Ringkamp, M., Ober-Blöbaum, S., Leyendecker, S.: *On the time transformation of mixed integer optimal control problems using a consistent fixed integer control function.* Mathematical Programming, 1–31 (2016). DOI 10.1007/s10107-016-1023-5

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@Article{ringkamp2016time,
author="Ringkamp, Maik and Ober-Bl{\"o}baum, Sina and Leyendecker, Sigrid",
title="On the time transformation of mixed integer optimal control problems
using a consistent fixed integer control function",
journal="Mathematical Programming",
year="2016",
pages="1--31",
abstract="Nonlinear control systems with instantly changing dynamical behavior
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valued in contrast to a control function that is allowed to have continuous
values. The discretization of a mixed integer optimal control problem (MIOCP)
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control consistent fixed integer control functions allow a better accuracy in
the resulting trajectories, in particular in the computed switching times. The
method is verified on analytical and numerical examples.",
issn="1436-4646",
doi="10.1007/s10107-016-1023-5",
url="http://dx.doi.org/10.1007/s10107-016-1023-5"
}
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Noname manuscript No.
(will be inserted by the editor)

On the time transformation of mixed integer optimal control problems using a consistent fixed integer control function

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Received: 5 May 2015 / Accepted: 29 April 2016

Abstract Nonlinear control systems with instantly changing dynamical behavior can be modeled by introducing an additional control function that is integer valued in contrast to a control function that is allowed to have continuous values. The discretization of a mixed integer optimal control problem (MIOCP) leads to a non differentiable optimization problem and the non differentiability is caused by the integer values. The paper is about a time transformation method that is used to transform a MIOCP with integer dependent constraints into an ordinary optimal control problem (OCP). Differentiability is achieved by replacing a variable integer control function with a fixed integer control function and a variable time allows to change the sequence of active integer values. In contrast to other contributions, so called control consistent fixed integer control functions are taken into account here. It is shown that these control consistent fixed integer control functions allow a better accuracy in the resulting trajectories, in particular in the computed switching times. The method is verified on analytical and numerical examples.

Keywords time transformation · control consistency · optimal control · mixed integer optimal control · hybrid systems · switched dynamical systems

Mathematics Subject Classification (2000) 90C11 · 34H05

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1 Introduction

In the optimal control of mixed integer control systems, the aim is to find feasible state and continuous as well as integer valued control trajectories such that an a priori defined objective functional is optimized. Different differential equations are selected by changing the integer dependent right-hand side instantly with a changing integer control value. Such problems occur for example in the optimal control of moving vehicles [28,16,17,39,26,19], where a different right-hand side represents the dynamical behavior of the vehicle within a different driving mode, e.g., a gear of a car. Other MIOCPs include integer control functions to treat contact problems in robotics [7,8,43] or processes involving valves [40,6]. Some of the authors use binary instead of integer control functions. Due to the huge variety of different scientific backgrounds, the terminology often differs while the problem classes are related. For example, instead of the optimal control of mixed integer control systems [16,17, 39,26,19], some authors prefer to write optimal control of hybrid systems [7,8,43], or of switched dynamical systems [21].

In ordinary OCPs the dynamical behavior does not change instantly and OCPs do not include integer valued control functions. The numerical methods of choice to locally solve many OCPs in practice are discretize-then-optimize approaches using a gradient based optimization method (e.g., cf. [15,33]) to solve the discretized OCP (e.g., [5,30,34,18,12,13,10]). Applying a discretize-then-optimize approach to a MIOCP leads to a mixed integer nonlinear program (MINLP) that is in general not differentiable with respect to the integer variables, thus gradient based optimization methods cannot be applied to solve the MINLP at once. MINLPs combine combinatorial and nonlinear optimization and are known to be hard to solve globally [32,38]. In practice, often local solutions of the MIOCP are accepted (c.f. [7] Remark 6).

Different approaches are used throughout the literature that numerically solve a MIOCP by decoupling the problem in a discrete (i.e. in our setting integer) and a continuous part. This allows to solve the purely continuous parts in an inner procedure by a discretize-then-optimize approach using gradient based optimization methods. The integer parts are optimized by an outer procedure. In [7], the MIOCP is decoupled by discretizing the set of possible switching times and states. The result is a set of purely continuous OCPs, each corresponding to a fixed integer control value. The best combination of the computed optimal trajectories approximates the optimal solution of the MIOCP and is found by a graph search. Another method applied in [21] uses a fixed switching sequence (i.e. a fixed integer control function), solves the resulting OCP and changes the integer values partially. The procedure is repeated until no feasible change of the integer values exists that reduces the objective. Other approaches avoid a decoupling in discrete and continuous parts by a relaxation of discrete variables. Branch and bound tree search methods are often used [7,20,16] to solve MIOCPs numerically. They are a mixture of a decoupling and a relaxation. Each node of the tree represents a MIOCP that is solved while the integer control function is fixed at some time nodes and relaxed at the remaining time nodes. Further integer values are fixed if the tree depth increases, until each integer value is fixed at the leafs. Subtrees are not investigated if the computed lower bound is greater or equal to the current upper bound. Computing the lower bound is a crucial part of the procedure and has a huge influence on the computational effort, since global optimality cannot be guaranteed for the solution of general nonconvex OCPs and too coarse underestimates avoid the pruning of subtrees. Similarly, the selection of an appropriate initial guess is a crucial part in numerical optimal control methods that rely on a pure relaxation without a decoupling and a well chosen initial guess can lead to a global solution, even though local optimization methods are applied. In general global optimality can not be guaranteed except if the discretized problem is convex (c.f. [15,32] for details on global optimization). In [38,4,39,9], a binary control function allows to change the right-hand sides instantly. The used relaxation guarantees that the optimized relaxed binary control trajectory can be approximated arbitrarily close by a binary control, c.f. [38,39] for theoretical aspects and the consideration of different rounding strategies to yield the desired binary values from the relaxed values.

The method that is developed here relies on a transformation into a purely continuous problem and originates back to [11, p. 47]. There, a time transformation t_w is used to prove a maximum principle and later, e.g., in [28] or [17], to solve a MIOCP numerically. A comparison of the method with a branch and bound approach is done in [17] and reveals a significant reduction of computational time. An improvement in the accuracy of the computed switching times is the main contribution of this paper. The time transformation method utilizes a fixed integer control function $\bar{v}_{N,n}$ instead of the variable integer control function v to avoid integer optimization variables. The time interval I is partitioned into N major intervals I_j and each $I_j = [t_{j-1}, t_j[$ into n minor intervals $I_j^i = [\tau_j^{i-1}, \tau_j^i[$. The function $\bar{v}_{N,n}$ is defined to be constant on each of the minor intervals I_j^i and the time transformation t_w allows to scale the minor intervals. A different sequence of integer control values is achieved by scaling some of the minor intervals to zero. The above mentioned right-hand side F is defined on the isolated values $v(t) \in \mathscr{V} = \{1, \ldots, n_{\mathscr{V}}\} \subseteq \mathbb{N}$ and does not need to be defined on relaxed values, nor to be differentiable with respect to the corresponding component. An extension to the integer dependent domains D_l respectively integer dependent mixed state-control constraints g is made here, as briefly introduced by the authors in [37] and recently published by Palagachev and Gerdts in [36]. Former publications that use the time transformation utilize a fixed integer control function $\bar{v}_{N,n}$ with the sequence of integer values $(1, \ldots, n_{\mathscr{V}})$ for each major interval I_j. The approach presented here takes other sequences into account. A property for fixed integer control functions named control consistency is introduced. An upper bound for the number of needed major time intervals I_j is identified providing that a minimal switching time distance ΔT_{min} is supposed and that the utilized fixed integer control function $\bar{v}_{N,n}$ is control consistent (CC). Counterexamples show that the number of needed major intervals and therefore also the number of needed discretization variables is unbounded if the fixed integer control function is not CC (NCC) as it is utilized in prior publications. As a result, using a NCC fixed integer control function with an insufficient number of major intervals can lead to a gap in the value of the objective function as well as in the computed trajectories. This is in contrast to a CC fixed integer control function, where the number of needed major intervals is a priori known, provided that ΔT_{min} is known, or at least given as a tolerance.

Section 2 delivers definitions regarding MIOCPs. In Section 3 the time transformation as in [17] is introduced formally and extended to problems involving integer dependent constraints. CC fixed integer control functions are defined and their advantages over NCC fixed integer control functions are highlighted theoretically and in academic examples. The resulting time transformed MIOCP (TMIOCP) is discretized in Section 4 and the computation of a numerical example is presented in Section 5. Section 6 concludes the paper and gives a short outlook.

2 Mixed integer optimal control

In the following section, the MIOCP is introduced. Nonlinear control systems with a finite number $n_{\mathscr{V}} \in \mathbb{N}$ of different right-hand sides $F_l(x,u), l \in \mathscr{V} = \{1,\dots,n_{\mathscr{V}}\}$ are used to model dynamical behavior that can change instantly. An integer valued control function $v \in \mathscr{L}^\infty(I,\mathscr{V})$ controls which of the right-hand sides F_l is active at which time, resulting in a mixed integer differential equation $\dot{x} = F(x,u,v)$ with $F(x,u,l) = F_l(x,u)$ for all $l \in \mathscr{V}$. Here, the values $l \in \mathscr{V}$ of v are selected without loss of generality as natural numbers (c.f. Remark 1), each right-hand side F_l does not need to depend explicitly on the specific value $l \in \mathscr{V}$. The functions $x \in \mathscr{W}^{1,\infty}(I,\mathbb{R}^{n_x})$ and $u \in \mathscr{L}^\infty(I,\mathbb{R}^{n_u})$ represent the continuous state and the continuous control of the nonlinear control system $\dot{x} = F(x,u,l)$ for each $l \in \mathscr{V}$ and $I = [t_0,t_f]$ represents a time interval. Similarly, the corresponding domain $D_l = \{(x,u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} | \, g(x,u,l) \leq 0\}$ of the system can change with a changing value $l \in \mathscr{V}$ of the integer control function v. That allows to restrict the active right-hand side F_l to a specific domain D_l what is important, for example in contact dynamics, where the switch from one right-hand side F_{l^1} to another right-hand side F_{l^2} is caused by a changing domain.

Remark 1 Other authors (e.g., [28,36]) use a finite set of vectors $\{v_1,\dots,v_{n_{\mathscr{V}}}\} \subseteq \mathbb{R}^{n_v}$ with $n_v \geq 1$. The here used natural numbers $1,\dots,n_{\mathscr{V}} \in \mathbb{N}$ can be interpreted as an enumeration of these vectors $v_1,\dots,v_{n_{\mathscr{V}}}$.

