insegnante delle scuole superiori, Professoressa Annamaria Lopiano. Le sue lezioni hanno fatto sbocciare in me l'amore per questa materia. I suoi continui stimoli l'hanno fatto crescere. Il suo supporto mi ha spinto a scegliere questo percorso. Sulla stessa lunghezza d'onda, voglio ringraziare di vero cuore la Professoressa Maria Petitto, che mi ha sempre valorizzato, coinvolgendomi nei più disparati e diversi progetti a scuola e anche oltre pur di far crescere la mia passione per la materia.

Un grazie particolarmente sentito va alla Professoressa Maria Grazia Falzone, che posso permettermi di chiamare zia in quanto segue. Appuntamento fisso di ogni ritorno a casa, le chiacchierate con lei sono sempre state di ispirazione per me, il suo entusiasmo sempre contagioso, la sua attenzione nei miei confronti sempre genuina. E poi, vuoi mettere avere una zia che legge e capisce i tuoi paper?

Infine, un posto speciale in questa sezione non può che essere occupato da mia mamma e mio papà, Maria Assunta e Luigi. Loro sono le fondamenta della mia vita, il mio costante ed eterno supporto. Sono loro che mi hanno insegnato, tra le innumerevoli cose, ad essere curioso, a credere in me, ad inseguire i miei sogni, ad essere libero. In tutti questi anni, ho sentito la loro forza dietro di me, a coprirmi le spalle. Sono orgoglioso di loro.

Abstract

Quantum theory is one of the most successful theories in modern physics. It has been extensively tested in the microscopic domain, always with remarkable success. Yet, these tests do not currently extend in a conclusive way to all physical systems and across all scales. This leaves open the question of whether quantum theory provides a truly universal description of reality, especially in regimes where direct experimental access is limited, such as the gravitational field or complex macroscopic biological systems.

This Thesis addresses this question through the study of hybrid systems, where a well-controlled quantum sector interacts with an unknown sector whose underlying description may or may not be quantum. It contributes to the emerging framework of information-theoretical witnesses of non-classicality, where the quantum sector is used as a probe to test the nature of the unknown system under minimal assumptions.

We first examine spatial entanglement-based witnesses, in which the ability of an unknown system to mediate entanglement between two quantum probes by local means provides a sufficient condition for its non-classicality. We extend this framework by quantitatively showing that locally mediated quantum correlations lower bound the non-commutativity of the observables of the mediator, within quantum theory.

We then introduce a novel class of single-probe temporal witnesses, where non-classicality is inferred from the temporal evolution of a probe under a global conservation law. We prove that quantum coherent evolution under such a constraint can only be mediated by a non-classical system, and propose an application to light-harvesting processes in photosynthetic complexes.

Finally, we develop a temporal entanglement-based witness. Here, the sufficient condition for the non-classicality of the unknown system is the ability to mediate temporal entanglement relative to the probe's local conserved quantity and a non-commuting observable. This result provides a temporal extension of the spatial entanglement-based witness, representing a promising route for the development of the next generation of tests to probe the universality of quantum theory in previously inaccessible fields such as gravity and biology.

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The quantization of a given system implies also the quantization of any other system to which it can be coupled.
Quantum theory must be immediately extended to all physical systems.

— Bryce DeWitt, *The Quantization of Geometry*

1 Introduction

The problem of whether a physical theory provides a universal description of reality is a long-standing question in the history of physics. Prior to the advent of quantum mechanics, classical physics was widely regarded as the ultimate framework for describing nature. This belief is perhaps best summarised in the famous statements attributed to Michelson and Lord Kelvin, according to which the fundamental laws of physics had already been discovered, leaving only refinements of precision and just two "clouds" on the horizon of twentieth-century physics [1, 2]. In hindsight, these two clouds marked the origin of two of the most profound revolutions in physics. The resolution of one led to the development of general relativity, which, despite its conceptual novelty, remains fully consistent with the classical paradigm that dominated at the time. The resolution of the other, instead, gave rise to quantum theory, whose implications radically overturned the view that classical physics could provide a complete and universal description of reality.

Quantum theory is now one of the best explanatory theories in modern physics. Introducing peculiar phenomena such as superposition and entanglement, it provides extraordinarily accurate predictions at the smallest scales and across those macroscopic phenomena that trivially rely on their microscopic constituents. However, the difficulty in observing precisely these phenomena at progressively larger scales leaves open the question of the effective domain of applicability of quantum theory.

In other words, does quantum theory provide a universal description of all physical systems at all scales? Or does classical behaviour fundamentally describe reality above a certain scale?

In this Thesis, we explore these issues by searching for indirect signatures of the non-classical nature of physical systems using information-theoretical methods, namely by analysing how physical systems process and exchange information. These signatures are known as *information-theoretical witnesses of non-classicality*.

1.1 The Problem of Hybrid Systems

Many of the questions concerning the universality of quantum theory can be reduced to the study of a particular kind of system, which we shall call a *hybrid system*. A hybrid system consists of two interacting sectors:

- a *quantum sector* Q, composed of systems that are well described by quantum theory (such as electrons, photons or, more generally, qubits);
- an *unknown sector* M, whose underlying physical description may or may not be quantum (e.g., classical).

From a historical perspective, one can find instances of hybrid systems in Heisenberg's analysis of charged quantum particles interacting with the electromagnetic field, which took place at a time when it was not yet clear whether the field itself should be quantised [3]. Similarly, the measurement problem in quantum mechanics involves a quantum system interacting with a measurement apparatus whose status, quantum or classical, has been the subject of long-standing debate [4, 5]. In contemporary physics, hybrid systems appear in several key research directions. The problem of quantum gravity is perhaps the most prominent example, involving the interaction between quantum matter and a gravitational field whose quantum nature remains under investigation. Likewise, quantum biology explores the possibility that non-trivial quantum effects may play a role in biological processes, raising the question of whether such complex and macroscopic systems can sustain genuinely quantum behaviour when excited by a quantum system, for example a photon.

The problem when studying hybrid systems is whether it is possible to make consistent predictions when a fully classical sector couples to a fully quantum one. Answering this question would therefore contribute to the understanding of quantum theory as a universal description of reality by extending its domain of applicability to include the investigated system M.

Given the relevance of hybrid systems for the foundations of physics, several *hybrid schemes* have been proposed to solve this problem. A hybrid scheme is a model that aims to consistently couple a quantum sector with a classical sector, capturing both quantum and classical dynamics. These schemes must satisfy a number of requirements, firstly identified by Boucher and Traschen in [6], to be fully consistent in a broader physical picture. Therefore, different hybrid schemes adopt different *model-dependent assumptions* in an attempt to meet these requirements (see Ref. [7] for a review).

Among the consistency requirements, the most commonly violated are standard conservation laws, namely conservation of additive quantities defined on the hybrid system such as energy and momentum. For example, attempts to define the dynamics through a hybrid bracket constructed as a linear combination of classical Poisson brackets and quantum commutators lead to a loss of antisymmetry. This, in turn, results in the non-commutativity of time-independent Hamiltonians, implying energy non-conservation (as well as non-positivity of the classical state) [8]. Many alternative formulations of hybrid brackets aimed at restoring conservation laws introduce different inconsistencies, ranging from time-irreversible dynamics to non-positivity of the resulting quantum density matrices [9–15]. Another example is represented by the Koopman-von Neumann representation of classical mechanics, in which classical states are described by classical wave functions of the Hilbert space of square-integrable complex-valued functions. In this framework, the classical degrees of freedom can be encoded into those of two quantum systems [16–18]. Specific choices of the gauge potential defining the covariant Liouvillian introduce unobservable classical terms in the hybrid dynamics [19], leading either to violation of the energy conservation or to the failure to recover the correct classical and quantum

equations of motion in isolation [20–23]. A final example is given by completely positive (CP) hybrid schemes, where irreversibility is introduced in the linear theory to preserve positivity of probabilities in the classical and quantum sectors [24–26]. These schemes include a mechanism responsible for the classicality of one of the sectors in the hybrid [27, 28]. The key element of this mechanism is decoherence, mathematically described by coarse-graining procedures such as coupling to a thermal bath or to a measuring device, whose degrees of freedom are subsequently traced out [29, 30]. To maintain positivity, decoherence must be accompanied by diffusion in the classical phase space, equivalent to introducing stochasticity into the hybrid dynamics [31]. However, accounting for the backreaction of the quantum sector on the classical one imposes a trade-off: maintaining coherence for a finite time requires injecting a corresponding amount of noise into the classical sector, and vice versa [32]. As a consequence, any non-zero backreaction is tied to a minimal dissipation budget even in closed hybrid systems, which leads to the violation of conservation laws and to the failure of Noether's theorem [33].

While many of these arguments show inconsistencies when a fully classical sector is coupled to a quantum one, their model dependence prevents a conclusive answer about the impossibility of constructing a consistent hybrid theory. This is because, as illustrated by the examples above, it is always possible to modify the model to overcome the inconsistency of the previous one. It therefore remains possible that the right model to describe a particular hybrid has to be discovered yet, and that this model can work for one hybrid system but not for all of them.

Another relevant line of reasoning approaches the problem arguing that the coupling of a quantum system to a classical one inevitably leads to the spread of the quantum features in the classical sector, making it impossible to consistently couple the two [34]. This argument was originally proposed by Bryce DeWitt and is now known as the totalitarian property of quantum theory [35]. However, this and similar arguments have also been criticised on two main grounds. The first is that, like in the case of hybrid schemes, the totalitarian property as currently formulated relies on very strong assumptions, particularly the unitarity of the quantum sector

as well as the particular model dependence of the coupling between the quantum and classical sectors. The second is that testing these claims requires full quantum control of the system that is argued to be quantised, but the existence of such a control is what the test is supposed to probe, creating a loophole in the argument.

The most direct way to test the non-classicality of the system M would be to directly measure it and verify the existence of two incompatible degrees of freedom. However, the very motivation for considering hybrid systems is that the "unknown" sector M is often inaccessible or extremely difficult to probe in isolation. For example, when considering the gravitational field as M, the 42 orders of magnitudes separating the gravitational fine structure constant from the electromagnetic fine structure constant would imply a waiting time of 10^{36} seconds before a graviton (the hypothetical quantum of gravitational field) is spontaneously emitted by an atom. This is 19 orders of magnitude more than the age of the universe [36].

Therefore, the problem of the hybrid system calls for the introduction of novel techniques that can allow one to infer the non-classical nature of the potentially classical sector M in a conclusive way. These tests (i) should rely on minimal assumptions obeyed by the laws of physics, not just by quantum or classical theory. In doing so, they can rule out *all* the classical descriptions of M that obey that set of assumptions [37], not only the particular chosen model; (ii) should not require any quantum control on the system under investigation M, as their purpose is precisely to establish the existence of such a control [38].

This leads us to the introduction of the main topic of this Thesis: information-theoretical witnesses of non-classicality and their application to hybrid systems.

1.2 Witnesses of Non-Classicality

The idea of developing a class of tests satisfying the requirements (i) and (ii) was first introduced by Marletto and Vedral in Refs. [37, 38], inspired by Constructor Theory (CT) [39]. CT is a research programme with the goal of generalising quantum computation beyond the quantum formalism to include all physical processes, constrained solely by the laws of physics. To this end, it introduces principles

based on statements about which transformations are possible and impossible on a physical system, and why. These transformations are referred to as *tasks*. Therefore, in CT, statements about tasks supplement existing dynamical laws, enabling predictions in regimes where those laws are unable to make progress alone, and can be applied to all relevant physical systems independently of their scale. Hybrid systems, as discussed in Chapter 1.1, are characterised in general by an unspecified hybrid dynamics (it may not be classical, nor quantum, but potentially more general) and by an unspecified scale (they may be composed of macroscopic systems interacting with microscopic ones, as in the case of biology). Therefore, the principle-based approach provided by CT represents a promising route to make consistent predictions about them.

In Ref. [37], Marletto and Vedral proposed a generalisation of the DeWitt's totalitarian property using the constructor theory of information [40]. Within the framework of *superinfomation theories*, which allow one to unify classical, quantum and post-quantum theories under the same language of how they can encode and process information, they showed that if a superinformation medium Q interacts with an information medium M via copy-like interactions, the latter must also embody superinformation. Differently from DeWitt's argument, this result relies on general, dynamics- and scale-independent assumptions, specifically the principles of locality and of interoperability of information (see Chapter 2.2.2), together with deterministic dynamics. Note here that these assumptions are simply related to the embodiment and transmission of information between physical systems, which then limits which tasks are possible or impossible on a system. Therefore, we shall refer to them as *information-theoretical*. In Ref. [38], they proposed a protocol to test this theorem by coupling the information medium M to a qubit Q and measuring only M's classical variable. However, this test still required control over M, the system under investigation.

Building on these ideas, information-theoretical tests have recently been proposed in the field of quantum information theory [41], and subsequently generalised beyond quantum theory within constructor theory [42] and generalised probability theories

[43]. The development of new tests of this kind, as well as their application to various hybrid systems of interest, are active areas of research in the field of information-theoretical witnesses of non-classicality. This Thesis represents a contribution to such a field.

The setup one considers is that of a hybrid system, where a quantum sector Q is coupled to a sector M whose features are under investigation.

The feature of M these tests want to assess is its *non-classicality*. The notion of non-classicality adopted here follows the definition introduced in Ref. [42] and represents a generalisation of the uncertainty principle beyond quantum theory. Informally, a system is non-classical when its description requires a minimum of two variables that cannot be distinguished to an arbitrarily high degree of accuracy by the same measuring apparatus, i.e., simultaneously and in a single shot. We shall refer to such variables as "incompatible". One of these variables, which we shall denote by Z, corresponds to the classical variable of the system under consideration, necessary for it to couple to the quantum sector of the hybrid. For example, the variable $\hat{\sigma}_z$ identifying the computational basis in a qubit can play such a role. The additional variable, denoted by V, is required to support information-processing tasks that cannot be accounted for within a classical description. As we shall see in this Thesis, a prominent example of such behaviour is the ability to mediate entanglement, whether in space (between distinct systems) or in time (between different temporal instances of a system). However, the states of V may not be physically preparable or perfectly distinguishable; therefore, it may not be a fully fledged observable of the system.

This statement can be made precise within the constructor theory of information [40], without relying on the quantum formalism. We only need four ingredients, each building on the previous one. The first is the *attribute*, defined as a set of states of a system sharing a given property. The tasks introduced above are transformations between input and output attributes. A set of disjoint attributes defines a *variable*. For a qubit, $\hat{\sigma}_z$ is a variable with two attributes each consisting of a single state, $\{|0\rangle, |1\rangle\}$. The second is an *information variable*, namely a variable

whose attributes can be copied and permuted, like the values of a classical bit. The third is *distinguishability*: a variable is distinguishable if its attributes can be mapped onto those of an information variable, i.e., if they can be told apart by a measurement, and an *observable* is an information variable such that, whenever its measurement returns a sharp value, the system indeed has the corresponding attribute. The fourth is *incompatibility*: two variables are incompatible if their union is not distinguishable, i.e., if no single measuring apparatus can tell apart all their attributes at once. For example, in a qubit the eigenstates of $\hat{\sigma}_z$ and those of $\hat{\sigma}_x$ define two incompatible observables. With these ingredients, non-classicality can be defined concisely as follows [42], leaving the complete formulation to Appendix A:

Definition 1 (Non-classicality)**.** *A system M is non-classical if it possesses an observable Z and another variable V, disjoint from Z, such that:*

(i) Z and V are incompatible;

(ii) V is dynamically relevant: it can be correlated with a variable of another system S_1 so that the resulting variable of $S_1 \oplus M$ is distinguishable by measuring incompatible observables of $S_1 \oplus M$, but not by measuring observables of S_1 only.

Definition 1 makes no reference to quantum theory. It is phrased in terms of which tasks are possible, namely copying, permuting and distinguishing attributes, so that no dynamical law or scale is invoked: information variables identify what is classical in a system, distinguishability takes the place of orthogonality, and the incompatibility of two observables, i.e., the non-distinguishability of their union, generalises the uncertainty principle. A classical system, by contrast, is one whose information variables are all compatible with one another, since the union of any two of them is again an information variable, hence a distinguishable one. The definition therefore applies equally to classical, quantum and post-quantum systems, as needed for a sector M whose underlying description is not known in advance.

Within quantum theory, Definition 1 specialises as follows. Two quantum observables can be distinguished simultaneously, in a single shot and to an arbitrary

accuracy, if and only if they commute. Therefore, condition (i) becomes $[\hat{Z}, \hat{V}] \neq 0$ and non-classicality is expressed by the principle of complementarity: M must be described by at least two non-commuting variables. Condition (ii), in turn, gives M the information capability of sustaining entanglement: since V is dynamically relevant, when S_1 and M interact the correlation between V and a variable of S_1 gives rise to entangled states of $S_1 \oplus M$, which cannot be told apart by measuring S_1 alone, but can be by measurements involving both systems. A non-classical M must therefore be able to become entangled with another quantum system, for example a quantum probe, as considered in this Thesis. This is the form of the definition used in the proofs of Theorems 1, 2 and 3, where a classical mediator is a system with the single variable $\hat{\sigma}_z$. Verifying directly that a given M satisfies Definition 1 would, however, require measuring its incompatible variables, which is precisely what is hard to do for an unknown sector that is often inaccessible (Chapter 1.1).

To avoid the loopholes mentioned above, it is essential to devise protocols to *indirectly* infer the non-classicality of M without requiring control over it. In a hybrid system, the quantum sector Q can serve this purpose, acting as a *probe* to test M's non-classicality. The key idea is that of a *witnessing task*: an information-processing task that can only be possible on Q if M employs at least two incompatible degrees of freedom to mediate it under minimal assumptions, i.e., if M is non-classical. The possibility of the chosen task would provide a sufficient condition for M's non-classicality; however, its impossibility would make the test inconclusive: it could mean that M is classical after all, but it could also be compatible with a failure of the assumptions for the hybrid. The possibility of the witnessing task is verified by measurements performed on Q alone, without requiring control over M. This motivates the term "witness", borrowed from quantum information theory [44].

A *witness of non-classicality* is therefore defined by the witnessing task together with the minimal set of assumptions under which the task implies the non-classicality of M. The fewer and more physically motivated the assumptions required to make the chosen task a witnessing task are, the more general the witness of non-classicality will be. This is because, with fewer assumptions, it is possible to consider a larger

class of classical models for describing the investigated M. Consequently, a successful witness of non-classicality would rule out all the models in that class, providing a more robust witness of the non-classicality of M. An example of how additional assumptions restrict the class of models being tested, and hence weaken the generality of the witness, is to consider the assumption of a specific form of coupling between the two sectors of a hybrid system. In that case, the witnessing task would only exclude those semiclassical models that reproduce that particular interaction. Any model based on a different coupling would fall outside the scope of the witness and would therefore remain a viable explanation of the observed behaviour. It is precisely for this reason that information-theoretical principles, which minimise such assumptions, provide a particularly powerful framework for witnessing non-classicality.

1.3 Thesis Structure and Key Messages

In this Thesis, we investigate how information-theoretical tasks such as the mediation of spatial and temporal quantum correlations can be used as witnessing tasks to infer the non-classicality of physical systems. We further apply these witnesses to relevant examples of hybrid systems, with particular emphasis on biological systems, more precisely on light-harvesting complexes. The remainder of the Thesis is organised as follows.

In Chapter 2, we introduce the General Witness Theorem (GWT), presented as Theorem 1, which provides a two-probe, spatial entanglement-based witness of non-classicality. We begin by discussing the minimal assumptions underlying the GWT: the principle of locality, the principle of interoperability of information and the possibility for the investigated system to be a mediator of interactions between the probes. We then provide a proof of the GWT within the quantum formalism and complement it with a quantitative analysis linking the amount of quantum correlations locally mediated by M between the two quantum probes to its non-classicality, expressed as a function of M's non-commuting variables. Finally, we comment on the implications of the GWT for hybrid systems, highlighting its theoretical robustness as well as its experimental challenges.

Chapter 3 introduces a novel class of single-probe, temporal witnesses of non-classicality. We first motivate the need for a single-probe variant of the argument and clarify how it differs from existing approaches. To compensate for the absence of the second probe, we introduce and justify an extra assumption: the conservation of a global additive quantity in the hybrid. A formal proof of the temporal witness of non-classicality within the formalism of quantum theory (Theorem 2) will follow: it is impossible for a classical system to mediate quantum coherent evolution of a quantum probe under a dynamics conserving a global additive quantity. Finally, we propose an application of Theorem 2 to the light-harvesting stage of photosynthesis in biological systems, supporting the claim that quantum effects play a non-trivial role in the process. To investigate the possibility of the witnessing task in noisy systems, we propose a collision model describing coherent transport in a complex system interacting with an environment characterised by tunable non-Markovianity.

In Chapter 4 we develop a different temporal witness of non-classicality which, under the same assumptions of Theorem 2, relies on M's capability of mediating temporal entanglement on the quantum probe (relative to the local conserved quantity and a non-commuting observable) through a global energy-conserving dynamics. We first introduce the notion of temporal entanglement by connecting it to Leggett-Garg tests of macrorealism and their relation to temporal Bell inequalities. We then provide a formal proof of the temporal entanglement-based witness of non-classicality (Theorem 3) within the quantum formalism. We interpret Theorem 3 as the temporal extension of the GWT in quantum theory, highlighting the symmetric role spatial and temporal quantum correlations play in witnessing non-classicality and conjecturing a novel connection between the principle of locality and energy conservation. The Chapter concludes with a proof-of-principle experimental implementation of Theorem 3 using a three-qubit nuclear magnetic resonance quantum computer.

Finally, Chapter 5 summarises the results presented in the Thesis and outlines possible future research directions in the field of information-theoretical non-classicality witnesses and their applications.

Two central themes permeate this Thesis.

The first theme concerns the status of quantum theory as a universal physical theory [35]. Hybrid systems naturally arise when probing regimes where the domain of applicability of quantum theory is questioned. Establishing general and robust arguments capable of ruling out semiclassical descriptions is therefore essential if quantum theory is to be extended as a universal framework valid across all physical scales. In this context, witnesses of non-classicality emerge as a particularly powerful tool, as they allow one to exclude broad classes of semiclassical models under minimal and physically motivated assumptions. From this perspective, semiclassical models retain an important operational role as effective descriptions, useful for calculations and numerical simulations, but they are not viable candidates for a fundamental description of reality.

The second theme concerns the methodological role of information theory in achieving this level of generality. A defining feature of the witnesses developed in this Thesis is their information-theoretical formulation. Rather than relying on detailed dynamical models, these witnesses are based on the possibility or impossibility of specific information-processing tasks, such as the generation of spatial or temporal quantum correlations, between subsystems. This approach minimises the underlying assumptions, restricting them to general physical principles obeyed by all currently known physical theories. This is particularly relevant in temporal witnesses of non-classicality, introduced precisely to extend already existing single-probe non-classicality tests in information-theoretical terms. This reformulation enhances their generality, allowing them to rule out broader classes of classical models, and thereby strengthens their role as tools for probing the fundamental nature of hybrid systems.

1.4 Publications

Many of the results in this Thesis have been published in the works listed below.

- Chapter 2 is based on:

1. Giuseppe Di Pietra, Vlatko Vedral, and Chiara Marletto. "On the Role of Locality in the Bose-Marletto-Vedral Effect", Chapter 3 in *Quantum Gravity and Computation: Information, Pregeometry, and Digital Physics*, Rickles, D., Arsiwalla, X.D., and Elshatlawy, H. eds. Routledge, 2025.
2. Salvatore Raia, Giuseppe Di Pietra and Chiara Marletto. "Role of non-classicality in mediated spatial quantum correlations", *Physical Review A* **113** 052218 (2026).

- Chapter 3 is based on:

1. Giuseppe Di Pietra and Chiara Marletto. "Temporal witnesses of non-classicality and conservation laws", *Journal of Physics A: Mathematical and Theoretical* **56** 265305 (2023).
2. Giuseppe Di Pietra, Vlatko Vedral, and Chiara Marletto. "Temporal witnesses of non-classicality in a macroscopic biological system", *Scientific Reports* **14** 20094 (2024).
3. Simone Rijavec and Giuseppe Di Pietra[1]. "Tunable non-Markovian dynamics in a collision model: an application to coherent transport", *New Journal of Physics* **27** 043003 (2025).

- Chapter 4 is based on:

1. Giuseppe Di Pietra, Gaurav Bhole, James Eaton, Andrew J. Baldwin, Jonathan A. Jones, Vlatko Vedral and Chiara Marletto, "Temporal Entanglement and Witnesses of Non-Classicality", arXiv: 2506.15474 (2025). Accepted for publication in *Physical Review Letters*.

The following work was completed during the DPhil but is not included in this Thesis:

[1]Joint first authorship

- Giuseppe Di Pietra, Fabrizio Piacentini, Ettore Bernardi, Ekaterina Moreva, Carmine Napoli, Ivo Pietro Degiovanni, Marco Genovese, Vlatko Vedral, Chiara Marletto. "The Bose-Marletto-Vedral experiment with nanodiamond interferometers: an insight on entanglement detection", arXiv:2410.19601 (2024)

2

Spatial Entanglement-Based Witness of Non-Classicality

The content of this chapter is partially based on [45, 46]. [45] was developed in collaboration with Salvatore Raia and Chiara Marletto; [46] was developed in collaboration with Vlatko Vedral and Chiara Marletto.

The challenging problem of hybrid systems, the study of systems in which a quantum sector Q interacts with an unknown sector M whose non-classical nature is disputed, requires a shift in perspective to be addressed. Instead of focusing on direct tests of non-classical effects on M, often beyond experimental reach, one can use the quantum sector Q to *indirectly* infer M's non-classicality (namely that its description requires a minimum of two non-compatible degrees of freedom). Specifically, one shall look for a task that is possible on Q only if M is non-classical, under the appropriate assumptions. Such a task would be called a "witnessing task" for M's non-classicality, and together with the right assumptions it determines a *witness of non-classicality*.

In this Chapter, we shall introduce the first proposed information-theoretical witness of non-classical effects in an unknown system. Originally proposed by

Bose *et al.* [47] and, independently, by Marletto and Vedral [48] in the context of gravity, the argument relies on the investigated system's capability of inducing entanglement between two spatially separated quantum probes. If the unknown system M can mediate entanglement between the quantum probes Q_A and Q_B by *local means* only, then it must be non-classical. We shall first explain why the quantum sector in this witness requires two quantum probes by introducing the mental experiment proposed by Feynman to prove the quantum nature of gravity, noting its criticalities and the effects of a second mass on its predictions: the generation of entanglement between the probes, mediated by the gravitational field (a phenomenon now known in the literature as gravitationally-induced entanglement, gravitationally-mediated entanglement, or the Bose-Marletto-Vedral effect [41]). We shall then introduce the assumptions that make this task a witnessing task for a witness of non-classicality beyond the specific example of gravity: the principle of locality, the principle of interoperability of information and the possibility for M to act as a mediator of interactions. These assumptions are the minimal required by a dynamics-independent witness, capable of inferring the non-classical properties of M following the general definition given in Chapter 1.2 (Definition 1): M must be described by a minimum of two variables that cannot be simultaneously distinguished to an arbitrarily high degree of accuracy. In particular, we will reflect on the importance of the principle of locality as phrased in the witness and clarify the meaning of the "local means" required by the witness. Thanks to these assumptions, we shall formulate the *General Witness Theorem* (GWT) [42] and provide a formal proof of the witness within the formalism of quantum theory, for which the definition of non-classicality coincides with that of quantum features. Finally, we shall support the GWT in quantum theory with a quantitative connection between the amount of mediated quantum correlations between the quantum probes and the non-classicality of the unknown system M. By using this result, we shall discuss the consequences of a successful (and unsuccessful) witness of non-classicality on the foundations of the hybrid system.

2.1 Detection of Quantum Effects in Gravity via Two Quantum Masses

General relativity and quantum theory are the two most successful theories of modern physics. Both are formulated as theories describing reality at all scales, encompassing all physical systems. However, they have been extensively tested only in some specific domains. The former has been tested at scales where the gravitational interactions become comparable with all the other interactions in nature, the latter at the atomic and subatomic scales, where the gravitational interactions are indeed irrelevant for all practical purposes. Therefore, in their respective domain, both theories have been proven to be unchallenged. However, there is a striking difference between the two: general relativity is fundamentally classical. It does not allow for those phenomena typical of quantum mechanics, such as superposition and entanglement, as it does not have the physical support of non-compatible variables necessary to show them.

Given this fundamental incompatibility between the two most important theories of physics so far, quantum matter in a gravitational field constitutes one of the hybrid systems that are at the heart of this Thesis. Thus, applying witnesses of non-classicality to this hybrid system can represent one promising strategy to devise experimentally-feasible tests of non-classical properties of the gravitational field. It is precisely this direction that led to the introduction of the first witness of non-classicality, based on the entangling power of a quantum gravitational field between two spatially separated quantum masses: the Gravitationally-Induced Entanglement (GIE), or Bose-Marletto-Vedral (BMV) effect [47, 48].

GIE improves on a famous thought experiment proposed by Richard Feynman during the conference on quantum gravity that took place in Chapel Hill, North Carolina, in 1957 [49]. During a discussion with Hermann Bondi about the possibility of distinguishing between a genuinely quantum superpositions and merely classical mixtures of gravitational states, Feynman proposed a mental experiment involving a single mass in the gravitational field generated by a ball. The mass is split by a Stern-Gerlach interferometer in a superposition of two locations, leading to different

displacement of the ball depending on the branch of the superposition. When the interferometer is closed—Feynman argued—the appearance of an interference picture would prove the quantum nature of gravity, as only a genuinely quantum system can undergo the process. However, the statement is only partially correct. Feynman's experiment, even if performed successfully, would show that the phenomenology is *compatible* with a quantum gravitational field, but would not exclude the possibility that a classical gravitational field was in fact coupled to the mass. This is because it is the *mass*, not the gravitational field itself, that undergoes interference in this experiment. The phases required for the interference of the states of the mass can equally be generated by a classical field. It follows that the gravitational field is coupling linearly to the mass superposed across different locations at that scale, not collapsing it, even if classical. In other words, the experiment does not require the gravitational field itself to be in a non-classical state.

Feynman's experiment is not proving that the gravitational field is quantum because it does not leave entanglement generation between the mass and the gravitational field itself as the *only* possible explanation for the interference. Thus, the experiment with a single mass is not revealing the existence of (at least) two non-compatible variables describing the gravitational field, i.e., it is not proving its non-classicality.

To do so, one must reveal that the superposition of the mass position has determined an entangled state with the gravitational field by preparing it in a non-classical state as well. This would imply that the dynamics of the gravitational field has engaged a minimum of two non-compatible degrees of freedom, showing its non-classicality according to Definition 1 adopted in this Thesis (see Chapter 1.2). Inspired by analogous experiments for the electromagnetic field, which is known to be a non-classical system, the idea is to use a *second* mass to reveal this entanglement. The first mass, prepared in a superposition of two locations, generates a superposition of classical states of the gravitational field following Feynman's idea. If we now introduce a second mass, acting as a probe of this interaction, it will in turn be placed in a particular non-classical state, depending on the exact superposition the field is in

[50]. If the second probe mass is itself placed in a superposition of two locations, the gravitational field state would be influenced by the superposition of both the masses, ultimately determining entanglement between the two. This gravitationally-induced entanglement between the two masses is specifically the phenomenon one should look for in the quantum sector to witness the non-classical nature of the gravitational field. This phenomenon, if observed under appropriate assumptions, will provide a witness of the quantum nature of gravity in the low-energy regime [47, 48].

This reasoning, described in the context of quantum gravity for historical purposes[1], leads to a more general conclusion: one possible way to provide a witness of non-classicality in a hybrid system $Q \oplus M$ is to consider a quantum sector Q composed of *two* quantum probes, one acting as the "preparing probe" and the other as the "revealing probe", and looking at M's capability of mediating entanglement between those [42, 43]. The relevant questions now are (1) what assumptions make it a witnessing task, as required by a witness of non-classicality, and (2) what is the level of generality this argument can achieve. We shall answer these in the next Section, presenting the general theorem known as General Witness Theorem (GWT) which backs this *spatial* entanglement-based witness of non-classicality. We shall discuss the assumptions behind it in a completely dynamics-independent formulation, without relying on the formalism of quantum theory, thus commenting on the generality of this theorem as a witness of non-classicality beyond quantum theory [42]. We shall also provide a formal proof of the GWT *within the formalism of quantum theory*, and leave the rigorous dynamics-independent proof to Appendix A.

2.2 Spatial Entanglement: Sufficient Condition for Non-Classicality

Consider a hybrid system composed of the system under investigation M and a quantum sector Q consisting of two quantum probes Q_A and Q_B. The system M is unknown and, as such, it is important to relax any dynamical assumptions

[1] As mentioned before, the research field of information-theoretical witnesses of non-classicality arose precisely in the context of quantum gravity.

about it to achieve a fully general witness of its non-classical properties. In doing so, one can allow for the possibility that a non-classical theory beyond quantum theory may be required to describe M, and consequently the entire hybrid $Q \oplus M$. Furthermore, consistently with the spirit of a witness of non-classicality, the more general the assumptions backing it, the larger the class of classical descriptions of M that a successful witness would rule out. Therefore, having a dynamics-independent witness becomes essential for the argument to be robust and conclusive about M's non-classicality.

Marletto and Vedral have identified the minimal set of assumptions required by a two-probe, spatial entanglement-based witness of non-classicality. Importantly, these assumptions are satisfied by all the currently known physical theories, including general relativity and quantum theory, while not relying on their specific mathematical formalism. Within the information-theoretical research programme of constructor theory [39, 40], they provided a dynamics-independent argument known as the General Witness Theorem (GWT) [42]. The argument relies on the general notion of non-classicality introduced in Definition 1 and formalised in Appendix A. Informally, a system is non-classical if its description requires a minimum of two distinct physical variables that cannot be distinguished to an arbitrarily high degree of accuracy by the same measuring device. This can be seen as a more general formulation of the Heisenberg uncertainty principle, therefore it generalises the property of quantum complementarity to a system that does not possess the full computational power of a quantum one. For a quantum system, these non-compatible variables correspond to "non-commuting" variables. Therefore, the GWT provides the theoretical foundation for the recently proposed tests to witness the non-classical properties of the gravitational field [41] and in photosynthetic complexes [51].

We now introduce the assumptions underlying the GWT.

2.2.1 The Assumptions I: Principle of Locality

In order for the entanglement generated between Q_A and Q_B to act as a *witnessing task* for the witness of non-classicality we are exploring in this Chapter, it is necessary to ensure that it is solely due to the system we are investigating, M. Otherwise, the two probes Q_A and Q_B could become entangled because of other phenomena, such as direct interactions or other spurious effects acting at the global level of the hybrid, irrespective of M's non-classicality. In such cases, the witness would be invalidated.

For this reason, the first crucial assumption underlying the GWT is the *Principle of Locality*. We can formalise the principle in a dynamics-independent way as follows [40, 46]:

Principle 1 (Principle of Locality)**.** *The state of a system is a description of it that satisfies two properties: (i) any attribute of a system, at any given time t, is a fixed function of the system state and (ii) any state of a composite system $Q_A \oplus Q_B$ is an ordered pair of states (a, b) of Q_A and Q_B, with the property that if a task is performed on Q_A only, then the state b of the substrate Q_B is not changed thereby.*

The importance of Principle 1 in avoiding possible loopholes in the GWT, in which direct or effective interactions between the probes are responsible for the entanglement without engaging the investigated system [52], motivates a more detailed discussion of its meaning and its relation to other notions of locality in the literature [46].

Informally, Principle 1 is the combination of the separability condition with no action at a distance [53]. The first allows one to distinguish between multiple subsystems, so that the composite system of Q_A and Q_B can be written as $Q_A \oplus Q_B$. The second ensures that given a partition of a system into subsystems Q_A and Q_B, a dynamical transformation that operates only on Q_A cannot change the states of Q_B. Note that here the notion of "state" refers to the *complete specification* of the state of affairs of a given system, not only to what is empirically accessible through measurements of that system [54, 55].

In quantum theory, the principle of locality is also called *Einstein's locality*. Often it is confused with a different notion of locality called *Bell's locality*. Bell's locality refers to the possibility of describing a set of data with a local hidden-variable theory, expressed as a stochastic, real-valued theory, thus based on *c-numbers*. When Bell's inequalities are violated in a given experiment, one can therefore rule out a c-number-based, stochastic description of reality, that is local in the sense of Einstein's locality. Quantum theory is Bell non-local; however, it is a *q-number-based*, deterministic description of reality that satisfies the principle of locality (as stated in Principle 1). Thus, Bell's non-locality of quantum theory should not be misunderstood as a violation of the principle of locality as stated above, but rather as a violation of Bell's locality. From now on, when referring to "locality", we will always mean the property required by the locality principle formulated in Principle 1, not Bell's locality.

The Principle of No-Signalling. The principle of no-signalling states that a dynamical transformation that operates only on one subsystem Q_A cannot change the *observable properties* of another subsystem Q_B.

In quantum theory, one can express this principle as follows:

$$[\hat{U}_A, \rho_B] = 0 \tag{2.1}$$

where $\hat{U}_A$ is a unitary transformation happening on the subsystem Q_A alone, while $\rho_B = \mathrm{Tr}_{Q_A}(\rho_{AB})$ is the reduced density operator of the subsystem Q_B, which can be expressed as a linear combination of the generators of its observables algebra. For example, if one considers Q_B to be a qubit, then $\rho_B = \frac{1}{2}\left(\mathbb{I} + \alpha\hat{\sigma}_x + \beta\hat{\sigma}_y + \gamma\hat{\sigma}_z\right)$, where $\hat{\sigma}_i$, $i \in \{x, y, z\}$ are the Pauli operators and $\mathbb{I}$ is the identity operator, generators of the qubit's observables algebra. Here, $\alpha, \beta, \gamma \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$.

The principle of no-signalling and the principle of locality are related. When considering a theory with 1:1 dynamics, including quantum theory, a general theorem shows that the two statements are equivalent [55]. This is because the 1:1 dynamics determines the *completeness* of the theory, which in turn allows the description of the complete state of the system, as required by the principle of locality, starting

from its observable properties, as mandated by the no-signalling principle. In general, while the principle of locality implies the principle of no-signalling, the converse is not true. This is because the principle of no-signalling only focuses on the locally empirically accessible features of a system. An example of a theory that satisfies no-signalling, but is non-local, is Bohmian Mechanics, [56]. This fact becomes important for the witness of non-classicality, which, in its most general form [42], relies on the principle of locality, not on the principle of no-signalling.

Notice also that the above two principles concern notions such as "observables", "variables" and "dynamical transformations", which can be expressed in many different formalisms. Hence, while these principles are satisfied by non-relativistic quantum theory, quantum field theory, and special and general relativity, they are formulated *independently of* any specific formalism. This is the power of principles as general rules that can constrain particular dynamics, rather than being derived within a particular dynamical law.

Finally, note that the principle of no-signalling can also be phrased, within specific dynamical laws, as requiring a finite speed of propagation for signals between space-like separated systems. Here we have opted for a formulation that is dynamics-independent and does not explicitly refer to speed of propagation.

Axiom of Microcausality. In quantum field theory, the property of *microcausality* is often taken as an *axiom*. This property states that the allowed quantum observables (operators) associated with space-like separated systems must commute. In other words,

$$\left[\hat{O}_A, \hat{O}_B\right] = 0, \tag{2.2}$$

for any two quantum observables $\hat{O}_A$ and $\hat{O}_B$ respectively pertaining to two space-like separated systems Q_A and Q_B. Note that when we talk about space-like separated systems we implicitly refer to a *particular formalism*, that of special relativity. Moreover, by expressing the condition in terms of a commutator, we are committing to the formalism of quantum theory. In passing, we note that Eq. 2.2 is a property of non-relativistic quantum theory, where the quantum observables of independent

subsystems are required to commute with each other. This property implies that both non-relativistic quantum theory and quantum field theory satisfy the principle of locality as stated above: since $\hat{O}_A$ commutes with $\hat{O}_B$ and the two theories are complete, operating a dynamical transformation on the observable $\hat{O}_B$ of the subsystem Q_B cannot modify the observable $\hat{O}_A$ of the subsystem Q_A, i.e., its state, as required by the principle of locality. Finally, we emphasise that microcausality as formulated above is *not* a general principle, because it is formulated within the specific formalism of quantum theory and special relativity.

Axiom of Lorentz-Covariance. In special relativity, allowed dynamical variables must satisfy the *axiom of Lorentz-covariance*. This means that they must transform properly under Lorentz transformations so that physical variables are described by either scalars, tensors, or spinors. Dynamical laws are then said to be *Lorentz-covariant* if they are expressed in terms of Lorentz-covariant variables only. A dynamical law with this property satisfies the principle of relativity, according to which all physical laws must make the same predictions for identical experiments in any two inertial frames. Some physical quantities such as scalars are also *Lorentz invariant*, which means that they take the same form in all reference frames. When a law satisfies this property it must also satisfy the principles of locality and no-signalling, as the maximum speed of propagation for dynamical perturbations is the speed of light. The two principles as formulated above refer to "instantaneous" modification of the state or the observable of the other subsystems and do not refer to any speed of propagation or other conditions in space and time. Similar concepts can be defined in general relativity, where one refers to *local Lorentz-covariance* (local in spacetime). The same considerations about the relation with the principle of locality and no-signalling hold in that case.

Once more, we emphasise that Lorentz-covariance is a dynamics-dependent formal requirement, which only holds within special relativity. It is implied by microcausality, but it is less general than the locality or the no-signalling principles as formulated earlier.

Universality. There are also more specific notions of locality, related to the particular structure of dynamical interactions. For instance, there is the property that a given unitary acting on an n-partite system can be approximated arbitrarily well by a set of unitaries operating on one and two systems. In the field of quantum information, this property is known as the *universality of two- and one-qubit gates*, which was proven by a sequence of results in the 1990s (see [57–59]). This property can also be extended to continuous dynamics, considering that the latter can be approximated arbitrarily well by the dynamics of discrete systems. These well-known results were also recently termed *subsystem locality*, [60, 61]. These properties are, once more, formalism-specific and thus more narrow than the principles discussed earlier.

In conclusion, the principle of locality as phrased in Principle 1 is a plausible candidate for the most general notion of locality in physics, considering its relation to the other relevant notions that we have reviewed in this Section. It is also the minimal notion of locality on which the witness of non-classicality expressed by the GWT must rely [42]: we shall see in the rest of the Chapter where this assumption enters the demonstration and what are the loopholes generated by its misunderstanding.

2.2.2 The Assumptions II: Principle of Interoperability of Information

Following the logic introduced in Chapter 2.1, the first probe Q_A prepared in a non-classical state allows the unknown system M to enter a corresponding state, presumably non-classical too. This state will then be picked up by the second probe Q_B, and vice versa, in order to establish the entanglement [50]. For this to occur, the system Q_A must be capable of *transferring information* to the system M, and similarly M to Q_B. Within any subsidiary theory such as quantum theory, this is allowed by the possibility to couple the systems in some way that depends on the dynamics of that theory. For example, in a two-qubit quantum circuit, one can prepare one of the two in the state of the other by means of a SWAP gate: $\text{SWAP} = \frac{1}{2}(\mathbb{I} \otimes \mathbb{I} + \hat{\sigma}_x \otimes \hat{\sigma}_x + \hat{\sigma}_y \otimes \hat{\sigma}_y + \hat{\sigma}_z \otimes \hat{\sigma}_z)$. The SWAP gate transfers the

information between the two qubits, both systems capable of storing and processing information in quantum theory, thanks to their coupling in the $\hat{\sigma}_{x,y,z}$ basis.

To express this property irrespective of the physical details of the two systems, we need the *Principle of Interoperability of Information*. The principle can be elegantly formulated within the information-theoretical formalism of constructor theory, whose essential elements have been introduced in Appendix A [40]:

Principle 2 (Principle of Interoperability of Information)**.** *If Q_A and Q_B are two information media with information variables X_1 and X_2 respectively, then their composite system $Q_A \oplus Q_B$ is an information medium too with information variable $X_1 \times X_2$,*

where $\times$ denotes the Cartesian product between the two sets. Since $Q_A \oplus Q_B$ is an information medium with $X_1 \times X_2$ as an information variable, by the property in Eq. A.1 one can copy the attributes of $X_1 \in Q_A$ to Q_B thanks to the interaction between the two systems. This interaction explains the Cartesian product $\times$, and can be stated without referring to any dynamical assumptions. Thus, informally, Principle 2 expresses the property that classical information must be copiable from one system—here the information medium—to another irrespective of their physical details. Notice here that the definition of information variable, as stated in Eq. A.2, implies also the possibility to do computations on the system Q_A and Q_B. The principle of locality in Principle 1 maintains the possibility to perform permutations of the attributes of Q_A without affecting Q_B even when considering the composite system $Q_A \oplus Q_B$.

The principle of interoperability of information is satisfied by all the currently known theories in physics, both classical and quantum, as it is necessary for the testability of a theory. Thanks to Principle 2 it is possible to copy the information about a substrate into a second one that one would call the "measurer" and, therefore, to test the theory. This is also possible thanks to the principle of locality in Principle 1.

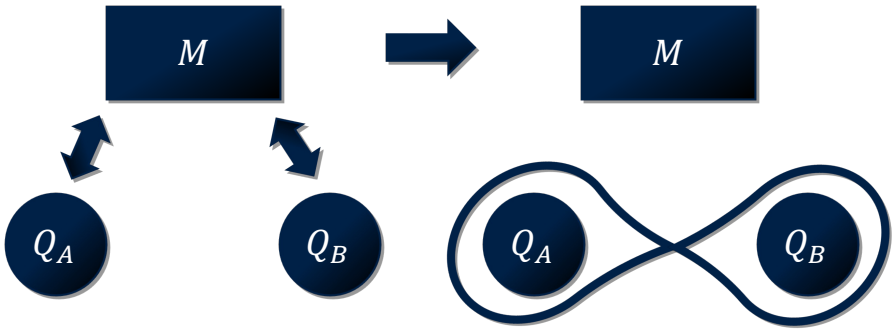


Figure 2.1: Schematic representation of the GWT, Theorem 1. Two spatially separated quantum probes Q_A and Q_B are only locally coupled to the investigated system M. The generation of entanglement mediated by M is a sufficient witness of its non-classicality.

2.2.3 The Assumptions III: M is a Mediator

The goal of a witness of non-classicality is to reveal the non-compatible variables of the investigated system M by looking for a task performed on the quantum sector and whose possibility is due to the dynamical involvement of these variables. Therefore, the only one responsible for this task must be the system M. While the principle of locality (Principle 1) ensures that no interactions can occur between the quantum probes Q_A and Q_B if acting on one alone, a further assumption is necessary to ensure that the party responsible for their interactions is indeed M.

This is the purpose of the third assumption of the GWT: the system M acts as a *mediator of interactions* between the quantum probes Q_A and Q_B. This assumption can be verified independently by other experiments involving the system M. An example for the case of gravity are the experiments where gravity, acting as a classical system, mediates the evolution of neutrons (also known as COW experiments [62]).

Notice that this assumption is required when we want to provide a dynamics-independent formalisation of a witness of non-classicality, as it is the case for the GWT, but is automatically satisfied when considering the quantum formalism. In fact, it can be shown that by assuming unitarity, locality and quantum field theory, a field (such as gravity) must act as a mediator of interactions [63]. Given this assumption, in what follows, we shall call the unknown system M the "mediator".

2.2.4 The Theorem: General Witness Theorem

Taken together, the general assumptions in Chapter 2.2.1, Chapter 2.2.2, and Chapter 2.2.3 represent the minimal set of requirements to provide a dynamics-independent two-probe spatial entanglement-based witness of non-classicality, the GWT, see Fig. 2.1 [42]:

Theorem 1 (General Witness Theorem)**.** *If M can entangle Q_A and Q_B by locally interacting with each of them, then M is non-classical.*

Proof. We provide here a proof of the theorem within the quantum formalism [48] and leave the most general, dynamics-independent proof to Appendix A. Notice first that assuming the formalism of quantum theory automatically grants the assumptions we discussed in the previous Sections, thus representing an additional assumption in the set of minimal requirements to prove the GWT.

