Rigidity of Automorphic Galois Representations over CM Fields

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Abstract

We show the vanishing of adjoint Bloch–Kato Selmer groups of automorphic Galois representations over CM fields. This proves their rigidity in the sense that they have no deformations which are de Rham. In order for this to make sense we also prove that automorphic Galois representations over CM fields are de Rham themselves. Our methods draw heavily from the ten author paper, where these Galois representations were studied extensively. Another crucial piece of inspiration comes from the work of P. Allen who used the smoothness of certain local deformation rings in characteristic zero to obtain rigidity in the polarized case.

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Chapter 1

Introduction

One of the protagonists of modern number theory is the group of automorphisms of an algebraic closure of the field of rational numbers $\mathbb{Q}$. This group is well-defined up to inner automorphisms and is also called the absolute Galois group of $\mathbb{Q}$. We denote it by $G_{\mathbb{Q}}$. It follows that isomorphism classes of linear representations of $G_{\mathbb{Q}}$, also known as Galois representations, are well-defined. As is explained in [Tay04], they are the subject of many influential theorems and conjectures. In this thesis we prove special cases of one such conjecture due to Bloch–Kato [BK07], which relates invariants of Galois representations to L-values. The impatient reader may skip to section 1.5 for the precise result we can prove. Otherwise, sections 1.1-1.4 introduce the subjects studied in this thesis. In section 1.6 we provide an outline of the proof of our main result.

1.1 Galois Representations

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. Recall that $G_{\mathbb{Q}} = \mathrm{Aut}(\overline{\mathbb{Q}})$ is equipped with a natural topology which has a basis of open neighbourhoods of the identity given by $\{G_F < G_{\mathbb{Q}} : F \subset \overline{\mathbb{Q}} \text{ finite extension of } \mathbb{Q}\}$, where $G_F := \{g \in G_{\mathbb{Q}} : g|_F = \mathrm{id}_F\}$. This is a profinite topology on $G_{\mathbb{Q}}$, inducing a natural isomorphism of topological groups

$$G_{\mathbb{Q}} \longrightarrow \varprojlim_{F/\mathbb{Q} \text{ finite Galois}} \mathrm{Gal}(F/\mathbb{Q}).$$

Throughout this section we fix a field $F \subset \overline{\mathbb{Q}}$ which is a finite extension of $\mathbb{Q}$ and a prime number p which splits completely in F. We will consider continuous representations of G_F acting on finite, free modules over $\mathbb{Q}_p$, $\mathbb{Z}_p$ or $\mathbb{F}_p$.

Remark 1.1.1. Everything in this section also has straightforward generalisations to representations of G_F, with coefficients in finite extensions of $\mathbb{Q}_p$, $\mathbb{Z}_p$ or $\mathbb{F}_p$. If we allow the coefficients to be big enough we also do not need to require that p splits in F, but we chose this setup to simplify the exposition.

For every prime ideal v of $\mathcal{O}_F$, let F_v denote the v-adic completion of F and let $k(v)$ be the residue field of v. Given a choice of field homomorphism $\iota : \overline{\mathbb{Q}} \to \overline{F}_v$, there exists a closed embedding $G_{F_v} \to G_F$ which only depends on ι up to conjugation. The inertia group $I_v < G_{F_v}$ is defined to be the subgroup of elements which act trivially on the residue field of $\overline{F}_v$. The natural map $G_{F_v} \to G_{k(v)}$ is surjective by a theorem going back to Frobenius. Thus, there is a short exact sequence

$$1 \to I_v \to G_{F_v} \to G_{k(v)} \to 1.$$

Moreover, recall that $\mathrm{Frob}_v : x \mapsto x^{q_v}$ is a topological generator of $G_{k(v)} \cong \widehat{\mathbb{Z}}$, where $q_v = |k(v)|$.

Let A be one of $\mathbb{Q}_p$, $\mathbb{Z}_p$ or $\mathbb{F}_p$. If $\rho : G_F \to \mathrm{GL}_n(A)$ is a continuous representation, the isomorphism classes of the restrictions $\rho|_{G_{F_v}}$ and $\rho|_{I_v}$ are well-defined. We say that ρ is unramified at v if $\rho|_{I_v}$ is trivial. In this case, $\rho|_{G_{F_v}}$ factors through $G_{k(v)}$, hence the characteristic polynomial

$$\det(T \cdot \mathrm{id} - \rho(\mathrm{Frob}_v)) \in A[T]$$

is well-defined. Moreover, if $A \in \{\mathbb{Q}_p, \mathbb{F}_p\}$ and ρ is continuous and unramified at $v \notin S$ for a finite set of primes S, then we can combine the Chebotarev density theorem with [Bou62, ch. 8, §12, n° I, prop. 3] to show that the semisimplification of ρ is determined up to isomorphism by the sequence of polynomials $(\det(T \cdot \mathrm{id} - \rho(\mathrm{Frob}_v)))_{v \notin S}$.

Given a field isomorphism $\iota : \overline{\mathbb{Q}_p} \cong \mathbb{C}$ and a Galois representation $\rho : G_F \to \mathrm{GL}_n(\mathbb{Q}_p)$ which is unramified outside a finite set of primes S, one can package these polynomials into the L-function:

$$L^S(\rho, s) := \prod_{v \notin S} (\iota \circ \det)(\mathrm{id} - \iota^{-1}(q_v^{-s}) \cdot \rho(\mathrm{Frob}_v))^{-1}.$$

A priori, this is just a formal infinite product over the places of F. However, it is conjectured that if ρ "comes from geometry", then $L^S(\rho, s)$ converges in some right half plane and even has a meromorphic continuation to the entire complex plane [FM95, Conjecture 3b]. This is known for automorphic Galois representations, which we introduce below. Moreover, with more work one can also define factors at primes $v \in S$ and define an L-function $L(\rho, s)$ by multiplying $L^S(\rho, s)$ with a finite product. However, we will not need this here.

Example 1.1.2. Here are two examples of such L-functions.

- There is a unique character $\chi_p : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ such that for all $\sigma \in G_{\mathbb{Q}}$ and p-power roots of unity ξ, we have $\sigma(\xi) = \xi^{\chi_p(\sigma)}$. This is a one-dimensional Galois representation which is unramified outside p and for each prime $v \neq p$ we have $T - \chi_p(\mathrm{Frob}_v) = T - v$. The character χ_p is called the p-adic cyclotomic character. The definition of the L-function in this case is independent of ι and reduces to

$$L^{\{p\}}(\chi_p, s) = \prod_{v \neq p} (1 - v^{1-s})^{-1} = (1 - p^{1-s})\zeta(s-1)$$

  and in fact $L(\chi_p, s) = \zeta(s-1)$, converges for $\Re s > 2$, where ζ denotes the Riemann zeta function.

- If $X/\mathbb{Q}$ is an algebraic variety, then $H^i_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)$ is a continuous Galois representation. If X admits a smooth proper model over $\mathbb{Z}[S^{-1}]$ and $p \in S$, then this representation is unramified outside S. When $X = E$ is an elliptic curve over $\mathbb{Q}$ which has good reduction outside S and $v \notin S$, then one can show that

$$\det(T \cdot \mathrm{id} - \mathrm{Frob}_v | H^1_{et}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)) = T^2 - v^{-1} a_v T + v^{-1},$$

  where $a_v = v + 1 - \#E(\mathbb{F}_v)$. From this equation we can also read off that the determinant of $H^1_{et}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)$ is χ_p^{-1}. The corresponding L-function is again independent of ι and

$$L^S(H^1_{et}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_p), s) = \prod_{v \notin S} (1 - a_v v^{-1-s} + v^{-1-2s})^{-1}$$

  converges for $\Re s > 1/2$ by the Hasse bound.

In almost all interesting cases a continuous Galois representation $\rho : G_F \to \mathrm{GL}_n(\mathbb{Q}_p)$ will be ramified at the primes $v \mid p$ and the restriction $\rho|_{G_{F_v}}$ usually contains a lot of information. For example, Fontaine [Fon82] defined an additive functor

$$D_{dR} : \mathbf{Rep}_{G_{\mathbb{Q}_p}}(\mathbb{Q}_p) \to \mathbf{Fil}_{\mathbb{Q}_p}$$

which assigns a $\mathbb{Q}_p$-vector space with exhaustive and separated filtration to a continuous p-adic representation of $G_{\mathbb{Q}_p}$. It satisfies the following properties:

- $\dim D_{dR}(V) \leq \dim V$.

- If $V = H^i_{et}(X_{\mathbb{Q}_p}, \mathbb{Q}_p)$, then $D_{dR}(V) = H^i_{dR}(X/\mathbb{Q}_p)$ with its Hodge filtration. (This was first conjectured by Fontaine and eventually proved in full generality by Faltings.)

If the inequality $\dim D_{dR}(V) \leq \dim V$ is an equality, then V is called de Rham. In this case the integers i such that $F^i D_{dR}(V) \neq F^{i+1} D_{dR}(V)$ are called the Hodge–Tate weights of V. The category of de Rham representations is abelian and the composition

$$\text{(forget the filtration)} \circ D_{dR}$$

is exact and faithful. A global Galois representation $\rho : G_F \to \mathrm{GL}_n(\mathbb{Q}_p)$ is called de Rham if $\rho|_{G_{\mathbb{Q}_v}}$ is de Rham for all $v \mid p$. Note that this makes sense since $F_v \cong \mathbb{Q}_p$ as p is totally split by assumption.

Example 1.1.3. If E/F is an elliptic curve and $v \mid p$ is a place above p, then $V = H^1(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)$ is de Rham and $D_{dR}(V|_{G_{F_v}}) = H^1_{dR}(E/F_v) \cong (F^1 = H^0(E_{F_v}, \Omega^1_E) \subset H^1_{dR}(E/F_v) = F^0)$. This implies that $V|_{F_v}$ has Hodge-Tate weights $\{0, 1\}$.

Furthermore, it will sometimes be useful to know that there is a $\mathbb{Q}_p$-algebra B_{dR} with $G_{\mathbb{Q}_p}$-action such that $B_{dR}^{G_{\mathbb{Q}_p}} = \mathbb{Q}_p$ and $D_{dR}(V) := (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$, where invariants are taken with respect to the diagonal action. This ring was defined by Fontaine and is called the de Rham period ring. There are also the related crystalline and semistable period rings and their corresponding functors, which play an important role in the classification of p-adic Galois representations.

Definition 1.1.4. We say that a de Rham representation V of G_F is rigid if every short exact sequence

$$0 \to V \to W \to V \to 0$$

of de Rham representations splits.

The terminology will be justified by deformation theory later. Furthermore, by expressing Ext groups in terms of cocycles one can show that V is rigid if and only if the natural homomorphism on continuous group cohomology

$$H^1(G_{\mathbb{Q}}, \mathrm{ad}\, V) \longrightarrow \prod_{v|p} H^1(G_{F_v}, B_{dR} \otimes_{\mathbb{Q}_p} \mathrm{ad}\, V)$$

is injective. In general, we define the geometric Bloch–Kato Selmer group of a Galois representation $\rho : G_F \to \mathrm{GL}_n(\mathbb{Q}_p)$ by

$$H^1_g(G_F, \rho) := \ker \left(H^1(G_F, \rho) \to \prod_{v|p} H^1(G_{F_v}, B_{dR} \otimes_{\mathbb{Q}_p} \rho) \right)$$

so that V is rigid if and only if $H^1_g(G_F, \operatorname{ad} V) = 0$. Usually, one also requires that the elements of H^1_g are almost everywhere unramified, but in the cases we study later this will be automatic.

Example 1.1.5. Let $V = \mathbb{Q}_p$ have the trivial G_F-action. Then V is de Rham and rigid. To see that it is rigid, let

$$0 \to V \to W \to V \to 0$$

be a de Rham extension. The action of G_F on W can be written in a suitable basis as

$$\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$

for some continuous group homomorphism $\alpha : G_F \to \mathbb{Q}_p$. We wish to prove that $\alpha = 0$. One can use [BK07, Prop 3.8 and Corollary 3.8.4] to show that

$$\ker(H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p) \to H^1(G_{\mathbb{Q}_p}, B_{dR})) = \ker(H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p) \to H^1(I_p, \mathbb{Q}_p)).$$

Thus, the fact that W is de Rham implies that $\alpha|_{I_v} = 0$ for all $v \mid p$. If $\alpha \neq 0$, then $\overline{\mathbb{Q}}^{\ker \alpha}$ would be an infinite abelian pro-p extension of F which is unramified at all places above p. This contradicts class field theory, hence $\alpha = 0$.

Example 1.1.6. Let $\mathbb{Q}_p(-1)$ denote a one-dimensional $\mathbb{Q}_p$-vector space on which $G_{\mathbb{Q}}$ acts via multiplication by χ_p^{-1}. The representation $V = \mathbb{Q}_p \oplus \mathbb{Q}_p(-1)$ is de Rham but not rigid. To see that it is not rigid, it suffices to construct a non-split extension

$$0 \to \mathbb{Q}_p \to W \to \mathbb{Q}_p(-1) \to 0$$

such that W is de Rham. This can be done using Kummer theory. Consider the field $L = \mathbb{Q}(p^{1/p^\infty}, \mu_{p^\infty})$. Fix compatible choices of primitive p-power roots of unities ζ_{p^n} and p-power roots p^{1/p^n}. There is the continuous homomorphism

$$\operatorname{Gal}(L/\mathbb{Q}) \to \operatorname{GL}_2(\mathbb{Q}_p) : g \mapsto \begin{bmatrix} 1 & c(g) \\ 0 & \chi_p(g)^{-1} \end{bmatrix},$$

where $c(g) \in \mathbb{Z}_p$ is the unique element such that $g(p^{1/p^n}) = \zeta_{p^n}^{c(g)} p^{1/p^n}$ for all n. This construction yields a representation W sitting inside the desired non-split extension. Moreover, using Tate uniformization, one can show that the action of $G_{\mathbb{Q}_p}$ on W is isomorphic to the action of $G_{\mathbb{Q}_p}$ on $H^1_{et}(E_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$, where $E/\mathbb{Q}_p$ is the elliptic curve given by the equation

$$y^2 + xy = x^3 - \sum_{n \geq 1} \frac{5n^3 p^n}{1 - p^n} x - \sum_{n \geq 1} \frac{(5n^3 + 7n^5)p^n}{12(1 - p^n)}.$$

It follows that W is de Rham.

1.2 Galois Deformations

If G is a finitely generated (discrete) group, then there exists a finite type affine $\mathbb{Z}$-scheme X whose functor of points is

$$X(A) = \{\rho : G \to \mathrm{GL}_n(A) \text{ group homomorphism}\}.$$

The conjugation action of PGL_n on X gives rise to the moduli stack X/PGL_n which parametrises isomorphism classes of representations of G. Let $U \subset X$ be a PGL_n-stable open subset such that U/PGL_n is a scheme. One can show that closed points of U are Schur irreducible representations, i.e. they only have scalar endomorphisms. Given a field k and a representation $\rho : G \to \mathrm{GL}_n(k)$ whose corresponding point lies in U, we can define the space of deformations of ρ as the formal completion of U_k/PGL_n at the closed point corresponding to ρ. Then the deformation space is $\mathrm{Spf}\, R_\rho$ for some complete Noetherian local ring R_ρ with residue field k and captures infinitesimal information of families of representations passing through ρ.

In the setting of Galois representations it is harder to define a stack like X/PGL_n because we have to incorporate the profinite topology on G into the definition. However, Mazur [Maz89] observed that one can still define a completely analogous deformation theory of Galois representations. The idea is that it suffices to describe $\mathrm{Hom}(R_\rho, A)$ for all local Artinian rings with residue field k to "know" R_ρ by Yoneda's lemma.

So let G be a profinite group and $\overline{\rho} : G \to \mathrm{GL}_n(\mathbb{F}_p)$ a continuous representation. Let $\mathbf{Art}_{\mathbb{Z}_p}$ denote the category of local Artinian $\mathbb{Z}_p$-algebras with residue field $\mathbb{F}_p$. Consider the functor

$$\begin{aligned} \mathcal{D}_{\overline{\rho}} : &\mathbf{Art}_{\mathbb{Z}_p} \to \mathbf{Set} \\ &A \mapsto \{\rho_A : G \to \mathrm{GL}_n(A) \mid \rho_A \equiv \overline{\rho} \pmod{\mathfrak{m}_A}\}/\sim, \end{aligned}$$

where $\rho_A \sim \rho'_A$ if they are conjugate by an element of $\ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(A/\mathfrak{m}_A))$. Here ρ_A is required to be continuous when A is given the discrete topology. Elements of $\mathcal{D}_{\overline{\rho}}(A)$ are called deformations of ρ to A.

If G satisfies a certain finiteness hypothesis [Maz89, §1.1] and $\overline{\rho}$ is absolutely irreducible, then there exists a complete Noetherian local $\mathbb{Z}_p$-algebra $R_{\overline{\rho}}$ with residue field $\mathbb{F}_p$ and an isomorphism of functors

$$\begin{aligned} \mathrm{Hom}(R_{\overline{\rho}}, A) &\to \mathcal{D}_{\overline{\rho}}(A) \\ f &\mapsto \rho^{univ} \otimes_{R_{\overline{\rho}}, f} A, \end{aligned}$$

where $\rho^{univ} : G \to \mathrm{GL}_n(R_{\overline{\rho}})$ is called the universal deformation of $\overline{\rho}$.

Similarly, let $\mathbf{Art}_{\mathbb{Q}_p}$ denote the category of local $\mathbb{Q}_p$-algebras with residue field $\mathbb{Q}_p$. For a continuous representation $\rho : G \to \mathrm{GL}_n(\mathbb{Q}_p)$, we define

$$\begin{aligned}\mathcal{D}_\rho : &\mathbf{Art}_{\mathbb{Q}_p} \to \mathbf{Set} \\ &A \mapsto \{\rho_A : G \to \mathrm{GL}_n(A) \mid \rho_A \equiv \rho \pmod{\mathfrak{m}_A}\}/\sim,\end{aligned}$$

where A is equipped with the topology given by its structure as a finite dimensional $\mathbb{Q}_p$-vector space. If ρ is absolutely irreducible, and $\overline{\rho}$ denotes the reduction of ρ (mod p), then there exists an isomorphism (see 5.1.4)

$$\mathcal{D}_\rho(A) \cong \mathrm{Hom}((R_{\overline{\rho}})^\wedge_{\mathfrak{p}}, A),$$

where $\mathfrak{p}$ is the kernel of the homomorphism $R_{\overline{\rho}} \to \mathbb{Z}_p$ corresponding to a lattice of ρ. In this case we define $R_\rho := (R_{\overline{\rho}})^\wedge_{\mathfrak{p}}$.

Let F and p be a number field and a totally split prime as above. For a finite set S of primes of F we define $G_{F,S} := \mathrm{Gal}(F_S/F)$, where $F_S \subset \overline{\mathbb{Q}}$ is the maximal extension of F which is unramified outside S. A representation of G_F factors through $G_{F,S}$ if and only if it is unramified outside S. To apply Mazur's result it is important that we work with $G_{F,S}$ instead of G_F because it has better finiteness properties. So set $G = G_{F,S}$ and let $\rho : G_F \to \mathrm{GL}_n(\mathbb{Q}_p)$ be a de Rham Galois representation which is unramified outside S. If ρ satisfies $H^2(G_{F_v}, \mathrm{ad}\,\rho) = 0$ for all $v \in S$ such that $v \nmid p$, then one can use local Tate duality to show $H^1_g(G_{F,S}, \mathrm{ad}\,\rho) = H^1_g(G_F, \mathrm{ad}\,\rho)$, where

$$H^1_g(G_{F,S}, \mathrm{ad}\,\rho) := \ker(H^1(G_{F,S}, \mathrm{ad}\,\rho) \to \prod_{v|p} H^1(G_{F_v}, B_{dR} \otimes_{\mathbb{Q}_p} \mathrm{ad}\,\rho)).$$

The representations of our main result will satisfy this condition. In this case ρ is rigid if and only if $H^1_g(G_{F,S}, \mathrm{ad}\,\rho) = 0$.

Consider the subfunctor

$$\mathcal{D}^{dR}_\rho(A) := \{\rho_A \in \mathcal{D}_\rho(A) : \rho_A \text{ is de Rham}\} \subset \mathcal{D}_\rho(A).$$

One can show that it is represented by a quotient ring $R^{dR}_\rho = R_\rho/I$. Its tangent space is naturally identified with the geometric Bloch–Kato Selmer group via

$$\begin{aligned}\mathcal{D}^{dR}_\rho(\mathbb{Q}_p[\varepsilon]/(\varepsilon^2)) &\cong H^1_g(G_{F,S}, \mathrm{ad}\,\rho) \\ \tilde{\rho} &\mapsto \frac{d}{d\varepsilon}\tilde{\rho}\end{aligned}$$

Thus, ρ is rigid if and only if $R_\rho^{dR} = \mathbb{Q}_p$ if and only if $|\mathcal{D}_\rho^{dR}(A)| = 1$ for every $A \in \mathbf{Art}_{\mathbb{Q}_p}$. This finally explains the word rigid since $\operatorname{Spf} R_\rho^{dR}$ is the deformation space of ρ and an object is rigid if it cannot be deformed. In our main theorem we will prove that the Galois representation of interest is rigid by proving that R_ρ^{dR} is a field.

1.3 Automorphic Representations

The methods of this thesis rely on the Langlands program, which allows us to relate Galois representations to a completely different type of representation, called automorphic representations. They have a much more analytic flavour than the Galois side and are typically infinite dimensional representations of reductive groups.

To define them, we first have to introduce the space of automorphic forms and before that the ring of adeles. Let F be a number field and if v is a place of F, i.e. an equivalence class of absolute values on F, we let F_v be the completion. It is either a finite extension of a p-adic field or one of $\mathbb{R}$ or $\mathbb{C}$ by Ostrowski's theorem. In the latter case v is called archimedean. For a finite set of places S of F containing the archimedean ones, we define the topological ring

$$\mathbb{A}_{F,S} := \prod_{v \in S} F_v \times \prod_{v \notin S} \mathcal{O}_{F_v},$$

where $\mathcal{O}_{F_v} := \{x \in F_v : |x|_v \leq 1\}$. The topological ring

$$\mathbb{A}_F := \varinjlim_S \mathbb{A}_{F,S}$$

is called the ring of adeles of F and is naturally a F-algebra equipped with the norm $|x| = \prod_v |x_v|_v$. Since each $\mathbb{A}_{F,S}$ is locally compact, so is $\mathbb{A}_F$. Given a finite type affine F-scheme $X = \operatorname{Spec} F[x_1, \ldots, x_n]/I$, we endow $X(\mathbb{A}_F)$ with its subspace topology in $\mathbb{A}_F^n$. This is independent of the generators $x_1, \ldots, x_n$. In particular, if G/F is an affine algebraic group, then $G(\mathbb{A}_F)$ is a locally compact topological group. From now on we take $G = \mathrm{GL}_n$ but the theory applies to any connected reductive group over F.

Let $K_v \subset \mathrm{GL}_n(F_v)$ be a maximal compact subgroup for each place v of F. The product $K = \prod_v K_v$ is a maximal compact subgroup of $\mathrm{GL}_n(\mathbb{A}_F)$. The following is a reformulation of the definition in [BJ79]. An automorphic form for GL_n/F is a function $f : \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F) \to \mathbb{C}$ satisfying the following four conditions:

(a) For every $g_0 \in \mathrm{GL}_n(\mathbb{A}_F)$ there exists an open neighbourhood U of g_0 and a smooth function $f_\infty^U : \mathrm{GL}_n(F \otimes \mathbb{R}) \to \mathbb{C}$ such that $f(g) = f_\infty^U(g_\infty)$ for all $g \in U$.

(b) There exist constants $C, B > 0$ such that $|f(g)| < C\|g\|^B$ for all $g \in \mathrm{GL}_n(\mathbb{A}_F)$, where $\|g\| = \max_i |x_i(g)|$ and the x_i are some choice of affine coordinates for GL_n.

(c) The span of the (right) K-translates of f is finite dimensional.

(d) Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{GL}_n(F \otimes \mathbb{R})$ and $Z(U(\mathfrak{g}))$ the center of its universal enveloping algebra. The span of $\{Df : D \in Z(U(\mathfrak{g}))\}$ is finite dimensional.

If f moreover satisfies

$$\int_{N(F)\backslash N(\mathbb{A}_F)} f(nx)\, dn = 0$$

for all $x \in \mathrm{GL}_n(\mathbb{A}_F)$ and all unipotent radicals N of proper parabolic F-subgroups $P < \mathrm{GL}_n$, then f is called cuspidal. Here dn refers to a Haar measure on the locally compact group $N(\mathbb{A}_F)$.

Example 1.3.1. Given a classical modular form $f : \mathbb{H} \to \mathbb{C}$ of weight k and level $\Gamma_0(N)$, one obtains an automorphic form φ_f for $\mathrm{GL}_2/\mathbb{Q}$ via the formula

$$\varphi_f(\gamma \iota_\infty(g_\infty) u) = j(g_\infty, i)^{-k} f(g_\infty \cdot i),$$

where $g_\infty \in \mathrm{GL}_2(\mathbb{R}) \xrightarrow{\iota_\infty} \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ and $u \in K_0(N)$, where $K_0(N) \subset \mathrm{GL}_2(\hat{\mathbb{Z}})$ is the subgroup of matrices which are upper triangular modulo N. The function j is defined as $j\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, z\right) = cz + d$ and we extend f to $\mathbb{H}^\pm = \mathbb{C} \setminus \mathbb{R}$ by $f(\overline{z}) = f(z)$. One can use the transformation properties of f and strong approximation for SL_2 to show that this formula defines a unique automorphic form φ_f. Moreover, one can prove that f is cuspidal if and only if φ_f is.

We denote the space of automorphic forms for GL_n/F by $\mathcal{A}(\mathrm{GL}_n/F)$ and the space of cuspidal automorphic forms by $\mathcal{A}^\circ(\mathrm{GL}_n/F)$. Both are infinite dimensional complex vector spaces which carry the following actions:

- For an archimedean place v of F, the maximal compact subgroup K_v acts by right translation on $\mathcal{A}(\mathrm{GL}_n/F)$. Moreover, the Lie algebra $\mathfrak{g}_v$ of $G(F_v)$ acts by differentiation on $\mathcal{A}(\mathrm{GL}_n/F)$ in a way which is compatible with K_v. This is called a $(\mathfrak{g}_v, K_v)$-module [Wal79, §2.12].

- For a non-archimedean place v of F, the whole group $\mathrm{GL}_n(F_v)$ acts on the space $\mathcal{A}(\mathrm{GL}_n/F)$ by right translation. Moreover, this action is smooth, i.e. every automorphic form has an open stabilizer in $\mathrm{GL}_n(F_v)$.

Alternatively, we say that $\mathcal{A}(\mathrm{GL}_n/F)$ is a $(\mathfrak{g}_\infty, K_\infty)\times\mathrm{GL}_n(\mathbb{A}_F^\infty)$-module, where $\mathfrak{g}_\infty = \mathrm{Lie}(\mathrm{GL}_n(F\otimes\mathbb{R}))$, $K_\infty = \prod_{v \text{ arch}} K_v$ and $\mathbb{A}_F^\infty$ is the ring of finite adeles, which is defined exactly like $\mathbb{A}_F$ but leaving out the archimedean places, so that $\mathbb{A}_F = (F\otimes\mathbb{R})\times\mathbb{A}_F^\infty$.

Finally we can make the following definition

Definition 1.3.2. An automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ is by definition an irreducible $(\mathfrak{g}_\infty, K_\infty)\times\mathrm{GL}_n(\mathbb{A}_F^\infty)$-module which is a subquotient of $\mathcal{A}(\mathrm{GL}_n/F)$. It is called cuspidal if it is a subquotient of $\mathcal{A}^\circ(\mathrm{GL}_n/F)$.

Example 1.3.3. If f is a classical cuspidal newform, then φ_f generates a cuspidal automorphic representation $\Pi_f \subset \mathcal{A}^\circ(\mathrm{GL}_2/\mathbb{Q})$.

Let Π be an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Flath's tensor product theorem [Fla79] shows that for each place v of F, there is a representation Π_v associated with Π, which is unique up to isomorphism such that

- For every archimedean place v, Π_v is an irreducible $(\mathfrak{g}_v, K_v)$-module.
- For every non-archimedean place v, Π_v is a smooth irreducible representation of $\mathrm{GL}_n(F_v)$.
- There exists a finite set of places S containing the archimedean ones such that for $v \notin S$, there exists $0 \neq f_v \in \Pi_v^{K_v}$ and

$$\Pi \cong \varinjlim_{S'\supset S} \bigotimes_{v\in S'} \Pi_v,$$

where the transition maps in the colimit are

$$\bigotimes_{v\in S'} \Pi_v \longrightarrow \bigotimes_{v\in S''} \Pi_v : x \mapsto x \bigotimes_{v\in S''\setminus S'} f_v.$$

We will also use the notations $\Pi_\infty := \bigotimes_{v \text{ arch}} \Pi_v$ and

$$\Pi^\infty = \varinjlim_{S'\supset S} \bigotimes_{\substack{v\in S \\ v \text{ non-arch}}} \Pi_v$$

so that $\Pi \cong \Pi_\infty \otimes \Pi^\infty$.

For an archimedean place v we can attach an infinitesimal character

$$\xi_v : Z(U(\mathfrak{g}_v)) \to \mathbb{C}$$

to Π_v by [Wal79, 3.21]. We say that an automorphic representation Π is cohomological if there exists an irreducible algebraic representation V of GL_n / F such that for each archimedean place v, the action of $Z(U(\mathfrak{g}_v))$ on $V \otimes_F F_v$ is given by multiplication with ξ_v. Such representations are automatically algebraic in the sense of [Clo90, Definition 1.8]. The name cohomological will be justified by comparison to the cohomology of locally symmetric spaces below.

For a non-archimedean place v, we say that Π is unramified at v if $\Pi_v^{K_v} \neq 0$. In this case $\Pi_v^{K_v}$ is a simple module over the spherical Hecke algebra, which is defined as

$$\mathcal{H}(\mathrm{GL}_n(F_v), K_v) = \{f \in C_c(\mathrm{GL}_n(F_v)) \mid \forall k_1, k_2 \in K_v,\ f(k_1 g k_2) = f(g)\},$$

where $C_c(\mathrm{GL}_n(F_v))$ denotes the set of compactly supported functions $\mathrm{GL}_n(F_v) \to \mathbb{C}$. The identity of $\mathcal{H}(\mathrm{GL}_n(F_v), K_v)$ is the characteristic function of K_v. The product structure is the convolution

$$(f_1 * f_2)(g) = \int_{\mathrm{GL}_n(F_v)} f_1(x) f_2(x^{-1}g)\, dx,$$

where dx refers to the Haar measure on $\mathrm{GL}_n(F_v)$ such that K_v has measure one. The isomorphism class of Π_v is uniquely determined by the module structure on $\Pi_v^{K_v}$. Moreover, there is the Satake isomorphism [Car79, Theorem 4.1]

$$\mathcal{H}(\mathrm{GL}_n(F_v), K_v) \to \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n}.$$

Thus the isomorphism class of Π_v is determined by the set $\{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{C}^\times$ such that

$$\Pi_v^{K_v} \cong \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]/(x_1 - \alpha_1, \ldots, x_n - \alpha_n).$$

Let $t(\Pi_v)$ denote the conjugacy class of the diagonal matrix with entries $(\alpha_1, \ldots, \alpha_n)$. It determines Π_v up to isomorphism and is called the Satake parameter of Π_v.

Now we can again define an L-function of an automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ which is unramified outside S as

$$L^S(\Pi, s) = \prod_{v \notin S} \det(1 - q_v^{-s} t(\Pi_v))^{-1}.$$

As opposed to L-functions associated with Galois representations, it is known that $L^S(\Pi, s)$ converges on some half plane $\Re s > c$ and has meromorphic continuation to the entire complex plane. If Π is cuspidal and $n > 1$, then $L^S(\Pi, s)$ is even an entire function [GJ72, Theorem 13.8].

Finally, we can make the connection to Galois representations. Choose a prime p and an isomorphism $\iota : \overline{\mathbb{Q}}_p \to \mathbb{C}$. If $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is a continuous Galois representation and Π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, we say that ρ is associated with Π if for all but finitely many places v, we have the following equality of characteristic polynomials

$$(\iota \circ \det)(T \cdot \mathrm{id} - \rho(\mathrm{Frob}_v)) = \det(T \cdot \mathrm{id} - q_v^{w/2} t(\Pi_v)) \in \mathbb{C}[T],$$

where w is an integer that only depends on Π_∞. Explicitly, if v is an archimedean place of F and $\xi_v : Z(U(\mathfrak{g}_v)) \to \mathbb{C}$ denotes the infinitesimal character of Π_v, then we have the equation

$$n\left(w + \frac{1-n}{2}\right) = \xi_v(E),$$

where $E \in \mathfrak{g}_v$ is the identity matrix (see also [Clo90, §4.3.2]). In terms of the L-function, ρ is associated with Π if and only if

$$L^S(\rho, s) = L^S(\Pi, s - w/2)$$

for a sufficiently large set of places S. As mentioned in the previous section, this uniquely determines the isomorphism class of the semisimplification ρ^{ss}.

Example 1.3.4. A classical cuspidal newform f of level N, weight k and character ψ generates a cohomological cuspidal automorphic representation Π_f of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ if and only if $k \geq 2$. In this case $w = k-1$ and it was proven by Deligne that the Galois representation associated with Π_f exists as a direct summand of the étale cohomology of a modular curve.

