

NON-CLASSICAL PROPOSITIONAL CALCULI

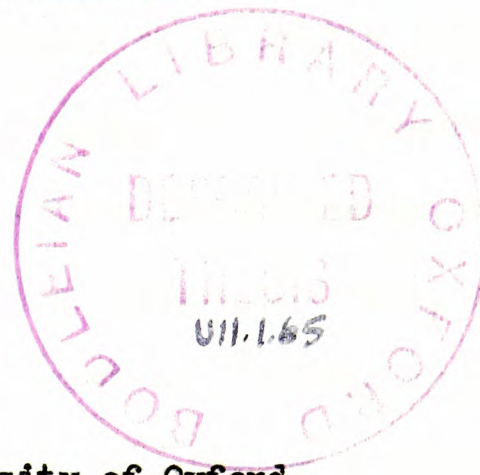
ABSTRACT

There exist well-known varieties of implication, such as strict, intuitionist, three-valued and rigorous, which are non-classical in the sense of being more restrictive than material implication. But there exists also a type of implication, intuitively plausible, which is non-classical not only in being more restrictive, but in satisfying certain theses which are classically false. These theses are exceedingly venerable, dating back to Aristotle and Boethius, but, despite their plausibility, have been generally rejected by logicians since. It has not been noticed, however, that in Sextus Empiricus reference is made to a species of Stoic implication which fits them perfectly.

In this work formal recognition is given to this species of implication, known as connexive implication. It is shown that none of the well-known systems of propositional logic is connexive, and a new system is accordingly constructed. A proof of consistency is given, and a number of problems posed for further investigation.

NON-CLASSICAL PROPOSITIONAL CALCULI

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'The study of Triangles and Circles interferes not with the study of Minds. Nor does the Student in the mean while suppose himself advancing in Wisdom, or the Knowledg of Himself or Mankind. All he desires, is to keep his Head sound, as it was before. And well, he thinks indeed, he has come off, if by good fortune there be no Crack made in it. As for other Ability or Improvement in the Knowledg of human Nature or the World; he refers himself to other Studys and Practice. Such is the Mathematician's Modesty and good Sense. But for the Philosopher, who pretends to be wholly taken up in considering his higher Facultys, and examining the Powers and Principles of his Understanding; if in reality his Philosophy be foreign to the Matter profess'd; if it goes beside the mark, and reaches nothing we can truly call our Interest or Concern; it must be somewhat worse than mere Ignorance or Idiotism. The most ingenious way of becoming foolish, is by a System.'

Shaftesbury, Advice to an Author, 3 i.

PREFACE

This work owes its existence to the conviction (not shared by most logicians) that there remains much original work to be done in propositional logic. Hence there are not many acknowledgements to be made in this preface. Professors Arthur Prior and Alan Anderson both helped and encouraged me, and I am indebted to Nuel Belnap for reading and making suggestions on the manuscript. I think I have learned as much from my students, Lung-Ock Chung and Patrick Schindler of McGill, as they have from me. Finally I want to thank Mrs. Sharpe, who did the typing, and Ann Griffin, who proof-read and made the index.

Storrs McCall

Pittsburgh,
September 21, 1963.

NOTATION

Polish notation will be used throughout:

<u>Cpq</u>	for if <u>p</u> then <u>q</u>
<u>Np</u>	not <u>p</u>
<u>Kpq</u>	<u>p</u> and <u>q</u>
<u>Apq</u>	<u>p</u> or <u>q</u>
<u>Epq</u>	<u>p</u> if and only if <u>q</u> .

The lower case letters p, q, r, s, t, and their capitalized counterparts for implications, will be used as propositional variables, and u, v, w, x, y, z (plus capitals) will be used as metalogical variables ranging over well-formed formulae.

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CHAPTER ONE

INTRODUCTORY

1. Introduction

The logic of propositions, as employed by Aristotle, systematized by the Stoics, rediscovered by the Scholastics, ignored by the later Schoolmen and by Leibniz, discovered once again by Frege, and finally brought to its present position of eminence in Western thought by the authors of Principia Mathematica, is the subject of this monograph.

It is misleading, however, to speak of the logic of propositions, as if this long historical process resulted in the gradual revelation of one and only one logical system. On the contrary, the history of logic reveals that there has been continuous and profound disagreement about the central notions of implication, negation, conjunction and alternation; that is, about the very subject matter of propositional logic. It is the contention of the present author that these disagreements are fruitful, in that they stimulate further investigation, and that they are far from being resolved, so that we may look forward to continuing controversy and development in the field.

Roughly speaking, there are two sectors in which logical research takes place. First (and foremost these days) there is the tracing and elaboration of relations existing among the formal analogues of certain concepts used in reasoning and argument; e.g. implication, negation, necessity, quantity, identity. These formal analogues, represented by symbolic operators, find their place in a wide variety of deductive systems, each one granting slightly different logical powers to the operators in question. What limits are placed on these powers in the

various systems are determined entirely by the structure of the systems themselves: in investigating these powers, as reflected in the various structural properties of the systems, the logician is studying a subject matter which is as 'hard', as determinate, and as independent of his own thought processes as the crystalline structure of a diamond.

Secondly, logical research may concern itself, not only with formal properties and powers of the symbolic analogues of certain concepts, but with such analogues as fitting representatives of the concepts in question. The properties of a formal operator such as material implication are unalterable, although there may still be some of its properties of which we are ignorant. But if dissatisfaction with material implication is expressed, on the grounds that it fails to conform to our intuitive notion of implication, then we are free to seek different formal systems, with different implication analogues. The motives behind this search differ from those prompting us to explore formal systems for their own sake. We explore the latter, as we climb mountains, because they are there, but we search among these innumerable formal systems because we wish to find the one, or ones, which fit our intuitions best. This quest is not independent of the philosophical opinions and prejudices of individual logicians.

It would be remarkable if each of these branches^h of logical investigation did not complement the other, and in this work both will be pursued. Certain natural requirements for a formal analogue of implication will be discussed, together with past attempts to produce formal systems embodying one, and new attempts will be made, resulting in some new systems. In addition, some purely formalistic problems connected with these systems will be considered.

2. Historical Survey

Aristotle must be regarded as the founder of Western logic. But his syllogistic differs radically from the logic of propositions. Probably the best way to introduce anyone to propositional logic is to invite him to consider the difference between the following syllogism in Barbara:

All B is A
All C is B,
therefore, all C is A,¹

and the following inference-schema of the Stoics:

If the first then the second,
the first,
therefore the second.

The most striking difference between the two lies in the fact that only terms, such as 'men', 'animals', 'Greeks', etc., can be meaningfully substituted for the variables A, B, C of the syllogism, while only propositions, such as 'it is day', 'it is light', etc., can be meaningfully substituted for the variables 'the first' and 'the second' of the Stoic inference-schema. It is to Łukasiewicz that we are indebted for having pointed this out, and for having reinstated the Stoics as the recognized founders of the logic of propositions.²

Aristotle, however, though his syllogistic is a logic of terms, made use of many individual theses of propositional logic. One of his

¹I ignore for the purposes of this example Łukasiewicz's insistence that Aristotle always states his syllogistic moods in the form of a conditional proposition rather than an inference: See Aristotle's Syllogistic (AS), p. For differing opinions in any case see J.L. Austin's review of Łukasiewicz in *Mind* 61 (1952), pp. 397-8, and A.N. Prior, Formal Logic (FL), 2nd edition, Oxford 1962, p. 116. Łukasiewicz himself is not consistent on this matter, for he occasionally uses 'premiss' and 'conclusion' where he should be using 'antecedent' and 'consequent'.

²Łukasiewicz, Zur Geschichte der Aussagenlogik, *Erkenntnis* 5 (1935), pp. 111-131.

favourite methods of proving the validity of one syllogistic mood from another is by 'indirect reduction': a form of reductio ad absurdum reasoning whereby the joining of the contradictory of the conclusion of a syllogism to one of the premisses yields a new conclusion which contradicts the other premiss.¹ Put in Stoic terms, this mode of reasoning would allow us to derive from any inference-schema of the form 'The first and the second, therefore the third' an inference-schema of the form 'The first and not the third, therefore not the second'. Without derivations of this sort, appropriated from propositional logic, Aristotle's system of syllogisms would be no system at all. For this reason propositional logic, or the 'theory of deduction', is the most basic part of any logical, mathematical or scientific theory.

Another feature of Stoic propositional logic is its articulation in a system, a system in which each valid inference-schema is derived or proven from a certain number of 'indemonstrables'. This way of treating logic, like Euclid's treatment of geometry, we call its axiomatization in a deductive system, the difference here being that it is, in a manner of speaking, deduction itself that is being axiomatized. Judging from what we know of the history of logic, it appears that, although virtuosity in the subject reached new heights in the Scholastics' discussions of individual theses and inferences, the very idea of axiomatizing propositional logic lay fallow from the time of the Stoics until 1879, when it was rediscovered by Frege.

¹It is in this way that Aristotle proves the mood Baroco by reducing it to Barbara. See An pr. 27a37-27b1.

Frege axiomatized propositional logic. Furthermore, although he was probably unaware of it, his conception of implication is precisely the same as that of Philo the ^{Megarian} ~~Stoic~~ - an implication of the form 'If p then q ' is true for all instances of p and q except when p is true and q false.¹ Frege's axiomatization takes the notions of implication and negation as primitive, and defines conjunction and alternation in terms of them. His axioms, from which he proves further propositional theses with the help of the rules of substitution and modus ponens, may be written in Łukasiewicz's symbolism as follows:

1. $CpCqp$
2. $CCpCqrCCpqCpr$
3. $CCpCqrCqCpr$
4. $CCpqCNqNp$
5. $CNNpp$
6. $CpNNp$.

The axiomatic basis provided by Whitehead and Russell in the first volume of Principia Mathematica (1910) is different from Frege's, and represents a retrograde step in one respect. The authors take the notions of (weak) alternation and negation as primitive, but their axioms all contain the defined notion of implication, no doubt for reasons of greater perspicuity. Certainly it is natural that any axiomatic basis for the theory of deduction should contain the central notion of implication, and for this reason implication/negation bases ('C-N bases') are in the writer's opinion to be preferred to A-N bases. Frege's C-N basis contains redundancies, but in 1929 Łukasiewicz produced the following set of C-N axioms, each one of which is independent in the sense of not being derivable

¹See the summary of Frege's propositional logic, with an explanation of his symbolism, in W. and M. Kneale, The Development of Logic (DL), Oxford 1962, p. 478 ff.

from the others by means of the rules of substitution and modus ponens:¹

1. CCpqCCqrCpr
2. CCNppp
3. CpCNpq

Whitehead and Russell refer in their introduction to the properties of consistency and completeness, but give no proof that their system possesses either.² This was left to such workers as Post, who in 1921 published a proof that:

- (a) The propositional logic of Principia Mathematica is consistent (i) in the sense that no expression and its own negation are both provable in the system, and also (ii) in the sense that the simple expression p (from which any expression is derivable by substitution) is not provable;
- (b) The system is complete in the sense that if any expression, not already provable, were added to it, it would become inconsistent in sense (ii).³

The importance of these results, of which more will be said in section 50, cannot be over-emphasized. From the way Whitehead and Russell wrote, it is apparent that they believed their system was at least consistent. By a happy accident it turned out in fact to be so, but it is extraordinary that, even in the introduction to the second edition of

¹See Łukasiewicz, AS p. 80, for an account of the provenance of these axioms.

²A.N. Whitehead and B. Russell, Principia Mathematica, vol. 1, Cambridge 1910, p. 12. The terms they use are 'coherence' and 'adequacy'.

³E.L. Post, Introduction to a general theory of elementary propositions, American Journal of Mathematics, 43 (1921), pp. 171-3. Kneale (DL p. 689) reports that Bernays had obtained similar results slightly earlier than Post. The standard reference for these proofs is to Hilbert and Ackermann, Principles of Mathematical Logic, New York 1950, (translation of the German edition of 1928).

Principia, published in 1925, no reference is made either to Post or to any consistency or completeness proof. Nevertheless, it is to Whitehead and Russell that we look, and shall no doubt always look, for the definitive description of what has come to be known as the classical or two-valued propositional calculus.

3. Alternatives to classical implication

The definition of implication urged by Frege and by Philo is that of material implication. But from the earliest days alternatives to material implication were considered. The following passage from Sextus Empiricus, by now a locus classicus in histories of logic, illustrates with what keenness the question of implication was debated in the fourth century B.C. - so much so that it was said the very crows on the roof-tops croaked about what conditionals were true.

(1. Material implication)

'Philo says that a sound conditional is one that does not begin with a truth and ^{and} with a falsehood, e.g. when it is day and I am conversing, the statement "If it is day, I am conversing."

(2. Diodorean implication)

'But Diodorus says it is ^{and} one that neither could nor can begin with a truth and with a falsehood. According to him the conditional statement just quoted seems to be false, since when it is day and I have become silent it will begin with a truth and end with a falsehood. But the following statement seems to be true: "If atomic elements of things do not exist, then atomic elements of things do exist." For he maintains it will always begin with the false antecedent "Atomic elements of things do not exist" and end with the true consequent "Atomic elements of things do exist."

(3. Connexive implication)¹

'And those who introduce the notion of connexion say that a conditional is sound when the contradictory of its consequent is incompatible with its antecedent. According to them the conditionals mentioned above are unsound, but the following is true: "If it is day, it is day."

(4. Inclusive implication)¹

'And those who judge by implication² say that a true conditional is one whose consequent is contained potentially in its antecedent. According to them the statement "If it is day, it is day" and similarly every conditional which is repetitive will apparently be false; for it is impossible for a thing to be contained in itself.'³

If Philo's conception of implication is precisely that of material implication, Diodorus' is very close to strict implication. We can say, if we wish, that a true material implication is one that does not begin with a truth and end with a falsehood; a true strict implication is one that cannot begin with a truth and end with a falsehood.⁴ Diodorus would disagree with most philosophers, however, about what it is for something to be possible or impossible. For him, what neither was nor is nor will be true is impossible;⁵ thus an

¹The terms 'connexive' and 'inclusive' are Bochenski's, taken from his History of Formal Logic (HFL) (translated from the German by I. Thomas), Notre Dame 1961, pp. 118-9.

²Bochenski has: 'Those who judge (implication) by what is implicit...'

³Sextus Empiricus, Hyp. Pyrrh. ii, 110-12, translated by Kneale, DL, pp. 128-9.

⁴'p strictly implies q' is defined as 'it is impossible that p is true and q false'. See C.I. Lewis, A Survey of Symbolic Logic, (SSL), Berkeley 1918, p. 292.

⁵B. Mates, Stoic Logic, Berkeley 1953, p. 37. Note that on Diodorus' own view one of the two words 'could' and 'can' in his definition of a true conditional is redundant, for 'was possible' = 'is possible'. See Kneale, DL, p. 132. [This is questionable - S.M.]

integer which never has been and never will be written down is impossible of being written down. This is not Lewis's conception of possibility, but apart from this Diodorean and strict implication are the same.

The third type of implication described by Sextus is of a fascinating but little-known variety: it will in fact be the basis for the new formal systems investigated in this work. It has been given the name of 'connexive' implication in virtue of the fact that a true conditional of this sort asserts, or comes to be in virtue of, a logical connexion or relation of incompatibility between the antecedent and the negation of the consequent. Mrs. Kneale gives some evidence that this type of implication was adopted by Chrysippus,¹ and Mates actually calls it Chrysippean,² but if it deserves to be equal in interest and importance to material, strict and rigorous³ implication, as I believe it does, then perhaps 'connexive' is a better word. As is shown below, 'connexive' implication turns out to be identical with what Nelson in 1930 called 'intensional' implication, but for the benefit of those philosophers who boggle at the adjective 'intensional' (though, as I shall show in section 49, there is no need for alarm) I use 'connexive'.

Whatever we call it, it is clear that we cannot identify Sextus' third type of implication with strict implication, as both Mates

¹See Martha Hurst, Implication in the fourth century B.C., Mind. 44 (1935), p. 491, and Kneale, DL, p. 129. Diogenes Laërtius gives a full description of this type of implication, with examples, in the paragraph following his statement of Chrysippus' definition of conditional propositions, and describes no other type of implication (Lives of Eminent Philosophers, vii, 48).

²Mates, p. 49.

³For 'rigorous' implication, see section 20.

and Bochenski seem to do.¹ For Sextus makes it plain that the conditional exemplifying the second type, 'If atomic elements of things do not exist, then atomic elements of things do exist' could not exemplify the third type. In fact no conditional of the form 'If not- $\underline{x}$, then $\underline{x}$ ' could, since the negation of the consequent would not be incompatible with the antecedent. But there are many such conditionals which hold for strict implication, e.g. when $\underline{x}$ is a necessary proposition, and hence intensional implication differs from strict.

Not much has been made, in the subsequent development of logic, of Sextus' fourth kind of implication. Mates points out that, in ruling out repetitive implications of the form 'If $\underline{x}$, then $\underline{x}$ ', of which the Stoics apparently made use, it could scarcely have been adopted by them, and Mrs. Kneale conjectures that it was of Peripatetic origin, remarking that it is quite possible to view repetitive implications as limiting cases of necessary connexion, such extensions for the purpose of simplicity being not uncommon in mathematics.² There is an isolated instance of a similar view of implication in Wittgenstein's Tractatus, where he says, 'If one proposition follows from another, the latter says more than the former, and the former less than the latter. If $\underline{p}$ follows from $\underline{q}$ and $\underline{q}$ from $\underline{p}$, then they are one and the same proposition.'³ This view puts forward implication as excluding equivalence, and would seem to call for a non-classical logic in which, in Polish notation, CCpqNCqp

¹Mates, p. 49; Bochenski, p. 119.

²Mates, p. 49; Kneale, DL, pp. 129 and 134.

³Tractatus Logico-Philosophicus 5.14-5.141, translation of Pears and McGuinness, London 1961.

is asserted and Cpp rejected, but so far as I know no work has yet been done on such a system.

Returning to the third type of implication, we find a description of it in a list of possible types of implication assembled by Paul of Venice in the 15th century: 'Tenthly people say that for the truth of a conditional it is required that the opposite of the consequent be incompatible with the antecedent.'¹ In the twentieth century, advocates of this ^{view} have included such philosophers as Nelson and Strawson. Nelson defines 'p intensionally implies q' as 'p is inconsistent with the contradictory of q' and uses the expression 'p entails q' as a synonym for these,² although his definition of entailment is not taken directly from Moore, with whom the term seems to have originated. In a paper written in 1919, in which he criticizes Whitehead's and Russell's use of 'material implication' as a name for the relationship between p and q in 'It is not the case that p is true and q false', Moore says:

'We require, first of all, some term to express the converse of that relation which we assert to hold between a particular proposition q and a particular proposition p, when we assert that q follows from or is deducible from p. Let us use the term "entails" to express the converse of this relation. We shall then be able to say truly that "p entails q", when and only when we are able to say truly that "q follows from p" or "is deducible from p", in the sense in which the conclusion of a syllogism in Barbara follows from the two premisses, taken as one conjunctive proposition;

¹See Bochenski, p. 196.

²E.J. Nelson, Intensional Relations, Mind 39 (1930), pp. 440-453. Nelson's views will be discussed further in section 14.

or in which the proposition "This is coloured"
follows from "This is red." : 1

This notion of entailment is probably what Nelson had in mind when he spoke of it as 'a necessary connexion between meanings'.

The definition of implication in terms of inconsistency is taken up again by Strawson in defining ' S_1 entails S_2 ' as ' S_1 and not- S_2 is inconsistent'.² Put in this way, the definition of entailment is not unlike Lewis's definition of strict implication, which we may write as ' S_1 and not- S_2 is impossible'. For Strawson, the two in fact come to the same thing, for he maintains that the contradictory of an inconsistent statement is a necessary statement, and consequently to say that S_1 entails S_2 is to say that not-(S_1 and not- S_2) is logically necessary. Abbreviating 'not-(S_1 and not- S_2)' to ' $S_1 \supset S_2$ ' we get that ' S_1 entails S_2 ' = Df ' $S_1 \supset S_2$ is logically necessary',³ which is just the definition of strict implication. However, for reasons given earlier, I do not think that implication defined in terms of necessity and possibility should be confused with implication defined in terms of inconsistency (Sextus' second and third types of implication). The basis of this confusion lies, as Nelson saw, in regarding inconsistency as something attributable to a single proposition, rather than as a relationship between propositions.

¹G.E. Moore, External and Internal Relations, reprinted in his Philosophical Studies, London 1922, p. 291. Moore's definition of entailment is exactly what Mill offers as the meaning of a hypothetical proposition; thus "If A is B, C is D" is found to be an abbreviation of the following: "The proposition C is D, is a legitimate inference from the proposition A is B" (A System of Logic, London 1843, vol. i, p. 111). Cf. also C.I. Lewis's 'ordinary meaning of "implies" - for which " p implies q " is equivalent to " q can validly be inferred from p ".' (Implication and the algebra of logic (IAL), *Mind* 21 (1912), p. 529.)

²P.F. Strawson, Introduction to Logical Theory, London, 1952, p. 20.

³Strawson, p. 23.

4. Implication, entailment and conditional propositions

In this section, I shall make some general comments on the connexion between implication, entailment and conditional propositions. When Moore introduced ' p entails q ' to mean ' q is deducible from p ', he distinguished sharply between this and 'it is not the case that p is true and q false', but he also distinguished sharply between the latter and both 'If p then q ' and the ordinary sense of ' p implies q '.¹ Now it seems that Moore's characterization of entailment is equally well applicable to what we ordinarily understand by implication, though not of course to 'material implication', and so there is no great objection to identifying ' p implies q ' with ' q is deducible from p '. In fact the defenders of strict implication defend it in exactly this way - by attempting to show that strict implication is the converse of deducibility.² But there seems on the face of it to be a considerable difference between implication on the one hand and a conditional proposition on the other.

Firstly, implication appears to be a relation between propositions rather than a proposition itself. This is ~~true~~, but logicians often refer, in the case where p implies q , to the conditional proposition 'If p then q ' as an implication, thus making the word do duty both for the relation and the proposition expressing it. It is clear, though, that as far as its actual use in argument and reasoning is concerned, possession of a conditional proposition 'If p then q ' is in no way inferior to knowledge of the relationship of implication between the propositions p and q to which it corresponds. In the latter case, given p ,

¹Moore, pp. 295-6.

²See C.I. Lewis and C.H. Langford, Symbolic Logic, New York 1932, p. 235 ff.

we may infer q simpliciter; in the former case, given p and 'If p then q' we may infer q by the rule of modus ponens. Secondly, although all implications may be written as 'if...then' propositions, not all such propositions express implications. This point has been made by numerous logicians. Mitchell, for example, in discussing the proposition 'If it rains, the match will be cancelled', remarks that the falling of rain does not imply, but is causally related to, the cancellation of a cricket match.¹ Nor does the notorious 'I can if I choose' assert that choosing implies ability. Hence non-implicational 'if...then' propositions exist. However, this point having been made, 'if...then' expressions and symbolic expressions Cxy will generally hereafter be called implications.

5. Critique of material implication

What now of material implication? Those who recommend it as an adequate formal analogue of implication usually stress that it carries at least the minimum burden that any implication-operator must, namely that it leads us only from truth to truth, and never from truth to falsehood.² Furthermore, it is said, the totality of true material conditions^{al} mirrors all those instances in which considerations of (non-modal) propositional logic alone would lead us to argue validly from one proposition p to another proposition q; although, as for example in the case of the paradoxes of material implication, this totality may also mirror other instances in which we would not wish to argue from p to q. Hence, it is urged, we should view material implication as the

¹D. Mitchell, An Introduction to Logic, London 1962, p. 62.

²'The essential property that we require of implication is this: "What is implied by a true proposition is true".' (Whitehead and Russell, p. 94).

broadest possible type of implication-operator, convenient and useful for many purposes, and capable of having limitations and restrictions placed upon it so as to make it useful for others. Strict implication may indeed be regarded as a restricted or limited type of material implication, since there are no true strict implications which are not also true material implications, although there are many true material implications which are not true strict implications.

To this defence of material implication two replies may be made. Firstly, mere breadth does not make an implication-operator, and secondly, for all its breadth, material implication may not be broad enough.

To illustrate the first objection, we shall consider an example offered by Anderson and Belnap. It is often said by the defenders of material implication that there is nothing odd about the system if one views it as a calculus of disjunction, conjunction and negation rather than of implication. Even the 'paradox' of material implication, 'If p , then if q then p ', really only means 'Either not- p , or else not- q or p ', and there is nothing paradoxical about this. The main thing is that the system never leads from truth to falsehood.

Now, Anderson and Belnap say:

'Let us imagine a logician who offers the following formalization as an explication or reconstruction of implication in formal terms. In addition to the rule of the modus ponens and the rule of substitution⁷ he takes as primitive the following three axioms:

Cp p
CCp q CCq r Cpr
CCp q Cqp.

One might find those who would object that 'if...then---' doesn't seem to be symmetrical, and that the third axiom is objectionable. But our logician has an answer to that.

"There is nothing paradoxical about the third axiom; it is just a matter of understanding the formalism properly. 'If p then q ' means simply 'Either p and q are both true, or else they are both false', and if we understand the arrow in that way, then our rule will never allow us to infer a false proposition from a true one, and moreover all the axioms are evidently logical truths. The implication relations of this system may not exactly coincide with the intuitions of naive, untutored folk, but it is quite adequate for my needs, and for the rest of us who are reasonably sophisticated. And it has the important property, common to all kinds of implication, of never leading from truth to falsehood." ¹

The lesson to be learned from this example is plain: merely possessing the weakest property common to all kinds of implication, namely that of never leading from truth to falsehood, does not make an operator an implication-operator. The fact that other operators, which we would refuse to call implication-operators, have this property, shows what additional burdens are borne covertly by material implication.

To make the second objection stick, we must show that, despite its comprehensiveness, there are actually some implications which our intuitions would tell us are true, but which lie outside the system of material implication. It is harder to show this, that material implication lacks something, than it is to show the other, that it proves too much, but the end result may be more telling. Thus Mitchell:

'Until logicians succeed in bringing to light laws of propositional logic which lie outside the system of the calculus (and which the notation of the calculus is inadequate to express), we may assume it to be satisfactory.'²

Leaving aside the matter of notation, which I cannot see the point of, we are presented with a clear challenge. If we can produce

¹A.R. Anderson and N.D. Belnap, jr., The Pure Calculus of Entailment (PCE), Journal of Symbolic Logic (JSL), 27 (1962), p. 19 ff. I have taken the liberty of changing the symbolism to Polish symbolism.

²Mitchell, p. 67.

one or more cases of genuine implication within propositional logic, which are at the same time not cases of material or classical implication, then we shall have produced something with a real claim to being an 'alternative logic'. Such a logic would more faithfully reflect our common intuitions concerning the notion of implication, and as such would have more claim to the title 'logic' than the classical two-valued calculus. I believe that examples of such non-classical implication relations can be found, and that in fact one such example has been known and discussed since the time of Aristotle. Its history will be traced in the next chapter.

CHAPTER TWO

THE HISTORY OF A SUPPOSED ERROR

6. Aristotle

In the second book of the Prior Analytics we find a very interesting passage, in which Aristotle seems to be saying that it is never possible for a proposition to be implied by its own negation. The passage is as follows:

'It is impossible that the same thing should be necessitated by the being and by the not-being of the same thing. I mean, for example, that it is impossible that B should necessarily be great if A is white, and that B should necessarily be great if A is not white. . . . For if B is not great A cannot be white. But if, when A is not white, it is necessary that B should be great, it necessarily results that if B is not great, B itself is great. But this is impossible.'¹

What Aristotle is trying to show here, as Lukasiewicz explains, is that two implications of the form 'If p then q' and 'If not-p then q' cannot both be true. For the first implication yields, by transposition, 'If not-q then not-p', and this together with the second gives 'If not-q then q' in virtue of the fact that implication is a transitive relation. But this conclusion is according to Aristotle impossible; not only in the case of a thing's being great, since it is clear from the context that Aristotle meant his argument to apply to all propositions indifferently, but in every case. A proposition cannot be implied by its own negation.

I shall in future refer to the assertion, that no proposition can be implied by its own negation, as Aristotle's first thesis, and to

¹An. pr. 57b3-14; translation in Lukasiewicz, AS, pp. 49-50.

the assertion, that no proposition and its contradictory can both imply some other proposition, as Aristotle's second thesis.

Aristotle's view as expressed in the above passage from the Prior Analytics have been criticized by both Łukasiewicz and Kneale. First a minor point: it is not clear how Aristotle's example serves his purpose, which is to show that in a syllogism with false premisses the conclusion may be true, but its truth cannot be necessitated by them.¹ But, more importantly, both Łukasiewicz and Kneale disagree with Aristotle on 'If not- p then p '. Łukasiewicz states that a proposition of this form may in certain cases be true, and when it ^{is} yields p by the law 'If (if not- p , then p), then p ', known as the law of Clavius or the consequentia mirabilis.² Kneale conjectures that Aristotle may have propounded his objections to the possibility of a proposition and its contradictory both implying the same conclusion in connexion with certain arguments put forward by the Megarian school, who were precursors of the Stoics and opponents of Aristotle.³ The Megarians, being followers of the Eleatic tradition, may have been interested in reductio ad absurdum arguments, and the pair of implications which Aristotle refused to admit are those required as premisses for a constructive version of the reductio ad absurdum.

¹See W. Kneale, Aristotle and the Consequentia Mirabilis (ACM), *Journal of Hellenic Studies*, 77 part i (1957), pp. 62-66.

²Łukasiewicz, AS, pp. 50 and 80.

³Kneale, ACM, p. 66.

The reductio is:

If p then q ,
 if p then not- q ,
 not both q and not- q
 therefore not- p

and its constructive counterpart is:

If p then q ,
 if not- p then q ,
 either p or not- p ,
 therefore q .

Hence if the Megarians attempted to put forward arguments of the latter type, they might have met with Aristotle's insistence that their first two premisses could not simultaneously be true. Be that as it may, Kneale sides with Łukasiewicz in maintaining that there are occasions where a proposition can be implied by its own contradictory, namely those where the proposition in question is a necessary truth, and points out that in a surviving fragment of his early work Protrepticus Aristotle argues in exactly the way the Megarians may have:

If we ought to philosophize, then we ought to philosophize,
 if we ought not to philosophize, then we ought to philo-
 sophize (i.e. in order to justify this view),
 therefore in any case we ought to philosophize.

To sum up, Aristotle maintains that it is not possible for a proposition to be implied by its own contradictory - a view which, incidentally, accords with those whom Sextus reports as advocating 'connexive' implication - and Łukasiewicz and Kneale disagree. Aristotle's view, if correct, would entail that in practice there could be no employment of either the constructive version of reductio ad absurdum or, as we shall see in the next section, of reductio ad absurdum itself.

7. Boethius

We find in Boethius' De Syllogismo hypothetico, a work written between 510 and 523 A.D. containing an elaborate system of inference-schemata in propositional logic, a number of inferences which depend for their validity on something very like Aristotle's second thesis, namely that which asserts the incompatibility of the two implications

- (i) (If p then q
(If not-p then q.

If we transpose these implications we get:

- (ii) (If not-q then not-p
(If not-q then p,

which we may re-write as follows:

- (iii) (If p then q
(If p then not-q.

Hence if Aristotle maintains the incompatibility of the propositions (i), it is only a step from there to the incompatibility of the propositions (iii), which incompatibility rules out in practice any employment of the (destructive) reductio ad absurdum of section 6. It is incompatibility of type (iii) that Boethius makes use of in some of his hypothetical syllogisms, and the assertion of this incompatibility I shall call Boethius' first thesis.

Boethius divided his inference-schemata into eight classes, which are reproduced by Dürr.¹ The first inference-schema of the second group of the second of these classes, which the original text gives as follows:

'Si est A, cum sit B, est C; . . . atqui cum sit B, non est C; non est igitur A.'²

¹K. Dürr, The Propositional Logic of Boethius, Amsterdam 1951.

²C. Prantl, Geschichte der Logik im Abendlande, Leipzig 1855, Vol. 1, p. 706 note 157; Dürr, p. 37.

may be transliterated thus:

If p , then if q then r ,
if q then not- r ,
therefore, not- p .

The reasoning which led Boethius to assert the validity of this inference-schema was presumably this. Since the two implications 'if q then r ' and 'if q then not- r ' are incompatible, the second premiss contradicts the consequent of the first premiss. Hence we get, by modus tollens, the contradictory of the antecedent of the first premiss, namely 'not- p '.

Is reasoning of this kind correct? Not, certainly, if we interpret the 'if...then' of Boethius' inference-schema as material, strict, or any other hitherto known variety of implication. As Dürr points out, the thesis CKCpCqrCqNrNp is not to be found in Principia Mathematica. But the reasoning may be correct nonetheless. In any case it is not a temporary aberration on Boethius' part, since exactly similar arguments lie behind the seven other schemata of group two, class two.¹

In group two, class three, a different type of inference is made. The first schema is as follows:

If (if p then q) then r ,
not- r ,
therefore, if p then not- q .²

Here Boethius seems to be arguing from 'It is not the case that if p then q ' (got by modus tollens from the two premisses) to 'If p then not- q '.

Although in many ways not so intuitively compelling as the inference from

¹We signal, but reject, the attempt of R. van den Driesche to bring these schemata into line with Principia Mathematica by interpreting the second (though not the first) of the cum's in the Latin text above as a conjunction- rather than as an implication-operator (Sur le 'De syllogismo hypothetico' de Boèce, *Methodos* 1 (1949), pp. 293-307).

²Dürr, p. 42.

'If p then q ' to 'It is not the case that if p then not- q ', which reflects the incompatibility of 'If p then q ' and 'If p then not- q ', this inference has some weight of tradition behind it. Aristotle in the De Interpretatione, for example, in trying to show that when something is necessary it is also possible, says that if it did not follow that it were possible ($NCpq$) the opposite would follow, namely that it were impossible ($CpNq$).¹ And the formula $CNCpqCpNq$ holds for material implication, though not for strict.

As we have seen, Boethius' first thesis may be stated as Cpq 's implying $NCpNq$. The converse of this, $NCpq$'s implying $CpNq$, I shall call Boethius' second thesis. The first implication entails that Cpq and $CpNq$ are contraries, the second that they are sub-contraries, and the two together that they are contradictories.

In the syllogisms of group two, class seven Boethius combines his two theses.

If (if p then q) then (if r then s),
 if r then not- s
 therefore, if p then not- q .²

Here 'If r then not- s ' is being taken to imply 'It is not the case that if r then s ', which with the first premiss yields 'It is not the case that if p then q ' by modus tollens, which in turn implies 'If p then not- q '. From the frequency with which this type of reasoning occurs, we may conclude that Boethius was convinced of the mutual contradictoriness of the pair of implications 'If p then q ' and 'If p then not- q '.

¹De int. 22b28-32.

²Dürr, p. 56.

8. A conjecture concerning Stoic logic

The question of the extent to which Boethius owes his ideas about propositional logic to the Stoics is discussed by Dürr, who comes to the conclusion that it was the Peripatetics rather than the Stoics who served as Boethius' model. The principal evidence he presents for this view is that Boethius' system in De syllogismo hypothetico does not seem either to be based on or to take account of the five 'undemonstrated' inference-schemata of Chrysippus (see sect. 2). These are the following:

- I. If the first, then the second,
the first,
therefore, the second.
- II. If the first, then the second,
not the second,
therefore, not the first.
- III. Not both the first and the second,
the first,
therefore, not the second.
- IV. Either the first or the second,
the first,
therefore, not the second.
- V. Either the first or the second,
not the first,
therefore, the second.

Now Boethius' system in De syllogismo hypothetico contains schemata of types I, II, IV, and V above (groups one and two of class one; class eight), but no schema of type III. Therefore it would seem, as Dürr says, that Boethius did not use Chrysippus' system as a model, for, had he done so, he would scarcely have dispensed with the third indemonstrable unless he recognized that it was superfluous. But we find no evidence that he did so recognize it. Further reasons for thinking that Boethius' system was of Peripatetic origin are to be found in the absence of schemata corresponding to reductio ad absurdum arguments, popular with the Stoics, and

in Boethius' espousal of his first thesis, which is a natural consequence of Aristotle's second thesis.

This being said, however, there remain other equally strong reasons for thinking Boethius' logic to be Stoic-inspired, at least in part. The second of Boethius' works on propositional logic which Dürr takes up is his commentary on Cicero's Topics, and there Boethius does give Chrysippus' five indemonstrables. Furthermore, we find the classic Stoic examples of these schemata in concrete form: 'If it is day, it is light; it is day, therefore, it is light'.¹ Here the Stoic influence is unmistakable, and in what follows is presented the thread of an admittedly very tenuous line of argument leading to the conclusion that the incompatibility of 'If p then q' and 'If p then not-q' may well have found its way into some versions of the Stoic dialectic too.