Let us assume, for simplicity and without loss of generality, that Q_A and Q_B are two qubits. We describe the qubits using the descriptors formalism, which stems from the Heisenberg picture of quantum theory [64, 65]. In this formalism, the qubit Q_A is described by a vector of *descriptors*:

$$\begin{aligned}\hat{\vec{q}}^{(A)}(t_0) &\coloneqq \left(\hat{q}_x^{(A)}(t_0), \hat{q}_y^{(A)}(t_0), \hat{q}_z^{(A)}(t_0)\right) \\ &= \left(\hat{\sigma}_x \otimes \mathbb{I}_M \otimes \mathbb{I}_B, \hat{\sigma}_y \otimes \mathbb{I}_M \otimes \mathbb{I}_B, \hat{\sigma}_z \otimes \mathbb{I}_M \otimes \mathbb{I}_B\right), \end{aligned} \tag{2.3}$$

where $\hat{\sigma}_k$, $k = x, y, z$, are the Pauli operators and $\mathbb{I}_i$, $i = M, B$, is the identity operator on the unknown system M and on the qubit Q_B. Similarly:

$$\begin{aligned}\hat{\vec{q}}^{(B)}(t_0) &\coloneqq \left(\hat{q}_x^{(B)}(t_0), \hat{q}_y^{(B)}(t_0), \hat{q}_z^{(B)}(t_0)\right) \\ &= \left(\mathbb{I}_A \otimes \mathbb{I}_M \otimes \hat{\sigma}_x, \mathbb{I}_A \otimes \mathbb{I}_M \otimes \hat{\sigma}_y, \mathbb{I}_A \otimes \mathbb{I}_M \otimes \hat{\sigma}_z\right), \end{aligned} \tag{2.4}$$

is the vector of descriptors of the qubit Q_B. The descriptors allow us to keep track of the evolution of each qubit's *local algebra of observables* while acting on the full Hilbert space. Note that $\left[\hat{q}_k^{(i)}, \hat{q}_l^{(j)}\right] = 0$, $\forall\, i \neq j$ and $k, l = x, y, z$, which implies that quantum observables of two different, non-interacting subsystems must commute.

With this picture in mind, we now prove the GWT in quantum theory. Since the witness provides a *sufficient* condition for the non-classicality of the mediator M, we assume that at the end of the protocol the two quantum probes Q_A and Q_B are entangled.

The proof proceeds by contradiction. Let us assume M to be a classical system. Its vector of descriptors then has a single component,

$$\hat{q}_z^{(M)}(t_0) \coloneqq \mathbb{I}_A \otimes \hat{\sigma}_z \otimes \mathbb{I}_B, \tag{2.5}$$

reflecting the existence of a single observable for it, as per our definition of classicality. We identify this observable with the Pauli Z operator $\hat{\sigma}_z$. At time t, the quantum probe Q_A interacts with the mediator M. The most general state of the system $Q_A \oplus M$ is:

$$\rho(t) = \frac{1}{4}\left(\mathbb{I} + \vec{r}_A \cdot \hat{\vec{q}}^{(A)}(t) + s_z \hat{q}_z^{(M)}(t) + \vec{t}_A \cdot \hat{\vec{q}}^{(A)}(t)\hat{q}_z^{(M)}(t)\right), \tag{2.6}$$

where $\vec{r}_A$ and $\vec{t}_A$ are real-valued vectors, $s_z \in \mathbb{R}$, and $\hat{\vec{q}}^{(A)}(t)$ and $\hat{q}_z^{(M)}(t)$ are functions of $\hat{\vec{q}}^{\,(A)}(t_0)$ in Eq. 2.3 and $\hat{q}_z^{(M)}(t_0)$ in Eq. 2.5 only, respectively. This state, interpreted as a two-qubit state, is *separable*, meaning that no quantum correlations can be generated between the quantum probe Q_A and a classical mediator M [66]. Therefore, it would be impossible to generate entanglement between Q_A and Q_B at the end of the protocol, since the final state would remain separable too. This contradicts the initial assumption of the proof that the probes would become entangled. The reason why $\rho(t)$ in Eq. 2.6 is separable lies in the classicality of M: in order to locally mediate entanglement, the mediator needs another observable that does not commute with $\hat{\sigma}_z$. Hence, observing entanglement between Q_A and Q_B at the end of a protocol performed *by local means only* implies that M must be non-classical, according to our definition of non-classicality.

The descriptors formalism in quantum theory clarifies the role of Principle 1 and the consequences of dropping it for the GWT. Let us modify the GWT setup and suppose that the quantum probes Q_A and Q_B can interact directly, or that some non-local mechanism allows their coupling. This means that the descriptors

of the quantum probe Q_A at time t would become a function of those of Q_B at the previous time t_0 in Eqs. 2.3 and 2.4, i.e., $\hat{\vec{q}}^{(A)}(t) = f(\hat{\vec{q}}^{(A)}(t_0), \hat{\vec{q}}^{(B)}(t_0))$. Under these conditions, the state in Eq. 2.6 can result in an entangled state for Q_A and Q_B at the end of the protocol even with a classical mediator M. Therefore, observing the entanglement between the quantum probes would no longer lead to the conclusion that M must be non-classical: there would be no contradiction between observing Q_A and Q_B entangled and the mediator M having only a single variable. In fact, the entanglement in question could have been generated already before Q_A's interaction with M, irrespectively of the mediator. Thus, the assumption of locality is essential to "force" M to engage (at least) two non-commuting variables to entangle Q_A and Q_B: one is used to entangle with Q_A, the other to transmit the quantum correlations between the two quantum probes. □

Theorem 1 provides a *sufficient* condition for the non-classicality of M: if M-mediated entanglement between Q_A and Q_B is observed under the assumptions in Chapters 2.2.1, 2.2.2 and 2.2.3, it can be concluded that M must have involved another variable that is not compatible with the classical one. Consequently, all classical models of M which are compatible with these assumptions would be ruled out by a successful experimental implementation of the GWT. In this sense, the GWT is akin in spirit to Bell's theorem, where the violation of Bell's inequalities rules out all local hidden variable models for the tested theory [67]. In Bell tests, one can quantify the amount of entanglement between the two parties by looking at the violation of the Bell's inequalities: two maximally entangled qubits will maximally violate the inequality, reaching the Cirel'son bound $2\sqrt{2}$ [68]. Following this connection, it is of interest to develop a similar *quantitative* link between the amount of entanglement mediated by the channel M and its non-classicality. We shall explore this possibility in the following Section.

2.3 Quantifying the Mediator's Non-Classicality in Quantum Theory

Suppose that an experiment based on the GWT—for example, its gravitational application, the GIE—is successful, and a certain amount of entanglement is mediated between the probes Q_A and Q_B. Can this be quantitatively linked to the non-classicality of the mediator M? In this Section, we show that quantifying the locally mediated entanglement using appropriate entanglement measures provides a lower bound to the non-classicality of the channel M [45]. The analysis is performed within the formalism of quantum theory.

Consider a general hybrid system composed of the unknown and inaccessible system M locally coupled to the two quantum probes Q_A and Q_B. We assume here that all the systems in the hybrid are defined in finite-dimensional Hilbert spaces. The assumption of the principle of locality in Principle 1, within the formalism of quantum theory, implies that the total Hamiltonian of the hybrid takes the form:

$$\hat{H} = \hat{H}_{AM} + \hat{H}_{MB}, \tag{2.7}$$

where $\hat{H}_{AM}$ and $\hat{H}_{MB}$ describe the interactions between the mediator M and the quantum probes Q_A and Q_B, respectively. The dynamics of the hybrid is generated by Eq. 2.7 and is unitary:

$$\hat{U}_{AMB} = e^{-i(\hat{H}_{AM}+\hat{H}_{MB})t}, \tag{2.8}$$

with the associated unitary channel defined as:

$$\Lambda_{AMB}(\rho) := \hat{U}_{AMB}(t)\,\rho\,\hat{U}^{\dagger}_{AMB}(t). \tag{2.9}$$

To quantify the entanglement mediated by M while retaining the highest possible level of generality, we follow the seminal works in this direction [69, 70] and consider a broad class of quantum correlation measures G. Specifically, G is assumed to be a *gd*-continuous function, a property shared by commonly used quantum correlation measures such as entropy of entanglement and logarithmic negativity [71, 72]. A

quantum correlation measure G is *gd*-continuous when it is monotonic under local processing and satisfies the continuity condition:

$$|G(x) - G(y)| \leq g(d(x, y)), \tag{2.10}$$

where g is an invertible monotonically increasing function with the property $\lim_{s\to 0} g(s) = 0$ and d is a distance between states x and y [70].

We then define the increase in quantum correlation between the two quantum probes Q_A and Q_B as:

$$\Delta G_{corr\ A:B} := G_{A:B}(\rho_t) - \mathcal{B}\left(G(\rho_0)\right). \tag{2.11}$$

The quantity $G_{A:B}(\rho_t)$ denotes the amount of accessible (i.e., measurable) quantum correlation in the evolved two probes' state $\rho_t = \mathrm{Tr}_M\left[\Lambda_{AMB}(\rho_{AMB}(0))\right]$ after the interaction. $\mathcal{B}(G(\rho_0))$ represents the maximum possible initial quantum correlations in the hybrid given the initial state of the probes $\rho_0 = \mathrm{Tr}_M[\rho_{AMB}(0)]$:

$$\mathcal{B}\Big(G(\rho_0)\Big) = \sup_{\sigma_{AM}} G_{A:M}(\sigma_{AM}) + \sup_{\sigma_{MB}} G_{M:B}(\sigma_{MB}) + I_{A:B}(\rho_0), \tag{2.12}$$

where σ_{AM} and σ_{MB} are states of the subsystems $Q_A \oplus M$ and $M \oplus Q_B$, respectively, and $I_{A:B}(\rho_0) = \inf_{\sigma_A\otimes\sigma_B} g\left(d(\rho_0, \sigma_A \otimes \sigma_B)\right)$, with σ_A (resp. σ_B) generic initial state of the probe Q_A (resp. Q_B).

Previous works have adopted different notions of non-classicality, and have used Eq. 2.11 to quantify M's non-classicality accordingly. Refs. [73, 74] use M-mediated quantum correlations to witness that the mediator must have explored non-orthogonal *states* during the dynamics. Refs. [69, 70] extend these results by focusing on non-classicality of *interactions* instead. Specifically, the notion of non-classicality they adopt is characterised by the non-commutativity of *interaction Hamiltonians*, generalised to *map decomposability*: classical interactions are defined by commuting Hamiltonians. As discussed in Chapter 1.2, the notion of non-classicality we adopt in this Thesis is more fundamental as it is related to *non-compatible variables*[2] describing the system, namely variables that cannot be measured to an arbitrarily

[2]Within the quantum formalism, we would identify non-compatible variables as non-commuting.

high degree of accuracy by the same measuring device in a single shot. Therefore, we want now to extend these tools by connecting the non-classicality of interactions to the *non-commutativity of the mediator's variables* [45].

2.3.1 Characterisation of Non-Classical Interactions

The first step towards achieving the goal of this Section is to prove that the increase in quantum correlations between Q_A and Q_B, locally mediated by M, provides a lower bound to a function of the commutator between the local interaction Hamiltonians $\hat{H}_{AM}$ and $\hat{H}_{MB}$, defining non-classicality of interactions.

To do so, let us introduce the notion of a decomposable map [69, 70]. Decomposability formalises the idea that the mediated dynamics can be realised in a single sequential "round": first an operation involving $Q_A \oplus M$, followed by an operation involving $M \oplus Q_B$. A map that acts on the tripartite system, $\lambda_{AMB}(\rho)$, is *continuously decomposable* if there exist two CPTP maps λ_{AM} and λ_{MB} such that:

$$\lambda_{AMB}(\rho) = \lambda_{MB} \circ \lambda_{AM}(\rho) \tag{2.13}$$

and the decompositions $\lambda_{AM}, \lambda_{MB}$ commute. In what follows, we shall refer to "continuously decomposable" simply as "decomposable". In the non-classicality witness scenario, one cannot fully determine the map induced by the local interactions in Eq. 2.7, as the mediator is unknown and not directly accessible. One can only access the marginal map $\Lambda_{AB} = \mathrm{Tr}_M[\Lambda_{AMB}]$. The global map Λ_{AMB} can be found among the Stinespring dilations of Λ_{AB}, with an appropriate dimension m. If the interactions are classical, there exists a Stinespring dilation of Λ_{AB} of dimension $m \geq d_M$ that is decomposable, where d_M is the dimension of the mediator's Hilbert space. We call the space of all such maps' dilations $\overline{DEC}(m)$, and we can summarise this implication as:

$$[\hat{H}_{AM}, \hat{H}_{MB}] = 0 \implies \Lambda_{AB} \in \overline{DEC}(m). \tag{2.14}$$

Crucially, the non-decomposability of the induced map Λ_{AB} admits a direct dynamical interpretation, as it implies the non-classicality of the mediated interaction:

$$\Lambda_{AB} \notin \overline{DEC}(m) \iff [\hat{H}_{AM}, \hat{H}_{MB}] \neq 0. \tag{2.15}$$

With these definitions in place, it can be shown that if the total quantum correlation shared between Q_A and Q_B increases, then the marginal map Λ_{AB} is non-decomposable [69, 70]:

$$\Delta G_{corr\ A:B} > 0 \implies \Lambda_{AB} \notin \overline{DEC}(m). \tag{2.16}$$

We remark here that $\Delta G_{corr\ A:B}$ depends only on quantities related to the two probes Q_A and Q_B, which can be measured in the laboratory. Moreover, no additional assumption is made on $\hat{H}$, $\hat{H}_{AM}$ and $\hat{H}_{MB}$ beyond locality and finite dimensionality. Therefore, the connection between the increase of quantum correlations and the non-classicality of interactions is valid for any of the allowed local Hamiltonians.

Let us now consider a dilation $\widetilde{\Lambda}$ of the marginal map Λ_{AB} and a dilation $\widetilde{\lambda}$ of the decomposable marginal map $\lambda_{AB} \in \overline{DEC}(m)$. It can be shown that the distance between the states on the hybrid $Q_A \oplus M \oplus Q_B$ obtained when applying $\widetilde{\Lambda}$ and $\widetilde{\lambda}$ respectively is lower bounded by the inverse function of the g characterising the chosen quantum correlation measure G [70]:

$$g^{-1}(\Delta G_{corr\ A:B}) \leq d\left(\widetilde{\Lambda}(\rho_0 \otimes \sigma_M), \widetilde{\lambda}(\rho_0 \otimes \sigma_M)\right), \tag{2.17}$$

where ρ_0 is the initial state of the subsystem composed of the two quantum probes Q_A and Q_B, and σ_M is the initial state of the mediator M in the dilation. Following the demonstration provided in Appendix B.2 when considering unitary evolutions, the above distance can be further bounded by the deviation of the true global evolution of $Q_A \oplus M \oplus Q_B$ from the decomposed evolution obtained by concatenating the evolutions generated by $\hat{H}_{AM}$ and $\hat{H}_{MB}$:

$$g^{-1}(\Delta G_{corr\ A:B}) \leq \left\| \hat{U}_{AMB}(t) - \hat{U}_{AM}(t)\, \hat{U}_{MB}(t) \right\|_\infty \tag{2.18}$$

where $\hat{U}_{AMB}(t)$ is defined in Eq. 2.8, while $\hat{U}_{AM}(t)$ and $\hat{U}_{MB}(t)$ are defined as:

$$\hat{U}_{AM}(t) := e^{-it\hat{H}_{AM}}, \qquad \hat{U}_{MB}(t) := e^{-it\hat{H}_{MB}}, \tag{2.19}$$

and $\| \cdot \|_\infty$ is the spectral norm.

To make the dependence on non-commutativity explicit, we consider the Zassenhaus factorisation of the global unitary,

$$\hat{U}_{AMB}(t) = \hat{U}_{AM}(t)\,\hat{U}_{MB}(t)\,\prod_{n=2}^{\infty}\exp\Big[(-it)^n\hat{Z}_n\Big], \tag{2.20}$$

and define the associated Zassenhaus correction unitary:

$$\hat{U}_C(t) := \prod_{n=2}^{\infty}\exp\Big[(-it)^n\hat{Z}_n\Big] = e^{-it\hat{H}_C}, \tag{2.21}$$

where $\hat{H}_C$ is the corresponding Hermitian generator of $\hat{U}_C(t)$. As shown in Appendix B.1, this allows one to express the deviation from decomposability as:

$$\Big\|\hat{U}_{AMB}(t) - \hat{U}_{AM}(t)\hat{U}_{MB}(t)\Big\|_\infty = \Big\|\hat{U}_C(t) - \mathbb{I}\Big\|_\infty, \tag{2.22}$$

which admits a closed functional form in terms of $\hat{H}_C$:

$$\Big\|\hat{U}_C - \mathbb{I}\Big\|_\infty = 2\,\Big\|\sin\Big(\frac{t}{2}\hat{H}_C\Big)\Big\|_\infty. \tag{2.23}$$

This result establishes a quantitative relation that links $g^{-1}\Big(\Delta G_{\text{corr }A:B}\Big)$ directly to a function of the non-classicality of interactions:

$$g^{-1}\Big(\Delta G_{\text{corr }A:B}\Big) \le 2\Big\|\sin\Big(\frac{t}{2}\,\hat{H}_C\Big)\Big\|_\infty \qquad \forall\ t. \tag{2.24}$$

$\hat{H}_C$ is completely determined by the interaction Hamiltonians $\hat{H}_{AM}$ and $\hat{H}_{MB}$, and it vanishes whenever they commute, i.e., $[\hat{H}_{AM}, \hat{H}_{MB}] = 0 \Rightarrow \hat{H}_C = 0$. In the short-time regime, a second-order expansion in t yields an especially transparent form of the bound, since it makes the dependence on the commutator $[\hat{H}_{AM}, \hat{H}_{MB}]$ fully explicit:

$$g^{-1}\Big(\Delta G_{\text{corr }A:B}\Big) \le 2\Big\|\sin\Big(\frac{it^2}{4}\,[\hat{H}_{AM}, \hat{H}_{MB}]\Big)\Big\|_\infty. \tag{2.25}$$

This inequality relates the gain in quantum correlations between Q_A and Q_B to a function of the commutator $[\hat{H}_{AM}, \hat{H}_{MB}]$. Equivalently, by the same expansion, the leading non-trivial order in t implies that the generator entering Eq. 2.24 is:

$$\hat{H}_C \approx \frac{it}{2}\,[\hat{H}_{AM}, \hat{H}_{MB}]. \tag{2.26}$$

Taken together, inequalities Eq. 2.24 and Eq. 2.25 establish a quantitative link between the gain in quantum correlations of the two probes and the non-classicality of the underlying interactions, in the setting where the mediator is unknown and experimentally inaccessible. We can summarise the result of this Section by introducing the function:

$$f(H) := g\left(2\left\|\sin\left(\frac{it^2}{4}H\right)\right\|_\infty\right) \tag{2.27}$$

and rewriting Eq. 2.25 as:

$$\Delta G_{corr\ A:B} \leq f\left(\left[\hat{H}_{AM}, \hat{H}_{MB}\right]\right), \tag{2.28}$$

where the function f is defined once the correlation measure G (hence, the function g characterising its continuity) and the interaction time t are fixed. The quantum correlations locally mediated by the system under investigation M between the quantum probes Q_A and Q_B are therefore upper bounded by a function that depends only on the commutator of the interaction Hamiltonians, i.e., on the non-classicality of the interactions.

It is worth making a few remarks on this result before proceeding to its extension to non-commuting variables. As shown in Appendix B.1, if the interactions are classical, namely if $[\hat{H}_{AM}, \hat{H}_{MB}] = 0$, then the Zassenhaus correction is trivial, $\hat{U}_C(t) = \mathbb{I}$, and hence $\hat{H}_C = 0$ (see Eq. B.13). Consequently, since $\sin\left(\frac{t}{2}\hat{H}_C\right) = 0$, the right-hand side of Eq. 2.24 vanishes, and we obtain $g^{-1}\left(\Delta G_{\text{corr}\,A:B}\right) \leq 0$. Since g is monotonically increasing, invertible, and satisfies $\lim_{s\to 0} g(s) = 0$, it follows that:

$$\Delta G_{\text{corr}\,A:B} \leq 0. \tag{2.29}$$

In particular, the condition in Eq. 2.16 cannot be satisfied.

Conversely, if an experiment reveals an increase in quantum correlations, namely $\Delta G_{\text{corr}\,A:B} > 0$, then, by the properties of g recalled above, $g^{-1}\left(\Delta G_{\text{corr}\,A:B}\right) > 0$. By Eq. 2.24, this can only occur if the right-hand side is strictly positive, hence $\hat{H}_C \neq 0$. Using Eq. B.14 in Appendix B.1, we conclude that the interaction Hamiltonians cannot commute, i.e., $[\hat{H}_{AM}, \hat{H}_{MB}] \neq 0$. This is consistent with the

explicit second-order form in Eq. 2.25, where the leading non-trivial contribution is governed by the commutator $[\hat{H}_{AM}, \hat{H}_{MB}]$.

A further remark concerns the size and achievability of the right-hand side of (2.24). Since $\|\sin(X)\|_\infty \leq 1$ for any Hermitian operator X, the bound is limited as:

$$2\left\|\sin\left(\frac{t}{2}\hat{H}_C\right)\right\|_\infty \leq 2, \tag{2.30}$$

so that its maximum is independent of the specific choice of $\hat{H}_{AM}$ and $\hat{H}_{MB}$. Moreover, for any fixed $\hat{H}_C \neq 0$, this maximal value can be achieved by an appropriate choice of the interaction time t. As shown in Appendix B.2 (see Eq. B.34), the earliest time at which this occurs is controlled by the largest eigenvalue of $\hat{H}_C$, and scales as $t_* = \pi/\max_j |\lambda_j|$, where $\{\lambda_j\}$ are the eigenvalues of $\hat{H}_C$. In this sense, while the maximal value of the bound is independent of the interaction Hamiltonians, the shortest interaction time needed to approach it depends on the spectral structure of $\hat{H}_C$.

Finally, in the short-time regime where $\hat{H}_C \approx \frac{it}{2}[\hat{H}_{AM}, \hat{H}_{MB}]$ (see Eq. B.38 in Appendix B.2), it is natural to expect that larger non-commutativity, as quantified for instance by $\|[\hat{H}_{AM}, \hat{H}_{MB}]\|_\infty$, tends at leading order to increase the initial growth rate of the right-hand side of the bound in Eq. 2.25 as a function of t, and may therefore allow the maximum to be approached more rapidly. A systematic characterisation of these time scales is beyond the scope of this Thesis.

2.3.2 From Non-Classical Interactions to Non-Classical Mediator

We now prove that the inequality Eq. 2.24 can be expressed as a function of the commutators of the variables of the mediator M. In doing so, we connect the non-classicality of interactions with the non-classicality of the mediator, as defined in this Thesis, proving the latter to be more fundamental. The shift is crucial for understanding the GWT: what makes the channel M capable of mediating entanglement between the two quantum probes Q_A and Q_B by local means only is its non-classicality, namely the existence and dynamical involvement of at least two non-commuting degrees of freedom, not the specific local interactions with the probes.

We begin by defining the quantum probes Q_A, Q_B and the mediator M. Under the assumption of finite dimensions, we have two possible very general choices for the Hilbert spaces $\mathcal{H}_{Q_A}, \mathcal{H}_M, \mathcal{H}_{Q_B}$. The first is to consider a generic d-dimensional Hilbert space, while the second possibility is to choose a tensor product of qubit Hilbert spaces $\mathcal{H}_{(2)} \otimes \ldots \otimes \mathcal{H}_{(2)}$, that would allow us to use a certain number of qubits to simulate any quantum system of finite dimension [58]. To simplify the notation and without loss of generality, we consider the mediator M to be a chain of T qubits with Hilbert space dimension $d_M = 2^T$, while the probes Q_A and Q_B to be d_A, d_B dimensional systems described by Hilbert spaces with $SU(N)$ operator algebra. Hence, the most general $\hat{H}_{AM}, \hat{H}_{MB}$ Hamiltonians can be written as:

$$\hat{H}_{AM} = \sum_{i,j_1,j_2,\ldots j_T} \alpha_{i,j_1,j_2,\ldots j_T} (\hat{a}_i \otimes \hat{m}_{j_1} \otimes \hat{m}_{j_2} \otimes \ldots \otimes \hat{m}_{j_T} \otimes \mathbb{I}_B), \tag{2.31}$$

$$\hat{H}_{MB} = \sum_{l_1,l_2,\ldots,l_T,k} \beta_{l_1,l_2,\ldots,l_T,k} (\mathbb{I}_A \otimes \hat{m}_{l_1} \otimes \hat{m}_{l_2} \otimes \ldots \otimes \hat{m}_{l_T} \otimes \hat{b}_k), \tag{2.32}$$

where α and β are real-valued coefficients, $\hat{a}_i$ and $\hat{b}_k$ are $SU(d_A)$ and $SU(d_B)$ operators including the identity; $\{\hat{m}_{j_i}\}_{j_i=0,\ldots,3} = \{\mathbb{I}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$ is the set of the Pauli operators including the identity for the i-th qubit of the mediator, with $i = 1, 2, ..., T$. The indices j and l distinguish the mediator's operators appearing in $\hat{H}_{AM}$ and $\hat{H}_{MB}$, respectively. The identity operators appearing in the tensor products reflect the *locality* condition on the interactions. This avoids any direct interaction terms between Q_A and Q_B, thus enforcing locality.

Using this general structure, the commutator of the interaction Hamiltonians can be expanded as follows (see Appendix B.3):

$$\begin{aligned}\left[\hat{H}_{AM}, \hat{H}_{MB}\right] = \sum_{i,j_1,j_2,\ldots j_T,l_1,l_2,\ldots,l_T,k} & \alpha_{i,j_1,j_2,\ldots j_T} \beta_{l_1,l_2,\ldots,l_T,k} \cdot \\ & \hat{a}_i \otimes \Big([\hat{m}_{j_1}, \hat{m}_{l_1}] \otimes \hat{m}_{j_2}\hat{m}_{l_2} \otimes \ldots \otimes \hat{m}_{j_T}\hat{m}_{l_T} \\ & + \hat{m}_{l_1}\hat{m}_{j_1} \otimes [\hat{m}_{j_2}, \hat{m}_{l_2}] \otimes \ldots \otimes \hat{m}_{j_T}\hat{m}_{l_T} \\ & + \hat{m}_{l_1}\hat{m}_{j_1} \otimes \hat{m}_{l_2}\hat{m}_{j_2} \otimes \ldots \otimes [\hat{m}_{j_T}, \hat{m}_{l_T}] \Big) \otimes \hat{b}_k. \end{aligned} \tag{2.33}$$

Eq. 2.33 shows that the commutator $\left[\hat{H}_{AM}, \hat{H}_{MB}\right]$ can be non-zero only if at least one commutator between the mediator observables appearing in the two interactions

is different from zero, i.e., only if there exists at least one pair of observables such that $[\hat{m}_{j_r}, \hat{m}_{l_r}] \neq 0$ for some mediator-qubit $r \in \{1, ..., T\}$. Equivalently, if all mediator observables that appear in the interaction terms commute with each other, then the interaction Hamiltonians commute. In other words, non-commutativity at the level of the interaction Hamiltonians requires non-commuting variables in the mediator.

However, the converse does not generally hold: a mediator characterised by multiple non-commuting degrees of freedom does not imply $[\hat{H}_{AM}, \hat{H}_{MB}] \neq 0$. This is because the interaction Hamiltonians may include only a commuting subset of the mediator's observables, or cancellations may occur. To explain the logic of the implication, let us provide a simple example. Let M be a single qubit, so that it can be described by non-commuting observables (e.g., $[\hat{\sigma}_x, \hat{\sigma}_z] = -2i\hat{\sigma}_y$). Choose local couplings involving only the mediator observable $\hat{\sigma}_z$:

$$\hat{H}_{AM} = \hat{a}_i \otimes \hat{\sigma}_z \otimes \mathbb{I}_B, \qquad \hat{H}_{MB} = \mathbb{I}_A \otimes \hat{\sigma}_z \otimes \hat{b}_k, \tag{2.34}$$

with generic Hermitian operators $\hat{a}_i$ and $\hat{b}_k$ acting on the two probes. Then $[\hat{H}_{AM}, \hat{H}_{MB}] = 0$ even though the mediator is non-classical. This example is important to clarify the role of the witness as a *sufficient condition* for M's non-classicality. While a successful entanglement generation between the two probes under the appropriate assumptions conclusively infers M's non-classicality, an unsuccessful witnessing task makes the test inconclusive. This is because the mediator M may still be a non-classical system, but in that particular configuration is not engaging a minimum of two non-compatible variables in the dynamics with Q_A and Q_B, making it impossible for the witnessing task to reveal them.

Furthermore, Eq. 2.33 shows that the presence of a single non-vanishing commutator for one of the mediator subsystems is sufficient to ensure that the commutator of the interaction Hamiltonians is different from zero. Therefore, a single quantum system within a composite classical-quantum mediator M is sufficient to mediate quantum correlations between the probes Q_A and Q_B. This conclusion is particularly relevant to justify the application of the GWT in scenarios where the mediator is a complex system composed of many interacting subsystems, for example

photosynthetic biological systems. An application of this non-classicality witness where the mediator is a green sulphur bacterium has been proposed by Krisnanda and collaborators in Ref. [51].

The result derived in this Section quantitatively connects the amount of quantum correlations locally mediated between the quantum probes Q_A and Q_B to the non-classicality of the mediator M:

$$\Delta G_{corr\ A:B} \leq f\left(\left[\hat{H}_{AM}, \hat{H}_{MB}\right]\right) \equiv f_H\left([\hat{m}_{AM}, \hat{m}_{MB}]\right) \tag{2.35}$$

where $[\hat{m}_{AM}, \hat{m}_{MB}]$ denotes the set of commutators between the mediator variables $\{[\hat{m}_{j_r}, \hat{m}_{l_r}]\}_{r=1,...,T}$ appearing in Eq. 2.33. This result is completely general within the quantum formalism, as no further assumptions are made about the interactions between the probes and the mediator. Importantly, the functional form of f_H is given by Eq. 2.27, but its explicit expression can be obtained only after specifying the coefficients $\alpha_{i,j_1,...,j_T}$ and $\beta_{l_1,...,l_T,k}$ of the interaction Hamiltonians $\hat{H}_{AM}$ and $\hat{H}_{MB}$. If these are known, for example in scenarios where the coupling between the probes and the channel can be engineered, one can fix the quantum correlation measure G and the interaction time t to estimate the minimum amount of non-classicality required by the channel to mediate the measured quantum correlations. Conversely, if the quantum channel M is fully characterised, the relation in Eq. 2.35 can provide a theoretical tool for computing the maximum amount of quantum correlations that can be mediated between two nodes in a network, without the need for prior experimental testing.

Furthermore, Eq. 2.35 implies that no classical system, defined as a system described by a single variable, can mediate entanglement between two quantum probes *by local means only* (see Principle 1). In fact, if $[\hat{m}_{j_i}, \hat{m}_{l_i}] = 0\ \forall\ i = 1, 2, ..., T$, then $f_H([\hat{m}_{AM}, \hat{m}_{MB}]) = 0$, and this implies that $\Delta G_{corr\ A:B} \leq 0$. Any model that predicts entanglement generation while maintaining that the mediator is classical must therefore abandon at least one assumption of the GWT. This is the case of claims that classical gravity can mediate entanglement between two quantum masses (see for example [60, 75, 76]), where some non-local effects are responsible for the

direct interaction between the masses, ultimately responsible for the entanglement generation [46, 77–79], or where the gravitational field is not considered as a mediator of the interactions between Q_A and Q_B [80, 81]. Another important class of models that can be falsified by this result are superselected ones, for example Ref. [19]. In these models, an additional constraint (the superselection rule) limits the possibility of dynamically involving multiple non-compatible variables describing the mediating system, making it effectively classical. Eq. 2.35 shows that if an increase in the quantum correlations between the two probes Q_A and Q_B is recorded under the assumptions discussed in Chapters 2.2.1, 2.2.2 and 2.2.3, then the mediator M must have engaged multiple non-compatible variables in the dynamics, thereby ruling out this particular form of superselection rules. Alongside superselected models, Koopmanian systems would be ruled out by a successful two-probe spatial entanglement-based witness of non-classicality. In this approach to hybrid classical-quantum systems, different commuting variables of a classical system are encoded into multiple quantum systems, whose other, non-compatible degrees of freedom are hidden [16]. However, when coupled to fully fledged quantum systems, in order to account for their back-reaction it is necessary to engage precisely those hidden variables of the quantum systems used to encode the classical one [82]. Eq. 2.35 makes this explicit: if an increase in quantum correlations between Q_A and Q_B is recorded in an experiment satisfying the minimal assumptions of the GWT, the Koopmanian system must have engaged a minimum of two non-commuting degrees of freedom.

For a complete review of the implications of a successful GWT in the context of gravity, we refer the interested reader to Ref. [41].

2.4 Summary

In this Chapter we have presented a two-probe witness of non-classicality that relies on the capability of the system under investigation M to locally mediate entanglement between two quantum probes Q_A and Q_B: the General Witness Theorem in Theorem 1. If successful, one can infer that M must be described by at

least two non-compatible variables, i.e., variables that cannot be simultaneously measured to an arbitrarily degree of accuracy. Within the formalism of quantum theory, these are non-commuting variables.

The GWT relies on three very general assumptions, which make it a completely dynamics-independent argument. The first assumption is Principle 1, the principle of locality, understood here as the impossibility of modifying the complete state of a subsystem by acting on a second, separated one only. We have compared this notion of spatial locality with other subsidiary theory-dependent ones commonly used in the literature, such as the principle of no-signalling, the axioms of microcausality and of Lorentz-covariance and the notions of subsystem locality. By distinguishing the different notions of locality, we have argued why Principle 1 is more general and why it is important to adopt this minimal notion of locality for the GWT: it provides a robust theorem that proves the non-classicality of the investigated system even beyond quantum theory. The second assumption is Principle 2, the principle of interoperability of information, which informally implies the possibility of copying classical information from the probes to the investigated system and vice versa. This assumption, necessary for the testability of a theory, was elegantly phrased in the information-theoretical framework of constructor theory of information. This avoids committing to any subsidiary dynamical theory to study the unknown sector M. The third assumption is that the investigated system M acts as a mediator of interaction between the quantum probes Q_A and Q_B, in order for it to be responsible for the witnessing task the GWT is based on: the mediation of entanglement between the probes.

Under these general assumptions, we have introduced the GWT in Theorem 1 and reproduced its proof within the formalism of quantum theory (and in general, using constructor theory, in Appendix A). Within the quantum formalism, we have developed a quantitative support to the GWT with the aim of connecting the amount of quantum correlations mediated by the investigated system M between the probes Q_A and Q_B to the non-classicality of M. We have shown that the quantum correlation mediated between Q_A and Q_B, quantified by an appropriate

correlation function G in a broad class of entanglement measures, lower bounds a function of M's non-classical variables only. This function is uniquely defined once the correlation measure G is chosen and the interaction time is fixed. This quantitative relation, summarised by Eq. 2.35, shows that only a non-classical mediator can induce entanglement between two spatially separated quantum probes under the assumptions of Theorem 1. In fact, if the system M were classical, $f_H([\hat{m}_{AM}, \hat{m}_{MB}]) = 0$ as all its variables would commute. Therefore, $\Delta G_{corr\ A:B} \leq 0$: it would not be possible to increase the quantum correlations between Q_A and Q_B. Thus, the GWT is a sufficient condition for M's non-classicality: observing $\Delta G_{corr\ A:B} > 0$ implies the non-classicality of the mediator, while the converse makes the witness inconclusive.

The GWT, thanks to its generality, has been proposed as a robust theorem to design applications of the non-classicality witness in relevant hybrid systems, such as photosynthetic complexes and gravity. The application of the GWT to the latter is known as the gravitationally-induced entanglement or the Bose-Marletto-Vedral effect and is now among the most promising experiments to detect quantum effects of the gravitational field. Several platforms are currently competing as the most suitable to implement the quantum probes capable of interacting gravitationally. Among them, nitrogen-vacancy (NV) centres in nanodiamonds in trapped configurations are promising candidates to realise the Stern-Gerlach interferometers necessary to create the spatial superpositions of the probes, let the probes interact gravitationally and close the spatial superposition to enable the entanglement readout [83, 84]. However, other platforms may prove useful to design GIE experiments, and we refer the reader to Ref. [41] for an overview.

All of these platforms share common challenges in the implementation of the GWT, the most relevant of which is enforcing the principle of locality as mandated by Principle 1. As discussed in this Chapter, if the two quantum probes interact directly, or if any non-local mechanism allows them to couple independently of the mediator M, the entanglement observed at the end of the protocol would be generated because of these direct interactions. Therefore, the GWT would

be inconclusive about the non-classicality of the investigated system, since the generation of quantum correlations would fail to reveal the existence of a second non-compatible variable describing it. One way to address this challenge is to rethink the quantum sector interacting with the unknown system and consider it as composed of a *single* quantum probe Q. In this way, there would be no more direct interactions between two quantum probes to exclude beforehand, and the witnessing task should be sought in the *temporal* evolution of Q mediated by the investigated system M.

Devising a novel class of single-probe, temporal witnesses of non-classicality is not only interesting because it may ease the experimental implementability of these arguments. As we have noted in this Chapter, the construction of a witness of non-classicality looks deep into the information-theoretical foundations of the non-classical theories describing the hybrid. The choice of a particular witnessing task is dictated by its ability to reveal the single-shot indistinguishability of the investigated system's variable, if successful under the appropriate assumptions. The more minimal the assumptions, the broader the class of theories it embraces, the more general is the argument. In the case of the GWT, this role is revealed by the *spatial* quantum correlations M mediates between two spatially separated quantum probes, and the assumptions in Chapter 2.2 are general enough to incorporate subsidiary theories beyond the quantum formalism. The possibility to investigate whether this role can be played by *temporal* quantum correlations too is fascinating, both because it may reformulate previously proposed single-probe arguments in information-theoretical terms and because it may achieve the level of robustness and generality characterising two-probe arguments such as the GWT. Furthermore, having such a general argument with a single probe would stimulate the study of the relevance of temporal quantum correlations in quantum information processing and beyond. The symmetrical role of mediated temporal and spatial quantum correlations in revealing the non-classicality of a physical system can also provide insights on the structure of spacetime in superinformation theories, including quantum theory.

We shall focus on the construction of this new class of single-probe, temporal witnesses of non-classicality in the next Chapters.

3

Temporal Witness of Non-Classicality

The content of this chapter is partially based on [85–87]. [85] was developed in collaboration with Chiara Marletto. [86] was developed in collaboration with Vlatko Vedral and Chiara Marletto. [87] was developed in collaboration with Simone Rijavec.

In Chapter 2, we have discussed the first example of a witness of non-classicality: the spatial entanglement-based witness of non-classicality. This argument relies on the entangling power of the unknown system (the "mediator") between two spatially separated quantum probes to infer its non-classicality. Here, non-classicality is defined as the existence of a minimum of two non-compatible variables in the description of the mediator, see Chapter 1.2. Specifically, the General Witness Theorem in Theorem 1 states that if a system M can mediate entanglement between two quantum probes Q_A and Q_B by local means only, then M must be non-classical. While theoretically robust, relying on minimal information-theoretical assumptions which go beyond quantum theory, its experimental implementation presents several challenges. The one common to all the different platforms is the need to exclude all the direct interactions between the two quantum probes, as discussed in Chapter 2.

In this Chapter, we introduce a novel class of witnesses of non-classicality motivated by the need to overcome these experimental challenges. The main

idea is to remove one of the two probes required by the GWT, and formulate a novel theorem with a *single* quantum probe instead. The witnessing task in these scenarios must be looked for in the *time evolution* of the probe, so we shall refer to them as *temporal witnesses of non-classicality*. To account for the removal of the second quantum probe, temporal witnesses of non-classicality require an additional assumption, which we shall identify in the *conservation of a global, additive quantity* throughout the evolution of the hybrid system. We shall provide a formal proof of the argument within the quantum formalism and propose an application of this novel theorem to the light-harvesting stage of photosynthesis. We also present a collision model to investigate the possibility of the witnessing task and the role the environment has in favouring long-lived quantum coherence. The model will be connected to the real components of a biological system, suggesting a possible experiment to conclusively prove the quantum nature of this crucial biological process. Together with the two-probe witness proposed in Ref. [51], this represents another application of the non-classicality witnesses methodology to the continuously-growing field of quantum biology.

3.1 Towards a Single Probe Non-Classicality Witness

Since Feynman's seminal proposal to witness quantum effects in gravity using a single mass, which we already discussed in Chapter 2, several other proposals have been put forward in the field.

The first example is the experiment performed by Page and Geilker, where they used a single superposed mass to test the predictions of a *particular* semiclassical gravity theory: quantum field theory in curved spacetime including the gravitational field's backreaction on a superposed mass [88]. A Cavendish device is coupled to a quantum particle that controls its preparation in one of two configurations: clockwise and counterclockwise torques. When the quantum particle is prepared in a superposition in the control variable basis, the two masses of the Cavendish device get entangled to the controlling mass, resulting in a mass distribution whose effects

on the net torque of the device can be used as a witnessing task. According to this semiclassical theory, the net torque on the Cavendish device is expected to be null. This is because the theory predicts that the gravitational field associated with this mass distribution is an *average* of the two gravitational fields, each generated by one of the configurations. If the gravitational field is quantum instead, there would be a non-zero torque in each branch of the entangled global state including both the Cavendish device and the controlling mass. This is what the actual experiment recorded, conclusively ruling out quantum field theory in curved spacetime as a fundamental theory of gravity. However, the conclusions of this experiment about the quantum nature of gravity are not as strong as the ones that would come from a successful BMV experiment: while it can consolidate the role of quantum field theory in curved spacetime as a useful approximation for calculations, it cannot rule out other classical descriptions of gravity such as gravitational collapse theory. This is because the quantum nature of the probe is not operationally involved in this experiment: the quantum probe has already undergone decoherence when it controls the device, suggesting that even preparing the Cavendish device by tossing a fair coin would have led to the same experimental conclusions. For a single-probe experiment to be a witness of non-classicality, the quantum probe must be *actively* involved in the witnessing task, not just as a controlling device to prepare the investigated system in different configurations, but as a device undergoing genuinely quantum processes *mediated by* the investigated system.

Another example is the experiment performed by Colella, Overhauser and Werner (COW) involving neutron interferometry in the Earth's gravitational field [62]. Here, a single neutron is sent across an interferometer, where a first beam splitter prepares it in a superposition of two locations. Gravity induces a phase shift between the two branches of the superposition, which can be modulated by tilting the angle ϕ of the plane of the interferometer with respect to the position parallel to Earth's surface. By changing the angle ϕ, one can change the local phase kicked in by the gravitational field in one of the interferometer's arms, leading the neutron to the state $\alpha|0\rangle + \beta e^{i\Phi}|1\rangle$, where $\Phi = \frac{2\pi gm}{h^2} A\lambda \sin\phi$. The relative phase Φ, measured to

different degrees of accuracy, is proportional to the tilt angle ϕ, the neutron mass m and de Broglie wavelength λ, and the area enclosed by the interferometer A, with the proportionality constant including the gravitational coupling constant g and the Planck's constant h. Detecting this phase is an important signature of the fact that gravity behaves consistently with any other external field in nonrelativistic quantum mechanics: it couples to a superposed mass respecting linearity and preserving neutron's coherence (i.e., without collapsing the superposition) even at neutron's scale. Thus, this experiment successfully rules out collapse-based model due to gravity, another *particular* description of the gravitational field in a quantum world. In fact, for the same reason we have discussed for Feynman's proposal, a classical model of the gravitational field would perfectly account for the generated local phase after the first beam splitter of the interferometer. Said differently, a semiclassical theory of gravity coupled to quantum matter would explain this phenomenon, and the COW experiment *cannot* act as a single-probe witness of non-classicality for gravity. Although the COW experiment correctly exploits the neutron as the single-probe intended by a witness of non-classicality (the task here is the generation of the local phase in the neutron's state), it fails in setting the stage for the gravitational field to exploit its non-compatible variables when mediating such a task. In fact, the gravitational field here is generated by the Earth, whose state is not in a superposition. This makes a semiclassical model sufficient to explain the phenomenology we are observing in this class of experiments.

One solution to such a problem is to include a second probe prepared in a superposition of two locations and generate a gravitational field in a non-classical state. This forces gravity to exploit the non-compatible variables when mediating the witnessing task, as discussed in Chapter 2.

Another solution can be to look for a *particular* phenomenon that can be treated *both* classically and quantumly, with distinguishable effects in the two cases. In Ref. [89], the identified phenomenon is the self-gravitational interaction of a Bose-Einstein condensate (BEC). Treating the gravitational field as a classical or a

quantum system in the Newtonian regime leads to two different Hamiltonians:

$$\hat{H}_{QG} = -\frac{1}{2}Gm^2 \int d^3\vec{r}d^3\vec{r}' \frac{|\psi(\vec{r}')|^2|\psi(\vec{r})|^2}{|\vec{r}-\vec{r}'|}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} \tag{3.1}$$

$$\hat{H}_{CG} = Gm \int d^3\vec{r}|\psi(\vec{r})|^2\Phi[\psi](t,\vec{r})\hat{a}^\dagger\hat{a} \tag{3.2}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators for the condensate, $\psi(\vec{r})$ is the wavefunction of the condensed atom of mass m, G is the gravitational constant and Φ is the classical gravitational potential obtained as a quantum average of the quantised version of Poisson's equation $\hat{\Phi}[\psi](\vec{r}) = -Gm \int d^3\vec{r}' \frac{|\psi(\vec{r}')|^2}{\vec{r}-\vec{r}'}\hat{a}^\dagger\hat{a}$. Of the two, only the former can create non-Gaussianity in the quantum field of matter. This is because non-Gaussianity is created by processes that are non-quadratic in quantum operators [90]. Eq. 3.1 is quartic in the field operators: such a Hamiltonian is capable of generating non-Gaussianity in the quantum state of the condensate. By contrast, when gravity is treated classically as in Eq. 3.2, the gravitational potential is sourced by the expectation value of the density operator, resulting in an effective mean-field Hamiltonian that is quadratic in the quantum operators and therefore preserves Gaussianity. Consequently, if non-Gaussianity is generated in a Bose-Einstein condensate with only self-gravitational interactions, the classical description of this phenomenon will be ruled out because it is not capable of generating non-Gaussian states. This approach provides a solid ground to "force" the dynamical involvement of the two non-compatible variables required for a system to be considered a quantum system, and it can be applied to other scenarios different from gravity. However, as described above, the classical description of the phenomenon relies on a *particular* model compared to its quantised version, leaving other classical models still viable.

The examples above illustrate three recurring challenges for single-probe witnesses of non-classicality: (i) the quantum probe is often not actively involved in the witnessing task, serving merely as a control or preparation device instead; (ii) the observed signatures can frequently be reproduced by classical or semiclassical mediators; and (iii) the witnessing task is not formulated in a fully general, theory-independent manner.

These considerations suggest that a genuine single-probe witness of non-classicality must be grounded in the *evolution* of the probe itself, and must identify dynamical features that cannot be reproduced by *any* classical mediator satisfying a minimal set of physically motivated assumptions. In the following, we introduce a novel class of such witnesses—*temporal witnesses of non-classicality*—in which the witnessing task is encoded in the *quantum coherent evolution* of a single quantum probe. By requiring the *conservation of a global additive quantity* during the interaction, this framework overcomes the limitations of previous proposals and provides a general, information-theoretical, single-probe criterion for non-classicality in hybrid systems.

3.2 Quantum Coherent Evolution: Sufficient Condition for Non-Classicality

Let us consider a hybrid system in which an unknown sector M interacts with a single quantum probe Q. The crucial difference with respect to the hybrid setting analysed in Chapter 2 is that the quantum sector now lacks one of the two probes.

In two-probe non-classicality witnesses, the second probe plays a crucial operational role: by interacting with the mediator M after the first probe, it accesses the non-classical state of M and thus encodes information about the incompatibility of its variables. This, in turn, makes it possible to certify the non-classicality of M through the entanglement established between the two probes, thanks to the principle of locality [50].

In the single-probe setting, the absence of a second probe must be compensated by introducing an additional assumption that replaces the missing operational step. One possible physically well-motivated choice is the *conservation of a global additive quantity* in the hybrid system. As we shall show, this additional condition is sufficient to constrain the dynamics in such a way that the *quantum coherent evolution* of the single probe Q in the basis of the conserved quantity cannot be accounted for by a classical mediator. This identifies the witnessing task for temporal witnesses of non-classicality and naturally addresses challenges (i), (ii), and (iii) identified above, providing a theoretically robust single-probe argument.

Before formalising this assumption and analysing its role in the witnessing protocol, it is useful to summarise the assumptions underlying the different witness constructions discussed in this Thesis. The GWT (Theorem 1) is formulated within the constructor theory of information and relies on minimal assumptions such as the principle of locality (Principle 1), the principle of interoperability of information (Principle 2), and the possibility for the investigated system to act as a mediator, discussed in Chapter 2.2.3. By contrast, the temporal witness is derived within the formalism of quantum theory and includes, as a sufficient additional assumption, the exact conservation of a global additive quantity. Importantly, the quantum formalism already entails the principles assumed by the GWT. Therefore, the temporal witness is general within quantum theory, while the GWT retains a broader, theory-independent scope.

The assumptions underlying the two constructions are summarised in Table 3.1.