Definition 2 Mixed integer optimal control problem MIOCP
Let $[t_0,t_f] \subseteq \mathbb{R}_0^+$ be a closed interval, $F_l : \tilde{D}_l \to \mathbb{R}^{n_x}$, $g_0 : \tilde{D}_0 \to \mathbb{R}^{n_{g_0}}$ and $g_l : \tilde{D}_l \to \mathbb{R}^{n_g}$ continuously differentiable functions for all $l \in \mathscr{V} = \{1,2,\dots,n_{\mathscr{V}}\} \subseteq \mathbb{N}$ and $g(x,u,l) = g_l(x,u)$ for $(x,u) \in \tilde{D}_l \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}, l \in \{0\} \cup \mathscr{V}$. Then, a *mixed integer optimal control problem* in short *MIOCP* is defined as follows

$$\min_{x,u,v} \quad J(x,u,v) = \int_{t_0}^{t_f} B(x(t),u(t),v(t))\, dt \tag{1}$$

$$\text{s. t.} \quad \dot{x}(t) = F(x(t),u(t),v(t)) \quad \text{for a.e. } t \in [t_0,t_f] \tag{2}$$

$$g_0(x(t),u(t)) \leq 0 \quad \text{for a.e. } t \in [t_0,t_f] \tag{3}$$

$$g(x(t),u(t),v(t)) \leq 0 \quad \text{for a.e. } t \in [t_0,t_f] \tag{4}$$

$$r(x(t_0),x(t_f)) = 0 \tag{5}$$

$$v(t) \in \mathscr{V} \quad \text{for a.e. } t \in [t_0,t_f]. \tag{6}$$

Here, J is called the *objective functional* and is defined by continuously differentiable functions $B_l : \tilde{D}_l \to \mathbb{R}$ with $B(x,u,l) = B_l(x,u), l \in \mathscr{V}$. Equation (5) is called *boundary condition* and is defined by a continuously differentiable function $r : \tilde{X}_0 \times \tilde{X}_0 \to \mathbb{R}^{n_r}$.

Remark 3 Equation (3) can in principle be included in (4). The equations are separated here because they are treated differently in the time transformation in Section 3. For each $l \in \{0\} \cup \mathscr{V}$, the set $\tilde{D}_l \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ is supposed to be a sufficiently large superset of $D_l = \{(x,u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} | \, g_l(x,u) \leq 0\}$. The later used numerical optimization method must ensure that $(x,u) \in \tilde{D}_l$ is satisfied at every iteration to ensure that the used functions are defined and differentiable. The x-part of the superset is in the following shortly denoted as $\tilde{X}_l = \{x \in \mathbb{R}^{n_x} | \exists u \in \mathbb{R}^{n_u} : (x,u) \in \tilde{D}_l\}$.

Definition 4 Feasible solution
For a given MIOCP a function (x^*, u^*, v^*) with $x^* \in \mathscr{W}^{1,\infty}([t_0,t_f],\mathbb{R}^{n_x})$, $u^* \in \mathscr{L}^\infty([t_0,t_f],\mathbb{R}^{n_u})$ and $v^* \in \mathscr{L}^\infty([t_0,t_f],\mathscr{V})$ is called *feasible solution*, *feasible trajectory* or short *feasible* if it fulfills the equations (2)-(6). The function x^* is called the *(continuous) state*, u^* the *(continuous) control*, and v^* the *integer (or discrete) control function*.

Remark 5 Here, the vector space $\mathscr{L}^\infty(I,\mathbb{R}^n)$ is the space of all functions f with $f : I \to \mathbb{R}^n$ (Lebesgue-) measurable and essentially bounded, thus $\|f\|_\infty := \text{ess sup}_{t\in[t_0,t_f]} \|f(t)\| < \infty$. The vector space $\mathscr{W}^{1,\infty}(I,\mathbb{R}^n)$ is defined as the space of all functions f with $f : I \to \mathbb{R}^n$ absolutely continuous and where f and the derivative $f^{(1)}$ are essentially bounded, so $\|f\|_{1,\infty} := \max\{\|f\|_\infty, \|f^{(1)}\|_\infty\} < \infty$. For more details, e.g., confer the definitions given in [18, p. 60].

Let us assume that the feasible solution (x^*, u^*, v^*) of a MIOCP has a discrete control v^* that is constant on both sub intervals $]T_0,T_1[,]T_1,T_2[\subseteq [t_0,t_f], T_0 < T_1 < T_2$, with $v^*(t_1) = l^1 \neq l^2 = v^*(t_2)$ for a.e. $t_1 \in]T_0,T_1[, t_2 \in]T_1,T_2[$. Then, T_1 is called a switching time. The exact definition of all switching times and the sequence of all the corresponding integer values l_k is given below.

Definition 6 For a discrete control $v : [t_0,t_f] \to \mathscr{V}$, let $n_s \in \mathbb{N}$, $T_0 := t_0, T_{n_s+1} := t_f$ and $T_k \in [t_0,t_f]$ for $k \in \{1,\ldots,n_s\}$ such that

- $T_{k-1} < T_k$ for $k \in \{1,\ldots,n_s+1\}$
- $v(\tau)$ is constant for a.e. $\tau \in]T_{k-1},T_k[, k \in \{1,\ldots,n_s+1\}$
- $v(\tau_k) \neq v(\tau_{k+1})$ for a.e. $\tau_k \in]T_{k-1},T_k[, \tau_{k+1} \in]T_k,T_{k+1}[, k \in \{1,\ldots,n_s\}$.

The set $\{T_k\}_{k=1}^{n_s}$ is called the *switching time set*, every T_k is called a *switching time* and $(l_k)_{k=1}^{n_s+1}$ with $l_k = v(\tau_k)$ for a.e. $\tau_k \in]T_{k-1},T_k[$ the *switching sequence*. An example with $n_s = 3$ switching times is depicted in Figure 1.

Remark 7 The definition implicitly restricts the number of switching times n_s to be finite. That is appropriate, in order to gain an approximation of the discrete control v by a finite representation later. Moreover, discrete controls v with so called Zeno behavior (meaning an unlimited number of switching times on a finite time interval) are excluded by that restriction.

Fig. 1: An example of $n_s = 3$ switching times $T_1, T_2, T_3 \in [t_0, t_f]$ with the corresponding switching sequence $(1, 2, 3, 1)$, where each color represents one of the integer values.

If a switching sequence is given and only the computation of optimal switching times is desired, the resulting discretized OCP is an NLP (as in [25, 14, 27, 13]). Thus, s allowed switching sequences lead to s different NLPs which can in principle be solved in parallel, to yield an optimal solution of the MIOCP. In practice a discretized OCP with N time steps and $n_{\mathscr{V}}$ possible integer values for each step leads to $s = n_{\mathscr{V}}^N$ possible switching sequences and therefore a large number of parallel processes. Doing this means to decouple the combinatorial problem (discrete part) from the optimization (continuous part). Instead we consider the mixed problem without knowing the order of the switching sequence. The switching sequence is optimized here by including v as an optimization parameter in the MIOCP. The problem is time transformed in the next section, in order to replace the discrete control v that is feasible only on the isolated integer values $v(t) \in \mathscr{V}$, by a control $w(t) \in \mathbb{R}$ that has feasible values on a connected subset of $\mathbb{R}$ such that gradient based minimization methods as SQP or interior point methods (e.g., cf. [33], or [15]) can be used to solve the discretized problem.

3 Time transformation

The time transformation in combination with a fixed integer control function $\bar{v}_{N,n}$ and a partition of the time interval I in major I_j and minor intervals I_j^i is introduced in Section 3.1, similar as in [17]. There, the considered systems have a single mixed state-control domain

$$D_0 = \{(x, u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \,|\, g_0(x, u) \leq 0\} \tag{7}$$

for all the integer values $l \in \mathscr{V}$ and the mixed state-control constraint function g in Equation (4), that depends also on $l \in \mathscr{V}$, is excluded. A generalization to a DAE of index-one is given in [19] and [18] and a generalization that also includes Equation (4) is given here, in Section 3.2 and in [29, 37, 36]. That leads to systems that may have a different mixed state-control domain

$$D_l = \{(x, u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \,|\, g(x, u, l) \leq 0\} \tag{8}$$

for each $l \in \mathscr{V}$. Additional relaxed binary control functions are needed for the generalization in [29], this is avoided here as well as in [37, 36]. Section 3.3 gives an answer to the question, under which condition it is possible to transform a given integer control function v to a fixed integer control function $\bar{v}_{N,n}$. The needed property is defined as *control consistency* and a rigorous proof shows that for any discrete control function v a time transformation t_w exists such that the CC fixed integer control function $\bar{v}_{N,n}$ equals the transformed discrete control function $v \circ t_w$ almost everywhere on *active* minor intervals I_j^i. Examples show that this is not always true for a fixed integer control function that is not consistent to the integer control function v and that the lack of consistency can lead to the need of arbitrary many minor intervals or a solution with a worse objective value as the solution of the MIOCP.

3.1 Time transformation for the MIOCP with integer independent constraints

The MIOCP from Definition 2 without Equation (4) is transformed to an auxiliary optimal control problem that does not depend on a discrete valued control function v in the following, similar as in [17]. Therefore, the time interval I is partitioned into several minor and major intervals, and a fixed integer control function $\bar{v}_{N,n}$ is selected that assigns an integer value to each of the minor intervals, as depicted in Figure 2. Then, the time is transformed by a control w that allows to deactivate some of the minor intervals I_j^i such that the length of these time transformed intervals $t_w(I_j^i)$ is equal to zero. The time transformed minor intervals with positive length are active and applying the fixed integer control function $\bar{v}_{N,n}$ leads to the sequence of active integer values. An example is illustrated in Figure 3. Finally, the auxiliary optimal control problem is defined in Definition 20. To introduce the time transformation accurately and to allow rigorous proofs, some definitions are necessary first.

Definition 8 Ordered partition
For a given compact interval $I \subseteq \mathbb{R}$ representing the time, an *ordered partition of I into $N \in \mathbb{N}$ intervals* is defined as a set $\{I_j\}_{j=1}^N$ of non empty intervals $I_j \subseteq I$ with

- $I_{j_1} \cap I_{j_2} = \emptyset$ for all $j_1, j_2 \in \{1,\dots,N\}$ with $j_1 \neq j_2$
- $\bigcup_{j=1}^{N} I_j = I$
- $\sup I_j = \inf I_{j+1}$ for all $j \in \{1,\dots,N-1\}$.

In words, the intervals I_j are disjoint, their union equals the complete interval I and they are ordered such that the upper bound of an interval equals the lower bound of the interval with the next index.

Example 9 For a given switching time set $\{T_k\}_{k=1}^{n_s}$ the set $\{I_k\}_{k=1}^{n_s+1}$ with $I_1 = [t_0, T_1[$, $I_{n_s+1} = [T_{n_s}, t_f]$ and $I_k = [T_{k-1}, T_k[$ for $k \in \{2,\dots,n_s\}$ is an ordered partition of $[t_0, t_f]$.

Remark 10 The ordered partition in Example 9 is desired, but unknown a priori. Instead in the following, always ordered partitions $\{I_j\}_{j=1}^N$ with intervals I_j that have the same length ΔI_j are used. Thus, if ΔI is the length of I and N the number of intervals, then the length of each interval I_j is $\Delta I_j = \frac{\Delta I}{N}$.

Definition 11 Ordered partition in major and minor intervals (of the same length)
For a given compact interval $I \subseteq \mathbb{R}$ with length ΔI, an *ordered partition of I in N major and n minor intervals (of the same length)* is defined as a set $\{I_j^i\}_{j=1,i=1}^{N,n}$ with

- $\Delta I_j := \text{length}\{I_j\} = \frac{\Delta I}{N}$ for all $j \in \{1,\dots,N\}$

- $\Delta I_j^i := \text{length}\{I_j^i\} = \frac{\Delta I_j}{n}$ for all $j \in \{1,\dots,N\}, i \in \{1,\dots,n\}$
- $\{I_j\}_{j=1}^N$ is an ordered partition of I
- $\{I_j^i\}_{i=1}^n$ is an ordered partition of I_j for all $j \in \{1,\dots,N\}$.

