Stochastic PDEs and Weakly Interacting Particle Systems

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Statement of Originality

I declare that all work is my own, unless otherwise stated, explicitly cited or commonly known.

The main results of Part I were obtained as part of a joint work between myself and Fabian Harang of the University of Oslo.

Contents

Abstract

We study two problems relating to weakly interacting particle systems in the presence of exogenous noise. Part I presents a pathwise regularisation by noise result for McKean–Vlasov equations with singular interaction kernels. Using ideas from the theory of averaged fields and non-linear Young integrals we obtain well-posedness of the McKean–Vlasov systems, particle approximations and mean field convergence.

In Part II we study a family of semi-linear, convection-diffusion SPDEs that are closely related to PDEs coming from the theory of collision-less kinetics. We study these equations in the presence of additive space-time white noise. In one dimension we show global well-posedness and exponential ergodicity for an equation with a cubic non-linearity and repulsive sign choice. In two dimensions we show local well-posedness for a renormalised equation.

Chapter 1

Introduction

This thesis is presented in two parts, both concerned with studying the effect of randomness on dynamic models incorporating weak interactions. Part I contains a regularisation by noise result for McKean–Vlasov equations with singular interaction kernels. This result is obtained using recent approaches to path-by-path regularisation by noise for O/SDEs and the theory of non-linear Young equations, [35, 48, 64, 22]. We study both the particle dynamics and mean field approximations of weakly interacting systems and we obtain well-posedness and validity of the mean field limit for a perturbed McKean–Vlasov equation. In Part II we study a family of convection-diffusion SPDEs forced by additive space-time white noise and with non-local dependence in the convection term. Without the additive noise the associated PDEs are closely related to the mean-field diffusions studied in Part I. In this part we obtain global well-posedness and exponential ergodicity for a repulsive equation forced by additive space-time white noise in one dimension. We also obtain local well-posedness for the same equations in two dimensions and discuss the issues encountered in extending these results to global well-posedness. Precise statements of these results, with appropriate preliminaries, are presented at the start of Chapter 2 and Chapter 6 respectively.

By a weakly interacting system we mean a collection of dynamic particles, moving freely in space that influence each other by forces acting at a distance. Sometimes this paradigm is referred to as collision-less kinetics, in contrast to collisional kinetics, such as the Boltzmann model for an ideal gas. In a relatively general form, such systems can be described by the family of trajectories

$$\begin{aligned} \dot{X}^i(t) &= V^i(t), & \dot{V}^i(t) &= \frac{1}{N}\sum_{j=1}^{N} K(X^i(t), X^j(t)), \\ X^i(0) &= x_0^i, & V^i(0) &= v_0^i, \end{aligned} \tag{1.0.1}$$

where $K(x, y) \neq 0$ for $x \neq y$. A typical pairwise interaction is the Coulomb potential, $K(x, y) \sim \frac{x-y}{|x-y|^d}$. In this case, depending on a sign choice in front of the interaction, (1.0.1), describes either the Vlasov–Poisson model for plasmas or the Jeans–Vlasov model of galactic dynamics. Many variants of such models exist in the literature, including, but not limited to; incorporating additional interactions, coupling to other classical equations e.g. Vlasov–Maxwell or Vlasov–Einstein systems and including random fluctuations, [55, 54, 4, 74]. A common simplification of (1.0.1) is to consider the so-called infinite friction limit, where we assume that velocity equals acceleration. In this case we re-write the particle system in the form

$$\begin{aligned} \dot{X}^i(t) &= \frac{1}{N}\sum_{j=1}^{N} K(X^i(t), X^j(t)), \\ X^i(0) &= x_0^i. \end{aligned} \tag{1.0.2}$$

A canonical example of (1.0.2) is the system of vortex dynamics associated to the Euler equations in 2-dimensions, [85]. In this case the interaction is given by the Biot–Savart law, $K(x, y) \sim \frac{(x-y)^\perp}{|x-y|^2}$ where $(z_1, z_2)^\perp = (-z_2, z_1)$ and each X^i is assigned a weight a_i describing its rotation. We will focus on models associated with dynamics of the kind (1.0.2). However, many questions we address would also be relevant in the case of second order systems, (1.0.1).

It is also natural to include randomness in the dynamics of (1.0.1) and (1.0.2). This can be for the purpose of accounting for un-modelled background effects, errors in measurement or to study invariant behaviour of the dynamical system. Adding i.i.d Brownian trajectories to each particle in (1.0.2) results in the system of coupled stochastic differential equations,

$$\begin{aligned} \mathrm{d}X_t^i &= \frac{1}{N}\sum_{j=1}^{N} K(X_t^i, X_t^j)\,\mathrm{d}t + \mathrm{d}B_t^i, \\ X_0^i &= x_0^i. \end{aligned} \tag{1.0.3}$$

In [87], H. P. McKean elucidated a connection between the formal limit $N \to \infty$ of the (1.0.3) (see (1.0.4) below) and semi-linear parabolic equations. This work can be said to have started the now well-developed study of McKean–Vlasov diffusions, written in the general form

$$\begin{aligned} \mathrm{d}X_t &= b(t, X, \mu)\,\mathrm{d}t + \sigma(t, X, \mu)\,\mathrm{d}B_t, \quad \mu = \mathcal{L}(X), \\ X_0 &\sim x_0. \end{aligned} \tag{1.0.4}$$

McKean's idea was to draw on the formula obtained by R. Feynman and M. Kac, [77] to propose, and for some cases prove, a connection between (1.0.4) and non-linear parabolic PDEs of the kind

$$\partial_t u - \nabla\cdot(\sigma(u)\nabla u) - \nabla\cdot(u b(u)) = 0, \quad u|_{t=0} = \mathcal{L}(x_0). \tag{1.0.5}$$

Since these early works, this field has now developed into a broad subject which interacts with many areas of mathematics, as well as the physical and applied sciences. From a mathematical stand-point, the three perspectives: as particle systems, (1.0.3), as representative dynamics (1.0.4) and as a PDE, (1.0.5), lend the theory to being approached from multiple directions and with a variety of techniques. A non-exhaustive list of commonly considered questions in the theory are as follows:

(i) **Particle Dynamics:** Well-posedness, blow-up criteria, long time behaviour and calculation of statistical quantities of the system (1.0.3), or approximations thereof, [24, 58, 42, 38]. As an analytical tool, particle systems of this kind can also be used to simulate solutions of both (1.0.4) and (1.0.5). In this case the interaction and model parameters can be chosen more carefully so as to balance approximation accuracy and convergence rate with efficiency or other requirements, [102, 1].

(ii) **McKean–Vlasov SDE:** Well-posedness and behaviour of the distribution dependent diffusion, (1.0.4), [110, 36, 63]. Recently, in a number of works a pathwise approach to such problems has been developed, extending the theory of rough paths to distribution dependent problems, [6, 21]. As well as analysing the dynamics themselves, validity of the mean field approximation given by (1.0.3) is commonly of interest, [101, 44, 43, 28].

(iii) **McKean–Vlasov PDE:** As PDE problems in their own right, one is often concerned with well-posedness, regularity and long time behaviour for (1.0.5), [20, 16, 19]. In addition, establishing the connection between (1.0.3), (1.0.4) and (1.0.5) is often seen as an important goal. Establishing this link often intersects with typical questions from PDE theory. For example in the attractive Keller–Segel model on $\mathbb{R}^2$, it is known that if the interaction is multiplied by a coefficient above a given threshold, then the dynamics blow-up in finite time from all L^1 initial data with mass 1. In [24] it was shown that the particle system observes the same phase transition, with collisions between particles occurring almost surely in finite time when the parameter is above the same threshold.

The two topics presented in this thesis fit into this framework in the following way. In Part I we study the well-posedness and mean field approximation for a class of McKean–Vlasov equations and their associated particle systems, following a path-by-path regularisation by noise approach. In Part II, we study well-posedness and long-time properties of SPDEs related to semi-linear PDEs of the form (1.0.5). This latter result is presented in part as a first step in studying more general stochastic perturbations of such PDEs as well as studying invariant behaviours of these equations.

We conclude this introductory chapter by detailing some notation that we use throughout the thesis.

1.1 Notation

We use $\lesssim$ to indicate that an inequality holds up to a constant depending on quantities that we do not keep track of or are fixed throughout. When we do wish to emphasise the dependence on certain quantities, we either write, $\lesssim_{K,v}$ or define $C := C(\alpha, p, d) > 0$ and write $\leq C$. We allow these constants to change from line to line without further comment. We write $\cdot \vee \cdot$ for the maximum between two quantities and $\cdot \wedge \cdot$ for the minimum.

For mappings $f : \mathbb{R}^d \to \mathbb{R}$, we write $\nabla f = (\partial_1 f, \ldots, \partial_d f) \in \mathbb{R}^d$ for the gradient, the vector of partial derivatives and for mappings $g : \mathbb{R}^d \to \mathbb{R}^d$, we write $\nabla \cdot g = \sum_{i=1}^d \partial_i g_i$, for the sum of partial derivatives which defines the divergence. For vectors $k \in \mathbb{N}^d$ we use standard multi-index notation and for any $a \geq 0$ we write $C^a(\mathbb{R}^d)$ for the set of all functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $D^k f := \prod_{i=1}^d \partial_{k_i} f$ is continuous for all $|k| \leq a$. When $a = \infty$ we define $C^\infty(\mathbb{R}^d) := \cup_{a \geq 0} C^a(\mathbb{R}^d)$. We write $C^a_c(\mathbb{R}^d)$ for the set of C^a functions with compact support on $\mathbb{R}^d$ and $C^a_b(\mathbb{R}^d)$ for the set of bounded C^a functions. We write $\mathcal{S}(\mathbb{R}^d)$ for the space of Schwarz functions $f : \mathbb{R}^d \to \mathbb{R}$ such that for any $m \geq 0$ and $a \geq 0$ we have

$$\sup_{|k|=a} \sup_{x \in \mathbb{R}^d} |x^m D^k f(x)| < \infty.$$

This equips the space $\mathcal{S}(\mathbb{R}^d)$ with the structure of a Fréchet space and we write $\mathcal{S}'(\mathbb{R}^d)$ for its dual, the space of tempered distributions. When we wish to make the target space explicit in denoting a functions space, we write, for example $C(\mathbb{R}^d; \mathbb{R}^d)$

for the continuous maps $f : \mathbb{R}^d \to \mathbb{R}^d$, or $C([0,T];\mathbb{R}^d)$ for the set of continuous mappings $f : [0,T] \to \mathbb{R}^d$. When the context is clear we will sometimes drop the explicit dependence on domain and range, so as to lighten notation.

For $p \geq 1$, we write $L^p(\mathbb{R}^d)$ for the usual spaces of p-integrable real functions and when $p = \infty$, the space of essentially bounded functions. For a, $p \geq 1$, we write $W^{a,p}(\mathbb{R}^d)$ for the space of functions with p-integrable weak derivatives up to order a. For $p \in [1,\infty)$ we write ℓ^p for the set of sequences $(f_m)_{m=1}^{\infty}$ such that $\sum_{m=1}^{\infty}|f_m|^p < \infty$ and $\sup_{m\geq 1}|f_m| < \infty$ for $p = \infty$. In both parts we work predominantly in the scale of Hölder–Besov spaces, which we denote by $\mathcal{B}^{\alpha}_{p,q}$ for $\alpha \in \mathbb{R}$ and $p, q \in [1,\infty)$. We define these spaces slightly differently in each context, see Sections 2.4 and 5.2.

In Part II we work on the d-dimensional torus which we understand as the half open box $[0,1)^d$ with sides appropriately identified. As such $\mathbb{T}^d$ is a compact domain with no boundary and has volume 1. We extend the definitions above, writing $C^a(\mathbb{T}^d)$ for the spaces of periodic a-times differentiable functions and $\mathcal{S}'(\mathbb{T}^d)$ for the space of tempered distributions acting on $C^\infty(\mathbb{T}^d)$ and similarly for $L^p(\mathbb{T}^d)$, $W^{a,p}(\mathbb{T}^d)$ and other spaces of periodic functions defined in Part II.

Part I

Pathwise Regularisation of McKean–Vlasov Equations with Singular Potentials

Chapter 2

Introduction

A McKean–Vlasov equation is typically understood as a stochastic Itô differential equation whose coefficients are allowed to depend on the law of the solution itself. A general form of this type of problem is to solve

$$\begin{cases} dX_t = a(t, X, \mu)\, dt + b(t, X, \mu) dB_t, \\ \mu = \mathcal{L}(X), \\ X_0 = x_0 \in \mathbb{R}^d, \end{cases}$$

with appropriate mappings $a(t, X, \mu) \in \mathbb{R}^d$, $b(t, X, \mu) \in \mathbb{R}^d \otimes \mathbb{R}^d$, B_t a d-dimensional Brownian motion and $\mathcal{L}(X)$ denoting the law of the solution. Such models have found applications in a diverse range of subjects, including physics, biology, the social sciences and recently the modelling of neural networks, [74, 63, 18, 44, 56, 20, 45, 106]. As such, the study of such models constitutes both an important area of research as well as presenting interesting mathematical challenges.

In this part of the thesis we present a regularisation by noise approach for generalised McKean–Vlasov equations with additive noise and convolutional interaction. We write these equations in the form

$$\begin{cases} dX_t = (K * \mu_t)(X_t)\, dt + dB_t, \\ \mu_t = \mathcal{L}(X_t), \\ X_0 \sim \mathcal{L}(x_0), \end{cases} \tag{2.0.1}$$

where $K \in \mathcal{S}'(\mathbb{R}^d)$ is a possibly singular interaction kernel, B is a random path in $\mathcal{C}_T^\eta$ for $\eta > 0$ and x_0 is an $\mathbb{R}^d$ random variable. Closely related to (2.0.1) is the system of interacting particles, written for $i = 1, \ldots, N$, in the form

$$\begin{cases} dX_t^i = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} K(X_t^i - X_t^j)\, dt + dB_t^i, \\ X_0^i = x_0^i \sim \mathcal{L}(x_0), \end{cases} \tag{2.0.2}$$

where we assume either K is bounded, or $K = 0$ on the diagonal. In many situations one expects the particle system (2.0.2) to converge in law to a solution of (2.0.1). It is a classical result that this convergence holds when $K \in W^{1,\infty}(\mathbb{R}^d)$ and $(B_t^i)_{i=1}^N$ are i.i.d Brownian motions. An overview and discussion of the wider theory of McKean–Vlasov equations is given in the monograph of A-S. Sznitman, [101]. In many important situations, however, it is physically relevant and mathematically interesting to consider interactions that are less regular than Lipschitz, [44, 43, 45, 75]. Despite results in some specific cases, the threshold of $K \in W^{1,\infty}$ remains in general.

We show that given a distributional interaction kernel $K \in \mathcal{B}_{p,q}^\sigma(\mathbb{R}^d)$, with $\sigma \in (-\infty, 1)$, by including an additional, suitably rough term into equation (2.0.1), the well-posedness and mean-field approximation result can be recovered. Letting $t \mapsto Z_t \in C([0,T]; \mathbb{R}^d)$ be a given continuous path, such that $Z_0 = 0$, we consider the new problem

$$\begin{cases} \mathrm{d}X_t = (K * \mu_t)(X_t)\ \mathrm{d}t + \mathrm{d}B_t + \mathrm{d}Z_t, \\ \mu_t = \mathcal{L}(X_t - Z_t), \\ X_0 \sim \mathcal{L}(x_0), \end{cases} \tag{2.0.3}$$

with B, x_0 as above. We give a precise notion of a solution to this problem in Section 2.3 below. Defining the remainder process, $Y_t := X_t - Z_t$, we see that it formally solves

$$\begin{cases} \mathrm{d}Y_t = (K(\,\cdot\, + Z_t) * \mu_t)\,(Y_t)\ \mathrm{d}t + \mathrm{d}B_t, \\ \mu_t = \mathcal{L}(Y_t), \\ Y_0 \sim \mathcal{L}(x_0). \end{cases} \tag{2.0.4}$$

We mostly work with the process Y, which we show enjoys both better regularity and stability properties than X. These results constitute the regularisation by noise phenomenon in our context. We will see that the paths $t \mapsto Z_t$ for which we obtain a regularising effect are less than 1/2-Hölder continuous. As a result, we interpret the drift term as a non-linear Young integral, see Subsection 3.1.1.

The idea that noise can regularise certain ill-posed dynamics stems back to the early works of A. K. Zvonkin and A. J. Veretennikov, [113, 107], where PDE theory and Itô calculus were employed to obtain, among other results, strong well-posedness, in a probabilistic sense for SDEs of the kind

$$\mathrm{d}X_t = f(X_t)\ \mathrm{d}t + \mathrm{d}B_t, \tag{2.0.5}$$

with $f \in L^\infty(\mathbb{R}^d)$. This is in contrast to the Cauchy–Lipschitz theory for ODEs where the sharp threshold for well-posedness, requires $b \in W^{1,\infty}(\mathbb{R}^d)$. Since these early works, the subject has received a great deal of attention, with authors exploring similar results for time dependent drift, [82], methods based on Malliavin calculus, [88] and considering more general noise terms, [7]. Many of these topics as well as a discussion of selection by noise for ODEs and regularisation/selection by noise for PDEs are surveyed in [41]. In [35], A. M. Davie presented a pathwise approach to the same problem. This is in contrast to the other works mentioned, which are all based on probabilistic tools and involve averaging over the random perturbation. Davie established that for, (2.0.5) with $b \in L^\infty(\mathbb{R}^d)$ and almost every path B of the Brownian motion, there exists a unique continuous path $X \in C([0,T];\mathbb{R}^d)$ satisfying (2.0.5) in an integral sense. This notion of pathwise regularisation has recently been extensively developed, see [22, 64, 48, 49, 50, 52] for a selection of works in this direction. Our main result is presented from a pathwise perspective, and we give more background to the approach in Section 2.5. For now we give a brief intuition behind this regularising effect in the context of McKean–Vlasov equations.

Consider the particle system, in mean field scaling, associated to (2.0.4)

$$\begin{cases} \mathrm{d}Y_t^i = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} K(Y_t^i - Y_t^j + Z_t)\,\mathrm{d}t + \mathrm{d}B_t, \\ Y^i = x_0^i, \end{cases} \tag{2.0.6}$$

and let us assume that K is singular at the origin. Then, let $Z_t \in \mathbb{R}^d$ be fixed, we have $K(Y_t^i - Y_t^j + Z_t) \to \infty$ as $Y_t^i - Y_t^j + Z_t \to 0$. So the singularity in the equation remains. However, if $t \mapsto Z_t$ is path, whose trajectories oscillate sufficiently fast then we may imagine that the event $|Y_t^i - Y_t^j + Z_t| < \varepsilon$ occurs with a very small probability. If this probability decreases sufficiently fast in $\varepsilon \to 0$, compared with the blow-up rate of K near zero, then we may hope to show that the drift terms of (2.0.6) are suitably bounded. The theory of averaged fields and non-linear Young equations makes this idea rigorous and extends it to more general regularising process.

Regularisation by noise results for McKean–Vlasov equations and associated particle systems have been obtained in a number of specific settings. In [42] the authors demonstrate that the vortex dynamics associated to the Euler equations in two dimensions are globally well-posed from all initial configurations, when perturbed by suitable *turbulent* noise. This is in contrast to the well-posedness from almost every initial configuration obtained by C. Marchioro, M. Pulvirenti for the deterministic vortex system, see [85]. A selection by noise result for a system of Vlasov point charges in one dimension was obtained in [38]. The weak convergence result obtained in [44] can also be seen as a regularisation by noise phenomenon, as the same result does not hold for the corresponding deterministic system. Finally, we mention the work of V. Marx, [86], which in a very different direction, concerns regularisation by noise for McKean–Vlasov equations viewed as differential equations in the space of probability measures perturbed by the Wasserstein diffusion. To the best of our knowledge the result presented here is the first to consider a pathwise regularisation of McKean–Vlasov equations in the sense of Davie.

Methodologically, we focus primarily on Y, solving (2.0.4). Passing to the process X, solving (2.0.3), is carried out using the identity $X = Y - Z$. Concerning (2.0.4), we handle the random dynamics coming from B in a pathwise manner, using the framework of [27]. As a consequence we are free to make very few assumptions on the processes B, $(B^i)_{i=1}^N$. In particular we are able to exploit the trick of Tanaka, [103], allowing us to reformulate the mean field approximation result in terms of stability of the equation with respect to the noise. We highlight that this is only possible because we do not allow σ to depend on (X,μ). To handle the drift term $(K(\,\cdot\,+Z_t)*\mu_t)(Y_t)$, we extend the theory of non-linear Young integration, first presented in [22], to include measure dependent integrands. We present this extension in Section 3.1.1.

2.1 Structure and Outline

In the remainder of this chapter we detail the additional notation used in this part of the thesis, present the main results and discuss some useful preliminaries. In Section

2.5 we recap some ideas from the theory of path-by-path regularisation, averaged fields and non-linear Young integration. In section 2.6 we recall the definitions and some properties of the Wasserstein distances on the space of probability measures. We also define a notion of Hölder continuity for time dependent measure valued flows which is a central tool in proving our results. Section 2.4 contains a review of Hölder-Besov spaces and homogeneous distributions. Chapter 3 is entirely concerned with proving the three main results stated in Section 2.3. This is done in two steps. Firstly we consider the abstract non-linear Young equation (3.1.2) and then in Sections 3.1 and 3.2 we prove analogues of Theorems 2.3.2 and 2.3.5 for the abstract equation, which makes no reference to K or Z. Section 3.3 contains the proof of our main results, in which we relate the abstract theorems proved for (3.1.2) to the perturbed McKean–Vlasov problem (2.3.2). Finally, in Chapter 4 we discuss some specific models to which our result applies as well as some open questions for future work.

2.2 Notation

For p, $q \in [1, \infty]$ and $\alpha \in \mathbb{R}$ we write $\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^d)$ (resp. $\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^d; \mathbb{R}^n)$) to denote the associated spaces of real valued (resp. $\mathbb{R}^n$ valued) Hölder–Besov functions, equipped with the norm $\|\cdot\|_{\mathcal{B}^{\alpha}_{p,q}}$ which is defined in Section 2.4. We often simplify notation by writing $\mathcal{C}^{\alpha}(\mathbb{R}^d) := \mathcal{B}^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$ for any $\alpha \in \mathbb{R}$ and $d \geq 1$. For $\alpha \in \mathbb{R}$ we write $\mathcal{H}^{\alpha}(\mathbb{R}^d)$ (resp. $\mathcal{H}^{\alpha}(\mathbb{R}^d; \mathbb{R}^n)$) for the Hilbertian Sobolev spaces of real (resp. $\mathbb{R}^n$) valued functions, also defined in Section 2.4. For brevity we often drop the domain and range dependence when the context is clear.

For E a Banach space we write $\mathcal{C}^{\alpha}(\mathbb{R}_+; E)$, with $\alpha \in (0, 1)$, for the space of α-Hölder maps $X : \mathbb{R}_+ \to E$. For $T > 0$ and maps $X : [0, T] \to E$ we write $\mathcal{C}^{\alpha}_T E := \mathcal{C}^{\alpha}([0, T]; E)$ for these spaces and $\mathcal{C}^{\alpha}_{[s,t]} E$ for maps $X : [s, t] \to E$ with $[s, t] \subset \mathbb{R}_+$. When $E = \mathbb{R}^d$ we simply write $\mathcal{C}^{\alpha}_T$ (resp. $\mathcal{C}^{\alpha}_{[s,t]}$). For a mapping $X : [0, T] \to E$ and any $0 \leq s < t \leq T$ we write $X_{s,t} := X_t - X_s$ to denote the increment. With a slight abuse of notation, we will also denote by $X_{s,t}$ a two parameter function $X : [0,T]^2 \to E$. For $n \in \mathbb{N}$ we define the n-simplex Δ_n^T by

$$\Delta_n^T := \{(s_1, \ldots, s_n) \in [0,T]^n | \, s_1 \leq \ldots \leq s_n\}.$$

Then for $X : [0,T]^2 \to E$, when we say that $[X]_{\alpha;T} < \infty$, we mean that

$$[X]_{\alpha;T} := \sup_{(s,t)\in\Delta_2^T} \frac{\|X_{s,t}\|_E}{|t-s|^\alpha} < \infty.$$

To convert this into a proper norm we define $\|X\|_{\mathcal{C}_T^\alpha} := \|X_0\|_E + [X]_{\alpha;T}$. When $X_0 = 0$ by convention we always measure the path in $[\,\cdot\,]_{\alpha;T}$. When $\alpha = 0$ we write $C_T(E)$ (resp. $C_{[s,t]}(E)$) for the space of continuous mappings $[0,T] \to E$ (resp. $[s,t] \to E$). When $E = \mathbb{R}^d$ for concision we only write $\mathcal{C}_T^\alpha$, $C_[s,t]$ etc. For a space time function $\Gamma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, when the context is clear, for $\alpha, \gamma \in \mathbb{R} \times \mathbb{R}_+$, we use the notation

$$\|\Gamma\|_{\gamma,\alpha} := \|\Gamma\|_{\mathcal{C}_T^\gamma \mathcal{C}^\alpha}.$$

For $(E, \mathcal{E})$ a Hausdorff topological space which we always equip with its Borel sigma algebra we let $\mathcal{M}(E)$ denote the set of real valued, signed Radon measures on E and we write $\mathcal{P}(E)$ for the set of probability measures on E. For $\mu \in \mathcal{M}(E)$ there exists a pair of measures (μ^+, μ^-) such that at least one is finite, they have disjoint support and $\mu^+(A) \geq 0$ and $\mu^-(A) \leq 0$ for any Borel set $A \subseteq E$. Then we define the total variation of $\mu \in \mathcal{M}(E)$ by $|\mu| := \mu^+ - \mu^-$. Given an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable map $X : \Omega \to E$ we write $\mathcal{L}(X)$ to designate the law of X which is a probability measure on E. The law is defined to be the probability measure $\mu \in \mathcal{P}(E)$ such that the identity

$$\mathbb{E}[f(X)] = \int_E f \, \mathrm{d}\mu$$

holds for all $f : E \to \mathbb{R}$ continuous and bounded. Given two Banach spaces E, F, a Borel measurable mapping $\pi : E \to F$ and a measure $\mu \in \mathcal{M}(E)$, then we define the push-forward of μ by π to be $\pi\#\mu := \mu(\pi^{-1}\,\cdot\,) \in \mathcal{P}(F)$. For any measurable mapping $g : F \to \mathbb{R}$ such that $g \circ \pi : E \to \mathbb{R}$ is $\mathrm{d}\mu$ integrable then the push-forward satisfies

$$\int_F g \; \mathrm{d}\pi\#\mu = \int_E g \circ \pi \; \mathrm{d}\mu.$$

For any $p \geq 1$, and E a Banach space, we write $L^p(\Omega; E)$ for the set of measurable maps $X : \Omega \to E$ such that $\mathbb{E}\left[\|X\|_E^p\right] < \infty$. We say that a sequence of probability measures, $(\mu_n)_{n\geq 0} \subset \mathcal{P}(E)$, converges weakly to $\mu \in \mathcal{P}(E)$ and write $\mu_n \rightharpoonup \mu$, if

$$\int_E \varphi \; \mathrm{d}\mu_n \to \int_E \varphi \; \mathrm{d}\mu, \quad \forall\, \varphi \text{ continuous and bounded.} \tag{2.2.1}$$

We write $\mathcal{W}_{p,E}(\mu, \nu)$ for the p-Wasserstein distance between two probability measures μ, ν in $\mathcal{P}_p(E)$, the space of probability measures with p-finite moments. We give a detailed definition of these distances and discussion of their properties in Section 2.6.

2.3 Main Results

Before stating our main results, we give a rigorous definition of solution to the perturbed McKean–Vlasov equation (2.0.3).

We fix $T > 0$, $(\gamma, \eta) \in \left(\frac{1}{2}, 1\right) \times (0, \infty)$ such that $\eta + \gamma > 1$, and $(\Omega, \mathcal{F}, \mathbb{P})$ an abstract probability space.

Definition 2.3.1. Let $q, p \geq 1$, $(\xi, B) \in L^q(\Omega; \mathbb{R}^d) \times L^p(\Omega; \mathcal{C}_T^\eta)$, with B a zero at zero path and assume we are given $K \in \mathcal{S}'(\mathbb{R}^d; \mathbb{R}^d)$ and zero at zero path $Z \in C_T$ such that the associated averaged field, ΓK (see Def. 2.5.1), is contained in $\mathcal{C}_T^\gamma \mathcal{C}^2(\mathbb{R}^d; \mathbb{R}^d)$. Then we say that Y is a solution to (2.0.3) if $Y \in L^{q\wedge p}(\Omega; \mathcal{C}_T^{\gamma\wedge\eta})$ and Y solves the generalised McKean–Vlasov equation

$$\begin{cases} Y_t = \xi + \displaystyle\int_0^t \left(\Gamma_{\mathrm{d}r} K * \mu_r\right)(Y_r) + B_t, \\ \mu_r = \mathcal{L}(Y_r). \end{cases} \tag{2.3.1}$$

where the drift term is properly defined as a measure dependent non-linear Young integral in Section 3.1.1, see Lemma 3.1.5 for example and in particular for an explanation of the notation $\Gamma_{\mathrm{d}s}K$.

Proofs of Theorems 2.3.2 and 2.3.5 and Corollary 2.3.9 stated below, are completed in Section 3.3.

Theorem 2.3.2. *Let $\sigma \in \mathbb{R}$, $q, r \in [1, \infty]$. Assume we are given a pair $(\xi, B) \in L^1(\Omega; \mathbb{R}^d) \times L^p(\Omega; \mathcal{C}^\eta_T)$ for all $p \geq 1$, such that $B_0 = 0$, a $K \in \mathcal{B}^\sigma_{r,q}(\mathbb{R}^d; \mathbb{R}^d)$ and a zero at zero path $Z \in C_T$ such that the associated averaged field $\Gamma K \in \mathcal{C}^\gamma_T \mathcal{C}^2$. Then there exists a unique solution $Y \in L^1(\Omega; \mathcal{C}^{\eta\wedge\gamma}_T)$ to the equation*

$$\begin{cases} \mathrm{d}Y_t = (K(\cdot + Z_t) * \mu_t)(Y_t)\, \mathrm{d}t + \mathrm{d}B_t, \\ \mu_t = \mathcal{L}(Y_t), \\ Y_0 = \xi, \end{cases} \tag{2.3.2}$$

in sense of Definition 2.3.1. Furthermore, if (ξ^1, B^1), $(\xi^2, B^2) \in L^1(\Omega; \mathbb{R}^d) \times L^p(\Omega; \mathcal{C}^\eta_T)$ for all $p \geq 2$ with $(\xi^1 - \xi^2) \perp (B^1, B^2)$ and , and Y^1, $Y^2 \in L^1(\Omega; \mathcal{C}^{\eta\wedge\gamma}_T)$ are the corresponding solutions to (2.3.2), then there exists a constant $C = C(T, \Gamma, \gamma, \eta) > 0$ such that,

$$\begin{aligned} \mathcal{W}_{1;\mathcal{C}^{\eta\wedge\gamma}_T}(\mathcal{L}(Y^1), \mathcal{L}(Y^2)) \leq C\Big(& \mathcal{W}_{1;\mathbb{R}^d}(\mathcal{L}(\xi^1), \mathcal{L}(\xi^2)) \\ & + \mathcal{W}_{2;\mathcal{C}^\eta_T}(\mathcal{L}(B^1), \mathcal{L}(B^2))\Big). \end{aligned} \tag{2.3.3}$$

Remark 2.3.3. From (2.3.3), since Z is held fixed for both equations we see that, for a different constant $C := C(\mathbb{E}\left[[B^1]^{2p}_{\eta;T}\right] \vee \mathbb{E}^2\left[[B^2]^{2p}_{\eta;T}\right] T, \Gamma, \gamma, \eta) > 0$,

$$\begin{aligned} \mathcal{W}_{1;C_T}(\mathcal{L}(X^1), \mathcal{L}(X^2)) \leq C\Big(& \mathcal{W}_{1;\mathbb{R}^d}(\mathcal{L}(x^1_0), \mathcal{L}(x^2_0)) \\ & + \mathcal{W}_{2;\mathcal{C}^\eta_T}(\mathcal{L}(B^1), \mathcal{L}(B^2)) + \|Z\|_{C_T}\Big). \end{aligned} \tag{2.3.4}$$

That is we only establish approximate stability for (2.3.2) with Z fixed. This is to be expected since we allow K to be possibly very rough and so in general there is no reason for X to be stable. However, (2.3.4) raises the question of introducing processes Z^1, Z^2 and then studying the stability between X^i in the triple $(x_0^i, B^i, Z^i)_{i=1}^2$. We do not address this here but leave it as an open question for future work.

Remark 2.3.4. Regarding the random inputs to (2.3.2) we remark that our results hold equally well with $B = 0$. Therefore, although we refer to these equations as generalised McKean–Vlasov equations, in the case $B = 0$ they are closer to the characteristics of a mean field transport equation.

With the above general theorem at hand we turn to some specific applications. First we describe the implications of Theorem 2.3.2 in the context of mean field approximations to (2.3.2). We use the trick of Tanaka, [103], and the fact we hold the volatility constant to obtain Theorem 2.3.5 below.

Theorem 2.3.5 (Mean Field Approximation)**.** *Let $N \in \mathbf{N}$ and $(\xi, B) \in L^1(\Omega; \mathbb{R}^d) \times L^p(\Omega; \mathcal{C}_T^\eta)$, $(\xi_0^{(N)}, B^{(N)}) \in L^1(\Omega; \mathbb{R}^{Nd}) \times L^p(\Omega; (\mathcal{C}_T^\eta)^N)$ for all $p \geq 1$ be such that $B_0 = 0$ and $B_0^{(N)} = 0$. In addition assume that for some $\tilde{p} \geq 1$*

$$\lim_{N\to\infty} \left(\mathcal{W}_{1;\mathbb{R}^d} \left(\mathcal{L}_N(\xi^{(N)}(\omega)), \mathcal{L}(\xi) \right) + \mathcal{W}_{\tilde{p};\mathcal{C}_T^\eta} \left(\mathcal{L}_N(B^{(N)}(\omega)), \mathcal{L}(B) \right) \right) = 0, \quad \mathbb{P}\text{-}a.s.$$

Then with Z and Γ as in Theorem 2.3.2, for any $N \geq 1$ there exists a unique solution $Y^{(N)} := (Y^i)_{i=1}^N \in L^1(\Omega; (\mathcal{C}_T^{\eta\wedge\gamma})^N)$ to the particle system

$$Y_t^i = \xi^i + \frac{1}{N} \sum_{j=1}^N \int_0^t \Gamma_{\mathrm{d}r} K(Y_r^i - Y_r^j) + B_t^i, \qquad \textit{for } i = 1, \ldots, N. \tag{2.3.5}$$

Furthermore, letting Y be the unique solution to (2.3.1) and assuming $\xi \perp B$,

$$\lim_{N\to\infty} \mathcal{W}_{1;\mathcal{C}_T^{\eta\wedge\gamma}} \left(\mathcal{L}_N(Y^{(N)}(\omega)), \mathcal{L}(Y) \right) = 0 \quad \mathbb{P}\text{-}a.s.$$

Remark 2.3.6. The assumption of convergence in Wasserstein of the data $(\xi^{(N)}, B^{(N)})$ is satisfied if for example the empirical measure converges weakly $\mathbb{P}$-a.s. and sufficiently high moments are uniformly integrable. In particular, if the sequences

$(x_0^i, B^i)_{i=1}^N$ are i.i.d then one obtains the required convergence. See [27, Lem. 53 & 54] for details.

Remark 2.3.7. It is clear from the proofs of Theorems 2.3.2 & 2.3.5 that if in addition one assumes the initial data to have finite q moments for some $q > 1$ then the respective solutions to the McKean–Vlasov equation and particle system will have finite q moments in $\mathcal{C}_T^{\eta\wedge\gamma}$. However, so as not to unnecessarily complicate the notation further we restrict ourselves to L^1 initial data. Furthermore, from the steps of the proofs one can see that we do not necessarily require all moments of B to be finite, but only p moments for p that can be arbitrarily large depending on some parameters chosen in the proofs. Informally, there is a trade off between the regularising effect coming from Z and the number of finite moments we require B to have. Since we are mostly interested in leveraging the maximum possible regularising effect for simplicity we ask for B to have all finite moments. For more details see Remark 3.1.4.

Since Theorems 2.3.2 and 2.3.5 are both quite general in nature, we specify our results to a particular class of McKean–Vlasov equations, that includes some physically relevant models.

Corollary 2.3.8. *Let $p \geq 1$, $x_0 \in L^p(\Omega; \mathbb{R}^d)$, $(B_t)_{t\in[0,T]}$ be a Brownian motion carried by $(\Omega, \mathcal{F}, \mathbb{P})$ and K be a homogeneous distribution of order $\sigma < 0$. Then let $(Z_t)_{t\in[0,T]}$ be a sample path of an fBm with Hurst parameter $H \in (0,1)$, carried by a separate probability space, $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. If $H < \frac{1}{4-2\sigma}$, there exists a set of full measure $\tilde{\mathcal{N}} \subseteq \tilde{\Omega}$, independent of K, such that for all $\tilde{\omega} \in \tilde{\mathcal{N}}$, choosing $Z = Z(\tilde{\omega})$, there exists a unique solution $X \in L^p(\Omega; C_T)$ to (2.3.2), and the stability estimates (2.3.3) and (2.3.4) both hold.*

Corollary 2.3.9. *Let $\sigma < 0$, $\xi \in L^1(\Omega; \mathbb{R}^d)$ K be a homogeneous kernel of degree σ and $B \in L^p(\Omega; \mathcal{C}_T^{1/2-})$ for all $p \geq 1$. Then let $(Z_t)_{t\in[0,T]}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$ on a possibly different probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. If $H < \frac{1}{4-2\sigma}$, then there exists a set $\tilde{\mathcal{N}} \subset \tilde{\Omega}$ of full measure, such that* *for all $\tilde{\omega} \in \tilde{\mathcal{N}}$, $Z = Z(\tilde{\omega}) : [0,T] \to \mathbb{R}^d$ is a continuous path and under suitable remaining assumptions, the results of Theorems 2.3.2 and 2.3.5 apply.*

Remark 2.3.10. Regarding the threshold on the Hurst parameter, we note that in contrast to the results obtained in [64], for a general distribution $f \in \mathcal{B}^{\sigma}_{2,2}(\mathbb{R}^d;\mathbb{R}^d)$ which requires a Hurst parameter $H < \frac{1}{4+2d-2\sigma}$, the threshold of $H < \frac{1}{4-2\sigma}$ in Corollary 2.3.9 is dimension independent. This is due to the fact that K is assumed to be a homogeneous distribution; see Definition 2.4.8 and therefore one has $K \in \mathcal{B}^{\sigma+d/p}_{p,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ for any d, $p \geq 1$. See Subsection 2.4.3 for details.

2.4 Hölder–Besov Spaces

We recall some definitions and standard analysis regarding the scale of Hölder–Besov spaces on $\mathbb{R}^d$. We refer to [5] for more details. We define the Fourier transform on $L^1(\mathbb{R}^d)$ by setting,

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,\mathrm{d}x.$$

It is standard that $\mathcal{F}$ fixes $\mathcal{S}(\mathbb{R}^d)$. We define the inverse transform,

$$\mathcal{F}^{-1}f(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} f(\xi)e^{-ix\cdot\xi}\,\mathrm{d}\xi,$$

and extend both definitions to the tempered distributions by duality. For $T \in \mathcal{S}'(\mathbb{R}^d)$ and any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we define $\mathcal{F}T$ to be the distribution such that

$$\langle \mathcal{F}T, \varphi\rangle := \langle T, (2\pi)^d\mathcal{F}^{-1}\varphi\rangle.$$

As a result, $\mathcal{F}$ is an automorphism of $\mathcal{S}'(\mathbb{R}^d)$ and we have the equality

$$\|f\|_{L^2(\mathbb{R}^d)} = (2\pi)^{\frac{d}{2}}\|\mathcal{F}f\|_{L^2(\mathbb{R}^d)}, \quad \forall\, f \in L^2(\mathbb{R}^d).$$

We also recall the definition of a Fourier multiplier. For any $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\varphi : \mathbb{R}^d \to \mathbb{R}$ measurable and with at most polynomial growth, we define,

$$\varphi(D)f := \mathcal{F}^{-1}(\varphi \mathcal{F} f) = \left(\mathcal{F}^{-1}\varphi\right) * f,$$

where D is a placeholder for the derivative operator on $\mathbb{R}^d$.

2.4.1 Dyadic Partition of Unity and Littlewood-Paley Blocks

We let $\tilde{\chi}$, $\chi \in C_c^\infty(\mathbb{R}^d)$ be such that

1. $\operatorname{supp} \tilde{\chi} \subset B_{\frac{4}{3}}(0)$ and $\operatorname{supp} \chi \subset B_{\frac{8}{3}}(0) \setminus B_{\frac{3}{4}}(0)$,
2. $\tilde{\chi}(\zeta) + \sum_{k=0}^{\infty} \chi(2^{-k}\zeta) = 1$, for all $\zeta \in \mathbb{R}$.

The existence of such a dyadic partition of unity is shown in [5, Prop. 2.10]. For $k \geq 0$ we define $\chi_k(\,\cdot\,) := \chi(2^{-k}\,\cdot\,)$ and set $\chi_k = 0$ for all $k < -1$.

For any $f \in \mathcal{S}'(\mathbb{R}^d)$ we define the inhomogeneous Littlewood–Paley blocks by setting,

$$\begin{aligned} \Delta_{-1} f &:= \tilde{\chi}(D) f = \tilde{h} * f, \\ \Delta_k f &:= \chi_k(D) f = h(2^k\,\cdot\,) * f, \quad \forall\, k \geq 0, \end{aligned} \tag{2.4.1}$$

where $\tilde{h} = \mathcal{F}^{-1}\tilde{\chi}$ and $h = \mathcal{F}^{-1}\chi$. Since $\tilde{h}$, $h \in \mathcal{S}(\mathbb{R}^d)$, the operators Δ_k map L^p to L^p for any $p \in [1,\infty]$ with norms independent of p and k.

Definition 2.4.1 (Inhomogeneous Besov Spaces). For $\alpha \in \mathbb{R}$ and $p, q \in [1,\infty]$, the inhomogeneous Besov space $\mathcal{B}^\alpha_{p,q}(\mathbb{R}^d)$ is defined by

$$\mathcal{B}^\alpha_{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \,:\, \|f\|_{\mathcal{B}^\alpha_{p,q}(\mathbb{R}^d)} := \left(\sum_{k \geq -1} 2^{kq\alpha} \|\Delta_k f\|^q_{L^p} \right)^{\frac{1}{q}} < \infty \right\}.$$

Sometimes to lighten notation we write $\mathcal{B}^\alpha_p := \mathcal{B}^\alpha_{p,\infty}$. For $p = q = \infty$ we use the notation

$$\mathcal{C}^\alpha(\mathbb{R}^d) := \Big\{ f \in \mathcal{S}' \,:\, \|f\|_{\mathcal{B}^\alpha_p(\mathbb{R}^d)} := \sup_{k\geq -1} 2^{k\alpha} \|\Delta_k f\|_{L^\infty(\mathbb{R}^d)} < \infty \Big\}.$$

For $\alpha > 0$ and not an integer these spaces agree with the usual spaces of α-Hölder continuous functions. When $p = q = 2$ we use the special notation $\mathcal{H}^\alpha = \mathcal{B}^\alpha_{2,2}$ to denote the Hilbertian Sobolev spaces, on which an equivalent norm is given by the expression

$$\|f\|_{\mathcal{H}^\alpha} := \|(1+|\cdot|)^\alpha \mathcal{F}(f)\|_{L^2}.$$

The Hölder–Besov spaces enjoy a number of useful properties, some of which we list below. Proofs of the following statements can be found in [5].

(i) Embeddings: for $\alpha \in \mathbb{R}$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$ one has,

$$\|\cdot\|_{\mathcal{B}^\alpha_{p_2,q_2}} \lesssim \|\cdot\|_{\mathcal{B}^{\alpha+d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_1,q_1}}. \tag{2.4.2}$$

We also have the following, continuous embeddings,

$$\|f\|_{\mathcal{B}^\alpha_{p,q}} \lesssim \|f\|_{\mathcal{B}^{\alpha'}_{p,q}} \quad \alpha < \alpha' \in \mathbb{R}, \tag{2.4.3}$$

$$\|f\|_{\mathcal{B}^\alpha_{p,q}} \lesssim \|f\|_{\mathcal{B}^\alpha_{p,q'}} \quad q > q' \in [1,\infty], \tag{2.4.4}$$

$$\|f\|_{\mathcal{B}^\alpha_{p,q}} \lesssim \|f\|_{\mathcal{B}^{\alpha'}_{p,q'}} \quad \alpha < \alpha' \in \mathbb{R}, \forall\, q \leq q' \in [1,\infty] \tag{2.4.5}$$

and the embedding $\mathcal{B}^{\alpha'}_{p,q} \hookrightarrow \mathcal{B}^\alpha_{p,q}$ of (2.4.3) is compact.

(ii) Relations to L^p spaces: For $p \in [1,\infty]$ one has,

$$\|f\|_{L^p} \lesssim \|f\|_{\mathcal{B}^0_{p,1}}, \quad \|f\|_{\mathcal{B}^0_{p,\infty}} \lesssim \|f\|_{L^p}.$$

Lemma 2.4.2 (Young's convolution inequality). *For $\alpha, \beta \in \mathbb{R}$, let $f \in \mathcal{C}^\beta_p$ and $g \in \mathcal{C}^\alpha_q$, and let $r \in [1,\infty]$ be defined through the relation $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$. Then the* *following inequality holds*

$$\|f * g\|_{\mathcal{B}_r^{\alpha+\beta}} \lesssim \|f\|_{\mathcal{B}_p^\beta} \|g\|_{\mathcal{B}_q^\alpha}. \tag{2.4.6}$$

2.4.2 Functions of Hölder Continuity on Intervals of the Real Line

The next lemma gives a useful criteria for extending a local control on the Hölder continuity of a path to a global one. We provide a sketch proof but it can also be found as [46, Excersise 4.24], with full proof.

Lemma 2.4.3. *Let E be a Banach space, $\alpha \in (0,1)$, $h > 0$ and $X : [0,T] \to E$. Suppose that there exists a constant $M > 0$ such that for any $t \in [0, T-h]$ we have $[X]_{\alpha;[t,t+h]} \leq M$. Then X is α-Hölder continuous on $[0,T]$. In particular, we have that*

$$[X]_{\alpha;[0,T]} \leq M(1 \vee 2h^{\alpha-1})T^{1-\alpha}.$$

Proof. We need to show that for any $0 \leq s \leq t \leq T$ then $\frac{\|X_{s,t}\|_E}{|t-s|^\alpha} \leq M(1 \vee 2h^{\alpha-1})T^{1-\alpha}$. In the case when $|t-s| \leq h$ there is nothing to prove, so let $|t-s| \geq h$. Define $t_i = (s+ih) \wedge t$ for $i \in \mathbf{N}$. Note that for $N \geq (t-s)/h$, $t_N = t$, and that $t_{i+1} - t_i \leq h$ for all $i \in \mathbf{N}$. Therefore, we have

$$\|X_{s,t}\|_E \leq \sum_{0 \leq i < (t-s)/h} \|X_{t_i,t_{i+1}}\|_E \leq Mh^\alpha \left(1 + \frac{t-s}{h}\right) \leq 2Mh^\alpha \frac{t-s}{h} \leq 2Mh^{\alpha-1}|t-s|^\alpha T^{1-\alpha},$$

which concludes the proof. □

2.4.3 Besov Regularity of Homogeneous Distributions

In Section 3.3 we discuss applications of our general result (2.3.2), to some specific McKean–Vlasov problems where K is a given homogeneous distribution. In this subsection we discuss the regularity of these distributions in Hölder–Besov spaces. This allows us to characterise the range of regularising paths Z for which our regularisation result holds.

For $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\lambda > 0$ we define the dilation

$$\varphi_\lambda(x) := \lambda^{-d}\varphi\left(\lambda^{-1}x_1, \dots, \lambda^{-1}x_d\right). \tag{2.4.7}$$

Definition 2.4.4 (Homogeneous Distribution). We say that $K \in \mathcal{S}'(\mathbb{R}^d)$ is homogeneous of degree $\sigma \in \mathbb{R}$ if for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\lambda > 0$

$$\langle K, \varphi_\lambda\rangle = \lambda^\sigma \langle K, \varphi\rangle. \tag{2.4.8}$$

Replacing $\mathbb{R}^d$ with the punctured domain $\mathbb{R}^d \setminus \{0\}$ we can instead define the notion of homogeneous distributions in $\mathcal{S}'(\mathbb{R}^d \setminus \{0\})$. For any $\sigma \in \mathbb{R}\setminus\mathbb{Z}_{\leq -d}$, all homogeneous distributions on $\mathcal{S}'(\mathbb{R}^d \setminus \{0\})$ of order σ are of the form

$$\tilde{K}_\sigma(x) = f\left(\frac{x}{|x|}\right)|x|^\sigma, \tag{2.4.9}$$

where $f \in \mathcal{S}'(\mathbb{S}^{d-1})$ is a distribution on the d-dimensional unit sphere. If $\sigma > -d$ then $\tilde{K}_\sigma(x)$ extends uniquely to a homogeneous distribution $K_\sigma \in \mathcal{S}'(\mathbb{R}^d)$ without modification. However, for $\sigma \leq -d$ the question of extending $\tilde{K}_\sigma$ to a distribution on the un-punctured plane is more complicated. For a full discussion see [71, Sec. 3.2].

When $\sigma < -d$ and not an integer there exists a unique extension K_σ defined by

$$\langle K_\sigma, \varphi\rangle = \int_{\mathbb{R}^d} \tilde{K}_\sigma(x)\left(\varphi(x) - P^{\lfloor -\sigma-d\rfloor}_{\varphi;0}(x)\right)\,\mathrm{d}x \tag{2.4.10}$$

where $P^k_{\varphi;0}$ is the Taylor polynomial to order $k-1$ of φ at 0. This is proved as [71, Theorem 3.2.3]. We refer to (2.4.10) as the principle value extension of $\tilde{K}_\sigma$.

For $\sigma \in \mathbb{Z}_{\leq -d}$ the formula (2.4.10) does define an extension of $\tilde{K}_\sigma$ but it is not unique, since can always add any linear combination of sufficiently high derivatives of the Dirac, [71, Thm. 3.2.4]. For $\sigma = -n$ with $n \in \mathbb{Z}_{\geq d}$ we choose to define the extension of $\tilde{K}_n$ by the formula

$$\langle K_n, \varphi \rangle = \int_{\mathbb{R}^d} \tilde{K}_n(x) \left(\varphi(x) - P^{n-d}_{\varphi;0}(x) \right) \, \mathrm{d}x + \sum_{|a|=n-d} D^a \varphi(0).$$

Defining the convolution of a distribution in the usual way, we see that for all $\sigma < 0$ we have

$$\begin{aligned}(K_\sigma * \varphi)(x) := &\int_{\mathbb{R}^d} \tilde{K}_\sigma(x-y) \left(\varphi(y) - P^{\lfloor -\sigma - d \rfloor}_{\varphi;0}(y) \right) \, \mathrm{d}y \\ &+ \mathbb{1}_{\{\sigma \in \mathbb{Z}_{\leq -d}\}} \sum_{|a|=\sigma-d} D^a \varphi(0),\end{aligned} \tag{2.4.11}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. The Fourier transform of a homogeneous distribution is nicely described by the following theorem.

Theorem 2.4.5. *Let $K \in \mathcal{S}'(\mathbb{R}^d)$ be a homogeneous distribution of order $\sigma \in \mathbb{R}$, then $\mathcal{F}K \in \mathcal{S}'(\mathbb{R}^d)$ and is a homogeneous distribution of order $-\sigma - d$.*

Proof. See the proof of [71, Theorem 7.1.16] □

This result suggests that K_σ should be controlled in a suitable space of negative regularity, in fact it almost immediately follows that $K_\sigma \in \mathcal{H}^{\sigma+\frac{d}{2}}$. Below we give a self contained proof that for $\sigma < 0$, $K_\sigma \in \mathcal{B}^{\sigma+\frac{d}{p}}_{p,\infty}$ for any $p \in [1, \infty]$.

Proposition 2.4.6. *Let $\sigma < 0$ and $\tilde{K}_\sigma \in \mathcal{S}'(\mathbb{R}^d \setminus \{0\})$ be the distribution described by (2.4.9) and let K_σ be its principle value extension to $\mathcal{S}'(\mathbb{R}^d)$ defined in (2.4.10). Then $K_\sigma \in \mathcal{B}^{\sigma+\frac{d}{p}}_{p,\infty}(\mathbb{R}^d)$ for any p, $q \in [1, \infty]$.*

Proof. From the Besov embeddings (2.4.2), for any $p, q \in [1, \infty]$ we have that

$$\|K_\sigma\|_{\mathcal{B}^{\sigma+\frac{d}{p}}_{p,\infty}} \lesssim \|K_\sigma\|_{\mathcal{B}^{\sigma+d}_{1,\infty}},$$

so we concentrate on showing that $K_\sigma \in \mathcal{B}^{\sigma+d}_{1,\infty}$. From Theorem 2.4.5 for any $k \geq 0$

we have

$$\Delta_k K_\sigma = 2^{-(\sigma+d)k} \Delta_0 K_\sigma.$$

Therefore, we have

$$\sup_{k\geq 0} 2^{(\sigma+d)k} \|\Delta_k K_\sigma\|_{L^1} = \|\Delta_0 K_\sigma\|_{L^1}. \tag{2.4.12}$$

So it suffices to show that $\|\Delta_{-1}K_\sigma\|_{L^1}$, $\|\Delta_0 K_\sigma\|_{L^1}$ are both finite. We choose a smooth, cut-off function $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\text{supp}(\psi) = B_1(0)$ and $\|\psi\|_{C_c^\infty(\mathbb{R}^d)} \leq 1$. Then we write,

$$\tilde{K}_\sigma = \psi \tilde{K}_\sigma + (1-\psi)\tilde{K}_\sigma := \tilde{K}_{\sigma,0} + \tilde{K}_{\sigma,1},$$

and define the principle value extensions $K_{\sigma,0}$ and $K_{\sigma,1}$ analogously. Then we divide the proof into two cases, $-d < \sigma < 0$ and $\sigma \leq -d$.

First consider the case $-d < \sigma < 0$. We directly have that $K_{\sigma,0} \in L^1(\mathbb{R}^d)$ so since Δ_{-1} and Δ_0 are both bounded maps $L^p \to L^p$ we have $\Delta_{-1}K_{\sigma,0}$, $\Delta_0 K_{\sigma,0} \in L^1(\mathbb{R}^d)$. Regarding the part supported away from the origin, using that h, $\tilde{h}$ decay faster than any polynomial we also have $\Delta_{-1}K_{\sigma,1}$, $\Delta_0 K_{\sigma,1} \in L^1(\mathbb{R}^d)$.

When $\sigma \leq -d$ the situation is reversed. In this case we see that $K_{\sigma,1} \in L^1(\mathbb{R}^d)$ directly and so by the boundedness of Δ_0 and Δ_{-1} as mappings $L^p \to L^p$ we have $\Delta_{-1}K_{\sigma,1}$, $\Delta_0 K_{\sigma,1} \in L^1(\mathbb{R}^d)$. Regarding the compactly supported term the proofs for $\Delta_{-1}K_{\sigma,0}$ and $\Delta_0 K_{\sigma,0}$ are very similar so we only present the -1 block. Using Taylor's theorem we have,

$$\begin{aligned}
\Delta_{-1}K_{\sigma,0}(x) &\leq \int_{B_1(x)} |x-y|^\sigma \left|\tilde{h}(y) - P^{\tilde{h}}_{k;y}(x)\right| \,\mathrm{d}y + \mathbb{1}_{\sigma\in\mathbb{Z}_{\leq -d}} D^{|\sigma|-d}\tilde{h}(x) \\
&\leq \|D^{k+1}\tilde{h}\|_{L^\infty(B_1(x))} \left(\int_{B_1(x)} |x-y|^{\sigma+k+1} \,\mathrm{d}y + 1 \right) \\
&\lesssim (1+|x|^{k+1})^{-1} \|\tilde{h}\|_{k+1,\mathcal{S}},
\end{aligned}$$

where we used the fact that $\sigma + k + 1 > -d$ to evaluate the integral. The last line is integrable over $\mathbb{R}^d$ and so we have $\Delta_{-1}K_{\sigma,0} \in L^1(\mathbb{R}^d)$. Applying the same argument to $\Delta_0 K_{\sigma,0}$ we have $\|\Delta_0 K_{\sigma,0}\|_{L^1} < \infty$.

In conclusion, for any $\sigma < 0$ we have

$$\|K_\sigma\|_{\mathcal{B}^{\sigma+d}_{1,\infty}} = \sup_{k\geq -1} 2^{(\sigma+d)k}\|\Delta_k K_\sigma\|_{L^1} \leq \sup_{k\in\{0,1\}} \|\Delta_k K_\sigma\|_{L^1} < \infty.$$

□

Remark 2.4.7. Using the Besov embedding (2.4.5), for any $\varepsilon > 0$ we also have that $K_\sigma \in \mathcal{B}^{\sigma+\frac{d}{2}-\varepsilon}_{p,q}$ for any $q \in [1,\infty)$.

2.5 Averaged Fields and Pathwise regularisation of ODEs

Let $f \in \mathcal{B}^\beta_{p,q}(\mathbb{R}^d)$ for $\beta \in \mathbb{R}$, $p, q \in [1,\infty]$ and $Z : [0,T] \to \mathbb{R}^d$ be a possibly random path on an abstract probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Then consider the formal, integral equation,

$$\tilde{X}^\xi_t = \xi + \int_0^t f(\tilde{X}^\xi_r)\,\mathrm{d}r + Z_t. \tag{2.5.1}$$

In [22], the authors show that if Z is sufficiently irregular (exact meaning to be explained later) then (2.5.1) can be interpreted rigorously and is pathwise well-posed, even when f is only a distribution. More specifically the authors show that if Z is a fractional Brownian motion with Hurst parameter $H \in (0,1)$ then for a given $f \in \mathcal{C}^{-\frac{1}{2H}+2}$ there exists a full measure set $\tilde{\mathcal{N}} \in \tilde{\Omega}$ (depending on f) such that for all $\tilde{\omega} \in \tilde{\mathcal{N}}$ there exists a unique solution to (2.5.1) driven by $Z(\tilde{\omega})$. This result has recently been developed further in [64, 48]. In this section we will give a short introduction to the methodology and ideas behind such regularisation of ordinary differential equations, which will in subsequent sections be applied to McKean–Vlasov problems.

The first step is to reformulate (2.5.1) by defining $X_t := \tilde{X}_t - Z_t$, which we ask to solve,

$$X_t^{\xi} = \xi + \int_0^t f(X_r^{\xi} + Z_r)\,\mathrm{d}r, \tag{2.5.2}$$

where again the drift is only to be understood formally for now. We then define a new distribution, for any $0 \leq s < t \leq T$ and $x \in \mathbb{R}^d$, setting

$$\langle \Gamma_t f, \varphi(\,\cdot\, - x)\rangle := \int_0^t \langle f, \varphi(\,\cdot\, - x - Z_r)\rangle\,\mathrm{d}r, \quad . \tag{2.5.3}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. We refer to $\Gamma_t f$ as an averaged distribution and denote the time increment by $\Gamma_{s,t} f = \Gamma_t f - \Gamma_s f$.

Definition 2.5.1 (Averaged distributions). For $\beta \in \mathbb{R}$, $p, q \in [1, \infty]$, let $f \in \mathcal{B}^{\beta}_{p,q}(\mathbb{R}^d)$. We say that $\Gamma f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ defined by (2.5.3), is an *averaged distribution* if $t \mapsto \Gamma_t f \in \mathcal{C}^{\gamma}_T \mathcal{C}^{\alpha}(\mathbb{R}^d)$ for some $\gamma > 1/2$ and $\alpha \geq \beta$. If $\Gamma f \in C^{\gamma}_T \mathcal{C}^{\alpha}(\mathbb{R}^d)$ for all $f \in \mathcal{B}^{\beta}_{p,q}(\mathbb{R}^d)$ then by an abuse of notation we also define the averaging operator $\Gamma : \mathcal{B}^{\beta}_{p,q}(\mathbb{R}^d) \to \mathcal{C}^{\gamma}_T \mathcal{C}^{\alpha}(\mathbb{R}^d)$, where the evaluation is given by (2.5.3).

Remark 2.5.2. Note that Γ in general depends on a given path $Z : [0,T] \to \mathbb{R}^d$, however in this article we are not concerned with the properties of Γ w.r.t Z, we only assume we can build a sufficiently regular averaged distribution from some set of paths Z. Therefore we only write Γ, and say that Γ is associated to the path Z when necessary.

Definition 2.5.3. Let $\beta \in \mathbb{R}$, $p, q \in [1, \infty]$ and $\rho > 0$. A path $Z : [0,T] \to \mathbb{R}^d$ is called *ρ-regularising on $\mathcal{B}^{\beta}_{p,q}(\mathbb{R}^d)$* if there exists a $\gamma > 1/2$ such that the averaging operator Γ associated to Z satisfies $\Gamma f \in \mathcal{C}^{\gamma}_T \mathcal{C}^{\beta+\rho}(\mathbb{R}^d)$ for every $f \in \mathcal{B}^{\beta}_{p,q}(\mathbb{R}^d)$. If the path Z is such that for any $f \in \mathcal{B}^{\beta}_{p,q}(\mathbb{R}^d)$ with $\beta \in \mathbb{R}$ and $p, q \in [1, \infty]$, the averaged field $\Gamma f \in \mathcal{C}^{\gamma}_T \mathcal{C}^{\alpha}(\mathbb{R}^d)$, for any $\alpha \in \mathbb{R}$, we say that Z is *infinitely regularising.*

Remark 2.5.4. Any continuous path $(Z_t)_{t\in[0,T]}$ is 0-regularising, in the sense that for any function $f \in \mathcal{C}^{\beta}$ with $\beta \in \mathbb{R}$, it follows that $\Gamma f \in \mathcal{C}^1_T \mathcal{C}^{\beta}(\mathbb{R}^d)$. With this knowledge, interpolation reveals that time regularity of Γf, when Γ is associated to a ρ-regularising path, can be traded for spatial regularity. To see this, since

$\Gamma f \in \mathcal{C}_T^{1/2}\mathcal{C}^{\beta+\rho}(\mathbb{R}^d) \cap \mathcal{C}_T^1\mathcal{C}^\beta(\mathbb{R}^d)$, it follows by interpolation in Besov spaces (see e.g. [5, Thm. 2.80]) that for any $\theta \in [0,1]$

$$\|\Gamma_{s,t}f\|_{\mathcal{C}^{\beta+\theta\rho}} \leq \|\Gamma_{s,t}f\|_{\mathcal{C}^{\beta+\rho}}^{\theta}\|\Gamma_{s,t}f\|_{\mathcal{C}^{\beta}}^{1-\theta}.$$

Thus, for any $\gamma \in [\frac{1}{2}, 1]$ it follows that $\Gamma f \in \mathcal{C}_T^{\gamma}\mathcal{C}^{\beta+2\rho(1-\gamma)}$.

From now on, we assume that given f we are able to find Z sufficiently regularising that $\Gamma_{s,t}f$ is a genuine function and so we may drop the test function in (2.5.3). In this case we refer to Γf as an *averaged field*. This assumption will be justified below. In the case that Z is a random path, the averaged field can be written as the integral of $f(x+z)$ against the occupation measure, $\mu_t(z)$, of the path $t \mapsto Z_t$,

$$\Gamma_{0,t}f(x) = \int_0^t f(x+Z_r)\,\mathrm{d}r = \int_{\mathbb{R}^d} f(x+z)\,\mathrm{d}\mu_t(z). \tag{2.5.4}$$

Assuming the occupation measure has a density, $L_t \in L^1(\mathbb{R}^d)$, we can re-write (2.5.4) as a convolution,

$$\Gamma_{0,t}f(x) = f * \bar{L}_t(x) \quad \text{where} \quad \bar{L}_t(x) := L_t(-x). \tag{2.5.5}$$

Therefore, one approach to defining the averaged field is to first obtain regularity estimates on the occupation measure μ_t and then define Γf as in (2.5.5). We outline a few known results on the regularity of the averaged fields Γf and the regularity of the local times, L_t, associated to certain Gaussian processes. For a deeper discussion on occupation measures and local times, see the survey paper [51].