For a prime $v \nmid pN$, let $a_v \in \mathbb{C}$ be the normalised Fourier coefficient of f. Then the Galois representation ρ associated with Π_f satisfies

$$(\iota \circ \det)(T \cdot \mathrm{id} - \rho(\mathrm{Frob}_v)) = T^2 - a_v T + v^{k-1}\psi(v).$$

1.4 Locally Symmetric Spaces

For each open compact subgroup $K < \mathrm{GL}_n(\mathbb{A}_F^\infty)$ we define the double quotient topological space

$$X^K_{\mathrm{GL}_n/F} := \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/\mathbb{R}^\times_{>0} K_\infty \cdot K.$$

When $F=\mathbb{Q}$, then $X^K_{\mathrm{GL}_2/F}$ is closely related to modular curves. Here is what happens when $F=\mathbb{Q}(\sqrt{-2})$ and

$$K(\mathfrak{n})=\left\{k\in \mathrm{GL}_2(\widehat{\mathbb{Z}[\sqrt{-2}]}) : k\equiv \mathrm{id} \pmod{\mathfrak{n}}\right\},$$

where $\widehat{\mathbb{Z}[\sqrt{-2}]}$ is the profinite completion of $\mathbb{Z}[\sqrt{-2}]$ and $\mathfrak{n}<\mathbb{Z}[\sqrt{-2}]$ is an ideal such that $2\notin\mathfrak{n}$. We have $F\otimes\mathbb{R}=\mathbb{C}$ and

$$\mathrm{GL}_2(\mathbb{C})/\mathbb{R}^\times K_\infty \cong \mathbb{H}_3 := \{(x,y,z)\in\mathbb{R}^3 : z>0\}.$$

Thus, by strong approximation for SL_2, the connected components of $X^K_{\mathrm{GL}_2/F}$ are indexed by the ray class group $F^\times\backslash(\mathbb{A}_F^\infty)^\times/\det(K(\mathfrak{n}))\cong \mathrm{Cl}^{\mathfrak{n}}(F)$. Since F has class number one and $K(\mathfrak{n})<K(1)$ is a normal subgroup, each connected component is isomorphic to the Bianchi manifold $\Gamma\backslash\mathbb{H}_3$, where

$$\Gamma=\mathrm{GL}_2(F)\cap K(\mathfrak{n})=\left\{\gamma\in\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}]) : \gamma\equiv\mathrm{id} \pmod{\mathfrak{n}}\right\}.$$

Suppose that $K=\prod_v K_v$ decomposes as a product of open compact subgroups $K_v<\mathrm{GL}_n(\mathbb{A}_F^\infty)$, then there exists a finite set of places S such that K_v is conjugate to $\mathrm{GL}_n(\mathcal{O}_{F_v})$ for $v\notin S$. Hence, the vector space $H^\bullet(X^K_{\mathrm{GL}_n/F},\mathbb{C})$ is a module over

$$\mathcal{H}(\mathrm{GL}_n(F_v),K_v)\cong\mathbb{C}[x_{v,1}^{\pm1},\dots,x_{v,n}^{\pm n}]^{S_n}$$

for all $v\notin S$. Moreover, these actions commute with each other and generate a commutative ring $T_K^S\subset\mathrm{End}(H^\bullet(X^K_{\mathrm{GL}_n/F},\mathbb{C}))$.

Following [ACC+23], we call a number field F CM if there exists $\sigma\in\mathrm{Aut}(F)$ such that for all field homomorphisms $\iota:F\to\mathbb{C}$, we have $\iota\circ\sigma=c\circ\iota$, where $c\in\mathrm{Aut}(\mathbb{C})$ is the complex conjugation. Note that this includes totally real fields (when $\sigma=\mathrm{id}$) and one can show that subfields of CM fields are CM. We denote the maximal totally real subfield of F by F^+. From now on F is always CM.

Let Π be a cuspidal cohomological automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Suppose for simplicity that the action of $Z(U(\mathfrak{g}_\infty))$ on Π_∞ is trivial. It follows from [FS98, Theorem 2.3] that for each open compact subgroup $K<\mathrm{GL}_n(\mathbb{A}_F^\infty)$ we have a T_K^S-equivariant direct summand

$$(\Pi^\infty)^K\subset H^i(X^K_{\mathrm{GL}_n/F},\mathbb{C})$$

for $i\in[q_0,q_0+l_0]$, where $q_0=[F^+:\mathbb{Q}]n(n-1)/2, l_0=[F^+:\mathbb{Q}]n-1$ if $F\neq F^+$ and $l_0=[F^+:\mathbb{Q}]\lfloor(n+1)/2\rfloor-1$ if $F=F^+$. This implies that there exists a homomorphism $f : T_K^S \to \mathbb{C}$ such that $\{f(x_{v,i})\}$ coincides with the set of eigenvalues of the Satake parameter of Π_v for all $v \notin S$. In words, the system of Hecke eigenvalues associated with Π occurs in the cohomology of the locally symmetric space, justifying the adjective "cohomological" from above.

This topological structure allows us to also define Hecke modules over $\mathbb{Z}$ which play the role of "integral automorphic representations". They are crucial for the study of p-adic phenomena and provide a link to Galois deformation theory. Namely for each prime p, we still have a commutative subring

$$\mathbb{T}_K^S \subset \mathrm{End}_{\mathbf{D}(\mathbb{Z}_p)}(R\Gamma(X^K_{\mathrm{GL}_n/F}, \mathbb{Z}_p))$$

which is generated by the action of local Hecke algebras $\mathcal{H}_v$ for $v \notin S$. But now $\mathcal{H}_v$ is defined as $\mathbb{Z}_p$ valued, compactly supported, locally constant functions on $\mathrm{GL}_n(F_v)$ which are K_v-bi-invariant. (As opposed to $\mathbb{C}$-valued as above.) Given a choice of embedding $\iota : \mathbb{Q}_p \to \mathbb{C}$, the algebra $\mathbb{T}_K^S \otimes_\iota \mathbb{C}$ is isomorphic to T_K^S.

Since $\mathbb{T}_K^S$ is a finite $\mathbb{Z}_p$-algebra, the natural map

$$\mathbb{T}_K^S \to \prod_{\mathfrak{m}} \mathbb{T}_{K,\mathfrak{m}}^S,$$

where $\mathfrak{m}$ ranges over the maximal ideals of $\mathbb{T}_K^S$, is an isomorphism. Since derived categories are idempotent complete, we get a similar decomposition

$$R\Gamma(X^K_{\mathrm{GL}_n/F}, \mathbb{Z}_p) = \bigoplus_{\mathfrak{m}} R\Gamma(X^K_{\mathrm{GL}_n/F}, \mathbb{Z}_p)_{\mathfrak{m}}$$

Hence we fix a maximal ideal $\mathfrak{m} < \mathbb{T}_K^S$ from now on.

After possibly increasing S, the work of Scholze [Sch15] implies that there exists a continuous semisimple Galois representation $\overline{\rho}_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbb{T}_K^S/\mathfrak{m})$ such that for each $v \notin S$, the map $\mathcal{H}_v \to \mathbb{T}_K^S/\mathfrak{m}$ is compatible with the characteristic polynomial of $\overline{\rho}_{\mathfrak{m}}$ in an analogous way as in the previous section. See [ACC$^+$23, Theorem 2.3.5] for the precise statement. Moreover, if $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible, then there exists a nilpotent ideal $I < \mathbb{T}_{K,\mathfrak{m}}^S$ and a continuous representation

$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbb{T}_{K,\mathfrak{m}}^S/I)$$

such that the characteristic polynomial of $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ is compatible with $\mathcal{H}_v \to \mathbb{T}_{K,\mathfrak{m}}^S/I$ for all $v \notin S$. It will be important that $I^\delta = 0$ for some δ which only depends on $[F : \mathbb{Q}]$ and n. If $R_{\overline{\rho}_{\mathfrak{m}}}$ is the deformation ring defined above, then this representation induces a surjective local ring homomorphism

$$R_{\overline{\rho}_{\mathfrak{m}}} \to \mathbb{T}_{K,\mathfrak{m}}^S/I.$$

Since prime ideals of $\mathbb{T}_{K,\mathfrak{m}}$ are in bijection with Hecke eigensystems occuring in $H^\bullet(X^K_{\mathrm{GL}_n/F}, \mathbb{Z}_p)_{\mathfrak{m}}$, one can interpret the induced map

$$\operatorname{Spec} \mathbb{T}^S_{K,\mathfrak{m}} \to \operatorname{Spec} R_{\overline{\rho}_{\mathfrak{m}}}$$

as attaching Galois representations to automorphic representations via the Langlands correspondence. Hence, the ring homomorphism $R_{\overline{\rho}_{\mathfrak{m}}} \to \mathbb{T}^S_{K,\mathfrak{m}}/I$ can be thought of as an integral refinement of the Langlands correspondence. It will be the most important object in our method.

1.5 Statement of Results

Let F be a CM number field. In this thesis we prove, under certain assumptions, that the Galois representations associated with cohomological cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ are rigid.

The existence of such Galois representations was proven in [HLTT16] and [Sch15]. For an isomorphism $\iota : \overline{\mathbb{Q}}_p \to \mathbb{C}$ and a cohomological cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ we let

$$r_\iota(\Pi) : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$$

denote the Galois representation from the main theorem of [HLTT16].

Given a partition $S_p = S_{cris} \cup S_{ss}$ of the places of F lying above p, a finite set of places $S \supset S_p$ of F and a continuous representation $\rho : G_{F,S} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$, we define the "mixed" Bloch–Kato Selmer group $H^1_{fg}(G_{F,S}, \operatorname{ad}\rho)$ as the kernel of the natural map

$$H^1(G_{F,S}, \operatorname{ad}\rho) \to \prod_{v \in S_{cris}} H^1(G_{F,S}, \operatorname{ad}\rho \otimes B_{cris}) \times \prod_{v \in S_{ss}} H^1(G_{F,S}, \operatorname{ad}\rho \otimes B_{dR}).$$

It is sandwiched between the usual Bloch–Kato Selmer groups as follows

$$H^1_f(G_{F,S}, \operatorname{ad}\rho) \subset H^1_{fg}(G_{F,S}, \operatorname{ad}\rho) \subset H^1_g(G_{F,S}, \operatorname{ad}\rho).$$

Both the inclusions here are equalities if the Weil–Deligne representations $WD(\rho|_{G_{F_v}})$ are generic [All16, Definition 1.1.2] for all $v \in S$. Now we can state our main theorem.

Theorem 1.5.1 (= Theorem 5.3.5). *Let $F \subset L$ be CM fields and let $\rho : G_F \to \mathrm{GL}_n(E)$ be a continuous representation, where $E \subset \overline{\mathbb{Q}}_p$ is a finite extension of $\mathbb{Q}_p$ containing the images of all field homomorphisms $L \to \overline{\mathbb{Q}}_p$. Moreover, let Π be a* *cohomological cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_L)$ *and let* S *be a finite set of places of* F*, stable under complex conjugation, containing the archimedean ones and those above* p *such that* Π_v *is unramified for all* v *not lying above a place of* S*. Let* $S_p = S_{cris} \cup S_{ss}$ *be a partition of the places of* F *lying above* p *which is stable under complex conjugation such that the following are satisfied.*

(a) $p > n$*;*

(b) There exists an isomorphism $\iota : \overline{\mathbb{Q}}_p \to \mathbb{C}$ *such that* $\rho|_{G_L} \otimes_E \overline{\mathbb{Q}}_p \cong r_\iota(\Pi)$*;*

(c) The residual representation $\overline{\rho}|_{G_L}$ *is absolutely irreducible and decomposed generic [ACC+23, Definition 4.3.1]. Moreover,* $\zeta_p \notin L$ *and* $\overline{\rho}|_{G_{L(\zeta_p)}}$ *has enormous image [ACC+23, Definition 6.2.28], where* $\zeta_p \in \overline{L}$ *is a primitive pth root of unity;*

(d) If v *lies above a place in* $S \setminus S_p$*, then the Weil–Deligne representation* $\mathrm{WD}(\rho|_{G_{L_v}})$ *is generic [All16, Definition 1.1.2];*

(e) If v *lies above a place in* S_{cris}*, then* Π_v *is unramified. If* v *lies above a place in* S_{ss}*, then* $\rho|_{G_{L_v}}$ *is de Rham and for any finite extension* L'_v/L_v*,* $\mathrm{WD}(\rho|_{G_{L'_v}})$ *is generic.*

Then the Bloch–Kato Selmer group $H^1_{fg}(G_{F,S}, \mathrm{ad}\,\rho)$ *vanishes, i.e.* ρ *is rigid. Moreover,* $\rho|_{G_{L_v}}$ *is crystalline for all* v *lying above a place in* S_{cris}*.*

Remark 1.5.2. There are two main new ingredients that allow us to prove this theorem. The first one is a generalisation of the degree-shifting argument from [ACC+23, section 4], where we allow ramification at p and semistable Galois representations. As a result we obtain the weak semistable local-global compatibility theorem 4.3.1. See the outline below and chapter 4 for more details.

The second ingredient is a Taylor–Wiles patching lemma that uses the smoothness of certain characteristic 0 deformation rings to eliminate nilpotent ideals in patched Hecke algebras. The idea of using the smoothness comes from [All16]. See the outline below for more context and lemma 2.6.4 for the precise statement.

The local assumptions *(d)* and *(e)* are as general as I could manage with these methods but they will become unnecessary when more precise local-global compatibility theorems are proven. Such results would also imply that one can replace H^1_{fg} with H^1_g in the conclusion of the theorem.

Using the techniques of Miagkov–Thorne [MT22] it should also be possible to replace “enormous image” with “adequate image” [Tho12, Definition 2.3] in assumption *(c)*.

The assumption *(a)* comes from [ACC+23] and is only needed for the existence of Taylor–Wiles (Proposition 5.3.2) and for the construction of the associated complexes of Hecke modules (Proposition 5.3.3).

Together with the potential automorphy theorems of [ACC+23] the main theorem implies the more digestible

Corollary 1.5.3 (= Corollary 5.3.6)**.** *Let A/F be an elliptic curve without complex multiplication over a CM field F and $p \geq 7$ a prime such that $\zeta_p \notin F$ and the image of G_F in $\mathrm{Aut}(A[p])$ contains $\mathrm{SL}_2(\mathbb{F}_p)$. Let*

$$V_pA = \Big(\varprojlim A[p^n](\overline{F})\Big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

be the p-adic G_F-representation attached to A. Then V_pA is rigid, or equivalently any short exact sequence

$$0 \to V_pA \to V \to V_pA \to 0,$$

where V is a de Rham G_F-representation, splits. Furthermore, if $F/\mathbb{Q}$ is Galois and n is a positive integer, such that $p > 2n+3$, then the symmetric power $\mathrm{Sym}^n(V_pA)$ is rigid, too.

Remark 1.5.4. Serre's open image theorem [Ser72, assertion (6)] shows that for a given elliptic curve without CM, all but finitely many primes satisfy the conditions of the corollary. To check them at a particular prime one can often use the LMFDB [LMF20].

More generally, the recent theorem [Qia22, Theorem 1.4] gives many instances where the conditions of our main theorem are satisfied. In particular, it implies

Corollary 1.5.5. *Let F be a CM field and let $\rho : G_F \to \mathrm{GL}_n(E)$ be a continuous representation, where $E \subset \overline{\mathbb{Q}}_p$ is a finite extension of $\mathbb{Q}_p$ containing the images of all field homomorphisms $F \to \overline{\mathbb{Q}}_p$. Let S be a finite set of places of F containing those where ρ is ramified. Suppose the following are satisfied.*

(a) $p > n$;

(b) For all places $v \mid p$, $\rho|_{G_{F_v}}$ is potentially semistable and ordinary with regular Hodge–Tate weights [Qia22, Definition 1.3]. Moreover, for all finite extensions F'_v/F_v, $\mathrm{WD}(\rho|_{G_{F'_v}})$ is generic;

(c) The residual representation $\overline{\rho}$ is absolutely irreducible and decomposed generic as defined in [ACC+23, Definition 4.3.1]. Moreover, $\zeta_p \notin F$ and $\overline{\rho}|_{G_{F(\zeta_p)}}$ has enormous image as defined in [ACC+23, Definition 6.2.28];

(d) There exists $\sigma \in G_F \setminus G_{F(\zeta_p)}$ such that $\overline{\rho}(\sigma)$ is a scalar;

(e) If $v \in S \setminus S_p$, then the Weil–Deligne representation $\mathrm{WD}(\rho|_{G_{F_v}})$ is generic.

Then $H^1_g(G_{F,S}, \mathrm{ad}\,\rho) = 0$.

Along the way we also need to prove a weak form of semistable local-global compatibility and as a consequence we obtain

Theorem 1.5.6 (= Theorem 4.3.3)**.** *Let F be a CM field, $\iota : \mathbb{C} \to \overline{\mathbb{Q}_p}$ an isomorphism and Π a cohomological cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. If the reduction $\overline{r_\iota(\Pi)}$ is absolutely irreducible and decomposed generic, then for every place $v \mid p$ of F the representation $r_\iota(\Pi)|_{G_{F_v}}$ is potentially semistable (hence also de Rham) with Hodge–Tate weights*

$$HT_\tau := \{\lambda_{\tau,1} + n - 1, \lambda_{\tau,2} + n - 2, \ldots, \lambda_{\tau,n}\},$$

where $\lambda \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_p)}$ is the unique dominant weight for GL_n/F such that the infinitesimal character of Π_∞ coincides with the infinitesimal character of $(V_\lambda \otimes_{\mathcal{O},\iota} \mathbb{C})^\vee$ (see definition 3.4.4). If Π_v and Π_{v^c} are unramified, then $r_\iota(\Pi)|_{G_{F_v}}$ is crystalline.

Remark 1.5.7. Further local-global compatibility results in the crystalline case, which also treat torsion Galois representations, were recently proven in [CN23].

Let us provide some context for our results. We begin by explaining how the main theorem is predicted by a combination of the Fontaine–Mazur conjecture [FM95, Conjecture 1] and the Bloch–Kato conjecture [BK07, §5], to which [Bel09] is an introduction.

Let F be a CM field and assume for simplicity that p is prime which splits completely[1] in F. Let $\rho : G_F \to \mathrm{GL}_n(\mathbb{Q}_p)$ be an absolutely irreducible Galois representation which is unramified outside a finite set of places and de Rham at all places above p. Since $\mathrm{ad}\,\rho$ is semisimple, the Fontaine–Mazur conjecture implies that $\mathrm{ad}\,\rho$ is isomorphic to a subquotient of a Tate twist of a p-adic étale cohomology group of a variety over F and that its L-function $L(\mathrm{ad}\,\rho, s)$ has meromorphic continuation to the whole complex plane [FM95, Conjecture 3b]. Moreover, we assume that the subquotient corresponds to a "motive" of weight zero over F to which we can apply the Bloch–Kato conjecture. In particular, the conjectures stated in [FPR94, §4.2.2] together with [BK07, Proposition 3.8 and Corollary 3.8.4] imply that

$$\dim_{\mathbb{Q}_p} H^1_g(G_F, \mathrm{ad}\,\rho) - \dim_{\mathbb{Q}_p} H^0(G_F, \mathrm{ad}\,\rho) = \mathrm{ord}_{s=0} L(\mathrm{ad}\,\rho(1), s) = \mathrm{ord}_{s=1} L(\mathrm{ad}\,\rho, s).$$

[1] We also treat general primes p and Galois representations with coefficients in $\overline{\mathbb{Q}}_p$ in the main text.

Since ρ is absolutely irreducible, Schur's lemma implies that $\dim_{\mathbb{Q}_p} H^0(G_F, \operatorname{ad}\rho) = 1$. If $\rho \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p = r_\iota(\Pi)$ for some cuspidal cohomological automorphic representation Π, then [Cog08, Lecture 9 §3] shows that

$$\operatorname{ord}_{s=1} L(\operatorname{ad}\rho, s) = \operatorname{ord}_{s=1} L(\Pi \times \widetilde{\Pi}, s) = -1.$$

Hence the Bloch–Kato Selmer group $H^1_g(G_F, \operatorname{ad}\rho)$ must vanish, as confirmed by our main theorem.

When ρ is conjugate self-dual, many cases of this rigidity have been proven in the last 20 years but even for these our theorem often says something new because previous results only treat the subspace of $H^1_g(G_F, \operatorname{ad}\rho)$ corresponding to extensions which are conjugate self-dual themselves.

Recent results in the conjugate self-dual case were proven in [All16], with even more generalisations (but still under the conjugate self-dual assumption) in [NT20]. Besides confirming cases of the Bloch–Kato conjecture, these results have also led to exciting applications to symmetric power functoriality [NT21a, NT21b, NT22].

Without the conjugate self-dual assumption very little was known except for the paper [CG20] which shows how conjectures on local-global compatibility imply the expected rigidity and also manages to show a special case unconditionally. In this work, we obtain rigidity in a large class of cases without the conjugate self-dual assumption and without having to prove the full conjectures on local-global compatibility.

We continue the tradition of using automorphy lifting theorems to prove rigidity and use the 10 author paper [ACC+23] which recently established many cases of automorphy lifting for Galois representations over CM fields which are not necessarily conjugate self-dual. However, due to some nilpotent ideals in Hecke algebras their results do not directly imply the result we seek. This is one issue we address in our proof. Moreover, the techniques of [ACC+23] require that ρ satisfies either Fontaine–Laffaille or ordinary conditions at p. We weaken this and require much milder conditions on $\rho|_{G_{F_v}}$ for the places $v \mid p$.

1.6 The Method

Let us sketch the proof of the main theorem. Certain things we say here will be slightly wrong in order to simplify the exposition but we hope that nothing essential will be misrepresented. Fix a CM field F and a cohomological cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$. We wish to show that the associated Galois representation $\rho := r_\iota(\Pi)$ is rigid. Assume for the sake of the introduction that the action of

$Z(U(\mathfrak{g}))$ on Π_∞ is trivial, that p splits completely in F and that ρ has coefficients in $\mathbb{Q}_p \subset \overline{\mathbb{Q}}_p$.

Let $K = \prod_v K_v < \mathrm{GL}_n(\mathbb{A}_F^\infty)$ be an open compact subgroup such that $(\Pi^\infty)^K$ is a direct summand of $H^\bullet(X^K_{\mathrm{GL}_n/F}, \mathbb{C})$. Let S be a finite set of places of F such that for all $v \notin S$, the representation ρ is unramified at v and $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$. As discussed above, there exists a maximal ideal

$$\mathfrak{m} < \mathbb{T}^S_K \subset \mathrm{End}_{\mathbb{Z}_p}(R\Gamma(X^K_{\mathrm{GL}_n/F}, \mathbb{Z}_p))$$

such that $\overline{\rho} = \overline{\rho}_{\mathfrak{m}}$. If $\overline{\rho}$ is absolutely irreducible, then there exists a nilpotent ideal $I < \mathbb{T}^S_{K,\mathfrak{m}}$ and a deformation

$$\rho_{\mathfrak{m}} \to \mathrm{GL}_n(\mathbb{T}^S_{K,\mathfrak{m}}/I)$$

of $\overline{\rho}_{\mathfrak{m}}$ such that the Frobenius characteristic polynomials are compatible with the local Hecke action for $v \notin S$. Assume for simplicity that $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ for all places $v \mid p$. We prove in chapter 4 that $\rho_{\mathfrak{m}}$ is torsion crystalline with Hodge–Tate weights contained in $\{0, 1, \ldots, 2n-1\}$ at all places $v \mid p$ in the terminology of [Liu07]. See chapter 5 for more precise references.

The Galois Representation is Crystalline

Here we sketch why $\rho_{\mathfrak{m}}$ is torsion crystalline. This property is crucial for the proof of both the theorems stated above. We use that if we have a finite free $\mathbb{Z}_p$-algebra A with a surjection $A \twoheadrightarrow \mathbb{T}^S_K/I$ and a continuous representation

$$\tilde{\rho} : G_{F,S} \to \mathrm{GL}_{2n}(A)$$

such that $\tilde{\rho} \otimes \mathbb{Q}_p$ is crystalline with Hodge–Tate weights $\{0, 1, \ldots, 2n-1\}$, then the representation $\tilde{\rho} \otimes_A \mathbb{T}^S_{K,\mathfrak{m}}/I$ is torsion crystalline. If there is an isomorphism

$$\tilde{\rho} \otimes_A \mathbb{T}^S_{K,\mathfrak{m}}/I \cong \rho_{\mathfrak{m}} \oplus \rho_{\mathfrak{m}}^{c,\vee}(1-2n),$$

then $\rho_{\mathfrak{m}}$ is torsion crystalline with Hodge–Tate weights contained in $\{0, 1, \ldots, 2n-1\}$. In this way we are able to reduce the problem to the case of conjugate self-dual Galois representations for which a lot more is known, because they appear in the étale cohomology of some Shimura variety. So we have to find such a $\tilde{\rho}$.

In section 3.4 we define an auxiliary unitary similitude group $\widetilde{G}/F^+$ and the Siegel parabolic $P \subset \widetilde{G}$ whose Levi group is isomorphic to $\mathrm{Res}_{F/F^+}\mathrm{GL}_n$. Let $\widetilde{\mathbb{T}}$ denote the $\mathbb{Z}_p$-algebra generated by the local spherical Hecke algebras associated with $\widetilde{G}$ for all $v \notin S$. The unnormalised Satake transform is a homomorphism of Hecke algebras $\widetilde{\mathbb{T}} \to \mathbb{T}$, where $\mathbb{T}$ is the analogous ring for GL_n/F. The operation $\rho_{\mathfrak{m}} \mapsto \rho_{\mathfrak{m}} \oplus \rho_{\mathfrak{m}}^{c,\vee}(1-2n)$ corresponds to viewing $H^\bullet(X^K_{\mathrm{GL}_n/F}, \mathbb{Z}_p)_{\mathfrak{m}}$ as a $\widetilde{\mathbb{T}}$-module via the morphism $\widetilde{\mathbb{T}} \to \mathbb{T}$.

Let $\widetilde{\mathfrak{m}}$ be the preimage of $\mathfrak{m}$ in $\widetilde{\mathbb{T}}$. Let A be the faithful quotient of $\widetilde{\mathbb{T}}$ acting on $H^d(X^{\widetilde{K}}_{\widetilde{G}}, \mathbb{Z}_p)_{\widetilde{\mathfrak{m}}}$, where $X^{\widetilde{K}}_{\widetilde{G}}$ is a suitable locally symmetric space associated with $\widetilde{G}$ and d is the middle degree. Just as in [ACC+23], the algebra A is free over $\mathbb{Z}_p$ as a consequence of the main theorem of [CS19]. Moreover, we have an A-valued crystalline Galois representation $\tilde{\rho}$ by the known results on local-global compatibility for conjugate self dual automorphic Galois representations.

The existence of a Hecke equivariant map $A \to \mathbb{T}^S_{K,\mathfrak{m}}/I$ for some nilpotent ideal I is equivalent to the existence of an integer $m \geq 1$ such that

$$\mathrm{Ann}_{\widetilde{\mathbb{T}}}(H^d(X^{\widetilde{K}}_{\widetilde{G}}, \mathbb{Z}_p)_{\widetilde{\mathfrak{m}}})^m \subset \mathrm{Ann}_{\widetilde{\mathbb{T}}}(H^i(X^K_{\mathrm{GL}_n/F}, \mathbb{Z}_p)_{\mathfrak{m}})$$

for all i. In section 4.1 we introduce the notation $H^i(X^K_{\mathrm{GL}_n/F}, \mathbb{Z}_p)_{\mathfrak{m}} \prec H^d(X^{\widetilde{K}}_{\widetilde{G}}, \mathbb{Z}_p)_{\widetilde{\mathfrak{m}}}$ for this relation. Up to nilpotents, $A \prec B$ means that the Hecke eigensystems of A occur in B.

By an argument contained in [ACC+23, section 4] and outlined in section 3.4 below, to find our map $A \to \mathbb{T}^S_{K,\mathfrak{m}}/I$, it suffices to show that

$$H^i(X^K_{\mathrm{GL}_n/F}, \mathbb{Z}_p)_{\mathfrak{m}} \prec H^d(X_P^{P\cap\widetilde{K}}, \mathbb{Z}_p)_{\widetilde{\mathfrak{m}}},$$

for a certain $(\mathbb{R}/\mathbb{Z})^{n^2[F^+:\mathbb{Q}]}$-bundle $f : X_P^{P\cap\widetilde{K}} \to X^K_{\mathrm{GL}_n/F}$. In the Fontaine–Laffaille case discussed in [ACC+23], it turns out that the Leray-Serre spectral sequence

$$E_2^{i,j} := H^i(X^K_{\mathrm{GL}_n/F}, R^j f_* \mathbb{Z}_p)_{\widetilde{\mathfrak{m}}} \implies H^{i+j}(X_P^{P\cap\widetilde{K}}, \mathbb{Z}_p)_{\widetilde{\mathfrak{m}}}$$

degenerates at the second page and in this way one can obtain H^i as a Hecke equivariant direct summand of H^d since $R^j f_*\mathbb{Z}_p$ is non-zero for $0 \leq j \leq n^2[F^+ : \mathbb{Q}]$.

This is where we proceed differently from [ACC+23] and work with the higher pages as well, since it is not known if the spectral sequence degenerates in general. Following an idea which originates in [CN23], we show that for each $m \geq 1$ one can make suitable choices of level subgroups $K(m), \widetilde{K}(m)$, such that the spectral sequence

$$\overline{E}_2^{i,j} = H^i(X^{K(m)}_{\mathrm{GL}_n/F}, R^j f_*\mathbb{Z}/p^m\mathbb{Z})_{\widetilde{\mathfrak{m}}} \implies H^{i+j}(X_P^{P\cap\widetilde{K}(m)}, \mathbb{Z}/p^m\mathbb{Z})_{\widetilde{\mathfrak{m}}}$$

does degenerate at the second page and the local systems $R^j f_*\mathbb{Z}/p^m\mathbb{Z}$ are constant for all j. The existence of $K(m), \widetilde{K}(m)$ is a completely formal once we set things up correctly and ultimately boils down to the fact that if a profinite group G acts on a perfect complex of $\mathbb{Z}/p^m\mathbb{Z}$-modules, then some open subgroup will act trivially in the derived category, see lemma 4.2.2. If p is sufficiently split in F, we can ensure for any fixed place $v \mid p$, that $K(m)_v = K_v$ and $\widetilde{K}(m)_v = \widetilde{K}_v$. Thus, we will be able to conclude that $\rho_\mathfrak{m}$ is torsion crystalline at v and then we can repeat the argument for every $v \mid p$.

Let $I_r^{i,j}$ denote the image of the natural map $E_r^{i,j} \to \overline{E}_r^{i,j}$. With some algebra involving the differentials of the spectral sequence $E_r^{i,j}$ we prove in lemma 4.1.8 that

$$I_r^{i,j} \prec I_{r+1}^{i,j} \oplus I_r^{i+r,j-r+1}.$$

Thus, we can prove by reverse induction on i that for all j

$$I_r^{i,j} \prec I_\infty^{i,j} \oplus H^d(X_P^{P\cap\widetilde{K}}, \mathbb{Z}_p)_{\widetilde{\mathfrak{m}}}.$$

In particular, $I_r^{i,d-i} \prec H^d(X_P^{P\cap\widetilde{K}}, \mathbb{Z}_p)_{\widetilde{\mathfrak{m}}}$. And sending $m \to \infty$ gives that

$$H^i(X_{\mathrm{GL}_n/F}^{K(m)}, \mathbb{Z}_p)_\mathfrak{m} \prec H^d(X_P^{P\cap\widetilde{K}(m)}, \mathbb{Z}_p)_{\widetilde{\mathfrak{m}}},$$

and some devissage involving the Hochschild-Serre spectral sequence proves the same claim for K instead of $K(m)$.

Finally, one deduces that $\rho_\mathfrak{m}$ is torsion crystalline. From this it is immediate that ρ is crystalline. However, to show that it has the correct Hodge–Tate weights, we need to use heavier tools than in [ACC+23, section 4] since there is no notion of Hodge–Tate weight for torsion crystalline representations. Instead we use the theory of crystalline pseudodeformation rings [WE18] to prove that $\rho \oplus \rho^{c,\vee}(1-2n)$ has Hodge–Tate weights equal to $\{0, 1, \dots, 2n-1\}$. By compatibility with determinants we deduce ρ must have Hodge–Tate weights $\{0, 1, \dots, n-1\}$, see 4.3.3 for the details. This is a different approach to the one taken in [CN23], where stronger results on the torsion representation were proven.

The Galois Representation is Rigid

Let us return to rigidity. We keep the assumption that S contains no places above p. We know that ρ is crystalline and in particular de Rham, hence it makes sense to say that ρ is rigid. By the main theorem of [Liu07], there is a quotient $R_{\overline{\rho}}^{cris}$ of $R_{\overline{\rho}}$, parametrising deformations which are torsion crystalline with Hodge–Tate weights contained in $\{0, 1, \dots, 2n-1\}$. Since $\rho = r_\iota(\Pi)$ is associated with Π and

$$\mathbb{T}_{K,\mathfrak{m}}[1/p] \subset \mathrm{End}_{\mathbb{Q}_p}(H^\bullet(X_{\mathrm{GL}_n/F}^K, \mathbb{Q}_p))$$

is reduced, we have a commutative square

$$\begin{array}{ccc} R^{cris}_{\overline{\rho}} & \longrightarrow & \mathbb{T}^S_{K,\mathfrak{m}}/I \\ \big\downarrow f_1 & & \big\downarrow f_2 \\ \overline{\mathbb{Q}}_p & \xrightarrow{\ \iota\ } & \mathbb{C} \end{array}$$

where $\rho = f_1 \circ \rho^{univ}$ and f_2 describes the action of $\mathbb{T}_{K,\mathfrak{m}}$ on

$$(\Pi^\infty)^K \subset H^\bullet(X^K_{\mathrm{GL}_n/F}, \overline{\mathbb{Q}}_p) \otimes_{\overline{\mathbb{Q}}_p,\iota} \mathbb{C}.$$

Let $\mathfrak{q} = \ker(f_2)$ and $\mathfrak{p} = \ker(f_1)$. One can show that if $WD(\rho|_{G_{F_v}})$ is generic for places $v \mid p$, we have isomorphisms

$$R^{dR}_\rho \cong (R^{cris}_{\overline{\rho}})^\wedge_{\mathfrak{p}} \qquad\qquad \mathbb{T}^\wedge_{K,\mathfrak{q}} \cong \overline{\mathbb{Q}}_p.$$

See 5.1.7 for the first one. The second one follows from the fact that

$$H^\bullet(X^K_{\mathrm{GL}_n/F}, \overline{\mathbb{Q}}_p)_{\mathfrak{q}} \otimes_{\overline{\mathbb{Q}}_p,\iota} \mathbb{C}$$

is isomorphic to a direct sum of copies of the simple Hecke module $(\Pi^\infty)^K$ which can be proven with the decomposition in [FS98, §2.2]. The non-generic case is the reason why we only show the vanishing of the group H^1_{fg} in our main theorem below.