	GWT	**Temporal Witness**
Framework	CT of information	Quantum formalism
Locality	✓	Included in quantum formalism
Interoperability	✓	Included in quantum formalism
Mediator assumption	Required	Automatically satisfied
Conservation law	×	Exact global additive conservation
Number of probes	Two	One
Level of generality	Theory-independent	Quantum-theoretical

Table 3.1: Assumptions underlying the two classes of non-classicality witnesses.

3.2.1 The Extra Assumption: Conservation Laws

To physically motivate the sufficient extra assumption, it is instructive to revisit some of the difficulties encountered by earlier approaches to hybrid classical-quantum systems. As discussed in Chapter 1.1, attempts to provide a consistent classical–quantum description of a hybrid system $Q \oplus M$ often result in violations of conservation laws. More generally, consistently coupling a classical system to a quantum one appears to be incompatible with the exact enforcement of conservation laws [21, 33].

However, the objective of a witness of non-classicality is precisely to infer the non-classical properties of the mediator M indirectly, by probing it with a quantum system Q. Here, as formalised in Chapter 1, we define a system as *non-classical* if it is described by a minimum of two distinct physical variables that cannot be measured to arbitrarily high accuracy simultaneously, i.e., by the same measuring device. Importantly, the dynamics-independence of this notion of non-classicality makes it more general than other definitions. In quantum theory, which will underpin the argument at this stage, these variables are non-commuting.

From this perspective, enforcing a conservation law as the additional assumption underpinning the theorem emerges as a natural strategy. If coupling a quantum probe Q to a genuinely classical mediator M leads to violations of conservation laws, then probing M while enforcing such a conservation law should be sufficient to reveal its non-classicality.

This intuition is further supported by the way conservation laws are implemented in quantum theory, under the assumption that unitary quantum mechanics applies universally, across all physical scales. In quantum theory, symmetries associated with additive conserved quantities hold not merely on average, but exactly, branch by branch, when quantum states are expressed in the eigenbasis of the conserved observable [91]. In other words, conservation laws in quantum theory apply to q-number observables, rather than to their expectation values. Consequently, if a quantum system prepared in a superposition of eigenstates of a conserved quantity is coupled to another system, exact conservation at the level of the composite system requires the interaction to generate entanglement between them. In each branch of the entangled state, any change in the value of the conserved quantity in the probe is exactly compensated by a corresponding change in the other system.

In a hybrid setting, this implies that a genuinely classical mediator cannot consistently interact with a quantum probe while exactly conserving an additive quantity, if the quantum system is in a superposition in the conserved quantity's basis. Therefore, enforcing exact conservation mandates that the mediator be described quantum mechanically, with at least one variable incompatible with the

conserved observable. This phenomenon is known as the *totalitarian property* of quantum theory[1]: any system that interacts with a quantum system while exactly obeying conservation laws must itself be describable quantum mechanically [35]. It is precisely this property that we aim to exploit in the construction of a single-probe witness of non-classicality: while in two-probe schemes the role of accessing and certifying the non-classical state of the mediator is played by the second quantum probe, in the single-probe construction this role is effectively taken over by the exact conservation of an additive quantity.

The remaining question is then: how do we witness this effect? What form should a witnessing task take under this assumption?

3.2.2 The Witnessing Task: Quantum Coherent Evolution

Suppose that (i) the quantum probe Q is initially prepared in an eigenstate of a local observable associated with the global conserved additive quantity, and (ii) that it undergoes a *coherent evolution* into a superposition of eigenstates of that same observable through its interaction with the mediator M only. Such a transformation corresponds to a rotation of Q into a basis that does not commute with the conserved quantity.

Under the assumption of exact conservation, this evolution cannot occur autonomously on Q. This is because any local violation of the conserved quantity on Q would not be compensated by M, thus violating the conservation law at the global level. Exact conservation therefore requires that M compensates the changes in the conserved quantity associated with the different components of Q's superposition, branch by branch. This compensation is possible only if M possesses at least two variables that are incompatible with one another. In other words, the mediator must engage non-commuting variables in order to mediate the coherent evolution of the probe. Hence, if a coherent rotation of Q in the conserved basis is observed while the conservation law holds exactly, the mediator M cannot be classical. In this sense, the conservation law is playing the same role as the

[1]This notion should not be confused with Gell-Mann's "totalitarian principle", according to which anything not forbidden is compulsory.

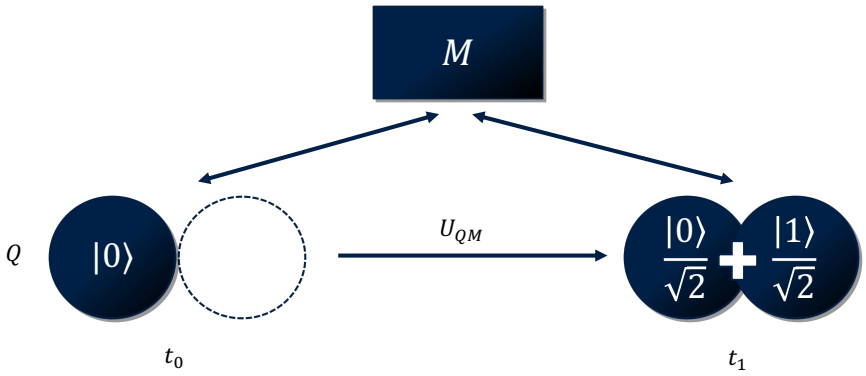


Figure 3.1: Schematic representation of the temporal witness of non-classicality, Theorem 2. For illustrative purposes, the probe Q is a qubit and the local conserved quantity is $\hat{Z}_Q = \hat{\sigma}_z \otimes \mathbb{I}_M$. The probe is prepared in $|0\rangle$, eigenstate of $\hat{Z}_Q$, at time t_0. The evolution U_{QM} conserves the global additive quantity $\hat{Z}_Q + \hat{Z}_M$, where $\hat{Z}_M = \mathbb{I}_Q \otimes \hat{\sigma}_z$. At time t_1, the probe is in a state that does not commute with that at time t_0, in this case $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

principle of locality in the GWT: it ensures that M is the one responsible for the witnessing task, thereby revealing its non-classicality.

3.2.3 The Theorem: Temporal Witness of Non-Classicality

This observation naturally defines a *temporal witness of non-classicality* within the formalism of quantum theory, see Fig. 3.1 [85]:

Theorem 2 (Temporal Witness of Non-Classicality)**.** *If a system M can induce a quantum coherent evolution of the quantum probe Q, then M must be non-classical, provided that a global additive observable of the system $Q \oplus M$ is conserved.*

Proof. Let us consider without loss of generality the quantum probe Q to be a qubit, with its computational basis defined by $\hat{Z}_Q = \hat{\sigma}_z \otimes \mathbb{I}_M$ where $\hat{\sigma}_z$ is the Pauli operator and $\mathbb{I}_M$ is the identity operator on the Hilbert space of M. We prepare Q in an eigenstate of $\hat{Z}_Q$, meaning that $\hat{Z}_Q$ is *sharp* at time t_0. We prove Theorem 2 by contradiction. Therefore, we assume (i) M to be a classical bit, described by the single variable $\hat{Z}_M = \mathbb{I}_Q \otimes \hat{\sigma}_z$, so that the set of classical states is included in the span of its eigenstates, forming a vector space; (ii) that the witnessing task is possible, i.e., the quantum probe Q undergoes a quantum coherent evolution in the

$\hat{Z}_Q$ basis, so that its final state does not commute with the initial one and that $\hat{Z}_Q$ becomes unsharp at time $t_1 > t_0$; and (iii) that the evolution $\hat{U}_{QM}$ of the hybrid system conserves the global additive quantity $\hat{Z}_Q + \hat{Z}_M$, that is,

$$\left[\hat{U}_{QM}, \hat{Z}_Q + \hat{Z}_M\right] = 0. \tag{3.3}$$

The most general Hamiltonian $\hat{H}_{QM}$ generating the evolution $\hat{U}_{QM}$ under these assumptions has the form:

$$\hat{H}_{QM} = a\hat{Z}_Q + b\hat{Z}_M + c\hat{Z}_Q\hat{Z}_M + r\mathbb{I}_{(4)}, \tag{3.4}$$

where $\mathbb{I}_{(4)} = \mathbb{I}_Q \otimes \mathbb{I}_M$ and $a, b, c, r \in \mathbb{R}$ are referred to as *classical* coefficients of the Hamiltonian, as they appear when the sector M in the hybrid is classical. However, we notice that $\left[\hat{H}_{QM}, \hat{Z}_Q\right] = 0$, and therefore $\hat{H}_{QM}$ *cannot* generate a non-trivial evolution of $\hat{Z}_Q$. This is in contradiction with (ii): if $\hat{Z}_Q$ is initially sharp, it cannot be made unsharp by $\hat{H}_{QM}$. The contradiction arises from assumption (i), namely that M is a classical system. Hence, we conclude that M must have been a non-classical system, as claimed by the temporal witness of non-classicality. □

Theorem 2 provides a *sufficient* condition for the non-classicality of an unknown system M: if a quantum coherent evolution of the probe Q is observed in the conserved quantity basis, then M must be non-classical. The argument is independent of the initial state of the mediator M, as the conservation law is enforced at the level of the operator. This is an important feature in the hybrid system scenario: since M is an unknown system, we should not assume to have direct control on its initial state during the protocol we envisage.

The result can also be generalised to a quantum channel operating on Q and M, under the assumptions that the channel can be extended to a unitary transformation on three systems Q, M, and M' and that the conservation law holds for the quantity $\hat{Z}_Q + \hat{Z}_M + \hat{Z}_{M'}$. Two cases can be distinguished here. The first is when M' is, like M in the proof of Theorem 2, a classical system. This corresponds to a situation in which the channel on Q and M is generated by interacting with an effectively classical environment. In this case, the extension of Theorem 2 is

straightforward, and one finds that no quantum coherent evolution can be mediated in the conserved quantity basis of Q. Hence, the temporal witness of non-classicality applies successfully in this scenario. The second is when M' is a quantum system instead, for example, a quantum system in the environment. In this scenario, Q may interact coherently with M', which could be mediating the coherent evolution instead of M. The temporal witness of non-classicality would no longer be applicable directly on M in this case, but would inform on the quantum properties of the composite $M \oplus M'$: it is $M \oplus M'$ that must be described by a minimum of two non-commuting variables. However, Theorem 2 does not allow one to discriminate which system is supplying them. Addressing this question would require a specific model of the hybrid system, and is therefore beyond the scope of the witness. Nevertheless, this observation can still provide a good insight in complex systems, for example biological systems, where the traditional role of the environment as a source of decoherence is challenged by experimental observations. A successful temporal witness of non-classicality in this scenario would indicate that the quantum degrees of freedom of the composite are not suppressed by the coupling of the system M with its environment M'. We shall see how this becomes relevant in the next Section. Notice that this observation would be analogous to a situation where, in the GWT in Theorem 1, the two quantum probes were allowed to interact via another quantum system that is not the mediator. In that case the witness does not allow one to conclude anything about the mediator, but a similar conclusion can be drawn about the composite system.

Another interesting connection can be made between Theorems 1 and 2 when representing the quantum probe Q in second quantisation. The quantum coherent evolution of the probe Q from one basis to some other non-commuting basis, used as a witnessing task in Theorem 2, corresponds to the creation of *mode-entanglement* on Q [92]. If this entanglement is mediated via M (and it must be so, because of the conservation law), then using the spatial entanglement-based witness in Theorem 1 one can conclude directly that M is non-classical. The conservation law in this case is a sufficient condition that implies the impossibility of building a gate

that entangles the two modes in Q directly, in the same way as the principle of locality (Principle 1) avoids direct entangling interactions between the probes Q_A and Q_B. This single-particle entanglement interpretation of the temporal witness of non-classicality provides another element to conclude that the conservation law plays an analogous role to the principle of locality in spatial entanglement-based witnesses of non-classicality.

Consider now a case in which the unknown sector M is indeed a quantum system. One may then ask which unitaries are compatible with the conservation law. By imposing the condition Eq. 3.3 for both a quantum Q and a quantum M, described as qubits, we get:

$$\hat{H}_{QM} = a\hat{Z}_Q + b\hat{Z}_M + c\hat{Z}_Q\hat{Z}_M + f(\hat{X}_Q\hat{X}_M + \hat{Y}_Q\hat{Y}_M) + g(\hat{X}_Q\hat{Y}_M - \hat{Y}_Q\hat{X}_M) + r\mathbb{I}_{(4)}, \quad (3.5)$$

where $f, g \in \mathbb{R}$ are the *non-classical* coefficients of $\hat{H}_{QM}$, as they appear in the Hamiltonian only when M is a non-classical system too. Here $\hat{O}_\alpha$ represents the component $\hat{O} \in \{\hat{X}, \hat{Y}, \hat{Z}\}$ of the qubit $\alpha \in \{Q, M\}$ (e.g., $\hat{X}_Q = \hat{\sigma}_x \otimes \mathbb{I}_M$, $\hat{X}_M = \mathbb{I}_Q \otimes \hat{\sigma}_x$).

An elementary example of a unitary that satisfies this requirement is the SWAP gate. If M is already in a non-classical state, for example an eigenstate of $\hat{X}_M$, then an interaction described by $\text{SWAP} = \frac{1}{2}\left(\mathbb{I}_{(4)} + \hat{X}_Q\hat{X}_M + \hat{Y}_Q\hat{Y}_M + \hat{Z}_Q\hat{Z}_M\right)$ would swap the state of Q and M, thus preparing Q in an eigenstate of $\hat{X}_Q$ when it was initially in an eigenstate of $\hat{Z}_Q$. This necessarily involves a quantum coherent evolution in the $\hat{Z}_Q$ basis, mediated by M, and therefore represents a simple proof-of-principle of the temporal witness of non-classicality. M is non-classical as it must be preparable in an eigenstate of the variable $\hat{X}_M$, which is incompatible with $\hat{Z}_M$, and it is engaging the non-commuting variables $\hat{X}_M, \hat{Y}_M, \hat{Z}_M$ to mediate the evolution. More general Hamiltonians allowed by Eq. 3.3 include also *exchange interactions* $\hat{H}_{QM} \propto \left(\hat{S}_{Q+}\hat{S}_{M-} + \hat{S}_{Q-}\hat{S}_{M+}\right)$, where $\hat{S}_{\alpha\pm} = \left(\hat{X}_\alpha \pm i\hat{Y}_\alpha\right)$ with $\alpha \in \{Q, M\}$. In all such cases, M engages non-commuting variables to mediate a non-trivial coherent evolution on Q, consistently with the statement of the theorem.

Until now, we have considered *additive* conservation laws as assumptions underlying the temporal witness of non-classicality. This is consistent with the central tenets of fundamental laws (e.g., energy and momentum conservation laws), which are indeed additive in subsystems. Moreover, additive conservation laws are typically assumed as background knowledge in experimental tests, as they have been independently verified. However, the level of generality required for a witness of non-classicality motivates us to consider the possibility that, in a more general non-classical theory beyond quantum physics, conservation laws may take a different mathematical form. In this spirit, in Appendix C, we shall explore the possibility of formulating the temporal witness of non-classicality by considering a general conservation law, function of the classical observables of the systems $\hat{Z}_{Q,M}$ only but not necessarily additive. We show that the temporal witness of non-classicality can still hold in this more general setting, and support the claim by presenting three proof-of-principle examples in which a quantum probe Q interacts with progressively more complex systems, from a bit to a harmonic oscillator.

In the next Section, we shall introduce the first proposed application of Theorem 2 to a realistic hybrid system, specifically in the context of quantum biology. We will discuss how the temporal witness of non-classicality can be applied to the light-harvesting stage of photosynthesis in biological complexes, extending the methodology of non-classicality witnesses to the investigation of non-trivial quantum effects of biological processes [86]. Another interesting application of Theorem 2 to the gravitational field was proposed in Ref. [93].

3.3 Temporal Witness of Non-Classicality in a Biological System

Over the past centuries, the high complexity of biological systems has discouraged scientists from describing them using physics and mathematics methodologies. Without experimental devices to probe the structure of living beings at a sufficiently fine scale, any attempt to employ sets of differential equations to make predictions about a given biological process resulted in a mathematically *intractable* problem

[94]. However, at the beginning of the twentieth century, the improved experimental resolution provided by new microscopes and techniques enabled scientists to explore progressively more microscopic details of biological systems. This development led to the increasing application of physics and mathematics to biology. From statistical mechanics to thermodynamics, different *classical* theories have been employed to describe biological systems and processes with remarkable predictive power.

In the same years, physics itself was profoundly shaken by the advent of quantum theory, which proved extremely successful in describing reality at the atomic and subatomic scale. It was therefore natural for the founding fathers of quantum theory to ask whether quantum mechanics might play a significant role in biology, or if biological systems should instead be regarded as the clearest manifestation of classical reality. This marked the dawn of a new discipline at the interface between quantum theory and biology: quantum biology, as Jordan named it in 1932 [95]. The origin of quantum biology is conventionally associated with the publication of the highly influential book "What is Life?", written by Schrödinger in 1944 [96]. The field has been characterised by multiple setbacks and subsequent recoveries over the years [97], often driven by the enthusiasm of the scientific community in the wake of increasingly significant technological developments in fields such as two-dimensional spectroscopy (2DES) [98].

One of the focuses of quantum biology is the study of *non-trivial* quantum effects in biological systems [99]. Clearly, all those quantum phenomena arising from the atomic and molecular structure of matter, for example electronic structures or intermolecular forces, are as relevant to biological systems as they are to any other artificial material. Therefore, being properties of reality at the microscopic level, these phenomena have already been incorporated in those fields of biology describing biological systems as ultimately classical, such as molecular biology. We shall thus call these "trivial" quantum effects in this context.

However, differently from artificial complex systems, biological ones are by definition *open* systems. "Warm, wet and noisy", the fluctuations of their environments are expected to quickly suppress any quantum dynamics via decoherence [100]. By

"non-trivial" quantum effects we refer to phenomena typically associated with the microscopic domain—such as quantum coherence, tunnelling and entanglement—that persist despite the fast decoherence induced by the open nature of biological systems and play an essential role in explaining or driving a biological process. In other words, these quantum processes must have been harnessed by biology over the course of evolution to improve the functionality of specific biological functions [94].

In recent years, several proposals for non-trivial quantum effects in biology have emerged, ranging from the radical pair model for magnetoreception in bird navigation to proton tunnelling in DNA. We refer the interested reader to Refs. [101, 102] for a review of these developments.

Here, we focus on a specific biological process for which the role of quantum coherence in explaining its remarkable efficiency has been debated for decades: light-harvesting in photosynthesis, the process powering up life.

3.3.1 Light-Harvesting in Photosynthesis

The process of photosynthesis begins when light-harvesting antennas in photosynthetic systems absorb sunlight to create an electronic excitation. This excitation is then transferred to a reaction centre, where charge separation initiates the chemical transformations of photosynthesis, known as *dark reactions* [103]. The transfer is made possible by the electronic couplings between the different molecules forming the antenna, and occurs at a faster rate than the fluorescence relaxation induced by the interaction with the environment. A variety of biological structures and pigment constituents are responsible for the process across different photosynthetic organisms, ranging from the symmetric ring-like light-harvesting complexes of purple bacteria to the less ordered chromophore arrangements found in green plants and cyanobacteria, reflecting evolutionary adaptation to diverse physiological requirements and natural environments [104, 105]. Despite these structural differences, the efficiency characterising this process, *electronic energy transfer* (EET), is remarkably high across all the light-harvesting complexes, with peaks of above 95% in the *Fenna-Matthews-Olson* (FMO) complex of green sulphur

bacteria and carotenoids in purple bacteria [106, 107]. This means that almost all the absorbed photons are capable of initiating the dark reactions in the reaction centre. Understanding how such high efficiency is maintained despite the "warm, wet and noisy" environment characterising life, remains a topic of active debate [102, 108].

Historically, EET has been described as a series of "jumps", within fully classical theories. One of the most important examples is the Förster resonance energy transfer (FRET) theory in the regime of weak inter-site electronic coupling compared to the system-bath coupling, where transport proceeds via incoherent hopping between localised excitations [109, 110]. Other examples are Redfield-type master equations in the opposite limit where system–bath coupling is weak and excitonic states are strongly delocalised, providing an effective description of relaxation between exciton eigenstates [111]. Between these two limits lies a broad intermediate regime, characteristic of many biological complexes, in which neither approximation— both based on second order perturbative methods—is strictly valid. This has motivated the development of Modified Redfield theories, each tailored to capture specific intermediate parameter regimes, spectral densities, or timescale separations relevant to different molecular environments [112]. All these approaches are intrinsically quantum, as they incorporate those "trivial" quantum effects needed to describe matter at the molecular level. However, they do not attribute a fundamental role to quantum processes in EET, which is instead described as arising from stochastic hopping between sites. For this reason, they focus only on the diagonal term of the density matrix describing the biological complex [113–115].

However, following the discovery of long-lived quantum coherence in the FMO complex and marine algae both at cryogenic and physiological temperatures [116–119], a strong case has been made for non-trivial quantum effects to play a significant role in EET. This has led to the proposal of several environment-assisted transport mechanisms, in which quantum coherence and an appropriate level of environmental noise enhance EET efficiency [120–124], as well as to the development of numerical methods to simulate the phenomenon including coherence such as hierarchical equations of motion [125]. These approaches provide different explanations of how

quantum coherence could drive the transport process and of how it can persist over surprisingly long times and distances within the protein and water environment surrounding the complexes [113, 126]. We refer the interested reader to Ref. [127] for a broader comparison between classical and quantum models for EET.

Despite the experimental evidence, the long tradition of classical descriptions of EET has led several researchers to attribute the "quantum beats" observed in 2DES spectra to vibrational modes of the electronic ground state of the protein system. Therefore, there remains a vast consensus that those classical theories such as FRET and Modified Redfield are enough to explain the remarkable efficiency of EET [128–133]. As a result, whether quantum coherence plays a non-trivial role in the transport phenomena preceding dark reactions is still controversial [134–137].

Settling this debate requires a shift in perspective.

3.3.2 Temporal Witness of Non-Classicality as a New Paradigm

From a theoretical perspective, the diversity of modelling approaches reflects a genuine physical challenge. Photosynthetic systems operate across a wide range of coupling strengths, reorganisation energies, and environmental timescales, often placing them outside the regime of validity of any single theoretical framework. As a result, identifying the "correct" dynamical model among competing classical and quantum descriptions requires introducing system-specific assumptions. These assumptions inevitably restrict the scope of the model and leave room for alternative approaches that, under different assumptions, may provide a better account of the experimental observations. In other words, this approach fails in revealing a conclusive inadequacy of classical theories in describing EET.

From the experimental point of view, the high complexity of biological systems creates an important challenge in accessing and measuring them, limiting the pool of photosynthetic complexes that can be investigated in detail. This means that the available experimental data are limited by the technological capability of measuring those systems, without the possibility of disproving competing theoretical descriptions with innovative protocols.

It is precisely at this juncture that temporal witnesses of non-classicality become relevant. The witness framework addresses a logically prior question: does EET in biological systems require a minimum of two non-commuting variables in order to be successfully performed? A positive answer to this question would automatically exclude *all* possible classical descriptions of the phenomenon obeying the assumptions of the witness. Indeed, according to our definition of non-classicality (see Chapter 1.2, Definition 1), a classical model requires the contribution of a *single* variable from the systems. The more general the assumptions underlying the witness, the broader the class of classical theories that would be ruled out by a successful non-classicality witness.

This line of reasoning is not entirely new to the field of quantum biology. For example, Refs. [138, 139] propose Leggett-Garg tests of quantumness in biological systems to assess the quantum nature of their temporal evolution in a model-independent way. While the logic is akin to the one of our non-classicality witnesses schemes, these tests require multiple time-separated measurements on the biological system under investigation. Even setting aside the "clumsiness loophole" at this stage (see Chapter 4 for a more detailed discussion of this problem), constructing a Leggett-Garg inequality requires direct experimental access and control of the highly complex biological system at multiple times. This requirement makes their implementation in the laboratory particularly challenging. Another example is given by Refs. [140, 141], where light is used to probe the quantum features of living green sulphur bacteria, therefore providing a realisation of a hybrid system in the context of biology. At the heart of the witness is the Rabi splitting observed when light is strongly coupled to the bacteria. One possible explanation of this phenomenon is indeed the entanglement generated between the electromagnetic field and the biological system. However, the conclusion stems from the construction of a *particular* quantum model to describe the experiment. Hence, it does not rule out possible completely classical analyses of the phenomenon. This is because the assumptions on which this argument relies are too stringent to provide a broad and conclusive non-classicality witness.

These limitations are overcome by the temporal witness of non-classicality in Theorem 2. By focusing on the ability of the investigated system to mediate a quantum-coherent evolution of a controllable probe under exact conservation laws, the temporal witness provides a model-independent criterion. It rules out all classical descriptions consistent with the conservation law and does not require measurements on the biological system itself, but only on the quantum probe used to test it. Therefore, a successful witnessing task implies that the underlying transport mechanism cannot be described in terms of a single classical variable. Instead, it must involve at least two incompatible degrees of freedom, thereby establishing the intrinsically non-classical nature of the mediator. Crucially, this conclusion holds independently of the specific dynamical details, even in the presence of the "warm, wet and noisy" protein environment.

We now explain how the light-harvesting stage can be mapped onto the single-probe setting of Theorem 2 [86].

A standard simplifying regime, well motivated in many light-harvesting scenarios [142], is the *single-excitation manifold*: the probability of creating more than one excitation in the relevant network is negligible during the time window of interest. Thus, the effective Hilbert space is spanned by "site" basis states $\{|n\rangle\}$ representing an excitation localised on site n. In this regime, "coherent transport" can be understood as the persistence of off-diagonal terms between site states (or, equivalently, as the ability of the excitation to occupy superpositions of distinct sites as it evolves), rather than as a purely classical hopping process [113].

The key ingredients of the temporal witness are: (1) a quantum probe Q whose evolution can be controlled and measured; (2) an unknown system M (the "mediator") that interacts with the probe; (3) a global additive conserved quantity that constrains the joint evolution; and (4) a witnessing task in which M induces a quantum-coherent evolution of Q from an eigenstate of the conserved observable to a state in which that observable is unsharp. In the photosynthetic setting considered here, the natural identification is the following. The controllable quantum probe Q is taken to be the photon associated with the initial absorption event (and,

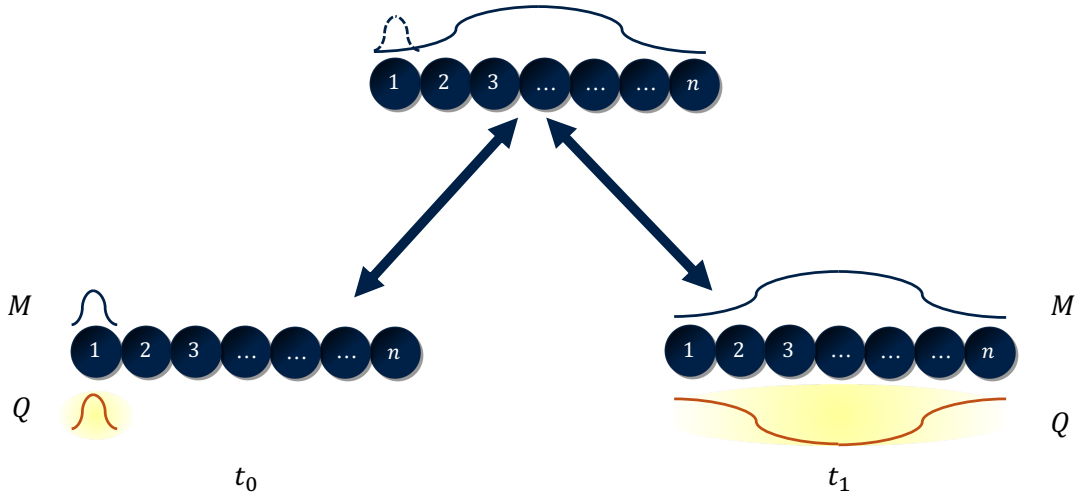


Figure 3.2: Schematic representation of the application of the temporal witness of non-classicality to EET. From the bottom left, moving clockwise: a photon Q excites a single site of the complex, where it is initially localised. Following the absorption, an exciton M is created on the same site. EET allows the exciton to propagate through the biological system. The exciton then recombines, leading to the emission of the photon Q. Observing a delocalisation of the photon's position with an interference experiment, under global energy conservation, would provide a witness of the non-classicality of the exciton's evolution.

crucially, with a final read-out after the transport process). The mediator M is the exciton supported by the pigment network, together with those molecular degrees of freedom that are inseparable from the excitonic dynamics at the relevant scales, most notably vibrational and environmental modes that can exchange energy with the electronic subsystem while respecting global conservation.

Operationally, the witness is implemented through a protocol made of three stages. Initially, the photon is absorbed by one of the sites to create an exciton. Then, the exciton propagates across the biological complex. Finally, the photonic degree of freedom is re-emitted (or otherwise mapped back onto an accessible optical mode), so that the probe can be measured interferometrically in the site basis (see Fig. 3.2). In this picture, the witnessing task is not the coherence of the exciton *per se*, but rather the ability of the excitonic dynamics to mediate a coherent transformation on the photonic probe that can be certified by interference. The conceptual advantage of formulating the task in this way is that the experimenter

does not need to exercise microscopic control over the biological complex. Instead, the protocol requires control and measurement primarily on the probe Q, while the complex acts as the unknown mediator.

The physical assumption underpinning the argument is the conservation of energy, phrased in the $\{|n\rangle\}$ basis since we are considering the single-excitation manifold. In the most general setting relevant to an open biological system, energy conservation must be imposed globally over the probe, the exciton, and the environment, rather than locally on individual subsystems. This is fully consistent with the witness framework: as remarked in Chapter 3.2.1, the conservation law is enforced at the level of the global observable (a q-number constraint in the quantum formalism). It is precisely this requirement of exact conservation that forces the mediator to compensate the coherent evolution of the probe, branch-by-branch.

With these identifications, the logical structure of the witness becomes transparent. Suppose that the photon Q is initially directed onto a single site of the biological system, i.e., it is prepared in an energy eigenstate (sharp energy). If, after undergoing the protocol described above, it is found in a coherent superposition of distinct energy-localised states, then the exciton M must have provided the incompatible variables required to mediate such a coherent transformation under exact global energy conservation. In other words, if the biological complex can induce such a coherent evolution of the probe while respecting energy conservation, then it cannot be described by a single classical variable: it must be non-classical in the precise information-theoretical sense introduced in Chapter 1 (Definition 1).

To make the witnessing task explicit, consider the effective "site" description of the light-harvesting network in the single-excitation regime. The successful execution of the protocol corresponds, in essence, to the probe being mapped onto a coherent superposition of alternatives associated with different sites. A minimal

way to express the target probe state after recombination is therefore [86]:

$$\begin{aligned}\rho_p =& p_{0,0} \left|00...00\right\rangle \left\langle 00...00\right| + p_{1,0} \left|10...00\right\rangle \left\langle 00...00\right| + p_{0,1} \left|00...00\right\rangle \left\langle 10...00\right| + ... \\ &+ p_{N-1,N} \left|00...10\right\rangle \left\langle 00...01\right| + p_{N,N-1} \left|00...01\right\rangle \left\langle 00...10\right| + p_{N,N} \left|00...01\right\rangle \left\langle 00...01\right| \\ =& \sum_{n,m=1}^{N} c_n c_m^* \left|1_n\right\rangle \left\langle 1_m\right| \end{aligned} \tag{3.6}$$

where $|1_n\rangle$ denotes the probe localised in the mode associated with site n and the presence of stable relative phases between the amplitudes c_n is what enables an interference-based verification of coherence. The crucial point is not the detailed structure of the coefficients (which is system-dependent), but that the transformation from an initially sharp state of Q to a state of the above form constitutes a coherent evolution of the probe that must be mediated by M under the conservation constraint.

The next step is therefore to demonstrate, within a general and controllable theoretical framework, that the witnessing task can indeed be realised under realistic noise mechanisms compatible with warm biological environments.

3.3.3 A Collision Model to Illustrate the Possibility of the Witnessing Task

We now propose a collision model to illustrate the possibility of the witnessing task under the aforementioned conditions, thereby providing a scheme to experimentally test non-classicality in a macroscopic, complex, and open system by applying Theorem 2 [86, 87].

The model is not intended to describe exciton transfer in a real biopolymer, for which further refinements depending on the specific biological system under investigation would be required. Rather, it should be intended as an abstract scenario designed to illustrate how the temporal witness of non-classicality would operate in complex systems interacting with a noisy environment. As such, it shall represent the groundwork upon which a realistic experiment can be constructed by incorporating system-specific details. To achieve this goal, the purpose of the model is to study the non-Markovian information dynamics of a chain of qubits interacting with an

environment, focusing specifically on the coherence of information transport along a chain of interacting qubits [87]. This extends the relevance of the model beyond biological-related applications, placing it within the broader context of coherent transport in open quantum systems described via collision models [143–149].

In what follows, we consider an isotropic, one-dimensional chain C of N qubits. We shall refer to these qubits as "system qubits" $S_n,\ n = 1, ..., N$. In the Heisenberg picture, the system qubit S_n is described by $\{\hat{X}_n, \hat{Y}_n, \hat{Z}_n\}$, where $\hat{X}_n = \mathbb{I}^{n-1} \otimes \hat{\sigma}_x \otimes \mathbb{I}^{N-n}$, $\hat{Y}_n = \mathbb{I}^{n-1} \otimes \hat{\sigma}_y \otimes \mathbb{I}^{N-n}$, $\hat{Z}_n = \mathbb{I}^{n-1} \otimes \hat{\sigma}_z \otimes \mathbb{I}^{N-n}$ and $\hat{\sigma}_i,\ i \in \{x, y, z\}$ are the Pauli operators describing its spin components.

The chain is initially prepared in the state $\rho_C = |00...0\rangle \langle 00...0|$ as no excitation has been created yet. A photon Q, described by the quantum observables $\{\hat{X}^p, \hat{Y}^p, \hat{Z}^p\}$, interacts with the first system qubit S_1 and creates a *single* excitation M initially *localised* on it. We assume here that the probability of creating a second excitation in the chain is negligible, consistently with [150]. Therefore, the initial state of the chain is:

$$\rho_C(0) = |10...0\rangle \langle 10...0| . \tag{3.7}$$

The proposed collision model [87] describes the excitation dynamics using a *Heisenberg Hamiltonian*:

$$\hat{H}_{chain} = \frac{1}{2} \sum_{n=1}^{N-1} J^n_{chain} \left(\hat{X}_n \hat{X}_{n+1} + \hat{Y}_n \hat{Y}_{n+1} + \hat{Z}_n \hat{Z}_{n+1} \right), \tag{3.8}$$

with $J^n_{chain} \in \mathbb{R}\ \forall n = 1, ..., N$. By changing J^n_{chain}, this model can capture different physical systems characterised by different interaction strengths between their constituents. Moreover, the model can be easily generalised to explore other phenomena, such as anisotropies in the chain, next-nearest neighbours coupling, multiple dimensions, and effects of external fields or systems [146]. Furthermore, this Hamiltonian satisfies the conservation law required for the application of the temporal witness of non-classicality, as $\left[\hat{H}_{chain}, \sum_{n=1}^{N} \hat{Z}_n\right] = 0$.

We model the environment as a reservoir of N interacting "reservoir qubits" Q_l, $l = 1, ..., N$. Their interaction is described by a Heisenberg Hamiltonian:

$$\hat{H}_{res} = \frac{1}{2} \sum_{l=1}^{N-1} J_{res}^{l} \left(\hat{\mathcal{X}}_l \hat{\mathcal{X}}_{l+1} + \hat{\mathcal{Y}}_l \hat{\mathcal{Y}}_{l+1} + \hat{\mathcal{Z}}_l \hat{\mathcal{Z}}_{l+1} \right), \tag{3.9}$$

with coupling constant $J_{res}^{l} \in \mathbb{R}$ $\forall l = 1, ..., N$ and $\hat{\mathcal{X}}_l$, $\hat{\mathcal{Y}}_l$, $\hat{\mathcal{Z}}_l$ the Pauli operators of the l-th reservoir qubit, defined as above. Since we are interested in how the information dynamics of the system is affected by a loss of information, we initialise the reservoir qubits in the maximally mixed state:

$$\xi = \bigotimes_{l=1}^{N} \xi_l = \left[\frac{\mathbb{I}}{2} \right]^{\otimes N}, \tag{3.10}$$

which represents a state containing no information.

We describe here the system-environment interactions as collisions of each system qubit with its *nearest reservoir qubit* via the *partial swap* [151]:

$$\text{PSWAP}_m(\eta) = \cos\eta \; \mathbb{I} + i \sin\eta \; \text{SWAP}_m. \tag{3.11}$$

The SWAP operator acts on the m-th system and reservoir qubits, and $\eta \in [0, \pi/2]$ is the *interaction* or *decoherence* parameter. Since PSWAP can be written as

$$\text{PSWAP}_m(\eta) = e^{i\eta/2} e^{i\eta/2(\hat{X}_m \hat{\mathcal{X}}_m + \hat{Y}_m \hat{\mathcal{Y}}_m + \hat{Z}_m \hat{\mathcal{Z}}_m)}, \tag{3.12}$$

all the interactions in the model up to now are fundamentally of the same type. However, we prefer to maintain a continuous description in time for the chain and reservoir interactions to avoid introducing temporal order into the pairwise interactions between their qubits.

Since all the interactions considered so far are unitary and the same reservoir qubits keep interacting with the chain, the model is intrinsically non-Markovian. Unlike other non-Markovian collision models where the reservoir qubits that have interacted with the system are discarded at some point [152–156], we propose to control the *degree* of non-Markovianity of the model by applying a *quantum depolarising channel* to the *fixed* and *rigid* reservoir qubits after each collision. If ρ_{CR}

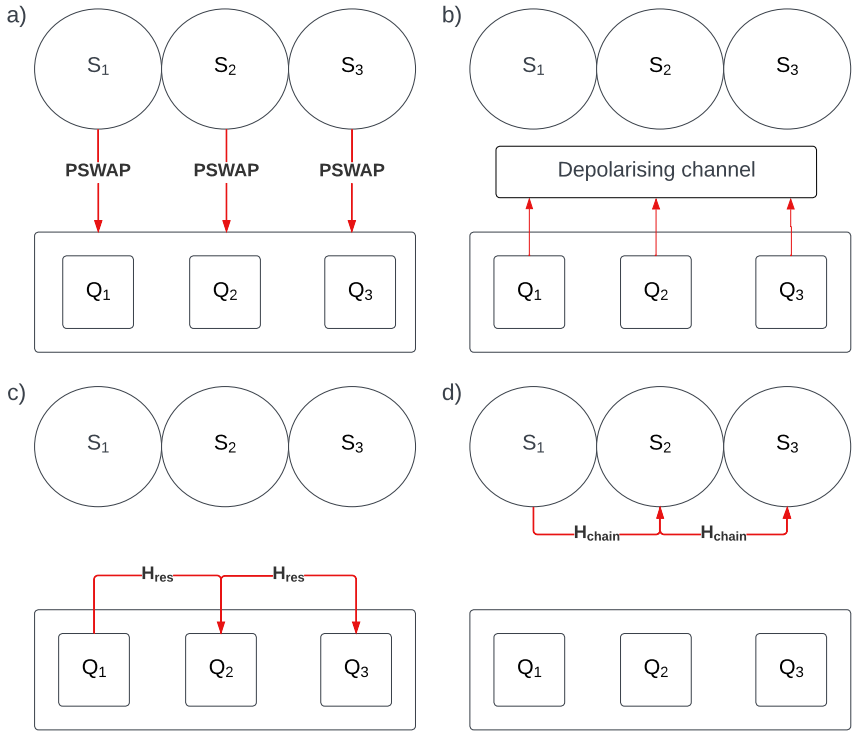


Figure 3.3: Four steps of the protocol with three chain qubits and three reservoir qubits. a) exchange step: each system qubit interacts with its nearest reservoir qubit via a partial swap; b) depolarising step: the depolarising channel is applied to the reservoir qubits; c) transfer via reservoir: unitary evolution of the reservoir qubits with $\hat{H}_{res}$; d) transfer via chain: unitary evolution of the system qubits with $\hat{H}_{chain}$.

is the state of the composite system made by chain and reservoir, the depolarising channel for a single reservoir qubit can be expressed as:

$$\Delta_{\Omega}^{m}(\rho_{CR}) = \sum_{i=0}^{3} \hat{K}_i^m \rho_{CR} \hat{K}_i^{m\dagger}, \tag{3.13}$$

where $\hat{K}_i^m$ are the Kraus operators $\hat{K}_0^m := \sqrt{1-3\Omega/4}\,\mathbb{I}_m$, $\hat{K}_1^m := \sqrt{\Omega/4}\,\hat{\mathcal{X}}_m$, $\hat{K}_2^m := \sqrt{\Omega/4}\,\hat{\mathcal{Y}}_m$, $\hat{K}_3^m := \sqrt{\Omega/4}\,\hat{\mathcal{Z}}_m$, and $0 \leq \Omega \leq 1$. Since the $\hat{K}_i^m$ for different m commute, the order in which they are applied is irrelevant. The study of the depolarising channel as a tool to control the environment's degree of non-Markovianity and how this relates to the information loss in the chain are presented in Appendix D.1. In particular, we show how Ω acts as a parameter controlling the degree of non-Markovianity of the reservoir when tuned from $\Omega = 0$ (non-Markovian environment) to $\Omega = 1$ (Markovian environment) [87].

The protocol consists of four steps (Fig. 3.3).

1. The system qubits interact with the nearest reservoir qubits via a partial swap (Eq. 3.11) with probability η (Fig. 3.3a). We will call this step the "exchange step".

2. The depolarising channel is applied to the reservoir qubits (Fig. 3.3b). This step controls the degree of non-Markovianity of the environment. We will call this step the "depolarising step".

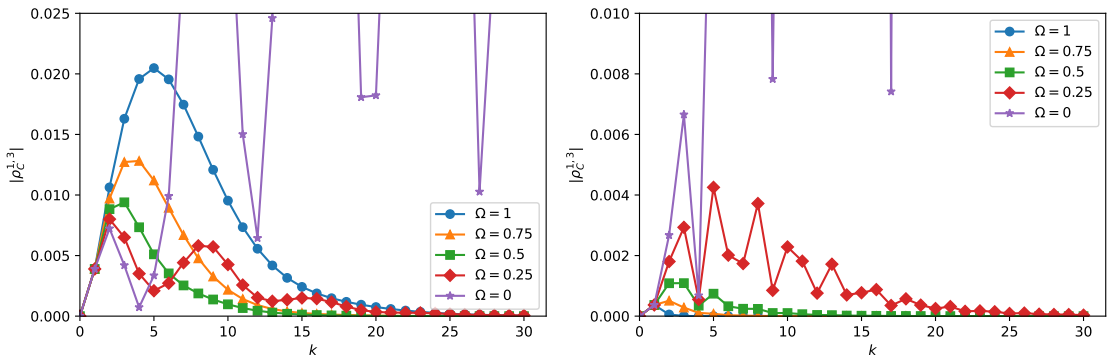
3. The reservoir qubits evolve according to the Hamiltonian of Eq. 3.9 for a time interval Δt (Fig. 3.3c). We will call this step the "transfer via reservoir".

4. The system qubits evolve according to the Hamiltonian of Eq. 3.8 for a time interval Δt (Fig. 3.3d). We will call this step the "transfer via chain".

All the steps of the protocol can be repeated indefinitely.

We apply the model described above to study the coherence of the transport of information along the chain, initialised in the state in Eq. 3.7: $\rho_C(0) = |100...0\rangle\langle 100...0|$. Applying the protocol described above, the excitation spreads through the chain and the environment. After repeating the protocol k times, we trace out the environment and extract the coherence between the first and last qubit of the chain $\rho_C^{1,N}(k)$, that is, the coefficient of the term $|100\ldots0\rangle\langle 00\ldots01|$ in the density matrix. We take $|\rho_C^{1,N}(k)|$ as an indicator of the coherence of the transport across the chain. We choose to study this parameter as the goal of this model is to show that the excitation transport along the chain occurs *coherently*, a feature that other entries of ρ_C such as the population $\rho_C^{N,N}$ would not capture.

For simplicity, we consider a model with three chain qubits. We implemented the protocol as a quantum circuit in Qiskit [157] as follows. The exchange phase consists of a sequence of local unitaries PSWAP as in Eq. 3.11 for each of the system qubits S_i; the depolarising step is implemented as a depolarising noise model on the identity gate, which applies the quantum channel in Eq. 3.13; the transfer phases are implemented as unitaries on the chain and the reservoir with a time



(a) $\eta = 0.4$ **(b)** $\eta = 1.2$

Figure 3.4: Coherence element $|\rho_C^{1,3}(k)|$ over multiple iterations of the protocol for different values of Ω. The case $\Omega = 0$ represents the *unrealistic* scenario of a fully unitary evolution. In this case, the coherence oscillates wildly, so the vertical axis has been truncated to improve the visibility of the graphs.

step $\Delta t = 0.01$. We choose $J^n_{chain} = J_{chain} = 10$, $\forall n$ and $J^m_{res} = J_{res} = 1$, $\forall m$. This choice represents a scenario where the chain is the most efficient medium for the propagation of the excitation.

We first apply the protocol 30 times and study how $|\rho_C^{1,3}(k)|$ varies for different values of Ω when $\eta = 0.4$ and $\eta = 1.2$. We select these values of η as representative of the chain-environment "weak" and "strong" coupling regimes, respectively. This is because η represents the probability of swapping the chain and reservoir qubits during the interaction described by Eq. 3.11: when $\eta < \frac{\pi}{4} \approx 0.8$, the probability is lower than 50%; when $\eta > \frac{\pi}{4} \approx 0.8$ the probability is higher than 50%. After studying the temporal behaviour of $|\rho_C^{1,3}(k)|$ at fixed η, we investigate the effect of η on the coherence of the transport. We do so by examining the maximum coherence $\max_k |\rho_C^{1,3}(k)|$ as a function of η for different values of Ω. We then restrict again to the cases $\eta = 0.4$ and $\eta = 1.2$ to study how the ratio J_{res}/J_{chain} affects $\max_k |\rho_C^{1,3}(k)|$.

In Fig. 3.4 we plot $|\rho_C^{1,3}(k)|$ as a function of the number of iterations of the protocol k for different values of Ω. We consider two cases: (a) $\eta = 0.4$ (weak coupling regime) in Fig. 3.4a, and (b) $\eta = 1.2$ (strong coupling regime) in Fig. 3.4b.

Before analysing the two cases separately, we notice that in both scenarios $|\rho_C^{1,3}(k)|$ is not affected by Ω for $k = 1$, i.e., when considering a protocol with a single iteration. This is because each protocol iteration contains a single interaction

between the chain and the environment (described by PSWAP in Eq. 3.11), so there is no further exchange of information after the application of the depolarising channel. As a result, the degree of non-Markovianity of the environment is irrelevant to the coherent transport along the chain.

In both Fig. 3.4a and Fig. 3.4b, the case $\Omega = 0$ corresponds to a fully unitary evolution of the global system chain plus environment. In this case, no information is lost, and no steady state can be reached. Such a scenario is unrealistic in most ordinary physical systems, where some information is irreparably lost over time. For this reason, we will consider only $\Omega \geq 0.25$ in the following discussion. In the cases examined in Figs. 3.4a-3.4b we see that $|\rho_C^{1,3}(k)|$ is quickly damped and reaches zero within 30 iterations of the protocol. We expect this to happen for any $\Omega \neq 0$ due to the depolarising channel losing information at each protocol iteration, although the damping occurs more slowly for values of Ω close to zero.

Focusing now on Fig. 3.4a, we observe that, in the weak coupling regime $\eta = 0.4$, higher values of Ω result in greater coherence. This is noteworthy, as one may expect more information loss to be associated with decreasing amounts of coherence within the system. Instead, we observe that, for $0.25 \leq \Omega \leq 1$, a more Markovian environment better assists coherent transport. A detailed account of information loss in this model is provided in Appendix D.1 [87]. The situation is inverted in the strong coupling regime. In Fig. 3.4b, we observe that the coherence is immediately suppressed for higher values of Ω while it is higher and more persistent for highly *non-Markovian* environments.