In Boethius' commentary we find, in addition to Cicero's erroneous seventh schema 'Non et hoc et illud; non autem hoc; illud igitur',² a new schema, corresponding to Boethius' second thesis. At least, we find, not the schema itself, but a group of concrete examples, of which the following is one:

It is not the case that if it is day it is not light,
it is day,
therefore, it is light.³

Boethius places the inference, of which this is an example, third in his list of seven, so it appears that he may have intended it to replace Cicero's third, which Cicero repeated as number six. The fact that it

¹Dürr, p. 67.

²Kneale, DL, p. 179.

³Dürr, p. 69.

continues the day/light motif indicates that it may be Stoic. What I shall show is that, if we add the schema corresponding to Boethius' second thesis to Chrysippus' five, and use another schema of an obvious nature which probably found its place among the 'innumerable' Stoic inferences, we may derive the required incompatibility.

We start with the schema derived from the following example of Boethius:

It is not the case that if he is awake, he snores,
he is awake,
therefore, he does not snore.¹

The schema is:

- (i) Not (if p then q),
 p
 therefore, not-q.

We now use the 'principle of conditionalization', which the Stoics knew and made use of,² to derive from (i) the following true conditional:

- (ii) If not (if p then q) then (if p then not-q).

Next we require the following schema, which we shall assume to have been known to the Stoics:

- (iii) If p then q,
 if r then s,
 either p or r,
 therefore, either q or s.

The substitution-instance of this schema that we use is:

- (iv) If (if p then q) then (if p then q),
 if not (if p then q) then (if p then not-q),
 either (if p then q) or not (if p then q),
 therefore, either (if p then q) or (if p then not-q).

¹Dürr, p. 69.

²Mates, p. 74 ff.

Here the first premiss is a 'repetitive' conditional such as is used in the Stoic 'sign' argument reported by Sextus, the second is (ii) above, and the third is a logical truism also used in the 'sign' argument, and by Kneale in his reconstruction of some Stoic proofs.¹ Consequently the conclusion,

(v) Either (if p then q) or (if p then not- q),

will be a true disjunction.

We are now in a position to derive the incompatibility of the two conditionals mentioned above. Since the incompatibility is non-classical, and since all the results we have so far obtained are classically valid, we must make use of the only non-classical schema among the five indemonstrables, schema ^{IV}~~iv~~. Schema ^{IV}~~iv~~ holds only for strong disjunction, not for (classical) weak. We proceed as follows:

(vi) Either (if p then q) or (if p then not- q),
if p then q ,
therefore, not (if p then not- q).

Here what we have done is to show that if 'If p then q ' is true, 'If p then not- q ' must be false: the first thesis of Boethius. Admittedly it is pure guesswork whether anyone in antiquity actually constructed such a proof, but it is interesting to know that, by adding schema (i) to his five indemonstrables, Chrysippus probably could have. Whether or not he actually could have we cannot say, until we know more about how he constructed his proofs.

I shall conclude this section with some questions. Firstly, did schema (i) form part of the Stoic system? If not, who originally proposed it? Did Boethius, or some Peripatetic, propose it for the purpose of showing not only that CpNq followed from NCpq, but also that NCpq

¹Mates, p. 81; Kneale, DL, p. 168.

followed from CpNq? Finally, even without schema (i), did the Stoics profess the truth of the disjunctive proposition (v), from which the incompatibility of Cpq and CpNq follows by the fourth indemonstrable? These are questions ^{whose answer} which further investigation into the sources for our knowledge of Stoic logic may reveal.

9. Abelard

Boethius' use of the proposition 'If p then not-q' to contradict the proposition 'If p then q' in his hypothetical syllogisms has since met with almost universal disapproval among logicians. His principal expositor and commentator in the Middle Ages, Peter Abelard, tries to correct Boethius' supposed error in his Dialectica, a work completed shortly before Abelard's death in 1142. Abelard ~~says~~ that the denial or 'destruction' of a conditional is accomplished not through the denial of any of its components, but through the denial of the whole proposition. Accordingly, he changes Boethius' group two class two syllogism to an inference of which the following is a concrete example:

'Si est homo, cum est animatum est animal, sed non cum est animatum est animal, ergo non est homo.'¹

Here Abelard is replacing Boethius' syllogism by one which does not require passing from CpNq to NCpq, as it would if the second premiss were, à la Boethius, 'Sed cum est animatum non est animal', and in discussing group two class three syllogisms Abelard dispenses with Boethius' second thesis by giving examples which do not require passing from NCpq to CpNq.² These

¹Abelard, Dialectica, ed. L.M. de Rijk, Assen, 1956, pp. 506-7.

²Abelard, p. 509.

expressions are not, according to Abelard, mutually contradictory.

Yet Abelard's thinking does not itself seem to be completely clear on this point. In trying to construct a proof of the law of transposition for conditionals, according to which 'If p then q ' implies 'If not- q then not- p ', he seems to make just the same sort of step which he criticizes Boethius for making. The proof, by reductio, is similar to that constructed by Aristotle to demonstrate his second thesis, and proceeds as follows:

'Let us posit this as true: if there is man there is animal, and doubt about this: if there is not animal there is not man, i.e. whether 'animal' negated negates 'man'. We shall confirm this in the following way. Either 'animal' negated negates 'man' or negated it admits 'man', so that it may happen that when 'animal' is denied of something man may exist in that thing. Suppose it be conceded that when 'animal' is denied, man may persist; yet it was formerly conceded that 'man' necessarily requires 'animal', viz. in the consequence: if there is man there is animal. And so it is contingent¹ that what is not animal, be animal; for what the antecedent admits, the consequent admits. . . . But this is impossible'²

Schematically, Abelard's proof runs like this:

Given	(i) If p then q
it follows that	(ii) If not- q then not- p
For if not (ii), then	(iii) If not- q then p ,
and (iii) together with	
(i) gives	(iv) If not- q then q ,
which is impossible.	

In line (iii) Abelard is committing the very mistake of which he accuses Boethius - in fact he is making use of Boethius' second thesis. Furthermore, on line (iv) he is assuming Aristotle's first thesis, which is easily derivable, as we shall see, from Boethius' first. Hence Abelard

¹Or better, 'it happens'. Boethius occasionally used 'contingere' in the sense of 'accidere'. See A. Becker-Freyseng, Die Vorgeschichte des philosophischen Terminus 'contingens', Heidelberg 1938, p. 17.

²Abelard, p. 289, translated by I. Thomas in Bochenski, HFL, p. xi.

himself is scarcely disentangled from the difficulties he finds in Boethius.

10. Kilwardby

We find in the commentary on the Prior Analytics of Robert Kilwardby, who lived in the century after Abelard, a new criticism of Aristotle's second thesis, asserting the incompatibility of 'If p then q ' and 'If not- p then q '. Kilwardby gives two examples of pairs of such propositions which are not incompatible:

- (i) If you are seated, God exists
If you are not seated, God exists
- (ii) If you are seated, then either you are seated or
you are not seated
If you are not seated, then either you are seated¹
or you are not seated.

The first pair is true because 'God exists', being a necessary proposition, follows from anything - quia necessarium sequitur ad quodlibet: an early formulation of the positive paradox of strict implication. But here we must distinguish, Kilwardby says, two kinds of implication; consequentia essentialis or naturalis, and consequentia accidentalis. In the former the consequent must be 'understood' (*intelligitur*) in the antecedent, and such is not the case with 'If you are seated, God exists'. The latter is a consequentia accidentalis, 'et de tali non intelligendum est sermo Aristotelis'.

The second pair, on the other hand, (in which we see the first explicit introduction of weak disjunction), are both consequentiae naturales, and here it does seem as if the same thing can follow from two

¹I. Thomas, Maxims in Kilwardby, *Dominican Studies* 7 (1954), p. 137. The passage in question is quoted in Kneale, DL, pp. 275-6, and an excellent discussion provided.

contradictory propositions. Kilwardby tries to defend Aristotle's position nonetheless, saying that the philosopher intended only to deny that the same proposition could follow from two contradictories in virtue of the same part of itself (gratia eiusdem in ipso). But it is doubtful that Aristotle intended any such thing, and Kilwardby seems to be leaning over backward here. It appears we must accept the fact that the type of implication for which Aristotle's thesis holds cannot consistently admit of conditionals of the form 'If p , then either p or q '.

11. Pseudo-Scotus

In the last section, it appeared^c that a certain type of compound proposition could be said to be 'implied' by both of two mutually contradictory propositions, in the sense of occurring as the consequent of two of Kilwardby's consequentiae naturales. Only for an uncompounded proposition might Aristotle's thesis possibly hold true. This result is reaffirmed in the work of the unknown logician referred to as Pseudo-Scotus, who divides consequentiae into formales and materiales according as they do or do not hold true for all substitutions of the non-syncategorematic words they contain. Pseudo-Scotus shows at length how a contradiction can imply anything in a consequentia formalis, and from this it is only a step, as Kneale shows, to the thesis that 'Any proposition which is itself formally necessary as being the disjunction of two contradictories follows formally from any proposition whatever',¹ so that here again Aristotle's second thesis seems to be mistaken.

This, essentially, is in Kilwardby, but Pseudo-Scotus' original contribution lies in enlarging the class of propositions which purportedly

¹Kneale, DL, p. 283.

can follow from anything at all; not it is true in consequentia formali, but in consequentia materiali bona ut nunc. What we have are the familiar paradoxes of material implications: e.g. 'Omnis propositio vera sequitur ad quamcumque aliam propositionem in bona consequentia materiali ut nunc',¹ where a 'consequentia materialis bona ut nunc' is a ~~true~~ material conditional. If Aristotle's second thesis fails for strict implication, because of the paradoxes of strict implication, it fails a fortiori for material implication. In fact his first thesis fails as well, since if a necessary (true) proposition is strictly (materially) implied by any proposition at all, it is implied by its own negation. Hence the maintaining of Aristotle's theses requires us to accept a form of implication that is neither material nor strict. If we do not do this, his theses simply remain errors.

12. Euclid, Clavius and Saccheri

Historians of logic are indebted to Łukasiewicz and to Professor Kneale for investigating the use which philosophers and mathematicians since the earliest times have made of reasoning in which a proposition is purportedly deduced from its own contradictory. Euclid, for example, puts forward a deduction which seems to be of this pattern in proving the theorem whose statement in modern terminology is:

If a, b and n are natural numbers such that a is prime and a is a factor of bⁿ, then a is a factor of b.²

Euclid proves this by assuming that a is not a factor of b. But, then, since a is a factor of b.bⁿ⁻¹, a must be a factor of bⁿ⁻¹ by a previous

¹Kneale, DL, p. 281.

²Kneale, ACM, p. 63.

theorem, which we shall call T. Repeating this argument, Euclid shows that $\underline{a}$ must be a factor of $\underline{b}^{n-2}$, $\underline{b}^{n-3}$, etc., and so finally of $\underline{b}$. Here a contradiction has resulted from the assumption, which must therefore be false.

Kneale puts forward the following schema as the logical skeleton of Euclid's argument:

If not- $\underline{p}$ then $\underline{p}$,
 if not- $\underline{p}$ then not- $\underline{p}$
 not both $\underline{p}$ and not- $\underline{p}$,
 therefore not-not- $\underline{p}$,
 therefore $\underline{p}$.

If we omit the second and third premisses, which are logical truths, and telescope the last step, we get

If not- $\underline{p}$ then $\underline{p}$,
 therefore $\underline{p}$,

which Clavius and Saccheri, two Jesuits of the 16th and 17th centuries respectively, took to be the kernel of Euclid's proof. Łukasiewicz has in fact given the name 'law of Clavius' to the conditional CCNppp corresponding to the above argument.¹

The question is, does Euclid in fact establish the first premiss of the above proposed schemata, the premiss 'If not- $\underline{p}$ then $\underline{p}$ ', or does he proceed more conventionally, simply deducing a contradiction from the assumption that the number $\underline{a}$ is not a factor of $\underline{b}$? Certainly doing the former has an air of the miraculous about it:

'And this has never been done by anyone; nay it seems clearly impossible, and is the most wonderful thing that has been discovered since the beginning of the world, namely to prove something from its opposite,'²

¹Łukasiewicz, Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls, (PB), Comptes rendus de la Société des Sciences et des Lettres de Varsovie, Cl. III, Vol. 23 (1930), pp. 51-77. Łukasiewicz also claims to have found the name 'consequentia mirabilis' in the works of some Polish Jesuits.

²Cardan, De Proportionibus, quoted in Kneale, DL, p. 347.

and I think it may reasonably be doubted that Euclid has done it. What has Euclid done? Surely what he has done is to take the assumption that a is not a factor of b, plus the assumptions that a is prime, a factor of bⁿ etc., plus the previously proved theorem T, plus various other truths such as that bⁿ = b.bⁿ⁻¹, and deduce from these something which contradicts the first assumption. Schematically, we might say that from premisses not-A₁, A₂, . . . A_n Euclid deduced A₁. Now the premisses A₂, . . . A_n are either background assumptions or logical truths or already proved, so it might look as if Euclid were deducing A₁ from not-A₁ alone. In fact, this is no doubt how it did look to Clavius and Saccheri. But it is not so.

Let us consider another example from Saccheri, quoted by Kneale, where the number of background assumptions is minimal and the appearance of a consequentia mirabilis consequently greater. Saccheri wishes to show that the syllogistic mood AEE in the first figure is invalid by (a) assuming its validity, and then (b) deducing its invalidity. To this effect he constructs the following syllogism:

Every syllogism with a universal major and an affirmative minor premiss yields a conclusion in the first figure.
 But no syllogism of the pattern AEE has a universal major and an affirmative minor premiss.
 Therefore no syllogism of the pattern AEE yields a conclusion in the first figure.¹

Now this syllogism is of mood AEE in the first figure, and its premisses are true, so that if it were valid, its conclusion would be true. But its conclusion is that it is invalid. Once more we have a situation in which several premisses not-A₁, A₂ . . . A_n entail A₁, but

¹Kneale, DL, p. 346.

this time each of $A_2 \dots A_n$ has the appearance of a logical truth (e.g. that every syllogism with a universal major and an affirmative minor yields a conclusion in the first figure). So we are tempted to progress from

$$\vdash CKK \dots KNA_1 A_2 \dots A_n A_1$$

$$\vdash A_2$$

..

.

$$\vdash A_n$$

to

$$\vdash CNA_1 A_1.$$

But, as we shall see, this step is not always permissible in the theory of deduction. For example, in the system E which Anderson and Belnap propose as a formalization of the notion of entailment, we have that

$$\vdash CKCp p C p q C p K p q$$

$$\vdash C p p,$$

but we do not have $\vdash CCp q C p K p q.$ ¹

We conclude therefore that even Saccheri's example, whose background assumptions are, perhaps, logically true, is not a genuine example of a consequentia mirabilis. Hence, in the absence to date of any case of a proposition being entailed by its own contradictory, Aristotle's thesis stands.

13. Lewis Carroll

In a short and amusing note in Mind of 1894, Lewis Carroll presents what is in essence precisely the argument of Boethius' group two class two hypothetical syllogisms, and shows that it leads to paradoxical

¹This last expression is rejected for E by matrix 3 of section 20 (put $p = 4$, $q = 5$).

results.¹ Carroll's argument is as follows.

Uncle Joe and Uncle Jim are going to a barbershop run by Allen, Brown and Carr, and Uncle Jim hopes that Carr will be in to shave him. Uncle Joe says he can prove he will be in by an argument having as premisses two hypotheticals. First, if Carr is out, then if Allen is out, Brown must be in (since otherwise there'd be nobody to mind the shop). Secondly if Allen is out Brown is out (since Allen, after a recent bout of fever, always takes Brown with him). Letting 'A' stand for 'Allen is out', 'B' for 'Brown is out', etc., we have:

(i) If C then (if A then not-B)

(ii) If A then B,

and these two premisses, according to Uncle Joe, imply not-C, because of the incompatibility of the two hypotheticals 'If A then B' and 'If A then not-B'. The result is, of course, paradoxical, because under the stated conditions Carr can perfectly well be out when the other two are in, or even when Allen alone is in. The question is, at which point is Uncle Joe's argument fallacious?

What Burks and Copi call the 'received' solution is that of Johnson and Russell. According to them, the two hypotheticals 'If A then B' and 'If A then not-B' are not incompatible: they may in fact both be true when A is false, as is the case in classical two-valued logic. Hence we cannot infer not-C by modus tollens. The thought underlying this solution - that 'If A then not-B' does not properly negate 'If A then B' -

¹A Logical Paradox, Mind N.S.3 (1894), pp. 436-8. This paper has been discussed by A. Sidgwick, Mind 3, p. 582 and 4, p. 143; W.E. Johnson, Mind 3, p. 583 and 4, pp. 143-4; B. Russell, The Principles of Mathematics, Cambridge 1903, p. 18n.; E.E. Constance Jones, Mind 14, pp. 146-8, and 576-8; an anonymous "W" (Johnson?), Mind 14, pp. 292-3; A.W. Burks and I.M. Copi, Mind 59 (1950), pp. 219-22; J.C.C. McKinsey, JSL 15 (1950), pp. 222-3; A.W. Burks, Mind 60, pp. 377-8.

is identical with that which led Abelard to correct Boethius' syllogisms.

Burks and Copi, however, disagree. When interpreted as 'causal' implications rather than as material implications, the two hypotheticals above are in their opinion incompatible, and this is in general true of 'causal' implication. To take another example:

'If one politician argues that if the Conservatives win the election in 1950 then Britain's economic situation will improve, and another argues that if the Conservatives win in 1950 then Britain's economic situation will not improve, there is a genuine disagreement. It would indeed be an over-zealous proponent of material implication who would expect the disputants to agree that another Labour victory at the polls would make both their statements true!' ¹

For Burks and Copi, then, the inference from Cpq to NCpNq - Boethius' first thesis - holds for causal implication, a form of implication which, they maintain, Uncle Joe's hypotheticals exemplify. Hence the fallacy in the argument must be sought elsewhere.

The fallacy, according to them, lies in an impropriety in the statement of the first premiss. Uncle Joe has

(iii) If Carr is out, then if Allen is out Brown is in,
but in fact the conditions of the problem permit only

(iv) If Carr is out and if Allen is out, then Brown is in;
nor does (iv) either mean the same as or imply (iii). In other words, the principle of exportation CCKpqrCpCqr does not hold for causal implication.

Why not? To see this, let us return to the example of the election of 1950. Both sides would admit, Burks and Copi say, that changes in the overall world situation would falsify their hypotheticals. Thus if a new world war or a deep depression occurred in 1950, then a Conservative victory would probably not produce any economic improvement. Conversely,

¹Burks and Copi, p. 220.

if extensive gold deposits were found on English public land, then (even) a Conservative win would issue in a period of prosperity. Hence both disputants require that, apart from the election, other conditions remain equal.

Let us abbreviate 'other conditions remain equal' by 'E', 'the Conservatives win the 1950 election' by 'F', and 'Britain's economic situation improves in 1950' by 'G'. The two opposed positions then are:

(v) If (E and F) then G.

(vi) If (E and F) then not-G.

But now suppose that other conditions do remain equal, namely that E.

If we could proceed by exportation from (v) and (vi) to

(vii) If E, then if F then G

and (viii) If E, then if F then not-G,

we would be able to detach the two consequents 'If F then G' and 'If F then not-G' of (vii) and (viii), asserting that economic improvement will or will not follow upon a Conservative victory unconditionally. But neither of the disputants wish to say this, even if other conditions do remain equal, and so the passage from (v) and (vi) to (vii) and (viii) must be denied. Uncle Joe's statement of the first premiss of the Barbershop argument should be in the form of (iv) rather than of (iii); he cannot proceed from (iv) to (iii); hence he cannot arrive at his conclusion.

As we shall see later, there are many systems of propositional logic, the systems of strict implication among others, in which the principle of exportation fails. Burks, in another paper, puts forward a system of this type, an attempt at a formalization of the notion of causal implication. This system rejects exportation for causal conditionals,

but does not, surprisingly, allow for the unrestricted inference from 'A causally implies B' to 'It is not the case that A causally implies not-B', which figured so strikingly in the earlier paper.¹

To sum up, Lewis Carroll's supposed paradox appears at first sight to demonstrate the erroneousness of taking 'If A then B' and 'If A then not-B' as incompatible, but a solution to it may be found which allows us to retain the incompatibility. And Burks and Copi have shown that there is a species of implication which actually seems to demand this incompatibility, although its incorporation into a formal system is not an easy matter.

14. E.J. Nelson

Reference was made in section 3 to E.J. Nelson, who in 1930 put forward the notion of 'intensional' implication. In this section it will be shown that Nelson's variety of implication is the same as the third put forward by Sextus, which was called 'connexive' implication, and that it allows for the truth of both Aristotle's theses and the first of Boethius.

Nelson begins by attacking C.I. Lewis's notion of consistency, according to which two propositions p and q are consistent if it is possible that they both be true.² It follows, Nelson says, that an impossible proposition, such as ' $2+2 \neq 4$ ', is inconsistent with every proposition, including itself. And this is not so, since 'from the mere

¹A.W. Burks, The Logic of Causal Propositions, Mind 60 (1951), pp. 363-382. The inference would obtain only in the case where A was 'causally possible' (p. 377).

²Lewis, SSL, p. 293.

fact that p is false or impossible it cannot be determined that it is inconsistent with q . The meanings of both propositions are required to determine the relation.¹ Similar considerations lead Nelson to reject Lewis's notion of strict implication, which has as one of its consequences that an impossible proposition implies any proposition. Like consistency, implication is 'essentially relational: it depends upon the meaning of both propositions.'²

As an alternative to Lewis's notions, Nelson accordingly offers a new primitive relation of consistency, symbolized as ' poq ', and a concept of 'intensional implication', or 'entailment', defined in such a way that p entails q if and only if p is inconsistent with the contradictory of q (symbolized here by ' $\neg q$ '):

$$pEq = \neg(po\neg q) \quad \text{Df.}$$

As will be seen, this definition of implication is the same as that of Sextus's third variety, and is to be contrasted with Lewis's definition of strict implication:

$$p \rightarrow q = \neg \Diamond(p \cdot \neg q) \quad \text{Df.}$$

Nelson's definition of entailment leads naturally to the most distinctive law characterizing his system:

$$(pEq) E (poq)$$

This does not hold true for Lewis's systems, since if p is impossible, it implies any proposition q without necessarily being consistent with it. Nelson's law is certainly plausible, however, and is seen to be

¹Nelson, p. 443.

²Nelson, p. 446. Hence his term 'intensional implication'.

equivalent to Boethius' first thesis when we insert the definition of entailment:

$$(\underline{pE}q) E \neg(\underline{pE}\neg q).$$

Furthermore, given the law of identity $\underline{pEp}$, Nelson's law yields the following even more plausible thesis by substitution and modus ponens:

$$\underline{pop},$$

the latter being equivalent to Aristotle's first thesis:

$$\neg(\underline{pE}\neg p)^1.$$

To Nelson must go the credit, then, for being the first logician to construct a formal analogue for, and trace the properties of, Chrysippean or connexive implication.

It is worth noting as a postscript that, while Nelson rejects strict in favour of connexive implication, Lewis started out by advocating something very close to connexive implication himself. In his earliest published work on the subject - a criticism of material implication - Lewis attempts to define a more satisfactory type of implication in terms of what he calls 'intensional disjunction'.² A disjunction is intensional if, one of its disjuncts being supposed false, we are in consistency bound to suppose the other true (e.g. 'Either Matilda does not love me or I am beloved'); it is extensional if both disjuncts may be false ('Either Caesar died or the moon is made of green cheese'). What Lewis even here calls 'strict' implication is defined à la Whitehead and Russell as the

¹These theses will all be found in Nelson, p. 449. G.H. von Wright, in his Logical Studies, London 1957, p. 89 ff., presents an operator 'M', denoting conditional possibility or consistency, with similar powers. Von Wright has as a thesis $\underline{M}(p/p) \rightarrow \sim \underline{M}(\sim p/p)$, but not, for some reason, $\underline{M}(p/q) \rightarrow \sim \underline{M}(\sim p/q)$.

²Lewis, IAL, p. 523.

intensional disjunction of the negation of the antecedent with the consequent, and, although Lewis fails to notice this, we have once more Aristotle's thesis that no proposition can be implied by its own contradictory. For 'not- p implies p ' would have to come from ' p or p ', and the latter is not an intensional disjunction. It must be merely a coincidence, though a tantalizing one, that Lewis describes his new variety of implication as 'the Aristotelian "implies" ' (IAL, p. 523).

15. R.B. Angell

Although Nelson formulated Aristotle's and Boethius' non-classical theses in symbols, and offered a handful of axioms from which he deduced a handful of theorems, he made no serious attempt to incorporate his insights into a full-fledged logical system. Such an attempt has been made by R.B. Angell, who however does not claim to be constructing a system of entailment, but one containing an operator more akin to Burks' and Copi's causal implication.¹

Taking as primitive notions the operators ' $\neg$ ' for 'not', ' $\cdot$ ' for 'and' and ' $\rightarrow$ ' for 'if...then', Angell offers ten axioms, the first nine of which are classical when we substitute material implication for ' $\rightarrow$ ', but the tenth of which is Boethius' first thesis:

$$(i) \quad (p \rightarrow q) \rightarrow \neg(p \rightarrow \neg q).$$

This axiom, as we have seen, gives precise expression to the doctrine of the incompatibility of the conditionals 'If p then q ' and 'If p then not $\neg q$ ', which is referred to by Angell as 'the principle of subjunctive contrariety'.

¹R.B. Angell, A propositional logic with subjunctive conditionals, forthcoming in The Journal of Symbolic Logic.

Why subjunctive? The answer is that Angell seems to regard the incompatibility of the above conditionals as holding only if the conditionals are subjunctive, with contrary-to-fact antecedents. Actually, Angell confines himself to saying that in such cases, the incompatibility is 'at least plausible', and to support his view offers examples very like that of Burks and Copi:

If the match had been scratched, it would have lighted.
If the match had been scratched, it would not have lighted.

If we had followed a different policy towards Germany in the 1920's, the second World War would not have occurred.
If we had followed a different policy towards Germany in the 1920's, the second World War would still have occurred.¹

Certainly the case of subjunctive conditionals provides the most striking contrast with that of material conditionals, since any material conditional with a false antecedent is true, but Angell would surely also extend his principle of contrariety to causal cases which are not subjunctive, such as Strawson's, which Angell quotes with approval:

If it rains, the match will be cancelled.
If it rains, the match will not be cancelled.²

With Angell's agreement, we may therefore regard his system as a formalization of causal implication. But we need not stop there. Angell does not raise the question of whether his axiom (i) holds for implication tout court, possibly because of being over-awed by the fact that (i) runs counter to material implication. But Nelson has raised the question, as have Aristotle and Boethius, and since Angell's axiom fits their intuitions to a T it seems unkind not to regard his system as a formalization of what we have called connexive implication.

¹Angell, §5.

²Strawson, p. 85; Angell, §5.

Angell's system is more developed than Nelson's: he proves some hundred theorems from his axioms. But it has more important virtues as well. The danger of introducing an axiom such as (i) is that, being non-classical, it may generate an inconsistency when added to other axioms. As was seen in section 2, any non-classical formula added to the complete classical calculus will result in a contradiction. Hence if the logician who introduces axiom (i) is worried, as logicians should be, about matters of consistency, he should produce a proof that the resulting system is consistent. This Angell does, to his great credit. The following truth-matrices, for the operators ' $\neg$ ', ' $\cdot$ ' and ' $\rightarrow$ ', satisfy all the axioms of Angell's system in the sense that they assign to the axioms a 'designated' value (starred on the matrix) under all possible assignments of the values 0, 1, 2, 3 to the individual propositional variables the axioms contain:

Matrix 1	$p \neg p$	$(p \cdot q) 0 \ 1 \ 2 \ 3$	$(p \rightarrow q) 0 \ 1 \ 2 \ 3$
	* 0 3	* 0 1 0 3 2	* 0 1 2 3 2
	* 1 2	* 1 0 1 2 3	* 1 2 1 2 3
	2 1	2 3 2 3 2	2 1 2 1 2
	3 0	3 2 3 2 3	3 2 1 2 1

Further, the matrices satisfy the rules of substitution (no values occur in the matrices for the operators apart from the possible values 0, 1, 2, 3 of the variables) and modus ponens (if two formulae x and $x \rightarrow y$ both receive designated values, then y receives a designated value). Hence all theorems of the system have designated values ('are designated'). Finally, if all theorems are designated, i.e. take the values 0 or 1, then their negations take the values 3 or 2, i.e. are undesigned. Hence no formula and its own negation can both be theorems. Also, the simple variable p is not a theorem. Therefore, the system is consistent.

Angell, as a result, has produced a formal system of connexive implication that is demonstrably consistent. It is not the only such system, as we shall see, and its relations to other formal systems of implication must be explored. For more on Angell's system, see section 55.

16. Aristotle's error: a fresh approach

In this chapter we have considered at length Aristotle's and Boethius' non-classical theses, and have tried to answer the question, whether they hold true of implication as we ordinarily conceive it. One or two philosophical objections to the theses have been removed, and a variety of implication found with which they are in accord. But there is one voice which has not yet been heard - that of the person who reasons, but is philosophically naive. To sound his opinions, a questionnaire was devised which included concrete examples of Boethius' first thesis and a version of Aristotle's first - that no proposition implies its own negation - in a list of (a) propositions which might or might not be true, and (b) arguments which might or might not be valid. To avoid difficulties about variables, each item was written in concrete form; e.g. 'If Hitler is dead, then Hitler is dead' - true or false? A 'don't know' box was added to discourage guessing. The questionnaire was given in September 1962 to 89 students in the first lecture of the elementary logic course at McGill University, and half an hour provided in which to answer it. The students, who did not sign their names, and did not receive a mark, had previously had no logic, and, in the majority of cases, no philosophy of any kind. Though no doubt not exactly representative of those of the man in the street, their answers do give the logician a glimpse of how the world outside views matters. Table 1 below reproduces

the questionnaire, and table 2 summarizes the answers to it. The final column in table 2 is explained below.

TABLE 1

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Logical Questionnaire

- A. Whether Hitler is dead or not, are the following statements true?
1. If Hitler is dead then Hitler is dead.
2. If Hitler is dead then either Hitler is dead or there is life on Venus.
3. If Mussolini is dead then (Mussolini is dead and Hitler is dead).
4. If Hitler is dead and smoking produces cancer then Hitler is dead.
5. If Hitler is dead then (if the moon is made of green cheese then Hitler is dead).
6. If Hitler is dead then (if Hitler is not dead then shrimps whistle).
7. It is not the case that (if Hitler is dead then Hitler is not dead).
8. If Hitler is dead then (if Hitler is not dead then Hitler is dead).
9. If (if Hitler is not dead then Hitler is dead) then Hitler is dead.
10. If (if Hitler is dead then von Rümer is a liar) then (if von Rümer is not a liar then Hitler is not dead).
11. If (if Hitler is dead then von Rümer is a liar) then it is not the case that (if Hitler is dead then von Rümer is not a liar).

- B. Are the following arguments valid?
12. Either snow is white or my eyes deceive me. My eyes do not deceive me. Therefore, snow is white.
13. If the fires are lit the battle is over. The fires are not lit. Therefore, the battle is not over.
14. You can't both have your cake and eat it too. You've eaten it. Therefore, you can't have it.
15. If the fires are lit the battle is over. Therefore, it is false that if the fires are lit the battle is not over.
16. If the fires are lit the battle is over. Therefore, if the battle is not over the fires are not lit.

[illegible]

TABLE 2REPLIES

Question	Yes	No	Don't Know	Not Answered	Percentage Answering 'Yes'	Percentage of experts answering 'Yes' to tendentious questions
1	86	2	0	1	97%	
2	36	46	7	0	40	73%
3	5	78	6	0	6	
4	69	18	1	1	78	91
5	36	46	6	1	40	73
6	11	65	11	2	12	11
7	78	6	3	2	88	100
8	19	60	10	0	21	18
9	33	39	15	2	37	55
10	54	29	5	1	61	
11	75	7	4	3	84	100
12	81	7	1	0	91	
13	55	31	3	0	62	
14	63	23	2	1	71	
15	76	12	1	0	85	100
16	68	18	2	1	77	

The answers to the questionnaire are exceedingly interesting, especially from the point of view of how they reflect on material implication, and on the theses of Aristotle and Boethius. In what follows I shall construct a 'popularity ordering' of the statements held to be truest and the arguments most valid, and add my comments. But before beginning it is as well to anticipate a question sure to be asked by those reflecting on the results: how much importance should be attached to the logical opinions of admittedly total amateurs? The answer to this question depends on how much logical acumen the amateurs display. As an indicator of this, the questions on the sheet fell into two groups.

(a) Numbers 1, 3, 10, 12, 13, 14 and 16 were more or less straightforward, with right-or-wrong answers. They served as a measure of the answerer's ability. The performance of the class in answering these questions was not exactly alpha, questions 1, 3 and 12 each being answered correctly by 88% of the class or more, but those who succeeded with question 13 numbering only 35%. The students who got every one of this group right constituted only 12%; these I designated 'experts'.

(b) The remainder of the questions were tendentious, and the answers to them correspondingly more interesting. Table 2 compares the opinion of the 'experts' on these questions with that of the rest. Those who place little trust in the philosophical beliefs of the average man may, if they wish, pay more attention to these statistics than to the others.

I turn now to consider, in order of mass preference, the different truth- and inference-schemata exemplified by the questions.

<u>Schema no.</u>	<u>Subscribed to by</u>	<u>Name of schema</u>	<u>Remarks</u>
1	97%	Law of identity	Obvious, but see Sextus number four.
12	91%	<u>Modus tollendo ponens</u>	Obvious.
7	88%	Aristotle)	Surprise choice in view of the history of these schemata. The experts' judgement even more favourable.
15	85%	Rule of Boethius)	
11	84%	Boethius)	
4	78%	Law of Conjunctive Simplification	Violates strict criterion of relevance (See sect. 24)
16	77%	Rule of transposition	Fundamental; introduced largely for comparison with 15.
14	71%	Chrysippus' 3rd indemonstrable	Obvious
13	62%	Denying the antecedent.	A fallacy unrecognized by a surprisingly large number.
10	61%	Law of transposition	Fundamental; cf. 16.
5	40%	Positive paradox of material implication	Violates criterion of relevance (see sect. 19). More favoured by experts.
2	40%	Law of addition	See Kilwardby. Analogous to 4.
9	37%	<u>Consequentia mirabilis</u>	See section 12.
8	21%	Combination of both positive and negative paradoxes of material implication	Seemingly more difficult to accept than 5.
6	12%	Negative paradox of material implication	Not as popular as 5. ¹
3	6%		Fortunately most saw through this obvious fallacy.

¹This result confirms A.N. Prior's views as to the relative plausibility of the positive and negative paradoxes of material implication. See Prior, FL, n. 259.

Perhaps not very much weight will be attached to these statistics. Logicians on the whole probably adopt Socrates' attitude toward the opinion of the many, especially when they reflect that 80% of college freshmen appear to see no objection in passing from 'Some A is B' to 'Some A is not B'. But still it is interesting to speculate on what moves the student's mind to assent more readily to Aristotle's and Boethius' theses than to most of the classical schemata. If one wishes to argue, as I do, that these theses have suffered an unjust historical fate, then statistical evidence will be grist to his mill. But only part of the grist; the main burden will lie in showing that there exist systems of propositional logic containing the theses which, though non-classical, are otherwise perfectly respectable. This will be the task of succeeding chapters.

CHAPTER THREE

THE PARADOXES OF IMPLICATION

17. Paradoxes of material implication

It was W.E. Johnson who first used the expression 'paradox of implication', explaining that a paradox of this sort arises when a logician proceeds step by step, using accepted principles or formulae, until a formula is reached which conflicts with common sense.¹ In classical logic, the two formulae which do most notoriously conflict with common sense in this way - CpCqp (a true proposition is implied by any proposition), and CpCNpq (a false proposition implies any proposition) - are known as the paradoxes of material implication.

The counter-intuitiveness of the paradoxes has been stressed by many philosophers,² and their harmlessness urged by many more,³ usually on the grounds sketched in section 5. In this section I wish only to indicate the different forms in which the paradoxes may occur. Later I shall consider their essence.

Firstly, we may have paradoxical formulae which involve all four of the principal operators of propositional logic - implication, negation,

¹W.E. Johnson, Logic, part 1, Cambridge, 1921, p. 39. Moore, p. 295, speaks of the 'paradoxical results' of identifying implication with material implication.

²See for example Lewis, IAL, p. 529, and SSL, p. 326; Moore, p. 295; Strawson, p. 88; Prior, FL, p. 19; Anderson and Belnap, PCE, p. 21.