Thus, to show that ρ is rigid, it suffices to show that the map

$$(R^{cris}_{\overline{\rho}})^\wedge_{\mathfrak{p}} \to \mathbb{T}^\wedge_{K,\mathfrak{q}}$$

is an isomorphism. This is what we will prove using a variant of the Taylor–Wiles patching method introduced in [CG18]. In fact, the Langlands philosophy even predicts an isomorphism $R^{cris}_{\overline{\rho}} \to \mathbb{T}^S_{K,\mathfrak{m}}$, but this seems to be out of reach with our current tools.

Let us simplify notations and set $R = R^{cris}_{\overline{\rho}}$ and $\mathbb{T} = \mathbb{T}^S_{K,\mathfrak{m}}$. Rather than proving something about the homomorphism $R \to \mathbb{T}/I$ directly, we put it in a countable family of morphisms $R_m \to \mathbb{T}_m/I_m$. The rings R_m are defined as crystalline deformation rings which allow ramification at an auxiliary set of primes Q_m. The rings $\mathbb{T}_m$ are defined as Hecke algebras with a bigger level K_m at Q_m. See the proof of theorem 5.3.5 for the definitions. This is also the same setup as in [ACC+23].

Both R_m and $\mathbb{T}_m$ are local S_m-algebras, where $S_m = \mathbb{Z}_p[y_1, \ldots, y_r]/((1+y_i)^{p^m} - 1)$ and there is a surjective map of S_m-algebras

$$R_m \to \mathbb{T}_m/I_m$$

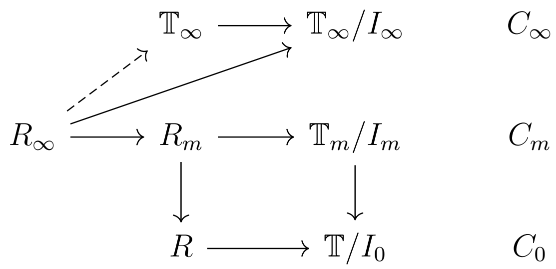
for some nilpotent ideal I_m such that there exists a $\delta \geq 1$, independent of m, with $I_m^\delta = 0$. Let $S_\infty = \mathbb{Z}_p[[y_1, \ldots, y_r]]$ and $\mathfrak{a} = (y_1, \ldots, y_r)$. For each m we also have the perfect complex of $\mathbb{Z}_p$-modules $C_m = R\Gamma(X^{K_m}_{\mathrm{GL}_n/F}, \mathbb{Z}_p)_\mathfrak{m}$ on which $\mathbb{T}_m$ acts. Finally, there also exists a complete Noetherian $\mathbb{Z}_p$-algebra R_∞ and surjections

$$R_\infty \twoheadrightarrow R_m$$

for all m. The following key properties hold

- $R_m/\mathfrak{a} = R$.
- $C_m \otimes S_m/\mathfrak{a} \cong C_0$ and $C_0[1/p] = (C_0)_\mathfrak{a}$ is concentrated in degrees $[q_0, q_0 + l_0]$.
- The localisation of R_∞ at the preimage of $\mathfrak{p} < R$ is a regular ring of dimension $r - l_0$. In the main proof we deduce this using [All16, section 1.2].

In chapter 2 we explain how to define a perfect complex of S_∞ modules C_∞ such that $C_\infty \otimes S_\infty/\mathfrak{a} = C_0$ which acts as the limit of the complexes C_m, even though there are no transition maps. The construction of such limits is known as Taylor–Wiles patching. Let $\mathbb{T}_\infty \subset \mathrm{End}(C_\infty)$ be the Hecke algebra acting on C_∞. The maps $R_\infty \to \mathbb{T}_m/I_m$ can be patched to a map $R_\infty \to \mathbb{T}_\infty/I_\infty$ for some nilpotent ideal. The situation is summarised in the following diagram.



There is also a natural S_∞-algebra homomorphism $\mathbb{T}_\infty \to \mathbb{T}$. The dashed arrow exists after localising R_∞ and $\mathbb{T}_\infty$ at $\mathfrak{q}$ since the localisation of R_∞ is a regular ring. However, it is not necessarily a S_∞-algebra morphism! Working with this arrow is the main innovation of our proof.

A lemma due to Calegari–Geraghty [CG18, Lemma 6.2] shows that $(C_\infty)_\mathfrak{a}$ is a projective resolution of the $(S_\infty)_\mathfrak{a}$-module $M := H^{q_0+l_0}(C_\infty)_\mathfrak{a}$ and the Auslander–Buchsbaum formula implies that M has $(S_\infty)_\mathfrak{a}$-depth $r - l_0$. Some commutative algebra in section 2.6 shows that $M_\mathfrak{q}$ viewed as a $R_{\infty,\mathfrak{p}}$-module via the dashed arrow also has depth $r - l_0 = \dim R_{\infty,\mathfrak{p}}$. Applying the Auslander–Buchsbaum formula again, we see that $M_\mathfrak{q}$ is free over $R_{\infty,\mathfrak{p}}$.

As $R_{\infty,\mathfrak{p}}$ is reduced, it follows that $M_\mathfrak{q}/I_\infty$ is a faithful $R_{\infty,\mathfrak{p}}$-module and that the map $R_{\infty,\mathfrak{p}} \to \mathbb{T}_{\infty,\mathfrak{q}}/I_\infty$ is an isomorphism. In previous approaches one would have only been able deduce that $M_\mathfrak{q}/I_\infty$ is a nearly faithful $\mathbb{T}_{\infty,\mathfrak{q}}$-module [Tay08, Definition 2.1], i.e. has nilpotent annihilator. This is why it was important to lift the map from $R_{\infty,\mathfrak{p}}$. To go back down to $R \to \mathbb{T}$ we can use the isomorphism

$$M_\mathfrak{q}/\mathfrak{a}M_\mathfrak{q} = H^{q_0+l_0}(C_0)_\mathfrak{q}.$$

It implies that $\mathrm{Ann}_{\mathbb{T}_{\infty,\mathfrak{q}}}(M_\mathfrak{q}/\mathfrak{a}M_\mathfrak{q})$ is the maximal ideal of $\mathbb{T}_{\infty,\mathfrak{q}}$. Consequently,

$$\mathfrak{a}M_\mathfrak{q} = \mathfrak{m}_{\mathbb{T}_{\infty,\mathfrak{q}}}M_\mathfrak{q}.$$

Now we can use that $M_\mathfrak{q}/I_\infty$ is faithful to apply lemma 2.5.9 and deduce that

$$\mathfrak{a} + I_\infty = \mathfrak{m}_{\mathbb{T}_{\infty,\mathfrak{q}}}.$$

But from the patching construction we also know that $\mathfrak{a} + I_\infty$ is identified with $\ker(R_{\infty,\mathfrak{p}} \to R_\mathfrak{p})$ under the isomorphism $R_{\infty,\mathfrak{p}} \to \mathbb{T}_{\infty,\mathfrak{q}}/I_\infty$. This shows that the map $R_\mathfrak{p} \to \mathbb{T}_\mathfrak{q}$ is an isomorphism! Finally, it follows that $R_\rho^{dR} = R_\mathfrak{p}^\wedge$ is a field and ρ is rigid.

1.7 Notation

If F is a number field, i.e. a finite extension of $\mathbb{Q}$, and v is a place of F and S is a finite set of places of F, then we denote by

F_v	the completion of F at v
$\mathbb{A}_F$	the ring of adeles of F
$\mathbb{A}_F^\infty$	the ring of finite adeles of F
$\mathbb{A}_F^S$	the ring of S-adeles of F
F_S	the maximal extension of F, unramified outside a set of places S
$G_{F,S}$	the Galois group $\mathrm{Gal}(F_S/F)$
$\mathcal{O}_F$	the ring of integers of F
$\mathcal{O}_{F_v}$	the ring of integers of F_v

Moreover, for a general field F we let $\overline{F}$ be a separable closure and $G_F := \mathrm{Gal}(\overline{F}/F)$. For any place v of a number field F, the choice of an embedding $\overline{F} \hookrightarrow \overline{F_v}$ gives rise to an embedding of absolute Galois groups $G_{F_v} \hookrightarrow G_F$. The conjugacy class of $G_{F_v} \hookrightarrow G_F$ is independent of choices, hence it makes sense to restrict the isomorphism class of a representation of G_F to G_{F_v}.

We let B_{dR} denote Fontaine's de Rham period ring. See [Ber04] for an introduction. If p is a prime, E is a finite extension of $\mathbb{Q}_p$, F is a number field such that

$|\operatorname{Hom}(F,E)| = |\operatorname{Hom}(F,\overline{\mathbb{Q}}_p)|$ and $\rho : G_F \to \mathrm{GL}_n(E)$ is a continuous representation, then we say that ρ is de Rham if for each place $v \mid p$, the restriction $\rho|_{G_{F_v}}$ is de Rham. Assuming that ρ is de Rham, we denote by $\mathrm{WD}(\rho|_{G_{F_v}})$ the Weil–Deligne representation attached to $\rho|_{G_{F_v}}$ for any place v of F. This construction is recalled in [All16, 1.1.4 and 1.1.6].

If F is any field in which the prime p is invertible, then for a continuous representation $\rho : G_F \to \mathrm{GL}_n(E)$ and $n \in \mathbb{Z}$, we define the Tate twist $\rho(n) := \rho \otimes_{\mathbb{Q}_p} \chi_{cyc}^n$, where $\chi_{cyc} : G_F \to \mathbb{Q}_p^\times$ is the cyclotomic character.

Chapter 2

Abstract Patching

This chapter consists of pure commutative algebra and forms the foundation of the main argument. Its purpose is to prove lemma 2.6.4 which allows us to deduce the isomorphism $R_x \cong \mathbb{T}_x$ mentioned in the introduction. To do so we employ a variant of the Taylor–Wiles patching method [TW95]. The main modifications that we need were proposed in [CG18] but we make a few further modifications in the main lemma below to deal with nilpotent ideals. Moreover, we use ultrafilters as a convenient organisational tool, as first introduced in this context in [Sch18]. See also the exposition in [Man21, section 4].

2.1 Non-Canonical Limits

It is common to define the p-adic integers as

$$\mathbb{Z}_p = \lim_{\leftarrow} \mathbb{Z}/p^n\mathbb{Z}.$$

Implicit in this definition is the system of reduction maps

$$\mathbb{Z}/p\mathbb{Z} \leftarrow \mathbb{Z}/p^2\mathbb{Z} \leftarrow \mathbb{Z}/p^3\mathbb{Z} \leftarrow \dots$$

which is really what defines the projective limit. The goal of patching is to be able to define similar objects *without* reference to such a system of morphisms. To the disciple of category theory this is of course blasphemy. However, sometimes in nature we encounter sequences of objects (M_n) without any maps between them but we still could use a limit object M_∞. The construction of such a limit is exactly the purpose of patching.

Let us sketch the idea. First, suppose that each M_n is a $\mathbb{Z}/p^n\mathbb{Z}$-module such that M_n is generated by at most C elements for some constant C. We cannot write $M_\infty =$

$\lim_{\leftarrow} M_n$ since we have no maps between the M_n. However, for each $e \geq 1$ there are only finitely many isomorphism classes of $\mathbb{Z}/p^e\mathbb{Z}$-modules which are generated by at most C elements. Hence the sequence $(M_n/p^e)_{n\geq 1}$ contains at least one isomorphism class infinitely often.

In particular, choosing $e = 1$, there is a sequence $(n_k^1)_{k\geq 1}$ and a $\mathbb{Z}/p\mathbb{Z}$-module M^1 such that $M^1 \cong M_{n_k^1}/p$ for all $k \geq 1$. Next, there is a subsequence $(n_k^2)_{k\geq 1}$ of $(n_k^1)_{k\geq 1}$ and a $\mathbb{Z}/p^2\mathbb{Z}$-module M^2 such that $M^2 \cong M_{n_k^2}/p^2$ for all k. Moreover, we have an isomorphism $M^2/p \cong M_{n_1^2}/p \cong M^1$. Now we can continue inductively and find a subsequence $(n_k^{j+1})_{k\geq 1}$ of $(n_k^j)_{k\geq 1}$ such that there exists a module M^{j+1} with $M^{j+1} \cong M_{n_k^{j+1}}/p^{j+1}$ for all $k \geq 1$ and an isomorphism $M^{j+1}/p^j \cong M^j$. Let $f^{j+1} : M^{j+1} \to M^j$ be defined as the composition $M^{j+1} \to M^{j+1}/p^j \cong M^j$ and let

$$M_\infty = \lim \left(M^1 \xleftarrow{f^2} M^2 \xleftarrow{f^3} M^3 \leftarrow \dots \right).$$

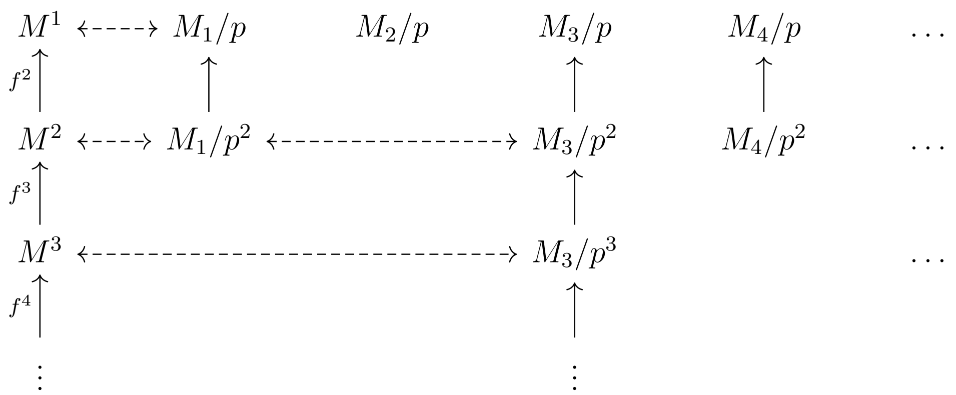


Figure 2.1: Construction of M_∞

This construction is illustrated in figure 2.1 for some example of a sequence of modules $(M_n)_{n\geq 1}$. The jth row of the figure displays the sequence of modules $(M_{n_k^j}/p^j)_{k\geq 1}$, which are all isomorphic to M^j. The dashed arrows are isomorphisms chosen in a way such that the diagram commutes and in our example we have

$$(n_k^1)_{k\geq 1} = (1, 2, 3, 4, \dots)$$
$$(n_k^2)_{k\geq 1} = (1, 3, 4, \dots)$$
$$(n_k^3)_{k\geq 1} = (1, 3, \dots)$$

The module M_∞ constructed in this way of course depends on many choices of subsequences. But that is not a problem in the applications as long as we make consistent choices. Ultraproducts turn out to be a very convenient tool to achieve this consistency.

Definition 2.1.1. Let X be a non-empty set and $\mathcal{P}(X)$ its power set. A non-empty subset $\mathcal{F} \subset \mathcal{P}(X)$ is called an ultrafilter on X if the following are satisfied.

- If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
- If $A \in \mathcal{F}$ and $B \supset A$, then $B \in \mathcal{F}$.
- If $A \subset X$, then exactly one of A and $X \setminus A$ belongs to $\mathcal{F}$.

For example, given an element $x \in X$, we have the ultrafilter

$$\mathcal{F}_x := \{A \in \mathcal{P}(X) : x \in A\}.$$

An ultrafilter $\mathcal{F}$ on X is called *non-principal* if there is no $x \in X$ such that $\mathcal{F} = \mathcal{F}_x$. Equivalently, an ultrafilter $\mathcal{F}$ is non-principal if it contains no finite sets. It can be shown with Zorn's lemma that non-principal ultrafilters exist whenever X is infinite. Given an ultrafilter $\mathcal{F}$ on $\{1, 2, 3, \dots\}$ and sets M_n, we define the ultraproduct

$$\left(\prod_{n \geq 1} M_n\right)_{\mathcal{F}} := \left(\prod_{n \geq 1} M_n\right)_{/\sim},$$

where $(x_n) \sim (y_n)$ if $\{n : x_n = y_n\} \in \mathcal{F}$. If $\mathcal{F} = \mathcal{F}_{n_0}$ is principal, then the projection to M_{n_0} induces a bijection $\left(\prod_{n \geq 1} M_n\right)_{\mathcal{F}} \cong M_{n_0}$. Hence ultraproducts are most interesting when $\mathcal{F}$ is non-principal.

In this case we still have the key property that for any finite partition

$$X = \bigcup_i X_i,$$

exactly one of the (necessarily infinite) X_i belongs to $\mathcal{F}$. This can be used to show that if all the M_n have cardinality bounded by a constant, then the cardinality of $\left(\prod_{n \geq 1} M_n\right)_{\mathcal{F}}$ is bounded by the same constant and is in bijection with M_n for infinitely many n (but not induced by a projection!). Moreover, for maps $f_n : M_n \to N_n$ there is a canonical induced map $\left(\prod_{n \geq 1} M_n\right)_{\mathcal{F}} \to \left(\prod_{n \geq 1} N_n\right)_{\mathcal{F}}$. Hence it makes sense to form the limit

$$\varprojlim_{e \geq 1} \left(\prod_{n \geq 1} M_n / p^e\right)_{\mathcal{F}}$$

for any sequence of $\mathbb{Z}_p$-modules M_n. This behaves very similarly to the construction of M_∞ via subsequences above but only relies on one choice: the non-principal ultrafilter $\mathcal{F}$. Below we will define the patching functor as precisely this limit. In the applications we make one choice of such a filter in the beginning and never change it. This hides the non-canonical nature of the patching method quite well and saves the paper from many indices.

2.2 Ultraproducts of Finite Sets

Let X be an infinite set and $\mathcal{F}$ a non-principal ultrafilter on it. Let $\mathbf{Set}_M$ be the category of sets of cardinality bounded by $M \in \mathbb{N}$. Then we have a functor

$$\mathrm{ulim}_{\mathcal{F}} : \prod_{\alpha \in X} \mathbf{Set}_M \to \mathbf{Set}_M : (S_\alpha) \mapsto \left(\prod_{\alpha \in X} S_\alpha \right)_{\mathcal{F}}$$

To see that this is well-defined, note that there are only finitely many isomorphism classes of objects in $\mathbf{Set}_M$, one for each non-negative integer $\leq M$. This yields the partition

$$X = \bigcup_{i=0}^{M} X_i,$$

where $X_i = \{\alpha \in X : |S_\alpha| = i\}$. Let j be the unique index such that $X_j \in \mathcal{F}$. For each $\alpha \in X_j$, let $f_\alpha : S_\alpha \cong \{1, 2, \ldots, j\}$ be a bijection. Then there is a unique bijection

$$f : \left(\prod_{\alpha \in X} S_\alpha \right)_{\mathcal{F}} \to \{1, 2, \ldots, j\}$$

such that for all $(s_\alpha)_{\alpha \in X} \in \prod_{\alpha \in X} S_\alpha$

$$\{\beta \in X_j : f_\beta(s_\beta) = f((s_\alpha)_{\alpha \in X})\} \in \mathcal{F}.$$

Alternatively, having cardinality bounded by M is expressible as a first order formula, hence Łoś' theorem [Hod08, Theorem 9.5.1] implies that the ultraproduct of such sets has again cardinality bounded by M. One can consider the same functor for groups, rings, modules over a finitely generated ring, etc. since there are only finitely many isomorphisms classes of groups, rings, modules over a finitely generated ring, etc. with cardinality bounded by M.

Lemma 2.2.1. *Let A be a ring of finite cardinality. There is a unique ring isomorphism $f : \left(\prod_{\alpha \in X} A\right)_{\mathcal{F}} \to A$ such that for each $(x_\alpha) \in \prod_{\alpha \in X} A$ we have*

$$\{\beta : x_\beta = f((x_\alpha))\} \in \mathcal{F}.$$

Proof. Given $(x_\alpha)_\alpha \in \prod_{\alpha\in X} A$ there exists a unique $y \in A$ such that $\{\alpha \in X : x_\alpha = y\} \in \mathcal{F}$ since we have the finite disjoint union

$$X = \bigcup_{y\in A} \{\alpha \in X : x_\alpha = y\}.$$

This uniquely characterises f as the function $f((x_\alpha)) := y$. It is clear that $(x_\alpha) \sim 0$ if and only if $f((x_\alpha)) = 0$. Moreover, f admits a section $s : A \to \prod_{\alpha\in X} A : a \mapsto (a)_{\alpha\in X}$. Hence f is a bijection $(\prod_\alpha A)_\mathcal{F} \to A$. If $f((x_\alpha)) = y$ and $f((z_\alpha)) = w$, then

$$\{\alpha : x_\alpha = y\} \cap \{\alpha : z_\alpha = w\} \subset \{\alpha : x_\alpha z_\alpha = yw\}.$$

Since filters are closed under finite intersection and supersets, we have $f((x_\alpha z_\alpha)) = yw$ and similarly $f((x_\alpha + z_\alpha)) = y + w$. Thus, f is a ring homomorphism. We already saw that it is bijective, hence f is an isomorphism. □

Lemma 2.2.2. *Let A be a local ring of finite cardinality with maximal ideal $\mathfrak{m}$. Let f be the map from lemma 2.2.1. The composition $g : \prod_{\alpha\in X} A \to \left(\prod_{\alpha\in X} A\right)_\mathcal{F} \xrightarrow{f} A$ is a localisation*[1] *of $\prod_{\alpha\in X} A$ at $g^{-1}(\mathfrak{m})$ in $\prod_{\alpha\in X} A$.*

Proof. Firstly, note that the complement of $g^{-1}(\mathfrak{m})$ maps to units under g since A is local. Now let $h : \prod_{\alpha\in X} A \to B$ be any ring homomorphism such that the complement of $g^{-1}(\mathfrak{m})$ is mapped to $B^\times$. Appealing to the universal property of quotients it remains to show that $\ker g \subset \ker h$. If $(x_\alpha) \in \ker g$, then $S := \{\alpha : x_\alpha = 0\} \in \mathcal{F}$. Let 1_S be the characteristic function on S, i.e. the element of $\prod_{\alpha\in X} A$ such that $1_{S,\alpha} = 1$ if $\alpha \in S$ and zero otherwise. By assumption $1_S \cdot (x_\alpha) = 0$. Moreover $g(1_S) = 1 \notin \mathfrak{m}$, hence $1_S \notin g^{-1}(\mathfrak{m})$. Finally, $h((x_\alpha)) = h(1_S)^{-1} \cdot h(1_S(x_\alpha)) = 0$ and so $\ker g \subset \ker h$. □

Lemma 2.2.3. *Let A be a local ring of finite cardinality. The functor*

$$\prod_{\alpha\in X} \mathbf{Mod}_A \to \mathbf{Mod}_{(\prod_{\alpha\in X} A)_\mathcal{F}} : \prod_{\alpha\in X} M_\alpha \mapsto \left(\prod_{\alpha\in X} M_\alpha\right)_\mathcal{F}$$

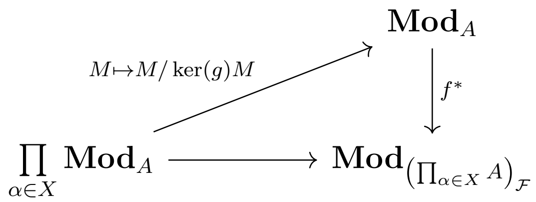
is exact.

Proof. Let f be the isomorphism from lemma 2.2.1 and $g = f \circ q$, where

$$q : \prod_{\alpha\in X} A \to \left(\prod_{\alpha\in X} A\right)_\mathcal{F}$$

[1] Meaning that it satisfies the universal property of localisation.

is the quotient map. We have the commutative diagram of functors



The functor f^* is an equivalence of categories and we have shown in lemma 2.2.2 that $M \mapsto M/\ker(g)M$ is a localisation, hence an exact functor. Thus, the functor in question is exact itself. □

Definition 2.2.4. Let A be a ring. A family $(M_\alpha)_{\alpha \in X}$ of A-modules is called *bounded* if there exists an integer s such that each M_α is a quotient of A^s.

Lemma 2.2.5. *Let A be a ring of finite cardinality and $(M_\alpha)_{\alpha \in X}$ a bounded family of A-modules. There exists a set $S \in \mathcal{F}$ such that for $\beta \in S$, there exist isomorphisms $f_{\alpha\beta} : M_\alpha \cong M_\beta$ for all $\alpha \in S$. Given S and $f_{\alpha\beta}$, there is a unique isomorphism*

$$f : \left(\prod_{\alpha \in X} M_\alpha \right)_{\mathcal{F}} \cong M_\beta$$

satisfying

$$\{\gamma \in S : f((m_\alpha)_{\alpha \in X}) = f_{\gamma\beta}(m_\gamma)\} \in \mathcal{F}$$

for all $(m_\alpha) \in \left(\prod_{\alpha \in X} M_\alpha\right)_{\mathcal{F}}$.

Proof. If each M_α is a quotient of A^s, then $|M_\alpha| \leq |A|^s$. There are finitely many isomorphism classes of A-modules of cardinality at most $|A|^s$. Choose a set of representatives $\mathcal{R}$ for them. Since $\mathcal{R}$ is finite, there exists a unique $N \in \mathcal{R}$ such that

$$S = \{\alpha \in X : M_\alpha \cong N\} \in \mathcal{F}.$$

Let $\beta \in S$ and choose isomorphisms $f_{\alpha\beta} : M_\alpha \to M_\beta$ for $\alpha \in S$. This proves the first part of the claim. It remains to find the uniqueness isomorphism f. The uniqueness is clear from the definition of the equivalence relation determined by $\mathcal{F}$. For the existence, note that if $(m_\alpha) \in \prod_{\alpha \in X} M_\alpha$, then there exists a unique $n \in M_\beta$ such that

$$\{\alpha \in S : f_{\alpha\beta}(m_\alpha) = n\} \in \mathcal{F}$$

and mapping $(m_\alpha) \mapsto n$ does the job. □

Lemma 2.2.6. *Let A be a ring of finite cardinality and $(M_\alpha)_{\alpha\in X}, (N_\alpha)_{\alpha\in X}$ bounded families of A-modules. The natural map*

$$\left(\prod_{\alpha\in X}\mathrm{Hom}_A(M_\alpha,N_\alpha)\right)_{\mathcal{F}} \to \mathrm{Hom}_A\left(\left(\prod_{\alpha\in X}M_\alpha\right)_{\mathcal{F}},\left(\prod_{\alpha\in X}N_\alpha\right)_{\mathcal{F}}\right)$$

is an isomorphism.

Proof. By lemma 2.2.5 there are sets $S_1, S_2 \in \mathcal{F}$ such that for $\beta \in S := S_1 \cap S_2$ we have isomorphisms $x_{\alpha\beta} : M_\alpha \cong M_\beta$ and $y_{\alpha\beta} : N_\alpha \cong N_\beta$ for all $\alpha \in S$. These give rise to isomorphisms $x : \left(\prod_{\alpha\in X} M_\alpha\right)_{\mathcal{F}} \cong M_\beta$ and $y : \left(\prod_{\alpha\in X} N_\alpha\right)_{\mathcal{F}} \cong N_\beta$ satisfying

$$\{\gamma\in S : x((m_\alpha)_{\alpha\in X}) = x_{\gamma\beta}(m_\gamma)\}\in\mathcal{F} \qquad \{\gamma\in S : y((n_\alpha)_{\alpha\in X}) = y_{\gamma\beta}(n_\gamma)\}\in\mathcal{F}.$$

Applying the second part of lemma 2.2.5 to $(\mathrm{Hom}_A(M_\alpha,N_\alpha))_{\alpha\in X}$, $\beta\in S$ and $z_{\alpha\beta}(h) = y_{\alpha\beta}\circ h\circ x_{\alpha\beta}^{-1}$ we obtain an isomorphism

$$z : \left(\prod_{\alpha\in X}\mathrm{Hom}_A(M_\alpha,N_\alpha)\right)_{\mathcal{F}} \to \mathrm{Hom}_A(M_\beta,N_\beta)$$

satisfying

$$\{\gamma\in S : z((h_\alpha)_{\alpha\in X}) = z_{\gamma\beta}(h_\gamma)\}\in\mathcal{F}.$$

The lemma now follows from the diagram

$$\begin{array}{ccc}
\left(\prod_{\alpha\in X}\mathrm{Hom}_A(M_\alpha,N_\alpha)\right)_{\mathcal{F}} & \longrightarrow & \mathrm{Hom}_A\left(\left(\prod_{\alpha\in X}M_\alpha\right)_{\mathcal{F}},\left(\prod_{\alpha\in X}N_\alpha\right)_{\mathcal{F}}\right)\\
\big\downarrow z & & \big\downarrow h\mapsto y\circ h\circ x^{-1}\\
\mathrm{Hom}_A(M_\beta,N_\beta) & \xrightarrow{\quad\mathrm{id}\quad} & \mathrm{Hom}_A(M_\beta,N_\beta)
\end{array}$$

which commutes because if $m\in M_\beta$ and $h_\alpha\in\mathrm{Hom}_A(M_\alpha,N_\alpha)$, then for all $\gamma\in S$, we have

$$(y_{\gamma\beta}\circ h_\gamma\circ x_{\gamma\beta}^{-1})(m) = z_{\gamma\beta}(h_\gamma)(m).$$

By intersecting the three elements of $\mathcal{F}$ constructed in this proof, we find a set $T\in\mathcal{F}$, such that for $\gamma\in T$ the left hand side equals $(y\circ(h_\alpha)_{\alpha\in X}\circ x^{-1})(m)$ and the right hand side equals $z((h_\alpha)_{\alpha\in X})(m)$. □

2.3 Patching Functor

Consider a complete Noetherian local ring R with maximal ideal $\mathfrak{m}$ and finite residue field $k := R/\mathfrak{m}$. We define the patching functor by

$$\mathbf{P}_R : \prod_{\alpha\in X}\mathbf{Mod}_R \to \mathbf{Mod}_R : (M_\alpha)_{\alpha\in X} \mapsto \varprojlim_{n\geq 1}\left(\prod_{\alpha\in X} M_\alpha/\mathfrak{m}^n\right)_{\mathcal{F}},$$

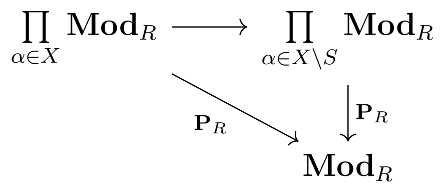
where $M_\alpha/\mathfrak{m}^n$ is short for $M_\alpha/\mathfrak{m}^n M_\alpha$. The R-module structure is given by acting diagonally and makes $\mathbf{P}_R$ into an R-linear functor. Note that this functor also depends on the ultrafilter $\mathcal{F}$ but we omit it from the notation. This is a special case of a *cataproduct* as defined in [Sch10, Chapter 8].

Lemma 2.3.1. *If M is a finitely generated R-module, then*

$$\mathbf{P}_R\left((M)_{\alpha\in X}\right) \cong M.$$

Proof. Since k is finite and M is finitely generated, $M/\mathfrak{m}^n$ is finite, too. By lemma 2.2.1 there is a natural isomorphism $\left(\prod_{\alpha\in X} M/\mathfrak{m}^n\right)_{\mathcal{F}} \cong M/\mathfrak{m}^n$ for each n. Now we note that M is complete as it is finitely generated and taking the inverse limit yields the desired isomorphism. □

Lemma 2.3.2. *For each finite set $S \subset X$ the we obtain a non-principal ultrafilter $\mathcal{F}^S = \{A \subset X \setminus S : A \in \mathcal{F}\}$ on $X \setminus S$. With respect to this ultrafilter, the functor $\mathbf{P}_R$ factors through the category $\prod_{\alpha\in X\setminus S} \mathbf{Mod}_R$.*



Proof. This follows from the fact that $\mathcal{F}$ is non-principal. □

Lemma 2.3.3 (Selection Lemma)**.** *If (M_α) is a bounded family of R-modules, then*

$$\mathbf{P}_R\left(M_\alpha\right) = \varprojlim M_{\alpha_n}/\mathfrak{m}^n$$

for some sequence α_n of elements of X and transition maps

$$f_n : M_{\alpha_{n+1}}/\mathfrak{m}^{n+1} \to M_{\alpha_{n+1}}/\mathfrak{m}^n \xrightarrow{\sim} M_{\alpha_n}/\mathfrak{m}^n.$$

Proof. Since k is finite and $\mathfrak{m}$ is finitely generated, $R/\mathfrak{m}^n$ is finite, too. Hence this is a corollary to lemma 2.2.5. □

Lemma 2.3.4. *Let A be a ring and (M_n, ϕ_{ij}) a countable inverse system of finite length A-modules and let N be a finitely presented A-module. The natural map*

$$\left(\varprojlim M_n\right) \otimes_A N \to \varprojlim \left(M_n \otimes_A N\right)$$

is an isomorphism.