So far, we have considered how $|\rho_C^{1,3}(k)|$ is affected by Ω for *fixed* values of the interaction probability η, one in the weak coupling regime and the other in the strong coupling regime. To check how these results depend on the value of η, in Fig. 3.5 we plot the maximum of the coherence $\max_k |\rho_C^{1,3}(k)|$ computed over 30 protocol iterations as a function of η for different values of $\Omega \geq 0.25$. The results are consistent with what was shown above: coherence slightly increases with the *Markovianity* of the environment in the weak coupling regime, while a *non-Markovian* environment is preferred in the strong coupling regime. The transition between these two regimes

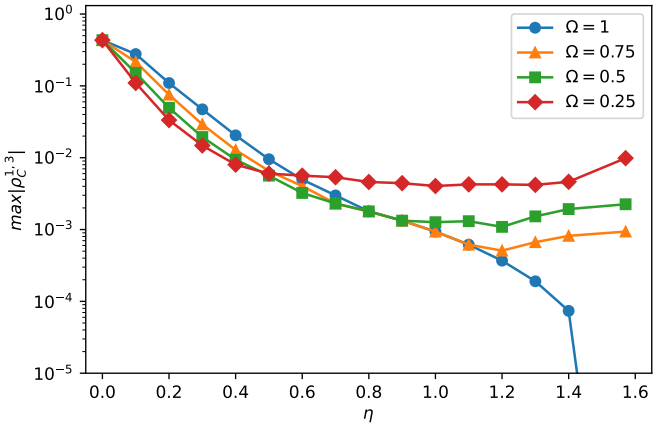
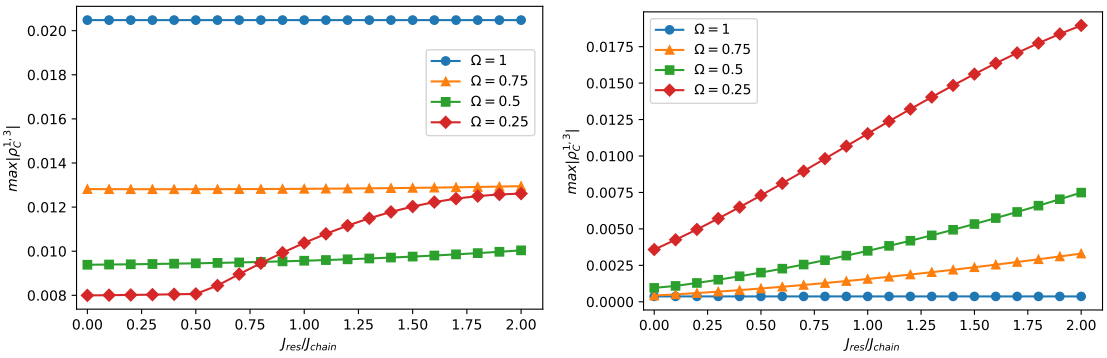


Figure 3.5: Maximum coherence $\max_k |\rho_C^{1,3}(k)|$ over $k = 30$ protocol iterations as a function of the system-environment coupling η, for different values of Ω when $J_{chain} = 10$, $J_{res} = 1$. The case $\Omega = 1$ tends to zero when $\eta \to \frac{\pi}{2}$, so the y-axis has been truncated to improve the visibility of the results.

happens for intermediate values of the coupling probability, which depend on Ω. For $\eta = 0$, since there is no coupling between the chain and the environment, $\max_k |\rho_C^{1,3}(k)|$ does not depend on Ω. For η close to $\frac{\pi}{2}$, i.e., when the interaction probability is close to 1, the coherence goes to zero for $\Omega = 1$. In this case, the coherence is immediately suppressed during the first iteration of the protocol because the excitation is entirely transferred to the reservoir and erased in the first two steps of the protocol (exchange and depolarising). For $\Omega \neq 1$, $\max_k|\rho_C^{1,3}|$ is always non-zero and, after reaching a minimum for some intermediate values of η, it increases as η tends to $\frac{\pi}{2}$, with higher values of coherence associated with lower values of Ω.

The results presented so far are based on simulations with $J_{chain} = 10$ and $J_{res} = 1$. As we mentioned above, this corresponds to a scenario in which the chain is the most efficient medium for the propagation of the excitation. While this situation is the most physically realistic, in Fig. 3.6a we explore how the results depend on the ratio J_{res}/J_{chain}. Specifically, we plot $\max_k |\rho_C^{1,3}(k)|$ as a function of J_{res}/J_{chain} in the weak ($\eta = 0.4$) and strong ($\eta = 1.2$) coupling regimes for $J_{chain} = 10$ and different values of $\Omega \geq 0.25$. As expected, J_{res}/J_{chain} does not affect $\max_k |\rho_C^{1,3}(k)|$ when $\Omega = 1$ since no transfer of information along the environment is possible when all the information stored in the environment is completely lost at each protocol iteration.



(a) $\eta = 0.4$ **(b)** $\eta = 1.2$

Figure 3.6: Maximum coherence $\max_k |\rho_C^{1,3}(k)|$ over $k = 30$ protocol iterations as a function of the ratio between the reservoir qubits and the chain qubits couplings J_{res}/J_{chain}, for $J_{chain} = 10$ and for different values of the degree of non-Markovianity Ω.

Excluding the case $\Omega = 1$, in Fig. 3.6a we observe a low sensitivity of $\max_k |\rho_C^{1,3}(k)|$ to changes in J_{res}/J_{chain}, with higher values of Ω initially associated with higher levels of coherence, consistently with Figs. 3.4-3.5. The low sensitivity to J_{res}/J_{chain} indicates that the chain is mainly responsible for the coherent transport of the excitation. For $\Omega = 0.25$, we observe a clear-cut change in behaviour for $J_{res}/J_{chain} = 0.5$. Beyond this point, $\max_k |\rho_C^{1,3}(k)|$ increases more rapidly with increasing values of J_{res}/J_{chain}, suggesting a more prominent role of the intra-environment interactions in the coherent transport.

The situation is different in Fig. 3.6b, where we observe that, apart from the case $\Omega = 1$ discussed above, $\max_k |\rho_C^{1,3}(k)|$ increases consistently with J_{res}/J_{chain}, indicating that the environment plays an active role in the coherent transport of the excitation. This happens because the environment is more likely to interact with the chain in the strong coupling regime and is thus more likely to assist in the propagation of the excitation. Accordingly, higher degrees of non-Markovianity result in higher values of coherence since the information in the environment is less likely to be lost.

To complete the study, in Appendix D.2 we analyse the effects of information dynamics on the populations $|\rho_C^{1,1}|$ and $|\rho_C^{3,3}|$, namely the coefficients of the terms $|100\rangle\langle 100|$ and $|001\rangle\langle 001|$.

In conclusion, when the chain is the most efficient medium for the propagation

of the excitation, i.e., $J_{res}/J_{chain} \ll 1$, and when $0.25 \leq \Omega \leq 1$, we have seen that non-Markovianity improves the coherence of the transport for strong system-environment couplings but hinders it if the coupling is weak.

3.3.4 Closing remarks

Considering these results from the perspective of the application of the temporal witness of non-classicality in Theorem 2 to the biological scenario is particularly instructive. Consider for example the Fenna-Matthew-Olson (FMO) complex, one of the most well studied complexes in quantum biology [158, 159]. The FMO is a protein complex that connects the light-harvesting antennas (also known as *chromophores*) to the reaction centre of green sulphur bacteria, and is therefore responsible for their EET. It consists of a trimer, each unit of which is composed of eight bacteriochlorophyll a (BChl-a) molecules embedded in a protein scaffolding representing its biological environment. In the FMO complex, each BChl-a molecule can sustain the single exciton M created after the absorption of a single photon Q from solar light [100], and can therefore be modelled as a system qubit in the chain description introduced above. The reservoir qubits would correspond instead to the phonons associated with the vibrations of the ground state of the protein environment in which each BChl-a is embedded. Within this picture, a Markovian reservoir describes a biological environment that does *not* actively participate in the coherent exciton transport on the biopolymer, but acts only as a source of noise and decoherence. By contrast, a non-Markovian reservoir corresponds to a biological environment that actively contributes to the transport process on the biopolymer.

The maximum coherence $\max_k|\rho_C^{1,3}(k)|$ in Fig. 3.5 shows that a finite amount of coherence is retained in the long-time limit of the protocol, except for the $\Omega = 1$ scenario when η is close to $\frac{\pi}{2}$ as discussed in Chapter 3.3.3. This suggests that the witnessing task required by Theorem 2 would *successfully* be achieved, even in the presence of decoherence. Furthermore, the possibility to tune the non-Markovianity of the environment through the parameter $\Omega \in [0, 1]$ allows for a more refined analysis of the role of the protein environment in the transport process.

Specifically, non-Markovianity enhances the coherence of the transport only when the system and the environment are strongly interacting, see Fig. 3.4b and Fig. 3.5. This is because, in the weak coupling regime, the biological system does not exchange sufficient information about the delocalised exciton with its surrounding environment for the latter to act as a quantum memory on which the system can rely to restore its coherence. As a result, the vibrations of the protein scaffolding are not strong enough to maintain the coherence of the exciton's position across the chain. Conversely, in a strong coupling regime, a significant amount of the information about the exciton's position will be exchanged with the protein environment, exciting its vibrational degrees of freedom. These vibrations will consequently be retrieved at the subsequent interaction, thereby sustaining the coherence of the exciton. In other words, the closer the evolution is to unitary, the more the exchange between the electronic and vibrational degrees of freedom of the biological system in its protein surrounding will contribute to maintain the coherence in the exciton delocalisation across the chromophores in the light-harvesting complex.

Surprisingly, non-Markovianity hinders the coherence of the transport if the coupling between the system and the environment is weak, as shown by Fig. 3.4a and Fig. 3.5. This suggests that a light-harvesting complex may prefer a more Markovian protein pigment to enhance the exciton coherent delocalisation if the amount of information transmitted is limited, a phenomenon worth exploring to study the beneficial impact on exciton-mediated transport phenomena [128, 130–133].

Note that these two mechanisms may not be mutually exclusive. For example, in the FMO photosynthetic complex [159], the electronic degrees of freedom of the BChl-a molecules (the chain in our model) can couple both to the phonons of the molecular environment (non-Markovian) and to an external thermal bath (Markovian environment). Our results may therefore capture both of these situations, suggesting that the coexistence of a weakly coupled thermal bath with a strongly coupled non-Markovian environment might determine an optimal condition for coherent information transport.

In conclusion, using the temporal witness of non-classicality one could demonstrate that the exciton mediating the photon coherent delocalisation is non-classical. Consequently, the process governing its transport along the biopolymer must itself be non-classical, ruling out all classical frameworks compatible with the assumptions in Table 3.1. This conclusion remains valid even when the protein environment is included as part of the system M mediating the coherent delocalisation of the photon: the quantum features characterising the process will be proven to persist even at the level of the full complex, macroscopic system. The generality of our witnessing scheme provides a theoretical backbone for the design of a real experiment implementing the temporal non-classicality witness in a biological system, to pin down conclusively the role of quantum effects in exciton transfer and other energy transfer processes. For example, the structure of the environment presented in this work is similar to the one considered in Ref. [160], where molecular dynamics simulations are employed to study the coupling between the protein environment and the vertical excitation energies in a real biological system. The scenario also closely corresponds to experiments on exciton transfer in polydiacetylene, see e.g., Ref. [150].

3.4 Summary

The temporal witness of non-classicality in Theorem 2 represents the first example of an information-theoretical, single-probe argument to indirectly infer the non-classical properties of an unknown system M, defined as the existence and dynamical involvement of a minimum of two non-compatible variables. Motivated by the experimental challenges associated with two-probe witnesses such as the GWT in Theorem 1, we instead consider here a single-probe scenario.

The argument exploits the ability of a system M to mediate a quantum-coherent evolution of a probe Q under the exact conservation law of a global observable. In particular, the evolution of the probe from an eigenstate of a conserved observable to a superposition of its eigenstates provides a sufficient condition to certify the mediator's non-classicality.

We have proposed an application of the temporal witness of non-classicality to the phenomenon of excitation transport in photosynthetic systems, with the aim of conclusively identifying the quantum properties of the process even in a complex environment such as the protein scaffold arranging the light-harvesting complex. This would represent the first application of the temporal witness of non-classicality in Theorem 2 to the field of quantum biology, providing a new methodology to study non-trivial quantum effects in biological systems.

By relying on minimal assumptions, namely the conservation of a global additive quantity and the formalism of quantum theory (see Table 3.1), Theorem 2 provides a robust theoretical argument to demonstrate the non-classical nature of M. In doing so, it rules out all possible classical descriptions of the mediator that are compatible with the assumptions of the witness. In fact, as shown by the proof of the theorem as well as by the toy models in Appendix C, the possibility of the witnessing task implies that M must engage more than one non-commuting variable in mediating the quantum coherent evolution of Q. Any semiclassical description of the hybrid $Q \oplus M$ would involve only a single variable for M, according to the definition of classicality adopted here, and would therefore be insufficient to explain the coherent rotation. An example of a class of models that would be ruled out by the temporal witness of non-classicality are the hybrid classical-quantum dynamics that satisfy the conservation laws as a consistency condition, which we discussed in Chapter 1.

From a theoretical perspective, this behaviour is captured by the so-called quantum totalitarian property, which states that, under exact global conservation laws, quantum features of a probe Q must spread to the classical sector of a hybrid system. The temporal witness of non-classicality is precisely built upon such conservation laws. In the witnessing protocol, the probe is rotated into a basis that does not commute with the locally conserved observable. This creates the conditions under which the totalitarian property becomes operational and allows M to mediate the coherent transformation. In this sense, conservation of an additive quantity plays a role analogous to the principle of locality in the GWT: it makes M reveal its non-compatible observables when interacting with a quantum sector.

The last consideration connecting conservation laws to the principle of locality suggests that Theorem 2 should not be viewed merely as an alternative route to providing a simpler experimental implementation of two-probe witnesses of non-classicality, but as a complement to the spatial arguments in the *temporal domain*. However, as currently formulated, the temporal witness of non-classicality has the drawback of relying on the quantum formalism. This is manifested by the formulation of the conservation law: it is defined by the commutator of the corresponding quantum observables, a distinctive element belonging to the quantum formalism. To achieve a theoretically robust temporal extension to the GWT, we must relax the assumption of the quantum dynamics, and formulate the temporal witness of non-classicality in a dynamics-independent framework such as the one provided by constructor theory, specifically constructor theory of thermodynamics [161] given the required conservation law. This would provide a General Witness Theorem *in Time*, capable of witnessing the non-classicality of M even beyond quantum theory with a single quantum probe. The first step towards this generalisation is to reconsider the witnessing task. In the next Chapter, we will reformulate the temporal witness of non-classicality in terms of temporal quantum correlations, exploiting the symmetry between temporal and spatial correlations in quantum theory, and introducing a *temporal entanglement-based witness of non-classicality*.

4

Temporal Entanglement-Based Witness of Non-Classicality

The content of this Chapter is based on [162], done in collaboration with Gaurav Bhole, James Eaton, Andrew Baldwin, Jonathan Jones, Vlatko Vedral and Chiara Marletto. The theoretical argument was entirely developed by myself, starting from an idea by Chiara Marletto and Vlatko Vedral. The experiment to confirm the theoretical prediction was performed by Gaurav Bhole, James Eaton, Jonathan Jones and Andrew Baldwin.

In the previous Chapter we examined the temporal witness of non-classicality, in which the quantum coherent evolution of the probe system Q, under a global conservation law, served as the witnessing task for the non-classicality of the unknown mediator M (see Theorem 2). We restate the Theorem here for the reader's convenience [85]:

Theorem (Temporal Witness of Non-Classicality)**.** *If a system M can induce a quantum coherent evolution of the quantum probe Q, then M must be non-classical, provided that a global additive observable of the system $Q \oplus M$ is conserved.*

While this approach overcomes the key experimental challenge of enforcing spatial locality, it leaves several conceptual questions open and lacks the theoretical robustness of the General Witness Theorem (GWT) backing two-probe spatial entanglement-based witnesses discussed in Chapter 2, restated here for convenience [42]:

Theorem (General Witness Theorem)**.** *If M can entangle Q_A and Q_B, by locally interacting with each of them, then M is non-classical.*

A direct comparison between the assumptions underlying the two theorems is presented in Table 3.1.

This raises a natural question: can one construct a temporally formulated, single-probe witness that matches the generality of the GWT, while retaining the practical advantages of the temporal setting?

In this Chapter, we take a first step in this direction by developing a novel formulation of the temporal witness of non-classicality, based on M's capability of mediating *temporal entanglement* during the evolution of Q, relative to the probe's local conserved quantity and a non-commuting observable. This new *temporal entanglement-based witness of non-classicality* qualifies as the extension of the General Witness Theorem within the quantum formalism to the temporal domain, and identifies the mediation of both spatial and temporal quantum correlations as witnessing tasks. Considering temporal quantum correlations as part of the witnessing task strengthens the single-probe argument within quantum theory, making it more robust in its conclusions about the non-classicality of M. In particular, it allows one to characterise the non-compatible variables as *observables* of the unknown system.

We shall begin by reviewing the role temporal quantum correlations have played so far in quantum theory, in particular in the study of macroscopic systems, by considering the Leggett-Garg inequalities and related tests of quantumness. The important clumsiness loophole behind these tests will motivate the connection between Leggett-Garg inequalities and temporal Bell inequalities, leading to the

notion of entanglement in time. We shall then provide a formal proof of the temporal entanglement-based witness of non-classicality within the quantum formalism and support it with a proof-of-principle experiment using a three-qubit NMR quantum computer. Finally, we shall discuss the importance of having both a spatial and a temporal entanglement-based witness of non-classicality to understand the symmetrical role of spatial and temporal quantum correlations in quantum theory, and to shed light on the meaning of locality in time thanks to the connection between the assumptions of the two arguments.

4.1 Leggett-Garg Inequalities and Temporal Entanglement

One of the distinguishing features of quantum theory is quantum coherence, the ability of a quantum system to maintain a well-defined phase difference between different states in a superposition. Quantum coherence was the core of the temporal witness of non-classicality introduced in the previous Chapter, and what Leggett and Garg sought to test in macroscopic systems in their seminal 1985 work [163]. In this Section, we introduce the notion of a *macroscopic system* they formulated, its connections with classical mechanics, and the tool they derived to test whether a system behaves according to this definition: the Leggett-Garg Inequalities (LGIs) [164]. A violation of the LGIs during the temporal evolution of a physical system is a signature of the discrepancy from such a definition. Furthermore, we analyse the connection between LGIs and Bell's inequalities, clarifying why they are often referred to as *temporal* Bell's inequalities. We deepen this connection in the two-time correlation scenario and derive the idea of entanglement *in time* [165], which will play a central role in the formulation of the *temporal* entanglement-based witness of non-classicality.

4.1.1 The Leggett-Garg Inequalities

The first step in deriving a tool to distinguish between *microscopic* systems, obeying the laws of quantum theory, and *macroscopic*, classical systems is to codify the

intuitive notion of classical behaviour in a set of assumptions. Leggett and Garg identified three principles for this purpose [163, 166]:

1. **Macroscopic Realism *per se* (MRps)**: A macroscopic system with two or more macroscopically distinct states available to it will at all times *be* in one or the other of these states.

2. **Noninvasive measurability at the macroscopic level (NIM)**: It is possible, in principle, to determine the state of the system with arbitrarily small perturbation on its subsequent dynamics.

3. **Induction**: The outcome of a measurement on the system cannot be affected by what will or will not be measured on it later.

Taken together, these three assumptions capture the intuition that macroscopic systems behave *classically* [167]: a macroscopic system *is always* in one and only one of the macroscopic states available, and the measurement process simply reveals the pre-existing and well-defined property of the system in that state without affecting neither its future nor its past. Under these conditions, a Schrödinger's cat must be either dead or alive, never both, and this fact cannot be modified by measurements done at earlier or later times [164].

However, what counts as a "macroscopically distinct" state is itself ambiguous and theory-dependent. Several proposals have aimed to quantify quantum macroscopicity—some based on the distinguishability of superposition branches and coherence in a particular basis, others on their usefulness in interferometry, or on metrics like the quantum Fisher information and decoherence sensitivity (see Ref. [168] for a comprehensive review). To address this ambiguity, Maroney and Timpson propose reframing the notion of macroscopic distinctness not in terms of states, but in terms of *macroscopic observables* [169]. Macroscopic observables are postulated to possess definite values at all times, with values determined by the system's underlying *complete state*. By complete state, we mean here the state describing all the properties of the system both measurable and non-measurable [54,

55]. Within this framework, two states are macroscopically distinct if, and only if, they assign different values to at least one such observable. This shift clarifies the formal content of MRps: it implies that measurements on macroscopic systems reveal pre-existing, well-defined *values* belonging to *macroscopically observable properties*. Furthermore, this allows one to show that one can derive LGIs without assuming MRps in full: *realism* is sufficient combined with NIM (and Induction) [170].

Performing these measurements on a macroscopic system must not alter either its subsequent or previous evolution; they must not "disturb" the system itself. This is the idea behind NIM and Induction: a measurement of a living Schrödinger's cat cannot kill it nor in the past nor in the future, and explains why the macroscopic properties of the system are "pre-existing". It perfectly embodies the behaviour we expect of a macroscopic system obeying the laws of classical physics, for this assumption has been identified as a "natural corollary" of MRps by Leggett himself [166].

The two assumptions allow us to derive another aspect about the elements of reality that we would measure in a macroscopic system: they must be real-valued quantities, so-called c-numbers. If this were not the case, a measurement on the system would entangle it with the measurer, causing a branching structure ultimately resulting in a disturbance of the investigated system itself. Moreover, if these observables must be described by c-numbers, then they must all be *compatible* with each other: one can measure all the macroscopic properties of the physical system under investigation simultaneously and with arbitrarily high precision. This is exactly the definition of classicality we introduced and formalised in Chapter 1 (Definition 1), phrased in a model-dependent scenario. Thus, Leggett and Garg's notion of classicality is an instance of the more general definition of classicality adopted in this Thesis. For this reason, we shall refer to a system obeying the assumptions above as a "macrorealistic" or "classical" system.

Nevertheless, while Induction is generally not contested in the literature, several discussions have developed around NIM giving rise to the so-called *clumsiness loophole* [171]. From an experimental point of view, it is very challenging to ensure that a measurement has not been invasive on the investigated system, thus has

not altered its macroscopic state. For this reason, a violation of the LGIs can always be ascribed to the experimenter being "clumsy" and disturbing the system under investigation, undermining the validity of the LGI test. Several proposals have been advanced to mitigate the effect of this loophole. However, we shall see in the remainder of this Chapter how the proposed temporal entanglement-based witness of non-classicality avoids the loophole by relying on a quantum probe rather than on direct measurements of the system.

The discussions about the meaning of macroscopic realism and the clumsiness loophole are part of a more general, on-going debate among scientists which can be summarised by the question: "what does the violation of the LGIs actually tell us about macroscopic systems"? We shall not delve into the full debate here (we refer the interested reader to [169, 172–174]), and we shall conform to the interpretation of MR, NIM and Induction as a definition of "classicality", so that a violation of the LGIs can be interpreted as a deviation from the classical behaviour. If the temporal evolution of a physical system violates LGIs, the statistics of the measurements cannot be described by a classical joint probability distribution. This implies that the macroscopic properties revealed in the measurement processes cannot be described by c-numbers: q-numbers are required at the macroscopic scale too. The extent of this scale is then quantified by the macroscopicity measures mentioned above.

Formally, assuming MRps, NIM, and Induction, one can derive a set of temporal inequalities that any classical system must obey, the Leggett-Garg Inequalities. We now present the inequality in the four-time scenario. Consider a physical system obeying the assumptions above (MRps, NIM and Induction). We measure its dichotomic observable $\hat{Q} = \pm 1$ at four successive times $t_1 < ... < t_4$ and denote the outcome at each time t_i as $\hat{Q}(t_i) = \hat{Q}_i$, $i = 1, 2, 3, 4$. The formal consequence of MRps, NIM and Induction is that it is possible to describe the correlation functions $E_{ij} = \langle \hat{Q}_i \hat{Q}_j \rangle$, computed for different pairs of time arguments $i, j = 1, 2, 3, 4$, with a single classical joint probability distribution. This implies that [164]:

$$|E_{21} - E_{32} + E_{43} + E_{41}| \leq 2. \tag{4.1}$$

Eq. 4.1 is the (four-time) Leggett-Garg inequality. It can be shown that quantum systems violate this inequality. For example, a qubit undergoing oscillations between the states with $\hat{Q} = \pm 1$ can achieve a maximum value of the sum of correlation functions equal to $2\sqrt{2}$ [164, 175].

4.1.2 From Leggett-Garg to Temporal Bell Inequalities

While the LGIs framework tests the classicality of macroscopic systems via temporal correlations of a single observable $\hat{Q}$, one can consider a temporally structured analogue of the spatial Bell test in which the observer has a choice between different observables at different times: $\hat{A}_{1,2}$ at time t_1 and $\hat{B}_{1,2}$ at a later time t_2. This leads to the so-called temporal Bell scenario, which mirrors the structure of the original Clauser-Horne-Shimony-Holt (CHSH) inequality [176]. By mapping $\hat{Q}(t_1) = \hat{B}_1$, $\hat{Q}(t_2) = \hat{A}_1$, $\hat{Q}(t_3) = \hat{B}_2$ and $\hat{Q}(t_4) = \hat{A}_2$, one gets:

$$\mathcal{B} := \left| E(\hat{A}_1, \hat{B}_1) - E(\hat{A}_1, \hat{B}_2) + E(\hat{A}_2, \hat{B}_1) + E(\hat{A}_2, \hat{B}_2) \right| \leq 2, \tag{4.2}$$

mathematically equivalent to the spatial CHSH inequality.

The equivalence between Eq. 4.2 and the CHSH inequality goes beyond a purely formal analogy when one considers two-fold correlations: the geometry of two-point spatial and temporal correlations in the space generated by the Pauli operators is symmetrical [177, 178]. In this scenario, Brukner and collaborators derived the *temporal Bell inequality* in Eq. 4.2 under the following assumptions [165]:

1. **Realism**: The measurement results are determined by "hidden" properties the particles carry prior to and independent of observation.

2. **Locality in time**: The results of measurement performed at time t_2 are independent of any measurement performed at some earlier or later time t_1.

It can be shown that quantum mechanics violates the temporal Bell inequality. Consider a physical system on which two dichotomic observables $\hat{A} = \pm 1$ and $\hat{B} = \pm 1$ are measured at times t_1 and $t_2 > t_1$, respectively. The projection operators onto the $+1$ or -1 eigenspaces of the two observables are $\hat{\Pi}_A^{\pm} = \frac{1 \pm \hat{A}}{2}$

and $\hat{\Pi}_B^{\pm} = \frac{1\pm\hat{B}}{2}$. Considering $|\psi\rangle$ as the initial state of the system, the correlator describing the correlation between successive measurements of $\hat{A}$ and $\hat{B}$ with outcomes $r, s \in \{+1, -1\}$ is given by [175]:

$$E(\hat{A},\hat{B}) = \sum_{r,s} rs \left\langle \psi \left| \frac{1+r\hat{A}}{2} \cdot \frac{1+s\hat{B}}{2} \cdot \frac{1+r\hat{A}}{2} \right| \psi \right\rangle = \frac{1}{2} \left\langle \{\hat{A},\hat{B}\} \right\rangle, \tag{4.3}$$

where $\{\cdot,\cdot\}$ denotes the anticommutator. If the system under investigation is a qubit, then $\hat{A} = \vec{a}\cdot\hat{\vec{\sigma}}$ and $\hat{B} = \vec{b}\cdot\hat{\vec{\sigma}}$, where $\vec{a}, \vec{b}$ are unit vectors and $\hat{\vec{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is the vector of Pauli matrices. Substituting these observables into the correlator in Eq. 4.3, one gets:

$$E(\hat{A},\hat{B}) = \frac{1}{2} \left\langle \{\hat{A},\hat{B}\} \right\rangle = \vec{a}\cdot\vec{b}. \tag{4.4}$$

Eq. 4.4 shows that the correlation function for two successive measurements of a qubit depends only on the chosen observables. Crucially, it does not depend on its initial state $|\psi\rangle$.

One can now compute the Bell quantity $\mathcal{B}^Q$ using the correlators in Eq. 4.4 [165]:

$$\mathcal{B}^Q = \left| \vec{a}_1\cdot(\vec{b}_1 - \vec{b}_2) + \vec{a}_2\cdot(\vec{b}_1 + \vec{b}_2) \right| \tag{4.5}$$

and it is easy to see that by appropriately choosing $\vec{a}_1 = \frac{1}{\sqrt{2}}(\vec{b}_1 - \vec{b}_2)$ and $\vec{a}_2 = \frac{1}{\sqrt{2}}(\vec{b}_1 + \vec{b}_2)$ one gets $\mathcal{B}^Q = 2\sqrt{2}$: quantum mechanics violates the temporal Bell inequality with a maximum violation equal to $2\sqrt{2}$, the (temporal) Cirel'son bound [68]. Thus, quantum mechanics can violate the Bell inequality when considering two-fold temporal correlations, with the same boundaries discovered for spatial correlation in the same two-fold scenario. In analogy to the spatial scenario, these temporal correlations are referred to as entanglement *in time* [165, 179–181]. Here the quantum correlations are not between subsystems, but between different temporal instances of the same system. Importantly, the temporal entanglement is relative to the observables $\hat{A}_{1,2}$ and $\hat{B}_{1,2}$.

A violation of the temporal Bell inequality implies that q-numbers, elements of reality, must be not only *spatially*, but also *temporally* local. Let us deepen the meaning of the assumptions behind this conclusion.

While the assumption of realism is the same we discussed in Chapter 4.1.1, locality in time summarises NIM and Induction with the goal of phrasing the crucial principle of locality behind the spatial Bell inequality (Principle 1) in the temporal domain: given a partition of a system into subsystems S_1 and S_2, a dynamical transformation that operates only on S_1 cannot change the state of S_2 [46]. To avoid confusion, we shall refer to it as principle of locality *in space*. Two important points must be made here. The first is that locality in time as phrased in Ref. [165] is relative to measurement *results*, so to the *observable* properties of the system's state. In the principle of locality in space, the "state" refers to the *entire* state of affairs of the subsystem, i.e., to its *complete* state. This makes the principle of locality in space a stronger statement than the current principle of locality in time. The second is that the principle of locality in space, when considered within a given dynamics, is backed by special relativity, which ensures that when acting on two space-like separated subsystems S_1 and S_2, the effects on S_1 cannot influence S_2 during the duration of the experiment [182]. Thus, special relativity readily closes the so-called "communication loophole" [183], the spatial equivalent of the clumsiness loophole introduced in the context of the LGIs. However, no such physical principle can be invoked to ensure the principle of locality in time, leaving open the clumsiness loophole and introducing the problem of *non-locality in time* for temporal correlations [184, 185].

Entanglement in space plays a crucial role in the GWT (Theorem 1): the mediation of spatial quantum correlations between two spatially separated quantum probes allows one to infer the non-classicality of the mediator under minimal assumptions. It is therefore of interest to investigate whether the use of entanglement in time on a single probe as a witnessing task for the non-classicality of the investigated mediator can lead to an equally robust temporal witness of non-classicality. This possibility will be explored in the next Section.

4.2 Temporal Entanglement: Sufficient Condition for Non-Classicality

As in the temporal witness of non-classicality introduced in Chapter 3, we consider here a hybrid system composed of the investigated sector M interacting with a *single* quantum probe Q, within the formalism of quantum theory. In this framework, we identify "non-classicality" with "quantum features", namely the necessary description of a system by a minimum of two non-commuting variables (see Chapter 1 for a more formal definition of non-classicality, Definition 1). Importantly, the hybrid system $Q \oplus M$ evolves under a dynamics that conserves a global, additive quantity, for example the total energy of the hybrid—which we shall consider for the remainder of the Chapter. This is the key assumption underlying the temporal witness of non-classicality, as explained in Chapter 3.2.1. For a summary and comparison of the assumptions behind the different classes of non-classicality witnesses considered in this Thesis, see Table 3.1. Under these assumptions, we want now to investigate whether the mediation of temporal entanglement can act as a witnessing task for the non-classicality of M.

4.2.1 Temporal Bell Inequality for the Hybrid System

To determine the temporal Bell inequality in this setting, we first compute the temporal correlation function for consecutive measurements performed on a quantum system Q interacting with an unknown system M. We measure the relevant dichotomic variables $\hat{A}$ and $\hat{B}$ on the quantum system Q at times t_0 and $t > t_0$, respectively. We emphasise here that no measurements are performed on the unknown system M: this is the system one wants to investigate, and therefore it is in general inaccessible. Instead, we probe its properties through the temporal quantum correlations it induces on Q under energy-conserving dynamics, thanks to the witness of non-classicality.

The quantum correlation function is given by [165]:

$$E(\hat{A}, \hat{B}) = \sum_{k,l=\pm 1} k \cdot l \cdot \mathrm{Tr}\left(\rho \hat{P}^k_{\hat{A}}\right) \cdot \mathrm{Tr}\left(\rho^k_{\hat{A}} \hat{P}^l_{\hat{B}}\right) \tag{4.6}$$

where $\rho = \rho_Q \otimes \rho_M$ is the initial state of the hybrid system $Q \oplus M$, $\hat{P}^k_{\hat{A}}$ and $\hat{P}^l_{\hat{B}}$ are the projectors onto the eigenspaces corresponding to the outcomes k and l, and $\rho^k_{\hat{A}}$ denotes the state of the hybrid system after the variable $\hat{A}$ of the quantum system Q has been measured at time t_0 with outcome k. It can be shown that in this scenario the quantum correlation function is:

$$E(\hat{A}, \hat{B}) = \frac{1}{2}\mathrm{Tr}\left(\left\{\hat{A}, \hat{B}\right\} \rho_Q \otimes \rho_M\right), \tag{4.7}$$

where $\left\{\hat{A}, \hat{B}\right\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ is the anticommutator between the chosen observables $\hat{A}$ and $\hat{B}$. The correlator in Eq. 4.7 depends on the *initial state of the hybrid system* $\rho_Q \otimes \rho_M$: whether the quantum probe Q violates the temporal Bell's inequality depends on the initial state of the unknown system M inducing its temporal evolution.

Since we want the evolution to be *energy-conserving* (to comply with the assumptions of the temporal witness) and in a temporal Bell test we have the freedom to choose the observables measured at times t_0 and t, it is convenient to adopt the Heisenberg picture to include the temporal evolution of the hybrid system between the two measurements:

$$\hat{B} = \hat{U}^\dagger \hat{A} \hat{U} = e^{i\hat{H}t} \hat{A} e^{-i\hat{H}t} \tag{4.8}$$

where $[\hat{H}, \hat{E}_{tot}] = 0$, with $\hat{E}_{tot}$ the total energy of the hybrid system. For example, considering a hybrid system made up by a qubit Q and a bit M, the total energy will be $\hat{E}_{tot} = \hat{\sigma}^Q_z \otimes \mathbb{I}_M + \mathbb{I}_Q \otimes \hat{\sigma}^M_z$, where $\hat{\sigma}^{Q,M}_z$ is the Pauli operator and $\mathbb{I}_{Q,M}$ is the unit operator. Importantly, we notice here that in choosing the observables $\hat{B}_{1,2}$ as in Eq. 4.8 one can, from the experimental point of view, measure the probe Q in the same basis at both times. This means that one does not need to know the evolution of the hybrid to measure $\hat{B}_{1,2}$, consistently with the goal of a witness of non-classicality, and with important simplifications on the experimental setup.

Using the quantum correlation function in Eq. 4.7, one can construct the temporal Bell inequality in Eq. 4.2 for a hybrid system. In our case, $\hat{A}_i$, $\hat{B}_j$, $i, j = 1, 2$, are the observables measured on the quantum probe Q at times t_0 and $t > t_0$ respectively, see Fig. 4.1.

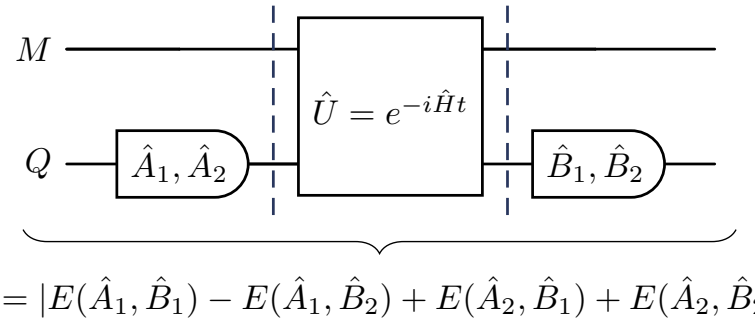


Figure 4.1: Schematic description of the process to compute the temporal Bell quantity $\mathcal{B}$. Two chosen variables of the quantum probe Q, $\hat{A}_1$ and $\hat{A}_2$, are measured at time t_0. The hybrid system $Q \oplus M$ evolves under the global energy-conserving dynamics generated by $\hat{H}$. The two time-evolved variables $\hat{B}_1 = \hat{U}^\dagger \hat{A}_1 \hat{U}$ and $\hat{B}_2 = \hat{U}^\dagger \hat{A}_2 \hat{U}$ are then measured at time $t > t_0$. Repeating this experiment allows one to construct $\mathcal{B}$ and check whether it violates the classical bound $\mathcal{B} = 2$. This will be a witness of M's non-classicality.

4.2.2 The Witnessing Task: Temporal Entanglement

We now show that a violation of the temporal Bell inequality, when considering an energy-conserving dynamics for the hybrid system $Q \oplus M$ and measuring the local energy and a non-commuting observable on Q, is a witness of the non-classicality of M. To do so, we consider a quantum mediator and analyse under which conditions on M the temporal Bell inequality (Eq. 4.2) cannot be violated. We identify two different conditions: (1) M is fully described by a single variable, identified by the coupling with Q as the classical variable; (2) M is prepared in a state ρ_M^{class} such that $[\rho_M^{\text{class}}, \hat{E}_{tot}] = 0$, i.e., an eigenstate of the conserved quantity or in a classical mixture of these eigenstates. We define ρ_M^{class} as a *classical* state for M[1]. Condition (1) corresponds to the definition of classicality adopted in this Thesis (see Definition 1 for a more formal notion), while (2) to a non-classical M (i.e., described by at least two non-commuting variables) initialised in a classical state. This will be essential to show that only a non-classical mediator prepared in a non-classical initial state will be capable of mediating an evolution that violates Eq. 4.2, hence

[1]Notice here that the classical basis, namely the one that couples the two sectors in the hybrid, coincides with the one defining the local observable participating in the global conservation. This is because the conservation law is part of the set of assumptions for the temporal witnesses, see Table 3.1.

of mediating temporal entanglement on Q. To do so, it must dynamically involve at least two non-commuting variables.

Without loss of generality, we consider the quantum probe Q to be a qubit whose algebra is generated by $\{\mathbb{I}_Q, \hat{\sigma}_x^Q, \hat{\sigma}_y^Q, \hat{\sigma}_z^Q\}$, where $\hat{\sigma}_i^Q \ i \in \{x, y, z\}$ are the Pauli operators and $\mathbb{I}_Q$ the identity. Similarly, M is a qubit whose algebra is generated by $\{\mathbb{I}_M, \hat{\sigma}_x^M, \hat{\sigma}_y^M, \hat{\sigma}_z^M\}$. Considering the composite system $Q \oplus M$, we define $\hat{X}_Q = \hat{\sigma}_x^Q \otimes \mathbb{I}_M, \hat{Y}_Q = \hat{\sigma}_y^Q \otimes \mathbb{I}_M, \hat{Z}_Q = \hat{\sigma}_z^Q \otimes \mathbb{I}_M$, and similarly for $\hat{X}_M, \hat{Y}_M, \hat{Z}_M$. The most general global energy-conserving Hamiltonian can then be written as:

$$\hat{H}_{QM} = a\hat{Z}_Q + b\hat{Z}_M + c\hat{Z}_Q\hat{Z}_M + f(\hat{X}_Q\hat{X}_M + \hat{Y}_Q\hat{Y}_M) + g(\hat{X}_Q\hat{Y}_M - \hat{Y}_Q\hat{X}_M) + r\mathbb{I}_{(4)}, \quad (4.9)$$

where $\mathbb{I}_{(4)} = \mathbb{I}_Q \otimes \mathbb{I}_M$. One can identify three blocks in $\hat{H}_{QM}$, associated with the different total energies the hybrid $Q \oplus M$ can acquire:

$$\hat{H}_{QM} = \begin{pmatrix} a+b+c & 0 & 0 & 0 \\ 0 & a-b-c & 2(f+ig) & 0 \\ 0 & 2(f-ig) & -a+b-c & 0 \\ 0 & 0 & 0 & -a-b+c \end{pmatrix} = \begin{pmatrix} E_{00} & 0 & 0 & 0 \\ 0 & & & 0 \\ 0 & & \hat{H}_{\text{sub}} & 0 \\ 0 & 0 & 0 & E_{11} \end{pmatrix}, \quad (4.10)$$

where $E_{00} = a+b+c,\ E_{11} = -a-b+c$. The non-trivial 2×2 block on $\text{span}\{|01\rangle, |10\rangle\}$ is $\hat{H}_{\text{sub}} = -c\,\mathbb{I} + \Delta\,\hat{\sigma}_z^{\text{sub}} + 2f\,\hat{\sigma}_x^{\text{sub}} - 2g\,\hat{\sigma}_y^{\text{sub}}$, where $\hat{\sigma}_{x,y,z}^{\text{sub}}$ are Pauli matrices acting on the two-dimensional subspace, and we defined $\Delta := a - b,\ \Omega := \sqrt{\Delta^2 + 4(f^2+g^2)}$.

Hence, it is convenient to define the projectors onto the subspaces associated with the total conserved quantity, corresponding to sectors with total excitation number $N = 0, 1, 2$:

$$\begin{aligned} \hat{P}_0 &:= \hat{\Pi}_0^Q \otimes \hat{\Pi}_0^M = |00\rangle\langle 00|, \\ \hat{P}_1 &:= \hat{\Pi}_0^Q \otimes \hat{\Pi}_1^M + \hat{\Pi}_1^Q \otimes \hat{\Pi}_0^M = |01\rangle\langle 01| + |10\rangle\langle 10|, \\ \hat{P}_2 &:= \hat{\Pi}_1^Q \otimes \hat{\Pi}_1^M = |11\rangle\langle 11|. \end{aligned} \quad (4.11)$$

Global energy conservation, the assumption under which we have derived Eq. 4.9, implies that both $\hat{H}_{QM}$ and the corresponding unitary evolution $\hat{U}_{QM} := e^{-i\hat{H}_{QM}t}$ are block diagonal with respect to the decomposition $\hat{P}_0 \oplus \hat{P}_1 \oplus \hat{P}_2$:

$$\left[\hat{U}_{QM}, \hat{P}_n\right] = 0, \qquad n = 0, 1, 2. \quad (4.12)$$

Consequently, for any operator $\hat{O}$ on $Q \oplus M$ and any $m, n \in \{0, 1, 2\}$,

$$\hat{P}_m(\hat{U}_{QM}^\dagger \hat{O} \hat{U}_{QM})\hat{P}_n = \hat{U}_{QM}^\dagger(\hat{P}_m \hat{O} \hat{P}_n)\hat{U}_{QM}. \tag{4.13}$$

In particular, if $\hat{P}_m \hat{O} \hat{P}_n = 0$, then $\hat{P}_m(\hat{U}_{QM}^\dagger \hat{O} \hat{U}_{QM})\hat{P}_n = 0$: the unitary evolution cannot create new connections between the energy sectors. Furthermore, with respect to this decomposition, the unitary has the form:

$$\hat{U}_{QM} = e^{-itE_{00}}\hat{P}_0 + \hat{P}_1 \hat{U}_{\text{sub}} \hat{P}_1 + e^{-itE_{11}}\hat{P}_2, \tag{4.14}$$

where $\hat{U}_{\text{sub}} = e^{itc}\left(\cos(\Omega t)\,\mathbb{I} - i\,\frac{\sin(\Omega t)}{\Omega}\,(\Delta\,\hat{\sigma}_z^{\text{sub}} + 2f\,\hat{\sigma}_x^{\text{sub}} - 2g\,\hat{\sigma}_y^{\text{sub}})\right)$.

Lemma 1. *For the Hamiltonian $\hat{H}_{QM}$ in Eq. 4.9 and the choice $\hat{A}_1 = \hat{Z}_Q$, $\hat{A}_2 = \hat{X}_Q$ with a generic initial state ρ_Q for the quantum probe Q, the temporal Bell quantity $\mathcal{B}$ satisfies $\mathcal{B} \leq 2$ if M is classical or if M is initialised in a classical state ρ_M^{class}.*

Proof. [2] The proof is a selection-rule argument based on excitation number, thus on the energy of the hybrid $Q \oplus M$. Consider an arbitrary initial state for the probe Q:

$$\rho_Q = \frac{1}{2}\left(\mathbb{I}_Q + r_x\hat{\sigma}_x^Q + r_y\hat{\sigma}_y^Q + r_z\hat{\sigma}_z^Q\right), \qquad \vec{r} \in \mathbb{R}^3, |\vec{r}| \leq 1. \tag{4.15}$$

Let $\hat{O}$ be any Hermitian operator on $Q \otimes M$ expanded in the Pauli basis of Q:

$$\hat{O} = \mathbb{I}_Q \otimes \hat{O}_0^M + \sum_{\beta \in \{x,y,z\}} \hat{\sigma}_\beta^Q \otimes \hat{O}_\beta^M, \tag{4.16}$$

where $\hat{O}_0^M$ and $\hat{O}_\beta^M$ are the coefficients of $\mathbb{I}_Q$ and $\hat{\sigma}_\beta^Q$ in $\hat{O}$, respectively. They are thus operators defined on the Hilbert space of M containing, in general, all the components $\{\mathbb{I}_M, \hat{\sigma}_x^M, \hat{\sigma}_y^M, \hat{\sigma}_z^M\}$.

Using $\{\hat{\sigma}_\alpha, \mathbb{I}\} = 2\hat{\sigma}_\alpha$, $\{\hat{\sigma}_\alpha, \hat{\sigma}_\beta\} = 2\delta_{\alpha,\beta}\mathbb{I}$ with $\alpha, \beta \in \{x, y, z\}$, one has:

$$\begin{aligned}
\frac{1}{2}\left\{\hat{\sigma}_\alpha^Q \otimes \mathbb{I}_M,\, \hat{O}\right\} &= \frac{1}{2}\left\{\hat{\sigma}_\alpha^Q \otimes \mathbb{I}_M,\, \left(\mathbb{I}_Q \otimes \hat{O}_0^M + \sum_{\beta \in \{x,y,z\}} \hat{\sigma}_\beta^Q \otimes \hat{O}_\beta^M\right)\right\} \\
&= \frac{1}{2}\left\{\hat{\sigma}_\alpha^Q \otimes \mathbb{I}_M,\, \mathbb{I}_Q \otimes \hat{O}_0^M\right\} + \frac{1}{2}\left\{\hat{\sigma}_\alpha^Q \otimes \mathbb{I}_M,\, \sum_{\beta \in \{x,y,z\}} \hat{\sigma}_\beta^Q \otimes \hat{O}_\beta^M\right\} \\
&= \hat{\sigma}_\alpha^Q \otimes \hat{O}_0^M + \mathbb{I}_Q \otimes \hat{O}_\alpha^M.
\end{aligned} \tag{4.17}$$

[2] The proof of Lemma 1 is rather technical and spans the following pages, up to p. 102. It may be omitted on a first reading without loss of continuity; the reader may proceed directly to Section 4.2.3.

Here $\hat{O}_0^M$ is the operator acting on M representing the coefficient of $\mathbb{I}_Q$ in $\hat{O}$. Similarly, $\hat{O}_\alpha^M$ is the coefficient of $\hat{\sigma}_\alpha^Q$ in $\hat{O}$, which has been selected by the anticommutator: $\frac{1}{2}\{\hat{\sigma}_\alpha^Q \otimes \mathbb{I}_M, \sum_\beta \hat{\sigma}_\beta^Q \otimes \hat{O}_\beta^M\} = \sum_\beta \frac{1}{2} 2\delta_{\alpha,\beta} \mathbb{I}_Q \otimes \hat{O}_\beta^M = \mathbb{I}_Q \otimes \hat{O}_\alpha^M$.