Here, each $I_j = \bigcup_{i=1}^{n} I_j^i$ is called a *major interval* and each I_j^i is called a *minor interval*.

In the following the boundaries of the intervals are always shortly denoted by $t_{j-1} := \inf I_j, t_j := \sup I_j$ and $\tau_j^{i-1} := \inf I_j^i, \tau_j^i := \sup I_j^i$. The next definition is used to describe a function that assigns a specific integer value $l^i \in \mathscr{V}$ to each of the minor intervals I_j^i.

Definition 12 Fixed integer control function
For a given set $\mathscr{V} = \{1,\dots,n_{\mathscr{V}}\} \subset \mathbb{N}$ and an ordered partition $\{I_j^i\}_{j=1,i=1}^{N,n}$ of an interval I in N major and n minor intervals, a function $\bar{v}_{N,n} : I \to \mathscr{V}$ that is constant on each minor interval I_j^i is called a fixed integer control function.

Remark 13 A fixed integer control function $\bar{v}_{N,n}$ is an integer control function with $\bar{v}_{N,n} \in \mathscr{L}^\infty(I,\mathbb{R})$, in particular it fulfills the set constraint (6) for $I = [t_0, t_f]$.

The following example of a fixed integer control function is used in [17].

Example 14 Let $I = [t_0, t_f]$ and $\left(I_j^i\right)_{j=1,i=1}^{N,n}$ be an ordered partition in major and minor intervals with $n = n_{\mathscr{V}}$. Then define $l^i := i$ for all $i \in \{1,\dots,n\}$ and moreover for all minor intervals I_j^i the fixed integer control function $\bar{v}_{N,n} \in \mathscr{L}^\infty(I,\mathscr{V})$ as

$$\bar{v}_{N,n}(\tau) := l^i \Leftrightarrow \tau \in I_j^i. \tag{9}$$

The expression is well-defined because for each $\tau \in I$ there exists exactly one minor interval I_j^i with $\tau \in I_j^i$. The sequence $\left(l^i\right)_{i=1}^n$ is the switching sequence on each major interval I_j and the switching sequence $\left(l_j^i\right)_{j=1,i=1}^{N,n}$ on I can be defined analogously. Figure 2 depicts the fixed integer control function for an example with $n_{\mathscr{V}} = 3$ minor and $N = 3$ major intervals.

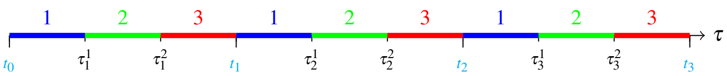

Fig. 2: Fixed integer control function for an example with three integer values.

Replacing the variable discrete control function v in the MIOCP by a fixed integer control function $\bar{v}_{N,n}$ would lead to a fixed switching sequence. Using a controllable time transformation $t_w : I \to I$ in addition, allows to influence the length of each time transformed minor interval $t_w(I_j^i)$ and therefore how much time is spent in the corresponding system with integer value $l^i \in \mathscr{V}$. A controllable time transformation is defined in the following.

Definition 15 Time transformation
Let $\bar{v}_{N,n} \in \mathscr{L}^\infty(I,\mathscr{V})$ be a fixed integer control function and let $w \in \mathscr{L}^\infty(I,\mathbb{R})$ fulfill the properties

$$w(\tau) \geq 0 \qquad \text{for a.e. } \tau \in I, \tag{10}$$

$$\Delta I_j = \int_{I_j} w(s)ds. \tag{11}$$

Then, a time transformation $t_w : I \to I$ is defined by

$$t_w(\tau) := t_0 + \int_{t_0}^{\tau} w(s)ds \tag{12}$$

and therefore absolutely continuous with $t'_w = \frac{dt_w}{d\tau}(\tau) = w(\tau)$ almost everywhere on I according to the fundamental theorem of calculus. Thus, $t_w \in \mathscr{W}^{1,\infty}(I,I)$ holds. The function w is called a *time control*.

Remark 16 Equation (10) assures that the transformed time does not move backwards and Equation (11) assures that $t_w(I_j) = I_j$ holds and therefore that the major interval length is fixed. This assures that a certain accuracy of the approximated trajectories can be guaranteed and prevents that corresponding major time nodes t_{j-1} and t_j coincide in the later numerical computation of a discretized version. Numerical computations of the example in Section 5 reveal that leaving out Equation (11) in practice can lead to undesirably large time steps in the middle of the trajectories as also stated for the numerical example in [17, p. 173].

Remark 17 The inverse function of the time transformation t_w is defined here as in [17, p. 173]

$$\tau_w(t) := \inf\{\tau \in I | t_w(\tau) = t\}. \tag{13}$$

Remark 18 The time control w is assumed to be constant on each minor interval I_j^i, in a later discretized version. Moreover, in Theorem 26 such a piecewise constant time control is constructed to prove that a fixed integer control function $\bar{v}_{N,n}$ equals a time transformed variable integer control $v \circ t_w$ almost everywhere on active minor intervals I_j^i. Therefore the following notation is introduced here

$$w_j^i := w(\tau) \text{ for } \tau \in I_j^i. \tag{14}$$

Then, the time transformation (12) can be given explicitly for each $\tau \in I_j^i$ as

$$t_w(\tau) = t_{j-1} + \sum_{k=1}^{i-1} \Delta I_j^k w_j^k + (\tau - \tau_j^{i-1}) w_j^i. \tag{15}$$

Figure 3 illustrates the transformed time and the fixed integer control function from Example 14 for possible constants w_j^i and moreover depicts the resulting active minor intervals I_j^i. The term active is properly defined in the following.

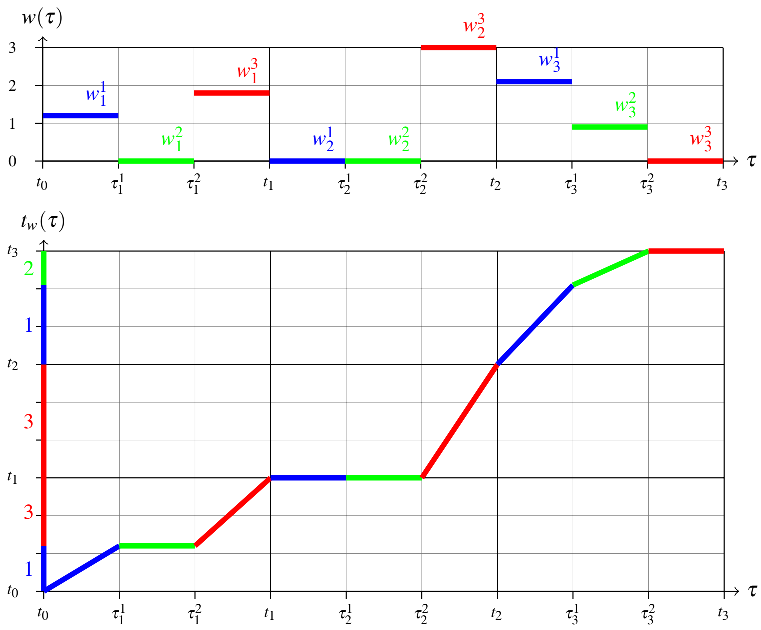

Fig. 3: Transformed time $t_w(\tau)$ over τ for possible constants w_j^i for an example with three integer values. The bottom picture further illustrates the resulting sequence of active integer values on the t_w-axis.

Definition 19 Active integer values
For a given time transformation $t_w \in \mathscr{W}^{1,\infty}(I,I)$ with fixed integer control function $\bar{v}_{N,n} \in \mathscr{L}^{\infty}(I,\mathscr{V})$ and ordered partition in major and minor intervals $\{I_j^i\}_{j=1,i=1}^{N,n}$, a minor interval $I_j^i \subseteq I$ is called *active* if the time transformed minor interval $t_w(I_j^i)$ has nonzero length

$$t_w(\tau_j^i) - t_w(\tau_j^{i-1}) = \int_{I_j^i} w(s)ds. \tag{16}$$

Otherwise, I_j^i is called inactive. Accordingly, the corresponding integer value $l_j^i = \bar{v}_{N,n}(\tau), \tau \in I_j^i$ is called active in the switching sequence $\left(l_j^i\right)_{j=1,i=1}^{N,n}$.

In the case that w is constant for each minor interval I_j^i, it follows from Definition 19 that I_j^i is active if and only if $w_j^i > 0$. For the fixed integer control function from Example 14 with e.g. three integer values and $w_j^2 = 0$, no time is spent in the system corresponding to integer value 2 and the sequence of active integer values in the major interval I_j is $(1,3)$ if $w_j^1, w_j^3 > 0$. Figure 4 illustrates the possible switches in between a major time interval. So far, just switches from a lower integer value to a higher integer value are possible in a major time interval. A switch from a higher to a lower number is only possible at a time node t_j at the boundary between two major

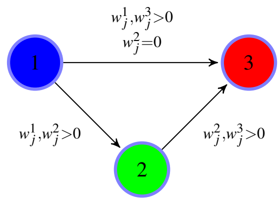

Fig. 4: Possible switches in a major time interval by using the time transformation for an example with three integer values.

intervals I_j and I_{j+1}. As a result, at least $n_{\mathscr{V}}$ major intervals are necessary to achieve the sequence of active integer values $(n_{\mathscr{V}}, \ldots, 1)$. In Section 3.3, fixed integer control functions $\bar{v}_{N,n}$ are introduced for which a switch from a lower to a higher number as well as a switch from a higher to a lower number is possible in the interior of each major interval.

The motivation of solving a time transformed problem instead of the MIOCP as defined in Section 2 is that replacing the discrete valued control function v by the real valued time control w in combination with the fixed integer control $\bar{v}_{N,n}$ leads, after a discretization, to an ordinary optimization problem instead of a mixed integer optimization problem.