Example 2.5.5. Let Z be a fractional Brownian motion, on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, with Hurst parameter $H \in (0,1)$ and $f \in \mathcal{C}^\beta$. In [22] it is shown that there exists a set $\tilde{\mathcal{N}} \subset \tilde{\Omega}$ of full measure depending on the distribution f and Z, and a $\gamma > 1/2$, such that for all $\tilde{\omega} \in \tilde{\mathcal{N}}$, the averaged field $\Gamma f \in \mathcal{C}_T^{\gamma}\mathcal{C}^{\beta+\rho}(\mathbb{R}^d)_{loc}(\mathbb{R}^d)$ for any $\rho < 1/(2H)$. Note that this result does not define an average operator in sense of Definition 2.5.1, as the set of full measure, $\tilde{\mathcal{N}}$, depends explicitly on f, and thus the regularising effect does not necessarily hold for all $f \in \mathcal{C}^\beta(\mathbb{R}^d)$ simultaneously.

In two recent publications [48, 49], Galeati and Gubinelli prove that infinitely regularising paths are prevalent in C_T. The concept of prevalence was earlier used by Hunt, [72], to prove that almost all continuous paths are nowhere differentiable. In [48] it is also shown that for any $\delta > 0$, the $\frac{1}{2\delta}$-regularising paths are prevalent in $\mathcal{C}_T^{\delta-\varepsilon}$ for any $\varepsilon > 0$, [48, Thm. 1]. This result makes rigorous the heuristic that more irregular paths Z lead to more regularising averaging operators Γ.
In the next proposition we give a concrete example of a criterion that guarantees the regularising effect of a given process. This condition has been applied to obtain regularisation results in [64, Thm. 17] and [49].

Proposition 2.5.6. *Let $(Z_t)_{t\in[0,T]}$ be a continuous Gaussian process, such that for some $\zeta \in (0, \frac{1}{d})$*

$$\inf_{t\in[0,T]} \inf_{s\in[0,t]} \inf_{z\in\mathbb{R}^d;\,|z|=1} \frac{z^T \mathrm{Var}(Z_t|\mathcal{F}_s) z}{|t-s|^{2\zeta}} > 0.$$

Then there exists a $\gamma > 1/2$ such that the associated local time $L : [0,T]\times\mathbb{R}^d \to \mathbb{R}_+$ is contained in $\mathcal{C}_T^\gamma \mathcal{H}^\rho(\mathbb{R}^d)$ for any $\rho < \frac{1}{2\zeta} - \frac{d}{2}$, $\mathbb{P}$-a.s..

Proof. See [64, Thm. 17]. □

If $(Z_t)_{t\in[0,T]}$ is assumed to be a fractional Brownian motion, it is shown in [64, 49] that the associated local time is ρ-regular in space for any $\rho \in (0, \frac{1}{2H} - \frac{d}{2})$. We summarize this in the following proposition.

Proposition 2.5.7. *Let $Z : [0,T]\times\Omega \to \mathbb{R}^d$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$. For a vector field $f \in \mathcal{H}^\beta(\mathbb{R}^d)$ with $\beta \in \mathbb{R}$, let $\Gamma f : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be defined as in (2.5.3). Then there exists a set of full measure, $\tilde{\mathcal{N}} \subset \tilde{\Omega}$, depending only on Z, such that for all $\tilde{\omega} \in \tilde{\mathcal{N}}$ and any $\gamma \in (\frac{1}{2}, 1)$,*

$$\Gamma(\tilde{\omega}) f \in \mathcal{C}_T^\gamma \mathcal{C}^{\beta + \frac{1-\gamma}{H} - d(1-\gamma)}(\mathbb{R}^d).$$

Moreover, the mapping $f \mapsto \Gamma f$ defines an average operator on $\mathcal{H}^\beta(\mathbb{R}^d)$.

Proof. A full proof is given in the proof of [64, Thm. 17 and Rem. 18], however, we give a quick sketch using the local time approach. Using Proposition 2.5.6 we see that the local time L associated to $(Z_t)_{t\in[0,T]}$ is contained in $\mathcal{C}^\gamma_T\mathcal{H}^\rho(\mathbb{R}^d)$, $\tilde{\mathbb{P}}$-a.s. for some $\rho \in (0, \frac{1}{2H} - \frac{d}{2})$ and $\gamma \in (\frac{1}{2}, 1)$, then an application of Young's convolution inequality for Besov spaces, (2.4.6), gives

$$\|f * L_{s,t}\|_{\mathcal{C}^{\beta+\alpha}} \leq \|f\|_{\mathcal{H}^\beta}\|L\|_{\mathcal{C}^\gamma_T\mathcal{H}^\alpha}|t-s|^\gamma \tag{2.5.6}$$

Thus since $\Gamma f = f * \bar{L}$ where $\bar{L}_t(x) = L_t(-x)$, as seen in (2.5.5), and using the fact that the Sobolev regularity of $\bar{L}$ is identical to that of L, it follows that the path Z is ρ-regularising according to Definition 2.5.3. An application of the interpolation shown in Remark 2.5.4 completes the proof. □

Note that in contrast to Example 2.5.5 the full measure set, $\tilde{\mathcal{N}}$, here does not depend on f. However, the regularity gain is lower, at almost $\frac{1}{2H} - \frac{d}{2}$, as opposed to almost $\frac{1}{2H}$.

As the concept and regularity of averaging operators as given in Definition 2.5.1 is by now well established, and the examples of explicit paths which provide a regularising effect is vast, for the rest of this text we do not deal with particular paths but rather assume that the average operator Γ can be built from a suitable path. In Section 4.1 we provide some concrete examples with Z a fractional Brownian motion to highlight the degree of roughness one might expect to require in certain cases of classical interest.

Once it is established that Γ is an operator from $\mathcal{B}^\beta_{p,q}(\mathbb{R}^d) \to \mathcal{C}^\gamma_T\mathcal{C}^{\beta+\rho}(\mathbb{R}^d)$ for some $\rho > -\beta$ and $\gamma > 1/2$, we return to the ODE (2.5.1). The idea now is to use the spatial regularity of Γf to ensure well-posedness of the reformed equation (2.5.2). To do so we employ the method of non-linear Young integrals, introduced in [22] and also employed in [64, 48, 50]. A more general survey can be found in [47].

Considering a path $Y \in \mathcal{C}_T^{\gamma'}$ with $\gamma + \gamma'((\beta+\rho)\wedge 1) > 1$, one defines

$$\int_0^t \Gamma_{\mathrm{d}r} f(Y_r) := \lim_{|\mathcal{D}|\to 0} \sum_{[u,v]\in\mathcal{D}} \Gamma_{u,v} f(Y_u), \tag{2.5.7}$$

where $\mathcal{D}$ is any partition of the given time interval and $|\mathcal{D}|$ is the maximal increment size in $\mathcal{D}$. An application of the Sewing lemma, known from the theory of rough paths (cf. [46, Lem. 4.2]), proves that this integral is well defined. Indeed, setting $\Xi_{s,t} = \Gamma_{s,t} f(Y_s)$ then we see that the abstract integral

$$\mathcal{I}(\Xi)_t - \mathcal{I}(\Xi)_s = \lim_{|\mathcal{D}|\to 0} \sum_{[u,v]\in\mathcal{D}} \Xi_{u,v}$$

is well defined, if for all $(s,t) \in \Delta_T^2$

$$|\Xi_{s,t}| \lesssim |t-s|^{\delta_1}, \qquad \text{and} \qquad |\delta_u \Xi_{s,t}| \lesssim |t-s|^{\delta_2}$$

where $\delta_1 \in (0,1)$, $\delta_2 > 1$ and for $u \in [s,t]$, $\delta_u \Xi_{s,t} := \Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}$. It is readily checked in our case that

$$\delta_u \Xi_{s,t} = \Gamma_{u,t} f(Y_s) - \Gamma_{u,t} f(Y_u).$$

So invoking the assumption that $\Gamma f \in \mathcal{C}_T^{\gamma} \mathcal{C}^{\beta+\rho}(\mathbb{R}^d)$ it holds that for any $x, y \in \mathbb{R}^d$ and $(s,t) \in \Delta_2^T$,

$$|\Gamma_{s,t} f(x) - \Gamma_{s,t} f(y)| \lesssim |x-y|^{(\beta+\rho)\wedge 1} |t-s|^{\gamma},$$

and thus

$$|\delta_u \Xi_{s,t}| \lesssim [Y]_{\gamma'} |t-u|^{\gamma} |u-s|^{((\beta+\rho)\wedge 1)\gamma'} \lesssim |t-s|^{((\beta+\rho)\wedge 1)\gamma' + \gamma}.$$

Since $(\rho \wedge 1)\gamma' + \gamma > 1$ by assumption, we conclude that the integral (2.5.7) is well defined. We will refer to this construction as the non-linear Young integral (NLYI) due to the structure of the integrand.

In the coming sections we will use the concept of the averaging operator Γ to give meaning to McKean–Vlasov equations. This leads us to consider non-linear Young integrals constructed to coincide with integrals of the form

$$\int_0^t (K * \mu_r)(Y_r + Z_r)\,\mathrm{d}r, \tag{2.5.8}$$

where $\mu \in \mathcal{P}(\mathcal{C}_T^{\gamma'})$, $Y \in \mathcal{C}_T^{\gamma'}$ and $K \in \mathcal{B}^{\beta}_{p,q}(\mathbb{R}^d)$ for $\beta \in \mathbb{R}$. The regularising path $Z : [0,T] \to \mathbb{R}^d$ we will take to be deterministic and sufficiently regularising such that ΓK, as defined in (2.5.3), is contained in $\mathcal{C}_T^{\gamma}\mathcal{C}^{\alpha}$ for any $T > 0$, some $\gamma > 1/2$ and $\alpha \geq 2$. From Proposition 2.5.7 we see that this assumption is not vacuous. Indeed, we can always choose a sample path of a fractional Brownian motion (on a different probability space) with Hurst parameter $H \in (0,1)$ as small as we want (this can now be seen as a deterministic path), so that $\beta + \frac{1}{2H} - \frac{d}{2} > 2$. We mention that, much like in the theory of rough paths, we require 1-degree more regularity than the spatial Lipschitz property on ΓK in order to obtain stability of solutions. By analogy with (2.5.7) our first task will be to construct the non-linear Young integral

$$\int_0^t \Gamma_{\mathrm{d}r} K * \mu_r(Y_r) := \lim_{|\mathcal{D}|\to 0} \sum_{[u,v]\in\mathcal{D}} \Gamma_{u,v} K * \mu_u(Y_u). \tag{2.5.9}$$

We again use the sewing lemma to show that the non-linear integral on the right hand side is well defined in a Young sense, however, for this purpose we require a notion of Hölder continuity for the measure valued flow $t \mapsto \mu_t$. This is discussed in Section 2.6 below. We note that in the sequel we will view K as fixed for a given interacting particle system, so for notational ease we will collapse $\Gamma_{\mathrm{d}r} K$ to $\Gamma_{\mathrm{d}r}$.

2.6 Wasserstein Distances and Hölder Regularity of Measure Flows

As we saw in the construction of the non-linear Young integral, (2.5.7), it was important that the path $Y : [0,T] \to \mathbb{R}^d$ was sufficiently regular. Since we are concerned with defining non-linear Young integrals with measure valued integrands, as in (2.5.9), we will require a notion of time regularity for measure valued flows. In this section we recap some well known material concerning the notion of Wasserstein distances between probability measures and employ them to make rigorous a notion of Hölder continuity for measure valued flows. Similar ideas were applied in [21].

If (E, d_E) is a metric space for any $\mu \in \mathcal{M}(E)$ and $p \geq 1$ we define the p^{th}-moment of $\mu \in \mathcal{M}(E)$ by the expression

$$\int_E d_E(\xi, x)^p \, \mathrm{d}|\mu|(x), \qquad \text{for some } \xi \in E.$$

For $p \in [1, \infty)$ we let $\mathcal{M}_p(E)$ denote the set of real valued Radon measures with finite p^{th}-moment and we denote the subspace of zero mass Radon measures by $\mathcal{M}^0(E) := \{\mu \in \mathcal{M}(E) \,:\, \mu(E) = 0\}$ (resp. $\mathcal{M}^0_p(E) := \big\{\mu \in \mathcal{M}(E) \,:\, \mu(E) = 0, \mu \text{ has finite } p^{\text{th}} \text{ moment } \big\}$). We write $\mathcal{P}_p(E)$ for the probability measures with finite p^{th} moment on E. For $(E, d_E), (F, d_F)$ a pair of metric spaces and $p, q \in [1, \infty]$ we write $\mathcal{P}_{p,q}(E \times F)$ for the set of probability measures, μ, on $E \times F$ whose first marginals, $\mu|_E$, lie in $\mathcal{P}_p(E)$ and whose second marginals, $\mu|_F$, lie in $\mathcal{P}_q(F)$.

Definition 2.6.1 (Wasserstein Distances). Let (E, d_E) be a Polish space, and $\mathcal{P}_p(E)$ be as above. Then we may equip $\mathcal{P}_p(E)$ with the distance,

$$\mathcal{W}_{p;E}(\mu, \nu) := \left(\inf_{m \in \Pi(\mu,\nu)} \iint_{E \times E} d_E(x,y)^p \, \mathrm{d}m(x,y) \right)^{\frac{1}{p}}, \tag{2.6.1}$$

where $\Pi(\mu, \nu) \subseteq \mathcal{P}_{p,p}(E \times E)$ is the set of measures on the product space with first marginal equal to μ and second marginal equal to ν.

Note that the above definition makes no assumption on an underlying abstract probability space(s) giving rise to the measures μ, $\nu \in \mathcal{P}_p(E)$.

Remark 2.6.2. For μ, $\nu \in \mathcal{P}_p(E)$ the metric $\mathcal{W}_{p;E}$ can be equivalently characterised in terms of E valued random variables on a fixed probability space. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X, Y be any measurable mappings X, $Y : \Omega \to E$ such that $\mathbb{E}\left[|X|^p\right] < \infty$ and $\mathbb{E}\left[|Y|^p\right] < \infty$, with $\mu = \mathcal{L}(X)$ and $\nu = \mathcal{L}(Y)$, then,

$$\mathcal{W}_{p;E}(\mu, \nu) = \inf_{X,Y} \mathbb{E}\left[d(X, Y)^p\right]^{\frac{1}{p}},$$

where the infimum is taken over all X, Y as above.

The Wasserstein metrics play an important role in the study of McKean–Vlasov equations.

Proposition 2.6.3. *Let (E, d_E) be a polish space and for any $p \geq 1$ let $\mathcal{W}_{p;E}$ be the distance defined by (2.6.1). Then the following all hold:*

(i) The distance $\mathcal{W}_{p;E}$ satisfies the properties of a metric on $\mathcal{P}_p(E)$. Furthermore $(\mathcal{P}_p, \mathcal{W}_{p;E})$ is itself polish.

(ii) For any pair μ, $\nu \in \mathcal{P}_p(E)$ there exists a measure $\bar{m} \in \Pi(\mu, \nu)$ such that,

$$\mathcal{W}_{p;E}(\mu, \nu) = \left(\iint_{E\times E} d_E(x, y)^p \, \mathrm{d}\bar{m}(x, y)\right)^{\frac{1}{p}}.$$

(iii) Let $(\mu^n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{P}_p(E)$, then the following are equivalent:

(a) There exists a $\mu \in \mathcal{P}_p(E)$ such that $\lim_{n\to\infty} \mathcal{W}_{p;E}(\mu^n, \mu) = 0$

(b) The sequence converges weakly to $\mu \in \mathcal{P}_p(E)$ and there exists an $e_0 \in E$ such that

$$\lim_{k\to\infty} \int_{E\setminus B_k(\xi)} d(e_0, x)^p \, \mathrm{d}\mu^n(x) = 0, \text{ uniformly in } n \in \mathbf{N}.$$

We refer the reader to [3, 108] for more details. Point (i) in particular is proved in [108, Ch. 1] and Point (iii) is proved as [3, Prop. 7.1.5].

The $\mathcal{W}_{1;E}$ metric will play a central role in our analysis. By the Kantorovich–Rubinstein duality (Theorem 2.6.4 below) we see that $\mathcal{W}_{1;E}$ can be written as the restriction of a norm on the linear space $\mathcal{M}_1(E)$ to $\mathcal{P}_1(E)$. This allows us to define a notion of Hölder continuous measure flows $t \mapsto \mu_t \in \mathcal{P}_1(\mathbb{R}^d)$, see Definition 2.6.7 below.

For a complete metric space E and a map $\varphi : E \to \mathbb{R}$ we define the Lipschitz constant of φ by setting

$$[\varphi]_{\mathrm{lip}(E)} := \sup_{x \neq y \in E} \frac{|\varphi(x) - \varphi(y)|}{d_E(x, y)},$$

and then we define the set

$$\mathrm{lip}_1(E) := \{\varphi : E \to \mathbb{R} \,:\, [\varphi]_{\mathrm{lip}} \leq 1\}\,.$$

For $\mu \in \mathcal{M}_1^0(E)$ we define its Lipschitz dual norm by the expression

$$\|\mu\|_{\mathrm{lip}^*(E)} := \sup_{\varphi \in \mathrm{lip}_1(E)} \int_E \varphi \,\mathrm{d}\mu. \tag{2.6.2}$$

Given $\mu, \nu \in \mathcal{P}_1(E)$, while $\mu - \nu \notin \mathcal{P}(E)$ the difference is in $\mathcal{M}_1^0(E)$ and so $\|\mu - \nu\|_{\mathrm{lip}^*(E)}$ is well defined. The Kantorovich–Rubinstein theorem states that this quantity is equal to the 1-Wasserstein distance.

Theorem 2.6.4 (Kantorovich–Rubinstein Duality)**.** *Let (E, d_E) be a Polish space and $\mathrm{lip}_1(E)$ be as defined above. Then for all μ, $\nu \in \mathcal{P}_1(E)$ we have the equality*

$$\mathcal{W}_{1;E}(\mu, \nu) = \|\mu - \nu\|_{\mathrm{lip}^*(E)}. \tag{2.6.3}$$

Furthermore, it does not affect the norm on the right hand side if we further restrict the supremum to all $\varphi \in lip_1(E) \cap C_b(E)$.

Proof. See the proof of [108, Th. 1.14]. □

Remark 2.6.5. Although Theorem 2.6.4 is referred to as the Kantorovich–Rubesntein duality the dual quantity in our case is actually the Lipschitz dual norm. This discrepancy is resolved when E is a compact metric space. We define the $\mathrm{KR}(E)$ norm on $\mathcal{M}_1(E)$ by the expression

$$\|\mu\|_{\mathrm{KR}(E)} = \sup_{\varphi \in \mathrm{Lip}_1(E)} \int_E \varphi \, \mathrm{d}\mu, \qquad \mathrm{Lip}_1(E) := \{\varphi : E \to \mathbb{R} \, : \, \|\varphi\|_C + [\varphi]_{\mathrm{lip}} \leq 1\} .$$

Then it is easily seen that when E has finite diameter it is equivalent to restrict the supremum to $\varphi : E \to \mathbb{R}$ such that $[\varphi]_{\mathrm{lip}} < 1$ and $\varphi(x) = 0$ for some $x \in E$. Then, since for any μ, $\nu \in \mathcal{P}_1(E)$ the difference $\mu - \nu$ is in $\mathcal{M}_1^0(E)$ and so integrates constants to zero, one has

$$\|\mu - \nu\|_{\mathrm{lip}*(E)} = \|\mu - \nu\|_{\mathrm{KR}(E)}.$$

Remark 2.6.6. It is a classical result that unless E is a finite space, any complete metric on $\mathcal{M}(E)$ is equivalent to the total variation metric, which metrizes the topology of strong convergence. Therefore it is clear that neither $\big(\mathcal{M}_1(E), \|\cdot\|_{\mathrm{lip}*(E)}\big)$ nor $\big(\mathcal{M}_1(E), \|\cdot\|_{\mathrm{KR}(E)}\big)$ are complete metric spaces. However, combining Theorem 2.6.4 and Point (i) of Proposition 2.6.3 one sees that $(\mathcal{P}_1(E), \|\cdot\|_{\mathrm{lip}*(E)})$ is complete. Since $(\mathcal{C}_T^\beta(\mathbb{R}^d), \|\cdot\|_\beta)$ is a Banach space all the results of the previous section apply to the Wasserstein metrics $\mathcal{W}_{p;\mathcal{C}_T^\beta}$. In particular the space $(\mathcal{P}_1(\mathcal{C}_T^\beta), \mathcal{W}_{1;\mathcal{C}_T^\beta})$ is itself a Polish space when equipped with the metric,

$$\mathcal{W}_{1;\mathcal{C}_T^\beta}(\mu, \nu) = \|\mu - \nu\|_{\mathrm{lip}^*(\mathcal{C}_T^\beta)}.$$

From now on, when E is a Polish space, unless otherwise specified we always treat $\mathcal{P}_p(E)$ as being equipped with the metric $\mathcal{W}_{p;E}$. When the context is clear we will simply write $\|\cdot\|_{\mathrm{lip}^*}$, dropping the explicit dependence on E.

2.6.1 Hölder Regularity of Measure Valued Flows

Let $f \in C_T$, then for every $t \in [0,T]$ we define the projection $\pi_t : C_T \to \mathbb{R}^d$ to be the map such that $\pi_t f := f_t$. Then for $\mu \in \mathcal{M}(C_T)$, we set $\mu_t := \pi_t \# \mu \in \mathcal{M}(\mathbb{R}^d)$. Using the definition of the push-forward, we see that if $\mu \in \mathcal{P}(\mathcal{C}_T^\beta)$, then

$$\int_{\mathbb{R}^d} |x|^p \, \mathrm{d}\mu_t(x) = \int_{\mathcal{C}_T^\beta} |f_t|^p \, \mathrm{d}\mu(f) \lesssim \int_{\mathcal{C}_T^\beta} \|f\|_{\beta;T}^p \, \mathrm{d}\mu(f), \quad \forall\, t \in [0,T]. \tag{2.6.4}$$

So $\mu \in \mathcal{P}_p(\mathcal{C}_T^\beta) \Rightarrow \mu_t \in \mathcal{P}_p(\mathbb{R}^d)$ for every $t \in [0,T]$. In particular the $\mathrm{lip}^*(\mathbb{R}^d)$ norm of μ_t is well defined for every $t \in [0,T]$. We use this fact to define a notion of Hölder continuity for $\mathcal{P}_1(\mathbb{R}^d)$ valued measure flows.

Definition 2.6.7. Let $\beta \in (0,1)$, $0 \le s < t < \infty$ and $[s,t] \ni u \mapsto \mu_u \in \mathcal{M}_1(\mathbb{R}^d)$ be a flow of Radon measures. Then we say that $(\mu_u)_{u \in [s,t]}$ is β-Hölder continuous if

$$[\mu]_{\beta;[s,t]} := \sup_{u \neq v \, \in [s,t]} \frac{\|\mu_v - \mu_u\|_{\mathrm{lip}^*(\mathbb{R}^d)}}{|v-u|^\beta} < \infty. \tag{2.6.5}$$

We write $\mathcal{C}^\beta_{[s,t]}\mathcal{P}_1(\mathbb{R}^d) := \mathcal{C}^\beta([s,t]; \mathcal{P}_1(\mathbb{R}^d), ||| \cdot ; \cdot |||_{\beta;[s,t]})$ for the space of $\mathcal{P}_1(\mathbb{R}^d)$ valued flows, equipped with the metric,

$$|||\mu;\nu|||_{\beta;[s,t]} := \|\mu_0 - \nu_0\|_{\mathrm{lip}^*(\mathbb{R}^d)} + [\mu - \nu]_{\beta;[s,t]}.$$

We use the unusual notation $||| \cdot ; \cdot |||_{\beta;[s,t]}$ since the space of $\mathcal{P}_1(\mathbb{R}^d)$ valued flows is not linear. As with real valued Hölder continuous maps, we retain the convention that if $[s,t] = [0,T]$ for some $T > 0$, we simply write $[\,\cdot\,]_{\beta;T}$, $||| \cdot , \cdot |||_{\beta;T}$ and $\mathcal{C}^\eta \mathcal{P}_1(\mathbb{R}^d)$. For $\mu \in \mathcal{C}^\beta([0,T]; \mathcal{P}_1(\mathbb{R}^d))$ and $\beta' \in (0,\beta)$, one has $[\mu]_{\beta';[s,t]} \le |t-s|^{\beta-\beta'}[\mu]_{\beta;[s,t]}$ for any $[s,t] \subseteq [0,T]$.

Theorem 2.6.8. *The push-forward of the projection map π_t gives a continuous embedding from $(\mathcal{P}_1(\mathcal{C}_T^\beta), \mathcal{W}_{1;\mathcal{C}_T^\beta})$ into $(\mathcal{C}_T^\beta \mathcal{P}_1(\mathbb{R}^d), ||| \cdot ; \cdot |||_{\beta;T})$ and for μ, $\nu \in \mathcal{P}_1(\mathcal{C}_T^\beta)$,*

$$|||\mu;\nu|||_{\beta;T} \le \mathcal{W}_{1;\mathcal{C}_T^\beta}(\mu,\nu). \tag{2.6.6}$$

Proof. Let $\mu,\, \nu \in \mathcal{P}_1(\mathcal{C}^\beta_T)$ and we define the associated measure flows $(\mu_t)_{t\in[0,T]},\ (\nu_t)_{t\in[0,T]} \subset \mathcal{P}_1(\mathbb{R}^d)$ via the projection $\pi_t : \mathcal{C}^\beta_T \to \mathbb{R}^d$. From (2.6.4) we see that for all $t \in [0,T]$, $\mu_t,\, \nu_t \in \mathcal{P}_1(\mathbb{R}^d)$. Then let $\varphi \in \mathrm{lip}_1(\mathbb{R}^d)$ and using the push-forward, we have

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(x)\ \mathrm{d}(\mu_t(x) - \nu_t(x) - \mu_s(x) + \nu_s(x)) &= \int_{\mathcal{C}^\beta_T} (\varphi(f_t) - \varphi(f_s))\ \mathrm{d}(\mu(f) - \nu(f)) \\ &\leq \int_{\mathcal{C}^\beta_T} |f_t - f_s|\, \mathrm{d}(\mu(f) - \nu(f)),\end{aligned}$$

where we used the fact that $\varphi \in \mathrm{lip}_1(\mathbb{R}^d)$ in the last line. Dividing by $|t-s|^\beta$ and taking the supremum over $s \neq t\ \in [0,T]$, we have the bound,

$$\begin{aligned}\sup_{t\neq s\in[0,T]} \frac{1}{|t-s|^\beta} \int_{\mathbb{R}^d} \varphi(x)\ \mathrm{d}(\mu_t(x) - \nu_t(x) - \mu_s(x) + \nu_s(x)) &\leq \int_{\mathcal{C}^\beta_T} [f]_{\beta;T}\ \mathrm{d}(\mu(f) - \nu(f)) \\ &\leq \sup_{\phi\in\mathrm{lip}_1(\mathcal{C}^\beta_T)} \int_{\mathcal{C}^\beta_T} \phi(f)\ \mathrm{d}(\mu(f) - \nu(f)),\end{aligned}$$

where the last inequality follows since $[\,\cdot\,]_\beta$ is a lip_1 function on $\mathcal{C}^\beta_T$. Therefore we have

$$[\mu - \nu]_{\beta;T} \leq \|\mu - \nu\|_{\mathrm{lip}*(\mathcal{C}^\beta_T)} = \mathcal{W}_{1;\mathcal{C}^\beta_T}(\mu,\nu).$$

By the same steps, we have

$$\begin{aligned}\|\mu_0 - \nu_0\|_{\mathrm{lip}^*(\mathbb{R}^d)} + [\mu - \nu]_{\beta;T} &\leq \int_{\mathcal{C}^\beta_T} (|f_0| + [f]_{\beta;T})\ \mathrm{d}(\mu(f) - \nu(f)) \\ &= \int_{\mathcal{C}^\beta_T} \|f\|_{\beta;T}\, \mathrm{d}(\mu(f) - \nu(f)),\end{aligned}$$

from which (2.6.6) follows. □

Lemma 2.6.9. *The metric space* $(\mathcal{C}^\beta_T \mathcal{P}_1(\mathbb{R}^d), |\!|\!|\cdot;\cdot|\!|\!|_{\beta;T})$ *is complete.*

Proof. From Proposition 2.6.3 we have that $(\mathcal{P}_1(\mathbb{R}^d), \|\cdot\|_{\mathrm{lip}^*})$ is a complete metric space. Therefore, a minor modification of the usual proof that the space of real valued α-Hölder functions is complete shows that $(\mathcal{C}^\beta_T \mathcal{P}_1(\mathbb{R}^d), |\!|\!|\cdot;\cdot|\!|\!|_{\beta;T})$ is complete.

□

At last we mention two useful results regarding the interplay between Wasserstein distances of time marginals and Hölder regularity of measure valued flows.

Lemma 2.6.10. *For any two probability measures $\mu, \nu \in \mathcal{P}(\mathcal{C}^\beta_T)$, and $t \in [0,T]$, we have that*

$$\mathcal{W}_{1,\mathbb{R}^d}(\mu_t, \nu_t) \leq \mathcal{W}_{1;\mathbb{R}^d}(\mu_0, \nu_0) + T^\beta \mathcal{W}_{1;\mathcal{C}^\beta_T}(\mu, \nu). \tag{2.6.7}$$

Proof. From (2.6.3), for any $t > 0$, we see that

$$\begin{aligned}\mathcal{W}_{1;\mathbb{R}^d}(\mu_t, \nu_t) &= \|\mu_t - \nu_t\|_{\mathrm{KR}(\mathbb{R}^d)} \\ &\leq \|\mu_0 - \nu_0\|_{\mathrm{KR}(\mathbb{R}^d)} + T^\beta [\mu - \nu]_{\beta;T} \\ &\leq \mathcal{W}_{1;\mathbb{R}^d}(\mu_0, \nu_0) + T^\beta \mathcal{W}_{1;\mathcal{C}^\beta_T}(\mu, \nu).\end{aligned}$$

□

The next lemma is a variation of Lemma 2.4.3 for measure valued flows.

Lemma 2.6.11. *Let $T > 0$, $\alpha \in (0,1)$ and μ^1, $\mu^2 \in \mathcal{P}(C^\alpha_T)$. Suppose there exists a constant $M > 0$ and $h \in (0,T]$ such that for any $t \in [0, T-h]$,*

$$\mathcal{W}_{1;C^\alpha_{[t,t+h]}}(\mu^1, \mu^2) \leq M. \tag{2.6.8}$$

Then

$$\mathcal{W}_{1;C^\alpha_T}(\mu^1, \mu^2) \leq \mathcal{W}_{1;\mathbf{R}^d}(\mu^1_0, \mu^2_0) + M(1 \vee 2h^{-1}T). \tag{2.6.9}$$

Proof. We first note that for μ^1, $\mu^2 \in \mathcal{P}(\mathcal{C}^\alpha_T)$ there exists an optimal coupling given by random variables Y^1, $Y^2 \in \mathcal{C}^\alpha_T$ on a common probability space such that

$$\mathcal{W}_{1;\mathcal{C}^\alpha_{[t,t+h]}}(\mu^1, \mu^2) = \mathbb{E}\left[\|Y^1 - Y^2\|_{\alpha;[t,t+h]}\right] = \mathbb{E}\left[|Y_t - Y_t|_{\mathbf{R}^d}\right] + \mathbb{E}\left[[Y^1 - Y^2]_{\alpha;[t,t+h]}\right],$$

and in particular,

$$\mathcal{W}_{1;\mathcal{C}^\alpha_T}(\mu^1, \mu^2) = \mathbb{E}\left[\|Y^1 - Y^2\|_{\alpha;T}\right] = \mathcal{W}_{1;\mathbf{R}^d}(\mu^1_0, \mu^2_0) + \mathbb{E}\left[[Y^1 - Y^2]_{\alpha;T}\right].$$

From (2.6.8) it follows that for any $t \in [0, T-h]$,

$$\mathbb{E}\left[[Y^1 - Y^2]_{\alpha;[t,t+h]}\right] \le M$$

and so in order to conclude it suffices to show that for a given random variable X, if

$$\mathbb{E}\left[[X]_{\alpha;[t,t+h]}\right] \le M,$$

uniformly over $t \in [0, T-h]$, then

$$\mathbb{E}\left[[X]_{\alpha;T}\right] \le M\left(1 \vee 2h^{-1}T\right).$$

To this end, we will follow a similar procedure as used in the proof of Lemma 2.4.3. We begin to observe that

$$\mathbb{E}\left[[X]_{\alpha;T}\right] = \mathbb{E}\left[\sup_{\substack{0\le s<t\le T\\ |t-s|<h}} \frac{|X_{s,t}|}{|t-s|^\alpha}\right] + \mathbb{E}\left[\sup_{\substack{0\le s<t\le T\\ |t-s|\ge h}} \frac{|X_{s,t}|}{|t-s|^\alpha}\right] \le M + \mathbb{E}\left[\sup_{\substack{0\le s<t\le T\\ |t-s|\ge h}} \frac{|X_{s,t}|}{|t-s|^\alpha}\right].$$

To control the second term let us define $t_i = (s+ih) \wedge t$ for $i \in \mathbf{N}$ so that for $N \ge (t-s)/h$, $t_N = t$ and $|t_{i+1} - t_i| \le h$ for all $i = \{1, \dots, N-1\}$. Also note that since $\sup_{t\in[0,T-h]} \mathbb{E}[[X]_{\alpha;[t,t+h]}] \le M$ the random variables $[X]_{\alpha;[t,t+h]}$ is $\mathbb{P}$-a.s. finite. We therefore have,

$$\begin{aligned}
\mathbb{E}\left[\sup_{\substack{0\le s<t\le T\\ |t-s|\ge h}} \frac{|X_{s,t}|}{|t-s|^\alpha}\right] &\le \mathbb{E}\left[\sup_{\substack{0\le s<t\le T\\ |t-s|\ge h}} \frac{1}{|t-s|^\alpha} \sum_{0\le i<(t-s)/h} [X]_{t_i,t_i+h} h^\alpha\right] \\
&\le \sum_{0\le i\le T/h} \mathbb{E}\left[[X]_{t_i,t_i+h}\right] \\
&\le 2h^{-1}TM,
\end{aligned}$$

where in the last passage we have used that $(1 + \frac{T}{h}) \le 2Th^{-1}$. This concludes the proof. □

Chapter 3

Pathwise Regularisation of McKean–Vlasov Equations with Singular Potentials

In this chapter we employ the results on non-linear Young integration discussed in Section 2.5 and the notions of Hölder continuous measure flows introduced in Section 2.6 to demonstrate existence, uniqueness and stability for non-linear Young equations of McKean–Vlasov type. The non-linear Young equations we consider are generalised examples of (2.3.1) and in Section 3.3 we show how the abstract results here apply to equations in the form of (2.3.1).

3.1 Well-Posedness and Stability of Distribution Dependent non-linear Young Equations

For the rest of this section we fix $T > 0$, $(\gamma, \eta) \in \left(\frac{1}{2}, 1\right) \times (0, \infty)$ such that

$$(\eta \wedge \gamma) + \gamma > 1, \tag{3.1.1}$$

and $(\Omega, \mathcal{F}, \mathbb{P})$ an abstract probability space. All laws of random variables will be taken with respect to $\mathbb{P}$. These are the same standing assumptions as those made at the beginning of Section 2.3.

The equations we consider in this section are of the form

$$Y_t = \xi + \int_0^t (\Gamma_{\mathrm{d}r} * \mu_r)(Y_r) + B_t, \quad \mu = \mathcal{L}(Y), \tag{3.1.2}$$

where $\Gamma \in \mathcal{C}_T^\gamma \mathcal{C}^\alpha(\mathbb{R}^d)$ for some $\alpha \geq 1$, is a given function, and the drift term is rigorously defined in Subsection 3.1.1 below. Throughout this section Γ will be assumed to be a given space-time function, not necessarily an averaged field of any particular kernel. In Section 3.3 we show how such Γ can be built from a wide range of distributions $K \in \mathcal{S}'(\mathbb{R}^d)$ and local times associated to regularising paths $Z \in C_T$. We fix a solution concept for (3.1.2).

Definition 3.1.1. Let $\alpha \geq 1$, $\Gamma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be such that $\Gamma \in \mathcal{C}_T^\gamma \mathcal{C}^\alpha(\mathbb{R}^d)$ and $(\xi, B) \in L^1(\Omega; \mathbb{R}^d \times \mathcal{C}_T^\eta)$, with $B_0 = 0$. Then we say that a random variable $Y : \Omega \to \mathcal{C}_T^{\eta\wedge\gamma}$ is a solution to the non-linear Young equation of McKean–Vlasov type if for any $t \in (0,T]$ the identity,

$$Y_t(\omega) = \xi(\omega) + \int_0^t (\Gamma_{\mathrm{d}r} * \mu_r)(Y_r(\omega)) + B_t(\omega), \quad \mu = \mathcal{L}(Y) \tag{3.1.3}$$

holds for $\mathbb{P}$-a.a. $\omega \in \Omega$, where the integral is understood as a non-linear Young integral with measure dependence, properly defined in Lemma 3.1.5.

For any $s \in (0,T)$ and $h \in (0, T-s]$, given the interval $[s, s+h] \subset (0,T]$ and data $(x_s, B) \in L^1(\Omega; \mathbb{R}^d \times \mathcal{C}_T^\eta)_{[s,s+h]}$ we say that $Y : \Omega \to \mathcal{C}_T^{\eta\wedge\gamma}$ is a solution to the non-linear Young equation on $[s, s+h]$ if for any $t \in [s, s+h]$ the identity,

$$Y_t(\omega) = x_s(\omega) + \int_s^t (\Gamma_{\mathrm{d}r} * \mu_r)(Y_r(\omega)) + B_t(\omega) - B_s(\omega), \quad \mu = \mathcal{L}(Y)|_{[s,s+h]}, \tag{3.1.4}$$

holds for $\mathbb{P}$-a.a. $\omega \in \Omega$.

It is immediate from the definition that any solution to the generalised McKean–

Vlasov problem (3.1.2) satisfy the semi-group property. More precisely, if for any $s \leq t \in [0,T]$ and $(\xi, B) \in \mathbb{R}^d \times \mathcal{C}^\eta_T$ as above, we let

$$S_{[s,t]}(\xi, B) := \xi + \int_s^t \Gamma_{\mathrm{d}r} * \mu_r(Y_r) + B_t - B_s, \quad \mu = \mathcal{L}(Y)$$

then the identity

$$S_{[0,t]}(\xi, B) = S_{[s,t]}\left(S_{[0,s]}(\xi, B), B\right),$$

holds $\mathbb{P}$-almost surely.

Remark 3.1.2. It follows that if B is a Markov process on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]})$ then any solution $Y : \Omega \times [0,T] \to \mathbb{R}^d$ will be too. However, it is also easily seen that in this setting the solution Y cannot satisfy the strong Markov property, since the law of the stopped process is not equal to the law of the un-stopped process evaluated at the random time.

The main result of this section is the following abstract equivalent of Theorem 2.3.2.

Theorem 3.1.3. *Let $\alpha \geq 2$, $\Gamma \in \mathcal{C}^\gamma_T \mathcal{C}^\alpha(\mathbb{R}^d)$, satisfying the assumptions of Lemma 3.1.7 below, and $(\xi, B) \in L^1(\Omega; \mathbb{R}^d) \times L^p(\Omega; \mathcal{C}^\eta_T)$, such that $B_0 = 0$. Then there exists a unique solution $Y \in L^1(\Omega; \mathcal{C}^{\eta\wedge\gamma}_T)$ to the non-linear Young equation of McKean–Vlasov type, (3.1.2), in the sense of Definition 3.1.1.*
Furthermore, if $(\xi^1, B^1), (\xi^2, B^2) \in L^1(\Omega; \mathbb{R}^d) \times L^p(\Omega; \mathcal{C}^\eta_T)$, for any $p \geq 1$, are two pairs of data, then given the corresponding solutions $Y^1, Y^2 \in L^1(\Omega; \mathcal{C}^{\eta\wedge\gamma}_T)$, defining $\mu^1 = \mathcal{L}(Y^1)$, $\mu^2 = \mathcal{L}(Y^2)$ and for any $\beta \in (1-\gamma, \eta\wedge\gamma)$ choose $q = \frac{2-\beta}{\gamma-\beta}$, there exists a constant

$$C := C\left(T, \mathbb{E}\left[[B^1]^{2q}_{\eta;T}\right] \vee \mathbb{E}\left[[B^2]^{2q}_{\eta;T}\right], \Gamma, \gamma, \eta, \beta\right) > 0$$

such that

$$\mathcal{W}_{1;\mathcal{C}^\beta_T}(\mu^1, \mu^2) \leq C\left(\mathcal{W}_{1;\mathbb{R}^d}(\mathcal{L}(\xi^1), \mathcal{L}(\xi^2)) + \mathcal{W}_{2;\mathcal{C}^\eta_T}(\mathcal{L}(B^1), \mathcal{L}(B^2))\right). \tag{3.1.5}$$

Remark 3.1.4. Note here that while the constant C depends only on the $2q$ moments of B for $q = \frac{2-\beta}{\gamma-\beta}$, since $\beta \in (1-\gamma, \eta\wedge\gamma)$ is arbitrary q can in fact be arbitrarily large hence the requirement for for B, B^1, B^2 to have all finite moments.

In the remainder of this section we first extend the definition of the non-linear Young integral to integrands involving functions convolved with the time marginals of a measure flow. We also obtain stability estimates on the non-linear Young integral (NLYI) with respect to the measure and spatial trajectory. This is all done in Subsection 3.1.1. Then we prove Theorem 3.1.3 in two stages; firstly in Subsection 3.1.2 we freeze a path-measure $\mu \in \mathcal{P}_1\left(\mathcal{C}_T^{\eta\wedge\gamma}\right)$ and demonstrate existence and uniqueness of solutions Y^μ to the dynamics of (3.1.2) with μ fixed. Then using the stability in measure of the NLYI we show by a fixed point argument the existence of unique solutions to the full McKean–Vlasov type non-linear Young equation and the associated stability bound (3.1.5).

3.1.1 Non-Linear Young Integration for Measure Dependent Integrands

We extend the notion of non-linear Young integration to include measure dependent integrands. We make use of the results presented in Section 2.6. For completeness we include proofs of many results even if they closely reflect those already obtained in the literature for NLYI without measure dependence.

Lemma 3.1.5. *Let $\alpha \geq 1$, $\Gamma : [0,T] \times \mathbb{R}^d$ be in $\mathcal{C}_T^\gamma \mathcal{C}^\alpha(\mathbb{R}^d)$ and be such that for all $s < t \in [0,T]$ and $x, y \in \mathbb{R}^d$*

$$\begin{aligned} &\text{(i)} \qquad |\Gamma_{s,t}(x)| + |\nabla \Gamma_{s,t}(x)| \lesssim |t-s|^\gamma \\ &\text{(ii)} \qquad |\Gamma_{s,t}(x) - \Gamma_{s,t}(y)| \lesssim |t-s|^\gamma |x-y|. \end{aligned} \tag{3.1.6}$$

Let $\beta > 0$ be such that $\gamma + \beta > 1$ and assume we are given $\mu \in \mathcal{C}_T^\beta \mathcal{P}_1(\mathbb{R}^d)$ and $Y \in \mathcal{C}_T^\beta$. Then there exists a unique path

$$t \mapsto \int_0^t (\Gamma_{\mathrm{d}r} * \mu_r)(Y_r) \in \mathcal{C}^\gamma([0,T], \mathbb{R}^d)$$

constructed as

$$\int_0^t \Gamma_{\mathrm{d}r} * \mu_r(Y_r) := \lim_{|\mathcal{D}|\to 0} \sum_{[u,v]\in\mathcal{D}} (\Gamma_{u,v} * \mu_u)(Y_u), \tag{3.1.7}$$

where $\mathcal{D}$ is a partition of $[0,t]$ with maximal resolution $|\mathcal{D}|$. Moreover, there exists a constant $C > 0$ such that for all $s < t \in [0,T]$

$$\left| \int_s^t \Gamma_{\mathrm{d}r} * \mu_r(Y_r) - \Gamma_{s,t} * \mu_s(Y_s) \right| \le C|s-t|^{\gamma+\beta} \|\Gamma\|_{\gamma,\alpha} \big([Y]_{\beta;[s,t]} + [\mu]_{\beta;[s,t]} \big) . \tag{3.1.8}$$

Remark 3.1.6. For $\beta < \gamma$ the condition $\gamma + \beta > 1$ required by the statement of Lemma 3.1.5 can be relaxed to the condition $\gamma + \beta(\alpha \wedge 1) > 1$, for any $\alpha > 0$, where $\Gamma \in \mathcal{C}_T^\gamma \mathcal{C}^\alpha(\mathbb{R}^d)$, see e.g. [64]. However, in subsequent sections we require $\alpha \ge 2$ in order to obtain the necessary stability estimates, see Lemma 3.1.7, so we directly impose the simpler requirement above.

Proof. We define $\delta_u f_{s,t} := f_{s,t} - f_{s,u} - f_{u,t}$ and $\Xi_{u,v} := \Gamma_{u,v} * \mu_u(Y_u)$, and recall from the sewing lemma, [46, Lem. 4.2], that if

$$|\Xi_{u,v}| \lesssim |v-u|^{\delta_1} \qquad \text{and} \qquad |\delta_z \Xi_{u,v}| \lesssim |v-u|^{\delta_2}, \tag{3.1.9}$$

for $\delta_1 \in (0,1)$ and $\delta_2 > 1$, uniformly in $z \in [u,v]$, then there exists a unique limit of the Riemann sums $\sum_{[u,v]\in\mathcal{D}} \Xi_{u,v}$, along a decreasing sequence of partitions $\mathcal{D}$ of $[0,t]$. In this case there exists a unique function $\mathcal{I}(\Xi)(t)$ such that

$$\mathcal{I}(\Xi)(t) := \lim_{|\mathcal{D}|\to 0} \sum_{[u,v]\in\mathcal{D}} \Xi_{u,v}.$$

We begin by showing the first inequality in (3.1.9). From condition (i) of (3.1.6), Γ is globally bounded in space and is γ-regular in time, therefore we have that

$$|\Xi_{u,v}| = |\Gamma_{u,v} * \mu_u(Y_u)| \le \int_{\mathbb{R}^d} |\Gamma_{u,v}(Y_u - y)| \mu_u(\mathrm{d}y) \le \|\Gamma\|_{\gamma,\alpha} |v-u|^\gamma, \tag{3.1.10}$$

where we used that $\mu_u(\mathbb{R}^d) = 1$. Thus the first bound in (3.1.9) holds. To prove the second inequality in (3.1.9), using the additivity of $t \mapsto \Gamma_t$, we observe that for

$u \leq z \leq v$

$$\delta_z \left(\Gamma_{u,v} * \mu_u(Y_u)\right) = \Gamma_{z,v} * (\mu_u - \mu_z)(Y_u) + (\Gamma_{z,v} * \mu_z(Y_u) - \Gamma_{z,v} * \mu_z(Y_z)). \quad (3.1.11)$$

Considering the second term of (3.1.11), we again use the fact that $\mu_t(\mathbb{R}^d) = 1$ for all $t \in [0, T]$ to obtain by application of (3.1.6) that

$$\begin{aligned} |\Gamma_{z,v} * \mu_z(Y_u) - \Gamma_{z,v} * \mu_z(Y_z)| &\leq \int_{\mathbb{R}^d} |\Gamma_{z,v}(Y_u - y) - \Gamma_{z,v}(Y_z - y)| \mu_z(\mathrm{d}y) \\ &\leq \sup_{y \in \mathbb{R}^d} |\Gamma_{z,v}(Y_u - y) - \Gamma_{z,v}(Y_z - y)| \\ &\leq \|\Gamma\|_{\gamma,\alpha} [Y]_{\beta;[s,t]} |v - u|^{\gamma+\beta}, \end{aligned} \quad (3.1.12)$$

where we have used that $|v - z| \vee |z - u| \leq |v - u|$ in the last line. For the first term of (3.1.11) we first argue that the function,

$$\mathbb{R}^d \ni y \mapsto \varphi(y) := \frac{1}{|v - u|^\gamma \|\Gamma\|_{\gamma,\alpha}} \Gamma_{z,v}(Y_u - y),$$

is 1-Lipschitz continuous. Using (ii) of (3.1.6) we directly find, for $x \neq y \in \mathbb{R}^d$

$$|\varphi(x) - \varphi(y)| \leq \frac{|v - z|^\gamma}{|v - u|^\gamma} \leq 1,$$

where we again used that $|v - z| \vee |u - z| \leq |v - u|$. Hence, using (2.6.5),

$$|\Gamma_{z,v} * (\mu_u - \mu_z)(Y_u)| = |v - u|^\gamma \|\Gamma\|_{\gamma,\alpha} \left| \int_{\mathbb{R}^d} \varphi(y) \, \mathrm{d}(\mu_u - \mu_z)(y) \right| \leq |v - u|^{\gamma+\beta} \|\Gamma\|_{\gamma,\alpha} [\mu]_{\beta;[s,t]}. \quad (3.1.13)$$

Thus since $\gamma + \beta > 1$ by assumption, we conclude that the integral in (3.1.7) is well defined. Moreover, again using [46, Lem. 4.2], we directly obtain the inequality (3.1.8). □

In addition to its construction it will be useful to have estimates on the stability of the non-linear Young integral constructed in (3.1.7) with respect to the path Y and the measure flow μ. The next lemma establishes these bounds under an additional regularity assumption.

Lemma 3.1.7. *Let $\alpha \geq 2$, $\Gamma : [0,T] \times \mathbb{R}^d$ be in $\mathcal{C}^\gamma_T \mathcal{C}^\alpha(\mathbb{R}^d)$ be such that for all $s < t \in [0,T]$ and $x, y \in \mathbb{R}^d$*

$$\begin{aligned} &\text{(i)} \qquad |\Gamma_{s,t}(x)| + |\nabla \Gamma_{s,t}(x)| \lesssim |s-t|^\gamma \\ &\text{(ii)} \qquad |\Gamma_{s,t}(x) - \Gamma_{s,t}(y)| \lesssim |s-t|^\gamma |x-y| \\ &\text{(iii)} \qquad |\nabla\Gamma_{s,t}(x) - \nabla\Gamma_{s,t}(y)| \lesssim |s-t|^\gamma |x-y|. \end{aligned} \tag{3.1.14}$$

Let $\beta > 0$ be such that $\gamma + \beta > 1$ and M_1, $M_2 > 0$. Then assuming we are given two measure flows $\mu, \tilde{\mu} \in \mathcal{C}^\beta_T \mathcal{P}_1(\mathbb{R}^d)$, and two paths $Y, \tilde{Y} \in \mathcal{C}^\beta_T$ such that for any $0 \leq s < t \leq T$, $[\mu]_{\beta;[s,t]} \vee [\tilde{\mu}]_{\beta;[s,t]} \leq M_1$ and $[Y]_{\beta;[s,t]} \vee [\tilde{Y}]_{\beta;[s,t]} \leq M_2$, then

$$\begin{aligned} &\left| \int_s^t [\Gamma_{\mathrm{d}r} * \mu_r(Y_r) - \Gamma_{\mathrm{d}r} * \tilde{\mu}_r(\tilde{Y}_r)] - \Gamma_{s,t} * \mu_s(Y_s) + \Gamma_{s,t} * \tilde{\mu}_s(\tilde{Y}_s) \right| \\ &\qquad \lesssim |s-t|^{\gamma+\beta} \|\Gamma\|_{\gamma,\alpha} (1 + M_1 + M_2) \, \|Y - \tilde{Y}\|_{\beta;[s,t]} \\ &\qquad\quad + |s-t|^{\gamma+\beta} \|\Gamma\|_{\gamma,\alpha} (1 + M_2) \, \|\!|\mu; \tilde{\mu}|\!\|_{\beta;[s,t]} \end{aligned} \tag{3.1.15}$$

where the integral is interpreted as the non-linear Young integral given in Lemma 3.1.5.

Remark 3.1.8. It is crucial that the pre-factors on the right hand side that do not depend on the differences $\mu - \tilde{\mu}$ and $Y - \tilde{Y}$ only depend on the semi-norms in the relevant quantities. Later this is important in proving existence and uniqueness results for the non-linear Young equations as it allows us to obtain contraction bounds independently of the initial data.

Proof of Lemma 3.1.7. Set $\Xi_{s,t} := \Gamma_{s,t} * \mu_s(Y_s) - \Gamma_{s,t} * \tilde{\mu}_s(\tilde{Y}_s)$. To prove (3.1.15) we follow the same strategy as in the proof of Lemma 3.1.5, by invoking the sewing lemma. Recall from [46, Lem. 4.2], that if for any $(s,u,t) \in \Delta^T_3$ the following inequalities are satisfied

$$|\Xi_{s,t}| \lesssim |s-t|^{\delta_1} \qquad \text{and} \qquad |\delta_u \Xi_{s,t}| \lesssim |s-t|^{\delta_2}, \tag{3.1.16}$$

for some $\delta_1 \in (0,1)$ and $\delta_2 > 1$, then

$$\left| \int_s^t \left[\Gamma_{\mathrm{d}r} * \mu_r(Y_r) - \Gamma_{\mathrm{d}r} * \tilde{\mu}_r(\tilde{Y}_r) \right] - \Gamma_{s,t} * \mu_s(Y_s) + \Gamma_{s,t} * \tilde{\mu}_s(\tilde{Y}_s) \right| \lesssim [\delta \Xi]_{\delta_2;[s,t]} |s-t|^{\delta_2},$$

where we have used the notation

$$[\delta \Xi]_{\delta_2;[s,t]} := \sup_{(r,u,v) \in \Delta_3^{[s,t]}} \frac{|\delta_u \Xi_{r,v}|}{|r-v|^{\delta_2}}.$$

We begin by splitting Ξ into two functions, setting

$$\Xi^1_{s,t} := \Gamma_{s,t} * \mu_s(Y_s) - \Gamma_{s,t} * \mu_s(\tilde{Y}_s), \quad \Xi^2_{s,t} := \Gamma_{s,t} * (\mu_s - \tilde{\mu}_s)(\tilde{Y}_s).$$

Similar steps as in the proof of the bound (3.1.10) in Lemma 3.1.5 show that the first bound of (3.1.16) holds for both Ξ^1, Ξ^2, with $\delta_1 = \gamma$. Therefore we concentrate on showing that $|\delta_u \Xi^1_{s,t}|$ and $|\delta_u \Xi^2_{s,t}|$ are both controlled by $|s-t|^{\gamma+\beta}$.

By the fundamental theorem of calculus we can write Ξ^1 as

$$\Xi^1_{s,t} = \int_0^1 \nabla \Gamma_{s,t} * \mu_s(\rho Y_s + (1-\rho)\tilde{Y}_s)\,\mathrm{d}\rho \cdot (Y_s - \tilde{Y}_s).$$

It is readily checked that for $0 \le s \le u \le t \le T$

$$\delta_u \Xi^1_{s,t} = \int_0^1 \nabla \Gamma_{s,u} * \mu_u(\rho Y_u + (1-\rho)\tilde{Y}_u)\,\mathrm{d}\rho \cdot (Y_u - \tilde{Y}_u) - \int_0^1 \nabla \Gamma_{s,u} * \mu_t(\rho Y_t + (1-\rho)\tilde{Y}_t)\,\mathrm{d}\rho \cdot (Y_t - \tilde{Y}_t).$$

By adding and subtracting $\int_0^1 \nabla \Gamma_{s,u} * \mu_u(\rho Y_u + (1-\rho)\tilde{Y}_u) d\rho \cdot (Y_t - \tilde{Y}_t)$ in the equality above, we obtain the two differences

$$\mathfrak{D}^1_{s,u,t} = \left(\int_0^1 \nabla \Gamma_{s,u} * \mu_t(\rho Y_t + (1-\rho)\tilde{Y}_t)\,\mathrm{d}\rho - \int_0^1 \nabla \Gamma_{s,u} * \mu_u(\rho Y_u + (1-\rho)\tilde{Y}_u)\,\mathrm{d}\rho \right) \cdot (Y_t - \tilde{Y}_t)$$
$$\mathfrak{D}^2_{s,u,t} = \int_0^1 \nabla \Gamma_{s,u} * \mu_u(\rho Y_u + (1-\rho)\tilde{Y}_u)\,\mathrm{d}\rho \cdot (Y_t - \tilde{Y}_t - Y_u + \tilde{Y}_u).$$

Considering $\mathfrak{D}^1$, we add and subtract the term $\int_0^1 \nabla \Gamma_{s,u} * \mu_u(\rho Y_t + (1-\rho)\tilde{Y}_t)\,\mathrm{d}\rho$, and define,

$$\mathfrak{D}^{1,1}_{s,u,t} := \int_0^1 \nabla\Gamma_{s,u} * (\mu_t - \mu_u)(\rho Y_t + (1-\rho)\tilde{Y}_t)\,\mathrm{d}\rho \cdot (Y_t - \tilde{Y}_t),$$
$$\mathfrak{D}^{1,2}_{s,u,t} := \int_0^1 \Big(\nabla\Gamma_{s,u} * \mu_u(\rho Y_t + (1-\rho)\tilde{Y}_t) - \nabla\Gamma_{s,u} * \mu_u(\rho Y_u + (1-\rho)\tilde{Y}_u)\Big)\,\mathrm{d}\rho \cdot (Y_t - \tilde{Y}_t)$$

In order to bound the term $\mathfrak{D}^{1,1}_{s,u,t}$ we use a similar argument as we used to obtain (3.1.13). Using (iii) of (3.1.19) we first see that for any $\rho \in [0,1]$, the map

$$\mathbb{R}^d \ni y \mapsto \varphi_\rho(y) := \frac{1}{|t-s|^\gamma \|\Gamma\|_{\gamma,\alpha}} \nabla\Gamma_{s,u}(\rho Y_t + (1-\rho)\tilde{Y}_t - y),$$

is 1-Lipschitz continuous. It then follows, again from (2.6.5), that

$$\begin{aligned} |\mathfrak{D}^{1,1}_{s,u,t}| &\le |t-s|^\gamma \|\Gamma\|_{\gamma,\alpha} \left| \int_0^1 \int_{\mathbb{R}^d} \varphi_\rho(y)\ \mathrm{d}(\mu_u - \mu_t)(y)\ \mathrm{d}\rho \right|\ |Y_t - \tilde{Y}_t| \\ &\le |t-s|^{\gamma+\beta} \|\Gamma\|_{\gamma,\alpha} [\mu]_{\beta;[s,t]}\ \|Y - \tilde{Y}\|_{\beta;[s,t]}. \end{aligned}$$

We now bound the $\mathfrak{D}^2$ term; again using similar steps as in the proof of (3.1.12), we have

$$\|\mathfrak{D}^2_{s,u,t}\| \lesssim |t-s|^{\gamma+\beta} \|\Gamma\|_{\gamma,\alpha} \|Y - \tilde{Y}\|_{\beta;[s,t]}.$$

Combining the bounds for $\mathfrak{D}^1$ and $\mathfrak{D}^2$ gives

$$|\delta_u \Xi^1_{s,t}| \lesssim |t-s|^{\gamma+\beta} \|\Gamma\|_{\gamma,\alpha} (1 + M_1 + M_2)\ \|Y - \tilde{Y}\|_{\beta;[s,t]}. \tag{3.1.17}$$

Concerning the bound on $|\delta_u \Xi^2_{s,t}|$, we first divide the expression into two parts and then invoking inequalities similar to (3.1.12) and (3.1.13) we obtain,

$$\begin{aligned} |\delta_u \Xi^2_{s,t}| &\lesssim |\Gamma_{u,t} * (\mu_s - \tilde{\mu}_s - \mu_u + \tilde{\mu}_u)(\tilde{Y}_s)| + |\Gamma_{u,t} * (\mu_u - \tilde{\mu}_u)(\tilde{Y}_t) - \Gamma_{u,t} * (\mu_u - \tilde{\mu}_u)(\tilde{Y}_u)| \\ &\lesssim |t-s|^{\gamma+\beta} \|\Gamma\|_{\gamma,\alpha} \Big(1 + [\tilde{Y}]_{\beta;[s,t]}\Big)\ |||\mu; \tilde{\mu}|||_{\beta;[s,t]}. \end{aligned} \tag{3.1.18}$$

So combining (3.1.17) and (3.1.18), we see that

$$|\delta_u \Xi_{s,t}| \lesssim |t-s|^{\gamma+\beta} \|\Gamma\|_{\gamma,\alpha}(1+M_1+M_2) \|Y-\tilde{Y}\|_{\beta;[s,t]} + |t-s|^{\gamma+\beta} \|\Gamma\|_{\gamma,\alpha}(1+M_2) \|\!|\mu;\tilde{\mu}|\!\|_{\beta;[s,t]}.$$

This shows that the second of the two conditions in (3.1.16) is satisfied, with $\delta_2 = \gamma+\beta > 1$. The estimate (3.1.15) then also follows from the sewing lemma, [46, Lem. 4.2]. □

Corollary 3.1.9. *Let $\alpha \geq 2$, $\beta > 0$, be such that $\gamma+\beta > 1$, M_1, $M_2 > 0$ and $\Gamma \in \mathcal{C}^\gamma_T \mathcal{C}^\alpha(\mathbb{R}^d)$, satisfying the assumptions of Lemma 3.1.7. Then, given μ, $\tilde{\mu} \in \mathcal{C}^\beta_T \mathcal{P}_1(\mathbb{R}^d)$ and Y, $\tilde{Y} \in \mathcal{C}^\beta_T$, such that for any $0 \leq s < t \leq T$, we have $[\mu]_{\beta;[s,t]} \vee [\tilde{\mu}]_{\beta;[s,t]} \leq M_1$ and $[Y]_{\beta;[s,t]} \vee [\tilde{Y}]_{\beta;[s,t]} \leq M_2$,*

$$\begin{aligned}\left| \int_s^t \Gamma_{\mathrm{d}r} * \mu_r(Y_r) - (\Gamma_{\mathrm{d}r} * \tilde{\mu}_r)(\tilde{Y}_r) \right| &\lesssim_{\gamma,\beta} |t-s|^{\gamma+\beta} \|\Gamma\|_{\gamma,\alpha}(1+M_1+M_2)\|Y-\tilde{Y}\|_{\beta;[s,t]} \\ &\quad + |t-s|^{\gamma+\beta} \|\Gamma\|_{\gamma,\alpha}(1+M_2)\|\!|\mu;\tilde{\mu}|\!\|_{\beta;[s,t]} \\ &\quad + |t-s|^{\gamma} \|\Gamma\|_{\gamma,\alpha} \Big(|Y_s - \tilde{Y}_s| + \|\mu_s - \tilde{\mu}_s\|_{KR} \Big).\end{aligned} \tag{3.1.19}$$

Proof. This follows directly from Lemma 3.1.7 in combination with the triangle inequality, where we observe that

$$\begin{aligned}\left| \int_s^t \Gamma_{\mathrm{d}r} * \mu_r(Y_r) - \Gamma_{\mathrm{d}r} * \tilde{\mu}_r(\tilde{Y}_r) \right| &\leq \left| \Gamma_{s,t} * \mu_s(Y_s) - \Gamma_{s,t} * \tilde{\mu}_s(\tilde{Y}_s) \right| \\ &\quad + \left| \int_s^t \Big[\Gamma_{\mathrm{d}r} * \mu_r(Y_r) - \Gamma_{\mathrm{d}r} * \tilde{\mu}_r(\tilde{Y}_r) \Big] - \Gamma_{s,t} * \mu_s(Y_s) + \Gamma_{s,t} * \tilde{\mu}_s(\tilde{Y}_s) \right|,\end{aligned}$$

where the estimate for the first term on the right hand side is found by similar procedures as done in the proof of Lemma 3.1.7, and a bound for the second is given in (3.1.15). □

3.1.2 Existence and Uniqueness under Frozen Measure Flow

The next theorem provides pathwise existence and uniqueness of (3.1.3) in the presence of a frozen measure flow.