Considering now the correlator in Eq. 4.7 and using Eq. 4.17, one gets:

$$\begin{aligned} E(\hat{\sigma}_\alpha^Q \otimes \mathbb{I}_M,\ \hat{O}) &= \frac{1}{2}\mathrm{Tr}\left[\rho_Q \otimes \rho_M \left\{\hat{\sigma}_\alpha^Q \otimes \mathbb{I}_M,\ \hat{O}\right\}\right] \\ &= \mathrm{Tr}\left[\rho_Q \otimes \rho_M \left(\hat{\sigma}_\alpha^Q \otimes \hat{O}_0^M + \mathbb{I}_Q \otimes \hat{O}_\alpha^M\right)\right] \\ &= \mathrm{Tr}\left[\rho_Q \otimes \rho_M \left(\hat{\sigma}_\alpha^Q \otimes \hat{O}_0^M\right)\right] + \mathrm{Tr}\left[\rho_Q \otimes \rho_M \left(\mathbb{I}_Q \otimes \hat{O}_\alpha^M\right)\right]. \end{aligned} \tag{4.18}$$

One can work separately on the two summands. Starting with the first, expanding the general initial state ρ_Q as in Eq. 4.15 and using the property of the trace $\mathrm{Tr}[A \otimes B] = \mathrm{Tr}[A]\mathrm{Tr}[B]$:

$$\begin{aligned} \mathrm{Tr}\left[\rho_Q \otimes \rho_M \left(\hat{\sigma}_\alpha^Q \otimes \hat{O}_0^M\right)\right] &= \mathrm{Tr}\left[\rho_Q\, \hat{\sigma}_\alpha^Q \otimes \rho_M\, \hat{O}_0^M\right] \\ &= \mathrm{Tr}_Q\left[\frac{1}{2}\left(\mathbb{I}_Q + r_x\hat{\sigma}_x^Q + r_y\hat{\sigma}_y^Q + r_z\hat{\sigma}_z^Q\right)\hat{\sigma}_\alpha^Q\right]\mathrm{Tr}_M\left[\rho_M \hat{O}_0^M\right] \\ &= r_\alpha \mathrm{Tr}_M\left[\rho_M\, \hat{O}_0^M\right]; \end{aligned} \tag{4.19}$$

since $[\hat{\sigma}_\alpha, \hat{\sigma}_\beta] = 2i\epsilon_{\alpha,\beta,\gamma}\hat{\sigma}_\gamma$, $\alpha, \beta, \gamma \in \{x, y, z\}$, the only term giving a non-null trace in the first factor will be the one corresponding to $\hat{\sigma}_\alpha^Q$, so that $\mathrm{Tr}\left[\frac{1}{2} r_\alpha \hat{\sigma}_\alpha^Q \hat{\sigma}_\alpha^Q\right] = r_\alpha$. Moving to the second, since the Pauli operators are traceless, one gets:

$$\mathrm{Tr}\left[\rho_Q \otimes \rho_M \left(\mathbb{I}_Q \otimes \hat{O}_\alpha^M\right)\right] = \mathrm{Tr}\left[\rho_Q \mathbb{I}_Q \otimes \rho_M \hat{O}_\alpha^M\right] = \mathrm{Tr}_M\left[\rho_M \hat{O}_\alpha^M\right]. \tag{4.20}$$

Including Eqs. 4.19 and 4.20 in Eq. 4.18, one finally obtains:

$$E(\hat{\sigma}_\alpha^Q \otimes \mathbb{I}_M,\ \hat{O}) = r_\alpha\, \mathrm{Tr}_M\left[\rho_M\, \hat{O}_0^M\right] + \mathrm{Tr}_M\left[\rho_M\, \hat{O}_\alpha^M\right], \tag{4.21}$$

with $\alpha \in \{x, y, z\}$.

Let $\hat{A}_1 = \hat{\sigma}_z^Q \otimes \mathbb{I}_M$, $\hat{A}_2 = \hat{\sigma}_x^Q \otimes \mathbb{I}_M$, and $\hat{B}_{1,2} = \hat{U}_{QM}^\dagger \hat{A}_{1,2} \hat{U}_{QM}$. With this choice, one can now compute the four correlators entering $\mathcal{B}$, starting from the cross ones, $E(\hat{A}_1, \hat{B}_2)$ and $E(\hat{A}_2, \hat{B}_1)$, for M classical or initialised in a classical state.

— $\mathbf{E(\hat{A}_1, \hat{B}_2)}$ **for classical M or for M prepared in** $\rho_{\mathbf{M}}^{\mathrm{class}}$**.**

Consider $\hat{A}_1 = \hat{\sigma}_z^Q \otimes \mathbb{I}_M$ (corresponding to $\alpha = z$) and $\hat{B}_2 = \hat{U}_{QM}^\dagger(\hat{\sigma}_x^Q \otimes \mathbb{I}_M)\hat{U}_{QM}$.

Since $\hat{\sigma}_x^Q$ flips Q while leaving M unchanged, it cannot connect $N = 0$ directly to $N = 2$ and it has no diagonal blocks in the excitation decomposition:

$$\hat{P}_0(\hat{\sigma}_x^Q \otimes \mathbb{I}_M)\hat{P}_2 = \hat{P}_2(\hat{\sigma}_x^Q \otimes \mathbb{I}_M)\hat{P}_0 = 0, \qquad \hat{P}_n(\hat{\sigma}_x^Q \otimes \mathbb{I}_M)\hat{P}_n = 0 \ (n = 0, 1, 2). \tag{4.22}$$

By block invariance in Eq. 4.13, the same holds for $\hat{B}_2 = \hat{U}_{QM}^\dagger(\hat{\sigma}_x^Q \otimes \mathbb{I}_M)\hat{U}_{QM}$:

$$\hat{P}_0\hat{B}_2\hat{P}_2 = \hat{P}_2\hat{B}_2\hat{P}_0 = 0, \qquad \hat{P}_n\hat{B}_2\hat{P}_n = 0 \ (n = 0, 1, 2). \tag{4.23}$$

Expand $\hat{B}_2$ in Pauli operators on Q:

$$\hat{B}_2 = \mathbb{I}_Q \otimes (\hat{B}_2)_0^M + \sum_{\beta \in \{x,y,z\}} \hat{\sigma}_\beta^Q \otimes (\hat{B}_2)_\beta^M. \tag{4.24}$$

The coefficients $(\hat{B}_2)_0^M$ and $(\hat{B}_2)_z^M$ are constrained by the diagonal elements of $\hat{B}_2$ in the computational basis $\{|qm\rangle\}$, where $q, m \in \{0, 1\}$ label Q and M. Each $|qm\rangle$ belongs to a single sector $\hat{P}_n$ ($|00\rangle \in \hat{P}_0$, $|01\rangle, |10\rangle \in \hat{P}_1$, $|11\rangle \in \hat{P}_2$), so Eq. 4.23 gives $\langle qm|\hat{B}_2|qm\rangle = 0$. On the other hand, $\hat{\sigma}_x^Q$ and $\hat{\sigma}_y^Q$ have no diagonal elements in the basis of Q, while $\hat{\sigma}_z^Q|q\rangle = (-1)^q|q\rangle$, so Eq. 4.24 yields:

$$0 = \langle qm|\hat{B}_2|qm\rangle = \langle m|(\hat{B}_2)_0^M|m\rangle + (-1)^q \langle m|(\hat{B}_2)_z^M|m\rangle, \qquad q, m \in \{0, 1\}. \tag{4.25}$$

For each m, the cases $q = 0$ and $q = 1$ are two independent linear conditions on $\langle m|(\hat{B}_2)_0^M|m\rangle$ and $\langle m|(\hat{B}_2)_z^M|m\rangle$; their sum and their difference give:

$$\hat{\Pi}_m^M(\hat{B}_2)_0^M\hat{\Pi}_m^M = 0, \qquad \hat{\Pi}_m^M(\hat{B}_2)_z^M\hat{\Pi}_m^M = 0, \qquad m = 0, 1. \tag{4.26}$$

This implies that $(\hat{B}_2)_0^M$ and $(\hat{B}_2)_z^M$ contain only terms proportional to $\hat{\sigma}_x^M$ and $\hat{\sigma}_y^M$, which connect different energy sectors by exciting M, i.e.,

$$(\hat{B}_2)_0^M, \ (\hat{B}_2)_z^M \in \mathrm{span}\{\hat{\sigma}_x^M, \hat{\sigma}_y^M\}. \tag{4.27}$$

Evaluating now:

$$E(\hat{A}_1, \hat{B}_2) = r_z \, \mathrm{Tr}_M[\rho_M(\hat{B}_2)_0^M] + \mathrm{Tr}_M[\rho_M(\hat{B}_2)_z^M] \tag{4.28}$$

one can conclude that (1) if M is a classical system, according to the definition of classicality here adopted, its only variable would be the classical variable $\hat{\sigma}_z^M$, which

makes $E(\hat{A}_1, \hat{B}_2) = 0$; (2) if M is initialised in a classical state ρ_M^{class}, meaning that $[\rho_M^{\text{class}}, \hat{E}_{tot}] = [\rho_M^{\text{class}}, \hat{\sigma}_z^Q \otimes \mathbb{I}_M + \mathbb{I}_Q \otimes \hat{\sigma}_z^M] = 0$, then both the traces in Eq. 4.28 will vanish as they will involve traces of Pauli operators, which vanish, since they are traceless.

— $\mathbf{E(\hat{A}_2, \hat{B}_1)}$ **for classical M or for M prepared in** $\rho_{\mathbf{M}}^{\text{class}}$**.**
Consider $\hat{A}_2 = \hat{\sigma}_x^Q \otimes \mathbb{I}_M$ (corresponding to $\alpha = x$) and $\hat{B}_1 = \hat{U}_{QM}^\dagger(\hat{\sigma}_z^Q \otimes \mathbb{I}_M)\hat{U}_{QM}$. The operator $\hat{\sigma}_z^Q \otimes \mathbb{I}_M$ is diagonal in the computational basis and conserves the total excitation number, hence it has only diagonal blocks:

$$\hat{P}_m(\hat{\sigma}_z^Q \otimes \mathbb{I}_M)\hat{P}_n = 0 \quad \text{for } m \neq n. \tag{4.29}$$

By block invariance, $\hat{B}_1$ satisfies the same property:

$$\hat{P}_m \hat{B}_1 \hat{P}_n = 0 \quad \text{for } m \neq n. \tag{4.30}$$

Expand $\hat{B}_1$ in Pauli operators on Q:

$$\hat{B}_1 = \mathbb{I}_Q \otimes (\hat{B}_1)_0^M + \sum_{\beta \in \{x,y,z\}} \hat{\sigma}_\beta^Q \otimes (\hat{B}_1)_\beta^M. \tag{4.31}$$

Consider the term $\hat{\sigma}_x^Q \otimes (\hat{B}_1)_x^M$, appearing in the second summand in Eq. 4.21 when $\hat{A}_2 = \hat{\sigma}_x^Q \otimes \mathbb{I}_M$ (and, therefore, $\alpha = x$). The operators $\mathbb{I}_Q$ and $\hat{\sigma}_z^Q$ do not flip Q, whereas $\langle \bar{q}|\hat{\sigma}_x^Q|q\rangle = 1$ and $\langle \bar{q}|\hat{\sigma}_y^Q|q\rangle = i(-1)^q$, with $\bar{q} = 1 - q$. Eq. 4.31 thus gives, for $q, m \in \{0, 1\}$,

$$\langle \bar{q}m|\hat{B}_1|qm\rangle = \langle m|(\hat{B}_1)_x^M|m\rangle + i(-1)^q \langle m|(\hat{B}_1)_y^M|m\rangle. \tag{4.32}$$

The states $|qm\rangle$ and $|\bar{q}m\rangle$ belong to different sectors ($N = q + m$ and $N = \bar{q} + m$), so the left-hand side vanishes by Eq. 4.30. Summing over $q = 0, 1$ the term in $(\hat{B}_1)_y^M$ cancels, and one finds $\hat{\Pi}_m^M (\hat{B}_1)_x^M \hat{\Pi}_m^M = 0$ for $m = 0, 1$. This implies, as before,

$$(\hat{B}_1)_x^M \in \text{span}\{\hat{\sigma}_x^M, \hat{\sigma}_y^M\}. \tag{4.33}$$

Moving now to the $\mathbb{I}_Q$ coefficient of $\hat{B}_1$ expansion, appearing in the first summand of Eq. 4.21, one notices from Eq. 4.31 that this component can be extracted by tracing out the probe Q:

$$(\hat{B}_1)_0^M = \frac{1}{2}\text{Tr}_Q[\hat{B}_1] = \frac{1}{2}\text{Tr}_Q[\hat{U}_{QM}^\dagger(\hat{\sigma}_z^Q \otimes \mathbb{I}_M)\hat{U}_{QM}]. \tag{4.34}$$

Using the block form of $\hat{U}_{QM}$ in Eq. 4.14 and the fact that $\hat{\sigma}_z^Q \otimes \mathbb{I}_M$ acts as $+1$ on $\hat{P}_0$, -1 on $\hat{P}_2$, and as $\hat{\sigma}_z^{\rm sub} := \hat{P}_1(\hat{\sigma}_z^Q \otimes \mathbb{I}_M)\hat{P}_1$ on the $\hat{P}_1$ sector, one finds:

$$\begin{aligned}
2(\hat{B}_1)_0^M &= \mathrm{Tr}_Q[|e^{-itE_{00}}|^2\hat{P}_0] - \mathrm{Tr}_Q[|e^{-itE_{11}}|^2\hat{P}_2] + \mathrm{Tr}_Q[(\hat{P}_1\hat{U}_{\rm sub}^\dagger\hat{P}_1)(\hat{\sigma}_z^Q \otimes \mathbb{I}_M)(\hat{P}_1\hat{U}_{\rm sub}\hat{P}_1)] \\
&= |e^{-itE_{00}}|^2\mathrm{Tr}_Q[\hat{P}_0] - |e^{-itE_{11}}|^2\mathrm{Tr}_Q[\hat{P}_2] + \mathrm{Tr}_Q[(\hat{U}_{\rm sub}^\dagger\hat{P}_1\hat{P}_1)(\hat{\sigma}_z^Q \otimes \mathbb{I}_M)(\hat{P}_1\hat{P}_1\hat{U}_{\rm sub})] \\
&= |e^{-itE_{00}}|^2\hat{\Pi}_0^M - |e^{-itE_{11}}|^2\hat{\Pi}_1^M + \mathrm{Tr}_Q[\hat{U}_{\rm sub}^\dagger\hat{P}_1(\hat{\sigma}_z^Q \otimes \mathbb{I}_M)\hat{P}_1\hat{U}_{\rm sub}] \\
&= \hat{\Pi}_0^M - \hat{\Pi}_1^M + \mathrm{Tr}_Q[\hat{U}_{\rm sub}^\dagger\hat{\sigma}_z^{\rm sub}\hat{U}_{\rm sub}] \\
&= \hat{\Pi}_0^M - \hat{\Pi}_1^M + \mathrm{Tr}_Q[(1-2\mathcal{T})\hat{\sigma}_z^{\rm sub} + \beta_x\hat{\sigma}_x^{\rm sub} + \beta_y\hat{\sigma}_y^{\rm sub}] \\
&= \hat{\Pi}_0^M - \hat{\Pi}_1^M + (1-2\mathcal{T})(\hat{\Pi}_1^M - \hat{\Pi}_0^M) = 2\mathcal{T}\,\hat{\sigma}_z^M
\end{aligned} \tag{4.35}$$

where we have used $\mathrm{Tr}_Q[\hat{P}_0] = \hat{\Pi}_0^M$, $\mathrm{Tr}_Q[\hat{P}_2] = \hat{\Pi}_1^M$ and $\mathrm{Tr}_Q[\hat{\sigma}_z^{\rm sub}] = \hat{\Pi}_1^M - \hat{\Pi}_0^M$. $\mathrm{Tr}_Q[\hat{\sigma}_{x,y}^{\rm sub}] = 0$ since $\hat{\sigma}_{x,y}^{\rm sub}$ map $|01\rangle \leftrightarrow |10\rangle$, and thus flip the state of Q, they are off-diagonal in Q and are annihilated by the partial trace Tr_Q . The coefficient of $\hat{\sigma}_z^{\rm sub}$ is obtained considering that $\hat{U}_{\rm sub}$ in Eq. 4.14 rotates the Bloch vector of the two-level subspace by an angle $2\Omega t$ about $\vec{n} = (2f, -2g, \Delta)/\Omega$: the z component of the rotated $\hat{\sigma}_z^{\rm sub}$ is $n_z^2 + (1-n_z^2)\cos(2\Omega t) := 1 - 2\mathcal{T}$, hence $\mathcal{T} = 4(f^2+g^2)\sin^2(\Omega t)/\Omega^2 \in [0,1]$. Notice that $\mathcal{T} = |\langle 10|\hat{U}_{QM}|01\rangle|^2$: it is the probability that $\hat{U}_{QM}$ maps $|01\rangle$ into $|10\rangle$, hence that the energy moves across the two components of the hybrid $Q \oplus M$.

Evaluating in conclusion:

$$E(\hat{A}_2, \hat{B}_1) = \mathrm{Tr}_M[\rho_M(\hat{B}_1)_x^M] + r_x\mathrm{Tr}_M[\rho_M(\hat{B}_1)_0^M] = r_x\mathcal{T}\,\mathrm{Tr}_M[\rho_M\hat{\sigma}_z^M] \tag{4.36}$$

one gets that (1) for M classical or M prepared in a classical state $\mathrm{Tr}_M[\rho_M(\hat{B}_1)_x^M] = 0$ by Eq. 4.33 and (2) $(\hat{B}_1)_0^M = \mathcal{T}\,\hat{\sigma}_z^M$, which is in general non-vanishing. Hence $E(\hat{A}_2, \hat{B}_1)$ vanishes for classical M ($f = g = 0$, hence $\mathcal{T} = 0$), but not in general for M prepared in a classical state. For example, for $\hat{U}_{QM}$ equal to the swap gate ($\mathcal{T} = 1$), Q in $|+\rangle$ and M in $|0\rangle$ one has $\hat{B}_1 = \hat{Z}_M$ and $E(\hat{A}_2, \hat{B}_1) = 1$.

— **$\mathbf{E(\hat{A}_1, \hat{B}_1)}$ and $\mathbf{E(\hat{A}_2, \hat{B}_2})$ for classical M or for M prepared in $\rho_{\mathbf{M}}^{\rm class}$.** We start with $E(\hat{A}_1, \hat{B}_1)$. Consider $\hat{A}_1 = \hat{\sigma}_z^Q \otimes \mathbb{I}_M$ and $\hat{B}_1 = \hat{U}_{QM}^\dagger(\hat{\sigma}_z^Q \otimes \mathbb{I}_M)\hat{U}_{QM}$. Both $\hat{Z}_Q$ and $\hat{B}_1$ are block diagonal, and $\frac{1}{2}\{\hat{\sigma}_z^{\rm sub}, \hat{\sigma}_{x,y}^{\rm sub}\} = 0$; therefore $\frac{1}{2}\{\hat{A}_1, \hat{B}_1\} =$

$\hat{P}_0 + \hat{P}_2 + (1-2\mathcal{T})\hat{P}_1 = \mathbb{I} - 2\mathcal{T}\hat{P}_1$, and

$$E(\hat{A}_1, \hat{B}_1) = \frac{1}{2}\mathrm{Tr}\left[\{\hat{A}_1, \hat{B}_1\}\rho_Q \otimes \rho_M\right] = \mathrm{Tr}\left[(\mathbb{I} - 2\mathcal{T}\hat{P}_1)\rho_Q \otimes \rho_M\right]$$
$$= 1 - \mathcal{T}(1 - r_z \mathrm{Tr}_M[\rho_M \hat{\sigma}_z^M]), \tag{4.37}$$

since $\mathrm{Tr}[(\rho_Q \otimes \rho_M)\hat{P}_1] = \frac{1}{2}(1+r_z)(\rho_M)_{11} + \frac{1}{2}(1-r_z)(\rho_M)_{00} = \frac{1}{2}(1 - r_z \mathrm{Tr}_M[\rho_M \hat{\sigma}_z^M])$.

Moving now to $E(\hat{A}_2, \hat{B}_2)$, consider $\hat{A}_2 = \hat{\sigma}_x^Q \otimes \mathbb{I}_M$ and $\hat{B}_2 = \hat{U}_{QM}^\dagger(\hat{\sigma}_x^Q \otimes \mathbb{I}_M)\hat{U}_{QM}$. Setting $\alpha = x$ and $\hat{O} = \hat{B}_2$ in Eq. 4.21, one has

$$E(\hat{A}_2, \hat{B}_2) = r_x \,\mathrm{Tr}_M[\rho_M (\hat{B}_2)_0^M] + \mathrm{Tr}_M[\rho_M (\hat{B}_2)_x^M]. \tag{4.38}$$

The first term vanishes: $(\hat{B}_2)_0^M \in \mathrm{span}\{\hat{\sigma}_x^M, \hat{\sigma}_y^M\}$ by Eq. 4.27, and these operators have no diagonal elements in the eigenbasis of $\hat{\sigma}_z^M$, in which ρ_M is diagonal for M classical or prepared in ρ_M^{class}. The correlator reduces to the second term, where the initial state of the probe no longer appears: $E(\hat{A}_2, \hat{B}_2)$ does not depend on ρ_Q. The coefficient of $\hat{\sigma}_x^Q$ in $\hat{B}_2$ is extracted by tracing out the probe, $(\hat{B}_2)_x^M = \frac{1}{2}\mathrm{Tr}_Q[(\hat{\sigma}_x^Q \otimes \mathbb{I}_M)\hat{B}_2]$. Since ρ_M is classical, only the diagonal elements of $(\hat{B}_2)_x^M$ in the basis $\{|0\rangle, |1\rangle\}$ of M matter. For $m \in \{0,1\}$, using that $\hat{\sigma}_x^Q$ flips Q and that $\hat{B}_2$ is Hermitian,

$$\langle m|(\hat{B}_2)_x^M|m\rangle = \frac{1}{2}\sum_q \langle qm|(\hat{\sigma}_x^Q \otimes \mathbb{I}_M)\hat{B}_2|qm\rangle = \frac{1}{2}\sum_q \langle \bar{q}m|\hat{B}_2|qm\rangle = \mathrm{Re}\, z_m, \tag{4.39}$$
$$z_m := \langle 0m|\hat{B}_2|1m\rangle = (\hat{U}_{QM}|0m\rangle)^\dagger(\hat{\sigma}_x^Q \otimes \mathbb{I}_M)(\hat{U}_{QM}|1m\rangle), \tag{4.40}$$

with $\bar{q} = 1-q$. The elements z_m follow from the action of $\hat{U}_{QM}$ on the computational basis. By Eq. 4.14, $|00\rangle$ and $|11\rangle$ are the only states in their sectors, so that $\hat{U}_{QM}|00\rangle = u_{00}|00\rangle$ and $\hat{U}_{QM}|11\rangle = u_{11}|11\rangle$; $\hat{U}_{QM}$ mixes $|01\rangle$ and $|10\rangle$ instead, so that $\hat{U}_{QM}|01\rangle = u_{01}|01\rangle + v_{01}|10\rangle$ and $\hat{U}_{QM}|10\rangle = u_{10}|10\rangle + v_{10}|01\rangle$. Here, $u_{qm} := \langle qm|\hat{U}_{QM}|qm\rangle$, $v_{01} := \langle 10|\hat{U}_{QM}|01\rangle$ and $v_{10} := \langle 01|\hat{U}_{QM}|10\rangle$. Using $\hat{\sigma}_x^Q|00\rangle = |10\rangle$, $\hat{\sigma}_x^Q|01\rangle = |11\rangle$ (and conversely) and the orthonormality of the basis, each overlap reduces to a single term:

$$z_0 = (u_{00}|00\rangle)^\dagger(\hat{\sigma}_x^Q \otimes \mathbb{I}_M)(u_{10}|10\rangle + v_{10}|01\rangle) = \bar{u}_{00}\,\langle 00|(u_{10}|00\rangle + v_{10}|11\rangle) = \bar{u}_{00}\,u_{10}, \tag{4.41}$$
$$z_1 = (u_{01}|01\rangle + v_{01}|10\rangle)^\dagger(\hat{\sigma}_x^Q \otimes \mathbb{I}_M)(u_{11}|11\rangle) = (\bar{u}_{01}\langle 01| + \bar{v}_{01}\langle 10|)u_{11}|01\rangle = \bar{u}_{01}\,u_{11}. \tag{4.42}$$

Hence, with $p_m := \langle m|\rho_M|m\rangle$,

$$E(\hat{A}_2, \hat{B}_2) = p_0 \,\mathrm{Re}(\bar{u}_{00}u_{10}) + p_1 \,\mathrm{Re}(\bar{u}_{01}u_{11}). \tag{4.43}$$

Explicitly, Eq. 4.14 gives $u_{01} = e^{itc}\Big[\cos(\Omega t) - i\,\frac{\Delta}{\Omega}\sin(\Omega t)\Big]$ and $u_{10} = e^{itc}\Big[\cos(\Omega t) + i\,\frac{\Delta}{\Omega}\sin(\Omega t)\Big]$, so that $z_0 = e^{it(a+b+2c)}\Big[\cos(\Omega t)+i\frac{\Delta}{\Omega}\sin(\Omega t)\Big]$ and $z_1 = e^{it(a+b-2c)}\Big[\cos(\Omega t)+ i\frac{\Delta}{\Omega}\sin(\Omega t)\Big]$. For classical M this is $E(\hat{A}_2, \hat{B}_2) = p_0\cos(2t(a+c)) + p_1\cos(2t(a-c))$, i.e., the rotation of $\hat{\sigma}_x^Q$ about z by an angle that depends on the state of M.

For the remainder of the proof only the moduli are needed. One has $|u_{00}| = |u_{11}| = 1$, and since the block of $\hat{U}_{QM}$ on the $N = 1$ sector is a 2×2 unitary, $|v_{01}| = |v_{10}|$ and $|u_{01}|^2 = |u_{10}|^2 = 1 - |v_{01}|^2 = 1 - \mathcal{T}$. Therefore $|z_0| = |z_1| = \sqrt{1-\mathcal{T}}$ and, since $|\mathrm{Re}\, z| \leq |z|$ and $p_0 + p_1 = 1$,

$$|E(\hat{A}_2, \hat{B}_2)| \leq p_0|z_0| + p_1|z_1| = \sqrt{1-\mathcal{T}}. \tag{4.44}$$

The phases of z_0 and z_1 depend on $a + b$, c and t and are unconstrained, so that only this modulus is relevant in what follows.

Collecting the results, one gets:

$$\begin{aligned}\mathcal{B} &= |E(\hat{A}_1, \hat{B}_1) + E(\hat{A}_2, \hat{B}_1) + E(\hat{A}_2, \hat{B}_2)| \\ &\leq |E(\hat{A}_1, \hat{B}_1) + E(\hat{A}_2, \hat{B}_1)| + |E(\hat{A}_2, \hat{B}_2)|.\end{aligned} \tag{4.45}$$

First, using Eq. 4.44, one has $|E(\hat{A}_2, \hat{B}_2)| \leq \sqrt{1-\mathcal{T}}$. Then, from Eqs. 4.36 and 4.37 and since $|\mathrm{Tr}[\rho_M \hat{\sigma}_z^M]| \leq 1$ and $(r_x + r_z)^2 \leq 2(r_x^2 + r_z^2) \leq 2$, one can conclude that $E(\hat{A}_1, \hat{B}_1) + E(\hat{A}_2, \hat{B}_1) = 1 - \mathcal{T}\left[1 - \mathrm{Tr}[\rho_M \hat{\sigma}_z^M](r_x + r_z)\right] \in [\,1 - (1+\sqrt{2})\mathcal{T},\, 1 + (\sqrt{2}-1)\mathcal{T}\,]$ has modulus at most $1 + (\sqrt{2}-1)\mathcal{T}$ for $\mathcal{T} \in [0, 1]$. Therefore, one obtains:

$$\mathcal{B} \leq h(\mathcal{T}) := 1 + (\sqrt{2}-1)\mathcal{T} + \sqrt{1-\mathcal{T}} \leq 2, \tag{4.46}$$

since $h'(\mathcal{T}) = (\sqrt{2}-1) - \frac{1}{2\sqrt{1-\mathcal{T}}} < 0$ on $[0, 1)$, so that $\mathcal{B} \leq h(0) = 2$. In particular, for classical M ($\mathcal{T} = 0$) the bound reads $\mathcal{B} \leq 1 + 1 = 2$. This concludes the proof.

□

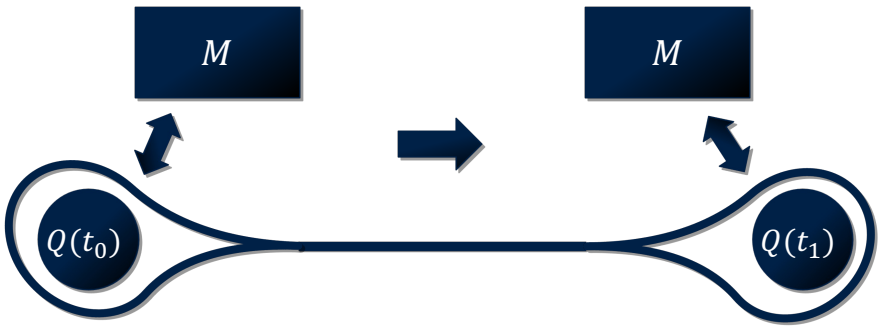


Figure 4.2: Schematic representation of the temporal entanglement-based witness of non-classicality, Theorem 3. A single quantum probe Q is coupled to the investigated system M in a way such that, by assumption, the global dynamics is energy conserved. The generation of temporal entanglement relative to the measured observables and mediated by M is a sufficient witness of its non-classicality.

Given Lemma 1, $\mathcal{B} \leq 2$ whenever M is classical or is prepared in a classical initial state ρ_M^{class}. This implies that only a non-classical M prepared in a non-classical initial state can induce temporal entanglement relative to $\hat{Z}_Q$ and $\hat{X}_Q$ on the quantum probe Q in the composite system $Q \oplus M$.

4.2.3 The Theorem: Temporal Entanglement-Based Witness of Non-Classicality

This allows us to restate the temporal witness of non-classicality as follows (see Fig. 4.2). Assuming the conservation of the energy on the system $Q \oplus M$ [162]:

Theorem 3 (**Temporal entanglement-based witness of non-classicality**). *If M can generate temporal entanglement on Q, relative to its local energy and a non-commuting observable, then M is non-classical.*

Theorem 3 represents the central result of this Chapter, connecting witnesses of non-classicality in the temporal domain with the theory of entanglement in time. It proves that the entanglement *in time* induced by M is a sufficient condition for its non-classicality, namely the existence and dynamical involvement of a minimum of two non-commuting variables, provided that a global quantity on $Q \oplus M$ is conserved during the evolution of the hybrid system and that one of the observables

is the probe's energy. The argument is general within quantum theory, as it does not rely on any model describing the hybrid $Q \oplus M$.

This framework supports tests of non-classicality based on the violation of Leggett-Garg inequalities (LGIs), while extending their applicability to multi-system configurations. As discussed in Chapter 4.1, the temporal Bell inequality in Eq. 4.2, involving two-fold temporal correlations with two observables, is equivalent to the four-time LGI of Eq. 4.1. Conventional LGI tests, however, require sequential measurements on the systems under investigation and, in deriving the inequality from distinct experimental configurations, they rely on the NIM assumption. This requirement opens the *clumsiness loophole*, frequently invoked to question the validity of LGI-based demonstrations of quantumness: an observed violation of LGIs can be attributed to an invasive measurement, rather than to the failure of MRps. In this light, one interpretation of these tests is that they demonstrate that certain measurements fail to be non-disturbing at the *operational or statistical level*—that is, inserting the measurement alters the observable statistics of subsequent measurements—without necessarily ruling out MRps [169]. A substantial body of work has sought to address this issue either by engineering measurement schemes designed to approximate NIM (e.g., ideal negative-result strategies) [163], by isolating NIM as the sufficient assumption to derive LGIs (the so-called *no-signalling in time* or *nondisturbance* conditions) so as to avoid interpretation conflicts with MRps [186, 187], or by explicitly quantifying invasiveness and thereby shrinking the loophole (e.g., "adroit" measurement protocols) [171], often at the cost of additional modelling assumptions or residual loopholes [174]. Theorem 3 overcomes this limitation in a conceptually different way: it relocates the LGI-sensitive sequential measurement structure onto a well-characterised auxiliary quantum system Q, for which measurement disturbance is part of the controlled quantum description. Temporal entanglement witnessed through LGI violations on Q can then be used as a robust *indirect* witness of non-classical behaviour in a system M whose quantum nature remains undetermined, without requiring NIM to hold on M itself, as it is *never* measured at any stage of the protocol. This contrasts with standard

loophole-mitigation strategies, as the burden of "measurement credibility" is shifted from the unknown system to an explicitly quantum probe, enabling inference about M through experimentally accessible statistics of Q.

The temporal entanglement framework substantially strengthens Theorem 3 compared to Theorem 2, by demonstrating that a successful temporal entanglement-based witness of non-classicality requires the mediator M to be initially prepared in a *non-classical* state—that is, an eigenstate of an observable that does not commute with the classical variable of the system. For example, if M is a qubit, this means being in an eigenstate of $\hat{X}_M$ or $\hat{Y}_M$ when the classical variable is $\hat{Z}_M$. This result resolves a key limitation of the temporal witness in Theorem 2, where semi-classical models compatible with the assumptions of Theorem 2 could, in principle, still account for the observed coherent dynamics of Q. A typical example is that of a spin-$\frac{1}{2}$ particle interacting with an effectively classical magnetic field: in such cases, detecting temporal entanglement in Q would imply that prior interactions between M and its environment prepared M in a non-classical state, thereby ruling out any semi-classical description of the hybrid system, even in the temporal scenario.

Furthermore, this result qualifies the non-commuting variables of M as *observables* of the unknown system within the quantum formalism, as Theorem 3 indirectly confirms that M (1) was prepared in one of their eigenstates and (2) needs at least two of these variables to successfully execute the witnessing protocol on Q. This conclusion is consistent with the effects that one of the two probes in the GWT has on the investigated system M: preparing it in a non-classical state. In two-probe witnesses of non-classicality, the probe Q_A is prepared in a superposition precisely for M to enter a non-classical state. With the example of the BMV experiment, the mass Q_A enters a superposition precisely for the gravitational field to be prepared in a superposition too. The non-classical state of M, prepared in this way, will in turn be picked up by the second quantum probe Q_B and indirectly witnessed through the two-probe entanglement (see Chapter 2.1). In a single-probe setting, the role of Q_A must be played by the environment, which is responsible for preparing M in the non-classical state that is subsequently accessed through the temporal

evolution of Q, leading to temporal entanglement. This consideration suggests an improvement to the experimental design of a temporal entanglement-based witness of non-classicality. With the help of an external quantum system prepared in a superposition in the conserved-quantity basis (which we shall refer to as the "preparation probe"), one can let M enter the non-classical state necessary for the temporal entanglement to be mediated during the evolution of Q. However, differently from the GWT, the preparation probe is then discarded, preserving the single-probe distinctive character of Theorem 3.

It is important to emphasise that the conclusion about the observable nature of M's non-commuting variables strictly holds when assuming the quantum formalism (see Table 3.1). In more general formulations beyond quantum theory, where the mathematical structure of Hermitian operators that guarantees single-shot distinguishability no longer applies, further considerations would be required to qualify the variable as an observable of the system. For example, in constructor theory, a variable qualifies as an observable if and only if each of its attributes $\mathbf{x}$ coincides with the set of all and only those states that necessarily yield the sharp outcome x under every possible measurement of that variable; equivalently, observability obtains precisely when the defining attributes of the variable are fixed points of the theory's distinguishability structure, being determined entirely by which distinguishing tasks are possible and which are impossible. A formal definition of observable in CT is provided in Appendix A.

4.2.4 A temporal extension of the GWT in quantum theory

By replacing Q's quantum coherent evolution (Theorem 2) with the generation of entanglement *in time* on Q as the witnessing task, the connection between witnesses of non-classicality and the theory of temporal entanglement strengthens the identification of Theorem 3 as the temporal version of the GWT in Theorem 1 within the quantum formalism. This has several interesting consequences, which we now discuss.

First, it shows that mediating two-point quantum correlations, equivalently in space and time, is a sufficient task to infer the non-classicality of an unknown system, under the appropriate assumptions. This opens the possibility of extending recently proposed methods for *quantifying* the non-classicality of the mediator of spatial quantum correlations to the temporal domain. In particular, it has been shown that a suitable function of the spatial quantum correlations established between two spatially separated quantum probes provides a lower bound on the non-classicality of the mediator's local interactions with each probe [69, 70]. We have extended this result to the mediator's non-classicality, as discussed in Chapter 2.3.2 [45]. Theorem 3 suggests that analogous tools can be applied to *temporal* quantum correlations mediated by an unknown system, provided that appropriate temporal counterparts of spatial entanglement measures are introduced. This, in turn, establishes desirable mathematical criteria, such as *gd-continuity* [70], that temporal entanglement measures should satisfy in order to construct such quantifiers.

Second, this framework provides a novel and robust route for designing experiments in fields such as gravity and biology, where applications of the GWT have already been proposed but remain experimentally challenging due to the requirement of enforcing spatial locality [41, 46, 51]. By replacing spatial entanglement with temporal entanglement as the witnessing mechanism, our approach enables a conclusive test of the non-classicality of M using a *single* quantum probe. This eliminates the need for two *non-directly interacting* quantum probes, making Theorem 3 particularly advantageous for experimental implementations. Consider for example the scenario discussed in Chapter 3.3.4, where we proposed an application of the temporal witness of non-classicality in Theorem 2 to light-harvesting in biological systems. A photon excites the light-harvesting complex of a photosynthetic organism (e.g., a chromophore in a green sulphur bacterium), creating a Frenkel exciton initially localised on that chromophore. The FMO complex, which connects the chromophores to the reaction centre of the bacterium via the BChl-a molecules, enables the remarkably efficient transport of the exciton to funnel the energy required by the dark reactions behind photosynthesis. In order to show that the near-unit

efficiency of this process is genuinely due to non-trivial quantum phenomena[3], specifically long-lived quantum coherence, Theorem 3 suggests focusing on the capability of the exciton M's evolution to mediate temporal entanglement between the absorbed photon $Q(t_0)$, which initiates the process, and the emitted photon $Q(t)$ after transport. To construct the temporal Bell inequality in Eq. 4.2 in this scenario, one should measure the four bases $\hat{A}_1 = \hat{\sigma}_z^Q \otimes \mathbb{I}_M$, $\hat{A}_2 = \hat{\sigma}_x^Q \otimes \mathbb{I}_M$ and $\hat{B}_{1,2} = \hat{U}^\dagger \hat{A}_{1,2} \hat{U}$, with no measurements required on the biological complex at any stage. A violation of the classical bound $\mathcal{B} = 2$ would imply the non-classicality of the process and rule out all the possible classical explanations compatible with the assumptions of Theorem 3. Importantly, this would provide the first experimental demonstration of a biological system responsible for the mediation of temporal entanglement on a quantum system, stimulating research into its capability of entangling two spatially separated quantum systems, as mandated by the GWT [51].

Finally, the symmetry between Theorem 1 and Theorem 3, highlighted by their respective witnessing tasks, can be pushed one step further: not only do the *conclusions* align, but the *assumptions* that render the two witnesses conclusive also play analogous logical roles. In Theorem 3, conclusiveness relies on imposing the *exact conservation of a global additive quantity* (e.g., energy) on the hybrid system $Q \oplus M$ throughout the entire evolution responsible for generating entanglement in time on the probe Q. Operationally, this requirement rules out "spurious" explanations in which Q acquires the relevant temporal correlations through a spontaneous (or effectively uncontrolled) local evolution, rather than through a genuinely non-classical mediation by M [165, 175]. This mirrors the role of the *principle of locality* in Theorem 1: allowing direct Q_A–Q_B interactions, or more generally non-local semiclassical mechanisms, would trivially enable the probes to become spatially entangled irrespective of any non-classicality of the mediator [46]. In both cases, the assumption serves to exclude explanations in which the observed correlations arise without genuine mediation.

[3]For a more detailed discussion of what "non-trivial" quantum phenomena means in Quantum Biology, see Chapter 3.3

This correspondence suggests a notion of *temporal locality*, parallel to spatial locality. In the spatial case, locality forbids the possibility of changing the state of a subsystem by acting on another one with which it does not directly interact. In the temporal case, the analogous requirement is that a non-trivial change in the state of the probe cannot arise "for free" during the evolution: it must be mediated by an ancillary system and accompanied by a compensating change that preserves a global constraint at all intermediate times.

Within the formalism of quantum theory, such a constraint naturally takes the form of an exact conservation law. This motivates the following conjecture: a conservation-based formulation of temporal locality.

Principle 3 (**Conservation-based temporal locality**). *A task on a system S is said to be local in time if its implementation admits an ancilla A such that a global additive observable $G_{S\oplus A}$ is conserved throughout the evolution.*

Principle 3 expresses the property of temporal locality in terms of *counterfactuals*: it is *impossible* to change the state of a system with a given value of the conserved quantity G_S to another state with a different value without side effects on another system A. The side effect is that any such change must be compensated by the mediator, so that the global quantity remains conserved at each instant.

In our scenario, the ancilla corresponds to the system M, which mediates the dynamics of the quantum probe Q. A change in the state of Q that would modify the value of the locally defined quantity G_Q cannot occur in isolation, but must be accompanied by a corresponding change in M to ensure exact conservation at the global level. To do so, as discussed in Chapter 3.2.1, M must engage a second non-commuting variable, revealing its non-classicality. This property is then witnessed through the temporal evolution of Q alone.

This logic is akin to the role the principle of locality in space has in the GWT (Theorem 1). Principle 1 states that it is *impossible* to change the state of a subsystem by acting on the state of a different one. Therefore, a change in the state of Q_A cannot occur because of Q_B, if they are not directly interacting. For

it to be possible, some side effects are necessary on the system M, with which both Q_A and Q_B are coupled. The final entangled state of $Q_A \oplus Q_B$ reveals the non-classicality of M.

However, differently from the GWT, both Theorem 2 and Theorem 3 are formulated within the standard quantum formalism, where unitary dynamics of closed systems guarantees exact conservation laws at all times. Beyond quantum theory, this need not be the case: one may envisage transformations for which conservation holds only on average over time, rather than at each intermediate instant. In such a setting, the formulation of temporal locality as an exact conservation constraint may require further refinements, and its generalisation constitutes an open problem. We shall speculate more on this in the Conclusions.

Compared with earlier temporal notions of locality, which we have discussed in Chapters 4.1.1 and 4.1.2, the conjectured Principle 3 is structurally distinct but conceptually complementary. In the Leggett–Garg tradition, temporal "locality" is typically expressed through constraints on how measurement choices and interventions at one time can affect statistics at other times, namely NIM and Induction [164]. Likewise, temporal Bell–CHSH approaches formalise "locality in time" as the possibility that a measurement on a system influences the *same* system at a later (or earlier) time [165, 180]. In this sense, LG/temporal-CHSH notions are related to observable properties of the system. The state they deal with is not the complete state of the system, but simply its observable component. By contrast, Principle 3 is formulated at a more general level: it is a constraint on the physical *tasks* admissible in the system, and can be used to underpin (but is not reducible to) particular operational criteria. The tasks affect the complete state of the system, encompassing both observable and non-observable properties. In this sense, the state considered by Principle 1 and Principle 3 is the same: the complete state of affairs of the system.

4.2.5 Experimental Emulation with NMR

To confirm the predicted effect of the theory, an emulation of the quantum network shown in Fig. 4.1 was performed using NMR qubits [188]. Previous emulations

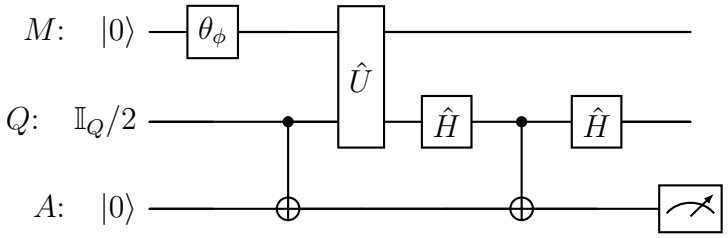


Figure 4.3: A typical circuit to simulate temporal entanglement in a three-qubit NMR system; this example shows the case of a $\hat{\sigma}_z$ measurement followed by a $\hat{\sigma}_x$ measurement. For more details see the text.

run on the same platform were used to confirm the predictions of the GWT, as presented in Ref. [189].

The temporal entanglement-based witness requires a three-spin NMR system. This is because of the lack of true measurements, which requires using an ancilla qubit to store and combine partial results [190]. The other two qubits will represent the quantum probe Q and the mediator M.

A typical circuit for the NMR implementation is shown in Fig. 4.3. Here the ancilla qubit A begins in the state $|0\rangle$, while the mediator qubit M starts in some desired pure state which can be prepared from $|0\rangle$ using a θ_ϕ single-qubit rotation. The quantum system qubit Q is initially prepared in the maximally mixed state, $\mathbb{I}_Q/2$. Quantum measurements of $\hat{\sigma}_z$ are emulated using a simple CNOT gate, while measurements of $\hat{\sigma}_x$ are emulated using a CNOT gate preceded and followed by Hadamard gates. Finally the measurement gate was emulated by applying a 90° excitation pulse to spin A, and then observing the NMR spectrum.

A proof-of-principle of Theorem 3 requires three different scenarios for the mediator M: (i) a classical M; (ii) a quantum M prepared in a classical initial state (i.e., a state such that $[\rho_M, \hat{E}_{tot}] = 0$); (iii) a quantum M prepared in any non-classical state. Only in the latter case do we expect to observe a violation of the temporal Bell inequality in Eq. 4.2. This is because the temporal entanglement-based witness of non-classicality is a sufficient condition for the non-classicality of M: if one observes $\mathcal{B} > 2$ under the assumptions in Table 3.1, one can conclude

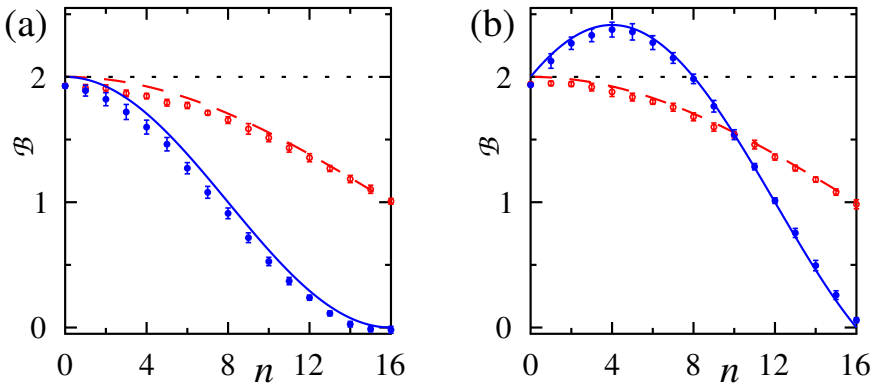


Figure 4.4: Data from our simulation experiments. The lines indicate theoretical calculations while the points indicate experimental measurements, and red dashed lines/open points indicate the classical mediator (mixing Hamiltonian with $f = 0$), while blue solid lines/filled points indicate the non-classical mediator (mixing Hamiltonian with $f = 1$). Error bars were estimated by repetition. Note that for classical mediator the Bell inequality $\mathcal{B}$ never exceeds 2, indicated by the dotted line, while for non-classical mediator deviations are seen when M starts in a non-classical pure state, shown in (b), but not when it starts in a classical pure state, shown in (a). The observed damping in the experimental results, which causes these to deviate from theoretical calculations, particularly at long mixing times, reflects a build up of errors arising from imperfect implementations of the logic gates.

that M must be a non-classical system, namely described by a minimum of two non-commuting variables, prepared in a non-classical initial state.

We recall here that, under the assumptions of Theorem 3, the most general Hamiltonian generating the evolution of $Q \oplus M$ can be expressed as in Eq. 4.9. $a, b, c \in \mathbb{R}$ are the "classical" coefficients, as they are the only terms in $\hat{H}_{QM}$ that remain non-vanishing when the mediator M is a classical bit. In this case, according to the definition of non-classicality in Chapter 1, M would be described by the single variable $\hat{Z}_M$. For the same reason, $f, g \in \mathbb{R}$ are the "non-classical" coefficients as they arise only when M possesses non-commuting variables (in the case of a qubit, $\hat{X}_M$ and $\hat{Y}_M$).

To investigate the three scenarios described above, experiments were performed for three different initial states of M, for two different mixing Hamiltonians, and over a range of times. M is prepared in an effective state $|0\rangle$, which was used first directly (a classical pure state), secondly after the application of a 90°_x pulse (a non-

classical pure state, corresponding to $|+i\rangle$, the eigenstate of the observable $\hat{Y}_M = \mathbb{I}_Q \otimes \hat{\sigma}_y^M$), and thirdly after converting to the maximally mixed state (classical) by the application of a 90°_x pulse followed by a crush gradient. The mixing Hamiltonians both had $a = b = g = 0$ and $c = 1$, but varied by choosing $f = 0$ (simulating a classical M) or $f = 1$ (a non-classical M). Experiments were performed for a range of mixing times, corresponding to both classical and non-classical behaviour for the non-classical mixing Hamiltonian applied to a non-classical pure state. Experiments were performed for each of the four possible measurement combinations. The individual measured integrals were normalised against the simplest case (no mixing and two $\hat{\sigma}_z$ measurements) and then combined offline to produce the desired temporal Bell inequality, $\mathcal{B}$, shown in Eq. 4.2. The results are depicted in Fig. 4.4 and shown as dots: the filled points represent the non-classical mixing Hamiltonian $f = 1$, while the open points the classical mixing Hamiltonian $f = 0$. They are consistent with the theoretical predictions discussed in Chapter 4.2: the solid lines represent the calculations for a non-classical mixing Hamiltonian, the dashed lines for a classical mixing Hamiltonian.