Proof. Let $A^r \to A^s \to N \to 0$ be a finite presentation of N. Since tensor products are right exact we have an exact sequence

$$\left(\varprojlim M_n\right) \otimes_A A^r \to \left(\varprojlim M_n\right) \otimes_A A^s \to \left(\varprojlim M_n\right) \otimes_A N \to 0 \qquad (2.3.0.1)$$

On the other hand the finite length assumption implies that $\{\ker(M_n \otimes_A A^r \to M_n \otimes_A A^s)\}$ is a Mittag-Leffler system, hence

$$\varprojlim (M_n \otimes_A A^r) \to \varprojlim (M_n \otimes_A A^s) \to \varprojlim (M_n \otimes_A N) \to 0 \qquad (2.3.0.2)$$

is exact. Note that the map in the claim of the lemma is an isomorphism when $N = A^d$ for some finite d as can be checked directly by viewing the two limits as submodules of $(\prod_n M_n)^d \cong \prod_n M_n^d$. Hence the natural map from sequence 2.3.0.1 to sequence 2.3.0.2 induces the desired isomorphism. □

Lemma 2.3.5. *If (M_α) is a bounded family of R-modules, then there are natural isomorphisms*

$$\mathbf{P}_R\left(M_\alpha\right)/\mathfrak{m}^n \cong \mathbf{P}_R(M_\alpha/\mathfrak{m}^n) \cong \left(\prod_{\alpha\in X} M_\alpha/\mathfrak{m}^n\right)_{\mathcal{F}}$$

Proof. Firstly, note that under isomorphism from lemma 2.2.1 the ideal $\mathfrak{m} \subset R/\mathfrak{m}^n$ is mapped to $\left(\prod_{\alpha\in X} \mathfrak{m}\right)_{\mathcal{F}}$, hence $\left(\prod_{\alpha\in X} M_\alpha/\mathfrak{m}^n\right)_{\mathcal{F}}/\mathfrak{m}^k = \left(\prod_{\alpha\in X} M_\alpha/(\mathfrak{m}^n + \mathfrak{m}^k)\right)_{\mathcal{F}}$. Moreover, each $\left(\prod_{\alpha\in X} M_\alpha/\mathfrak{m}^n\right)_{\mathcal{F}}$ is finite length over R by lemma 2.2.5. Now lemma 2.3.4 yields the result. □

Lemma 2.3.6. *$\mathbf{P}_R$ is right exact on the category of bounded families of R-modules.*

Proof. Suppose we are given a short exact sequence

$$0 \to \prod_{\alpha\in X} A_\alpha \to \prod_{\alpha\in X} B_\alpha \to \prod_{\alpha\in X} C_\alpha \to 0$$

Using lemma 2.2.3 and the fact that tensor products are right exact we already have exact sequences

$$\left(\prod_{\alpha\in X} A_\alpha/\mathfrak{m}^n\right)_{\mathcal{F}} \to \left(\prod_{\alpha\in X} B_\alpha/\mathfrak{m}^n\right)_{\mathcal{F}} \to \left(\prod_{\alpha\in X} C_\alpha/\mathfrak{m}^n\right)_{\mathcal{F}} \to 0$$

for each n. Since (A_α) is bounded, a Mittag-Leffler argument shows that we also have exactness in the limit. □

Lemma 2.3.7. *If (M_α) is bounded, then $\mathbf{P}_R\left(M_\alpha\right)$ is finitely generated.*

Proof. Let s be sufficiently large and choose for each α, a surjection $f_\alpha : R^s \to M_\alpha$. Then by lemma 2.3.1 and lemma 2.3.6 we have a surjection

$$R^s \cong \mathbf{P}_R(R^s) \xrightarrow{\mathbf{P}_R(f_\alpha)} \mathbf{P}_R(M_\alpha). \qquad \square$$

Lemma 2.3.8. *If (M_α) and (N_α) are bounded families of R-modules, then there is a natural isomorphism*

$$\mathrm{Hom}_R\left(\mathbf{P}_R\left(M_\alpha\right), \mathbf{P}_R\left(N_\alpha\right)\right) \cong \varprojlim_n \left(\prod_{\alpha\in X} \mathrm{Hom}_R(M_\alpha/\mathfrak{m}^n, N_\alpha/\mathfrak{m}^n)\right)_{\mathcal{F}}$$

Proof. For any finitely generated R-modules M and N, the natural map

$$\mathrm{Hom}_R(M, N) \to \varprojlim_n \mathrm{Hom}_R(M/\mathfrak{m}^n, N/\mathfrak{m}^n)$$

is an isomorphism. Thus, lemmas 2.3.7 and 2.3.5 imply that the natural map

$$\mathrm{Hom}_R\left(\mathbf{P}_R\left(M_\alpha\right), \mathbf{P}_R\left(N_\alpha\right)\right) \to \varprojlim_n \mathrm{Hom}_R\left(\left(\prod_{\alpha\in X} M_\alpha/\mathfrak{m}^n\right)_{\mathcal{F}}, \left(\prod_{\alpha\in X} N_\alpha/\mathfrak{m}^n\right)_{\mathcal{F}}\right)$$

is an isomorphism. Now lemma 2.2.6 finishes the proof. $\square$

Lemma 2.3.9. *Let R' be another complete Noetherian local ring with maximal ideal $\mathfrak{m}'$ and finite residue field k' and suppose that $(M_\alpha)_{\alpha\in X}$ is a collection of R'-modules together with local ring homomorphisms $f_\alpha : R \to R'/\mathrm{Ann}_{R'}(M_\alpha)$ which allow us to also view M_α as an R-module. If (M_α) is a bounded family of R-modules, then the natural map $\mathbf{P}_R(M_\alpha) \to \mathbf{P}_{R'}(M_\alpha)$ is an isomorphism of R-modules, where the R-module structure on the second term comes from functoriality of $\mathbf{P}_{R'}$.*

Proof. We first show surjectivity. The assumption yields the inclusions

$$\mathfrak{m}^n M_\alpha \subset (\mathfrak{m}')^n M_\alpha$$

for all $n \geq 1$ and $\alpha \in X$. Note that the kernel of the surjective R-module map

$$\left(\prod_{\alpha\in X} M_\alpha/\mathfrak{m}^n\right)_{\mathcal{F}} \twoheadrightarrow \left(\prod_{\alpha\in X} M_\alpha/(\mathfrak{m}')^n\right)_{\mathcal{F}}$$

has finite length for each $n \geq 1$. Hence a Mittag-Leffler argument shows that the map $\mathbf{P}_R(M_\alpha) \to \mathbf{P}_{R'}(M_\alpha)$ is surjective, too. Similarly to lemma 2.3.3 we can write the map in question as the limit (with R'-linear transition maps) of the maps

$$M_{\alpha_n}/\mathfrak{m}^n \to M_{\alpha_n}/(\mathfrak{m}')^n$$

for some sequence α_n. Now let

$$(m_n) \in \lim_{\leftarrow} M_{\alpha_n}/\mathfrak{m}^n$$

be an element of the kernel. Then $m_n \in (\mathfrak{m}')^n M_{\alpha_n}$ for each $n \geq 1$ and by using the transition maps $M_{\alpha_{n+1}} \to M_{\alpha_n}$ we see that $m_n \in \bigcap_{k\geq 1} (\mathfrak{m}')^k M_{\alpha_n} + \mathfrak{m}^n M_{\alpha_n}$. Since M_{α_n} is finite over R', this implies that $m_n \in \mathfrak{m}^n M_{\alpha_n}$. Thus, the map is injective, too. □

2.4 Derived Patching Functor

Keep the notations from the previous section. We have shown in lemma 2.3.6 that $\mathbf{P}_R$ is a right exact functor on the category of bounded families of R-modules, viewed as a full subcategory of $\prod_{\alpha\in X} \mathbf{Mod}_R$. Unfortunately, this category is not necessarily abelian since it is not necessarily closed under subobjects. Thus, the standard methods of homological algebra do not apply here. Instead we will only prove the minimal functoriality property needed below but with more care it should be possible to develop a more general derived patching functor.

Definition 2.4.1. Let $e : X \to \mathbb{Z}_{\geq 1} \cup \{\infty\}$ be a function. Then we say a family $(M_\alpha)_{\alpha\in X}$ in $\prod_{\alpha\in X} \mathbf{Mod}_R$ is an e-family if $\mathfrak{m}^{e(\alpha)} M_\alpha = 0$. Equivalently, $(M_\alpha)_{\alpha\in X}$ is an e-family if it comes from an object in $\prod_{\alpha\in X} \mathbf{Mod}_{R/\mathfrak{m}^{e(\alpha)}}$. We say it is e-free if each M_α is a free $R/\mathfrak{m}^{e(\alpha)}$-module, where we use the convention that $\mathfrak{m}^\infty = 0$.

Lemma 2.4.2. *Let $e : X \to \mathbb{Z}_{\geq 1} \cup \{\infty\}$ be a function such that $\{\alpha \in X : e(\alpha) > N\} \in \mathcal{F}$ for all integers N. If (M_α) is a bounded family of R-modules, then the natural map $(M_\alpha) \to (M_\alpha/\mathfrak{m}^{e(\alpha)})$ induces an isomorphism $\mathbf{P}_R(M_\alpha) \to \mathbf{P}_R(M_\alpha/\mathfrak{m}^{e(\alpha)})$.*

Proof. By lemma 2.3.7 we know that $\mathbf{P}_R(M_\alpha/\mathfrak{m}^{e(\alpha)})$ is a finitely generated R-module. Hence with lemma 2.3.5 we find isomorphisms

$$\begin{aligned}
\mathbf{P}_R(M_\alpha/\mathfrak{m}^{e(\alpha)}) &\cong \lim_N \mathbf{P}_R(M_\alpha/\mathfrak{m}^{e(\alpha)} M_\alpha)/\mathfrak{m}^N \\
&\cong \lim_N \mathbf{P}_R(M_\alpha/\mathfrak{m}^{\min(N,e(\alpha))} M_\alpha) \\
&\cong \lim_N \mathbf{P}_R(M_\alpha/\mathfrak{m}^N M_\alpha) \\
&\cong \mathbf{P}_R(M_\alpha)
\end{aligned}$$

whose inverse is the map induced by $(M_\alpha) \to (M_\alpha/\mathfrak{m}^{e(\alpha)})$. □

Lemma 2.4.3. *Let $e : X \to \mathbb{Z}_{\geq 1} \cup \{\infty\}$ be a function such that $\{\alpha \in X : e(\alpha) > N\} \in \mathcal{F}$ for all integers N. If (M_α) is a bounded e-free family of R-modules, then $\mathbf{P}_R(M_\alpha)$ is a finitely generated free R-module.*

Proof. By lemma 2.3.7 we know that $\mathbf{P}_R(M_\alpha)$ is a finitely generated R-module. Moreover, lemma 2.3.5 implies that

$$\mathbf{P}_R(M_\alpha)/\mathfrak{m}^N \cong \mathbf{P}_R(M_\alpha/\mathfrak{m}^N)$$

is a free $R/\mathfrak{m}^N$ module for each $N \geq 1$. Hence $\mathbf{P}_R(M_\alpha)$ is free over R, too. □

Lemma 2.4.4. *Suppose that $(C_\alpha^\bullet)_{\alpha \in X}$ is a bounded complex of bounded e-free families of R-modules, where $e : X \to \mathbb{Z}_{\geq 1} \cup \{\infty\}$ is a function such that $\{\alpha \in X : e(\alpha) > N\} \in \mathcal{F}$ for all integers N, then*

- $\mathbf{P}_R(C_\alpha^\bullet)$ *is a bounded complex of finitely generated free R-modules.*
- *If $(D_\alpha)_{\alpha \in X}$ is another such complex and $f_\alpha^\bullet : C_\alpha^\bullet \to D_\alpha^\bullet$ is a sequence of nullhomotopic maps, then $\mathbf{P}_R(f_\alpha^\bullet)$ is nullhomotopic.*

Proof. The first bullet point is lemma 2.4.3. The second bullet point holds because $\mathbf{P}_R$ is an additive functor. □

2.5 Depth and Dimension

Here we state some results from commutative algebra that will be used to prove dimension inequalities in the main lemma.

Definition 2.5.1. Let R be a Noetherian local ring. If M is a finitely generated module over R, then we define $\mathrm{depth}_R(M)$ as the supremum of the lengths of sequences of elements $x_1, \ldots, x_n \in \mathfrak{m}_R$ such that $M/(x_1, \ldots, x_n)M \neq 0$ and x_i is not a zero divisor on $M/(x_1, \ldots, x_{i-1})$.

Lemma 2.5.2. *Let R be a Noetherian local ring and M a non-zero finitely generated R-module, then* $\mathrm{depth}_R(M)$ *is the smallest integer i such that* $\mathrm{Ext}^i_R(R/\mathfrak{m}_R, M) \neq 0$.

Proof. See [Sta23, Lemma 00LW]. □

Lemma 2.5.3 (Auslander–Buchsbaum formula)**.** *Let R be a Noetherian local ring and M a finitely generated R-module of finite projective dimension, then*

$$\mathrm{pd}_R(M) + \mathrm{depth}_R(M) = \mathrm{depth}(R)$$

Proof. See [Sta23, Proposition 090V]. □

Lemma 2.5.4. *Let $l_0 \geq 0$ and q_0 be integers and S a Cohen–Macaulay local ring of dimension $n \geq l_0$. Let $C^\bullet$ be a bounded complex of finite free S-modules which is not exact. Suppose that $H^i(C^\bullet/\mathfrak{m}_S)$ is non-zero only if $i \in [q_0, q_0 + l_0]$. Then $\dim_S H^*(C^\bullet) \geq \dim S - l_0$ and if equality holds, then $H^i(C^\bullet)$ is non-zero only if $i = q_0 + l_0$ and $H^{q_0+l_0}(C^\bullet)$ has projective dimension l_0.*

Proof. The idea goes back to [CG18]. See [KT17, Lemma 2.9] for a detailed proof. □

Lemma 2.5.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $R \to S$ a finite ring map. If $\mathfrak{n}$ is a maximal ideal of S and A is a finitely generated S-module, then*

$$\mathrm{depth}_R(A) \leq \mathrm{depth}_{S_\mathfrak{n}}(A_\mathfrak{n}).$$

Proof. See [Sta23, Lemma 0AUK]. □

Lemma 2.5.6. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $\widehat{R}$ its $\mathfrak{m}$-adic completion. Then for any finitely generated R-module M we have*

$$\mathrm{depth}_R(M) = \mathrm{depth}_{\widehat{R}}(M \otimes_R \widehat{R})$$

Proof. Since R is Noetherian, $\widehat{R}$ is a flat R-module and so [Sta23, Lemma 0338] applies. □

Lemma 2.5.7. *Let S, R be Noetherian local rings and $f : S \to R$ a local ring homomorphism. If M is an R-module which is finite over S, then*

$$\mathrm{depth}_S(M) = \mathrm{depth}_R(M)$$

Proof. This proof is from [DG67, IV. 16.4.8] but we include it for convenience. First suppose that $\mathrm{depth}_S(M) = 0$, then $0 \neq \mathrm{Hom}_S(S/\mathfrak{m}_S, M)$. Now

$$P := \mathrm{Hom}_S(S/\mathfrak{m}_S, M) \subset \mathrm{Hom}_S(S, M) = M$$

is a sub S-module of M. Since M is finite over S, P has finite S-length. But P is also naturally an R-submodule of M and has finite length over R since the action of S factors through R. Hence P has a simple R-submodule which must be isomorphic to $R/\mathfrak{m}_R$, hence $\mathrm{Hom}_R(R/\mathfrak{m}_R, M) \neq 0$ and $\mathrm{depth}_R(M) = 0$ as well.

If $\mathrm{depth}_S(M) > 0$, then we may choose an element $x \in \mathfrak{m}_S$ such that $M/f(x)M \neq 0$ and $f(x)$ is not a zero divisor on M. Then $\mathrm{depth}_S(M/xM) = \mathrm{depth}(M) - 1$ and $\mathrm{depth}_R(M/f(x)M) = \mathrm{depth}(M) - 1$, so the claim follows by induction. □

Lemma 2.5.8. *Let R be a Noetherian ring and $S \subset R$ a Noetherian subring over which R is integral. Then* $\dim(S) = \dim(R)$.

Proof. See [Mat80, 13.C]. ☐

Lemma 2.5.9. *Let k be a field and $R = k[[x_1, \dots, x_n]]$. If $\mathfrak{a} < R$ is an ideal and M is a finitely generated faithful R-module such that $\mathfrak{a}M = \mathfrak{m}M$, then $\mathfrak{a} = \mathfrak{m}$, where $\mathfrak{m} = (x_1, \dots, x_n)$ is the maximal ideal of R.*

Proof. We can write $\mathfrak{a} + \mathfrak{m}^2 = I + \mathfrak{m}^2$, where $I < R$ is an ideal generated by linear polynomials. It suffices to show that $I = \mathfrak{m}$ because then

$$\mathfrak{a}/\mathfrak{m}\mathfrak{a} \twoheadrightarrow \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{m}^2) = I/(I \cap \mathfrak{m}^2) = \mathfrak{m}/\mathfrak{m}^2$$

which implies that $\mathfrak{a} = \mathfrak{m}$ by Nakayama's lemma.

By construction $\mathfrak{a} \subset I + \mathfrak{m}^2$, hence

$$\mathfrak{m}M = \mathfrak{a}M \subset IM + \mathfrak{m}^2 M \subset \mathfrak{m}M.$$

It follows that $\mathfrak{m}(M/IM) = \mathfrak{m}^2(M/IM)$. By Nakayama's lemma this implies that $\mathfrak{m}(M/IM) = 0$, i.e. $\mathfrak{m}M \subset IM$. Now [AM69, Proposition 2.4] implies that for each $x \in \mathfrak{m}$, the endomorphism $\phi_x : M \to M : m \mapsto xm$ satisfies an equation of the form

$$\phi_x^k + a_{k-1}\phi_x^{k-1} + \cdots + a_0 = 0$$

with $a_i \in I$. Since M is faithful, x itself satisfies the same equation and $x^k \in I$. Note that $I < R$ is prime since it is generated by linear elements, hence $x \in I$ as well. Thus, $I = \mathfrak{m}$ as desired. ☐

2.6 Main Lemma

We begin with a standard lemma which forms a crucial ingredient of the argument below.

Lemma 2.6.1 (Formal Implicit Function Theorem)**.** *Let E be a field of characteristic 0 and R be a regular local E-algebra. Then for every complete noetherian local E-algebra T and local E-algebra map $R \to T/I$, where $I \subset T$ is a nilpotent ideal, there exists a lift $R \to T$ as indicated in the following diagram.*

$$\begin{array}{ccc} & & T \\ & \nearrow & \downarrow \\ R & \longrightarrow & T/I \end{array}$$

Proof. By [Sta23, Lemma 07EJ] the map $E \to R$ is formally smooth in the $\mathfrak{m}_R$-adic topology. Now the lemma follows directly from [Sta23, Lemma 07NJ]. $\square$

Example 2.6.2. Consider a non-constant polynomial $F \in \mathbb{R}[x, y]$ with a solution $(x_0, y_0) \in \mathbb{R}^2$, and consider the ring $R = \mathbb{R}[x, y]_{(x-x_0,y-y_0)}/(F)$ and take $T = \mathbb{R}[t]/(t^N)$ and $I = (t^2)$ for some $N > 1$. Then the lemma follows from the implicit function theorem since R is regular if and only if $dF_{(x_0,y_0)}$ is surjective.

Definition 2.6.3. Let R be a local ring with maximal ideal $\mathfrak{m}$. A complex $C^\bullet$ of R-modules is called minimal if the differentials of $C^\bullet \otimes_R R/\mathfrak{m}$ are zero.

Lemma 2.6.4 (Main Lemma)**.** *Let $S_\infty = W[[x_1, \ldots, x_r]]$, where W is a complete discrete valuation ring with finite residue field of characteristic p and uniformizer ϖ and set $\mathfrak{a} = (x_1, \ldots, x_r)$ and $\mathfrak{a}_N = ((1+x_1)^{p^N} - 1, \ldots, (1+x_r)^{p^N} - 1)$ and $S_N = S_\infty/\mathfrak{a}_N$ for $N \geq 0$. Suppose we are given*

(1) A complete noetherian local W-algebra R_∞.

(2) For each $N \geq 0$, a local S_N-algebra R_N which is a quotient of R_∞ such that $R_N/\mathfrak{a} = R_0$.

(3) For each $N \geq 0$, a minimal complex $C_N^\bullet$ of free S_N-modules such that $C_N^\bullet/\mathfrak{a} = C_0^\bullet$ and $C_0^\bullet$ is finitely generated.

(4) Integers q_0, l_0 such that $C_0^\bullet[1/p]$ is not exact and concentrated in degrees $[q_0, q_0+l_0]$ and $\dim R_\infty[1/p] + l_0 \leq \dim(S_\infty)_\mathfrak{a} = r$.

(5) For each $N \geq 0$, a commutative S_N-subalgebra $T_N \subset \operatorname{End}_{\mathbf{D}(S_N)}(C_N^\bullet)$ whose image in $\operatorname{End}_{\mathbf{D}(S_0)}(C_0^\bullet)$ is contained in T_0.

(6) A constant $\delta > 0$ such that for each $N \geq 0$, there is an ideal I_N of T_N satisfying $I_N^\delta = 0$ with I_0 equal to the nilradical of T_0 and such that there exists a surjective S_N-algebra homomorphism $R_N \twoheadrightarrow T_N/I_N$ making the square

$$\begin{array}{ccc} R_N & \longrightarrow & T_N/I_N \\ \downarrow & & \downarrow \\ R_0 & \longrightarrow & T_0/I_0 \end{array}$$

commute.

(7) A maximal ideal $\mathfrak{q}$ of $T_0[1/p]$ with image $\mathfrak{p}$ in R_0 under the map

$$(\operatorname{Spec} T_0[1/p])_{red} \to (\operatorname{Spec} T_0)_{red} \cong (\operatorname{Spec}(T_0/I_0))_{red} \to \operatorname{Spec} R_0 \hookrightarrow \operatorname{Spec} R_\infty$$

such that $\widehat{(R_\infty)_\mathfrak{p}}$ is regular and $(T_0)_\mathfrak{q}$ is a field.

Then $\dim \widehat{(R_\infty)_\mathfrak{p}} = r - l_0$ and there is an isomorphism $(R_0)_\mathfrak{p} \cong (T_0)_\mathfrak{q}$ which is compatible with the map $R_0 \to T_0/I_0$.

Proof. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{Z}_{\geq 0}$ and set

$$C^\bullet_\infty := \mathbf{P}_{S_\infty}(C^\bullet_N), \quad T'_\infty := \mathbf{P}_{S_\infty}(T_N), \quad I'_\infty := \mathbf{P}_{S_\infty}(I_N).$$

Then

(a) Using (3) and lemma 2.4.4 we find that $C^\bullet_\infty$ is a minimal complex of finitely generated free S_∞-modules. By right exactness of $\mathbf{P}_{S_\infty}$ we have $C^\bullet_\infty \otimes_{S_\infty} S_0 = C^\bullet_0$.

(b) From (5) we obtain a commutative square for each $N \geq 1$

$$\begin{array}{ccc} T_N & \longrightarrow & \operatorname{End}_{\mathbf{D}(S_N/\mathfrak{m}_{S_N}^{e(N)})}(C^\bullet_N/\mathfrak{m}^{e(N)}) \\ \downarrow & & \downarrow \\ T_0 & \longrightarrow & \operatorname{End}_{\mathbf{D}(S_0/\mathfrak{m}_{S_0}^{e(N)})}(C^\bullet_0/\mathfrak{m}^{e(N)}) \end{array}$$

and by the second bullet point of lemma 2.4.4 we obtain a commutative square

$$\begin{array}{ccc} T'_\infty & \longrightarrow & \operatorname{End}_{\mathbf{D}(S_\infty)}(C^\bullet_\infty) \\ \downarrow & & \downarrow \\ T_0 & \longrightarrow & \operatorname{End}_{\mathbf{D}(S_0)}(C^\bullet_0) \end{array}$$

We denote the image of $T'_\infty \to \operatorname{End}_{\mathbf{D}(S_\infty)}(C^\bullet_\infty)$ by T_∞. Since the family $(C^\bullet_N)$ is bounded (as $C^\bullet_N/\mathfrak{m}_{S_N} \cong C_0/\varpi$), we find that $C^\bullet_\infty$ is a finitely generated S_∞-module by lemma 2.3.7 and so is $\operatorname{End}_{\mathbf{D}(S_\infty)}(C^\bullet_\infty)$. Hence T_∞ is a finite S_∞-algebra as S_∞ is Noetherian. From the diagram we also conclude that the image of T_∞ in $\operatorname{End}_{\mathbf{D}(S_0)}(C^\bullet_0)$ is contained in T_0.

(c) We define I_∞ as the image of I'_∞ in T_∞. It satisfies $I^\delta_\infty = 0$ since that can be described by a first order formula which holds for each I_N.

(d) By the right exactness of $\mathbf{P}_{R_\infty}$, we find a W-algebra surjection $R_\infty \twoheadrightarrow \mathbf{P}_{R_\infty}(R_N)$. Moreover, the S_∞-algebra homomorphism $\mathbf{P}_{R_\infty}(R_N) \twoheadrightarrow \mathbf{P}_{R_\infty}(T_N/I_N)$ is surjective. Lemma 2.3.9 gives a natural isomorphism of S_∞-algebras $\mathbf{P}_{R_\infty}(T_N/I_N) \cong \mathbf{P}_{S_\infty}(T_N/I_N) \cong T'_\infty/I'_\infty$. Together with the commutative diagram from assumption (6) we obtain a commmutative diagram

$$\begin{array}{ccccc} R_\infty & \twoheadrightarrow & \mathbf{P}_{R_\infty}(R_N) & \twoheadrightarrow & T_\infty/I_\infty \\ \big\downarrow\!\!\downarrow & & & & \big\downarrow \\ R_0 & & \twoheadrightarrow & & T_0/I_0 \end{array}$$

where all arrows except those starting at R_∞ are S_∞-linear.

We apply lemma 2.5.4 to $(C^\bullet_\infty)_{\mathfrak{a}}$, noting that $(C^\bullet_\infty)_{\mathfrak{a}}/\mathfrak{a} = C^\bullet_0[1/p]$ is concentrated in degrees $[q_0, q_0 + l_0]$. Hence $\dim_{(S_\infty)_{\mathfrak{a}}} H^{q_0+l_0}((C^\bullet_\infty)_{\mathfrak{a}}) \geq \dim(S_\infty)_{\mathfrak{a}} - l_0$ and if equality holds, then $H^{q_0+l_0}((C^\bullet_\infty)_{\mathfrak{a}})$ has projective dimension l_0.

On the other hand, since $C^\bullet_\infty$ is finitely generated, the map

$$(S_\infty)_{\mathfrak{a}}/\operatorname{Ann}_{(S_\infty)_{\mathfrak{a}}}(H^{q_0+l_0}(C^\bullet_\infty)_{\mathfrak{a}}) \hookrightarrow (T_\infty)_{\mathfrak{a}}/\operatorname{Ann}_{(T_\infty)_{\mathfrak{a}}}(H^{q_0+l_0}(C^\bullet_\infty)_{\mathfrak{a}})$$

is finite and with lemma 2.5.8 we conclude

$$\dim_{(S_\infty)_{\mathfrak{a}}} H^{q_0+l_0}(C^\bullet_\infty)_{\mathfrak{a}} = \dim_{(T_\infty)_{\mathfrak{a}}} H^{q_0+l_0}(C^\bullet_\infty)_{\mathfrak{a}}.$$

Since Krull dimension decreases with surjective ring homomorphisms and is insensitive to nilpotent thickenings we find that

$$\dim_{(S_\infty)_{\mathfrak{a}}} H^{q_0+l_0}(C^\bullet_\infty)_{\mathfrak{a}} \leq \dim(T_\infty)_{\mathfrak{a}} = \dim(T_\infty)_{\mathfrak{a}}/I_\infty \leq \dim \mathbf{P}_{R_\infty}(R_N)_{\mathfrak{a}}.$$

Moreover, $\mathbf{P}_{R_\infty}(R_N)_{\mathfrak{a}}$ is a localisation of $\mathbf{P}_{R_\infty}(R_N)[1/p]$ since $p \notin \mathfrak{a}$. Hence

$$\dim \mathbf{P}_{R_\infty}(R_N)_{\mathfrak{a}} \leq \dim \mathbf{P}_{R_\infty}(R_N)[1/p].$$

With the bound above and the surjection $R_\infty \to \mathbf{P}_{R_\infty}(R_N)$ we conclude

$$\dim_{(S_\infty)_{\mathfrak{a}}} H^{q_0+l_0}(C^\bullet_\infty)_{\mathfrak{a}} \leq \dim R_\infty[1/p] \leq \dim(S_\infty)_{\mathfrak{a}} - l_0,$$

hence we must have equalities throughout. Thus, the projective dimension of the $(S_\infty)_{\mathfrak{a}}$-module $H^{q_0+l_0}((C^\bullet_\infty)_{\mathfrak{a}})$ is l_0 and by the Auslander–Buchsbaum formula (lemma 2.5.3), we find that $\operatorname{depth}_{(S_\infty)_{\mathfrak{a}}} H^{q_0+l_0}((C^\bullet_\infty)_{\mathfrak{a}}) = \dim(S_\infty)_{\mathfrak{a}} - l_0 = \dim R_\infty[1/p]$. Now $(C^\bullet_\infty)_{\mathfrak{a}}$ is a projective resolution of $H^{q_0+l_0}(C^\bullet_\infty)_{\mathfrak{a}}$ so that in fact we can view $(T_\infty)_{\mathfrak{a}} \subset \operatorname{End}_{(S_\infty)_{\mathfrak{a}}}(M_{\mathfrak{a}})$, where $M := H^{q_0+l_0}(C^\bullet_\infty)$.

Denote the kernel of $T_\infty \to T_0/\mathfrak{q}$ also by $\mathfrak{q}$, then $\mathfrak{a}T_\infty \subset \mathfrak{q}$ and the image of $\mathfrak{p} < R_\infty$ in T_∞/I_∞ equals $\mathfrak{q}$ as can be seen from the diagram

$$\begin{array}{ccccc} R_\infty & \longrightarrow & R_0 & \longrightarrow & R_0/\mathfrak{p} \\ \downarrow\!\!\downarrow & & \downarrow & & \downarrow \\ T_\infty/I_\infty & \longrightarrow & T_0/I_0 & \longrightarrow & T_0/\mathfrak{q} \end{array}$$

Thus, we obtain a surjective local map $(R_\infty)_\mathfrak{p} \to (T_\infty)_\mathfrak{q}/I_\infty$, hence also a surjective local map on completions $\widehat{(R_\infty)_\mathfrak{p}} \to \widehat{(T_\infty)_\mathfrak{q}/I_\infty}$. To simplify the notation we define

$$\begin{aligned} \widehat{R} &:= \widehat{(R_\infty)_\mathfrak{p}}, \\ \widehat{T} &:= \widehat{(T_\infty)_\mathfrak{q}}, \\ \widehat{I} &:= \ker(\widehat{(T_\infty)_\mathfrak{q}} \to \widehat{(T_\infty)_\mathfrak{q}/I_\infty}). \end{aligned}$$

Since $I_\infty^\delta = 0$, we have $(I_\infty + \mathfrak{q}^n)^\delta \subset \mathfrak{q}^n$ for all $n \geq 1$ and thus $\widehat{I}^\delta = 0$ as well. Since $\widehat{R} = \widehat{(R_\infty)_\mathfrak{p}}$ is a regular $W[1/p]$-algebra, we can apply Lemma 2.6.1 to choose a lifting

$$\begin{array}{ccc} & & \widehat{T} \\ & \nearrow & \downarrow \\ \widehat{R} & \longrightarrow & \widehat{T}/\widehat{I} \end{array}$$

We use it to equip $\widehat{M} := M_\mathfrak{a} \otimes_{(T_\infty)_\mathfrak{a}} \widehat{(T_\infty)_\mathfrak{q}}$ with a $\widehat{R}$-module structure. This is the key step for dealing with the nilpotent ideals.

Note that $\widehat{R} \to \widehat{T}$ is a finite local ring map since $\widehat{R} \to \widehat{T}/\widehat{I}$ is surjective and $\widehat{I}$ is nilpotent and finitely generated, i.e. $\widehat{T}$ is a quotient of $\widehat{R}[\epsilon_1, \dots, \epsilon_t]/(\epsilon_i^N)$. Hence lemma 2.5.7 implies that

$$\operatorname{depth}_{\widehat{R}} \widehat{M} = \operatorname{depth}_{\widehat{T}} \widehat{M} = \operatorname{depth}_{(T_\infty)_\mathfrak{q}}(M_\mathfrak{a} \otimes_{(T_\infty)_\mathfrak{a}} (T_\infty)_\mathfrak{q})$$

where the second equality follows from lemma 2.5.6. Now lemma 2.5.5 implies

$$\operatorname{depth}_{\widehat{R}} \widehat{M} \geq \operatorname{depth}_{(S_\infty)_\mathfrak{a}} M_\mathfrak{a} \geq \dim R_\infty[1/p] \geq \dim(R_\infty)_\mathfrak{p} = \dim \widehat{R}.$$

Applying the Auslander–Buchsbaum formula to the regular ring $\widehat{R}$, we find that $\widehat{M}$ is a projective, hence free $\widehat{R}$-module. It is non-zero since $\mathfrak{q}$ is in the support of $\widehat{M}/\mathfrak{a}$ as C_0 is not exact. In particular, $\dim \widehat{R} = \dim R_\infty[1/p] = r - l_0$.

Since $\widehat{M}$ is free over $\widehat{R}$, the map $\widehat{R} \to \widehat{T}$ is injective. The regularity of $\widehat{R}$ implies that it is reduced, hence $\widehat{R} \to \widehat{T}/\widehat{I}$ is injective as well. By construction it is surjective and factors through the map $\widehat{\mathbf{P}_{R_\infty}(R_N)_\mathfrak{p}} \to \widehat{T}/\widehat{I}$. Thus, we have isomorphisms

$$\widehat{R} \cong \widehat{\mathbf{P}_{R_\infty}(R_N)_\mathfrak{p}} \cong \widehat{T}/\widehat{I}$$

and we equip $\widehat{R}$ with an S_∞-algebra structure via these isomorphisms.

$(T_0)_\mathfrak{q}$ is a field by assumption (7), i.e. $\mathfrak{q}(T_0)_\mathfrak{q} = 0$. Recall that from (b) above we know that the image of T_∞ in $\mathrm{End}_{S_0}(C_0)$ is contained in T_0. Consequently, $\mathfrak{q}T_\infty$ acts as 0 on

$$\widehat{H^{q_0+l_0}(C_0)_\mathfrak{q}} \cong \widehat{M}/\mathfrak{a}\widehat{M}$$

and $\mathrm{Ann}_{\widehat{T}}(\widehat{M}/\mathfrak{a}\widehat{M})$ must be the maximal ideal of $\widehat{T}$. Thus,

$$\mathfrak{m}_{\widehat{T}}\widehat{M} \subset \mathfrak{a}\widehat{M} \subset \mathfrak{m}_{\widehat{T}}\widehat{M}.$$

In particular, we have an equality of $\widehat{R}$-modules

$$\mathfrak{a}\left(\widehat{M}/\widehat{I}\widehat{M}\right) = \mathfrak{m}_{\widehat{R}}\left(\widehat{M}/\widehat{I}\widehat{M}\right).$$

The module $\widehat{M}/\widehat{I}\widehat{M}$ is faithful over $\widehat{R}$ since if $r \in \widehat{R}$ kills it, then any lift $\tilde{r} \in \widehat{T}$ satisfies $\tilde{r}\widehat{M} \subset \widehat{I}\widehat{M}$. Now [AM69, Proposition 2.4] shows that $\tilde{r}^k \in \widehat{I}$ for some $k \geq 1$. Hence $r^k = 0$ and the regularity of $\widehat{R}$ implies that $r = 0$.