As discussed in the beginning of Section 3, the method described here so far and in [17] is restricted to solve problems with a single mixed state-control domain D_0. That means, the function g from Definition 2 is not included in the time transformed optimal control problem. Thus, the time transformed MIOCP with integer independent state-control constraints results in:

Definition 20 Time transformed MIOCP with integer independent state-control constraints
For a MIOCP with the notation as in Definition 2, a fixed integer control function (Definition 12) $\bar{v}_{N,n} \in \mathscr{L}^\infty(I, \mathscr{V})$ and a time transformation (Definition 15) $t_w \in \mathscr{W}^{1,\infty}(I, I)$ with time control $w \in \mathscr{L}^\infty(I, \mathbb{R})$ is used to define the *time transformed mixed integer optimal control problem with integer independent constraints*:

$$\min_{x,u,w} \quad J^*(x,u,w) = \int_I w(\tau) B(x(\tau), u(\tau), \bar{v}_{N,n}(\tau))\, d\tau \tag{17}$$

$$\text{s. t.} \quad \dot{x}(\tau) = w(\tau) F(x(\tau), u(\tau), \bar{v}_{N,n}(\tau)) \quad \text{for a.e. } \tau \in I \tag{18}$$

$$g_0(x(\tau), u(\tau)) \leq 0 \quad \text{for a.e. } \tau \in I \tag{19}$$

$$r(x(t_0), x(t_N)) = 0 \tag{20}$$

$$w(\tau) \geq 0 \quad \text{for a.e. } \tau \in I \tag{21}$$

$$\Delta I_j = \int_{I_j} w(s) ds. \tag{22}$$

3.2 Time transformation for the MIOCP with integer dependent constraints

In extension to Section 3.1 in the following section, the time transformation is applied to the more general case of systems that may have a different mixed state-control domain D_l for each integer value $l \in \mathscr{V}$ similar as in [37,36]. Here, $(x(\tau), u(\tau)) \in D_{l^i_j}$ must hold only if integer value $l^i_j \in \mathscr{V}$ and therefore the corresponding minor interval I^i_j is active. Thus, the equation

$$g(x(\tau), u(\tau), \bar{v}_{N,n}(\tau)) \leq 0 \qquad \text{for a.e. } \tau \in I^i_j \tag{23}$$

must be fulfilled if I^i_j is active and needs not to be fulfilled if I^i_j is inactive. It follows from Definition 19 that $w(\tau) = 0$ for a.e. $\tau \in I^i_j$ if and only if I^i_j is inactive. Therefore, by multiplying the function g with $w(\tau)$ the resulting equation

$$w(\tau) g(x(\tau), u(\tau), \bar{v}_{N,n}(\tau)) \leq 0 \qquad \text{for a.e. } \tau \in I^i_j \tag{24}$$

is always fulfilled if I^i_j is inactive and equivalent to Equation 23 if I^i_j is active. As a result, Equation (24) is included in the time transformed MIOCP and assures that $(x(\tau), u(\tau)) \in D_{l^i_j}$ holds only if the minor interval I^i_j is active. The time transformed MIOCP with integer independent state-control constraints results in:

Definition 21 Time transformed MIOCP with integer dependent state-control constraints (TMIOCP)
A *time transformed mixed integer optimal control problem with integer dependent constraints*, or short TMIOCP is defined as in Definition 20, but with the additional constraint:

$$w(\tau) g(x(\tau), u(\tau), \bar{v}_{N,n}(\tau)) \leq 0 \qquad \text{for a.e. } \tau \in I. \tag{25}$$

Remark 22 Equation (25) is a generalization of Equation (19) and therefore g_0 is theoretically not necessary here. On the other hand, whenever possible rewriting (25) as (19) is recommended for a numerical computation of a discretization as in Section 5.

The influence of different fixed integer control functions $\bar{v}_{N,n}$ is analyzed in the following section.

3.3 Time transformation with CC fixed integer control function

In the following section, control consistency is introduced and an example of a CC fixed integer control function is given in Example 24. As supposed in Remark 7 the number of switching times n_s of the integer control functions $v \in \mathscr{L}^\infty$ is assumed to be finite. If in addition, the smallest distance

$$\Delta T_{min} := \min_{k \in \{1,\dots,n_s-1\}} \{T_{k+1} - T_k\} \tag{26}$$

of all switching times T_k is known, a partition of the time interval I into N major intervals $I_j, j \in \{1,\ldots,N\}$ of the same length ΔI_j can be selected such that each closed interval $\overline{I_j}$ contains at most one of the switching times T_k. Therefore, select the number of major intervals $N \in \mathbb{N}$ such that

$$\Delta I_j = \frac{\Delta I}{N} < \Delta T_{min}. \tag{27}$$

In Theorem 26, a time transformation t_w is constructed such that a given fixed integer control function $\bar{v}_{N,n}$ equals the time transformed integer control $v \circ t_w$ almost everywhere on active minor intervals I_j^i. Therefore, the fixed integer control function $\bar{v}_{N,n}$ has to be consistent to v, what is defined in the following. Moreover, N is selected such that at most one of the switching times T_k is in each major interval I_j, even though more switching times can in principle be achieved with $\bar{v}_{N,n}$ (c.f. Remark 29 for details). Such an N can in general be selected for a given smallest distance ΔT_{min} according to Equation (27). If the smallest switching time distance is unknown a priori, it is common usage to assume a smallest switching time distance ΔT_{min} as a tolerance (e.g., in [38, p. 14] the switching time distance is restricted by specifying a finite set of possible switching times) for a later numerical approximation. The resulting total number of needed minor intervals for the consistent $\bar{v}_{N,n}$ is then bounded by Nn. Example 30 gives an integer control function v for which the transformation to a CC fixed integer control function $\bar{v}_{N,n}$ is possible using a total number of $Nn = 3$ minor intervals, but the total number of necessary minor intervals is unbounded if the NCC fixed integer control function from Example 14 is used instead. Example 31 illustrates that a TMIOCP with a CC fixed integer control function can be used to yield the same analytic solution as the MIOCP. However, the selected TMIOCP with a fixed integer control function that is NCC yields a different solution with a higher objective value even though the total number of minor intervals is higher then in the case of consistency.

Definition 23 Control consistency
A fixed integer control function $\bar{v}_{N,n} \in \mathscr{L}^\infty(I,\mathscr{V})$ is *consistent* to a discrete control $v \in \mathscr{L}^\infty(I,\mathscr{V})$, if for all $j \in \{1,\ldots,N\}$ and a.e. $\tilde{t}_1, \tilde{t}_2 \in I_j$ with $\tilde{t}_1 < \tilde{t}_2$ exist $\tau_1, \tau_2 \in I_j$ with

- $\tau_1 < \tau_2$
- $\bar{v}_{N,n}(\tau_1) = v(\tilde{t}_1), \bar{v}_{N,n}(\tau_2) = v(\tilde{t}_2)$.

The function $\bar{v}_{N,n}$ is called *control consistent* if it is consistent to each discrete control $v \in \mathscr{L}^\infty(I,\mathscr{V})$, that has at most one switch per major interval I_j.

The fixed integer control function from Example 14 is for example not consistent to a control v with one switching time T_1 in the interior $\mathring{I}_j$ of I_j and $v(\tilde{t}_1) = 2 > 1 = v(\tilde{t}_2)$ for $\tilde{t}_1 < T_1 < \tilde{t}_2$ with $\tilde{t}_1, \tilde{t}_2 \in I_j$, because $\bar{v}_{N,n}(\tau_1) \leq \bar{v}_{N,n}(\tau_2)$ holds for all $\tau_1, \tau_2 \in I_j$ with $\tau_1 < \tau_2$. The following example shows a fixed integer control function $\bar{v}_{N,n}$ that is consistent to each discrete control $v : I \to \mathscr{V}$ that has at most one switch per major interval I_j.

Example 24 Let $I = [t_0, t_f]$ and $\left(I_j^i\right)_{j=1,i=1}^{N,n}$ be an ordered partition in major and minor intervals with $n = 2n_{\mathscr{V}} - 1$. Then, define the fixed integer control function $\bar{v}_{N,n} \in \mathscr{L}^\infty(I, \mathscr{V})$ as

$$\bar{v}_{N,n}(\tau) := \begin{cases} i \Leftrightarrow \tau \in I_j^i & \text{for } i \in \{1, \dots, n_{\mathscr{V}}\} \\ n+1-i \Leftrightarrow \tau \in I_j^i & \text{for } i \in \{n_{\mathscr{V}}+1, \dots, n\}. \end{cases} \tag{28}$$

Figure 5 depicts that fixed integer control function for an example with $n_{\mathscr{V}} = 3$ integer values and $N = 3$ major and $n = 5$ minor intervals. Figure 6 illustrates that a switch

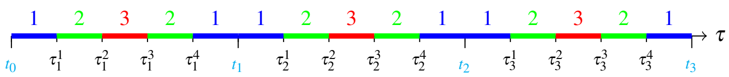

Fig. 5: Fixed integer control function that is consistent to each control that has one switch per major interval, illustrated for an example with three integer control variables.

from an integer value $l_j^a \in \mathscr{V}$ to each other integer value $l_j^b \in \mathscr{V}$ is possible for the specific $\bar{v}_{N,n}$ in the interior of each major interval I_j. Here again w is assumed to be constant on each minor interval I_j^i.

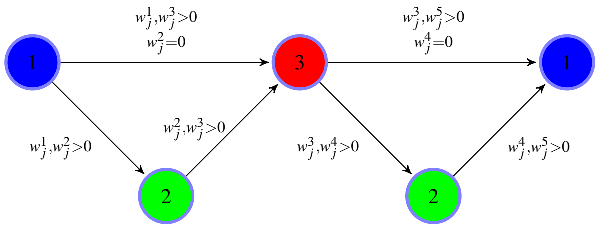

Fig. 6: Possible switches in the interior of a major interval by using the time transformation for an example with three integer values.

Remark 25 Another example of a CC fixed integer control function can be achieved by composing a permutation $\pi : \mathscr{V} \to \mathscr{V}$ with the function $\bar{v}_{N,n}$ from Example 24 to $\pi \circ \bar{v}_{N,n} \in \mathscr{L}^\infty(I, \mathscr{V})$. For example the fixed integer control function that is composed with the permutation $\pi : \mathscr{V} \to \mathscr{V}, \pi(l) = n_{\mathscr{V}} + 1 - l$ is depicted in Figure 7.

In the following main theorem of this contribution, a time transformation t_w is constructed such that the given CC fixed integer control function $\bar{v}_{N,n}$ equals the time transformed integer control $v \circ t_w$ almost everywhere on active minor intervals I_j^i.

Theorem 26 *Let $v \in \mathscr{L}^\infty(I, \mathscr{V}), I = [t_0, t_f]$ be an integer control with switching time set $\{T_k\}_{k=1}^{n_s}$ and $N \in \mathbb{N}$ such that v has at most one switching time in each major interval I_j (e.g., as selected in Equation* (27)*). Moreover let $\bar{v}_{N,n} \in \mathscr{L}^\infty(I, \mathscr{V})$ be a fixed*

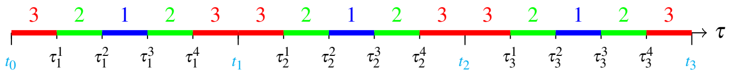

Fig. 7: CC fixed integer control function resulting from the fixed integer control function from Example 24 composed with the permutation from Remark 25, illustrated for an example with three integer control variables.

integer control function, that is consistent to v. Then, a time control $w \in \mathscr{L}^\infty(I,\mathbb{R})$ that is constant on each minor interval I_j^i exists such that the corresponding time transformation $t_w \in \mathscr{W}^{1,\infty}(I,I)$ fulfills

$$\bar{v}_{N,n}(\tau) = v(t_w(\tau)) \tag{29}$$

for all $\tau \in \{\tilde{\tau} \in I |\, w(\tilde{\tau}) > 0, t_w(\tilde{\tau}) \notin \{T_1,\dots,T_{n_s}\}\}$. Moreover it holds that the set

$$\{\tilde{\tau} \in I |\, w(\tilde{\tau}) > 0, t_w(\tilde{\tau}) \in \{T_1,\dots,T_{n_s}\}\} \tag{30}$$

is a set of measure zero.