Theorem 3.1.10. *Let $\alpha \geq 2$, $\Gamma \in \mathcal{C}^{\gamma}_T\mathcal{C}^{\alpha}(\mathbb{R}^d)$, satisfy the assumptions of Lemma 3.1.7, $(\xi, B) \in L^1(\Omega;\mathbb{R}^d) \times L^1(\Omega;\mathcal{C}^{\eta}_T)$, with $B_0 = 0$ and $\mu \in \mathcal{P}_1(\mathcal{C}^{\eta\wedge\gamma}_T)$. Then for $\mathbb{P}$-a.a. $\omega \in \Omega$ there exists a unique solution $Y^{\mu}(\omega) \in \mathcal{C}^{\eta\wedge\gamma}_T(\mathbb{R}^d)$ to the equation*

$$Y^{\mu}_t(\omega) = \xi(\omega) + \int_0^t \Gamma_{\mathrm{d}r} * \mu_r(Y^{\mu}_r(\omega)) + B_t(\omega). \tag{3.1.20}$$

Proof. First, let $(B_t)_{t\in[0,T]} := (B_t(\omega))_{t\in[0,T]}$ be a realisation of B, finite in $\mathcal{C}^{\eta}_T$ and we define the measure valued flow $t \mapsto \mu_t \in \mathcal{C}^{\gamma\wedge\eta}_T\mathcal{P}_1(\mathbb{R}^d)$ by setting $\mu_t = \pi_t\#\mu$. Then for any $\beta \in (1-\gamma, \eta\wedge\gamma)$, $x \in \mathbb{R}^d$ and $\bar{T} \in [0,T]$ we define the ball in $\mathcal{C}^{\beta}_T(\mathbb{R}^d)$,

$$\mathfrak{B}_{\bar{T};x} := \left\{ Y \in \mathcal{C}^{\beta}_T(\mathbb{R}^d) \,:\, Y_0 = \xi,\, [Y]_{\beta;\bar{T}} \leq 1 \right\},$$

We equip $\mathfrak{B}_{\bar{T};x}$ with the structure of a complete metric space via the Hölder semi-norm $[\,\cdot\,]_{\beta;[0,\bar{T}]}$. Then we define the solution map, $\Phi_{\bar{T}}(Y)$, by setting, for every $Y \in \mathfrak{B}_{\bar{T};\xi}$,

$$\Phi_{\bar{T}}(Y)_t := \xi + \int_0^t (\Gamma_{\mathrm{d}r} * \mu_r)(Y_r) + B_t, \qquad \text{for all } t \in (0,\bar{T}].$$

We first check that there exists a $T_0 > 0$ such that Φ_{T_0} leaves the ball $\mathfrak{B}_{T_0;\xi}$ invariant. Adding and subtracting the term $\Gamma_{s,t} * \mu_s(Y_s)$, and then using the fact that we have $\gamma + \beta > 1$ to invoke the bounds on the non-linear Young integral from Lemma 3.1.5, for any $0 \leq s < t \leq \bar{T}$ we have

$$\begin{aligned}\left|\int_s^t \Gamma_{\mathrm{d}r} * \mu_r(Y_r)\right| &\leq \left|\int_s^t \Gamma_{\mathrm{d}r} * \mu_r(Y_r) - \Gamma_{s,t} * \mu_s(Y_s)\right| + |\Gamma_{s,t} * \mu_s(Y_s)| \\ &\lesssim_{\gamma,\beta} |t-s|^{\gamma+\beta}\|\Gamma\|_{\gamma,\alpha}\big([Y]_{\beta;\bar{T}} + [\mu]_{\beta;T}\big) + |t-s|^{\gamma}\|\Gamma\|_{\gamma,\alpha},\end{aligned} \tag{3.1.21}$$

so that for any $Y \in \mathfrak{B}_{\bar{T};x}$ we have

$$[\Phi_{\bar{T}}(Y)]_{\beta;\bar{T}} \lesssim_{\gamma,\beta} \bar{T}^{\gamma}\|\Gamma\|_{\gamma,\alpha}(1 + [\mu]_{\beta;T}) + \bar{T}^{\gamma-\beta}\|\Gamma\|_{\gamma,\alpha} + \bar{T}^{\eta-\beta}[B]_{\eta;T}. \tag{3.1.22}$$

Therefore we see that choosing $\bar{T} := T_0 > 0$ sufficiently small we ensure that

$\Phi(\mathfrak{B}_{T_0;x}) \subseteq \mathfrak{B}_{T_0;x}$. The initial condition is satisfied due to the positive regularity of the integral and B.

The next step is to show that $\Phi_{\bar{T}}$ is a contraction on $\mathfrak{B}_{\bar{T};x}$ for some $\bar{T} \leq T_0$. For $Y, \tilde{Y} \in \mathfrak{B}_{T_0;x}$,

$$\left[\Phi_T(Y) - \Phi_T(\tilde{Y})\right]_{\beta;\bar{T}} = \left[\int_0^{\cdot} [\Gamma_{\mathrm{d}r} * \mu_r(Y_r) - \Gamma_{\mathrm{d}r} * \mu_r(\tilde{Y}_r)]\right]_{\beta;\bar{T}}.$$

For any $0 \leq s < t \leq \bar{T}$, from Corollary 3.1.9, using the fact that $Y_0 - \tilde{Y}_0 = 0$ to replace the norm with the semi-norm, we have

$$\begin{aligned}\left| \int_s^t [\Gamma_{\mathrm{d}r} * \mu_r(Y_r) - \Gamma_{\mathrm{d}r} * \mu_r(\tilde{Y}_r)] \right| &\lesssim_{\gamma,\beta} |s-t|^{\gamma+\beta} \|\Gamma\|_{\gamma,\alpha}(2 + [\mu]_{\eta\wedge\gamma;T})[Y - \tilde{Y}]_{\beta;\bar{T}} \\ &\quad + |s-t|^{\gamma} \|\Gamma\|_{\gamma,\alpha} |Y_t - \tilde{Y}_t|,\end{aligned}$$

furthermore, for $t \in [0, \bar{T}]$ we have

$$|Y_t - \tilde{Y}_t| \leq |Y_0 - \tilde{Y}_0| + t^{\beta}[Y - \tilde{Y}]_{\beta;t},$$

so, using that $Y_0 = \tilde{Y}_0 = x$, we have

$$\begin{aligned}\left| \int_s^t [\Gamma_{\mathrm{d}r} * \mu_r(Y_r) - \Gamma_{\mathrm{d}r} * \mu_r(\tilde{Y}_r)] \right| &\lesssim_{\gamma,\beta} |t-s|^{\gamma+\beta} \|\Gamma\|_{\gamma,\alpha}(2 + [\mu]_{\eta\wedge\gamma;T})[Y - \tilde{Y}]_{\beta;\bar{T}} \\ &\quad + |t-s|^{\gamma} \bar{T}^{\beta} \|\Gamma\|_{\gamma,\alpha} [Y - \tilde{Y}]_{\beta;\bar{T}},\end{aligned}$$

Taking the $\beta-$Hölder semi-norm of both sides we have that

$$[\Phi_{\bar{T}}(Y) - \Phi_{\bar{T}}(\tilde{Y})]_{\beta;\bar{T}} \lesssim_{\gamma,\beta} \bar{T}^{\gamma} \|\Gamma\|_{\gamma,\alpha} \left(1 + [\mu]_{\eta\wedge\gamma;T}\right) [Y - \tilde{Y}]_{\beta;\bar{T}}. \tag{3.1.23}$$

So choosing $\bar{T} = T_1 \in (0, T_0]$ sufficiently small, Φ is a contraction on $\mathfrak{B}_{T_1;x}$. It follows by the Banach fixed point theorem, that there exists a unique solution to (3.1.20) contained in $\mathfrak{B}_{T_1;x}$. Furthermore, since the bounds (3.1.22) and (3.1.23) do not depend on $|x|$, we may extend the solution to a further time interval by defining the new ball $\mathfrak{B}_{T_1+T;Y_{T_1}}$ for $\bar{T} > 0$ and then repeating the above arguments. Thus this solution can be extended to any interval $[0,T] \subset \mathbb{R}_+$. Finally, we observe that by extending (3.1.22) all the way up to $\beta = \eta \wedge \gamma$ we see that the solution constructed above a is $\mathbb{P}$-measurable mapping $\Omega \ni \omega \mapsto Y^\mu(\omega) \in \mathcal{C}_T^{\eta\wedge\gamma}$ and furthermore that $\mathcal{L}(Y_0^\mu) = \mathcal{L}(\xi)$. So the proof is complete. □

For later analysis it will be useful to obtain control on the growth of this solution, Y in $\mathcal{C}_T^{\eta\wedge\gamma}$ and its law in $\mathcal{C}_T^{\eta\wedge\gamma}\mathcal{P}_1(\mathbb{R}^d)$. The following lemma collects these controls.

Lemma 3.1.11 (Growth Control)**.** *Let $\alpha \geq 2$, $\Gamma \in \mathcal{C}_T^\gamma \mathcal{C}^\alpha(\mathbb{R}^d)$, satisfy the assumptions of Lemma 3.1.7, $\mu \in \mathcal{P}_1(\mathcal{C}_T^{\eta\wedge\gamma})$ and $(\xi, B) \in L^1(\Omega;\mathbb{R}^d) \times L^1(\Omega;\mathcal{C}_T^\eta)$ with $B_0 = 0$. Then writing Y^μ for the associated solution to (3.1.20), for any $\beta \in (1-\gamma, \eta\wedge\gamma)$, there exists a deterministic constant $C := C(\gamma,\eta,\beta) > 0$ and a $\theta := \theta(\gamma,\eta,\beta) > 0$ such that for $\mathbb{P}$-a.a. $\omega \in \Omega$,*

$$[Y^\mu]_{\beta;T}(\omega) \leq CT^\theta \left(1 + [\mu]_{\beta;T} + [B]_{\eta;T}(\omega)\right) \left(1 \vee \|\Gamma\|_{\gamma,\alpha}^2\right). \tag{3.1.24}$$

In addition, for a new, deterministic constant $C := C(\gamma,\eta,\beta) > 0$, we have that

$$[\mathcal{L}(Y^\mu)]_{\beta;T} \leq CT^\theta \left(1 + [\mu]_{\beta;T} + \mathbb{E}[[B]_{\eta;T}]\right) \left(1 \vee \|\Gamma\|_{\gamma,\alpha}^2\right). \tag{3.1.25}$$

Proof. We begin by proving (3.1.24). Let $h \in (0, 1\wedge T)$ and $t \in [0, T-h]$ then for any $s \in [t, t+h]$, we have

$$|Y_{s,t}^\mu| \leq \left| \int_s^t (\Gamma_{\mathrm{d}r} * \mu_r)(Y_r^\mu) \right| + |B_{s,t}|.$$

From (3.1.21) we have that

$$|Y_{s,t}^\mu| \lesssim_{\gamma,\beta} |t-s|^{\gamma+\beta} \|\Gamma\|_{\gamma,\alpha} \left([Y^\mu]_{\beta;[s,t]} + [\mu]_{\beta;T}\right) + |t-s|^\gamma \|\Gamma\|_{\gamma,\alpha} + |B_{s,t}|.$$

Taking the β-Hölder semi-norm, over $[t, t+h]$, on both sides we obtain that

$$[Y^\mu]_{\beta;[t,t+h]} \lesssim_{\gamma,\beta} h^\gamma \|\Gamma\|_{\gamma,\alpha} [Y^\mu]_{\beta;[t,t+h]} + T^\gamma \|\Gamma\|_{\gamma,\alpha} [\mu]_{\beta;T} + T^{\gamma-\beta} \|\Gamma\|_{\gamma,\alpha} + T^{\eta-\beta} [B]_{\eta;T}.$$

We define $\bar{h} > 0$ by the formula

$$\bar{h} := \left(\frac{1}{2C\|\Gamma\|_{\gamma,\alpha}}\right)^{\frac{1}{\gamma}} \wedge 1 \wedge T, \tag{3.1.26}$$

with $C > 0$ the implied proportionality constant above and so it follows that,

$$[Y^\mu]_{\beta;[t,t+\bar{h}]} \lesssim_{\gamma,\beta} T^\gamma \|\Gamma\|_{\gamma,\alpha} [\mu]_{\beta;T} + T^{\gamma-\beta} \|\Gamma\|_{\gamma,\alpha} + T^{\eta-\beta} [B]_{\eta;T}.$$

Applying Lemma 2.4.3 there exists a $C := C(\gamma, \eta, \beta) > 0$ and a $\theta := \theta(\gamma, \eta, \beta) > 0$ such that

$$[Y^\mu]_{\beta,T} \leq CT^\theta \left(1 + [\mu]_{\beta;T} + [B]_{\eta;T}\right) \left(1 \vee \|\Gamma\|_{\gamma,\alpha}^{1+\frac{1-\beta}{\gamma}}\right).$$

Since $0 < \frac{1-\beta}{\gamma} < 1$, (3.1.24) follows. To prove (3.1.25), first we observe that

$$\begin{aligned}
[\mathcal{L}(Y^\mu)]_{\beta;T} &= \sup_{\varphi \in \mathrm{lip}_1(\mathbb{R}^d)} \sup_{t \neq s \in [0,T]} \frac{1}{|t-s|^\beta} \left| \int_{\mathbb{R}^d} \varphi(y) \, \mathrm{d} \left(\mathcal{L}(Y^\mu_t) - \mathcal{L}(Y^\mu_s)\right)(y) \right| \\
&= \sup_{\varphi \in \mathrm{lip}_1(\mathbb{R}^d)} \sup_{t \neq s \in [0,T]} \frac{1}{|t-s|^\beta} \left| \mathbb{E}\left[\varphi(Y^\mu_t) - \varphi(Y^\mu_s)\right] \right| \\
&\leq \mathbb{E}\left[[Y^\mu]_{\beta;T}\right],
\end{aligned} \tag{3.1.27}$$

where in the penultimate line we used that $\varphi \in \mathrm{lip}_1(\mathbb{R}^d)$ and Jensen's inequality. Applying (3.1.24) inside the expectation gives (3.1.25). □

Remark 3.1.12. Taking $\theta = 0$ and $\beta = \eta \wedge \gamma$ in (3.1.24) shows that $Y^\mu \in L^1(\Omega; \mathcal{C}_T^{\eta\wedge\gamma})$. Obtaining higher moment bounds follows in the same vein, only requiring us to assume that $(\xi, B) \in L^p(\Omega; \mathbb{R}^d) \times L^p(\Omega; \mathcal{C}^\eta_T)$ and $\mu \in \mathcal{P}_p(\mathcal{C}_T^{\eta\wedge\gamma})$ for the same $p \geq 1$.

3.1.3 Stability of Frozen Measure Flow Solutions

We define the solution map for the frozen measure flow equation, (3.1.20), for every $\mu \in \mathcal{P}_1(\mathcal{C}_T^{\eta\wedge\gamma})$, setting

$$\begin{aligned} S_T^\mu : \mathbb{R}^d \times \mathcal{C}_T^\eta \times &\to \mathcal{C}_T^{\eta\wedge\gamma} \\ (\xi, B) &\mapsto Y^\mu, \end{aligned} \tag{3.1.28}$$

where Y^μ is the solution to the NLYE, (3.1.20), constructed in Theorem 3.1.10.

Lemma 3.1.13 (Stability Control). *Let $\alpha \geq 2$, $\Gamma \in \mathcal{C}_T^\gamma \mathcal{C}^\alpha(\mathbb{R}^d)$, satisfy the assumptions of Lemma 3.1.7, $\beta \in (1-\gamma, \eta\wedge\gamma)$ and $p = \frac{2-\beta}{\gamma-\beta}$. Then let (ξ, B, μ), $(\tilde{\xi}, \tilde{B}, \tilde{\mu}) \in L^1(\Omega;\mathbb{R}^d) \times L^{2p}(\Omega;\mathcal{C}_T^\eta) \times \mathcal{P}_1(\mathcal{C}_T^{\eta\wedge\gamma})$ be two pairs of input triples, such that $(\xi - \tilde{\xi}) \perp (B, \tilde{B})$, $B_0 = \tilde{B}_0 = 0$ and suppose there exists a constant $M > 0$ such that*

$$[\mu]_{\beta;T} \vee [\tilde{\mu}]_{\beta;T} \leq M.$$

Then setting $Y := S_T^\mu(\xi, B)$, $\tilde{Y} := S_T^{\tilde{\mu}}(\tilde{x}_0, \tilde{B})$ and defining the strictly positive random variable

$$\mathfrak{G} := (1 + M + [B]_{\eta;T} \vee [\tilde{B}]_{\eta;T}),$$

there exists a constant $C := C(\gamma, \beta, \|\Gamma\|_{\gamma,\alpha}) > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\|Y - \tilde{Y}\|_{\beta;T} \leq C\mathfrak{G}^p T^{2-\beta}\left([B - \tilde{B}]_{\beta;T} + \|\mu;\tilde{\mu}\|_{\beta;T} + |\xi - \tilde{\xi}|\right) \tag{3.1.29}$$

Furthermore, we have that

$$\begin{aligned} \left|\left|\left|\mathcal{L}(Y);\mathcal{L}(\tilde{Y})\right|\right|\right|_{\beta;T} \leq CT^{2-\beta}&\left(\mathbb{E}\left[\mathfrak{G}^p\right] \|\mu;\tilde{\mu}\|_{\beta;T} + \mathbb{E}\left[\mathfrak{G}^{2p}\right]^{\frac{1}{2}} \mathbb{E}\left[[B - \tilde{B}]^2_{\beta;T}\right]^{\frac{1}{2}}\right) \\ &+ (1 + CT^{2-\beta}\mathbb{E}\left[\mathfrak{G}^p\right])\mathbb{E}\left[|\xi - \tilde{\xi}|\right]. \end{aligned} \tag{3.1.30}$$

Proof. For some $h \in (0,1)$, $\tau \in [0, T-h]$ and $s < t \in [\tau, \tau+h]$, we see that

$$|Y_{s,t} - \tilde{Y}_{s,t}| \leq \left|\int_s^t \Gamma_{\mathrm{d}r} * \mu_r(Y_r) - \Gamma_{\mathrm{d}r} * \tilde{\mu}_r(\tilde{Y}_r)\right| + |B_{s,t} - \tilde{B}_{s,t}|. \tag{3.1.31}$$

Applying Corollary 3.1.9, and using the definition of $\mathfrak{G}$, and $\theta > 0$ from Lemma 3.1.11,

$$\left[\int \Gamma_{\mathrm{d}r} * \mu_r(Y_r) - \Gamma_{\mathrm{d}r} * \tilde{\mu}_r(\tilde{Y}_r)\right]_{\beta;[\tau,\tau+h]} \lesssim_{\gamma,\beta} h^{\gamma}(1 \vee \|\Gamma\|^3_{\gamma,\alpha})\mathfrak{G}\left(\|Y - \tilde{Y}\|_{\beta;[\tau,\tau+h]} + \|\|\mu;\tilde{\mu}\|\|_{\beta;[\tau,\tau+h]}\right)$$
$$+ h^{\gamma-\beta}\|\Gamma\|_{\gamma,\alpha}\left(\|Y - \tilde{Y}\|_{\infty;[\tau,\tau+h]} + \sup_{s\in[\tau,\tau+h]}\|\mu_s - \tilde{\mu}_s\|_{\mathrm{KR}}\right).$$

Using the standard bound for supremum and Hölder norms, we have that

$$\|Y - \tilde{Y}\|_{\infty;[\tau,\tau+h]} \lesssim |Y_\tau - \tilde{Y}_\tau| + [Y - \tilde{Y}]_{\beta;[\tau,\tau+h]} = \|Y - \tilde{Y}\|_{\beta;[\tau,\tau+h]},$$

and similarly we can bound the term $\sup_{s\in[\tau,\tau+h]}\|\mu_s - \tilde{\mu}_s\|_{\mathrm{KR}}$. Thus, using that $\beta < \gamma$, we obtain

$$\left[\int \Gamma_{\mathrm{d}r}*\mu_r(Y_r) - \Gamma_{\mathrm{d}r}*\tilde{\mu}_r(\tilde{Y}_r)\right]_{\beta;[\tau,\tau+h]} \lesssim_{\gamma,\beta} h^{\gamma-\beta}(1\vee\|\Gamma\|^3_{\gamma,\alpha})\mathfrak{G}\left(\|Y - \tilde{Y}\|_{\beta;[\tau,\tau+h]} + \|\|\mu;\tilde{\mu}\|\|_{\beta;[\tau,\tau+h]}\right),$$

where we have used that $h < 1$. Inserting this bound in (3.1.31) and using that $\eta > \beta$, it follows that there exists a constant C depending on γ and β such that

$$[Y - \tilde{Y}]_{\beta;[\tau,\tau+h]} \leq [B - \tilde{B}]_{\beta;T} + Ch^{\gamma-\beta}(1 \vee \|\Gamma\|^3_{\gamma,\alpha})\mathfrak{G}\left(\|Y - \tilde{Y}\|_{\beta;[\tau,\tau+h]} + \|\|\mu;\tilde{\mu}\|\|_{\beta;T}\right), \tag{3.1.32}$$

where we have used that $h \leq T$. Adding $|Y_\tau - \tilde{Y}_\tau|$ on both sides of the above inequality, we have

$$\begin{aligned}&\|Y - \tilde{Y}\|_{\beta;[\tau,\tau+h]}\\ &\quad\leq [B - \tilde{B}]_{\beta;T} + Ch^{\gamma-\beta}(1 \vee \|\Gamma\|^3_{\gamma,\alpha})\mathfrak{G}\left(\|Y - \tilde{Y}\|_{\beta;[\tau,\tau+h]} + \|\|\mu;\tilde{\mu}\|\|_{\beta;T}\right) + |Y_\tau - \tilde{Y}_\tau|\end{aligned} \tag{3.1.33}$$

Then we choose $h = \bar{h} \in (0, 1 \wedge T)$ according to

$$\bar{h} = \left(\frac{1}{2C(1 \vee \|\Gamma\|^3_{\gamma,\alpha})\mathfrak{G}}\right)^{\frac{1}{\gamma-\beta}} \wedge 1 \wedge T,$$

where $C > 0$ is the proportionality constant in (3.1.33). This gives the bound

$$\|Y - \tilde{Y}\|_{\beta;[\tau,\tau+\bar{h}]} \leq 2[B - \tilde{B}]_{\beta;T} + |||\mu; \tilde{\mu}|||_{\beta;T} + 2|Y_\tau - \tilde{Y}_\tau|.$$

In order to get a global estimate we first proceed for integer multiple values of $\tau = k\bar{h}$. For $\tau = 0$ we directly have $|Y_\tau - \tilde{Y}_\tau| = |x - \tilde{x}|$ while for $\tau = \bar{h}$ we have the bound

$$|Y_{\bar{h}} - \tilde{Y}_{\bar{h}}| \leq \|Y - \tilde{Y}\|_{\beta;[0,\bar{h}]} \leq 2[B - \tilde{B}]_{\beta;T} + |||\mu; \tilde{\mu}|||_{\beta;T} + 2|\xi - \tilde{\xi}|.$$

Inserting this, we see that

$$\|Y - \tilde{Y}\|_{\beta;[\bar{h},2\bar{h}]} \leq 6[B - \tilde{B}]_{\beta;T} + 3|||\mu; \tilde{\mu}|||_{\beta;T} + 4|\xi - \tilde{\xi}|$$

repeating this argument, using that Y and $\tilde{Y}$ are continuous on $[0, T]$ and that for any $\tau \in [0, T - \bar{h}]$, there exists a $k \in \mathbf{N}$ such that $\tau \in [k\bar{h}, (k+1)\bar{h}]$ we may conclude that on any interval $[\tau, \tau + \bar{h}] \subset [0, T]$,

$$\|Y - \tilde{Y}\|_{\beta;[\tau,\tau+\bar{h}]} \leq \frac{6T}{\bar{h}} \left([B - \tilde{B}]_{\beta;T} + |||\mu; \tilde{\mu}|||_{\beta;T} + |\xi - \tilde{\xi}|\right)$$

Inserting the explicit value for $\bar{h}$ we find that

$$\|Y - \tilde{Y}\|_{\beta;[\tau,\tau+\bar{h}]} \leq 6T(2C(1 \vee \|\Gamma\|^3_{\gamma,\alpha})\mathfrak{G})^{\frac{1}{\gamma-\beta}} \left([B - \tilde{B}]_{\beta;T} + |||\mu; \tilde{\mu}|||_{\beta;T} + |\xi - \tilde{\xi}|\right)$$

An application of Lemma 2.4.3 then reveals that

$$\|Y - \tilde{Y}\|_{\beta;T} \leq 6(2C(1 \vee \|\Gamma\|^3_{\gamma,\alpha})\mathfrak{G})^{\frac{2-\beta}{\gamma-\beta}} T^{2-\beta} \left([B - \tilde{B}]_{\beta;T} + |||\mu; \tilde{\mu}|||_{\beta;T} + |\xi - \tilde{\xi}|\right)$$

By the same argument as used in the derivation of inequality (3.1.27), we have that

$$\left|\left|\left|\mathcal{L}(Y); \mathcal{L}(\tilde{Y})\right|\right|\right|_{\beta;T} \leq \mathbb{E}\left[|\xi - \tilde{\xi}|\right] + \mathbb{E}\left[\|Y - \tilde{Y}\|_{\beta;T}\right].$$

Then applying (3.1.29) inside the second expectation, using that $(x-\tilde{x}) \perp \mathfrak{G}$ due to the fact that x, $\tilde{x}$ are independent from $(B, \tilde{B})$, and applying Hölder's inequality, we obtain (3.1.30).

□

3.1.4 McKean–Vlasov Fixed Point

We now show that we can close the fixed point $\mu = \mathcal{L}(Y^\mu)$ and in doing so obtain a solution (Y, μ) to the full abstract McKean–Vlasov problem (3.1.2). For $p \geq 1$ and any $\nu := \mathcal{L}(\xi, B) \in \mathcal{P}_{1,p}(\mathbb{R}^d \times \mathcal{C}^\eta_T)$ with $B_0 = 0$, we define the map

$$\begin{aligned} \Psi(\nu, \,\cdot\,) : \mathcal{P}_1\left(\mathcal{C}^{\eta\wedge\gamma}_T\right) &\to \mathcal{P}_1\left(\mathcal{C}^{\eta\wedge\gamma}_T\right) \\ \mu &\mapsto \mathcal{L}(Y^\mu) := S^\mu_T \# \mathcal{L}(\xi, B), \end{aligned} \tag{3.1.34}$$

for S^μ_T as defined in (3.1.28). Given this set up, we prove the following theorem.

Theorem 3.1.14. *Let $\alpha \geq 2$, $\Gamma \in \mathcal{C}^\gamma_T \mathcal{C}^\alpha(\mathbb{R}^d)$, satisfy the assumptions of Lemma 3.1.7, $\beta \in (1-\gamma, \eta \wedge \gamma)$, $p = \frac{2-\beta}{\gamma-\beta}$ and $\nu \in \mathcal{P}_{1,p}(\mathbb{R}^d \times \mathcal{C}^\eta_T)$ be as above. Then there exists a unique $\bar{\mu} \in \mathcal{P}_1(\mathcal{C}^{\eta\wedge\gamma}_T)$ such that $\Psi(\nu, \bar{\mu}) = \bar{\mu}$. As a result $\bar{\mu}$ solves the McKean–Vlasov problem*

$$Y_t = \xi + \int_0^t (\Gamma_{\mathrm{d}r} * \bar{\mu}_r)(Y_r) + B_t, \quad \bar{\mu} = \mathcal{L}(Y). \tag{3.1.35}$$

Proof. For any $t \in [0, T]$, $\nu \in \mathcal{P}_{1,p}(\mathbb{R}^d \times \mathcal{C}^\eta_T)$ we define the set

$$\mathfrak{B}_t := \left\{ \mu \in \mathcal{P}_1(\mathcal{C}^\beta_T) |\, \mu_0 = \nu|_{\mathbb{R}^d},\, [\mu]_{\beta;t} \leq 1 \right\}$$

which is a complete metric space under $[\,\cdot\,]_{\beta;t}$ (due to the fact that all elements start in $\mu_0 = \nu|_{\mathbb{R}^d}$). We first show that there exists a $T_0 \in [0, T]$ such that $\Psi(\nu, \,\cdot\,)$ leaves $\mathfrak{B}_{T_0}$ invariant. We let $(\xi, B) \sim \nu$ and $Y = S^\mu_T(\xi, B)$. From Lemma 3.1.11, specifically (3.1.25) we have

$$[\mathcal{L}(Y)]_{\beta;t} \leq Ct^{\theta}\left(1+\mathbb{E}\left[[B]_{\eta;T}\right]\right)\left(1 \vee \|\Gamma\|^2_{\gamma,\alpha}\right),$$

for some $\theta := \theta(\gamma, \eta, \beta) > 0$ and where we used that $\mu \in \mathfrak{B}_t$ ensures $[\mu]_{\beta;t} \leq 1$. So choosing $T_0 > 0$ sufficiently small we see that $[\mathcal{L}(Y)]_{\beta;T_0} \leq 1$. Furthermore, it is immediate from the proof of Theorem 3.1.10 that $\mathcal{L}(Y_0) = \nu|_{\mathbb{R}^d}$ so we conclude that $\Psi(\nu, \mathfrak{B}_{T_0}) \subseteq \mathfrak{B}_{T_0}$.

Now we show that there exists some $T_1 \in (0, T_0]$ such that $\Psi(\nu, \cdot\,)$ is a contraction on $\mathfrak{B}_{T_1}$. Let μ^1, $\mu^2 \in \mathfrak{B}_{T_0}$ and $Y^1 = S^{\mu^1}(\xi, B)$, $Y^2 = S^{\mu^2}(\xi, B)$ be distinct. Then using (3.1.30), from Lemma 3.1.13, with $\mathfrak{G} := (2 + [B]_{\eta;T})$ (note that now Y^1 and Y^2 both starts in ξ with same random noise B, and so the independence condition $(\xi - \xi) \perp (B, B)$ of Lemma 3.1.13 is trivially satisfied), we have that for all $t \in (0, T_0]$,

$$\left|\left|\left|\mathcal{L}(Y); \mathcal{L}(\tilde{Y})\right|\right|\right|_{\beta;t} \lesssim_{\gamma,\eta,\beta} t^{\gamma-\beta} C\mathbb{E}\left[\mathfrak{G}^p\right] |||\mu; \tilde{\mu}|||_{\beta;t}.$$

Since we assume B is p-integrable we can choose $t := T_1 \in (0, T_0]$ sufficiently small to obtain that $\Psi(\nu, \cdot\,)$ is a contraction on $\mathfrak{B}_{T_1}$. Applying the Banach fixed point theorem it follows that there exists a unique fixed point $\bar{\mu} \in \mathfrak{B}_{T_1}$ such that $\Psi(\nu, \bar{\mu}) = \bar{\mu}$ on $[0, T_1]$.

We now show that we can extend this solution to the whole interval $[0, T]$. Note that both T_0 and T_1 were chosen independently of $\nu|_{\mathbb{R}^d}$, the initial distribution. From Lemma 3.1.11 we have that $\bar{\mu}_{T_1} \in \mathcal{P}_1(\mathbb{R}^d)$ and so we can define a new family of sets in $\mathcal{P}_1(\mathcal{C}^\beta)$ by setting

$$\mathfrak{B}^1_{[T_1,T_1+t]} := \left\{\mu \in \mathcal{P}_1\left(\mathcal{C}^\beta_{[T_1,T_1+T]}\right) \,:\, \mu_{T_1} = \bar{\mu}_{T_1},\, [\mu]_{\beta;[T_1,T_1+t]} \leq 1\right\},$$

for any $t \in [0, T - T_1]$. We may then repeat the same argument as above, now considering the solution map $\Psi(\bar{\mu}_{T_1} \otimes \nu|_{\mathcal{C}^\eta_T}, \mu) := S^\mu_{[T_1, T_1+T]}(\mathcal{L}(\bar{\mu}_{T_1}, B))$. Since T_1 was chosen independently of the initial distribution, by the same arguments we obtain a new fixed point $\bar{\mu} \in \mathfrak{B}^1_{[T_1, 2T_1]}$. Note that

$$\mathcal{L}(Y^{\bar{\mu}_{2T_1}}) = S^{\bar{\mu}_{2T_1}}_{[T_1,2T_1]} \# \bar{\mu}_{T_1} = S^{\bar{\mu}_{2T_2}}_{[T_1,2T_2]} \# S^{\mu_{T_1}}_{[0,T_1]} \# \nu,$$

and thus there exists a unique solution to (3.1.35) on $[0, 2T_2]$. This procedure can be iterated to any interval $[kT_1, (k+1)T_1 \wedge T] \subset [0, T]$, and so we conclude that there exists a unique solution to (3.1.35) on $[0, T]$.

Finally, using Theorem 3.1.10 we see that $Y^{\bar{\mu}} \in \mathcal{C}^{\eta\wedge\gamma}_T$ and so from Lemma 3.1.11 and Remark 3.1.12 we have that $\bar{\mu} \in \mathcal{P}_1(\mathcal{C}^{\eta\wedge\gamma}_T)$. This concludes the proof. □

We now define the fixed point map

$$\begin{aligned} \bar{\Psi} : \mathcal{P}_{1,2p}(\mathbb{R}^d \times \mathcal{C}^\eta_T) &\to \mathcal{P}_1\left(\mathcal{C}^{\eta\wedge\gamma}_T\right) \\ \nu &\mapsto \bar{\mu} := \bar{\Psi}(\nu, \bar{\mu}), \end{aligned} \tag{3.1.36}$$

for $p \geq 1$, where $\bar{\mu} = \Psi(\nu, \bar{\nu})$, with Ψ defined by (3.1.34).

3.1.5 Stability of the Fixed Point Law

In this section we investigate the stability of the solution to (3.1.35) with respect to the joint law of the initial data and the driving noise.

First we introduce some notation. Given two probability spaces, $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$, $(\Omega^2, \mathcal{F}^2, \mathbb{P}^2)$ we write $\mathbb{E}^1$ (resp. $\mathbb{E}^2$) for the expectation over Ω^1 (resp. Ω^2) with respect to $\mathbb{P}^1$ (resp. $\mathbb{P}^2$) and $\mathbb{E}^{1,2}$ for the expectation over $\Omega^1 \times \Omega^2$ with respect to $\mathbb{P}^1 \times \mathbb{P}^2$.

Theorem 3.1.15. *Let $\alpha \geq 2$, $\Gamma \in \mathcal{C}^\gamma_T \mathcal{C}^\alpha(\mathbb{R}^d)$, satisfy the assumptions of Lemma 3.1.7, $\beta \in (1-\gamma, \eta \wedge \gamma)$, $p = \frac{2-\beta}{\gamma-\beta}$ and two possibly different probability spaces $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$, $(\Omega^2, \mathcal{F}^2, \mathbb{P}^2)$. Assume that we are given $(\xi^1, B^1) \in L^1(\Omega^1; \mathbb{R}^d) \times L^{2p}(\Omega^1; \mathcal{C}^\eta_T)$*

and $(\xi^2, B^2) \in L^1(\Omega^2;\mathbb{R}^d) \times L^{2p}(\Omega^2;\mathcal{C}_T^\eta)$ *such that* $B_0^1 = B_0^2 = 0$ *and* $(\xi^1 - \xi^2) \perp (B^1, B^2)$*, with* $\nu^i = \mathcal{L}(\xi^i, B^i) \in \mathcal{P}_{1,2p}(\mathbb{R}^d \times \mathcal{C}_T^\eta)$ *for* $i = 1, 2$*, and letting* $\mu^1, \mu^2 \in \mathcal{P}_1(\mathcal{C}_T^{\eta\wedge\gamma})$ *be the associated solutions to* (3.1.2)*, then there exists a constant* $C := C\left(\mathbb{E}^1\left[[B^1]_{\eta;T}^{2p}\right] \vee \mathbb{E}^2\left[[B^2]_{\eta;T}^{2p}\right], \Gamma, \gamma, \eta, \beta\right) > 0$ *such that*

$$\mathcal{W}_{1;\mathcal{C}_T^\beta}(\mu^1, \mu^2) \leq C\left(\mathcal{W}_{1;\mathbb{R}^d}(\mathcal{L}(\xi^1), \mathcal{L}(\xi^2)) + \mathcal{W}_{2;\mathcal{C}_T^\eta}(\mathcal{L}(B^1), \mathcal{L}(B^2))\right). \tag{3.1.37}$$

Proof. We let $\mu^i = \bar{\Psi}^i(\nu^i)$, where $\bar{\Psi}$ is defined by (3.1.36). By Point (ii) of Proposition 2.6.3 there exist optimal transport plans $(m_0, m) \in \Pi(\nu^1|_{\mathbb{R}^d}, \nu^2|_{\mathbb{R}^d}) \times \Pi(\nu^1|_{\mathcal{C}_T^\eta}, \nu^2|_{\mathcal{C}_T^\eta})$ such that

$$\mathcal{W}_{1;\mathbb{R}^d}(\nu^1|_{\mathbb{R}^d}, \nu^2|_{\mathbb{R}^d}) + \mathcal{W}_{2;\mathcal{C}_T^\eta}(\nu^1|_{\mathcal{C}_T^\eta}, \nu^2|_{\mathcal{C}_T^\eta}) = \mathbb{E}_{m_0}\left[|\xi^1 - \xi^2|\right] + \mathbb{E}_m\left[[B^1 - B^2]_{\eta;T}^2\right]^{\frac{1}{2}}.$$

Defining

$$\bar{m} := m_0 \otimes m \in \mathcal{P}_1\left((\mathbb{R}^d \times \mathbb{R}^d) \times (\mathcal{C}_T^\eta \times \mathcal{C}_T^\eta)\right),$$

using the definition of 1-Wasserstein distance on $\mathcal{C}_t^\beta$, and since $\mathcal{P}_1(\mathcal{C}_t^\eta) \subset \mathcal{P}_1(\mathcal{C}_t^\beta)$, we have that

$$\mathcal{W}_{1;\mathcal{C}_t^\beta}(\mathcal{L}(Y^1), \mathcal{L}(Y^2)) \leq \mathbb{E}_{\bar{m}}\left[\|Y^1 - Y^2\|_{\beta;t}\right], \quad \text{for any } t \in [0, T] \tag{3.1.38}$$

Using Lemma 3.1.11, specifically (3.1.25), for $i = 1, 2$, any $t \in (0, T]$, and some $\theta := \theta(\gamma, \eta, \beta) > 0$, we have that

$$[\mu^i]_{\beta;t} \leq Ct^\theta\left(1 + [\mu^i]_{\beta;t} + \mathbb{E}^i\left[[B^i]_{\eta;T}\right]\right)\left(1 \vee \|\Gamma\|_{\gamma,\alpha}^2\right).$$

So now, choosing $t = T_0 \in (0, T]$ defined by

$$T_0 := \left(\frac{1}{2C\left(1 \vee \|\Gamma\|_{\gamma,\alpha}\right)}\right)^{\frac{1}{\theta}},$$

we see that, for a new constant $C := C(\Gamma, \gamma, \beta) > 0$

$$[\mu^i]_{\beta;T_0} \leq C\left(1 + \mathbb{E}^i\left[\left[B^i\right]_{\eta;T}\right]\right), \quad \text{for } i = 1, 2. \tag{3.1.39}$$

We now define the strictly positive random variable on $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1) \times (\Omega^2, \mathcal{F}^2, \mathbb{P}^2)$,

$$\mathfrak{G} := \left(1 + \mathbb{E}^1\left[\left[B^1\right]_{\eta;T}\right] \vee \mathbb{E}^2\left[\left[B^2\right]_{\eta;T}\right] + [B^1]_{\beta;T} \vee [B^2]_{\beta;T}\right) \geq 1.$$

Note in particular that both $\mathfrak{G}$ and T_0 are independent of initial data ξ^1, ξ^2. So, using that $(\xi^1 - \xi^2) \perp_m \mathfrak{G}$, and applying (3.1.29) of Lemma 3.1.13, for any $t \in [0, T_0]$, we have that

$$\begin{aligned}\mathbb{E}_{\bar{m}}\left[\|Y^1 - Y^2\|_{\beta;t}\right] \lesssim_{\gamma,\eta,\beta,\alpha,\Gamma} t^{2-\beta}\mathbb{E}_m\left[\mathfrak{G}^p\right] \left|\left|\left|\mu^1; \mu^2\right|\right|\right|_{\beta;t} + T^{2-\beta}\mathbb{E}_m\left[\mathfrak{G}^{2p}\right]^{\frac{1}{2}}\mathbb{E}_m\left[[B^1 - B^2]^2_{\eta;T}\right]^{\frac{1}{2}} \\ + \mathbb{E}_m\left[\mathfrak{G}^p\right]\mathbb{E}_{m_0}\left[|\xi^1 - \xi^2|\right].\end{aligned}$$

Note that $\mathbb{E}_m$ denotes an integration over the product space $\Omega^1 \times \Omega^2$. From Theorem 2.6.8, we have that

$$\left|\left|\left|\mu^1; \mu^2\right|\right|\right|_{\beta;t} \leq \mathcal{W}_{1;\mathcal{C}_t^\beta}(\mu^1, \mu^2),$$

so in turn

$$\begin{aligned}\mathbb{E}_{\bar{m}}\left[\|Y^1 - Y^2\|_{\beta;t}\right] \lesssim_{\gamma,\eta,\beta,\alpha,\Gamma} t^{2-\beta}\mathbb{E}_m\left[\mathfrak{G}^p\right] \mathcal{W}_{1;\mathcal{C}_t^\beta}(\mu^1, \mu^2) + T^{2-\beta}\mathbb{E}_m\left[\mathfrak{G}^{2p}\right]^{\frac{1}{2}}\mathbb{E}_m\left[[B^1 - B^2]^2_{\eta;T}\right]^{\frac{1}{2}} \\ + \mathbb{E}_m\left[\mathfrak{G}^p\right]\mathbb{E}_{m_0}\left[|\xi^1 - \xi^2|\right],\end{aligned}$$

where $\sigma = \sigma(\gamma, \eta, \beta) > 0$ is the same as in Lemma 3.1.13. Now choose $t = T_1 \in (0, 1 \wedge T_0)$ according to

$$T_1 := \left(\frac{1}{2C(1 \vee \|\Gamma\|^4_{\gamma,\alpha})\mathbb{E}_m\left[\mathfrak{G}^p\right]}\right)^{\frac{1}{2-\beta}} \wedge 1 \wedge T_0,$$

where $C := C(\gamma, \eta, \beta) > 0$ is the proportionality constant above. So then using

(3.1.38), we have, for a new constant $C := C(T, \Gamma, \gamma, \eta, \beta) > 0$,

$$\mathcal{W}_{1;\mathcal{C}^{\beta}_{T_1}}(\mu^1, \mu^2) \le C\left(\mathbb{E}_m\left[\mathfrak{G}^{2p}\right]^{\frac{1}{2}} \mathbb{E}_m\left[[B^1 - B^2]^2_{\eta;T}\right]^{\frac{1}{2}} + \mathbb{E}_m\left[\mathfrak{G}^{p}\right] \mathbb{E}_{m_0}\left[|\xi^1 - \xi^2|\right]\right).$$

So then we chose $\bar{m} = m_0 \otimes m$ to be the optimal transport for ν^1, ν^2, we have that

$$\mathcal{W}_{1;\mathcal{C}^{\beta}_{T_1}}(\mu^1, \mu^2) \le C\mathbb{E}_m\left[\mathfrak{G}^{2p}\right]^{\frac{1}{2}}\left(\mathcal{W}_{1;\mathbb{R}^d}(\nu^1|_{\mathbb{R}^d}, \nu^2|_{\mathbb{R}^d}) + \mathcal{W}_{2;\mathcal{C}^{\eta}_T}(\nu^1|_{\mathcal{C}^{\eta}_T}, \nu^2|_{\mathcal{C}^{\eta}_T})\right), \tag{3.1.40}$$

where we used the ordering of moments for the second expectation. Since T_0, T_1 were chosen independently of $\nu^1|_{\mathbb{R}^d}$ and $\nu^2|_{\mathbb{R}^d}$, we can iterate this procedure to find that on the interval $[T_1, 2T_1]$, where now $\nu^1 := \mathcal{L}(Y^1_{T_1}, B^1) \in \mathcal{P}_{1,2p}(\mathbb{R}^d \times \mathcal{C}^{\eta}_T)$ and $\nu^2 := \mathcal{L}(Y^2_{T_1}, B^2) \in \mathcal{P}_{1,2p}(\mathbb{R}^d \times \mathcal{C}^{\eta}_T)$ we have

$$\mathcal{W}_{1;\mathcal{C}^{\beta}_{[T_1,2T_1]}}(\mu^1, \mu^2) \le C\mathbb{E}_m\left[\mathfrak{G}^{2p}\right]^{\frac{1}{2}}\left(\mathcal{W}_{1;\mathbb{R}^d}(\mathcal{L}(Y^1_{T_1}), \mathcal{L}(Y^2_{T_1})) + \mathcal{W}_{2;\mathcal{C}^{\eta}_T}(\nu^1|_{\mathcal{C}^{\eta}_T}, \nu^2|_{\mathcal{C}^{\eta}_2})\right). \tag{3.1.41}$$

From Lemma 2.6.10 we have that

$$\mathcal{W}_{1;\mathbb{R}^d}(\mathcal{L}(Y^1_{T_1}), \mathcal{L}(Y^2_{T_1})) \le \mathcal{W}_{1;\mathbb{R}^d}(\mathcal{L}(\xi^1), \mathcal{L}(\xi^2)) + T_1^{\beta}\mathcal{W}_{1;\mathcal{C}^{\beta}_{T_1}}(\mu^1, \mu^2)$$

so inserting (3.1.40) in the above inequality yields that, for a new constant $C := C(T, T_1, \Gamma, \gamma, \eta, \beta)$, we have

$$\mathcal{W}_{1;\mathcal{C}^{\beta}_{[T_1,2T_1]}}(\mu^1, \mu^2) \le C\mathbb{E}_m[\mathfrak{G}^{2p}]\left(\mathcal{W}_{1;\mathbb{R}^d}(\mathcal{L}(\xi^1), \mathcal{L}(\xi^2)) + \mathcal{W}_{2;\mathcal{C}^{\eta}_T}(\nu^1|_{\mathcal{C}^{\eta}_T}, \nu^2|_{\mathcal{C}^{\eta}_2})\right).$$

We can repeat this procedure for any interval $[kT_1, (k+1)T_1] \subset [0, T]$, to give,

$$\mathcal{W}_{1;\mathcal{C}^{\beta}_{[kT_1,(k+1)T_1]}}(\mu^1, \mu^2) \le C\mathbb{E}_m[\mathfrak{G}^{2p}]^{\frac{k+1}{2}}\left(\mathcal{W}_{1;\mathbb{R}^d}(\mathcal{L}(\xi^1), \mathcal{L}(\xi^2)) + \mathcal{W}_{2;\mathcal{C}^{\eta}_T}(\nu^1|_{\mathcal{C}^{\eta}_T}, \nu^2|_{\mathcal{C}^{\eta}_2})\right).$$

Since there are only finitely many intervals of this kind inside $[0, T]$ this estimate can be made uniform in k and using the continuity of the measure flow, also to any sub-interval of $[0, T]$ of length T_1. Therefore, Lemma 2.6.11 implies that there exists a constant $C = C(T, T_1, \mathbb{E}_m[\mathfrak{G}^{2p}], \Gamma, \gamma, \eta, \beta) > 0$ such that

$$\mathcal{W}_{1;\mathcal{C}_T^\beta}(\mu^1, \mu^2) \leq C \left(\mathcal{W}_{1;\mathbb{R}^d}(\mathcal{L}(\xi^1), \mathcal{L}(\xi^2)) + \mathcal{W}_{2;\mathcal{C}_T^\eta}(\mathcal{L}(B^1), \mathcal{L}(B^2)) \right),$$

which concludes the proof. □

3.2 Mean Field Limit of the Abstract Particle System

We apply the results of the previous section to show convergence of the non-linear Young particle system,

$$Y_t^i = \xi^i + \frac{1}{N} \sum_{j=1}^{N} \int_0^t \Gamma_{\mathrm{d}r}(Y_r^i - Y_r^j) + B_t^i, \tag{3.2.1}$$

for $i = 1, \ldots, N$ to (3.1.2). As in [27] we only assume convergence in law of the idiosyncratic noise vectors $(B_t^i)_{i=1,\ldots,N}$ to some B_t. That is we do not require any independence or exchangeability of the vectors $(B_t^i)_{i=1,\ldots,N}$, although of course either or both could be ingredients to showing the convergence. Our approach makes use of a trick of Tanaka, [103], which was also employed in [27]. The idea is to re-cast the mean field approximation as a stability result by a transformation of the underlying probability space.

As in the introduction to [21] and Section 3 of [27], we begin by building, for any $N \geq 1$, the probability space $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$, by setting,

$$\Omega_N := \{1, \ldots, N\}, \quad \mathcal{F}_N := 2^{\Omega_N}, \quad \mathbb{P}_N := \frac{1}{N} \sum_{i=1}^{N} \delta_i.$$

where 2^{Ω_N} is the power set of Ω_N and δ_i is the Kronecker delta. So we can easily identify any N-tuple, $(Y^i)_{i=1,\ldots,N} \subset E^N$ with a random variable $Y^{(N)} : \Omega_N \to E$ defined such that $Y^{(N)}(i) = Y^i$. Furthermore, the law of $Y^{(N)}$ as an E valued random variable, on $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ is given by the empirical measure,

$$\mathcal{L}_N\left(Y^{(N)}\right) := \frac{1}{N}\sum_{i=1}^{N}\delta_{Y_i},$$

where the delta is now a Dirac mass on E. Using this construction we can associate to the random vectors

$$(\xi^{(N)}, B^{(N)}) = ((\xi^1, B^1), \dots, (x^N, B^N)) \in (\mathbb{R}^d \times \mathcal{C}_T^{\eta})^N \text{ and } Y^{(N)} = (Y^1, \dots, Y^N) \in (\mathcal{C}_T^{\gamma\wedge\eta})^N$$

the empirical measures

$$\mathcal{L}_N(\xi^{(N)}, B^{(N)}) \in \mathcal{P}_1(\mathbb{R}^d \times \mathcal{C}_T^{\eta}) \text{ and } \mathcal{L}_N(Y^{(N)}) \in \mathcal{P}_1(\mathcal{C}_T^{\eta\wedge\gamma})$$

which define their laws on $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$. From this point of view we can rewrite the particle system (3.2.1) in the more familiar form

$$Y_t^{(N)} = \xi_0^{(N)} + \int_0^t \left(\Gamma_{\mathrm{d}r} * \mathcal{L}_N(Y^{(N)})_r\right)(Y_r^{(N)}) + B_t^{(N)}. \tag{3.2.2}$$

We refer to equation (3.2.2) as the empirical McKean–Vlasov problem. Now we state and prove the following theorem, which is essentially Theorem 21 of [27] modified to our setting.

We fix $T > 0$, $(\Omega, \mathcal{F}, \mathbb{P})$ an abstract probability space, γ, η as in (3.1.1), $\alpha \geq 2$ and $\Gamma \in \mathcal{C}_T^{\gamma}\mathcal{C}^{\alpha}(\mathbb{R}^d)$ satisfying the assumptions of Lemma 3.1.7. Then we have the following mean field approximation result.

Theorem 3.2.1. *Let $\beta \in (1-\gamma, \eta\wedge\gamma)$, $p = \frac{2-\beta}{\gamma-\beta}$ and $\nu = \mathcal{L}(\xi, B) \in \mathcal{P}_{1,2p}(\mathbb{R}^d \times \mathcal{C}_T^{\eta\wedge\gamma})$ be such that $B_0 = 0$ and $\xi \perp B$. For any $N \in \mathbf{N}$ also assume we have $\left(\xi_0^{(N)}, B^{(N)}\right) \in L^1(\Omega; \mathbb{R}^{Nd}) \times L^{2p}(\Omega; (\mathcal{C}_T^{\eta})^N)$ a family of random variables with $B_0^{(N)} = 0$. Then the following statements hold:*

(i) For every $N \in \mathbf{N}$ and $\mathbb{P}$-a.a. $\omega \in \Omega$ there exists a unique solution $Y^{(N)}(\omega) \in$

$(\mathcal{C}_T^{\eta\wedge\gamma})^N$ *to the empirical McKean–Vlasov problem,* (3.2.2)*. Furthermore the mapping* $\omega \mapsto Y^{(N)}(\omega)$ *is* $\mathcal{F}$ *measurable.*

(ii) There exists a $Y \in \mathcal{P}_1(\mathcal{C}_T^{\eta\wedge\gamma})$ *such that for* $\mathbb{P}$*-a.a.* $\omega \in \Omega$*,* $Y(\omega)$ *solves the dynamics of* (3.1.35)*.*

(iii) There exists a constant $C := C\left(T, \Gamma, \mathbb{E}\left[[B]_{\eta;T}^{2p}\right] \vee \mathbb{E}\left[\mathbb{E}_N\left[[B^{(N)}]_{\eta;T}^{2p}\right]\right], \gamma, \eta, \beta\right) > 0$*, such that for all* $N \geq 1$*,* $\mathbb{P}$*-a.s. we have the bound*

$$\mathcal{W}_{1;\mathcal{C}_T^\beta}\left(\mathcal{L}_N(Y^{(N)})(\omega), \mathcal{L}(Y)\right) \leq C\Bigg(\mathcal{W}_{1;\mathbb{R}^d}\Big(\mathcal{L}_N(\xi_0^{(N)})(\omega), \mathcal{L}(\xi)\Big) + \mathcal{W}_{2;\mathcal{C}_T^\eta}\left(\mathcal{L}_N\left(B^{(N)}\right)(\omega), \mathcal{L}(B)\right)\Bigg). \tag{3.2.3}$$

Remark 3.2.2. The independence condition $\xi \perp B$ imposed in Theorem 3.2.1 is a consequence of the independence condition $(\zeta - \xi) \perp (\tilde{B}, B)$ required by Theorem 3.1.15, where $\zeta \sim \mathcal{L}(\xi^{(N)})(\omega)$ and $\tilde{B} \sim \mathcal{L}(B^{(N)})(\omega)$. Since we fix $\omega \in \Omega$ for the random variables $(\xi^{(N)}, B^{(N)})$, treating them as random variables on the space Ω_N defined below, amounts to requiring $\xi \perp B$ w.r.t. $\mathbb{P}$.

Proof. The existence and uniqueness statements of points (i) and (ii) are direct consequences of Theorem 3.1.14 with inputs $(\xi_0^{(N)}, B^{(N)})$ and (ξ, B) on the probability spaces $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ and $(\Omega, \mathcal{F}, \mathbb{P})$ respectively. The requirement that $\mathcal{L}_N(\xi_0^{(N)}, B^{(N)}) \in \mathcal{P}_{1,2}(\Omega_N; (\mathbb{R}^d \times \mathcal{C}_T^\eta)^N)$ is seen to be satisfied since for $\mathbb{P}$-a.a. $\omega \in \Omega$

$$\mathbb{E}_N\left[|\xi_0^{(N)}(\omega)|\right] + \mathbb{E}_N\left[[B^{(N)}(\omega)]_{\eta;T}^2\right] = \frac{1}{N}\sum_{i=1}^N |\xi^i(\omega)| + \frac{1}{N}\sum_{i=1}^N [B^i(\omega)]_{\eta;T}^2 < \infty.$$

The measurabillity assertion of point (i) follows from the continuity of the solution map $\bar{\Psi}_N : \mathcal{P}_1^N(\mathbb{R}^d \times \mathcal{C}_T^\eta) \to \mathcal{P}_1^N(\mathcal{C}_T^\eta)$, so that $\omega \mapsto \mathcal{L}_N(Y^{(N)}(\omega))$ is $\mathcal{F}$ measurable so that in turn $\omega \mapsto Y^{(N)}(\omega) = S^{\mathcal{L}_N(Y^{(N)}(\omega))}(\xi_0^{(N)}(\omega), B^{(N)}(\omega))$ is also $\mathcal{F}$ measurable.

The mean field approximation result of (iii) now follows directly from Theorem 3.1.15.

□

3.3 Proofs of Main Results

We collect the proofs of Theorem 2.3.2, Theorem 2.3.5 and Corollary 2.3.9. We combine the results of, for example [64] or [48], which ensure the existence of sufficiently regularising paths, with the abstract results of, Theorem 3.1.3 and Theorem 3.2.1, along with the preliminary results from Section 2.6.

Proof of Theorem 2.3.2. In order to prove Theorem 2.3.2 we need to ensure that given $K \in \mathcal{B}^{\sigma}_{q,r}(\mathbf{R}^d)$ there exists a $Z_t \in C([0,T];\mathbb{R}^d)$ such that

$$\Gamma_{s,t} := K * L_{s,t},$$

satisfies the requirements of Theorem 3.1.3. This can be done using the results of [64], or [48]. More specifically, from Proposition 2.5.7 we see that we can choose $t \mapsto Z_t$ to be an fBm with Hurst parameter sufficiently low. $\square$

Proof of Theorem 2.3.5. In order to prove Theorem 2.3.5, with $\Gamma_{s,t} := K * L_{s,t}$ as above, it suffices to apply Theorem 3.2.1. Since by assumption

$$\lim_{N\to\infty} \mathcal{W}_{1;\mathbb{R}^d\times\mathcal{C}^{\eta}_T}\left(\mathcal{L}_N(\xi^{(N)}_0, B^{(N)}), \mathcal{L}(\xi, B)\right) = 0,$$

it follows from (3.2.3) that for any $\beta \in (1-\gamma, \gamma\wedge\eta)$ we have $\mathcal{W}_{1;\mathcal{C}^{\beta}_T}(\mathcal{L}_N(Y^{(N)}), \mathcal{L}(Y)) \to 0$ as $N \to \infty$. Then, applying Proposition 2.6.3 we obtain weak convergence of $\mathcal{L}_N(Y^{(N)})$ to $\mathcal{L}(Y)$. $\square$

Proof of Corollary 2.3.9. Let $K \in \mathcal{S}'(\mathbb{R}^d)$ be a homogeneous distribution of order $\sigma < 0$ (see Def. 2.4.8) and $Z \in C([0,T];\mathbb{R}^d)$ distributed according to the law of an fBm with Hurst parameter $H \in (0,1)$ on a separate probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. So then from Proposition 2.4.6 and Remark 2.4.7, for any $\varepsilon > 0$ we have that $K \in \mathcal{B}^{\sigma+\frac{d}{2}-\varepsilon}_{2,2} = \mathcal{H}^{\sigma+\frac{d}{2}-\varepsilon}$. From Proposition 2.5.7 we see that for Γ the averaging operator associated to Z, there exists a set of full measure $\tilde{\mathcal{N}} \subseteq \tilde{\Omega}$ such that for all $\tilde{\omega} \in \tilde{\mathcal{N}}$, $\Gamma K(\tilde{\omega}) \in \mathcal{C}^{\gamma}_T\mathcal{C}^{\sigma+\frac{1-\gamma}{H}-d\left(\frac{1}{2}-\gamma\right)}$ for any $\gamma \in \left(\frac{1}{2}, 1\right)$. Since we assume that B

takes values in $\mathcal{C}_T^{1/2-\varepsilon}$ for any $\varepsilon > 0$, almost surely, we have $1/2 - \varepsilon + \gamma > 0$ for any $\gamma \in \left(\frac{1}{2} + 2\varepsilon, 1\right)$. So then for all $H < \frac{1}{4-2\sigma}$ we have that $\Gamma K \in \mathcal{C}_T^\gamma \mathcal{C}^\alpha$ for some $\alpha > 2$, $\gamma \in (1/2, 1)$ and so the results of Theorems 2.3.2 and 2.3.5 both apply. $\square$

Chapter 4

Conclusions and Open Questions

We discuss some applications of our results in more detail and some further questions below.

4.1 Applications Involving Homogeneous Kernels

In the preceding chapter we demonstrated a regularisation by noise result for generalised McKean–Vlasov equations. Our particular focus has been on McKean–Vlasov equations with homogeneous interaction kernels. Many physically relevant McKean–Vlasov models involve interaction kernels given by homogeneous distributions of negative order. Using Corollary 2.3.9 we now exhibit some classical examples to which our method applies along with the necessary upper bound on the Hurst parameter of the regularising path.

(i) **Coulomb Potential:** Typically used in the Keller–Segel model for chemotaxis and mean field models of plasma dynamics and gravitational dynamics. One has

$$K(x) \sim \begin{cases} \operatorname{sign}(x), & d = 1, \\ \frac{x}{|x|^d}, & d \geq 2. \end{cases}$$

In this case, choosing a fixed sign in front of the kernel often leads to diverging behaviours. With a positive sign, the interaction is referred to as repulsive and with a negative sign it is referred to as attractive or gravitational. As a general rule, the repulsive equation is better behaved although this is not a rigorous statement. For example the Fokker–Planck equation associated to the mean field limit in the repulsive case is globally well-posed in all dimensions. In $d = 1$ it is likely that the propagation of chaos result could be achieved by more direct means than ours in both repulsive and attractive cases, [67, 76]. Well-posedness of the limiting equation is well-known without regularisation in one dimension, irrespective of sign choice, [70, 95]. For $d \geq 2$ there are currently no known propagation of chaos results for the McKean–Vlasov system with the full kernel. In [53] the authors obtain propagation of chaos in two dimensions for $K(x) \sim \frac{1}{|x|^\alpha}$ with $\alpha \in (0, 1)$. In [65, 66] the authors obtain well-posedness and propagation of chaos results for the Keller–Segel model with the full kernel approximated by a cut-off. They also obtain a weaker convergence result, without propagation of chaos, similar to that of [44] for the full kernel.

In our setting, since the kernel is homogeneous of order $\sigma = -d+1$, our results hold with $H < \frac{1}{2+2d}$ with Z drawn independently of K.

(ii) **Biot–Savart Law:** Applied in the vorticity formulation of Euler and Navier–Stokes equations in $d = 2$.

$$K(x) \sim \frac{x^\perp}{|x|^2}, \quad x^\perp := (-x^2, x^1).$$

Since the kernel scales like the Coulomb potential in 2-dimensions our results hold for the same range of H. However, in this instance, due to the rotational structure of the kernel more is known in the un-regularised case. Well-posedness of the limiting equation is known in both the viscous and inviscid cases, [85]. In [43] the authors obtain propagation of chaos for the viscous vortex model - which corresponds in our setting to taking B a standard Brownian motion. A quantitative propagation of chaos result is also obtained in this setting in [75]. For the non-viscous model well-posedness of the particle system and propagation of chaos is shown in [85] for Lebesque almost all initial configurations.

(iii) **Power Law Potentials:** Generalising the Coulomb potential are power law potentials of the kind

$$K_\alpha(x) \sim \frac{1}{|x|^\alpha}, \quad \alpha > 0.$$

Common applications are in Cucker–Smale flocking models, [45, 23], in the Hartree–Fock approximation of many body quantum systems, [54] and in crystallisation models, see (v) below, amongst others. Here our results apply with $H < \frac{1}{4+2\alpha}$ for Z drawn independently of K. In this more general case some results are known in the un-regularised case. For the many body quantum system - which is quite different from our setting as it concerns the mean field of N PDEs, rather than O/SDEs - propagation of chaos holds for $K \in L^2_{\text{loc}}(\mathbb{R}^d) \cap C(\mathbb{R}^d \setminus \{0\})$, [55, Thm. 1.10.2]. In the case of SDEs, the results of [53], already discussed, prove the propagation of chaos for $\alpha \in (0, 1)$ with $d = 2$. Finally we mention that with additional assumptions, kernels of this type are incorporated into the framework of [75] which obtains propagation of chaos results.

(iv) **The Dirac:** Setting

$$K(x) \sim \delta_0(x),$$

our results apply with $H < \frac{1}{4+2d}$ with Z drawn independently of K. In [101], Sznitman studied a particle approximation of the one dimensional Burgers equation with the Dirac interaction kernel as the interaction kernel. Propagation of chaos and well-posedness results were shown in this case without additional regularisation.