³E.g. W.V. Quine, Methods of Logic, New York 1950, p. 16; I. Copi, Introduction to Logic, New York 1953, p. 233; P. Suppes, Introduction to Logic, Princeton 1957, p. 7. Johnson's explanation of their harmlessness (p. 43) is somewhat different: it follows along the lines of Russell's statement that 'inference will only in fact take place when the proposition "not-p or q" is known otherwise than through knowledge of not-p or knowledge of q'. (Introduction to Mathematical Philosophy, London 1919, p. 153). That is, in actual inference we never conclude that p implies q from the fact that p is false or q true.

conjunction and alternation. Examples of such formulae are CqApNp and CKpNpq, the first exemplifying^f the paradox that a true proposition, ApNp, is implied by any proposition, and the second that a false proposition, KpNp, implies anything. We have also, of course, CKpNpAqNq. Secondly there are paradoxical formulae involving only the operators implication and negation, such as CpCNpq and CNCppq, the latter taking NCpp as an example of an impossible proposition. Finally, paradoxical formulae exist, such as CpCqp and CqCqp, which involve implication alone. Hence even the simplest of all logical systems, those of pure implication, may contain implicational paradoxes.

18. Paradoxes of strict implication

Since its very introduction - specifically as a means of avoiding the paradoxes of material implication - strict implication has been known to engender similar paradoxes of its own.¹ In strict implication we have CLpCqp (a necessary proposition is implied by any proposition) and CLNpCqp (an impossible proposition implies any proposition), and many philosophers have believed that the occurrence of these paradoxes makes strict implication as unsuitable to serve as a formal analogue of implication as material implication. Alternatively, many philosophers have not. It is instructive to consider the reasons pro and con, for the debate throws a great deal of light on what we normally mean by implication.

Lewis originally defended the paradoxes on the grounds that the necessary principles of logic and mathematics are presupposed by any proposition in the sense that if they were false, it would be false.

¹See Lewis, The calculus of strict implication, Mind 23 (1914), pp. 240-47.

That is, it implies them.¹ Nelson and Duncan-Jones, attacking, object that for any proposition to imply another, there must be a relation holding between the meanings of the propositions which is lacking between 'Wellington defeated Napoleon' and ' $2+2 = 4$ '.² Prior and Bennett, defending, state that it is precisely this relation between meanings, of which Nelson speaks, that does hold between necessary and impossible propositions on the one hand, and the totality of all propositions on the other.³ Thus Prior:

'For some reason [Nelson] does not consider the possibility that what Lewis's paradoxes show is precisely that necessary and impossible propositions as such have a definite inner connexion with all propositions whatever.'

But none of these writers succeeds in making clear what this 'inner connexion' may be. In fact, it is most doubtful that it exists. The possibility of reductio ad absurdum proofs, for example, seems to count against it. As von Wright remarks:

'It is essential to such proof that an impossible proposition should entail exactly such and such consequences, and not anything whatever.'⁴

This point is driven home convincingly by Anderson and Belnap, in a passage which I shall quote in the next section. In sum, the critics of the paradoxes of strict implication_x have, in my opinion, had the best

¹ Lewis, SSL, p. 338.

² Nelson, op. cit., A.E. Duncan-Jones, Is strict implication the same as entailment? Analysis 2 (1935), pp. 70-78.

³ Prior, FL, p. 196; J. Bennett, Meaning and implication, Mind 63 (1954), pp. 451-63.

⁴ von Wright, p. 174.

of it, and have shown that, although Lewis may have corrected some of the faults of the horse-show^e as an implication operator, he has not corrected them all.

Before proceeding to consider what faults remain, let us see what forms the paradoxes of strict implication can take. The forms stated above involve L, the necessity operator. But exactly equivalent forms can be found which do not involve L. This possibility rests on the fact that, in all the systems S1-5 of strict implication, a proposition p's being necessary is equivalent to its being strictly implied by its own negation. In symbols:

$$\underline{Lp} = \underline{CNpp}^1$$

Because of this equivalence, we may in fact define necessity in the Lewis systems in terms of strict implication and negation, these latter being taken as primitive. Reasons will be offered in section 25 for considering the Lewis systems to be best presented in this fashion. In the meantime, we may reformulate the paradoxes of strict implication as CCNppCqp and, using double negation, CCpNpCpq respectively. Since, in S1-5, ApNp and Cpp are examples of necessary propositions, and KpNp and NCpp of impossible propositions, the positive paradox also appears in the forms CqApNp and CqCqp, and the negative paradox in the forms CKpNpq and CNCppq, as was the case with material implication.

¹This equivalence appears as theorem 18.14 in Lewis and Langford, p. 160. The operator 'C' will henceforth be used purely and simply as an implication operator, its strength (whether material, strict, etc.,) being determined by the context.

19. The essence of paradox

What have the paradoxes of material and strict implication in common? That they conflict with common sense, yes, but what makes them conflict with common sense? The answer proposed by Anderson and Belnap, in a series of papers constituting the most important recent contribution to the theory of implication, is that the paradoxes involve fallacies of relevance. To introduce this type of fallacy, and to see how it relates to the paradoxes, we shall consider one of Anderson's and Belnap's examples:

'Imagine, if you can, a situation as follows. A mathematician writes a paper on Banach spaces, and after proving a couple of theorems he concludes with a conjecture. As a footnote to the conjecture, he writes: "In addition to its intrinsic interest, this conjecture has connections with other parts of mathematics which might not immediately occur to the reader. For example, if the conjecture is true, then the first order functional calculus is complete; whereas if it is false, then it implies that Fermat's last conjecture is correct." The editor replies that the paper is obviously acceptable, but he finds the final footnote perplexing; he can see no connection whatever between the conjecture and the "other parts of mathematics", and none is indicated in the footnote. So the mathematician replies, "Well, I was using 'if...then---' and 'implies' in the way that logicians have claimed I was: the first order functional calculus is complete, and necessarily so, so anything implies that fact - and if the conjecture is false it is presumably impossible, and hence implies anything. And if you object to this usage, it is simply because you have not understood the technical sense of 'if...then---' worked out so nicely for us by logicians." And to this the editor counters: "I understand the technical bit all right, but it is simply not correct. In spite of what most logicians say about us, the standards maintained by this journal require that the antecedent of an 'if...then---' statement must be relevant to the conclusion drawn. And you have given no evidence that your conjecture about Banach spaces is relevant, either to the completeness theorem or to Fermat's conjecture." ¹

The form of argument that the mathematician in this example vainly employs is that of the paradoxes of strict implication. In the

¹Anderson and Belnap, PCE, p. 33.

symbolic formulation of these paradoxes, the lack of relevance comes out plainly: in the formula CKpNpq , for example, the antecedent KpNp and the consequent q may be propositions whose meanings are totally unconnected. This is reflected in the lack of any variable shared by the two, the occurrence of the same variable in different parts of the same schema being the device used in propositional logic to secure continuity of meaning. If antecedent and consequent share no variable, values may be assigned to the variables they do contain in such a way that there is no connexion in meaning between the two, which thus become entirely irrelevant to one another. Therefore, to avoid fallacies of relevance, the obvious way is to ensure that, in every true implication, antecedent and consequent share a variable. This is precisely what Anderson and Belnap do, as will be shown in the next section.

To conclude this section, we shall show that it is essentially the satisfaction of the condition of relevance that is demanded by another requirement for entailment that is occasionally encountered in the literature. This is the requirement, originally stated I think by Russell, and reiterated by Johnson and von Wright, that

' p entails q , if and only if, by means of logic, it is possible to know the truth of $p \rightarrow q$ without coming to know the falsehood of p or the truth of q '.¹

In other words, for q to be deducible from p , it is neither necessary nor sufficient that p alone should be false or impossible, or that q alone should be true or necessary. The essential thing is that there should be a connexion between the two, and this is the condition of relevance.

¹von Wright, p. 181. Compare the quotation from Russell in the footnote to section 17, and Johnson, p. 47.

20. The avoidance of paradox

The system E of Anderson and Belnap is the first intensively studied propositional logic which, inasmuch as it excludes fallacies of relevance, demonstrably avoids paradox. The system is a version, with fewer axioms, of one originally put forward in 1956 by Wilhelm Ackermann, and has since been the subject of numerous papers by Anderson and Belnap.¹ In Anderson's review of Ackermann's paper, it is stated that there have been several previous attempts by logicians to formulate systems which avoid paradox (see the references there to the work of Emch, Vredenduin, Halldén, Yonemitsu and Sugihara), but the resulting systems have generally been constructed in an ad hoc fashion, by juggling axioms, and have been too weak to be of much interest. The system E, on the other hand, has much to recommend it as a formalization of entailment that is both strong and natural. 'Rigorous', translating Ackermann's 'streng', is the name Anderson and Belnap have given to the type of implication it

¹W. Ackermann, Begründung einer strengen Implikation, JSL 21 (1956), pp. 113-128; A.R. Anderson, review of Ackermann, JSL 22, pp. 327-8; A.R. Anderson and N.D. Belnap, jr., A modification of Ackermann's "rigorous implication" (abstract), JSL 23, pp. 457-8; Anderson and Belnap, Modalities in Ackermann's "rigorous implication", JSL 24, pp. 107-111; Belnap, Pure rigorous implication as a sequenzenkalkül, (abstract), JSL 24, pp. 282-3; Belnap, A formal analysis of entailment (FAE), Technical report no. 7, Office of Naval Research, New Haven, 1960; Anderson, Belnap, and J.R. Wallace, Independent axiom schemata for the pure theory of entailment (IAS), Zeitschrift für math. Logik und Grundlagen der Math. 6 (1960), pp. 93-95; Anderson, Completeness theorems for the systems E of entailment and EQ of entailment with quantification (CTE), *ibid.*, pp. 201-216; Belnap, Entailment and relevance (ER), JSL 25, pp. 144-6; Belnap and Wallace, A decision procedure for the system E₁ of entailment with negation (DP), Technical report no. 11, Office of Naval Research, New Haven, 1961; Anderson and Belnap, Enthymemes, *The Journal of Philosophy* 58 (1961), pp. 713-723; Anderson and Belnap, Tautological entailments, *Philosophical Studies* 13 (1962), pp. 9-24; Anderson and Belnap, *PCE*.

contains.

Ackermann attempted to show that his system avoided paradox by arguing from matrices. Slightly modified to conform with the standard presentation of matrices in this work (as truth-values the integers beginning with 1, designated (starred) values the lowest), his matrices for implication, negation and conjunction are as follows:

Matrix 2

C	1	2	3	4	5	6	N
* 1	3	6	6	6	6	6	6
* 2	3	3	6	6	6	6	5
* 3	3	3	3	6	6	6	4
4	3	6	6	3	6	6	3
5	3	3	6	3	3	6	2
6	3	3	3	3	3	3	1

K	1	2	3	4	5	6
* 1	1	2	3	4	5	6
* 2	2	2	3	5	5	6
* 3	3	3	3	6	6	6
4	4	5	6	4	5	6
5	5	5	6	5	5	6
6	6	6	6	6	6	6

These matrices satisfy all the axioms and rules of E, which will be given in section 29, and are hence adequate for the system. Ackermann, in trying to explicate the notion of the 'logischer Zusammenhang' which he maintains to hold between the antecedent x and the consequent y of any thesis Cxy of his system Π' , uses the matrices to prove the following theorem:

No formula of the form CxCyz is provable in Π' , unless x itself contains the operator C.

Ackermann's argument is as follows. Suppose CxCyz is provable. Then it will take the value 3 for all values of its variables, and its component Cyz will take the values 3 or 6. Suppose further that x contains no occurrences of C. Then it will take, for some values of its variables, one of the values 2 or 5. But C23 = C26 = C53 = C56 = 6, so that CxCyz is no longer designated. Hence x contains occurrences of C. This suffices to rule out, as theses, paradoxical formulae such as

CpCqp, CpCNpq, CpCqq, CKpNpq and even CpCpp, since in all these cases p might contain no occurrences of C, and q might be an implication. Hence both Π' and E are to this extent paradox-free.

But Ackermann's matrices do not suffice to show that the systems are paradox-free in that they avoid fallacies of relevance. They do not, for example, show that the paradoxical formula CCqrCpp is not a thesis. In fact they are even adequate for Lewis's S3 in strict implication, negation and conjunction. Hence they will not suffice to demonstrate the desired 'logischer Zusammenhang' which Ackermann doubtless had in mind. The following matrices of Belnap's do, however, show this:¹

Matrix 3

	C	1	2	3	4	5	6	7	8	N
*	1	1	8	8	8	8	8	8	8	8
*	2	1	2	8	8	7	8	7	8	7
*	3	1	8	3	8	6	6	8	8	6
*	4	1	2	3	4	5	6	7	8	5
	5	1	8	8	8	4	8	8	8	4
	6	1	8	3	8	3	3	8	8	3
	7	1	2	8	8	2	8	2	8	2
	8	1	1	1	1	1	1	1	1	1

	K	1	2	3	4	5	6	7	8
*	1	1	2	3	4	5	6	7	8
*	2	2	2	4	4	7	8	7	8
*	3	3	4	3	4	6	6	8	8
*	4	4	4	4	4	8	8	8	8
	5	5	7	6	8	5	6	7	8
	6	6	8	6	8	6	6	8	8
	7	7	7	8	8	7	8	7	8
	8	8	8	8	8	8	8	8	8

By means of these matrices, which satisfy the axioms and rules of E (and Π'), the following theorem is provable:

No formula Cxy is provable in E (or Π') unless x and y share a variable.

To show this, assume that x and y share no variable. Then let the variables of x each take the value 3 and those of y the value 2.

x will take as its value either 3 or 6, and y either 2 or 7.

But C32 = C37 = C62 = C67 = 8. Hence if Cxy is provable, x and y

¹Belnap, ER, modified in the same way as matrix 2.

share a variable. That is to say, the system E avoids fallacies of relevance.

There is another sense in which rigorous implication conforms to standards of relevance, as has been shown by Anderson.¹ Anderson constructs a subproof formulation of E in the style of Fitch (see section 27), and stipulates, using a method which allows him to keep track of hypotheses, that a formula is provable from hypotheses only if each of the hypotheses is actually used in the proof. In the classical definition of 'proof from hypotheses' this is not so, for there it is said that $\underline{x}, \underline{x}_1, \underline{x}_2, \dots, \underline{x}_n$ is a proof of $\underline{x}_n$ from the hypothesis $\underline{x}$ if each $\underline{x}_i$ either is an axiom or follows from previous items by one of the rules of the system. This definition allows the following to be a proof:

- | | | |
|----|-------|--------------------------------|
| 1. | q | hypothesis |
| 2. | Cpp | axiom |
| 3. | CqCpp | 1,2 rule of conditional proof. |

Here, according to the definition, we 'proved' Cpp on line 2, but, as Anderson and Belnap remark, it is 'crashingly obvious' that we did not prove it from the hypothesis q.² Anderson's method of labelling hypotheses, so that the label recurs in association with every formula derived from each hypothesis, avoids 'proofs' of this kind, and gives a meaning to the notion of relevance which neatly complements Belnap's criterion of variable-sharing.

¹Anderson, CTE.

²Anderson and Belnap, PCE, p. 34. The same criticism could, ^{be} and doubtless was, brought against the following argument by the Stoics:

If it is day, it is light,
it is day,
Dion is walking,
therefore, it is light.

The Stoics regarded this argument as fallacious, although Mates, objecting, thinks that Sextus, in reporting it, must have been following an inferior handbook (Mates, p. 83).

21. Paradoxes of necessity

It was noted in the previous section that Ackermann's matrix did not suffice to show that the system of rigorous implication satisfied the criterion of relevance. Ackermann's matrix does show something, however, namely that the system avoids what Anderson and Belnap call 'paradoxes of necessity'. The thought behind the recognition of these as paradoxes is that if an entailment proposition is true at all, it is true by logical necessity. Paradox occurs when a proposition which is contingently true, depending for its truth upon some factual state of affairs, entails a proposition depending for its truth upon logical considerations, such as an entailment proposition. For example, consider the propositions 'Snow is white' and 'That snow is white entails that snow is white'. They are both true, but we would hardly say that the first entailed the second, since the colour of snow seems irrelevant to the ^{truth}~~truth~~ of the latter.¹ Hence we should exclude CpCpp as a thesis holding for entailment, even though it satisfies the relevance criterion of variable-sharing. Ackermann's theorem of the previous section shows that no thesis of the form CxCyz, where x is a propositional variable which may take as its value a contingent proposition, is provable in E.

Anderson and Belnap do not, I think, succeed in keeping the two fallacies of relevance and necessity intuitively separate, as is betrayed by the acknowledged irrelevance of p to Cpp. But in formal terms they are clear enough. What makes them of particular interest is

¹Anderson and Belnap, PCE, p. 45.

the fact that two other well-known systems each avoid one of the fallacies, but that E, which is contained in them, is unique in avoiding them both. For the pure implicational fragments of these systems, this may be shown by matrices which are small and easily handled. Firstly, Church's weak positive implicational calculus¹ avoids fallacies of relevance, as is shown by the following matrix, adapted from Sobocinski:²

Matrix 4

	C	1	2	3	4
*	1	1	4	4	4
*	2	1	2	3	4
	3	1	4	2	4
	4	1	1	1	1

To show the impossibility of $\vdash \underline{Cxy}$ where $\underline{x}$ and $\underline{y}$ share no variable, put the variables of $\underline{x}$ equal to 1, and those of $\underline{y}$ equal to 2. $C12 = 4$. Secondly, the pure strict implicational fragment of S3 avoids fallacies of necessity, as may be seen from the following adequate matrix:

Matrix 5

	C	1	2	3
*	1	2	3	3
*	2	2	2	3
	3	2	2	2

Here $\underline{Cx} \underline{Cyz} = 3$ whenever $\underline{x}$, as a propositional variable, takes the value 1. But only the pure implicational fragment of E, which is contained in Church's system and in S3, avoids both fallacies. This is true also when we join negation axioms to these systems.

¹A. Church, The weak theory of implication, in Kontrolliertes Denken, Munich 1951.

²B. Sobocinski, A note concerning the many-valued propositional calculi, The Notre Dame Journal of Formal Logic 2 (1961), pp. 127-8.

22. Three types of proposition which lead to paradox

I wish now to look at the paradoxes of relevance in a more generalized way, and offer the following definition:

A logical system contains 'paradoxes of relevance' if it permits the derivation of information about all propositions from information about some propositions.

This definition does not differ essentially from Belnap's criterion of variable-sharing, as will be seen. Perhaps the best way of showing how the refusal to pass from statements about some propositions to statements about all lies at the root of our abhorrence of paradox is to consider an example from Łukasiewicz. In 1930, this author, investigating the properties of the Aristotelian concept of contingency (symbolized by 'Q'), discovered that if we join the law $CQpQNp$, stating that if any proposition is contingent, then so is its contradictory, to Leśniewski's law $C\hat{p}C\hat{N}p\delta q$ for truth-functions (' δ ' denoting an arbitrary function), then we can derive the consequence $CQpQq$.¹ Consider this formula. Universal quantifiers always being implicitly understood to be prefixed to the free-variable formulae of propositional logic, its meaning is $\forall p \forall q CQpQq$, namely $C\forall p Qp \forall q Qq$ - if any proposition is contingent, then every proposition is contingent. This is certainly paradoxical. Here, then, we see the real point behind variable-sharing in any true implication. It prevents the attaching of an existential quantifier to the antecedent and a universal quantifier to the consequent; hence the impermissible inference from what concerns one proposition to what concerns all propositions cannot be carried out.

¹Łukasiewicz's derivation, in PB §4, amounts to this. Prior, FL, p. 191, derives the formula explicitly.

Granted that this impermissible inference is of the essence of paradoxes of relevance, we note that there are three main types of proposition which can give rise to it.

(a) What I shall call self-inconsistent propositions, and their counterparts vacuous propositions. These propositions give rise to paradox directly, without the intermediary of other propositions. Self-inconsistent propositions imply everything; they include, for example, NCpp and KpNp in classical logic, since we have CNCppq and CKpNpq as classical theses. Vacuous propositions, on the other hand, are implied by anything; they include, again in classical logic, Cpp and ApNp.

(b) Secondly, what I shall call self-defeating propositions, and their counterparts Clavian¹ propositions. These propositions do not give rise to paradox directly, but require paradoxical theses - specifically the paradoxes of strict implication. As was seen in section 18, the negative and positive versions of these paradoxical theses may be written as CCpNpCpq and CCNppCqp respectively. Of their antecedents, CpNp asserts that p is self-defeating, and CNpp that p is Clavian. That is to say, any proposition which implies its own negation is self-defeating, and such a proposition, together with the negative paradox of strict implication, engenders a paradox. Similarly, any proposition which is implied by its own negation is Clavian.

(c) Finally we have false propositions and true propositions. It seems odd to speak of them as paradoxogenic, but, like self-defeating and Clavian propositions, they require extra theses in order to be noxious. These are the paradoxes of material implication, which, together with any

¹From the law of Clavius.

false or any true proposition, produce paradox.

Using this division of paradox-engendering propositions, we may now proceed to classify the different systems of propositional logic according to the paradoxes they contain.

23. The pattern of increasing relevance

In order to avoid paradoxes of relevance in a system, one must avoid self-inconsistent and vacuous propositions; one must either avoid self-defeating and Clavian propositions, or avoid the paradoxes of strict implication, or both; and one must avoid the paradoxes of material implication, since one cannot retain them without at the same time retaining false and true propositions.¹ The classical propositional calculus does none of these things. But every other known system does one or the other, and hence, to a greater or a lesser degree, ~~falls~~ ^{approaches} ~~short of~~ the ideal of relevance and the avoidance of paradox. I shall give in this section a table of different systems, each containing operators for implication and negation alone, and show how they fare vis à vis the paradoxical and other related theses.

First, though, certain powers possessed by the theses should be explained.

(i) Any system containing the paradoxes of strict implication (p.s.i. theses) will be able to convert self-defeating propositions into

¹One can, of course, retain the règles corresponding to the paradoxes of material implication, and have no true propositions (theses) in the system at all.

self-inconsistent ones, and Clavian into vacuous. For if we have CCpNpCpq and some self-defeating proposition $\underline{x}$, we have CxNx , and hence by detachment Cxq , which means that $\underline{x}$ is self-inconsistent. Similarly for Clavian propositions.

(ii) The paradoxes of material implication (p.m.i. theses) are able to convert false propositions into self-inconsistent ones, and true propositions into vacuous, by a process similar to that above. Nor is it even necessary that p.m.i. theses should be used for this purpose, the p.m.i. rules corresponding to the theses (i.e. $\vdash \underline{x} \rightarrow \vdash \text{CNxy}$, $\vdash \underline{x} \rightarrow \vdash \text{Cyx}$) sufficing. Once a self-inconsistent proposition has been obtained, it is easy to show that it is also self-defeating, since if we have Cxq we get by substitution CxNx .

(iii) Since vacuous propositions are always Clavian and true, Clavian propositions vacuous under p.s.i. theses, and true propositions vacuous and Clavian under p.m.i. theses, the only question remaining is, under what conditions are Clavian propositions true, and self-defeating propositions false? Not under any of the standard paradoxical theses, it turns out, but under the law of Clavius CCNpppp , and its counterpart CCpNpNp . Hence, although these theses violate no hitherto-propounded criterion of relevance, they are included in the table. The reason is that they provide for one of the linkages between the three types of proposition whose severing seems to be part and parcel of a progressively increasing degree of relevance.

The table below shows whether or not each of the theses or rules in question is contained in each of 10 systems of propositional logic. Note that it is only a question of the implication-negation (C-N)

fragments of the systems in each case. Nomenclature is as follows: PC is classical logic; CN3-5 are the C-N fragments of S3-5; Ł3V is Łukasiewicz's original three-valued system of 1920;¹ ICN is the C-N fragment of intuitionist logic; MCN is the C-N fragment of Johansson's Minimal Calculus;² WCN is Church's weak positive implicational calculus plus the strong negation axioms CCNqNpCpq, CpNNp, CCNppp; IN is the C-N fragment of Anderson's and Belnap's E; and IEN is a fragment, contained in IN, of the C-N system characterized by Angell's matrix of section 15. These systems will be discussed more fully in chapters 4 and 5. Occasional entries in the table are justified in footnotes below.

¹Łukasiewicz, PB, §6.

²I. Johansson, Der Minimalalkül, ein reduzierter intuitionistischer Formalismus, Compositio Mathematica, 4 (1936), pp. 119-136. See also JSL 2(1937), p. 47, and Prior, FL, p. 259. This fragment turns out to be identical with Kolmogoroff's calculus, for which see S. Jaśkowski, On the rules of suppositions in formal logic, Studia Logica 1 (1934), pp. 5-32.

TABLE 3

System	Self- incon. props.	Vacuous props.	Self- def. props.	Clavian props.	P.s.i. the- ses		P.m.i. theses		P.m.i. rules		<u>CCpNpNp</u>	<u>CCNppp</u>
					neg.	pos.	neg.	pos.	neg.	pos.		
PC	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes
CN5	yes ¹	yes	yes	yes	yes	yes	no	no	yes ²	yes	yes	yes
CN4	yes ¹	yes	yes	yes	yes	yes	no	no	yes ²	yes	yes	yes
CN3	yes ³	yes	yes ⁴	yes	yes	yes	no	no	no ⁵	no	yes	yes
ICN ⁶	yes	yes	yes	yes	no	no	yes	yes	yes	yes	no	no
ICN	yes ⁷	yes	yes	yes	yes ⁸	no ⁸	yes	yes	yes	yes	yes ⁹	no ⁹
MCN	? ¹⁰	yes	yes ¹¹	yes	?	no	no ¹²	yes	no ¹²	yes	yes ¹³	no
WCN	no ¹⁴	no	yes ¹⁵	yes	no ¹⁶	no	no	no	no	no	yes	yes
IN	no ¹⁴	no	yes ¹⁵	yes	no ¹⁶	no	no	no	no	no	yes	yes
ICN ¹⁷	no	no	no	no	no	no	no	no	no	no	no	no

Notes to table 3

¹Both CqCpp and CNCppq are theses of S4 and S5, though not of S3.

²In S4 and S5 we have that if $\vdash \underline{x}$, then $\vdash \underline{Lx}$, i.e. $\vdash \underline{CNxx}$, whence by the p.s.i. thesis $\vdash \underline{Cqx}$. Similarly for the negative rule.

³From self-defeating propositions by p.s.i. theses.

⁴See section 53.

⁵As a counterexample to this rule, we have in S3 $\vdash \underline{Cpp}$, but not $\vdash \underline{CNCppq}$. Similarly for the positive rule.

⁶All the theses and rules in $\mathcal{L}3V$ may be checked by the following matrix:

	C	1	2	3	N
* 1	1	1	2	3	3
2	1	1	1	2	2
3	1	1	1	1	1

⁷From p.m.i. theses.

⁸We have $\vdash \underline{CCpNpCpq}$ in ICN, since $\underline{CpNp} = \underline{Np}$ and we have $\vdash \underline{CNpCpq}$. But we do not have $\vdash \underline{CCNppCqp}$, because $\underline{CNpp} = \underline{NNp}$, and $\underline{CNNpCqp} = \underline{CqC\cancel{M}Npp}$ is rejected because its substitution $\underline{CCppCNNpp}$ is rejected. For these equivalences and rejected formulae see S. McCall, A simple decision procedure for one-variable implication/negation formulae in intuitionist logic, Notre Dame Journal of Formal Logic 3 (1962), pp.120-2.

⁹See McCall, op. cit.

¹⁰Uncertain whether MCN contains any.

¹¹We have for example in MCN $\vdash \underline{CqCpp}$, whence $\vdash \underline{CNCppNa}$, whence $\vdash \underline{CNCppNNCpp}$.

¹²MCN is defined as ICN minus CNpCpq. The failure of the corresponding rule is implied in Johansson's discussion, pp. 126-7.

¹³See Systems of propositional calculus, (mimeographed, 3 pages), University of Canterbury, New Zealand.

¹⁴Such theses would violate relevance-matrix 3, section 20, which satisfies WCN as well as IN.

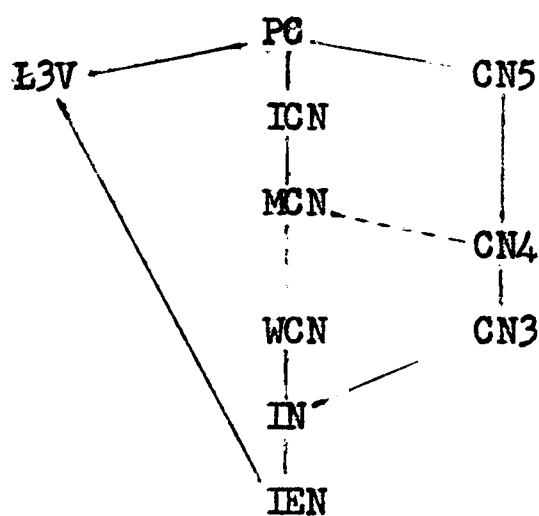
¹⁵See section 54.

¹⁶These theses, together with self-defeating theses, would produce self-inconsistent theses.

¹⁷IEN is contained in IN, and hence lacks what it lacks. What it lacks in addition is rejected by matrix 1, section 15.

To conclude this section I shall list the various relations of containment which hold among the 10 systems of table 3. A solid line, sloping downward, indicates that the lower system is contained wholly in the upper; a dotted line, that its pure implicational fragment is contained.

Table 4



IEN, it will be noted, occupies a position slightly ahead of IN in the direction of increasing relevance, although it is arguable that, since both

are characterized by variable-sharing across entailments, this represents no real gain. In the next section, however, IEN plus conjunction will be shown to adhere to a stricter form of relevance than E does.

24. A stricter relevance

We find in Nelson some well-taken objections on grounds of relevance to the formulae CKppq and CpApq, asserted as theses in E.

'It cannot be asserted that ^{the} conjunction of p and q entails p, for q may be totally irrelevant to and independent of p, in which case p and q do not entail p, but it is only p that entails p. I can see no reason for saying that p and q entail p, when p alone does and q is irrelevant, and hence does not function as a premiss in the entailing. . . .

'Furthermore, "p entails p or q" cannot be asserted on logical grounds, because from an analysis of p we cannot derive the propositional function "p or q" where q is a variable standing for just any other propositional function whatsoever.'¹

Plainly Nelson is here adopting a stricter notion of relevance than Anderson and Belnap, and it is precisely those theses which he rejects that are rejected by Angell's matrix. In other respects, however, Nelson is more liberal than Anderson and Belnap, for he allows the principle of the antilogism, CKpqrCKpNrNq, which they reject. It is easy to see why they must reject it, once they accept the law of simplification, since it would allow them to pass from CKppq to CKpNpNq, which violates the criterion of variable-sharing. Von Wright apparently has qualms about the principle of the antilogism's holding for entailment, but does not voice them.² It is hard to think what they might be.

¹Nelson, pp. 447-8. Lewis (IAL, pp. 528-30), rejects CpApq while accepting CKppq, though on different grounds.

²von Wright, p. 172 note. See also Duncan-Jones, p. 75 ff., for a rather confused discussion of the matter, in which CKpqr is said to 'invoke', rather than entail, CKpNrNq. Duncan-Jones' only reason for saying this seems to be to prevent the passing from CKppq, which he accepts, to CKpNpq, which he rejects.

Again, in giving the outlines of a system satisfying a stricter criterion of relevance than Anderson's and Belnap's E, reference must be made to the inference schema known as modus tollendo ponens. This schema, the fifth Stoic indemonstrable, plays a part in Pseudo-Scotus' demonstration that a contradiction, such as 'Socrates exists and Socrates does not exist', implies anything. Schematically, Pseudo-Scotus' proof is as follows:-¹

1. KpNp	hypothesis
2. p	1 law of simplification
3. Apq	2 law of addition
4. Np	1 law of simplification
5. q	3,4 <u>modus tollendo ponens</u>

To prevent this derivation, which yields a paradoxical result, Anderson and Belnap forbid modus tollendo ponens. Angell's matrices, on the other hand, allow modus tollendo ponens and forbid the laws of simplification and addition. It is hard, bearing in mind Nelson's strictures against the latter, together with the intuitive obviousness of modus tollendo ponens, not to side with the matrices against E in adopting a stricter criterion of relevance.

When we come to consider systems containing conjunction once more, reference will be made to Nelson's stricter criterion of relevance. But for the next two chapters we shall be concerned almost exclusively with systems in implication and negation alone, the aim being to discover which, if any, can consistently admit Aristotle's and Boethius' theses. The answer is, as we shall see, only those which progress furthest in the direction indicated in table 3 - furthest toward the ideal of relevance.

¹See Kneale, DL, p. 282.

CHAPTER FOUR

C, C-N AND C-N-K LOGICS

25. Propositional logic and modal logic

Table 3 of section 23 made reference to ten systems of propositional logic, all of them containing only operators for implication and negation. In this chapter certain of these implication/negation logics, together with their pure implicational fragments and the C-N-K systems of which they form part, will be presented in a natural way, facilitating their use in performing deductions. The presentation will also make it easier to see how the various systems stand vis à vis relevance and the possible introduction of Aristotle's and Boethius' theses.

A number of the better-known systems of propositional logic, for example S1-5, L3V and Ackermann's Π , were originally designed as modal logics, containing operators for necessity and possibility. In S1-5 the operator ' $\Diamond$ ', for 'possibly', and in Π the constant ' $\wedge$ ' ('das Absurde'), were taken as primitive. But it was soon discovered that these modal notions could be defined in terms of other notions, following the pattern of L3V. Łukasiewicz credits Tarski with having, in 1921, proposed the definition of Mp as CNpp in that system,¹ and the definition referred to in section 18 of Lp as CNpp (and hence of Mp as NCpNp, since M = NLN) in the Lewis systems was originally proposed by Lemmon et al.² In fact, these authors show, two possible definitions of Lp exist in S5; that mentioned above, and that of Lp as CCpppp. Of these, CNpp is the more usual definition of necessity, Lewis himself having spoken from the

¹Łukasiewicz, PB, §7. Despite Łukasiewicz's attempts to give it one, the definition has little intuitive justification.

²E.J. Lemmon, C.A. Meredith, D. Meredith, A.N. Prior, and I. Thomas, Calculi of pure strict implication, 1956 (mimeographed).

beginning of a necessary proposition as one implied by its own denial, and of an impossible proposition as one which implies it.¹ But the definition of Lp as CCppp is taken up again by Anderson and Belnap, who show that in E, which lacks the constant $\wedge$, the modality structure of ' $\Box$ ' can be exactly reproduced by defining Lp in that way, instead of as CNp $\wedge$, as Ackermann does.

The point of these remarks is that modal logics need not cease to be modal logics when viewed as C-N logics. Most of the modal features characterizing the Lewis systems, for example, are mirrored in those of their theses which contain only strict implication and negation: to CLpp there corresponds CCNppp, etc. In fact there are good reasons for thinking that the Lewis systems are best formulated with strict implication primitive instead of possibility, since they may then be exhibited as proper parts of classical logic instead of as containing it.² Lewis originally intended strict implication to be a narrower notion than material implication, and this way of formulating his systems has the effect of allowing the differences between strict and material implication to emerge as differences between each of S1-5 and PC, rather than within S1-5. Of course, any system which contains negation and conjunction ipso facto contains material implication, which is definable in terms of them, but if we build up the Lewis systems by beginning with strict implication, and then adding, successively, negation and conjunction, material implication will not be definable until the main characteristics distinguishing S1-5 from PC have already been established. It is true that Lewis himself proves numerous theses

¹Lewis, SSL, pp. 337-8.

²See Lemmon, New foundations for Lewis modal systems, JSL 22 (1957) pp. 176-86, for S1-5 conceived as additions to PC.

which contain both implication operators, but in many ways these only serve to confuse the reader. It does not help ^{us} as much, for example, to know that in S1 p strictly implies that q materially implies p .¹ We may therefore lay it down as a good principle, not to mix different kinds of implication within the same logical system.

There is one **other** advantage to looking at modal logics, where we can, as propositional logics, and this lies in being able to provide some sort of an answer to those who find modal concepts puzzling, confused, or not fit subject matter for logical treatment. If defined in terms of the more basic concepts of propositional logic, modal notions are presumably as clear as the former. Of course, this answer will not satisfy those whose dissatisfaction extends beyond modality to envelop the whole of logic, with the possible exception of set-theory. Thus Martin:

'It is a defensible view that there is no such thing as pure logic. There is no subject matter for logic itself. There are only non-logical subject matters suitably described in given systems.'²

But there are answers to this. The notion of implication, for example, is the very glue which holds any axiomatized system, mathematical or otherwise, together. The study of the notion of implication is, if anything, the study of pure logic.

It will be the position of the writer, in short, that the most fundamental of all logical systems are those of pure implication, and the next most fundamental are those of implication and negation. Hence the

¹Theorem 15.2 of Lewis and Langford, p. 142.