By the Cohen structure theorem we can apply lemma 2.5.9 to the $\widehat{R}$-module $\widehat{M}/\widehat{I}\widehat{M}$ and find that

$$\mathfrak{a}\widehat{R} = \mathfrak{m}_{\widehat{R}}.$$

Hence $(R_0)_\mathfrak{p} = \widehat{(R_0)_\mathfrak{p}} = \widehat{R}/\mathfrak{a}\widehat{R}$ is a field and the natural map $(R_0)_\mathfrak{p} \to (T_0)_\mathfrak{q}$ must be injective. By construction it is compatible with the initial $R_0 \to T_0/I_0$, hence also surjective. $\square$

Chapter 3

Locally Symmetric Spaces

In this chapter we define the spaces we need for the main argument. Their cohomology will carry the Hecke eigensystems corresponding to the Galois representations we wish to study. This is essentially a recollection of results from [BS73] and [NT16].

3.1 The Symmetric Space of a Linear Algebraic Group

The starting point for our definitions is a linear algebraic group G, defined over a number field F. Moreover, for the remainder of this section we assume that we have picked a maximal compact subgroup $K_\infty \subset G(F \otimes \mathbb{R})$.

Definition 3.1.1. Let $R_u G$ be the unipotent radical of G and let S_G be the identity component of the real points of the $\mathbb{Q}$-split part of the centre of $\mathrm{Res}_{F/\mathbb{Q}}(G/R_u G)$.

Example 3.1.2. If $G = \mathrm{GL}_n/F$, where F is any number field, then $S_G \cong \mathbb{R}^\times_{>0}$ embedded as scalar matrices in $\mathrm{GL}_n(F \otimes \mathbb{R})$.

Definition 3.1.3. Let G be a connected linear algebraic group over a number field F. For a section $s : (G/R_u G)(F \otimes \mathbb{R}) \to G(F \otimes \mathbb{R})$ we define the symmetric space (a smooth real manifold)

$$X_{G,s} := G(F \otimes \mathbb{R})/K_\infty s(S_G).$$

Example 3.1.4. Here are some examples of these symmetric spaces

1. If $G = \mathbb{G}_m$ over F, then $R_u G = 1$ and we can choose s as the identity. Then we find a diffeomorphism

$$X_{G,s} \to \mathbb{R}^{N_\infty}/\mathbb{R}(1, 1, \ldots, 1) : x \mapsto (\log |x|_i)_{i \in N_\infty},$$

where N_∞ denotes the set of equivalence classes of archimedean absolute values on F. Note that the regulator of F is the volume of $\mathcal{O}_F^\times \backslash X_{G,s}$, where $X_{G,s}$ is given its euclidean metric.

2. If $G = \mathrm{GL}_2/\mathbb{Q}$, $s = \mathrm{id}$ and $K_\infty = O_2(\mathbb{R})$, then

$$X_{G,s} \to \mathbb{H} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{a\varepsilon i + b}{c\varepsilon i + d}$$

is a diffeomorphism, where $\mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$ and $\varepsilon = \mathrm{sign}(ad - bc)$.

3. If $G = \mathrm{GL}_n/\mathbb{Q}$, $K_\infty = O_n(\mathbb{R})$, $s = \mathrm{id}$ and $Y \subset M_n(\mathbb{R})$ is the set of positive definite, symmetric matrices of determinant 1, then

$$X_{G,s} \to Y : A \mapsto A^T A |\det A|^{-2/n}$$

is a diffeomorphism.

4. If $G = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \subset \mathrm{GL}_3$ over $F = \mathbb{Q}$, $K_\infty = \{\pm 1\} \times O_2(\mathbb{R})$ and $s : G/R_u G \cong \mathrm{GL}_1 \times \mathrm{GL}_2 \to \mathrm{GL}_3$ is the natural embedding, then

$$X_{G,s} \to \mathbb{H} \times \mathbb{R}^2 : \begin{bmatrix} 1 & t & u \\ 0 & a & b \\ 0 & c & d \end{bmatrix} \mapsto \left(\frac{a\varepsilon i + b}{c\varepsilon i + d}, \frac{t}{|ad - bc|^{1/2}}, \frac{u}{|ad - bc|^{1/2}} \right)$$

is a diffeomorphism, where $\varepsilon = \mathrm{sign}(ad - bc)$.

Proposition 3.1.5. *Let G be a reductive group over F and $P \subset G$ a parabolic subgroup. Then $P(F \otimes \mathbb{R})$ contains a unique Levi subgroup stable under the Cartan involution associated with K_∞. Thus, there is a unique section $s : (P/R_u P)(F \otimes \mathbb{R}) \to P(F \otimes \mathbb{R}) \subset G(F \otimes \mathbb{R})$ whose image is stable under the Cartan involution.*

Proof. See [BS73, Corollary 1.9]. □

Definition 3.1.6. Let G be a reductive group over F and let K_∞ be a maximal compact subgroup of $G(\mathbb{R} \otimes F)$. For a parabolic subgroup $P \subset G$ the proposition provides us with a canonical choice of section s and we will simply write

$$X_P := X_{P,s}$$

for this choice of s.

Definition 3.1.7. Let $P \subset G$ be a parabolic subgroup of a reductive group over F, then we define $A_P := s(S_P)/S_G$, where s is the section stable under Cartan involution.

Lemma 3.1.8. *If $P \subset G$ is a parabolic subgroup of a reductive group over F, then there is a natural diffeomorphism $X_P \cong X_G/A_P$.*

Proof. We have $G(F \otimes \mathbb{R}) = P(F \otimes \mathbb{R})K_\infty$ by the Iwasawa decomposition. Hence naturally $X_G \cong P(F \otimes \mathbb{R})/(K_\infty \cap P(F \otimes \mathbb{R}))S_G$. Now $K_\infty \cap P(F \otimes \mathbb{R})$ is a maximal compact subgroup of $P(F \otimes \mathbb{R})$, hence $X_G/A_P \cong X_P$. □

Example 3.1.9. Let $P = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \subset \mathrm{GL}_2/\mathbb{Q}$. Then $A_P = \left\{ \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} : t > 0 \right\}$ and under the identification $X_{\mathrm{GL}_2} = \mathbb{H}$ from above we have

$$(x + iy) \cdot \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} = x + ity,$$

hence we can see how $X_{\mathrm{GL}_2}/A_P = \mathbb{R} = X_P$.

Definition 3.1.10. If A is a finite product of copies of $\mathbb{R}^\times_{>0}$, we define a manifold with corners $\overline{A}$ by replacing each $\mathbb{R}^\times_{>0}$ factor with $\mathbb{R}^\times_{>0} \cup \{\infty\}$. We let A act on $\overline{A}$ in the obvious way fixing ∞.

Definition 3.1.11. Given topological spaces X, Y and a topological group G acting on the right on X and on the left on Y, we define $X \times_G Y := (X \times Y)/G$, where $g \in G$ acts diagonally.

Lemma 3.1.12. *Let $P \subset G$ be a parabolic subgroup of a reductive group over F, then the topological space $Y_P := X_G \times_{A_P} \overline{A}_P$ is a disjoint union*

$$Y_P = \bigcup_{P \subset Q} X_Q$$

over all parabolic subgroups of G containing P and the X_Q are naturally identified with locally closed subsets of Y_P.

Proof. See [BS73, §5]. □

Example 3.1.13. Here are two examples of Y_P for different P.

1. Let $P = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \subset \mathrm{GL}_2/\mathbb{Q}$. Then $A_P = \left\{ \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} : t > 0 \right\} \cong \mathbb{R}^\times_{>0}$ and so we have

   $$Y_P = X_{\mathrm{GL}_2} \times_{A_P} A_P \cup X_{\mathrm{GL}_2} \times_{A_P} \{\infty\} = X_{\mathrm{GL}_2} \cup X_{\mathrm{GL}_2}/A_P = X_{\mathrm{GL}_2} \cup X_P.$$

   From the previous example we know that the A_P orbits in X_{GL_2} are of the form $\gamma_x(t) = x + it$ and to each such orbit we add a limit at $t = \infty$ to obtain Y_P.

2. Let $B = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \subset \mathrm{GL}_3 /\mathbb{Q}$ and $P = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix}, Q = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$. Then

$$A_B \cong \mathbb{R}^\times_{>0} \times \mathbb{R}^\times_{>0} \cong \left\{ \begin{bmatrix} st^{-2} & 0 & 0 \\ 0 & st & 0 \\ 0 & 0 & ts^{-2} \end{bmatrix} : s, t > 0 \right\}$$

and $A_P = \{(s, 1) \in A_B\}$ and $A_Q = \{(1, t) \in A_B\}$. Hence $\overline{A}_B \cong A_B \cup A_B/A_P \cup A_B/A_Q \cup A_B/A_B$ so that

$$Y_B = X_G \cup X_P \cup X_Q \cup X_B.$$

$X_B \subset Y_B$ is closed, $X_G \subset Y_B$ is open and dense, and $\partial X_P = \partial X_Q = X_B$. Combinatorially, X_G is the interior of a polyhedron Y_B, X_P and X_Q are faces and X_B is an edge. These closure relations can also be read off from the lattice of parabolic subgroups containing B:

$$\begin{array}{ccccc} & & P & & \\ & \nearrow & & \searrow & \\ B & & & & G \\ & \searrow & & \nearrow & \\ & & Q & & \end{array}$$

Theorem 3.1.14 (Borel–Serre)**.** *Let G be a reductive group over F. Then the disjoint union over parabolic subgroups of G*

$$\overline{X}_G := \bigcup_{P \subset G} X_P$$

carries the structure of a manifold with corners such that for each P, the set

$$\bigcup_{P \subset Q} X_Q$$

is open and has the topology given by $X_G \times_{A_P} \overline{A}_P$. Moreover, the action of $G(F)$ on X_G extends to $\overline{X}_G$.

Proof. See [BS73, §7.1 and Proposition 7.6]. □

3.2 Quotients by Congruence Subgroups

The symmetric spaces X_G and their Borel–Serre compactifications $\overline{X}_G$ carry an action by $G(F)$. This action places us in the subject of number theory. The Hecke actions that we study in this thesis make use of the family of congruence subgroups of $G(F)$. In particular we are interested in the cohomology of quotients $\Gamma\backslash X_G$ and $\Gamma\backslash\overline{X}_G$ for congruence subgroups $\Gamma < G(F)$ and how it varies with Γ.[1] In our case, Γ usually acts freely and X_G is contractible so that the cohomology of $\Gamma\backslash X_G$ is isomorphic to the group cohomology of Γ.

We introduce some notation to define congruence subgroups and the corresponding locally symmetric spaces in modern (adelic) terms. Let F be a number field. Then we denote the adeles of F by $\mathbb{A}_F$ and the finite adeles by $\mathbb{A}_F^\infty$. Let G be a linear algebraic group over F and define

$$\mathfrak{X}_G := G(F)\backslash(X_G \times G(\mathbb{A}_F^\infty)) \qquad \overline{\mathfrak{X}}_G := G(F)\backslash(\overline{X}_G \times G(\mathbb{A}_F^\infty)),$$

where $G(F)$ acts diagonally by left multiplication and $G(\mathbb{A}_F^\infty)$ is given the *discrete* topology. Now $\mathfrak{X}_G$ is a topological space with a right action by the discrete group $G(\mathbb{A}_F^\infty)$. In particular for any open compact subgroup $K \subset G(\mathbb{A}_F^\infty)$, we can define

$$X_G^K := \mathfrak{X}_G/K \qquad \overline{X}_G^K := \overline{\mathfrak{X}}_G/K.$$

Proposition 3.2.1. *Let G be a reductive group over F and let $\mathcal{P}$ be a set of representatives of $G(F)$ conjugacy classes of F-rational parabolic subgroups of G. There is a $G(\mathbb{A}_F^\infty)$-equivariant stratification*

$$\overline{\mathfrak{X}}_G \cong \bigcup_{P\in\mathcal{P}} \mathfrak{X}_P \times_{P(\mathbb{A}_F^\infty)} G(\mathbb{A}_F^\infty)$$

Proof. By [BS73, Proposition 7.6], the natural map $X_P \to X_{gPg^{-1}}$ is an isomorphism for all parabolic subgroups P and $g \in G(F)$. Since each parabolic group is its own stabilizer under the conjugation action of $G(F)$ we find that

$$\overline{\mathfrak{X}}_G \cong G(F)\backslash\left(\bigcup_{P\subset G} X_P \times G(\mathbb{A}_F^\infty)\right) \cong \bigcup_{P\in\mathcal{P}} P(F)\backslash(X_P \times G(\mathbb{A}_F^\infty))$$

Now there is a straightforward identification

$$P(F)\backslash(X_P \times G(\mathbb{A}_F^\infty)) \cong \mathfrak{X}_P \times_{P(\mathbb{A}_F^\infty)} G(\mathbb{A}_F^\infty)$$

and the claim follows. □

[1] The first of these quotients is called an arithmetic locally symmetric space and the second is its Borel–Serre compactification.

Corollary 3.2.2. *Let G be a reductive group over F, $P \subset G$ a proper parabolic subgroup and $M = P/R_uP$. There are $P(\mathbb{A}_F^\infty)$-equivariant maps*

$$\begin{array}{ccc} \overline{\mathfrak{X}}_G \setminus \mathfrak{X}_G =: \partial\mathfrak{X}_G & \xleftarrow[i_P]{} & \mathfrak{X}_P \\ & & \big\downarrow f_P \\ & & \mathfrak{X}_M \end{array}$$

where $P(\mathbb{A}_F^\infty)$ acts on $\mathfrak{X}_M$ via the quotient $M(\mathbb{A}_F^\infty)$ and $f_P : \mathfrak{X}_P \to \mathfrak{X}_M$ is the map induced by the morphism $P \to M$. Moreover, if $N = R_uP$, then f_P is a $N(F)\backslash N(\mathbb{A}_F)$-bundle.

Definition 3.2.3. An open compact subgroup $K < G(\mathbb{A}_F^\infty)$ is said to be neat if all of its elements are neat. An element $g = (g_v)_v \in G(\mathbb{A}_F^\infty)$ is said to be neat if the intersection $\bigcap_v \Gamma_v$ is trivial, where $\Gamma_v \subset \overline{\mathbb{Q}}^\times$ is the torsion subgroup of the subgroup of $\overline{F}_v^\times$ generated by the eigenvalues of g_v under the faithful representations of G_{F_v}.

Proposition 3.2.4. *Let G/F be a reductive group. If K is neat, then X_G^K is a smooth manifold and $\overline{X}_G^K$ is a smooth compact manifold with corners. X_G^K is the interior of $\overline{X}_G^K$ and the inclusion $X_G^K \hookrightarrow \overline{X}_G^K$ is a homotopy equivalence.*

Proof. See [BS73, Theorem 9.3] and [BS73, Lemma 8.3.1] for the homotopy equivalence part. □

Definition 3.2.5. An open compact subgroup $K < G(\mathbb{A}_F^\infty)$ is called good if it is neat and of the form $K = \prod_v K_v$.

Lemma 3.2.6. *Let $K = \prod_v K_v < \mathrm{GL}_n(\mathbb{A}_F^\infty)$ be an open compact subgroup and let v be a place of F such that v does not divide $\prod_{j=2}^{n^{C(n)[F:\mathbb{Q}]+1}} \Phi_j(1)$, where Φ_j denotes the jth cyclotomic polynomial and $C(n)$ is the number of index $\leq n$ subgroups of the free group on two generators. If K_v is contained in the subgroup $Iw_{v,1} < \mathrm{GL}_n(\mathcal{O}_{F_v})$ of matrices which are unipotent and upper triangular mod ϖ_v, then K is neat.*

Proof. Before beginning the proof, we explain $C(n)$ is finite. The subgroups of the free group on two generators of index $\leq n$ are in bijection with covering spaces of degree $\leq n$ of the wedge of two circles. Any such covering space is homeomorphic to a graph with at most n vertices and at most $2n$ edges. There is a finite set of such graphs and each graph admits at most finitely many covering maps to the wedge of two circles. Consequently, $C(n)$ is finite.

If $\alpha \in \overline{F}_v^\times$ is an eigenvalue of $g_v \in Iw_{v,1}$ under some faithful representation of GL_n / F_v, then $|\alpha - 1|_v < 1$. Suppose $\zeta \in \Gamma_v$ is a non-trivial element of prime order p. We will derive a contradiction, proving that $\Gamma_v = 1$.

The element ζ is a product of eigenvalues $\alpha_i \in \overline{F}_v^\times$ of elements $g_v^{(i)} \in Iw_{v,1}$ and we also have $|\zeta - 1|_v < 1$. This implies that p is the residue characteristic of F_v and that

$$\prod_{j=2}^{n^{C(n)}[F:\mathbb{Q}]+1} |\Phi_j(\zeta)|_v = \prod_{j=2}^{n^{C(n)}[F:\mathbb{Q}]+1} |\Phi_j(1)|_v = 1.$$

Hence $p > n^{C(n)}[F : \mathbb{Q}] + 1$ and $[\mathbb{Q}_p(\zeta) : \mathbb{Q}_p] = p - 1 > n^{C(n)}[F : \mathbb{Q}]$.

Since the tame inertia group of F_v is pro-generated by two elements, it has at most $C(n)$ open subgroups of index at most n. Consequently, there are at most $C(n)$ tamely ramified extensions of F_v of degree $\leq n$. Note that $\zeta = \prod_{i=1}^N \alpha_i$ for some eigenvalues α_i which satisfy $[F_v(\alpha_i) : F_v] \leq n$ since if α_i is the image of $g_v^{(i)} \in Iw_{v,1}$ under some faithful representation, then α_i can be expressed as a product of the eigenvalues of $g_v^{(i)}$, which satisfy the characteristic polynomial of $g_v^{(i)}$ of degree n. In particular $F_v(\zeta)$ is contained in the composite of the tamely ramified subextensions $F_v(\alpha_i)^{tr}/F_v$. This implies that

$$[F_v(\zeta) : \mathbb{Q}_p] \leq n^{C(n)}[F : \mathbb{Q}] < [\mathbb{Q}_p(\zeta) : \mathbb{Q}_p]$$

contradicting the inclusion $\mathbb{Q}_p(\zeta) \subset F_v(\zeta)$. □

Definition 3.2.7. Let K be a good open compact subgroup of $G(\mathbb{A}_F^\infty)$ and S a finite set of places such that K_v is hyperspecial for $v \notin S$. If V is a $\mathbb{Z}[K_S]$-module, where $K_S := \prod_{v \in S} K_v$, then we let $\mathcal{V}$ denote the $K_S \times G(\mathbb{A}_F^S)$-equivariant constant sheaf on $\overline{\mathfrak{X}}_G$ associated with V. In other words, if we identify V with an equivariant sheaf on a point, then its pull back to $\overline{\mathfrak{X}}_G$ under the constant map is denoted by $\mathcal{V}$. By abuse of notation we also let $\mathcal{V}$ be the same sheaf restricted to $\partial\mathfrak{X}_G$ or $\mathfrak{X}_G$.

Definition 3.2.8. Let G be a group and X a topological space with G-action. The category of G-equivariant sheaves of abelian groups on X is denoted $\mathbf{Sh}^G(X)$. Let Y be a topological space equipped with trivial G-action. If $f : X \to Y$ is a G-equivariant map and $\mathcal{F}$ is a G-equivariant sheaf on X, then we define a sheaf $f_*^G\mathcal{F}$ on Y by the formula

$$f_*^G\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))^G$$

for any open $U \subset Y$.

Lemma 3.2.9. *If $K < G(\mathbb{A}_F^\infty)$ is a good subgroup and $p^K : \mathfrak{X}_G \to X_G^K$ is the quotient map, then $p_*^K : \mathbf{Sh}^{K_S \times G(\mathbb{A}_F^S)}(\mathfrak{X}_G) \to \mathbf{Sh}(X_G^K)$ is exact and preserves injectives.*

Proof. Since the forgetful functor $\mathbf{Sh}^{K_S \times G(\mathbb{A}_F^S)}(\mathfrak{X}_G) \to \mathbf{Sh}^K(\mathfrak{X}_G)$ is exact and preserves injectives, it suffices to show that $p_*^K : \mathbf{Sh}^K(\mathfrak{X}_G) \to \mathbf{Sh}(X_G^K)$ is exact and preserves injectives. But this is even an equivalence of abelian categories by [NT16, Lemma 2.17] and in particular exact. □

3.3 Hecke Algebras

The locally symmetric spaces we just introduced carry many cohomological correspondences that give their cohomology the structure of a module over a Hecke algebra. The rings T_N and the complexes C_N appearing in our application of lemma 2.6.4 will be a special case. For an introduction to Hecke algebras of p-adic groups we refer to [Car79].

Definition 3.3.1. Let G be a locally profinite group and K an open compact subgroup of G, then we define a ring with underlying set

$$\mathcal{H}(G, K) := \{f : G \to \mathbb{Z} \mid f \text{ has cpct support and } \forall k_1, k_2 \in K,\ f(k_1 g k_2) = f(g)\}.$$

The addition is pointwise and the multiplication is given by

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) d\mu(x),$$

where μ is the left Haar measure on G such that $\mu(K) = 1$. This ring has a 1 given by the indicator function of $K \subset G$. For $g \in G$, we denote the the indicator function of the double coset KgK by $[KgK] \in \mathcal{H}(G, K)$.

Lemma 3.3.2. *For any $\mathbb{Z}[G]$-module M, the invariants M^K form a module over the ring $\mathcal{H}(G, K)$ via the formula*

$$f * m = \int_G f(g) gm \, d\mu(g)$$

Proof. See [NT16, Lemma 2.3]. □

Proposition 3.3.3. *Let G be a reductive group over F and K a good open compact subgroup of $G(\mathbb{A}_F^\infty)$ and S a finite set of places such that K_v is hyperspecial for all $v \notin S$. The Hecke algebra $\mathcal{H}(G(\mathbb{A}_F^S), K^S)$ acts by natural transformations on each of the functors*

- $\mathbf{D}(\mathbf{Sh}^{K_S \times G(\mathbb{A}_F^S)}(\mathfrak{X}_G)) \to \mathbf{D}(\mathbb{Z}) : A \mapsto R\Gamma(X_G^K, p_*^K A)$
- $\mathbf{D}(\mathbf{Sh}^{K_S \times G(\mathbb{A}_F^S)}(\partial\mathfrak{X}_G)) \to \mathbf{D}(\mathbb{Z}) : A \mapsto R\Gamma(\partial X_G^K, p_*^K A)$

such that for injective objects I, the action on $\Gamma(X_G^K, p_*^K I) = \Gamma(\mathfrak{X}_G, I)^K$ is the one given by lemma 3.3.2.

Proof. This follows from the fact that p_*^K is exact and preserves injectives, see lemma 3.2.9. Namely, if we choose an injective resolution $A \to I^\bullet$, then

$$R\Gamma(X_G^K, p_*^K A) \cong \Gamma(X_G^K, p_*^K I^\bullet) \cong \Gamma(\mathfrak{X}_G, I^\bullet)^K$$

carries a Hecke action by lemma 3.3.2. This is well-defined since injective resolutions are unique up to homotopy. □

Lemma 3.3.4. *Let G be a reductive group over F and K a good open compact subgroup of $G(\mathbb{A}_F^\infty)$ and S a finite set of places such that K_v is hyperspecial for all $v \notin S$. There is an isomorphism*

$$\mathcal{H}(G(\mathbb{A}_F^S), K^S) \cong \bigotimes_{v \notin S}{}' \mathcal{H}(G(F_v), G(\mathcal{O}_{F_v}))$$

and each $\mathcal{H}(G(F_v), G(\mathcal{O}_{F_v}))$ is commutative.

Proof. See [Fla79] for the version with complex coefficients. The $\mathbb{Z}$-coefficient statement follows from this since $\mathcal{H}(G(\mathbb{A}_F^S), K^S)$ is a free $\mathbb{Z}$-module. □

Definition 3.3.5. Let G be a reductive group over F and K a good open compact subgroup of $G(\mathbb{A}_F^\infty)$ and S a finite set of places such that K_v is hyperspecial for all $v \notin S$. Suppose $P \subset G$ is a parabolic subgroup with levi $M \subset P$, then we use the previous lemma to define

$$r_P : \mathcal{H}(G(\mathbb{A}_F^S), K^S) \to \mathcal{H}(P(\mathbb{A}_F^S), K \cap P(\mathbb{A}_F^S))$$
$$r_M : \mathcal{H}(P(\mathbb{A}_F^S), K \cap P(\mathbb{A}_F^S)) \to \mathcal{H}(M(\mathbb{A}_F^S), K \cap M(\mathbb{A}_F^S))$$

with local components given by [NT16, 2.2.3 and 2.2.4] for each $v \notin S$. Moreover, we set $\mathrm{Sat} = r_M \circ r_P$.

Definition 3.3.6. We abbreviate the following functors

- $E_G^K : \mathbf{D}(\mathcal{O}[K_S]) \to \mathbf{D}(\mathbf{Sh}^{K_S \times G(\mathbb{A}_F^S)}(\mathfrak{X}_G)) \to \mathbf{D}(\mathbb{Z}) : V \mapsto R\Gamma(X_G^K, p_*^K \mathcal{V})$
- $\partial E_G^K : \mathbf{D}(\mathcal{O}[K_S]) \to \mathbf{D}(\mathbf{Sh}^{K_S \times G(\mathbb{A}_F^S)}(\partial\mathfrak{X}_G)) \to \mathbf{D}(\mathbb{Z}) : V \mapsto R\Gamma(\partial X_G^K, p_*^K \mathcal{V})$

Proposition 3.3.7. *Let G be a reductive group over F and $P \subset G$ a parabolic subgroup and let $M \subset P$ be a Levi subgroup. Suppose K is a good open compact subgroup of $G(\mathbb{A}_F^\infty)$ such that $M(\mathbb{A}_F^\infty) \cap K \to (P(\mathbb{A}_F^\infty) \cap K)/(U(\mathbb{A}_F^\infty) \cap K)$ is an isomorphism, where $U = R_u P$ is the unipotent radical of P. Let S be a finite set of places containing those above p, such that K_v is hyperspecial for $v \notin S$. If V is a continuous $\mathbb{Z}_p[K_S]$-module which is finitely generated over $\mathbb{Z}_p$, then there is a commutative diagram*

$$\begin{array}{ccc}
E_M^{K\cap M} R\Gamma((U\cap K)_S, V) & \xrightarrow{\mathrm{Sat}(T)} & E_M^{K\cap M} R\Gamma((U\cap K)_S, V) \\
\uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle \phi} \\
E_P^{K\cap P}(V) & \xrightarrow{\quad r_P(T) \quad} & E_P^{K\cap P}(V) \\
\uparrow{\scriptstyle i_P^*} & & \uparrow{\scriptstyle i_P^*} \\
\partial E_G^K(V) & \xrightarrow{\quad T \quad} & \partial E_G^K(V)
\end{array}$$

where T is any element of $\mathcal{H}(G(\mathbb{A}_F^S), K^S)$, ϕ is an isomorphism and $R\Gamma((U \cap K)_S, -)$ denotes the complex computing continuous $(U\cap K)_S$-group cohomology, seen as modules over $\mathbb{Z}_p[M \cap K]$.

Proof. By corollary 3.2.2 we obtain a $(K \cap P)_S \times P(\mathbb{A}_F^S)$-equivariant diagram

$$\begin{array}{ccc}
\partial \mathfrak{X}_G & \xleftarrow[i_P]{} & \mathfrak{X}_P \\
& & \downarrow{\scriptstyle f_P} \\
& & \mathfrak{X}_M
\end{array}$$

where f_P is a $U(F)\backslash(U(F \otimes \mathbb{R}) \times U(\mathbb{A}_F^\infty))$-bundle. Now we have a natural transformation of functors

$$\mathrm{Res}^{K_S \times G(\mathbb{A}_F^S)}_{(K\cap P)_S \times P(\mathbb{A}_F^S)} \Gamma(\partial \mathfrak{X}_G, \mathcal{F}) \to \Gamma(\mathfrak{X}_P, i_P^* \mathcal{F})$$

where $\mathcal{F}$ runs through the $K_S \times G(\mathbb{A}_F^S)$-equivariant sheaves on $\mathfrak{X}_G$. Hence [NT16, Lemma 2.4] and taking derived functors gives us the lower square of the theorem.

Let $V = \lim V/p^m$ be a continuous $\mathbb{Z}_p[K_S]$-module, finite over $\mathbb{Z}_p$. Each V/p^m is a discrete, smooth $\mathbb{Z}/p^m\mathbb{Z}[K]$-module and

$$R\Gamma(U \cap K, R\Gamma(\mathfrak{X}_P, \mathcal{V}/p^m)) = R\Gamma(\mathfrak{X}_M, Rf_{P*}^{U\cap K}(\mathcal{V}/p^m)). \tag{3.3.0.1}$$

Now for any open subset $W \subset \mathfrak{X}_M$ and smooth $\mathbb{Z}/p^m\mathbb{Z}[(P \cap K)_S]$-module V/p^m we have

$$f_{P*}(\mathcal{V}/p^m)(W) = \{h : \pi_0(f_P^{-1}(W)) \to V/p^m\}$$

with $(U \cap K)$-action given by $(u \cdot h)(x) = uh(xu)$. If $w \in \mathfrak{X}_M$, then $f_P^{-1}(w) = U(F)\backslash(U(F \otimes \mathbb{R}) \times U(\mathbb{A}_F^\infty))$ is homeomorphic to a disjoint union of copies of $U(F \otimes \mathbb{R})$, which is contractible as U is unipotent. Hence $R^i f_{P*} = 0$ for $i > 0$. By strong approximation, we have that for all $w \in W$, $f_P^{-1}(w)/(U \cap K) \cong (U(F) \cap K)\backslash U(F \otimes \mathbb{R})$ is connected since U is unipotent. Thus, we can also write

$$f_{P*}^{U \cap K} \mathcal{V}/p^m(W) = \{h : \pi_0(W) \to (V/p^m)^{U \cap K}\} = \{h : \pi_0(W) \to (V/p^m)^{(U \cap K)_S}\}$$

Hence $f_{P*}^{U \cap K}(\mathcal{V}/p^m)$ is the constant sheaf associated with $(V/p^m)^{(U \cap K)_S}$. Consequently, we obtain a natural map

$$s^* R\Gamma((U \cap K)_S, V/p^m) \to R f_{P*}^{U \cap K}(\mathcal{V}/p^m)$$

where s is the constant map. We wish to show that this is a quasi-isomorphism. We may check it on stalks, so let $x \in \mathfrak{X}_M$ and W_α a basis of contractible open neighbourhoods of x. If $\mathcal{I}$ is an injective $(U \cap K)$-equivariant sheaf on $\mathfrak{X}_M$, then we claim that $\mathcal{I}(W_\alpha)$ is an injective $(U \cap K)$-module. To see this, note that

$$\mathrm{Hom}^{U \cap K}(A, \mathcal{I}(W_\alpha)) = \mathrm{Hom}_{\mathbf{Sh}^{U \cap K}(W_\alpha)}(\mathcal{A}, \mathcal{I}|_{W_\alpha}) = \mathrm{Hom}_{\mathbf{Sh}^{U \cap K}(\mathfrak{X}_M)}(i_! \mathcal{A}, \mathcal{I}),$$

where $i : W_\alpha \to \mathfrak{X}_M$ is the inclusion and $\mathcal{A}$ is the constant $U \cap K$-equivariant sheaf on W_α associated with A. But $i_!$ is exact and $\mathcal{I}$ is injective, so we have written $\mathrm{Hom}^{U \cap K}(-, \mathcal{I}(W_\alpha))$ as the composition of two exact functors. Now choose an injective resolution

$$f_{P*}(\mathcal{V}/p^m) \to \mathcal{I}^\bullet,$$

then $\mathcal{I}^\bullet(W_\alpha)$ is exact by contractibility of W_α and thus it is an injective resolution of $\mathcal{V}/p^m(f_P^{-1}(W_\alpha))$ and

$$R\Gamma(W_\alpha, R f_{P*}^{U \cap K}(\mathcal{V}/p^m)) = \mathcal{I}^\bullet(W_\alpha)^{U \cap K}.$$

Since W_α is contractible and the $R^i f_{P*}^{U \cap K}(\mathcal{V}/p^m)$ are locally constant we see that the cohomology of the left hand side is independent of α. Taking the colimit over all α, which is exact and commutes with $U \cap K$-invariants, we find the stalk

$$R^i f_{P*}^{U \cap K}(\mathcal{V}/p^m)_x = \varinjlim H^i\left(\mathcal{I}^\bullet(W_\alpha)^{U \cap K}\right) = H^i\left(\mathcal{I}^\bullet(W_{\alpha_0})^{U \cap K}\right)$$

for any fixed α_0. But $\mathcal{V}/p^m(f_P^{-1}(W_{\alpha_0})) = (f_{P*}\mathcal{V}/p^m)_x$ and $\mathcal{I}^\bullet(W_{\alpha_0})$ is an injective resolution of $\mathcal{V}/p^m(f_P^{-1}(W_{\alpha_0}))$ and so

$$R^i f_{P*}^{U \cap K}(\mathcal{V}/p^m)_x = H^i_{disc}(U \cap K, (f_{P*}\mathcal{V}/p^m)_x),$$

where H^i_{disc} denotes *discrete* group cohomology. The description of the fibres implies $f_{P*}(\mathcal{V}/p^m)_x \cong \mathrm{Ind}_{U(F)\cap K}^{U(\mathbb{A}_F^\infty)\cap K}(V/p^m)$ and by Shapiro's lemma $R^i f_{P*}^{U\cap K}(\mathcal{V}/p^m)_x = H^i_{disc}(U(F)\cap K, V/p^m)$. Now to verify the quasi-isomorphism in question it suffices to check that the natural map

$$H^i_{cont}((U\cap K)_S, V/p^m) \to H^i_{disc}(U(F)\cap K, V/p^m)$$

is an isomorphism. To see this, first observe that $H^i_{cont}((U\cap K)_S, V/p^m) \to H^i_{cont}(U\cap K, V/p^m)$ is an isomorphism since V/p^m is a finite abelian p-group and $(U\cap K)_S$ contains the pro-p part of $U\cap K$.

Further, we can find a finite index subgroup of $U(F)\cap K$ which admits a filtration with free abelian quotients. Now using [Ser02, I. §2.6 Exercise 2] we see that $U(F)\cap K$ is good because $\mathbb{Z}^r$ is good for any r, where a group G is called good if $H^i_{cont}(\hat{G}, M) \to H^i_{disc}(G, M)$ is an isomorphism for all i, whenever M is a finite abelian group with continuous $\hat{G}$-action. It follows that $H^i_{cont}(U\cap K, V/p^m) \to H^i_{disc}(U(F)\cap K, V/p^m)$ is an isomorphism, as required.