(v) **The Lennard–Jones Potentials:** Applied in particle simulations of crystallisation, [37, 104], the family of interactions

$$K_{p,2p}(x) \sim |x|^{-2p} - 2|x|^{-p}, \quad p > 0,$$

are known as Leonard–Jones potentials. Formally, these kernels converge to the Heitmann–Radin kernel

$$K_{HR}(x) := \begin{cases} \infty, & |x| < 1, \\ -1, & |x| = 1, \\ 0, & |x| > 1. \end{cases}$$

Intuitively speaking, K_{HR} acts to separate particles at distance 1 from each other. The typical approach to theories of crystallisation is to study static minimizers of the free energy associated to the interaction, [39, 37, 104]. Therefore these models do not directly fit into our framework, however, we propose it might be interesting to consider dynamic approximations to crystalline structures, using our regularisation by noise approach. Using the Leonard–Jones potential, for fixed $p > 0$ our results apply with $H < \frac{1}{4+2p}$ and Z independent of K. In this setting we may vary $p \in [p_{\min}, p_{\max}]$ without changing Z provided we choose $H < \frac{1}{4+p_{\max}}$.

4.2 Open Questions

We now comment on some open questions following on from the results presented so far.

4.2.1 Vanishing Regularisation

As mentioned in Section 2.3 in the setting presented we only obtain approximate stability for the true solution to the McKean–Vlasov problem. More precisely, if we introduce a small parameter $\varepsilon > 0$ and define the regularised problem

$$\begin{cases} dX^\varepsilon_t = (K * \mu_t)(X^\varepsilon_t)\ \mathrm{d}t + \mathrm{d}B_t + \varepsilon\,\mathrm{d}Z_t, \\ \mu_t = \mathcal{L}(X_t - \varepsilon Z_t), \\ X_0 \sim \mathcal{L}(x_0), \end{cases} \tag{4.2.1}$$

then for (x^1_0, B^1), (x^2_0, B^2) two pairs of input our result gives,

$$\mathcal{W}_{1;C_T}(\mathcal{L}(X^{\varepsilon;1}), \mathcal{L}(X^{\varepsilon;2})) \leq C(\varepsilon)\left(\mathcal{W}_{1;\mathbb{R}^d}(\mathcal{L}(x^1_0), \mathcal{L}(x^2_0)) + \mathcal{W}_{2;\mathcal{C}^\eta_T}(\mathcal{L}(B^1), \mathcal{L}(B^2))\right) + \varepsilon\|Z\|_{C_T},$$

where, if $K \in \mathcal{B}^\alpha_{\infty,\infty}$ for $\alpha < 2$, the constant $C(\varepsilon) \to \infty$ as $\varepsilon \to 0$. As a result we stress that we do not obtain the full mean field limit in the presence of singular kernels.

However, it would be natural to ask if in more specific situations it would be possible to control this loss of stability, or indeed remove the parameter entirely. For example, in some of the cases outlined above, although the kernel may be singular, the mean field limit problem is well-posed and it is only the finite particle system that may be ill-posed. The regularised particle system, for $i = 1, \ldots, N$,

$$X^i_t = x^i_0 + \frac{1}{N}\sum_{j=1}^N \int_0^t K(X^i_r - X^j_r - \varepsilon(N)Z_r)\ \mathrm{d}r + B^i_t + \varepsilon(N)Z_t, \tag{4.2.2}$$

with $K(x) = \pm\chi\frac{x}{2\pi|x|^2}$, corresponds to a regularisation of either the repulsive or attractive Keller–Segel model. In the repulsive case, or for $\chi < 8\pi$ in the attractive case, the Fokker–Planck equation associated to the mean field equation is known to be globally well-posed. In either of these cases therefore it is reasonable to expect the particle system to converge as well, [44]. Therefore, it would be interesting to ask if, in our setting, we could let $\varepsilon(N) \to 0$ as $N \to \infty$ in (4.2.2).

In a similar direction, for some models mean field approximation results have been obtained with a vanishing cut-off approximation of the kernel. In these situations the typical desired rate of cut-off is proportional to $N^{-\frac{1}{d}}$ which is the average distance between N uniformly distributed particles in a d-dimensional box. Although our approximation does not correspond to a sharp cut-off of the equation, if we renormalise the parameter, setting $\bar{\varepsilon} = \frac{\varepsilon}{\|Z\|_{C_T}}$ for Z and $T > 0$ fixed, then the perturbation has most effect for particles at distance no more than $\bar{\varepsilon} \in (0,1)$. This motivates studying the stability of the system (4.2.2) with $\varepsilon(N) \sim N^{-\frac{1}{d}}$.

4.2.2 Particles with Idiosyncratic Regularising Paths

The fact we only obtain approximate stability for the true solutions X is due do the additional perturbation we make inside the law in (4.2.1). This perturbation is necessary as without it the regularising path Z would be cancelled out in the equation for Y. This is similar to the observed fact that identical common noise cannot regularise particle systems, it simply shifts the points of collision to random locations. In the classical propagation of chaos results for trajectories driven by i.i.d Brownian motions, the interaction is smoothed by the rough individual trajectories that average out in the mean field limit to a single average noise. Extending this idea to our setting, it would be interesting to consider the particle system

$$X_t^i = x_0^i + \frac{1}{N}\sum_{j=1}^{N}\int_0^t K(X_r^i - X_r^j)\,\mathrm{d}r + Z_t^i, \qquad \text{for } i = 1,\dots,N, \tag{4.2.3}$$

where we have absorbed the B^i into Z^i. Writing $Y^i := X^i - Z^i$ gives the new system

$$Y_t^i = x_0^i + \frac{1}{N}\sum_{j=1}^{N}\int_0^t K(Y_r^i - Y_r^j - Z_r^i - Z_r^j)\,\mathrm{d}r, \qquad \text{for } i = 1,\dots,N. \tag{4.2.4}$$

Then, using the formalism of averaged fields we define $L^{i,j}$ to be the local time of the process $Z^i - Z^j$ and let $\Gamma^{i,j}K = K * L^{i,j}$ be the associated averaged field. So writing (4.2.4) as a non-linear Young integral we have

$$Y_t^i = x_0^i + \frac{1}{N}\sum_{j=1}^{N}\int_0^t \Gamma^{i,j}_{dr} K(Y_r^i - Y_r^j), \qquad \text{for } i = 1, \dots, N. \tag{4.2.5}$$

For fixed N and $(Z^i)_{i=1}^N$ drawn independently from a suitable distribution, the system (4.2.5) is seen to be well posed. However, the stability of these solutions, with respect to the vector of general drivers appears to be a very difficult question. Obtaining such a result would mirror the propagation of chaos for particles driven by idiosyncratic Brownian noise as in [101].

4.2.3 Second Order Systems

Finally we mention an extension that is likely to be simpler than the previous two. For many physical systems it is in fact more natural to model particles in position and velocity rather than just position as we have done so far. A typical example is the Vlasov–Poisson system for which the N-particle dynamics are written

$$\begin{cases} \mathrm{d}X_t^i = V_t^i \,\mathrm{d}t, \\ \mathrm{d}V_t^i = \frac{1}{N}\sum_{j=1}^N K(X_t^i - X_t^j)\,\mathrm{d}t \qquad \text{for } i = 1, \dots, N, \end{cases} \tag{4.2.6}$$

with the kernel of most interest being the Coulomb potential. The system (4.2.6) can be adapted to fit into our existing framework, by defining the phase space variable $W_t^i = (X_t^i, V_t^i) \in \mathbb{R}^{2d}$ and viewing the interaction as a kernel with support only in the space variable. However, this ignores the second order structure of (4.2.6). Since the singular interaction takes as input the spatial variable, but acts on the velocity variable, the dynamics of (4.2.6) are in a sense less singular than those for the corresponding first order system. For example propagation of chaos results have been attained for the Vlasov–Poisson system with quantitative cut-off, [83, 68] without any Brownian forcing, as compared with the Keller–Segel model which is a McKean–Vlasov system and no such result has been obtained. It would therefore be interesting to explore how our method of regularisation would apply for second order systems and possibly to understand if the same well-posedness and propagation of chaos results can be achieved using less irregular paths.

Part II

Convection-Diffusion SPDEs with Additive Space-Time White Noise

Chapter 5

Introduction

In this half of the thesis we study a class of convection-diffusion SPDEs on $\mathbb{T}^d$, forced by additive space-time white noise and with non-local dependence in the convection term. The prototypical equation we consider is formally written

$$\begin{cases} \partial_t u - \Delta u = \pm \chi \nabla \cdot (u^{m-1} \nabla \rho_u) + \xi, & \text{on } \mathbb{R}_+ \times \mathbb{T}^d, \\ -\Delta \rho_u = \mathcal{P} u, & \text{on } \mathbb{T}^d, \end{cases} \tag{5.0.1}$$

where ξ is a spatially mean free space-time white noise defined in Section 5.4.1 below. Regarding the convection term, $\chi > 0$ is a positive constant and $m \in \mathbb{N}_{\geq 2}$ is a positive integer. When m is even we interpret the power by the expression $u^{m-1} = |u| u^{m-2}$. On the right hand side of the elliptic equation, $\mathcal{P}u$ denotes the projection to the periodic functions with zero mass on $\mathbb{T}^d$. As per the discussion presented in Section 5.4.1 below, we cannot expect (5.0.1) to be well posed in any space of positive regularity when $d \geq 2$. Therefore, in this case we must give proper meaning to the non-linear term $u^{m-1}\nabla \rho_u$. We handle this issue with regards to local well-posedness when $d = 2$ in Chapter 7 and discuss possible approaches to local well-posedness in higher dimensions in Chapter 8.

Before discussing some motivations for studying (5.0.1) we note some interesting features of the equation. Firstly, due to the non-local term, while it is reasonable to expect solutions to be Markov processes in time, we cannot expect the solution to obtain a space-time Markov property as has recently been observed in the case of reaction diffusion SPDEs with polynomial non-linearities, [90, 89, 25]. However, we do expect some features of reaction diffusion SPDEs to hold for (5.0.1). Expanding the non-linear term, assuming for now that all terms are sufficiently regular, we have

$$\pm\chi\nabla\cdot(u^{m-1}\nabla\rho_u) = \mp\chi u^{m-1}\mathcal{P}u \pm \nabla u^{m-1}\cdot\nabla\rho_u.$$

Therefore, as in the case of reaction-diffusion type SPDEs, we expect different global behaviour of the equation depending on the sign choice $\pm\chi$. In accordance with the connection between (5.0.1) and interacting particle systems discussed below we refer to the choice $+\chi$ as the repulsive regime and $-\chi$ as the attractive regime. In general we expect the repulsive equation to enjoy some damping properties and in Chapter 6 we demonstrate that for $d = 1$ and $m = 3$ the first equation of (5.0.1) in fact does *come down from infinity*. However, we also note that due to the additional transport term the non-linearity does not define a dissipative mapping in any L^2 type space. This is a possible explanation for the apparent difficulty in extending the global well-posedness result to $d = 2$, as discussed in Chapter 7. The behaviour of the attractive equation is more subtle and we have not been able to establish conclusive results. Comparison with the deterministic PDE analogues suggest that at least for $d = 1$ we might still expect global well-posedness when $m = 2$ but that this should fail for higher dimensions and higher powers.

5.0.1 The Keller–Segel Model of Chemotaxis

The system (5.0.1) is closely related to the Keller–Segel model for chemotaxis. Introduced in [79], the Keller–Segel model was proposed as an effective description for large collections of cellular sized objects moving in suspension according to a mean field chemical potential. A canonical example is the motion of white blood cells around a wound or infection. In general the cells move randomly throughout the body, however, upon encountering foreign bodies the cells secrete a chemical that draws more white blood cells to that location. The Keller–Segel system models both the chemical secretion and cellular motions as diffusion equations, with the chemical density driven by the cell density and cell motion driven by the resulting chemical potential. A minimal version of the system can be written

$$\begin{cases} \partial_t u - \sigma_1 \Delta u = \pm \chi \nabla \cdot (u \nabla c), & \text{on } \mathbb{R}_+ \times \mathbb{T}^d \\ \partial_t \rho - \sigma_2 \Delta \rho = \mathcal{P} u - \rho, & \text{on } \mathbb{R}_+ \times \mathbb{T}^d, \\ u|_{t=0} = u_0, \ \rho|_{t=0} = \rho_0, & \text{on } \mathbb{T}^d. \end{cases} \tag{5.0.2}$$

The system (5.0.2) is referred to as the parabolic-parabolic Keller–Segel model and a discussion of its properties and variants can be found in [69]. In many cases it is reasonable to assume that the diffusion of the chemical potential occurs on a significantly faster time scale than that of the cells themselves and this leads to the even simpler parabolic-elliptic Keller –Segel system

$$\begin{cases} \partial_t u - \Delta u = \pm \chi \nabla \cdot (u \nabla \rho_u), & \text{on } \mathbb{R}_+ \times \mathbb{T}^d, \\ -\Delta \rho_u = \mathcal{P} u, & \text{on } \mathbb{T}^d, \\ u|_{t=0} = u_0, & \text{on } \mathbb{T}^d. \end{cases} \tag{5.0.3}$$

Equipped with non-negative initial data integrating to 1, (5.0.3) is naturally interpreted as a McKean–Vlasov equation for the cellular density $u(t,x)$. Furthermore, (5.0.3) arises in other contexts as a model for planetary dynamics, [12, 13]. We note that when $m = 2$ and $\xi \equiv 0$, for non-negative initial data (5.0.1) is exactly (5.0.3), so we can think of our SPDE as a stochastic generalisation of this model. The study of the deterministic systems (5.0.2) and (5.0.3) has been an active area of research since their proposal in the early $70's$. Comprehensive surveys are given in [69] and [97]. Focussing on the parabolic-elliptic equation we summarise the main known properties, referring to [70, 16, 97, 30, 14] for more details. We assume $u_0 \geq 0$, $\int_{\mathbb{T}^d} u_0 = 1$ and by implication $u_0 \in L^1(\mathbb{T}^d)$. Many of the properties below rely on the propagation of both the mean and the L^1 norm in this setting.

- Formally any solution to (5.0.3) decreases the energy

$$\mathscr{E}[u_t] := \int_{\mathbb{T}^d} u_t \ln u_t \pm \frac{\chi}{2} \int_{\mathbb{T}^d} u_t \rho_{u_t}, \tag{5.0.4}$$

  and (5.0.3) can be viewed as a gradient flow in the space of probability measures with respect to $\mathcal{E}$, in the sense of Otto [96, 14].

- In $d = 1$ the system (5.0.3) is globally well-posed independent of $\chi > 0$ and the sign choice, [69, 95]. Only the attractive equation displays non-trivial stable solutions.

- For $d \geq 1$ the repulsive equation is globally well-posed independent of χ and the sign choice. The only stable solutions are the constant solutions and they are global attractors for (5.0.4). However, variant or hybrid models can display more novel behaviour, for example [78, 26].

- In $d = 2$ the attractive equation is globally well-posed for $\chi < 8\pi$ and blows-up in finite time for $\chi > 8\pi$. The behaviour when $\chi = 8\pi$ is discussed in [15].

- For $d = 3$ global behaviour of the attractive equation is not determined by $\chi > 0$ alone. Regarding well-posedness, there exists a threshold $K_*(\chi) > 0$ such that if $\|u_0\|_{L^{3/2}} < K_*$ then global smooth solutions exist, while for u_0 failing this condition and also satisfying a moment condition, solutions blow-up in finite time is known, see [97, Sec. 2.1] and [30]. It is an open question to establish a global well-posedness/blow-up dichotomy as is known for the two dimensional case.

The difference in behaviour across dimension, not observed in polynomial reaction diffusion equations, arises due to the decreasing integrability of the Coulomb potential in higher dimensions. Let $\nabla\mathcal{G} \sim |x|^{d-1}$, denote the Coulomb potential on $\mathbb{T}^d$, so that we may write $\nabla\rho_u = \nabla\mathcal{G} * u$. We have $\nabla\mathcal{G} \in L^p(\mathbb{T}^d)$ for all $p < \frac{d}{d-1}$, which is decreasing in d. The L^∞ bound when $d = 1$ is crucial in the proof of global well-posedness in the attractive regime in one dimension. This highlights a phenomenon we also observe in our stochastic equation: even though the behaviour of (5.0.3) is broadly governed by the sign of the quadratic term $\mp u(u - \bar{u})$, the transport term, which involves the kernel $\nabla \mathcal{G}$, also plays a role.

5.0.2 Our Setting and Motivation

Our motivation to study (5.0.1) comes from two directions. Firstly, in relation to the PDE analogues of the equation, it is clear from the discussion above that the fundamental structure of (5.0.1) leads to interesting and varied behaviour that are tied to physically relevant processes. Understanding the persistence or modification of these behaviours in a stochastic setting is therefore a natural question. From a mathematical perspective, additive noise is in many ways the simplest stochastic perturbation of a parabolic PDE and has been the first step in studying stochastic versions of a variety of equations in the past, [31, 98, 32]. In our setting there are a number of advantages to studying the additive noise equation. Firstly additive noise provides the most direct route to studying ergodic properties of the equation and it enables us to consider space-time white noise, which further simplifies the ergodic analysis. In addition, by fixing the form of the noise we are able to focus on studying the behaviour of the non-linear part in higher dimensions and under rougher drivers. On the other hand, following the recent advances in SPDE theory, [59, 57], systems such as (5.0.1) offer an avenue to studying novel kinds of SPDE with non-local dependence. In Chapter 8 we discuss an application of the theory of regularity structures to well-posedness of (5.0.1) in $d = 3$ with $m = 3$. The main issue here is to incorporate the regularising effect of the elliptic term $\nabla \rho_u$.

Finally we mention one contrast with another well studied stochastic system. The Navier–Stokes equations in $d = 2$, written in vorticity form, appear very similar to a vectorial version of (5.0.1) with $m = 2$. We write the SNS equation in vorticity form as

$$\partial_t v - \Delta v = \nabla \cdot (v \mathcal{K} * v) + \xi, \tag{SNS}$$

where $\mathcal{K}(x) \sim \frac{x^\perp}{|x|^2}$ is the Biot–Savart law. Global well-posedness for the two dimensional stochastic Navier–Stokes (SNS) equation was obtained in [98]. The main difference in the case of (SNS) is that the vector field, $\mathcal{K} * v$ is divergence free, so that testing the right hand side with the solution gives

$$\langle \nabla \cdot (v_t \mathcal{K} * v_t), v_t \rangle = -\frac{1}{2} \langle \mathcal{K} * v_t, \nabla v_t^2 \rangle = 0,$$

and it is seen that the damping of the Laplacian is sufficient to obtain global well-posedness. Since $\nabla \rho_u$ is by definition not divergence free the analysis of (5.0.1) involves additional terms.

Despite the motivations discussed above, the additive noise equation (5.0.1) does present some drawbacks. Firstly, from a modelling perspective, interpreting (5.0.3) as a Fokker–Planck equation it would be natural to consider a stochastic version which maintains sign and mass preservation. While we can ensure mass preservation of the additive equation, by working with a mean free noise, the additive noise equation does not preserve sign. This leads to our second departure from (5.0.3) which is to consider powers $m \geq 2$ and to impose the absolute value for even powers. As we saw above the behaviour of the equation is in many cases dependent on the sign choice in front of the non-linear term therefore it is important to keep the sign inside the non-linearity consistent with the sign of the solution. A second issue of the additive equation is that is not clear how one might exploit the natural gradient flow structure of (5.0.3). A resolution to this and the first issue mentioned, would be to consider a stochastic perturbation of the kind

$$\partial_t u - \Delta u = \nabla \cdot (u \nabla \rho_u) + \nabla \cdot (F(u)\xi). \tag{5.0.5}$$

When $F(u) = \sqrt{u}$ and ξ is understood as a space-time white noise, (5.0.5) is a Dean–Kawasaki type equation and is related to the Wasserstein diffusion, [29, 109, 81]. SPDEs of this type present significant mathematical challenges and despite some very recent and interesting work very little is as yet known. On the other hand for ξ a less irregular noise and smoother $F(u)$, (5.0.5) is relatively tractable and in Chapter 8 we discuss some ideas in this direction.

5.0.3 Our Results

In the following two chapters we present analysis of (5.0.1) in $d = 1$ and $d = 2$. In both dimensions we focus on the case $m = 3$. The main reason for this is that it is the lowest power for which current techniques allows us to make sense of the non-linearity in $d = 2$. In both dimensions we obtain pathwise, local well-posedness of the equation, without appealing to the sign choice. In $d = 2$ we employ the notion of Wick powers to interpret the non-linearity. Although we do not present it explicitly, it is clear that the local analysis in $d = 1$ extends to all $m \geq 2$. For $d = 2$, following the ideas of [105], it would also be possible to extend the local result for all m odd. In $d = 1$ and for $m = 3$ we also obtain global well-posedness existence and uniqueness of an invariant measure with tail estimates and exponential rate of mixing for the repulsive equation. Extending this result to higher, odd, powers also seems very possible, however it is not clear how to extend the method to even powers. We present this result in detail in Chapter 6. Although it seems reasonable to expect global well-posedness for the repulsive equation in $d = 2$ we have not been able to achieve this result and a discussion of this issue is included in Chapter 7. The main results are formally presented at the beginning of each relevant chapter, following the introduction of suitable preliminaries in the remainder of this chapter.

5.1 Notation

We let $\mathbb{T}^d$ be the d-dimensional torus which we interpret as the half open box $[0,1)^d$ with sides identified appropriately. We write $\mathbb{R}_+$ for the half open real interval $[0,\infty)$. Solutions to our evolution equations will be mappings $u : I \times \mathbb{T}^d \to \mathbb{R}$, for $I \subset \mathbb{R}_+$ a bounded interval. For a mapping $f : [0,T] \times \mathbb{T}^d \to \mathbb{R}$, if we write f_t we mean the $\mathbb{T}^d \to \mathbb{R}$ mapping that obtains at this instance of $t \in [0,T]$. We write $\mathcal{S}'(\mathbb{T}^d)$ for the space of, periodic, tempered distributions acting on $C^\infty(\mathbb{T}^d)$, the space of infinitely differentiable, periodic functions. For $p \in [1,\infty)$ we let $L^p(\mathbb{T}^d)$ be the usual space of all p-integrable distributions and $L^\infty(\mathbb{T}^d)$ the usual space of essentially bounded functions. For $k \in \mathbb{N}$ we write $C^k(\mathbb{T}^d)$ for the set of k-times continuously differentiable functions on $\mathbb{T}^d$.

For $f \in \mathcal{S}'(\mathbb{T}^d)$ we write $\mathcal{P}f := f - \bar{f}$ where $\bar{f} := \langle 1, f\rangle_{L^2(\mathbb{T}^d)}$, for the projection to the space of mean-free periodic distributions. For $\mathfrak{m} \in \mathbb{R}$ and any of the function spaces above on $\mathbb{T}^d$ we write for example $\mathcal{S}'_{\mathfrak{m}}(\mathbb{T}^d), C^k_{\mathfrak{m}}(\mathbb{T}^d), \mathcal{C}^\alpha_{\mathfrak{m}}, L^p_{\mathfrak{m}}(\mathbb{T}^d)$ for the corresponding sets such that $\langle f, 1\rangle = \mathfrak{m}$ as well. In Section 6.4.1 we work with square integrable space-time functions for which we define $L^2_0(\mathbb{R}_+ \times \mathbb{T}^d)$ for those square integrable functions on $\mathbb{R}_+ \times \mathbb{T}^d$ such that for all $t \in \mathbb{R}_+$, $\langle f_t, 1\rangle = 0$. When the context is clear we drop the dependence of these function spaces on the domain in order to lighten notation.

For E a Banach space and $T > 0$, we write $C_T E := C([0,T];E)$ for the set of continuous mappings $f : [0,T] \to E$. For $\eta > 0$ a positive parameter we let $C_{\eta;T}E := C_\eta((0,T];E)$ be the set of similar mappings finite under the norm

$$\|f\|_{C_{\eta;T}E} = \sup_{t\in(0,T]} (t^\eta \wedge 1)\|f_t\|_E.$$

We define $\mathcal{H} : (0,\infty) \times \mathbb{T}^d \to \mathbb{R}$ to be the fundamental solution to the heat equation on $\mathbb{T}^d$, which solves

$$\partial_t \mathcal{H} - \Delta \mathcal{H} = \delta_{t=0}, \qquad \text{on } \mathbb{R}_+ \times \mathbb{T}^d.$$

For $(t, x) \in \mathbb{R} \times \mathbb{T}^d$ we write the fundamental solution as

$$\mathcal{H}_t(x) := \frac{1}{\left(\sqrt{4\pi t}\right)^d} \sum_{n \in \mathbb{Z}^d} e^{-\frac{|x-n|^2}{4t}} \mathbb{1}_{(0,\infty)}(t). \tag{5.1.1}$$

We define $\mathcal{G}$ to be the fundamental solution of Poisson's equation on the torus, which solves

$$-\Delta \mathcal{G} = \mathcal{P}\delta_0. \tag{5.1.2}$$

Finally we highlight that in Chapter 6 we use x both as a subscript to denote spatial derivatives, as in $\partial_x u$, and for the initial condition of our SPDE, $x \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}}$. If we write $u_t(x)$ we mean solution to our SPDE, with initial condition x, at time $t \in [0, T]$.

5.2 Hölder-Besov Spaces on $\mathbb{T}^d$

We recall the construction and some well-known properties of Hölder–Besov spaces defined through the theory of Littlewood–Paley decompositions. A more detailed exposition can be found in [5].

Let $(e_m)_{m \in \mathbb{Z}^d}$ be the basis of $L^2(\mathbb{T}^d)$ given by the trigonometric functions $e_m(x) := e^{2\pi i m \cdot x}$. If f, g are complex valued and square integrable we write,

$$\langle f, g \rangle = \int f(x)\overline{g(x)}\, dx.$$

Then for $f \in L^1(\mathbb{T}^d)$ and $m \in \mathbb{Z}^d$ we define the Fourier transform by

$$\mathcal{F}f(m) := \hat{f}_m = \langle f, e_m \rangle,$$

and for real valued functions, f, we have the symmetry condition $\hat{f}_{-m} = \overline{\hat{f}_m}$. Since $e_m \in C^\infty(\mathbb{T}^d)$, for any $f \in \mathcal{S}'(\mathbb{T}^d)$, $\hat{f}_m$ is defined by duality.

We define the inverse transform, for $f \in \ell^1(\mathbb{Z}^d)$, by

$$\mathcal{F}^{-1} f(x) = \sum_{m \in \mathbb{Z}^d} f_m e_{-m}(x).$$

For $z \in \mathbb{R}^d$, $r > 0$ and $0 < r_1 < r_2 < \infty$ we denote by $B_r(z)$ the ball of radius r centred at z and by $\mathcal{A}_{r_1,r_2}(z) = B_{r_2}(z) \setminus B_{r_1}(z)$ the annulus of inner radius r_1 and outer radius r_2, centred at z. Then we consider a partition of unity given by two functions $\tilde{\chi}$, $\chi \in C_c^\infty(\mathbb{R}^d)$, such that:

1. $\operatorname{supp} \tilde{\chi} \subset B_{\frac{4}{3}}(0)$ and $\operatorname{supp} \chi \subset \mathcal{A}_{\frac{3}{4},\frac{8}{3}}(0)$,
2. $\tilde{\chi}(z) + \sum_{k=0}^\infty \chi(2^{-k} z) = 1$, for all $z \in \mathbb{R}^d$.

The existence of such a dyadic partition of unity is shown in [5, Prop. 2.10]. We define $\chi_{-1} := \tilde{\chi}$, $\chi_k(\,\cdot\,) := \chi(2^{-k}\,\cdot\,)$ for all $k \geq 0$ and set $\chi_k = 0$ for all $k < -1$.

For $f \in C^\infty(\mathbb{T}^d)$ we denote its k^{th} Littlewood-Paley block by,

$$\Delta_k f := \mathcal{F}^{-1}(\chi_k \hat{f}) = \sum_{m \in \mathbb{Z}^d} \chi_k(m) \hat{f}_m e_{-m}, \text{ for } k \geq -1.$$

Observe, for $k \geq -1$, we can write $\Delta_k f(x) = (h_k * f)(x)$ where, using the modulation property of the Fourier transform, one has

$$h_{-1} := \mathcal{F}^{-1}\tilde{\chi}, \quad h_0 := \mathcal{F}^{-1}\chi, \quad h_k := \mathcal{F}^{-1}\chi_k = 2^{kd} h(2^k\,\cdot\,), \qquad \text{for } k \geq 1.$$

By duality we extend this definition to $f \in \mathcal{S}'(\mathbb{T}^d)$. Then, for $\alpha \in \mathbb{R}$ and $p,\, q \in [1, \infty]$ we define the non-homogeneous Besov norm for $f \in C^\infty(\mathbb{T}^d)$ by the expression

$$\|f\|_{B^\alpha_{p,\,q}(\mathbb{T}^d)} := \left\| \left(2^{\alpha k} \|\Delta_k f\|_{L^p(\mathbb{T}^d)}\right)_k \right\|_{l^q(\mathbb{Z})}, \tag{5.2.1}$$

and denote by $\mathcal{B}^\alpha_{p,\,q}(\mathbb{T}^d)$ the completion of $C^\infty(\mathbb{T}^d)$ with respect to (5.2.1). Defining $\mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)$ as the completion of $C^\infty(\mathbb{T}^d)$ under (5.2.1) ensures that all Besov spaces are separable. For the Besov spaces $\mathcal{B}^{\alpha}_{\infty,\infty}$ we write $\mathcal{C}^{\alpha}$. When $\alpha > 0$ and not an integer, an equivalent norm is given by

$$\|f\|_{\mathcal{C}^{\alpha}(\mathbb{T}^d)} = \|f\|_{C(\mathbb{T}^d)} + [f]_{\alpha} := \sup_{x\in\mathbb{T}^d} |f(x)| + \sup_{x\neq y\in\mathbb{T}^d} \frac{|f(x) - P^{\lfloor\alpha\rfloor}_{f,x}(y)|}{|x-y|^{\alpha}}, \tag{5.2.2}$$

where $P^{\lfloor\alpha\rfloor}_{f,x}$ is the Taylor polynomial of f to degree $\lfloor\alpha\rfloor$ at the point x. The equivalence between the semi-norm, for $\alpha \in (0,1)$, and the homogeneous Besov norm is proved in [5, Th. 2.36].

5.2.1 Embeddings, Duality and Derivatives

We recall various properties of the inhomogeneous Hölder–Besov spaces as defined above. We remark that since we work on exclusively on the torus in this section we have directly defined the periodic Besov spaces $\mathcal{B}^{\alpha}_{p,q}(\mathbb{T}^d)$. For many of the results stated below we refer to [5] for proofs, where the same results are stated for Besov spaces on $\mathbb{R}^d$. However, in many cases the passage to obtaining the same results for Besov spaces on the torus can be made by similar arguments. We refer to [91, App. A] and [57, App. A] where the same issue is discussed. A key tool in many proofs is Bernstein lemma.

Lemma 5.2.1 (Bernstein Lemma - [5, Lem. 2.1]). *Let $A = \{\zeta \in \mathbb{R}^d : r < |\zeta| < R\}$ be a given annulus and $B = \{\zeta \in \mathbb{R}^d : |\zeta| < R\}$ be a given ball and $n \in \mathbb{N}$. Then, there exists a constant $C := C(k) > 0$ such that for all $p, q \in [1,\infty]$ with $q \geq p$, $f \in L^p(\mathbb{R}^d)$, $\lambda > 0$ we have*

$$supp(\hat{f}) \subset \lambda B \Rightarrow \sup_{\substack{\sigma\in\mathbb{N}^d\\|\sigma|=n}} \|\partial^{\sigma} f\|_{L^q} \leq C\lambda^{n+d(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p}, \tag{5.2.3}$$

$$supp(\hat{f}) \subset \lambda A \Rightarrow C^{-1}\lambda^n\|f\|_{L^p} \leq \sup_{\substack{\sigma\in\mathbb{N}^d\\|\sigma|=n}} \|\partial^{\sigma} f\|_{L^p} \leq C\lambda^n\|f\|_{L^p}. \tag{5.2.4}$$

Theorem 5.2.2 (Besov Embeddings - [5, Prop. 2.71]). *Let $\alpha \leq \beta \in \mathbb{R}$, $q \in [1,\infty]$*

and $p \geq r \in [1, \infty]$, be such that

$$\beta = \alpha + d\left(\frac{1}{r} - \frac{1}{p}\right),$$

then there exists a constant $C := C(\alpha, \beta, q, p, r) > 0$ such that

$$\|f\|_{\mathcal{B}^{\alpha}_{p,q}} \leq C\|f\|_{\mathcal{B}^{\beta}_{r,q}}. \tag{5.2.5}$$

For $q_1 \geq q_2 \in [1, \infty]$ and α, p as above there exists a constant $C := C(q_1, q_2) > 0$ such that

$$\|f\|_{\mathcal{B}^{\alpha}_{p,q_1}} \leq C\|f\|_{\mathcal{B}^{\alpha}_{p,q_2}}. \tag{5.2.6}$$

Furthermore, for any $\beta > \alpha$ and $p, q \in [1, \infty]$ the embedding $\mathcal{B}^{\beta}_{p,q} \hookrightarrow \mathcal{B}^{\alpha}_{p,q}$ is compact.

Remark 5.2.3. In particular, for any $p \in [1, \infty]$, there exists a constant $C := C(p) > 0$ such that

$$C^{-1}\|f\|_{\mathcal{B}^{0}_{p,\infty}} \leq \|f\|_{L^p} \leq C\|f\|_{\mathcal{B}^{0}_{1,\infty}}.$$

Theorem 5.2.4 (Effect of Derivatives)**.** *Let $\alpha \in \mathbb{R}$, p, $q \in [1, \infty]$ and $k \in \mathbb{N}_{>0}$. Then there exists a constant $C := C(k) > 0$ such that for any $\sigma \in \mathbb{N}^d$ one has that*

$$\|D^{\sigma} f\|_{\mathcal{B}^{\alpha-|\sigma|}_{p,q}} \leq C\|f\|_{\mathcal{B}^{\alpha}_{p,q}}. \tag{5.2.7}$$

Proof. Since the derivative is linear, for any $k \geq -1$, we have that $\Delta_k(D^{\sigma} f) = D^{\sigma}(\Delta_k f)$. So then from Lemma 5.2.1 we have that

$$\|\Delta_k(D^{\sigma} f)\|_{L^p} = \|D^{\sigma}(\Delta_k f)\|_{L^p} \lesssim 2^{k|\sigma|}\|\Delta_m f\|_{L^p}, \tag{5.2.8}$$

from which the result follows. □

Theorem 5.2.5 (Besov Poincaré - [105, Prop. A.9])**.** *Let $\alpha \in (0, 1)$. Then there exists a constant $C := C(\alpha) > 0$ such that*

$$\|f\|_{\mathcal{B}^{\alpha}_{1,1}} \leq C\left(\|f\|^{1-\alpha}_{L^1}\|\nabla f\|^{\alpha}_{L^1} + \|f\|_{L^1}\right) \tag{5.2.9}$$

Theorem 5.2.6 (Duality Pairing - [5, Prop. 2.76]). *For $\alpha \in \mathbb{R}$, p, $q \in [1, \infty]$ and $p', q' \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$, there exists a constant $C := C(\alpha, p, q) > 0$ such that*

$$|\langle f, g \rangle| \leq C \|f\|_{\mathcal{B}^{\alpha}_{p,q}} \|g\|_{\mathcal{B}^{-\alpha}_{p',q'}}. \tag{5.2.10}$$

5.2.2 Paradifferential Calculus

Recall that for $f, g \in \mathcal{S}'(\mathbb{T}^d)$ we have formal the decomposition

$$f = \sum_{k \geq -1} \Delta_k f, \quad g = \sum_{j \geq -1} \Delta_j g.$$

Therefore, formally, we may write the product as

$$fg = \sum_{k, j \geq -1} \Delta_k f \Delta_j g.$$

The idea of Bony's decomposition is to decompose this sum into three terms that can be handled for distributions of suitable regularity.

Definition 5.2.7 (Bony's Decomposition). Let f, $g \in \mathcal{S}'(\mathbb{T}^d)$. Then we define the paraproducts between f and g by

$$f \otimes g := \sum_{k < j-1} \Delta_k f \Delta_j g, \quad f \otimes g := \sum_{k > j+1} \Delta_k f \Delta_j g,$$

and the resonant product between f and g by

$$f \circleddash g := \sum_{|k-j| \leq 1} \Delta_k f \Delta_j g.$$

Given these definitions we decompose the formal product as

$$fg = f \otimes g + f \circleddash g + f \otimes g.$$

Of these terms, we see below that the paraproducts are always well defined while the resonant term is only well defined on a subset of sufficiently regular distributions. When it is well defined, we will see that it also defines a bi-linear operator on its domain. This allows us to extend the usual pointwise product on $C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$ to a bi-linear form on suitable pairs of Besov spaces.

Theorem 5.2.8 (Paraproduct Bounds - [5, Th. 2.82, Th. 2.85 & Cor. 2.86])**.** *Let $\alpha, \beta \in \mathbb{R}$, $q \in [1, \infty]$ and $p, p_1, p_2 \in [1, \infty]$ be such that*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}.$$

Then:

(i) If $\alpha + \beta > 0$, the mapping $(f, g) \mapsto f \odot g$ extends to a continuous bi-linear map $\mathcal{B}^{\alpha}_{p_1,q} \times \mathcal{B}^{\beta}_{p_2,q} \to \mathcal{B}^{\alpha+\beta}_{p,q}$.

(ii) The mapping $(f, g) \mapsto f \varolessthan g$ extends to a continuous bilinear map $L^{p_1} \times \mathcal{B}^{\beta}_{p_2,q} \to \mathcal{B}^{\beta}_{p,q}$.

(iii) If $\alpha < 0$, then the mapping $(f, g) \mapsto f \varolessthan g$ extends to a continuous bi-linear map $\mathcal{B}^{\alpha}_{p_1,q} \times \mathcal{B}^{\beta}_{p_2,q} \to \mathcal{B}^{\alpha+\beta}_{p,q}$.

(iv) If $\alpha < 0 < \beta$ are such that $\alpha + \beta > 0$, then the mapping

$$(f, q) \mapsto fg := f \varolessthan g + f \odot g + f \varogreaterthan g$$

extends to a continuous bi-linear map $\mathcal{B}^{\alpha}_{p_1,q} \times \mathcal{B}^{\beta}_{p_2,q} \to \mathcal{B}^{\alpha}_{p,q}$.

(v) If $\alpha > 0$, then the mapping $(f, g) \mapsto fg$ extends to a continuous bi-linear map $\mathcal{B}^{\alpha}_{p_1,q} \times \mathcal{B}^{\alpha}_{p_2,q} \to \mathcal{B}^{\alpha}_{p,q}$. Furthermore for $p_3, p_4 \in [1, \infty]$ also satisfying

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p},$$

there exists a $C := C(\alpha, p, q) > 0$ such that

$$\|fg\|_{\mathcal{B}^{\alpha}_{p,q}} \leq C \left(\|f\|_{L^{p_1}} \|g\|_{\mathcal{B}^{\alpha}_{p_2,q}} + \|f\|_{\mathcal{B}^{\alpha}_{p_3,q}} \|g\|_{L^{p_4}} \right). \tag{5.2.11}$$

As a consequence of these properties we have the following theorem regarding products of Hölder–Besov distributions.

Theorem 5.2.9 (Product Bounds)**.** *Let $q \in [1,\infty]$, α, $\beta \in \mathbb{R}$ be such that $\alpha+\beta > 0$ with $\beta > 0$ and let p_1, p_2, $p \in [1,\infty]$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ then the usual point-wise product $C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d) \ni (f,g) \mapsto f \times g$ extends as a continuous bi-linear map from $\mathcal{B}^{\alpha}_{p_1,q}(\mathbb{T}^d) \times \mathcal{B}^{\beta}_{p_2,q}(\mathbb{T}^d) \to \mathcal{B}^{\alpha\wedge\beta}_{p,q}(\mathbb{T}^d)$.*

In particular, for $\alpha > 0$ and $\nu \in [0,1]$, there exists a constant $C := C(\alpha,\beta,p,q) > 0$ such that

$$\|fg\|_{\mathcal{B}^{\alpha}_{p,q}} \leq C\|f\|_{\mathcal{B}^{\alpha}_{\frac{p}{\nu},q}}\|g\|_{\mathcal{B}^{\alpha}_{\frac{p}{1-\nu},q}} \leq C\|f\|_{\mathcal{B}^{\alpha+(1-\nu)\frac{d}{p}}_{p,q}}\|g\|_{\mathcal{B}^{\alpha+\nu\frac{d}{p}}_{p,q}} \tag{5.2.12}$$

and for $\alpha < 0 < \beta$ with $\alpha+\beta > 0$, $\nu \in [0,1]$ there exists a constant $C := C(\alpha,\beta,p,q) > 0$ such that

$$\|fg\|_{\mathcal{B}^{\alpha}_{p,q}} \leq C\|f\|_{\mathcal{B}^{\alpha}_{\frac{p}{\nu},q}}\|g\|_{\mathcal{B}^{\beta}_{\frac{p}{1-\nu},q}} \leq C\|f\|_{\mathcal{B}^{\alpha+(1-\nu)\frac{d}{p}}_{p,q}}\|g\|_{\mathcal{B}^{\beta+\nu\frac{d}{p}}_{p,q}}. \tag{5.2.13}$$

5.2.3 Parabolic and Elliptic Regularity Estimates

We define heat semi-group by its action on Fourier space. For $f \in L^1(\mathbb{T}^d)$ and $t > 0$, we set

$$e^{t\Delta}f := \mathcal{F}^{-1}\left(e^{-t|\cdot|^2}\hat{f}.\right) = \mathcal{H}_t * f,$$

where,

$$\mathcal{H}_t(x) = \sum_{m\in\mathbb{Z}^d} e^{-4\pi^2|m|^2t}e_m(x)\mathbb{1}_{(0,\infty)}(t), \tag{5.2.14}$$

and extend this formula by duality. We refer to [92, Prop. 5 & 6] for a proof of the following theorem.

Theorem 5.2.10 (Regularising Effect of the Heat Flow - [92, Prop. 5 & 6])**.** *Let α, $\beta \in \mathbb{R}$ and $p \geq r \in [1,\infty]$ and $q \in [1,\infty]$. Then, if $\beta \leq \alpha \leq \beta+2$, there exists a* *constant* $C := C(d, \alpha, \beta, p, r, q) > 0$, *such that, uniformly over* $t > 0$,

$$\|e^{t\Delta} f\|_{\mathcal{B}^{\alpha}_{p,q}} \leq C t^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{p}\right)-\frac{1}{2}(\alpha-\beta)} \|f\|_{\mathcal{B}^{\beta}_{r,q}}. \tag{5.2.15}$$

Secondly, if $0 \leq \beta - \alpha \leq 2$ *then there exists a* $C := C(\alpha, \beta, p, q) > 0$ *such that for a any* $t > 0$,

$$\|(1 - e^{t\Delta}) f\|_{\mathcal{B}^{\alpha}_{p,q}} \leq C t^{\frac{\beta-\alpha}{2}} \|f\|_{\mathcal{B}^{\beta}_{p,q}}. \tag{5.2.16}$$

Remark 5.2.11. We remark that since we have defined the Hölder-Besov spaces as the closure of $C^{\infty}(\mathbb{T}^d)$ under the norm; for any $\alpha \in \mathbb{R}$, p, $q \in [1, \infty]$, if $f \in \mathcal{B}^{\alpha}_{p,q}(\mathbb{T}^d)$, we have that

$$\lim_{t \to 0} \|(1 - e^{t\Delta}) f\|_{\mathcal{B}^{\alpha}_{p,q}} = 0.$$

To see this we note that the set of smooth functions for which the same limit holds is a closed set, using (5.2.16). Therefore, upon taking the closure we see that the limit must hold for all $f \in \mathcal{B}^{\alpha}_{p,q}(\mathbb{T}^d)$ under our definition.

We also have the following elliptic regularity estimate. First we consider functions with Fourier transform supported in an annulus.

Lemma 5.2.12. *Let* $0 < r_1 < r_2 < \infty$, $\mathcal{A}_{r_1,r_2}(0)$ *be an annulus,* $p \geq 1$ *and* $\lambda > 0$. *Then there exist* $C := C(r_1, r_2) > 0$ *such that*

$$\mathit{supp}(\mathcal{F}(f)) \subset \lambda \mathcal{A}_{r_1,r_2}(0) \quad \Rightarrow \quad \|\Delta^{-1} f\|_{L^p} \leq C \lambda^{-2} \|f\|_{L^p}.$$

Proof. Let $\phi \in C^{\infty}_c(\mathbb{R}^d)$ be such that $\phi \equiv 1$ on $\mathcal{A}_{r_1,r_2}(0)$ and zero outside a neighbourhood of $\mathcal{A}_{r_1,r_2}(0)$. Then we have

$$\begin{aligned}\Delta^{-1} f &:= \mathcal{F}^{-1}\left(\phi\left(\frac{\cdot}{\lambda}\right) \frac{1}{4\pi^2 |\cdot|^2} (\mathcal{F} f)(\,\cdot\,)\right) \\ &= \mathcal{F}^{-1}\left(\phi\left(\frac{\cdot}{\lambda}\right) \frac{1}{4\pi^2 |\cdot|^2}\right) * f.\end{aligned}$$

By Young's convolution inequality,

$$\|\Delta^{-1} f\|_{L^p} \leq \left\| \mathcal{F}^{-1}\left(\phi\left(\frac{\cdot}{\lambda}\right) \frac{1}{4\pi^2 |\cdot|^2} \right) \right\|_{L^1} \|f\|_{L^p}.$$

Therefore, it suffices show that there exists a constant $C > 0$, independent of $\lambda \geq 1$, such that

$$\left\| \mathcal{F}^{-1}\left(\phi\left(\frac{\cdot}{\lambda}\right) \frac{1}{4\pi^2 |\cdot|^2} \right) \right\|_{L^1} \leq C\lambda^{-2}.$$

By definition, the function $\mathcal{F}^{-1}(\phi(\,\cdot\,/\lambda))$ has Fourier transform supported in an annulus slightly larger than $\lambda\mathcal{A}_{r_1,r_2}(0)$. So we now apply [5, Lem. 2.2] to the Fourier multiplier $\sigma(m) := \frac{1}{4\pi^2|m|^2}$ and the function $\mathcal{F}^{-1}(\phi(\,\cdot\,/\lambda))$ which gives us a constant $C > 0$ such that

$$\begin{aligned} \left\| \mathcal{F}^{-1}\left(\phi\left(\frac{\cdot}{\lambda}\right) \frac{1}{4\pi^2 |\cdot|^2} \right) \right\|_{L^1} &\leq C\lambda^{-2} \left\| \mathcal{F}^{-1}\left(\phi\left(\frac{\cdot}{\lambda}\right)\right) \right\|_{L^1} \\ &= C\lambda^{-2} \left\| \mathcal{F}^{-1}\phi \right\|_{L^1}. \end{aligned}$$

Since $\|\mathcal{F}^{-1}\phi\|_{L^1} < \infty$ by assumption the claim is shown. □

Theorem 5.2.13. *Let $f \in C^\infty(\mathbb{T}^d)$ and be such that $\langle f, 1\rangle = 0$. Then if $-\Delta\rho := f$, for any $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$, there exists a constant $C := C(d, p, q) \geq 0$ such that*

$$\|\rho\|_{\mathcal{B}^{\alpha}_{p,q}} \leq C\|f\|_{\mathcal{B}^{\alpha-2}_{p,q}}. \tag{5.2.17}$$

Proof. From Lemma 5.2.12, for any $k \geq 0$ we have

$$\|\Delta_k \rho\|_{L^p} = \|\Delta_k \Delta^{-1} f\|_{L^p} = \|\Delta^{-1}\Delta_k f\|_{L^p} \leq C2^{-2k}\|\Delta_k f\|_{L^p}. \tag{5.2.18}$$

Regarding the low frequency block, since we assumed $\langle f, 1\rangle = 0$ we have $(\mathcal{F}f)(0) = 0$. In turn $(\mathcal{F}\rho)(0) = 0$ and so, $\Delta_{-1}\rho(x) = \Delta_{-1}f(x)$. Therefore, there exists a $C := C(d) > 0$, such that for any $k \geq -1$ one has,

$$2^{\alpha k}\|\Delta_k \rho\|_{L^p} \leq C2^{(\alpha-2)k}\|\Delta_k f\|_{L^p},$$

from which the claim follows. $\square$

Remark 5.2.14. In the following we are mostly concerned with $\nabla\rho_f$ where $-\Delta\rho_f = f - \bar{f}$. Combining (5.2.17) with (5.2.7) and (5.2.5) we see that, for $\alpha \in \mathbb{R}$, $p \geq r \in [1,\infty]$ and $q \in [1,\infty]$,

$$\|\nabla\rho_f\|_{\mathcal{B}^{\alpha}_{p,q}} \lesssim \|f\|_{\mathcal{B}^{\alpha-1+d\left(\frac{1}{r}-\frac{1}{p}\right)}_{r,q}}. \tag{5.2.19}$$

5.3 Elements of Ergodic Theory

We recall some elements of ergodic theory for dynamical systems on infinite dimensional spaces. We refer to [33] for more details.

Let E be a separable Banach space and $\mathcal{E}$ be its Borel sigma algebra. We write $\mathcal{P}(E)$ for the set of Borel, probability measures on $(E,\mathcal{E})$ and $\mathcal{M}(E)$ for the set of Borel, measures on E. We denote by $\mathcal{B}_b(E)$ (resp. $\mathcal{C}_b(E)$) the set of bounded, real, Borel functions on E (resp. continuous, bounded, real Borel functions on E). For every $\mu \in \mathcal{M}(E)$ and $\Phi \in \mathcal{B}_b(E)$, the pairing $\langle \mu, \Phi\rangle := \int_E \Phi(x)\,\mu(\mathrm{d}x)$ is well defined. For μ, $\nu \in \mathcal{P}(E)$ we define the total variation distance

$$\|\mu-\nu\|_{\mathrm{TV}} := \frac{1}{2} \sup_{\substack{\Phi\in\mathcal{B}_b(E)\\ \|\Phi\|_{L^\infty}\leq 1}} |\langle \mu-\nu, \Phi\rangle|. \tag{5.3.1}$$

An equivalent formulation is given by viewing the total variation distance as an optimal transport cost, with cost function $\mathbb{1}_{x\neq y}$. We have

$$\|\mu-\nu\|_{\mathrm{TV}} := \frac{1}{2} \inf_{\pi\in\Pi(\mu,\nu)} \iint_{E\times E} (1-\mathbb{1}_{\{0\}}(x-y))\,\pi(\mathrm{d}x,\mathrm{d}y), \tag{5.3.2}$$

where $\Pi(\mu,\nu) \subset \mathcal{P}(E\times E)$ is the set of coupling measures between μ and ν.

Definition 5.3.1 (Markovian Transition Function). We say that $P_t(x,A)_{t\geq 0}$, defined for every $t \geq 0$, $x \in E$ and $A \in \mathcal{E}$ is a Markovian transition function on E if

1. $P_t(x,\,\cdot\,) \in \mathcal{P}(E)$ for every $t \geq 0$, $x \in E$,

2. $P_t(\,\cdot\,, A) : E \to \mathbb{R}$ is $\mathcal{E}$ measurable for every $t \geq 0$, $A \in \mathcal{E}$,

3. $P_{t+s}(x, A) = \int_E P_s(y, A)\, P_t(x, \mathrm{d}y)$, for every $t, s \geq 0$, $x \in E$, $A \in \mathcal{E}$,

4. $P_0(x, A) = \mathbb{1}_A(x)$, for every $x \in E$, $A \in \mathcal{E}$.

To every Markovian transition function, $P_t(\,\cdot\,,\,\cdot\,)$, we associate a Markovian semi-group of linear operators, for which we use the same notation, $P_t : \mathcal{B}_b(E) \to \mathcal{B}_b(E)$, by the expression

$$P_t\Phi(x) = \int_E \Phi(y)P_t(x, \mathrm{d}y), \quad \forall\, \Phi \in \mathcal{B}_b(E),\ t \geq 0.$$

We also define the adjoint semi-group, P_t^*. For any $\mu \in \mathcal{P}(E)$, $A \in \mathcal{E}$ and $t \geq 0$, we define

$$P_t^*\mu(A) := \langle P_t\mathbb{1}_A, \mu\rangle.$$

We say that a measure $\nu \in \mathcal{P}(E)$ is invariant for $(P_t)_{t\geq 0}$ if

$$P_t^*\nu = \nu, \quad \forall t \geq 0.$$

Definition 5.3.2 (Stochastically Continuous Transition Function). We say that a Markovian transition function is stochastically continuous if, for every $r > 0$ and $x \in E$

$$\lim_{t\to 0} P_t(x, B_r(x)) = 1. \tag{5.3.3}$$

For the associated Markovian semi-group an equivalent condition for stochastic continuity is that

$$\lim_{t\to 0} P_t\Phi(x) = \Phi(x), \quad \forall\, \Phi \in \mathcal{C}_b(E).$$

See [33, Prop. 2.1.1].

Definition 5.3.3 (Feller Semi-Group). We say that a stochastically continuous Markovian semi-group, $(P_t)_{t\geq 0}$, is a Feller semi-group, if for any $\Phi \in \mathcal{C}_b(E)$ and $t \geq 0$, one has that $P_t\Phi \in \mathcal{C}_b(E)$.

If $(P_t)_{t\geq 0}$ is a stochastically continuous, Markovian semi-group, then for every $x \in E$ and $T > 0$,

$$R_t \mathbb{1}_A(x) := \frac{1}{t}\int_0^t P_s \mathbb{1}_A(x)\ \mathrm{d}s,$$

defines a probability measure on E. We define the adjoint, R_T^*, by setting, for every $\mu \in \mathcal{P}(E)$, $A \in \mathcal{E}$ and $T > 0$,

$$R_t^* \mu(A) := \int_E R_t \mathbb{1}_A(x)\mu(\mathrm{d}x).$$

It follows that for any $\Phi \in \mathcal{B}_b(E)$ and $\mu \in \mathcal{P}(E)$ we have

$$\langle R_t^* \mu, \Phi\rangle = \frac{1}{t}\int_0^t \langle P_s^* \mu, \Phi\rangle\ \mathrm{d}s.$$

We recall Prokhorov's theorem regarding the weak convergence of measures. First we define a tight family of measures.

Definition 5.3.4 (Tight Family of Measures). We say that a family of probability measures $\mathfrak{M} \subseteq \mathcal{P}(E)$ is tight, if for any $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset E$ such that for any $\mu(K_\varepsilon^c) < \varepsilon$, for any $\mu \in \mathfrak{M}$.

Theorem 5.3.5 (Prokhorov's Theorem)**.** *For E a separable Banach space, a collection of probability measures $\mathfrak{M} \subset \mathcal{P}(E)$ is tight if and only if it is relatively, sequentially compact in $\mathcal{P}(E)$ with respect to the topology of weak convergence.*

In Chapter 6 we build the semi-group associated to the infinite dimensional dynamics of a stochastic evolution equation. Our primary interest is to establish criteria of the semi-group that ensure existence and uniqueness of an invariant measure for the dynamics. We outline these criteria below. The Krylov–Bogoliubov theorem gives criteria for existence of an invariant measure.

Theorem 5.3.6 (Krylov–Bogoliubov Existence Theorem)**.** *Let $(P_t)_{t\geq 0} : \mathcal{C}_b(E) \to \mathcal{C}_b(E)$ be a Feller semi-group on E. If for some $\mu \in \mathcal{P}(E)$ there exists a sequence $t_k \nearrow \infty$, we have that $R_{t_k}^* \mu \rightharpoonup \nu \in \mathcal{P}(E)$, then ν is invariant for P_t.*

Proof. See the proof of [33, Th. 3.1.1]. □

Theorem 5.3.6 ensures the existence of an invariant measure for P_t, however, it leaves open the possibility of multiple invariant measures. Doob's theorem (Theorem 5.3.9 below) shows that irreducibility plus a regularity condition on P_t is enough to ensure uniqueness of the invariant measure.

Definition 5.3.7 (Irreducible Semi-Group)*.* We say that a semi-group $(P_t)_{t\geq 0}$ is irreducible if for any $x \in E$, and non-empty and open $A \in \mathcal{E}$, $P_t \mathbb{1}_A(x) > 0$.

Definition 5.3.8 (Strong Feller Semi-Group)*.* We say that a semi-group $(P_t)_{t\geq 0}$ posses the strong Feller property if $P_t : \mathcal{B}_b(E) \to \mathcal{C}_b(E)$.

Theorem 5.3.9 (Doob's Theorem - [33, Prop. 4.1.1 & Th. 4.2.1])**.** *Let $(P_t)_{t\geq 0}$ be an irreducible, strong Feller semi-group. Then given a measure $\nu \in \mathcal{P}(E)$, invariant for P_t, it holds that,*

1. *for all $x \in E$ and A in $\mathcal{E}$ we have*

$$\lim_{t\to\infty} P_t \mathbb{1}_A(x) = \nu(A), \tag{5.3.4}$$

2. *ν is the unique invariant measure for P_t,*

3. *for any $x \in E$ and $t > 0$, ν is equivalent to $P_t^* \delta_x$.*

5.4 White Noise and the Stochastic Heat Equation

We provide a very brief summary of some definitions and key concepts from the theory of Gaussian processes on Hilbert spaces. A very thorough survey including a more general discussion of Gaussian measure theory and its application to SPDEs is given in [59, 34].

5.4.1 White Noise, the Stochastic Heat Equation and its Wick Powers

Definition 5.4.1 (White Noise). Given an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we say that a family of Gaussian random variables $(\xi(\varphi))_{\varphi\in L^2(\mathbb{R}_+\times\mathbb{T}^d)}$ (resp. $(\zeta(\phi))_{\phi\in L^2(\mathbb{T}^d)}$) defines a spatially mean free space-time white noise (resp. mean free spatial white noise) if

1. $\mathbb{E}[\xi(\varphi)] = 0$, $\mathbb{E}[\xi(\varphi)\xi(\varphi')] = \langle \varphi, \varphi'\rangle_{L^2(\mathbb{R}_+\times\mathbb{T}^d)}$ for all φ, $\varphi' \in L^2(\mathbb{R}_+ \times \mathbb{T}^d)$,
2. $\mathbb{E}[\zeta(\phi)] = 0$, $\mathbb{E}[\zeta(\phi)\zeta(\phi')] = \langle \phi, \phi'\rangle_{L^2(\mathbb{T}^d)}$ for all ϕ, $\phi' \in L^2(\mathbb{T}^d)$,
3. $\langle \xi, \psi \otimes 1\rangle_{L^2(\mathbb{R}_+\times\mathbb{T}^d)} = \langle \zeta, 1\rangle_{L^2(\mathbb{T}^d)} = 0$ for all $\psi \in L^2(\mathbb{R}_+)$, $\mathbb{P}$ - almost surely.

Letting $(e_m)_{m\in\mathbb{Z}^d}$ be the orthonormal basis of $L^2(\mathbb{T}^d)$ defined in Section 5.2 and $(W^m_t)_{m\in\mathbb{Z}^d}$ (resp. $(W^m)_{m\in\mathbb{Z}^d}$) be a family of independent real valued Brownian motions (resp. i.i.d real normal random variables) then for $\varphi \in L^2(\mathbb{R}_+ \times \mathbb{T}^d)$ (resp. $\phi \in L^2(\mathbb{T}^d)$) we define

$$\tilde{\xi}(\varphi) := \sum_{m\in\mathbb{Z}^d} \int_0^\infty \hat{\varphi}_m(t)\, dW^m_t, \quad \tilde{\zeta}(\phi) := \sum_{m\in\mathbb{Z}^d} \hat{\phi}_m W^m.$$

The first summands are understood as Itô stochastic integrals and with convergence of both sums taking place in $L^2(\Omega, \mathbb{P})$. Then $\tilde{\xi}$, $\tilde{\zeta}$ satisfy Items 1, 2 of Definition 5.4.1. In order to satisfy Item 3 we project onto the subspace of mean free distributions by defining

$$\xi(\varphi) := \tilde{\xi}(\varphi) - \int_0^\infty \hat{\varphi}_0(t) dW^0_t, \quad \zeta(\phi) := \tilde{\zeta}(\phi) - \hat{\phi}_0 W^0.$$

Although $\xi(\varphi)$ and $\zeta(\phi)$ are well-defined random variables, and so ξ, ζ define random distributions, it is readily checked that ξ, ζ are almost surely not members of $L^2(\mathbb{R}\times \mathbb{T}^d)$, $L^2(\mathbb{T}^d)$. However, both are contained in any strictly larger Hilbert space. An explicit example is given by the Hilbert–Schmidt extension.[1] For $\varphi \in L^2(\mathbb{R} \times$

[1] Let H be a Hilbert space and $(e_n)_{n\geq 0}$ be an orthonormal basis of H. Then the Hilbert–Schmidt

$\mathbb{T}^d$), $\phi \in L^2(\mathbb{T}^d)$ we write the evaluations $\xi(\varphi)$ and $\zeta(\phi)$ in the convenient form of stochastic integrals,

$$\xi(\varphi) = \int_{\mathbb{R}} \int_{\mathbb{T}^d} \varphi(t, x) \xi(\mathrm{d}t, \mathrm{d}x), \quad \zeta(\phi) = \int_{\mathbb{T}^d} \phi(x)\, \zeta(\mathrm{d}x), \tag{5.4.1}$$

even though ξ and ζ are almost surely not measures. The theory of stochastic integrals for Hilbert space valued processes is properly developed in [34, Ch. 4] and the quantities (5.4.1) are well defined, but only on a set of full measure $\Omega' \subset \Omega$ that depends on the functions φ, ϕ.