²R.M. Martin On Knowing, believing, thinking, The Journal of Philosophy 59 (1962), pp. 586-600. See p. 589.

presentation of various $\underline{C}$ and $\underline{C-N}$ logics in this chapter. Even though systems may be devised in which both implication and negation are defined in terms of, for example, an incompatibility operator, such as the Sheffer stroke-function, this is merely a notational strategem whose purpose is logical economy, and the derivation of theses from axioms in such a system is not intuitively explicable in terms of incompatibility, but makes covert use of disguised implication-relations. The definition of modal notions in terms of implication, or implication and negation, is optional in propositional systems, and the modal structure obtained is arbitrary in that different definitions may yield different results within the same system. For example, the different definitions $\underline{CNpp}$ and $\underline{CCpppp}$ of $\underline{Lp}$ in $S3$ result in $S3$'s having 42 distinct modalities under one definition and 14 under the other. Finally, conjunction axioms will be added to the $\underline{C-N}$ systems constructed in this chapter, partly because of their intrinsic interest, and partly in order to bring the systems up from the level of ' $\underline{C-N}$ fragments' to their full traditional strength.

26. Different formulations of the system E

The first system to be dealt with is the system E, for which Anderson and Belnap have devised three different formulations. The first of these, the traditional axiomatic formulation, has the advantage of being easy to work with when dealing with matrices. The second, Fitch's subproof formulation, yields a most natural and intuitive way of constructing proofs within the system. Furthermore, it is not difficult to correlate these two formulations with one another, and they will be used in presenting the

other systems considered in this chapter as well. Anderson's and Belnap's third formulation is of the Gentzen type, and is useful for certain more esoteric purposes, notably for proving things about the system. Its rules, for implication alone, are given in section 58.

In presenting the systems of this chapter, I shall begin with implication, then add negation and, in some cases, conjunction. This procedure raises the question, whether the bases provided for the C and the C-N fragments are complete with respect to the systems as a whole; that is, whether they enable us to derive every C or C-N thesis contained in the full C-N-K system. Such completeness results exist for many of the systems of this chapter, and will be referred to in the relevant sections.

27. The system I

The system I, conjectured by Anderson and Belnap to be the pure implicational fragment of the system E, and proved to be so by Chung and Schindler,¹ is defined by the following axioms, together with the rules of substitution and modus ponens:

- | | |
|----------------|-----------|
| 1. CCpqCCqrCpr | (Syl) |
| 2. CCpCpqCpq | (Hilbert) |
| 3. CCCppqq | (Belnap) |

The names attached to the axioms are for convenience in mentioning them:

'Belnap' is so called because that logician succeeded in the important task of deriving the weak law of commutation CCpCQrCQCpr² from it together with

¹Lung-Ock Chung, Investigation into certain implication-negation fragments of propositional logic, McGill University thesis, 1963; P.F. Schindler, The formalization of implication in propositional logic, McGill University thesis, 1963.

²Following Lemmon et al., capital letters P, Q, etc. are used as propositional variables taking as their values only formulae which are implications, e.g. Crs, CpCqr, etc.

Syl.¹ The independence of the three axioms is demonstrated in Anderson, Belnap and Wallace.

Corresponding to the above axiomatic formulation of I, there is an equivalent subproof formulation I* in the style of Fitch, as modified by Anderson.² Briefly, Fitch's method of proving any thesis Cxy is to assume x as an hypothesis, and to prove y from it using only the rules of the system. To prove CxCyz requires proving Cyz on hypothesis x, i.e. z on hypotheses x and y. Each hypothesis is placed at the head of a separate subproof, as will be shown. Suppose we wish to prove CpCqq. We proceed as follows:

1.	p	hyp
2.	q	hyp
3.	q	2 rep
4.	Cqq	2,3 CP
5.	CpCqq	1,4 CP

Here lines 1 and 2 are hypotheses; 3 follows from 2 by the rule of repetition; the second hypothesis is absorbed in line 4 by the rule of conditional proof,³ the subproof headed by that hypothesis being then closed; finally the first hypothesis is absorbed in line 5.

Anderson's modification of Fitch's subproof format is to introduce a method of keeping track of all the hypotheses actually used in a given proof, and then restricting the rule of conditional proof so that Cxy will be derivable only if x is used in the derivation of y. The above proof of CpCqq would not hold under Anderson's modification, since the

¹A proof of Weak Comm is given in appendix 4. See Prior, FL, p. 319, for the origin of some of these names.

²See F.B. Fitch, Symbolic Logic, New York 1952; Anderson, CTE.

³The name of this rule comes from Suppes, p. 28.

first hypothesis is not used in the derivation of line 4. Anderson's method of keeping track of hypotheses is to assign what I shall call a dependence-numeral to each hypothesis, further items on the proof accumulating dependence-numerals according to which hypotheses have been used in their derivation. The precise rules for the system I^* are as follows, where the variables $\underline{a}$, $\underline{b}$, $\underline{c}$, ... denote dependence-numerals or sets of dependence-numerals, and ' $(\underline{a})\underline{x}$ ' denotes the propositional formula $\underline{x}$ with which is associated the dependence-numeral or set of numerals $\underline{a}$. ' $\underline{a}+\underline{b}$ ' denotes the union of $\underline{a}$ and $\underline{b}$, ' $\underline{a}\underline{b}$ ' their intersection, and ' $\neg \underline{a}$ ' the complement of $\underline{a}$.

- hyp An hypothesis may be introduced at any line of a proof. The first such hypothesis in any proof receives the dependence-numeral (1), and the subproof which it heads is called the first subproof. An hypothesis introduced into the first subproof receives the numeral (2), and so on.
- rep $(\underline{a})\underline{x}$ may be repeated within the same subproof.
- reit $(\underline{a})\underline{x}$ may be reiterated from subproof n into subproof $n+m$, provided that $\underline{x}$ is an implication.
- MP From items $(\underline{a})\underline{x}$ and $(\underline{b})\underline{Cxy}$ in subproof n derive $(\underline{a}+\underline{b})\underline{y}$ in subproof n .
- CP From the hypothesis $(\underline{n})\underline{x}$ and the item $(\underline{n}+\underline{a})\underline{y}$ in subproof n derive $(\underline{a})\underline{Cxy}$ in subproof $n-1$.

The following proof of the weak law of commutation illustrates the use of these rules:

(1)	1.	CpCQr	hyp
(2)	2.	Q	hyp
(3)	3.	p	hyp
(1)	4.	CpCQr	1 reit
(1,3)	5.	CQr	3,4 MP
(2)	6.	Q	2 reit
(1,2,3)	7.	r	5,6 MP
(1,2)	8.	Cpr	3,7 CP
(1)	9.	CQCpr	2,8 CP
$\wedge$	10.	CCpCQrCQCpr	1,9 CP

Theses of the system depend on $\wedge$, the empty set of hypotheses. Note that the above proof will not work for the strong law of commutation CCpCqrCqCpr (Comm), since the reiteration at line 6 requires that Q be an implication.

It is not difficult to show that the two systems I and I^* are identical in the sense that the same class of theses is derivable in each. To show that I^* contains I , we prove the axioms of I in I^* and note that the rule of modus ponens occurs in I^* as MP. The rule of substitution in I is dispensable if we reinterpret the axioms as containing the metalogical variables x, y, z, ... instead of the propositional variables p, q, r, ... This converts the three axioms of I into three axiom schemata, standing for, in effect, an infinity of axioms of the three forms, and each of these axiom schemata is provable in I^* . We retain the old formulation of I with propositional variables and rule of substitution only for the sake of familiarity.

To show that I contains I^* , we show that to every proof in I^* there corresponds a proof in I . This is accomplished by Anderson's technique of successively collapsing the subproofs of I^* , beginning with the innermost, so that every item, or a transformation of every item, eventually finds its way into the zero or 'main' proof, which is a proof

of I.¹ Anderson first defines a quasi-proof in I^* as a proof in which the rule of reiteration is relaxed to allow reiteration of theses into any subproof. This relaxation does not enlarge the class of theses of I^* , since it obviously makes no difference whether, if we use a thesis T in subproof n , we prove T in n or prove it elsewhere and reiterate it into n . Anderson then proves the following:

Theorem Let P be a proof or quasi-proof of I^* , and let Q be an m th and innermost subproof of P . Let $(a_1)y_1, (a_2)y_2, \dots, (a_n)y_n$ be the items of Q , of which the first, $(m)x$, is the hypothesis. Let $(a_i)y_i'$ be $(a_i)y_i$ or $(a_i-m)Cxy_i$ according as m is not, or is, in a_i , and let P' be the result of replacing the subproof Q in P by the sequence of items $(a_i)y_i'$ in subproof $m-1$. Then we may reiterate theses of I into P' , and apply MP and rep, in such a way as to convert P' into a quasi-proof of I^* .

Proof We require the following theses of I for reiterating into P' :

- | | |
|-----|--------------------------|
| T1. | Cpp |
| T2. | CCpCqrCCpqCpr |
| T3. | CCpCqrCQCpr ² |

A The proof requires firstly that we show, by induction on n , that the occurrence of each of the $(a_i)y_i'$ in subproof $m-1$ is justified.

Basis $n = 1$. $(a_1)y_1'$ is $(m)x$; hence m is in a_1 and $(a_1)y_1'$ is $\wedge Cxx$. We treat this item as a reiteration of T1.

¹Anderson's argument, in CTE, applies to the full calculus E, but is easily adapted to I.

²For a proof that T1-3 are theses of I, see appendix 5.1.

- Induction step Assume that the theorem is proved for each $(a_1)y_1$ up to a certain item $(a)y$, and consider the justification of $(a)y$.
- Case 1. $(a)y$ comes by rep. Then the induction hypothesis guarantees the theorem for $(a)y$.
- Case 2. $(a)y$ comes by reit. Then m is not in a , so $(a)y'$ is $(a)y$. We treat $(a)y$ either as a repetition, or as a reiteration from a subproof of lower order.
- Case 3. $(a)y$ comes by MP from $(b)z$ and $(c)Czy$, where $a = b+c$. We distinguish four subcases.
- Case 3.1 m is in both b and c . Then $(b)z'$ is $(b-m)Cxz$, and $(c)Czy'$ is $(c-m)Cx Czy$. Reiterate T2 and use MP twice to get, successively, $(c-m)CCxz Cxy$ and $(c-m+b-m)Cxy$, i.e. $(a-m)Cxy$, i.e. $(a)y'$.
- Case 3.2 m is in b but not in c . Then $(b)z'$ is $(b-m)Cxz$, and $(c)Czy'$ is $(c)Czy$. Reiterate axiom 1 of I and use MP twice to get $(a-m)Cxy$, i.e. $(a)y'$.
- Case 3.3 m is in c but not in b . Then $(b)z'$ is $(b)z$ and $(c)Czy'$ is $(c-m)Cx Czy$. Since m is not in b , $(b)z$ must come by reit¹, and hence be an implication, say $(b)Cuv$. Then $(c-m)Cx Czy$ will be $(c-m)Cx CCuvy$. Reiterate T3 and use MP twice to obtain $(a-m)Cxy$, i.e. $(a)y'$.
- Case 3.4 m is in neither b nor c . Then MP gives $(a)y$, i.e. $(a)y'$.
- Case 4. $(a)y$ comes by CP. Impossible, Q being an innermost subproof.
- B. To complete the proof of the theorem, we must show that the item $(a_n-m)Cxy_n$ in subproof $m-1$, which was treated as a consequence of Q in P , can now be treated as a consequence

¹ Not necessarily: it might come by MP. Use the lemma of section 34.

of the items $(a_i)y_i'$ in P' . But we see immediately that the last of these items is just $(a_n-m)Cxy_n$, since the rule CP requires that the last item in a subproof depend on the hypothesis. So we treat $(a_n-m)Cxy_n$ in subproof $m-1$ as a repetition of this item.

This completes the proof of the theorem. We note that the proof or quasi-proof with which the theorem began now has one less subproof, and repeat the whole procedure until every subproof of order greater than zero has gone. There remains a sequence of theses of I, with MP and rep as the sole rules. Adding repetition to I does not change that system; hence I contains I^* .

28. The system IN

As in the case of the system I, the completeness of the following axiomatic basis for the implication/negation fragment IN of E follows from the work of Chung and Schindler:

- | | | |
|----|-------------|-----------|
| 1. | CCpqCCqrCpr | |
| 2. | CCpCpqCpq | |
| 3. | CCCppqq | |
| 4. | CCNpqCNqp | (Trans 3) |
| 5. | CpNNp | (Doub 1) |
| 6. | CCNppp | (Clavius) |

The three negation axioms added to I are each independent of the other five axioms, as may be shown by the following matrices:

Matrix 6 (axiom 4 fails for $p = 2, q = 1$)

	C	1	2	N
*	1	1	2	1
	2	1	1	1

Matrix 7 (axiom 5 fails for $p = 2$)

	C	1	2	3	N
*	1	1	3	3	3
	2	1	1	3	1
	3	1	1	1	1

Matrix 8 (axiom 6 fails for $p = 2$)

	C	1	2	3	N
*	1	1	3	3	3
	2	1	1	3	2
	3	1	1	1	1

I have not, however, succeeded in showing the independence of the pure C axioms under negation, with the exception of axiom 1, which is independent by a result of Diamond and McKinsey according to which any complete axiomatization of classical C-N logic must contain at least one three-variabled axiom.¹

The subproof formulation IN^* of the system IN is constructed by adding the following negation rules to the rules hyp, rep, reit, MP and CP of I^* :

- MT From items (a)Ny and (b)Cxy in subproof n derive (a+b)Nx in subproof n.
- DNI From (a)x derive (a)NNx in subproof n.
- DNE From (a)NNx derive (a)x in subproof n.
- RA From the hypothesis (n)x and items (n+a)y and (n+b)Ny in subproof n derive (a+b)Nx in subproof n-1.

It is easy to show that IN^* contains IN . I shall not prove the converse, which involves a theorem corresponding to that of section 27. The proof

¹

See Anderson, Belnap and Wallace.

proceeds as in Anderson, CTE, and is made slightly more complicated by the fact that in case 3.3 an item $(\underline{b})\underline{z}$ may occur in an innermost subproof $\underline{m}$, and $\underline{m}$ not be in $\underline{b}$, without $\underline{z}$ being an implication. Anderson overcomes this difficulty by means of a lemma (CTE, p. 207; see also section 34).

29. The system E

The full system E, in implication, negation and conjunction, is formed by adding certain conjunction axioms to IN. For convenience in stating the axiomatic basis of E, we make use of the following definitions:

Df.L: $Lx = CCxxx$

Df.A: $Axy = NKNxNy$

The axioms of E are:

1. $CCpqCCqrCpr$
2. $CCpCpqCpq$
3. $CCCpqq$
4. $CCNpqCNqp$
5. $CpNNp$
6. $CCNppp$
7. $CKpqp$
8. $CKpqq$
9. $CKCpqCprCpKqr$
10. $CKLpLqLKpq$
11. $CKpAqrAKpqr,$

and the rules (in addition to substitution) are modus ponens and adjunction: $\vdash \underline{x}, \vdash \underline{y} \rightarrow \vdash \underline{Kxy}$.

The subproof formulation E^* of E is formed by adding three rules for conjunction to IN^{*}. These are:

KI From items $(a)x$ and $(a)y$ in subproof n derive $(a)Kxy$ in subproof n.

KE From $(a)Kxy$ in subproof n derive $(a)x$ in subproof n.
From $(a)Kxy$ in subproof n derive $(a)y$ in subproof n.

dist From $(a)KxAyz$ in subproof n derive $(a)AKxyz$ in subproof n.

For the proof that E and E^* are one and the same system, see Anderson, CTE.

30. Church's weak calculus WC

If we take all the theorems of I which hold only for capitalized propositional variables, standing for implications, and replace these variables by ordinary propositional variables, we obtain Church's weak positive implicational calculus. The axioms for WC are as follows:

1. CCpqCCqrCpr
2. CCpCpqCpq
3. CpCCpqq (Pon)
4. Cpp (Id)

Church's original axioms for WC included Comm in place of Pon, but the derivation of the former from the latter is given in appendix 5.2. The same matrices as those used by Church suffice to show the independence of the four axioms.

The principal, in fact the only, difference between I and WC is that the latter permits fallacies of necessity. Thus I allows only Weak Pon, CPCCPqq, in place of Pon. This difference is reflected in the fact that any matrix which is (a) adequate for I, and (b) allows for a propositional variable p no value which is not also allowable for an implication Cpq, is also adequate for WC. For example, the matrix

Matrix 9

	C	1	2	3	4
* 1	1	4	4	4	4
* 2	1	2	4	4	4
3	1	4	2	4	4
4	1	1	1	1	1

is adequate for I, and distinguishes I from WC (CpCCpqq fails for $p = q = 3$), but if we change the value of C23 from 4 to 3 we get matrix 4, which is adequate for both I and WC. Matrices which allow, for functions, all the values of their arguments, make no distinction between capitalized and small-letter variables.

To construct a subproof formulation WC^* of WC , we need only change the rule of reiteration to allow for fallacies of necessity. Thus we have, for WC^* ,

reit₁ (a)x may be reiterated from subproof n into subproof n+m.

31. The system WCN

We may add strong axioms of negation to Church's calculus without collapsing that calculus into classical logic, as would happen in the case of the intuitionist calculus. The following axioms shall be taken as defining the system WCN:

1. $CCpqCCqrCpr$
2. $CCpCpqCpq$
3. $CpCCpqq$
4. $CCNpNqCqp$ (Trans 4)
5. $CpNNp$
6. $CCNppp$

Trans 4 is used, instead of IN's axiom Trans 3, in order that the axiom Id of WC be derivable (Trans 4 and Doub 1 give $CNNpp$ (Doub 2); Doub 2, Doub 1, and Syl give Id). I have not been able to show the independence of each negation axiom of WCN. It has been proved by Schindler and Chung that the pure implicational fragment of WCN is WC . A matrix showing the distinctness of WCN from intuitionist logic and from any of the systems of strict implication is matrix 4 plus strong negation, namely

Matrix 10

	C	1	2	3	4	N
*	1	1	4	4	4	4
*	2	1	2	3	4	3
	3	1	4	2	4	2
	4	1	1	1	1	1

Although it contains fallacies of necessity, WCN satisfies exactly the same criterion of relevance as the system E. This is shown by the fact that Belnap's relevance matrix 3 of section 20 satisfies WCN. The subproof formulation of WCN, WCN^* , consists of the rules hyp, rep, reit₁, MP and CP of WC , plus the rules MT, DNI, DNE and RA.

32. The system C3

We come now to a system which avoids fallacies of necessity, but which admits, to a limited degree, fallacies of relevance. This is the pure strict implicational fragment C3 of Lewis's S3, axiomatized as follows:

1. CCpqCCqrCpr
2. CCpCpqCpq · (Weak Comm'd Simp)
3. CQCpp

That this is a complete axiomatization of C3 was hypothesized by Lemmon et al. and proved by Hacking, in whose paper is to be found a Gentzen-type system shown to be complete with respect to the pure implicational fragment of S3.¹ That C3 contains no fallacies of necessity is proved in section 34, this being also true of the whole system S3. A slightly stronger theorem concerning necessity is proved, for the system CN3, in the next section. Note that axiom 3 allows the violation, for implicative (strict) antecedents, of the criterion of relevance.

The construction of C3*, the subproof formulation of C3, requires joining to the rules hyp, rep, reit, MP and CP of the system I* a sixth rule, namely

add From the hypothesis (n)x, where x is an implication,
and the item (a)y in subproof n+m, derive (a+n)y in
subproof n+m.

This rule, analogous to the Gentzen rule of weakening, allows for the 'dependence' of an item in a subproof on any strict hypothesis. Its use in proving axiom 3 above is as follows:

¹I.M. Hacking, Strict implication and natural deduction, Trinity College, Cambridge, 1960 (typescript).

(1)	1.	Q	hyp
(2)	2.	p	hyp
(2)	3.	p	2 rep
$\wedge$	4.	Cpp	2,3 CP
(1)	5.	Cpp	4 add
$\wedge$	6.	CQCpp	1,5 CP

Hence $C3^*$ contains C3. To show the converse, we first of all derive the axioms of I in C3 (see appendix 5.3), and then follow the same course of argument as in section 27. The only difference comes when the rule add occurs as the reason for one of the items of subproof $\underline{m}$. We then have the following case:

Case 5. (a)y comes by add from the hypothesis (k)z and the item (a-k)y. We have two subcases:

Case 5.1 m is not in a, so (a)y' is (a)y. Then reiterate (k)z and the thesis $\wedge CzCyy$ and use MP twice to get, successively, (k)Cyy and (a)y, i.e. (a)y'.

Case 5.2 m is in a, so (a)y' is (a-m)Cxy. Then, reiterate (k)z and the thesis $\wedge CzCCxyCxy$ and use MP twice to get (a-m)Cxy, i.e. (a)y'.

The theses $\underline{CzCyy}$ and $\underline{CzCCxyCxy}$ are both in C3, since $\underline{z}$ is an implication, and hence C3 contains $C3^*$.

33. The system CN3

It is conjectured that the following axioms are complete for CN3, the implication/negation fragment of S3:

1. CCpqCCqrCpr
2. CCpCpqCpq
3. CQCpp
4. CCNpqCNqp
5. CpNNp
6. CCNppCqp.

Note the replacement of the law of Clavius among IN's axioms by axiom 6, one of the paradoxes of strict implication. 6 is independent of 1-5 and of Clavius, as is shown by

Matrix 11

	C	1	2	3	4	N
*	1	2	4	4	4	4
*	2	2	2	4	4	3
	3	2	2	2	4	2
	4	2	2	2	2	1

(put $p = 2$, $q = 1$). Clavius is easily gotten from 6 by substituting C_{pp} for q and commuting. A strong theorem implying the absence of fallacies of necessity in CN3 is the following:

Theorem

Disregarding double negations, if any thesis of the form $CxCyz$ is provable in CN3, then x is an implication.

Proof

Consider Parry's matrix, adequate for CN3:¹

Matrix 12

	C	1	2	3	4	N
*	1	2	4	4	4	4
*	2	2	2	4	4	3
	3	2	4	2	4	2
	4	2	2	2	2	1

Suppose that $CxCyz$ is provable, and that x is not an implication. Then either x is a propositional variable, or the negation of a p.v., or the negation of an implication. In each of these cases x will somewhere take the values 1 or 3, whereas Cyz is always 2 or 4, and $C12 = C32 \overset{=C14}{\wedge} = C34 = 4$, which contradicts the assumption that $CxCyz$ is provable.

This result applies also to the system IN, for which matrix 12 is adequate.

The presence of 6 among the axioms of CN3 makes for difficulties in constructing its subproof formulation of CN3⁴. These difficulties

¹ See Lewis and Langford, p. 493.

could be solved by the adoption of some special RA rules, but it is not worthwhile going into the matter here, especially as 6 can be proved by existing rules in $S3^*$ and $CN4^*$.

34. The system S3

In this section we shall derive the complete system S3 by adding conjunction axioms to CN3. The full set of axioms defining S3 is:

1. CCpqCCqrCpr
2. CCpCpqCpq
3. CQCpp
4. CCNpqCNqp
5. CpNNp
6. CCNppp
7. CKppp
8. CKpqq
9. CCpqCCprCpKqr
10. CKpNKpqNq,

plus the usual rules of substitution and modus ponens. S3 avoids fallacies of necessity in the sense that, if a thesis of the form CxCyz is provable in S3, x must contain at least one occurrence of the operator C. This is provable using Ackermann's argument of section 20, as applied to matrix 12 plus conjunction matrix 13 of section 40.

Axioms 1-10 are, with the exception of axiom 6, ~~which is strengthened so as to provide a more plausible basis for the C-N fragment of S3~~, all to be found in Lemmon's C-N-K basis for S3.¹ Hence they form a complete basis for S3. The task of this section will be to construct a subproof formulation $S3^*$ of S3.

We define $S3^*$ by the rules hyp, rep, reit, MP, CP, add, MT, DNI, DNE, RA, KI and KE, plus the following additional rule for conjunction:²

¹See Lemmon et al., and Prior, FL, p. 315. Axiom 6, CCNppp, is provable from the remainder (see Lemmon et al., p. 19), and so may be omitted from the list.

²'MP' for modus tollendo ponens, which is what the rule would be if phrased in terms of disjunction.

MTP

From items (a)NKxy and (a)x in subproof n derive (a)Ny
in subproof n.

It will be noted that, leaving aside add and MTP, $S3^*$ contains just those rules which Anderson gives for his subproof formulation E^* of E, with the exception of the rule dist. However, the thesis corresponding to dist, CKpAgrAKpqr, is a thesis of $S1$, since every classically valid implication with no implicational components is also a valid strict implication. Hence $S3$ contains the thesis, hence dist is derivable in $S3^*$, hence $S3^*$ contains E^* , hence $S3$ contains E.

That $S3^*$ contains $S3$ is seen by proving the axioms of $S3$ in $S3^*$. The proof that $S3$ contains $S3^*$ closely follows Anderson's proof that E contains E^* . We require first the following theses and rules in $S3$, where, for this section only, we use the sign Lx as an abbreviation for CCxxx (Anderson's definition of necessity in E).

T1.	CCpqLCpq	CTE, p. 207
T2.	CCpqCLpLq	"
T3.	CCpqCLNqLNp	"
T4.	CLpLNNp	"
T5.	CLNNpLp	"
T6.	CCCppqCCqqq	Appendix 5.3
T7.	CCCqqqCCppq	" 5.3
T8.	CCCppqCCpprCCppKqr	Axiom 9
T9.	CLqCLrLKqr	T8, T6, T7
T10.	CLKpqLp	CTE, p. 207
T11.	CLKpqLq	"
T12.	CLKpNKpqLNq	Axiom 10, T2
T13.	Cpp	CTE, p. 202
T14.	CCpCqrCLqCpr	p. 208
T15.	CCpCqrCCpNrCpNq	"
T16.	CCpqCCrNqCrNp	"
T17.	CLNrCCpCqrCpNq	"
T18.	CCpqCNqNp	p. 203
T19.	CCpqCpNNq	p. 208
T20.	CCpNNqCpq	"
T21.	CCpKqrCpq	"
T22.	CCpKqrCpr	"
T23.	CCpCqrCCrsCpCqs	Appendix 5.3
T24.	CCCspCCsqCsKpqCCCsKpqCCKpqr	T23
	CsrCCspCCsqCCKpqrCsr	

T25.	CCspCCsqCCKpqrCsr	T24, 9, 1
T26.	CCKpqrCCspCCsqCsr	T25, Weak Comm
T27.	CCpqCCpNKqrCpNr	T26, 10
T28.	CCpqCCpNqNp	CTE, p. 208.

R1. $\vdash x \rightarrow \vdash Lx$

Proof

T1 and T2 provide an inductive proof for R1. That this is so becomes clear if we regard the axioms of S3 as axiom-schemata, with modus ponens as the sole rule. Each thesis of S3 is thus necessary because it is either an implication (T1), or implied by a necessary formula (T2).

R2. $\vdash x \rightarrow \vdash CCppx$

Proof

(1) $\vdash x$	
(2) $\vdash CCxxx$	(1) R1
(3) $\vdash CCppx$	(2) T7

R3. $\vdash x, \vdash y \rightarrow \vdash Kxy$

Proof

(1) $\vdash x$	
(2) $\vdash y$	
(3) $\vdash CCppx$	(1) R2
(4) $\vdash CCppy$	(2) R2
(5) $\vdash CCppKxy$	(3), (4) T8
(6) $\vdash Kxy$	(5) T13

We now proceed to prove the lemma and the theorem required for the proof that S3 contains S3*.

Lemma¹

Let Q be an mth and innermost subproof of a proof or quasi-proof P of S3*, and let (a)y be an item of Q such that m is not in a. Then it is possible to reiterate into Q theses of S3* from which, using MP and KI, we can obtain (a)Ly as an item of Q.

Proof

By induction.

Basis

Let (a)y be the first item of Q such that m is not in a. Then the reason for (a)y must be reiteration, and (a)y is

¹Anderson, CTE, p. 207, slightly modified.

either an implication or a thesis of S_3^* . If the former, use T1 and MP to get $(a)Ly$. If the latter, note that R1 gives $(a)Ly$ as a thesis of S_3^* , and $(a)Ly$ may be reiterated into Q.

- Induction step Assume that the lemma holds for each item of Q up to a certain item $(a)y$, and consider the justification for $(a)y$.
- Case 1. rep. Then the induction hypothesis guarantees the lemma for $(a)y$.
- Case 2. reit. As in the basis for the induction.
- Case 3. MP. $(a)y$ comes from $(b)z$ and $(c)Czy$, where by the inductive hypothesis we may obtain $(b)Lz$. Reiterate T2 and use MP twice to get $(a)Ly$.
- Case 4. CP. Impossible, Q being innermost.
- Case 5. add. As for rep.
- Case 6. MT. Similar to case 3, using T3.
- Case 7. DNI. Use T4.
- Case 8. DNE. Use T5.
- Case 9. RA. Impossible.
- Case 10. KI. Use T9.
- Case 11. KE. Use T10 or T11.
- Case 12. MTP. $(a)Ny$ comes from $(a)NKzy$ and $(a)z$. Use KI to get $(a)KzNKzy$; by case 10 we can obtain $(a)LKzNKzy$. Reiterate T12 and get $(a)LNy$.

This completes the proof of the lemma, and we proceed now to the theorem. We note first that the rules rep, MP, MT, DNI, DNE, KE and MTP of S_3^* are all either primitive or easily derived rules of S_3 . The rule KI is derivable as R3. Hence these rules, applied to theses of S_3 , lead always

to theses of S_3 . The rules hyp, reit and add are not applicable in the 'main' proof of S_3^* . Hence it remains only to show that any thesis of S_3^* which has CP or RA as its reason is also a thesis of S_3 , and this is proved by Anderson's theorem.

Theorem Exactly as stated in section 27, substituting S_3 and S_3^* for I and I^* respectively, and allowing the derived rule R3 in S_3 .

Proof As in section 27, with the following changes and additions.

Case 3.3 Since m is not in b , by the lemma we can get $(b)Lz$ as an item of Q . Then reiterate T14 and use MP to get $(a-m)Cxy$, i.e. $(a)y'$.

Case 5. add. Use the same argument as in section 32, reiterating axiom 3.

Case 6. MT. Similar to case 3. Case 6.1 uses T15, case 6.2 T16, case 6.3 T17 and the lemma, and case 6.4 T18.

Case 7. DNI. Use T19.

Case 8. DNE. Use T20.

Case 9. RA. Impossible.

Case 10. KI. $(a)y = (a)Kzw$ comes from $(a)z$ and $(a)w$.

Case 10.1 m is in a . Then $(a)z'$ is $(a-m)Cxz$ and $(a)w'$ is $(a-m)Cxw$. Reiterate axiom 9 to get $(a-m)CxKzw$, i.e. $(a)y'$.

Case 10.2 m is not in a . Then use KI or R3 in subproof $m-1$.

Case 11. KE. Use T21 and T22.

Case 12. MTP. $(a)y = (a)Nw$ comes from $(a)NKzw$ and $(a)z$.

Case 12.1 m is in a . Then $(a)NKzw'$ is $(a-m)CxNKzw$ and $(a)z'$ is $(a-m)Cxz$. Reiterate T27 to get $(a-m)CxNw$, i.e. $(a)y'$.

Case 12.2 m is not in a . Then $(a)NKzw'$ is $(a)NKzw$ and $(a)z'$ is $(a)z$. Use KI or R3 in subproof $m-1$ to get $(a)KzNKzw$, and reiterate axiom 10 to get $(a)Nw$, i.e. $(a)y'$.

To complete the proof of the theorem, we must show that the item $(a-m)z$ in subproof $m-1$ which was treated as a consequence

of Q in P can now be treated as a consequence of the items $(a_i)y_i'$ in P' . There are two cases:

- Case 1. $(a-m)z$ comes by CP. See section 27.
- Case 2. $(a-m)z$ comes by PA, and is hence $(a-m)Nx$. The foregoing inductive proof guarantees that for some y and some b and c , we have $(b-m)Cxy$ and $(c-m)CxNy$, where $a = b+c$. Then reiterate T28 and use MP twice to get $(a-m)Nx$, i.e. $(a-m)z$.

This completes the proof of the the theorem. We proceed to eliminate every subproof in this way until we are left with nothing but a sequence of theses of $S3$, with rep, MP and the derived rule R3 as the sole rules. Hence to every proof in $S3^*$ there corresponds a proof in $S3$ - i.e. $S3$ contains $S3^*$.

35. The system C4

In progressing from $C3$ to the stronger system $C4$, we progress to a system which allows both fallacies of necessity and fallacies of relevance. The axioms of $C4$ are the following:-

1. $CCpqCCqrCpr$
2. $CCqCpqCpq$
3. $CqCpp$ (Comm'd Simp)

and their completeness for the pure strict implicational fragment of $S4$ follows from the work of Hacking, and from that of Chung and Schindler. Axiom 3 is a formula which violates both relevance and necessity.

The subproof formulation of $C4^*$ of $C4$ will entirely do away with the use of relevance numerals. Let us indicate the absence of this relevance device by priming. Then leaving out add, which concerned relevance exclusively, the rules of $C4^*$ will be hyp' , rep' , $reit'$, MP' and CP' . Note that $reit'$ still retains the condition that the reiterated formula be an

implication. To give an example of a proof in $C4^*$, the following is a derivation of Weak Simp, one of the theses that distinguishes $C4$ from $C3$:

1.	P	hyp'
2.	q	hyp'
3.	P	1 reit'
4.	Cqp	2,3 CP'
5.	CPCqP	1,4 CP'

It is a simple matter to show that $C4^*$ contains $C4$. To show that $C4$ contains $C4^*$, we need the following simplified version of Anderson's Theorem

Let P be a proof or quasi-proof of $C4^*$, and let Q be an m th and innermost subproof of P . Let $y_1, y_2, \dots, y_n$ be the items of Q , of which the first, x , is the hypothesis.

Let P' be the result of replacing the subproof Q in P by the sequence of items Cxy_i in subproof $m-1$. Then we may reiterate theses of $C4$ into P' , and apply MP' and rep' , in such a way as to convert P' into a quasi-proof of $C4^*$.

Proof

We require the following theses of $C4$ for reiterating into P' :

- T1. Cpp
- T2. CPCqP
- T3. CCpCqrCCprCpq

A. The proof requires firstly that we show, by induction on n , that the occurrence of each of the items Cxy_i in subproof $m-1$ is justified.

Basis $n = 1$. Cxy_1 is Cxx . We treat this item as a reiteration of T1.

Induction step Assume that the theorem is proved for each y_i up to a certain y , and consider the justification of y .

Case 1. y comes by rep' . Then the induction hypothesis guarantees the theorem for y .

Case 2. $reit'$. Then y is an implication (we note that the only theses required for reiteration, T1-3, are implications). Reiterate

T2 and use MP' to get Cxy.

Case 3.

y comes by MP' from z and Czy. Then by the induction hypothesis we already have Cxz and CxCzy as items of P'. Reiterate T3 and use MP' twice to get Cxy.

Case 4.

CP. Impossible, Q being innermost.

B.

The item Cxy_m is a subproof m-1, which was a consequence of Q in P, is now seen to occur on the last item of the sequence Cxy₁ in P'. Hence we treat it as a repetition of this item, converting P' into a quasi-proof of C4*.

This is all we require for the proof that C4 contains C4*.

36. The system CN4

Adding IN's negation axioms to C4 gives CN4:

1. CCpqCCqrCpr
2. CCpCpqCpq
3. CqCpp
4. CCNpqCNqp
5. CpNNp
6. CCNppp,

which has been shown by Chung and Schindler to be the complete strict implication/negation fragment of S4. The independence of axioms ~~3~~ 1 and 4- is provable ~~as in section 31, and that of 1~~ as in section 28.

We formulate the system CN4* by adding the rules MT', DNI', DNE' and RA' to C4*. The paradoxes of strict implication are provable in CN4* as follows (for this reason IN's weaker axiom 6 could be used in place of CN3's 6):

1.	CNpp	hyp'
2.	q	hyp'
3.	Np	hyp'
4.	CNpp	1 reit'
5.	p	3,4 MP'
6.	NNp	3,5 RA'
7.	p	6 DNE'
8.	Cqp	2,7 CP'
9.	CCNppCqp	1,8 CP'

1.	CpNp	hyp'
2.	p	hyp'
3.	Nq	hyp'
4.	p	hyp'
5.	CpNp	1 reit'
6.	Np	4,5 MP'
7.	Np	4,6 RA'
8.	CNqNp	3,7 CP'
9.	NNp	2 DNI'
10.	NNq	8,9 MT'
11.	q	10 DNE'
12.	Cpq	2,11 CP'
13.	CCpNpCpq	1,12 CP'

There is no difficulty in showing that CN_4 and CN_4^* contain one another.