By taking derived $K\cap P$-invariants in (3.3.0.1) we obtain a natural isomorphism

$$\phi_m : E_M^{K\cap M}(R\Gamma((U\cap K)_S, V/p^m)) \to E_P^{K\cap P}(V/p^m).$$

Moreover, [NT16, Lemma 2.7] implies that ϕ_m satisfies $\mathrm{Sat}(T)\circ\phi = \phi\circ r_P(T)$ for all $T \in \mathcal{H}(G(\mathbb{A}_F^S), K^S)$.

Note that since $(U\cap K)_S$ is a topologically finitely generated profinite group, the groups $H^i((U\cap K)_S, V/p^m)$ are finite for all i and m. This implies that the inverse system $(H^i((U\cap K)_S, V/p^m))_{m\geq 1}$ is Mittag-Leffler for all i. Since the standard continuous cochain complex $C^\bullet((U\cap K)_S, V)$ is the limit of the standard continuous cochain complexes $C^\bullet((U\cap K)_S, V/p^m)$, we deduce that the natural map

$$R\Gamma((U\cap K)_S, V) \to \lim_m R\Gamma((U\cap K)_S, V/p^m)$$

is an isomorphism. Both the functors $E_M^{K\cap M}$ and $E_P^{K\cap P}$ commute with limits since they are a compositions of right adjoint functors. Thus, the ϕ_m converge to the desired isomorphism

$$\phi : E_M^{K\cap M}(R\Gamma((U\cap K)_S, V)) \to E_P^{K\cap P}(V).$$ □

Remark 3.3.8. The previous proof is very similar to the arguments used in the proof of [NT16, Theorem 4.2].

3.4 The Quasi-Split Unitary Group

Let F/F^+ be an imaginary CM field with complex conjugation $c \in \mathrm{Gal}(F/F^+)$. Let $n \geq 2$ be an integer and

$$J_n = \begin{bmatrix} 0 & \Psi_n \\ -\Psi_n & 0 \end{bmatrix},$$

where Ψ_n has 1's on the anti-diagonal and 0's elsewhere. We consider the hermitian pairing

$$\mathcal{O}_F^{2n} \times \mathcal{O}_F^{2n} \to \mathcal{O}_F : (x, y) \mapsto x^T J_n y^c$$

and use it to define the $\mathcal{O}_{F^+}$-group scheme $\widetilde{G}$ with functor of points

$$\widetilde{G}(R) = \{g \in \mathrm{GL}_{2n}(\mathcal{O}_F \otimes_{\mathcal{O}_{F^+}} R) : g^T J_n g^c = J_n\}.$$

We define the Siegel parabolic $P \subset \widetilde{G}$ as the closed subgroup scheme preserving the subspace $\mathcal{O}_F^n \oplus 0^n \subset \mathcal{O}_F^{2n}$, and $G \subset P$ as the subgroup scheme which preserves both $\mathcal{O}_F^n \oplus 0^n$ and $0^n \oplus \mathcal{O}_F^n$.

Lemma 3.4.1. *There is an isomorphism* $\mathrm{Res}^F_{F^+} \mathrm{GL}_n \cong G$ *given on R-points by*

$$g \mapsto \begin{bmatrix} \Psi_n (g^c)^{-T} \Psi_n & 0 \\ 0 & g \end{bmatrix}$$

Proof. See [NT16, Lemma 5.1]. □

Now for a good open compact subgroup $\widetilde{K} \subset \widetilde{G}(\mathbb{A}^\infty_{F^+})$ satisfying the conditions of proposition 3.3.7 with the Siegel parabolic P, the following diagram summarises the relation between the locally symmetric space attached to $\widetilde{G}$ and the one attached to G.

$$\begin{array}{ccccccc} X_P^{\widetilde{K}\cap P} & \hookrightarrow & \partial X_{\widetilde{G}}^{\widetilde{K}} & \hookrightarrow & \overline{X}_{\widetilde{G}}^{\widetilde{K}} & \hookleftarrow & X_{\widetilde{G}}^{\widetilde{K}} \\ \downarrow & & & & & & \\ X_G^{\widetilde{K}\cap G} & & & & & & \end{array} \tag{3.4.0.1}$$

Here is a sketch of the roles that these spaces will play later. We started with an automorphic representation of GL_n/F, which will give rise to some cohomology classes on X_G^K. Hence we will need to associate Galois representations to Hecke modules appearing in the singular cohomology of X_G^K. The cohomology of $X_{\widetilde{G}}^{\widetilde{K}}$ is known to have good properties by [CS19]. More precisely, in degree $d = 1 + \dim X_G$, a certain direct summand $H^d(X_{\widetilde{G}}^{\widetilde{K}}, \mathbb{Z}_p)_{\widetilde{\mathfrak{m}}} \subset H^d(X_{\widetilde{G}}^{\widetilde{K}}, \mathbb{Z}_p)$ is p-torsion free, hence can be computed in terms of automorphic representations on $\widetilde{G}$ as in [FS98]. Now we use [Gol14, Appendix] to attach Galois representations of G_F to these automorphic representations.

Thus, we would like to pass from $H^q(X_G^K)$ to $H^d(X_{\widetilde{G}}^{\widetilde{K}})_{\widetilde{\mathfrak{m}}}$ by walking along the arrows of diagram (3.4.0.1). The vertical arrow is addressed in proposition 3.3.7. The comparison between $H^\bullet(X_P^{\widetilde{K}\cap P})$ and $H^\bullet(\partial X_{\widetilde{G}}^{\widetilde{K}})$ is done in proposition 3.4.10. Finally to reach $X_{\widetilde{G}}^{\widetilde{K}}$ from $\partial X_{\widetilde{G}}^{\widetilde{K}}$ we use the vanishing theorem of [CS19].

Then it remains to show why it is enough to study only the degree d cohomology on $X_{\widetilde{G}}^{\widetilde{K}}$. This is done by the degree-shifting argument from [ACC+23] which we generalise in the next section. Before we can execute this strategy we need to set up some more definitions.

Definition 3.4.2. If S is a finite set of places of F, then we define the ring

$$\mathbb{T}^S := \mathcal{H}(\mathrm{GL}_n(\mathbb{A}_F^S), \prod_{v\notin S} \mathrm{GL}_n(\mathcal{O}_{F_v}))$$

which is commutative by lemma 3.3.4. For a ring R we also set $\mathbb{T}_R^S := \mathbb{T}^S \otimes_{\mathbb{Z}} R$. Similarly we let $\overline{S}$ denote the set of places of F^+ lying below a place of S and define $\widetilde{\mathbb{T}}^{\overline{S}}$ as $\mathcal{H}(\widetilde{G}(\mathbb{A}_{F^+}^{\overline{S}}), \prod_{\overline{v}\notin\overline{S}} \widetilde{G}(\mathcal{O}_{F_v}))$.

Definition 3.4.3. Keep the conditions from the previous definition. For $v \notin S$ we let $P_v(z) \in \mathbb{T}^S[z]$ be the polynomial with the same name defined in [ACC+23, 2.2.5]. Similarly for $\overline{v} \notin \overline{S}$, we let $\widetilde{P}_v(z) \in \widetilde{\mathbb{T}}^{\overline{S}}[q_{\overline{v}}^{-1}, z]$ be the other polynomial defined in [ACC+23, 2.2.6].

Let E be a finite extension of $\mathbb{Q}_p$ which contains all the p-adic embeddings of F and let $\mathcal{O}$ be the ring of integers of E with uniformiser ϖ. Choose a decomposition $\mathrm{Hom}(F, E) = H \cup Hc$. The restriction of a field homomorphism to F^+ induces a bijection $\alpha : H \cong \mathrm{Hom}(F^+, E)$.

Let $T \subset G$ be the standard torus of $\mathrm{Res}_{F^+}^F \mathrm{GL}_n$. Note that T splits over E, $T(F^+ \otimes_{\mathbb{Q}_p} E) = \prod_{\widetilde{\tau}:F\to E}(E^\times)^n$, $G(F^+ \otimes_{\mathbb{Q}_p} E) = \prod_{\widetilde{\tau}:F\to E} \mathrm{GL}_n(E)$ and $\widetilde{G}(F^+ \otimes_{\mathbb{Q}_p} E) = \prod_{\tau:F^+\to E} \mathrm{GL}_{2n}(E)$.

And with these identifications, the inclusion $G \to \widetilde{G}$ is given by

$$\prod_{\widetilde{\tau}:F\to E} \mathrm{GL}_n(E) \to \prod_{\tau:F^+\to E} \mathrm{GL}_{2n}(E) : (g_{\widetilde{\tau}})_{\widetilde{\tau}} \mapsto \left(\begin{bmatrix} \Psi_n g_{\alpha(\tau)c}^{-T} \Psi_n & 0 \\ 0 & g_{\alpha(\tau)} \end{bmatrix}\right)_\tau$$

Thus, $(\lambda_{\widetilde{\tau},1}, \dots, \lambda_{\widetilde{\tau},n}) \in X^\bullet(T(E \otimes_{\mathbb{Q}_p} F^+)) \cong (\mathbb{Z}^n)^{\mathrm{Hom}(F,E)}$ is dominant for G if and only if

$$\lambda_{\widetilde{\tau},1} \geq \cdots \geq \lambda_{\widetilde{\tau},n}$$

for all $\widetilde{\tau} : F \to E$. It is dominant for $\widetilde{G}$ if and only if

$$-\lambda_{\widetilde{\tau}c,n} \geq \cdots \geq -\lambda_{\widetilde{\tau}c,1} \geq \lambda_{\widetilde{\tau},1} \geq \cdots \geq \lambda_{\widetilde{\tau},n}$$

Definition 3.4.4. If $\widetilde{\lambda} \in X^\bullet(T(E \otimes_{\mathbb{Q}_p} F^+))$ is dominant for $\widetilde{G}$, then we can also see $\widetilde{\lambda}$ as a character $(\mathbb{G}_{m,\mathcal{O}}^n)^{\mathrm{Hom}(F,E)} \to \mathbb{G}_{m,\mathcal{O}}$ and we can consider the $\mathcal{O}$-points of the algebraic induction

$$V_{\widetilde{\lambda}} := \left(\mathrm{Ind}_{\overline{B}_{2n}^{\mathrm{Hom}(F^+,E)}}^{\mathrm{GL}_{2n}^{\mathrm{Hom}(F^+,E)}} \widetilde{\lambda} \right) (\mathcal{O}),$$

where $\overline{B}_{2n} \subset \mathrm{GL}_{2n}$ is the opposite of the standard Borel over $\mathcal{O}$. We view $V_{\widetilde{\lambda}}$ as a continuous module over $\mathcal{O}[\prod_{\overline{v}|p} \widetilde{G}(\mathcal{O}_{F^+,\overline{v}})]$ via the maps $\widetilde{G}(\mathcal{O}_{F^+,\overline{v}}) \to \widetilde{G}(\mathcal{O}) \cong \mathrm{GL}_{2n}(\mathcal{O})$ induced by the field embeddings $F^+ \to E$. Moreover, $V_{\widetilde{\lambda}} = \bigotimes_{\overline{v}|p} V_{\widetilde{\lambda}_{\overline{v}}}$, where $\widetilde{\lambda}_{\overline{v}} \in (\mathbb{Z}^n)^{\overline{S}_{\overline{v}}}$ is the projection of $\widetilde{\lambda}$ and $\overline{S}_{\overline{v}}$ is the set of embeddings $F^+ \to E$ inducing $\overline{v}$. The heighest weight $\mathcal{O}[\prod_{v|p} \mathrm{GL}_n(\mathcal{O}_{F,v})]$-modules V_λ for weights λ which are dominant for G are defined similarly.

By [Jan03, I.5.12] $V_{\widetilde{\lambda}}$ can be identified with the global sections of a line bundle on a projective $\mathcal{O}$-scheme. Thus, $V_{\widetilde{\lambda}}$ is a finite free $\mathcal{O}$-module such that $V_{\widetilde{\lambda}}[1/p]$ is the irreducible representation of $\widetilde{G}(F \otimes_{\mathbb{Q}_p} E)$ of highest weight $\widetilde{\lambda}$.

Definition 3.4.5. Let S be a set of places of F. We say that S satisfies (Σ_p) if

- S is finite and contains the archimedean places and those above p.
- S is stable under complex conjugation.
- Let v be a finite place of F not contained in S, and let ℓ be its residue characteristic. Then either S contains no ℓ-adic places of F and ℓ is unramified in F, or there exists an imaginary quadratic field $F_0 \subset F$ in which ℓ splits.

Theorem 3.4.6. *Let S be a set of places of F satisfying (Σ_p) and let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^S$ occuring in the support of the module $H^\bullet(X^K_{\mathrm{GL}_n/F}, \mathcal{V}_\lambda)$, for some good open compact subgroup K such that $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ for $v \notin S$. Then there exists a continuous semisimple representation $\overline{\rho}_\mathfrak{m} : G_{F,S} \to \mathrm{GL}_n(\mathbb{T}^S/\mathfrak{m})$ such that*

$$\det(z \cdot \mathrm{id} - \overline{\rho}_\mathfrak{m}(\mathrm{Frob}_v)) = P_v(z) \in (\mathbb{T}^S/\mathfrak{m})[z] \qquad \forall\, v \notin S$$

Note that this uniquely characterises $\overline{\rho}_\mathfrak{m}$ by semisimplicity.

Proof. See [ACC+23, Theorem 2.3.5]. □

Definition 3.4.7. A maximal ideal of $\mathbb{T}^S$ occuring in the support of $H^\bullet(X^K_{\mathrm{GL}_n/F}, \mathcal{V}_\lambda)$, for some good open compact subgroup K such that $K_v = \mathrm{GL}_n(\mathcal{O}_F)$ for $v \notin S$, is called non-Eisenstein if $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible.

Theorem 3.4.8. *Let K be a good open compact subgroup of $\mathrm{GL}_n(\mathbb{A}_F^\infty)$ and denote by $\mathbb{T}^S(R\Gamma(X^K_{\mathrm{GL}_n/F}, \mathcal{V}_\lambda))$ the image of $\mathbb{T}^S$ in the endomorphisms of $R\Gamma(X^K_{\mathrm{GL}_n/F}, \mathcal{V}_\lambda)$, where S is a finite set of places containing those above p such that $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ for all $v \notin S$. If $\mathfrak{m} < \mathbb{T}^S$ is a non-Eisenstein maximal ideal, then*

$$\mathbb{T}^S(R\Gamma(X^K_{\mathrm{GL}_n/F}, \mathcal{V}_\lambda))_{\mathfrak{m}}[1/p]$$

is either zero or a product of fields.

Proof. Let $\iota : \overline{\mathbb{Q}}_p \cong \mathbb{C}$ be an isomorphism. In the same way as in the proof of [ACC+23, Theorem 2.4.10] one can show that in the decomposition of [FS98, §2.2] only the cuspidal terms survive after localising at the non-Eisenstein ideal $\mathfrak{m}$. More precisely there exist integers $m(\pi) \geq 0$ and a direct sum of Hecke modules

$$\bigoplus_{i \geq 0} H^i(X^K_{\mathrm{GL}_n/F}, \mathcal{V}_\lambda)_{\mathfrak{m}}[1/p] \otimes_{\mathbb{Q}_p, \iota} \mathbb{C} \cong \bigoplus_{\pi} \left((\pi^\infty)^K\right)^{m(\pi)},$$

where π runs through the cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ whose central character ξ satisfies $\xi|_{\mathbb{R}_{>0}} = \xi_{\iota\lambda}^{-1}|_{\mathbb{R}_{>0}}$, where $\xi_{\iota\lambda}$ is the central character of the irreducible algebraic representation of $\mathrm{GL}_n(F \otimes \mathbb{R})$ of highest weight $\iota\lambda$. In particular, $\bigoplus_{i \geq 0} H^i(X^K_{\mathrm{GL}_n/F}, \mathcal{V}_\lambda)_{\mathfrak{m}}[1/p]$ is a semisimple $\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_F^S), K^S)$-module. Hence, its endomorphism algebra is either zero or a product of division algebras. The ring

$$\mathbb{T}^S(R\Gamma(X^K_{\mathrm{GL}_n/F}, \mathcal{V}_\lambda))_{\mathfrak{m}}[1/p]$$

is a commutative subalgebra of this endomorphism algebra because we have an isomorphism $R\Gamma(X^K_{\mathrm{GL}_n/F}, \mathcal{V}_\lambda)_{\mathfrak{m}}[1/p] \cong \bigoplus_{i \geq 0} H^i(X^K_{\mathrm{GL}_n/F}, \mathcal{V}_\lambda)_{\mathfrak{m}}[1/p]$ of objects of $\mathbf{D}(\mathbb{Q}_p)$. Hence, the ring in question is either zero or a product of fields. □

Theorem 3.4.9. *Let S be a set of places of F satisfying (Σ_p) and let $\widetilde{\mathfrak{m}}$ be a maximal ideal of $\widetilde{\mathbb{T}}^S$ occuring in the support of the module $H^\bullet(X^{\widetilde{K}}_{\widetilde{G}}, \mathcal{V}_{\widetilde{\lambda}})$, for some good open compact subgroup $\widetilde{K}$ such that $\widetilde{K}_v = \widetilde{G}(\mathcal{O}_{F_v^+})$ for $v \notin \overline{S}$. Then there exists a continuous semisimple representation $\overline{\rho}_{\widetilde{\mathfrak{m}}} : G_{F,S} \to \mathrm{GL}_{2n}(\widetilde{\mathbb{T}}^S/\widetilde{\mathfrak{m}})$ such that*

$$\det(z \cdot \mathrm{id} - \overline{\rho}_{\widetilde{\mathfrak{m}}}(\mathrm{Frob}_v)) = \widetilde{P}_v(z) \in (\widetilde{\mathbb{T}}^S/\widetilde{\mathfrak{m}})[z] \qquad \forall\, v \notin S$$

Note that this uniquely characterises $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ by semisimplicity.

Proof. See [ACC+23, Theorem 2.3.8]. □

Proposition 3.4.10. *Let S be a set of places of F satisfying (Σ_p) and $\widetilde{K} < \widetilde{G}(\mathbb{A}_{F^+}^\infty)$ a good open compact subgroup such that $\widetilde{K}_{\overline{v}} = \widetilde{G}(\mathcal{O}_{F_{\overline{v}}^+})$ for $\overline{v} \notin \overline{S}$ and $G(\mathbb{A}_F^\infty) \cap \widetilde{K} \to (P(\mathbb{A}_{F^+}^\infty) \cap \widetilde{K})/(U(\mathbb{A}_{F^+}^\infty) \cap \widetilde{K})$ is an isomorphism. If $\widetilde{\lambda}$ is a dominant weight for $\widetilde{G}$, λ is the corresponding dominant weight for G, $\mathfrak{m}$ is a non-Eisenstein ideal of $\mathbb{T}^S(R\Gamma(X_G^K, p_*^K\mathcal{V}_\lambda))$ and $\widetilde{\mathfrak{m}} = \mathrm{Sat}^{-1}(\mathfrak{m}) \subset \widetilde{\mathbb{T}}^{\overline{S}}$, then the map from proposition 3.3.7 makes $E_P^{P\cap\widetilde{K}}(V_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}$ a $\widetilde{\mathbb{T}}^{\overline{S}}$-equivariant direct summand of $\partial E_{\widetilde{G}}^{\widetilde{K}}(V_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}$, where P is the Siegel parabolic. Moreover, the natural map*

$$R\Gamma_c(X_P^{\widetilde{K}\cap P}, p_*^{\widetilde{K}\cap P}\mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} \to R\Gamma(X_P^{\widetilde{K}\cap P}, p_*^{\widetilde{K}\cap P}\mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}$$

is an isomorphism, where $\widetilde{\mathbb{T}}^{\overline{S}}$ acts via the map r_P.

Proof. Let us begin with the second part of the claim. Consider

$$\overline{X}_P^{\widetilde{K}\cap P} := \left(\bigcup_{Q\subset P} \mathfrak{X}_Q \times_{Q(\mathbb{A}_F^\infty)} P(\mathbb{A}_F^\infty)\right)/(\widetilde{K}\cap P)$$

as a locally closed subset of $\overline{X}_{\widetilde{G}}^{\widetilde{K}}$. Then the open embedding

$$X_P^{\widetilde{K}\cap P} \hookrightarrow \overline{X}_P^{\widetilde{K}\cap P}$$

is a homotopy equivalence (see [BS73, Lemma 8.3.1]) and the corresponding excision distinguished triangle is

$$R\Gamma_c(X_P^{\widetilde{K}\cap P}, p_*^{\widetilde{K}\cap P}\mathcal{V}_{\widetilde{\lambda}}) \to R\Gamma(X_P^{\widetilde{K}\cap P}, p_*^{\widetilde{K}\cap P}\mathcal{V}_{\widetilde{\lambda}}) \to R\Gamma(\overline{X}_P^{\widetilde{K}\cap P} \setminus X_P^{\widetilde{K}\cap P}, p_*^{\widetilde{K}\cap P}\mathcal{V}_{\widetilde{\lambda}}) \xrightarrow{[1]}$$

Thus, for the second part it suffices to show that $R\Gamma(\overline{X}_P^{\widetilde{K}\cap P} \setminus X_P^{\widetilde{K}\cap P}, p_*^{\widetilde{K}}\mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} = 0$. For the first part, we note that the Siegel parabolic P is a maximal parabolic subgroup, hence theorem 3.1.14 and proposition 3.2.1 show that

$$\mathfrak{X}_P \times_{P(\mathbb{A}_{F^+}^\infty)} \widetilde{G}(\mathbb{A}_{F^+}^\infty) \subset \partial\mathfrak{X}_{\widetilde{G}}$$

is an open subset. For a parabolic subgroup $P \subset \widetilde{G}$ we define

$$\widetilde{X}_P^{\widetilde{K}} := (\mathfrak{X}_P \times_{P(\mathbb{A}_{F^+}^\infty)} \widetilde{G}(\mathbb{A}_{F^+}^\infty))/\widetilde{K} \subset \partial X_{\widetilde{G}}^{\widetilde{K}}.$$

By the Iwasawa decomposition we have $P(F_v^+)\widetilde{K}_v = \widetilde{G}(F_v^+)$ for all $v \notin \overline{S}$, hence get a $\widetilde{G}(\mathbb{A}_{F^+}^{\overline{S}}) \times \widetilde{K}_{\overline{S}}$-stable disjoint union

$$\mathfrak{X}_P \times_{P(\mathbb{A}_{F^+}^\infty)} \widetilde{G}(\mathbb{A}_{F^+}^\infty) = \bigcup_\alpha \mathfrak{X}_P \times_{P(\mathbb{A}_{F^+}^\infty)} P(\mathbb{A}_{F^+}^\infty) g_\alpha \widetilde{K}$$

for some set of representatives $g_\alpha \in \prod_{v\in\overline{S}} \widetilde{G}(F_v^+)$. After quotienting by $\widetilde{K}$, it follows that there is a $\widetilde{\mathbb{T}}^{\overline{S}}$-equivariant decomposition

$$R\Gamma(\widetilde{X}_P^{\widetilde{K}}, p_*^{\widetilde{K}}\mathcal{V}_{\widetilde{\lambda}}) = \bigoplus_\alpha R\Gamma(X_P^{P\cap g_\alpha \widetilde{K} g_\alpha^{-1}}, p_*^{P\cap g_\alpha\widetilde{K}g_\alpha^{-1}}\mathcal{V}_{\widetilde{\lambda}}).$$

We have an excision distinguished triangle

$$R\Gamma_c(\widetilde{X}_P^{\widetilde{K}}, p_*^{\widetilde{K}}\mathcal{V}_{\widetilde{\lambda}}) \xrightarrow{i_{P!}} R\Gamma(\partial X_{\widetilde{G}}^{\widetilde{K}}, p_*^{\widetilde{K}}\mathcal{V}_{\widetilde{\lambda}}) \to R\Gamma(\partial X_{\widetilde{G}}^{\widetilde{K}} \setminus \widetilde{X}_P^{\widetilde{K}}, p_*^{\widetilde{K}}\mathcal{V}_{\widetilde{\lambda}}) \xrightarrow{[1]}$$

and we will show that $i_{P!}$ localised at $\widetilde{\mathfrak{m}}$ is an isomorphism. Let $\widetilde{i}_P : \widetilde{X}_P^{\widetilde{K}} \hookrightarrow \partial X_{\widetilde{G}}^{\widetilde{K}}$, be the inclusion. Then the localization of $\widetilde{i}_P^*$ is also an isomorphism, because the composition of $\widetilde{i}_P^*$ after $i_{P!}$ is the natural map

$$R\Gamma_c(\widetilde{X}_P^{\widetilde{K}}, p_*^{\widetilde{K}}\mathcal{V}_{\widetilde{\lambda}}) \to R\Gamma(\widetilde{X}_P^{\widetilde{K}}, p_*^{\widetilde{K}}\mathcal{V}_{\widetilde{\lambda}})$$

and we will show that the localisation of this map is an isomorphism. The first part of the proposition follows from this because we already observed that the complex $R\Gamma(X_P^{P\cap\widetilde{K}}, p_*^{P\cap\widetilde{K}}\mathcal{V}_{\widetilde{\lambda}})$ is a $\widetilde{\mathbb{T}}^{\overline{S}}$-equivariant direct summand of $R\Gamma(\widetilde{X}_P^{\widetilde{K}}, p_*^{\widetilde{K}}\mathcal{V}_{\widetilde{\lambda}})$. In conclusion, it suffices to show that for any choice of $\widetilde{K}$ we have

$$R\Gamma(\partial X_{\widetilde{G}}^{\widetilde{K}} \setminus \widetilde{X}_P^{\widetilde{K}}, p_*^{\widetilde{K}}\mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} = 0 = R\Gamma(\overline{X}_P^{\widetilde{K}\cap P} \setminus X_P^{\widetilde{K}\cap P}, p_*^{\widetilde{K}}\mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}},$$

since then $i_{P!}$ is an isomorphism by the triangle above. Using proposition 3.2.1 we can do more excision to reduce this to showing that the compactly supported complexes $R\Gamma_c(X_Q^{\widetilde{K}\cap Q}, p_*^{\widetilde{K}\cap Q}\mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}$ vanish for all proper parabolic subgroups $Q \neq P$ containing the standard Borel B and that $R\Gamma(X_B^{\widetilde{K}\cap B}, p_*^{B\cap\widetilde{K}}\mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}$ vanishes. Using the long exact sequence in cohomology attached to

$$0 \to \mathcal{V}_{\widetilde{\lambda}} \to \mathcal{V}_{\widetilde{\lambda}} \to \mathcal{V}_{\widetilde{\lambda}}/\varpi\mathcal{V}_{\widetilde{\lambda}} \to 0,$$

we see that it suffices to show the vanishing of $R\Gamma_c(X_Q^{\widetilde{K}\cap Q}, p_*^{\widetilde{K}\cap Q}(\mathcal{V}_{\widetilde{\lambda}}/\varpi\mathcal{V}_{\widetilde{\lambda}}))_{\widetilde{\mathfrak{m}}}$ and $R\Gamma(X_B^{\widetilde{K}\cap B}, p_*^{B\cap\widetilde{K}}(\mathcal{V}_{\widetilde{\lambda}}/\varpi\mathcal{V}_{\widetilde{\lambda}}))_{\widetilde{\mathfrak{m}}}$. Let $\widetilde{K}' < \widetilde{K}$ be a small enough good open normal subgroup such that $\widetilde{K}'_v = \widetilde{K}_v$ for all places $v \nmid p$ and $p_*^{\widetilde{K}'}(\mathcal{V}_{\widetilde{\lambda}}/\varpi\mathcal{V}_{\widetilde{\lambda}})$ is trivial. Then we have Hochschild-Serre spectral sequences

$$H^i(\widetilde{K}/\widetilde{K}', H_c^j(X_Q^{\widetilde{K}'\cap Q}, p_*^{\widetilde{K}\cap Q}(\mathcal{V}_{\widetilde{\lambda}}/\varpi\mathcal{V}_{\widetilde{\lambda}}))_{\widetilde{\mathfrak{m}}}) \implies H_c^{i+j}(X_Q^{\widetilde{K}\cap Q}, p_*^{\widetilde{K}\cap Q}(\mathcal{V}_{\widetilde{\lambda}}/\varpi\mathcal{V}_{\widetilde{\lambda}}))_{\widetilde{\mathfrak{m}}}$$
$$H^i(\widetilde{K}/\widetilde{K}', H^j(X_B^{\widetilde{K}\cap B}, p_*^{B\cap\widetilde{K}}(\mathcal{V}_{\widetilde{\lambda}}/\varpi\mathcal{V}_{\widetilde{\lambda}}))_{\widetilde{\mathfrak{m}}}) \implies H^{i+j}(X_B^{\widetilde{K}\cap B}, p_*^{B\cap\widetilde{K}}(\mathcal{V}_{\widetilde{\lambda}}/\varpi\mathcal{V}_{\widetilde{\lambda}}))_{\widetilde{\mathfrak{m}}}$$

which reduce the problem to showing that

$$R\Gamma_c(X_Q^{\widetilde{K}\cap Q}, p_*^{\widetilde{K}\cap Q}\underline{k})_{\widetilde{\mathfrak{m}}} = 0 = R\Gamma(X_Q^{\widetilde{K}\cap Q}, p_*^{\widetilde{K}\cap Q}\underline{k})_{\widetilde{\mathfrak{m}}}$$

where $Q \neq P$, k is a finite field of characteristic p and we have replaced $\widetilde{K}$ by $\widetilde{K}'$. By duality it is enough to only show that

$$R\Gamma(X_Q^{\widetilde{K}\cap Q}, p_*^{\widetilde{K}\cap Q}\underline{k})_{\widetilde{\mathfrak{m}}} = E_Q^{Q\cap\widetilde{K}}(k)_{\widetilde{\mathfrak{m}}} = 0$$

Now proposition 3.3.7 implies that

$$E_Q^{Q\cap\widetilde{K}}(k) \cong E_M^{M\cap\widetilde{K}}(R\Gamma((U\cap\widetilde{K})_S, k)),$$

where U is the unipotent radical of Q and M is a Levi subgroup of Q. By shrinking $\widetilde{K}$ even further we can assume that $R\Gamma((U\cap\widetilde{K})_S, k)$ is a complex of trivial $M\cap\widetilde{K}$-modules. Now $E_M^{M\cap\widetilde{K}}(R\Gamma((U\cap\widetilde{K})_S, k))$ is a direct sum of shifts of $E_M^{M\cap\widetilde{K}}(k)$. Hence it suffices to show that $E_M^{M\cap\widetilde{K}}(k)_{\widetilde{\mathfrak{m}}} = 0$, where we see $E_M^{M\cap\widetilde{K}}(k)$ as a $\widetilde{\mathbb{T}}^S$-module via the map Sat. After possibly shrinking $\widetilde{K}$ we can assume that $\widetilde{K}\cap M = \prod_{i=1}^t K_i$, where $M = \prod_{i=1}^t M_i$ and each K_i is a good open compact subgroup of $M_i(\mathbb{A}_{F^+}^\infty)$. Now there is an isomorphism

$$\mathcal{H}(M(\mathbb{A}_{F^+}^S), \widetilde{K}\cap M) \to \bigotimes_{i=1}^t \mathcal{H}(M_i(\mathbb{A}_{F^+}^S), K_i)$$

Hence if $\mathfrak{m}' < \mathcal{H}(M(\mathbb{A}_{F^+}^S), \widetilde{K}\cap M)$ is a maximal ideal in the support of $E_M^{M\cap\widetilde{K}}(k)$ such that $\mathrm{Sat}^{-1}(\mathfrak{m}') = \widetilde{\mathfrak{m}}$, then there exist maximal ideals $\mathfrak{m}_i < \mathcal{H}(M_i(\mathbb{A}_{F^+}^S), K_i)$ such that $\mathfrak{m}'$ is the product of these $\mathfrak{m}_i$ in the sense that $\mathfrak{m}'$ is generated by the images of the $\mathfrak{m}_i$ under the maps

$$\mathcal{H}(M_i(\mathbb{A}_{F^+}^S), K_i) \to \bigotimes_{j=1}^t \mathcal{H}(M_j(\mathbb{A}_{F^+}^S), K_j) \to \mathcal{H}(M(\mathbb{A}_{F^+}^S), \widetilde{K}\cap M).$$

Similarly we have a compatible factorisation

$$E_M^{\widetilde{K}\cap M}(k) = \bigotimes_{i=1}^t E_{M_i}^{K_i}(k).$$

Hence each $\mathfrak{m}_i$ is in the support of $E_{M_i}^{K_i}(k)$ and there exist semisimple Galois representations $\overline{\rho}_{\mathfrak{m}_i}$ by [ACC+23, Theorem 2.3.5 and Theorem 2.3.8]. Since $\mathrm{Sat}^{-1}(\mathfrak{m}') = \widetilde{\mathfrak{m}}$, a computation like [NT16, Lemma 4.6] shows that if $Q \neq P$ is a proper parabolic subgroup, then $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ has at least 3 simple factors. But from the computation of $\mathrm{Sat}(\widetilde{P}_v)$ after [NT16, 5.3] we know that $\overline{\rho}_{\widetilde{\mathfrak{m}}} \cong \overline{\rho}_{\mathfrak{m}} \oplus \overline{\rho}_{\mathfrak{m}}^{c,\vee}(1-2n)$ has only 2 simple factors by the non-Eisenstein assumption. Thus, there are no maximal ideals $\mathfrak{m}' < \mathcal{H}(M(\mathbb{A}_{F^+}^S), \widetilde{K}\cap M)$ in the support of $E_M^{M\cap\widetilde{K}}(k)$ such that $\mathrm{Sat}^{-1}(\mathfrak{m}') = \widetilde{\mathfrak{m}}$. Consequently, $\widetilde{\mathbb{T}}^S(E_M^{M\cap\widetilde{K}}(k))/\widetilde{\mathfrak{m}}$ has no maximal ideals. This is absurd, hence $\widetilde{\mathfrak{m}}$ is not in the support of $E_M^{M\cap\widetilde{K}}(k)$ as desired. □

Chapter 4

Hunting Hecke Eigenvalues

Our basic strategy for accessing Galois representations is by finding suitable congruences of a system of Hecke eigenvalues to another one, where the attached Galois representations are known to exist and have the desired properties. In the notation of the discussion after diagram (3.4.0.1) we would like to show that the Hecke eigensystem on $H^q(X_G^K, \mathcal{V}_\lambda)$ occurs in $H^d(X_{\widetilde{G}}^{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})$, where $\widetilde{\lambda}$ is a suitable weight of $\widetilde{G}$ and λ is the weight of G corresponding to $\widetilde{\lambda}$. We achieve something slightly weaker but sufficient in corollary 4.2.7 below. After that the properties about Galois representations follow from the general theorems on integral p-adic Hodge theory of [Liu07].