Proof In the following, the time control w is defined separately for each major interval I_j, $j \in \{1,\dots,N\}$ to construct a representation of v by $\bar{v}_{N,n}$. The control consistency is used to select minor intervals I_j^i of I_j in the correct order in which the fixed integer control function $\bar{v}_{N,n}$ has the same value as the integer control function v. A time control w is defined that scales the selected minor intervals such that the possibly existing switching time $T_k \in I_j$ coincides for $\bar{v}_{N,n}$ and v and that scales all the remaining minor intervals to zero. With more details, two cases are considered:

1) It exists a switching time in the interior of I_j, thus $\exists k \in \{1,\dots,n_s\}$ with $T_k \in \mathring{I}_j$: Due to the selection of N, there is no switching time in $[t_{j-1},T_k[$ and $]T_k,t_j]$ and therefore v is constant on $[t_{j-1},T_k] \setminus \{T_1,\dots,T_{n_s}\}$ and constant on $[T_k,t_j] \setminus \{T_1,\dots,T_{n_s}\}$. Using that $\bar{v}_{N,n}$ is consistent to v and that $\bar{v}_{N,n}$ is constant on each minor interval, it follows that minor intervals $I_j^{i_1}, I_j^{i_2} \subset I_j$ exist with $i_1 < i_2$ and

$$\bar{v}_{N,n}(\tau) = v(\tilde{t}) \qquad \forall \tau \in I_j^{i_1}, \tilde{t} \in [t_{j-1},T_k] \setminus \{T_1,\dots,T_{n_s}\} \tag{31}$$
$$\bar{v}_{N,n}(\tau) = v(\tilde{t}) \qquad \forall \tau \in I_j^{i_2}, \tilde{t} \in [T_k,t_j] \setminus \{T_1,\dots,T_{n_s}\}. \tag{32}$$

Define the piecewise constant time control

$$w(\tau) := \begin{cases} \frac{T_k - t_{j-1}}{\Delta I_j^{i_1}} & \text{if } \tau \in I_j^{i_1} \\ \frac{t_j - T_k}{\Delta I_j^{i_2}} & \text{if } \tau \in I_j^{i_2} \\ 0 & \text{otherwise} \end{cases} \tag{33}$$

for all $\tau \in I_j$. Then, with the time transformation from Equation (15) *it follows for $\tau \in \bar{I}_j$*

$$t_w(\tau) = \begin{cases} t_{j-1} & \text{if } \tau \in [t_{j-1}, \tau_j^{i_1-1}] \\ t_{j-1} + (\tau - \tau_j^{i_1-1}) \frac{T_k - t_{j-1}}{\Delta I_j^{i_1}} & \text{if } \tau \in I_j^{i_1} \\ T_k & \text{if } \tau \in [\tau_j^{i_1}, \tau_j^{i_2-1}] \\ T_k + (\tau - \tau_j^{i_2-1}) \frac{t_j - T_k}{\Delta I_j^{i_2}} & \text{if } \tau \in I_j^{i_2} \\ t_j & \text{if } \tau \in [\tau_j^{i_2}, t_j], \end{cases} \tag{34}$$

and therefore

$$t_w(\tau) \in \begin{cases} [t_{j-1}, T_k] & \forall \tau \in I_j^{i_1} \\ [T_k, t_j] & \forall \tau \in I_j^{i_2}. \end{cases} \tag{35}$$

It follows, that the equations (31) *and* (32) *can be combined to*

$$\bar{v}_{N,n}(\tau) = v(t_w(\tau)) \qquad \forall \tau \in \{\tilde{\tau} \in I_j^{i_1} \cup I_j^{i_2} | t_w(\tilde{\tau}) \notin \{T_1, \dots, T_{n_s}\}\}. \tag{36}$$

The time control is defined such that $w(\tau) > 0 \Leftrightarrow \tau \in I_j^{i_1} \cup I_j^{i_2}$, therefore it holds

$$\bar{v}_{N,n}(\tau) = v(t_w(\tau)) \tag{37}$$

for all $\tau \in \{\tilde{\tau} \in I_j | w(\tilde{\tau}) > 0, t_w(\tilde{\tau}) \notin \{T_1, \dots, T_{n_s}\}\}$. Moreover according to Equation (33) *and* (34)

$$\{\tilde{\tau} \in I_j | w(\tilde{\tau}) > 0, t(\tilde{\tau}) \in \{T_1, \dots, T_{n_s}\}\} \tag{38}$$

$$\subseteq \left(I_j^{i_1} \cup I_j^{i_2}\right) \cap \left([t_{j-1}, \tau_j^{i_1-1}] \cup [\tau_j^{i_1}, \tau_j^{i_2-1}] \cup [\tau_j^{i_2}, t_j]\right) \tag{39}$$

$$\subseteq \{\tau_j^{i_1-1}, \tau_j^{i_1}, \tau_j^{i_2-1}, \tau_j^{i_2}\} \tag{40}$$

is a set of measure zero.

2) *A switching time in the interior of I_j does not exists, thus $\nexists k \in \{1, \dots, n_s\}$ with $T_k \in \mathring{I}_j$: The proof for this case can be done analogously to case* 1) *with the main difference that just one of the minor intervals $I_j^{i_1}$ is scaled by $w(\tau) = n > 0$ for $\tau \in I_j^{i_1}$ and all the others by $w(\tau) = 0$ for $\tau \in I_j \setminus I_j^{i_1}$.*

In both cases $w(\tau)$ is constant on each minor interval with $w(\tau) \geq 0$ and for all $j \in \{1, \dots, N\}$ equation (11) *is fulfilled because*

$$\int_{I_j} w(s) ds = \sum_{i=1}^{n} \Delta I_j^i w_j^i = \Delta I_j. \tag{41}$$

Remark 27 The equation $\bar{v}_{N,n}(\tau) = v(t_w(\tau))$ holds for a.e. $\tau \in I_j^i$ if the minor interval I_j^i is active and cannot be guaranteed to hold for $\tau \in I_j^i$ with $t_w(\tau) \in \{T_1, \dots, T_{n_s}\}$. The reason is that the function $\bar{v}_{N,n}$ is constant on the left-closed and right-open minor intervals I_j^i and the function v is constant on the interior of $t_w(I_j^i)$, but possibly different on the boundary.

Remark 28 A conclusion of Theorem 26 is that for all integer control functions v and for all consistent fixed integer control functions $\bar{v}_{N,n}$ there exists a time transformation t_w such that the following holds

$$l_j^i = v(t) \text{ for a.e. } t \in t_w(I_j^i), \tag{42}$$

with $l_j^i = \bar{v}_{N,n}(\tau)$ for $\tau \in I_j^i$. This follows directly from Theorem 26 for those minor intervals I_j^i with $w(\tau) > 0$ for $\tau \in I_j^i$ and it avoids the definition of the set $\{\tilde{\tau} \in I | w(\tilde{\tau}) > 0, t_w(\tilde{\tau}) \notin \{T_1, \ldots, T_{n_s}\}\}$. For those minor intervals I_j^i with $w(\tau) = 0$ for $\tau \in I_j^i$ (i.e. for inactive minor intervals) it holds $t_w(I_j^i)$ is a set of measure zero and therefore Equation (42) is also true. An alternative and even shorter conclusion is that $v \circ t_w$ equals $\bar{v}_{N,n}$ almost everywhere on active minor intervals I_j^i.

Remark 29 On the one hand, the assumption that v has at most one switch per major interval can be relaxed for specific fixed integer control functions, resulting in a lower number of needed major intervals N. For example for $\bar{v}_{N,n}$ from Example 24, a function v with the switching sequence $(1,2,1)$ in a major interval can be transformed to $\bar{v}_{N,n}$. Thus, instead of $N = 2$ major intervals for the two switches $(1,2)$ and $(2,1)$, $N = 1$ major interval is enough. On the other hand, if v has the switching sequence $(2,1,2)$ in one major interval, a transformation for $N = 1$ major interval cannot be found and $N = 2$ major intervals are needed instead. In general, v and its switching sequence are not known a priori and N is unknown. If the minimal switching time distance ΔT_{min} is known, or a priori given as a tolerance, a number of major intervals N can be selected as in Equation (27) to guarantee that v can be transformed to a CC $\bar{v}_{N,n}$.

The following example gives an answer to the question whether v can be transformed to $\bar{v}_{N,n}$ if the switching time T_1 is at the boundary of a major interval I_j and how many major intervals N are necessary for that. First, a time control for the CC fixed integer control function from Example 24 is selected such that v can be transformed to $\bar{v}_{N,n}$ as proved in Theorem 26, respectively Remark 28. Then, it is exemplified for the not consistent fixed integer control function from Example 14, that the transformation is possible if the switching time T_1 is at the boundary of a major interval. However, the number of needed major intervals N that yields to a switching time at the boundary turns out to be unbounded. In contrast, $N = 1$ major intervals are enough for the CC fixed integer control function in the example. A high number of major intervals can result in a higher number of discretization points in the computation of the discretized problem. The discretization and the number of needed points is discussed more specifically in the next section.

Example 30 Let $v \in \mathscr{L}^\infty(I, \mathscr{V}), I = [0,1], \mathscr{V} = \{1,2\}$ be an integer control function with one switching time T_1 in the interior of the interval $\mathring{I}$ with

$$v(t) = \begin{cases} 2 & \text{for } t \in [0, T_1] \\ 1 & \text{for } t \in \,]T_1, 1]. \end{cases} \tag{43}$$

First, let $\bar{v}_{N,3}$ be the CC fixed integer control function as in Example 24. Here, in particular that means $N = 1$ major interval and $n = 3$ minor intervals are enough for the only switching time T_1. By selecting the time transformation as in the proof of Theorem 26 it follows that $v \circ t_w$ equals $\bar{v}_{1,3}$ almost everywhere on active minor intervals I_j^i and the total number of needed minor intervals is $Nn = 3$. Then, assume that the used fixed integer control function is not consistent to v. As illustrated in Figure 4 for the case of 3 integer values, v cannot be transformed to such a fixed integer control function, if the switching time is in the interior of a major interval I_j. That is always the case for $T_1 \in [0,1] \setminus \mathbb{Q}$, because the values at the boundary of the major intervals given by $t_j = \frac{j}{N}, j \in \{0,\dots,N\}$ are rational. Thus, assume that $T_1 \in [0,1] \cap \mathbb{Q}$ and for simplicity here let $T_1 = \frac{1}{N}$ for an arbitrary $N \in \mathbb{N}$. Then, the partition of I into N major intervals leads to one major interval with switching time T_1 at the boundary. The resulting total number of minor intervals is then Nn and as the switching time $T_1 = \frac{1}{N}$ and therefore N is selected arbitrarily, the total number of needed minor intervals can be arbitrary high. For example in the case of the fixed integer control function $\bar{v}_{N,2}$ from Example 14 with $n = 2$, $v \circ t_w$ equals $\bar{v}_{N,2}$ almost everywhere on active minor intervals I_j^i if the time transformation t_w with $w(\tau) = 2$ for $\tau \in I_1^2 \cup \bigcup_{j=2}^{N} I_j^1$ and $w(\tau) = 0$ for $I_1^1 \cup \bigcup_{j=2}^{N} I_j^2$ is used.