A key object in the study of SPDEs forced by white noise is the stochastic convolution. This allows us to define the notion of a mild solution to linear SPDEs. We define the filtration

$$(\tilde{\mathcal{F}}_t)_{t\geq 0} := \sigma\left(\left\{\xi(\varphi) \,:\, \varphi \in L^2(\mathbb{R}_+ \times \mathbb{T}^2),\, \varphi|_{(t,\infty)} \equiv 0\right\} \cup \left\{\zeta(\varphi) \,:\, \phi \in L^2(\mathbb{T}^2)\right\}\right),$$

and let $(\mathcal{F}_t)_{t\geq 0}$ denote its usual augmentation. Now, for any $t > 0$ we define the dynamic stochastic convolution

$$\mathcal{I}_t(\phi) := \int_0^t \int_{\mathbb{T}^d} (\mathcal{H}_{t-r} * \phi)(x)\, \xi(\mathrm{d}r, \mathrm{d}x) \quad \forall \phi \in L^2(\mathbb{T}^d), \tag{5.4.2}$$

where $\mathcal{H}$ is the periodic heat kernel defined in (5.1.1). We see that $\mathcal{I}_t(\phi)$ is adapted to $(\mathcal{F}_t)_{t>0}$, the filtration outside of $t = 0$. Writing the convolution in terms of its Fourier series, using the fact that we have chosen the white noise to be spatially mean free, we see that

$$\mathcal{I}_t(\phi) = \sum_{m\neq 0} \int_0^t e^{-4\pi^2|m|^2(t-r)}\, \hat{\phi}_m \, \mathrm{d}W_r^m$$

We consider the long time behaviour of $\mathcal{I}_t$. Using Item 1 of Definition 5.4.1 and the

extension, $\tilde{H}$, is defined to the completion of H under the norm $\|x\|_{\tilde{H}}^2 := \sum_{n=1}^{\infty} \frac{1}{n^2} \langle x, e_n \rangle^2$. It can be checked that the inclusion map $\iota : \tilde{H} \to H$ is trace class.

semi-group property of the heat kernel, for any ϕ, $\phi' \in L^2(\mathbb{T}^d)$ we have

$$\begin{aligned}
\mathbb{E}\left[\mathcal{I}_t(\phi)\mathcal{I}_t(\phi')\right] &= \int_0^t \int_{\mathbb{T}^d} (\mathcal{H}_{2t-2s} * \phi)(x)\phi'(x)\, \mathrm{d}x\, \mathrm{d}s \\
&= \sum_{m\neq 0} \int_0^t e^{-8\pi^2|m|^2(t-s)} \hat{\phi}_m \hat{\phi'}_m\, \mathrm{d}s \\
&= \sum_{m\neq 0} \frac{1}{8\pi^2|m|^2}\left(1-e^{-8\pi^2|m|^2 t}\right)\hat{\phi}_m \hat{\phi'}_m \\
&\overset{t\to\infty}{\longrightarrow} \sum_{m\neq 0} \frac{1}{8\pi^2|m|^2}\hat{\phi}_m \hat{\phi'}_m \\
&= \int_{\mathbb{T}^d} \mathcal{G} * \phi(x)\phi'(x)\, \mathrm{d}x,
\end{aligned}$$

where $\mathcal{G}$ is the fundamental solution of Poisson's equation on the torus. Since the first two moments of a Gaussian random variable specify the random variable uniquely in law, we define the time homogeneous, random distribution, $\mathcal{I}_\infty$ to be such that, for any ϕ, $\phi' \in L^2(\mathbb{T}^d)$,

$$\mathbb{E}\left[\mathcal{I}_\infty(\phi)\right] = 0, \quad \mathbb{E}\left[\mathcal{I}_\infty(\phi)\mathcal{I}_\infty(\phi')\right] = \int_{\mathbb{T}^d} (\mathcal{G} * \phi)(x)\phi'(x)\, \mathrm{d}x.$$

We write μ_{GFF} for the law of $\mathcal{I}_\infty$, which we will see below defines a member of $\mathcal{P}(\mathcal{C}_0^\alpha(\mathbb{T}^d))$ for all $\alpha < 1 - \frac{d}{2}$. For now we may think of it as a probability measure on the Hilbert–Schmidt extension of $L^2(\mathbb{T}^d)$. It turns out that μ_{GFF} is the unique invariant measure for the process $t \mapsto \mathcal{I}_t$. For convenience we write $\mathcal{I}_\infty$ in a similar form as $\mathcal{I}_t$. Define $\phi \mapsto \mathcal{G}^{\frac{1}{2}} * \phi$ to be the functional square root of the operator $\phi \mapsto \mathcal{G} * \phi$. For $d \geq 2$, $\mathcal{G}^{\frac{1}{2}}$ can be written in the form of a periodic Riesz potential. For convenience, we formally write

$$\mathcal{I}_\infty(\phi) = \int_{\mathbb{T}^2} (\mathcal{G}^{\frac{1}{2}} * \phi)(x)\zeta(\mathrm{d}x), \quad \forall \phi \in L^2(\mathbb{T}^d). \tag{5.4.3}$$

The sum $\mathcal{I}_\infty(e^{t\Delta}\,\cdot\,) + \mathcal{I}_t$ defines a stationary $(\mathcal{F}_t)_{t\geq 0}$ adapted process. That is $\mathcal{I}_\infty(e^{t\Delta}\,\cdot\,) + \mathcal{I}_t$ is $\mathcal{F}_t$ measurable for all $t \geq 0$ and $\mathcal{L}(\mathcal{I}_\infty(e^{t\Delta}\,\cdot\,) + \mathcal{I}_t)$ does not depend on $t > 0$. We use these definitions to define a solution to the stationary stochastic heat equation (SHE).

Definition 5.4.2 (Stationary SHE). For $T > 0$ we say that $v \in \mathcal{S}'([0,T] \times \mathbb{T}^2)$ is a solution to the stationary stochastic heat equation, written formally as

$$\begin{cases} \partial_t v - \Delta v = \xi, & \text{on } (0,T] \times \mathbb{T}^d \\ v|_{t=0} \sim \mu_{\mathrm{GFF}}, & \text{on } \mathbb{T}^d, \end{cases} \tag{5.4.4}$$

if the identity

$$v_t(\phi) := \mathcal{I}_\infty(e^{t\Delta}\phi) + \mathcal{I}_t(\phi)$$

holds $\mathbb{P}$ - almost surely for all $t \in [0,T]$ and $\phi \in \mathcal{S}(\mathbb{T}^d)$.

5.4.2 Regularity of Stochastic Processes

We employ the linear equation, (5.4.4) as a building block for studying non-linear stochastic evolution equations. In order to do so it is important to obtain regularity estimates for stochastic processes built from ξ. We employ a Kolmogorov type regularity result based on the Littlewood–Paley decomposition of Section 5.2 that is suited to distributions continuous in time but possibly rough in space. We use a version of [92, Lem. 9 & 10].

Lemma 5.4.3 (Kolmogorov Continuity - [92, Lem. 9 & 10]). *Let $(t,\phi) \mapsto Z_t(\phi)$ be a map from $\mathbb{R}_+ \times L^2(\mathbb{T}^d) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ that is linear and continuous in ϕ. Assume that for some $p > 1$, $\alpha' \in \mathbb{R}$, $\kappa' > \frac{1}{p}$ and any $T > 0$, there exists a constant $C > 0$, such that for all $k \geq -1$, $x \in \mathbb{T}^d$ and $s, t \in [0,T]$,*

$$\mathbb{E}\left[|Z_t(h_k(\,\cdot\, - x))|^p\right] \leq C^p 2^{-k\alpha' p}, \tag{5.4.5}$$

$$\mathbb{E}\left[|Z_t(h_k(\,\cdot\, - x)) - Z_s(h_k(\,\cdot\, - x))|^p\right] \leq C^p |t-s|^{\kappa' p} 2^{-k(\alpha'-\kappa)p}. \tag{5.4.6}$$

Then there exists a random distribution $\tilde{Z} \in \mathcal{C}^\kappa(\mathbb{R}_+; \mathcal{B}^\alpha_{p,p})$ for all $\alpha < \alpha' - \kappa' - \frac{d}{p}$ and

$\kappa < \kappa' - \frac{1}{p}$ such that, for all $t \in [0,T]$ and $\phi \in \mathcal{S}(\mathbb{T}^d)$,

$$Z_t(\phi) = \tilde{Z}_t(\phi), \quad \mathbb{P} - a.s.$$

Furthermore, for every $T > 0$, there exists a constant $C := C(T, \alpha, \alpha', p) > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\tilde{Z}_t\|^p_{\mathcal{B}^{\alpha}_{p,p}}\right] \leq C.$$

Proof. See [92, Lem. 9 & 10]. □

Remark 5.4.4. The same result holds for $\phi \mapsto Z(\phi)$ as a time homogeneous map from $L^2(\mathbb{T}^d) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$.

5.4.3 Iterated Stochastic Integrals and Wick Products

We begin by applying 5.4.3 to treat $\mathcal{I}_t$ and $\mathcal{I}_\infty$. Since both are examples of stochastic integrals they obey Nelson's estimate, Lemma 5.4.10 below, with $n = 1$ as well as the Itô isometry (5.4.15). We use these tools to demonstrate that $\mathcal{I}_t$ satisfies the criteria of Lemma 5.4.3. A similar argument holds for the free field $\mathcal{I}_\infty$.

Since the law of the white noise is translation invariant it suffices to consider $x = 0$ in Lemma 5.4.3. So then for $k \geq -1$, $p \in [2, \infty)$, $t > 0$ and setting $x = 0$ in (5.4.5), we have

$$\begin{aligned}
\mathbb{E}\left[|\mathcal{I}_t(h_k)|^2\right] &= \int_0^t \|\chi_k \hat{\mathcal{H}}_{t-r}\|^2_{\ell^2} \,\mathrm{d}r \\
&= \int_0^t \left\langle \chi_k \hat{\mathcal{H}}_{t-r}, \chi_k \hat{\mathcal{H}}_{t-r} \right\rangle_{\ell^2} \mathrm{d}r \\
&= \sum_{\substack{m\in\mathrm{supp}(\chi_k)\\ m\neq 0}} \int_0^t e^{-8\pi^2|m|^2(t-r)} \,\mathrm{d}r \\
&= \sum_{\substack{m\in\mathrm{supp}(\chi_k)\\ m\neq 0}} \frac{1}{8\pi^2|m|^2}\left(1 - e^{-8\pi^2|m|^2 t}\right). \qquad (5.4.7)
\end{aligned}$$

Therefore, applying (5.4.16) below and in combination with (5.4.7), for fixed $t > 0$

and $k \geq 0$ we have, for any $p \geq 2$,

$$\mathbb{E}\left[|\mathcal{I}_t(h_k)|^p\right]^{\frac{2}{p}} \lesssim_p 2^{kd} \sup_{|m| \in \left(\frac{3}{4}2^k, \frac{8}{3}2^k\right)} \frac{1}{8\pi^2|m|^2}\left(1 - e^{-8\pi^2|m|^2 t}\right)$$
$$\lesssim_p 2^{-k(2-d)}. \tag{5.4.8}$$

Using a similar argument for $k = -1$, we conclude that for any $p \geq 2$, there exists a $C := C(p) > 0$ such that

$$\mathbb{E}\left[|\mathcal{I}_t(h_k)|^p\right] \leq C^{\frac{p}{2}} 2^{-k\left(1-\frac{d}{2}\right)p} \tag{5.4.9}$$

for all $k \geq -1$. To show continuity in time we set $p = 2$ and then for $0 \leq s < t \leq T$ and every $\gamma \in [0, 1)$ we have the bounds,

$$\begin{aligned}
\mathbb{E}\left[|\mathcal{I}_{0,t}(h_k) - \mathcal{I}_{0,s}(h_k)|^2\right] &\lesssim \sum_{\substack{m \in \operatorname{supp}(\mathcal{F}h_k) \\ m \neq 0}} \left(\int_s^t e^{-(t-r)8\pi|m|^2} + \int_0^s (1 - e^{-(t-s)8\pi|m|^2}) e^{-(s-r)8\pi|m|^2}\right) \\
&= \sum_{\substack{m \in \operatorname{supp}(\mathcal{F}h_k) \\ m \neq 0}} \frac{1}{8\pi^2|m|^2}(1 - e^{-(t-s)8\pi|m|^2}) \left(e^{-s8\pi|m|^2} + (1 - e^{-s8\pi|m|^2})\right) \\
&\lesssim \sum_{\substack{m \in \operatorname{supp}(\mathcal{F}h_k) \\ m \neq 0}} \frac{|t-s|^\gamma}{(8\pi^2|m|^2)^{1-\gamma}} \\
&\lesssim |t-s|^\gamma 2^{-k(1-2\gamma)},
\end{aligned}$$

Therefore, for $0 < s < t < T$ and $\gamma \in [0, 1)$, there exists a constant $C := C(T) > 0$ such that

$$\mathbb{E}\left[|\mathcal{I}_t(h_k) - \mathcal{I}_s(h_k)|^2\right] \leq C|t-s|^\gamma 2^{-k(2-d-2\gamma)}.$$

So then using Nelon's estimate again, for any $p \geq 2$, there exists a constant $C := C(p, T) > 0$ such that for all $k \geq -1$, we have

$$\mathbb{E}\left[|\mathcal{I}_t(h_k) - \mathcal{I}_s(h_k)|^p\right] \leq C|t-s|^{\frac{\gamma}{2}p} 2^{-k(1-\frac{d}{2}-\gamma)p}.$$

Theorem 5.4.5. *For any $T > 0$ there exists a modification of the stochastic convolution, for which we also use the notation, $[0,T] \ni t \mapsto \mathcal{I}_t$, such that for all $\kappa \in [0,1)$,*

$$\mathcal{I} \in \mathcal{C}^{\frac{\kappa}{2}}([0,T]; \mathcal{C}_0^{1-\frac{d}{2}-\kappa}(\mathbb{T}^d)), \quad \mathbb{P}\text{ - a.s.}$$

Furthermore, for any $p \geq 2$, there exists a constant $C := C(T,d,p,\kappa) > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\mathcal{I}_t\|^p_{\mathcal{C}^{1-\frac{d}{2}-\kappa}}\right] < C. \tag{5.4.10}$$

Proof. Applying Lemma 5.4.3 along with the preceding discussion for $\kappa' = \frac{\gamma}{2}$ and $\alpha' = 1 - \frac{d}{2} - \gamma$, with $\gamma \in [0,1)$ we obtain the existence of a modification, for which we also write $\mathcal{I}$, bounded in $\mathcal{C}^\kappa(\mathbb{R}_+; \mathcal{B}^\alpha_{p,p})$ for all $\alpha < 1 - \frac{d}{2} - \gamma - \frac{d}{p}$ and $\kappa < \frac{\gamma}{2} - \frac{1}{p}$, for any $p \geq 2$. Then taking p sufficiently large we apply the Besov embedding, (5.2.5), $\mathcal{B}^{\alpha+\frac{d}{p}}_{p,p}(\mathbb{T}^d) \hookrightarrow \mathcal{C}^\alpha(\mathbb{T}^d)$ in order to conclude. □

Remark 5.4.6. An almost identical result holds for the stationary convolution $\mathcal{I}_\infty$. It suffices to observe that $\widehat{\mathcal{G}^{1/2}} \approx \frac{1}{|m|}$ so that the Littlewood–Paley blocks obey the same scaling as in (5.4.9). Therefore, we also have a modification, for which we use the same notation, such that $\mathcal{I}_\infty \in \mathcal{C}^{1-\frac{d}{2}-\kappa}(\mathbb{T}^d)$ $\mathbb{P}$ - a.s. for all $\kappa > 0$ and there exists a constant $C := C(d,p,\kappa) > 0$ such that

$$\mathbb{E}\left[\|\mathcal{I}_\infty\|^p_{\mathcal{C}^{1-\frac{d}{2}-\kappa}}\right] < C.$$

Corollary 5.4.7. *Let $T > 0$ and $\alpha < 1 - \frac{d}{2}$. Then the process $[0,T] \ni t \mapsto v_t$ admits a modification, for which we use the same notation, such that $v \in C([0,T]; \mathcal{C}^\alpha(\mathbb{T}^d))$ $\mathbb{P}$ -a.s. In addition, for any $\kappa \in (0,1)$, $p \geq 2$, there exists a constant $C := C(T,\alpha,\kappa,p) > 0$ such that*

$$\mathbb{E}\left[\sup_{t\neq s\in[0,T]} \frac{\|v_t - v_s\|^p_{\mathcal{C}^\alpha}}{|t-s|^{p\kappa}}\right] < C. \tag{5.4.11}$$

Furthermore, for any $s, t \in [0,T]$ we have $v_t \overset{law}{=} v_s$.

Proof. By changing variables in the convolution, we have that $\mathcal{I}_\infty(e^{t\Delta}\phi) = e^{t\Delta}\mathcal{I}_\infty(\phi)$, so for any $t > 0$, $\alpha < 1 - \frac{d}{2}$ and $\gamma \in [0, 2)$, using Theorem 5.4.5, we have

$$\|v_t\|_{\mathcal{C}^\alpha} \le \|\mathcal{I}_t\|_{\mathcal{C}^\alpha} + t^{-\frac{\gamma}{2}}\|\mathcal{I}_\infty\|_{\mathcal{C}^{\alpha-\gamma}} < \infty.$$

Similarly, for any $\kappa \in (0, 1)$, using Theorem 5.4.5, we see that there exists a constant $C := C(T, \alpha, \kappa, p) > 0$, such that

$$\begin{aligned}\mathbb{E}\left[\sup_{s\neq t\in[0,T]} \frac{\|v_t - v_s\|^p_{\mathcal{C}^{\alpha-\kappa}}}{|t-s|^{\kappa p}}\right] &\lesssim \mathbb{E}\left[\sup_{s\neq t\in[0,T]} \frac{\|\mathcal{I}_t - \mathcal{I}_s\|^p_{\mathcal{C}^{\alpha-\kappa}}}{|t-s|^{\kappa p}}\right] + \mathbb{E}\left[\sup_{s\neq t\in[0,T]} \frac{\|(e^{t\Delta} - e^{s\Delta})\mathcal{I}_\infty\|^p_{\mathcal{C}^{\alpha-\kappa}}}{|t-s|^{\kappa p}}\right] \\ &< C.\end{aligned}$$

To show stationarity it suffices to consider the covariance. For any $t > 0$ we have

$$\begin{aligned}\mathbb{E}[v_t(\phi)^2] &= \mathbb{E}[\mathcal{I}_t(\phi)^2] + \mathbb{E}[\mathcal{I}_\infty(e^{t\Delta}\phi)^2] \\ &= \sum_{\substack{m\in\mathbb{Z}^2\\ m\neq 0}} \frac{1}{8\pi^2|m|^2}(1 - e^{-8\pi^2|m|^2 t})|\hat{\phi}_m|^2 + \sum_{\substack{m\in\mathbb{Z}^2\\ m\neq 0}} \frac{1}{8\pi^2|m|^2} e^{-8\pi^2|m|^2 t})|\hat{\phi}_m|^2 \\ &= \sum_{\substack{m\in\mathbb{Z}^2\\ m\neq 0}} \frac{1}{8\pi^2|m|^2}|\hat{\phi}_m|^2,\end{aligned}$$

so that for any $t > 0$, $v_t \sim \mu_{\text{GFF}}$. □

In analysing the SPDE in Chapter 6 it is convenient to work with Markov processes. To this end, for any $0 < s < t < \infty$ we set

$$v_{s,t} := v_t - e^{(t-s)\Delta} v_s \tag{5.4.12}$$

which has the following properties.

Theorem 5.4.8. *Let $d \ge 1$ and $T > 0$. Then for every $p \ge 2$, $t_0 \ge 0$ and $\alpha < 1 - \frac{d}{2}$,* *there exists a $\theta := \theta(\alpha) > 0$ and $C := C(T, \alpha, p) > 0$ such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|v_{t_0,t_0+t}\|^p_{\mathcal{C}^\alpha}\right] \leq Ct^{p\theta}, \tag{5.4.13}$$

and for any $\kappa \in (0, 1)$,

$$\mathbb{E}\left[\sup_{t\neq s\in[0,T]} \frac{\|v_{t_0,t_0+t} - v_{t_0,t_0+s}\|^p_{\mathcal{C}^{\alpha-\kappa}}}{|t-s|^{p\kappa}}\right] < C. \tag{5.4.14}$$

Finally, for any $h > 0$, and s, $t \in [0, T - h]$ we have that $v_{t,t+h} \stackrel{law}{=} v_{s,s+h}$.

Proof. Since $v_{t_0,t_0} = 0$ the growth bound (5.4.13) follows from (5.4.14). To show the space-time regularity we observe that

$$\|v_{t_0,t_0+t} - v_{t_0,t_0+s}\|_{\mathcal{C}^{\alpha-k}} \leq \|v_{t_0+t} - v_{t_0+s}\|_{\mathcal{C}^{\alpha-\kappa}} + \|\left(e^{t\Delta} - e^{s\Delta}\right) v_{t_0}\|_{\mathcal{C}^{\alpha-\kappa}}.$$

So then combining (5.4.11) with Theorem 5.2.10 we obtain the space-time bound, (5.4.14).

To show stationarity of the processes $t \mapsto v_{t,t+h}$ for fixed $h > 0$, we see that for any $\phi \in L^2(\mathbb{T}^2)$ we have

$$\begin{aligned}
\mathbb{E}\left[v_{t,t+h}(\phi)^2\right] &= \mathbb{E}\left[v_{t+h}(\phi)^2\right] - 2\mathbb{E}\left[e^{h\Delta}v_t(\phi)v_{t+h}(\phi)\right] + \mathbb{E}\left[\left(e^{h\Delta}v_t(\phi)\right)^2\right] \\
&= \mathbb{E}\left[\mathcal{I}_{t+h}(\phi)^2\right] - 2\mathbb{E}\left[\mathcal{I}_t(e^{h\Delta}\phi)\mathcal{I}_{t+h}(\phi)\right] + \mathbb{E}\left[\mathcal{I}_t(e^{h\Delta}\phi)^2\right] \\
&= \sum \frac{1}{8\pi^2|m|^2}\left(1 - e^{-8\pi^2|m|^2(t+h)}\right)|\hat{\phi}_m|^2 \\
&\quad - \sum \frac{1}{8\pi^2|m|^2}\frac{1}{8\pi^2|m|^2}\left(e^{-8\pi^2|m|^2h} - e^{-8\pi^2|m|^2(t+h)}\right)|\hat{\phi}_m|^2 \\
&= \sum \frac{1}{8\pi^2|m|^2}\left(1 - e^{-8\pi^2|m|^2h}\right)|\hat{\phi}_m|^2.
\end{aligned}$$

Therefore, since $v_{t,t+h}$ is Gaussian, $\mathcal{L}(v_{t,t+h})$ depends only on $h > 0$. □

In summary, from Corollary 5.4.7 we see that in $d = 1$ the stochastic heat equation $v_t := \mathcal{I}_\infty(e^{t\Delta}\cdot) + \mathcal{I}_t$ is almost surely a genuine space-time function. However, in any higher spatial dimension, although the mapping v_t defines a genuine distribution, the SHE almost surely cannot be realised as a genuine function in space. Since we cannot expect any solution to a semi-linear parabolic PDE forced by the space-time white noise to have better local regularity than v we see that while non-linear operations when $d = 1$ are in principle well-defined, in higher dimensions the same is not true.

In order to handle this issue we, define a priori, a finite number of stochastic objects, built from the noise, that are sufficient to at least locally describe the solution to our given non-linear SPDE. In our case we only need to define an object to take the place of the square v^2 when $d = 2$. We take a brief divergence to the introduce the idea of iterated stochastic integrals and discuss some properties that we employ. A more detailed discussion of this topic can be found in [61, 94].

5.4.4 Iterated Stochastic Integrals

Let $f : (\mathbb{R} \times \mathbb{T}^d)^n \to \mathbb{R}$ be a square integrable, symmetric function, in the sense that $f(z_1, \ldots, z_n) = f(\sigma(z_1, \ldots, z_n))$ for any permutation $\sigma \in S_n$. We define the iterated stochastic integral of f, see [94, Ch.1], for which we write

$$I^n(f) := \int_{(\mathbb{R}\times\mathbb{T}^d)^n} f(z_1, \ldots, z_n)\, \xi(\mathrm{d}z_1 \otimes \ldots \otimes \mathrm{d}z_n).$$

Similarly, for any symmetric $g \in L^2(\mathbb{T}^{dn})$ we write

$$I^n_\infty(g) := \int_{\mathbb{T}^{dn}} g(x_1, \ldots, x_n) \zeta(\mathrm{d}x_1 \otimes \ldots \otimes \mathrm{d}x_n),$$

for the iterated integral of g.

For $n \geq 1$ we set

$$\mathfrak{H}_n := \{I^n(f) \,:\, f \in L^2((\mathbb{R}\times\mathbb{T}^d)^n),\ f \text{ is symmetric}\} \cup \left\{I^n_\infty(g) \,:\, g \in L^2(\mathbb{T}^{dn}),\ g \text{ is symmetric}\right\}$$

$$\mathfrak{H}_0 = \mathbb{R}.$$

The set $\mathfrak{H}_n$ is referred to as the n^{th} homogeneous Wiener chaos.

Theorem 5.4.9 (Wiener Decomposition - Thm. 1.1.1 [94])**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the abstract Wiener space, carrying the white noises, ξ, ζ. Then we have the identity*

$$L^2(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{n\geq 0} \mathfrak{H}_n.$$

Furthermore, for any $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ there exist $(f_n)_{n\geq 1}$, $(g_n)_{n\geq 1}$ symmetric functions such that

$$X = \mathbb{E}[X] + \sum_{n\geq 1} I^n(f_n) + \sum_{n\geq 1} I^n_\infty(g_n).$$

Iterated stochastic integrals also obey a kind of Itô isometry. For $f \in L^2((\mathbb{R}\times\mathbb{T}^d)^n)$ or $g \in L^2(\mathbb{T}^{dn})$ both symmetric functions and $I^n(f)$, $I^n(g)$ their n-fold iterated integral one has

$$\mathbb{E}[I^n(f)^2] = n!\,\|f\|^2_{L^2((\mathbb{R}\times\mathbb{T}^d)^n)}, \quad \mathbb{E}[I^n_\infty(g)^2] = n!\,\|g\|^2_{L^2(\mathbb{T}^{dn})}. \tag{5.4.15}$$

We also have Nelson's estimate, which mirrors the equivalence of moments for finite dimensional Gaussian measures.

Lemma 5.4.10 (Nelson's Estimate)**.** *Let $n \geq 1$ and $p \in [2, \infty)$ and $Z \in \oplus_{k\leq n}\mathfrak{H}_k$. Then there exists a $C := C(n,p) > 0$ such that*

$$\mathbb{E}\left[|Z|^p\right]^{\frac{1}{p}} \leq C\mathbb{E}\left[Z^2\right]^{\frac{1}{2}}. \tag{5.4.16}$$

Furthermore, if $Z \in \mathfrak{H}_n$ then $C(n,p) = (p-1)^{\frac{n}{2}}$.

Proof. See proof of [93, Prop. 3.3]. □

5.4.5 Wick Square of the SHE on $\mathbb{T}^2$

After this brief digression we return to our issue of interpreting non-linear operations of v for $d \geq 2$. We restrict ourselves to defining the square in for $d = 2$. We see from the previous discussion that random variables defined as iterated integrals of symmetric functions at least have a hope of being well defined. Borrowing ideas from the path integral formulation of Quantum Field Theory, a natural approach is to define the Wick product by removing the anti-symmetric part of the product. Considering $\mathcal{I}_t^2$, for any $\phi \in L^2(\mathbb{T}^2)$ we have

$$\begin{aligned}\mathcal{I}_t(\phi)^2 &= \left(\int_0^t \int_{\mathbb{T}^2} (\mathcal{H}_{t-s_1} * \phi)(x_1)\xi(\mathrm{d}x_1, \mathrm{d}s_1)\right) \left(\int_0^t \int_{\mathbb{T}^2} (\mathcal{H}_{t-s_2} * \phi)(x_2)\xi(\mathrm{d}x_2, \mathrm{d}s_2)\right) \\ &= \iint_{([0,t]\times\mathbb{T}^2)^2} (\mathcal{H}_{t-s_1} * \phi)(x_1)\,(\mathcal{H}_{t-s_2} * \phi)(x_2)\xi(\mathrm{d}s_1 \otimes \mathrm{d}s_2, \mathrm{d}x_1 \otimes \mathrm{d}x_2) \\ &\quad + \iint_{([0,t]\times\mathbb{T}^2)^2} (\mathcal{H}_{t-s_1} * \phi)(x_1)\,(\mathcal{H}_{t-s_2} * \phi)(x_2)\delta_{(s_1=s_2, x_1=x_2)}\xi(\mathrm{d}x_1, \mathrm{d}s_1)\xi(\mathrm{d}x_2, \mathrm{d}s_2).\end{aligned}$$

We see that the first term is a member of the 2nd homogeneous Wiener chaos and is the iterated integral of the symmetric, square integrable function $(\mathcal{H}_{t-s_1} * \phi)(x_1)\,(\mathcal{H}_{t-s_2} * \phi)(x_2)$. As such Nelson's estimate applies. The second term is not an iterated integral and we do not have tools to evaluate its higher moments. However we see that just takings its first moment gives

$$\int_0^t \int_{\mathbb{T}^2} (\mathcal{H}_{t-s} * \phi)^2(x)\ \mathrm{d}x\ \mathrm{d}s = \sum_{\substack{m\in\mathbb{Z}^2 \\ m\neq 0}} \frac{1}{8\pi^2|m|^2} \left(1 - e^{-8\pi^2|m|^2 t}\right) \hat{\phi}_m.$$

If $\mathrm{supp}(\hat{\phi}) \subset B_n(0)$ then this quantity diverges like $\log(n)$. When $d \geq 3$, the same term diverges like n^{d-2}. Therefore, we see that the Wick square of $\mathcal{I}_t$ can be obtained by subtracting a diverging term. Formally we write

$$\mathcal{I}_t^{:2:} = \mathcal{I}_t^2 - \mathbb{E}[\mathcal{I}_t^2].$$

More precisely, for any $\varepsilon > 0$ letting ρ_ε be a mollifier on $\mathbb{T}^2$ whose Fourier transform is compactly supported in $B_{\frac{1}{\lfloor\varepsilon\rfloor}}(0)$ and for any $x \in \mathbb{T}^d$, we set

$$\mathcal{I}^{\varepsilon}_t(x) = \mathcal{I}_t(\rho_\varepsilon(\,\cdot\, - x)), \qquad \text{and} \qquad \mathcal{I}^{:2:}_{\varepsilon;t}(x) = \mathcal{I}^{\varepsilon}_t(x)^2 - \mathbb{E}\left[(\mathcal{I}^{\varepsilon}_t(x))^2\right].$$

By our previous calculation we see that

$$\mathcal{I}^{:2:}_{\varepsilon;t}(x) = \iint_{([0,t]\times\mathbb{T}^2)^2} (\mathcal{H}_{t-s_1} * \rho_\varepsilon)(x - x_1)\,(\mathcal{H}_{t-s_2} * \rho_\varepsilon)(x - x_2)\xi(\mathrm{d}s_1 \otimes \mathrm{d}s_2, \mathrm{d}x_1 \otimes \mathrm{d}x_2).$$

Taking the same approach to the stationary integral we define

$$\mathcal{I}^{\varepsilon}_\infty(x) := \mathcal{I}_\infty(\rho_\varepsilon(\,\cdot\, - x)), \qquad \text{and} \qquad \mathcal{I}^{:2:}_{\varepsilon;\infty}(x) := \mathcal{I}^{\varepsilon}_\infty(x)^2 - \mathbb{E}\left[\mathcal{I}^{\varepsilon}_\infty(x)^2\right].$$

Therefore, for $\varepsilon > 0$ and v_t the unique solution to (5.4.4), we may define

$$v_{\varepsilon;t}(x) = v_t(\rho_\varepsilon(\,\cdot\, - x)), \quad \text{and} \quad v^{:2:}_{\varepsilon;t} := v_{\varepsilon;t}(x)^2 - \mathbb{E}\left[v_{\varepsilon;t}(x)^2\right].$$

Finally, taking the limit $\varepsilon \to 0$, justified by Theorem 5.4.11 below, we define the distribution

$$v^{:2:}_t(\phi) = \mathcal{I}^{:2:}_\infty(e^{t\Delta}\phi) + 2\mathcal{I}_\infty(e^{t\Delta}\phi)\mathcal{I}_t(\phi) + \mathcal{I}^{:2:}_t(\phi).$$

Note that since ξ and ζ are independent, the middle term is always well defined and can be controlled using the same technique as for $\mathcal{I}_t$ itself. However, for $d < 4$ it is also pathwise well defined since $\mathcal{I}_\infty(e^{t\Delta}\,\cdot\,) \in \mathcal{C}^{3-\frac{d}{2}-}(\mathbb{T}^d)$ almost surely.

Systematically generalising this approach to higher powers can be done using the Hermite polynomials. If we set $\mathfrak{C}_\varepsilon := \mathbb{E}\left[v_{\varepsilon;\infty}(0)\right] \in \mathbb{R}$, then by stationarity of v_ε, we have already defined

$$v^{:2:}_{\varepsilon;t} := v^2_{\varepsilon;t} - \mathfrak{C}_\varepsilon = H_2(v_{\varepsilon;t}, \mathfrak{C}_\varepsilon), \tag{5.4.17}$$

where $H_2(x, c) = x^2 - c$ is the second Hermite polynomial. For any $n \geq 1$ the n^{th}

Hermite polynomial is defined recursively by the formula

$$H_{n+1}(x,c) = xH_n(x,c) - (n-1)cH_{n-2}(x,c), \quad H_{-1}(x,c) := 0, \quad H_0(x,c) := 1.$$

It turns out that when $d = 2$, the n^{th} Wick power, defined by setting $v_{\varepsilon;t}^{:n:} := H_n(v_{\varepsilon;t}, \mathfrak{C}_\varepsilon)$, converges to a well defined random distribution and the sequence is uniformly bounded in $\mathcal{C}^\alpha(\mathbb{T}^2)$ for any $\alpha < 0$, [105]. A rigorous construction of higher powers in $d = 3$ can be found for example in [93] and [60].

Theorem 5.4.11. *Let $p \geq 2$, $\alpha < 0$ and $T > 0$. Then for $n = 1, 2$ the process $t \mapsto v_t^{:n:}$ admits a modification, for which we also write $t \mapsto v_t^{:n:}$, such that $v^{:n:} \in C([0,T]; \mathcal{C}^\alpha(\mathbb{T}^2))$. In addition there exists $\theta := \theta(\alpha) \in (0,1)$ and $C := C(T, \alpha, p) > 0$ such that*

$$\mathbb{E}\left[\sup_{s,\, t\in[0,T]} \frac{\|v_t^{:n:} - v_s^{:n:}\|_{\mathcal{C}^\alpha}^p}{|t-s|^{p\theta}}\right] \leq C. \tag{5.4.18}$$

Furthermore, for any $t > 0$,

$$\lim_{\varepsilon \searrow 0} \|v_t^{:n:} - v_{\varepsilon;t}^{:n:}\|_{\mathcal{C}^\alpha} = 0,$$

and both $v_t^{:n:}$ and $v_{\varepsilon;t}^{:n:}$ are stationary processes.

Proof. For $n = 1$ we have already shown regularity and stationarity, see Corollary 5.4.7. The approximation result holds by continuity of the mollifier in Hölder–Besov spaces. To show regularity of the square we focus on $\mathcal{I}_t^{:2:}$, since similar arguments hold for $\mathcal{I}_\infty^{:2:}(e^{t\Delta}\phi)$. Using (5.4.15) and the semi-group property of the heat kernel, we have, for any $\alpha < 0$,

$$
\begin{aligned}
\mathbb{E}\left[\mathcal{I}_t^{:2:}(h_k)^2\right] &= 2\iint_{([0,T]\times\mathbb{T}^2)^2} (\mathcal{H}_{t-s_1} * h_k)^2(x_1)(\mathcal{H}_{t-s_2} * h_k)^2(x_2)\, \mathrm{d}s_1\, \mathrm{d}x_1\, \mathrm{d}s_2\, \mathrm{d}x_2 \\
&= 2\iint_{\mathbb{T}^4} h_k(x_1)h_k(x_2)\left(\int_0^t \mathcal{H}_{2t-2r}(x_1 - x_2)\, \mathrm{d}r\right)^2 \mathrm{d}x_1\, \mathrm{d}x_2 \\
&= 2\sum_{\substack{m_1,m_2\in\mathbb{Z}^2\setminus\{0\}\\ m=m_1+m_2}} \frac{1}{(8\pi^2|m_1|^2)(8\pi^2|m_2|^2)}\left(1 - e^{-8\pi^2|m_1|^2 t}\right)\left(1 - e^{-8\pi^2|m_2|^2 t}\right)|\chi_k(m)|^2 \\
&\leq 2\sum_{\substack{m_1,m_2\in\mathrm{supp}(\chi_k)\setminus\{0\}\}\\ m=m_1+m_2}} \frac{1}{(8\pi^2|m_1|^2)(8\pi^2|m_2|^2)} \\
&\lesssim \sum_{\substack{m\in\mathrm{supp}(\chi_k)\\ m\neq 0}} \frac{1}{(8\pi^2|m|^2)^{1+\alpha}} \\
&\leq 2^{-\alpha k}.
\end{aligned}
$$

We refer to [105, Lem. C.2 & Cor. C.3] for a proof of the penultimate estimate. Therefore applying Lemma 5.4.3 we obtain the required regularity of $\mathcal{I}_t^{:2:}$. Showing Hölder continuity in time follows in a similar manner as for $\mathcal{I}_t$, Theorem 5.4.5. Applying the same steps to $\mathcal{I}_\infty^{:2:}(e^{t\Delta}\,\cdot\,)$ and using the regularity established for $\mathcal{I}_t$ and $\mathcal{I}_\infty(e^{t\Delta}\,\cdot\,)$ in the mixed term, gives

$$
\begin{aligned}
\|v_t^{:2:}\|_{\mathcal{C}^\alpha} &\leq \|\mathcal{I}^{:2:}(e^{t\Delta}\,\cdot\,)\|_{\mathcal{C}^{2\alpha}} + 2\|\mathcal{I}_\infty(e^{t\Delta}\,\cdot\,)\mathcal{I}_t\|_{\mathcal{C}^\alpha} + \|\mathcal{I}_t^{:2:}\|_{\mathcal{C}^\alpha} \\
&\leq \|\mathcal{I}^{:2:}(e^{t\Delta}\,\cdot\,)\|_{\mathcal{C}^{2+\alpha}} + 2\|\mathcal{I}_\infty(e^{t\Delta}\,\cdot\,)\|_{\mathcal{C}^{2+\alpha}}\|\mathcal{I}_t\|_{\mathcal{C}^\alpha} + \|\mathcal{I}_t^{:2:}\|_{\mathcal{C}^\alpha} \\
&< \infty.
\end{aligned}
$$

To show stationarity of $t \mapsto v_t^{:2:}$, we compute its covariance.

$$
\begin{aligned}
\frac{1}{2}\mathbb{E}[v_t^{:2:}(\phi)^2] &= \frac{1}{2}\mathbb{E}\left[\left(\mathcal{I}_\infty^{:2:}(e^{t\Delta}\phi) + 2\mathcal{I}_\infty(e^{t\Delta}\phi)\mathcal{I}_t(\phi) + \mathcal{I}_t^{:2:}(\phi)\right)^2\right] \\
&= \frac{1}{2}\mathbb{E}\left[\mathcal{I}_\infty^{:2:}(e^{t\Delta}\phi)^2\right] + 2\mathbb{E}\left[\mathcal{I}_\infty(e^{t\Delta}\phi)\mathcal{I}_\infty^{:2:}(e^{t\Delta}\phi)\mathcal{I}_t(\phi)\right] + 2\mathbb{E}\left[\mathcal{I}_\infty(e^{t\Delta}\phi)\mathcal{I}_t(\phi)\mathcal{I}_t^{:2:}(\phi)\right] \\
&\quad + 2\mathbb{E}\left[\mathcal{I}_\infty(e^{t\Delta}\phi)^2\mathcal{I}_t(\phi)^2\right] + \mathbb{E}\left[\mathcal{I}_\infty^{:2:}(e^{t\Delta}\phi)\mathcal{I}_t^{:2:}(\phi)\right] + \frac{1}{2}\mathbb{E}\left[\mathcal{I}_t^{:2:}(\phi)^2\right] \\
&= \frac{1}{2}\mathbb{E}\left[\mathcal{I}_\infty^{:2:}(e^{t\Delta}\phi)^2\right] + 2\mathbb{E}\left[\mathcal{I}_\infty(e^{t\Delta}\phi)^2\right]\mathbb{E}\left[\mathcal{I}_t(\phi)^2\right] + \frac{1}{2}\mathbb{E}\left[\mathcal{I}_t^{:2:}(\phi)^2\right],
\end{aligned}
$$

where the last equality follows from the independence of ξ and ζ and the fact that the terms $\mathcal{I}_t$, $\mathcal{I}_\infty$, $\mathcal{I}_t^{:2:}$, $\mathcal{I}_\infty^{:2:}$ all have expectation 0. So then applying (5.4.15) and writing these three terms in Fourier coefficients we see that the last line is independent of t by a similar cancellation as occurred for v_t.

The convergence of $v_\varepsilon^{:2:} \to v_t^{:2:}$ holds by a similar argument as in the proof of [105, Prop. 2.3]. □

Recalling that, for any $0 < s < t < \infty$ we defined

$$v_{s,t} := v_t - e^{(t-s)\Delta} v_s.$$

We define the Wick square of this process by the expression

$$v_{s,t}^{:2:} := v_t^{:2:} - 2(e^{(t-s)\Delta} v_s) v_t + \left(e^{(t-s)\Delta} v_s\right)^2.$$

The process $v_{s,t}^{:2:}$ is not a member of the 2nd homogenous Wiener chaos and nor does it converge to $v_s^{:2:}$ as $t \to s$ in $\mathcal{C}^\alpha(\mathbb{T}^2)$. However, for any $t > s$ it does define a genuine distribution, bounded in a suitable Hölder–Besov space. We will see below that while v, $v^{:2:}$ are stationary processes, $v_{s,\cdot}$, $v_{s,\cdot}^{:2:}$ are Markov and for any $h > 0$ and $s, t \in [0, T-h]$ we have $v_{t,t+h} \overset{\text{law}}{=} v_{s,s+h}$ and $v_{t,t+h}^{:2:} \overset{\text{law}}{=} v_{s,s+h}^{:2:}$. We also define the approximations, for any $\varepsilon > 0$, by setting

$$v_{s,t}^\varepsilon := v_t^\varepsilon - e^{(t-s)\Delta} v_s^\varepsilon, \quad v_{\varepsilon;s,t}^{:2:} := v_{\varepsilon;t}^{:2:} - 2\left(e^{(t-s)\Delta} v_s^\varepsilon\right) v_t^\varepsilon + \left(e^{(t-s)\Delta} v_s^\varepsilon\right)^2.$$

We collect the following results.

Theorem 5.4.12. *Let $n = 1, 2$. Then for every $p \geq 2$, $T > 0$, $t_0 \geq 0$, $\alpha \in (-1, 0)$ and $\alpha' > 0$, there exists a $\theta := \theta(\alpha, \alpha') > 0$ and $C := C(T, \alpha, \alpha', p, n) > 0$ such that,*

$$\mathbb{E}\left[\sup_{t \in [0,T]} \left(t^{(n-1)\alpha' p} \|v_{t_0, t_0+t}^{:n:}\|_{\mathcal{C}^\alpha(\mathbb{T}^2)}^p\right)\right] \leq C t^{p\theta}, \tag{5.4.19}$$

and we have

$$\lim_{\varepsilon\searrow 0}\mathbb{E}\left[\sup_{t\in[0,T]}\left(t^{(n-1)\alpha' p}\|v^{:n:}_{\varepsilon;t_0,t_0+t}-v^{:n:}_{t_0,t_0+t}\|^p_{\mathcal{C}^\alpha(\mathbb{T}^2)}\right)\right]=0. \tag{5.4.20}$$

In addition, for any $\delta>0$ and $\kappa\in(0,1)$ there exists a constant $C(T,\delta,p,\kappa,\alpha)>0$ such that

$$\mathbb{E}\left[\sup_{t\neq s\in[0,T]}\frac{\|v^{:n:}_{t_0,t_0+\delta+t}-v^{:n:}_{t_0,t_0+\delta+s}\|^p_{\mathcal{C}^{\alpha-\kappa}}}{|t-s|^{p\kappa}}\right]<C. \tag{5.4.21}$$

Finally, for any $h>0$ and $s,t\in[0,T-h]$ we have

$$v_{t,t+h}\overset{law}{=}v_{s,s+h},\qquad v^{:2:}_{t,t+h}\overset{law}{=}v^{:2:}_{s,s+h}.$$

Proof. For $n=1$ we have already shown the regularity and stationarity results in Theorem 5.4.8. The approximation result holds by continuity of the mollifier in Hölder–Besov spaces.

For $n=2$ we let $\bar\alpha>-\alpha$. Then using (5.2.13) and (5.2.15) we have that

$$\left\|\left(e^{t\Delta}v_{t_0}\right)^2\right\|_{\mathcal{C}^\alpha}\le\|e^{t\Delta}v_{t_0}\|_{\mathcal{C}^{\bar\alpha}}\|e^{t\Delta}v_{t_0}\|_{\mathcal{C}^\alpha}\lesssim t^{-\frac{\overline{\alpha}-\alpha}{2}}\|v_{t_0}\|^2_{\mathcal{C}^\alpha}.$$

For the mixed term we have

$$\|\left(e^{t\Delta}v_{t_0}\right)v_{t_0+t}\|_{\mathcal{C}^\alpha}\le\|e^{t\Delta}v_{t_0}\|_{\mathcal{C}^{\overline{\alpha}}}\|v_{t_0+t}\|_{\mathcal{C}^\alpha}\lesssim t^{-\frac{\overline{\alpha}-\alpha}{2}}\|v_{t_0}\|_{\mathcal{C}^\alpha}\|v_{t_0+t}\|_{\mathcal{C}^\alpha}.$$

Therefore, using the fact that v_{t_0} and v_{t_0+t} are independent, we see that for $\alpha'>\frac{\bar\alpha-\alpha}{2}$, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\left(t^{\alpha' p}\|v^{:2:}_{t_0,t_0+t}\|^p_{\mathcal{C}^\alpha(\mathbb{T}^2)}\right)\right]&\lesssim t^{\alpha' p}\mathbb{E}\left[\sup_{t\in[0,T]}\|v^{:2:}_{t_0+t}\|_{\mathcal{C}^\alpha}\right]\\&\quad+t^{\left(\alpha'-\frac{\overline{\alpha}-\alpha}{2}\right)p}\mathbb{E}\left[\|v_{t_0}\|^p_{\mathcal{C}^\alpha}\right]\mathbb{E}\left[\sup_{t\in[0,T]}\|v_{t_0+t}\|^p_{\mathcal{C}^\alpha}\right]\\&\quad+t^{\left(\alpha'-\frac{\overline{\alpha}-\alpha}{2}\right)p}\mathbb{E}\left[\|v_{t_0}\|^2_{\mathcal{C}^\alpha}\right].\end{aligned}$$

We note that since $\alpha < 0$ is arbitrarily close to zero, we may take $\bar{\alpha}$, $\alpha' > 0$ arbitrarily small. Finally applying Theorem 5.4.11 gives (5.4.19) for some $\theta := \theta(\alpha, \alpha') > 0$.

Hölder continuity in time follows by a similar argument to the proof of (5.4.11) and the convergence of the approximations follows from Theorem 5.4.11 and the continuity of the heat semi-group.

To show stationarity of $t \mapsto v_{t,t+h}^{:2:}$ we compute its covariance and by a similar cancellation as occured for $v_{t,t+h}$ in (5.4.8), we see $\mathcal{L}(v_{t,t+h}^{:2:})$ depends only on $h > 0$. □

Chapter 6

Analysis in One Spatial Dimension

We study local and global well-posedness and ergodicity for the SPDE

$$\begin{cases} \partial_t u - \partial_{xx} u = \chi \partial_x (u^2 \partial_x \rho_u) + \xi, & \text{on } \mathbb{R}_+ \times \mathbb{T}, \\ -\partial_{xx} \rho_u = u - \bar{u}, & \text{on } \mathbb{T}, \\ u|_{t=0} = x, & \text{on } \mathbb{T}. \end{cases} \tag{6.0.1}$$

The first step is to obtain local well-posedness. This is completed in Section 6.1 where we also collect some useful properties of the solution for later analysis. We note that the local analysis does not depend on the positive sign choice and holds equally for the attractive equation. In Section 6.2 we employ a testing method to obtain a priori estimates on $\|u_t\|_{L^p}$ that are independent of the initial data and depend on the noise in a small time interval only. A key tool is the coming down from infinity property of the cubic ODE, $\dot{Y} = -Y^3$. Combining the a priori estimate with the local well-posedness enables us to obtain global well-posedness in Section 6.3. From this point we closely follow the strategy of [105] where the authors demonstrate exponential ergodicity for stochastic quantisation equations in two dimensions. Defining a Markov semi-group associated to (6.0.1), we employ the Krylov–Bogoliubov method to give the existence of an invariant measure associated to each initial condition. Then in Section 6.4 we prove that the semi-group is Lipschitz continuous and that the invariant measures all have full support in the target space. Combining these two results we show exponential ergodicity of the semi-group and so obtain a relaxation rate for the SPDE to its unique invariant measure. From the a priori bound we also obtain a tail estimate on the invariant measure.

6.0.1 Main Results

We summarise the main results obtained in this chapter. We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ carrying the space-time and space white noise pair (x, ξ) defined Section 5.4.1. We fix $\alpha_0 \in \left(-\frac{1}{2}, 0\right)$, $\alpha \in \left(0, \alpha_0 + \frac{1}{2}\right)$ and $\eta > 0$ such that,

$$\frac{\alpha - \alpha_0}{2} < \eta < \frac{1}{4}. \tag{6.0.2}$$

We recall that we write $\mathcal{C}^\alpha_{\mathfrak{m}}(\mathbb{T})$ for the space of α-Hölder distributions with spatial mean equal to $\mathfrak{m} \in \mathbb{R}$,

Theorem 6.0.1 (Global Well-Posedness)**.** *Let $\alpha_0 \in (-1/2, 0)$, $\mathfrak{m} \in \mathbb{R}$ and $x \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}}$ and $\chi > 0$. Then for any $T > 0$ and $\delta \in (0, 1/2)$, there exists a unique, probabilistically strong solution, $u(x)$, to (6.0.1) such that $\mathbb{P}$ - a.s. for all $t > 0$, $u_t(x) \in \mathcal{C}^{1/2-\delta}_{\mathfrak{m}}(\mathbb{T})$. Furthermore, the map*

$$\begin{aligned}\mathscr{S} : \mathcal{C}^{\alpha_0}_{\mathfrak{m}}(\mathbb{T}) \times C([0,T]; \mathcal{C}^\alpha_0(\mathbb{T})) &\to C_\eta((0,T]; \mathcal{C}^\alpha_{\mathfrak{m}}(\mathbb{T})) \\ (x, v) &\mapsto u(x),\end{aligned}$$

is continuous in both arguments.

We write $\mathcal{L}(u_t(x)) \in \mathcal{P}(\mathcal{C}^{1/2-\delta}_{\mathfrak{m}}(\mathbb{T}))$ for the law of the solution, started from $x \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}}$, at time $t > 0$.

Theorem 6.0.2 (Exponential Ergodicity)**.** *Let $\mathfrak{m} \in \mathbb{R}$ and $\delta \in (0, 1/2)$. Then there exists a unique measure $\nu \in \mathcal{P}(\mathcal{C}^{1/2-\delta}_{\mathfrak{m}}(\mathbb{T}))$ and a constant $c \in (0, 1)$ such that for all $t > 3$ and $x \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}}$, we have that*

$$\|\mathcal{L}(u_t(x)) - \nu\|_{TV} \leq c^{\lfloor \frac{t}{3} \rfloor}.$$

Furthermore, there exists a $\lambda > 0$ sufficiently small, such that for any even integer $p \in [1, \infty)$ and any $\delta \in (0, 1/2)$, we have that

$$\int_{\mathcal{C}_{\mathrm{m}}^{1/2-\delta}} \exp\left(\lambda \|u\|_{L^p}^{1-2\delta}\right) \nu(\mathrm{d}u) < \infty.$$

The proof of Theorem 6.0.1 is completed at the beginning of Section 6.3 and the proof of Theorem 6.0.2 is completed at the end of Section 6.4.

6.1 Local Well-Posedness

Throughout this section, we fix $T > 0$ and $\chi \in \mathbb{R}$ (not necessarily positive). From Theorem 5.4.8 we see that $\mathbb{P}$-a.s. the map $t \mapsto v_{0,t}$ is finite only in $C_T\mathcal{C}_0^\alpha$, for $\alpha < 1/2$. We therefore cannot expect to find solutions to (6.0.1) of any higher regularity. Obtaining the a priori estimates in Section 6.2 requires at least one degree of spatial regularity. Thus we decompose the solution into a regular and irregular part by defining $u_t := w_t + Z_t$. Here $t \mapsto Z_t$ is a deterministic function taking values in the Hölder–Besov space $\mathcal{C}_0^\alpha$ with zero spatial mean, which will play the role of a $\mathbb{P}$-a.s. realisation of $t \mapsto v_{0,t}$. The unknown, w, solves,

$$\begin{cases} \partial_t w - \partial_{xx} w = \chi \partial_x((w+Z)^2 \partial_x \rho_{w+Z}), & \text{on } \mathbb{R}_+ \times \mathbb{T}, \\ -\partial_{xx}\rho_{w+Z} = w - \bar{w} + Z, & \text{on } \mathbb{T}, \\ w|_{t=0} = x, & \text{on } \mathbb{T}. \end{cases} \tag{6.1.1}$$

For the rest of this section, we fix $x \in \mathcal{C}^{\alpha_0}(\mathbb{T})$ and $Z \in C_T\mathcal{C}_0^\alpha$.

We first show that under suitable regularity assumptions on w, the right hand side of the first equation in (6.1.1) is a well-defined element of $C_{\eta;T}\mathcal{C}^\alpha(\mathbb{T})$.

Lemma 6.1.1. *Let $w \in C_{\eta;T}\mathcal{C}^\alpha$. Then the map $w \mapsto \Psi w$, defined for any $t \in (0, T]$, by*

$$(\Psi w)_t := e^{t\Delta} x + \int_0^t e^{(t-s)\Delta} \chi \partial_x((w_s + Z_s)^2 \partial_x \rho_{w_s+Z_s}) \,\mathrm{d}s, \tag{6.1.2}$$

is well-defined from $C_{\eta;T}\mathcal{C}^\alpha$ *to itself.*

Proof. Applying (5.2.15), for any $t > 0$, we have that

$$\|e^{t\Delta}x\|_{\mathcal{C}^\alpha} \lesssim t^{-\frac{\alpha-\alpha_0}{2}}\|x\|_{\mathcal{C}^{\alpha_0}}. \tag{6.1.3}$$

Concerning the integral term, expanding the square and applying Theorem 5.2.9, along with (5.2.17) we obtain the bounds

$$\begin{aligned}
\|w_s^2\partial_x\rho_{w_s+Z_s}\|_{\mathcal{C}^\alpha} &\leq \|w\|_{\mathcal{C}^\alpha}^2\|\partial_x\rho_{w_s+Z_s}\|_{\mathcal{C}^\alpha} \lesssim \|w_s\|_{\mathcal{C}^\alpha}^3 + \|w_s\|_{\mathcal{C}^\alpha}^2\|Z_s\|_{\mathcal{C}^\alpha}\\
2\|w_sZ_s\partial_x\rho_{w_s+Z_s}\|_{\mathcal{C}^\alpha} &\leq 2\|w\|_{\mathcal{C}^\alpha}\|Z_s\|_{\mathcal{C}^\alpha}\|\partial_x\rho_{w_s+Z_s}\|_{\mathcal{C}^\alpha} \lesssim 2\|w_s\|_{\mathcal{C}^\alpha}^2\|Z_s\|_{\mathcal{C}^\alpha} + 2\|w_s\|_{\mathcal{C}^\alpha}\|Z_s\|_{\mathcal{C}^\alpha}^2\\
\|Z_s^2\partial_x\rho_{w_s+Z_s}\|_{\mathcal{C}^\alpha} &\leq \|Z_s\|_{\mathcal{C}^\alpha}^2\|\partial_x\rho_{w_s+Z_s}\|_{\mathcal{C}^\alpha} \lesssim \|w_s\|_{\mathcal{C}^\alpha}\|Z_s\|_{\mathcal{C}^\alpha}^2 + \|Z_s\|_{\mathcal{C}^\alpha}^3.
\end{aligned}$$

Combining these yields that

$$\|(w_s+Z_s)^2\partial_x\rho_{w_s+Z_s}\|_{\mathcal{C}^\alpha} \lesssim \sum_{k=0}^{3}\binom{3}{k}\|w_s\|_{\mathcal{C}^\alpha}^{3-k}\|Z_s\|_{\mathcal{C}^\alpha}^k. \tag{6.1.4}$$

Therefore, applying (5.2.15), (5.2.7) for any $s < t \in (0, T\wedge 1]$ we have

$$\begin{aligned}
\|e^{(t-s)\Delta}\partial_x((w_s+Z_s)^2\partial_x\rho_{w_s+Z_s})\|_{\mathcal{C}^\alpha} &\lesssim (t-s)^{-\frac{1}{2}}\|(w_s+Z_s)^2\partial_x\rho_{w_s+Z_s}\|_{\mathcal{C}^\alpha}\\
&\lesssim (t-s)^{-\frac{1}{2}}s^{-3\eta}\max_{k=0,\dots,3}\left\{\|w\|_{C_{\eta;T}\mathcal{C}^\alpha}^{3-k}\|Z\|_{C_T\mathcal{C}^\alpha}^k\right\}.
\end{aligned}$$

Since $\eta < \frac{1}{4}$ we may integrate $s^{-3\eta}$ near 0 and so for $t \in (0, T\wedge 1]$ we have

$$t^\eta\|(\Psi w)_t\|_{\mathcal{C}^\alpha} \lesssim t^{\eta-\frac{\alpha-\alpha_0}{2}}\|x\|_{\mathcal{C}^{\alpha_0}} + t^{\frac{1}{2}-2\eta}\max_{k=0,\dots,3}\left\{\|w\|_{C_{\eta;T}\mathcal{C}^\alpha}^{3-k}\|Z\|_{C_T\mathcal{C}^\alpha}^k\right\}. \tag{6.1.5}$$

where both exponents are positive due to (6.0.2) and the norms on the right hand side are finite by assumption. For $t > 1$ one may argue in almost exactly the same way, only splitting the time integral at $t = 1$ and replacing the multiplication by t^η with $(t\wedge 1)^\eta$. □

Definition 6.1.2 (Mild Solutions to (6.1.1)). We say that $w \in C_{\eta;T}\mathcal{C}^\alpha$ is a mild solution to (6.1.1) on $[0,T]$ (started from x and driven by Z) if for every $t \in (0,T]$,

$$w_t = e^{t\Delta}x + \int_0^t e^{(t-s)\Delta}\chi\partial_x\left((w_s+Z_s)^2\partial_x\rho_{w_s+Z_s}\right)\,\mathrm{d}s. \tag{6.1.6}$$

Remark 6.1.3. Lemma 6.1.1 demonstrates that for any solution, the right hand side of (6.1.6) is well-defined.

Theorem 6.1.4 (Local Well-Posedness of (6.1.1))**.** *Let* $\mathfrak{R} \geq 1$ *such that* $\|Z\|^3_{C_T\mathcal{C}^\alpha} + \|x\|_{\mathcal{C}^{\alpha_0}(\mathbb{T})} < \mathfrak{R}$*. Then there exists* $C > 0$*, independent of* $\mathfrak{R}$*,* x*, and* Z*, such that* (6.1.1) *has a unique mild solution* $w \in C_{\eta;T_*}\mathcal{C}^\alpha$ *where*

$$T_* = \left(\frac{1}{C\mathfrak{R}}\right)^{\frac{1}{\theta}} \wedge T, \quad \textit{with} \quad \theta := \left(\eta - \frac{\alpha-\alpha_0}{2}\right) \wedge \left(\frac{1}{2} - 2\eta\right). \tag{6.1.7}$$

Furthermore

$$\sup_{t\in(0,T_*]} t^\eta \|w_t\|_{\mathcal{C}^\alpha} \leq 1 \tag{6.1.8}$$

and

$$\lim_{t\to 0}\|w_t - x\|_{\mathcal{C}^{\alpha_0}} = 0. \tag{6.1.9}$$

Proof. Denoting

$$B_{T_*} := \left\{ w \in C((0,T_*];\mathcal{C}^\alpha(\mathbb{T})) \,:\, \sup_{t\in(0,T_*]} t^\eta\|w_t\|_{\mathcal{C}^\alpha} \leq 1 \right\},$$

we will show that

$$w \mapsto (\Psi w)_t := e^{t\Delta}x + \int_0^t e^{(t-s)\Delta}\chi\partial_x\left((w_s+Z_s)^2\partial_x\rho_{w_s+Z_s}\right)\,\mathrm{d}s,$$

is a contraction on B_{T_*} for T_* defined by (6.1.7) for $C > 0$ sufficiently large. By (6.1.5), for $w \in B_{T_*}$ and $t \in (0,T_*]$, there exists $C > 0$ such that

$$t^\eta\|(\Psi w)_t\|_{\mathcal{C}^\alpha} \leq Ct^{\left(\eta - \frac{\alpha-\alpha_0}{2}\right)\wedge\left(\frac{1}{2}-2\eta\right)}\mathfrak{R},$$

and so Ψ maps B_{T_*} into itself for T_* defined by (6.1.7). To show that Ψ is a contraction we let w, $\tilde{w} \in B_{T_*}$. For any $s \in (0, T_*]$, using similar steps as in the proof of (6.1.5), we have that

$$\begin{aligned}
&\|(w_s + Z_s)^2 \partial_x \rho_{w_s+Z_s} - (\tilde{w}_s + Z_s)^2 \partial_x \rho_{\tilde{w}_s+Z_s}\|_{\mathcal{C}^\alpha} \\
&\qquad \lesssim \|w_s - \tilde{w}_s\|_{\mathcal{C}^\alpha} \sum_{k=0}^{2} \binom{2}{k} \left(\|w_s\|_{\mathcal{C}^\alpha} \vee \|\tilde{w}_s\|_{\mathcal{C}^\alpha}\right)^{2-k} \|Z_s\|_{\mathcal{C}^\alpha}^k \\
&\qquad \lesssim \mathfrak{R} s^{-2\eta} \|w_s - \tilde{w}_s\|_{\mathcal{C}^\alpha}.
\end{aligned}$$

So then, for any $t \in (0, T_*]$, we have

$$\begin{aligned}
t^\eta \|(\Psi w)_t - (\Psi \tilde{w})_t\|_{\mathcal{C}^\alpha} &\leq t^\eta \int_0^t \|e^{(t-s)\Delta} \chi \partial_x ((w_s + Z_s)^2 \partial_x \rho_{w_s+Z_s} \\
&\qquad\qquad - (\tilde{w}_s + Z_s)^2 \partial_x \rho_{\tilde{w}_s+Z_s})\|_{\mathcal{C}^\alpha} \, \mathrm{d}s \\
&\lesssim t^\eta \int_0^t (t-s)^{-\frac{1}{2}} \|(w_s + Z_s)^2 \partial_x \rho_{w_s+Z_s} - (\tilde{w}_s + Z_s)^2 \partial_x \rho_{\tilde{w}_s+Z_s}\|_{\mathcal{C}^\alpha} \, \mathrm{d}s \\
&\lesssim \mathfrak{R} t^{\frac{1}{2}-2\eta} \|w - \tilde{w}\|_{C_{\eta;t}\mathcal{C}^\alpha}.
\end{aligned}$$

It follows that there exists $C > 0$ such that, for T_* given by (6.1.7),

$$\begin{aligned}
\|\Psi w - \Psi \tilde{w}\|_{C_{\eta;T_*}\mathcal{C}^\alpha} &\leq C\mathfrak{R}(T_*)^{\frac{1}{2}-2\eta} \|w - \tilde{w}\|_{C_{\eta;T_*}\mathcal{C}^\alpha} \\
&\leq \frac{1}{2} \|w - \tilde{w}\|_{C_{\eta;T_*}\mathcal{C}^\alpha}.
\end{aligned}$$

Hence Ψ is a contraction on B_{T_*}, and therefore there exists a unique fixed point $w \in B_{T_*}$ of Ψ which, by construction, is a mild solution to (6.1.1) in the sense of Definition 6.1.2 and satisfies (6.1.8).

To show that w is the unique solution in all of $C_{\eta;T_*}\mathcal{C}^\alpha$, let $\tilde{w}$ be another mild solutions of (6.1.1). Then, by a similar argument as above, there exists a $\tilde{T}(\|w\|_{C_{\eta;T_*}\mathcal{C}^\alpha}, \|\tilde{w}\|_{C_{\eta;T_*}\mathcal{C}^\alpha}) =: \tilde{T} \in (0, T_*]$ such that $w, \tilde{w} \in B_{\tilde{T}}$. Since both must be fixed points of Ψ on $B_{\tilde{T}}$ we have that $w = \tilde{w}$ on $[0, \tilde{T}]$. Iterating the argument, using the same $\tilde{T}$ at each step, shows that $w = \tilde{w}$ on $[0, T_*]$.