4.1 Hecke Eigenvalues Occur in Things

Since we will have to obtain our congruences of Hecke eigenvalues through a somewhat involved argument by chasing through a spectral sequence and everything is "up to nilpotents", we introduce a notation to abbreviate the precise notion of Hecke congruence we mean.

Definition 4.1.1. For a ring T and T-module M, we set $T(M) := T/\operatorname{Ann}_T(M) \hookrightarrow \operatorname{End}(M)$.

Lemma 4.1.2. *If T is a ring and M and N are T-modules such that M is a subquotient of N, then $T(M)$ is a quotient of $T(N)$.*

Proof. If $t \in T$ annihilates N, then it also annihilates M. This implies that $T(M)$ is a quotient of $T(N)$. □

Definition 4.1.3. Let T be a ring and M, N be T-modules. We say that M *occurs in* N if $\operatorname{Ann}(N)^n \subset \operatorname{Ann}(M)$ for some $n \geq 1$. We write $M \prec N$. If we want to emphasize that the exponent $n = n(x)$ only depends on some variable x, we write $M \prec_x N$.

Lemma 4.1.4. *If $M \prec N$, then there is a natural surjection $T(N) \to T(M)/J$ for some nilpotent ideal J.*

Proof. By assumption we know that $\operatorname{Ann}(N)^n \subset \operatorname{Ann}(M)$, hence $T/\operatorname{Ann}(N)^n$ surjects onto $T/\operatorname{Ann}(M)$ and if J denotes image of $\operatorname{Ann}(N)$ in $T/\operatorname{Ann}(M)$, then $J^n = 0$ and $T(N)$ surjects onto $T(M)/J$. □

Lemma 4.1.5. *The relations $\prec$ and $\prec_x$ are transitive and reflexive.*

Proof. Clear. □

Lemma 4.1.6. *Let T be a ring and*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of T-modules. Then

$$M \prec M' \oplus M''.$$

Proof. We have

$$(\operatorname{Ann}(M') \cap \operatorname{Ann}(M''))^2 \subset \operatorname{Ann}(M') \operatorname{Ann}(M'') \subset \operatorname{Ann}(M)$$

and the result follows since $\operatorname{Ann}(M' \oplus M'') = \operatorname{Ann}(M') \cap \operatorname{Ann}(M'')$. □

Lemma 4.1.7. *Let T be a ring and $E_r^{p,q}, r \geq 2$ a first quadrant spectral sequence of T-modules. For any $r \geq 2$, $E_\infty^{p,q}$ is a subquotient of $E_r^{p,q}$.*

Proof. It suffices to show that $E_{r+1}^{p,q}$ is a subquotient of $E_r^{p,q}$. But this is the case by definition. □

Lemma 4.1.8. *Let T be a ring and let $E_r^{p,q}, r \geq 2$ and $\overline{E}_r^{p,q}, r \geq 2$ be spectral sequences of T-modules. Suppose that there is a morphism of spectral sequences $\phi_r : E_r \to \overline{E}_r$ and that the spectral sequence $\overline{E}_r$ is trivial, i.e. all its differentials $\overline{d}_r$ vanish for $r \geq 2$. Then the images $I_r^{p,q} := \phi_r(E_r^{p,q})$ satisfy*

$$I_r^{p,q} \prec I_{r+1}^{p,q} \oplus I_r^{p+r,q-r+1}.$$

Proof. Consider the short exact sequence

$$0 \to \underbrace{\phi_r(\ker(d_r^{p,q}))}_{=:A} \to I_r^{p,q} \to \underbrace{I_r^{p,q}/\phi_r(\ker(d_r^{p,q}))}_{=:B} \to 0.$$

Firstly, we analyse A. Note that by assumption we have $\phi_r \circ d_r^{p-r,q+r-1} = \overline{d}_r^{p-r,q+r-1} \circ \phi_r = 0$, hence $\phi_r : \ker(d_r^{p,q}) \to I_r^{p,q}$ gives a well-defined map $E_{r+1}^{p,q} \to I_r^{p,q}$ which coincides with ϕ_{r+1} if we identify $\overline{E}_r^{p,q}$ and $\overline{E}_{r+1}^{p,q}$. In conclusion

$$A = I_{r+1}^{p,q}.$$

On the other hand $E_r^{p,q}/\ker(d_r^{p,q})$ surjects onto B via ϕ_r. Moreover, $E_r^{p,q}/\ker(d_r^{p,q})$ embeds into $E_r^{p+r,q-r+1}$ via $d_r^{p,q}$. Hence B is a subquotient of $\phi_r(E_r^{p+r,q-r+1}) = I_r^{p+r,q-r+1}$. The claim now follows from lemma 4.1.6. □

4.2 Degree Shifting Revisited

In this section, we adapt the degree shifting argument of [ACC+23, Section 4] to the case when p ramifies in F. The difference is that a certain spectral sequence does not degenerate anymore. To address this we follow an idea which originated in [CN23] and show that for high enough level the differentials in the aforementioned spectral sequence are divisible by high powers of p. Moreover, thanks to an observation of Ana Caraiani this approach is also able to completely avoid Kostant's formula!

Definition 4.2.1. Let $\mathcal{A}$ be an abelian category and C a complex in $\mathbf{D}(\mathcal{A})$. C is called formal if it is quasi-isomorphic to the complex $H^\bullet(C)$ with zero differentials.

Lemma 4.2.2 (Caraiani–Newton)**.** *Let A be an Artinian ring. Let G be a profinite group and C be a bounded complex of smooth $A[G]$-modules such that C is perfect in $\mathbf{D}(A)$. Then there exists an open subgroup $H < G$ such that $C \in \mathbf{D}(\mathrm{Mod}_H^{sm}(A))$ is quasi-isomorphic to a complex of A-modules with trivial H-action.*

Proof. Choose a bounded complex P of finitely generated projective A-modules which is quasi-isomorphic to C in $\mathbf{D}(A)$. Let G act trivially on P. Now we have

$$\mathrm{Hom}_{\mathbf{D}(\mathrm{Mod}_H^{sm}(A))}(P, C) = \mathrm{Hom}_{\mathbf{D}(A)}(P, R\Gamma(H, C)) = H^0(P^\vee \otimes_A^{\mathbb{L}} R\Gamma(H, C)),$$

by [Sta23, Lemma 07VI]. But the H-action on $P^\vee$ is trivial, so we also have

$$\mathrm{Hom}_{\mathbf{D}(\mathrm{Mod}_H^{sm}(A))}(P, C) = H^0(R\Gamma(H, P^\vee \otimes_A^{\mathbb{L}} C))$$

Now choose a complex I of injective, smooth $A[G]$-modules representing C. Then

$$\mathrm{Hom}_{\mathbf{D}(\mathrm{Mod}_H^{sm}(A))}(P, C) = H^0((P^\vee \otimes_A^{\mathbb{L}} I)^H).$$

Since filtered colimits are exact, we find that the natural map

$$\operatorname{colim}_{H<G} \operatorname{Hom}_{\mathbf{D}(\mathrm{Mod}_H^{sm}(A))}(P, C) \to \operatorname{Hom}_{\mathbf{D}(A)}(P, C)$$

is an isomorphism. Now the identity map on the right hand side has to lie in the image of $\operatorname{Hom}_{\mathbf{D}(\mathrm{Mod}_H^{sm}(A))}(P, C)$ for some open subgroup $H < G$, hence for such an H we have $P \cong C$ in $\mathbf{D}(\mathrm{Mod}_H^{sm}(A))$. □

Corollary 4.2.3. *Suppose C is a complex of smooth $A[G]$-modules such that C is perfect and formal in $\mathbf{D}(A)$, then there exists an open subgroup $H < G$ such that C is formal in $\mathbf{D}(\mathrm{Mod}_H^{sm}(A))$.*

Proof. With H as in the lemma, the quasi-isomorphism $C \cong H^\bullet(C)$ is H-equivariant. □

Remark 4.2.4. The lemma and its corollary replace [ACC+23, Lemma 4.2.3] following a suggestion of Peter Scholze. I learnt the proof from Ana Caraiani and James Newton and thank them for allowing me to include it here. I do not know in general when we can expect that the group cohomology complex $R\Gamma(U, \mathcal{O}/\varpi^m)$ is formal as a complex of $\mathcal{O}/\varpi^m[G]$-modules. It might be interesting to answer this "elementary" algebraic question.

Lemma 4.2.5. *Let $G = N \rtimes M$ be a profinite group. If A is a smooth $R[G]$-module such that there exists $R[G]$-modules B, C and a decomposition $A = B \oplus C$ as $R[M]$-modules, with $B^N = B$, then $B[0]$ is a direct summand of $R\Gamma(N, A)$ in the derived category of smooth $R[G/N]$-modules.*

Proof. We use inhomogeneous cochains to represent $R\Gamma(N, A)$. Namely, define a complex with graded pieces $C^i(N, A) := \{f : N^i \xrightarrow{cont} A\}$ and differentials

$$\begin{aligned} d : C^i(N,A) \to C^{i+1}(N,A) : f \mapsto df \\ df(g_0, \ldots, g_i) = g_0 f(g_1, \ldots, g_i) \\ + \sum_{j=1}^{i} (-1)^j f(g_0, \ldots, g_{j-1}g_j, \ldots, g_i) + (-1)^{i+1} f(g_0, \ldots, g_{n-1}) \end{aligned}$$

Then a standard argument shows that $C^\bullet(N, A)$ is quasi-isomorphic to $R\Gamma(N, A)$, where $\sigma \in G/N$ acts on $f \in C^i(N, A)$ by

$$(\sigma f)(g_1, \ldots, g_n) = \sigma f(\sigma^{-1} g \sigma, \ldots, \sigma^{-1} g \sigma).$$

Now let $e : A \to B$ be the projection, then we define $e' : C^\bullet(N, A) \to C^\bullet(N, A)$ by

$$e'f = \begin{cases} ef & f \in C^0(N, A) \\ 0 & f \in C^i(N, A), i > 0 \end{cases}$$

We have $e'd = de' = 0$ since each $b \in B$ is N-invariant , hence $db(n) = nb - b = 0$ for all $n \in N$. ☐

Let E be a finite extension of $\mathbb{Q}_p$ which contains all the p-adic embeddings of F and let $\mathcal{O}$ be the ring of integers of E. In the following, the symbol $\prec$ always refers $\widetilde{\mathbb{T}}^{\overline{S}}_{\mathcal{O}} = \widetilde{\mathbb{T}}^{\overline{S}} \otimes \mathcal{O}$-modules for some finite set of places S, as defined in the previous section.

Proposition 4.2.6. *Suppose we have a disjoint union $\overline{S}_p = \overline{S}_1 \cup \overline{S}_2$ and $S_p = S_1 \cup S_2$ a compatible decomposition. Suppose $\widetilde{K}$ is a good open compact subgroup of $\widetilde{G}(\mathbb{A}^\infty_{F^+})$ such that $G(\mathbb{A}^\infty_{F^+}) \cap \widetilde{K} \to (P(\mathbb{A}^\infty_{F^+}) \cap \widetilde{K})/(U(\mathbb{A}^\infty_{F^+}) \cap \widetilde{K})$ is an isomorphism, where U is the unipotent radical of P. Let $K = \widetilde{K} \cap G$ and $S \supset S_p$ a finite set of places such that $\widetilde{K}_v = \widetilde{G}(\mathcal{O}_{F^+_v})$ for $v \notin \overline{S}$. Let V_i be continuous $\mathcal{O}[K_{S_i}]$-modules for $i = 1, 2$, where $\mathcal{O}$ is the ring of integers in a finite extension of $\mathbb{Q}_p$ with uniformizer ϖ. Assume further that the V_i are finite free over $\mathcal{O}$. Let $\widetilde{V}_1$ be a continuous $\mathcal{O}[\widetilde{K}_{\overline{S}_1}]$-module, finite free over $\mathcal{O}$.*

Let $d = n^2[F^+ : \mathbb{Q}] = 1 + \dim X_{\mathrm{GL}_n/F}$ and suppose that the following conditions are satisfied:

(1) We have

$$\sum_{\overline{v} \in \overline{S}_2} [F^+_{\overline{v}} : \mathbb{Q}_p] > \frac{1}{2}[F^+ : \mathbb{Q}].$$

(2) We have a K-equivariant direct sum decomposition $\widetilde{V}_1 \cong W_1 \oplus W_2$ with an isomorphism of K-modules $V_1 \cong W_1$ such that $W_1 \subset \widetilde{V}_1^{\widetilde{K} \cap U}$.

Fix an integer $m \geq 1$. Then for $m' \geq m$ large enough, we have

$$H^q(X_G^{K(m')}, \mathcal{V}/\varpi^m) \prec_{n^2[F:\mathbb{Q}]} H^d(X_P^{\widetilde{K}(m')}, \widetilde{\mathcal{V}})$$

for $\lfloor d/2 \rfloor \leq q \leq d - 1$, where

$$\widetilde{K}(m') := \left\{ g \in \widetilde{K} : \forall \overline{v} \in \overline{S}_2,\ g \equiv \begin{bmatrix} \mathrm{id} & * \\ 0 & \mathrm{id} \end{bmatrix} \pmod{\varpi_{\overline{v}}^{m'}} \right\}$$
$$K(m') := \left\{ g \in K : \forall v \in S_2,\ g \equiv \mathrm{id} \pmod{\varpi_v^{m'}} \right\}$$

and $\mathcal{V}$ is the local system associated with $V_1 \otimes V_2$ and $\widetilde{\mathcal{V}}$ is the local system associated with $\widetilde{V}_1$.

Proof. By proposition 3.3.7 we have a $\widetilde{\mathbb{T}}^S \otimes \mathcal{O}$-equivariant isomorphism

$$R\Gamma(X_P^{\widetilde{K}\cap P}, \widetilde{\mathcal{V}}) = R\Gamma(X_G^K, p_*^K s^* R\Gamma((U \cap \widetilde{K})_{\overline{S}}, \widetilde{V}_1)).$$

Let $m' \geq m$ be an integer. Since $\widetilde{V}_1/\varpi^m$ is an abelian p-group, we find that

$$R\Gamma((U \cap \widetilde{K}(m'))_S, \widetilde{V}_1/\varpi^m) = R\Gamma((U \cap \widetilde{K}(m'))_{\overline{S}_p}, \widetilde{V}_1/\varpi^m)$$

and by the Künneth formula, this is equal to

$$R\Gamma((U \cap \widetilde{K})_{\overline{S}_1}, \widetilde{V}_1/\varpi^m) \otimes^{\mathbb{L}}_{\mathcal{O}/\varpi^m} R\Gamma((U \cap \widetilde{K}(m'))_{\overline{S}_2}, \mathcal{O}/\varpi^m).$$

Note that U is abelian for the Siegel parabolic P, hence $(U \cap \widetilde{K}(m')) \cong \mathbb{Z}_p^N$ for some N. Hence all the $H^j((U \cap \widetilde{K}(m'))_{\overline{S}_2}, \mathcal{O})$ are torsion free and

$$H^i((U \cap \widetilde{K}(m'))_{\overline{S}_2}, \mathcal{O}/\varpi^m) = H^i((U \cap \widetilde{K}(m'))_{\overline{S}_2}, \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^m.$$

The natural morphism

$$V_1 \otimes^{\mathbb{L}}_{\mathcal{O}} R\Gamma((U \cap \widetilde{K}(m'))_{\overline{S}_2}, \mathcal{O}) \to V_1/\varpi^m \otimes^{\mathbb{L}}_{\mathcal{O}/\varpi^m} R\Gamma((U \cap \widetilde{K}(m'))_{\overline{S}_2}, \mathcal{O}/\varpi^m),$$

induces a morphism of spectral sequences $E_r^{i,j} \to \overline{E}_r^{i,j}$, where

$$\begin{aligned} E_2^{i,j} &= H^i(X_G^{K(m')}, p_*^{K(m')} s^* V_1 \otimes H^j((U \cap \widetilde{K}(m'))_{\overline{S}_2}, \mathcal{O})) \\ \overline{E}_2^{i,j} &= H^i(X_G^{K(m')}, p_*^{K(m')} s^* V_1/\varpi^m \otimes H^j((U \cap \widetilde{K}(m'))_{\overline{S}_2}, \mathcal{O})/\varpi^m). \end{aligned}$$

By assumption, V_1 is a direct summand of $\widetilde{V}_1$, hence by lemma 4.2.5 also of $R\Gamma((U \cap \widetilde{K})_{\overline{S}_1}, \widetilde{V}_1)$. Consequently, $E_2^{i,j}$ converges to a direct summand of $H^{i+j}(X_P^{\widetilde{K}(m')}, \widetilde{\mathcal{V}})$. Similarly, $\overline{E}_2^{i,j}$ converges to a direct summand of $H^{i+j}(X_P^{\widetilde{K}(m')}, \widetilde{\mathcal{V}}/\varpi^m)$. In particular, $E_\infty^{i,j}$ is a subquotient of $H^{i+j}(X_P^{\widetilde{K}(m')}, \widetilde{\mathcal{V}})$, hence $E_\infty^{i,j} \prec H^{i+j}(X_P^{\widetilde{K}(m')}, \widetilde{\mathcal{V}})$.

By lemma 4.2.2 we see that we can choose m' large enough, so that $R\Gamma((U \cap \widetilde{K}(m'))_{\overline{S}_2}, \mathcal{O})/\varpi^m$ is formal in $\mathbf{D}(\mathrm{Mod}^{sm}_{K(m')}(\mathcal{O}/\varpi^m))$. Thus, for such m' the spectral sequence $\overline{E}_2^{i,j}$ degenerates (recall that all the constructions implicitly depend on m'). Hence the hypotheses of lemma 4.1.8 are satisifed. (Both E_r and $\overline{E}_r$ are Hecke equivariant since the construction of Grothendieck spectral sequences is functorial and we have proposition 3.3.7.) Consequently,

$$I_r^{i,j} \prec I_{r+1}^{i,j} \oplus I_r^{i+r,j-r+1},$$

where $I_r^{i,j}$ is the image of $E_r^{i,j}$ in $\overline{E}_r^{i,j}$. Additionally, it follows from a standard long exact sequence argument that the natural map

$$E_2^{i,j}/\varpi^m \to \overline{E}_2^{i,j}$$

is injective.

We prove the following claim by reverse induction on n:

- Let $n \geq \lfloor d/2 \rfloor$, then for all $q \geq n$ and $m \geq 1$, there exists an integer m_0 such that for all $m' \geq m_0$ we have

$$H^q(X_G^{K(m')}, \mathcal{V}/\varpi^m) \prec_{n^2[F:\mathbb{Q}]} H^d(X_P^{\widetilde{K}(m')}, \widetilde{\mathcal{V}})$$

For $n = d$, the claim is vacuous so we may assume that $\lfloor d/2 \rfloor \leq n \leq d-1$ and that the claim holds for $n+1$. We need to prove something for $q = n$. Fix an $m \geq 1$ and let m' be sufficiently large so that the spectral sequence $\overline{E}_2^{i,j}$ above degenerates and such that $K(m')$ acts trivially on V_2/ϖ^m and $H^\bullet((U \cap \widetilde{K}(m'))_{\overline{S}_2}, \mathcal{O})/\varpi^m$.

By assumption we have that

$$\dim \prod_{\overline{v} \in \overline{S}_2} U(F_{\overline{v}}^+) = \sum_{\overline{v} \in \overline{S}_2} n^2[F_{\overline{v}}^+ : \mathbb{Q}_p] > d/2 \geq d - q,$$

hence $H^{d-q}((U \cap \widetilde{K}(m'))_{\overline{S}_2}, \mathcal{O})$ is not zero and free over $\mathcal{O}$ by [ACC$^+$23, Lemma 4.2.2]. Thus, we may simply replace V_2 by $H^{d-q}((U \cap \widetilde{K}(m'))_{\overline{S}_2}, \mathcal{O})$ in the claim we are proving! Now we have $H^q(X_G^{K(m')}, \mathcal{V}/\varpi^m) = \overline{E}_2^{q,d-q}$ and the long exact sequence in cohomology associated with $0 \to \mathcal{V} \xrightarrow{\varpi^m} \mathcal{V} \to \mathcal{V}/\varpi^m\mathcal{V}$ implies that

$$\overline{E}_2^{i,j} \prec I_2^{q,d-q} \oplus H^{q+1}(X_G^{K(m')}, \mathcal{V})[\varpi^m].$$

For m'' large enough, we also have

$$H^{q+1}(X_G^{K(m')}, \mathcal{V})[\varpi^m] \prec H^{q+1}(X_G^{K(m')}, \mathcal{V}/\varpi^{m''}).$$

Moreover, by repeating the above application of lemma 4.1.8 at most $n^2[F : \mathbb{Q}]$ times (after which the bounded spectral sequence converges) we obtain

$$I_2^{q,d-q} \prec_{n^2[F^+:\mathbb{Q}]} I_\infty^{q,d-q} \bigoplus_{r \geq 2} I_r^{q+r,d-q-r+1}$$

By the inductive hypothesis we may choose m' large enough so that

$$I_r^{q+r,d-q-r+1} \prec \overline{E}_r^{q+r,d-q-r+1} \prec \overline{E}_2^{q+r,d-q-r+1} \prec H^d(X_P^{\widetilde{K}(m')}, \widetilde{\mathcal{V}})$$

and

$$H^{q+1}(X_G^{K(m')}, \mathcal{V}/\varpi^{m''}) \prec H^d(X_P^{\widetilde{K}(m')}, \widetilde{\mathcal{V}}).$$

We conclude the proof of the claim by using that $E_\infty^{q,d-q}$ is a subquotient of the module $H^d(X_P^{\widetilde{K}(m')}, \widetilde{\mathcal{V}})$ and $I_\infty^{q,d-q} \prec E_\infty^{q,d-q}$. □

Corollary 4.2.7. *Let $\overline{v}, \overline{v}'$ be distinct elements of $\overline{S}_p$, and let $\lambda \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,E)}$ be a dominant weight for G. Let $\widetilde{K} \subset \tilde{G}(\mathbb{A}_{F^+}^{\infty})$ be a good open compact subgroup and S a set of places of F satisfying (Σ_p) such that $\widetilde{K}_{\overline{v}} = \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})$ for $\overline{v} \notin \overline{S}$ and $G(\mathbb{A}_{F^+}^{\infty}) \cap \widetilde{K} \to (P(\mathbb{A}_{F^+}^{\infty}) \cap \widetilde{K})/(U(\mathbb{A}_{F^+}^{\infty}) \cap \widetilde{K})$ is an isomorphism. Suppose that the following conditions are satisfied:*

(1) For each embedding $\tau : F \hookrightarrow E$ inducing the place $\overline{v}$ of F^+, we have $-\lambda_{\tau c,1} - \lambda_{\tau,1} \geq 0$

(2) We have
$$\sum_{\overline{v}'' \in \overline{S}_p \setminus \{\overline{v}, \overline{v}'\}} [F^+_{\overline{v}''} : \mathbb{Q}_p] > \frac{1}{2}[F^+ : \mathbb{Q}].$$

(3) $\mathfrak{m} \subset \mathbb{T}^S$ is a non-Eisenstein maximal ideal in the support of $H^{\bullet}(X^K_{\mathrm{GL}_n/F}, \mathcal{V}_\lambda)$ such that $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ is decomposed generic, where $\widetilde{\mathfrak{m}} = \mathrm{Sat}^{-1}(\mathfrak{m})$.

Define a weight dominant weight $\widetilde{\lambda} \in (\mathbb{Z}^{2n})^{\mathrm{Hom}(F^+,E)}$ for $\widetilde{G}$ as follows: if $\tau : F^+ \to E$ does not induce either $\overline{v}$ or $\overline{v}'$, then $\widetilde{\lambda}_\tau = 0$. If τ induces $\overline{v}$, then we set
$$\widetilde{\lambda}_\tau = (-\lambda_{\tilde{\tau}c,n}, \ldots, -\lambda_{\tilde{\tau}c,1}, \lambda_{\tilde{\tau},1}, \ldots, \lambda_{\tilde{\tau},n}).$$

If τ induces $\overline{v}'$, then $\widetilde{\lambda}_\tau$ may be chosen arbitrarily from $\mathbb{Z}^{2n}_+$. Then for all integers m, there exists an integer $m' \geq m$ such that
$$H^q(X^K_G, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}} \prec_{n^2[F:\mathbb{Q}]} H^d(X^{\widetilde{K}(m')}_{\widetilde{G}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} \qquad \lfloor d/2 \rfloor \leq q \leq d-1$$
$$j_* H^q(X^K_G, \mathcal{V}_\lambda/\varpi^m)_{j^{-1}\mathfrak{m}} \prec_{n^2[F:\mathbb{Q}]} H^d(X^{\widetilde{K}(m')}_{\widetilde{G}}, \mathcal{V}^{\vee}_{\widetilde{\lambda}})_{\widetilde{j^{-1}\mathfrak{m}}} \qquad 0 \leq q < \lfloor d/2 \rfloor,$$

where
$$\widetilde{K}(m') := \left\{ g \in \widetilde{K} : \forall \overline{v}'' \neq \overline{v}, \overline{v}' \ g \equiv \begin{bmatrix} \mathrm{id} & * \\ 0 & \mathrm{id} \end{bmatrix} \pmod{\varpi^{m'}_{\overline{v}''}} \text{ and } g \equiv \mathrm{id} \pmod{\varpi^m_{\overline{v}'}} \right\}$$

and $j : \mathbb{T}^S \to \mathbb{T}^S$ is the isomorphism $[K^S g K^S] \mapsto [K^S g^{-1} K^S]$.

Proof. Let $V_{\widetilde{\lambda}}$ be the $\prod_{u|p} \widetilde{G}(\mathcal{O}_{F^+_u})$-module corresponding to $\widetilde{\lambda}$ from definition 3.4.4. Choose a decomposition $\mathrm{Hom}(F,E) = H \cup Hc$, where c denotes the complex conjugation of F. The restriction of a field homomorphism to F^+ induces a bijection $\alpha : H \cong \mathrm{Hom}(F^+, E)$. For $\tau \in H$, set
$$\lambda'_\tau = (\widetilde{\lambda}_{\alpha(\tau),n+1}, \ldots, \widetilde{\lambda}_{\alpha(\tau),2n})$$
$$\lambda'_{\tau c} = (-\widetilde{\lambda}_{\alpha(\tau),n}, \ldots, -\widetilde{\lambda}_{\alpha(\tau),1})$$

Now $\lambda'_\tau = \lambda_\tau$ for all τ which induce a place above $\overline{v}$ and by [NT16, 2.11], the $\prod_{u|p} G(\mathcal{O}_{F_u^+})$-module $V_{\lambda'}$ is a $\prod_{u|p} U(\mathcal{O}_{F_u^+})$-invariant direct summand of $V_{\widetilde{\lambda}}$. We apply proposition 4.2.6 with $\overline{S}_1 = \{\overline{v}, \overline{v}'\}$, $\overline{S}_2 = \overline{S}_p \setminus \overline{S}_1$, $\widetilde{K}' = \{g \in \widetilde{K} : g \equiv \mathrm{id} \pmod{\varpi_{\overline{v}'}^m}\}$ and $\widetilde{V}_1 = V_{\widetilde{\lambda}}$ and $V_1 = V_\lambda$ and $V_2 = \mathcal{O}$. Hence, for m' large enough we have

$$H^q(X_G^{K(m')}, \mathcal{V}_\lambda/\varpi^m) \prec_{n^2[F:\mathbb{Q}]} H^d(X_P^{K(m')}, \mathcal{V}_{\widetilde{\lambda}})$$

for $\lfloor d/2 \rfloor \leq q \leq d-1$, where

$$K(m') = \{g \in K : \forall \overline{v}'' \neq \overline{v}, \overline{v}',\ g \equiv \mathrm{id} \pmod{\varpi_{\overline{v}''}^{m'}} \text{ and } g \equiv \mathrm{id} \pmod{\varpi_{\overline{v}'}^{m}}\}.$$

We also know that for $m' \geq m$, there exist integers $a, b \geq 1$ such that we have isomorphisms of $K(m')$-modules

$$V_\lambda/\varpi^m \cong (V_{\lambda''}/\varpi^m)^a \qquad V_{\lambda'}/\varpi^m \cong (V_{\lambda''}/\varpi^m)^b,$$

where $\lambda''_\tau = \lambda_\tau$ for all τ which induce a place above $\overline{v}$ and $\lambda''_\tau = 0$ otherwise. To see this note that V_{λ_v}/ϖ^m and $V_{\lambda'_v}/\varpi^m$ are trivial $K(m')_v$-modules for all $v \nmid \overline{v}$. In conclusion, we have

$$H^q(X_G^{K(m')}, \mathcal{V}_\lambda/\varpi^m) \prec H^q(X_G^{K(m')}, \mathcal{V}_{\lambda'}/\varpi^m) \prec_{n^2[F:\mathbb{Q}]} H^d(X_P^{K(m')}, \mathcal{V}_{\widetilde{\lambda}})$$

for $\lfloor d/2 \rfloor \leq q \leq d-1$.

When $q < \lfloor d/2 \rfloor$, we apply [NT16, Proposition 2.10] to obtain a direct sum decomposition $V_{\widetilde{\lambda}} = W'_1 \oplus W'_2$ of $\prod_{u|p} G(\mathcal{O}_{F_u^+})$-modules, where the inclusion $W'_1 \to V_{\widetilde{\lambda}}$ is $\prod_{u|p} U(\mathcal{O}_{F_u^+})$-invariant and $W'_1 \cong V_{\lambda'}$. Moreover, the projection $V_{\widetilde{\lambda}} \to W'_1$ is invariant under the opposite unipotent group $w^{-1}\left(\prod_{u|p} U(\mathcal{O}_{F_u^+})\right) w$, where

$$w = \begin{bmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{bmatrix} \in \widetilde{G}(\mathcal{O}_{F^+}).$$

To see this it suffices to check the invariance for $V_{\widetilde{\lambda}}[1/p] \to W'_1[1/p]$. Now one can apply the main theorem of [Cab84] since we have $W'_1[1/p] = V_{\widetilde{\lambda}}[1/p]^U$ as in the proof of [NT16, Proposition 2.10].

Let $V_{\widetilde{\lambda}}^w$ be the $\prod_{u|p} \widetilde{G}(\mathcal{O}_{F_u^+})$-module which has the same underlying $\mathcal{O}$-module as $V_{\widetilde{\lambda}}$, but where $g \in \prod_{u|p} \widetilde{G}(\mathcal{O}_{F_u^+})$ acts as wgw^{-1}. We see that multiplication by w is an isomorphism

$$V_{\widetilde{\lambda}} \xrightarrow{w} V_{\widetilde{\lambda}}^w.$$

Then $W_1 := w(W'_1)$ is a direct summand of $V_{\widetilde{\lambda}}^w$ as a $\prod_{u|p} G(\mathcal{O}_{F_u^+})$-module since w normalises $\prod_{u|p} G(\mathcal{O}_{F_u^+})$. Moreover, the projection $V_{\widetilde{\lambda}}^w \to W_1$ is $\prod_{u|p} U(\mathcal{O}_{F_u^+})$-invariant since the projection $V_{\widetilde{\lambda}} \to W'_1$ is $w^{-1}\left(\prod_{u|p} U(\mathcal{O}_{F_u^+})\right) w$-invariant. Since

$W_1' \cong V_{\lambda'}$ and W_1 is simply the twist of W_1' by the outer automorphism $x \mapsto wxw^{-1}$ we find that $W_1 \cong V_{w\lambda'}$, where

$$w\lambda'_\tau := (-\widetilde{\lambda}_{\alpha(\tau),n}, \dots, -\widetilde{\lambda}_{\alpha(\tau),1})$$
$$w\lambda'_{\tau c} := (\widetilde{\lambda}_{\alpha(\tau),n+1}, \dots, \widetilde{\lambda}_{\alpha(\tau),2n}).$$

Dualising, we see that the inclusion

$$V_{w\lambda'}^\vee \cong W_1^\vee \hookrightarrow (V_{\widetilde{\lambda}}^w)^\vee \xrightarrow{w^T} V_{\widetilde{\lambda}}^\vee$$

is a $\widetilde{K} \cap U$-invariant direct summand of $V_{\widetilde{\lambda}}^\vee$ as required for proposition 4.2.6. Thus, we can find $m' \geq m$ such that

$$H^{d-1-q}(X_G^{K(m')}, \mathcal{V}_{w\lambda'}^\vee/\varpi^m) \prec_{n^2[F:\mathbb{Q}]} H^d(X_P^{\widetilde{K}(m')}, \mathcal{V}_{\widetilde{\lambda}}^\vee)$$

for $0 \leq q < \lfloor d/2 \rfloor$. With Verdier duality and [NT16, Proposition 3.7] we find an isomorphism of Hecke modules

$$\mathrm{Hom}(H_c^q(X_G^{K(m')}, \mathcal{V}_{w\lambda'}/\varpi^m), \mathcal{O}/\varpi^m) \cong H^{d-1-q}(X_G^{K(m')}, \mathcal{V}_{w\lambda'}^\vee/\varpi^m),$$

where $[K^S g K^S]$ on the right hand side acts as $[K^S g^{-1} K^S]$ on the left hand side. Since the left hand side has the same annihilator as $H_c^q(X_G^{K(m')}, \mathcal{V}_{w\lambda'}/\varpi^m)$, we have

$$j_* H_c^q(X_G^{K(m')}, \mathcal{V}_{w\lambda'}/\varpi^m) \prec H^{d-1-q}(X_G^{K(m')}, \mathcal{V}_{w\lambda'}^\vee/\varpi^m).$$

Moreover, for m' large enough, we find similarly to the case $q \geq \lfloor d/2 \rfloor$ that

$$H_c^q(X_G^{K(m')}, \mathcal{V}_{w\lambda}/\varpi^m) \prec H_c^q(X_G^{K(m')}, \mathcal{V}_{w\lambda'}/\varpi^m),$$

where $w\lambda_\tau = \lambda_{\tau c}$. For all $\overline{v} \notin \overline{S}$, we have $w \in \widetilde{K}_{\overline{v}}$, thus conjugation by w is the identity on the ring $\mathcal{H}(\widetilde{G}(F_{\overline{v}}^+), \widetilde{K}_{\overline{v}})$. On the other hand, w normalises $G(\mathcal{O}_{F_v})$ and maps V_λ to $V_{w\lambda}$, inducing an isomorphism of Hecke modules

$$H_c^q(X_G^{K(m')}, \mathcal{V}_\lambda/\varpi^m) \cong H_c^q(X_G^{wK(m')w^{-1}}, \mathcal{V}_{w\lambda}/\varpi^m).$$

Since $\mathfrak{m}$ is a non-Eisenstein ideal, we have $H_c^q(X, \mathcal{V})_\mathfrak{m} = H^q(X, \mathcal{V})_\mathfrak{m}$ for all q by [NT16, 4.2]. By the Hochschild–Serre spectral sequence and duality we have

$$H^q(X_G^K, \mathcal{V}_\lambda/\varpi^m)_\mathfrak{m} \prec_d \bigoplus_{i=q}^{d-1} H^i(X_G^{K(m')}, \mathcal{V}_\lambda/\varpi^m)_\mathfrak{m},$$

hence we obtain

$$H^q(X_G^K, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}} \prec_d H^d(X_P^{\widetilde{K}(m')}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} \qquad \lfloor d/2 \rfloor \le q \le d-1$$
$$j_* H^q(X_G^K, \mathcal{V}_\lambda/\varpi^m)_{j^{-1}\mathfrak{m}} \prec_d H^d(X_{P^-}^{w\widetilde{K}(m')w^{-1}}, \mathcal{V}_{\widetilde{\lambda}}^\vee)_{\widetilde{j^{-1}\mathfrak{m}}} \qquad 0 \le q < \lfloor d/2 \rfloor$$

and with proposition 3.4.10 and the main theorem of [CS19], we conclude that

$$H^q(X_G^K, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}} \prec_{n^2[F:\mathbb{Q}]} H^d(X_{\widetilde{G}}^{\widetilde{K}(m')}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} \qquad \lfloor d/2 \rfloor \le q \le d-1$$
$$j_* H^q(X_G^K, \mathcal{V}_\lambda/\varpi^m)_{j^{-1}\mathfrak{m}} \prec_{n^2[F:\mathbb{Q}]} H^d(X_{\widetilde{G}}^{w\widetilde{K}(m')w^{-1}}, \mathcal{V}_{\widetilde{\lambda}}^\vee)_{\widetilde{j^{-1}\mathfrak{m}}} \qquad 0 \le q < \lfloor d/2 \rfloor.$$

The claim follows from the $\widetilde{\mathbb{T}}^{\overline{S}}$-module isomorphism

$$H^d(X_{\widetilde{G}}^{w\widetilde{K}(m')w^{-1}}, \mathcal{V}_{\widetilde{\lambda}}^\vee) \cong H^d(X_{\widetilde{G}}^{\widetilde{K}(m')}, \mathcal{V}_{\widetilde{\lambda}}^\vee)$$

induced by conjugation by w. □

4.3 Weak Semistable Local-Global Compatiblity

We now use the results of the previous section to show a weak form of local-global compatibility in the semistable case using the ring defined in [Liu07]. We are able to show that the relevant Galois representations are torsion semistable and obtain a bound on their Hodge–Tate weights. With the results of [WE18] we also show that automorphic Galois representations are de Rham with the predicted Hodge–Tate weights. The proof of the following uses the determinant laws introduced in [Che14] and a similar argument as in the proof of [ACC+23, Proposition 4.4.6].