Theorem 26 can be used to show that the optimal solution (x,u,v) of a MIOCP can be transformed to a solution (x^*,u^*,w) of a TMIOCP with the same objective value $J^*(x^*,u^*,w)$ as $J(x,u,v)$, if the used fixed integer control function $\bar{v}_{N,n}$ is consistent to the optimal integer control function v. Therefore a time control function w as constructed in Theorem 26 is used to obtain the time transformation t_w and the transformed state and control

$$x^*(\tau) := x(t_w(\tau)), \quad u^*(\tau) := u(t_w(\tau)). \tag{44}$$

If $\bar{v}_{N,n}$ is not consistent to v, the transformation to (x^*,u^*,w) with the same value $J^*(x^*,u^*,w)$ as $J(x,u,v)$ is not always possible. Then, it holds that $J^*(x^*,u^*,w) > J(x,u,v)$ and the TMIOCP cannot be used to yield the exact analytical solution of the MIOCP. The following example confirms that and uses the inverse τ_w (as given in Remark 17) to yield a trajectory x of the MIOCP from a trajectory x^* of the TMIOCP by the back transformation $x(t) := x^*(\tau_w(t))$. Similarly, the inverse can be used to yield the corresponding continuous control trajectory $u^*(t) := u(\tau_w(t))$ and the integer control trajectory $v(t) := \bar{v}_{N,n}(\tau_w(t))$. In Example 31, the back transformed trajectories resulting from a TMIOCP with a CC fixed integer control function have the same analytic solution as the MIOCP. However, the selected TMIOCP with a fixed integer control function that is NCC yields a different solution with a higher objective value in the case that the transformation is not possible.

Example 31 Similar as in Example 30, let $I = [0,1]$ and consider the following MIOCP:

$$\min_{x,u,v} \quad J(x,u,v) = \int_0^1 x(t)\, dt \tag{45}$$

$$\text{s. t.} \quad \dot{x}(t) = 1 - v(t) + u(t) \quad \text{for a.e. } t \in I \tag{46}$$

$$-u(t) \leq 0 \quad \text{for a.e. } t \in I \tag{47}$$

$$u(t) - 1 \leq 0 \quad \text{for a.e. } t \in I \tag{48}$$

$$x(0) - \frac{2}{3} = 0 \tag{49}$$

$$x(1) - 1 = 0 \tag{50}$$

$$v(t) \in \{1,2\} \quad \text{for a.e. } t \in I. \tag{51}$$

The feasible trajectories are depicted in Figure 8 (a) and the optimal trajectories in (b) and (c), the optimal objective value is $J(x,u,v) = \frac{22}{36}$. The optimal integer control is

$$v(t) = \begin{cases} 2 & \text{for } t \in [0,T_1] \\ 1 & \text{for } t \in \,]T_1,1] \end{cases} \tag{52}$$

with $T_1 = \frac{1}{3}$. Figure 9 illustrates the optimal trajectories of the resulting TMIOCP for

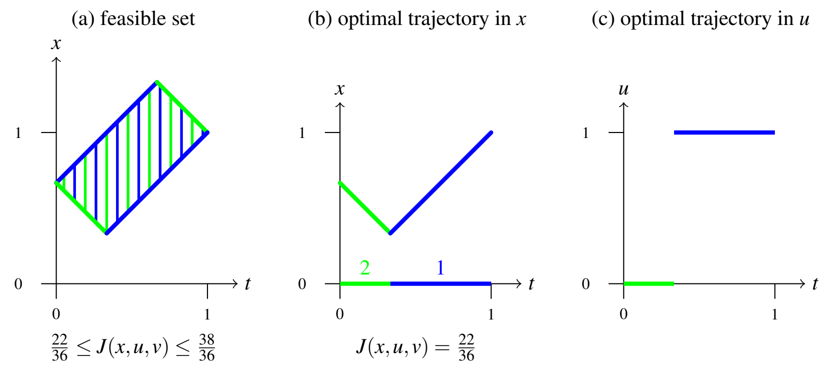

Fig. 8: The MIOCP from Example 31. (a) Feasible trajectories in the striped area. (b) Optimal state trajectory with integer control v on the t-axis. (c) Optimal continuous control trajectory.

different fixed integer control functions and Figure 10 the back transformed optimal trajectories. The first columns (a)-(b) represent the optimal trajectories using $\bar{v}_{1,2}$ and $\bar{v}_{2,2}$ from Example 14 and the third column (c) represents the optimal trajectories using the CC $\bar{v}_{1,3}$ from Example 24. The optimal trajectory $x^* \circ \tau_w$ using $\bar{v}_{1,3}$ equals the optimal trajectory for v of the MIOCP and can also be achieved with the NCC $\bar{v}_{3,2}$ from Example 14 what is not plotted here.

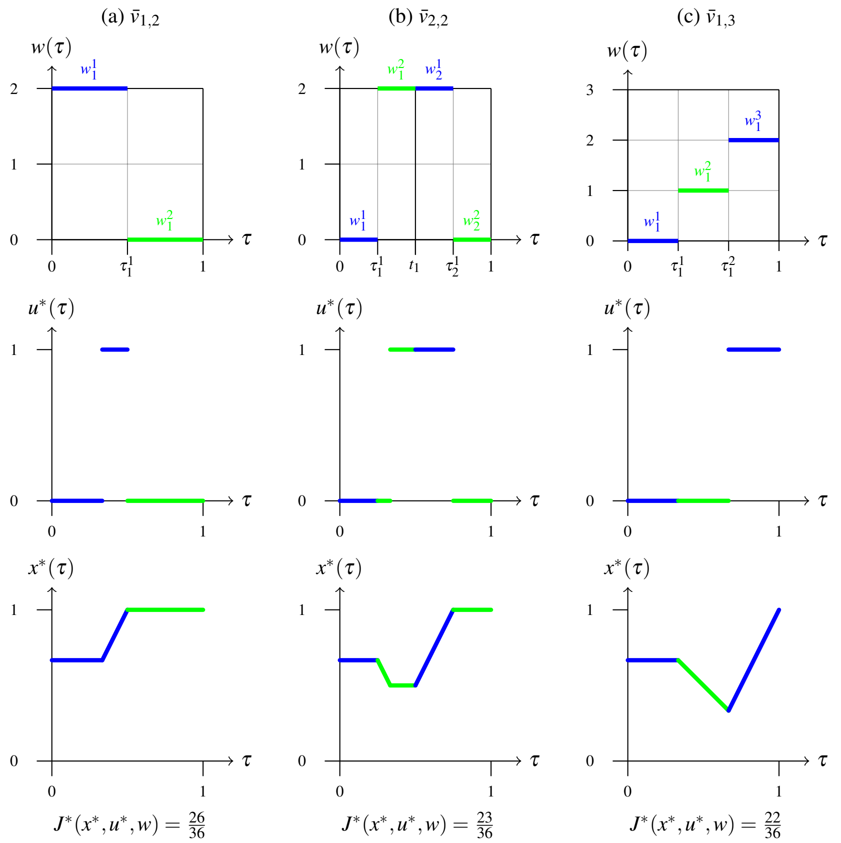

Fig. 9: Optimal solutions of the TMIOCP from Example 31 for the NCC $\bar{v}_{1,2}$ in column (a), the NCC $\bar{v}_{2,2}$ in column (b) and the CC $\bar{v}_{1,3}$ in column (c).

Remark 32 The resulting trajectories in Example 31 are analytically computed and without loss of generality the trajectories for the continuous controls u^* are assumed zero on those minor intervals I_j^i with $w(\tau) = 0$ for $\tau \in I_j^i$. Arbitrary other feasible values for u^* are possible there and lead to the same solution in x^*. Experimental numerical computations of the problem for the CC $\bar{v}_{1,3}$ using different initial guesses w_{init} for the discretized w and zero for the other discretized trajectories of x and u lead to the same solution trajectories as analytically computed. The used local optimization method is a MATLAB interior point algorithm where the necessary derivatives are approximated by finite differences. The computations indicate that the initially given switching sequence resulting from the discretized initial guess w_{init} changes during the optimization and therefore such a change is not avoided in general due to the time transformation method. However, this does not mean that a general discretized TMIOCP always converges to the global minimum, in particular for objectives with nonconvex discretized objective functions or feasible sets. The parameters of the different optimizations are represented in Table 1. Here, the first column denotes the

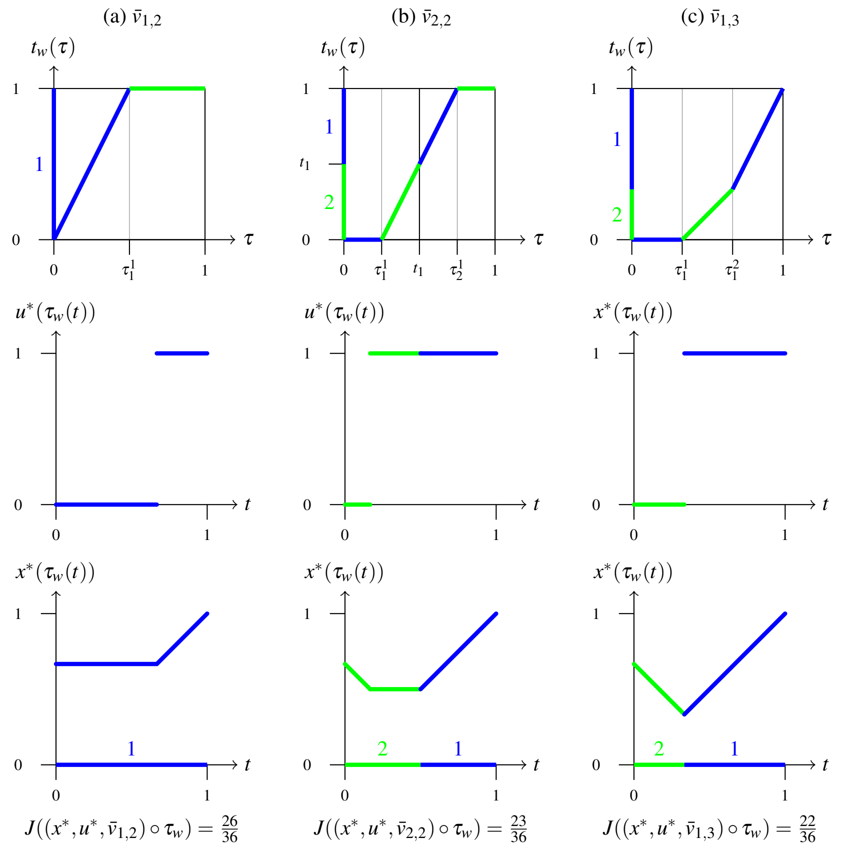

Fig. 10: Back transformed optimal solutions of the TMIOCP from Example 31 for the NCC $\bar{v}_{1,2}$ in column (a), the NCC $\bar{v}_{2,2}$ in column (b) and the CC $\bar{v}_{1,3}$ in column (c).