37. The system S4

Add to CN_4 the same conjunction axioms as those of S3:

7. CKppq
8. CKpqq
9. CCpqCCprCpKqr
10. CKpNKpqNq

and we get the system S4, distinguished from S3 by matrix 12 plus conjunction matrix 13 of section 40.

S_4^* is formed from CN_4^* by adding the rules KI', KE' and MTP'.

A simple extension of the theorem of section 35 shows that S4 contains S_4^* .

38. The system C5

C5, the strict implicational fragment of S5, has the following

axioms:

1. CCpqCCqrCpr
2. ~~CCpCpCp~~ CqCpp
3. CCCpPP

(Weak Peirce).

Its completeness is proved, using a Gentzen formulation of S5, by Hacking. It is also proved, in the earlier paper of Lemmon et al., by a method which involves reducing, in C5, every formula of implicational logic to standard form, and then showing that every formula not provable in C5 is not a thesis of S5. This method is used in section 52 for the axiomatization of a certain matrix.

It is notorious that the formula Peirce resists derivation in any system of 'natural' rules for pure implicational logic.¹ Hence, although we could doubtless introduce some such ad hoc rule as the following for C5*:

peir' From CCxyx in subproof n, where x is an implication,
 derive x in subproof n,

we shall not in fact construct any subproof formulation of C5 at all.

39. The system CN5

Join the usual axioms of negation to C5 to get CN5:

1. CCpqCCqrCpr
2. ~~CCpCpqCp~~ CqCp
3. CCCpPP
4. CCNpqCNqp
5. CpNNp
6. CCNppp.

Once more the independence of axioms 4-6 is provable as in section 28, and CN5 is conjectured to be the complete C-N fragment of S5.

In contrast to C5, a subproof formulation for CN5 is readily obtained by strengthening one of the rules of CN4* in a natural way. This is

¹See for example the highly artificial rule HE* of Leblanc's Gentzen-type treatment of propositional logic, which is necessary for the derivation of Peirce in pure implicational logic (H. Leblanc, Etudes sur les règles d'inférence dites 'règles de Gentzen', Dialogue 1 (1962), pp. 56-66.)

the rule reit' , which is replaced by

reit_2' From x in subproof n derive x in subproof $n+m$, provided
that x is either an implication or the negation of an
implication.

CN5^* therefore contains the rules hyp' , rep' , reit_2' , MP' , CP' , MT' , DNI' ,
 DNE' and RA' . Weak Peirce is proved in it as follows:

1.	CCPqP	hyp'
2.	NP	hyp'
3.	CCPqP	1 reit_2'
4.	NCPq	2,3 MT'
5.	P	hyp'
6.	Nq	hyp'
7.	NP	2 reit_2'
8.	P	5 reit_2'
9.	NNq	6,7,8 RA'
10.	q	9 DNE'
11.	CPq	5,10 CP'
12.	NNP	2,4,11 RA'
13.	P	12 DNE'
14.	CCCPqPP	1,13 CP'

To show that CN5 contains CN5^* requires, in addition to the thesis CPCqP of C4 , the thesis CNPCqNP of CN5 , whose derivation in CN5 is given in appendix 5.4. The argument follows that of section 35, except that in case 2 the thesis CNPCqNP allows us to deal with reiterations of negations of implications.

40. The system S5

Adding the usual conjunction axioms 7-10 of section 37 to CN5 gives S5 , the distinctness of which system from S4 is shown by the following matrices, taken from Lewis and Langford, p. 493:

Matrix 13

	C	1	2	3	4	N		K	1	2	3	4
*	1	1	4	3	4	4		1	1	2	3	4
*	2	1	1	3	3	3		2	2	2	4	4
	3	1	4	1	4	2		3	3	4	3	4
	4	1	1	1	1	1		4	4	4	4	4

$S5^*$ is formed from $CN5^*$ by adding the rules KI' , KE' and MTP' , and there are no special problems which arise in showing that $S5$ and $S5^*$ are identical.

41. The intuitionist implicational system IC

The intuitionist calculus is closely related to the Lewis systems, and this relationship emerges most forcibly when we phrase the latter in C, N and K. IC, the pure implicational fragment of intuitionist logic, is axiomatized as follows:

1. $CCpqCCqrCpr$
2. $CCpCpqCpq$
3. $CpCqp$ (Simp)

or, alternatively, by the following two axioms alone:

4. $CCpCqrCCpqCpr$ (Frege)
3. $CpCqp^1$

The completeness of these axiomatizations for the pure implicational intuitionist calculus follows from the Gentzen-style formulation of that calculus.² Matrix 4 (section 21) shows that IC is stronger than WC, in the sense that more can be proved in it, and the implicational part of matrix 13 that IC is stronger than C4.

A subproof formulation of IC is obtained by removing the last restriction upon reiteration. The rule required for IC^* is WC^* 's

¹The first basis is Hilbert's (hence the name of the second axiom), and the second Łukasiewicz's. See Prior, FL, p. 316.

²See for example S.C. Kleene, Introduction to Metamathematics, New York, 1952, pp.442-3.

reit_1 (see section 30) without reference to dependence; i.e. reit_1' . Thus IC^* will consist of the rules hyp' , rep' , reit_1' , MP' , and CP' . To prove that IC contains IC^* , we argue as in section 35, reiterating CpCqp to cover ad lib. reiterations into the innermost subproof.

42. Johansson's system MCN

IC may be strengthened to yield Johansson's minimal calculus. The following axioms have been put forward as complete for MCN , the C-N fragment of that calculus:¹

1. CCpCqrCCpqCpr
2. CpCqp (Simp)
3. CCpNqCqNp

Corresponding to MCN , there is a simple subproof formulation MCN^* , gotten by adding RA' to the rules of IC^* . Note the absence of MT' , DNI' and DNE' . The last is not wanted in MCN^* , since the formula CNNpp is not provable in MCN , but the former two are derivable from the other rules as follows:

<u>MT'</u> :	i.	Cxy	
	j.	Ny	
	k.	x	hyp'
	l.	Cxy	l reit'
	m.	y	k, l MP'
	n.	Ny	j reit ₁ '
	o.	Nx	k, m, n RA'
<u>DNI'</u> :	i.	x	
	j.	Nx	hyp'
	k.	x	l reit ₁ '
	l.	NNx	j, k RA'

The additional thesis required for the proof that MCN contains MCN^* , namely CCpqCCpNqNp , is derivable in MCN (see Prior, FL, p. 316).

¹The completeness of the axioms is asserted in Prior, FL, p. 316, although no proof is given. MCN is Kolmogoroff's calculus.

43. The system ICN

The axioms of the C-N fragment of intuitionist logic, ICN, are gotten by adding the negative paradox of material implication to the axioms of MCN:

1. CCpCqrCCpqCpr
2. CpCqp
3. CCpNqCqNp
4. CpCNpq (Duns Scotus).

Matrix 6 of section 28 shows ICN to be stronger than MCN.

If we wish to construct a subproof formulation of ICN, we must add an additional negation rule to MCN^{*}, namely:

RA₁' From items x and Nx in subproof n derive any item y
in subproof n.¹

Thus ICN^{*} will consist of hyp', rep', reit₁', MP', CP', RA' and RA₁'.

The last rule is used to prove Duns Scotus as follows:

- | | | |
|----|--------|-----------------------|
| 1. | p | hyp' |
| 2. | Np | hyp' |
| 3. | p | 1 reit ₁ ' |
| 4. | q | 2,3 RA ₁ ' |
| 5. | CNpq | 2,4 CP' |
| 6. | CpCNpq | 1,5 CP' |

To show that ICN contains ICN^{*} we need only, in addition to the theses of MCN, the thesis CCpqCCpNqCpr of ICN.

44. The full intuitionist calculus IPC

Since alternation is not definable in intuitionist logic in terms of conjunction and negation, we must add both C-K and C-A axioms to ICN to get the full intuitionist calculus. The following are Łukasiewicz's axioms:²

¹This rule is one of the negation rules found in Fitch, p.54.

²Łukasiewicz, On the intuitionistic theory of deduction, *Indagationes Mathematicae* 14 (1952); Prior, *FL* (1st edition), p. 308.

1. $CCpCqrCCpqCpr$
2. $CpCqp$
3. $CCpNqCqNp$
4. $CpCNpq$
5. $CKppq$
6. $CKpqq$
7. $CpCqKpq$
8. $CpApq$
9. $CqApq$
10. $CCprCCqrCApqr$

Matrix 13 of section 40, with Axy defined as NKNxNy, shows the distinctness from S4 and its fragments of all the intuitionist systems and fragments, although IC contains C4.

A subproof formulation of IPC requires rules for alternation as well as for disjunction. Hence we add to ICN^* KI' and KE' , plus the rules:

AI' From x derive Axy. From y derive Axy (all in subproof n).

AE' From Cxz , Cyz and Axy derive z, all in subproof n.

For the derivation of the corresponding rules (with dependency numerals) in E^* , where alternation is defined, see Anderson, CTE, p. 205. The rule MTP' is derivable in IPC^* as follows:

i.	x	
j.	NKxy	
k.	y	hyp'
l.	x	i reit ₁ '
m.	Kxy	k, l KI'
n.	NKxy	j reit ₁ '
o.	Ny	k, m, n RA'

There is no special difficulty in showing that IPC and IPC^* are identical.

45. The classical system PC

Since C, the pure implicational fragment of classical logic, contains the thesis Peirce, we shall not construct a subproof formulation for it, but merely give its axioms:

1. $CCpqCCqrCpr$
2. $CpCqp$
3. $CCCpqqp$ (Peirce)

The full classical calculus, PC, in which conjunction is definable in terms of implication and negation, has the following axioms:

1. CCpqCCqrCpr
2. CpCNpq
3. CCNppp.

Its distinctness from S5 is shown by matrix 26 of appendix 1 and from IPC by the matrices in Prior, FL, p. 243.

We may construct a subproof formulation PC^* of PC with the help of the rules hyp' , rep' , $reit_1'$, MP' , CP' , DNE' and RA' . This is a strengthening of ICN^* , since RA_1' is derivable in PC^* . A simpler version of PC^* , due to Jaśkowski, consists in replacing the three rules rep' , DNE' and RA' by the single rule RA_2' :

RA_2' From the hypothesis Nx and items y and Ny in subproof n derive x in subproof n-1¹.

The proofs of rep' , DNE' and RA' from RA_2' are as follows:

rep'

i.	x	
j.	Nx	hyp'
k.	x	$reit_1'$
l.	x	RA_2'

DNE'

i.	NNx	
j.	Nx	hyp'
k.	NNx	$reit_1'$
l.	x	RA_2'

RA' Instead of:

i.	x	write:
	:	
j.	y	
k.	Ny	
l.	Nx	RA'

i.	NNx	
i+l.	x	DNE'
	:	
j.	y	
k.	Ny	
l.	Nx	RA_2'

¹Jaśkowski, p. 11.

46. Summary of rules for subproof systems

To conclude this chapter, a resumé is presented of the various rules required for each of the subproof systems considered. For the relationships of containment existing among the systems see table 4, section 23, bearing in mind that E is contained in S3. Note that systems I^* to $S3^*$ require the use of dependence numerals, indicating the presence of some degree of relevance, while systems $C4^*$ to PC^* eschew relevance. Systems I^* to E^* , and $C3^*$ to $S5^*$ avoid, to some degree or other, fallacies of necessity.

Table 5

<u>System</u>	<u>Rules</u>
I^*	hyp, rep, reit, MP, CP.
IN^*	, MT, DNI, DNE, RA.
E^*	, KI, KE, dist.
WC^*	hyp, rep, reit ₁ , MP, CP.
WCN^*	, MT, DNI, DNE, RA.
$C3^*$	hyp, rep, reit, MP, CP, add.
$S3^*$	, MT, DNI, DNE, RA, KI, KE, MTP.
$C4^*$	hyp', rep', reit', MP', CP'.
$CN4^*$	, MT', DNI', DNE', RA'.
$S4^*$	, KI', KE', MTP'.
$CN5^*$	hyp', rep', reit ₂ ', MP', CP', MT', DNI', DNE', RA'.
$S5^*$	, KI', KE', MTP'.
IC^*	hyp', rep', reit ₁ ', MP', CP'.
MCN^*	, RA'.
ICN^*	, RA ₁ '.
IPC^*	, KI', KE', AI', AE'.
PC^*	hyp', reit ₁ ', MP', CP', RA ₂ '.

CHAPTER FIVE

THE QUEST FOR A CONNEXIVE CALCULUS

47. Connexive implication and the traditional systems

In the preceding chapters we have seen how Aristotle's thesis, NCNpp, and Boethius' thesis, CCpqNCpNq, demand for their satisfaction a type of implication which is precisely the third of those described by Sextus. This species of implication we have called *connexive*. Other species of implication - material, intuitionist, Curry's weak and the different degrees of strict - have been seen to engender various types of paradox, and a criterion has been produced, that of relevance, increasing degrees of which progressively reduce paradox. The currently well-known systems of propositional logic have been surveyed, with particular attention being paid to their pure implicational and implication/negation fragments, and they have been presented both in their traditional axiomatic and in a more natural subproof style. Certain properties of these systems have been noted, particularly the extent to which they allow or do not allow for what Anderson and Belnap call fallacies of relevance and of necessity.

It will be seen in this chapter that, with the exception of the system defined by Angell's matrix, which we do not yet know much about, none of the traditional logics captures the notion of *connexive* implication. What exactly is *connexive* implication? Sextus characterizes it by saying that a *connexive* conditional is sound when the contradictory of its consequent is incompatible with its antecedent. That is, we can never have a sound *connexive* conditional of the form CNxx, since here the contradictory of the consequent, Nx, is not incompatible with the antecedent. Nor does it seem possible to have two sound *connexive* conditionals of the forms

Cxy and $CxNy$, for if x is incompatible with Ny , it would seem intuitively that it could not be incompatible with y as well. But when we spell out the conditions for connexivity in this way, they are seen to be identical with the conditions under which Aristotle's and Boethius' theses are true. Hence we may take the truth of one or both of these theses as a mark (necessary and sufficient condition) of connexive implication.

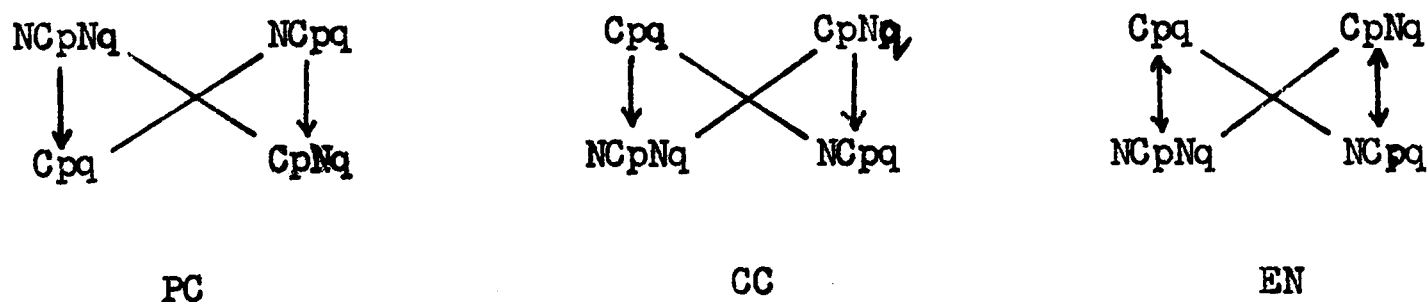
As was said, none of the traditional systems captures the idea of connexive implication. And if its presence is defined as being contingent upon the presence of Aristotle's or Boethius' theses, it would seem that no pure implicational system, lacking negation, could do so. But the quest for such a system is still, in a sense, the quest for a pure implicational system, for some implicational systems do, and so do not, admit of the consistent addition of $NCNpp$ and/or $CCpqNCpNq$ in the presence of other customary negation theses. The matter of consistency is crucial here, since anyone can construct and claim extravagant properties for inconsistent systems. The consistent $C-N$ system we seek, which serves to catch and domesticate the notion of connexive implication, we shall refer to as the connexive calculus (CC). Adding conjunction will result in a full propositional system CPC. As will be seen, the ultimate aim is completeness as well as consistency.

48. Classical logic and connexion

It is plain that material implication is not connexive. There are many formulae which are materially implied by their own negations, such as $ApNp$ and Cpp , and many which imply both of two mutually contradictory propositions, such as $KpNp$ and $NCpp$. Nevertheless, there is a germ of connexivity to be found even in material implication. Using the word 'negation'

in its strong (and proper) sense of 'contradictory', so that $\underline{x}$ is the negation of $\underline{Nx}$ just as much as $\underline{Nx}$ is of $\underline{x}$,¹ the classical thesis $\underline{CNCppCpp}$ becomes an instance both of a proposition's implying its own negation and of a proposition's being implied by its own negation. But in classical logic it is never the case that a true proposition implies its own negation, or that a false proposition is implied by its own negation - we have, on the contrary, $\underline{CpNCpNp}$ and $\underline{CNpNCNpp}$ as classical theses. $\underline{Cpp}$, for example, is true, and so is $\underline{NCCppNCpp}$. Hence Aristotle's thesis, which we can now also write as $\underline{NCxNx}$, holds for a restricted class of propositions $\underline{x}$ in classical logic, and the problem becomes that of extending its scope to all propositions.

The similarities between material and connexive implication, however, are overshadowed by their dissimilarities. In some respects connexive implication is more like the equivalence relation than like material implication: in fact the equivalence relation might be regarded as a blending of the mutually opposed extremes of material and connexive implication. This will become clearer from a diagram:



Here the arrows stand for implication-relations, and the diagonal lines join contradictories. In PC, $\underline{NCpNq}$ implies $\underline{Cpq}$ but not vice versa (cf. Boethius' second thesis of section 7). In CC, on the other hand, $\underline{Cpq}$ implies $\underline{NCpNq}$. Making use of the above diagrams' resemblance to the traditional square of

¹ See the note at the end of this section.

opposition, we may say that in CC the two formulae Cpq and CpNq are contraries, while in PC they are sub-contraries. In the third figure they are both, i.e. they are contradictories, in virtue of the two-way relation of mutual implication between Cpq and NCpNq. This is characteristic of equivalence, for in EN, the classical system of equivalence with negation, EEpqNEpNq is a thesis. In this sense, therefore, equivalence combines the features of both material and connexive implication. In addition we note that NCpNp, the thesis we seek in CC, is already present in EN as NEpNp.

It will be, as we shall see, a difficult problem to construct a formal system embodying connexive implication which is not at the same time an equivalential logic. Even a system containing such harmless-looking theses as the following:

- | | |
|----------------|-----------|
| 1. CCpqCCqrCpr | (Syl) |
| 2. CCpCqrCqCpr | (Comm) |
| 3. CCpqCNqNp | (Trans 1) |
| 4. CCNpqCNqp | (Trans 3) |
| 5. CCNpNqCqp | (Trans 4) |

will turn out to be an equivalential system if NCpNp is introduced.

To see this, we make the following deduction from theses 1-5. Proof notation is based on that of Łukasiewicz, AS, p.81, with substitutions omitted. '(w,x,y,RE)' on line z means that z is the result of replacing in w one expression by another shown to be equivalent to it through the implications x and y, it being noted that the rule for the replacement of equivalents is derivable by repeated application of theses 1, 3 and 6.

- | | |
|-------------|-----------------------|
| 2=C1--6. | CCqrCCpqCpr |
| 1=C1--7. | CCCCqrCprsCCpqs |
| 7=C7--8. | CCpCqrCCsqCpCsr |
| 7=C8--9. | CCqpCCsqCCprCsr |
| (9) | 10. CCqpCCpqCCpqCpq |
| (10,3,5,RE) | 11. CCqpCCpqCNCpqNCpq |
| (11,2,2,RE) | 12. CCqpCNCpqCCpqNCpq |
| (12,4,4,RE) | 13. CCqpCNCpqNCpqCpq |
| (13,2,2,RE) | 14. CNCpqNCpqCCqpCpq |

Note now that L_4 's antecedent is a substitution of $\underline{NpNp}$, and its consequent is $\underline{CCqpCpq}$, the characteristic thesis of equivalence. Joining $\underline{NpNp}$ to the system would allow this consequent to be detached, with the subsequent transformation of the system into an equivalential one. Needless to say this is undesirable: the connexive calculus should not be allowed to contain the thesis $\underline{CCpqCqp}$.

Note. At the beginning of this section it was stated that, properly speaking, the negation of any proposition p was its contradictory, so that p was as much the negation of $\underline{Np}$ as $\underline{Np}$ was of p . It is worth remarking (a) that in not every system can the symbol $\underline{Np}$ be understood in this way, namely as the contradictory of p , and (b) that understanding $\underline{Np}$ in this way (which I take to be the proper way) requires the assertion of both laws of double negation, namely $\underline{CpNNp}$ and $\underline{CNNpp}$.

As will be gathered, these remarks are partly polemical, directed against intuitionist logic. In that system $\underline{Np}$ cannot be understood as the contradictory of p , since this would require that the following four conditions be satisfied:

- (i) If p is true, $\underline{Np}$ is false
- (ii) If $\underline{Np}$ is true, p is false
- (iii) If p is false, $\underline{Np}$ is true
- (iv) If $\underline{Np}$ is false, p is true.

But, since even in intuitionist logic we would wish to say that $\underline{Np}$ had at least the minimum meaning of ' p is false' (some would wish to say that it had the stronger meaning of ' p is absurd' as well), we see that the formal counterparts of the above four conditions are:

- (i) $\underline{CpNNp}$
- (ii) $\underline{CNpNp}$
- (iii) $\underline{CNpNp}$
- (iv) $\underline{CNNpp}$.

Of these, the last does not hold in intuitionist logic, even though, in that logic, $\underline{NNp}$ should at least imply ' $\underline{Np}$ is false', the antecedent of condition (iv). Hence the negation of p is not to be understood in intuitionism as its contradictory, but rather as its contrary, i.e. as something satisfying only conditions (i) and (ii), but not (iii) and (iv).. This I take to be a mis-analysis of negation.

To reinforce the point that intuitionist logic allows certain expressions $\underline{Nx}$ to be false, without at the same time allowing $\underline{x}$ to be true, consider the formula $\underline{NCNNpp}$. This formula is false for all substitutions of its variables in IPC, since $\underline{CNCNNppq}$ is a thesis of IPC and if any formula $\underline{NCNNxx}$ were ever asserted, we could detach $\underline{q}$.¹ But, although $\underline{NCNNpp}$ is false, $\underline{CNNpp}$ is not true.

49. Strict implication and connexion

In order to avoid the unwelcome consequences of the derivation of the previous section, we must weaken one or more of the axioms with which we started. Most of the better-known implicational systems contain Syl, the most notable exception being S2, and so for the time being it will be retained (though see section 63). For the reasons given above we prefer to retain both laws of double negation, and hence all the laws of transposition as well, these laws being interdeducible given strong negation (though, once again, see section 63 for a CC with intuitionist negation). This leaves the law of commutation as the most obvious candidate for weakening, and we note that it is precisely this law, in addition to the law of simplification, which is weakened in the Lewis systems.

¹ $\underline{CNCNNppq}$ is a thesis because $\underline{CNCppq}$ is a thesis, and $\underline{NCNNpp}$ is equivalent to $\underline{NCpp}$.

In place of Comm, S5 has the following weakened laws of commutation: CCpCQrCQcpr (Weak Comm), CCpCqRCqCpR and CCpCNQrCNQCpr.¹ Unfortunately the presence of the last makes S5 unsuitable as a CC, since the deduction of section 48 can be carried through with it and Weak Comm in place of Comm. S4 lacks the second and third of S5's laws of commutation while retaining Weak Comm, but contains too strong a version of the law of simplification to support CC. S4 has Comm'd Simp, CqCp, and by substitution this yields CNCppCp. Hence neither S5 nor S4 will do.

More promising at first sight is S3, which has only Weak Comm'd Simp, CQCp, in place of Comm'd Simp, and for which CNCppCp is rejected by matrix 12 of section 33, here reproduced:

Matrix 12

	C	1	2	3	4	N
*	1	2	4	4	4	4
*	2	2	2	4	4	3
	3	2	4	2	4	2
	4	2	2	2	2	1

Better still, we see that the same matrix rules out, for CN3, any thesis of the form CNxx, where x is an implication (see the argument of section 33), and hence CN3 would provide a suitable basis for CC if we could be sure that it contained no theses of the form CxNx. That is, matrix 12 shows that CN3 contains no Clavian implications, i.e. implications x for which CNxx is true. The question is, does it contain any self-defeating implications?

This question will be answered in section 53. Before arriving there, however, there are certain interesting lines of investigation opened up by the consideration of such matrices as matrix 12, and something about non-classical logic can be learned by following them up.

¹For the second see Lemmon et al. The first and third are easily provable in CN5.

The three four-valued matrices which distinguish (a) CN3 from CN4, (b) CN4 from CN5, and (c) CN5 from PC, are matrices 12, 13, and 26 of appendix 1; they are found in Lewis and Langford, p. 493. Each is adequate for its own particular system (and for weaker systems) in the sense that it satisfies all the theses of the system. But none is characteristic for any of the systems in the sense that it not only satisfies all its theses, but also fails to satisfy all its non-theses. That is, each of the matrices satisfies more than the theses of its particular system. Nor is this just a feature of the particular 4 x 4 matrices which Lewis and Langford give: Dugundji has shown that no matrix with a finite number of truth-values can be characteristic for any of the systems S1-5,¹ and this result also holds for the C-N fragments of S3-5, S7 and S8, as may be shown by an adaptation of Dugundji's argument (see appendix 1). This does not, however, show that the calculi of strict implication are 'intensional' rather than 'extensional', if by 'extensional' ('intensional') we mean 'possessing (not possessing) a truth-value interpretation'.² The Lewis systems each possess a truth-value interpretation, i.e. a characteristic matrix, but the number of truth-values is infinite, rather than two (PC), three (E3V) or any other finite number.³ For all we know, the putative system CC may turn out to have

¹J. Dugundji, Note on a property of matrices for Lewis and Langford's calculi of propositions, JSL 5 (1940), pp. 150-1.

²The writer would ask here (a) What else can we mean? (b) If 'intensional' means 'not possessing a truth-value interpretation', does this make an intensional calculus inferior to an extensional one? Note also that Lindenbaum has shown that every sentential calculus possesses an infinite characteristic matrix, so that, strictly speaking, there is no such thing as an intensional calculus.

³See however T. Sugihara, Strict implication free from implicational paradoxes, Memoirs of the College of Liberal Arts, Fuku University, 1955, pp. 55-59, for the view that this infinity of values should be known as an infinity of statement-values rather than an infinity of truth-values.

a finite characteristic matrix, and for this reason the term 'connexive' rather than 'intensional' has been chosen to describe the type of implication it contains.

To return to the 4×4 matrices for CN3-5, their mere adequacy, i.e. their lack of characteristicity, is of interest for non-classical logic. Since, firstly, each of the matrices is consistent in that it cannot simultaneously satisfy any pair of propositions x and Nx , and, secondly, each satisfies more than the theses of its particular system, each therefore allows for consistent extensions of CN3-5. It is the extension of S3 known as S7 that is of particular significance here.

S7 is formed by adding the axiom MMp to S3.¹ When translated into C-N terms, MMp becomes NCMpNMp, i.e. NCNCpNpCpNp, and matrix 12 guarantees that this formula can be consistently added to S3. What interests us is that the formula NCNCpNpCpNp is non-classical: it does not hold for material implication. Furthermore it is a formula directly along the lines of those which should form part of CC, being a substitution of NCNpp. The formula NCpNp itself translates as Mp, so it would seem that if we could strengthen S7 a little more to include Mp as well as MMp, we would have the connexive calculus for which we are looking.

Adding Mp to a system of modal logic does not, on the face of it, seem to run completely counter to intuition - if in S7 every proposition p is possibly possible, why should every proposition not be possible? But in fact the technical difficulties of such an addition are great. There is a stronger system than S7, to be sure, namely the system S8, which adds IMMp. But this further addition is not, interestingly enough, in the

¹See Prior, Time and Modality, Oxford 1957, Appendix B, for some facts about the systems S6-8.

direction of CC, for its C-N translation is CCNCpNpCpNpNCNCpNpCpNp, a formula which is an instance of the forbidden CxNx. For the time being S7 will not be proposed as a connexive calculus, but only as an example of a consistent non-classical strengthening of a weaker-than-classical base.

50. Consistency and completeness

The possibility opened up by matrix 12, namely that of making a consistent non-classical addition to S3, yielding S7, requires that what is meant by consistency and its related notion of completeness be made clear. It is necessary to be precise, because since the work of Post there have been two distinct varieties of consistency referred to in the literature, and these have sometimes been confused. In order to keep them separate, we shall say that

(a) A system is consistent if it contains no pair of theses of the form x and Nx.

(b) A system is Post-consistent if the simple variable p is not a thesis in it.

Obviously, for systems containing negation, consistency implies Post-consistency, since in a system which is not Post-consistent, i.e. which is Post-inconsistent, any two theses x and Nx can be proved by substitution, but Post-consistency does not in general imply consistency except in systems which contain Duns Scotus, CpCNpq, or the corresponding rule. Equally obviously, the notions of consistency and inconsistency have no application to systems not containing negation: these systems may, however, be Post-consistent.

We may illustrate these notions by reference to the C-N logics for which certain matrices are characteristic. That of matrix 12, for

example, is consistent. But if we designate one more of its values:

Matrix 14

C	1	2	3	4	N
* 1	2	4	4	4	4
* 2	2	2	4	4	3
* 3	2	4	2	4	2
4	2	2	2	2	1

the system so defined is inconsistent. It remains, however, Post-consistent, unless we choose to designate the fourth value. Whether the system characterized by a matrix is consistent or not can be told at a glance.¹ What is much more difficult to determine is whether it satisfies non-classical formulae. This can in general be done only by axiomatizing the matrix, i.e. by providing a set of axioms and rules of inference from which all the formulae it satisfies may be derived.

This brings us to completeness, and here the first distinction to be made is between absolute and relative completeness. A system may be complete relative to any number of things: complete relative to a matrix (meaning that it contains as theses all formulae satisfied by the matrix); complete relative to another system, defined by different axioms or rules (meaning that it contains that system); complete for a fragment of another system (e.g. the C-N fragment, meaning that it contains that fragment). Absolute completeness, on the other hand, is defined in terms of consistency:

- (a) A system is strongly complete if any thesis added to it makes it inconsistent.

¹ It is not true that the system so characterized is consistent only if ordinary arithmetic is consistent, as is urged by Prior (FL, p. 231). We could substitute anything at all, such as apples and oranges, for the numbers in the matrices presented in this work, and stipulate, for example, that a formula was designated if it were assigned nothing but apples, according to the matrix's scheme. Consistency would then depend only on human infallibility in reading matrices, and the tendency of apples to retain their properties: i.e. not to turn into oranges. See also Kneale, DL, pp. 691-692 on this subject.

- (b) A system is Post-complete if any thesis added to it makes it Post-inconsistent.

Exclusive concentration on classical logic tends to obscure these distinctions, since classical logic is at once strongly complete, Post complete and complete relative to the following matrix:

Matrix 15

	C	1	2	N
*	1	1	2	2
	2	1	1	1

But, as will be seen, working with non-classical logics requires that the distinctions be observed.

51. Does a strongly complete non-classical system exist?

In section 49 it was seen that S7 and S8 constitute consistent non-classical strengthenings of S3. But the existence of matrix 12 shows that S8 is not strongly complete, for there are formulae satisfied by the matrix which can be consistently added to S8 (we know that these formulae are not already in S8, for, if they were, matrix 12 would be not only adequate but characteristic for the C-N fragment of S8, and in appendix 1 it is shown that no finite matrix characterizes CN8). Hence if S8 (or CN8) is not strongly complete, the question is, whether it can be made so. If it could, it would be a strongly complete system independent of PC, and as such the object of some theoretical interest.¹

We shall not here attempt to complete CN8. It is not even clear how this could be done, but it would seem to involve at least the axiomatization of matrix 12, itself a formidable task. Our interest is

¹Note that this question of a strong completion of S8 is different from that discussed by J.C.C. McKinsey in his paper On the number of complete extensions of the Lewis systems of sentential calculus, JSL 9 (1944), pp. 42-46. By 'complete' McKinsey means 'Post-complete'.

in the general question, whether there could exist a strongly complete non-classical system, since this is what CC aims at being. Fortunately, in answering this question, we are able to find an example much closer at hand.

There is a well-known strongly complete non-classical system, and this is the two-valued equivalential calculus with negation (EN). It is 'non-classical' only if considered as an implication/negation system: as an E-N system it is prototypal. Care is needed in attributing exactly the right kind of completeness to EN. Prior says that EN is not strongly complete, but by this he means not Post-complete.¹ The following is a proof that EN is strongly complete.

We define EN as the system for which the following matrix is characteristic:

Matrix 16

	E	1	2	N
*	1	1	2	2
	2	2	1	1

This matrix has been axiomatized (by Mihailescu in 1937 - see Prior, FL, p. 307), using the rules of substitution, E-detachment ($\vdash x, \vdash Ex \rightarrow \vdash y$) and the axioms:

1. $EEpqEErqEpr$
2. $EENpNqEpq,$

the first of these axioms being Łukasiewicz's, sufficient for the pure equivalential calculus.² However, to cut short the proof, we shall accept

¹Prior, FL, p. 307.

²Note that the proof in section 48 makes possible a new axiomatization of EN, using only classical C-N axioms plus Aristotle. The axioms are:

1. $CCpqCCqrCpr$
2. $CpCCpqq$
3. $CCNpNqCqp$
4. $CpNNp$
5. $NCpNp$

1 and 2 give Comm (see the deduction in appendix 5.2); 3 and 4 give Trans 1-4; as in section 48 we get CCpqCqp; and this applied to Syl and to Trans 4 gives Mihailescu's axioms.

the fact of being satisfied by matrix 16 as proof of EN thesishood, although incidentally the method of proof may be used to confirm that the axiomatization of the matrix is correct, by deducing all the EN theses used in it from Mihailescu's axioms.

The method is as follows. Theses of EN are used to reduce all (well-formed) E-N formulae to standard form. Inspecting different instances of this form reveals whether the formulae are EN theses or not: if they are, it is shown how they may be proven. If they are not, it is shown either how they are rejected by the matrix (if we are interested in proving that Mihailescu's axioms are complete), or how adding any one of them to EN makes EN inconsistent.

1. Reduction of E-N formulae to standard form

We note first that the rule for the replacement of equivalents holds in EN through repeated application of the theses

- (1) $EEpqEEqrEpr$
- (2) $EEqrEEpqEpr$
- (3) $EEpqENpNq$

Replacing any formula by one equivalent to it entails that if the latter is a thesis, then the former is a thesis, and if the latter is not a thesis, then the former is not a thesis. We sometimes write

$$x = y$$

to indicate that x and y are equivalent.

Consider now any E-N formula. It is of the form

$$E_1 = Ex_1Ex_2 \dots Ex_{n-1}x_n,$$

where $n = 1$ and the x_i are formulae. We call $x_1 \dots x_{n-1}$ antecedents and x_n the consequent. When each antecedent (if any exist) is a propositional variable distinct from the other antecedents, and the consequent is either a propositional variable or the negation of a

propositional variable, and distinct from the antecedents if there is more than one, we say that E_1 is in standard form.

If E_1 is not in standard form, it may be put into standard form as follows. First cancel all double negations in virtue of the equivalence

$$(4) \quad x = NNx,$$

obtaining a new expression E_2 equivalent to E_1 . If the consequent of E_2 is either a propositional variable or the negation of a propositional variable we re-name the expression E_3 . If not, the consequent is the negation of an equivalence, and we get E_3 by repeated applications of

$$(5) \quad NExy = ExNy.$$

If E_3 has no negative antecedents we re-name it E_4 . If it has we commute the negation signs to the right, beginning with those on the far left, by

$$(6) \quad ENxEyz = ExENyz,$$

and $(7) \quad ENxy = ExNy$

until they either disappear by cancellation of double negations or the last one qualifies the consequent.

If E_4 has no antecedents which are equivalences we re-name it E_5 . If it has, we simplify them by means of

$$(8) \quad EExyz = ExEyz,$$

repeating this process and the earlier ones where necessary to obtain E_5 .

If E_5 has no two antecedents the same it becomes E_6 . If not, we eliminate pairs of similar antecedents by first commuting them to the right:

$$(9) \quad ExEyz = EyExz$$

and then using

$$(10) \quad E\bar{E}xy = y$$

If E_6 has two or more antecedents, and if the consequent is one of them, or is the negation of one of them, we interchange the consequent, or the negation of the consequent, with one of the others by commuting it next to the consequent and using

$$(11) \quad Exy = Eyx$$

$$\text{or } (12) \quad E\bar{x}Ny = Ey\bar{N}x,$$

thus reverting to E_5 .