With a classical mixing Hamiltonian, which describes a classical M, the Bell quantity $\mathcal{B}$ is always bounded by 2, confirming that a classical mediator cannot induce temporal entanglement on the quantum probe Q. When considering a non-classical mixing Hamiltonian, thus a non-classical M, and we prepare M in a classical initial state (Panel (a) in Fig. 4.4), again we see no violation of the temporal Bell inequality ($\mathcal{B} \leq 2$): a non-classical M prepared in a classical initial state cannot entangle in time the quantum probe Q. It is only when the non-classical mediator is initialised in a non-classical state (Panel (b) in Fig. 4.4) that we get a Bell quantity $\mathcal{B}$ exceeding the classical boundary 2: only a non-classical mediator prepared in a non-classical initial state can entangle in time the quantum probe Q, as predicted by the theory.

Analytical calculations allows one to extend these results to different values of $f, g \in [-2, 2]$ when considering the three cases explored by the emulation: quantum M initialised in a non-classical state, specifically an eigenstate of $\hat{Y}_M$ (Fig. 4.5(a));

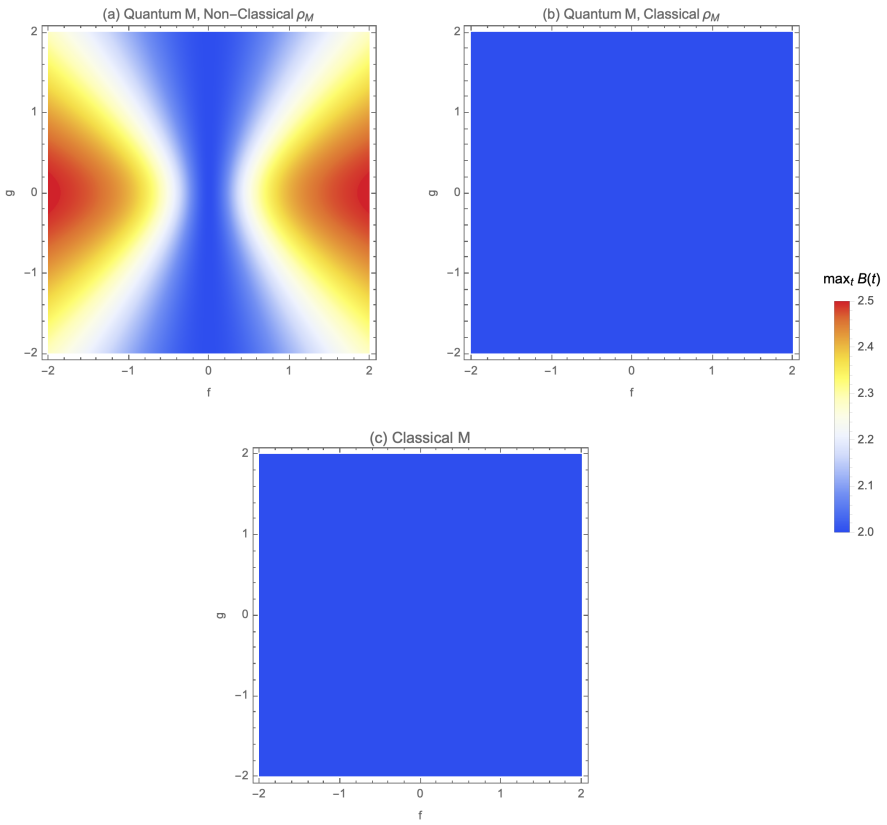


Figure 4.5: Analytical calculations showing the maximum of the Bell quantity $\mathcal{B}$ over the interaction time t for different combinations of the non-classical parameters $f, g \in [-2, 2]$ in Eq. 4.9. (a) A quantum M prepared in an eigenstate of $\hat{Y}_M$ mediates temporal entanglement on Q. (b) A quantum M prepared in a classical state and (c) a classical M cannot mediate temporal entanglement on Q.

quantum M initialised in a classical state (Fig. 4.5(b)); classical M (Fig. 4.5(c)). Figure 4.5 shows the maximum value of the Bell quantity $\mathcal{B}$ in Eq. 4.2 taken over the interaction time t between Q and M when $a = b = c = 0$ and $f, g \in [-2, 2]$. We restrict f, g to the interval $[-2, 2]$ for simplicity, as the unitarity of the interaction renders the dynamics periodic over a larger interval. The analytical results confirm the behaviour observed in the NMR emulation across a wide range of parameters in the general energy-conserving Hamiltonian in Eq. 4.9: temporal entanglement on Q can only be mediated by a non-classical M prepared in a non-classical initial state ρ_M, under a global energy-conserving dynamics.

4.3 Summary

In this Chapter, we have introduced a temporal entanglement-based witness of non-classicality, establishing a novel information-theoretical framework to infer the non-classical nature of an unknown system M in a hybrid setting. Building upon the temporal witness developed in Chapter 3, we have identified the mediation of temporal entanglement relative to the local energy on a single quantum probe as a witnessing task capable of providing a sufficient condition for non-classicality.

Theorem 3 establishes that if the evolution of a quantum probe Q, mediated by an unknown system M, violates the temporal Bell inequality in Eq. 4.2 under a global additive conservation law, then M must be non-classical. According to the definition of non-classicality adopted in this Thesis (see Chapter 1, Definition 1), this implies that M must be described by at least two non-commuting variables. Importantly, the witnessing task requires M to be prepared in a non-classical state, namely a state with coherence in the conserved quantity basis, strengthening the conclusions of the temporal witness introduced in Theorem 2. By shifting the witnessing task entirely onto the quantum probe Q, the protocol avoids direct measurements on the unknown system M, thereby circumventing the clumsiness loophole that affects traditional Leggett-Garg tests. Therefore, Theorem 3 provides an extension of non-classicality tests based on the violation of Leggett-Garg inequalities to multiple systems where the non-classicality of M is inferred indirectly from the statistics of Q.

We have supported the theoretical argument with an experimental proof-of-principle involving a three-qubit NMR system. The experiment demonstrates that the temporal Bell inequality is violated only when the mediator is both non-classical and prepared in a non-classical state, in full agreement with the predictions of the theory. Interestingly, this experiment represents an instance of temporal entanglement realised on a NMR quantum computer, suggesting a possible operational interpretation of the protocol as a method for generating such correlations in a quantum platform. In particular, one can engineer an energy-conserving dynamics in which the evolution of the qubit one wants to temporally entangle is mediated by a suitably prepared second, auxiliary qubit, as

illustrated in Fig. 4.3. This perspective opens new avenues for further investigations of temporal entanglement as a resource for quantum information processing, as first envisaged in Ref. [165].

The violation of the temporal Bell inequality is the condition we identify as defining entanglement *in time* [165, 175], which allows us to restate Theorem 3 in terms of M-mediation of temporal entanglement (relative to the energy of the probe and a non-commuting observable) under global additive quantity-conserving dynamics. Hence, Theorem 3 provides a *temporal extension* to the GWT (Theorem 1) within the quantum formalism, proving that both entanglement in space and in time can serve as a sufficient condition to witness the quantum nature of an unknown sector in a hybrid system, under the appropriate assumptions (see Table 3.1). This suggests a deeper connection between the principles underlying the two constructions, expressed by the conservation-based temporal locality in Principle 3. Moreover, it points towards the possibility of formulating a general *spatiotemporal* witness of non-classicality within a unified framework in which spatial and temporal witnesses of non-classicality can be treated on equal footing under the same underlying principles.

Importantly, Theorem 3 represents the first milestone for the derivation of the GWT *in Time* we envisaged in this Thesis. The next step in this research programme is to *relax* the assumption of the quantum formalism, in the same way Ref. [42] did in introducing the GWT (Theorem 1). We shall return to these questions in the next Chapter, where we summarise the results of this Thesis and outline possible future directions.

Per ora vado solo fuori fuoco
Come la fisica classica in tempi quantum

For now, I am just going off focus
like classical physics in quantum times

— Caparezza, *Come la musica elettronica*

5 Conclusions and Outlook

The question of whether quantum theory provides a universal description of physical reality remains one of the most profound open problems in modern physics. This Thesis has addressed this question from an operational and information-theoretical perspective, focusing on the study of hybrid systems in which a well-controlled quantum sector interacts with an unknown system whose underlying description may or may not be quantum. Within this setting, the central challenge is to devise methods capable of revealing non-classical features of the inaccessible sector without requiring direct control over it and under minimal assumptions.

The approach developed throughout this work is based on information-theoretical witnesses of non-classicality: protocols in which the possibility of a suitable information-processing task performed on a quantum probe provides a sufficient condition for the non-classicality of the system mediating the interaction. The shift in perspective, from dynamical modelling to information theoretical, task-based reasoning, has enabled the formulation of general, robust arguments that rely only on minimal and physically motivated assumptions.

The starting point of the analysis is the General Witness Theorem, which establishes that if an unknown system can mediate entanglement between two quantum probes by local means only, then it must be non-classical [42]. Building on this result, this Thesis has clarified the role of the assumptions underlying the

theorem, in particular by providing a detailed analysis of the notion of locality required for the argument to hold in a dynamics-independent framework [46]. The comparison between the minimal notion of locality as phrased in Principle 1 and other dynamics-dependent formulations is useful to close apparent loopholes where certain classical models of the investigated system can mediate entanglement between the quantum probes and invalidate the witness. In addition, within the quantum formalism, we have developed a quantitative extension of the witness, establishing a direct relation between the amount of entanglement mediated by the investigated system and the non-commutativity of its variables. Therefore, this result strengthens the witness by linking observable correlations to an intrinsic measure of non-classicality [45]. The notion of non-classicality that we adopted in this Thesis is, in quantum theory, that a non-classical system must be described by a minimum of two non-commuting variables.

Motivated by the experimental challenges associated with enforcing the principle of locality in two-probe scenarios by excluding their direct interactions, the second part of the Thesis has introduced a novel class of single-probe, information-theoretical witnesses of non-classicality. In this original approach, the witnessing task is encoded in the temporal evolution of a single quantum probe, rather than in spatial correlations between distinct systems. By imposing the exact conservation of a global additive quantity, we have shown within quantum theory that the ability of an unknown system to induce quantum-coherent evolution in the conserved quantity basis of the probe constitutes a sufficient condition for its non-classicality [85]. This result provides an operational realisation of the so-called totalitarian property of quantum theory, highlighting how conservation laws can play a role analogous to locality in constraining the dynamics of hybrid systems and therefore revealing the non-classicality of the unknown sector.

This temporal framework has been further developed by introducing a stronger witnessing task based on temporal entanglement. By establishing a connection between temporal Bell inequalities and energy-conserving mediated dynamics, we have shown that the mediation of temporal quantum correlations relative to the

local conserved quantity of the probe provides a more robust and structurally complete analogue of the spatial witness. This is because it supplies a sufficient condition that also constrains the initial state of the mediator, requiring it to be non-classical [162]. In this sense, the temporal entanglement-based witness can be regarded as the natural extension of the General Witness Theorem within the quantum formalism, revealing a deep symmetry between mediated spatial and temporal correlations in witnessing non-classicality.

Finally, the applicability of these ideas has been explored in the context of complex and realistic systems, most notably biological systems. In particular, the temporal witness has been proposed as a tool to address the longstanding debate on the role of quantum coherence in photosynthetic energy transport [86]. By formulating the problem in a model-independent way, the witness framework provides a route to exclude entire classes of classical descriptions under general assumptions, offering a new perspective on the non-trivial role of quantum effects in macroscopic and noisy environments.

The extension of the recently emerged field of information-theoretical witnesses of non-classicality to the temporal domain, as developed in this Thesis, opens several promising research directions both at the foundational and applied level.

The first natural development concerns the extension of temporal witnesses of non-classicality beyond the quantum formalism, a direction currently under investigation with Chiara Marletto. While the temporal entanglement-based witness provides a complete analogue of the General Witness Theorem within quantum theory, its current formulation still relies on quantum dynamical assumptions. Relaxing these assumptions has two important consequences. First, the dynamics need no longer be unitary, and conservation laws are no longer guaranteed to hold exactly at all times throughout the evolution. As a result, the totalitarian property of quantum theory no longer enforces a branching structure for the mediator. Moreover, the conjectured notion of temporal locality (Principle 3) must be accordingly extended to accommodate conservation laws that may hold only on average, admitting intermediate violations that do not affect the overall balance.

Second, conservation laws can no longer be formulated in terms of commutation relations, and must instead be expressed in a dynamics-independent way. A fully general formulation—what one may call a General Witness Theorem in time—requires recasting conservation laws within such a framework. Constructor theory of thermodynamics provides a natural candidate for this extension, offering a scale-independent formulation of conservation laws in terms of possible and impossible tasks [161]. Within this framework, conserved quantities are directly connected to equivalence classes of transformations and, crucially, to distinguishability properties of physical variables. Specifically, the variable corresponding to a conserved quantity is itself a distinguishable variable [191]. Therefore, all the properties of the constructor theory of information apply to such a variable as well. This reveals a deep connection between thermodynamics and the theory of information: conservation laws constrain which information-processing tasks are possible. Formulating the General Witness Theorem in time would build on this connection by explaining how conservation laws can reveal non-classicality in physical systems.

Closely related to this perspective is the possibility of exploiting this connection not only for foundational purposes, but also for the development of novel witnesses of non-classicality based on macroscopic thermodynamic observables. Recent works have shown that quantities such as heat exchange or heat capacity can reveal non-classical features, including entanglement and coherence, without requiring detailed microscopic control of the system [192, 193]. Moreover, these ideas have been extended to the temporal domain, where thermodynamic quantities can also witness temporal quantum correlations [194]. Such thermodynamic witnesses are particularly appealing as they rely on macroscopic quantities routinely measured in fields such as chemistry and condensed matter physics. In this sense, they can provide a complementary route to the information-theoretical witnesses discussed in this Thesis, potentially enabling experimentally accessible tests of non-classicality in complex systems where thermodynamical quantities are simpler to measure on ancillary systems.

A second important direction is the unification of spatial and temporal witnesses into a single spatiotemporal framework. The results of this Thesis strongly indicate that spatial and temporal quantum correlations play a symmetric role in revealing non-classicality in quantum theory. Formulating a unified spatiotemporal witness would not only generalise the existing theorems, but also provide a new tool to investigate the structure of correlations in quantum theory through the effects they produce when probing the non-classical nature of physical systems. Since the two witnesses are based on bipartite spatial and temporal quantum correlations, such a unification may benefit from formalisms that treat space and time on an equal footing [195], such as pseudo-density operators, constructed from correlations between subsystems across both space and time [177, 190, 196–199], or formalisms that introduce a detector-timer pair that interacts with a system to recover the tensor product structure, used to describe spatially separated systems, in the context of timelike events [200].

Among these possible formalisms, process tensors and quantum combs represent a promising candidate, thanks to their close relation to the temporal witnesses developed in this Thesis [201–203]. A process tensor describes the multi-time dynamics of a system S coupled to an uncontrolled environment E as the multilinear map from sequences of control operations on S alone to its output states. Its memory is captured by the correlations of the associated generalised Choi state, a many-body state in which temporal correlations are mapped onto spatial ones [203, 204]. This is precisely the setting of our witnesses, with the probe Q playing the role of S and the unknown system M that of E: only operations and measurements on Q are allowed. In Theorems 2 and 3, Q is measured at t_0 and at $t > t_0$ only, so all the statistics entering the witnesses are determined by the two-step process tensor of Q (or the two-tooth quantum comb). This structure would become relevant for initially correlated Q and M, or for witnesses involving more than two times, both natural extensions of this work. Notice that the formalism is agnostic about the nature of E: classical and quantum environments alike yield valid process tensors, and memory alone does not imply a non-classical carrier. In our witnesses, the

link between the dynamics of Q and the non-classicality of M is provided by the conservation law, which constrains the global dynamics of Q and M but is not made explicit by the process tensor of Q. Reformulating the witnesses in this language would thus require identifying which process tensors can be generated by a classical mediator under a conserved quantity, a question related to the distinction between classical and quantum memory in non-Markovian processes [205, 206], and could clarify how the temporal entanglement, witnessed by the violation of a temporal Bell inequality relative to the two chosen observables, relates to the temporal correlations encoded in the process tensor.

Furthermore, with the formulation of the General Witness Theorem in Time, the spatiotemporal witness of non-classicality could also be extended beyond quantum theory, potentially offering new insights into the interplay between information, causality, and the structure of spacetime in more general physical theories, including superinformation theories.

A third line of research concerns the systematic study of temporal quantum correlations as a resource for quantum information processing. While spatial entanglement has been extensively characterised and exploited, temporal entanglement remains comparatively less understood. The parallels highlighted in this work suggest that temporal quantum correlations may provide alternative or complementary resources for tasks such as communication, computation, and metrology. In this context, it would be particularly interesting to compare temporal entanglement with other forms of quantum correlations in time, such as atemporality [199], both to broaden the landscape of temporal quantum correlations and to identify novel operational tasks. These could, in turn, be employed as witnessing tasks in next-generation witnesses of non-classicality, potentially offering simpler and more experimentally accessible implementations than those currently available. Developing a full resource theory of temporal entanglement, building on early insights by Brukner and collaborators [165], represents a promising avenue in this direction.

Finally, the witness framework developed here lends itself naturally to applications in experimentally relevant hybrid systems. In particular, temporal witnesses

offer an alternative route to test non-classicality using a single quantum probe, thereby avoiding the need to control multiple probes and suppress their direct interactions. This feature may enable the next generation of single-probe non-classicality witnesses, complementing existing two-probe proposals such as those based on spatial entanglement. The price to pay for this simplification is the need to independently validate the conservation laws underpinning the protocol. However, these assumptions are well supported by extensive experimental evidence and established theoretical frameworks.

In the biological context discussed in this Thesis, the next step towards a concrete experimental realisation consists in developing system-specific models tailored to the chosen light-harvesting complex, in order to identify the relevant timescales and operational regimes required for implementing the witnessing protocol. This can benefit from previous analysis of the nine-level and seven-level models [120–124], describing one monomer in the trimer of the FMO complex discussed in Chapter 3.3.4 and applied to LG tests in [138]. Specifically, these models should be extended to incorporate the quantum probe, which constitutes the distinguishing feature of the temporal witness of non-classicality compared to more traditional tests requiring direct measurements on the biological system. Beyond energy transfer in photosynthesis, this new methodology calls for a broader investigation of how temporal witnesses could be applied to other biological processes where non-trivial quantum effects have been conjectured to play a functional role. Similarly, extending temporal witnesses to the gravitational field may provide new experimental avenues to probe its non-classical nature in regimes where controlling multiple massive probes remains particularly challenging.

In conclusion, the results presented in this Thesis highlight the power of information-theoretical methods as a framework to probe the non-classical nature of physical systems, contributing to the long-standing debate about the universality of quantum theory. By extending the notion of witnesses of non-classicality to the temporal domain and to single-probe scenarios, this work provides a significant step forward in the development of general, operational tools characterised by their

reliance on minimal assumptions. These methods not only enable more robust and conclusive tests of non-classicality for otherwise inaccessible systems, but also open new avenues for investigating the foundations of quantum theory and, more broadly, of superinformation theories. In particular, the interplay between spatial and temporal correlations emerging in this framework suggests a deeper structure underlying the manifestation of non-classical features, positioning information-theoretical witnesses as a powerful paradigm for both experimental exploration and foundational research in physics.

Appendices

A

The General Witness Theorem: Proof Beyond Quantum Theory

The content of this Appendix is based on Ref. [46], done in collaboration with Chiara Marletto and Vlatko Vedral.

This Appendix includes a summary of the proof of the General Witness Theorem within the constructor theory of information, first introduced by Marletto and Vedral in Ref. [42]. As explained in Chapter 2 of the main text, constructor theory allows one to provide a general formulation of the two-probe, spatial entanglement-based witness of non-classicality that does not rely on any dynamical or scale assumptions about the hybrid system. In fact, by shifting the focus from initial conditions and dynamical laws to possible and impossible tasks, constructor theory of information unifies the description of classical, quantum and post-quantum systems within broader *superinformation* theories. This is crucial for witnessing non-classicality in a hybrid system: a robust proof of M's non-classicality must rely on nothing more than principles of physics that hold across all the currently known dynamical formalisms.

To begin, we briefly summarise the basics of constructor theory (CT), referring the reader to Ref. [40] for the formal details. Constructor theory of information supplements the traditional physics viewpoint, based on trajectories, dynamical

laws and initial conditions at a given scale, with statements about which *tasks* are *possible* or *impossible* on a given *substrate*, namely the physical system on which transformations specified by the task can be performed. These tasks are specified in terms of ordered pairs of input/output *attributes*. An attribute of a substrate is the set of all states where said substrate has a given property. For example, considering a spin-$\frac{1}{2}$ particle in quantum theory, an attribute is "the z component of the spin is $+\frac{1}{2}$", and the state belonging to this attribute is $\{|0\rangle\}$. A task is said to be *impossible* if the laws of physics impose a limit on how accurately it can be performed. Otherwise it is said to be *possible* and one can devise an arbitrarily good approximation to a substrate that never fails in mapping any of the input attributes of the task to the correct output attributes and, most importantly, is capable of performing the task again: the *constructor*.

A set of disjoint attributes of a substrate defines a *variable*. Whenever a substrate is in a state with attribute $\mathbf{v} \in V$, where V is a variable, we say that V is *sharp* with value v. Considering again the example of the spin-$\frac{1}{2}$ particle in quantum theory, $Z = \{|0\rangle, |1\rangle\}$ is a variable formed by two single-state attributes; if the state of the spin is $|0\rangle$, the variable Z is sharp with value $+\frac{1}{2}$. Among the variables, one defines an *information variable* X as a set of disjoint *attributes* $\{\mathbf{x}\}$ for which the following tasks are possible:

$$\bigcup_{x \in X} \{(\mathbf{x}, \mathbf{x_0}) \to (\mathbf{x}, \mathbf{x})\}, \tag{A.1}$$

$$\bigcup_{x \in X} \{\mathbf{x} \to \Pi(\mathbf{x})\}. \tag{A.2}$$

The *copying task* in Eq. A.1 corresponds to copying the attributes $\mathbf{x} \in X$ of the first replica of the system onto the second, target system prepared in a blank attribute $\mathbf{x_0} \in X$. The *permutation task* in Eq. A.2 describes the possibility of performing a logically reversible computation on the variable X, represented as a permutation Π on the set of labels of the attributes in X. Together, the two tasks in Eqs. A.1 and A.2 imply that it is possible to perform *information processing* on the variable X. An *information medium* is a system possessing at least one information variable. A *superinformation medium* is an information medium with at least two information

variables, whose union is not an information variable. This means that the two information variables are complementary to one another, and cannot be copied by the same device. In Ref. [40], it is shown that superinformation media share all the qualitative properties of quantum systems.

A variable X of a substrate is said to be *distinguishable* if the task:

$$\bigcup_{x\in X}\{(\mathbf{x})\rightarrow(\mathbf{q_x})\} \tag{A.3}$$

is possible, where $\mathbf{q_x}$ is an information variable. In other words, the task in Eq. A.3 implies that the attributes of X can be mapped onto those of an information variable[1]. If a variable $\{\mathbf{x_0},\mathbf{x_1}\}$ is distinguishable, then the attribute $\mathbf{x_0}$ is distinguishable from the attribute $\mathbf{x_1}$. Therefore, for any attribute $\mathbf{x}$, it is possible to define the attribute $\bar{\mathbf{x}}$ as the union of all attributes that are distinguishable from $\mathbf{x}$. This provides a generalisation of the notion of orthogonal complement in a vector space. Extending this definition to the variable $X=\bigcup_{x\in X}\mathbf{x}$, one gets $\bar{X}=\overline{\bigcup_{x\in X}\mathbf{x}}$; if $\bar{X}$ is empty, then X is a *maximal* variable.

In the same way, one can construct $\bar{\bar{\mathbf{x}}}$ and define an *observable* as an information variable whose attributes have the property that $\bar{\bar{\mathbf{x}}}=\mathbf{x}$. The idea of observable in constructor theory of information is that of a variable, say Z, with the property that whenever a measurer of Z produces a sharp output z, the input substrate indeed has that attribute z. In quantum theory, the possibility of preparing a system in an eigenstate of one of its variables qualifies that variable as an observable.

Within this framework, the idea of non-classicality that is relevant to the GWT, stated concisely as Definition 1 in Chapter 1, is formalised as follows. A system is said to be *non-classical* if it is an information medium M, with maximal information observable Z, that also possesses another variable V *disjoint* from Z and with the *same cardinality* as Z, satisfying the following properties:

[1]In particular, if the result of the distinction process can be stored in a second replica of the substrate in a way such that the original substrate is not affected by the transformation, one obtains the *perfect measurement* task. This generalises in purely constructor-theoretic terms the idea of measurements. Importantly, the interoperability principle (Principle 2) implies that this task is always possible for any information medium.

1. There exists a superinformation medium S_1 and a distinguishable variable $E = \{\mathbf{e_j}\}$ of the joint substrate $S_1 \oplus M$, whose attributes $\mathbf{e_j} = \{(s_j, v_j)\}$ are sets of ordered pairs of states, where v_j is a state belonging to some attribute in V and s_j is a state of $\mathbf{S_1}$;

2. The union of V with Z is *not* a distinguishable variable;

3. The task of distinguishing the variable $E = \{\mathbf{e_j}\}$ is possible by measuring incompatible observables of a *composite superinformation medium* including $\mathbf{S_1}$, but impossible by measuring observables of $\mathbf{S_1}$ only.

The above conditions express the fact that a non-classical system has a classical variable (the information variable), which is also an observable, together with another dynamical variable that is necessary to mediate the generation of entanglement. This additional variable may or may not be directly measurable: its attributes may or may not be all preparable or single-shot distinguishable.

The proof of the GWT proceeds as follows. First, one considers three systems: A and B (two quantum probes), and M, the mediator, which has a "classical" observable Z, and no other known degrees of freedom (see Fig. 2.1). In the case of gravity, the classical basis of the gravitational field would correspond to the number operator or its energy. Let the descriptors of A be denoted by $\mathbf{q_A(t_0)}$, those of B by $\mathbf{q_B(t_0)}$, and the mediator's descriptors by $\mathbf{c(t_0)}$ [64]. Let us assume that the observables (or measurable properties) of the three systems at time t, as well as their joint observables, are a *fixed function* of the triplet $(\mathbf{q_A(t_0)}, \mathbf{c(t_0)}, \mathbf{q_B(t_0)})$. In quantum theory, this is given by the trace with the density operator of the initial state, which never changes and therefore can be incorporated into the definition of the function.

Assume that initially (at time $t = t_0$) the three systems are *uncorrelated*. For instance, in quantum theory, we would say that they are in a product state where a local observable of each subsystem is sharp with some value. In particular, the mediator will have Z sharp with value say z_0. We further assume that it is possible to run the experiment with two distinguishable initial conditions; for

instance, the case in which the two masses are prepared in some state s_+, where the descriptors are labelled as $\mathbf{q}^{+}_{\mathbf{A},\mathbf{B}}(\mathbf{t_0})$, and in another distinguishable state s_-, with descriptors $\mathbf{q}^{-}_{\mathbf{A},\mathbf{B}}(\mathbf{t_0})$.

Suppose that at time t_2, in both cases, A and B end up being entangled. This means that, at time t_2, they have been prepared in one of two entangled states, denoted $\mathbf{e}_+$ and $\mathbf{e}_-$, which are themselves distinguishable by measuring observables of A and B only. Therefore, the following task must be possible:

$$T_E \coloneqq \left\{\left(\mathbf{q}^{\pm}_{\mathbf{A}}(\mathbf{t_0}), \mathbf{c}(\mathbf{t_0}), \mathbf{q}^{\pm}_{\mathbf{B}}(\mathbf{t_0})\right) \to \mathbf{e}_{\pm}\right\}, \tag{A.4}$$

and the goal of the experiments proposed in Refs. [47, 48] is to show that T_E is indeed possible, by successfully generating entanglement between A and B. The principle of locality (Principle 1) here can be used to conclude that in both configurations the descriptors of A, M and B at time t_2 form an ordered triplet:

$$\mathbf{e}_{\pm} \equiv \left(\mathbf{q}^{\pm}_{\mathbf{A}}(\mathbf{t_2}), \mathbf{c}^{\pm}(\mathbf{t_2}), \mathbf{q}^{\pm}_{\mathbf{B}}(\mathbf{t_2})\right). \tag{A.5}$$

Now, we can use the assumption that A and B are not interacting directly, but the interaction is mediated by M (Assumption III in Chapter 2.2.3). To a first approximation, this means that the interaction occurs in two steps: first A interacts with the mediator M at time t_1, and then M interacts with the qubit B at time t_2. In terms of tasks, T_E in Eq. A.4 can thus be decomposed as:

$$T^{(1)}_E \doteq \left\{\left(\mathbf{q}^{\pm}_{\mathbf{A}}(\mathbf{t_0}), \mathbf{c}(\mathbf{t_0}), \mathbf{q}^{\pm}_{\mathbf{B}}(\mathbf{t_0})\right) \to \mathbf{r}_{\pm}\right\}, \tag{A.6}$$

performed on $A \oplus M$ only, and:

$$T^{(2)}_E \doteq \left\{\mathbf{r}_{\pm} \to \mathbf{e}_{\pm}\right\}, \tag{A.7}$$

performed on $M \oplus B$ only. Once more using the principle of locality, the state of the three systems at time t_1 must be described by one of two triplets, depending on whether the initial condition s_+ or s_- was used:

$$\mathbf{r}_{\pm} \equiv \left(\mathbf{q}^{\pm}_{\mathbf{A}}(\mathbf{t_1}), \mathbf{c}^{\pm}(\mathbf{t_1}), \mathbf{q}^{\pm}_{\mathbf{B}}(\mathbf{t_0})\right). \tag{A.8}$$

Locality is here used because we have considered that the descriptor of system B must not have changed since t_0, given that the interaction at t_1 only involves A and M.

The proof proceeds by showing that the descriptors $\mathbf{c}^{\pm}(\mathbf{t_1})$ are: 1) disjoint (set-wise) from one another, and from the classical states of M; 2) not distinguishable from the classical states of M; 3) not distinguishable from one another, yet the joint state of A and M is (using measurements that involve both A and M). These three properties make M non-classical, in that it has a dynamical variable V, made of the two descriptors $\{\mathbf{c}^{\pm}(\mathbf{t_1})\}$ which is disjoint from the classical observable, and yet is not distinguishable from the latter, in analogy with the requirements of Heisenberg's uncertainty principle. We refer the interested reader to Ref. [42] for a detailed proof of the above three points.

B

Locally Mediated Entanglement Lower Bounds the Non-Classicality of the Mediator

The content of this Appendix is based on Ref. [45], done in collaboration with Salvatore Raia and Chiara Marletto.

B.1 Zassenhaus Factorisation of $\hat{\mathbf{U}}_{\mathbf{AMB}}(\mathbf{t})$

In this Section of the Appendix, we express the unitary $\hat{U}_{AMB}$ in Eq. 2.8 in terms of its local components $\hat{U}_{AM}$ and $\hat{U}_{MB}$ (Eq. 2.19) and its Zassenhaus expansion.

Following Chapter 2, we consider a closed hybrid system composed of two quantum probes, Q_A and Q_B, and a mediator M only. Each subsystem is described by a finite-dimensional Hilbert space, denoted $\mathcal{H}_A$, $\mathcal{H}_M$, and $\mathcal{H}_B$, with dimensions $d_A = \dim(\mathcal{H}_A)$, $d_M = \dim(\mathcal{H}_M)$, and $d_B = \dim(\mathcal{H}_B)$, respectively. To enforce the principle of locality (Principle 1) and therefore exclude direct interactions between Q_A and Q_B, we assume that the system Hamiltonian has the mediated form:

$$\hat{H} = \hat{H}_{AM} + \hat{H}_{MB}. \tag{B.1}$$

In particular, no direct Q_A–Q_B coupling term is present. The unitary evolution

associated with Eq. B.1 is:

$$\hat{U}_{AMB}(t) := e^{-i\hat{H}t} = e^{-i(\hat{H}_{AM}+\hat{H}_{MB})t}. \tag{B.2}$$

We also define the local evolutions:

$$\hat{U}_{AM}(t) := e^{-i\hat{H}_{AM}t}, \qquad \hat{U}_{MB}(t) := e^{-i\hat{H}_{MB}t}. \tag{B.3}$$

Using the Zassenhaus expansion, one can factorise $\hat{U}_{AMB}$ as:

$$\hat{U}_{AMB} = \hat{U}_{AM}\,\hat{U}_{MB}\,\prod_{n=2}^{\infty}\exp\Big[(-it)^n\hat{Z}_n\Big], \tag{B.4}$$

where the Zassenhaus exponents $\hat{Z}_n$ are uniquely determined by the pair $\{\hat{H}_{AM}, \hat{H}_{MB}\}$. More precisely, for each $n \geq 2$, $\hat{Z}_n$ is a homogeneous Lie polynomial of degree n in the two generators $\hat{H}_{AM}$ and $\hat{H}_{MB}$ [207], i.e., a finite linear combination (with rational coefficients) of nested commutators between $\hat{H}_{AM}$ and $\hat{H}_{MB}$:

$$[\hat{V}_1, [\hat{V}_2, \ldots, [\hat{V}_{n-1}, \hat{V}_n]\ldots]] : \hat{V}_i \in \{\hat{H}_{AM}, \hat{H}_{MB}\}. \tag{B.5}$$

In particular, the leading Zassenhaus exponent is:

$$\hat{Z}_2 = -\frac{1}{2}\,[\hat{H}_{AM}, \hat{H}_{MB}]. \tag{B.6}$$

The unitary $\hat{U}_C(t)$, introduced in the main text in Eq. 2.21, is defined by the Zassenhaus correction appearing in Eq. B.4. We define:

$$\hat{U}_C(t) := \prod_{n=2}^{\infty}\exp\Big[(-it)^n\hat{Z}_n\Big] = \hat{U}^{\dagger}_{MB}(t)\,\hat{U}^{\dagger}_{AM}(t)\,\hat{U}_{AMB}(t), \tag{B.7}$$

and also the corresponding Hamiltonian generator $\hat{H}_C$ implicitly via:

$$\hat{U}_C(t) = e^{-it\hat{H}_C}. \tag{B.8}$$

Moreover, if the interaction Hamiltonians commute, then every nested commutator built from $\{\hat{H}_{AM}, \hat{H}_{MB}\}$, as introduced in Eq. B.5, vanishes. Since each $\hat{Z}_n$ is a homogeneous Lie polynomial of degree $n \geq 2$ in these two generators, this implies:

$$[\hat{H}_{AM}, \hat{H}_{MB}] = 0 \implies \hat{Z}_n = 0 \quad \forall\, n \geq 2, \tag{B.9}$$

and therefore each factor in the Zassenhaus product is the identity,

$$\exp\Big[(-it)^n \hat{Z}_n\Big] = \mathbb{I} \quad \forall\, n \geq 2. \tag{B.10}$$

Hence, the entire Zassenhaus correction is trivial,

$$\hat{U}_C(t) = \mathbb{I}, \qquad \text{and consequently} \qquad \hat{H}_C = 0. \tag{B.11}$$

Independently, in the commuting case $[\hat{H}_{AM}, \hat{H}_{MB}] = 0$, the Baker–Campbell–Hausdorff formula yields the exact factorisation of the global unitary:

$$\hat{U}_{AMB}(t) = e^{-it(\hat{H}_{AM}+\hat{H}_{MB})} = e^{-it\hat{H}_{AM}}\, e^{-it\hat{H}_{MB}} = \hat{U}_{AM}(t)\, \hat{U}_{MB}(t). \tag{B.12}$$

By the definition in Eq. B.7, this exact factorisation implies $\hat{U}_C(t) = \mathbb{I}$, and hence $\hat{H}_C = 0$, in agreement with the Zassenhaus-based argument above.

We therefore summarise these implications as:

$$[\hat{H}_{AM}, \hat{H}_{MB}] = 0 \implies \hat{U}_C(t) = \mathbb{I} \implies \hat{H}_C = 0. \tag{B.13}$$

Equivalently, a non-trivial Zassenhaus correction implies non-commuting interactions:

$$\hat{U}_C(t) \neq \mathbb{I} \implies \hat{H}_C \neq 0 \implies [\hat{H}_{AM}, \hat{H}_{MB}] \neq 0. \tag{B.14}$$

B.2 Connecting Mediated Quantum Correlations to Non-Classicality of Interactions

We derive here the main result of Chapter 2.3.1 (mathematically expressed in Eq. 2.24) starting from the distance-based inequality obtained in [70]. The argument proceeds in two steps: first, we show that the quantity $g^{-1}\Big(\Delta G_{\text{corr}\,A:B}\Big)$ can be upper bounded by the spectral norm of a suitable difference of unitaries, $\|\hat{U}-\hat{V}\|_\infty$, where $\hat{U}$ is the unitary evolution generated by $\hat{H} = \hat{H}_{AM}+\hat{H}_{MB}$ while $\hat{V}$ is generated by $\hat{H}_{AM}$ and $\hat{H}_{MB}$ (see Eq. 2.8 and Eq. 2.19). Second, we rewrite this difference of unitaries in terms of the Zassenhaus correction $\hat{U}_C(t)$ and its generator $\hat{H}_C$, showing that it reduces to the spectral norm of $\hat{U}_C(t) - \mathbb{I}$, which admits a closed functional form.

We begin the derivation by recalling the distance-based inequality introduced in Ref. [70]:

$$g^{-1}\left(\Delta G_{\text{corr }A:B}\right) \leq d\Big(\widetilde{\Lambda}\Big(\rho_{AB}(0)\otimes\sigma_M\Big), \widetilde{\lambda}\Big(\rho_{AB}(0)\otimes\sigma_M\Big)\Big). \tag{B.15}$$

It holds for arbitrary choices of the dilations $\widetilde{\Lambda}, \widetilde{\lambda}$ of the marginal map Λ_{AB} and of a decomposable map $\lambda_{AB} \in \overline{DEC}(m)$, respectively. In particular, since the right-hand side provides an upper bound for any admissible pair of dilations, we may take a convenient choice for $\widetilde{\Lambda}$ and for a decomposable $\widetilde{\lambda}$. We take the unitary dilation induced by the actual evolution of the hybrid $Q_A \oplus M \oplus Q_B$, and a decomposable unitary dilation defined on $Q_A \oplus M$ and $M \oplus Q_B$, namely:

$$\begin{aligned}
\widetilde{\Lambda}(\rho) &:= \Lambda_{AMB}(\rho) = \hat{U}_{AMB}\,\rho\,\hat{U}^{\dagger}_{AMB}, \\
\widetilde{\lambda}(\rho) &:= \lambda_{AMB}(\rho) = \Big(\hat{U}_{AM}\hat{U}_{MB}\Big)\,\rho\,\Big(\hat{U}^{\dagger}_{MB}\hat{U}^{\dagger}_{AM}\Big),
\end{aligned} \tag{B.16}$$

where we keep the time argument t implicit to ease the notation. We take d to be the trace distance:

$$d_{\text{tr}}(X,Y) := \tfrac{1}{2}\|X-Y\|_1. \tag{B.17}$$

For compactness, we define:

$$\hat{U} := \hat{U}_{AMB}, \quad \hat{V} := \hat{U}_{AM}\hat{U}_{MB}, \quad \rho := \rho_{AB}(0)\otimes\sigma_M. \tag{B.18}$$

With these choices, Eq. B.15 becomes:

$$g^{-1}\Big(\Delta G_{\text{corr }A:B}\Big) \leq d_{\text{tr}}\Big(\Lambda_{AMB}(\rho),\, \lambda_{AMB}(\rho)\Big) = \frac{1}{2}\left\|\hat{U}\rho\hat{U}^{\dagger} - \hat{V}\rho\hat{V}^{\dagger}\right\|_1, \tag{B.19}$$

which we can rewrite as:

$$\begin{aligned}
\frac{1}{2}\left\|\hat{U}\rho\hat{U}^{\dagger} - \hat{V}\rho\hat{V}^{\dagger}\right\|_1 &= \frac{1}{2}\Big\|(\hat{U}-\hat{V})\rho\,\hat{U}^{\dagger} + \hat{V}\rho\,(\hat{U}^{\dagger}-\hat{V}^{\dagger})\Big\|_1 \\
&\leq \frac{1}{2}\Big(\|(\hat{U}-\hat{V})\,\rho\,\hat{U}^{\dagger}\|_1 + \|\hat{V}\rho\,(\hat{U}^{\dagger}-\hat{V}^{\dagger})\|_1\Big),
\end{aligned} \tag{B.20}$$

where we have added and subtracted $\hat{V}\rho\hat{U}^{\dagger}$ and used the triangle inequality for $\|\cdot\|_1$. Using the property of Schatten norms $\|AXB\|_1 \leq \|A\|_\infty\|X\|_1\|B\|_\infty$ [208, 209], we obtain:

$$\Big(\|(\hat{U}-\hat{V})\rho\hat{U}^{\dagger}\|_1 + \|\hat{V}\rho(\hat{U}^{\dagger}-\hat{V}^{\dagger})\|_1\Big) \leq \Big(\|\hat{U}-\hat{V}\|_\infty\|\rho\|_1\|\hat{U}^{\dagger}\|_\infty + \|\hat{V}\|_\infty\|\rho\|_1\|\hat{U}^{\dagger}-\hat{V}^{\dagger}\|_\infty\Big). \tag{B.21}$$

Finally, since $\|\rho\|_1 = 1$, $\hat{U}$ and $\hat{V}$ are unitary so that $\|\hat{U}\|_\infty = \|\hat{V}\|_\infty = \|\hat{U}^\dagger\|_\infty = 1$, and $\|\hat{U}^\dagger - \hat{V}^\dagger\|_\infty = \|\hat{U} - \hat{V}\|_\infty$, we conclude

$$g^{-1}\Big(\Delta G_{\text{corr }A:B}\Big) \leq \|\hat{U} - \hat{V}\|_\infty. \tag{B.22}$$

We now express the right-hand side of Eq. B.22 in terms of the Zassenhaus correction $\hat{U}_C$ introduced in Appendix B.1. Using Eqs. B.4 and B.7:

$$\hat{U}_{AMB} = \hat{U}_{AM}\,\hat{U}_{MB}\,\hat{U}_C. \tag{B.23}$$

Therefore, given the definitions of $\hat{U}$ and $\hat{V}$ in Eq. B.18 we obtain:

$$\begin{aligned}\|\hat{U} - \hat{V}\|_\infty &= \Big\|\hat{U}_{AMB} - \hat{U}_{AM}\hat{U}_{MB}\Big\|_\infty \\ &= \Big\|\hat{U}_{AM}\hat{U}_{MB}\Big(\hat{U}_C - \mathbb{I}\Big)\Big\|_\infty \leq \Big\|\hat{U}_{AM}\hat{U}_{MB}\Big\|_\infty \Big\|\hat{U}_C - \mathbb{I}\Big\|_\infty \\ &= \Big\|\hat{U}_C - \mathbb{I}\Big\|_\infty,\end{aligned} \tag{B.24}$$

where we used the sub-multiplicative property of $\|\cdot\|_\infty$ and the fact that products of unitaries have unit spectral norms. Hence, we obtain:

$$g^{-1}\Big(\Delta G_{\text{corr }A:B}\Big) \leq \Big\|\hat{U}_C - \mathbb{I}\Big\|_\infty. \tag{B.25}$$

To manipulate the right-hand side, we recall the spectral norm definition:

$$\|X\|_\infty := \sqrt{\lambda_{\max}\Big(X^\dagger X\Big)}. \tag{B.26}$$

Evaluating the square of Eq. B.26 with $X = \hat{U}_C - \mathbb{I}$ yields

$$\begin{aligned}\Big\|\hat{U}_C - \mathbb{I}\Big\|_\infty^2 &= \lambda_{\max}\Big(\Big(\hat{U}_C - \mathbb{I}\Big)^\dagger\Big(\hat{U}_C - \mathbb{I}\Big)\Big) = \lambda_{\max}\Big(\Big(e^{it\hat{H}_C} - \mathbb{I}\Big)\Big(e^{-it\hat{H}_C} - \mathbb{I}\Big)\Big) \\ &= \lambda_{\max}\Big(2\mathbb{I} - e^{it\hat{H}_C} - e^{-it\hat{H}_C}\Big) = \lambda_{\max}\Big(2\Big(\mathbb{I} - \cos(t\hat{H}_C)\Big)\Big),\end{aligned} \tag{B.27}$$

where we used the definition in Eq. B.8 for $\hat{U}_C$ and the operatorial identity $e^{i\hat{Y}} + e^{-i\hat{Y}} = 2\cos(\hat{Y})$ for a Hermitian operator $\hat{Y}$. Applying the trigonometric identity $1 - \cos(\hat{Y}) = 2\sin^2(\hat{Y}/2)$ and extracting square roots gives:

$$\Big\|\hat{U}_C - \mathbb{I}\Big\|_\infty = 2\,\Big\|\sin\Big(\frac{t}{2}\hat{H}_C\Big)\Big\|_\infty, \tag{B.28}$$

and therefore we find Eq. 2.24:

$$g^{-1}\Big(\Delta G_{\mathrm{corr}\,A:B}\Big) \leq 2\left\|\sin\Big(\frac{t}{2}\hat{H}_C\Big)\right\|_\infty. \tag{B.29}$$

The closed form in Eq. B.29 allows one to characterise the maximal value of the bound and the shortest interaction time for which it can be reached. Since $\|\sin(\hat{X})\|_\infty \leq 1$ for any Hermitian operator $\hat{X}$, the right-hand side of Eq. B.29 is bounded by 2. Let $\{\lambda_j\}$ denote the eigenvalues of $\hat{H}_C$; we have:

$$\left\|\sin\Big(\frac{t}{2}\hat{H}_C\Big)\right\|_\infty = \max_j\left|\sin\Big(\frac{t}{2}\lambda_j\Big)\right|. \tag{B.30}$$

Fix j with $\lambda_j \neq 0$. The modulus $\left|\sin(\frac{t}{2}\lambda_j)\right|$ attains its maximum value 1 if and only if:

$$\frac{t}{2}\lambda_j = \frac{\pi}{2} + k\pi, \qquad k \in \mathbb{Z}, \tag{B.31}$$

since $\sin(\frac{\pi}{2} + k\pi) = (-1)^k$. Accordingly, the set of interaction times for which the j-th eigenvalue saturates the modulus is:

$$t_{j,k} = \frac{\pi + 2k\pi}{\lambda_j}, \qquad k \in \mathbb{Z}, \tag{B.32}$$

and the corresponding *smallest positive* time is obtained by taking $k = 0$ and considering the modulus,

$$t_j^* = \frac{\pi}{|\lambda_j|}. \tag{B.33}$$

Therefore, the earliest time at which the operator norm can reach its maximum is:

$$t_* := \min_j t_j^* = \min_j \frac{\pi}{|\lambda_j|} = \frac{\pi}{\max_j |\lambda_j|}. \tag{B.34}$$

Let $\lambda_{\max}$ be an eigenvalue such that $|\lambda_{\max}| = \max_j |\lambda_j|$. Then, at $t = t_*$, we have

$$\left|\sin\left(\frac{t_*}{2}\lambda_{\max}\right)\right| = 1, \quad \text{and hence} \quad \left|\sin\left(\frac{t_*}{2}\hat{H}_C\right)\right|_\infty = 1. \tag{B.35}$$

In particular, for $\hat{H}_C \neq 0$, the right-hand side of Eq. B.29 attains its maximal value 2 at $t = t_*$.

Finally, although Eq. B.29 is exact for all interaction times t, it is useful to extract a more explicit commutator-based expression in the short-time regime.

In particular, truncating $\hat{U}_C$, as defined in Eq. B.7, to the leading non-trivial Zassenhaus term ($n = 2$), yields:

$$\hat{U}_C = \exp\Big[(-it)^2 \hat{Z}_2\Big]. \tag{B.36}$$

Using the explicit expression for $\hat{Z}_2$ given in Eq. B.6, we therefore obtain:

$$\hat{U}_C = \exp\Big(-it\,\hat{H}_C\Big) \approx \exp\Big(\tfrac{t^2}{2}\,[\hat{H}_{AM}, \hat{H}_{MB}]\Big), \tag{B.37}$$

which implies

$$\hat{H}_C = \frac{it}{2}\,[\hat{H}_{AM}, \hat{H}_{MB}]. \tag{B.38}$$

Substituting this approximation into Eq. B.29 gives the explicit commutator form

$$g^{-1}\Big(\Delta G_{\text{corr }A:B}\Big) \leq 2\left\|\sin\Big(\frac{it^2}{4}\,[\hat{H}_{AM}, \hat{H}_{MB}]\Big)\right\|_\infty, \tag{B.39}$$

which matches the short-time version Eq. 2.25.