Table 1: Optimization parameters for Example 31 using the CC $\bar{v}_{1,3}$ and the NCC $\bar{v}_{3,2}$

w_{init}	$\bar{v}_{N,n}$	#var	#eq	#iter	CPU time	J
$(1,1,1)$	$\bar{v}_{1,3}$	10	6	25	0.10499s	0.61111
$(1,1)$	$\bar{v}_{3,2}$	19	11	34	0.18598s	0.61111
$(3,0,0)$	$\bar{v}_{1,3}$	10	6	27	0.11143s	0.61111
$(0,0,3)$	$\bar{v}_{1,3}$	10	6	20	0.085531s	0.61111
$(3,0)$	$\bar{v}_{3,2}$	19	11	119	0.5924s	0.61111
$(0,3,0)$	$\bar{v}_{1,3}$	10	6	23	0.090559s	0.61111
$(0,3)$	$\bar{v}_{3,2}$	19	11	44	0.23273s	0.61111
$(0,0,0)$	$\bar{v}_{1,3}$	10	6	27	0.10414s	0.61111
$(0,0)$	$\bar{v}_{3,2}$	19	11	34	0.18021s	0.61111

selected discretized values for the initial guess w_{init} in each major interval. It turns out that the number of needed iterations #iter and the CPU time is lower for the CC

$\bar{v}_{1,3}$ compared to the NCC $\bar{v}_{3,2}$, as well as the numbers of discretized variables #var and equality constraints #eq. A modified example with switching time $T_1 = \frac{1}{N}$ and $N > 3$ leads to bigger differences since at least the numbers #var and #eq are higher for the NCC case and fixed for the CC case (cf. Example 30).

4 Discretization of the TMIOCP

Forced Hamiltonian systems with a Hamiltonian $H_l : \tilde{X}_l \to \mathbb{R}$ and a controlled force $f_l : \tilde{D}_l \to \mathbb{R}^{n_q}$ for each integer value $l \in \mathscr{V}$ are investigated. The pure continuous differential equation for each integer value is defined as follows

$$\dot{q} = \frac{\partial H_l}{\partial p}(q,p), \tag{53}$$

$$\dot{p} = f_l(q,p,u) - \frac{\partial H_l}{\partial q}(q,p). \tag{54}$$

The functions $q : I \to \mathbb{R}^{n_q}$ and $p : I \to \mathbb{R}^{n_q}$ represent the position and the momentum of the dynamical system. A Hamiltonian is typically given by $H_l(q,p) = T_l(q,p) + V_l(q)$ with the systems kinetic energy $T_l(q,p) = \frac{1}{2} p^T \left(M_l(q)\right)^{-1} p$ and its potential energy $V_l(q)$. The matrix $M_l(q)$ is symmetric and positive definite and thus invertible. A simple example for a force function is the thrust of an engine [35], that is directly given by u with $f_l(q(t), p(t), u(t)) = u(t)$. Denoting the right-hand side of the equations (53) and (54) by $F(x,u,l)$ and the state $x = (q,p)$ leads after the time transformation (Section 3) to the differential equation (18).

The resulting infinite dimensional TMIOCP is discretized to yield a finite dimensional optimization problem that can be solved using gradient based optimization methods as SQP or interior point methods. Confer e.g. [33] or [15] for an introduction in gradient based optimization methods. Gerdts [17] uses an s-staged Runge Kutta method to discretize a differential equation that describes a double-lane-change manoeuvre of a car with different gears describing different systems. The continuous control function u is discretized there with one variable for each major interval I_j. In contrast to that, here the discretization has one variable for each minor interval I_j^i. The used discretization scheme is the midpoint rule and preserves geometric invariants of Hamiltonian systems [22, p. 190]. The influence of the time transformation on the preservation is not analyzed here, but planned in future works. Approximating q and p by linear functions and u and w by constant functions on each minor interval I_j^i, leads to a representation by six discretization points q_j^{i-1}, q_j^i, p_j^{i-1}, p_j^i, u_j^i, and w_j^i. Here, q_j^{i-1}, q_j^i and p_j^{i-1}, p_j^i represent values at the boundaries of I_j^i and u_j^i and w_j^i the constants on I_j^i. The midpoint of a linear approximation of q is then given by $q_j^{i-1/2} := \frac{q_j^{i-1}+q_j^i}{2}$ and analogously for p. Thus, the discretized differential equation reads

$$q_j^i = q_j^{i-1} + w_j^i \Delta I_j^i \frac{\partial H_{l_j^i}}{\partial p}\left(q_j^{i-1/2}, p_j^{i-1/2}\right) \tag{55}$$

$$p_j^i = p_j^{i-1} + w_j^i \Delta I_j^i \left[f^{l_j^i}\left(q_j^{i-1/2}, p_j^{i-1/2}, u_j^i\right) - \frac{\partial H_{l_j^i}}{\partial q}\left(q_j^{i-1/2}, p_j^{i-1/2}\right)\right] \tag{56}$$

with $l_j^i = \bar{v}_{N,n}\left(\frac{\tau_j^{i-1}+\tau_j^i}{2}\right)$. Assuming $(q_{j+1}^0, p_{j+1}^0) = (q_j^n, p_j^n)$, because (q_j^n, p_j^n) approximates position and momentum at the right boundary of I_j and (q_{j+1}^0, p_{j+1}^0) on the left boundary of I_{j+1}, leads to a discrete integration method for the complete interval I.

Similar, the discretization of the objective function J^* from Equation (17) is given by

$$\sum_{j=1}^{N}\sum_{i=1}^{n} w_j^i \Delta I_j^i B(x_j^{i-1/2}, u_j^i, l_j^i), \tag{57}$$

using the shorter notation $x_j^{i-1/2} = (q_j^{i-1/2}, p_j^{i-1/2})$. The integer dependent mixed state-control constraints (Equation (25)) result to

$$w_j^i g(x_j^{i-1}, u_j^i, l_j^i) \leq 0 \tag{58}$$

$$w_j^i g(x_j^i, u_j^i, l_j^i) \leq 0. \tag{59}$$

The first equation guarantees, that the approximated state x_j^{i-1} at the left boundary of the minor interval I_j^i is part of the domain $(x_j^{i-1}, u_j^i) \in D_{l_j^i}$ and the second equation guarantees that $(x_j^i, u_j^i) \in D_{l_j^i}$ holds for the approximated state x_j^i at the right boundary of the minor interval I_j^i. The Equation (22) that restricts the time control leads to

$$\Delta I_j = \sum_{i=1}^{n} w_j^i \Delta I_j^i \tag{60}$$

for each major interval I_j. The Equations (19), (20) and (21) lead to

$$g_0(x_j^{i-1}, u_j^i) \leq 0 \tag{61}$$

$$g_0(x_j^i, u_j^i) \leq 0 \tag{62}$$

$$r(x_1^0, x_N^n) = 0 \tag{63}$$

$$w_j^i \geq 0. \tag{64}$$

The Equations (58), (59) and (64) are called blocks of vanishing constraints [36], because the inequalities (58) and (59) vanish (i.e. are automatically fulfilled) if $w_j^i = 0$. Blocks of vanishing constraints are a generalization of vanishing constraints, where for the latter type of constraints the Equations (58) and (59) are replaced by a single inequality equation $w_j^i g(x_j^{i-1}, u_j^i, l_j^i) \leq 0$ with $g(x_j^{i-1}, u_j^i, l_j^i) \in \mathbb{R}$. In [2], it is revealed that necessary constraint qualifications such as the Mangasarian-Fromovitz constraint qualifications or the linear independence constraint qualifications are usually violated for optimization problems including vanishing constraints. Recently developed regularization techniques as, e.g. proposed in [1], [23] or [24] avoid problems caused by the violation of the constraint qualifications. Similar regularizations are used in [31] or [3] to solve optimization problems including equilibrium constraints or complementarity constraints that are also a generalization of vanishing constraints. The vanishing constraints are regularized here by replacing the zeros on the right-hand-side of the Equations (58) and (59) by a regularization parameter $r_1 > 0$ as in [36]. The solution trajectories for each regularization parameter $r_k, k \in \mathbb{N}$ are used as an initial guess for the next optimization with a lower parameter value $r_{k+1} < r_k$, until the desired problem with a value $r_{n_r} = 0$ is solved. The next section provides computational results for a first example.

5 Computational results

In the following section, an example is introduced, the hybrid mass oscillator with three springs. A similar problem with two springs instead of three is computed in [14] by a decoupling of the discrete and continuous parts. The discretized problem here is minimized by the gradient based IPOPT interior point method [41]. The needed derivatives for the minimization are provided by the automatic differentiation package ADOL-C [42]. All computations are obtained in a virtual machine with 3GB allocated RAM running on a MacBook with 2.6 GHz Intel Core i7 CPU.

A mass $M = 0.6\ kg$ is fixed on a linear spring with the spring constant $c = 1\frac{kg}{s^2}$. The gravitational acceleration is supposed to be $g = 10\frac{m}{s^2}$. Let $q : [t_0, t_f] \to \mathbb{R}$ represent the time dependent vertical position of the bottom of the mass. The spring is relaxed at the position $q_1^r = 0$, two further springs are mounted in parallel with relaxed positions $q_2^r = -1$ and $q_3^r = -2$. Both have the spring constant c and are not fixed to the mass, such that these springs are only active if the mass is below their relaxed position. Figure 11 (a) gives a sketch of the hybrid control system and Figure 11 (b) represents the possible switches between the systems. The resulting three Hamiltonians are given as

$$H_l(q,p) = \frac{p^2}{2M} + Mgq + \sum_{i=1}^{l} \frac{c}{2i}(q - q_i^r)^{2i} \tag{65}$$

for $l \in \mathscr{V} = \{1,2,3\}$. Thus, the differential equations are $\dot{p} = F_p(q,p,u,l)$ with

$$F_p(q,p,u,l) = u - Mg - c\sum_{i=1}^{l}(q - q_i^r)^{2i-1} \tag{66}$$

and $\dot{q} = F_q(q,p,u,l)$ with

$$F_q(q,p,u,l) = \frac{p}{M}. \tag{67}$$

Here, the control force is $f_l(q,p,u) = u$ for each $l \in \mathscr{V}$ and the control function $u(t) \in \mathbb{R}$ is bounded by the interval $[-Mg, Mg]$. That leads to the integer independent mixed state-control constraint

$$g_0(x,u) = (u - Mg, -Mg - u) \leq (0,0). \tag{68}$$

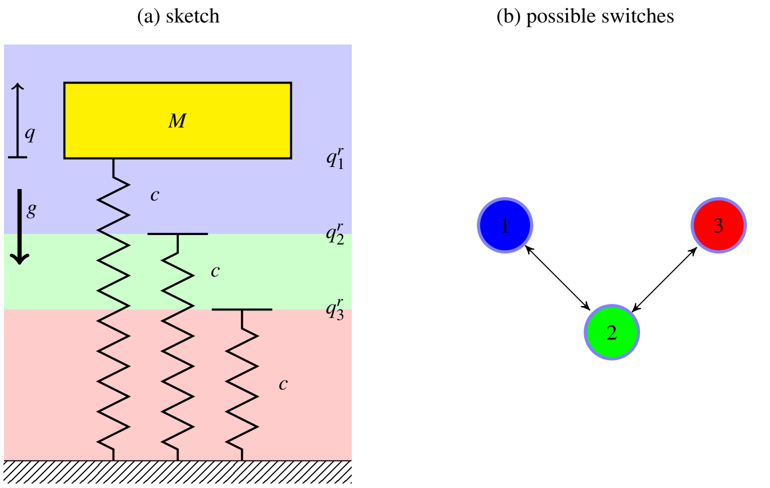

Fig. 11: The hybrid mass oscillator.