To show (6.1.9), observe first that $\lim_{t\to 0} \|e^{t\Delta} x - x\|_{\mathcal{C}^{\alpha_0}} = 0$ by Remark 5.2.11 and so it only remains to show that the integral term converges to zero in $\mathcal{C}^{\alpha_0}$. Consider $\tilde{\eta} \in \left(\frac{\alpha-\alpha_0}{2}, \eta\right)$. Since $C_{\tilde{\eta};T_*}\mathcal{C}^\alpha \hookrightarrow C_{\eta;T_*}\mathcal{C}^\alpha$, applying what we have proved so far to $(\alpha_0, \alpha, \tilde{\eta})$ in place of (α_0, α, η), we see that, for $S > 0$ sufficiently small, w is also the unique mild solution to (6.1.1) in $C_{\tilde{\eta};S}\mathcal{C}^\alpha$, and that $\sup_{t\in(0,S]} t^{\tilde{\eta}}\|w_t\|_{\mathcal{C}^\alpha} \leq 1$. Applying (5.2.15) and (6.1.4), for all $t \in (0, S]$

$$\begin{aligned}\int_0^t \|e^{(t-s)\Delta}\partial_x((w_s + Z_s)^2\partial_x\rho_{w_s+Z_s}\|_{\mathcal{C}^{\alpha_0}}\mathrm{d}s &\lesssim \int_0^t (t-s)^{-\frac{\alpha_0-\alpha+1}{2}} s^{-3\tilde{\eta}}\,\mathrm{d}s \\ &= t^{\frac{1}{2}+\frac{\alpha-\alpha_0}{2}-3\tilde{\eta}},\end{aligned}$$

where we used the fact that $\frac{\alpha_0-\alpha+1}{2} \vee 3\tilde{\eta} < 1$ to evaluate the integral. We now choose $\tilde{\eta}$ sufficiently close to $\frac{\alpha-\alpha_0}{2}$ so that $\frac{1}{2} + \frac{\alpha-\alpha_0}{2} - 3\tilde{\eta} > 0$, from which (6.1.9) follows. □

Lemma 6.1.5. *Suppose that $w \in C_{\eta;T}\mathcal{C}^\alpha$ is a mild solution to (6.1.1). Then for all $t_0 \in (0, T)$*

$$\sup_{t\neq s\in[t_0,T]} \frac{\|w_t - w_s\|_{\mathcal{C}^{\beta-2\kappa}}}{|t-s|^\kappa} < \infty. \tag{6.1.10}$$

Proof. Applying (5.2.15) we have that

$$\|e^{t\Delta}x\|_{\mathcal{C}^\beta} \lesssim t^{-\frac{\beta-\alpha_0}{2}}\|x\|_{\mathcal{C}^{\alpha_0}}. \tag{6.1.11}$$

Regarding the integral term, applying similar steps as in the proof of (6.1.5) and that by assumption $\frac{1+\beta-\alpha}{2} < 1$, we have that

$$\begin{aligned}\int_0^t \|e^{(t-s)\Delta}\partial_x\left((w_s + Z_s)^2\partial_x\rho_{w_s+Z_s}\right)\|_{\mathcal{C}^\beta}\,\mathrm{d}s &\lesssim \int_0^t (t-s)^{-\frac{1+\beta-\alpha}{2}} s^{-3\eta}\,\mathrm{d}s \\ &= t^{\frac{1}{2}-\frac{\beta-\alpha}{2}-3\eta}.\end{aligned} \tag{6.1.12}$$

It follows that

$$\sup_{t\in[t_0,T_*]} \|w_t\|_{\mathcal{C}^\beta} < \infty.$$

Now, let $0 < t_0 < s < t \leq T$ and $\kappa \in [0, 1)$. Applying the triangle inequality and using the semi-group property, we have that

$$\begin{aligned}\|w_t - w_s\|_{\mathcal{C}^{\beta-2\kappa}} &\leq \left\|(e^{(t-s)-1)\Delta} w_s\right\|_{\mathcal{C}^{\beta-2\kappa}} \\ &\quad + \int_s^t \left\|e^{(t-r)\Delta}\partial_x((w_r+Z_r)^2\partial_x\rho_{w_r+Z_r})\right\|_{\mathcal{C}^{\beta-2\kappa}} \, \mathrm{d}r \\ &=: I_1(s,t) + \int_s^t I_2(r,t) \, \mathrm{d}r.\end{aligned}$$

Using (5.2.15) and (5.2.16),

$$I_1(s,t) \lesssim \left\|\left(e^{(t-s)\Delta} - 1\right) w_s\right\|_{\mathcal{C}^{\beta-2\kappa}} \lesssim (t-s)^{\kappa}\|w_s\|_{\mathcal{C}^{\beta}}. \tag{6.1.13}$$

Since $\beta - 1 < \alpha$ and $\sup_{t\in[t_0,T_*]}\|Z_t\|_{\mathcal{C}^\alpha}+\|w_t\|_{\mathcal{C}^\alpha}< \infty$, applying (5.2.15) followed by (5.2.16) and (6.1.4), we obtain

$$I_2(r,t) \lesssim (t-r)^{-\frac{1+\beta-2\kappa-(\beta-1)}{2}} \left\|((w_r+Z_r)^2\partial_x\rho_{w_r+Z_r})\right\|_{\mathcal{C}^{\beta-1}} \lesssim (t-r)^{-1+\kappa}.$$

Therefore

$$\int_s^t I_2(r,t) \, \mathrm{d}r \lesssim (t-s)^{\kappa}. \tag{6.1.14}$$

Combining (6.1.13) and (6.1.14) we obtain (6.1.10). □

Lemma 6.1.6 (Mild Solutions are Weak Solutions)**.** *Let $t_0 \in [0,T)$ and $w \in C_{\eta;T}\mathcal{C}^\alpha$ be a mild solution to (6.1.1) on $[0,T]$. Then for any $\phi \in C^\infty(\mathbb{T})$ and $t \in [t_0,T]$,*

$$\langle w_t, \phi\rangle - \langle w_{t_0}, \phi\rangle = -\int_{t_0}^t \langle \partial_x w_s + \chi(w_s+Z_s)^2\partial_x\rho_{w_s+Z_s}, \partial_x\phi\rangle \, \mathrm{d}s. \tag{6.1.15}$$

Moreover, (6.1.15) holds for all $t_0 \in (0,T)$ and $\phi \in C^1(\mathbb{T})$.

Proof. From the mild form of the equation started from data w_{t_0} at $t = t_0$, it follows that for $\phi \in C^\infty(\mathbb{T})$

$$\begin{aligned}\int_{t_0}^t \langle w_s, \partial_{xx}\phi\rangle \, \mathrm{d}s = \int_{t_0}^t &\left\langle e^{s\Delta}w_{t_0}, \partial_{xx}\phi\right\rangle \, \mathrm{d}s \\ &+ \int_{t_0}^t\int_{t_0}^s \left\langle e^{(s-r)\Delta}\chi\partial_x((w_r+Z_r)^2\partial_x\rho_{w_r+Z_r}), \partial_{xx}\phi\right\rangle \, \mathrm{d}r \, \mathrm{d}s.\end{aligned}$$

Integrating by parts in the first term on the right hand side gives

$$\int_{t_0}^{t} \left\langle e^{s\Delta} w_{t_0}, \partial_{xx}\phi \right\rangle \, \mathrm{d}s = \int_{t_0}^{t} \langle \partial_{xx} e^{s\Delta} w_{t_0}, \phi \rangle \, \mathrm{d}s = \langle e^{t\Delta} w_{t_0}, \phi \rangle - \langle w_{t_0}, \phi \rangle. \tag{6.1.16}$$

Using the same argument for the second term on the right hand side we have that

$$\left\langle e^{(s-r)\Delta} \partial_x ((w_r + Z_r)^2 \partial_x \rho_{w_r+Z_r}), \partial_{xx}\phi \right\rangle = \partial_s \left\langle e^{(s-r)\Delta} \partial_x ((w_r + Z_r)^2 \partial_x \rho_{w_r+Z_r}), \phi \right\rangle .$$

Changing the order of integration gives

$$\begin{aligned}\int_{t_0}^{t} \int_{t_0}^{s} &\langle e^{(s-r)\Delta} \partial_x ((w_r + Z_r)^2 \partial_x \rho_{w_r+Z_r}), \partial_{xx}\phi \rangle \, \mathrm{d}r \, \mathrm{d}s \\ &= \int_{t_0}^{t} \langle e^{(t-r)\Delta} \partial_x ((w_t + Z_t)^2 \partial_x \rho_{w_t+Z_t}), \phi \rangle \, \mathrm{d}r \\ &\quad - \int_{t_0}^{t} \langle \partial_x ((w_r + Z_r)^2 \partial_x \rho_{w_r+Z_r}), \phi \rangle \, \mathrm{d}r.\end{aligned}$$

Putting this together with (6.1.16) gives that

$$\begin{aligned}\int_{t_0}^{t} \langle w_s, \partial_{xx}\phi \rangle \, \mathrm{d}s = \langle e^{t\Delta} w_{t_0}, \phi \rangle - \langle w_{t_0}, \phi \rangle + \int_{t_0}^{t} \langle e^{(t-r)\Delta} \chi \partial_x ((w_r + Z_r)^2 \partial_x \rho_{w_r+Z_r}), \phi \rangle \, \mathrm{d}r \\ - \int_{t_0}^{t} \langle \chi \partial_x ((w_r + Z_r)^2 \partial_x \rho_{w_r+Z_r}), \phi \rangle \, \mathrm{d}r.\end{aligned}$$

Rearranging and integrating the left hand side by parts once proves (6.1.15) for any $\phi \in C^\infty(\mathbb{T})$. For $t_0 \in (0, T)$, by Theorem 6.1.4, we have that $\sup_{t\in[t_0,T]} \|w_t\|_{C^1} < \infty$, so that the map $C^1(\mathbb{T}) \ni \phi \mapsto \int_{t_0}^{t} \langle \partial_x w_s + (w_s + Z_s)^2 \partial_x \rho_{w_s+Z_s}, \partial_x \phi \rangle \, \mathrm{d}s$ is continuous. Approximating $\phi \in C^1(\mathbb{T})$ by a smooth sequence gives the result. □

It follows from the proof of Lemma 6.1.6 that any solution to (6.1.1) has constant spatial mean.

Corollary 6.1.7. *Let $w \in C_{\eta;T}\mathcal{C}^\alpha$ be a mild solution to (6.1.1) on $[0, T]$. Then for any $t \in [0, T]$, $\bar{w}_t = \bar{x}$.*

Proof. Applying Lemma 6.1.6 with $\phi \equiv 1$ and $t_0 = 0$ gives $\bar{w}_t = \langle w_t, 1 \rangle = \langle x, 1 \rangle = \bar{x}$. □

6.2 A Priori Estimate on $\mathbb{T}$

We now turn towards establishing a priori estimates for the L^p norms of the solution. To achieve this we leverage the improved regularity of the remainder w along with Lemma 6.1.6 to obtain an evolutionary identity for $\|w_t\|^p_{L^p}$. Then making use of the positive sign in front of the non-linearity we establish an a priori bound on $\|w_t\|^p_{L^p}$ that is independent of the initial data.

We establish the evolutionary identity in Proposition 6.2.1 below, which we prove in a very similar way to [92, Prop. 14].

We recall that we have fixed $\alpha_0 \in \left(-\frac{1}{2}, 0\right)$, $\alpha \in \left(0, \alpha_0 + \frac{1}{2}\right)$, $\beta > 0$ and $\eta > 0$ such that,

$$\frac{\alpha - \alpha_0}{2} < \eta < \frac{1}{4}, \quad \beta \in (0, \alpha + 1).$$

In this section, we let $t \mapsto Z_t \in \mathcal{C}^\alpha_0$ be a continuous, zero at zero trajectory and for any $x \in \mathcal{C}^{\alpha_0}$ and we write $w(x)$ for a mild solution to

$$\begin{cases} \partial_t w - \partial_{xx} w = \chi \partial_x ((w + Z)^2 \partial_x \rho_{w+Z}), & \text{on } [0, T] \times \mathbb{T}, \\ -\partial_{xx} \rho_{w+Z} = w + Z - \bar{w}, & \text{on } \mathbb{T}, \\ w|_{t=0} = x, & \text{on } \mathbb{T}. \end{cases} \tag{6.2.1}$$

The same steps as the proofs of Theorem 6.1.4 and Corollary 6.1.7 show that there exists a $\bar{T} > 0$ such that a solution w to (6.2.1) exists in $\mathcal{C}^\kappa((0, \bar{T}]; \mathcal{C}^{\beta - 2\kappa}_{\bar{x}}(\mathbb{T}))$ for any $\kappa \in (0, 1)$.

Proposition 6.2.1 (Testing the Equation)**.** *Let $0 < t_0 < t \leq \bar{T}$, $p \geq 2$ an integer, $x \in \mathcal{C}^{\alpha_0}(\mathbb{T})$, and $w(x) \in C_{\eta;T}\mathcal{C}^\alpha$ be a mild solution to (6.1.1) on $[0, T]$. Then*

$$\begin{aligned} \frac{1}{p(p-1)} \left(\|w_t\|^p_{L^p} - \|w_{t_0}\|^p_{L^p}\right) = &- \int_{t_0}^t \langle w_s^{p-2} \partial_x w_s, \partial_x w_s \rangle \, \mathrm{d}s \\ &- \int_{t_0}^t \langle \chi (w_s + Z_s)^2 \partial_x \rho_{w_s + Z_s}, w_s^{p-2} \partial_x w_s \rangle \, \mathrm{d}s. \end{aligned} \tag{6.2.2}$$

Proof. Let $\beta \in (1, \alpha+1)$ and $\kappa \in (1/2, \beta/2)$. By Theorem 6.1.4,

$$\sup_{s \neq r \in [t_0,T]} \frac{\|w_s - w_r\|_{L^\infty}}{|s-r|^\kappa} \leq \sup_{s \neq r \in [t_0,T]} \frac{\|w_s - w_r\|_{\mathcal{C}^{\beta-2\kappa}}}{|s-r|^\kappa} < \infty. \tag{6.2.3}$$

Now, let $\varphi \in C^\infty(\mathbb{R})$ and observe that for any $t_0 \leq r < s \leq T$ we have the identity

$$\langle w_s, \varphi(w_s)\rangle - \langle w_r, \varphi(w_r)\rangle = \langle w_s, \varphi(w_r)\rangle - \langle w_r, \varphi(w_r)\rangle + \langle w_s, \varphi(w_s) - \varphi(w_r)\rangle.$$

So for any $t \in (t_0, T]$, and $n \geq 2$, defining a family of partitions, by setting $t_i = t_0 + i(t-t_0)/n$, for $i = 0, \dots, n$ and applying Lemma 6.1.6 with the test function $\varphi(w_{t_i})$, we have

$$\begin{aligned}\langle w_t, \varphi(w_t)\rangle - \langle w_{t_0}, \varphi(w_{t_0})\rangle = &- \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \langle \partial_x w_s + \chi(w_s + Z_s)^2 \partial_x \rho_{w_s+Z_s}, \partial_x \varphi(w_{t_i})\rangle \, \mathrm{d}s \\ &+ \sum_{i=0}^{n-1} \big\langle w_{t_{i+1}}, \varphi(w_{t_{i+1}}) - \varphi(w_{t_i})\big\rangle \\ &=: -I^n(t) + R^n(t).\end{aligned}$$

By continuity of $s \mapsto \partial_x \varphi(w_s)$ we see that

$$\lim_{n\to\infty} I_n(t) = \int_0^t \langle \partial_x w_s + \chi(w_s + Z_s)^2 \partial_x \rho_{w_s+Z_s}, \partial_x \varphi(w_s)\rangle \, \mathrm{d}s. \tag{6.2.4}$$

For $R^n(t)$, Taylor's formula gives

$$\begin{aligned}R^n(t) &= \sum_{i=0}^{n-1} \left[\langle w_{t_{i+1}}, (w_{t_{t+i}} - w_{t_i})\varphi'(w_{t_i})\rangle + \left\langle w_{t_{i+1}}, \int_{w_{t_i}}^{w_{t_{i+1}}} \varphi''(y)(w_{t_{i+1}} - y) \, \mathrm{d}y \right\rangle \right] \\ &=: R_1^n(t) + R_2^n(t).\end{aligned}$$

We show that $R_2^n(t)$ converges to zero. Using (6.2.3), we have the bound

$$|R_2^n(t)| \lesssim \sup_{s\in[t_0,t]} \|w_s\|_{L^\infty} \sup_{\substack{i=0,\dots,n-1 \\ y \in [w_{t_i}, w_{t_{i+1}}]}} |\varphi''(y)| \sum_{i=0}^{n-1} \|w_{t_{i+1}} - w_{t_i}\|_{L^\infty}^2 \overset{(6.2.3)}{\lesssim} \sum_{i=0}^{n-1} |t_{i+1} - t_i|^{2\kappa},$$

where we used that φ'' is continuous. Regarding $R_1^n(t)$, we may apply Lemma 6.1.6 once again to each bracket, this time with the test function $w_{t_{i+1}}\varphi'(w_{t_i})$, to give that

$$R_1^n(t) = -\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \langle \partial_x w_s + \chi(w_s+Z_s)^2\partial_x\rho_{w_s+Z_s}, \partial_x(w_{t_{i+1}}\varphi'(w_{t_i}))\rangle \,\mathrm{d}s.$$

So again, by regularity of the map $s \mapsto \partial_x w_s$ and φ, we have that

$$\lim_{n\to\infty} R^n(t) = -\int_{t_0}^{t} \langle \partial_x w_s + \chi(w_s+Z_s)^2\partial_x\rho_{w_s+Z_s}, \partial_x(w_s\varphi'(w_s))\rangle \,\mathrm{d}s. \tag{6.2.5}$$

So combining (6.2.4) and (6.2.5) gives,

$$\langle w_t, \varphi(w_t)\rangle - \langle w_{t_0}, \varphi(w_{t_0})\rangle = -\int_{t_0}^{t} \langle \partial_x w_r + \chi(w_r+Z_r)^2\partial_x\rho_{w_r+Z_r}, \\ (2\varphi'(w_r) + w_r\varphi''(w_r))\,\partial_x w_r\rangle \,\mathrm{d}r.$$

For $\varphi(x) := x^{p-1}$ we have $2\varphi'(x) + x\varphi''(x) = p(p-1)x^{p-2}$ and

$$\langle w_t, w_t^{p-1}\rangle - \langle w_{t_0}, w_{t_0}^{p-1}\rangle = \|w_t\|_{L^p}^p - \|w_{t_0}\|_{L^p}^p,$$

which gives (6.2.2). □

We now reintroduce the constant χ in the formulation of (6.0.1). For a given initial data, $x \in \mathcal{C}^{\alpha_0}$ and $u_t(x)$ solving (6.0.1), if $\bar{x} \neq 0$, defining $\tilde{u}_t := u_t/\bar{u}$ for all $t > 0$ and $\tilde{x} = x/\bar{x}$, we see that $\tilde{u}$ solves

$$\partial_t\tilde{u}_t - \partial_{xx}\tilde{u}_t = \bar{u}^2\partial_x(\tilde{u}_t^2\partial_x\rho_{\tilde{u}_t}) + \frac{1}{\bar{u}}\xi, \quad \tilde{u}|_{t=0} = \tilde{x}, \quad \text{and} \quad \bar{\tilde{x}} = 1.$$

So without loss of generality, for $\chi > 0$, we may consider the problem

$$\partial_t u_t - \partial_{xx}u_t = \chi\partial_x(u^2\partial_x\rho_{u_t}) + \xi, \quad u|_{t=0} = x, \quad \text{and} \quad \bar{x} = 1 \text{ or } 0.$$

It follows from Corollary 6.1.7, which applies equally to (6.2.1) that $\bar{w}_t \equiv \bar{u}_t \equiv \bar{u}_0$. Therefore, we also treat (6.2.1) with either $\bar{w} = 1$ or $\bar{w} = 0$ and $\chi > 0$.

Expanding the second term on the right hand side of (6.2.2) gives

$$-\langle \partial_x \rho_w, w^p \partial_x w \rangle = -\frac{1}{p+1}\langle \partial_x \rho_w, \partial_x w^{p+1}\rangle = -\frac{1}{p+1}\|w^{p+2}\|_{L^1} + \frac{1}{p+2}\langle \bar{w}, w^{p+1}\rangle.$$

In combination with the first term on the right hand side of (6.2.2) coming from the Laplacian we obtain an a priori estimate on $\|w_t\|_{L^p}^p$ that is independent of the initial data.

We recall the ODE comparison test, [105, Lem. 3.8], which is key in obtaining our a priori bound.

Lemma 6.2.2 (ODE Comparison). *Let $\lambda > 1$, c_1, $c_2 > 0$ and $f : [0, T] \to [0, \infty)$ be differentiable and satisfying*

$$f'(t) + c_1 f(t)^\lambda \le c_2, \tag{6.2.6}$$

for every $t \in [0, T]$. Then there exists a constant $C := C(\lambda) > 0$ such that for all $t > 0$

$$f(t) \le C \max\left\{ (tc_1)^{-\frac{1}{\lambda-1}}, \left(\frac{c_2}{c_1}\right)^{\frac{1}{\lambda}} \right\}. \tag{6.2.7}$$

Proof. Let $t > 0$ then either of the following must be the case:

(i) there exists some $s_0 \le t$ such that $f(s_0) \le \left(\frac{2c_2}{c_1}\right)^{\frac{1}{\lambda}}$,

(ii) for every $s \le t$, $f(s) > \left(\frac{2c_2}{c_1}\right)^{\frac{1}{\lambda}}$.

If the second holds, then, using (6.2.6) we see that

$$f'(s) + \frac{c_1}{2} f(s)^\lambda \le 0.$$

Solving the differential inequality on $[0, t]$ gives

$$f(t) \le \frac{f(0)}{\left(1 + (\lambda - 1)\frac{c_1}{2} f(0)^{\lambda-1} t\right)^{\frac{1}{\lambda-1}}} \lesssim (tc_1)^{-\frac{1}{\lambda-1}},$$

where the constant depends only on λ.

On the other hand, if the first case holds, we assume for a contradiction that $f(t) > \left(\frac{2c_2}{c_1}\right)^{\frac{1}{\lambda}}$ and we set

$$s^* = \sup\left\{ s < t \,:\, f(s) \le \left(\frac{2c_2}{c_1}\right)^{\frac{1}{\lambda}} \right\},$$

that is, $t > s^*$. Furthermore, since f is continuous, $f(s^*) = \left(\frac{2c_2}{c_1}\right)^{\frac{1}{\lambda}}$. Again, employing (6.2.6), we have for every $s \in (s^*, t]$ that

$$f'(s) + \frac{c_1}{2} f(s)^\lambda \le 0 \Rightarrow f'(s) \le 0.$$

Therefore,

$$f(t) = f(s^*) + \int_{s^*}^t f'(s) \, \mathrm{d}s \le f(s^*) = \left(\frac{2c_2}{c_1}\right)^{\frac{1}{\lambda}},$$

which is a contradiction and so the conclusion follows. □

Theorem 6.2.3 (A Priori Bound on the Remainder)**.** *Let $p \ge 2$ be an even integer, $\chi > 0$, $\mathfrak{m} \in \{0,1\}$, $x \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}}(\mathbb{T})$ and $w(x) \in C_{\eta;T}\mathcal{C}^\alpha$ be a mild solution to (6.1.1) on $[0,T]$. Then there exists a constant $C > 0$, independent of x, Z, p and χ,*

$$\|w_t(x)\|_{L^p} \le \max\left\{ (\chi t/4)^{-\frac{1}{2}}, C(p \vee \chi)^\gamma \|Z\|^{\frac{1}{\alpha}}_{C_t\mathcal{C}^\alpha}, C(p\vee\chi)^{\frac{p+1}{p+2}}\mathfrak{m} \right\}, \qquad (6.2.8)$$

where $\gamma = \frac{p(1+\alpha)+2+\alpha}{\alpha(p+2)}$.

Proof. Since $t \mapsto \partial_x w_t$ and $t \mapsto (w_t + Z_t)^2 \partial_x \rho_{w_t+Z_t}$ are both continuous mappings from $(0,T]$ into $L^\infty(\mathbb{T})$ by Theorem 6.1.4, we may differentiate (6.2.2) with respect to t in order to obtain

$$\frac{1}{p(p-1)} \frac{\mathrm{d}}{\mathrm{d}t} \|w_t\|^p_{L^p} = -\|w_t^{p-2} |\partial_x w_t|^2\|_{L^1} - \chi \langle (w_t + Z_t)^2 \partial_x \rho_{w_t+Z_t}, w_t^{p-2} \partial_x w_t \rangle,$$

where we used that p is even in the first term on the right hand side. Regarding the second term, since $f \mapsto \partial_x \rho_f$ is linear, we have

$$\begin{aligned}\langle (w_t + Z_t)^2 \partial_x \rho_{w_t+Z_t}, w_t^{p-2} \partial_x w_t\rangle &= \langle \partial_x \rho_{w_t}, w_t^p \partial_x w_t\rangle + \langle \partial_x \rho_{Z_t}, w_t^p \partial_x w_t\rangle \\ &\quad + 2\langle Z_t \partial_x \rho_{w_t+Z_t}, w_t^{p-1} \partial_x w_t\rangle + \langle Z_t^2 \partial_x \rho_{w_t+Z_t}, w_t^{p-2} \partial_x w_t\rangle.\end{aligned}$$

Integrating the first term by parts, using that $-\partial_{xx}\rho_{w_t} = w_t - \bar{x}$ by Corollary 6.1.7 and recalling that we have set $\bar{x} = \mathfrak{m}$,

$$\begin{aligned}\langle \partial_x \rho_{w_t}, w_t^p \partial_x w_t\rangle = \frac{1}{p+1}\langle \partial_x \rho_{w_t}, \partial_x w_t^{p+1}\rangle &= \frac{1}{p+1}\langle w_t - \mathfrak{m}, w_t^{p+1}\rangle \\ &= \frac{1}{p+1}\|w_t^{p+2}\|_{L^1} - \frac{1}{p+1}\langle \mathfrak{m}, w_t^{p+1}\rangle,\end{aligned}$$

Therefore, applying the chain rule in the remaining terms gives

$$\begin{aligned}\frac{1}{p(p-1)\chi}\frac{\mathrm{d}}{\mathrm{d}t}\|w_t\|_{L^p}^p = -\frac{1}{\chi}\||w_t^{p-2}|\partial_x w_t|^2\|_{L^1} - \frac{1}{p+1}\|w_t^{p+2}\|_{L^1} + \frac{1}{p+1}\langle \mathfrak{m}, w_t^{p+1}\rangle \\ + \frac{1}{p+1}\langle \partial_x \rho_{Z_t}, \partial_x w_t^{p+1}\rangle + \frac{2}{p}\langle Z_t \partial_x \rho_{Z_t}, \partial_x w_t^p\rangle \\ + \frac{1}{p-1}\langle Z_t^2 \partial_x \rho_{Z_t}, \partial_x w_t^{p-1}\rangle + \frac{2}{p}\langle Z_t, \partial_x \rho_{w_t} \partial_x w_t^p\rangle \\ + \frac{1}{p-1}\langle Z_t^2, \partial_x \rho_{w_t} \partial_x w_t^{p-1}\rangle.\end{aligned} \tag{6.2.9}$$

We demonstrate that all terms without a definite sign can be bounded by a constant multiple of the two negative terms,

$$\frac{1}{\chi}\|w_t^{p-2}|\partial_x w_t|^2\|_{L^1} + \frac{1}{p+1}\|w_t^{p+2}\|_{L^1} =: \frac{1}{\chi}A_t + \frac{1}{p+1}B_t.$$

From Theorems 5.2.2 and 5.2.13, for any $\alpha \in \mathbb{R}$ and $p, q \in [1, \infty]$, we have that

$$\|\partial_x \rho_w\|_{\mathcal{B}^\alpha_{p,q}} \lesssim \|w\|_{\mathcal{B}^{\alpha-\frac{1}{p}}_{1,q}}, \qquad \text{and} \qquad \|\partial_x \rho_w\|_{L^\infty} \lesssim \|w\|_{L^1}. \tag{6.2.10}$$

For any $0 < q \le p+2$, by Jensen's inequality we have

$$\|w_t^q\|_{L^1} \le B_t^{\frac{q}{p+2}}. \tag{6.2.11}$$

Furthermore, by applying Cauchy–Schwarz followed by (6.2.11), for $\frac{2}{p} < q \leq p+1$ we have

$$\|\partial_x w_t^q\|_{L^1} = q\|w_t^{q-1}\partial_x w_t\|_{L^1} \leq q\|w_t^{2q-p}\|_{L^1}^{\frac{1}{2}} A_t^{\frac{1}{2}} \leq q B_t^{\frac{2q-p}{2(p+2)}} A_t^{\frac{1}{2}}. \tag{6.2.12}$$

To keep track of dependence on p, we also make use of the following inequality which readily follows from Young's inequality for products: for $\gamma_1, \gamma_2 \in (1,\infty)$ such that $\frac{1}{\gamma_1} + \frac{1}{\gamma_2} = 1$, a, $b > 0$ and $c \geq 1$

$$ab < \frac{c^{\gamma_1 - 1}}{\gamma_1 - 1} a^{\gamma_1} + \frac{1}{c} b^{\gamma_2} \tag{6.2.13}$$

(in fact $ab < \frac{c^{\gamma_1-1}}{e(\gamma_1-1)} a^{\gamma_1} + \frac{1}{c} b^{\gamma_2}$). From now on we let $C > 0, c \geq 1$ be constants, that are independent of x, Z, p, and χ. Later, we will fix $c \geq 1$ sufficiently large at the end of the proof. If we write $\lesssim$ in an inequality below, the implied proportionality constant is equal to $C > 0$ which we take sufficiently large so that the inequality holds.

We work through the terms of (6.2.9) in order. For the third term of (6.2.9), applying Hölder, (6.2.11), (6.2.13) and using that $\mathfrak{m} \in \{0,1\}$, we have,

$$\frac{1}{p+1} |\langle \mathfrak{m}, w_t^{p+1} \rangle| \leq \frac{\mathfrak{m}}{p+1} \|w_t^{p+1}\|_{L^1} \leq \frac{\mathfrak{m}}{p+1} B_t^{\frac{p+1}{p+2}} \leq \frac{\mathfrak{m} c^{p+1}}{(p+1)^2} + \frac{1}{c(p+1)} B_t. \tag{6.2.14}$$

For the fourth term of (6.2.9), we integrate by parts to give

$$\begin{aligned} \frac{1}{p+1} |\langle \partial_x \rho_{Z_t}, \partial_x w_t^{p+1} \rangle| = \frac{1}{p+1} |\langle Z_t, w_t^{p+1} \rangle| &\overset{(5.2.10)\,(6.2.11)}{\lesssim} \frac{1}{p} \|Z\|_{C_t \mathcal{C}^\alpha} B_t^{\frac{p+1}{p+2}} \\ &\overset{(6.2.13)}{\leq} \frac{c^{p+1}}{p^2} \|Z\|_{C_t \mathcal{C}^\alpha}^{p+2} + \frac{1}{cp} B_t. \end{aligned} \tag{6.2.15}$$

Concerning the two remaining terms of (6.2.9) involving $\partial_x \rho_{Z_t}$, we have for $k = 1, 2$

$$
\begin{aligned}
\frac{1}{p+1-k}|\langle Z_t^k \partial_x \rho_{Z_t}, \partial_x w_t^{p+1-k}\rangle| \quad &\leq \quad \frac{1}{p+1-k}\|Z_t^k \partial_x \rho_{Z_t}\|_{L^\infty}\|\partial_x w_t^{p+1-k}\|_{L^1} \\
&\overset{(5.2.12),\,(5.2.19),\,(6.2.12)}{\lesssim} \quad \|Z_t\|_{\mathcal{C}^\alpha}^{k+1} B_t^{\frac{p+2-2k}{2(p+2)}} A_t^{\frac{1}{2}} \\
&\overset{(6.2.13)}{\leq} \quad c\|Z_t\|_{\mathcal{C}^\alpha}^{2(k+1)} B_t^{\frac{p+2-2k}{p+2}} + \frac{1}{c}A_t \\
&\overset{(6.2.13)}{\lesssim} \quad \frac{c^{\frac{p+2-k}{k}}}{p}\|Z\|_{C_t\mathcal{C}^\alpha}^{\frac{k+1}{k}(p+2)} + \frac{1}{c}B_t + \frac{1}{c}A_t.
\end{aligned}
\tag{6.2.16}
$$

(In the case $p = k = 2$, note that we do not need to apply (6.2.13) in the final line.) Combining (6.2.15), (6.2.16) and noting that $\frac{3}{2}(p+2)$ is the highest power of $\|Z\|_{C_t\mathcal{C}^\alpha}$ encountered so far, we have that

$$
\sum_{k=0}^{2}\binom{2}{k}\frac{1}{p+1-k}|\langle Z_t^k \partial_x \rho_{Z_t}, \partial_x w_t^{p+1-k}\rangle| \lesssim \frac{c^{p+1}}{p}\|Z\|_{C_t\mathcal{C}^\alpha}^{\frac{3}{2}(p+2)} + \frac{1}{c}B_t + \frac{1}{c}A_t. \tag{6.2.17}
$$

For the seventh term of (6.2.9), we first integrate by parts and apply the triangle inequality to obtain that

$$
\frac{1}{p}|\langle Z_t, \partial_x \rho_{w_t} \partial_x w_t^p\rangle| \leq \frac{1}{p}|\langle \partial_x Z_t, w_t^p \partial_x \rho_{w_t}\rangle| + \frac{1}{p}|\langle Z_t, w_t^{p+1}\rangle|. \tag{6.2.18}
$$

Since $\alpha \in (0, \frac{1}{2})$, using Theorems 5.2.9 and 5.2.5, we have

$$
\begin{aligned}
\|w_t^p \partial_x \rho_{w_t}\|_{\mathcal{B}^{1-\alpha}_{1,1}} \quad &\overset{(5.2.11)}{\lesssim} \quad \|w_t^p\|_{L^{\frac{1}{1-\alpha}}}\|\partial_x \rho_{w_t}\|_{\mathcal{B}^{1-\alpha}_{\frac{1}{\alpha},1}} + \|w_t^p\|_{\mathcal{B}^{1-\alpha}_{1,1}}\|\partial_x \rho_{w_t}\|_{L^\infty} \\
&\overset{(6.2.10),\,(5.2.5),\,(5.2.9)}{\lesssim} \quad \|w_t^p\|_{\mathcal{B}^{1-\alpha}_{1,1}}\|w_t\|_{L^1} \\
&\overset{(5.2.9)}{\lesssim} \quad \left(\|\partial_x w_t^p\|_{L^1}^{1-\alpha}\|w_t^p\|_{L^1}^{\alpha} + \|w_t^p\|_{L^1}\right)\|w_t\|_{L^1} \\
&\overset{(6.2.11)}{\lesssim} \quad \|\partial_x w_t^p\|_{L^1}^{1-\alpha} B_t^{\frac{p\alpha+1}{p+2}} + B_t^{\frac{p+1}{p+2}} \\
&\overset{(6.2.12)}{\lesssim} \quad p^{1-\alpha} B_t^{\frac{p(1+\alpha)+2}{2(p+2)}} A_t^{\frac{1-\alpha}{2}} + B_t^{\frac{p+1}{p+2}}.
\end{aligned}
\tag{6.2.19}
$$

Considering the first term of (6.2.18), applying (6.2.19) and then (6.2.13) twice with

$\gamma_1 = \frac{2}{1+\alpha}$ and then with $\gamma_1 = \frac{(p+2)(1+\alpha)}{2\alpha}$, and using that $p^{-\alpha} \leq 1$,

$$\begin{aligned}
\frac{1}{p}|\langle \partial_x Z_t, w_t^p \partial_x \rho_{w_t}\rangle| &\overset{(5.2.10)}{\lesssim} \frac{1}{p}\|\partial_x Z_t\|_{\mathcal{C}^{\alpha-1}}\|w_t^p \partial_x \rho_{w_t}\|_{\mathcal{B}^{1-\alpha}_{1,1}} \\
&\overset{(6.2.19)}{\lesssim} \|Z_t\|_{\mathcal{C}^\alpha}\left(B_t^{\frac{p(1+\alpha)+2}{2(p+2)}} A_t^{\frac{1-\alpha}{2}} + \frac{1}{p}B_t^{\frac{p+1}{p+2}}\right) \\
&\overset{(6.2.13)}{\lesssim} c\|Z\|_{C_t\mathcal{C}^\alpha}^{\frac{2}{1+\alpha}} B_t^{\frac{p(1+\alpha)+2}{(p+2)(1+\alpha)}} + \frac{c^{p+1}}{p^2}\|Z\|_{C_t\mathcal{C}^\alpha}^{p+2} \\
&\qquad + \frac{1}{c}A_t + \frac{1}{pc}B_t \\
&\overset{(6.2.13)}{\lesssim} \frac{c^{\frac{p(1+\alpha)+2+\alpha}{\alpha}}}{p}\|Z\|_{C_t\mathcal{C}^\alpha}^{\frac{p+2}{\alpha}} + \frac{p+1}{cp}B_t + \frac{1}{c}A_t. \qquad (6.2.20)
\end{aligned}$$

Concerning the second term of (6.2.18), we have

$$\frac{1}{p}|\langle Z_t, w_t^{p+1}\rangle| \overset{(5.2.10),\,(6.2.11)}{\lesssim} \frac{1}{p}\|Z\|_{C_t\mathcal{C}^\alpha} B_t^{\frac{p+1}{p+2}} \overset{(6.2.13)}{\lesssim} \frac{c^{p+1}}{p^2}\|Z\|_{C_t\mathcal{C}^\alpha}^{p+2} + \frac{1}{pc}B_t. \qquad (6.2.21)$$

Regarding the eighth and final term of (6.2.9),

$$\begin{aligned}
\frac{1}{p-1}|\langle Z_t^2, \partial_x \rho_{w_t} \partial_x w_t^{p-1}\rangle| &\overset{(5.2.10),\,(5.2.12)}{\lesssim} \frac{1}{p}\|Z\|_{C_t\mathcal{C}^\alpha}^2 \|\partial_x \rho_{w_t}\|_{L^\infty}\|\partial_x w_t^{p-1}\|_{L^1} \\
&\overset{(6.2.10),\,(6.2.12)}{\lesssim} \|Z\|_{C_t\mathcal{C}^\alpha}^2 B_t^{\frac{p}{2(p+2)}} A_t^{\frac{1}{2}} \\
&\overset{(6.2.13)}{\leq} c\|Z\|_{C_t\mathcal{C}^\alpha}^4 B_t^{\frac{p}{p+2}} + \frac{1}{c}A_t \\
&\overset{(6.2.13)}{\lesssim} \frac{c^{p+1}}{p}\|Z\|_{C_t\mathcal{C}^\alpha}^{2(p+2)} + \frac{1}{c}B_t + \frac{1}{c}A_t. \qquad (6.2.22)
\end{aligned}$$

So, since $p+1 \leq \frac{p(1+\alpha)+2+\alpha}{\alpha} =: \beta$ and $c \geq 1$, combining (6.2.20)–(6.2.22),

$$\sum_{k=1}^{2}\binom{2}{k}\frac{1}{p+1-k}|\langle Z_t^k, \partial_x \rho_{w_t}\partial_x w_t^{p+1-k}\rangle| \lesssim \frac{c^\beta}{p}\|Z\|_{C_t\mathcal{C}^\alpha}^{\frac{p+2}{\alpha}} + \frac{1}{c}B_t + \frac{1}{c}A_t. \qquad (6.2.23)$$

Since $\frac{1}{4(p-1)} \leq \frac{3/4}{p+1}$, combining (6.2.9) with (6.2.14), (6.2.17), (6.2.23) and choosing

$c \geq (p \vee \chi)C$ with $C > 0$ the sufficiently large implied constant, we obtain that

$$\frac{1}{\chi p(p-1)} \frac{\mathrm{d}}{\mathrm{d}t} \|w_t\|_{L^p}^p \leq \max\left\{ \frac{(C(p\vee\chi))^\beta}{p} \|Z\|_{C_t\mathcal{C}^\alpha}^{\frac{p+2}{\alpha}}, \frac{(C(p\vee\chi))^{p+1}\mathfrak{m}}{p^2} \right\} - \frac{1}{4(p-1)} \|w_t^{p+2}\|_{L^1}.$$

Using Jensen's inequality we have $(\|w_t\|_{L^p}^p)^{\frac{p+2}{p}} \leq \|w_t^{p+2}\|_{L^1}$, so that for $C > 0$ sufficiently large

$$\frac{\mathrm{d}}{\mathrm{d}t} \|w_t\|_{L^p}^p + \frac{\chi p}{4} (\|w_t\|_{L^p}^p)^{\frac{p+2}{p}} \leq \chi \max\left\{ p(C(p\vee\chi))^\beta \|Z\|_{C_t\mathcal{C}^\alpha}^{\frac{p+2}{\alpha}}, (C(p\vee\chi))^{p+1}\mathfrak{m} \right\}.$$

Applying Lemma 6.2.2 with $f(t) = \|w_t\|_{L^p}^p$ and $\lambda = \frac{p+2}{p}$, we obtain that, for $C > 0$ sufficiently large

$$\|w_t(x)\|_{L^p}^p \leq \max\left\{ (\chi t/4)^{-\frac{p}{2}}, (C(p\vee\chi))^{\beta\frac{p}{p+2}} \|Z\|_{C_t\mathcal{C}^\alpha}^{\frac{p}{\alpha}}, (C(p\vee\chi))^{\frac{p(p+1)}{p+2}} \mathfrak{m}/p^{\frac{p}{p+2}} \right\},$$

for every $t \in (0, T]$ and $x \in \mathcal{C}^{\alpha_0}$. The stated bound, (6.2.8), then follows after taking the p^{th} root on both sides and noticing that $\gamma = \frac{\beta}{p+2} \leq \frac{4+3\alpha}{4\alpha}$. □

We discuss some implications of the a priori bound (6.2.8).

Remark 6.2.4. Fixing $\chi > 0$, we see that in the limit $p \nearrow \infty$ the bound (6.2.8) behaves like

$$\sup_{x\in\mathcal{C}^{\alpha_0}} \|w_t(x)\|_{L^p} \leq C \max\left\{ \frac{1}{\sqrt{\chi t}}, p^\gamma \|Z\|_{C_t\mathcal{C}^\alpha}^{\frac{1}{\alpha}}, p^{\frac{p+1}{p+2}}\mathfrak{m} \right\}.$$

The maximum on the right hand side blows-up as p increases and so we see that this approach does not seem to allow us to obtain an a priori L^∞ bound independent of the initial data for (6.1.1).

Compare this with the cubic stochastic reaction diffusion equation in one dimension, where a similar method yields that, for C_1, $C_2 > 0$ independent of $p \geq 2$, we have

$$\frac{d}{dt}\|w_t\|_{L^p}^p + \frac{p}{2}\left(\|w_t\|_{L^1}^p\right)^{\frac{p+2}{p}} \leq C_1 \max\left\{C_2^{p+1}\|Z\|_{C_t\mathcal{C}^\alpha}^{p+2}, 1\right\}.$$

So then applying Lemma 6.2.2 we have, for possibly new constants C_1, $C_2 > 0$,

$$\frac{d}{dt}\|w_t\|_{L^p} \leq C_1 \max\left\{t^{-\frac{1}{2}}, C_2^{\frac{p+1}{p+2}} p^{-\frac{1}{p+2}}\|Z\|_{C_t\mathcal{C}^\alpha}, 1\right\}. \tag{6.2.24}$$

In contrast with (6.2.8), the right hand side of (6.2.24) is finite in the limit $p \nearrow \infty$, which would allow us, in this case to obtain an a priori bound on the L^∞ norm independent of the initial data. An alternative approach to directly obtaining the L^∞ a priori bound for the cubic stochastic reaction diffusion equation, for all $d < 4$, is given in [25]. However, it also seems that this method will not apply to our equation either.

Remark 6.2.5. We also obtain a tail bound on the law $\mathcal{L}(u_t(x))$ from (6.2.8). Since from Fernique's theorem there exists a $\lambda > 0$ sufficiently small such that $\mathbb{E}\left[e^{\lambda\|v_{0,\cdot}\|_{C_t\mathcal{C}^\alpha}^2}\right] < \infty$, letting $t \mapsto Z_t$ be given by the random trajectory $t \mapsto v_{0,t}$, for any even integer $p \geq 2$, and the same $\lambda > 0$ we have that

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\lambda\|u_t\|_{L^p}^{2\alpha}\right)\right] &\leq \mathbb{E}\left[\exp\left(\lambda\left(\|w_t\|_{L^p}^{2\alpha} + \|v_{0,\cdot}\|_{C_t\mathcal{C}^\alpha}^2\right)\right)\right] \\
&\lesssim_p \mathbb{E}\left[\exp\left(\lambda\|v_{0,\cdot}\|_{C_t\mathcal{C}^\alpha}^2\right)\right]^2 \\
&< \infty.
\end{aligned} \tag{6.2.25}$$

This tail bound displays two interesting properties. Firstly, since $\alpha \in (0, 1/2)$ the bound (6.2.25) gives that $\mathcal{L}(u_t)$ has heavier than Gaussian tails - of course this does not prove that the solution does not have lighter tails. Using this method we could expect to show, at best exactly Gaussian tails, since we are limited by the moments of $v_{0,\cdot}$. Secondly, taking a higher power $m \geq 3$ in the equation does not seem to improve this tail bound, at least using the method presented here. We see this by inspecting the proof of Theorem 6.2.3. Applying the same method, for any $m \geq 3$, we see that the term $|\langle Z_t^{m-2}, \partial_x \rho_{w_t} \partial_x w_t^p \rangle|$, results in the power $\|Z\|_{C_T\mathcal{C}^\alpha}^{\frac{p-1+m}{\alpha}}$ in the final bound. After applying Lemma 6.2.2 we see that we are left with the same power on $\|Z\|_{C_T\mathcal{C}^\alpha}$ as in (6.2.8).

Again we contrast this with known results for stochastic reaction diffusion equations. In [89] the authors demonstrate that solutions to

$$\partial_t u - \partial_{xx} u = -u^m + \xi,$$

is such that if, for $\lambda > 0$,

$$\mathbb{E}\left[\exp\left(\lambda \|\xi\|^2_{\mathcal{C}^{\alpha-2}(\mathbb{R}_+\times\mathbb{R})}\right)\right] < \infty,$$

then

$$\mathbb{E}\left[\exp\left(\lambda \|u_t\|_{L^\infty}^{2+(m-1)\alpha}\right)\right] < \infty. \tag{6.2.26}$$

In contrast with (6.2.25), the bound (6.2.26) improves both with increasing m and α.

Remark 6.2.6. Finally we observe that it does not seem we can directly apply our method for the equation with an even power on the right hand side. For example, when $m = 2$ the remainder solves

$$\partial_t w - \partial_{xx} w = \partial_x(|w + Z|\partial_x \rho_{w+Z}).$$

We can write the absolute value in the form $|w + Z| = \mathrm{sign}(w + Z)(w + Z)$ which allows us to isolate the damping term in the testing argument. However, in order to do so we integrate by parts which results in terms such as $|\langle \partial_x Z, w^{p+1} \partial_x \rho_w \rangle|$. The issue in handling such terms is that they involve both the highest power of w, and the negative regularity term $\partial_x Z$. So it is not clear how the testing method would apply in this case.

We conclude this section with a proof of global well-posedness for w.

Proposition 6.2.7. *Let $\mathfrak{m} \in \{0,1\}$, $x \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}}(\mathbb{T})$ and $Z \in C_T\mathcal{C}^{\alpha}_0$. Then there exists a unique mild solution $w(x) \in C_{\eta;T}\mathcal{C}^{\alpha}$ to (6.1.1). Furthermore, the map,*

$$\begin{aligned} \mathscr{S} : \mathcal{C}^{\alpha_0}_{\mathfrak{m}}(\mathbb{T}) \times C_T\mathcal{C}^{\alpha}_0(\mathbb{T}) &\to C_{\eta;T}\mathcal{C}^{\alpha}_{\mathfrak{m}}(\mathbb{T}) \\ (x, Z) &\mapsto u(x) := w(x) + Z, \end{aligned} \tag{6.2.27}$$

is jointly, locally Lipschitz.

Proof. We recall from Theorem 6.1.4 that there exists a $T_* \in (0,1)$, depending only on $\|x\|_{\mathcal{C}^{\alpha_0}}$ and $\|Z\|_{C_T\mathcal{C}^{\alpha}}$, such that a mild solution $w \in C_{\eta;T_*}\mathcal{C}^{\alpha}$ exists to (6.1.1). Without loss of generality let us assume $T > T_*$ and that we fix an even $p \in (-\frac{1}{\alpha_0}, \infty)$ so that $L^p(\mathbb{T}) \hookrightarrow \mathcal{C}^{\alpha_0}(\mathbb{T})$. In this case, it is clear that we can extend the solution for a positive time of existence so long as $\|w_t\|_{L^p}$ remains finite. However, by Theorem 6.2.3, $\|w_t\|_{L^p}$ is bounded above by a function of t independent of the initial data and so we may continue the solution to all of $[0,T]$. From Corollary 6.1.7, for $u_t := w_t + Z_t$, we have $\bar{u}_t = \bar{x} + \bar{Z}_t$, for all $t \in (0,T]$. Similarly, for all $t > 0$, $\|u_t\|_{\mathcal{C}^{\alpha}} \leq \|w_t\|_{\mathcal{C}^{\alpha}} + \|Z_t\|_{\mathcal{C}^{\alpha}}$. Hence the solution map (6.2.27) is well-defined and $\|u\|_{C_{\eta,T}\mathcal{C}^{\alpha}}$ depends only on $\|Z\|_{C_T\mathcal{C}^{\alpha}}$.
To continue the proof, we state the following

Lemma 6.2.8. *Consider $\mathfrak{R} > 0$ and define the set*

$$D_{\mathfrak{R}} := \{(x, Z) \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}} \times C_T\mathcal{C}^{\alpha} \;:\; \|x\|_{\mathcal{C}^{\alpha_0}} + \|Z\|_{C_T\mathcal{C}^{\alpha}} < \mathfrak{R}\}.$$

Then for $T_* = T_*(\mathfrak{R}) > 0$ sufficiently small, $\mathscr{S} \colon D_{\mathfrak{R}} \to C_\eta((0, 2T_*], \mathcal{C}^{\alpha}_{\mathfrak{m}})$ is K-Lipschitz where $K = K(\mathfrak{R}) > 0$.

Proof. Let $(x, Z), (\tilde{x}, \tilde{Z}) \in D_{\mathfrak{R}}$ and consider the corresponding solutions

$$\begin{aligned} u_t &= e^{t\Delta}x + \int_0^t e^{(t-s)\Delta}\chi\partial_x(u_s^2\partial_x\rho_{u_s})\,\mathrm{d}s + Z_t, \\ \tilde{u}_t &= e^{t\Delta}\tilde{x} + \int_0^t e^{(t-s)\Delta}\chi\partial_x(\tilde{u}_s^2\partial_x\rho_{\tilde{u}_s})\,\mathrm{d}s + \tilde{Z}_t. \end{aligned}$$

From Theorem 6.1.4, there exists $T_*(\mathfrak{R}) > 0$ such that $\|u\|_{C_{\eta;2T_*}\mathcal{C}^\alpha} \vee \|\tilde{u}\|_{C_{\eta;2T_*}\mathcal{C}^\alpha} \leq 2$. For $t \in (0, 2T_*]$, using Theorem 5.2.10 and similar bounds as in the proof of Theorem 6.1.4 we see that

$$\|u_t - \tilde{u}_t\|_{\mathcal{C}^\alpha} \lesssim t^{-\frac{\alpha-\alpha_0}{2}} \|x - \tilde{x}\|_{\mathcal{C}^{\alpha_0}} + t^{\frac{1}{2}-3\eta} \|u - \tilde{u}\|_{C_{\eta;t}\mathcal{C}^\alpha} + \|Z_t - \tilde{Z}_t\|_{\mathcal{C}^\alpha},$$

where the proportionality constant does not depend on $x, \tilde{x}, Z, \tilde{Z}$. So multiplying through by t^η and taking the supremum over $t \in (0, 2T_*]$ we have that

$$\|u - \tilde{u}\|_{C_{\eta;2T_*}\mathcal{C}^\alpha} \lesssim \|x - \tilde{x}\|_{\mathcal{C}^{\alpha_0}} + T_*^{\frac{1}{2}-2\eta} \|u - \tilde{u}\|_{C_{\eta;S}\mathcal{C}^\alpha} + \|Z - \tilde{Z}\|_{C_T\mathcal{C}^\alpha}.$$

By lowering T_* further, we obtain

$$\|u - \tilde{u}\|_{C_{\eta;2T_*}\mathcal{C}^\alpha} \lesssim \|x - \tilde{x}\|_{\mathcal{C}^{\alpha_0}} + \|Z - \tilde{Z}\|_{C_T\mathcal{C}^\alpha}. \qquad \square$$

We now prove that $\mathscr{S}$ is jointly locally Lipschitz. Consider $\mathfrak{R} > 1$, $D_\mathfrak{R}$, and $T_*(\mathfrak{R}) > 0$ as in Lemma 6.2.8. Then $\mathscr{S}$ is $K(\mathfrak{R})$-Lipschitz from $D_\mathfrak{R}$ to $C_\eta((0, 2T_*], \mathcal{C}^\alpha)$. In particular, the map $(x, Z) \mapsto u|_{[T_*,2T_*]} \in C([T_*, 2T_*], \mathcal{C}^\alpha)$ is $K(\mathfrak{R})$-Lipschitz. If $T \leq 2T_*$, then we are done. Hence, suppose $T > 2T_*$. We will prove that, for $\bar{T}_*(\mathfrak{R}) \in (0, T_*]$ sufficiently small, $u|_{[T_*+\bar{T}_*,T]} \in C([\bar{T}_*, T], \mathcal{C}^\alpha)$ is a Lipschitz function of (x, Z), from which the conclusion will follow.

To this end, consider $(x, Z) \in D_\mathfrak{R}$ and let u be the corresponding solution. By the a priori estimate, for $p \geq 2$ sufficiently large $\|u_t\|_{\mathcal{C}^{\alpha_0}} \lesssim_{\alpha_0,p} \|u_t\|_{L^p} \lesssim t^{-1/2} \vee \mathfrak{R}^{1/\alpha}$ for all $t > 0$. Therefore, there exists $\bar{\mathfrak{R}}(\mathfrak{R}) > 0$ such that $\|u_t\|_{\mathcal{C}^{\alpha_0}} \leq \bar{\mathfrak{R}}$ for all $t \in [T_*, T]$. Let $\bar{T}_*(\bar{\mathfrak{R}}) > 0$ be the constant from Lemma 6.2.8. Without loss of generality, we can assume that $\bar{T}_* \leq T_*$ and that $T = T_* + N\bar{T}_*$ for some integer $N \geq 2$. By the above remark, for all $1 \leq n < N$, $u|_{[T_*+n\bar{T}_*, T_*+(n+1)\bar{T}_*]} \in C([T_* + n\bar{T}_*, T_* + (n+1)\bar{T}_*], \mathcal{C}^\alpha)$ is a $K(\mathfrak{R})$-Lipschitz function of $(u_{T_*+(n-1)\bar{T}_*}, Z) \in \mathcal{C}^{\alpha_0} \times C_T\mathcal{C}^\alpha$. Combining these estimates together, it follows that $u|_{[T_*+\bar{T}_*,T]} \in C([\bar{T}_*, T], \mathcal{C}^\alpha)$ is a Lipschitz function of (x, Z). □

6.3 Global Well-Posedness and Invariant Measures

We now employ the a priori bound obtained in Section 6.2 to establish global existence of the process $u_t \in \mathcal{C}^\alpha(\mathbb{T})$ and an invariant measures for every $x \in \mathcal{C}^{\alpha_0}$ via the Krylov–Bogoliubov method.

We recall that for any $0 \leq s < t < \infty$, we defined the process

$$v_{s,t} := v_t - e^{(t-s)\Delta} v_s,$$

where $t \mapsto v_t$ is the solution to (5.4.4). We recall that $v_{s,t}$ defined in this way is bounded in the same regularity as v_t, is a Markov process with respect to $(\mathcal{F}_t)_{t \geq 0}$, is such that $v_{s,s} = 0$ and for any $h > 0$ the process $t \mapsto v_{t,t+h}$ is stationary. We again remove χ from our analysis, setting it equal to 1. We further recall that we have fixed, $\alpha_0 \in \left(-\frac{1}{2}, 0\right)$, $\alpha \in \left(0, \alpha_0 + \frac{1}{2}\right)$, $\beta > 0$ and $\eta > 0$ such that,

$$\frac{\alpha - \alpha_0}{2} < \eta < \frac{1}{4}, \quad \beta \in (1, \alpha + 1). \tag{6.3.1}$$

We also reset $\chi = 1$. We begin this section by completing the proof of Theorem 6.0.1.

Proof of Theorem 6.0.1. From Theorem 5.4.8, for any $T > 0$ and $\alpha < \frac{1}{2}$, $v_{0,\cdot} \in C_T \mathcal{C}_0^\alpha$ (in fact $v_{0,\cdot} \in \cap_{\kappa \in [0,1/2)} \mathcal{C}_T^\kappa \mathcal{C}_0^{\alpha - 2\kappa}$). Hence Proposition 6.2.7 implies that $\mathbb{P}$-a.s. there exists a unique mild solution $u(x) := w(x) + v_{0,\cdot} = \mathscr{S}(x, v_{0,\cdot})$ to (6.0.1) on $[0, T]$. □

Having established global well-posedness we define a semi-group associated to the dynamics. Recall that we denote by $\mathcal{B}_b(\mathcal{C}^{\alpha_0})$ (resp. $C_b(\mathcal{C}^{\alpha_0})$) the sets of bounded (resp. continuous, bounded) Borel maps $\mathcal{C}^{\alpha_0} \to \mathbb{R}$, both endowed with the norm

$$\|\Phi\|_{L^\infty} := \sup_{x \in \mathcal{C}^{\alpha_0}} |\Phi(x)|.$$

Then the semi-group associated to (6.0.1) is defined by the mapping $P_t : \mathcal{B}_b(\mathcal{C}^{\alpha_0}) \to \mathcal{B}_b(\mathcal{C}^{\alpha_0})$, with

$$P_t\Phi(x) := \mathbb{E}\left[\Phi(u_t(x))\right], \qquad \text{for all } \Phi \in \mathcal{B}_b(\mathcal{C}^{\alpha_0}),\ t \geq 0.$$

In the remainder of this section we show that u_t is a Markov process with Feller semi-group P_t which furthermore satisfies the necessary criteria to give existence of an invariant measure, $\nu_x \in \mathcal{P}_1(\mathcal{C}^{\alpha_0}_{\mathfrak{m}})$ for each $x \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}}$.

Lemma 6.3.1. *Let u be a mild solution to (6.0.1). Then for every $t \in [0,T]$ and $h \in (0, T-t)$ we have the identity*

$$u_{t+h} = \tilde{w}_{t,t+h} + v_{t,t+h}, \tag{6.3.2}$$

where $\tilde{w}_{t,t+h}$ solves

$$\tilde{w}_{t,t+h} = e^{h\Delta}u_t + \int_0^h e^{(h-r)\Delta}\partial_x((\tilde{w}_{t,t+r} + v_{t,t+r})^2\partial_x\rho_{\tilde{w}_{t,t+r}+v_{t,t+r}})\,\mathrm{d}r. \tag{6.3.3}$$

Proof. We first recall that

$$v_{t+h} = e^{h\Delta}v_t + v_{t,t+h},$$

where $v_{t,t+h}$ is independent of $\mathcal{F}_s$ for $s < t$. Then, for $t, h > 0$ we set $\tilde{w}_{t,t+h} := w_{t+h} + e^{h\Delta}v_{0,t}$, and we see that we have

$$u_{t+h} = w_{t+h} + v_{t+h} = \tilde{w}_{t,t+h} + v_{t,t+h},$$

where

$$\tilde{w}_{t,t+h} = e^{h\Delta}(e^{t\Delta}x + v_t) + \int_0^{t+h} e^{(t+h-r)\Delta}\partial_x((w_r + v_r)^2\partial_x\rho_{w_r,v_r})\,\mathrm{d}r.$$

Then, since

$$e^{t\Delta}x = w_t - \int_0^t e^{(t-r)\Delta}\partial_x((w_r + v_r)^2\partial_x\rho_{w_r+v_r})\,\mathrm{d}r$$

we have

$$e^{h\Delta}e^{t\Delta}x = e^{h\Delta}w_t - \int_0^t e^{(t+h-r)\Delta}\partial_x((w_r+v_r)^2)\,\mathrm{d}r,$$

and so

$$\begin{aligned}\tilde{w}_{t,t+h} &= e^{h\Delta}(w_t+v_t) + \int_t^{t+h} e^{(t+h-r)\Delta}\partial_x((w_r+v_r)^2\partial_x\rho_{w_r+v_r})\,\mathrm{d}r \\ &= e^{h\Delta}u_t + \int_0^h e^{(h-r)\Delta}\partial_x((w_{t+r}+v_{t+r})^2\partial_x\rho_{w_{t+r}+v_{t+r}})\,\mathrm{d}r. \qquad (6.3.4)\end{aligned}$$

So it only remains to show that

$$(w_{t+r}+v_{t+r})^2\partial_x\rho_{w_{t+r}+v_{t+r}} = (\tilde{w}_{t,t+r}+v_{t,t+r})^2\partial_x\rho_{\tilde{w}_{t,t+r}+v_{t,t+r}}.$$

Since the operator $f \mapsto \partial_x\rho_f$ is linear we directly have

$$\partial_x\rho_{w_{t+r}+v_{t+r}} = \partial_x\rho_{\tilde{w}_{t,t+r}+v_{t,t+r}}. \qquad (6.3.5)$$

For the quadratic term, expanding and then rearranging, we have

$$\begin{aligned}w_{t+r}^2 + 2w_{t+r}v_{t+r} + v_{t+r}^2 &= (\tilde{w}_{t,t+r} - e^{r\Delta}v_t)^2 + 2(\tilde{w}_{t,t+r} - e^{r\Delta}v_t)(e^{r\Delta}v_t + v_{t,t+r}) \\ &\quad + (e^{r\Delta}v_t + v_{t,t+r})^2 \\ &= \tilde{w}_{t,t+r}^2 + 2\tilde{w}_{t,t+r}v_{t,t+r} + v_{t,t+r}^2. \qquad (6.3.6)\end{aligned}$$

Combining (6.3.4) with (6.3.5) and (6.3.6) for $t \in [0,T]$ and $h \in (0, T-t)$ gives (6.3.3) which completes the proof of (6.3.2). □

Theorem 6.3.2. *Let $T > 0$, $\mathfrak{m} \in \mathbb{R}$ and $x \in \mathcal{C}_{\mathfrak{m}}^{\alpha_0}$. Then let $u(x) \in C_\eta((0,T];\mathcal{C}^\alpha(\mathbb{T}))$ be the unique solution to (6.0.1) as in Theorem 6.0.1. Then for every $p \in [2,\infty)$ and even, we have that*

$$\sup_{T>0}\sup_{x\in\mathcal{C}^{\alpha_0}}\sup_{t>0}\left(t^{\frac{p}{2}} \wedge 1\right)\mathbb{E}\left[\|u_t(x)\|_{L^p}^p\right] < \infty. \qquad (6.3.7)$$

Proof. Firstly, for $t \in (0,1)$, applying (6.2.8) and Corollary 5.4.7, we directly have that

$$\mathbb{E}\left[\|u_t\|_{L^p}^p\right] \leq \mathbb{E}\left[\|w_t\|_{L^p}^p\right] + \mathbb{E}\left[\|v_t\|_{L^p}^p\right] \lesssim t^{-\frac{p}{2}} + 1. \tag{6.3.8}$$

For $t > 1$ we employ the Markov decomposition (6.3.2) to give

$$\mathbb{E}\left[\|u_t\|_{L^p}^p\right] = \mathbb{E}\left[\|\tilde{w}_{t-1,t}\|_{L^p}^p\right] + \mathbb{E}\left[\|v_{t-1,t}\|_{L^p}^p\right].$$

Then, observing that $\tilde{w}_{t-1,t}$ solves (6.1.1) with initial condition $u_{t-1}(x)$ and driving path $Z_t = v_{t-1,t}$, by Theorem 6.2.3 we have

$$\mathbb{E}\left[\|\tilde{w}_{t-1,t}\|_{L^p}^p\right] \leq C_p \max\left\{\|v_{t-1,\cdot}\|_{C_{[t-1,t]}\mathcal{C}^\alpha}^{\frac{p}{\alpha}},\, 1\right\}.$$

So, since the law of $v_{t-1,t}$ does not depend on $t > 1$ we have,

$$\sup_{x\in\mathcal{C}^{\alpha_0}} \sup_{t>1} \|u_t(x)\|_{L^p}^p < \infty. \tag{6.3.9}$$

So combining (6.3.8) with (6.3.9) proves (6.3.7). □

In order to prove the Markov property for the semi-group we make use of the following representation theorem which can be found as [34, Prop. 1.12]. We include the statement for completeness.