B.3 Non-Commuting Variables are Necessary for Non-Classical Interactions

In this Section of the Appendix, we show how the commutator between the local interaction Hamiltonians $\hat{H}_{AM}, \hat{H}_{MB}$ depends on the non-classicality of the investigated system M. We recall that we define a system as *non-classical* when its description requires a minimum of two non-commuting observables, i.e., $\hat{m}_{j_1}, \hat{m}_{j_2}$ such that $[\hat{m}_{j_1}, \hat{m}_{j_2}] \neq 0,\ j_1 \neq j_2 \in \{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$.

We begin by writing the most general form of the local interaction Hamiltonians $\hat{H}_{AM}$ and $\hat{H}_{MB}$ used in the main text (Eqs. 2.31 and 2.32), when considering the two probes Q_A and Q_B as d_A, d_B-dimensional systems described by Hilbert spaces with $SU(N)$ operator algebra, and M is a chain of T interacting qubits:

$$\begin{aligned}
\hat{H}_{AM} &= \sum_{i,j_1,j_2,\ldots j_T} \alpha_{i,j_1,j_2,\ldots j_T}(\hat{a}_i \otimes \hat{m}_{j_1} \otimes \hat{m}_{j_2} \otimes \ldots \otimes \hat{m}_{j_T} \otimes \mathbb{I}_B) \\
\hat{H}_{MB} &= \sum_{l_1,l_2,\ldots,l_T,k} \beta_{l_1,l_2,\ldots,l_T,k}(\mathbb{I}_A \hat{\otimes} m_{l_1} \otimes \hat{m}_{l_2} \otimes \ldots \otimes \hat{m}_{l_T} \otimes \hat{b}_k).
\end{aligned} \tag{B.40}$$

We can now write the commutator:

$$[\hat{H}_{AM}, \hat{H}_{MB}] = \Bigg[\sum_{i,j_1,j_2,\ldots j_T} \alpha_{i,j_1,j_2,\ldots j_T} (\hat{a}_i \otimes \hat{m}_{j_1} \otimes \hat{m}_{j_2} \otimes \ldots \otimes \hat{m}_{j_T} \otimes \mathbb{I}_B), \sum_{l_1,l_2,\ldots,l_T,k} \beta_{l_1,l_2,\ldots,l_T,k} (\mathbb{I}_A \otimes \hat{m}_{l_1} \otimes \hat{m}_{l_2} \otimes \ldots \otimes \hat{m}_{l_T} \otimes \hat{b}_k) \Bigg]. \quad \text{(B.41)}$$

To simplify the notation, we factorise the sums and the coefficients out of the commutator by using its linearity:

$$[\hat{H}_{AM}, \hat{H}_{MB}] = \sum_{i,j_1,j_2,\ldots j_T,l_1,l_2,\ldots,l_T,k} \alpha_{i,j_1,j_2,\ldots j_T} \beta_{l_1,l_2,\ldots,l_T,k} \cdot \Bigg[\hat{a}_i \otimes \hat{m}_{j_1} \otimes \hat{m}_{j_2} \otimes \ldots \otimes \hat{m}_{j_T} \otimes \mathbb{I}_B, \mathbb{I}_A \otimes \hat{m}_{l_1} \otimes \hat{m}_{l_2} \otimes \ldots \otimes \hat{m}_{l_T} \otimes \hat{b}_k \Bigg]. \quad \text{(B.42)}$$

Using again the linearity of the commutator and distributivity of the tensor product, we can factorise out the probe operators as well:

$$[\hat{H}_{AM}, \hat{H}_{MB}] = \sum_{i,j_1,j_2,\ldots j_T,l_1,l_2,\ldots,l_T,k} \alpha_{i,j_1,j_2,\ldots j_T} \beta_{l_1,l_2,\ldots,l_T,k} \cdot \hat{a}_i \otimes [\hat{m}_{j_1} \otimes \hat{m}_{j_2} \otimes \ldots \otimes \hat{m}_{j_T}, \hat{m}_{l_1} \otimes \hat{m}_{l_2} \otimes \ldots \otimes \hat{m}_{l_T}] \otimes \hat{b}_k. \quad \text{(B.43)}$$

We can now focus on the commutator of the mediator observables. We use the following property of the commutator:

$$[\hat{A} \otimes \hat{B}, \hat{C} \otimes \hat{D}] = \hat{C}\hat{A} \otimes [\hat{B}, \hat{D}] + [\hat{A}, \hat{C}] \otimes \hat{B}\hat{D}, \quad \text{(B.44)}$$

where $\hat{A}, \hat{C} \in \mathcal{H}$, $\hat{B}, \hat{D} \in \mathcal{H}'$ with $\mathcal{H}$ and $\mathcal{H}'$ denoting two different Hilbert spaces. Setting $\hat{A} = \hat{m}_{j_1}$, $\hat{C} = \hat{m}_{l_1}$ and $\hat{B} = \hat{m}_{j_2} \otimes \ldots \otimes \hat{m}_{j_T}$, $\hat{D} = \hat{m}_{l_2} \otimes \ldots \otimes \hat{m}_{l_T}$, we obtain:

$$\begin{aligned} &[\hat{m}_{j_1} \otimes \hat{m}_{j_2} \otimes \ldots \otimes \hat{m}_{j_T}, \hat{m}_{l_1} \otimes \hat{m}_{l_2} \otimes \ldots \otimes \hat{m}_{l_T}] = \\ &= \hat{m}_{l_1}\hat{m}_{j_1} \otimes [\hat{m}_{j_2} \otimes \ldots \otimes \hat{m}_{j_T}, \hat{m}_{l_2} \otimes \ldots \otimes \hat{m}_{l_T}] + [\hat{m}_{j_1}, \hat{m}_{l_1}] \otimes \hat{m}_{j_2}\hat{m}_{l_2} \otimes \ldots \otimes \hat{m}_{j_T}\hat{m}_{l_T}. \end{aligned} \quad \text{(B.45)}$$

Now we can recursively apply the same property in Eq. B.44 to the commutator in the first term summand of the previous equation, coming to the final result for

the commutator of the Hamiltonians (Eq. 2.33 in the main text):

$$\begin{aligned}\left[\hat{H}_{AM}, \hat{H}_{MB}\right] = \sum_{i,j_1,j_2,\ldots j_T,l_1,l_2,\ldots,l_T,k} \alpha_{i,j_1,j_2,\ldots j_T}\beta_{l_1,l_2,\ldots,l_T,k}\ \cdot \\ \hat{a}_i \otimes \Big([\hat{m}_{j_1}, \hat{m}_{l_1}] \otimes \hat{m}_{j_2}\hat{m}_{l_2} \otimes \ldots \otimes \hat{m}_{j_T}\hat{m}_{l_T} \\ + \hat{m}_{l_1}\hat{m}_{j_1} \otimes [\hat{m}_{j_2}, \hat{m}_{l_2}] \otimes \ldots \otimes \hat{m}_{j_T}\hat{m}_{l_T} \\ + \hat{m}_{l_1}\hat{m}_{j_1} \otimes \hat{m}_{l_2}\hat{m}_{j_2} \otimes \ldots \otimes [\hat{m}_{j_T}, \hat{m}_{l_T}] \Big) \otimes \hat{b}_k. \end{aligned} \tag{B.46}$$

It is interesting to consider an example where Eq. B.46 takes an explicit form. Consider the case where mediator M consists of two qubits. The commutator reduces to:

$$\begin{aligned}[\hat{H}_{AM}, \hat{H}_{MB}] = \sum_{i,j_1,j_2,l_1,l_2,k} \alpha_{i,j_1,j_2}\beta_{l_1,l_2,k}\, \hat{a}_i \otimes \Big([\hat{m}_{j_1}, \hat{m}_{l_1}] \otimes \hat{m}_{j_2}\hat{m}_{l_2} \\ + \hat{m}_{l_1}\hat{m}_{j_1} \otimes [\hat{m}_{j_2}, \hat{m}_{l_2}] \Big) \otimes \hat{b}_k. \end{aligned} \tag{B.47}$$

To determine when the two-qubit interaction Hamiltonians commute, it is sufficient to analyse the mediator-dependent factor appearing in Eq. B.47:

$$[\hat{m}_{j_1}, \hat{m}_{l_1}] \otimes \hat{m}_{j_2}\hat{m}_{l_2} + \hat{m}_{l_1}\hat{m}_{j_1} \otimes [\hat{m}_{j_2}, \hat{m}_{l_2}]\,. \tag{B.48}$$

This is because the probe operators $\hat{a}_i$ and $\hat{b}_k$ factorise, so that $[\hat{H}_{AM}, \hat{H}_{MB}] \neq 0$ only if the factor in Eq. B.48 is non-vanishing. We therefore simplify the combination of commutators and products of the observables of the mediator into a more compact form:

$$\begin{aligned}&[\hat{m}_{j_1}, \hat{m}_{l_1}] \otimes \hat{m}_{j_2}\hat{m}_{l_2} + \hat{m}_{l_1}\hat{m}_{j_1} \otimes [\hat{m}_{j_2}, \hat{m}_{l_2}] \\ &= (\hat{m}_{j_1}\hat{m}_{l_1} - \hat{m}_{l_1}\hat{m}_{j_1}) \otimes \hat{m}_{j_2}\hat{m}_{l_2} + \hat{m}_{l_1}\hat{m}_{j_1} \otimes (\hat{m}_{j_2}\hat{m}_{l_2} - \hat{m}_{l_2}\hat{m}_{j_2}) \\ &= \hat{m}_{j_1}\hat{m}_{l_1} \otimes \hat{m}_{j_2}\hat{m}_{l_2} - \hat{m}_{l_1}\hat{m}_{j_1} \otimes \hat{m}_{j_2}\hat{m}_{l_2} + \hat{m}_{l_1}\hat{m}_{j_1} \otimes \hat{m}_{j_2}\hat{m}_{l_2} - \hat{m}_{l_1}\hat{m}_{j_1} \otimes \hat{m}_{l_2}\hat{m}_{j_2} \\ &= \hat{m}_{j_1}\hat{m}_{l_1} \otimes \hat{m}_{j_2}\hat{m}_{l_2} - \hat{m}_{l_1}\hat{m}_{j_1} \otimes \hat{m}_{l_2}\hat{m}_{j_2}. \end{aligned} \tag{B.49}$$

In the first two lines, we expand the commutators and distribute the tensor product over sums. After multiplying out, the second and third terms cancel. We then use the Pauli-algebra anti-commutation relations:

$$\hat{m}_{l_1}\hat{m}_{j_1} = 2\,\delta_{l_1 j_1}\,\mathbb{I} - \hat{m}_{j_1}\hat{m}_{l_1} \tag{B.50}$$

to get:

$$\begin{aligned} &= \hat{m}_{j_1}\hat{m}_{l_1} \otimes \hat{m}_{j_2}\hat{m}_{l_2} - (2\delta_{l_1 j_1}\mathbb{I} - \hat{m}_{j_1}\hat{m}_{l_1}) \otimes \hat{m}_{l_2}\hat{m}_{j_2} \\ &= \hat{m}_{j_1}\hat{m}_{l_1} \otimes (\hat{m}_{j_2}\hat{m}_{l_2} + \hat{m}_{l_2}\hat{m}_{j_2}) - 2\delta_{l_1 j_1}\mathbb{I} \otimes \hat{m}_{l_2}\hat{m}_{j_2}. \end{aligned} \tag{B.51}$$

Hence, in conclusion, we get:

$$[\hat{m}_{j_1}, \hat{m}_{l_1}] \otimes \hat{m}_{j_2}\hat{m}_{l_2} + \hat{m}_{l_1}\hat{m}_{j_1} \otimes [\hat{m}_{j_2}, \hat{m}_{l_2}] = \hat{m}_{j_1}\hat{m}_{l_1} \otimes \{\hat{m}_{j_2}, \hat{m}_{l_2}\} - 2\delta_{l_1 j_1}\mathbb{I} \otimes \hat{m}_{l_2}\hat{m}_{j_2}. \tag{B.52}$$

With Eq. (B.52) we can now characterise the conditions on the mediator's non-classicality to make the interactions classical, i.e., $[\hat{H}_{AM}, \hat{H}_{MB}] = 0$. This is conveniently done by distinguishing whether the first-qubit indices satisfy $l_1 \neq j_1$ or $l_1 = j_1$ (an equivalent split could be performed using the second mediator qubit). Starting from Eq. B.52, we distinguish two cases.

1. $l_1 \neq j_1$**.** This condition implies $\delta_{l_1 j_1} = 0$, leaving $\hat{m}_{j_1}\hat{m}_{l_1} \otimes \{\hat{m}_{j_2}, \hat{m}_{l_2}\}$ as the only term that may be non-vanishing in Eq. B.52. This is the case when:

$$\begin{cases} \hat{m}_{l_2} = \hat{m}_{j_2} = \mathbb{I} \\ \hat{m}_{l_2} = \hat{m}_{j_2} = \hat{\sigma}_x \\ \hat{m}_{l_2} = \hat{m}_{j_2} = \hat{\sigma}_y \\ \hat{m}_{l_2} = \hat{m}_{j_2} = \hat{\sigma}_z \end{cases} \quad \text{or} \quad \begin{cases} \hat{m}_{l_2} = \mathbb{I}, \ \hat{m}_{j_2} = \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z \\ \hat{m}_{l_2} = \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z, \ \hat{m}_{j_2} = \mathbb{I} \end{cases} \tag{B.53}$$

As a result, the second commutator appearing in Eq. B.48 vanishes, and this gives:

$$[\hat{m}_{j_1}, \hat{m}_{l_1}] \otimes \hat{m}_{j_2}\hat{m}_{l_2} + \hat{m}_{l_1}\hat{m}_{j_1} \otimes [\hat{m}_{j_2}, \hat{m}_{l_2}] = [\hat{m}_{j_1}, \hat{m}_{l_1}] \otimes \hat{m}_{j_2}\hat{m}_{l_2}. \tag{B.54}$$

Thus, in the case $l_1 \neq j_1$, the non-vanishing contribution to the mediator factor is due to the non-commutativity of the first-qubit operators, i.e., to $[\hat{m}_{j_1}, \hat{m}_{l_1}] \neq 0$.

2. $l_1 = j_1$**.** After some simple manipulations, Eq. B.48 reads:

$$[\hat{m}_{j_1}, \hat{m}_{l_1}] \otimes \hat{m}_{j_2}\hat{m}_{l_2} + \hat{m}_{l_1}\hat{m}_{j_1} \otimes [\hat{m}_{j_2}, \hat{m}_{l_2}] = \mathbb{I} \ \otimes [\hat{m}_{j_2}, \hat{m}_{l_2}]\,. \tag{B.55}$$

Therefore, in this second case, when $l_1 = j_1$, the mediator factor is non-vanishing only if the second-qubit contributes with non-commuting observables, i.e., only if $[\hat{m}_{j_2}, \hat{m}_{l_2}] \neq 0$.

In summary, in the two-qubit expansion in Eq. B.47, the commutator $[\hat{H}_{AM}, \hat{H}_{MB}]$ can be non-zero only if at least one pair of mediator Pauli operators, appearing in the two Hamiltonians, are non-commuting (either on the first or on the second mediator qubit). Therefore, the non-classicality of the mediator is necessary for the non-classicality of interactions.

C

Temporal Witness of Non-Classicality: Non-Additive Conservation Laws

The content of this Appendix is based on [85], done in collaboration with Chiara Marletto.

In this Appendix, we extend the temporal witness of non-classicality introduced in Theorem 2 to a more general scenario in which the global conserved quantity may not be necessarily additive, but rather a general function of the classical variables of the sectors in the hybrid $Q \oplus M$. This extension is important for applying Theorem 2 to probe non-classical features of the hybrid beyond quantum theory, where more general conservation laws may hold. Furthermore, it is important to prove that the assumption of *additive* conservation laws can already be relaxed within the quantum framework, as this helps to discriminate the role that each assumption in the Theorem plays for the conclusion of the witness. Specifically, this supports the important observation that *unitarity* is the most relevant property of quantum theory for the temporal witness of non-classicality within this formalism.

C.1 Proof of Theorem 2 with a general conservation law

Following Chapter 3, let Q be a qubit described by $\{\hat{X}_Q, \hat{Y}_Q, \hat{Z}_Q\} = \{\hat{\sigma}_x \otimes \mathbb{I}_M, \hat{\sigma}_y \otimes \mathbb{I}_M, \hat{\sigma}_z \otimes \mathbb{I}_M\}$ and let M be a classical system, namely a system described by a single variable $\hat{Z}_M$. Without loss of generality, we can assume M to be a bit, so that $\hat{Z}_M = \mathbb{I}_Q \otimes \hat{\sigma}_z$.

Let $F(\hat{Z}_Q, \hat{Z}_M)$ be a Hermitian observable on $Q \oplus M$ such that the dynamics of the composite system preserves $F(\hat{Z}_Q, \hat{Z}_M)$, namely

$$[\hat{U}_{QM}, F(\hat{Z}_Q, \hat{Z}_M)] = 0 \tag{C.1}$$

for all allowed unitary evolutions $\hat{U}_{QM}$ (equivalently, $[\hat{H}_{QM}, F(\hat{Z}_Q, \hat{Z}_M)] = 0$ at the level of the generator).

In the spectral decomposition of $\hat{Z}_M$, the operator F can be written as:

$$F(\hat{Z}_Q, \hat{Z}_M) = F_+(\hat{Z}_Q) \otimes \hat{P}_+ + F_-(\hat{Z}_Q) \otimes \hat{P}_-, \tag{C.2}$$

where $\hat{P}_\pm = \mathbb{I}_Q \otimes \frac{1}{2}(\mathbb{I}_M \pm \hat{\sigma}_z)$, and both $F_+(\hat{Z}_Q)$ and $F_-(\hat{Z}_Q)$ are functions of $\hat{Z}_Q$.

We now prove that, under the assumptions of Theorem 2 summarised in Table 3.1, any admissible Hamiltonian $\hat{H}_{QM}$ satisfying the *general* conservation law must commute with $\hat{Z}_Q$, i.e.,

$$[\hat{H}_{QM}, \hat{Z}_Q] = 0. \tag{C.3}$$

Consequently, no allowed evolution can map an eigenstate of $\hat{Z}_Q$ to an eigenstate of an observable not commuting with $\hat{Z}_Q$ (e.g. $\hat{X}_Q$). In particular, the witnessing task of generating $\hat{Z}_Q$-coherence on Q via interaction with M alone is impossible if M is classical, as mandated by the temporal witness of non-classicality. We proceed by characterising the most general form of the joint Hamiltonian compatible with the classicality of M, and then imposing the conservation law.

Since M is classical, its algebra of observables is generated by $\{\mathbb{I}_M, \hat{Z}_M\}$. Therefore, any Hermitian operator $\hat{O}$ on $Q \oplus M$ can be written as:

$$\hat{O} = \hat{O}_0 \otimes \mathbb{I}_M + \hat{O}_z \otimes \hat{Z}_M, \tag{C.4}$$

for some Hermitian operators $\hat{O}_0, \hat{O}_z$ acting on Q. This is true for the Hamiltonian of the hybrid $\hat{H}_{QM}$, too.

The condition $[\hat{H}_{QM}, F(\hat{Z}_Q, \hat{Z}_M)] = 0$ yields:

$$\begin{aligned}[\hat{H}_{QM}, F(\hat{Z}_Q, \hat{Z}_M)] &= [\hat{O}_0 \otimes \mathbb{I}_M + \hat{O}_z \otimes \hat{Z}_M,\ F_+(\hat{Z}_Q) \otimes \hat{P}_+ + F_-(\hat{Z}_Q) \otimes \hat{P}_-] \\ &= \Big([\hat{O}_0 + \hat{O}_z, F_+(\hat{Z}_Q)]\Big) \otimes \hat{P}_+ \ + \ \Big([\hat{O}_0 - \hat{O}_z, F_-(\hat{Z}_Q)]\Big) \otimes \hat{P}_-,\end{aligned} \tag{C.5}$$

where we have used $\hat{Z}_M \hat{P}_\pm = \pm \hat{P}_\pm$ and the linear independence of $\hat{P}_+$ and $\hat{P}_-$. Hence, $[\hat{H}_{QM}, F(\hat{Z}_Q, \hat{Z}_M)] = 0$ is equivalent to:

$$[\hat{O}_0 + \hat{O}_z, F_+(\hat{Z}_Q)] = 0, \qquad [\hat{O}_0 - \hat{O}_z, F_-(\hat{Z}_Q)] = 0. \tag{C.6}$$

Since Q is a qubit, any function of $\hat{Z}_Q$ has the same eigenspaces as $\hat{Z}_Q$. Therefore:

$$[\hat{O}_0 \pm \hat{O}_z, F_\pm(\hat{Z}_Q)] = 0 \quad \Longleftrightarrow \quad [\hat{O}_0 \pm \hat{O}_z, Z_Q] = 0 \tag{C.7}$$

Adding and subtracting these equations gives:

$$[\hat{O}_0, \hat{Z}_Q] = 0, \qquad [\hat{O}_z, \hat{Z}_Q] = 0. \tag{C.8}$$

It follows that:

$$[\hat{H}_{QM}, \hat{Z}_Q] = [\hat{O}_0, \hat{Z}_Q] \otimes \mathbb{I}_M + [\hat{O}_z, \hat{Z}_Q] \otimes \hat{Z}_M = 0, \tag{C.9}$$

so that $\hat{Z}_Q$ is itself conserved by the dynamics.

Since Z_Q is conserved, no unitary generated by $\hat{H}_{QM}$ can transform an eigenstate of $\hat{Z}_Q$ into an eigenstate of a non-commuting observable (such as $\hat{X}_Q$). Therefore, the witnessing task requiring the generation of quantum coherence in the $\hat{Z}_Q$ basis cannot be achieved through interaction with a classical system M even when considering a *general*, non-additive conservation law. This concludes the proof.

To illustrate the logic of Theorem 2 in this more general setting, we now consider for simplicity the conservation of the non-additive quantity $\hat{Z}_Q + \hat{Z}_M + \hat{Z}_Q \hat{Z}_M$. Under this constraint, the only allowed unitaries $\hat{U}_{QM}$ must satisfy:

$$[\hat{U}_{QM}, \hat{Z}_Q + \hat{Z}_M + \hat{Z}_Q \hat{Z}_M] = 0 \tag{C.10}$$

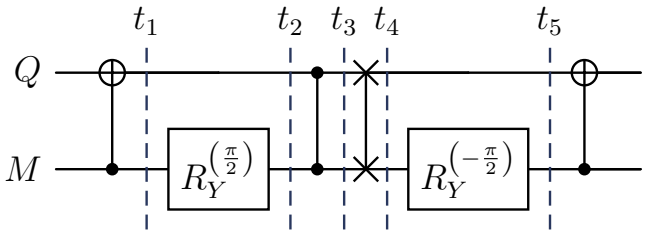


Figure C.1: A quantum circuit for the two-qubit proof-of-principle model of the temporal witness of non-classicality with a non-additive conservation law.

C.2 Proof-of-Principle Model with Two Qubits

We begin by discussing a sequence of two-qubit unitary gates that satisfy Eq. C.10 and can induce the witnessing task on Q through interaction with M only. This is a *proof-of-principle model* meant to illustrate the logic of the temporal witness, and it is important to note that the general argument supporting the witness did not assume any specific dynamics. To do so, we adopt the descriptors formalism in the Heisenberg picture [64].

Let $\hat{\vec{q}}^{(Q)} = \left(\hat{q}_x^{(Q)}, \hat{q}_y^{(Q)}, \hat{q}_z^{(Q)}\right) \coloneqq (\hat{\sigma}_x \otimes \mathbb{I}_M, \hat{\sigma}_y \otimes \mathbb{I}_M, \hat{\sigma}_z \otimes \mathbb{I}_M)$ be the vector of generators $\hat{q}_\alpha^{(Q)}$ of the algebra of observables of the qubits Q, where $\hat{\sigma}_\alpha$, $\alpha = x, y, z$, are the Pauli operators. Let $\hat{\vec{q}}^{(M)}$ be defined in a similar way. A quantum network that can perform the witnessing task is represented in Fig. C.1. First one applies a controlled-not (CNOT) gate using M as the control qubit. Then, one applies a rotation of an angle $\theta = \frac{\pi}{2}$ about the Y-axis. This is followed by a controlled-phase (CPHASE) gate, so that the qubits become entangled independently of the initial state of M. A SWAP gate is then applied. Finally, the same sequence of rotation (with angle $\theta' = -\frac{\pi}{2}$) and CNOT gates is applied to the system in order to disentangle the qubits. The gates have the following expressions:

$$\text{CNOT}_{Q,M}(t_i) = \frac{1}{2}\left(\mathbb{I} + \hat{q}_z^{(M)}(t_{i-1})\right) + \frac{1}{2}\left(\mathbb{I} - \hat{q}_z^{(M)}(t_{i-1})\right)\hat{q}_x^{(Q)}(t_{i-1}) \tag{C.11}$$

$$\text{CPHASE}_{Q,M}(t_i) = \frac{1}{2}\left(\mathbb{I} + \hat{q}_z^{(M)}(t_{i-1})\right) + \frac{1}{2}\left(\mathbb{I} - \hat{q}_z^{(M)}(t_{i-1})\right)\hat{q}_z^{(Q)}(t_{i-1}) \tag{C.12}$$

$$\text{R}^{(M)}_{Y_{(\pm\frac{\pi}{2})}}(t_i) = \frac{\sqrt{2}}{2}\left(\mathbb{I} \mp i\hat{q}_y^{(M)}(t_{i-1})\right) \tag{C.13}$$

	Qubit Q	Qubit M
t_0	$\left\{\hat{q}_x^{(Q)}, \hat{q}_y^{(Q)}, \hat{q}_z^{(Q)}\right\}$	$\left\{\hat{q}_x^{(M)}, \hat{q}_y^{(M)}, \hat{q}_z^{(M)}\right\}$
t_1	$\left\{\hat{q}_x^{(Q)}, \hat{q}_y^{(Q)}\hat{q}_z^{(M)}, \hat{q}_z^{(Q)}\hat{q}_z^{(M)}\right\}$	$\left\{\hat{q}_x^{(Q)}\hat{q}_x^{(M)}, \hat{q}_x^{(Q)}\hat{q}_y^{(M)}, \hat{q}_z^{(M)}\right\}$
t_2	$\left\{\hat{q}_x^{(Q)}, \hat{q}_y^{(Q)}\hat{q}_z^{(M)}, \hat{q}_z^{(Q)}\hat{q}_z^{(M)}\right\}$	$\left\{\hat{q}_z^{(M)}, \hat{q}_x^{(Q)}\hat{q}_y^{(M)}, -\hat{q}_x^{(Q)}\hat{q}_x^{(M)}\right\}$
t_3	$\left\{-\hat{q}_x^{(M)}, -\hat{q}_z^{(Q)}\hat{q}_y^{(M)}, \hat{q}_z^{(Q)}\hat{q}_z^{(M)}\right\}$	$\left\{\hat{q}_z^{(Q)}, \hat{q}_y^{(Q)}\hat{q}_x^{(M)}, -\hat{q}_x^{(Q)}\hat{q}_x^{(M)}\right\}$
t_4	$\left\{\hat{q}_z^{(Q)}, \hat{q}_y^{(Q)}\hat{q}_x^{(M)}, -\hat{q}_x^{(Q)}\hat{q}_x^{(M)}\right\}$	$\left\{-\hat{q}_x^{(M)}, -\hat{q}_z^{(Q)}\hat{q}_y^{(M)}, \hat{q}_z^{(Q)}\hat{q}_z^{(M)}\right\}$
t_5	$\left\{\hat{q}_z^{(Q)}, \hat{q}_y^{(Q)}\hat{q}_x^{(M)}, -\hat{q}_x^{(Q)}\hat{q}_x^{(M)}\right\}$	$\left\{-\hat{q}_z^{(Q)}\hat{q}_z^{(M)}, -\hat{q}_z^{(Q)}\hat{q}_y^{(M)}, -\hat{q}_x^{(M)}\right\}$
t_6	$\left\{\hat{q}_z^{(Q)}, -\hat{q}_y^{(Q)}, \hat{q}_x^{(Q)}\right\}$	$\left\{-\hat{q}_z^{(M)}, -\hat{q}_y^{(M)}, -\hat{q}_x^{(M)}\right\}$

Table C.1: Evolution of the system $Q \oplus M$ in the Heisenberg Picture. All the components are expressed as function of the descriptors at time t_0.

$$\text{SWAP}(t_i) = \frac{1}{2}\left(\mathbb{I} + \hat{q}_x^{(Q)}(t_{i-1})\hat{q}_x^{(M)}(t_{i-1}) + \hat{q}_y^{(Q)}(t_{i-1})\hat{q}_y^{(M)}(t_{i-1}) + \hat{q}_z^{(Q)}(t_{i-1})\hat{q}_z^{(M)}(t_{i-1})\right) \tag{C.14}$$

and the final unitary for the network is:

$$\hat{U}_{\text{net}} = \text{CNOT}_{Q,M} \cdot \text{R}^{(M)}_{Y_{(-\frac{\pi}{2})}} \cdot \text{SWAP} \cdot \text{CPHASE}_{Q,M} \cdot \text{R}^{(M)}_{Y_{(\frac{\pi}{2})}} \cdot \text{CNOT}_{Q,M}. \tag{C.15}$$

The evolution of the Heisenberg descriptors $\left\{\hat{q}_x^{(Q,M)}, \hat{q}_y^{(Q,M)}, \hat{q}_z^{(Q,M)}\right\}$ is reported in the Table C.1. In particular, note that the proposed unitary satisfies the condition in Eq. C.10, and the system Q cannot evolve without interacting with M. Moreover the Heisenberg picture allows us to see directly that the witnessing task is successfully performed independently of the initial state of the mediator M. Importantly, the unitary in Eq. C.15 involves three non-commuting degrees of freedom of the mediator M: $\hat{q}_x^{(M)}, \hat{q}_y^{(M)}, \hat{q}_z^{(M)}$. Hence, the witnessing task would have been impossible with a classical M.

C.3 Proof-of-Principle Model with a System of N Qubits

We now consider a more general case in which M consists of N qubits acting as mediators of the witnessing task on the system qubit Q. This can be achieved by exploiting the *quantum homogeniser*, a quantum device capable of "homogenising" the quantum state of the system with the state of the reservoir of N qubits to an arbitrary precision improving as N increases [151].

The initial state of the quantum probe Q is $\rho_0 = \frac{1}{2}(\mathbb{I} + \hat{q}_z^{(Q)})$, while each of the N qubits in the reservoir M is prepared in the state $\xi = \frac{1}{2}(\mathbb{I} + \hat{q}_x^{(M)})$, so that the global initial state is:

$$(\rho \otimes \xi^{\otimes N})_0 = \frac{1}{2}(\mathbb{I} + \hat{q}_z^{(Q)}) \otimes \left[\frac{1}{2}(\mathbb{I} + \hat{q}_x^{(M)})\right]^{\otimes N}. \tag{C.16}$$

The system qubit interacts with each qubit in the reservoir sequentially (one by one and only once) at each time step. The interaction is described by the unitary *partial* SWAP gate:

$$\text{PSWAP}(\eta) = \cos\eta\ \mathbb{I} + i \sin\eta\ \text{SWAP}, \tag{C.17}$$

where the SWAP gate is in Eq. C.14 and η represents the strength of the homogenisation. It can be shown that this is the *unique* unitary operation capable of performing the homogenisation task [151]. Notice moreover that PSWAP(η) satisfies the condition in Eq. C.10. After n interactions, the states of the qubit Q and of each reservoir qubit evolve as:

$$\rho^{(n)} = \cos^2\eta\rho^{(n-1)} + \sin^2\eta\xi + i\cos\eta\sin\eta\left[\xi, \rho^{(n-1)}\right] \tag{C.18}$$

$$\xi_n' = \cos^2\eta\xi + \sin^2\eta\rho^{(n-1)} - i\cos\eta\sin\eta\left[\xi, \rho^{(n-1)}\right] \tag{C.19}$$

which can be rewritten focusing on the terms proportional to ξ:

$$\rho^{(n)} = \sin^2\eta\sum_{k=0}^{n-1}\cos^{2k}\eta\xi + \rho_{rest}^{(n)} = \left(1 - \cos^{2n}\eta\right)\xi + \rho_{rest}^{(n)} \tag{C.20}$$

$$\xi_n' = \sin^2\eta\left(1 - \sin^2\eta\cos^{2(n-1)}\eta\right)\xi + \xi_{n,rest}. \tag{C.21}$$

where $\rho_{rest}^{(n)}$ and $\xi_{n,rest}$ denote the components of the density matrices of the system and reservoir qubit respectively that are not proportional to the initial state of the reservoir qubits ξ. Since the homogenisation is a *contractive* map [151], $\rho_{rest}^{(n)}$ converges monotonically to 0, implying that $\rho^{(N)} \to \xi$ with increasing accuracy as the number of qubits N in the reservoir grows. In this case, the non-classicality of the mediator is associated both with the interaction between M and Q (which involves all the non-commuting components of M) as well as with the state that M is initialised in. Therefore, the witnessing task would have been impossible with

a classical M or with a quantum M initialised in a classical state, namely a state such that $[\xi, \hat{Z}_Q + \hat{Z}_M + \hat{Z}_Q\hat{Z}_M] = 0$. This is consistent with the conclusions of the temporal entanglement-based witness of non-classicality in Chapter 4.

C.4 Proof-of-Principle Model with a Harmonic Oscillator

Finally, we provide a proof-of-principle model in which the mediator M is a harmonic oscillator whose creation and annihilation operators are denoted by $\hat{b}^\dagger$ and $\hat{b}$. We start from the unitary in Eq. C.15, introduced for the two-qubit scenario, and apply the *Holstein-Primakoff transformations* [210]:

$$\begin{cases} \hat{S}_z = \hbar\left(s - \hat{a}^\dagger\hat{a}\right) \\ \hat{S}^+ = \hbar\sqrt{2s}\sqrt{1 - \frac{\hat{a}^\dagger\hat{a}}{2s}}\,\hat{a} \\ \hat{S}^- = \hbar\sqrt{2s}\,\hat{a}^\dagger\sqrt{1 - \frac{\hat{a}^\dagger\hat{a}}{2s}} \end{cases} \tag{C.22}$$

where s is the particle's spin, $\hat{\vec{S}} = \left(\hat{S}_x, \hat{S}_y, \hat{S}_z\right)$ is the spin operator, and $\hat{S}^\pm = \hat{S}_x \pm i\hat{S}_y$ are the raising and lowering operators. We describe the qubit as a spin $s = \frac{1}{2}$ particle so that, reintroducing the usual descriptors $\hat{\vec{q}} = (\hat{q}_x, \hat{q}_y, \hat{q}_z)$ as above, we obtain (setting $\hbar = 1$ in what follows):

$$\begin{cases} \hat{q}_x = \sqrt{1 - \hat{a}^\dagger\hat{a}}\,\hat{a} + \hat{a}^\dagger\sqrt{1 - \hat{a}^\dagger\hat{a}} \\ \hat{q}_y = -\mathrm{i}\left(\sqrt{1 - \hat{a}^\dagger\hat{a}}\,\hat{a} - \hat{a}^\dagger\sqrt{1 - \hat{a}^\dagger\hat{a}}\right) \\ \hat{q}_z = 1 - 2\hat{a}^\dagger\hat{a}. \end{cases} \tag{C.23}$$

Applying this transformation to Eq. C.15, we find that the unitary for the proof-of-principle model is:

$$\begin{aligned} \hat{U} = \frac{1}{2}\Bigg[\left(-\frac{\sqrt{1 - \hat{b}^\dagger\hat{b}}\,\hat{b} + \hat{b}^\dagger\sqrt{1 - \hat{b}^\dagger\hat{b}}}{2} + \frac{1}{2} - \hat{b}^\dagger\hat{b}\right) \\ \cdot\left(\frac{\sqrt{1 - \hat{a}^\dagger\hat{a}}\,\hat{a} + \hat{a}^\dagger\sqrt{1 - \hat{a}^\dagger\hat{a}}}{2} + \frac{1}{2} - \hat{a}^\dagger\hat{a}\right)\Bigg] \end{aligned} \tag{C.24}$$

where we have neglected the constant terms. In Eq. C.24, the terms proportional to $\hat{a}^\dagger\hat{a}$ and $\hat{b}^\dagger\hat{b}$ describe the free evolution of the subsystems Q and M, respectively. The three interaction terms indicate that the harmonic oscillator acts as controller for

the spin-$\frac{1}{2}$ particle. Moreover, the non-commuting, incompatible degrees of freedom of the mediator are the number operator $\hat{b}^\dagger\hat{b}$ and the creation and annihilation operators $\hat{b}^\dagger$ and $\hat{b}$. These are involved in the interaction that makes the witnessing task possible on Q. This concludes the final generalisation to the continuous variable limit for the control system M. Independently of the dimensionality of the control system's Hilbert space, if it can induce a coherent rotation on the qubit, then it *must* be non-classical, provided that a global quantity on the system $Q \oplus M$ is conserved.

D

Non-Markovianity Characterisation and Population Dynamics in the Collision Model

The content of this Appendix is based on [87], done in collaboration with Simone Rijavec.

In this Appendix, we provide a characterisation of the non-Markovianity introduced in the collision model in Chapter 3.3.3, studying the effect of the parameter Ω on the degree of non-Markovianity and on the information loss on the chain. We then complete the study of the information transfer in the model by studying the population dynamics, described by the elements $|\rho_C^{1,1}(k)|$ and $|\rho_C^{3,3}(k)|$.

D.1 Non-Markovianity Characterisation and Information Loss

D.1.1 Characterisation of Non-Markovianity

We expect the parameter Ω to control the amount of information lost by the environment. When $\Omega = 0$, $\Delta_0^m(\rho_{CR})$ in Eq. 3.13 leaves ρ_{CR} unchanged, so that the global evolution of chain plus environment is unitary. When $\Omega = 1$, instead,

$\Delta_1^m(\rho_{CR})$ re-initialises the m-th reservoir qubit to the maximally mixed state and, if applied to all reservoir qubits, makes the environment Markovian. To quantify the effect of Ω on the degree of non-Markovianity, we use the measure proposed in Ref. [211]. Given two initial states of the system $\rho_1(0)$ and $\rho_2(0)$, the degree of non-Markovianity under the action of a dynamical map is measured as [211]:

$$\mathcal{N} = \max_{\{\rho_1(0),\rho_2(0)\}} \int_{\Sigma_+} \frac{\mathrm{d}}{\mathrm{d}t} D\left(\rho_1(t), \rho_2(t)\right) \mathrm{d}t, \tag{D.1}$$

where Σ_+ is the union of all subsets where $\frac{\mathrm{d}}{\mathrm{d}t} D\left(\rho_1(t), \rho_2(t)\right) > 0$ and

$$D\left(\rho_1(t), \rho_2(t)\right) = \frac{1}{2} ||\rho_1(t) - \rho_2(t)||_1, \tag{D.2}$$

is the trace distance between $\rho_1(t)$ and $\rho_2(t)$. Since in this work we consider a discrete process, we use a discretised version of Eq. D.1 [211]:

$$\mathcal{N} = \max_{\{\rho_{1,0},\rho_{2,0}\}} \sum_{k\in\Sigma_+} \left(D\left(\rho_{1,k}, \rho_{2,k}\right) - D\left(\rho_{1,k-1}, \rho_{2,k-1}\right)\right), \tag{D.3}$$

where the sum runs over all values of k such that $D\left(\rho_{1,k}, \rho_{2,k}\right) - D\left(\rho_{1,k-1}, \rho_{2,k-1}\right) > 0$, $\rho_{1,k}$ is the state of the system at the end of the k-th step obtained by tracing out the environment, and the maximisation is performed over all pairs of initial states $\{\rho_{1,0}, \rho_{2,0}\}$.

To illustrate the effect of the depolarising channel on the degree of non-Markovianity, we first restrict to a model with one system qubit and one reservoir qubit (1x1). We measure the non-Markovianity of this model by maximising Eq. D.3 over all the pairs of initial system states prepared in the following way:

$$\rho_{l,0} = \frac{1}{2} \begin{pmatrix} 1 + r_l \cos\left(\theta_l\right) & r_l \sin\left(\theta_l\right) e^{-i\Phi_l} \\ r_l \sin\left(\theta_l\right) e^{i\Phi_l} & 1 - r_l \cos\left(\theta_l\right) \end{pmatrix}, \tag{D.4}$$

with $r_l \in [0, 1]$, $\theta_l \in [0, \pi]$, and $\Phi_l \in [0, 2\pi]$, for $l = 1, 2$. The maximisation procedure becomes too complicated with a higher number of qubits. In that case, we use some of the insights on the states maximising Eq. D.3 drawn from the 1x1 case to choose a suitable pair of initial states.

In Fig. D.1, we plot the degree of non-Markovianity $\mathcal{N}$ induced by our protocol in two models with different numbers of qubits. The circles refer to a model with

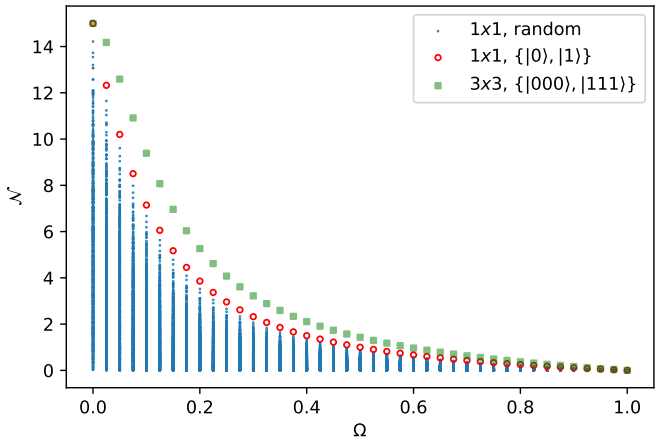


Figure D.1: Degree of non-Markovianity $\mathcal{N}$ as a function of Ω for two different models. $\mathcal{N}$ is calculated over 30 time steps with $\eta = \pi/2$. The blue circles correspond to values of $\mathcal{N}$ calculated with 1000 randomly generated pairs of initial system states in the 1x1 model. The red circles are the values of $\mathcal{N}$ obtained with the initial states $|0\rangle$ and $|1\rangle$. The green squares correspond to the values of $\mathcal{N}$ for the 3x3 model with initial states $|000\rangle$ and $|111\rangle$. Note that the values of $\mathcal{N}$ when $\Omega \approx 0$ are bounded by the finite number of time steps of our simulations rather than by the state of the system reaching a steady state. Increasing the number of time steps would increase these values.

one system qubit and one reservoir qubit (1x1), and the squares to the model with three system qubits and three reservoir qubits (3x3). We have calculated $\mathcal{N}$ over 30 temporal steps with $\eta = \pi/2$ for different values of Ω. The blue circles correspond to values of $\mathcal{N}$ for 1000 randomly generated pairs of initial states of the system in the 1x1 model. The red circles correspond to the initial states $|0\rangle$ and $|1\rangle$. The graph provides strong numerical evidence that $|0\rangle$ and $|1\rangle$ maximise $\mathcal{N}$ in Eq. D.3. During the course of the simulations, we noticed that any two antipodal states on the Bloch sphere would maximise $\mathcal{N}$. This observation, and the fact that the 3x3 model consists of identical copies of the 1x1 model, lead us to consider $|000\rangle$ and $|111\rangle$ as good candidates to maximise $\mathcal{N}$ in the 3x3 model. The green squares correspond to values of $\mathcal{N}$ for this pair of initial states and show a behaviour consistent with the 1x1 case. Due to the computational demand of simulating the 3x3 model with many different initial states, we consider only this pair of initial states. More extensive simulations are required to confirm that these initial states maximise $\mathcal{N}$ as the considerations made for the 1x1 model might not apply to the 3x3 model. Therefore,

the green squares can only be interpreted as a lower bound on the non-Markovianity of the 3x3 model, although we conjecture that they maximise it.

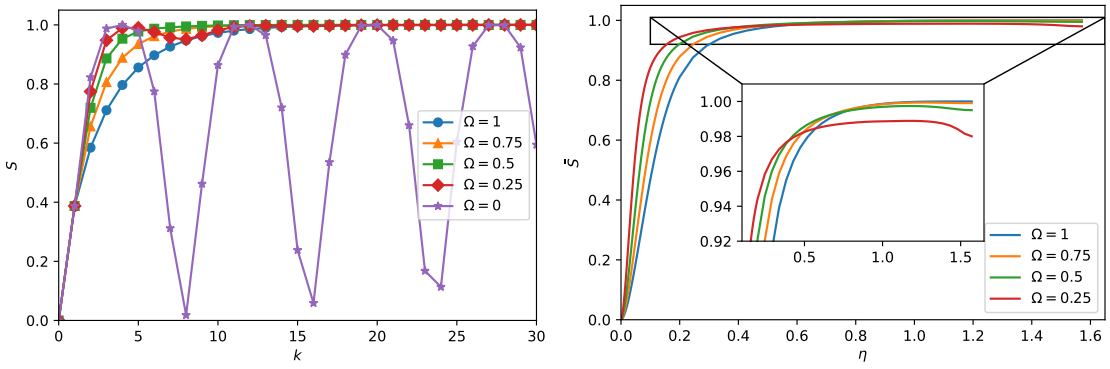
Fig. D.1 shows that the parameter Ω directly relates to the degree of non-Markovianity $\mathcal{N}$ of the models. $\mathcal{N}$ is maximal for $\Omega = 0$, that is when the evolution of system plus reservoir is unitary. $\mathcal{N}$ monotonically decreases with higher values of Ω and becomes null for $\Omega = 1$, i.e., when the reservoir is completely depolarised at each step. The results are similar for the 1x1 and 3x3 models and show that Ω is a good parameter to control the degree of non-Markovianity of the model.

D.1.2 Information Loss

Due to the interaction with the environment, the system qubits will lose information over time. We expect the information loss rate to be related to the degree of non-Markovianity. To quantify this effect, we study the behaviour of the von Neumann entropy of a system qubit in a 1x1 model. We initialise the qubit in the state $|1\rangle\langle 1|$, corresponding to a zero-entropy state (maximal information). In Fig. D.2a, we plot the von Neumann entropy $S(\rho_C(k))$ of the system qubit after the k-th step of the protocol for $\eta = 0.4$ and different values of Ω. We observe that the entropy tends to 1 (no information) unless $\Omega = 0$. If $\Omega = 0$, the system will lose some information to the environmental qubit (which is initialised in the maximally mixed state and thus has no initial information) in a periodic process, but the information will never be irretrievably lost due to the overall unitary dynamics of system plus environment. For any value of $\Omega > 0$, the qubit will eventually lose all its information. Moreover, we note that the entropy for higher values of Ω increases more slowly than for lower values of Ω, meaning that information loss is slower for lower degrees of non-Markovianity. Surprisingly, this suggests that Markovianity protects the qubit from information loss, despite being associated with more information loss from the environment.

To study how this effect depends on η, we consider the *time-average* entropy of the qubit, defined as:

$$\bar{S}(\eta, \Omega) = \frac{1}{K} \sum_{k=1}^{K} S(\rho_C(k)), \tag{D.5}$$



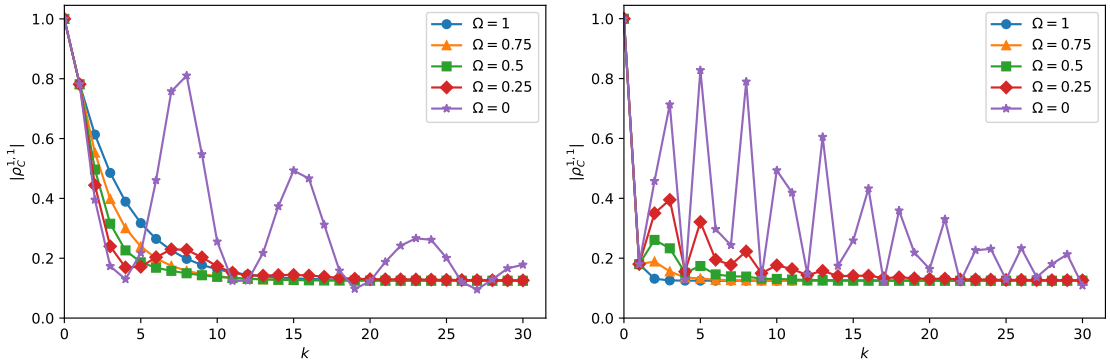
(a) **(b)**

Figure D.2: (a) Von Neumann entropy of the systems qubit $S(\rho_C(k))$ over multiple iterations of the protocol for $\eta = 0.4$ and different values of Ω. (b) Time-averaged entropy $\bar{S}$ as a function of η for different values of Ω. The average is taken over 30 steps.