What remains is an expression in standard form.

2. Treatment of standard forms

The possible types of standard form are as follows, where the meta-logical variables $x_1 - x_n$ denote distinct propositional variables:

- | | | |
|-------|--|----------|
| (i) | $E\bar{x}x$ | |
| (ii) | $E\bar{x}N\bar{x}$ | |
| (iii) | $E\bar{x}_1\bar{x}_2\cdots\bar{x}_{n-1}x_n$ |) n odd |
| (iv) | $E\bar{x}_1\bar{x}_2\cdots\bar{x}_{n-1}N\bar{x}_n$ | |
| (v) | like (iii) |) n even |
| (vi) | like (iv) | |

Of these six types, only (i) is asserted. The others are easily shown to be rejected by the matrix, by giving each antecedent x_i the value 1 and the consequent $\bar{x}_n$ or $N\bar{x}_n$ the value 2. Thus any axiomatic basis, such as Mihailescu's, from which (1) - (12) and (i) can be deduced, is complete relative to the matrix.

To show that if any of the types (ii) - (vi) were added to EN, the system would become inconsistent, we note that

(ii) is inconsistent with the EN thesis $NEpNp$.

(iii) and (iv) yield p and Np respectively when we identify

all their variables with p and cancel antecedents pairwise.

(v) and (vi) yield $EEppp$ and $EEppNp$ respectively when we identify all their variables with p except x_{n-1} , for which we substitute Epp . We then get p , or Np , by detachment.

This completes the proof that EN is strongly complete. Note that if to EN is added any expression yielding a standard form of type (ii), EN becomes inconsistent, but remains Post-consistent. This is the result referred to in Prior. The reason why, unlike PC, EN is strongly complete but not Post-complete, is that it is not functionally complete. If it were, we would be able to define the functions $\underline{C}$ and $\underline{N}$ in EN (which we cannot now do), and obtain the Post-complete PC.

52. The C-fragment of a non-classical matrix

In the last section we examined the axiomatization of a non-classical matrix, and saw that the result was a strongly complete system. In this section we shall axiomatize the implicational part of a matrix of greater interest from the point of view of connexive implication.

The matrix in question¹ is

Matrix 17

	C	1	2	3	N
*	1	2	3	3	3
*	2	2	2	3	2
	3	2	2	2	1

and it has, among others, the following properties:

- (a) It satisfies all the axioms and rules of C3, the C-fragment of S3, but not the characteristic thesis CqCpp of C4.
- (b) It satisfies the strong negation theses CCpNqCqNp, CNNpp and CCNpppp of CN3, though not the paradoxes of strict implication CCNppCqp and CCpNpCpq.
- (c) It satisfies the characteristic theses of S7 and S8.
- (d) It satisfies both NCpNp and CCpqNCpNq, and is hence connexive.
- (e) It does not satisfy CCpqCqp, and is hence not equivalential.
- (f) It is inconsistent, satisfying both Cpp and NCpp.
- (g) It is Post-consistent.

The prime disadvantage of matrix¹⁷ as a basis for CC is that it is inconsistent. However, the thought occurred that Aristotle and Boethius might be joined to the pure implicational part of the matrix without contradiction, and then, little by little, the other negation

¹This matrix was found by the author in searching for CC. Its implicational part is used as a distinguishing model by Trenchard More in his thesis Relations between implicational calculi (M.I.T. 1962), p. 91.

theses added until the system became strongly complete. Matrix 17 satisfies some very strong negation formulae indeed, such as $\underline{NCpq}$, and it was thought that perhaps these might be excluded, but double negation and the laws of transposition retained, without the system becoming inconsistent. As will be seen, this hope was eventually frustrated, but the axiomatization of the implicational part of the matrix is here presented in any case. Axiomatizations of finite matrices are rare enough,¹ and the result may be of interest, if only in order to answer the question, whether the pure implicational part of the matrix is 'classical' in the sense of satisfying no non-classical formulae.

Let us call every formula (rule) satisfied by the implicational part of matrix 17 and M3V thesis (rule). Then the technique of axiomatizing the matrix is, as with EN, to reduce every $\underline{C}$ formula to standard form, using only M3V theses and rules to do so. Some of these standard forms will be shown to be M3V theses. The rest will be shown to be rejected by the matrix. Collecting all the M3V theses and rules used in the demonstration will yield a system complete relative to the matrix, and this system will be later derived from four simple axioms plus the rules of substitution and modus ponens.

1. Terminology

Since matrix 17 distinguishes between the ranges of values which may be assumed by propositional variables and by implications (the latter never take the value 1), we shall use, in this section only, capital letters X, Y, Z, etc. to denote any formulae

¹The author can think of only six; those of the PC and EN two-valued matrices; that of L3V, L4V, ... etc. (although, so far as he knows, the $\underline{C}$ -fragments of these systems have never been axiomatized); that of some three-valued matrices adequate for IPC (Prior, FL, p. 250); that of Słupecki's three-valued matrix (Prior, FL (1st ed.), p. 308); and that of Sobocinski's three-valued matrix (The Journal of Computing Systems, 1 (1952)). In appendix 2 will be found an unverified conjecture of the author's concerning the axiomatization of matrices.

of M3V, and small letters x, y, z, etc. to denote propositional variables. The latter formulae are called simple, while implications are complex. If the letter X is intended to denote a complex formula, then this must be explicitly stated.

2. Equivalence and deductive equivalence

If two expressions X and Y are related in such a way that the formulae CXY and CYX are M3V theses, we say that X and Y are equivalent. We shall write:

$$X = Y$$

The rule for the replacement of equivalents holds in M3V, in virtue of Syl and Comm'd Syl:

- (1) CCXYCCYZCXZ
- (2) CCXYCCZXCZY

In certain instances an expression X will be said to be deductively equivalent to two expressions Y and Z. This occurs when the three theses CXY, CXZ and CYCZX hold. We write:

$$X = Y \ \& \ Z$$

In these instances it is evident that

- (a) If X is a thesis then Y and Z are both theses, so that
- (b) If either Y or Z is not a thesis, then X is not a thesis;
- (c) If Y and Z are both theses, then X is a thesis.

Hence, in deciding whether or not X is an M3V thesis, it will be sufficient to consider whether Y and Z are theses. In addition, if we wish to decide whether CWX is a thesis, it will be sufficient to consider whether CWY and CWZ are theses: this follows from (1) above and from

- (3) CCYCZXCCWYCCWZCWX.

3. Reduction to standard form

With this tool of deductive equivalence at our disposal, we are ready to establish a method for reducing any formula in M3V to standard form.

Every expression of M3V is of the form

$$E_1 = CX_1CX_2\dots CX_{n+m-1}x_m,$$

where $n \geq 0$, $m \geq 1$, the X_i are formulae, and x_m is a simple expression.

By means of the equivalence

$$(4) \quad CXCYZ = CYCxCZ \ \& \ CxCxx \qquad Y \text{ complex}$$

we may move each of the simple X_i to positions at the right hand side of E_1 , thus obtaining a number of new deductively equivalent expressions

$$E_2 = CX_1CX_2\dots CX_nCxx_1Cxx_2\dots Cxx_{m-1}x_m,$$

where the X_i are all complex.¹ In M3V we call the X_i antecedents, and we shall abbreviate the expression $Cxx_1Cxx_2\dots Cxx_{m-1}x_m$ to Z_1 , which we call the consequent.

We now take any antecedent X_i of the form CXY , with X and Y complex and $Y \neq Z_1$, and move it to the X_n position by repeated application of

$$(5) \quad CUCVW = CVCUW \qquad U, V \text{ complex.}$$

Then, by means of the equivalence

$$(6) \quad CCXYZ_1 = CCXZ_1Z_1 \ \& \ CYZ_1 \qquad X, Y \text{ complex}$$

we replace E_2 by two expressions not containing antecedents CXY . We repeat this procedure until no such antecedents CXY remain, thus

¹For example, where V, W and Y are complex:

$$\begin{aligned} CuCVCWCxCYZ &= CVCuCWCxCYZ \ \& \ CuCuu \\ &= CVCWCuCxCYZ \ \& \ CVCuCu & \& \ CuCuu \\ &= CVCWCuCYCxx \ \& \ CVCWCuCxCxx \ \& \ CVCuCu & \& \ CuCuu \\ &= CVCWCYCxCxx \ \& \ CVCWCuCu & \& \ CVCWCuCxCxx \ \& \ CVCuCu & \& \ CuCuu. \end{aligned}$$

obtaining a new expression E_3 .

If in E_3 there exist antecedents of the form CXy , X complex and $y \neq Z_1$, they may be moved to the X_n position and eliminated by means of

$$(7) \quad CCXyZ_1 = CCXZ_1Z_1 \ \& \ CCCyyyZ_1 \quad X \text{ complex},$$

the repetition of this process producing a new expression E_4 .

If E_4 contains antecedents CxY , Y complex, $Y \neq Cxx$ and $Y \neq Z_1$, they may be moved to the X_n position and eliminated by means of

$$(8) \quad CCxYZ_1 = CCxZ_1Z_1 \ \& \ CCxCxxCYZ_1 \quad Y \text{ complex},$$

thus yielding E_5 . It will be noted that the only antecedents now remaining which are not of the form CXZ_1 are either of the form $CwCww$, or of the form $CCxxx$, or of the form Cyz . We turn now to antecedents CXZ_1 .

If E_5 contains antecedents $CCXYZ_1$, X complex, they may be eliminated in E_6 by means of

$$(9) \quad CCCXYZ_1Z_1 = CXCCYZ_1Z_1 \quad X \text{ complex}.$$

E_6 's antecedents $CCxYZ_1$, Y complex and $Y \neq Cxx$, are eliminated in E_7 by the equivalence

$$(10) \quad CCCxYZ_1Z_1 = CCCxxxCCYZ_1Z_1 \ \& \ CCCxCxxZ_1Z_1 \quad Y \text{ complex}.$$

E_7 's antecedents $CCxyZ_1$ are eliminated in E_8 by

$$(11) \quad CCCxyZ_1Z_1 = CCCxxxCCyZ_1Z_1 \ \& \ CCCxCxxZ_1CCyCyyZ_1.$$

Now, if in E_8 Z_1 is complex, i.e. $Z_1 = Cx_1Z_2$, and if E_8 also contains an antecedent $Cx_1Cx_1x_1$, the latter is eliminated in E_9 by

$$(12) \quad CCx_1Cx_1x_1Cx_1Z_2 = CCx_1Cx_1x_1CCCx_1x_1x_1Z_2.$$

Finally, if in E_8 or E_9 Z_1 is complex, other antecedents of the form CxZ_1 are eliminated by the equivalence

$$(13) \quad CCxZ_1Z_1 = CCZ_1xx,$$

thus yielding E_{10} with Z_1 replaced by x , which becomes the new Z_1 . The steps leading from E_2 to E_{10} should now be repeated wherever possible. The effect is that antecedents CXZ_1 have been eliminated with the exception of those of the form $CCvCvvZ_1$, and of the form CyZ_1 where Z_1 is ~~complex~~ ^{simple}.

We now ensure that the effects of transitivity among antecedents are realized by moving any pairs of antecedents CXY and CYZ in E_8, E_9 or E_{10} to the X_{n-1} and X_n positions respectively, and adding a new antecedent CXZ in E_{11} by means of

$$(14) \quad CCXYCCYZZ_1 = CCXYCCYZCCXZZ_1.$$

To prevent the creation of new antecedents of the forms $CxCyy$ or $CCxxy$ by the application of (14), we eliminate such types as follows:

$$(15) \quad CxCyy = CxCxx$$

$$(16) \quad CCxxy = CCyyy.$$

This yields a standardized expression E_{12} with complex antecedents of the following forms only:

$$CCvCvvZ_1, CwCww, CCxxx, Cyz.$$

4. Treatment of standard forms

We are now in a position to start separating provable (asserted) M3V formulae from formulae which are rejected by the matrix. We proceed by cases.

Case 1. E_{12} includes among its antecedents either Z_1 , or the pair $CCvCvvZ_1, CvCvv$. Such expressions are asserted. To show this we note that the following are M3V theses:

(17) CZ_1Z_1

This expression, preceded by any number of complex antecedents, is still asserted in virtue of the repeated application of

(18) $CZCYZ$ Y,Z complex

and the rule of modus ponens.

(19) $CCvCvvCCCvCvvZ_1Z_1,$

and this expression too may be preceded by any number of complex antecedents.

In what follows we shall assume that the conditions of Case 1 do not hold. Further cases subdivide themselves according to the nature of the consequent Z_1 .

Case 2.

Z_1 is simple, i.e. is x_1 .

Case 2.1

The antecedents of E_{12} , if any, include either

(a) $CCx_1x_1x_1$; or (b) $CCx_1Cx_1x_1x_1$; or (c) the pair $CCxxx, Cxx_1$; or (d) the pair $CCvCvxx_1, Cvx_1$. E_{12} is then asserted. For proof we have as theses:

- (a) (20) $CCCx_1x_1x_1x_1$
- (b) (21) $CCCx_1Cx_1x_1x_1x_1$
- (c) (22) $CCCxxxCCxx_1x_1$
- (d) (23) $CCvCvxx_1CCvx_1x_1,$

each of which may be preceded by any number of antecedents.

Case 2.2

Lacking the conditions of Case 2.1, the expression E_{12} is rejected. To show this, we assign values to E_{12} 's variables in a certain specific order, resulting in the value 3 being assigned to the consequent and the value 2 to each antecedent. Since $C2C2...C23 = 3$, the whole expression will be rejected. We use the standard letters v, w, x, y and z for

each of the possible types of antecedent, viz:

$CCvCvvZ_1$, $CwCww$, $CCxxx$, Cyz ,

except that, where explicitly stated, y and z may do duty for all the remaining variables in E_{12} . We indicate sub-groups of the classes of variables by priming (e.g. w'). We assign values as follows, and in the following order, it being understood that each assignment concerns only those variables of the class in question not given a value by an earlier assignment.

Put $x_1 = 3$. This results in no antecedents receiving the value 3, since the only ones which could take this value are $CCx_1Cx_1x_1x_1$ and $CCx_1x_1x_1$, and these are not present in Case 2.2.

Put $v = 1$. We have among the antecedents no $CvCvv$ (see Case 1), nor Cvx_1 (see Case 2.1), and hence no antecedents taking the value 3.

Put $w' = 2$ for all pairs of antecedents $Cw'Cw'w'$, $CCw'w'w'$. We have in this case no $Cw'x_1$, since then by transitivity we would have $CCw'w'x_1$, i.e. $CCx_1x_1x_1$; nor Cvw' , since then we would have $CvCw'w'$, i.e. $CvCvv$.

Put $w = 3$ for all remaining $CwCww$. We have no Cvw , since then $CvCww$, i.e. $CvCvv$; nor $Cw'w$, since then $CCw'w'w$, i.e. $CCwww$.

Put $x = 1$ for all remaining $CCxxx$. We get no Cxx_1 , since then $CCxxx_1$, i.e. $CCx_1x_1x_1$; nor Cxw' , since then $CxCw'w'$, i.e. $CxCxx$; nor Cxw , since then $CxCww$, i.e. $CxCxx$.

Put $y = 3$ for all Cyx_1 . We have no Cvy , since no Cvx_1 ;
nor $Cw'y$, since no $Cw'x_1$; nor Cxy , since no Cxx_1 .

Put $z = 1$ for all other variables z . We have no Czx_1
(none left); nor Czw' , since no $CzCw'w'$; nor Czw , since no
 $CzCww$; nor Czy , since no Czx_1 .

Case 3.

$$Z_1 = Cx_1x_2$$

Case 3.1

The antecedents of E_{12} , if any, include (a) $CCx_2Cx_2x_2Cx_1x_2$,
or (b) the pair $CCvCvvCx_1x_2, Cvx_2$. Or (c) $x_1 = x_2$. E_{12} is
then asserted. To show this, we note that the following
are theses, and may be preceded by any number of antecedents:

- (a) (24) $CCCx_2Cx_2x_2Cx_1x_2Cx_1x_2$
- (b) (25) $CCvCvvCx_1x_2CCvx_2Cx_1x_2$
- (c) (26) Cx_1x_1 .

Case 3.2

Lacking the conditions of Case 3.1, the expression is
rejected as follows:

Put $x_1 = 1$. No $Cx_1Cx_1x_1$ (see the step leading from
 E_8 to E_9).

Put $x_2 = 2$. No $CCx_2Cx_2x_2Z_1$ (Case 3.1); no Cx_1x_2 (Case 1).

Put $v = 1$. No $CvCvv$; no Cvx_2 (Case 3.1).

Put $w = 2$. No Cx_1w , since no Cx_1Cww ; no Cvw , since no
 $CvCww$.

Put $x' = 2$ for all pairs $CCx'x'x'$, $Cx'x_2$. No Cx_1x' ,
since no Cx_1x_2 ; no Cvx' , ~~since no Cx_1x_2 ; no Cvx_1~~ , since
no Cvx_2 .

Put $x = 1$ for all remaining $CCxxx$. No Cxx_2 (none left);
no Cxw , since then $CxCww$, i.e. $CxCxx$, and these are
exhausted; no Cxx' , since no Cxx_2 .

Put $y = 2$ for all Cyx_2 . No Cx_1y , since no Cx_1x_2 ; no Cvy , since no Cvx_2 ; no Cxy , since no Cxx_2 .

Put $z = 1$ for all other variables z . No Czx_2 (none left); no Czw , since then $CzCww$; no Czx' , since no Czx_2 ; no Czy , since no Czx_2 .

Case 4.

$Z_1 = Cx_1Z_2$, Z_2 complex.

Lacking the conditions of Case 1, E_{12} is rejected as follows:

Put $x_1 = 1$. Hence $Cx_1Z_2 = 3$. No $Cx_1Cx_1x_1$ (see the step leading from E_8 to E_9).

Put $v = 1$. No $CvCvv$.

Put $w = 2$. No Cx_1w , since no Cx_1Cww ; no Cvw , since no $CvCww$.

Put $x = 1$. No Cxw , since then $CxCww$ and these are exhausted.

Put $y = 1$ for all other variables y . No Cyw , since then $CyCww$.

This completes the axiomatization of the implicational part of matrix 17. The M3V theses used in the axiomatization, namely theses (1) - (26), are all derived in appendix 6 from the following four axioms, using the rules of substitution and modus ponens:

1. $CCpqCCqrCpr$
2. $CCCpqpp$
3. $CQCpp$
4. $CCCpCqpCqp$.

It will be seen that these four axioms all hold for classical implication, hence the implicational part of matrix 17 satisfies no non-classical theses. By the same token, it is not, of course, Post-complete.

53. Failure of C3 as a base for CC

In section 49 it was noted that S3 contained no Clavian implications, i.e. (strict) implications x for which $CNxx$ was true. If in addition S3 contained no self-defeating implications, indicated by the lack of any thesis $CxNx$, its pure implicational part plus negation might have served as a base for CC.

Unfortunately, however, S3 does contain self-defeating implications, and their existence is demonstrable from some very simple axioms:

1. $CCpqCCqrCpr$
2. $CCCppqq$
3. $CQCpp$
4. $CCpNqCqNp$

The deduction proceeds as follows:

- (2) 5. $CCCppNCppNCpp$
- 4=C5--6. $CCppNCCppNCpp$
- 1=C3--C6--7. $CQNCCppNCpp$
- (7) 8. $CCCppNCppNCCppNCpp$

8 shows that $CCppNCpp$ is self-defeating. Hence C3 plus even the modest negation axiom Trans 2 is disqualified as a base for CC.¹

54. The failure of E

Slightly weaker than S3, as was seen in Chapter 4, is the system E. Specifically, E lacks C3's characteristic axiom $CQCpp$, and hence presents itself as the next obvious candidate for a base for CC. In this section it will be seen how E and IN (though not I) are disqualified as possible CC's.

¹It is conjectured that C2, which, though it does not have Syl, has the Rule of Syl $\vdash Cxy \rightarrow \vdash CCyzCxz$, is on a par with C3 in this respect.

The full E, with conjunction, is obviously disqualified from the start by its theses CKpNpp and CKpNpNp. The latter yields by transposition CpNKpNp, and this, together with the former, gives CKpNpNKpNp by Syl, which shows that KpNp is self-defeating.

IN is inconsistent with Boethius. Moreover this result, as will be seen, is obtainable without using any of the negation axioms of IN. That is, it is obtainable simply by introducing the operator N, and the thesis Boethius, into I. The following is the proof, starting from the theses Frege, Syl and Id of I:

1. CCpCqrCCpqCpr
2. CCpqCCqrCpr
3. Cpp
4. CCpqNCpNq (Boethius)
- (1) 5. CCCNppCNppCCCNppNpCCNppp
- 5=C3--6. CCCNppNpCCNppp
- (4) 7. CCCNpppNCCNppNp
- 2=C6--C7--8. CCCNppNpNCCNppNp
- 4=C3--9. NCCCNppNpNCCNppNp

Lines 8 and 9 show the inconsistency, hence Boethius cannot consistently be added to I.

Next it will be shown that Aristotle cannot be consistently added to IN. The proof of this, which involves showing that IN contains self-defeating implications, will be presented in subproof form:

(1)	1.	CCCNCppCppCppNCpp	hyp
(2)	2.	CCCNCppCppCppNCpp	hyp
(3)	3.	p	hyp
(3)	4.	p	3 rep
^	5.	Cpp	3,4 CP
^	6.	NNCpp	5 DNI
(2)	7.	NCCNCppCppCpp	2,6 MT
^	8.	CCCCNCppCppCppNCppNCCNCppCppCpp	2,7 CP
(2)	9.	CNCppCpp	hyp
(1)	10.	CCCNCppCppCppNCpp	1 reit
(3)	11.	CNCppCpp	hyp
(4)	12.	NCpp	hyp
(3)	13.	CNCppCpp	11 reit
(3,4)	14.	Cpp	12,13 MP
(3)	15.	NNCpp	12,14 RA
(3)	16.	Cpp	15 DNE
^	17.	CCNCppCppCpp	11,16 CP
(1)	18.	NCpp	10,17 MP
(1,2)	19.	Cpp	9,18 MP
(1)	20.	CCNCppCppCpp	9,19 CP
(1)	21.	NNCCNCppCppCpp	20 DNI
(1)	22.	NCCNCppCppCppNCpp	8,21 MP
^	23.	CCCCNCppCppCppNCppNCCNCppCppCppNCpp	1,22 CP

This proof shows that, in IN, the formula CCCNCppCppCppNCpp is self-defeating. Note that, in IN, the formula CCppNCpp is not self-defeating as it was in CN3: CCCppNCppNCCppNCpp is rejected for IN by matrix 10 of section 31. But IN contains more complex self-defeating propositions.

The one hope left is that the axioms of IN may be weakened so as to exclude all self-defeating propositions. This possibility is explored in the sequel.

55. Retreat to IEN

There is one weakening of IN that demonstrably excludes all self-defeating (and Clavian) propositions, and that is the fragment of IN satisfied by Angell's matrix. This matrix, matrix 1 of section 15, takes the following form when its C-N part is put into what we have adopted as standard formulation:

Matrix 18

	C	1	2	3	4	N
*	1	2	3	4	3	4
*	2	3	2	3	4	3
	3	2	3	2	3	2
	4	3	2	3	2	1

Exactly what fragment of IN this matrix satisfies, and what features the composition of the matrix displays, are questions which will be dealt with in this section.

Matrix 18 is an unusual-looking matrix. Its negation is classical, as indicated by the mirror-image negation values and the symmetry of the implicational part about the lower-left to upper-right diagonal (the 'CpNp diagonal'). Furthermore this diagonal is undesignated, indicating that the matrix is connexive in that it permits the consistent addition of NCpNp to the other formulae it satisfies. This latter property it shares with equivalence matrices, but matrix 18 is not an equivalence matrix. Equivalence matrices are symmetrical (at least with respect to mutually-implying values) about the upper-left to lower-right diagonal (the 'Cpp diagonal'), indicating that they satisfy CCpqCqp, but matrix 18 is not symmetrical in this way. This raises the possibility that matrix 18 is in some way compounded out of classical implication and classical equivalence.

In uncovering the origins of matrix 18 it will help to write it in a slightly different way, interchanging the values 1 and 2 and the values 3 and 4:

Matrix 19

	C	1	2	3	4	N
*	1	1	4	3	4	4
*	2	4	1	4	3	3
	3	1	4	1	4	2
	4	4	1	4	1	1

Matrix 19 should now be compared with the matrix formed by 'multiplying' together the matrix for classical implication, matrix 15, and the matrix

for classical equivalence, matrix 16, according to the method described in Łukasiewicz, AS, pp. 159-60. To keep the two matrices separate we begin with

$$* \begin{array}{c|cc|c} C & 1 & 2 & N \\ \hline 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 1 \end{array} \quad \text{and} \quad * \begin{array}{c|cc|c} E & 3 & 4 & N \\ \hline 3 & 3 & 4 & 4 \\ 4 & 4 & 3 & 3 \end{array}$$

and obtain

CE	(1,3)	(1,4)	(2,3)	(2,4)	N
(1,3)	(1,3)	(1,4)	(2,3)	(2,4)	(2,4)
(1,4)	(1,4)	(1,3)	(2,4)	(2,3)	(2,3)
(2,3)	(1,3)	(1,4)	(1,3)	(1,4)	(1,4)
(2,4)	(1,4)	(1,3)	(1,4)	(1,3)	(1,3)

which by renumbering becomes

Matrix 20	CE	1	2	3	4	N
	1	1	2	3	4	4
	2	2	1	4	3	3
	3	1	2	1	2	2
	4	2	1	2	1	1

Given the appropriate values as designated values, matrix 20 can be made to satisfy

- (a) PC
- (b) EN
- (c) The intersection of PC and EN
- ~~(d) The union of PC and EN.~~

The values in question are

- (a) 1 and 2 (since these are the original (1,3) and (1,4), 1 being designated for PC)
- (b) 1 and 3 (originally (1,3) and (2,3), 3 being designated for EN)
- (c) 1
- ~~(d) 1, 2 and 3 (note that this yields an inconsistent system).~~

The different systems related to matrix 20 are worth studying, though for the sake of simplicity only their pure implicational or equivalential fragments will be considered here. They will be presented in a sequence, leading up to the connexive system which is our main interest.

(1) The system C of pure classical implication, axiomatized by the rules of substitution and modus ponens (common to all the systems considered) and the axioms:

1. $CCpqCCqrCpr$
2. $CpCqp$
3. $CCCpqpp$.

(2) The system E of classical equivalence, whose sole axiom, written in C rather than E, is

1. $CCpqCCrqCpr$.

(3) The system CE of all those theses common to both C and E, i.e. of those formulae satisfied by the implicational part of matrix 20 with 1 designated. Axioms (Prior, FL, p. 308):

1. $CCpqCqrCpr$
2. $CqCpCpq$
3. $CCCpqCCpqq$.

(4) The system WC of Church:

1. $CCpqCCqrCpr$
2. $CCpCpqCpq$
3. $CpCCpqq$
4. Cpp .

(5) If we omit from WC all theses not holding true for equivalence - i.e. all theses in which one or more variables occur an odd number of times - we obtain WE, the intersection of WC and E. It is conjectured that WE may be axiomatized by dropping Hilbert from WC:

1. $CCpqCCqrCpr$
2. $CpCCpqq$
3. Cpp .

(6) The system I of Anderson and Belnap, namely WC lacking fallacies of necessity:

1. $CCpqCCqrCpr$
2. $CCpCpqCpq$
3. $CCCpqq$

(7) IE, the intersection of I and E, may be regarded either as WE minus fallacies of necessity or I minus non-equivalential theses. Just as matrix 9 of section 30, which is adequate for I, was formed from matrix 4, adequate for WC, by identifying certain values within the matrix, so matrix 21 will be formed by identifying certain values within matrix 20, which is characteristic for CE with 1 designated:

Matrix 21

	C	1	2	3	4	N
*	1	1	4	3	4	4
	2	4	1	4	3	3
	3	1	4	1	4	2
	4	4	1	4	1	1

We have now arrived precisely at Angell's matrix, with one less designated value, the implicational part of which is thus seen to be formed by weakening CE's matrix so as to exclude fallacies of necessity.¹ In short:

Matrix 21 : Matrix 20 (1 des) :: Matrix 9 : Matrix 4

and

IE : WE :: I : WC

It is conjectured that matrix 21 is adequate for IE, and that IE is axiomatized by

1. CCpqCCqrCpr
2. CCCppqq

~~(8) Although merely of formal interest, a system C+E could be constructed by uniting C and E. Axioms:~~

- ~~1. CCpqCCqrCpr~~
- ~~2. CpCqp~~
- ~~3. CCCpppp~~
- ~~4. CCpqCqp~~

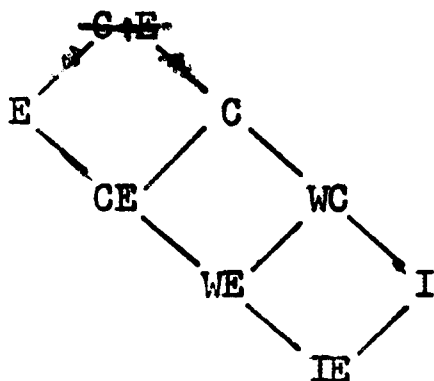
~~Matrix 20, with 1, 2 and 3 designated, shows that C+E is Post-consistent, and it is conjectured that C+E could be shown to be Post-complete by the~~

¹That matrix 21 excludes fallacies of necessity is seen by the fact that CxCyz takes the values 3 or 4 whenever x, as a propositional variable or its negation, takes the value 2.

~~method of section 51.~~

The various systems (1)-(7) exhibit the following containment-relations:

Table 6



The following results and conjectured results have now been arrived at. Matrix 21, which is Angell's matrix with one less designated value, has been shown to be created by weakening CE's matrix so as to exclude fallacies of necessity. It is conjectured that (as in the case of I as a weakening of WC) nothing else is excluded, so that Matrix 21 is adequate for all the equivalentially valid theses of I. We shall in fact assume (a) that IE is the class of theses of I satisfied by matrix 21; and (b) that IE is axiomatized simply by dropping Hilbert from the axioms of I.

Turning now to the system IEN, we define this as the intersection of EN and IN, and again assume that it is axiomatized by dropping all non-equivalential axioms from the basis for IN. This gives, as an assumed basis for IEN, the following axioms:

1. CCpqCCqrCpr
2. CCCppqq
3. CCNpqCNqp
4. CpNNp.

With this, a basis for CC has finally been arrived at. Matrix 21 satisfies Boethius as well as the axioms of IEN, rejects CCpqCqp, and is consistent.

Hence joining Boethius to axioms 1-4 above will produce a system IENB which, though still quite weak, qualifies nevertheless as a connexive calculus.

56. The system IENB in subproof form

It is not difficult to devise ways of strengthening IENB, whose axioms are as follows:

1. CCpqCCqrCpr
2. CCCppqq
3. CCNpqCNqp
4. CpNNp
5. CCpqNCpNq.¹

Matrix 21 satisfies many other formulae, such as CCppCqq, which may be consistently added to IENB without that system becoming equivalent. But before strengthening IENB it will be as well to consolidate whatever has been gained, and to see how strong the system is by setting what can and what cannot be proved in it. The best way of doing this is to construct a subproof formulation IENB.*

IENB* will consist of the rules hyp, rep, reit, CP, DNI and DNE of IN*, together with modifications of MP and MT and a new rule B:

MP₁ From items (a)x and (b)Cxy in subproof n derive (a+b)y in subproof n, provided ab = 0.

MT₁ From items (a)Ny and (b)Cxy in subproof n derive (a+b)Nx in subproof n, provided ab = 0.

B From (a)Cxy in subproof n derive (a)NCxNy in subproof n.

The modifications MP₁ and MT₁ prevent Hilbert from being proved, and the absence of RA excludes Clavius. Axioms 1-5 are easily provable in IENB*, and we allow MP₁ to be used when x and Cxy both 'depend' on $\wedge$, so IENB* satisfies modus ponens and hence contains IENB.

¹ Axiom 5 may be weakened slightly - see appendix 4.

To show that IENB contains IENB* we argue as in section 34.

The following theses in IENB are required, where, as in section 34, Lx is used as an abbreviation of CCxxx.

- T1. CCpqLCpq
- T2. CCpqCLpLq
- T3. CCpqCLNqLNp
- T4. CLpLNNp
- T5. CLNNpLp
- T6. CLCpqLNCpNq
- T7. Cpp
- T8. CCpCqrCLqCpr
- T9. CCpqCCrNqCrNp
- T10. CCpCqrCLNrCpNq
- T11. CCpqCNqNp
- T12. CCpqCpNNq
- T13. CCpNNqCpq
- T14. CCpCqrCpNCqNr

- R1. $\vdash x \rightarrow \vdash Lx$ (Proof from T1 and T2 - see section 34).

We now proceed to prove the lemma and the theorem required for showing that IENB contains IENB*.

Lemma As in section 34, substituting IENB* for $S3^*$, MP_1 for MP, and eliminating KI.

Proof As in section 34, omitting cases 5 and 9 - 12, changing cases 3 and 6, and adding case 13:

Case 3. MP_1 . (a)y comes from (b)z and (c)Czy, where $a = b+c$ and $bc = 0$. Then by the inductive hypothesis we may obtain (b)Lz, and we reiterate T2 and use MP_1 twice to obtain, successively, (c)CLzLy and (b+c)Ly, i.e. (a)Ly.

Case 6. MT_1 . Similar to case 3, using T3.

Case 13. B. (a)NCyNz comes from (a)Cyz. Reiterate T6 and use MP_1 to get (a)LNCyNz.

We proceed now to the proof of the theorem. Note first that the rules rep, MP_1 , MT_1 , DNI, DNE and B are all either primitive or easily

derived rules of IENB. Hence these rules, applied to theses of IENB, lead always to theses of IENB. It remains only to show that any thesis of IENB* which has CP as its reason is also a thesis of IENB, and this is proved by Anderson's theorem.

Theorem As in section 27, substituting IENB and IENB* for I and I* respectively.

Proof As in section 27, with the following changes and additions.

Case 3. (a)y comes by MP_1 from (b)z and (c)Czy, where $a = b+c$ and $bc = 0$. We distinguish four subcases.

Case 3.1 m is both b and c. Impossible, since $bc = 0$.

Case 3.2 m is in b but not in c. Then (b)z' is (b-m)Cxz, and (c)Czy' is (c)Czy. Reiterate axiom 1 and use MP_1 twice to get, successively, (b-m)CCzyCxy and (b-m+c)Cxy, i.e. (a-m)Cxy, i.e. (a)y'.

Case 3.3 m is in c but not in b. Then (b)z' is (b)z and (c)Czy' is (c-m)Cx Czy. By the lemma (since m is not in b) it is possible to reiterate theorems of IENB into P' and use MP_1 to obtain (b)Lz. Then reiterate T8 and use MP_1 twice to get, successively, (c-m)CLzCxy and (c-m+b)Cxy, i.e. (a-m)Cxy, i.e. (a)y'.

Case 3.4 m is in neither b nor c.
Then MP_1 gives (a)y, i.e. (a)y'.

Case 4. CP. Impossible.

Case 5. MT_1 . Similar to case 3, using T9, T10 and the lemma, and T11.

Case 6. DNI. Use T12.

Case 7. DNE. Use T13..

Case 8. B. Use T14.

In appendix 4 will be found the deduction of theses T1-14 from axioms 1-5: in fact from a base which includes a slightly weaker version of axiom 5. This completes the proof that IENB contains IENB*.

57. Strengthening the system IENB

IENB is a connexive calculus, but it is a weak one. To be sure, it could be strengthened by adding other axioms, satisfied by matrix 21, which are consistent with it, but this variety of strengthening is not of the sort which is, at present, of much interest. Matrix 21 satisfies no formulae which are not equivalential, and we know already that it is possible to strengthen IENB to the point where it is transformed into the strongly complete system EN. Since CC should be a system of implication, not of equivalence, it would be desirable to include in it at least one thesis which is not equivalential. This would guarantee that its strong completion, whatever it is, would not be EN.

The most obvious candidate for a plausible implication thesis which is not equivalential is Hilbert, ^{with} ~~which~~ its close companion Frege. Adding Hilbert to IE gives I, and so it is I that must now be scrutinized as a possible foundation for CC. IN will not do, the reason being that it contains Clavius, which produces a thesis of the form CxNx when added to IE plus strong negation, (note that the subproof derivation of section 54 holds for IENB*, with the exception of one use of RA). Hence the trick is to add Hilbert without adding Clavius, and this should not be impossible, since matrix 8 of section 28 shows that Clavius is not deducible from the other axioms of IN. It is interesting, too, that Clavius is just the thesis of which it was denied, in section 12, that there existed any concrete exemplifications.