Theorem 6.3.3 (Prop 1.12 [34])**.** *For $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ two measurable spaces and $\psi : E_1 \times E_2 \to \mathbb{R}$ be a bounded measurable function. Let ξ_1, ξ_2 be two random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ respectively, Then let $\mathcal{G} \subset \mathcal{F}$ be a sub-sigma algebra and assume ξ_1 is $\mathcal{G}$ - measurable. Then there exists a bounded, $\mathcal{E}_1 \times \mathcal{G}$ measurable function $\widehat{\psi} : E_1 \times \Omega \to \mathbb{R}$ such that,*

$$\mathbb{E}\left[\psi(\xi_1, \xi_2) \,|\, \mathcal{G}\right](\omega) = \widehat{\psi}(\xi_1(\omega), \omega), \quad \forall\, \omega \in \Omega.$$

If in addition $\xi_2 \perp \mathcal{G}$, then

$$\widehat{\psi}(x_1, \omega) = \widehat{\psi}(x_1) = \mathbb{E}\left[\psi(x_1, \xi_2)\right], \quad \forall\, x_1 \in E_1.$$

In conjunction with the Markov decomposition of the solution (6.3.2) we demonstrate the Markov property for $t \mapsto P_t$.

Theorem 6.3.4. *Let $x \in \mathcal{C}^{\alpha_0}(\mathbb{T}^2)$ and $u_t(x)$ be as defined in Theorem 6.3.1. Then for any $\Phi \in \mathcal{B}_b(\mathcal{C}^{\alpha_0})$ we have the identity*

$$\mathbb{E}\left[\Phi(u_{t+h}(x)) \,|\, \mathcal{F}_t\right] = P_h(\Phi(u_t(x)),$$

and in particular we see that $t \mapsto u_t$ is a Markov process with associated Markov semi-group P_t.

Proof. First, using the Markov decomposition (6.3.2) we see that

$$\Phi(u_{t+h}(x)) = \Phi(\tilde{w}_{t,t+h}(u_t(x)) + v_{t,t+h}) = \Phi(u_{t,t+h}(u_t(x))),$$

where $\tilde{w}_{t,t+h}(u_t(x))$ is the solution to the remainder equation (6.1.1) with initial condition $u_t(x)$ and driven by the vector $s \mapsto v_{t,t+s}$. We define

$$\psi(u_t(x), \underline{v}_{t,t+h}) := \Phi(\tilde{w}_{t,t+h}(u_t(x)) + v_{t,t+h})$$

and then we apply Proposition 6.3.3 with the random variables $u_t(x)$ and $v_{t,t+h}$ and using the sub-sigma algebra $\mathcal{F}_t$. Since $u_t(x)$ is $\mathcal{F}_t$ measurable and $v_{t,t+s} \perp \mathcal{F}_t$ we have that

$$\mathbb{E}\left[\Phi(u_{t+h}(x)) \,|\, \mathcal{F}_t\right] = \mathbb{E}[\Phi(u_{t,t+h}(u_t(x)))] = P_h(\Phi(u_t(x)))$$

and so the claim is shown. □

The following theorem shows that P_t is a Feller semi-group.

Lemma 6.3.5. *For any $\Phi \in \mathcal{C}_b(\mathcal{C}^{\alpha_0}(\mathbb{T}))$ and $t > 0$ we have that $P_t\Phi \in \mathcal{C}_b(\mathcal{C}^{\alpha_0}(\mathbb{T}))$ and for any $x \in \mathcal{C}^{\alpha_0}$, $\lim_{t\to 0}|P_t\Phi(x) - \Phi(x)|= 0$.*

Proof. By Theorem 6.3.1, $\mathcal{C}^{\alpha_0}(\mathbb{T}) \ni x \mapsto u_t(x) \in \mathcal{C}^{\alpha}(\mathbb{T})$ is $\mathbb{P}$-a.s. continuous for every $t > 0$. The fact that $P_t\Phi \in \mathcal{C}_b(\mathcal{C}^{\alpha_0}(\mathbb{T}))$ thus follows from the dominated convergence theorem. The fact that $\lim_{t\searrow 0}|P_t\Phi(x) - \Phi(x)| = 0$ for all $x \in \mathcal{C}^{\alpha_0}$, $\Phi \in \mathcal{C}_b(\mathcal{C}^{\alpha_0}(\mathbb{T}))$, similarly follows the dominated convergence theorem and (6.1.9). □

Theorem 6.3.6. *For any* $\Phi \in \mathcal{C}_b(\mathcal{C}^{\alpha_0})$ *and* $t > 0$ *we have that* $P_t\Phi \in \mathcal{C}_b(\mathcal{C}^{\alpha_0})$ *and* $\lim_{t\to 0}\|P_t\Phi - \Phi\|_{L^\infty} = 0$.

Proof. It suffices to prove almost sure continuity in $x \in \mathcal{C}^{\alpha_0}$ of the solution $t \mapsto u_t(x) \in \mathcal{C}^\alpha$ and then apply the dominated convergence theorem to bring the limit inside the expectation. We let x, $y \in \mathcal{C}^{\alpha_0}$ be such that $\|x - y\|_{\mathcal{C}^{\alpha_0}} \leq 1$ and $T > 0$ be arbitrary. Then we define the solutions to (6.0.1), started from x and y respectively by

$$u_t = e^{t\Delta}x + \int_0^t e^{(t-s)\Delta}\partial_x(u_s^2\partial_x\rho_{u_s})\,\mathrm{d}s + v_t,$$
$$\tilde{u}_t = e^{t\Delta}y + \int_0^t e^{(t-s)\Delta}\partial_x(\tilde{u}_s^2\partial_x\rho_{\tilde{u}_s})\,\mathrm{d}s + v_t.$$

Then, we let

$$\tau := \inf\{t \in (0, T] \,:\, \|u_t - \tilde{u}_t\|_{\mathcal{C}^\alpha} > 1\}, \quad \text{and} \quad M := \sup_{t\leq T}(t \wedge 1)^\eta\|u_t\|_{\mathcal{C}^\alpha}.$$

Therefore, for $t \leq \tau$ we have $\|\tilde{u}\|_{C_{\eta;t}\mathcal{C}^\alpha} \leq 1 + M$. So, using (5.2.15), for any $t \in (0, \tau]$ we have

$$\sup_{s\in(0,t]} s^\eta\|u_t - \tilde{u}_t\|_{\mathcal{C}^\alpha} \lesssim t^{\eta - \frac{\alpha-\alpha_0}{2}}\|x - y\|_{\mathcal{C}^{\alpha_0}} + t^{\frac{1}{2}-2\eta}(1 + M^2) \sup_{ts\in(0,t]} s^\eta\|u_t(x) - u_t(y)\|_{\mathcal{C}^\alpha}.$$

Therefore, there exists a $\theta := \theta(\alpha, \alpha_0, \eta) > 0$, such that choosing

$$t := T_1 = \left(\frac{1}{2CM(1 + M)}\right)^\theta \wedge \tau,$$

where $C > 0$ is the proportionality constant above, we have,

$$\sup_{s\in(0,T_1]} t^\eta\|u_t - \tilde{u}_t\|_{\mathcal{C}^\alpha} \leq \|x - y\|_{\mathcal{C}^{\alpha_0}}.$$

Iterating this procedure once, we have

$$\begin{aligned}\sup_{t\in(T_1,2T_1\wedge\tau]} t^{\eta}\|u_t-\tilde{u}_t\|_{\mathcal{C}^{\alpha}} &\leq C\|u_{T_1}-\tilde{u}_{T_1}\|_{\mathcal{C}^{\alpha_0}}\\ &\leq T_1^{-\eta}\sup_{t\in(0,T_1]} t^{\eta}\|u_t-\tilde{u}_t\|_{\mathcal{C}^{\alpha}}\\ &\leq T_1^{-\eta}\|x-y\|_{\mathcal{C}^{\alpha_0}},\end{aligned}$$

therefore, there exists a new $C := C(M,\theta) > 0$ such that

$$\sup_{t\in(0,2T_1\wedge\tau]} t^{\eta}\|u_t-\tilde{u}_t\|_{\mathcal{C}^{\alpha}}\leq (1+C)\|x-y\|_{\mathcal{C}^{\alpha_0}}.$$

So repeating the argument for a finite number of times, we find an $N\in\mathbb{N}_{>0}$ such that,

$$\sup_{t\in(0,T\wedge\tau]} t^{\eta}\|u_t-\tilde{u}_t\|_{\mathcal{C}^{\alpha}}\leq (1+NC)\,\|x-y\|_{\mathcal{C}^{\alpha_0}}.$$

Therefore, we have shown that the map $\mathcal{C}^{\alpha_0}\ni x\mapsto u(x)\in C_{\eta;T}\mathcal{C}^{\alpha}$ is Lipschitz for any $T>0$. Applying the dominated convergence theorem we obtain that $P_t\Phi\in\mathcal{C}_b(\mathcal{C}^{\alpha_0})$ for any $\Phi\in\mathcal{C}_b(\mathcal{C}^{\alpha_0})$.

To see that $\lim_{t\searrow 0}\|P_t\Phi-\Phi\|_{L^{\infty}}=0$ it suffices to show using Remark 5.2.11 that $\lim_{t\searrow 0}\|u_t(x)\|_{\mathcal{C}^{\alpha_0}}=\|x\|_{\mathcal{C}^{\alpha_0}}$. □

We now apply Theorem 5.3.6 with $\mu=\delta_x$ and so obtain for any $\mathfrak{m}\in\mathbb{R}$ and $x\in\mathcal{C}^{\alpha_0}_{\mathfrak{m}}$ an invariant measure $\nu_x\in\mathcal{P}(\mathcal{C}^{\alpha_0}_{\mathfrak{m}})$.

Theorem 6.3.7. *Let $\mathfrak{m}\in\mathbb{R}$. Then for every $x\in\mathcal{C}^{\alpha_0}_{\mathfrak{m}}$, there exists a measure $\nu_x\in\mathcal{P}(\mathcal{C}^{\alpha_0}_{\mathfrak{m}})$ and an increasing sequence of times $t_k\nearrow\infty$ such that,*

$$R^*_{t_k}\delta_x := \frac{1}{t_k}\int_0^{t_k} P^*_r\,\delta_x\,\mathrm{d}r \rightharpoonup \nu_x, \qquad \textit{as } k\to\infty.$$

Furthermore ν_x is invariant for P_t.

Proof. Our main goal is to show that there exists a diverging sequence of times

$(t_k)_{k>0}$ such that $R^*_{t_k}\delta_x$ is tight. Then applying [33, Corollary 3.1.2] we obtain the result.

We let $p > -\frac{1}{\alpha_0}$ be even and then applying Markov's inequality followed by Jensen's inequality, for every K, $t > 0$ and $x \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}}$, we find that

$$\mathbb{P}\left\{\|u_t(x)\|_{L^p} > K\right\} \leq \frac{1}{K}\mathbb{E}\left[\|u_t(x)\|_{L^p}\right] \leq \frac{1}{K}\mathbb{E}\left[\|u_t(x)\|^p_{L^p}\right]^{\frac{1}{p}}.$$

So then, for every $t > 1$, using (6.3.7), there exists a $C > 0$ such that

$$\begin{aligned}
\frac{1}{t}\int_0^t \mathbb{P}\left\{\|u_s(x)\|_{L^p} > K\right\} \mathrm{d}s &\leq \frac{1}{tK}\int_0^t \mathbb{E}\left[\|u_s(x)\|^p_{L^p}\right]^{\frac{1}{p}} \mathrm{d}s \\
&= \frac{1}{tK}\left(\int_0^1 \mathbb{E}\left[\|u_s(x)\|^p_{L^p}\right]^{\frac{1}{p}} \mathrm{d}s + \int_1^t \mathbb{E}\left[\|u_s(x)\|^p_{L^p}\right]^{\frac{1}{p}} \mathrm{d}s\right) \\
&\leq C\frac{1}{tK}\left(\int_0^1 \frac{1}{\sqrt{s}} \mathrm{d}s + (t-1)\right) \\
&\leq C\frac{1}{K}.
\end{aligned}$$

Now we define the set,

$$B_K := \{\varphi \in L^p(\mathbb{T}) \,:\, \|\varphi\|_{L^p} \leq K\},$$

which is compact in $\mathcal{C}^{\alpha_0}(\mathbb{T})$ by Theorem 5.2.2 and we have shown above that

$$R^*_t\delta_x(B^c_K) \leq \frac{1}{K}. \tag{6.3.10}$$

So letting $K_\varepsilon := \frac{C}{\varepsilon}$ and defining the sets B_{K_ε}, we conclude that $(R^*_t\delta_x)_{t>0}$ is a tight family in $\mathcal{P}(\mathcal{C}^{\alpha_0}_{\mathfrak{m}})$. Applying Prokhorov's theorem, Theorem 5.3.5, we see that there exists a subsequence $t_k \nearrow \infty$ such that $\lim_{k\to\infty} R^*_{t_k}\delta_x = \nu_x \in \mathcal{P}(\mathcal{C}^{\alpha_0}_{\mathfrak{m}})$. Then applying [33, Th. 3.1.1] completes the proof. □

6.4 Uniqueness of the Invariant Measure and Exponential Ergodicity

In the previous section we established that the semi-group associated to the one dimensional equation (6.0.1) satisfies the properties of being a Feller semi-group and for every initial condition there exists an invariant measure ν for the dynamics. In this section we demonstrate uniqueness of the invariant measure along with exponential convergence of the law to ν. The main steps are to establish the strong Feller property and irreducibility of P_t. We point out that these properties do not require global existence of the dynamics. Adding a point at infinity to the state space would allow us to establish these properties, in much the same way, for local dynamics up to a possibly finite explosion time.

We prove the strong Feller property in subsection 6.4.1 employing a similar strategy to that of [105, Sec. 5], in fact obtaining a local Lipschitz bound on the semi-group. To show irreducibility of P_t we show full support of the law on $\mathcal{C}_{\mathfrak{m}}^{1/2-\delta}(\mathbb{T})$, for any $\delta \in (0, 1/2)$ and $\mathfrak{m} = \bar{x}$. This is carried out in subsection 6.4.2. Finally we combine these results to show uniqueness of the invariant measure and exponential ergodicity of the dynamics in Subsection 6.4.3.

For technical reasons in this section we make new assumptions on the parameters, replacing (6.0.2). We fix $\alpha_0 \in \left(-\frac{1}{3}, 0\right)$, $\alpha \in \left(0, \alpha_0 + \frac{1}{3}\right)$ and $\eta > 0$ such that,

$$\frac{\alpha - \alpha_0}{2} < \eta < \frac{1}{6}. \tag{6.4.1}$$

We note that all previous results also hold for this more restrictive parameter range.

6.4.1 Strong Feller Property

We prove the strong Feller property in a similar way to [105, Sec. 5]. The strategy is to first take an approximation u_ε with compact support in Fourier space. We then obtain Fréchet differentiability of the approximate solution with respect to both the initial condition and the noise, locally in the size of v. With this in hand we prove a Bismut–Ellworthy–Li type formula, in the same manner as [105, Thm. 5.5], which allows us to relate the derivative in initial data to the derivative in noise. The final step is to use this relationship along with a priori control on the derivative of the equation in the initial condition to obtain a local Lipschitz bound on the semi-group.

The benefit of this approach is that it allows us to bypass the machinery of Malliavin calculus. Since our a priori bound on the solutions does not give sufficient integrability of the laws, we work locally on Ω, on sets where the noise is small, using our local well-posedness result to pass this smallness to the solution. We control the semi-group outside of this set using the probabilistic controls shown in Theorem 5.4.8. Applying this approach in the framework of Malliavin calculus would present significant complications. A common method for semi-linear SPDEs that does go through Malliavin calculus is to work with Yosida approximations to the non-linearity and local Malliavin differentiability, for example see [99]. However, as discussed in Chapter 5 our non-linearity is not dissipative in any L^2 type space, in the sense of [33, Sec. 5.5.1], and so this approach also does not apply directly.

For technical reasons in this section we replace (6.0.2) with the assumption that $\alpha_0 \in \left(-\frac{1}{3}, 0\right)$, $\alpha \in \left(0, \alpha_0 + \frac{1}{3}\right)$ and $\eta > 0$ such that,

$$\frac{\alpha - \alpha_0}{2} < \eta < \frac{1}{6}. \tag{6.4.2}$$

We note that all previous results also hold for this more restrictive parameter range and this restriction does not affect the main result as stated, see the proof of Theorem

6.0.2 at the conclusion of Section 6.4.3. Let $(e_m)_{m\in\mathbb{Z}}$, $e_m(x) = e^{i2\pi mx}$, be the usual Fourier basis elements of $L^2(\mathbb{T})$ and $(\Delta_k)_{k\geq -1}$ be the Littlewood–Paley projection operators, see Section 5.2 for more details. For $\varepsilon \in (0,1)$, we define $\Pi_\varepsilon(L^2(\mathbb{T}))$ as the space of real functions spanned by $\left\{(e_m)_{|m|<\frac{1}{\varepsilon}}\right\}$ and

$$\hat{\Pi}_\varepsilon := \sum_{-1\leq k\leq -\log_2(9\varepsilon)} \Delta_k\colon L^2(\mathbb{T}) \to \Pi_\varepsilon(L^2(\mathbb{T}^d)).$$

Observe that there exists $\ell_\varepsilon\colon \mathbb{Z} \to [0,1]$ with $\ell_\varepsilon(m) = 0$ for $|m|\geq \frac{1}{\varepsilon}$ such that $\mathcal{F}(\hat{\Pi}_\varepsilon f)(m) = \ell_\varepsilon(m)\mathcal{F}f(m)$ where $\mathcal{F}$ is the Fourier transform. Furthermore (see e.g. [105, p. 1213]),

i) $\|\hat{\Pi}_\varepsilon\|_{\mathrm{Op}(\mathcal{C}^\beta;\mathcal{C}^\beta)} := \sup_{\|u\|_{\mathcal{C}^\beta}\leq 1} \|\hat{\Pi}_\varepsilon u\|_{\mathcal{C}^\beta} \leq 1$ for all $\varepsilon \in (0,1)$ and $\beta \in \mathbb{R}$,

ii) for every $\beta \in \mathbb{R}$ and $\delta > 0$, there exists $C > 0$ such that for all $\varepsilon \in (0,1)$

$$\|\hat{\Pi}_\varepsilon f - f\|_{\mathcal{C}^{\beta-\delta}} \leq C\varepsilon^\delta \|f\|_{\mathcal{C}^\beta}.$$

We fix for the rest of the subsection $\delta > 0$

$$\alpha + \delta < \frac{1}{2}, \quad \eta - \frac{\alpha - \alpha_0 + \delta}{2} > 0, \quad \frac{1-\delta}{2} - 3\eta > 0. \tag{6.4.3}$$

Such a δ exists due to (6.4.2).

Consider now $Z \in C_T\mathcal{C}^{\alpha+\delta}$ and let $Z_{\varepsilon;t} := \hat{\Pi}_\varepsilon Z_t$. Consider likewise $x \in \mathcal{C}^{\alpha_0}$ and let $x_\varepsilon := \hat{\Pi}_\varepsilon x$. We define a smooth approximation to (6.0.1), with $\chi = 1$ and deterministic noise, by

$$u_{\varepsilon;t} = e^{t\Delta}x_\varepsilon + \int_0^t e^{(t-s)\Delta}\partial_x\hat{\Pi}_\varepsilon(u_{\varepsilon;s}^2\partial_x\rho_{u_{\varepsilon;s}})\,\mathrm{d}s + Z_{\varepsilon;t}, \tag{6.4.4}$$

where, as usual,

$$-\partial_{xx}\rho_{u_{\varepsilon;t}} = u_{\varepsilon;t} - \bar{u}_\varepsilon, \text{ on } \mathbb{T}.$$

Remark 6.4.1. We will later set, right before Theorem 6.4.7, $Z_\varepsilon = \hat{\Pi}_\varepsilon v$, where v is the SHE.

Theorem 6.4.2. *Let $T > 0$, $x \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}}(\mathbb{T})$, and $Z \in C_T\mathcal{C}^{\alpha+\delta}$. For every $\bar{\varepsilon} > 0$, there exists $\varepsilon_0(\bar{\varepsilon}, T, \|x\|_{\mathcal{C}^{\alpha_0}} + \|Z\|_{C_T\mathcal{C}^{\alpha+\delta}}) > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$ there exists a unique solution $u_\varepsilon \in C_T\Pi_\varepsilon(L^2(\mathbb{T}))$ to (6.4.4). Furthermore*

$$\|u_t - u_{\varepsilon;t}\|_{C_{\eta;T}\mathcal{C}^\alpha} \leq \bar{\varepsilon}. \tag{6.4.5}$$

Proof. It follows from Properties i) and ii) of the projection and the same steps as in the proof of Theorem 6.1.4 that for every $\varepsilon > 0$ there exists $T_\varepsilon(\|x_\varepsilon\|_{\mathcal{C}^{\alpha_0}} + \|Z_\varepsilon\|_{C_T\mathcal{C}^\alpha}) > 0$, such that $u_\varepsilon \in C_{\eta;T_\varepsilon}\mathcal{C}^\alpha$ solves (6.4.4) and satisfies $\|u_\varepsilon\|_{C_{\eta;T_\varepsilon}\mathcal{C}^\alpha} \leq 1$. The fact that $u_\varepsilon \in C_{T_\varepsilon}\Pi_\varepsilon(L^2(\mathbb{T}))$ follows from the same argument as the end of Theorem 6.1.4 since $x_\varepsilon \in C^\infty$.

To continue, we state the following

Lemma 6.4.3. *Consider $\mathfrak{R} > 0$ and define $D_{\mathfrak{R}}$ as in Lemma 6.2.8. There exists $T_*(\mathfrak{R}) > 0$ such that, if $(x_\varepsilon, Z_\varepsilon), (\tilde{x}, \tilde{Z}) \in D_{\mathfrak{R}}$, then*

$$\|\tilde{u}_t - u_{\varepsilon;t}\|_{C_{\eta;T_*}\mathcal{C}^\alpha} \leq \|x_\varepsilon - \tilde{x}\|_{\mathcal{C}^{\alpha_0-\delta}} + \varepsilon^\delta + \|\tilde{Z} - Z_\varepsilon\|_{C_T\mathcal{C}^\alpha},$$

where $\tilde{u} = \mathscr{S}(\tilde{x}, \tilde{Z})$ with $\mathscr{S}$ as in Proposition 6.2.7.

Proof. There exists $T_*(\mathfrak{R}) > 0$ such that $\|\tilde{u}\|_{C_{\eta;2T_*}\mathcal{C}^\alpha} + \|u_\varepsilon\|_{C_{\eta;2T_*}\mathcal{C}^\alpha} \leq 2$. For $t \in (0, 2T_*]$,

$$\begin{aligned}\|\tilde{u}_t - u_{\varepsilon;t}\|_{\mathcal{C}^\alpha} \leq \|e^{t\Delta}(\tilde{x} - x_\varepsilon)\|_{\mathcal{C}^\alpha} + \int_0^t \|e^{(t-s)\Delta}\partial_x(\tilde{u}_s^2\partial_x\rho_{\tilde{u}_s} - \hat{\Pi}_\varepsilon(u_{\varepsilon;s}^2\partial_x\rho_{u_{\varepsilon;s}}))\|_{\mathcal{C}^\alpha}\,\mathrm{d}s \\ + \|\tilde{Z}_t - Z_{\varepsilon;t}\|_{\mathcal{C}^\alpha}.\end{aligned}$$

Then

$$\|e^{t\Delta}(\tilde{x} - x_\varepsilon)\|_{\mathcal{C}^\alpha} \lesssim t^{-\frac{\alpha-\alpha_0+\delta}{2}}\|\tilde{x} - x_\varepsilon\|_{\mathcal{C}^{\alpha_0-\delta}},$$

and applying the triangle inequality and using Property ii) of the projection,

$$\begin{aligned}\|\tilde{u}_s^2\partial_x\rho_{\tilde{u}_s} - \hat{\Pi}_\varepsilon(u_{\varepsilon;s}^2\partial_x\rho_{u_{\varepsilon;s}})\|_{\mathcal{C}^{\alpha-\delta}} &\le \|\tilde{u}_s^2\partial_x\rho_{\tilde{u}_s} - u_{\varepsilon;s}^2\partial_x\rho_{u_{\varepsilon;s}}\|_{\mathcal{C}^\alpha} \\ &\quad + \|u_{\varepsilon;s}^2\partial_x\rho_{u_{\varepsilon;s}} - \hat{\Pi}_\varepsilon(u_{\varepsilon;s}^2\partial_x\rho_{u_{\varepsilon;s}})\|_{\mathcal{C}^{\alpha-\delta}} \\ &\lesssim s^{-2\eta}\|\tilde{u}_s - u_{\varepsilon;s}\|_{\mathcal{C}^\alpha} + \varepsilon^\delta s^{-3\eta} \\ &\le s^{-3\eta}\|\tilde{u} - u_\varepsilon\|_{C_{\eta;s}\mathcal{C}^\alpha} + \varepsilon^\delta s^{-3\eta}.\end{aligned}$$

Therefore, using that $\eta < \frac{1}{6}$, we may integrate $\|e^{(t-s)\Delta}\partial_x(\tilde{u}_s^2\partial_x\rho_{\bar{u}_s} - \hat{\Pi}_\varepsilon(u_{\varepsilon;s}^2\partial_x\rho_{u_{\varepsilon;s}}))\|_{\mathcal{C}^\alpha}$ from 0 to t and take suprema to obtain

$$\begin{aligned}\sup_{t\in(0,T_*]} t^\eta\|\tilde{u}_s - u_{\varepsilon;s}\|_{\mathcal{C}^\alpha} \lesssim T_*^{\eta - \frac{\alpha-\alpha_0-\delta}{2}}\|\tilde{x} - x_\varepsilon\|_{\mathcal{C}^{\alpha_0-\delta}} + T_*^{\frac{1-\delta}{2}-3\eta}\sup_{t\in(0,T_*]} t^\eta\|\tilde{u}_t - u_{\varepsilon;t}\|_{\mathcal{C}^\alpha} \\ + T_*^{\frac{1-\delta}{2}-2\eta}\varepsilon^\delta + T_*^\eta\|\tilde{Z} - Z_\varepsilon\|_{C_T\mathcal{C}^\alpha}.\end{aligned}$$

After potentially lowering $T_*(\mathfrak{R}) > 0$ further, we obtain

$$\sup_{t\in(0,T_*]} t^\eta\|\tilde{u}_t - u_{\varepsilon;t}\|_{\mathcal{C}^\alpha} \le \|\tilde{x} - x_\varepsilon\|_{\mathcal{C}^{\alpha_0-\delta}} + \varepsilon^\delta + \|\tilde{Z} - Z_\varepsilon\|_{C_T\mathcal{C}^\alpha}. \qquad \square$$

To conclude the proof of Theorem 6.4.2, observe that $\|x - \Pi_\varepsilon x\|_{\mathcal{C}^{\alpha_0-\delta}} \lesssim \varepsilon^\delta\|x\|_{\mathcal{C}^{\alpha_0}}$ and $\|Z_t - Z_{\varepsilon;t}\|_{\mathcal{C}^\alpha} \lesssim \varepsilon^\delta\|Z_t\|_{\mathcal{C}^{\alpha+\delta}}$ by Property ii). By the a priori estimate, for every $\mathfrak{R} > 0$ there exists $T_*(\mathfrak{R}) > 0$ as in Lemma 6.4.3 and $\bar{\mathfrak{R}}(\mathfrak{R}) > 0$ such that $\|u_t\|_{\mathcal{C}^{\alpha_0}} \le \bar{\mathfrak{R}}$ for all $t \in [T_*, T]$. Iteratively applying Lemma 6.4.3 as in the proof of Proposition 6.2.7, there exists $\varepsilon_0(\mathfrak{R}) > 0$ sufficiently small such that $\|u_{\varepsilon;t}\|_{\mathcal{C}^\alpha} \le 2\bar{\mathfrak{R}}$ on $[\bar{T}_*, T]$ for all $\varepsilon < \varepsilon_0$, from which (6.4.5) follows after another iterative application of Lemma 6.4.3. $\square$

By Theorem 6.4.2, for every $\mathfrak{R} > 0$ and $T > 0$, there exists $\varepsilon_0(\mathfrak{R}, T) > 0$ sufficiently small such that for all $\varepsilon \in (0, \varepsilon_0)$ the solution map to (6.4.4)

$$\begin{aligned}\mathscr{S}^\varepsilon \colon A_{\mathfrak{R}} &\to C_T\Pi_\varepsilon(L^2(\mathbb{T})) \\ (x, Z) &\mapsto u_\varepsilon,\end{aligned} \tag{6.4.6}$$

is well-defined and $\|u_\varepsilon - u\|_{C_{\eta;T}\mathcal{C}^\alpha} \le 1$, where $u = \mathscr{S}(x, Z)$ and $A_{\mathfrak{R}} := \{(x, Z\} \in$

$\mathcal{C}^{\alpha_0} \times C_T\mathcal{C}^{\alpha+\delta} \, : \, \|x\|_{\mathcal{C}^{\alpha_0}} + \|Z\|_{C_T\mathcal{C}^{\alpha+\delta}} < \mathfrak{R}\}$.

We now fix for the remainder of the section $x \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}}$, $T > 0$, and $Z \in C_T\mathcal{C}^{\alpha+\delta}$ with $Z_0 = 0$. We let $\varepsilon_0 > 0$ sufficiently small such that $\mathscr{S}^\varepsilon(x, Z)$ is well-defined and $\|u_\varepsilon - u\|_{C_{\eta;T}\mathcal{C}^\alpha} \leq 1$ for all $\varepsilon \in (0, \varepsilon_0)$. In the sequel we will always let $\varepsilon \in (0, \varepsilon_0)$.

We define $N_\varepsilon := \{m \in \mathbb{Z} \, : \, |m| < \frac{1}{\varepsilon}, \, m \neq 0\}$ and further assume that $Z_{\varepsilon;t} := \hat{\Pi}_\varepsilon Z_t$ is of the form

$$Z_{\varepsilon;t} = \sum_{m \in N_\varepsilon} \int_0^t \ell_\varepsilon(m) e^{-4\pi^2|m|^2(t-s)} \, \mathrm{d}\hat{B}_{m;s} \, e_m, \tag{6.4.7}$$

where $\{\hat{B}_m\}_{m \in \mathbb{Z}, m \neq 0}$ are functions $\hat{B}_m \in C([0, T], \mathbb{C})$ which satisfy the reality condition

$$\hat{B}_{-m} = \overline{\hat{B}_m}. \tag{6.4.8}$$

We denote

$$\hat{B}_\varepsilon := (\hat{B}_m)_{m \in N_\varepsilon} \in \bar{C}_T\mathbb{C}^{N_\varepsilon},$$

where $\bar{C}_T\mathbb{C}^{N_\varepsilon} \subset C_T\mathbb{C}^{N_\varepsilon}$ is the subspace of $(\hat{B}_m)_{m \in N_\varepsilon}$ which satisfies the reality condition (6.4.8).

Remark 6.4.4. We will henceforth identify $\bar{C}_T\mathbb{C}^{N_\varepsilon}$ with a subspace of $C_T C_0^\infty \subset C_T\mathcal{C}_0^{\alpha+\delta}$ by mapping $(\hat{B}_m)_{m \in N_\varepsilon}$ to Z_ε via (6.4.7). Observe that the integral in (6.4.7) is well-defined as a Riemann–Stieltjes integral for any $\hat{B} \in \bar{C}_T\mathbb{C}^{N_\varepsilon}$. For fixed $x \in \mathcal{C}^{\alpha_0}$, we will treat in this way $u_\varepsilon(x, Z) := \mathscr{S}^\varepsilon(x, Z)$ as a function of $\hat{B}_\varepsilon$ whenever it is well-defined.

We now prove the differentiability of $\mathscr{S}^\varepsilon$ with respect to both arguments separately, using $\mathcal{D}$ for derivatives with respect to $\hat{B}_\varepsilon$ and D for derivatives with respect to x. For $\mathfrak{R} \geq 1$, we recall the definition of $T_*(\mathfrak{R})$ given by (6.1.7).

Lemma 6.4.5 (Derivative in Noise)**.** *There exists an open neighbourhood $\mathcal{O}_{\hat{B}_\varepsilon} \subset \bar{C}_T\mathbb{C}^{N_\varepsilon}$ containing $\hat{B}_\varepsilon$ such that $u_\varepsilon(x, \cdot)$ is Fréchet differentiable as a mapping from $\mathcal{O}_{\hat{B}_\varepsilon}$ to $C_T\Pi_\varepsilon L^2(\mathbb{T})$. Furthermore, for any $f \in \bar{C}_T\mathbb{C}^{N_\varepsilon}$, such that $f|_{t=0} = 0$, the* *directional derivative $\mathcal{D}_f u_\varepsilon$ satisfies the equation*

$$\begin{aligned}\mathcal{D}_f u_{\varepsilon;t} = &\int_0^t e^{(t-s)\Delta}\partial_x \hat{\Pi}_\varepsilon \left(2u_{\varepsilon;s}\mathcal{D}_f u_{\varepsilon;s}\partial_x\rho_{u_{\varepsilon;s}} + u_{\varepsilon;s}^2\partial_x\rho_{\mathcal{D}_f u_{\varepsilon;s}}\right)\,\mathrm{d}s \\ &+ \sum_{m\in N_\varepsilon}\int_0^t \ell_\varepsilon(m)e^{-4\pi^2|m|^2(t-s)}\,\mathrm{d}f_{m;s}e_m.\end{aligned} \tag{6.4.9}$$

Finally, for $\varepsilon \in (0,\varepsilon_0)$, there exists a $C(T,\, \|x\|_{\mathcal{C}^{\alpha_0}},\, \|Z_\varepsilon\|_{C_T\mathcal{C}^\alpha}) > 0$ such that,

$$\|\mathcal{D}_f u_\varepsilon\|_{C_T\mathcal{C}^\alpha} \leq C\|f\|_{C_T\mathbb{C}^{N_\varepsilon}}. \tag{6.4.10}$$

Proof. Integration by parts implies that, for any $m\in\mathbb{Z}$ and $f\in C_T\mathbb{C}$ with $f_0 = 0$

$$\int_0^t e^{-4\pi^2|m|^2(t-s)}\,\mathrm{d}f_s = f_t - 4\pi^2|m|^2\int_0^t e^{-4\pi^2|m|^2(t-s)}f_s\,\mathrm{d}s.$$

It follows that Z_ε is a bounded, linear function of $\hat{B}_\varepsilon$ with values in $C_T\Pi_\varepsilon L_0^2(\mathbb{T})$, and so is continuously Fréchet differentiable. Furthermore, for any $f\in\bar{C}_T\mathbb{C}^{N_\varepsilon}$ with $f_0 = 0$

$$\mathcal{D}_f Z_{\varepsilon;t} = \sum_{m\in N_\varepsilon}\left(f_{m;t} - 4\pi^2|m|^2\int_0^t e^{-4\pi^2|m|^2(t-s)}f_{m;s}\,\mathrm{d}s\right)\ell_\varepsilon(m)e_m,$$

Regarding the approximate solution, $u_{\varepsilon;t}$, the mappings $h\mapsto h^2$ and $h\mapsto\partial_x\rho_h$ are Fréchet differentiable on $\Pi_\varepsilon(L^2(\mathbb{T}))$, so the map

$$\begin{aligned}\Phi_T\colon (z,(f_m)_{m\in N_\varepsilon}) \mapsto -z + e^{\cdot\,\Delta}x_\varepsilon + &\int_0^\cdot e^{(\cdot\,-s)\Delta}\partial_x\hat{\Pi}_\varepsilon\left(z_s^2\partial_x\rho_{z_s}\right)\,\mathrm{d}s \\ &+ \sum_{m\in N_\varepsilon}\int_0^\cdot \ell_\varepsilon(m)e^{-4\pi^2|m|^2(t-s)}\,\mathrm{d}f_{m;s}\,e_m,\end{aligned}$$

is Fréchet differentiable as a mapping $\Phi_T\colon(C_T\Pi_\varepsilon L^2(\mathbb{T}),\bar{C}_T\mathbb{C}^{N_\varepsilon})\to C_T\Pi_\varepsilon L^2(\mathbb{T})$ and is such that $\Phi_T(u_\varepsilon,\hat{B}_\varepsilon) = 0$. Moreover, writing $\mathscr{D}$ for the Fréchet derivative,

$$(\mathscr{D}\Phi_{T_*})(u_\varepsilon,\hat{B}_\varepsilon)(\,\cdot\,,0)\colon C_{T_*}\Pi_\varepsilon L^2(\mathbb{T})\to C_{T_*}\Pi_\varepsilon L^2(\mathbb{T})$$

is a Banach space isomorphism for $T_*(\|u_\varepsilon\|_{C_T\mathcal{C}^\alpha}) > 0$ sufficiently small. Applying the implicit function theorem for Banach spaces, [2, Thm. 2.3], we obtain that $u_\varepsilon(x,\cdot)|_{[0,T_*]}$ is Fréchet differentiable in a neighbourhood of $\hat{B}_\varepsilon$. Patching together a sufficient (but finite) number of intervals of length T_* to cover $[0,T]$, we obtain the first claim.

To derive (6.4.9), recall that the Fréchet and Gateaux derivatives of a Fréchet differentiable map agree. Hence, for any $f \in \bar{C}_T\mathbb{C}^{N_\varepsilon}$, $\mathcal{D}_f u_\varepsilon = \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathscr{S}^\varepsilon(x, \hat{B}_\varepsilon + \lambda f_\varepsilon)|_{\lambda=0}$ from which (6.4.9) follows now from the approximate equation, (6.4.4). We finally show (6.4.10). By the triangle inequality, the properties of $\hat{\Pi}_\varepsilon$, (5.2.19) and applying (5.2.15), for any $t \in [0,T]$,

$$\|\mathcal{D}_f u_{\varepsilon;t}\|_{\mathcal{C}^\alpha} \lesssim \|u_\varepsilon\|^2_{C_{\eta;t}\mathcal{C}^\alpha} \int_0^t s^{-\frac{1}{2}-2\eta} \|\mathcal{D}_f u_{\varepsilon;s}\|_{\mathcal{C}^\alpha}\,\mathrm{d}s + \sup_{s\in[0,t]} |f_s|_{\mathbb{C}^{N_\varepsilon}}.$$

Therefore, by Grönwall's inequality, there exists $C > 0$ such that

$$\|\mathcal{D}_f u_{\varepsilon;t}\|_{\mathcal{C}^\alpha} \lesssim \|f\|_{C_T\mathbb{C}^{N_\varepsilon}} \exp\left(Ct^{\frac{1}{2}-2\eta}\|u_\varepsilon\|^2_{C_{\eta;t}\mathcal{C}^\alpha}\right).$$

Due to the global existence of $u_\varepsilon \in C_{\eta;T}\mathcal{C}^\alpha$ (shown in Theorem 6.4.2 for $\varepsilon \in (0,\varepsilon_0)$ where ε_0 depends on $\|x\|_{\mathcal{C}^{\alpha_0}}, \|Z_\varepsilon\|_{C_T\mathcal{C}^\alpha}$), for a new constant $C(T, \|x\|_{\mathcal{C}^{\alpha_0}}, \|Z\|_{C_T\mathcal{C}^\alpha}) > 0$,

$$\|\mathcal{D}_f u_\varepsilon\|_{C_T\mathcal{C}^\alpha} \le C\|f\|_{C_T\mathbb{C}^{N_\varepsilon}}.$$

□

Regarding the derivative of $u_\varepsilon(x)$ with respect to the initial condition, for $g \in \mathcal{C}^{\alpha_0}$, we set $g_\varepsilon := \hat{\Pi}_\varepsilon g$ and then for any $0 \le s \le t \le T$ we let $J^\varepsilon_{s,t}g$ solve the equation

$$J^\varepsilon_{s,t}g = e^{(t-s)\Delta}g_\varepsilon + \int_s^t e^{(t-r)\Delta}\partial_x\hat{\Pi}_\varepsilon[2u_{\varepsilon;r}\partial_x\rho_{u_{\varepsilon;r}}J^\varepsilon_{s,r}g + u^2_{\varepsilon;r}\partial_x\rho_{J^\varepsilon_{s,r}g}]\,\mathrm{d}r. \qquad (6.4.11)$$

We show below that for any $x \in \mathcal{C}^{\alpha_0}(\mathbb{T})$ and $t \in [0,T]$, $J^\varepsilon_{0,t}g = D_g u_\varepsilon(x)$, the directional derivative of $u_\varepsilon(x)$ in x. Note that J^ε satisfies the flow property, that is for $0 \leq s \leq t \leq T$ one has $J^\varepsilon_{0,t} = J^\varepsilon_{s,t} J^\varepsilon_{0,s}$. In particular $J^\varepsilon_{s,s} = \mathrm{Id}$.

Lemma 6.4.6. *There exists an open neighbourhood $\mathcal{O}_{x_\varepsilon} \subset \Pi_\varepsilon(L^2(\mathbb{T}))$ containing x_ε such that $u_\varepsilon(\,\cdot\,, \hat{B}_\varepsilon)$ is Fréchet differentiable as a mapping from $\mathcal{O}_{x_\varepsilon}$ to $C_T\Pi_\varepsilon(L^2(\mathbb{T}))$. For any $g \in \mathcal{C}^{\alpha_0}$, the derivative is given by $D_g u_{\varepsilon;t}(x) = J^\varepsilon_{0,t} g$. Furthermore, setting $\mathfrak{R} = \|Z\|_{C_T\mathcal{C}^\alpha} + \|x\|_{\mathcal{C}^{\alpha_0}}$, there exists $C(\mathfrak{R}) > 0$ and $T_*(\mathfrak{R}) > 0$ such that for all $t \in (0, T_*]$*

$$\sup_{s\in[0,t]} s^\eta \|J^\varepsilon_{0,s} g\|_{\mathcal{C}^\alpha} \leq C\|g\|_{\mathcal{C}^{\alpha_0}}. \tag{6.4.12}$$

Proof. The same argument as in the proof of Lemma 6.4.5 shows that the map $\Pi_\varepsilon L^2_{\mathfrak{m}}(\mathbb{T}) \ni x_\varepsilon \mapsto u_\varepsilon(x) \in C_T\Pi_\varepsilon(L^2_{\mathfrak{m}}(\mathbb{T}))$ is Fréchet differentiable in a neighbourhood of x_ε. It is then readily checked that on $\mathcal{O}_{x_\varepsilon}$, for any $g \in \mathcal{C}^{\alpha_0}$ the Fréchet derivative is equal to the map $t \mapsto J^\varepsilon_{0,t} g$.

To prove (6.4.12), observe that

$$\|J^\varepsilon_{0,s} g\|_{\mathcal{C}^\alpha} \lesssim s^{-\frac{\alpha-\alpha_0}{2}} \|g\|_{\mathcal{C}^{\alpha_0}} + \|u_\varepsilon\|^2_{C_{\eta;s}\mathcal{C}^\alpha} \int_0^s (s-r)^{-\frac{1}{2}} r^{-2\eta} \|J^\varepsilon_{0,r} g\|_{\mathcal{C}^\alpha} \,\mathrm{d}r.$$

Therefore, for any $t \in (0, T]$,

$$\sup_{s\in[0,t]} s^\eta \|J^\varepsilon_{0,s} g\|_{\mathcal{C}^\alpha} \lesssim t^{\eta-\frac{\alpha-\alpha_0}{2}} \|g\|_{\mathcal{C}^{\alpha_0}} + t^{\frac{1}{2}-\eta} \|u_\varepsilon\|^2_{C_{\eta;t}\mathcal{C}^\alpha} \sup_{s\in[0,t]} s^\eta \|J^\varepsilon_{0,s} g\|_{\mathcal{C}^\alpha}.$$

Since $\|u_\varepsilon - u\|_{C_{\eta;T}\mathcal{C}^\alpha} \leq 1$, by Theorem 6.1.4 there exists a $T_*(\mathfrak{R}) \in (0,1)$ such that $\|u_\varepsilon\|_{C_{\eta;T_*}\mathcal{C}^\alpha} \leq 2$. Hence, for all $t \in (0, T_*]$,

$$\sup_{s\in[0,t]} s^\eta \|J^\varepsilon_{0,s} g\|_{\mathcal{C}^\alpha} \lesssim t^{\eta-\frac{\alpha-\alpha_0}{2}} \|g\|_{\mathcal{C}^{\alpha_0}} + t^{\frac{1}{2}-\eta} \sup_{s\in[0,t]} s^\eta \|J^\varepsilon_{0,s} g\|_{\mathcal{C}^\alpha},$$

so then choosing a sufficiently small time $t_1(\mathfrak{R}) \in (0, t]$,

$$\sup_{s\in[0,t_1]} s^\eta \|J^\varepsilon_{0,s} g\|_{\mathcal{C}^\alpha} \lesssim t_1^{\eta-\frac{\alpha-\alpha_0}{2}} \|g\|_{\mathcal{C}^{\alpha_0}}.$$

Repeating this procedure, we find a constant $C := C(\mathfrak{R}) > 0$ such that

$$\sup_{s\in[0,t]} s^{\eta}\|J^{\varepsilon}_{0,s}g\|_{\mathcal{C}^{\alpha}} \leq C\|g\|_{\mathcal{C}^{\alpha_0}}, \quad \forall\, t \in (0,T_*].$$

□

We finally reintroduce probability and consider in the remainder of the section a finite dimensional approximation $B_{\varepsilon;t}$ of the white noise defined by

$$B_{\varepsilon;t} := \sum_{m\in N_\varepsilon} e_m \hat{B}_{t;m}, \quad \hat{B}_{m;t} := \xi(\mathbb{1}_{[0,t]} \times e_m), \qquad \text{for } m \in N_\varepsilon.$$

Note that $(\hat{B}_m)_{m\in N_\varepsilon}$ is a family of complex valued Brownian motions which satisfy the reality condition (6.4.8). Our approximation of the SHE corresponding to (6.4.7) is then

$$v_{\varepsilon;s,t} := \hat{\Pi}_\varepsilon v_{s,t} = \sum_{m\in N_\varepsilon} \int_s^t \ell_\varepsilon(m) e^{-4\pi^2|m|^2(t-r)}\,\mathrm{d}\hat{B}_{m;r}\, e_m, \tag{6.4.13}$$

By Remark 6.4.4, since $\hat{B}_\varepsilon \in \bar{C}_T\mathbb{C}^{N_\varepsilon}$, $(s,t) \mapsto v_{\varepsilon;s,t}$ is well defined pathwise. Furthermore, by Property ii), there exists a $\mathbb{P}$-null set $\mathcal{N} \subset \Omega$ such that, fixing any realisation $\xi(\omega)$ for $\omega \in \Omega \setminus \mathcal{N}$ gives realisations of $v_\varepsilon(\omega)$, $v(\omega)$ and for which $v_{\varepsilon;0,\cdot}(\omega) \to v_{0,\cdot}(\omega)$ in $\mathcal{C}^{\kappa}_T\mathcal{C}^{\alpha-2\kappa}$ for every $\kappa \in [0,1/2)$. In the rest of the subsection, we will let v_ε denote the random path $t \mapsto v_{\varepsilon;0,t}$.
For each $\varepsilon > 0$, the Cameron–Martin space of $\hat{B}_\varepsilon$ is

$$\mathcal{CM}^{\varepsilon} = \bar{W}^{1,2}_0\left([0,T];\mathbb{C}^{N_\varepsilon}\right) := \Big\{ f \in L^2([0,T];\,\mathbb{C}^{N_\varepsilon}) \,:\, \partial_t f \in L^2\left([0,T];\,\mathbb{C}^{N_\varepsilon}\right), \\ f_{-m} = \bar{f}_m,\ f|_{t=0} = 0 \Big\}.$$

By the Sobolev embedding, $W^{1,2}(\mathbb{R}) \hookrightarrow \mathcal{C}^{1/2}(\mathbb{R})$, we see $\mathcal{CM}^{\varepsilon} \subset \bar{C}_T\mathbb{C}^{N_\varepsilon}$. Therefore, Lemma 6.4.5 applies with $f \in \mathcal{CM}^{\varepsilon}$. We also choose a smooth, compactly supported, cut-off function $\chi : \mathbb{R}_+ \to [0,1]$ such that $\chi(z) = 1$ for $z < \frac{1}{2}$ and $\chi(z) = 0$ for $z \geq 1$.

We introduce the notion of right sided derivatives, $\mathcal{D}^+$, of $\|Z\|_{C_T\mathcal{C}^\alpha}$, which, for $f \in \bar{C}_T\mathcal{C}^\alpha$, is defined by

$$\mathcal{D}_f^+\|Z\|_{C_T\mathcal{C}^\alpha} := \lim_{\lambda\searrow 0}\frac{\|Z+\lambda f\|_{C_T\mathcal{C}^\alpha} - \|Z\|_{C_T\mathcal{C}^\alpha}}{\lambda}.$$

Recall that in this section we set the interaction parameter in the equation to be 1 and therefore we hope there should be no confusion with writing χ for the cut-off function.

We introduce the notation $\mathcal{C}_b^1(\mathcal{C}^{\alpha_0})$ for the continuous, bounded, Borel measurable maps $\mathcal{C}^{\alpha_0} \to \mathbb{R}$ whose first Fréchet derivative are also bounded and continuous.

Theorem 6.4.7 (Bismut–Ellworthy–Li Formula)**.** *Let $T > 0$, $x \in \mathcal{C}_{\mathrm{m}}^{\alpha_0}$ and $\Phi \in \mathcal{C}_b^1(\mathcal{C}^{\alpha_0})$. Then for any $f^\varepsilon \in \mathcal{CM}^\varepsilon$ with $\partial_t f^\varepsilon$ an adapted process such that $\|\partial_t f^\varepsilon\|_{L^2([0,T];\mathbb{R}^{N_\varepsilon+1})} \leq C$, $\mathbb{P}$-a.s. for a deterministic constant $C := C(T)$ and $t \in (0,T]$, we have the identity*

$$\begin{aligned}
&\mathbb{E}[\mathcal{D}_{f^\varepsilon}\Phi(u_{\varepsilon;t}(x))\chi(\|v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha})] \\
&\qquad = \mathbb{E}\left[\Phi(u_{\varepsilon;t}(x))\chi(\|v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha})\int_0^t \partial_t f_s^\varepsilon \cdot \mathrm{d}\hat{W}_{\varepsilon;s}\right] \qquad (6.4.14)\\
&\qquad\quad - \mathbb{E}\left[\Phi(u_{\varepsilon;t}(x))\chi'(\|v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha})\mathcal{D}_{f^\varepsilon}^+\|v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha}\right],
\end{aligned}$$

where $\mathcal{D}_{f^\varepsilon}^+(\|Z\|_{C_T\mathcal{C}^\alpha})$ denotes the right sided derivative given explicitly by the expression

$$\mathcal{D}_{f^\varepsilon}^+\chi(\|Z\|_{C_T\mathcal{C}^\alpha}) := \lim_{\delta\searrow 0}\frac{\|Z+\delta f^\varepsilon\|_{C_T\mathcal{C}^\alpha} - \|Z\|_{C_T\mathcal{C}^\alpha}}{\delta}.$$

Proof. For any $f^\varepsilon \in \mathcal{CM}^\varepsilon$, and $\lambda > 0$ we recall we defined the shift operator, $\mathcal{T}_{f^\varepsilon}^\lambda \hat{W}^\varepsilon := \hat{W}^\varepsilon + \lambda f^\varepsilon$, from which we also define

$$\begin{aligned}
u_{\varepsilon;t}^{\lambda;f} &:= \mathscr{S}_t^\varepsilon\left(x, \mathcal{T}_{f^\varepsilon}^\lambda \hat{W}^\varepsilon\right),\\
\mathcal{T}_{f^\varepsilon}^\lambda v_{\varepsilon;0,t} &:= \sum_{\substack{|m|<\frac{1}{\varepsilon}\\ m\neq 0}}\int_0^t e^{-4\pi^2|m|^2(t-s)}\,\mathrm{d}\hat{W}_s^m\, e_m + \lambda\int_0^t e^{-4\pi^2|m|^2(t-s)}\partial_t f_s^\varepsilon\,\mathrm{d}s.
\end{aligned}$$

We now look for a measure $\mathbb{P}_\lambda$ such that the law of $\mathcal{T}^\lambda_{f^\varepsilon}\hat{W}^\varepsilon$ under $\mathbb{P}_\lambda$ is equal to the law of $\hat{W}^\varepsilon$ under $\mathbb{P}$. For such a measure, we would have the identity,

$$\frac{d}{d\lambda}\mathbb{E}_\lambda\left[\Phi(u^{\lambda;f^\varepsilon}_{\varepsilon;t}(x))\chi(\|\mathcal{T}^\lambda_{f^\varepsilon}v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha})\right]\bigg|_{\lambda=0+}=0,$$

where $\lambda=0+$ indicates that we only consider the limit $\lambda\searrow 0$. We derive the BEL formula, (6.4.14), from this identity.

To construct the measure $\mathbb{P}_\lambda$, we define $B^\lambda_t:=-\lambda\int_0^t\partial_t f^\varepsilon_s\cdot \mathrm{d}\hat{W}^\varepsilon_s$ and the exponential process,

$$A^\lambda_t:=\exp\left(B^\lambda_t-\frac{\lambda^2}{2}\int_0^t|\partial_t f^\varepsilon_s|^2\,\mathrm{d}s\right).$$

Novikov's condition is satisfied by our assumptions on $f^\varepsilon\in\mathcal{CM}^\varepsilon\cap L^\infty(\Omega;W^{1,2}([0,T];\mathbb{R}^{N_\varepsilon}))$ and so from [100, Ch. 8, Prop. 1.15] we see that A^λ_t is a strictly positive martingale, with expectation 1. We define $\mathbb{P}_\lambda$ by its Radon–Nikodym derivative, setting

$$\frac{\mathrm{d}\mathbb{P}_\lambda}{\mathrm{d}\mathbb{P}}:=A^\lambda_t, \tag{6.4.15}$$

and then we apply Girsanov's theorem, [100, Ch. 4, Th. 1.4], to see that for any $s\le t$, the random variable $\hat{W}^\varepsilon_s-\langle B^\lambda,\hat{W}^\varepsilon\rangle_s$ has the same law under $\mathbb{P}_\lambda$ as $\hat{W}^\varepsilon_s$ does under $\mathbb{P}$. Observing that $\langle B^\lambda,\hat{W}^\varepsilon\rangle_s=-\lambda f^\varepsilon_s$ we have found $\mathbb{P}_\lambda$ satisfying the necessary conditions. Therefore, writing $\mathbb{E}_\lambda[\,\cdot\,]=\mathbb{E}\left[\,\cdot\,A^\lambda_t\right]$ we have that

$$\frac{d}{d\lambda}\mathbb{E}\left[\Phi(u^{\lambda;f^\varepsilon}_{\varepsilon;t}(x))\chi(\|\mathcal{T}^\lambda_{f^\varepsilon}v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha})A^\lambda_t\right]\bigg|_{\lambda=0+}=0.$$

Assuming for now that we can pass the derivative inside the expectation we obtain

$$\begin{aligned}\mathbb{E}\left[\mathcal{D}^+_{f^\varepsilon}\Phi\left(u_{\varepsilon;t}(x)\right)\chi(\|v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha})A^0_t\right]&=-\mathbb{E}\left[\Phi\left(u_{\varepsilon;t}(x)\right)\chi\left(\|v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha}\right)\frac{d}{d\lambda}A^\lambda_t\bigg|_{\lambda=0+}\right]\\&\quad-\mathbb{E}\left[\Phi\left(u_{\varepsilon;t}(x)\right)\frac{d}{d\lambda}\chi\left(\|\mathcal{T}^\lambda_{f^\varepsilon}v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha}\right)\bigg|_{\lambda=0+}A^0_t\right].\end{aligned}$$

The same argument to show equation (5.16) in the proof of [105, Th. 5.5] applies in a simpler form, to show that

$$\frac{d}{d\lambda}\chi\left(\|\mathcal{T}^{\lambda}_{f^\varepsilon}v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha}\right) = \chi'(\|v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha})\mathcal{D}^+_{f^\varepsilon}\|v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha}.$$

Then since $A^0_t = 1$ and $\frac{d}{d\lambda}A^\lambda_t|_{\lambda=0+} = -\int_0^t \partial_s f^\varepsilon_s \cdot \mathrm{d}W^\varepsilon_s$ we obtain the identity

$$\begin{aligned}\mathbb{E}\left[\mathcal{D}^+_{f^\varepsilon}\Phi\left(u_{\varepsilon;t}(x)\right)\chi(\|v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha})\right] &= \mathbb{E}\left[\Phi\left(u_{\varepsilon;t}(x)\right)\chi\left(\|v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha}\right)\int_0^t \partial_t f^\varepsilon_s \cdot \mathrm{d}W^\varepsilon_s\right] \\ &\quad - \mathbb{E}\left[\Phi(u_{\varepsilon;t}(x))\chi'(\|v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha})\mathcal{D}^+_{f^\varepsilon}\|v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha}\right].\end{aligned}$$

Finally, since $\Phi(u_{\varepsilon;t}(x))$ is genuinely Fréchet differentiable the right sided derivative on the left hand side agrees with the full derivative and we obtain (6.4.14).

Demonstrating that we can pass the derivative inside the integral is done by showing equi-integrability of the corresponding difference quotients and follows by the same arguments as in the proof of [105, Thm. 5.5].

□

We now observe a relation between the derivative in the initial data and the derivative in the noise. For $f^\varepsilon \in \mathcal{CM}^\varepsilon$ and $g \in \mathcal{C}^{\alpha_0}$, writing $\mathcal{D}_{f^\varepsilon}u_{\varepsilon;t}$, solving (6.4.9) and $J^\varepsilon_{0,t}g$, solving (6.4.11), in mild form we obtain the identity

$$\mathcal{D}_f u_{\varepsilon;t}(x) = \sum_{|m|<\frac{1}{\varepsilon}}\int_0^t J^\varepsilon_{s,t}\partial_t f^m_s e_m \,\mathrm{d}s.$$

Therefore, choosing $f^\varepsilon \in \mathcal{CM}^\varepsilon$, such that, for each $|m|< \frac{1}{\varepsilon}$, $\partial_t f^m_s = \frac{1}{t}\langle J^\varepsilon_{0,s}g, e_m\rangle$ we see that

$$\mathcal{D}_{f^\varepsilon}u_{\varepsilon;t}(x) = J^\varepsilon_{0,t}g = D_g u_{\varepsilon;t}(x). \tag{6.4.16}$$

From Theorem 6.4.6 we see that this choice of f^ε is valid in $\mathcal{CM}^\varepsilon$.

Theorem 6.4.8. *Let $T > 1$, x, $y \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}}$ with $y \in B_1(x)$ and $t \in (0, T_*(\mathfrak{R})]$ with $\mathfrak{R} := 1 + 2\|x\|_{\mathcal{C}^{\alpha_0}}$. Then there exists a constant $C := C(\mathfrak{R}, \alpha, \alpha_0, \eta) > 0$ and* *exponent* $\theta := \theta(\eta) > 0$ *such that for any* $\Phi \in \mathcal{C}^1_b(\mathcal{C}^\alpha_{\mathfrak{m}})$,

$$|P_t\Phi(x) - P_t\Phi(y)| \leq C\frac{1}{t^\theta}\|\Phi\|_{L^\infty}\|x-y\|_{\mathcal{C}^{\alpha_0}} + 2\|\Phi\|_{L^\infty}\mathbb{P}(\|v_{0,\cdot}\|_{C_t\mathcal{C}^\alpha} > 1). \tag{6.4.17}$$

Proof. We define the semi-group for the approximate equation

$$P^\varepsilon_t\Phi(x) := \mathbb{E}\left[\Phi(u_{\varepsilon;t}(x))\right] \quad \forall\, \Phi \in \mathcal{B}_b(\mathcal{C}^{\alpha_0}_{\mathfrak{m}}), \quad t > 0.$$

By definition we have

$$|P^\varepsilon_t\Phi(x) - P^\varepsilon_t\Phi(y)| = \left|\mathbb{E}\left[\Phi(u_{\varepsilon;t}(x)) - \Phi(u_{\varepsilon;t}(y))\right]\right|.$$

Then using the triangle inequality we have,

$$\begin{aligned} |P^\varepsilon_t\Phi(x) - P^\varepsilon_t\Phi(y)| &\leq \left|\mathbb{E}\left[(\Phi(u_{\varepsilon;t}(x)) - \Phi(u_{\varepsilon;t}(y)))\chi(\|v_{\varepsilon;0,\cdot}\|_{C_t\mathcal{C}^\alpha})\right]\right| \\ &\quad + \left|\mathbb{E}\left[(\Phi(u_{\varepsilon;t}(x)) - \Phi(u_{\varepsilon;t}(x)))\left(1 - \chi(\|v_{\varepsilon;0,\cdot}\|_{C_t\mathcal{C}^\alpha})\right)\right]\right| \\ &=: I_1(t) + I_2(t). \end{aligned} \tag{6.4.18}$$

We now bound these terms for $t \in (0, T_*(\mathfrak{R})]$. To bound the second term we apply the triangle inequality to give that

$$I_2(t) \leq 2\|\Phi\|_{L^\infty}\mathbb{E}\left[(1 - \chi(\|v_{\varepsilon;0,\cdot}\|_{C_t\mathcal{C}^\alpha}))\right] \leq 2\|\Phi\|_{L^\infty}\mathbb{P}\left(\|v_{\varepsilon;0,\cdot}\|_{C_t\mathcal{C}^\alpha} > 1\right).$$

Regarding the first term, working with $\Phi \in \mathcal{C}^1_b(\mathcal{C}^{\alpha_0})$ for now, we use the fundamental theorem of calculus along with Fubini to write,

$$I_1(t) = \left|\int_0^1 \mathbb{E}\left[D_{x-y}\Phi(u_t(x + \lambda(x-y)))\chi(\|v_{\varepsilon;0,\cdot}\|_{C_t\mathcal{C}^\alpha})\right] d\lambda\right|.$$

We let $z_\lambda := x - \lambda(x-y) \in \mathcal{C}^{\alpha_0}$ and choose $f^\varepsilon \in \mathcal{CM}^\varepsilon$ such that

$$\partial_t f^\varepsilon_s = \frac{1}{t}(\langle J^\varepsilon_{0,s}(x-y), e_m\rangle)_{|m|<\frac{1}{\varepsilon}}\mathbb{1}_{s\leq\tau},$$

where $J^{\varepsilon}_{0,s}(x-y) = D_{x-y}u_{\varepsilon;s}(z_\lambda)$ and for notational ease, we have suppressed the dependence on z_λ in $J^{\varepsilon}_{0,s}(x-y)$. Note that we have $\|z_\lambda\|_{\mathcal{C}^{\alpha_0}} \leq \mathfrak{R}$ for all $\lambda \in (0,1)$, so the local bounds of Theorems 6.4.5 and 6.4.6 both hold uniformly in z_λ. So then using (6.4.16) and (6.4.14) we obtain the identity

$$\begin{aligned}
\mathbb{E}\left[\mathcal{D}_{f^\varepsilon}\Phi(u_{\varepsilon;t}(z_\lambda))\chi(\|v_{\varepsilon;0,\cdot}\|_{C_t\mathcal{C}^\alpha})\right] &= \frac{1}{t}\mathbb{E}\left[\Phi(u_{\varepsilon;t}(z_\lambda))\int_0^t \langle J^{\varepsilon}_{0,s}(x-y)_\varepsilon, \mathrm{d}W_{\varepsilon;s}\rangle\, \chi(\|v_{\varepsilon;0,\cdot}\|_{C_t\mathcal{C}^\alpha})\right] \\
&\quad - \frac{1}{t}\mathbb{E}\left[\Phi(u_{\varepsilon;t}(z_\lambda))\,\chi'(\|v_{\varepsilon;0,\cdot}\|_{C_t\mathcal{C}^\alpha})\mathcal{D}^+_{f^\varepsilon}\|v_{\varepsilon;0,\cdot}\|_{C_t\mathcal{C}^\alpha}\right] \\
&\leq \frac{1}{t}\|\Phi\|_{L^\infty}\Bigg(\mathbb{E}\left[\left|\int_0^t \langle J^{\varepsilon}_{0,s}(x-y)_\varepsilon, \mathrm{d}W_{\varepsilon;s}\rangle\, \chi(\|v_{\varepsilon;0,\cdot}\|_{C_t\mathcal{C}^\alpha})\right|\right] \\
&\qquad + \mathbb{E}\left[\left|\chi'(\|v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha})\mathcal{D}^+_{f^\varepsilon}\|v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha}\right|\right]\Bigg).
\end{aligned}$$

We bound the first term here by Cauchy-Schwarz and then Itô's isometry to give,

$$\begin{aligned}
\mathbb{E}\left[\left|\int_0^t \langle J^{\varepsilon}_{0,s}(x-y)_\varepsilon, \mathrm{d}W_{\varepsilon;s}\rangle\, \chi(\|v_{\varepsilon;0,\cdot}\|_{C_t\mathcal{C}^\alpha})\right|\right] &\leq \left(\mathbb{E}\left[\left(\int_0^{t\wedge\tau} \langle J^{\varepsilon}_{0,s}(x-y)_\varepsilon, \mathrm{d}W_{\varepsilon;s}\rangle\right)^2\right]\right)^{\frac{1}{2}} \\
&\leq \left(\mathbb{E}\left[\int_0^{t\wedge\tau^\varepsilon} \|J^{\varepsilon}_{0,s}(x-y)_\varepsilon\|^2_{L^2}\,\mathrm{d}s\right]\right)^{\frac{1}{2}} \\
&\leq t^{\frac{1}{2}-\eta}\mathbb{E}\left[\sup_{s\in(0,t\wedge\tau^\varepsilon)} s^\eta \|J^{\varepsilon}_{0,s}(x-y)_\varepsilon\|_{\mathcal{C}^\alpha}\right] \\
&\leq Ct^{\frac{1}{2}-\eta}\|x-y\|_{\mathcal{C}^{\alpha_0}},
\end{aligned}$$

where we used that $t \leq t\wedge\tau^\varepsilon$ and Theorem 6.4.6 in the penultimate and last lines. The constant, $C := C(\mathfrak{R}, \alpha, \alpha_0, \eta) > 0$, here is the same as that in (6.4.12). For the second term, since $\|\chi'\|_{L^\infty} < \infty$ and by definition, for any $f^\varepsilon \in C_T\Pi_\varepsilon(L^2(\mathbb{T}))$, we have

$$\mathcal{D}^+_{f^\varepsilon}\|v_{\varepsilon;0,\cdot}\|_{C_t\mathcal{C}^\alpha} := \lim_{\lambda\searrow 0}\frac{\|v_{\varepsilon;0,\cdot}+\lambda f^\varepsilon\|_{C_t\mathcal{C}^\alpha} - \|v_{\varepsilon;0,\cdot}\|_{C_t\mathcal{C}^\alpha}}{\lambda} \leq \|v_{\varepsilon;0,\cdot}\|_{C_t\mathcal{C}^\alpha}\|f^\varepsilon\|_{C_t\mathcal{C}^\alpha}.$$

So from Theorem 6.4.6, for our choice of f^ε we see that there exists a $C :=$

$C(\mathfrak{R}, \alpha, \alpha_0, \eta) > 0$ such that,

$$\mathbb{E}\left[\,\left|\,\chi'(\|v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha})\mathcal{D}^+_{f^\varepsilon}\|v_{0,\cdot}\|_{C_T\mathcal{C}^\alpha}\,\right|\,\right] \leq t^{1-\eta}\sup_{s\in[0,t]} s^\eta \|J^\varepsilon_{0,s}(x-y)\|_{\mathcal{C}^\alpha}|\chi'(\|v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha})|$$
$$\leq Ct^{1-\eta}\|x-y\|_{\mathcal{C}^{\alpha_0}},$$

where in the last line we used that $|\chi'(\|v_{\varepsilon;0,\cdot}\|_{C_T\mathcal{C}^\alpha})|< \infty$ uniformly. Therefore, for all $t \in (0, T_*(\mathfrak{R})]$, there exists a $\theta := \theta(\eta) > 0$ such that

$$I_1(t) \leq Ct^{-\theta}\|\Phi\|_{L^\infty}\|x-y\|_{\mathcal{C}^{\alpha_0}}.$$

So in conclusion, for all $t \in (0, T_*(\mathfrak{R})]$, there exists a constant $C := C(\mathfrak{R}, \alpha, \alpha_0, \eta) > 0$, and a $\theta := \theta(\eta) > 0$ such that

$$|P^\varepsilon_t\Phi(x) - P^\varepsilon_t\Phi(y)|\leq Ct^{-\theta}\|\Phi\|_{L^\infty}\|x-y\|_{\mathcal{C}^{\alpha_0}}+2\|\Phi\|_{L^\infty}\mathbb{P}(\|v_{\varepsilon;0,\cdot}\|_{C_t\mathcal{C}^\alpha}> 1).$$

From Theorem 6.4.2 for $\varepsilon < \varepsilon_0$, we have that $u_\varepsilon \to u \in C_{\eta;T}C^{\alpha_0}$ for any $T > 0$, $\mathbb{P}$ - almost surely and $v_{\varepsilon;0,\cdot}$ converges in law to $v_{0,\cdot}$. So applying the dominated convergence theorem we see that

$$|P_t\Phi(x) - P_t\Phi(y)|\leq t^{-\theta}\|\Phi\|_{L^\infty}\|x-y\|_{\mathcal{C}^{\alpha_0}}+2\|\Phi\|_{L^\infty}\mathbb{P}(\|v_{0,\cdot}\|_{C_t\mathcal{C}^\alpha}> 1),$$

for all $\Phi \in \mathcal{C}^1_b(\mathcal{C}^{\alpha_0})$. □

We now use the stochastic properties of v to obtain a bound on $\mathbb{P}(\|v_{0,\cdot}\|_{C_T\mathcal{C}^\alpha}> 1)$ and then we obtain a local Lipschitz bound for the dual semi-group in the total variation distance.