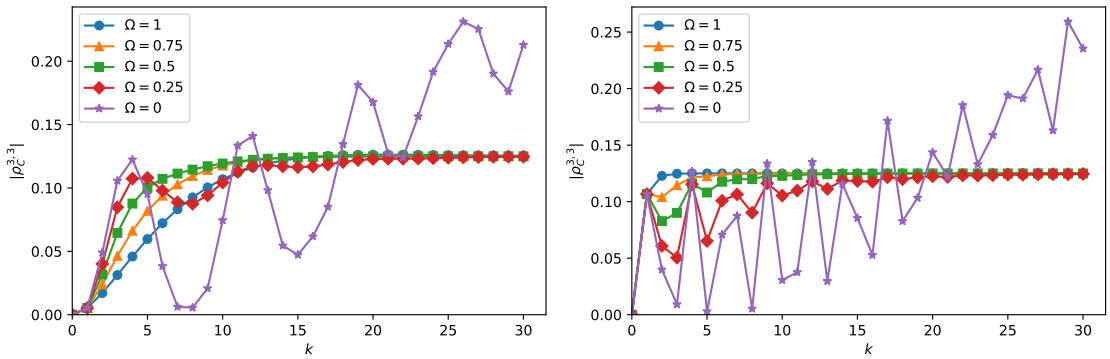
where the average is taken over the first K iterations of the protocol. We plot this quantity in Fig. D.2b for $K = 30$ and different values of Ω. We exclude $\Omega = 0$ as, in this case, the information is never entirely lost. Fig. D.2b shows that, for $0.25 \leq \Omega \leq 1$, lower degrees of non-Markovianity protect the qubit from information loss (lower values of $\bar{S}$) if system-environment interactions are weak (lower values of η). There is an inversion in this behaviour for higher values of η, although $\bar{S}$ is higher due to the qubit losing information to the environment more quickly in this regime. This protecting mechanism of Markovianity for small η is surprising given that Markovianity is associated with complete information loss from the environment. We therefore expect this effect to favour the coherent transport of information on a chain of qubits, as we have shown in Chapter 3.3.3.

D.2 Population Dynamics

To complete the study of the information dynamics in the collision model introduced in Chapter 3.3.3, in Fig. D.3 we plot $|\rho_C^{1,1}(k)|$ and $|\rho_C^{3,3}(k)|$ as a function of the number of iterations of the protocol k for different values of Ω, once again in the weak ($\eta = 0.4$, Figs. D.3a-D.3c) and strong ($\eta = 1.2$, Figs. D.3b-D.3d) coupling regimes identified in the main text, with $J_{chain} = 10$ and $J_{res} = 1$. We observe that the populations tend to 1/8 unless $\Omega = 0$, which corresponds to a fully unitary



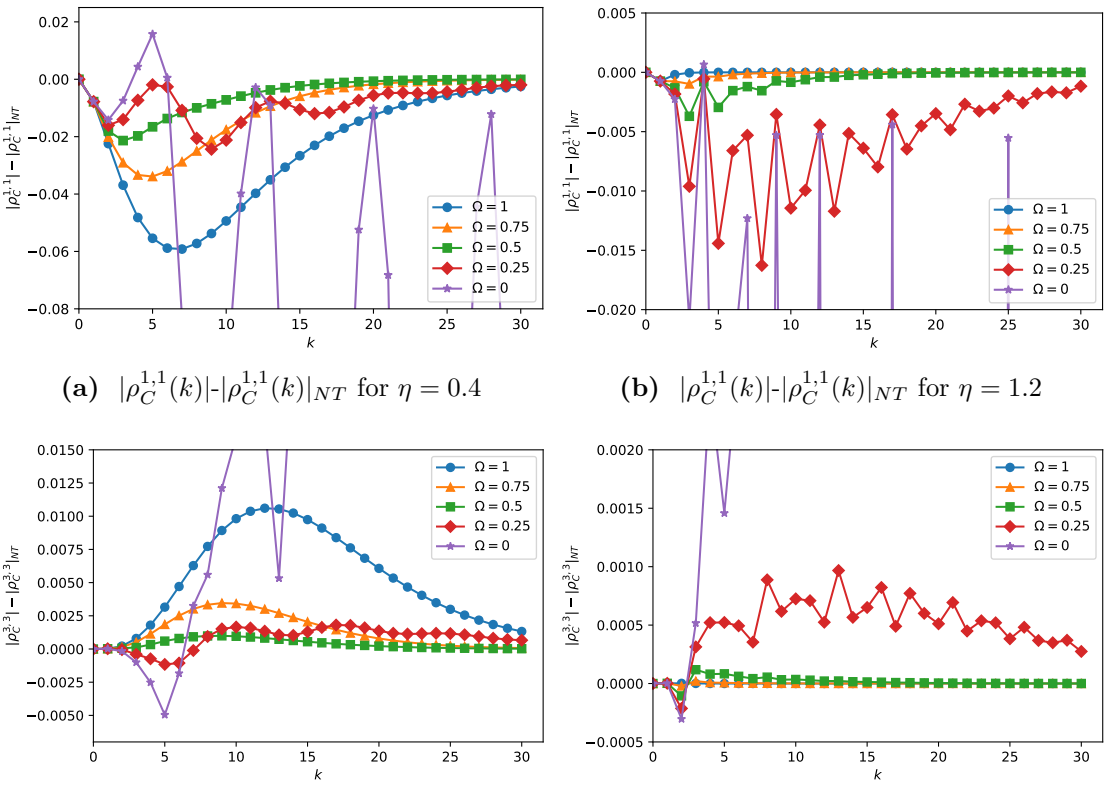
(a) Population of the first system qubit ($\eta = 0.4$) **(b)** Population of the first system qubit ($\eta = 1.2$)



(c) Population of the third system qubit ($\eta = 0.4$) **(d)** Population of the third system qubit ($\eta = 1.2$)

Figure D.3: Population elements $|\rho_C^{1,1}(k)|$ and $|\rho_C^{3,3}(k)|$ over multiple iterations of the protocol for different values of Ω.

process and therefore does not admit a steady state. The steady state for $\Omega \neq 0$ is the maximally mixed state $\mathbb{I}^{\otimes 3}/8$ determined by the repeated application of the depolarising channel. In our model, the populations for $\Omega \neq 0$ are not indicative of the location of the excitation. Instead, taken together with the coherences, they show that the information initially located on the first qubit is gradually lost. This is not a limiting aspect of our model; it simply reflects its focus on the *information* dynamics of the process. In fact, each reservoir qubit is initialised in a maximally mixed state, representing a qubit with no information. Because of the interaction with the chain, the system qubit can acquire a population equal to 1/8 even with no excitation transport occurring. To obtain specific information about the location of the excitation, non-Markovianity should be implemented with a different model, where the reservoir qubits are initialised in a state containing



(a) $|\rho_C^{1,1}(k)|$-$|\rho_C^{1,1}(k)|_{NT}$ for $\eta = 0.4$ **(b)** $|\rho_C^{1,1}(k)|$-$|\rho_C^{1,1}(k)|_{NT}$ for $\eta = 1.2$

(c) $|\rho_C^{3,3}(k)|$-$|\rho_C^{3,3}(k)|_{NT}$ for $\eta = 0.4$ **(d)** $|\rho_C^{3,3}(k)|$-$|\rho_C^{3,3}(k)|_{NT}$ for $\eta = 1.2$

Figure D.4: Difference between the populations $|\rho_C^{N,N}(k)|$ in the case $J_{chain} = 10$, $J_{res} = 1$ and the populations $|\rho_C^{N,N}(k)|_{NT}$ in the case $J_{chain} = J_{res} = 0$ (no transport). The difference is plotted for the first and third qubits of the chain over multiple iterations of the protocol, for $\eta = 0.4$ and $\eta = 1.2$, and with different values of Ω. The vertical axis has been truncated to improve the visibility of the graphs.

the information that the excitation is not located on it (see, for instance, [212]). Nevertheless, we expect that the excitation dynamics will also affect population changes during the initial iterations of the protocol.

To quantify this effect, in Fig. D.4 we plot the differences between the populations $|\rho_C^{1,1}(k)|$ and $|\rho_C^{3,3}(k)|$ of Fig. D.3 ($J_{chain} = 10$, $J_{res} = 1$), and the populations $|\rho_C^{1,1}(k)|_{NT}$ and $|\rho_C^{3,3}(k)|_{NT}$ in the case $J_{chain} = J_{res} = 0$, i.e., when there is no transfer of information along the chain or the reservoir and so the change in populations is entirely due to information loss. We interpret these population differences as an indicator of the population change caused solely by excitation transport, excluding the effects of information loss. We immediately note that—apart from the case $\Omega = 0$—the values of Fig. D.4 are much smaller than the

corresponding values in Fig. D.3, meaning that information loss is the dominant effect on population changes. Examining Fig. D.3 with this in mind, we observe that higher values of Ω correspond to a slower information loss in the weak coupling regime, signalled by a slower decay in the population of the first qubit (Fig. D.3a) and a faster increase in the population of the third qubit (Fig. D.3c). Conversely, we see that higher values of Ω determine a quicker information loss in the strong coupling regime (Figs. D.3b-D.3d). Thus, for $0.25 \leq \Omega \leq 1$, higher Markovianity mitigates information loss in the weak coupling regime and accelerates it in the strong coupling regime. This is also confirmed by the analysis of the information loss mechanism in this model, provided in Appendix D.1.

Let us now focus on the effect of the information transport on the population changes as quantified by Fig. D.4. In Figs. D.4a-D.4b, negative values indicate that the depopulation of the first qubit is quicker when there is transport on the chain and the reservoir ($J_{chain} \neq 0, J_{res} \neq 0$) compared to the case without transport ($J_{chain} = J_{res} = 0$). Similarly, positive values in Figs. D.4c-D.4d indicate that the transport contributes to the increase in the population. Importantly, the negative values are an effect of the initial state of the reservoir, which makes $|\rho_C^{3,3}(k)|_{NT} = \frac{1}{8}$ with a probability given by η, even if no transport occurs along the chain or along the reservoir. This choice follows the original motivation of the model: studying the coherent transport of information on the chain. Overall, Fig. D.4 points to an information transfer from the first to the third qubit before the information is lost in the long run (unless $\Omega = 0$ for which there is no information loss). Moreover, we note that, for $0.25 \leq \Omega \leq 1$, higher Markovianity generally enhances information transfer in the weak coupling regime and reduces it in the strong coupling regime. This is in line with the behaviour observed for the coherence of the transport (Fig. 3.5), meaning that this effect holds for both the amount of information transferred and its coherence.

References

[1] R. T. G., "Light waves and their uses", Nature **69**, 50–51 (1903).

[2] O. Passon, "Kelvin's clouds", American Journal of Physics **89**, 1037–1041 (2021).

[3] W. Heisenberg, *The physical principles of the quantum theory* (Courier Corporation, 1, 1949), 212 pp.

[4] N. Bohr, "The quantum postulate and the recent development of atomic theory", Nature **121**, 580–590 (1928).

[5] J. Von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, Heidelberg, 1996).

[6] W. Boucher and J. Traschen, "Semiclassical physics and quantum fluctuations", Physical Review D **37**, 3522–3532 (1988).

[7] D. R. Terno, "Classical-quantum hybrid models", in *Encyclopedia of mathematical physics (second edition)*, edited by R. Szabo and M. Bojowald, Second Edition (Academic Press, Oxford, 2025), pp. 57–72.

[8] A. Anderson, "Quantum backreaction on "classical" variables", Physical Review Letters **74**, 621–625 (1995).

[9] I. V. Aleksandrov, "The statistical dynamics of a system consisting of a classical and a quantum subsystem", Zeitschrift für Naturforschung A **36**, 902–908 (1981).

[10] V. I. Gerasimenko, "Dynamical equations of quantum-classical systems", Theoretical and Mathematical Physics **50**, 49–55 (1982).

[11] J. Caro and L. L. Salcedo, "Impediments to mixing classical and quantum dynamics", Physical Review A **60**, 842–852 (1999).

[12] Q. Zhang and B. Wu, "General approach to quantum-classical hybrid systems and geometric forces", Physical Review Letters **97**, 190401 (2006).

[13] L. L. Salcedo, "Statistical consistency of quantum-classical hybrids", Physical Review A **85**, 022127 (2012).

[14] V. Gil and L. L. Salcedo, "Canonical bracket in quantum-classical hybrid systems", Physical Review A **95**, 012137 (2017).

[15] M. Amin and M. A. Walton, "Quantum-classical dynamical brackets", Physical Review A **104**, 032216 (2021).

[16] B. O. Koopman, "Hamiltonian systems and transformation in hilbert space", Proceedings of the National Academy of Sciences **17**, 315–318 (1931).

[17] J. v. Neumann, "Zur operatorenmethode in der klassischen mechanik", Annals of Mathematics **33**, 587–642 (1932).

[18] J. V. Neumann, "Zusätze zur arbeit„zur operatorenmethode...", Annals of Mathematics **33**, 789–791 (1932).

[19] T. N. Sherry and E. C. G. Sudarshan, "Interaction between classical and quantum systems: a new approach to quantum measurement.i", Physical Review D **18**, 4580–4589 (1978).

[20] A. Peres and D. R. Terno, "Hybrid classical-quantum dynamics", Physical Review A **63**, 022101 (2001).

[21] D. R. Terno, "Inconsistency of quantum—classical dynamics, and what it implies", Foundations of Physics **36**, 102–111 (2006).

[22] A. Ahmadzadegan, R. B. Mann, and D. R. Terno, "Classicality of a quantum oscillator", Physical Review A **93**, 032122 (2016).

[23] F. Gay-Balmaz and C. Tronci, "Koopman wavefunctions and classical states in hybrid quantum–classical dynamics", Journal of Geometric Mechanics **14**, 559–596 (2022).

[24] L. Diósi, "Gravity-related spontaneous wave function collapse in bulk matter", New Journal of Physics **16**, 105006 (2014).

[25] J. Oppenheim, "A postquantum theory of classical gravity?", Physical Review X **13**, 041040 (2023).

[26] T. D. Galley, F. Giacomini, and J. H. Selby, "Any consistent coupling between classical gravity and quantum matter is fundamentally irreversible", Quantum **7**, 1142 (2023).

[27] L. Diósi and J. J. Halliwell, "Coupling classical and quantum variables using continuous quantum measurement theory", Physical Review Letters **81**, 2846–2849 (1998).

[28] L. Diósi, N. Gisin, and W. T. Strunz, "Quantum approach to coupling classical and quantum dynamics", Physical Review A **61**, 022108 (2000).

[29] W. H. Zurek, *Decoherence and the transition from quantum to classical – REVISITED*, arXiv:`quant-ph/0306072`.

[30] M. Schlossahuer, *Decoherence and the quantum-to-classical transition*, Frontiers Collection (Springer, Berlin, Heidelberg, 2007).

[31] L. Diósi, "Classical-quantum hybrid canonical dynamics and its difficulties with special and general relativity", Physical Review D **110**, 084052 (2024).

[32] J. Oppenheim, C. Sparaciari, B. Šoda, and Z. Weller-Davies, "Gravitationally induced decoherence vs space-time diffusion: testing the quantum nature of gravity", Nature Communications **14**, 7910 (2023).

[33] M. Hotta, S. Murk, and D. R. Terno, *Classical-quantum gravity as quantum gravity in disguise*, arXiv:`2506.15291`.

[34] L. Witten, *Gravitation: an introduction to current research* (Wiley, 1962), 504 pp.

[35] D. Deutsch, "Quantum theory as a universal physical theory", International Journal of Theoretical Physics **24**, 1–41 (1985).

[36] F. Dyson, "Is a graviton detectable?", International Journal of Modern Physics A **28**, 1330041 (2013).

[37] C. Marletto and V. Vedral, "Why we need to quantise everything, including gravity", npj Quantum Information **3**, Number: 1, 1–5 (2017).

[38] C. Marletto and V. Vedral, "Witnessing the quantumness of a system by observing only its classical features", npj Quantum Information **3**, 41 (2017).

[39] D. Deutsch, "Constructor theory", Synthese **190**, 4331–4359 (2013).

[40] D. Deutsch and C. Marletto, "Constructor theory of information", Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **471**, 20140540 (2015).

[41] C. Marletto and V. Vedral, "Quantum-information methods for quantum gravity laboratory-based tests", Reviews of Modern Physics **97**, 015006 (2025).

[42] C. Marletto and V. Vedral, "Witnessing nonclassicality beyond quantum theory", Physical Review D **102**, 086012 (2020).

[43] T. D. Galley, F. Giacomini, and J. H. Selby, "A no-go theorem on the nature of the gravitational field beyond quantum theory", Quantum **6**, 779 (2022).

[44] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, "Quantum entanglement", Reviews of Modern Physics **81**, 865–942 (2009).

[45] S. Raia, G. Di Pietra, and C. Marletto, "Role of nonclassicality in mediated spatial quantum correlations", Physical Review A **113**, 052218 (2026).

[46] G. Di Pietra, V. Vedral, and C. Marletto, "On the role of locality in the bose-marletto-vedral effect", in *Quantum gravity and computation: information, pregeometry, and digital physics*, edited by D. Rickles, X. D. Arsiwalla, and H. Elshatlawy, Num Pages: 14 (Routledge, 2025).

[47] S. Bose, A. Mazumdar, G. W. Morley, H. Ulbricht, M. Toroš, M. Paternostro, A. A. Geraci, P. F. Barker, M. S. Kim, and G. Milburn, "Spin entanglement witness for quantum gravity", Physical Review Letters **119**, 240401 (2017).

[48] C. Marletto and V. Vedral, "Gravitationally induced entanglement between two massive particles is sufficient evidence of quantum effects in gravity", Physical Review Letters **119**, 240402 (2017).

[49] D. Rickles and C. M. DeWitt, *The role of gravitation in physics: report from the 1957 chapel hill conference*, EOS – Sources (Max-Planck-Gesellschaft zur Förderung der Wissenschaften, Berlin, 14, 2011).

[50] C. Marletto and V. Vedral, "Witness gravity's quantum side in the lab", Nature **547**, 156–158 (2017).

[51] T. Krisnanda, C. Marletto, V. Vedral, M. Paternostro, and T. Paterek, "Probing quantum features of photosynthetic organisms", npj Quantum Information **4**, Number: 1, 1–7 (2018).

[52] A. Kent and D. Pitalúa-García, "Testing the nonclassicality of spacetime: what can we learn from bell–bose et al.-marletto-vedral experiments?", Physical Review D **104**, 126030 (2021).

[53] N. Tibau Vidal, "On the locality of indistinguishable quantum systems", DPhil Thesis (University of Oxford, 2023).

[54] R. W. Spekkens, "Evidence for the epistemic view of quantum states: a toy theory", Physical Review A **75**, 032110 (2007).

[55] P. Raymond-Robichaud, "A local-realistic model for quantum theory", Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **477**, 20200897 (2021).

[56] S. Goldstein, "Bohmian mechanics", in *The stanford encyclopedia of philosophy*, edited by E. N. Zalta and U. Nodelman, Summer 2024 (Metaphysics Research Lab, Stanford University, 2024).

[57] D. P. DiVincenzo, "Two-bit gates are universal for quantum computation", Physical Review A **51**, 1015–1022 (1995).

[58] D. E. Deutsch, A. Barenco, and A. Ekert, "Universality in quantum computation", Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences **449**, 669–677 (1997).

[59] A. Barenco, "A universal two-bit gate for quantum computation", Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences **449**, 679–683 (1997).

[60] E. Martín-Martínez and T. R. Perche, "What gravity mediated entanglement can really tell us about quantum gravity", Physical Review D **108**, L101702 (2023).

[61] A. Di Biagio, R. Howl, Č. Brukner, C. Rovelli, and M. Christodoulou, "Circuit locality from relativistic locality in scalar field mediated entanglement", Quantum **10**, 2046 (2026).

[62] R. Colella, A. W. Overhauser, and S. A. Werner, "Observation of gravitationally induced quantum interference", Physical Review Letters **34**, 1472–1474 (1975).

[63] D. Carney, Y. Chen, A. Geraci, H. Müller, C. D. Panda, P. C. E. Stamp, and J. M. Taylor, *Snowmass 2021 white paper: tabletop experiments for infrared quantum gravity*, arXiv:2203.11846.

[64] D. Deutsch and P. Hayden, "Information flow in entangled quantum systems", Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences **456**, 1759–1774 (2000).

[65] C. A. Bédard, "The ABC of deutsch–hayden descriptors", Quantum Reports **3**, Number: 2, 272–285 (2021).

[66] R. Horodecki, P. Horodecki, and M. Horodecki, "Violating bell inequality by mixed spin-1/2 states: necessary and sufficient condition", Physics Letters A **200**, 340–344 (1995).

[67] J. S. Bell, *Speakable and unspeakable in quantum mechanics: collected papers on quantum philosophy*, 2nd ed. (Cambridge University Press, Cambridge, 2004).

[68] B. S. Cirel'son, "Quantum generalizations of bell's inequality", Letters in Mathematical Physics **4**, 93–100 (1980).

[69] T. Krisnanda, R. Ganardi, S.-Y. Lee, J. Kim, and T. Paterek, "Detecting nondecomposability of time evolution via extreme gain of correlations", Physical Review A **98**, 052321 (2018).

[70] R. Ganardi, E. Panwar, M. Pandit, B. Woloncewicz, and T. Paterek, "Quantitative nonclassicality of mediated interactions", PRX Quantum **5**, 010318 (2024).

[71] M. J. Donald and M. Horodecki, "Continuity of relative entropy of entanglement", Physics Letters A **264**, 257–260 (1999).

[72] M. B. Plenio, "Logarithmic negativity: a full entanglement monotone that is not convex", Physical Review Letters **95**, 090503 (2005).

[73] T. Krisnanda, M. Zuppardo, M. Paternostro, and T. Paterek, "Revealing nonclassicality of inaccessible objects", Physical Review Letters **119**, 120402 (2017).

[74] S. Pal, P. Batra, T. Krisnanda, T. Paterek, and T. S. Mahesh, "Experimental localisation of quantum entanglement through monitored classical mediator", Quantum **5**, 478 (2021).

[75] M. J. W. Hall and M. Reginatto, "On two recent proposals for witnessing nonclassical gravity", Journal of Physics A: Mathematical and Theoretical **51**, 085303 (2018).

[76] J. Aziz and R. Howl, "Classical theories of gravity produce entanglement", Nature **646**, 813–817 (2025).

[77] E. Marconato and C. Marletto, "Vindication of entanglement-based witnesses of non-classicality in hybrid systems", Journal of Physics A: Mathematical and Theoretical **54**, 375301 (2021).

[78] C. Marletto and V. Vedral, *Classical gravity cannot mediate entanglement by local means*, arXiv:2510.19969.

[79] C. Marletto, J. Oppenheim, V. Vedral, and E. Wilson, *Classical gravity cannot mediate entanglement*, arXiv:2511.07348.

[80] J. A. Wheeler and R. P. Feynman, "Classical electrodynamics in terms of direct interparticle action", Reviews of Modern Physics **21**, 425–433 (1949).

[81] F. Hoyle and J. V. Narlikar, "A new theory of gravitation", Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences **282**, 191–207 (1964).

[82] C. Marletto and V. Vedral, *Can entanglement be mediated by a koopmanian system?*, arXiv:2507.11713.

[83] G. Di Pietra, F. Piacentini, E. Bernardi, E. Moreva, C. Napoli, I. P. Degiovanni, M. Genovese, V. Vedral, and C. Marletto, *The bose-marletto-vedral experiment with nanodiamond interferometers: an insight on entanglement detection*, arXiv:2410.19601.

[84] M. Vicentini, M. Bordin, E. Bernardi, E. Moreva, F. Piacentini, C. Napoli, I. P. Degiovanni, A. Manzin, and M. Genovese, *Table-top nanodiamond interferometer enabling quantum gravity tests*, arXiv:2405.21029.

[85] G. Di Pietra and C. Marletto, "Temporal witnesses of non-classicality and conservation laws", Journal of Physics A: Mathematical and Theoretical **56**, 265305 (2023).

[86] G. Di Pietra, V. Vedral, and C. Marletto, "Temporal witnesses of non-classicality in a macroscopic biological system", Scientific Reports **14**, 20094 (2024).

[87] S. Rijavec and G. Di Pietra, "Tunable non-markovian dynamics in a collision model: an application to coherent transport", New Journal of Physics **27**, 043003 (2025).

[88] D. N. Page and C. D. Geilker, "Indirect evidence for quantum gravity", Physical Review Letters **47**, 979–982 (1981).

[89] R. Howl, V. Vedral, D. Naik, M. Christodoulou, C. Rovelli, and A. Iyer, "Non-gaussianity as a signature of a quantum theory of gravity", PRX Quantum **2**, 010325 (2021).

[90] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, "Gaussian quantum information", Reviews of Modern Physics **84**, 621–669 (2012).

[91] C. Marletto and V. Vedral, "The quantum totalitarian property and exact symmetries", AVS Quantum Science **4**, 015603 (2022).

[92] M. O. Terra Cunha, J. A. Dunningham, and V. Vedral, "Entanglement in single-particle systems", Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **463**, 2277–2286 (2007).

[93] T. Feng, C. Marletto, and V. Vedral, *Conservation laws and the non-classicality of gravity*, arXiv:2311.08971.

[94] N. Lambert, Y.-N. Chen, Y.-C. Cheng, C.-M. Li, G.-Y. Chen, and F. Nori, "Quantum biology", Nature Physics **9**, 10–18 (2013).

[95] P. Jordan, "Die Quantenmechanik und die Grundprobleme der Biologie und Psychologie", Naturwissenschaften **20**, 815–821 (1932).

[96] E. Schrödinger, *What is life?: with mind and matter and autobiographical sketches*, Canto (Cambridge University Press, Cambridge, 1992).

[97] J. McFadden and J. Al-Khalili, "The origins of quantum biology", Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **474**, 20180674 (2018).

[98] A. Gelzinis, R. Augulis, V. Butkus, B. Robert, and L. Valkunas, "Two-dimensional spectroscopy for non-specialists", Biochimica et Biophysica Acta (BBA) - Bioenergetics **1860**, 271–285 (2019).

[99] M. Mohseni, Y. Omar, G. S. Engel, and M. B. Plenio, eds., *Quantum effects in biology* (Cambridge University Press, Cambridge, 2014).

[100] S. Huelga and M. Plenio, "Vibrations, quanta and biology", Contemporary Physics **54**, 181–207 (2013).

[101] A. Marais, B. Adams, A. K. Ringsmuth, M. Ferretti, J. M. Gruber, R. Hendrikx, M. Schuld, S. L. Smith, I. Sinayskiy, T. P. J. Krüger, F. Petruccione, and R. van Grondelle, "The future of quantum biology", Journal of The Royal Society Interface **15**, 20180640 (2018).

[102] Y. Kim, F. Bertagna, E. M. D'Souza, D. J. Heyes, L. O. Johannissen, E. T. Nery, A. Pantelias, A. Sanchez-Pedreño Jimenez, L. Slocombe, M. G. Spencer, J. Al-Khalili, G. S. Engel, S. Hay, S. M. Hingley-Wilson, K. Jeevaratnam, A. R. Jones, D. R. Kattnig, R. Lewis, M. Sacchi, N. S. Scrutton, S. R. P. Silva, and J. McFadden, "Quantum biology: an update and perspective", Quantum Reports **3**, Number: 1, 80–126 (2021).

[103] R. Blankenship, *Molecular mechanisms of photosynthesis* (Wiley, 2021).

[104] R. J. Cogdell, A. T. Gardiner, H. Hashimoto, and T. H. P. Brotosudarmo, "A comparative look at the first few milliseconds of the light reactions of photosynthesis", Photochemical & Photobiological Sciences: Official Journal of the European Photochemistry Association and the European Society for Photobiology **7**, 1150–1158 (2008).

[105] R. Croce and H. van Amerongen, "Light harvesting in oxygenic photosynthesis: structural biology meets spectroscopy", Science **369**, eaay2058 (2020).

[106] A. Klinger, D. Lindorfer, F. Müh, and T. Renger, "Living on the edge: light-harvesting efficiency and photoprotection in the core of green sulfur bacteria", Physical Chemistry Chemical Physics **25**, 18698–18710.

[107] R. J. Cogdell, N. W. Isaacs, T. D. Howard, K. McLuskey, N. J. Fraser, and S. M. Prince, "How photosynthetic bacteria harvest solar energy", Journal of Bacteriology **181**, 3869–3879 (1999).

[108] R.-H. Chen, J. Dong, W. Yang, Q. Ai, and G.-L. Long, *Quantum biology, quantum simulation and quantum coherent devices*, arXiv:2511.14363.

[109] T. Forster, "Energiewanderung und Fluoreszenz", Naturwissenschaften **33**, 166–175 (1946).

[110] T. Förster, "Transfer mechanisms of electronic excitation energy", Radiation Research Supplement **2**, 326–339 (1960).

[111] A. G. Redfield, "The theory of relaxation processes", in *Advances in magnetic and optical resonance*, Vol. 1 (Academic Press, 1, 1965), pp. 1–32.

[112] M. Yang and G. R. Fleming, "Influence of phonons on exciton transfer dynamics: comparison of the redfield, förster, and modified redfield equations", Chemical Physics **282**, 163–180 (2002).

[113] A. Chenu and G. D. Scholes, "Coherence in energy transfer and photosynthesis", Annual Review of Physical Chemistry **66**, 69–96 (2015).

[114] J. Seibt and T. Mančal, "Ultrafast energy transfer with competing channels: non-equilibrium förster and modified redfield theories", The Journal of Chemical Physics **146**, 10.1063/1.4981523 (2017).

[115] A. Trushechkin, "Calculation of coherences in förster and modified redfield theories of excitation energy transfer", The Journal of Chemical Physics **151**, 074101 (2019).

[116] G. S. Engel, T. R. Calhoun, E. L. Read, T.-K. Ahn, T. Mančal, Y.-C. Cheng, R. E. Blankenship, and G. R. Fleming, "Evidence for wavelike energy transfer through quantum coherence in photosynthetic systems", Nature **446**, Number: 7137, 782–786 (2007).

[117] H. Lee, Y.-C. Cheng, and G. R. Fleming, "Coherence dynamics in photosynthesis: protein protection of excitonic coherence", Science **316**, 1462–1465 (2007).

[118] E. Collini, C. Y. Wong, K. E. Wilk, P. M. G. Curmi, P. Brumer, and G. D. Scholes, "Coherently wired light-harvesting in photosynthetic marine algae at ambient temperature", Nature **463**, 644–647 (2010).

[119] G. Panitchayangkoon, D. Hayes, K. A. Fransted, J. R. Caram, E. Harel, J. Wen, R. E. Blankenship, and G. S. Engel, "Long-lived quantum coherence in photosynthetic complexes at physiological temperature", Proceedings of the National Academy of Sciences **107**, 12766–12770 (2010).

[120] A. Olaya-Castro, C. F. Lee, F. F. Olsen, and N. F. Johnson, "Efficiency of energy transfer in a light-harvesting system under quantum coherence", Physical Review B **78**, 085115 (2008).

[121] M. Mohseni, P. Rebentrost, S. Lloyd, and A. Aspuru-Guzik, "Environment-assisted quantum walks in photosynthetic energy transfer", The Journal of Chemical Physics **129**, 174106 (2008).

[122] M. B. Plenio and S. F. Huelga, "Dephasing-assisted transport: quantum networks and biomolecules", New Journal of Physics **10**, 113019 (2008).

[123] F. Caruso, A. W. Chin, A. Datta, S. F. Huelga, and M. B. Plenio, "Highly efficient energy excitation transfer in light-harvesting complexes: the fundamental role of noise-assisted transport", The Journal of Chemical Physics **131**, 105106 (2009).

[124] P. Rebentrost, M. Mohseni, I. Kassal, S. Lloyd, and A. Aspuru-Guzik, "Environment-assisted quantum transport", New Journal of Physics **11**, 033003 (2009).

[125] Y. Tanimura, "Numerically "exact" approach to open quantum dynamics: the hierarchical equations of motion (HEOM)", The Journal of Chemical Physics **153**, 020901 (2020).

[126] P. Rebentrost, M. Mohseni, and A. Aspuru-Guzik, "Role of quantum coherence and environmental fluctuations in chromophoric energy transport", The Journal of Physical Chemistry B **113**, 9942–9947 (2009).

[127] M.-J. Tao, N.-N. Zhang, P.-Y. Wen, F.-G. Deng, Q. Ai, and G.-L. Long, "Coherent and incoherent theories for photosynthetic energy transfer", Science Bulletin **65**, 318–328 (2020).

[128] V. Butkus, D. Zigmantas, L. Valkunas, and D. Abramavicius, "Vibrational vs. electronic coherences in 2d spectrum of molecular systems", Chemical Physics Letters **545**, 40–43 (2012).

[129] V. Perlík, C. Lincoln, F. Šanda, and J. Hauer, "Distinguishing electronic and vibronic coherence in 2d spectra by their temperature dependence", The Journal of Physical Chemistry Letters **5**, 404–407 (2014).

[130] R. Tempelaar, T. L. C. Jansen, and J. Knoester, "Vibrational beatings conceal evidence of electronic coherence in the FMO light-harvesting complex", The Journal of Physical Chemistry B **118**, 12865–12872 (2014).

[131] A. Halpin, P. J. M. Johnson, R. Tempelaar, R. S. Murphy, J. Knoester, T. L. C. Jansen, and R. J. D. Miller, "Two-dimensional spectroscopy of a molecular dimer unveils the effects of vibronic coupling on exciton coherences", Nature Chemistry **6**, Number: 3, 196–201 (2014).

[132] H.-G. Duan, P. Nalbach, V. I. Prokhorenko, S. Mukamel, and M. Thorwart, "On the origin of oscillations in two-dimensional spectra of excitonically-coupled molecular systems", New Journal of Physics **17**, 072002 (2015).

[133] M. Reppert and P. Brumer, "Quantumness in light harvesting is determined by vibrational dynamics", The Journal of Chemical Physics **149**, 234102 (2018).

[134] D. M. Wilkins and N. S. Dattani, "Why quantum coherence is not important in the fenna–matthews–olsen complex", Journal of Chemical Theory and Computation **11**, 3411–3419 (2015).

[135] H.-G. Duan, V. I. Prokhorenko, R. J. Cogdell, K. Ashraf, A. L. Stevens, M. Thorwart, and R. J. D. Miller, "Nature does not rely on long-lived electronic quantum coherence for photosynthetic energy transfer", Proceedings of the National Academy of Sciences **114**, 8493–8498 (2017).

[136] J. Cao, R. J. Cogdell, D. F. Coker, H.-G. Duan, J. Hauer, U. Kleinekathöfer, T. L. C. Jansen, T. Mančal, R. J. D. Miller, J. P. Ogilvie, V. I. Prokhorenko, T. Renger, H.-S. Tan, R. Tempelaar, M. Thorwart, E. Thyrhaug, S. Westenhoff, and D. Zigmantas, "Quantum biology revisited", Science Advances **6**, eaaz4888 (2020).

[137] E. Zerah Harush and Y. Dubi, "Do photosynthetic complexes use quantum coherence to increase their efficiency? probably not", Science Advances **7**, eabc4631 (2021).

[138] M. M. Wilde, J. M. McCracken, and A. Mizel, "Could light harvesting complexes exhibit non-classical effects at room temperature?", Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **466**, 1347–1363 (2009).

[139] C.-M. Li, N. Lambert, Y.-N. Chen, G.-Y. Chen, and F. Nori, "Witnessing quantum coherence: from solid-state to biological systems", Scientific Reports **2**, 885 (2012).

[140] D. Coles, L. C. Flatten, T. Sydney, E. Hounslow, S. K. Saikin, A. Aspuru-Guzik, V. Vedral, J. K.-H. Tang, R. A. Taylor, J. M. Smith, and D. G. Lidzey, "A nanophotonic structure containing living photosynthetic bacteria", Small **13**, 1701777 (2017).

[141] C. Marletto, D. M. Coles, T. Farrow, and V. Vedral, "Entanglement between living bacteria and quantized light witnessed by rabi splitting", Journal of Physics Communications **2**, 101001 (2018).

[142] F. Dubin, J. Berréhar, R. Grousson, T. Guillet, C. Lapersonne-Meyer, M. Schott, and V. Voliotis, "Optical evidence of a purely one-dimensional exciton density of states in a single conjugated polymer chain", Physical Review B **66**, 113202 (2002).

[143] D. V. Alexandrov and A. Y. Zubarev, "Transport phenomena in complex systems (part 1)", Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences **379**, 20200301 (2021).

[144] D. V. Alexandrov and A. Y. Zubarev, "Transport phenomena in complex systems (part 2)", Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences **380**, 20210366 (2022).

[145] M. Müller, S. Diehl, G. Pupillo, and P. Zoller, "Engineered open systems and quantum simulations with atoms and ions", in *Advances in atomic, molecular, and optical physics*, Vol. 61, edited by P. Berman, E. Arimondo, and C. Lin, Advances in Atomic, Molecular, and Optical Physics (Academic Press, 1, 2012), pp. 1–80.

[146] A. Civolani, V. Stanzione, M. L. Chiofalo, and J. Yago Malo, "Engineering transport via collisional noise: a toolbox for biology systems", Entropy **26**, Number: 1, 20 (2024).

[147] H.-P. Breuer and F. Petruccione, *The theory of open quantum systems* (Oxford University Press, 25, 2007).

[148] S. Campbell and B. Vacchini, "Collision models in open system dynamics: a versatile tool for deeper insights?", Europhysics Letters **133**, 60001 (2021).

[149] F. Ciccarello, S. Lorenzo, V. Giovannetti, and G. M. Palma, "Quantum collision models: open system dynamics from repeated interactions", Physics Reports **954**, 1–70 (2022).

[150] F. Dubin, R. Melet, T. Barisien, R. Grousson, L. Legrand, M. Schott, and V. Voliotis, "Macroscopic coherence of a single exciton state in an organic quantum wire", Nature Physics **2**, Number: 1, 32–35 (2006).

[151] M. Ziman, P. Štelmachovič, V. Bužek, M. Hillery, V. Scarani, and N. Gisin, "Diluting quantum information: an analysis of information transfer in system-reservoir interactions", Physical Review A **65**, 042105 (2002).

[152] F. Ciccarello, G. M. Palma, and V. Giovannetti, "Collision-model-based approach to non-markovian quantum dynamics", Physical Review A **87**, 040103 (2013).

[153] R. McCloskey and M. Paternostro, "Non-markovianity and system-environment correlations in a microscopic collision model", Physical Review A **89**, 052120 (2014).

[154] S. Kretschmer, K. Luoma, and W. T. Strunz, "Collision model for non-markovian quantum dynamics", Physical Review A **94**, 012106 (2016).

[155] Z.-X. Man, Y.-J. Xia, and R. Lo Franco, "Temperature effects on quantum non-markovianity via collision models", Physical Review A **97**, 062104 (2018).

[156] T. Saha, A. Das, and S. Ghosh, "Quantum homogenization in non-markovian collisional model", New Journal of Physics **26**, 023011 (2024).

[157] A. Javadi-Abhari, M. Treinish, K. Krsulich, C. J. Wood, J. Lishman, J. Gacon, S. Martiel, P. D. Nation, L. S. Bishop, A. W. Cross, B. R. Johnson, and J. M. Gambetta, *Quantum computing with qiskit*, 2024.

[158] R. E. Fenna and B. W. Matthews, "Chlorophyll arrangement in a bacteriochlorophyll protein from chlorobium limicola", Nature **258**, 573–577 (1975).

[159] B. W. Matthews, R. E. Fenna, M. C. Bolognesi, M. F. Schmid, and J. M. Olson, "Structure of a bacteriochlorophyll *a*-protein from the green photosynthetic bacterium *Prosthecochloris aestuarii*", Journal of Molecular Biology **131**, 259–285 (1979).

[160] C. Olbrich, J. Strümpfer, K. Schulten, and U. Kleinekathöfer, "Theory and simulation of the environmental effects on FMO electronic transitions", The Journal of Physical Chemistry Letters **2**, 1771–1776 (2011).

[161] C. Marletto, *Constructor theory of thermodynamics*, arXiv:1608.02625.

[162] G. Di Pietra, G. Bhole, J. Eaton, A. J. Baldwin, J. A. Jones, V. Vedral, and C. Marletto, "Temporal entanglement and witnesses of nonclassicality", Physical Review Letters, 10.1103/h1rh-cld4 (2026).

[163] A. J. Leggett and A. Garg, "Quantum mechanics versus macroscopic realism: is the flux there when nobody looks?", Physical Review Letters **54**, 857–860 (1985).

[164] C. Emary, N. Lambert, and F. Nori, "Leggett–garg inequalities", Reports on Progress in Physics **77**, 016001 (2013).

[165] C. Brukner, S. Taylor, S. Cheung, and V. Vedral, *Quantum entanglement in time*, arXiv:quant-ph/0402127.

[166] A. J. Leggett, "Testing the limits of quantum mechanics: motivation, state of play, prospects", Journal of Physics: Condensed Matter **14**, R415 (2002).

[167] F. Benatti, G. C. Ghirardi, and R. Geassi, "Testing macroscopic quantum coherence", Il Nuovo Cimento B (1971-1996) **110**, 593–610 (1995).

[168] F. Fröwis, P. Sekatski, W. Dür, N. Gisin, and N. Sangouard, "Macroscopic quantum states: measures, fragility, and implementations", Reviews of Modern Physics **90**, 025004 (2018).

[169] O. J. E. Maroney and C. G. Timpson, *Quantum- vs. macro- realism: what does the leggett-garg inequality actually test?*, arXiv:1412.6139.

[170] O. J. E. Maroney, "Measurements, disturbances and the quantum three box paradox", Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics **58**, 41–53 (2017).

[171] M. M. Wilde and A. Mizel, "Addressing the clumsiness loophole in a leggett-garg test of macrorealism", Foundations of Physics **42**, 256–265 (2012).

[172] L. Clemente and J. Kofler, "Necessary and sufficient conditions for macroscopic realism from quantum mechanics", Physical Review A **91**, 062103 (2015).

[173] R. Hermens and O. J. E. Maroney, "Constraints on macroscopic realism without assuming non-invasive measurability", Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics **63**, 50–64 (2018).

[174] D. Schmid, "A review and reformulation of macroscopic realism: resolving its deficiencies using the framework of generalized probabilistic theories", Quantum **8**, 1217 (2024).

[175] T. Fritz, "Quantum correlations in the temporal clauser–horne–shimony–holt (CHSH) scenario", New Journal of Physics **12**, 083055 (2010).

[176] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, "Proposed experiment to test local hidden-variable theories", Physical Review Letters **23**, 880–884 (1969).

[177] Z. Zhao, R. Pisarczyk, J. Thompson, M. Gu, V. Vedral, and J. F. Fitzsimons, "Geometry of quantum correlations in space-time", Physical Review A **98**, 052312 (2018).

[178] T. Zhang, O. Dahlsten, and V. Vedral, "Different instances of time as different quantum modes: quantum states across space-time for continuous variables", New Journal of Physics **22**, 023029 (2020).

[179] M. Nowakowski, "Quantum entanglement in time", AIP Conference Proceedings **1841**, 020007 (2017).

[180] D. Rajan, "Quantum entanglement in time", thesis (Open Access Te Herenga Waka-Victoria University of Wellington, 1, 2020).

[181] L. Castellani, "History entanglement entropy", Physica Scripta **96**, 055217 (2021).

[182] T. Scheidl, R. Ursin, J. Kofler, S. Ramelow, X.-S. Ma, T. Herbst, L. Ratschbacher, A. Fedrizzi, N. K. Langford, T. Jennewein, and A. Zeilinger, "Violation of local realism with freedom of choice", Proceedings of the National Academy of Sciences **107**, 19708–19713 (2010).

[183] J. F. Clauser and A. Shimony, "Bell's theorem. experimental tests and implications", Reports on Progress in Physics **41**, 1881 (1978).

[184] E. Adlam, "Spooky action at a temporal distance", Entropy **20**, Number: 1, 41 (2018).

[185] L. Letertre, "Temporal nonlocality from indefinite causal orders", The British Journal for the Philosophy of Science, 10.1086/734907 (2025).

[186] J. Kofler and Č. Brukner, "Condition for macroscopic realism beyond the leggett-garg inequalities", Physical Review A **87**, 052115 (2013).

[187] G. C. Knee, K. Kakuyanagi, M.-C. Yeh, Y. Matsuzaki, H. Toida, H. Yamaguchi, S. Saito, A. J. Leggett, and W. J. Munro, "A strict experimental test of macroscopic realism in a superconducting flux qubit", Nature Communications **7**, 13253 (2016).

[188] J. A. Jones, "Controlling NMR spin systems for quantum computation", Progress in Nuclear Magnetic Resonance Spectroscopy **140-141**, 49–85 (2024).

[189] G. Bhole, J. A. Jones, C. Marletto, and V. Vedral, "Witnesses of non-classicality for simulated hybrid quantum systems", Journal of Physics Communications **4**, 025013 (2020).

[190] J. F. Fitzsimons, J. A. Jones, and V. Vedral, "Quantum correlations which imply causation", Scientific Reports **5**, 18281 (2015).

[191] C. Marletto, "The information-theoretic foundation of thermodynamic work extraction", Journal of Physics Communications **6**, 055012 (2022).

[192] A. de Oliveira Junior, J. B. Brask, and P. Lipka-Bartosik, "Heat as a witness of quantum properties", Physical Review Letters **134**, 050401 (2025).

[193] M. Wiesniak, V. Vedral, and C. Brukner, "Heat capacity as a witness of entanglement", Physical Review B **78**, 064108 (2008).

[194] M. Stamatova and V. Vedral, "Complex heat capacity as a witness of spatio-temporal entanglement", Physical Review A **114**, 022448 (2026).

[195] T. Zhang, O. Dahlsten, and V. Vedral, *Quantum correlations in time*, arXiv:2002.10448.

[196] C. Marletto, V. Vedral, S. Virzì, E. Rebufello, A. Avella, F. Piacentini, M. Gramegna, I. P. Degiovanni, and M. Genovese, "Theoretical description and experimental simulation of quantum entanglement near open time-like curves via pseudo-density operators", Nature Communications **10**, 182 (2019).

[197] C. Marletto, V. Vedral, S. Virzì, A. Avella, F. Piacentini, M. Gramegna, I. P. Degiovanni, and M. Genovese, "Temporal teleportation with pseudo-density operators: how dynamics emerges from temporal entanglement", Science Advances **7**, eabe4742 (2021).

[198] C. Marletto, V. Vedral, S. Virzì, E. Rebufello, A. Avella, F. Piacentini, M. Gramegna, I. P. Degiovanni, and M. Genovese, "Non-monogamy of spatio-temporal correlations and the black hole information loss paradox", Entropy **22**, 228 (2020).

[199] M. Song, V. Narasimhachar, B. Regula, T. J. Elliott, and M. Gu, "Causal classification of spatiotemporal quantum correlations", Physical Review Letters **133**, 110202 (2024).

[200] A. Iyer, E. O. Dias, and V. Vedral, "Signatures of causality and determinism in a quantum theory of events", Physical Review A **105**, L010202 (2022).

[201] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Quantum circuit architecture", Physical Review Letters **101**, 060401 (2008).

[202] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Theoretical framework for quantum networks", Physical Review A **80**, 022339 (2009).

[203] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, "Non-markovian quantum processes: complete framework and efficient characterization", Physical Review A **97**, 012127 (2018).

[204] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, "Operational markov condition for quantum processes", Physical Review Letters **120**, 040405 (2018).

[205] C. Giarmatzi and F. Costa, "Witnessing quantum memory in non-markovian processes", Quantum **5**, 440 (2021).

[206] S. Milz, "When is a non-markovian quantum process classical?", Physical Review X **10** (2020).

[207] F. Casas, A. Murua, and M. Nadinic, "Efficient computation of the zassenhaus formula", Computer Physics Communications **183**, 2386–2391 (2012).

[208] R. Schatten, *Norm ideals of completely continuous operators* (Springer, Berlin, Heidelberg, 1960).

[209] M. Nielsen and I. Chuang, *Quantum computation and quantum information: 10th anniversary edition* (Cambridge University Press, 2010).

[210] T. Holstein and H. Primakoff, "Field dependence of the intrinsic domain magnetization of a ferromagnet", Physical Review **58**, 1098–1113 (1940).

[211] H.-P. Breuer, E.-M. Laine, and J. Piilo, "Measure for the degree of non-markovian behavior of quantum processes in open systems", Physical Review Letters **103**, 210401 (2009).

[212] R. Uzdin, S. Gasparinetti, R. Ozeri, and R. Kosloff, "Markovian heat sources with the smallest heat capacity", New Journal of Physics **20**, 063030 (2018).