It should be noted, however, that an I-based CC cannot simply be a strengthening of IENB. It was shown in section 54 that the system I plus the negation operator could not consistently admit of the addition of Boethius. However, it was left open whether I might not consistently admit Aristotle. In section 59, it will be shown that it can: in the next section a Gentzen-type formulation of I as a sequenzenkalkül will be given, the aim being to prove a lemma about I.

58. The system I as a sequenzenkalkül

Belnap and Wallace, in collaboration with Kripke, have presented I as a sequenzenkalkül.¹ In this section a somewhat simplified version of their formulation will be considered, known as LI. For certain results concerning the system LI, I am indebted to Messrs. Lung-Ock Chung and Patrick Schindler.

In stating the postulates for LI, the variables $x, y, z, \dots$ will, as usual, be taken to range over formulae, and the variables $\alpha, \beta, \gamma, \dots$ over sequences of formulae. Any expression of the form $\alpha \vdash \beta$ is called a sequent, where α is the prosequent and β the consequent. In LI all consequents are restricted to exactly one member.

¹ Belnap and Wallace, DP; S.A. Kripke, The problem of entailment (abstract), JSL 24 (1959), p. 324.

Axiom schema $x \vdash x$, where x is a propositional variable.

Rules of inference

C
(Permutation)

$$\frac{\alpha, x, y, \beta \vdash z}{\alpha, y, x, \beta \vdash z}$$

W
(Contraction)

$$\frac{\alpha, x, x \vdash y}{\alpha, x \vdash y}$$

CL

$$\frac{\alpha \vdash x \quad \beta, y \vdash z}{\alpha, \beta, Cxy \vdash z}$$

CR

$$\frac{\alpha, x \vdash y}{\alpha \vdash Cxy}, \text{ where every member of } \alpha \text{ is an implication.}$$

The following is a derivation of the thesis Hilbert in LI:

$$\begin{array}{rcl} p \vdash p & q \vdash q & \\ \hline & & \text{CL} \\ p \vdash p & p, Cpq \vdash q & \\ \hline & & \text{CL} \\ & p, p, CpCpq \vdash q & \\ \hline & & \text{C} \\ & p, CpCpq, p \vdash q & \\ \hline & & \text{C} \\ & CpCpq, p, p \vdash q & \\ \hline & & \text{W} \\ & CpCpq, p \vdash q & \\ \hline & & \text{CR} \\ & CpCpq \vdash Cpq & \\ \hline & & \text{CR} \\ & \vdash CCpCpqCpq & \end{array}$$

Showing that LI contains I requires, firstly, proving the axioms of I in LI, and then showing that LI satisfies the rule of modus ponens by showing that the following rule of fusion of sequences, when

introduced into LI, is dispensable in that system:

$$\text{FS} \quad \frac{\alpha \vdash x \quad \beta \vdash y}{\alpha, \beta^* \vdash y}, \quad \text{where } \beta^* \text{ is like } \beta \text{ except for} \\ \text{containing one or more fewer} \\ \text{occurrences of } x.$$

Showing that I contains LI requires producing an interpretation for sequents of LI in terms of formulae of I. For a full proof of the identity of LI and I see the thesis of Schindler.

We now prove the following lemma, required for the next section:

Lemma

In no thesis of LI does any propositional variable occur less than twice.

Proof

(informally)¹ A variable occurs twice in each axiom. No rule of inference, with the exception of W, effects a reduction in the number of variables from line to line in a derivation. Moreover, extending to sequents Anderson's and Belnap's definition of 'antecedent part' and 'consequent part' (PCE, p. 49), we can show that, before contraction, given any pair of variables coming from the same axiom, one always occurs as an antecedent part and the other as a consequent part. Hence if two identical formulae occur in the same prosequent (the condition for contraction) none of the pairs of corresponding variables in the two formulae can come from the same axiom. For example, if two formulae Cpq and Cpq occur in the same prosequent, the two p's concerned

¹For a different proof of this lemma, based on matrices, see Anderson and Belnap's forthcoming book on Entailment.

cannot both come from the same axiom, for they are both consequent parts. Hence contraction cannot eliminate one of a pair of variables from the same axiom, leaving the other as the sole occurrence of that variable in the sequent.

59. I as a basis for CC

CC will not be presented as a sequenzenkalkül. The reason for this is, that it appears to be very difficult to adapt and to add negation rules to LI in such a way as to make the usual transposition and double negation theses provable, without making theses from which Clavius is deducible provable as well. For example, allowing the consequent to be empty and introducing NL and NR as versions of Belnap's and Wallace's rules for negation licences the following derivation, where the double lines indicate various permutations:

	$q \vdash q$	
	-----	NL
$p \vdash p$	$q, Nq \vdash$	
-----	-----	CL
	$p, q, CpNq \vdash$	
-----	-----	NR
$p \vdash p$	$p, q \vdash NCpNq$	
-----	-----	CL
	$p, p, Cpq \vdash NCpNq$	
=====	=====	C
	$Cpq, p, p \vdash NCpNq$	
-----	-----	W
	$Cpq, p \vdash NCpNq$	
-----	-----	CR
	$Cpq \vdash CpNCpNq$	
-----	-----	CR
	$\vdash CCpqCpNCpNq$	

The last line is a derivative of CCpqCCpNqNp, from which Clavius is easily gotten. Clavius is not deducible from the other axioms of IN, but a Gentzen-type system reflecting this fact is not yet forthcoming.

It will, nevertheless, be possible to prove, by the lemma of the previous section, that Aristotle is consistent with I plus strong negation.¹ This requires examining a series of systems ION, I₁N, I₂N, etc., about which the following facts will be demonstrated:

- (i) No system I_iN, $i \geq 0$, contains any theses of the type CxNx or CNxx.
- (ii) When the rule of modus ponens is added to it, no system I_iN, $i \geq 1$, contains any theses of the type CxNx or CNxx.
- (iii) The totality of all the theses of the systems I_iN, $i \geq 1$, when the rule of modus ponens is added to these systems, is contained in a certain axiomatic system INN.

We begin with the system ION, formed by adding the primitive operator N to I but not adding any axioms for negation. The only theses of ION containing negation will therefore be substitution instances of theses of I. By the previous result, that I contains no thesis with a singly-occurring variable, it is plain that any part of a thesis of ION beginning with the operator N is reproduced somewhere in the thesis.

We now prove the following lemma, defining the length of a formula as the number of variables it contains:

Lemma 1 In any thesis CxNy (CNxy) of ION, either x is Ny (y is Nx) or x is longer (shorter) than y.

Proof Consider the thesis CxNy. Since the consequent Ny is negative, Ny must be contained in x. If x is Ny, CxNy is a substitution instance of the thesis Cpp of I. If x is negative but not Ny, then x would have to also be contained in Ny, which is impossible. If x is an

¹ (Added later) This is incorrect. S.M.

implication, say Cuv , then Ny is contained either in u or in v . In both cases x is longer than y . A similar argument holds for the thesis $CNxy$.

We are now ready to add the effects of transposition and double negation to ION. This is done by allowing theses of ION to be transformed by the rule for the replacement of equivalents (RE) in conjunction with the following equivalences:

- (1) $x = NNx$
- (2) $Cxy = CNyNx$
- (3) $CxNy = CyNx$
- (4) $CNxy = CNyx$.

We obtain theses of the progressively stronger systems IlN , $I2N$, etc. as follows. Theses of IlN which are not theses of ION must always be gotten from theses of ION by one use of RE; theses of $I2N$ which are not theses of IlN must be gotten from theses of IlN by one use of RE; etc. Thus CCCqgNpNp is a thesis of ION (and of IlN , $I2N$, etc.), CCpNCqgNp is a thesis of IlN (and of $I2N$), CpNCpNCqg is a thesis of $I2N$, etc. We call each application of RE together with one of the equivalences (1)-(4) a transformation, and note that every transformation preserves the length of the transformed formulae. This leads to the following lemma:

Lemma 2 No system of the series IlN , $I2N$, etc. contains any theses of the forms $CxNx$ or $CNxx$.

Proof By lemma 1, ION contains no thesis $CxNx$. Such a thesis could be obtained in IlN only by a transformation of $CxNy$ or $CyNx$ in ION, where x is a transformation of y . But this is impossible, since all transformations preserve length, and, by lemma 1, if $CxNy$ ($CyNx$) is in ION, x is longer (shorter)

than y . It follows also that if any thesis $CxNy$ is in $I1N$, where $x \neq Ny$, then x is either longer or shorter than y . Hence no transformation will yield $CxNx$ ⁱⁿ ~~or~~ $I2N$. The same goes for $I3N$, $I4N$, etc. A similar argument applies in the case of the thesis $CNxx$.

It has now been shown that the systems $I1N$, $I2N$, etc. contain no instances of a proposition's implying its own negation. But these systems are still of rather limited interest, since they do not unrestrictedly satisfy the rule of modus ponens. For example, both Cpp and $CCppNCCppNCpp$ are theses of $I1N$, but $NCCppNCpp$ is not. However, it can be proved that, with the exception of $I1N$, they satisfy the rule of modus ponens for positive theses. In this section, a positive formula will be defined as a formula whose total number of prenex N's is zero or even, where a prenex N is initial or preceded only by N's. A negative formula has an odd total of prenex N's. Thus if x and Cxy are theses of any system IiN , $i > 1$, and x and y are both positive, then y is a thesis of IiN . This will be proved in two stages - first a weaker lemma for the system $I1N$, and then the full theorem for $I2N$. The proof for $I2N$ can be extended by induction to any other of the systems.¹

Lemma 3

If x and Cxy are theses of $I1N$, and if x and y are positive, then y is a thesis either of $I1N$ or of $I2N$.

Proof

There are four main cases.

Case 1.

x and Cxy are both theses of $I0N$. Then, since $I0N$ contains modus ponens, y will be a thesis of $I0N$, and hence of $I1N$.

Case 2.

x is in $I0N$ but Cxy is not.

¹(Added in Proof) The proofs of lemma 3 and the theorem below are not satisfactory, but I do not at present see how they should be altered.

Case 2.1

Cxy is transformed from $CNyNx$ in ION. But then (lemma 1) $x = y$, and y is a thesis of both ION and ILN.

Case 2.2

Cxy is transformed from $CNyNx'$, where $x \neq NNx'$. But then (lemma 1) $y = x'$, and hence y can be gotten in ILN since $x = NNy$ is in ION.

Case 2.3

Cxy is transformed from $CNy'Nx$, where $y \neq NNy'$. But then (lemma 1) $y' = x$, hence $y = NNx$ can be gotten in ILN.

Case 2.4

Cxy is transformed from $CNy'Nx'$, where $x = NNx'$ and $y = NNy'$. But then $y' = x'$ i.e. $NNy' = NNx'$, i.e. $y = x$, and hence y is in both ION and ILN.

Case 2.5

Cxy is transformed from Cxy' in ION, where y is a transformation of y' . But then y is in ILN.

Case 3.

x is not in ION but Cxy is. Then x is a transformation of x' in ION. But, since transformations are two-way equivalences, if there is a transformation which transforms the thesis x' of ION into the thesis x of ILN, the same transformation will transform the thesis Cxy of ILN into a thesis $Cx'y$ of ION. But then the reverse transformation of $Cx'y$, namely Cxy , would not be in ION, contrary to hypothesis. Hence case 3 is impossible.

Case 4.

Neither x nor Cxy is in ION.

Case 4.1

Cxy comes from $CNyNx$ in ION. But then (lemma 1) $x = y$, and y is a thesis of ILN.

Case 4.2

Cxy comes from $CNyNx'$, where $x = NNx'$. But then (lemma 1) $y = x'$ and hence y can be gotten in ILN since $x = NNy$ is in ILN.

Case 4.3

Cxy comes from $CNy'Nx$, where $y = NNy'$. But then $y' = x$, and hence $y = NNx$ can be gotten in ILN.

Case 4.4

Cxy comes from $CNy'Nx'$, where $x = NNx'$ and $y = NNy'$. But then $y' = x'$, i.e. $NNy' = NNx'$, i.e. $y = x$, and hence y is in ILN .

Case 4.5

Cxy comes from $Cx'y$ in ION , and x comes from x'' in ION , where x is a transformation of x' and x'' . But if there is a transformation which transforms the thesis $Cx'y$ of ION into a thesis Cxy of ILN , the same transformation will transform the thesis x of ILN into a thesis x' of ION . Hence x' is in ION , and hence, by modus ponens, y is in ION . Therefore y is in ILN .

Theorem

In $I2N$, where x and y are positive, $\vdash x, \vdash Cxy \rightarrow \vdash y$.

Proof

Similar to that of lemma 3.

Case 1.

x and Cxy are both theses of ILN . Then, by lemma 3, y is in $I2N$.

Case 2.

x is in ILN but Cxy is not.

Case 2.1

Cxy is transformed from $CNyNx$ in ILN . But $CCNyNx$ is in ILN , and two applications of modus ponens give y as a positive thesis of ILN and/or $I2N$.

Case 2.2

Cxy comes from $CNyNx'$, where $x = NNx'$. But $CCNyNx'Cxy$ is in ILN and two applications of modus ponens give y .

Cases 2.3 and 2.4. Cxy comes from $CNy'Nx$ or $CNy'Nx'$, where $x = NNx'$ and $y = NNy'$. But $CCNy'Nx$ and $CCNy'Nx'Cxy$ are ⁱⁿ _{$\wedge$} ILN , and modus ponens gives y .

Case 2.5

As in lemma 3.

Case 3.

As in lemma 3.

Case 4.

Neither x nor Cxy is in ILN . Then x comes from x'' in ILN , where x'' is positive.

Case 4.1 Cxy comes from CNyNx in I1N. But CCNyNx Cxy is in I1N, hence Cxy is in I1N, hence Cx'y is in I0N, hence by lemma 3, y is in I1N.

Cases 4.2 - 4.4. As in cases 2.2 - 2.4, resulting in Cxy being in I1N. But then (case 4.1) y is in I1N, hence in I2N.

Case 4.5 Cxy comes from Cx'y in I1N. As in lemma 3.

This completes the proof that, in any system IiN, $i \neq 1$, modus ponens holds for positive theses. We now add the unrestricted rule of modus ponens to each of these systems, resulting in the addition of certain negative theses to each. However, the addition of these negative theses will not bring about, by modus ponens, the addition of any new positive theses, since the system I1N, and indeed the system CN3, in which all the systems here being considered are contained, does not permit any theses of the form CNxCyz where x is positive (see section 33). Hence no system IiN, enriched by modus ponens, contains any thesis of the form CxNx or CNxx.

It remains to assemble the totality of theses derivable in all the systems IiN plus modus ponens, and to construct an axiomatic system which exactly coincides with this totality. The system, named INN, comprises as its basis the rules of substitution and modus ponens, and the following axioms:

1. CCpqCCqrCpr
2. CCpCpqCpq
3. CCCppqq
4. CCNpqCNqp
5. CpNNp

It is easy to show that INN contains all the theses of every system IiN plus modus ponens, since INN contains the rule for the replacement of

equivalents in the form

$$\begin{aligned} \vdash Cxy, \vdash Cyx &\rightarrow \vdash CCxzCyz \\ &\rightarrow \vdash CCzxCzy \\ &\rightarrow \vdash CNxNy \end{aligned}$$

and also contains the equivalences (1)-(4) above in the form of mutual implications. To show that the totality of the theses of every system $I \cap N$ plus modus ponens contains INN , note that this totality includes all the theses of I , and that $CCNpqCNqp$ and $CpNNp$ are theses of $I \cap N$, so that the totality contains INN . Hence INN also has the property that it contains no thesis of the forms $CxNx$ or $CNxx$. Hence I can serve as a basis for CC .

60. The system INC

It was seen in the last section that the system INN is connexive in that it contains no instances of a proposition's implying its own negation. Hence we may give formal recognition to this fact by adding Aristotle. The system INN will actually bear more than this addition with consistency, and the connexive system INC constructed here will include more non-classical theses than Aristotle.

If INN contains no instances of $CxNx$ or $CNxx$, it will contain no pairs of theses Cxy , $CxNy$ or Cxy , $CNxy$. That is, in INN a proposition never implies both of two contradictory propositions, and no proposition is implied by both of two contradictory propositions. This is so because $CxNy$ becomes $CyNx$ by transposition, and this together with Cxy gives $CxNx$ by Syl. Hence, given any thesis Cxy of INN , the thesis $NCxNy$ can be consistently added in the proven absence of $CxNy$. The system INC will therefore be INN strengthened not by the thesis but by the rule of Boethius; i.e. the system composed of the following:

<u>Rules</u>	(1)	Substitution
	(2)	<u>Modus ponens</u> : $\vdash x, \vdash Cxy \rightarrow \vdash y$
	(3)	Boethius: $\vdash Cxy \rightarrow \vdash NCxNy$
<u>Axioms</u>	1.	CCpqCCqrCpr
	2.	CCpCpqCpq
	3.	CCCppqq
	4.	CCNpqCNqp
	5.	CpNNp

Since the rule of Boethius generates nothing but negative theses, these will be the only additions made by INC to INN.

With INC we have achieved a system of the type we were seeking, namely a consistent non-classical connexive logic which is not at the same time an equivalential logic. Hence the purpose of this chapter has been achieved. The next chapter will discuss, besides peripheral matters, some further problems concerning the strengthening of INC into a connexive calculus that is strongly complete, possible departures toward different families of CC's, and the question of conjunction.

CHAPTER SIX

FURTHER PROBLEMS

61. IMNC, an extension of INC

In the next two sections ways and means of strengthening INC will be discussed, the aim being a connexive calculus that is strongly complete. As will be seen, there are two steps which can be taken toward this completion, but the second of these, discussed in section 62, may be undesirable.

The essential factors involved in the proof of section 59, that the system INN contained no instances of a proposition's implying its own negation, were (a) that the pure implicational base I contained no theses in which any variable occurred only once, and (b) that the addition of negative theses to the systems ILN, I2N, etc., by the use of modus ponens created no new positive theses. Any stronger system satisfying these conditions may be shown to be a system of connexive implication in exactly the same way.

At least one strengthening of I satisfies condition (a), namely that no thesis of it contains any singly-occurring variable. The sequenzenkalkül version of this system is formed by adding the rule M below to the rules C, W, CL and CR of LI (see section 58):

$$\begin{array}{c} \alpha \vdash x \quad \beta \vdash x \\ \text{M} \quad \hline \alpha, \beta \vdash x \end{array}$$

thus obtaining the system LIM.¹ The elimination theorem, rendering the

¹Much the same effect could also be obtained by the rule:

$$\begin{array}{c} \alpha, x \vdash y \\ \hline \alpha, x, x \vdash y \end{array}$$

A version of the rule M, called 'mingle', is to be found in M. Ohnishi and K. Matsumoto, A system for strict implication, Proceedings of the symposium on the foundations of mathematics, Katada, Japan, 1962, pp. 99-108.

fusion rule FS otiose, is provable for LIM along the same lines as it is proved by Belnap and Wallace for their system $LE_{\bar{I}}$ (see Belnap and Wallace, DP, pp. 5-11. The elimination theorem for LI is simpler, its sequents being all singular on the right, but exactly the same cases arise.) There are only two additional cases which arise for LIM, which are dealt with as follows, using Belnap's and Wallace's numbering, definition of rank, etc.

Case 2.1.1 The rank on the left is greater than one, and the left premiss in the fusion comes by a structural rule.

Case 2.1.1.3 The left premiss comes by M. We have:

$$\frac{\frac{\alpha \vdash x \quad \beta \vdash x}{\alpha, \beta \vdash x} M \quad \gamma \vdash y}{\alpha, \beta, \gamma^* \vdash y} FS$$

We transform this to the following, where the rank of each fusion is less than the original rank:

$$\frac{\frac{\frac{\alpha \vdash x \quad \gamma \vdash y}{\alpha, \gamma^* \vdash y} FS \quad \frac{\beta \vdash x \quad \gamma \vdash y}{\beta, \gamma^* \vdash y} FS}{\alpha, \gamma^*, \beta, \gamma^* \vdash y} M \quad \text{various permutations and contractions}}{\alpha, \beta, \gamma^* \vdash y}$$

Case 2.2 The rank on the left is equal to one, and the rank on the right is greater than one.

Case 2.2.1.3 The right premiss comes by M. We have:

$$\frac{\frac{\beta \vdash y \quad \gamma \vdash y}{\beta, \gamma \vdash y} M \quad \alpha \vdash x}{\alpha, \beta^*, \gamma^* \vdash y} FS$$

which occurrences of x may be deleted in β , in γ , or in both. Suppose x 's are deleted only in β . Then the

transformation is:

$$\frac{\alpha \vdash x \quad \beta \vdash y}{\alpha, \beta^* \vdash y} \text{ FS}$$

$$\frac{\alpha, \beta^* \vdash y \quad \gamma \vdash y}{\alpha, \beta^*, \gamma \vdash y} \text{ M}$$

Similarly if x's are deleted only in γ . If x's are

deleted in both β and γ we have the following transformation:

$$\frac{\alpha \vdash x \quad \beta \vdash y}{\alpha, \beta^* \vdash y} \text{ FS} \quad \frac{\alpha \vdash x \quad \gamma \vdash y}{\alpha, \gamma^* \vdash y} \text{ FS}$$

$$\frac{\alpha, \beta^* \vdash y \quad \alpha, \gamma^* \vdash y}{\alpha, \beta^*, \alpha, \gamma^* \vdash y} \text{ M}$$

$$\frac{\alpha, \beta^*, \alpha, \gamma^* \vdash y}{\alpha, \beta^*, \gamma^* \vdash y} \text{ various permutations and contractions}$$

Proof of the elimination theorem for LIM means that that system, defined by the rules C, W, CL, CR and M, satisfies the rule of modus ponens. We now prove the following lemma about LIM:

Lemma In no thesis of LIM does any propositional variable occur less than twice.

Proof As for the lemma of section 58. The rule M provides no way of bringing together in the same prosequent the two variables from a single axiom, and so the previous proof stands.

No interpretation of the system LIM in terms of the theses of an axiomatic system will be given, and so no axiomatic version of LIM will be provided. The thesis CPCPP, however, is provable in LIM, and so LIM will at least contain the axiomatic system formed by adding CPCPP to I. This yields the system IM, with axioms:

1. CCpqCCqrCpr
2. CCpCpqCpq
3. CCCppqq
4. CPCPP,

and it is in fact conjectured that IM is identical with LIM. This conjecture is made more plausible by the fact that IM has a very natural subproof formulation IM^* , formed by adding to the rules hyp, rep, reit, MP and CP of I^* the new rule plus:

plus From items (a)x and (b)x in subproof n derive
 (a+b)x in subproof n.

For proof of the identity of IM and IM^* , see appendix 3. IM has the following properties.

(a) It is stronger than I, matrix 4 of section 21, adequate for I, rejecting the fully written-out form CCpqCCpqCpq of CPCPP for $p = 2$, $q = 3$.

(b) It satisfies Anderson's and Belnap's criterion of relevance (i.e. variable-sharing). This is shown by the fact that IM is satisfied by the 4 x 4 matrix used by Anderson and Belnap as a relevance-matrix for I.¹ (This matrix, put in standard form, is

Matrix 22

	C	1	2	3	4
* 1	1	1	4	4	4
* 2	1	1	2	3	4
3	1	2	2	2	4
4	1	1	1	1	1

If, in Cxy, x and y share no variable, give all the variables in x the value 1, and all those in y the value 2, yielding C12 = 4.)

(c) It avoids fallacies of necessity, as is shown by the fact that IM is satisfied by the necessity-matrix 5 of section 21.

Thus IM possesses all the paradox-avoiding virtues of the system I, and is at the same time stronger than I. It differs from I mainly in not being contained in WC, but leaning rather on the side of

¹ Anderson and Belnap, PCE, p. 49.

C3, of one of whose characteristic theses CPCQP IM's axiom 4 is a substitution.

IM being contained in C3, IM plus strong negation will be contained in CN3, and therefore retain CN3's property of not including as theses any instances of a negative formula implying a positive one ('negative' and 'positive' in the sense of section 59). Hence IM plus strong negation satisfies condition (b) of the beginning of this section, and the whole argument of section 59 can be repeated for IM, yielding a proof that IMN is connexive, i.e. that the system IMNC, formed by adding CPCPP as a sixth axiom to the system INC of section 60, is consistent. IMNC is thus a consistent strengthening of INC. It has a subproof formulation $IMNC^*$ consisting of the rules hyp, rep, reit, MP, CP, plus, MT, DNI, DNE and a further rule B_1 :

B_1 From $\wedge Cxy$ derive $\wedge NCxNy$ in the main proof only.

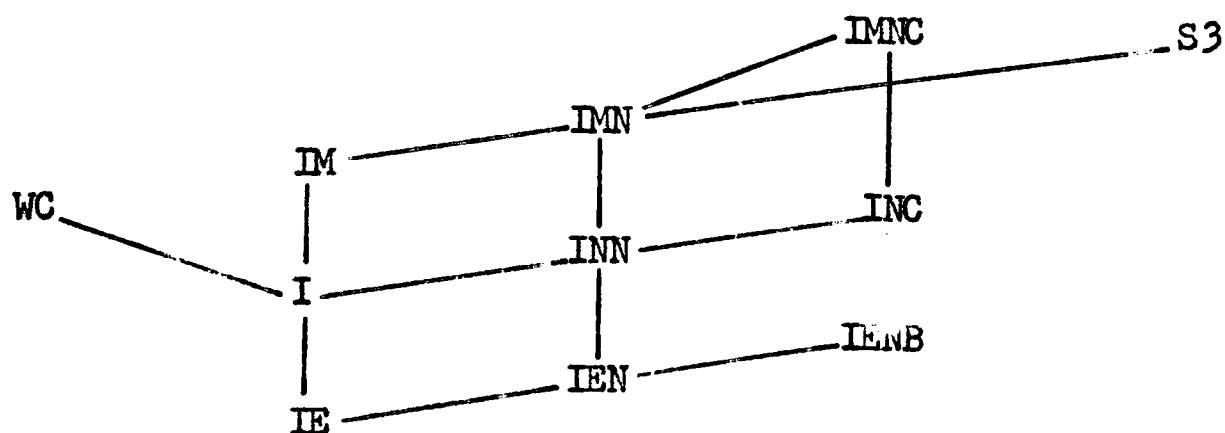
62. Further extensions of INC

In section 56 it was mentioned that the system IENB could be strengthened by the addition of the formula CCppCqq. Once IENB is strengthened to the level of INC, however, this possibility is no longer open. If Cpp and Cqq are to be equivalent, the substitution CCqCqqCqq of Hilbert will yield CCqCcppCcpp, a thesis with a singly-occurring variable. From there it is merely a step to contradiction with INC, for CCqCcppCcpp yields by substitution CCNCppCcppCcpp, and this thesis has only to be inserted at line 17 of the subproof derivation of section 54 to yield an instance of CxNx. It is interesting that relevance (which CCppCqq violates), the absence of singly-occurring variables, and connexivity go hand in hand once the level of INN is reached, and it is conjectured that in C-N systems of this strength they are logically inseparable.

One other possibility of strengthening INC (or IMNC), this time consistently, lies in capitalizing on the fact that in these systems no negative implication ever implies a positive one. Hence a new axiom could be consistently added; the non-classical axiom NCNPQ. This axiom is not quite as implausible as it looks. Anderson and Belnap in fact support its truth, saying that no one ever argues in the form 'A doesn't follow from B, hence C follows from D'.¹ And INC already contains its substitution NCNPP. However, in spite of all this, and in spite of the fact that the addition of NCNPQ might conceivably make IMNC strongly complete, it would be preferable, I think, to seek to complete IMNC by classical additions. This would leave the rule of Boethius as the only non-classical element in the system, a state of affairs more welcome to traditionalists. But I do not see at the moment how to construct any further classical C-N strengthening of IMNC.²

To conclude this section, a map is given of containment relations among the different connexive systems so far discussed. Further strengthenings of these systems, i.e. a possible strongly complete CC, must lie somewhere between Church's calculus and S3.

Table 7



¹See their forthcoming book. However, an argument of this sort would seem to be quite in order if, for example, the premiss 'Either A follows from B, or C follows from D' had been accepted.

²Two additions suggest themselves - CCNppCqp and CNCpNpCpp - but these are only possibilities.

63. Other sorts of connexive calculus

It was noted in section 48 that, in the presence of Syl and strong negation, Aristotle plus Comm gave the equivalential thesis CCpqCqp. Also, in section 54, it was shown that in the same conditions Aristotle plus Clavius resulted in a contradiction. However, in this section it will be shown that, if the background conditions of Syl and strong negation are relaxed, Comm and Clavius can each form part of connexive calculi that are not equivalential.

Firstly, the following matrix shows that Syl plus Comm plus weak intuitionist negation yields a non-equivalential connexive system:

Matrix 23

	C	1	2	3	N
* 1	1	1	2	3	2
2	2	2	1	2	1
3	1	2	1	1	2

The following is a partial list of the formulae satisfied and the formulae rejected by this matrix (which also satisfies the rule of modus ponens). No axiomatization of the matrix is attempted.

<u>Satisfied</u>		<u>Rejected</u>
CCpqCCqrCpr	)	CCpCpqCpq
CCpCqrCqCpr	) System WE	CCpqCqp
CpCCpqq	) of	CCCpqpp
Cpp	) Section 55	CpCqp
CCpqCNqNp	)	CNNpp
CCpNqCqNp	) Intuitionist	CCNppp
CpNNp	) negation	CCpNpNp
CCpqNCpNq	Boethius	

Secondly, a matrix shows that not Syl but the rule of Syl plus strong negation plus Clavius yields a consistent connexive logic:

Matrix 24

	C	1	2	3	4	N
* 1	1	1	4	3	4	4
* 2	3	1	3	3	3	3
3	1	4	1	4	4	2
4	3	1	3	1	1	1

The rule of modus ponens holds for this matrix, and the following formulae are respectively satisfied and rejected:

<u>Satisfied</u>	<u>Rejected</u>
$\vdash Cxy \rightarrow \vdash CCyzCxz$	$CCpqCCqrCpr$
$CCpCpqCpq$	$CCpCQrCQCpr$
$CCCppqq$	$CPCCPqq$
$CCCpqpp$	$CCpqCqp$
Cpp	$CQCpp$
$CCNpNqCqp$)strong	$\vdash Cxy \rightarrow \vdash NCxNy$
$CpNNp$)negation	
$CCNppp$	
$NCpNp$	

It is conjectured that matrix 24 is adequate for a weakening of C2, the pure strict implicational fragment of S2, which is analogous to the weakening I of C3. I does not contain C3's (and C2's) thesis QQCpp, which is rejected by matrix 24, and matrix 24 satisfies C2's rule of Syl, $\vdash Cxy \rightarrow \vdash CCyzCxz$.

64. 4 x 4 matrices: a computational approach

Matrix 24 should be compared with matrix 21 of section 55.

Both are 4 x 4 consistent connexive matrices satisfying modus ponens, Cpp, CCCppqq and strong negation. The principal differences are that matrix 21 satisfies Syl and Weak Comm, and rejects Hilbert and Clavius, while matrix 24 satisfies Hilbert, Clavius and the rule of Syl, but rejects Syl and Weak Comm. This raises the question, whether a connexive matrix could not be found which satisfied the common elements of matrices 21 and 24 plus Syl, Weak Comm, Hilbert and the rule of Boethius. Such a matrix would be adequate for INC, and would make the elaborate consistency proof of section 59 unnecessary.

As far as 4 x 4 matrices of a type to be defined below are concerned, the answer to this question is in the negative. That is,

there is no 4 x 4 consistent matrix of this type adequate for INC. This result was obtained by computing all possible 4 x 4 matrices, and then showing that none of the required type had the required properties. Since the number of different 4 x 4 matrices is very large - there are 4^{16} or about 4 billion such matrices - an IBM 1410 computer was used. Although the result was negative, a sketch of the method employed will be given.

To begin with, the number of matrices dealt with was radically reduced by the following stipulations, these stipulations determining the type of 4 x 4 matrix sought for.

- (i) The values 1 and 2 were to be set aside as designated values.
- (ii) The column for negation was to be the 'mirror image' (4321) of the argument column 1234, thus satisfying the laws of double negation and allowing the matrix to be consistent.
- (iii) The matrix was to be symmetrical about the CpNp diagonal (see section 55), thus satisfying the laws of transposition.
- (iv) The matrix was to have the four values in its upper right quadrant undesignated, thus satisfying modus ponens.
- (v) The Cpp diagonal was to be designated.
- (vi) The CpNp diagonal was to be undesignated, thus rejecting every instance of CxNx.

These stipulations reduced the total number of matrices of the required type to those having the following configuration, where 'D'

and 'U' denote designated and undesignated values respectively:

	C	1	2	3	4	N
*	1	D		U	U	4
*	2		D	U	U	3
	3		U	D		2
	4	U			D	1

Furthermore, since the matrix was to be symmetrical about the CpNp diagonal, the 10 values of that diagonal and of one of the triangular sections cut off by it determined the matrix. Of these 10 values, each had 2 possibilities open to it except C12, C21 and C31, which had 4. Hence the total number of matrices of the required type was $2^7 \times 4^3 = 2^{13} = 8192$.

The first task of the computer, therefore, was to compute these 8192 matrices. Then it tested each one of them to see whether it satisfied the following formulae and rules:

1. Syl and rule of Syl
2. Weak Comm
3. Hilbert
4. Boethius and rule of Boethius.

The results were, that not one of the 8192 matrices satisfied each one of these conditions, hence not one was adequate for INC. About 400 simultaneously satisfied Syl, Weak Comm and Boethius, but, of these, all but two also satisfied CCpqCqp, and hence were equivalence matrices. The two that were not equivalence matrices were matrix 18 and its twin, matrix 19, of section 55. Hence, if nothing else, the investigation showed that Angell, presumably by chance, had lighted on the only 4 x 4 matrix, of the required type and possessing the required properties, that was not an equivalence matrix.

I am indebted to my student, Mr. Patrick Schindler, for working out the computer program. The results could be made of greater interest if (a) it were demonstrated that the satisfaction of strong

negation theses demand mirror image (Boolean) negation and symmetry about the CpNp diagonal, or, at least, that every matrix satisfying strong negation which was not of this type was transformable into one that was, and (b) the program were amended so as to allow the value 2 to be undesignated in some matrices. With these improvements, it might be possible to determine whether INC possessed any 4 x 4 adequate matrix, and not merely one of the type considered in this section. And then (if anyone wished to write the program) the investigation could be extended to larger matrices.

65. Connexive implication and class-inclusion

Select from among the classical tautologies in C, N, K and A all those whose principal operator is C, or C preceded by any number of N's, and which contain no other occurrence of C. Call these classical first degree formulae. Then, interpreting negation as complementation, conjunction as intersection and alternation as union, it is well-known that each classical first degree formula can be interpreted as an assertion or a denial of class-inclusion. What is less well-known is that the first degree formulae corresponding to the inclusion relations of non-empty classes are connexive implications.

For example,

$$\neg a \subset a$$

is the class-inclusion corresponding to the first degree formula CNNpp;

$$\neg(a \cup a) \subset a \cup a$$

that corresponding to CNpNpApNp; and

$$\neg(a - a) \not\subset a - a$$

that corresponding to NCNKpNpKpNp. All these are classical. But if we

restrict ourselves to non-empty classes, class-inclusions of the second of the above types cease to exist, while the third exemplifies a type of inclusion that is universally true:

$$-a \not\subset a$$

The latter corresponds to NCNpp. Hence the interpretation of first degree connexive formulae requires, like the Aristotelian inference from 'All a is b' to 'Some a is b', that empty classes be excluded.

The simplest way to exclude them, when we retain complementation (to which there is nothing corresponding in Aristotle's syllogistic), is to exclude intersection and union, for the most natural reading of a-a is as a name for the null class. We might retain intersection and union, while retaining complementation, but this would require that we reject the highly intuitive laws

$$ab \subset a$$

and $ab \subset b,$

for the latter yield

$$a-a \subset a$$

and $a-a \subset -a;$

the second of these gives by transposition

$$a \subset -(a-a);$$

and we get by transitivity

$$a-a \subset -(a-a).$$

In view of these considerations it would seem preferable simply to take no account of intersection and union, and to say that first degree connexive C-N formulae may be interpreted as the theory of inclusion and complementation of non-empty classes. A completeness result concerning

this theory has been obtained by Shepherdson, who offers the following five axioms and rules as a basis:¹

1. $a \subset \neg\neg a$
2. $\neg\neg a \subset a$
3. $a \not\subset \neg a$
4. From $a \subset b$ and $b \subset c$ derive $a \subset c$
5. From $a \subset b$ derive $\neg b \subset \neg a$.