Theorem 6.4.9. *Let $x, y \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}}$ with $y \in B_1(x)$. Then there exists a $C > 0$, $\theta \in (0,1)$ and $\sigma > 0$ such that, for every $t \geq 1$,*

$$\|P^*_t\delta_x - P^*_t\delta_y\|_{TV} \leq C(1+\|x\|_{\mathcal{C}^{\alpha_0}})^\sigma\|x-y\|^\theta_{\mathcal{C}^{\alpha_0}}.$$

Proof. Recalling the definition of the total variation distance between probability measures, we have that

$$\|P_t^*\delta_x - P_t^*\delta_y\|_{\mathrm{TV}} = \sup_{\|\Phi\|_{L^\infty}\le 1} |\langle P_t^*\delta_x, \Phi\rangle - \langle P_t^*\delta_y, \Phi\rangle| = \sup_{\|\Phi\|_{L^\infty}\le 1} |P_t\Phi(x) - P_t\Phi(y)|.$$

Furthermore it follows from [33, Lem. 7.1.5] that (6.4.17) is equivalent to the statement,

$$\|P_t^*\delta_x - P_t^*\delta_{\tilde{x}}\|_{\mathrm{TV}} \le C\frac{1}{t^\theta}\|x-\tilde{x}\|_{\alpha_0} + 2\mathbb{P}(\|v\|_{C_t\mathcal{C}^\alpha} > 1).$$

From Markov's inequality and Theorem 5.4.8, we see that

$$\mathbb{P}(\|v\|_{C_t\mathcal{C}^\alpha} > 1) \lesssim t^{\theta_2},$$

for some $\theta_2 \in (0,1)$. Inserting this into (6.4.17) we obtain for any $t \in (0, T_*(\mathfrak{R})]$ and some $\theta_1 > 0$ that

$$\sup_{\|\Phi\|_{L^\infty}\le 1} |P_t\Phi(x) - P_t\Phi(y)| \lesssim t^{-\theta_1}\|x-y\|_{\mathcal{C}^{\alpha_0}} + t^{\theta_2}. \tag{6.4.19}$$

We note for $t, s > 0$

$$\sup_{\|\Phi\|\le 1} |P_{t+s}\Phi(x) - P_{t+s}\Phi(y)| \le \sup_{\|\Phi\|\le 1} |P_t\Phi(x) - P_t\Phi(y)|. \tag{6.4.20}$$

That is the total variation distance can only decrease in time. We let $\mathfrak{R} := 1 + 2\|x\|_{\mathcal{C}^{\alpha_0}}$ and define $T_*(\mathfrak{R}) \in (0,1)$ by (6.1.7) in Theorem 6.1.4. So then using (6.4.20) we see that we may restrict our attention to obtaining the result at $T_*(\mathfrak{R})$. From (6.4.19) we have

$$\sup_{\|\Phi\|_{L^\infty}\le 1} |P_{T_*(\mathfrak{R})}\Phi(x) - P_{T_*(\mathfrak{R})}\Phi(y)| \lesssim \inf_{t\in(0,T_*(\mathfrak{R})]} \left\{t^{-\theta_1}\|x-y\|_{\mathcal{C}^{\alpha_0}} + t^{\theta_2}\right\}. \tag{6.4.21}$$

The function $t \mapsto t^{-\theta_1}\|x-y\|_{\mathcal{C}^{\alpha_0}} + \frac{1}{r}t^{\theta_2}$ attains its minimum at,

$$t_0 := \left(\frac{\theta_1}{\theta_2} r \|x-y\|_{\mathcal{C}^{\alpha_0}}\right)^{\frac{1}{\theta_1+\theta_2}},$$

and is decreasing on $[0, t_0]$. Then it is either the case that $t_0 \leq T_*(\mathfrak{R})$ or $t_0 > T_*(\mathfrak{R})$. In the former case the bound (6.4.19) applies at t_0 and so combining this with (6.4.21) we obtain,

$$\begin{aligned}
\sup_{\|\Phi\|_{L^\infty}\leq 1} |P_{T_*(\mathfrak{R})}\Phi(x) - P_{T_*(\mathfrak{R})}\Phi(y)| &\lesssim t_0^{-\theta_1}\|x-y\|_{\mathcal{C}^{\alpha}} + t_0^{\theta_2} \\
&= \left(\frac{\theta_1}{\theta_2}\|x-y\|_{\mathcal{C}^{\alpha_0}}\right)^{\frac{-\theta_1}{\theta_1+\theta_2}} \|x-y\|_{\mathcal{C}^{\alpha_0}} \\
&\quad + \left(\frac{\theta_1}{\theta_2}\|x-y\|_{\mathcal{C}^{\alpha_0}}\right)^{\frac{\theta_2}{\theta_1+\theta_2}} \\
&= C\|x-y\|_{\mathcal{C}^{\alpha_0}}^{\frac{\theta_2}{\theta_1+\theta_2}}.
\end{aligned}$$

Therefore, we have

$$\sup_{\|\Phi\|_{L^\infty}\leq 1} |P_{T_*(\mathfrak{R})}\Phi(x) - P_{T_*(\mathfrak{R})}\Phi(y)| \lesssim \|x-y\|_{\mathcal{C}^{\alpha_0}}^{\frac{\theta_2}{\theta_1+\theta_2}}.$$

On the other hand, if $T_*(\mathfrak{R}) < t_0$ then, from (6.1.7), there exists a $\theta_3 > 0$, such that

$$\begin{aligned}
\sup_{\|\Phi\|_{L^\infty}\leq 1} |P_{T_*(\mathfrak{R})}\Phi(x) - P_{T_*(\mathfrak{R})}\Phi(y)| &\lesssim T_*(\mathfrak{R})^{-\theta_1}\|x-y\|_{\mathcal{C}^{\alpha_0}} + \frac{1}{r}T_*^{\theta_2} \\
&\leq T_*(\mathfrak{R})^{-\theta_1}\|x-y\|_{\mathcal{C}^{\alpha_0}} + t_0^{\theta_2} \\
&= \left(\frac{1}{C(1+\mathfrak{R})}\right)^{-\frac{\theta_1}{\theta_3}} \|x-y\|_{\mathcal{C}^{\alpha_0}} \\
&\quad + C\|x-y\|_{\mathcal{C}^{\alpha_0}}^{\frac{\theta_2}{\theta_1+\theta_2}},
\end{aligned}$$

where in the last line we used the explicit expressions for t_0 and $T_*(\mathfrak{R})$. So since we chose $\mathfrak{R} = 1 + 2\|x\|_{\mathcal{C}^{\alpha_0}}$, we have, for a new $\theta > 0$, and a $\sigma > 0$, that

$$\sup_{\|\Phi\|_{L^\infty} \leq 1} |P_{T_*(\mathfrak{R})}\Phi(x) - P_{T_*(\mathfrak{R})}\Phi(y)| \leq C(\mathfrak{R}, r, \theta, \sigma)(1 + \|x\|_{\mathcal{C}^{\alpha_0}})^{\sigma} \|x - y\|^{\theta}_{\mathcal{C}^{\alpha_0}}.$$

□

6.4.2 Full Support

We demonstrate that $u_T(x)$, and thus any invariant measure ν_x for P_t as in Theorem 5.3.6, has full support in $\mathcal{C}^{1/2-\delta}_{\bar{x}}(\mathbb{T})$ for any $\delta \in (0, 1/2)$. In this subsection we are not concerned with the behaviour of the solution near zero, so until the start of Section 6.4.3 we just consider $\alpha \in (0, 1/2)$ and $\mathfrak{m} \in \mathbb{R}$.

Let $L^2_0(\mathbb{R}_+ \times \mathbb{T})$ be the space of square integrable functions on $\mathbb{R}_+ \times \mathbb{T}$ such that for any $t \geq 0$, $\bar{f}_t = 0$. Then for any $T \geq 0$ we define

$$\mathscr{H}_T := \left\{ h \colon [0, T] \times \mathbb{T} \to \mathbb{R} \,:\, h_t = \int_0^t e^{(t-s)\Delta} f_s \,\mathrm{d}s,\ f \in L^2_0(\mathbb{R}_+ \times \mathbb{T}) \right\},$$

Note that by Theorems 5.2.2 and 5.2.10, $\mathscr{H}_T$ is continuously and densely embedded in $\{h \in C_T\mathcal{C}^{\alpha}_0 \,:\, h(0) = 0\}$. $\mathscr{H}_T$ is the Cameron–Martin space of $v := v_{0,\cdot}$ and the following is a direct consequence of the Cameron–Martin theorem.

Lemma 6.4.10. *Let $T > 0$ and $\mathcal{L}(v) = (v)\#\mathbb{P} \in \mathcal{P}(C_T\mathcal{C}^{\alpha})$ be the law of v. Then $supp(\mathcal{L}(v)) = \overline{\mathscr{H}_T}^{\|\cdot\|_{C_T\mathcal{C}^{\alpha}}}$.*

Proof. See [17, Thm. 3.6.1]. □

In the following theorem, we treat $\mathcal{L}(u_T(x))$ as a probability measures on $\mathcal{C}^{\alpha}_{\mathfrak{m}}(\mathbb{T})$.

Theorem 6.4.11. *Let $T > 0$, $x \in \mathcal{C}^{\alpha}_{\mathfrak{m}}(\mathbb{T})$. Then*

$$supp(\mathcal{L}(u_T(x))) = \mathcal{C}^{\alpha}_{\mathfrak{m}}(\mathbb{T}).$$

Proof. We first show

$$\overline{\{\mathscr{S}_T(x,h) \,:\, h \in \mathscr{H}_T\}}^{\|\cdot\|_{\mathcal{C}^\alpha}} \subseteq \operatorname{supp}(\mathcal{L}(u_T(x))). \tag{6.4.22}$$

Recall that the map $\mathscr{S}_T(x,\,\cdot\,)\colon C_T\mathcal{C}_0^\alpha(\mathbb{T}) \to \mathcal{C}_{\mathfrak{m}}^\alpha$ is continuous and $\mathscr{H}_T \subset C_T\mathcal{C}_0^\alpha(\mathbb{T})$. Consider now $h \in \mathscr{H}_T$. Then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|v - h\|_{C_T\mathcal{C}_0^\alpha} < \delta \;\Rightarrow\; \|u_T - \mathscr{S}_T(x,h)\|_{\mathcal{C}^\alpha} < \varepsilon,$$

and therefore

$$\mathbb{P}(\|u_T - \mathscr{S}_T(x,h)\|_{\mathcal{C}^\alpha} < \varepsilon) \geq \mathbb{P}(\|v - h\|_{C_T\mathcal{C}_0^\alpha} < \delta) > 0,$$

where the last inequality follows from Lemma 6.4.10 and this shows (6.4.22). It now suffices to show that,

$$\overline{\{\mathscr{S}_T(x,h) \,:\, h \in \mathscr{H}_T\}}^{\|\cdot\|_{\mathcal{C}^\alpha}} = \mathcal{C}_{\mathfrak{m}}^\alpha(\mathbb{T}).$$

Let $y \in C_{\mathfrak{m}}^\infty(\mathbb{T})$. Then for $t \in [0,T]$ define

$$u_t^y := e^{t\Delta}x + \frac{t}{T}(y - e^{T\Delta}x), \tag{6.4.23}$$

and

$$h_t^y := -\int_0^t e^{(t-s)\Delta}\partial_x((u_s^y)^2\partial_x\rho_{u_s^y})\,\mathrm{d}s + \frac{t}{T}(y - e^{T\Delta}x).$$

Since $x \in \mathcal{C}_{\mathfrak{m}}^\alpha$, we have $u^y \in C_T\mathcal{C}_{\mathfrak{m}}^\alpha$ and it also follows that $h^y \in C_T\mathcal{C}_0^\alpha$ with $h^y(0) = 0$. Furthermore, by construction,

$$\mathscr{S}_T(x,h^y) = u_T^y = y. \tag{6.4.24}$$

Approximating h^y by functions in $C_0^\infty([0,T]\times\mathbb{T}) \cap \mathscr{H}_T$ and using the density of $C^\infty(\mathbb{T})$ in $\mathcal{C}^\alpha(\mathbb{T})$ concludes the proof. □

6.4.3 Exponential Mixing

In Theorem 6.4.12 and Corollary 6.4.13 below we keep α_0, α satisfying (6.4.1) - the more restrictive parameter set from the start of Section 6.4.

Theorem 6.4.12. *There exists a $\lambda \in (0,1)$ such that for every x, $y \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}}$ and $t \geq 3$,*

$$\|P_t^*\delta_x - P_t^*\delta_y\|_{TV} \leq 1 - \lambda. \tag{6.4.25}$$

Proof. We let $R > 0$ and define a family of compact subsets of $\mathcal{C}^{\alpha_0}_{\mathfrak{m}}$, by setting

$$A_R := \{x \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}} \ : \ \|x\|_{\mathcal{C}^\alpha} \leq R\} . \tag{6.4.26}$$

Then, from Theorem 6.4.9, for every $a \in (0,1)$, there exists an $r := r(a) > 0$ such that for every $x, y \in \bar{B}_r(0) \subset \mathcal{C}^{\alpha_0}_{\mathfrak{m}}$ and $t \geq 1$

$$\|P_t^*\delta_x - P_t^*\delta_y\|_{\mathrm{TV}} \leq 1 - a.$$

Using Theorem 6.4.11, for any $x \in \mathcal{C}^{\alpha_0}$, we have that

$$P_1 \mathbb{1}_{\bar{B}_r(0)}(x) > 0. \tag{6.4.27}$$

From the strong Feller property of P_t, implied by Theorem 6.4.9, we have that $x \mapsto P_1 \mathbb{1}_{\bar{B}_r(0)}(x)$ is a continuous function, and from (6.4.27), bounded below by 0. Therefore, since A_R are compact, by the extreme value theorem, there exists a $b := b(R) > 0$ such that

$$\inf_{x \in A_R} P_1 \mathbb{1}_{\bar{B}_r(0)}(x) \geq b. \tag{6.4.28}$$

Then for $t \geq 0$ and x, $y \in A_R \setminus \bar{B}_r(0)$ we define the coupling measure

$$P_t^{x,y}(F \times G) = P_t(x; F) P_t(x; G), \quad \forall\, F,\, G \text{ borel in } \mathcal{C}^{\alpha_0}_{\mathfrak{m}}. \tag{6.4.29}$$

For a pair of probability measure μ^1, $\mu^2 \in \mathcal{P}(E)$, we recall that for a coupling measure $\pi \in \Pi(\mu^1, \mu^2)$, we have the identity

$$\iint_{E\times E} (\varphi(x) + \psi(y))\, \pi(\mathrm{d}x, \mathrm{d}y) = \int_E \int_E (\varphi(x) + \psi(y))\, \mu^1(\mathrm{d}x)\mu^2(\mathrm{d}y), \tag{6.4.30}$$

for any Borel measurable functions $\varphi, \psi : E \to \mathbb{R}$. Therefore, using the Markov property, for x, $y \in A_R$, $t \geq 2$ and $\Phi \in \mathcal{C}_b(\mathcal{C}^{\alpha_0}_{\mathfrak{m}})$, along with (6.4.30), we have that

$$\begin{aligned} |P_t\Phi(x) - P_t\Phi(y)| &= |\mathbb{E}\left[P_{t-1}\Phi(u_1(x)) - P_{t-1}\Phi(u_1(y))\right]| \\ &= \left| \iint \left(P_{t-1}\Phi(\tilde{x}) - P_{t-1}\Phi(\tilde{y})\right) P_1^{x,y}(\mathrm{d}\tilde{x}, \mathrm{d}\tilde{y}) \right|. \end{aligned}$$

Therefore, for $t \geq 2$, and any $x, y \in A_R \setminus \bar{B}_r(0)$,

$$\begin{aligned} \|P_t^*\delta_x - P_t^*\delta_y\|_{\mathrm{TV}} &\leq \|P_{t-1}^*\delta_x - P_{t-1}^*\delta_y\|_{\mathrm{TV}}\, P_1^{x,y}((\bar{B}_r(0) \times \bar{B}_r(0))^c) \\ &\quad + \|P_{t-1}^*\delta_x - P_{t-1}^*\delta_y\|_{\mathrm{TV}}\, P_1^{x,y}(\bar{B}_r(0) \times \bar{B}_r(0)) \\ &\leq P_1^{x,y}((\bar{B}_r(0) \times \bar{B}_r(0))^c) + (1-a)P_1^{x,y}(\bar{B}_r(0) \times \bar{B}_r(0)) \\ &\leq 1 - ab^2. \end{aligned}$$

From (6.3.7) and Markov's inequality, there exists an $R > 0$ such that

$$\inf_{x \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}}} \inf_{t \geq 1} \mathbb{P}\left[\|u_t(x)\|_{\mathcal{C}^\alpha} \leq R\right] > \frac{1}{2}. \tag{6.4.31}$$

So then we have, for $t \geq 3$,

$$\begin{aligned} \|P_t^*\delta_x - P_t^*\delta_y\|_{\mathrm{TV}} &\leq \|P_{t-1}^*\delta_x - P_{t-1}^*\delta_y\|_{\mathrm{TV}} P_1^{x,y}\left(\left(A_R \setminus \bar{B}_r(0) \times A_R \setminus \bar{B}_r(0)\right)^c\right) \\ &\quad + \|P_{t-1}^*\delta_x - P_{t-1}^*\delta_y\|_{\mathrm{TV}} P_1^{x,y}(A_R \setminus \bar{B}_r(0) \times A_R \setminus \bar{B}_r(0)) \\ &\leq P_1^{x,y}\left(\left(A_R \setminus \bar{B}_r(0) \times A_R \setminus \bar{B}_r(0)\right)^c\right) \\ &\quad + (1-ab^2)P_1^{x,y}(A_R \setminus \bar{B}_r(0) \times A_R \setminus \bar{B}_r(0)) \\ &\leq 1 - \frac{ab^2}{4}. \end{aligned}$$

Therefore, for any $x,\, y \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}}$ and $t \geq 3$ we obtain

$$\|P_t^*\delta_x - P_t^*\delta_y\|_{\mathrm{TV}} \leq 1 - \frac{ab^2}{4}, \tag{6.4.32}$$

which proves (6.4.25) with $\lambda = \frac{ab^2}{4}$ and where we can choose $a \in (0,1)$ sufficiently small so that $\lambda \in (0,1)$. □

We remark that $t \geq 3$ here is arbitrary, but convenient since we work with a local time of existence $T_* \leq 1$. We now obtain exponential ergodicity of the semi-group, in analogue with [105, Cor. 6.6].

Corollary 6.4.13. *There exists a unique measure $\nu \in \mathcal{P}_1(\mathcal{C}^{\alpha_0}_{|x|})$, the semi-group $(P_t)_{t\geq 0}$ associated to (6.0.1) leaves ν invariant. Furthermore, supp$(\nu) = \mathcal{C}^{1/2-\delta}_m(\mathbb{T}^d)$ for any $\delta \in (0,1/2)$ and there exists a $\lambda > 0$ such that for every $\mu \in \mathcal{P}(\mathcal{C}^{\alpha_0}_{\mathfrak{m}})$ and $t \geq 3$*

$$\|P_t^*\mu - \nu\|_{TV} \leq (1-\lambda)^{\lfloor \frac{t}{3} \rfloor}\|\mu - \nu\|_{TV}. \tag{6.4.33}$$

Proof. Firstly, for any two $\mu^1,\, \mu^2 \in \mathcal{P}_1(\mathcal{C}^{\alpha_0}_{\mathfrak{m}})$ and $t > 0$, by definition we have

$$\begin{aligned}
\|P_t^*\mu^1 - P_t^*\mu^1\|_{\mathrm{TV}} &= \frac{1}{2}\sup_{\|\Phi\|_{L^\infty}\leq 1}\left|\iint_{(\mathcal{C}^{\alpha_0}_{\mathfrak{m}})^2} P_t\Phi(x) - P_t\Phi(y)\,\mu^1(\mathrm{d}x)\mu^2(\mathrm{d}y)\right| \\
&\leq \frac{1}{2}\sup_{\|\Phi\|_{L^\infty}\leq 1}\iint_{(\mathcal{C}^{\alpha_0}_{\mathfrak{m}})^2} |P_t\Phi(x) - P_t\Phi(y)|\;\pi(\mathrm{d}x,\mathrm{d}y) \\
&\leq \|P_t^*\delta_x - P_t^*\delta_y\|_{\mathrm{TV}}\left(1 - \pi\left(\{(x,x)\,:\, x \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}}\}\right)\right)
\end{aligned}$$

for any $\pi \in \Pi(\mu^1,\mu^2)$. So then from (6.4.25) we have, for $t \geq 3$,

$$\|P_t^*\mu^1 - P_t^*\mu^1\|_{\mathrm{TV}} \leq\; (1-\lambda)(1 - \pi\left(\{(x,x)\,:\, x \in \mathcal{C}^{\alpha_0}_{\mathfrak{m}}\}\right)).$$

So then from (5.3.2) we have

$$\|P_t^*\mu^1 - P_t^*\mu^1\|_{\mathrm{TV}} \leq (1-\lambda)\|\mu^1 - \mu^2\|_{\mathrm{TV}}, \tag{6.4.34}$$

for all $t \geq 3$. This ensures that we have a unique invariant measure, since any distinct invariant measures of P_t must be mutually singular ([33, Prop. 3.2.5]). From Theorem 6.4.11 we have that $\text{supp}(\nu) = \mathcal{C}_{\mathfrak{m}}^{1/2-\delta}$ for any $\delta \in (0, 12)$. So then letting $\nu \in \mathcal{P}(\mathcal{C}_{\mathfrak{m}}^{1/2-\delta})$ be the unique invariant measure, we have, for any $t > 3$ and $\mu \in \mathcal{P}\left(\mathcal{C}_{\mathfrak{m}}^{\alpha_0}\right)$,

$$\|P_t^*\mu - \nu\|_{\mathrm{TV}} \leq (1-\lambda)\|P_{t-3}^*\mu - \nu\|_{\mathrm{TV}}, \tag{6.4.35}$$

from which (6.4.33) follows. □

We now complete the proof of Theorem 6.0.2.

Proof of Theorem 6.0.2. Let α_0, α, η satisfy the criteria of (6.0.2) - the larger admissible set of parameters, and $\alpha', \alpha_0', \eta'$ satisfy (6.4.1) and be such that $\alpha_0' > \alpha_0$. Then applying Corollary 6.4.13 with $\alpha_0', \alpha', \eta'$ we find for every $\mu \in \mathcal{P}(\mathcal{C}_{\mathfrak{m}}^{\alpha_0'})$ there exists a unique invariant measure ν such that $\lim_{t\to\infty}\|P_t^*\mu - \nu\|_{\mathrm{TV}(\mathcal{C}^{\alpha_0'})} = 0$. From Theorem 6.0.1, given $x \in \mathcal{C}^{\alpha_0}$, for any $t > 0$, $u_t(x) \in \mathcal{C}_{\mathfrak{m}}^{\alpha}(\mathbb{T}) \subset \mathcal{C}_{\mathfrak{m}}^{\alpha_0'}(\mathbb{T})$. Therefore, setting $\mu = \mathcal{L}(u_t(x))$, in Corollary 6.4.13, we have

$$\begin{aligned}
\|P_{2t} * \delta_x - \nu\|_{\mathrm{TV}(\mathcal{C}^{\alpha_0})} &= \|P_t^* P_t^* \delta_x - \nu\|_{\mathrm{TV}(\mathcal{C}^{\alpha_0})} \\
&= \|P_t^* \mathcal{L}\left(u_t(x)\right) - \nu\|_{\mathrm{TV}(\mathcal{C}^{\alpha_0})} \\
&\leq (1-\lambda)^{\lfloor \frac{t}{3} \rfloor} \|\mathcal{L}\left(u_t(x)\right) - \nu\|_{\mathrm{TV}(\mathcal{C}^{\alpha_0'})} \\
&\leq (1-\lambda)^{\lfloor \frac{t}{3} \rfloor},
\end{aligned}$$

where in the penultimate line we use that $\mathcal{C}_b(\mathcal{C}^{\alpha_0}) \subset \mathcal{C}_b(\mathcal{C}^{\alpha_0'})$ and in the last line that the total variation distance is bounded by 1.

Finally from (6.2.25) in Remark 6.2.5 we recall that there exists a $\lambda > 0$, such that for any even $p \geq 2$, $t > 0$ and $\delta \in (0, 1/2)$,

$$\mathbb{E}\left[\exp\left(\lambda \|u_t\|_{L^p}^{1-2\delta}\right)\right] < \infty. \tag{6.4.36}$$

Propagating this moment bound to ν completes the proof. □

Chapter 7

Analysis in Two Spatial Dimensions

Having considered local and global well-posedness of the repulsive equation in one spatial dimension, we proceed to consider the model in two spatial dimensions. Formally we write the two dimensional repulsive equation in the form

$$\begin{cases} \partial_t u - \Delta u = \chi \nabla \cdot (u^{:2:} \nabla \rho_u) + \xi, & \text{on } \mathbb{R}_+ \times \mathbb{T}^2, \\ -\Delta \rho_u = u - \bar{u}, & \text{on } \mathbb{T}^2 \\ u|_{t=0} = u_0. \end{cases} \tag{7.0.1}$$

Referring to our discussion in Section 5.4.1 we recall that the stochastic heat equation cannot be solved in any space of positivity regularity on $\mathbb{R}_+ \times \mathbb{T}^2$. Therefore, we cannot expect to be able to interpret the product u^2 in the usual way. This is the reason for the introduction of the notation $u^{:2:}$ which mirrors the notation for the Wick products of v introduced in Section 5.4.1. In order to give an informal understanding we make the same decomposition as in Chapter 6. We set $u = w + v_{0,\cdot}$, where v solves (5.4.4) but now we require w to solve

$$\begin{cases} \partial_t w - \Delta w = \chi \nabla \cdot ((w + 2wv_{0,\cdot} + v_{0,\cdot}^{:2:}) \nabla \rho_{w+v_{0,\cdot}}), & \text{on } \mathbb{R}_+ \times \mathbb{T}, \\ -\Delta \rho_{w+v_{0,\cdot}} = w + v_{0,\cdot} - \bar{w}, & \text{on } \mathbb{T}, \\ w|_{t=0} = u_0, \end{cases} \tag{7.0.2}$$

where $v_{0,t}^{:2:}$ here is the Wick square of $v_{0,t}$ defined in Section 5.4.1. Therefore, taking a smooth approximation of the noise ξ_ε and using the decomposition $u_\varepsilon = w_\varepsilon - v_{\varepsilon;0,\cdot}$ and identity (5.4.17) we see that there exists a diverging sequence $\mathfrak{C}_\varepsilon \to \infty$ such that

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = \chi \nabla \cdot ((u_\varepsilon)^2 \nabla \rho_{u_\varepsilon}) + \mathfrak{C}_\varepsilon u_\varepsilon + \xi_\varepsilon, \tag{7.0.3}$$

We understand u as the limit of the approximations u_ε solving (7.0.3). Since u does not solve a self-contained equation our primary object of study is the remainder $w = u + v_{0,\cdot}$. Recalling that, for $t > 0$, $v_{0,t}$, $v_{0,t}^{:2:}$ are bounded in $\mathcal{C}^\alpha(\mathbb{T}^3))$ for any $\alpha < 0$, we expect to solve (7.0.2) inside $C((0,\infty);\mathcal{C}^{1+\alpha}(\mathbb{T}^2))$. We make this precise in the next section, obtaining local well-posedness of (7.0.2). At the end of the chapter we present a discussion of the issues we have encountered in attempting to obtain global well-posedness of (7.0.1).

7.1 Local Well-Posedness on $\mathbb{T}^2$

As in Chapter 6 the local analysis presented below is independent of $\chi > 0$ and the sign choice in (7.0.2). Throughout this section we fix $\alpha_0 \in (-1/2, 0)$ and $\alpha \in (-1/2 - \alpha_0, 0)$ and then choose β, η, $\alpha' > 0$ such that

$$\beta + \alpha > 0, \quad \frac{\beta - \alpha_0}{2} < \eta < \frac{1}{4} - \frac{\beta - \alpha}{4}, \quad \alpha' < 2\eta. \tag{7.1.1}$$

We measure $u_0 \in \mathcal{C}^{\alpha_0}$ and the noise $(v_{0,\cdot}, v_{0,\cdot}^{:2:})$ in $C_T\mathcal{C}^\alpha \times C_{\alpha';T}\mathcal{C}^\alpha$. The remainder w will be measured in $C_{\eta;T}\mathcal{C}^\beta$ with blow-up rate η for t close to 0.

For a vector $\underline{Z} = (Z^{(1)}, Z^{(2)}) \in C([0,T];\mathcal{C}^\alpha(\mathbb{T}^2)) \times C((0,T];\mathcal{C}^\alpha(\mathbb{T}^2))$, we fix the notation

$$F(w, \underline{Z}) := \sum_{k=0}^{2} \binom{2}{k} w^k Z^{(2-k)}.$$

and we define the norm

$$|||\underline{Z}|||_{\alpha,\alpha';T} := \max\left\{ \|Z^{(1)}\|_{C_T\mathcal{C}^\alpha}, \sup_{t\in(0,T]} t^\eta \|Z^{(2)}_t\|_{\mathcal{C}^\alpha} \right\}. \tag{7.1.2}$$

From Theorem 5.4.12 we recall that the pair of Markov processes $\underline{v}_{0,t} := (v_{0,t}, v^{:2:}_{0,t})$ is almost surely finite in $|||\cdot|||_{\alpha,\alpha';T}$ for $\alpha < 0 < \alpha'$ as in (7.1.1).

Definition 7.1.1 (Mild Solutions to (7.0.2)). Given $u_0 \in \mathcal{C}^{\alpha_0}$, and a positive time $T > 0$, we say that $w \in C((0,T);\mathcal{C}^\beta(\mathbb{T}^2))$ is a mild solution to (7.0.2) on $[0,T] \times \mathbb{T}^2$ if the identity,

$$w_t = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \nabla \cdot \left((F(w_s, \underline{v}_{0,s}) \nabla \rho_{w_s + v_{0,s}} \right) ds$$

holds $\mathbb{P}$-almost surely for every $t \in (0,T)$.

Theorem 7.1.2 (Local Well-Posedness of (7.0.2))**.** *Fix T, $\mathfrak{R} > 0$, $\underline{v}_{0,\cdot} \in C_T\mathcal{C}^\alpha \times C_{\alpha';T}\mathcal{C}^\alpha$ and let $u_0 \in \mathcal{C}^{\alpha_0}$ be such that $\|u_0\|_{\mathcal{C}^{\alpha_0}(\mathbb{T})} < \mathfrak{R}$. Then for $\alpha, \alpha', \eta, \beta$ satisfying (6.0.2) and $\underline{v} \in C_T\mathcal{C}^\alpha \times C_{\alpha';T}\mathcal{C}^{2\alpha}$ there exists a $T_*(\mathfrak{R}, |||\underline{v}_{0,\cdot}|||_{\alpha',\alpha;T}) \in (0,T)$ such that (7.0.2) has a unique mild solution, $w \in C_\eta([0,T_*];\mathcal{C}^\beta(\mathbb{T}^2))$. Furthermore the solutions satisfies the bound*

$$\sup_{t\in[0,T^*]} t^\eta \|w_t\|_{\mathcal{C}^\beta(\mathbb{T}^2)} < 1.$$

If $|||\underline{v}_{0,\cdot}|||_{\alpha',\alpha;T} < 1$ then there exist $\theta := \theta(\eta, \alpha, \alpha_0)$, $C > 0$ independent of $\mathfrak{R}$ such that,

$$T^* = \left(\frac{1}{C(1+\mathfrak{R})} \right)^{\frac{1}{\theta}}. \tag{7.1.3}$$

Finally, for any $t_0 \in (0, T_*]$ and $\kappa \in (0,1)$ we have that

$$\sup_{t \neq s \in [t_0, T_*]} \frac{\|w_t - w_s\|_{\mathcal{C}^{\beta-2\kappa}}}{|t-s|^\kappa} < \infty. \tag{7.1.4}$$

Proof. For $\bar{T} > 0$ we define the set

$$B_{\bar{T}} := \left\{ w \in C((0,\bar{T}];\mathcal{C}^\beta(\mathbb{T}^2));\ \sup_{t\in(0,\bar{T}]} t^\eta \|w_t\|_{\mathcal{C}^\beta} < 1 \right\}.$$

We define $\tau > 0$ be the maximal time such that,

$$\tau := \inf \left\{ t > 0\ :\ |||\underline{v}_{0,\cdot}|||_{\alpha',\alpha;\tau} \geq 1 \right\}.$$

It suffices to show that the mapping

$$(\Psi w)_t := e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \nabla \cdot \big(F(w_s, \underline{v}_{0,s}) \nabla \rho_{w_s + v_{0,s}} \big)\, \mathrm{d}s,$$

is a contraction on $B_{T_* \wedge \tau}$, with T_* defined by (7.1.3). We begin by showing that $\Psi(B_{T_*\wedge\tau}) \subseteq B_{T_*\wedge\tau}$.

Step 1: Regarding the initial data, using (5.2.15) we obtain, for any $t > 0$ that

$$\|e^{t\Delta} u_0\|_{\mathcal{C}^\beta} \leq t^{-\frac{\beta-\alpha_0}{2}} \|u_0\|_{\mathcal{C}^{\alpha_0}}. \tag{7.1.5}$$

Regarding the non-linear term, we recall that since we chose $\alpha + \beta > 0$ and by assumption $2\alpha + 1 > 0$, we apply Theorem 5.2.9 and the embeddings (5.2.5) to see that we have

$$\begin{aligned}
\|w_s^2 \nabla \rho_{w_s+v_{0,s}}\|_{\mathcal{C}^\alpha} &\leq \|w\|_{\mathcal{C}^\beta}^2 \|\nabla \rho_{w_s+v_{0,s}}\|_{\mathcal{C}^{1+\alpha}} \\
2\|w_s v_{0,s} \nabla \rho_{w_s+v_{0,s}}\|_{\mathcal{C}^\alpha} &\leq 2\|w\|_{\mathcal{C}^\beta} \|v_{0,s}\|_{\mathcal{C}^\alpha} \|\nabla \rho_{w_s+v_{0,s}}\|_{\mathcal{C}^{1+\alpha}} \\
\|v_{0,s}^{:2:} \nabla \rho_{w_s+v_{0,s}}\|_{\mathcal{C}^\alpha} &\leq \|v_{0,s}^{:2:}\|_{\mathcal{C}^\alpha} \|\nabla \rho_{w_s+v_{0,s}}\|_{\mathcal{C}^{1+\alpha}}.
\end{aligned}$$

Combining these estimates with the bound (5.2.19), which yields,

$$\|\nabla\rho_{w_s+v_{0,s}}\|_{\mathcal{C}^{1+\alpha}}\lesssim \|w_s+v_{0,s}\|_{\mathcal{C}^\alpha}\leq \|w_s\|_{\mathcal{C}^\beta}+\|v_{0,s}\|_{\mathcal{C}^\alpha},$$

we obtain the bound

$$\begin{aligned}\|F(w_s,\underline{v}_{0,s})\|_{\mathcal{C}^\alpha} \leq \|w_s\|_{\mathcal{C}^\beta}^3&+3\|w_s\|_{\mathcal{C}^\beta}^2\|v_{0,s}\|_{\mathcal{C}^\alpha}\\ &+2\|w_s\|_{\mathcal{C}^\beta}\|v_{0,s}\|_{\mathcal{C}^\alpha}^2+\|v_{0,s}^{:2:}\|_{\mathcal{C}^\alpha}\|w_s\|_{\mathcal{C}^\beta}\\ &+\|v_{0,s}^{:2:}\|_{\mathcal{C}^\alpha}\|v_{0,s}\|_{\mathcal{C}^\alpha}.\end{aligned}$$

Therefore, applying (5.2.15), (5.2.7) followed by the triangle inequality, we have, for any $w\in B_{T_*\wedge\tau}$, $t\in(0,T_*\wedge\tau]$ and $s\in(0,t)$ that

$$\|e^{(t-s)\Delta}\nabla\cdot F(w_s,\underline{v}_{0,s})\|_{\mathcal{C}^\beta}\lesssim (t-s)^{-\frac{\beta-\alpha+1}{2}}\|F(w,\underline{v}_{0,s})\|_{\mathcal{C}^\alpha}\lesssim (t-s)^{-\frac{\beta-\alpha+1}{2}}s^{-3\eta}.$$

We note that since $\eta<\frac{1}{4}$ the quantity, $s^{-3\eta}$ is integrable. So in combination with (7.1.5), for any $t\in(0,T_*\wedge\tau)$ we have that

$$\begin{aligned}t^\eta\|(\Psi w)_t\|_{\mathcal{C}^\beta} &\leq t^\eta\|e^{t\Delta}u_0\|_{\mathcal{C}^\beta}+t^\eta\int_0^t\|e^{(t-s)\Delta}\nabla\cdot F(w_s,\underline{v}_{0,s})\|_{\mathcal{C}^\beta}\,\mathrm{d}s\\ &\lesssim t^{\eta-\frac{\beta-\alpha_0}{2}}\mathfrak{R}+t^{1-\frac{\beta-\alpha+1}{2}-2\eta}\\ &\lesssim t^{\left(\eta-\frac{\beta-\alpha_0}{2}\right)\wedge\left(\frac{1}{2}-\frac{\beta-\alpha}{2}-2\eta\right)}(\mathfrak{R}+1).\end{aligned}$$

where both exponents are positive due to (6.0.2). Therefore, for $T_*\in(0,1)$ defined by (6.1.7), with $\theta:=\left(\eta-\frac{\beta-\alpha_0}{2}\right)\wedge\left(\frac{1}{2}-\frac{\beta-\alpha}{2}-2\eta\right)$ and $C>0$ equal to the proportionality constant, we have that Ψ maps $B_{T_*(\mathfrak{R})\wedge\tau}$ into itself.

Step 2: To obtain a contraction we let w, $\tilde{w}\in B_{T_*(\mathfrak{R})\wedge\tau}$ be distinct. Then for any $s\in(0,T_*(\mathfrak{R})\wedge\tau]$, using similar arguments as in **Step 2** of the proof of Theorem 6.1.4, we have that

$$\|F(w_s,\underline{v}_{0,s})\nabla\rho_{w_s,v_{0,s}}-F(\tilde{w}_s,\underline{v}_{0,s})\nabla\rho_{\tilde{w}_s,v_{0,s}}\|_{\mathcal{C}^\alpha}\lesssim s^{-2\eta}\|w_s-\tilde{w}_s\|_{\mathcal{C}^\beta}.$$

So then, for any $t \in (0, T_*(\mathfrak{R}) \wedge \tau)$,

$$\begin{aligned} t^\eta \|(\Psi w)_t - (\Psi \tilde{w})_t\|_{\mathcal{C}^\beta} &\lesssim t^\eta \int_0^t \|e^{(t-s)\Delta} \nabla \cdot \left(F(w_s, \underline{v}_{0,s}) \nabla \rho_{w_s, v_{0,s}} - F(\tilde{w}_s, \underline{v}_{0,s}) \nabla \rho_{\tilde{w}_s, v_{0,s}} \right) \|_{\mathcal{C}^\beta} \, \mathrm{d}s \\ &\lesssim t^\eta \int_0^t (t-s)^{-\frac{\beta-\alpha+1}{2}} \|F(w_s, \underline{v}_{0,s}) \nabla \rho_{w_s, v_{0,s}} - F(\tilde{w}_s, \underline{v}_{0,s}) \nabla \rho_{\tilde{w}_s, v_{0,s}}\|_{\mathcal{C}^\alpha} \, \mathrm{d}s \\ &\lesssim t^\eta \int_0^t (t-s)^{-\frac{\beta-\alpha+1}{2}} s^{-2\eta} \|w_s - \tilde{w}_s\|_{\mathcal{C}^\beta} \, \mathrm{d}s \\ &\lesssim t^{\frac{1}{2} - \frac{\beta-\alpha}{2} - 2\eta} \|w - \tilde{w}\|_{C_{\eta;t}\mathcal{C}^\beta}. \end{aligned}$$

Therefore, using the definition of T_*, we have that

$$\|\Psi w - \Psi \tilde{w}\|_{C_{\eta; T_* \wedge \tau}\mathcal{C}^\beta} \lesssim (T_* \wedge \tau)^{\frac{1}{2} - \frac{\beta-\alpha}{2} - 2\eta} \|w - \tilde{w}\|_{C_{\eta;T_*}\mathcal{C}^\beta} < \|w - \tilde{w}\|_{C_{\eta;T_*}\mathcal{C}^\beta}.$$

Therefore we see that Ψ defines a contraction on $B_{T_* \wedge \tau}$ and so obtain a unique fixed point. By construction this is the unique mild solution to (7.0.2) in the sense of Definition 7.1.1 in $C_\eta((0, T_* \wedge \tau]; \mathcal{C}^\beta(\mathbb{T}^2))$. Finally, if $\|\!|\underline{v}_{0,\cdot}|\!\|_{\alpha,\alpha';T} < 1 \Rightarrow \tau \geq T$ then with T_* as in (6.1.7) we have that

$$\sup_{t \in [0, T_*]} t^\eta \|w_t\|_{\mathcal{C}^\beta} < 1.$$

To show continuity in the data $\underline{v}_{0,\cdot} \in C_T\mathcal{C}^\alpha \times C_{\alpha';T}\mathcal{C}^\alpha$ we follow similar steps as in the proof of Theorem 6.1.4. Hölder regularity of the remainder in t for positive times can be shown also using a similar argument as in **Step 4** of the proof of Theorem 6.1.4. □

7.2 Discussion of Global Well-Posedness on $\mathbb{T}^2$

We now specify our consideration to the repulsive equation. Considering the right hand side of the remainder equation (7.0.2), and assuming for simplicity that $\bar{u}_0 = 0$, we see that it can be written as

$$\nabla \cdot (F(w_t, \underline{v}_t) \nabla \rho_{w_t + v_t}) = -w_t^3 + \nabla(w^2) \cdot \nabla \rho_{w+v} + \nabla \cdot ((2wv + v^{:2:}) \nabla \rho_{w+v}).$$

By analogy with the one dimensional equation it seems natural to expect that the damping from the $-w^3$ term should lead to global well-posedness of (7.0.2). One approach would be repeat the testing method of Section 6.2. However, inspecting (7.1.1) we see that we only obtain local well-posedness for $\beta < 1+\alpha < 1$. Therefore neither of the terms $\langle \Delta w_t, w_t^{p-1}\rangle$, $\langle F(w_t, \underline{v}_t), \nabla w_t^{p-1}\rangle$ are finite. This issue does not arise in the Φ^4_2 model, where the remainder is found to be in $\mathcal{C}^\beta$ for $\beta \in (1,2)$, [92].

7.2.1 Paraproduct Decomposition

In three dimensions the Φ^4 model presents a similar difficulty, where the remainder is only locally well-posed in $\mathcal{C}^\beta$ for $\beta \in (0,1/2)$. The testing method was successfully carried out in this context in [91] by making use of Bony's decomposition defined in Section 5.2. Let $f,\, g \in \mathcal{B}^\alpha_{p_1,q}(\mathbb{T}^2) \times \mathcal{B}^\beta_{p_2,q}(\mathbb{T}^2)$ for $\alpha, \beta \in \mathbb{R}$ and $p_1, p_2, q \in [1,\infty]$. Then the product fg can be formally decomposed into

$$fg = \sum_{j<k-1} \Delta_j f \Delta_k g + \sum_{|k-j|\leq 1} \Delta_j f \Delta_k g + \sum_{j>k+1} \Delta_j f \Delta_k g, \tag{7.2.1}$$

we define

$$f \varolessthan g := \sum_{j<k-1} \Delta_j f \Delta_k g, \quad f \varogreaterthan g := \sum_{j>k+1} \Delta_j f \Delta_k g$$
$$f \circledequal g := \sum_{|k-j|\leq 1} \Delta_j f \Delta_k g.$$

Theorem 5.2.8 asserts that when $\alpha < 0 < \beta$ this decomposition is valid in $\mathcal{B}^\alpha_{p,q}$ only when $\alpha + \beta > 0$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. However, from Theorem 5.2.8 we also see that it is only the resonant term, $f \circledequal g$, that fails to be well defined if $\alpha + \beta \leq 0$, yet when it is well defined it is always of positive regularity. In contrast, the paraproduct terms, $f \varolessthan g$ and $f \varogreaterthan g$ are always well defined and carry the regularity of the factor sampled at the higher frequencies. For concision we define $f \mathbin{\underline{\varolessthan}} g := f \varolessthan g + f \circledequal g$ and $f \mathbin{\underline{\varogreaterthan}} g := g \mathbin{\underline{\varolessthan}} f$.

Considering (7.0.1), we can use this decomposition to define two new processes, $\mathcal{V}$, $\mathcal{W}$ such that $w = \mathcal{V} + \mathcal{W}$ and the pair solves the system

$$\partial_t \mathcal{V} - \Delta \mathcal{V} = \nabla \cdot (v^{:2:} \varogreaterthan \nabla \rho_{\mathcal{W}+\mathcal{V}+v}) + 2\nabla \cdot (v \varogreaterthan (\mathcal{W} + \mathcal{V})\nabla \rho_{\mathcal{W}+\mathcal{V}+v}) \tag{7.2.2}$$

$$\begin{aligned} \partial_t \mathcal{W} - \Delta \mathcal{W} = \nabla \cdot ((\mathcal{W} + \mathcal{V})^2 \nabla \rho_{\mathcal{W}+\mathcal{V}+v}) + 2\nabla \cdot (v \varolessthan (\mathcal{W} + \mathcal{V})\nabla \rho_{\mathcal{W}+\mathcal{V}+v}) \\ + \nabla \cdot (v^{:2:} \varolessthan \nabla \rho_{\mathcal{W}+\mathcal{V}+v}). \end{aligned} \tag{7.2.3}$$

Since the paraproducts $v^{:2:} \varogreaterthan \nabla \rho_{\mathcal{W}+\mathcal{V}+v}$ and $v \varogreaterthan (\mathcal{W} + \mathcal{V})$ carry the regularity of $v^{:2:}$ and v respectively we expect $\mathcal{V}$ to be locally well-posed only in $\mathcal{C}^{1+2\alpha}(\mathbb{T}^2)$), i.e. of the same regularity as w. However, we would expect to bound the terms on the right hand side of $\mathcal{W}$ in $\mathcal{C}^{1+2\alpha}(\mathbb{T}^2)$ so that $\mathcal{W}$ would be locally well-posed in $\mathcal{C}^{2+2\alpha}(\mathbb{T}^2)$. This would enable us to test the equation for $\mathcal{W}$ with itself.

Since $\mathcal{V}$ serves as an input to (7.2.3) a necessary first step is to obtain global well-posedness of $\mathcal{V}$. However, unlike in the case of Φ^4_3, the equation (7.2.2) for our first component $\mathcal{V}$, is not linear. In addition it does not contain any obvious damping terms and so it seems that one would need to take a different approach to obtaining global well-posedness in this case.

Chapter 8

Conclusions and Open Questions

Having presented the concrete results we have obtained regarding the additive noise equation in $d = 1, 2$ we now present some open questions and interesting avenues for further research. Some of these questions are motivated by our current understanding of the additive noise equation.

8.0.1 Global Existence of Attractive Equation with Additive Noise

For the classical parabolic–elliptic Keller–Segel model it is known that smooth solutions exist globally in both the attractive and repulsive regimes in 1 dimension. For two different proofs of this result in the case of the parabolic–parabolic equation see [95, 70]; both methods can be easily adapted to the parabolic-elliptic equation. The intuitive reason behind global well-posedness in both regimes is the boundedness of the interaction kernel in one dimension. Formally, $\partial_x \rho_u = \operatorname{sign} * u$ and so when applying Young's convolution inequality we have that $\|\partial_x \rho_u\|_{L^\infty} \leq \|u\|_{L^1}$. We already used this in proving the a priori estimate, Theorem 6.2.3. As a result either through energy estimates, as in [95] or using the mild formulation as in [70], one obtains, for u solving (5.0.3) and $p \geq 2$, that $\|u_t\|_{L^p} \leq \|u_0\|_{L^1}$. Since the equation preserves sign and mass, global well-posedness follows almost directly. However, applying this methodology in the additive noise case is not so straightforward. We consider the model with $m = 2$,

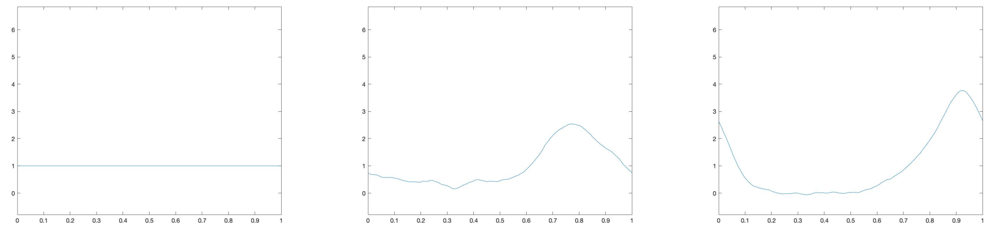
$$\begin{cases} \partial_t u - \partial_{xx} u = -\chi \partial_x(|u| \partial_x \rho_u) + \xi, \\ -\partial_{xx} \rho_u = \mathcal{P} u. \end{cases} \tag{8.0.1}$$

Although the solution to (8.0.1) is not sufficiently regular to test the equation with, using the mild solution approach of [70] we may obtain the bound

$$\sup_{t \in T} \|u_t\|_{L^p} \lesssim_{p,\gamma_1,\gamma_2} \|u_0\|_{L^1} + \sup_{t \in [0,T]} \|u_t\|_{L^1}^{\gamma_1} + T^{\gamma_2} + \|v\|_{C_T \mathcal{C}^\alpha}, \tag{8.0.2}$$

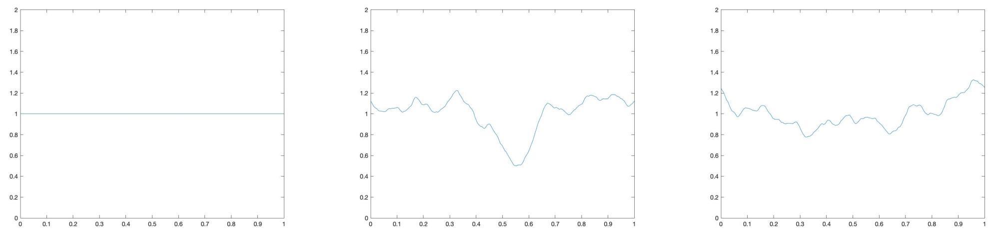
for $\gamma_1, \gamma_2 > 0$, $\alpha \in (0, 1/2)$ and T a local time of existence. Now the L^1 norm is almost surely not preserved and so something more must be done. This approach could be concluded given sufficiently strong, a priori control on the quantity $\|u_t\|_{L^1}$. One approach could be to extend the Itô type formula for the stochastic heat equation obtained in [8], to a Tanaka type formula for the non-linear SPDE, (8.0.1). Alternatively, working with the remainder we can test the equation with an approximation of the sign function and taking the expectation of (8.0.2), we see that we are would now be required to control probability that $|w_t + v_t|$ is very small. This also appears to be non-trivial, but could perhaps be approached using Malliavin calculus. From simulations the equation does appear to be globally well-posed and stable so it would be very interesting to obtain a proof. We also see from simulations that the stochastic equation obeys a phase transition for either high, or low χ. This is also the case for the deterministic equation, since the free energy only has non-trivial minimisers for χ sufficiently large. We present some simple simulations below, implemented using a backwards Euler-Muryama scheme on a regular space-time mesh. The time-space domain is $(t, x) \in [0, 1] \times [0, 1)$ and the mesh is 750×750 points.

Simulation of the Attractive Equation with $\chi = 50$:



(a) t=0 (b) t=0.1 (c) t=1

Simulation of the Attractive Equation with $\chi = 10$:



(a) t=0 (b) t=0.1 (c) t=1

8.0.2 Local Well-Posedness for $d > 2$

Regarding local well-posedness of (5.0.1) for $d > 2$ with space-time white noise one would need to introduce new tools, such as the theory of regularity structures or paracontrolled distributions. To explore this we discuss the criticality of (5.0.1) with $m = 3$ in $d = 3$ using the language of regularity structures. We lift the solution, the potential term and the noise, u, $\nabla \rho_u$, ξ to spaces of modelled distributions, writing U, $\mathcal{K}[U]$, Ξ for these local approximations. To each we assign a homogeneity by the following rules. The space-time white noise in $d = 3$ is almost surely controlled in $\mathcal{C}_\mathfrak{s}^{-\frac{5}{2}-}([0,T] \times \mathbb{T}^3)$, a scale of space-time Hölder spaces with parabolic scaling, so we set $|\Xi| = -\frac{5}{2}-$. Assuming that the solution to the semi-linear SPDE inherits the regularity of the stochastic heat equation we set $|U| = -\frac{1}{2}-$. Then leaving the regularity of the potential term floating for now, setting $|\mathcal{K}[U]|= \alpha$, the non-linear term is assigned the homogeneity $|\nabla \cdot (U^2\mathcal{K}[U])|= -(2-\alpha)-$. Therefore, the cubic equation is sub-critical, in the sense of [59, Ass. 8.3], provided $\alpha > -\frac{1}{2}$. So if the modelled distribution, $\mathcal{K}[U]$, has the same homogeneity as U, the three dimensional equation fails to be sub-critical. However, taking into account the regularising effect of the spatial convolution, we would expect to be able to solve the three dimensional equation locally. The issue in applying this regularising effect is that $\mathcal{K}[\,\cdot\,]$ cannot improve homogeneity/regularity in time. This issue touches on a wider subject in the theory of regularity structures, being the handling of systems with multiple scalings, and has already been encountered in a similar setting in [10, 11]. Using (5.0.1) as a working example it would be interesting to extend the theory of [59] more systematically to incorporate SPDEs with multiple scalings or anisotropic regularising kernels.

8.0.3 Conservative Noise Models

As discussed in Section 5.0.2, a somewhat more natural family of SPDEs associated to the Keller–Segel model are perturbations of the form

$$\partial_t u - \Delta u = \nabla \cdot (u \nabla \rho_u) + \nabla \cdot (F(u)\xi). \tag{8.0.3}$$

At least formally (8.0.3) preserves sign and mass of the initial data. In the language of [96], solutions to (8.0.3) stay on the manifold of probability measures, whereas the additive equation leaves this space for $t > 0$. From an analytical point of view, a benefit of many conservative noise models is that we would hope to be able to extend techniques that have already been applied successfully to the deterministic equation. In particular the gradient flow structure of the equation is preserved for both stochastic models discussed below. The downside of the conservative noise model is that handling the noise becomes more technical. For example, in the language of regularity structures, assuming $F(u) = u$, the equation is only sub-critical for $\xi \in \mathcal{C}_{\mathfrak{s}}^{\alpha}$ with $\alpha > -1$. This does not quite cover the full range of admissible noise terms - for example when ξ is a white noise in time, Itô calculus can be used to give meaning to the equation as a Banach space valued martingale. However, we see that space-time white noise is not tractable by either of these techniques. With this in mind we propose two models in the form of (8.0.3) that are amenable to current techniques and we think would be interesting to explore.

Since chemotaxis is usually observed for cells or organisms moving in suspension it is common to study the coupled Keller–Segel–Navier–Stokes model, [84, 9, 111, 112]. A simple stochastic model would be given by the system

$$\begin{cases} \partial_t u - \Delta u = \pm \chi \nabla \cdot (u \nabla \rho_u) + v \cdot \nabla u, \\ -\Delta \rho_u = \mathcal{P} u, \\ \partial_t v - \Delta v = - : (v \cdot \nabla) v : - \nabla p + \xi, \\ \nabla \cdot v = 0, \quad -\Delta p = \nabla \cdot (: (v \cdot \nabla) v :), \end{cases} \tag{8.0.4}$$

with ξ a divergence free, space-time noise on $L^2(\mathbb{R}^d; \mathbb{R}^d)$. Restricting ourselves to two dimensions, it is known that solutions to the stochastic Navier–Stokes equation, $v \in \mathcal{C}_{\mathfrak{s}}^{0-}(\mathbb{R}_+ \times \mathbb{T}^2)$, exist globally, [98]. Therefore, we would expect the first equation of (8.0.4) to be locally well-posed in $\mathcal{C}_{\mathfrak{s}}^{2-}$. All products, other than $(v \cdot \nabla) v$ which we replace with the Wick product, should therefore be classically well-posed. If this local picture can be properly established, a number of natural questions arise. Firstly, in the repulsive regime, $+\chi$, we expect the first equation to be globally well-posed, which would allow one to study its invariant behaviour, following [40, 62]. In the attractive regime, as in the deterministic case, attaining global well-posedness for $\chi < 8\pi$ should also be possible, in which case ergodic properties could also be investigated. Finally, in a number of recent works a relationship between enhanced mixing and suppression of chemotactic explosion has been found, [80, 73]. An interesting avenue to pursue would be to ask whether advection by the stochastic

Naiver–Stokes flow could have the same regularising effect.

A second family of stochastic Keller–Segel models with conservative noise that we expect would be interesting to study are models that consider stochastic perturbations in the chemical potential equation. In a general form consider the system

$$\begin{cases} \partial_t u - \Delta u \pm \chi \nabla \cdot (u \nabla \rho), \\ -\Delta \rho = \mathcal{P}\left(u + F(\rho, \xi)\right). \end{cases} \tag{8.0.5}$$

The simplest model of this type would be to set $F(\rho,\xi) \equiv \xi$ and ξ a spatially mean free noise. From simple power counting we would expect the system to be well posed for $\xi \in \mathcal{C}^\alpha_{\mathfrak{s}}$ for all $\alpha > -\frac{3}{2}$. This regime covers spatial white-noise in $d = 1$, 2. In the language of regularity structures we would expect the system to be sub-critical for $\alpha > -2$, which includes space-time white noise for $d = 1$ and spatial white noise for $d < 4$. More complicated models in this direction would be be to take $F(\rho,\xi) = \rho\xi$ or $F(\rho,\xi) = \nabla\cdot(\rho\xi)$. The former, with spatial noise, results in a PAM type equation for the chemical potential, while the latter is perhaps more natural in the context of the parabolic–parabolic model. In both of these situations the limiting equation in terms of criticality, becomes the equation for ρ. We would expect the multiplicative/PAM model to be classically well posed for $\xi \in \mathcal{C}^\alpha_{\mathfrak{s}}$ for $\alpha > -1$ and sub-critical for $\alpha > -2$. For the system with conservative noise in ρ, using these techniques we would have the same restrictions as hold for the equation with conservative noise in u. From a modelling perspective these SPDEs can be thought of as describing the evolution in an environment with randomly varying effects on the chemo-attractant/repulsant. From this viewpoint, the multiplicative equation $\rho\xi$ would perhaps be the the most interesting, giving the interpretation of the chemo-attractant/repulsant diffusing over a surface with localised strong absorbing or enhancing properties.

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