The Hamiltonian H_1 is active for $q(t) \in [q_2^r, \infty[$, H_2 for $q(t) \in [q_3^r, q_2^r]$, and H_3 for $q \in]-\infty, q_3^r]$. Thus, the regularized integer dependent mixed state-control constraints of the MIOCP for three regularization parameters $r_1 = 0.2, r_2 = 0.1$ and $r_3 = 0$ are as follows

$$g_1(x,u) = q_2^r - q \leq r_k \tag{69}$$

$$g_2(x,u) = (q - q_2^r, q_3^r - q) \leq (r_k, r_k) \tag{70}$$

$$g_3(x,u) = q - q_3^r \leq r_k. \tag{71}$$

The objective functional represents the control effort

$$J(x,u,v) = \frac{1}{2} \int_{t_0}^{t_f} u^2(t)\, dt \tag{72}$$

for the time interval $[t_0, t_f] = [0, 12]$. The problem is time transformed and the minimization using the NCC $\bar{v}_{N,3}$ from Example 14 is compared with the minimization using the CC $\bar{v}_{N,5}$ from Example 24. In both cases, the initial guess for all the discretized variables q, p, u, w is zero.

Two versions of this problem are considered. First, the initial state is the equilibrium of the first spring without a mass $(q_0^0, p_0^0) = (0,0)$ and the final position is free, such that the expected solution is a periodic oscillation for the position trajectory and constant zero for the control trajectory. The resulting state and control trajectories are plotted in Figure 12 for $N = 60$ major intervals. The result for the CC $\bar{v}_{N,5}$ has a control effort close to zero and the state trajectory oscillates almost as expected. However, the control effort for NCC $\bar{v}_{N,3}$ is rather high in the beginning and relatively low subsequently. The resulting position trajectory oscillates with a low amplitude between the second and the third domain and avoids switches from domain two to domain

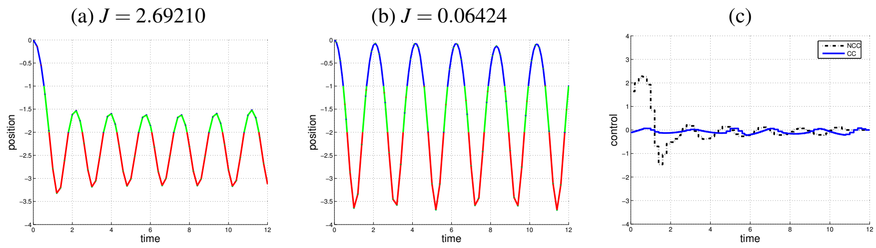

Fig. 12: Locally optimal discretized position q and control u trajectory of the hybrid mass oscillator with free final state and $N = 60$ major intervals. The computation of the TMIOCP is done with (a) the NCC $\bar{v}_{N,3}$, (b) the CC $\bar{v}_{N,5}$. The control trajectories for both are depicted in (c). Colors mark the positions with different numbers of active springs.

one at all. For increasing $N = 70, 80, \ldots$ for the NCC $\bar{v}_{N,3}$, the computed objective value gets closer to the objective value $J = 0.06424$ of the CC $\bar{v}_{60,5}$. For example for $N = 290$ the objective value is $J = 0.06555$. Thus, for a comparable objective value, this leads to $Nn = 870$ discretization intervals for the NCC and $Nn = 300$ intervals for the CC fixed integer control function.

Secondly, the approximated equilibrium in domain three is used as the final constraint $(q^n_N, p^n_N) = (-2.51, 0)$. The resulting optimal trajectories are plotted in Figure 13 for different fixed integer control functions. The accuracy of the computed trajec-

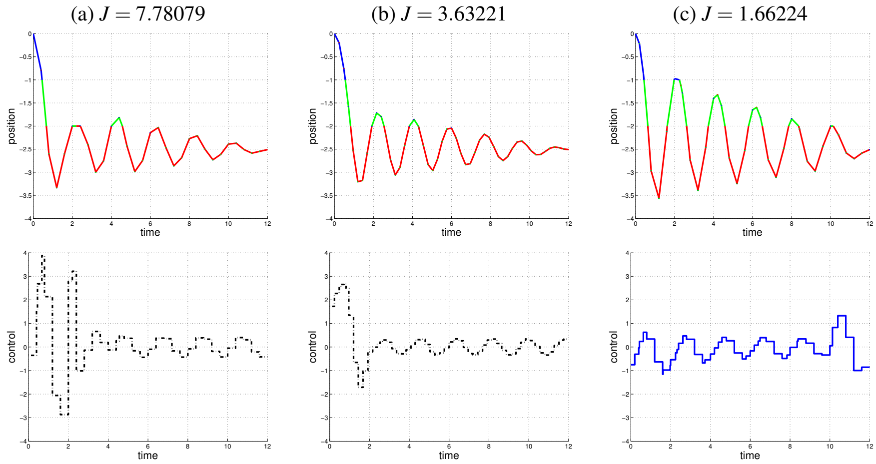

Fig. 13: Locally optimal discretized position q and control u trajectory of the hybrid mass oscillator with final state $(q^n_N, p^n_N) = (-2.51, 0)$. The computation of the TMIOCP is done with (a) the NCC $\bar{v}_{30,3}$, (b) the NCC $\bar{v}_{50,3}$ and (c) the CC $\bar{v}_{30,5}$. Colors mark the positions with different numbers of active springs.

Table 2: Optimization parameters for the hybrid mass oscillator with the NCC $\bar{v}_{N,3}$ and the CC $\bar{v}_{N,5}$. The numbers $\#iter_k$ represent the iterations for the regularization parameter r_k and the CPU time is combined for all three optimizations.

$\bar{v}_{N,n}$	$\#var$	$\#eq$	$\#ineq$	$\#iter_1$	$\#iter_2$	$\#iter_3$	J	CPU time
$\bar{v}_{10,3}$	123	74	80	173	18	59	31.77658	0.424s
$\bar{v}_{10,5}$	203	114	140	207	24	732	5.67604	2.559s
$\bar{v}_{20,3}$	243	144	160	231	54	106	5.26786	1.113s
$\bar{v}_{20,5}$	403	224	280	550	95	99	7.77714	3.686s
$\bar{v}_{30,3}$	363	214	240	191	24	177	7.78079	1.601s
$\bar{v}_{30,5}$	603	334	420	445	46	523	1.66224	7.828s
$\bar{v}_{40,3}$	483	284	320	171	34	75	2.49205	1.913s
$\bar{v}_{40,5}$	803	444	560	704	60	183	1.56475	9.069s
$\bar{v}_{50,3}$	603	354	400	320	130	251	3.63221	5.425s
$\bar{v}_{50,5}$	1003	554	700	244	62	169	2.49963	5.234s
$\bar{v}_{60,3}$	723	424	480	241	35	319	3.03275	4.812s
$\bar{v}_{60,5}$	1203	664	840	340	65	254	1.28626	9.918s

tories is in general identical if the discrete time nodes are identical and therefore the same total number of minor intervals possibly leads to the same accuracy. However, the accuracy is not guaranteed because time transformed minor time nodes may coincide. As shown in Table 2, the resulting objectives reveal a better solution for the CC $\bar{v}_{30,5}$ than for the NCC $\bar{v}_{50,3}$. The difference in the objective value is even bigger if the CC $\bar{v}_{30,5}$ is compared to the NCC $\bar{v}_{30,3}$. In this case, the guaranteed accuracy of the computed trajectories is the same because the major time nodes are identical. Further computations for different numbers of major intervals N reveal that the objective value is better for the CC compared to the NCC function in the most of the computed examples as can be seen in Table 2. All the computations converged to a local solution using an optimality tolerance of 10^{-8} and in all cases the constraint violation of the solution is below 10^{-11}. However, global optimality can not be guaranteed for the resulting optimized trajectories because a local optimizer is applied on a very rough initial guess. A similar objective value as for the CC $\bar{v}_{30,5}$ is achieved with the NCC $\bar{v}_{130,3}$ ($J = 1.67923$) resulting in a total of $Nn = 150$ compared to $Nn = 390$ minor intervals. The computational effort is not directly comparable for the same numbers of major intervals N, as the resulting trajectories are different. Comparing the CPU time for the computations with similar objective values reveals a huge advantage for the CC $\bar{v}_{30,5}$ with $7.828s$ compared to $25.888s$ for the NCC $\bar{v}_{130,3}$. The objective values (a) and the maximal constraint violations (b) are visualized for each iteration and all regularization steps in Figure 14 for the optimization with the NCC $\bar{v}_{30,3}, \bar{v}_{50,3}$ and the CC $\bar{v}_{30,5}$. The beginning of each optimization with an initial guess with a relatively low objective value that does not fulfill the constraints leads to increasing objective values while the constraint violation is decreasing. This is also the case after the optimization is restarted with a new regularization parameter r_k.

(a) J (b) $\|constraints\|_\infty$

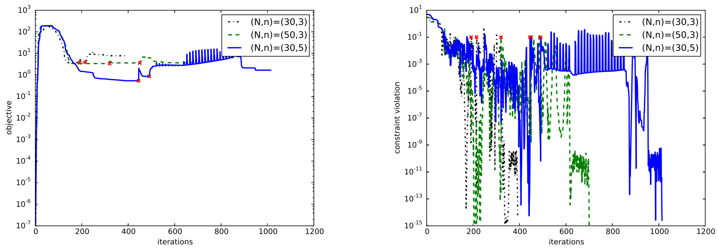

Fig. 14: Convergence history for the optimization with the NCC $\bar{v}_{30,3}, \bar{v}_{50,3}$ and the CC $\bar{v}_{30,5}$ of (a) the objective values and (b) the maximal constraint violation. The beginning of the optimization with a new regularization parameter is marked with a red cross.

6 Conclusions and outlook

A time transformation is introduced formally for integer independent mixed state-control constraints and then extended to the more general case of integer dependent mixed state-control constraints. The major contributions are Theorem 26, where it is shown that any variable integer control function v with at most one switch per major interval can be transformed to every consistent fixed integer control function $\bar{v}_{N,n}$. It is further shown that the transformation is in general not possible for a fixed integer control function $\bar{v}_{N,n}$ that is not consistent to v if the switching time is in the interior of a major interval. However, the selection of a higher number N of major intervals possibly enables the transformation also for a fixed integer control function $\bar{v}_{N,n}$ that is not consistent to v, but the needed number of major intervals N is in general unknown a priori and unbounded, leading to a possibly very high number of discretization variables of the discretized problem. Moreover, if N is selected too low, the corresponding objective value J^* of the TMIOCP is higher than the objective value J of the MIOCP. The discretized TMIOCP is introduced and used to numerically compute the hybrid mass oscillator example. It can be observed that the CC fixed integer control function leads to a better accuracy of the resulting trajectories, in particular the computed switching times are more accurate. The numerical examples reveal that trajectories with comparable objective values possibly can be computed with a significantly lower total number of discretization intervals if a CC instead of a NCC fixed integer control function is used.

In future works we want to consider more complex systems where switches are caused due to mechanical contact. Modeling such systems with integer dependent mixed state-control constraints may require specific numerical integration methods to discretize the differential algebraic equations and have to be tested. An adaptive refinement, starting with a low number of major intervals and keeping already computed integer values $l \in \mathscr{V}$ fixed whenever acceptable, could lead to reduced computing times and therefore allow the computation of more challenging systems.

Acknowledgements The authors would like to thank the anonymous reviewers for helpful comments and suggestions.

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