It is plain that each of the first degree formulae corresponding to this theory is connexive. Shepherdson calls the non-empty, non-universal classes of the theory 'Aristotelian' classes, and, in view of axiom 3, it is particularly appropriate to say that first degree connexive formulae represent the theory of inclusion and complementation of Aristotelian classes.

66. Connexive implication and conjunction

As was seen in section 54, the introduction of conjunction into a system of connexive implication is a dangerous business, since the common conjunction theses CKpqp and CKpqq, which are found even in E, yield instances of CxNx by Trans and Syl. However, there is a way of introducing conjunction into CC (yielding CPC) while still preserving its consistency. We shall not, of course, have CKpqp, but, bearing in mind what was said about this thesis in section 24, its omission is precisely what sets CPC's somewhat stricter criterion of relevance apart from E's. This criterion of relevance will be precisely stated at the end of this section.

¹J.C. Shepherdson, On the interpretation of Aristotelian syllogistic, JSL 21(1956), p. 141.

The method of introducing conjunction will be to add two conjunction rules to the sequenzenkalkül LIM of section 61:

$$\begin{array}{l}
 \text{KL} \quad \frac{\alpha, x, y \vdash z}{\alpha, Kxy \vdash z} \\
 \text{KR} \quad \frac{\alpha \vdash x \quad \beta \vdash y}{\alpha, \beta \vdash Kxy}
 \end{array}$$

The system composed of the rules C, W, M, CL, CR, KL and KR will be known as LIK. An elimination theorem is provable for LIK along the same lines as those in Belnap and Wallace. Once again, as with LIM, the addition of the rules KL and KR provide no way of bringing together in the same prosequent the two variables from a single axiom, and so we have the following

Theorem In no thesis of LIK does any propositional variable occur less than twice.

We now select from among the theses of LIK all those which are in S3, obtaining an axiomatic system IK. The following is a conjectured axiomatic basis for IK: it is offered with no guarantee that it is complete, or that some of the axioms are not redundant. The non-S3 theses of LIK are omitted because S3's property of lacking any negative formula which implies a positive one is needed in order to demonstrate the connexivity of IK plus strong negation.¹ Axioms for IK:

¹However, it may well be possible to prove that LIK plus strong negation possesses this property by arguing from the absence of fallacies of necessity in that system. (In IKON below no negative formula implies a positive one because IKON contains no theses of the form CxCyz, where x is a propositional variable, and this property is preserved in IKiN, $i \geq 1$.)

1. $CCpqCCqrCpr$
2. $CCpCpqCpq$
3. $CCCppqq$
4. $CCpCqrCKpqr$
5. $CKpqKqp$
6. $CKppp$
7. $CCpqCCprCpKqr$
8. $CCCpppCCCqKpq.$

Of these axioms, 7 and 8 are in S_3 but not in E . 8 means that the rule of adjunction, $\vdash x, \vdash y \rightarrow \vdash Kxy$, is derivable in IK , because of I 's 'rule of necessitation' $\vdash x \rightarrow \vdash CCxxx$. Note the absence of IM 's axiom CPCPP: it is derivable from 8 and 6.

Strong negation is now added to IK in exactly the same way it was added to I . That is, systems $IKON$, $IK1N$, $IK2N$, etc. are progressively constructed after the model of section 59. Lemmas 1 and 2 hold for $IKiN$. In addition, disjunction is added to $IKiN$, $i \geq 1$, by means of the following equivalences:

- (5) $Axy = NKNxNy$
- (6) $ANxy = NKxNy$
- (7) $AxNy = NKNxy$
- (8) $ANxNy = NKxy$

with the stipulation that, whenever any formula NKxy appears in any of these systems, the corresponding disjunction should immediately be substituted for it in the same system. Similarly if any negation of a disjunction appears. This guarantees that conjunctions and disjunctions will always be positive in $IKiN$, so that lemma 3 and the theorem of section 59 will be applicable to them. Thus adding modus ponens to these systems will at most produce new negative theses which are negations of implications, and by matrix 12 and the conjunction part of matrix 13 (adequate for S_3) we see that these new negative theses will not imply any new (positive) implicational theses. The upshot is, that no system $IKiN$ contains any thesis of the forms CxNx or CNxx.

Collecting together all the theses of all the systems IKiN, and restoring negations of conjunctions in place of disjunctions, we obtain the system IKN, i.e. IK plus the usual axioms of strong negation. Since IKN is connexive, we add the rule of Boethius, obtaining a full (though not of course strongly complete) connexive calculus with conjunction that will be called IKNC:

1. CCpqCCqrCpr
2. CCpCpqCpq
3. CCCppqq
4. CCpCqrCKpqr
5. CKpqKqp
6. CKppp
7. CCpqCCprCpKqr
8. CCCpppCCCqqqKpq
9. CCNpqCNqp
10. CpNNp
- (11) Boethius: $\vdash Cxy \rightarrow \vdash NCxNy$.

IKNC is the last formal system that will be presented here, although it is very tempting to try and strengthen it by the addition of theses such as of the LIK thesis CCCpppCqKpq, which would enable the rule MTP, $\vdash x, \vdash \underline{ANxy} \rightarrow \vdash y$, the law of the antilogism CCKpqrCKpNrNq, and the law of contradiction NKpNp to be proved.¹ This ^{latter} strengthening, like IKN, has many conjunction theses which are stronger than E's, and yet continues to satisfy E's criteria of relevance and necessity. This is seen by studying the rules of LIK: proceeding by cases it is possible to show that in no derivable sequent does there fail to be variable-sharing across the turnstile except when the prosequent is empty. Again, the use of the rule CR reveals that, if the consequent is an implication, the prosequent is either empty or contains implications itself. Hence the theses of LIK satisfy E's standards of relevance and necessity, and the transformations of strong negation do not change this.

¹This would be possible if the programme sketched in the previous footnote were carried out.

Reference was made at the beginning of this section to a stricter criterion of relevance. This is the stipulation that no thesis should contain a singly-occurring variable; a loose piece, as Anderson and Belnap put it, which can be jiggled about while the rest of the theorem remains the same. The implication-negation part of E, though not E itself, satisfies this stricter criterion. LIK plus strong negation does too, through its avoidance of theses like CKpqp and, with the definition of disjunction, CpApq. To sum up, relevance-and-necessity systems exist which differ from E, being weaker in some respects and stronger in others, and these are connexive systems. Furthermore, these connexive systems satisfy a yet stricter criterion of relevance, prohibiting the presence of any singly-occurring variable.

67. Conclusion

The main purpose of this work has been to show that there exists a variety of implication, recommended by intuition, which is non-classical not merely in being more restrictive than classical implication, but in satisfying certain theses which are classically false. It has been shown that these theses are exceedingly venerable, dating back to Aristotle and Boethius, and in the writings of Sextus Empiricus is to be found a species of implication which fits them perfectly, although it is not known whether the theses and Sextus' implication have ever previously been thought of as related. It is not clear from Sextus' writings what ancient logician, if any, espoused the type of implication in question, but some indications point to Chrysippus as its author. Following Bochenski, we have named it connexive implication.

The theses of Aristotle and Boethius, which entail that no proposition can imply its own contradictory, have not suffered a very kind historical fate. This is especially true since the publication of Principia Mathematica, since which time the theses have almost always been regarded as erroneous by those who have discussed them. On the other hand, there is to be found in the literature a thin but persistent stream of examples, now from one context, now from another, which seem to point to their truth.

What was needed in these circumstances was a formal treatment of connexive implication, enabling its properties to be exhibited with as much exactness and rigour as those of material implication. This the present work has gone some way to achieving: connexive implication has been shown to be captured in a consistent formal system that is strong enough to take its place among other well-known propositional calculi. Much still remains to be done, particularly in the matter of completing the connexive calculus and in finding an algebraic interpretation of it.

Łukasiewicz, writing in 1930, compared his discovery of three-valued logic to that of non-Euclidean geometry. The analogy, however, is not exact, for non-Euclidean geometry is not, as Łukasiewicz's three-valued implication/negation system is, a sub-system of the classical system. Instead it is independent of it. The epithet 'non-classical', like 'non-Euclidean', should be reserved for an autonomous independent formal system, and this is what the connexive calculus aims to be.

APPENDICES

Appendix

1. Proof that there is no finite characteristic matrix for the C-N fragments of S3-5, S7 and S8.
2. A conjecture concerning the axiomatization of matrices.
3. A subproof version IM^* of IM.
4. Theses of IENB
5. Various deductions in I, WC, C3 and CN5.
6. Theses of M3V.

APPENDIX 1

Proof that there is no finite characteristic matrix for the C-N fragments of S3-5, S7 and S8

The proof consists in producing a C-N formula satisfied by any finite matrix which is adequate for CN3. It is then shown that this formula is not in CN5 or CN8, since there are infinite matrices for CN5 and CN8 which reject it. Hence no finite matrix can be characteristic for CN3-5, CN7 or CN8.

Take any n -valued matrix M_n adequate for CN3, and consider the following formula F_{n+1} :

$$CCp_1p_2NCp_1p_2CCp_1p_3NCp_1p_3CCp_2p_3NCp_2p_3\ldots$$

$$CCp_{n-1}p_{n+1}NCp_{n-1}p_{n+1}NCCp_n p_{n+1}NCp_n p_{n+1}.$$

In this formula every possible pair-wise combination of the $n+1$ distinct variables $p_1 \ldots p_{n+1}$ occurs either in one of the antecedents $A_{ij} = CCp_i p_j NCp_i p_j$ or in the consequent $NA_{n,n+1} = NCCp_n p_{n+1} NCp_n p_{n+1}$, and for each A_{ij} we have that $i < j$.

It is clear that F_{n+1} will be satisfied by M_n . Since M_n contains only n values, every assignment of values to F_{n+1} will result either in (a) the consequent $NA_{n,n+1}$ having the same value as $NCCppNCpp$ would have for some value of p , or in (b) at least one of the antecedents A_{ij} having the same value as $CCppNCpp$ would have. If (a), we note that $CA_{n-1,n+1}NCCppNCpp$ is a thesis of CN3, since $CQNCCppNCpp$ is (see section 53), and hence that $CA_{n-1,n+1}NA_{n,n+1}$ must receive a designated value. Since in CN3 any implication implies a thesis which is an implication (recall that $\vdash CPCQP$), we see eventually that F_{n+1} must receive a designated value.

If (b), we move the antecedent A_{ij} in question to the right, using Weak Comm, until it is next to the consequent, and then use Trans 2 to make its negation the consequent, reverting to case (a).

We must now show that F_{n+1} is not in CN5 or CN8. This requires that we produce matrices with more than n values (i.e. matrices which are potentially infinite-valued) which reject F_{n+1} .

For CN5 the matrix needed is what Dugundji calls a 'Henle matrix', namely a Boolean $\underline{N-K}$ matrix with values $1, 2, \dots, m$, $m > n$, with M_p taking the value m when p is m , and the value 1 otherwise.¹ We convert this matrix into a $\underline{C-N-K}$ matrix by defining C_{pq} as $NM_k p N_q$, i.e. as $LNK_p N_q$. For example, consider the four-valued Boolean matrix for material implication and negation formed by multiplying the 2×2 classical matrix by itself:

Matrix 25

	C	1	2	3	4	N
*	1	1	2	3	4	4
	2	1	1	3	3	3
	3	1	2	1	2	2
	4	1	1	1	1	1

We convert this into a Henle matrix for strict implication by changing all the undesigned values into 4's (since $LNK_p N_q = 1$ when $NK_p N_q = 1$; otherwise $LNK_p N_q = m$). This gives

Matrix 26

	C	1	2	3	4	N
*	1	1	4	4	4	4
	2	1	1	4	4	3
	3	1	4	1	4	2
	4	1	1	1	1	1

A similar transformation can be used for a Boolean matrix of any number of values, and the resulting Henle matrix is adequate for CN5 (see Lewis and Langford, p. 501).

¹See Lewis and Langford, p. 501.

To show that an $n+1$ -valued Henle matrix rejects F_{n+1} , assign each variable p_i in F_{n+1} the value i . Each antecedent A_{ij} receives the value $C(n+1)N(n+1) = C(n+1)1 = 1$, since $i < j$, and the consequent the value $N(1) = n+1$. Hence F_{n+1} is not satisfied.

To show that F_{n+1} is not in CN8, we need what we shall call a 'Parry matrix', the four-valued version of which is the matrix, discovered by Parry, which distinguishes S3 from S4. An $n+1$ -valued Parry matrix is formed from an $n+1$ -valued Henle matrix by substituting the value 2 for the value 1 throughout the non-negative part of the matrix, and designating the value 2. Thus the four-valued C-N Parry matrix is:

Matrix 12

	C	1	2	3	4	N
*	1	2	4	4	4	4
*	2	2	2	4	4	3
	3	2	4	2	2	2
	4	2	2	2	2	1

It is conjectured (though not proved here) that every Parry matrix is adequate for CN3, the proof following Henle's proof that every Henle matrix is adequate for CN5. It is easy to see that every $n+1$ -valued Parry matrix also satisfies S8's characteristic thesis LMMp, since LMMp is CCNCpNpCpNpNCNCpNpCpNp, whose antecedent CNCpNpCpNp always takes the value $n+1$. Exactly the same argument as for the Henle matrix shows that the $n+1$ -valued Parry matrix rejects F_{n+1} .

APPENDIX 2

A conjecture concerning the axiomatization of matrices

Problem: to axiomatize any finite implication or implication/negation matrix satisfying the rule of modus ponens.

Possible method (outlined for implicational systems only). First, list all the non-equivalent formulae in one variable corresponding to the matrix ('C-p formulae'). For example, in classical two-valued logic there are two such formulae, p and Cpp. Second, construct the table (the 'C-p matrix') by which other C-p formulae may be reduced to the basic non-equivalent ones. In classical logic, the following is the table in question:

1. Cpp		C 1 2
2. p	*	1 1 2
		2 1 1

Thirdly, list the C-p theses required to reduce C-p formulae to the basic ones. Thus in classical logic CCppp (= C12) is equivalent to p (= 2) because of the theses CCl22 and C2C12. The following are the classical C-p reduction theses:

- | | | |
|----------------|-----|----------------|
| 1. CCCppCppCpp | and | 2. CCppCCppCpp |
| 3. CCCpppp | and | 4. CpCCppp |
| 5. CCpCppCpp | and | 6. CCppCpCpp |

Fourthly, take the most general theses, satisfied by the matrix, of which the C-p reduction theses are substitutions. For classical logic these are the following:

- | | |
|-----------------|-----------------|
| 1a. CCCppCqrCqr | 2a. CCpqCCqrCpr |
| 1b. CCCpqCrsCtt | 2b. CCqrCCpqCpr |
| 1c. CCCpqCqpCqp | 2c. CCpqCCrsCtt |
| | 2d. CCpqCCrsCpq |
| | 2e. CCpqCCrsCrs |
| 3a. CCCpppp | 4a. CpCCppq |
| 3b. CCCpppq | 4b. CpCCqrp |

5a. CCpCpqCpq

6a. CCpqCrCpq

5b. CCpCqrCss

6b. CCpqCrCss

We shall say that any set of axioms from which all of these generalizations of the C-p reduction theses can be derived by substitution and modus ponens is 'C-p complete', relative to the system.

It is to be noted that theses 1a - 6b include the following (or direct substitutions of them):

CCpqCCqrCpr

CpCqp

CCCpqpp,

and these three theses are known to provide a complete basis for classical implication. Hence we are led to offer the following:

Conjecture: Relative to a given implicational system, a set of axioms which is C-p complete is complete. I do not know how to prove this conjecture, except perhaps by showing that the supposition that **there** existed a true thesis in the system not derivable from the generalizations of the C-p reduction theses would lead to an absurdity - e.g. further C-p reductions. If true, the conjecture would provide a method of axiomatizing any finite matrix, since such a matrix permits only a finite number of irreducible C-p formulae. It would also provide a method of axiomatizing any system with a finite number of irreducible C-p formulae.

The conjecture holds true for (a) positive logic, (b) the implicational fragment C5 of S5, and, I think, (c) Łukasiewicz 3-valued implication.

(a) In positive logic the irreducible C-p formulae are Cpp and p, and the reduction theses include

$$CCpqCCqrCpr$$

$$CpCqp$$

$$CCpCpqCpq,$$

known to be complete for positive logic.

(b) In C5, which, incidentally, like positive logic, does not have a finite truth-value matrix, there are five irreducible C-p formulae, and the reduction theses include substitutions of the following complete basis for C5:

$$CCpqCCqrCpr$$

$$CqCpp$$

$$CCCPqPP.$$

(c) In the implicational part of Łukasiewicz 3-valued there are irreducible C-p formulae Cpp and p, and the most general reduction theses are all derivable from

$$CCpqCCqrCpr$$

$$CpCqp$$

$$CpCCpqq$$

$$CCCPqCqpCqp,$$

which I conjecture to be a complete basis for the system.

APPENDIX 3

A subproof version IM^* of IM

IM consists of substitution, modus ponens, and the axioms

1. $CCpqCCqrCpr$
2. $CCpCpqCp$
3. $CCCppq$
4. $CPCPP$

IM^* consists of the rules hyp, rep, reit, MP and CP of I^* , together with the rule plus:

plus From items (a)x and (b)x in subproof n derive (a+b)x in subproof n.

It is easy to show that IM^* contains IM . To show that IM contains IM^* we argue exactly as for the theorem of section 27, using in addition the following thesis of IM (easily derivable from axiom 4 and Syl),

T4. $CPCCCqPCqP$

and adding the following case:

Case 5. (a)y comes by plus from (b)y and (c)y, where $a = b+c$. We distinguish four subcases.

Case 5.1 m is in both b and c. Then (b)y' is (b-m)Cxy and (c)y' is (c-m)Cxy. Reiterate axiom 4 and use MP twice to get, successively, (b-m)CCxyCxy and (b-m+c-m)Cxy, i.e. (a-m)Cxy, i.e. (a)y'.

Case 5.2 m is in b but not in c. Then (b)y' is (b-m)Cxy and (c)y' is (c)y. Since m is not in c, (c)y must come by reit, and hence y must be an implication. Reiterate T4 and use MP twice to get (c+b-m)Cxy, i.e. (a-m)Cxy, i.e. (a)y'.

Case 5.3 m is in c but not in b. Like case 5.2.

Case 5.4

m is in neither b nor c. Then $(b)y'$ is $(b)y$ and $(c)y'$ is $(c)y$. But then y must come by reiteration and hence be an implication, so reiterate axiom 4 and use MP twice to get $(b \rightarrow c)y$. i.e. $(a)y'$.

APPENDIX 4

Theses of IENB

Theses T1-14 of section 56 are deduced from the following five axioms of IENB:

1. CCpqCCqrCpr
 2. CCCppqq
 3. CCNpqCNqp
 4. CpNNp
 5. CCppNCpNp
-
- 1=C1--6. CCCCqqCpqCCcpqrCCqqrCCpqCCcpqrCCqqr
 - 6=C1--7. CCpqCCcpqrCCqqr
 - 1=C2--8. CCrrCCCqqr
 - 1=C8--9. CCCCCqqrCCcpqrrCCrrCCCpqrr
 - 1=C9--62--10. CCCCCqqrCCcpqrrCCcpqrr
 - 1=C1--C10--11. CCCCpqqrCCqqrCCcpqrr
 - 1=C7--C11--12. CCpqCCcpqrr¹ (T1)
 - 1=C12--13. CCCCCpqrrCsrCCpqCsr
 - 1=C1--C13--14. CCsCCpqqrCCpqCsr
 - (14) 15. CCpCQrCQCpr
 - 15=C1--16. CCqrCCpqCpr
 - (2) 17. CCCCppCpCpCpCp
 - (2) 18. CCCppCpCpCp
 - 17=C18--19. Cpp (T7)
 - 3=C19--20. CNNpp
 - 16=C4--21. CCpqCpNNq (T12)
 - 16=C20--22. CCpNNqCpq (T13)
 - (3) 23. CCNNpqCNqNp

¹I am indebted to Mr. Patrick Schindler for the proof given here of this thesis.

- 1=C20--24. CCpqCNNpq
- 1=C24--C23--25. CCpqCNqNp (T11)
- 1=C2--26. CCqqCCCppqq
- 15=C26--27. CCCppqCCqqq
- 16=C27--28. CCCpqCCppqCCpqCCqqq
- (1) 29. CCCpppCCpqCCppq
- 1=C29--C28--30. CCCpppCpqCCqqq
- 15=C30--31. CCpqCCCpppCCqqq (T2)
- 1=C25--C31--32. CCpqCCCNqNqNqCCNpNpNp (T3)
- 31=C4--33. CCCpppCCNNpNNpNNp (T4)
- 31=C20--34. CCCNNpNNpNNpCCppp (T5)
- (16) 35. CCNqNpCCrNqCrNp
- 1=C25--C35--36. CCpqCCrNqCrNp (T9)
- 1=C2--37. CCqrCCCqqqr
- 16=C37--38. CCpCqrCpCCCqqqr
- 1=C38--C15--39. CCpCqrCCCqqqCpr (T8)
- 16=C25--40. CCpCqrCpCnrNq
- 1=C40--C39--41. CCpCqrCCCNrNrNrCpNq (T10)
- (36) 42. CCpqCCpNqCpNp
- 1=C42--C25--43. CCpqCNCpNpNCpNq
- 39=C43--44. CCCNCpNpNCpNpNCpNpCCpqNCpNq
- 31=C5--C18--45. CCNCpNpNCpNpNCpNp
- 44=C45--46. CCpqNCpNq (Boethius)
- 31=C46--47. CCCCPqCpqCpqCCNCpNqNCpNqNCpNq (T6)
- 16=C46--48. CCpCqrCpNCqNr (T14)

APPENDIX 5

Various deductions in I, WC, C3 and CN5

5.1 Theses of I

- | | | |
|----------------|-----------------|---------------------------|
| 1. | CCpqCCqrCpr |) |
| 2. | CCpCpqCpq |)axioms |
| 3. | CCCpqq |) |
| 4. | Cpp | (thesis 19 of appendix 4) |
| 5. | CCpCqrCCQr | (thesis 15 of appendix 4) |
| 1=C1--6. | CCCCqrCprCCpqs | |
| 6=C6--7. | CCpCqrCCsqCpCsr | |
| 5=C1--8. | CCqrCCpqCpr | |
| 7=C8--9. | CCsCpqCCqrCsCpr | |
| (7) 10. | CCpCqrCCpqCpCpr | |
| 9=C10--C2--11. | CCpCqrCCpqCpr | |

5.2 Comm from Syl and Pon

- | | | |
|----------|-----------------|-------------------------|
| 1. | CCpqCCqrCpr | (Syl) |
| 2. | CpCCpqq | (Pon) |
| 3. | CCpCqrCCsqCpCsr | (thesis 7 of 5.1 above) |
| 3=C2--4. | CCsCpqCpCsq | (Comm) |

5.3 Theses of C3

- | | | |
|----------|-----------------------------|---------|
| 1. | CCpqCCqrCpr |) |
| 2. | CCpCpqCpq |)axioms |
| 3. | CQCqp |) |
| 1=C1--4. | CCCCqrCprCCpqs | |
| 4=C4--5. | CCpCqrCCsqCpCsr | |
| 5=C3--6. | CPCQP | |
| 1=C6--7. | CCCPQrCQr | |
| (4) 8. | CCCCqrCCqrrCCqrrCCCqrqCCqrr | |

- 8=C2--9. CCCqrqCCqrr
 (7) 10. CCCCPrPCCPrrCPCCPrr
 10=C9--11. CPCCPrr
 (11) 12. CCppCCCppqq
 3=C3--13. Cpp
 12=13--14. CCCppqq (Belnap)
 15. CCpCQrCQCpr (thesis 5 of 5.1)
 16. CCpCqrCCrsCpCqs (T23--thesis 9 of 5.1)
 6=C14--17. CCrrCCCppqq
 15=C17--18. CCCppqCCqqq (T6)
 15=C17--19. CCCqqqCCppq (T7)

5.4 Theses of CN5

1. CCpqCCqrCpr)
 2. CCpCQrCQCpr)
 3. CCCppqq)
 4. CPCqP)theses of CN4
 5. CCqrCCpqCpr)
 6. CCpCqrCCrsCpCqs)
 7. CCpqCNqNp)
 8. CCpNqCqNp)
 9. CCNpqCNqp)
 10. CCpCqRCqCpR (thesis of C5--see
 Lemmon et al., p.1)
 (4) 11. CQCNPQ
 1=C11--C9--12. CQCNPQ
 10=C12--13. CNQCQP
 (13) 14. CNQCQCCppNCpp
 1=C14--C2--15. CNQCCppCQNCpp
 1=C15--C3--16. CNQCQNCpp
 1=C16--C8--17. CNQCCppNQ
 (4) 18. CCppNQpCCppNQ
 1=C17--C18--19. CNQCpCCppNQ
 6=C19--C3--20. CNQCpNQ

APPENDIX 6

Theses of M3V

In section 52, an equivalence

$$(n) \quad X = Y \ \& \ Z$$

will be taken as consisting of the three implications

- (n.1) CXY
- (n.2) CXZ
- (n.3) $CYCZX.$

- 1. $CCpqCCqrCpr$
 - 2. $CCCPpp$
 - 3. $CQCp$
 - 4. $CCCPppCqpCqp$
-) axioms of M3V
)
)

$$(1) \ 5. \ CCpCpqCCCPppqCpq$$

$$1=C5--6. \ CCCCCppqCpqCpqCCpCpqCpq$$

$$5=C2--7. \ CCpCpqCpq \quad \text{(Hilbert)}$$

The axioms of C3 having now been obtained, certain theses of C3 will be listed, for the proof of which see appendices 5.1 and 5.3.

- 8. $CCqrCCpqCpr$ (2) of section 52
- 9. $CCpCqrCCpqCpr$
- 10. $CCpCqrCQCpr$ (4.1),(5.1),(5.2)
- 11. $CCCPQrCQr$ (6.2)
- 12. $CCCPppqCCqqq$ (16.1)
- 13. $CCCqqqCCppq$ (16.2)
- 14. Cpp (17),(26)
- 15. $CPCQP$ (18)
- 16. $CPCCPqq$ (19)
- 17. $CCCPppq$ (20)
- 18. $CCpCqrCCrsCpCqs$

The deductions now continue.

$$18=C1--19. \ CCCCprCCpqCCqrs$$

$$19=C2--20. \ CCCCpqrCCrpp$$

- 10=C19--21. CCpqCCCprsCCqrs
- (18) 22. CCCCpqppCCCPqqCCqpp
- 22=C2--23. CCCpqqCCqpp (13.1),(13.2)
- 1=C20--C23--24. CCCpqrCCprrr (6.1),(7.1),(8.1)
- 1=C8--C10--25. CCqrCPCCPqr
- (18) 26. CCCqrCPCCPqrCCCCPqrrCCqrCPr
- 26=C25--27. CCCCPrqrrCCqrCPr
- 1=C27--C10--28. CCCCPrqrrCPCCqrr (9.1)
- 8=C20--29. CCsCCpqrCsCCrpp
- 1=C29--C10--30. CCsCCpqrCCrpCsp
- (29) 31. CCCPCCqrsCCsqCPqCCPCCqrsCCCCPqss
- 31=C30--32. CCPCCqrsCCCCPqss
- (32) 33. CCPCCqrrCCCCPqrr (9.2)
- 8=C3--34. CCpQCpCp
- (4.2)
- 1=C34--35. CCpCprrCCpQr
- 1=C35--36. CCCCpQrrCCCPprrr (10.2)
- 1=C17--37. CCpqCCCpppq
- 1=C37--38. CCCCCpppqrrCCpqr
- 1=C38--39. CCCCpqrrCCCCpppqrr
- (28) 40. CCCCCpppqrrCCCPppCCqrr
- 1=C39--C40--41. CCCCpqrrCCCPppCCqrr (10.1),(11.1)
- (1) 42. CCpqCCqCqqCpCqq
- (3) 43. CCqqCp
- 8=C43--44. CCpCqqCpCp
- 18=C42--C44--45. CCpqCCqCqqCpCp
- (1) 46. CCCqCqqCpCpCCCCpCprrCCqCqqr
- 1=C45--C46--47. CCpqCCCPprrCCqCqqr
- 10=C47--48. CCCpCprrCCpqCCqCqqr

- 1=C48--C10--49. CCCpCpprCCqCqqCCpqr
- 18=C49--C16--50. CCCpCpprCCqCqqCCCCpqrrr
- 1=C50--C10--51. CCCpCpprCCCCpqrrCCqCqqr
- 10=C51--52. CCCCpqrrCCpCpprCCqCqqr (11.2)
- (15) 53. CCCpqCCqrsCCprCCpqCCqrs
- 1=C53--C10--54. CCCpqCCqrsCCpqCCprCCqrs
- 18=C54--C10--55. CCCpqCCqrsCCpqCCqrCCprs (14.1)
- 18=C8--56. CCCprsCCqrCCpqs
- (56) 57. CCCsCqrCCsqCsrCCpCqrCCspCCsqCsr
- 57=C9--58. CCpCqrCCspCCsqCsr (3)
- 58=C9--59. CCpCqCrsCCpCqrCpCqs
- 10=C59--60. CCpCqrCCpCqCrsCpCqs
- 60=C1--61. CCCpqCCqrCCprsCCpqCCqrs (14.2)
- 10=C37--62. CCCpppCCpqq (22)
- (2) 63. CCCpCpppp (21)
- (19) 64. CCCCpCppppCCpCppqCCqpp
- 64=C2--65. CCCpCppqCCqpp
- 1=C65--C23--66. CCCpCppqCCpqq (23)
- 1=C1--C10--67. CCPqCPCqrr
- (15) 68. CCCqrrCCqqCCqrr
- 1=C68--C33--69. CCCqrrCCCCqqqrr
- 18=C67--C69--70. CCPqCPCCCCqqqrr
- 1=C70--C10--71. CCPqCCCCqqqrCPr
- 18=C71--C16--72. CCPqCCCCqqqrCCCPrrr
- 1=C72--C10--73. CCPqCCCPrrCCCCqqqrr
- 10=C73--74. CCCPrrCCPqCCCCqqqrr

- 1=C74--C10--75. CCCPrCCCCqqqrCCPqr (7.3)
- (74) 76. CCCPrCCPQCCCCQQQrr
- 1=C37--77. CCCCCQQQrrCCQrr
- 18=C76--C77--78. CCCPrCCPQCCQrr
- 1=C78--C10--79. CCCPrCCQrCCPQr (6.3)
- (15) 80. CCCpppCQCCppp
- 18=C80--C17--81. CCCpppCQp
- 1=C81--82. CCCPqrCCCqqqr (7.2)
- 8=C8--83. CCCpCpPCQrCCpCpPCpQCpr
- 1=C83--C10--84. CCCpCpPCQrCCpQCCpCpPCpr
- 10=C9--C34--85. CCCpQCCpCpPCprCCpQCpr
- 1=C84--C85--86. CCCpCpPCQrCCpQCpr
- 10=C86--87. CCpQCCCPpPCQrCpr
- 1=C87--C16--88. CCpQCCCCCPpPCQrCprCprCpr
- 10=C88--89. CCCCCpCpPCQrCprCprCCpQCpr
- 1=C33--C89--90. CCCpCpPCCCQrCprCprCCpQCpr
- 8=C33--91. CCCpCpPCQCCrCprCprCCpCpPCCCQrCprCpr
- 1=C91--C90--92. CCCpCpPCQCCrCprCprCCpQCpr
- 10=C18--C23--93. CCCpCpPCQCCPrrrrCCpCpPCQCCrCprCpr
- 10=C69--94. CCCCCpppCpCCpqqq
- 10=C18--C94--95. CCCpCpPCQCCpppprCCpCpPCQCCPrrrr
- 1=C95--C93--96. CCCpCpPCQCCpppprCCpCpPCQCCrCprCpr
- 8=C10--97. CCCpCpPCCCpppCQrCCpCpPCQCCppppr
- (10) 98. CCCCCpppCCpCpPCQrCCpCpPCCCpppCQr
- 1=C98--C97--99. CCCCCpppCCpCpPCQrCCpCpPCQCCppppr
- 1=C99--C96--100. CCCCCpppCCpCpPCQrCCpCpPCQCCrCprCpr
- 10=C18--C23--101. CCCpCpPCQCCrCprCprCCpCpPCQCCPrrrr

10=C18--C94--102. CCCpCppCQCCCprrrCCpCppCQCCCpppr
1=C101--C102--103. CCCpCppCQCcrCprCprCCpCppCQCCCpppr
1=C100--C103--104. CCCCpppCCpCppCQrCCpCppCQCCCpppr
1=C104--C96--105. CCCCpppCCpCppCQrCCpCppCQCcrCprCpr
1=C105--C92--106. CCCCpppCCpCppCQrCCpCQpr
(10) 107. CCQCCCppprCCCpppCQr
8=C37--108. CCQCprCQCCCpppr
1=C108--C107--109. CCQCprCCCpppCQr
18=C109--C15--110. CCQCprCCCpppCCpCppCQr
18=C110--C15--111. CCQCprCCCpppCCppCCpCppCQr
18=C111--C10--112. CCQCprCCCpppCCpCppCCppCQr
1=C112--C106--113. CCQCprCCpCppCpCQr (4.3)
(15) 114. CCCCpppqCGppCCCCpppq
1=C114--C10--115. CCCCpppqCCCpppCGppq
1=C115--C15--116. CCCCpppqCGpCppCCCCpppCGppq
1=C116--C10--117. CCCCpppqCCCpppCCpCppCGppq
1=C117--C106--118. CCCCpppqCGpCppCpq
10=C118--119. CCpCppCCCCpppqCpq
18=C119--120. CCCpqrCCpCppCCCCpppqr
18=C120--C16--121. CCCpqrCCpCppCCCCCCpppqrrr
1=C121--C10--122. CCCpqrCCCCCCpppqrrCCpCpr
10=C122--123. CCCCCpppqrrCCCpqrCCpCpr
1=C33--C123--124. CCCCpppCCqrrCCCpqrCCpCpr
18=C124--C16--125. CCCCpppCCqrrCCGpqrCCCCpCprrr
1=C125--C10--126. CCCCpppCCqrrCCCCpCprrrCCCpqr (10.3)
(8) 127. CCCppQCCpCppCpQ
1=C15--C127--128. CQCCpCppCpQ

10=C128--129.	CCpCpCQCpQ	
18=C129--130.	CCCPQrCCpCpCQr	(8.2)
18=C86--C16--131.	CCCPpCpCQrCCpQCCCPrrr	
1=C131--C10--132.	CCCPpCpCQrCCCPrrCCpQr	
10=C132--133.	CCCPrrCCCPpCpCQrCCpQr	(8.3)
8=C37--134.	CCCPpCpCpCpCCpCpCCCPppq	(12.1)
58=C79--135.	CCsCCPrrCCsCQrCsCCPQr	
135=C16--C119--136.	CCpCpCpCCCPpCpCCCPpppCpCp	
10=C136--137.	CCCPpCpCCCPpppCpCCpCpCpCp	(12.2)
8=C43--138.	CCpCqCpCp	(15.1)
8=C43--139.	CCpCpCpCq	(15.2)

So far no use has been made of axiom 4. The following deductions employ it.

(4) 140.	CCCqCqCpCpCp	(24)
15=C140--141.	CCCPpppCqCqCpCpCp	
135=C141--C118--142.	CCCPpppCqCqCpCpCpCp	
18=C142--143.	CCCPqrCCCPpppCqCqCpCpCp	
18=C143--C16--144.	CCCPqrCCCPpppCCCCCqCqCpCpCprrr	
1=C144--C10--145.	CCCPqrCCCCCqCqCpCpCprrrCCCPpppqr	
10=C145--146.	CCCCCqCqCpCpCprrrCCCPqrCCCPpppqr	
1=C33--C146--147.	CCCqCqCCCPpCpCprrrCCCPqrCCCCpppqr	
1=C10--C147--148.	CCCPpCpCpCqCqCpCpCpqr ^C _A CCCPpppqr	
18=C148--C16--149.	CCCPpCpCpCqCqCpCpCpqrCCCCCpCpCprrr	
1=C149--C10--150.	CCCPpCpCpCqCqCpCCCCCpCpCprrrCCCPqr	
10=C150--151.	CCCCCpCpCpCpCpCpCqCqCpCpCprrr	
1=C33--C151--152.	CCCPpppCqCpCpCpCpCpCqCqCpCpCprrr	(11.3)
(1) 153.	CCqCpCpCq	

1=C4--C153--154. CCCpCppCqpCCprCqr
10=C154--155. CCprCCCPcppCqpCqr
1=C155--C24--156. CCprCCCPcppCqrCqr
10=C156--157. CCCpCppCqrCCprCqr

(25)

INDEX [by sections]

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