Theorem 4.3.1. *Suppose that S is a set of places of F satisfying (Σ_p) and that $|\overline{S}_p| > 2$ and let $\lambda \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,E)}$ be a dominant weight for G. Let $K \subset \mathrm{GL}_n(\mathbb{A}_F^\infty)$ be a good subgroup. Suppose that the following conditions are satisfied:*

(1) For each embedding $\tau : F \hookrightarrow E$, we have $-\lambda_{\tau c,1} - \lambda_{\tau,1} \ge 0$;

(2) For every place v of F above p, the group K_v contains the Iwahori Iw_v;

(3) We have

$$\sum_{\overline{v}'' \in \overline{S}_p \setminus \{\overline{v}, \overline{v}'\}} [F^+_{\overline{v}''} : \mathbb{Q}_p] > \frac{1}{2}[F^+ : \mathbb{Q}].$$

for all pairs of distinct places $\overline{v}, \overline{v}'$ of F^+ lying above p;

(4) $\mathfrak{m} \subset \mathbb{T}^S(R\Gamma(X_G^K, \mathcal{V}_\lambda))$ is a non-Eisenstein maximal ideal such that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible and decomposed generic,

then for every integer $m \geq 1$, there exists a continuous representation

$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbb{T}^S(R\Gamma(X_G^K, \mathcal{V}_\lambda/\varpi^m))_{\mathfrak{m}}/I),$$

such that

- *The representation $\rho_{\mathfrak{m}}$ is torsion semistable with Hodge–Tate weights contained in the interval $[-r, r]$, as defined in [Liu07], where r only depends on λ.*
- *If $v \mid p$ is a place such that $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ and $K_{v^c} = \mathrm{GL}_n(\mathcal{O}_{F_{v^c}})$, then $\rho_{\mathfrak{m}}$ is torsion crystalline at v.*
- *For every $v \notin S$, we have $\det(X - \rho_{\mathfrak{m}}(\mathrm{Frob}_v)) = P_v(X)$.*
- *I is a nilpotent ideal such that $I^x = 0$ for some x that only depends on $[F : \mathbb{Q}]$ and n.*

Proof. By [Sch15] we already have a nilpotent ideal J' and a representation $\rho_{\mathfrak{m}}$ with the right Frobenius eigenvalues. Thus, it is enough to show that after possibly increasing J', $\rho_{\mathfrak{m}}$ becomes torsion semistable with Hodge–Tate weights bounded by r for some r only depending on λ. We first do this under the additional assumption that $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ is decomposed generic.

It is enough to show the claim for one place $v \mid p$ of F at a time. So let us fix one and let $\overline{v}$ be the place of F^+ below v. Moreover, by assumption there exists a place $\overline{v}' \neq \overline{v}$ of F^+ above p. Now we let $\widetilde{K}$ be a good subgroup of $\widetilde{G}(\mathbb{A}_{F^+}^\infty)$ such that $\widetilde{K} \cap G = K$ and apply corollary 4.2.7 to $\overline{v}, \overline{v}'$ and a $\widetilde{\lambda}$ that is CTG [ACC+23, Definition 4.3.5] using [ACC+23, Lemma 4.3.6]. Note that this also implies that $\widetilde{\lambda}^\vee$, the highest weight of $V_{\widetilde{\lambda}}^\vee$, is CTG.

Consider the rings

$$\begin{aligned} \widetilde{A}(\widetilde{K}(m')) &:= \widetilde{\mathbb{T}}^S(H^d(X_{\widetilde{G}}^{\widetilde{K}(m')}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} \oplus H^d(X_{\widetilde{G}}^{\widetilde{K}(m')}, \mathcal{V}_{\widetilde{\lambda}}^\vee)_{\widetilde{j^{-1}\mathfrak{m}}}), \\ A(m, \lambda) &:= \mathbb{T}^S(R\Gamma(X_G^K, \mathcal{V}_\lambda/\varpi^m))_{\mathfrak{m}}. \end{aligned}$$

Corollary 4.2.7 gives a large enough $m' \geq m$ and a homomorphism

$$\widetilde{A}(\widetilde{K}(m')) \to \prod_{q=0}^{d-1} \mathbb{T}^S(H^q(X_G^K, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}})/J_q$$

for some nilpotent ideals J_q with exponent of nilpotency only depending on $[F : \mathbb{Q}]$ and n, such that the projections with $q \geq \lfloor d/2 \rfloor$ commute with $\mathrm{Sat} : \widetilde{\mathbb{T}}^S \to \mathbb{T}^S$ and the projections with $q < \lfloor d/2 \rfloor$ commute with $j \circ \mathrm{Sat}$. Thus, by composing with the map $j \times \cdots \times j \times \mathrm{id} \times \cdots \times \mathrm{id}$ we obtain a map

$$\Psi : \widetilde{A}(\widetilde{K}(m')) \to \prod_{q=0}^{d-1} \mathbb{T}^S(H^q(X_G^K, \mathcal{V}_\lambda/\varpi^m)_\mathfrak{m})/J'_q,$$

commuting with Sat, where $J'_q = j(J_q)$ for $q < \lfloor d/2 \rfloor$ and $J'_q = J_q$ otherwise. Now [ACC+23, Lemma 2.2.3] shows that the kernel J_{-1} of the natural map

$$A(m, \lambda) \to \prod_{q=0}^{d-1} \mathbb{T}^S(H^q(X_G^K, \mathcal{V}_\lambda/\varpi^m)_\mathfrak{m})$$

satisfies $J_{-1}^d = 0$. Let $I < A(m, \lambda)$ be the ideal generated by J', J_{-1} and the preimages of the J'_q for $0 \le q \le d - 1$. Then Ψ factors through a surjection

$$\widetilde{A}(\widetilde{K}(m')) \twoheadrightarrow A(m, \lambda)/I,$$

commuting with Sat. The degree of nilpotency of I still only depends on $[F : \mathbb{Q}]$ and n.

The main theorem of [CS19] implies that $\widetilde{A}(\widetilde{K}(m'))$ is $\mathcal{O}$-flat. Moreover, [ACC+23, Theorem 2.4.11] shows that $\widetilde{A}(\widetilde{K}(m')) \otimes_\mathcal{O} \overline{\mathbb{Q}}_p$ is semisimple and can be computed in terms of cuspidal automorphic representations of $\widetilde{G}$. Now by [ACC+23, Theorem 2.3.3] there exists a continuous representation

$$\widetilde{\rho}_{m'} : G_{F,S} \to \mathrm{GL}_{2n}(\widetilde{A}(\widetilde{K}(m')) \otimes_\mathcal{O} \overline{\mathbb{Q}}_p)$$

such that $\widetilde{\rho}_{m'}$ is semistable at v and v^c of Hodge–Tate weights

$$HT_\tau(\widetilde{\rho}) = \{-\lambda_{\tau c,n} + 2n - 1, \dots, -\lambda_{\tau c,1} + n, \lambda_{\tau,1} + n - 1, \dots, \lambda_{\tau,n}\}$$

Moreover, if $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ and $K_{v^c} = \mathrm{GL}_n(\mathcal{O}_{F_{v^c}})$, then we can choose $\widetilde{K}_{\overline{v}}$ to be hyperspecial as well and $\widetilde{\rho}_{m'}$ is crystalline at v and v^c. Let $r = \max_\tau \max_{\mu \in HT_\tau(\widetilde{\rho})} |\mu|$.

Set $\widetilde{A} := \widetilde{A}(\widetilde{K}(m'))$ and let $\widetilde{D}$ be the continuous $\widetilde{A}$-valued determinant of $G_{F,S}$ attached to $\widetilde{\rho}_{m'}$ by [Che14, 2.32]. The formation of kernels commutes with flat base change, so $\ker(\widetilde{D}) \otimes_\mathcal{O} \overline{\mathbb{Q}}_p \cong \ker(\widetilde{D} \otimes_\mathcal{O} \overline{\mathbb{Q}}_p)$. Hence [Che14, 2.12] implies that there is an algebra embedding

$$(\widetilde{A}[G_{F,S}]/\ker(\widetilde{D})) \otimes_\mathcal{O} \overline{\mathbb{Q}}_p = (\widetilde{A} \otimes_\mathcal{O} \overline{\mathbb{Q}}_p)[G_{F,S}]/\ker(\widetilde{D} \otimes_\mathcal{O} \overline{\mathbb{Q}}_p) \hookrightarrow M_{2n}(\widetilde{A} \otimes_\mathcal{O} \overline{\mathbb{Q}}_p).$$

Seen as a Galois representations, $\widetilde{A}[G_{F,S}]/\ker(\widetilde{D})$ is a subobject of $\widetilde{\rho}_{m'}^{2n} \cong M_{2n}(\widetilde{A} \otimes_\mathcal{O} \overline{\mathbb{Q}}_p)$. By [Che14, 1.18 (iii)] there is a natural surjection

$$\widetilde{A}[G_{F,S}]/\ker(\widetilde{D}) \to (A(m, \lambda)/I)[G_{F,S}]/\ker(\widetilde{D}_{A(m,\lambda)/I}).$$

Hence $(A(m,\lambda)/I)[G_{F,S}]/\ker(\widetilde{D}_{A(m,\lambda)/I})$ is a subquotient of $\widetilde{\rho}_{m'}^{2n}$ and also torsion semistable at v and v^c with weights contained in $[-r,r]$. Now by Chebotarev density theorem and the computation of $\mathrm{Sat}(\widetilde{P}_v)(X)$ right after [NT16, 5.3] we find

$$\widetilde{D}_{A(m,\lambda)/I} = D(\rho_{\mathfrak{m}} \oplus \rho_{\mathfrak{m}}^{c,\vee}(1-2n)).$$

We obtain an induced surjective $A(m,\lambda)$-algebra morphism

$$(A(m,\lambda)/I)[G_{F,S}]/\ker(\widetilde{D}) \to M_n(A(m,\lambda)/I)$$

mapping $g \in G_{F,S}$ to $\rho_{\mathfrak{m}}(g)$. Hence $\rho_{\mathfrak{m}}$ is a quotient of $(A(m,\lambda)/I)[G_{F,S}]/\ker(\widetilde{D})$ and also torsion semistable (resp. crystalline) with weights contained in $[-r,r]$.

Now we can remove the assumption that $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ is decomposed generic in exactly the same way as at the end of the proof of [ACC+23, 4.4.8]. □

Theorem 4.3.2 (Wang-Erickson). *Let $\mathbb{F}$ be a finite field of characteristic p and S a finite set of places of F containing S_p. Let $\overline{D}: G_{F,S} \to \mathbb{F}$ be a multiplicity-free determinant of dimension n and $S_p = S_{cris} \cup S_{ss}$ a partition and $\mu = (\mu_\tau)_{\tau: F \to \overline{\mathbb{Q}}_p}$ a tuple of n-element multisets of integers. Then there exists a quotient $R_{\overline{D}}^{cs,\mu}$ of the pseudodeformation ring $R_{\overline{D}}$ such that for any finite extension $E/\mathbb{Q}_p$, a homomorphism $f: R_{\overline{D}} \to E$ factors through $R_{\overline{D}}^{cs,\mu}$ if and only if the semisimple representation corresponding to f is semistable at all places above p with Hodge–Tate weights μ and crystalline at the places in S_{cris}.*

Proof. This is proved in the same way as [WE18, Theorem 7.9], except that we first consider the formal closed substacks

$$\mathcal{R}ep_{\overline{D}|_{G_{F_v}}}^{ss/cris,\mu_v} \subset \mathcal{R}ep_{\overline{D}|_{G_{F_v}}}$$

from [WE18, Theorem 6.7] for $v \mid p$ and form the fibre product

$$\begin{array}{ccc} \mathcal{R}ep_{\overline{D}}^{cs,\mu} & \longrightarrow & \mathcal{R}ep_{\overline{D}} \\ \downarrow & & \downarrow \\ \prod_{v\in S_{ss}} \mathcal{R}ep_{\overline{D}|_{G_{F_v}}}^{ss,\mu_v} \times \prod_{v \in S_{cris}} \mathcal{R}ep_{\overline{D}|_{G_{F_v}}}^{cris,\mu_v} & \longrightarrow & \prod_{v|p} \mathcal{R}ep_{\overline{D}|_{G_{F_v}}} \end{array}$$

The formal stack $\mathcal{R}ep_{\overline{D}} \to \mathrm{Spf}\, R_{\overline{D}}$ has an algebraization $\mathrm{Rep}_{\overline{D}} \to \mathrm{Spec}\, R_{\overline{D}}$, which is a good moduli space by [WE18, Theorem 3.8]. By [WE18, Theorem 3.16] formal GAGA holds for the morphism $\mathrm{Rep}_{\overline{D}} \to \mathrm{Spec}\, R_{\overline{D}}$, hence there exists a unique closed substack $\mathrm{Rep}_{\overline{D}}^{cs,\mu} \subset \mathrm{Rep}_{\overline{D}}$ whose formal completion along $\mathfrak{m}_{R_{\overline{D}}}$ is $\mathcal{R}ep_{\overline{D}}^{cs,\mu}$. Let $\mathrm{Spec}\, R_{\overline{D}}^{cs,\mu}$ be the scheme theoretic image of this substack in $\mathrm{Spec}\, R_{\overline{D}}$. Now one can check that $R_{\overline{D}}^{cs,\mu}$ has the desired property in a similar way to [WE18, Corollary 6.8 and Theorem 7.9]. □

Theorem 4.3.3. *Let F be a CM field, $\iota : \mathbb{C} \to \overline{\mathbb{Q}_p}$ an isomorphism and Π a cohomological cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. If the reduction $\overline{r_\iota(\Pi)}$ is absolutely irreducible and decomposed generic, then for every place $v \mid p$ of F $r_\iota(\Pi)|_{G_{F_v}}$ is potentially semistable (hence also de Rham) with Hodge–Tate weights*

$$HT_\tau := \{\lambda_{\tau,1} + n - 1, \lambda_{\tau,2} + n - 2, \ldots, \lambda_{\tau,n}\},$$

where $\lambda \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_p)}$ is the unique dominant weight for GL_n/F such that the infinitesimal character of Π_∞ coincides with the infinitesimal character of $(V_\lambda \otimes_{\mathcal{O},\iota} \mathbb{C})^\vee$ (see definition 3.4.4). If Π_v and Π_{v^c} are unramified, then $r_\iota(\Pi)|_{G_{F_v}}$ is crystalline.

Proof. Let S be the union of the places where Π is ramified and those lying above p or ∞. Let S_{cris} be the set of places $v \mid p$ such that Π_v and Π_{v^c} are unramified. We show that $r_\iota(\Pi)$ is semistable at all places above p and crystalline at places above S_{cris} when restricted to $G_{F'}$ for some finite extension F'/F which is totally split at all places in S_{cris}. By solvable base change (see section 5.2 below) we can find an extension F'/F which is totally split at all places in S_{cris} such that for the level of the base change of Π to $\mathrm{GL}_n(\mathbb{A}_{F'})$, the conditions *2* and *3* of theorem 4.3.1 are satisfied and S satisfies (Σ_p). We replace F by F' and Π by its base change. To ensure condition *1* we twist Π by a suitable power of $\|\cdot\| \circ \det$ and $r_\iota(\Pi)$ by the corresponding power of the cyclotomic character. Then Π is still cohomological cuspidal automorphic and $\overline{r_\iota(\Pi)}$ is still absolutely irreducible and decomposed generic. It suffices to prove the theorem for this twisted Π since a Galois representation is potentially semistable if and only if it has a Tate twist which is potentially semistable. By twisting with a suitable crystalline character we may also assume that $\overline{r_\iota(\Pi)} \not\cong \overline{r_\iota(\Pi)}^{c,\vee}(1-2n)$.

Now by the decomposition in [FS98, §2.2] there exists a good open compact subgroup $K < \mathrm{GL}_n(\mathbb{A}_F)$ such that the module $(\Pi^\infty)^K$ is a $\mathbb{T}^S$-equivariant direct summand of $R\Gamma(X^K_{\mathrm{GL}_n/F}, \mathcal{V}_\lambda) \otimes_\iota \mathbb{C}$, where $K_v = \mathrm{GL}_n(\mathcal{O}_{F,v})$ for $v \notin S$. In particular there is a non-trivial map $\mathbb{T}^S_{\mathcal{O}} \to \mathrm{End}((\Pi^\infty)^K)$ and since Π is irreducible, its kernel is a prime ideal $\mathfrak{p}$. After enlarging $\mathcal{O}$, we can assume that $\mathbb{T}^S_{\mathcal{O}}/\mathfrak{p} \cong \mathcal{O}$. Now let $\mathfrak{m}$ be any maximal ideal of $\mathbb{T}^S_{\mathcal{O}}$, containing $\mathfrak{p}$. The residual Galois representation associated with $\mathfrak{m}$ by [Sch15] is $\overline{r_\iota(\Pi)}$, hence condition *4* of theorem 4.3.1 is satisfied. We obtain a sequence of Galois representations

$$\rho_j : G_{F,S} \to \mathrm{GL}_n(\mathbb{T}^S(R\Gamma(X^K_G, \mathcal{V}_\lambda/\varpi^j))_{\mathfrak{m}}/I_j)$$

which are all torsion semistable with Hodge–Tate weights bounded independently of j and crystalline at all places in S_{cris}. Moreover, with theorem 4.3.2 we see that the proof of theorem 4.3.1 implies that the homomorphism α_j corresponding to the determinant $D = D(\rho_j \oplus \rho_j^{c,\vee}(1-2n))$ factors as follows

$$\begin{array}{ccc} R_{\overline{D}} & \xrightarrow{\alpha_j} & \mathbb{T}^S(R\Gamma(X_G^K, \mathcal{V}_\lambda/\varpi^j))_{\mathfrak{m}}/I_j \\ \downarrow & \overset{\beta_j}{\nearrow} & \\ R_{\overline{D}}^{cs,\mu} & & \end{array}$$

where

$$\mu_\tau = \{-\lambda_{\tau c,n} + 2n - 1, \dots, -\lambda_{\tau c,1} + n, \lambda_{\tau,1} + n - 1, \dots, \lambda_{\tau,n}\}.$$

Consider the product of the β_j

$$\beta : R_{\overline{D}}^{cs,\mu} \to \prod_j \mathbb{T}^S(R\Gamma(X_G^K, \mathcal{V}_\lambda/\varpi^j))_{\mathfrak{m}}/I_j.$$

By the computation of the characteristic polynomials of Frobenius elements we know that its image is contained in the image of the natural map

$$\mathbb{T}^S(R\Gamma(X_G^K, \mathcal{V}_\lambda))_{\mathfrak{m}} \to \prod_j \mathbb{T}^S(R\Gamma(X_G^K, \mathcal{V}_\lambda/\varpi^j))_{\mathfrak{m}}/I_j,$$

which has a nilpotent kernel itself. Hence we get a map $R_{\overline{D}}^{cs,\mu} \to \mathbb{T}^S(R\Gamma(X_G^K, \mathcal{V}_\lambda))_{\mathfrak{m}}/I$ for some nilpotent ideal I. Quotienting out by $\mathfrak{p}$ we obtain the determinant attached to

$$\widetilde{\rho} = r_\iota(\Pi) \oplus r_\iota(\Pi)^{c,\vee}(1-2n).$$

Now theorem 4.3.2 implies that $\widetilde{\rho}$ is semistable with Hodge–Tate weights μ and crystalline at all places in S_{cris}. It follows that $r_\iota(\Pi)$ is semistable with Hodge–Tate weights a subset of μ. Moreover, compatibility with central characters shows that

$$HT_\tau(\det r_\iota(\Pi)) = \left\{ \sum_{i=1}^n (\lambda_{\tau,i} + n - i) \right\}$$

By condition *1* of theorem 4.3.1 we have that $\sum_{i=1}^n (\lambda_{\tau,i}+n-i) < \sum_{i \in J}(\lambda_{\tau,i}+n-i)$ for every n elements subset $J_\tau \subset \mu$ such that $J \neq HT_\tau$. Thus, $HT_\tau(r_\iota(\Pi)) = HT_\tau$. $\square$

Chapter 5

Vanishing of the Bloch-Kato Selmer Group

In this chapter we combine the results of the previous sections to prove the main theorem. More concretely, we check that the conditions of lemma 2.6.4 are satisfied. The rings T_N will be special cases of the Hecke algebras discussed above. The rings R_N will be Galois deformation rings which we introduce now.

5.1 Galois Deformations

We now fix notations for Galois deformation rings and recall the necessary results. The basic ideas go back to [Maz89] but we also require p-adic Hodge theory conditions studied in [Kis08] and [Liu07] and the smoothness condition from [All16]. Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$, uniformizer ϖ and residue field $k = \mathcal{O}/\varpi\mathcal{O}$. Let F be a number field and S a finite set of places of F containing S_p, the places above p. We assume that E contains all p-adic embeddings of F.

Let G_F be the absolute Galois group of F and n a positive integer. Fix continuous homomorphisms $\rho : G_F \to \mathrm{GL}_n(\mathcal{O})$ and $\overline{\rho} : G_F \to \mathrm{GL}_n(k)$ which are unramified outside S and so that $\rho \equiv \overline{\rho} \pmod{\varpi}$. We are interested in the following two types of questions:

(A) Given an Artinian local $\mathcal{O}$-algebra A with an isomorphism $A/\mathfrak{m}_A \cong k$, what are the continuous (here A has the discrete topology) homomorphisms $\rho_A : G_F \to \mathrm{GL}_n(A)$ such that $\rho_A \equiv \overline{\rho} \pmod{\mathfrak{m}_A}$?

(B) Given an Artinian local E-algebra A with an isomorphism $A/\mathfrak{m}_A \cong E$, what are the continuous (here A has the topology induced by the E-vector space structure on A) homomorphisms $\rho_A : G_F \to \mathrm{GL}_n(A)$ such that $\rho_A \equiv \rho \pmod{\mathfrak{m}_A}$?

Question (A) asks about (framed) Galois deformations in the style of Mazur [Maz89]. It concerns integral questions and asks about congruences between Galois representations. Question (B) asks about (framed) Galois deformations in characteristic 0 like $\rho' : G_F \to \mathrm{GL}_n(E[t]/(t^m))$. Taking $A = E[t]/(t^2)$, this is equivalent to classifying all extensions $0 \to \rho \to \rho' \to \rho \to 0$, where ρ' is a continous Galois representation $G_F \to \mathrm{GL}_{2n}(E)$. Ultimately, we will deduce the vanishing of the Bloch–Kato Selmer group by analysing a case of question (B).

Let us make more precise definitions now. If A is an Artinian local $\mathcal{O}$-algebra with an isomorphism $A/\mathfrak{m}_A \cong k$, then a continous homomorphism $\rho_A : G_F \to \mathrm{GL}_n(A)$ such that $\rho_A \equiv \overline{\rho} \pmod{\mathfrak{m}_A}$ is called a *lift* of $\overline{\rho}$. Two such homomorphisms are *strictly equivalent* if there is an element $\gamma \in \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k))$ such that $\gamma\rho_A\gamma^{-1} = \rho'_A$. A strict equivalence class is called a *deformation* of $\overline{\rho}$. We use the analogous terminology in question (B) for lifts and deformations of ρ to Artinian local E-algebras A with an isomorphism $A/\mathfrak{m}_A \cong E$. We let $\mathcal{D}$ denote the functor that sends A to the set of all deformations of $\overline{\rho}$ over A and $\mathcal{D}^\square$ the the functor that sends A the set of all lifts of $\overline{\rho}$ over A. The analogous functors for question (B) are denoted by $\mathcal{D}_\rho$ and $\mathcal{D}^\square_\rho$.

If v is a place of F, a local lift at v is a continuous homomorphism $\rho_A : G_{F_v} \to \mathrm{GL}_n(A)$. We denote the functor that sends A to the set of local lifts of $\overline{\rho}$ over A by $\mathcal{D}^\square_v$. It is well-known that $\mathcal{D}^\square_v$ is represented by a complete Noetherian local $\mathcal{O}$-algebra $R^\square_v$ with universal lift $\rho^{\square,univ} : G_{F_v} \to \mathrm{GL}_n(R^\square_v)$.

Definition 5.1.1. A global deformation problem is a tuple

$$\mathcal{S} = (\overline{\rho}, S, \{\mathcal{D}_v\}_{v\in S}),$$

where $\overline{\rho}$ and S are as above and each $\mathcal{D}_v$ is a subfunctor of $\mathcal{D}^\square_v$ which is represented by a quotient $R^\square_v \to R_v$ and if $\rho_A \in \mathcal{D}_v(A)$ is strictly equivalent to $\rho'_A \in \mathcal{D}^\square_v$, then also $\rho'_A \in \mathcal{D}_v(A)$. For a subset $T \subset S$, we define $R^{T,loc}_{\mathcal{S}} := \widehat{\bigotimes}_{v\in T} R_v$.

Definition 5.1.2. A deformation $\rho_A \in \mathcal{D}(A)$ (resp. lift) is of *type* $\mathcal{S}$ if one element (equivalently all elements) of the strict equivalence class of $\rho_A|_{G_{F_v}}$ lie in $\mathcal{D}_v(A)$ for all $v \in S$. We note by $\mathcal{D}_\mathcal{S}$ the subfunctor of $\mathcal{D}$ of deformations of type $\mathcal{S}$. If $T \subset S$, then a *T-framed lift of type* $\mathcal{S}$ is a tuple $(\rho_A, \{\alpha_v\}_{v\in T})$, where ρ_A is a lift of $\overline{\rho}$ of type $\mathcal{S}$ and α_v are elements of $\ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k))$. Two T-framed lifts of type $\mathcal{S}$ are *strictly equivalent* if there exists $\gamma \in \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k))$ such that $\rho'_A = \gamma\rho_A\gamma^{-1}$ and $\alpha'_v = \gamma\alpha_v$. The functor of T-framed deformations of type $\mathcal{S}$, i.e. strict equivalence classes of T-framed lifts of type $\mathcal{S}$, is denoted by $\mathcal{D}^T_\mathcal{S}$.

Theorem 5.1.3. *If $\mathcal{S} = (\overline{\rho}, S, \{\mathcal{D}_v\}_{v\in S})$ is a global deformation problem, where $\overline{\rho}$ is absolutely irreducible and $T \subset S$, then $\mathcal{D}_{\mathcal{S}}$ and $\mathcal{D}_{\mathcal{S}}^T$ are representable by complete noetherian local $\mathcal{O}$-algebras $R_{\mathcal{S}}$ and $R_{\mathcal{S}}^T$ with residue fields isomorphic to k.*

Proof. This is an application of Schlessinger's criterion as explained in [Maz89]. □

Theorem 5.1.4. *Let $\mathcal{S} = (\overline{\rho}, S, \{\mathcal{D}_v^{\square}\}_{v\in S})$, where $\overline{\rho}$ is absolutely irreducible and let $[\rho] : R_{\mathcal{S}} \to E$ be the homomorphism corresponding to ρ. Then $\widehat{(R_{\mathcal{S}})_{[\rho]}}$ pro-represents the functor $\mathcal{D}_{\mathcal{S},\rho}$ which sends an Artinian local E-algebra A with residue field E to the set of deformations of ρ, unramified outside S.*

The tangent space of $\widehat{(R_{\mathcal{S}})_{[\rho]}}$ is canonically isomorphic to $H^1(G_{F,S}, \operatorname{ad}\rho)$.

Proof. See [Kis09, Lemma 2.3.3 and Proposition 2.3.5] for the identification of deformation rings. The tangent space computation follows from a standard argument which identifies

$$\mathcal{D}_{\mathcal{S},\rho}(E[t]/(t^2)) \cong \operatorname{Ext}^1_{E[G_{F,S}]}(\rho,\rho) \cong \operatorname{Ext}^1_{E[G_{F,S}]}(E, \operatorname{ad}\rho) \cong H^1(G_{F,S}, \operatorname{ad}\rho),$$

where the Ext groups are computed in the category of continuous $E[G_{F,S}]$-modules. □

Lemma 5.1.5. *If $\mathcal{S} = (\overline{\rho}, S, \{\mathcal{D}_v\}_{v\in S})$ is a global deformation problem, where $\overline{\rho}$ is absolutely irreducible and T a nonempty subset of S, then $R_{\mathcal{S}}^T \cong R_{\mathcal{S}}[[X_1, \dots, X_{n^2|T|-1}]]$.*

Proof. If A is an Artinian local $\mathcal{O}$-algebra with residue field k, then

$$\mathcal{D}_{\mathcal{S}}^T(A) = \{(\rho_A, (\alpha_v)_{v\in T}) : \rho_A \in \mathcal{D}_{\mathcal{S}}^{\square}(A)\}/\sim .$$

Since $\overline{\rho}$ is absolutely irreducible, there exists a universal deformation

$$\rho^{univ} : G_{F,S} \to \operatorname{GL}_n(R_{\mathcal{S}})$$

of $\overline{\rho}$. We define a morphism of functors

$$\Phi : \operatorname{Hom}(R_{\mathcal{S}}[[X_1, \dots, X_{n^2|T|-1}]], A) \to \mathcal{D}_{\mathcal{S}}^T(A)$$

by sending f to $(\rho^{univ} \otimes_{R_{\mathcal{S}},f} A, (\alpha_v)_{v\in T})$ with $\alpha_v = \operatorname{id} + \left[Y_{ij}^v\right]_{1\le i,j\le n}$ for $Y_{ij}^v \in \mathfrak{m}_A$ such that

- $Y_{00}^{v_0} = 0$ for a fixed $v_0 \in T$.
- $Y_{ij}^v = f(X_k)$, where $k = k(i,j,v)$ is a bijection between the indices $(i,j,v) \neq (0,0,v_0)$ and $k = 1, \dots, n^2|T| - 1$.

If $\Phi(f_1) \sim \Phi(f_2)$, then $\rho^{univ} \otimes_{R_{\mathcal{S}},f_1} A$ and $\rho^{univ} \otimes_{R_{\mathcal{S}},f_2} A$ are equivalent. This implies that $f_1|_{R_{\mathcal{S}}} = f_2|_{R_{\mathcal{S}}}$. Since $\overline{\rho}$ is absolutely irreducible, all its endomorphisms are scalar and $\Phi(f_1)$ and $\Phi(f_2)$ must be conjugate by an element $\gamma = (1+t)\,\mathrm{id}$, with $t \in \mathfrak{m}_A$. But $Y_{00}^{v_0}$ forces $t = 0$ and $f_1(X_k) = f_2(X_k)$ for all k. Thus $f_1 = f_2$ and Φ is injective.

On the other hand, given $(\rho_A, (\alpha_v)_{v\in T})$, one can conjugate by a the diagonal matrix $\gamma = (1+Y_{00}^{v_0})^{-1}\,\mathrm{id}$ so that $(\gamma\rho_A\gamma^{-1}, (\gamma\alpha_v)_{v\in T})$ is in the image Φ. Thus, Φ is also surjective. Now the claim follows from Yoneda's lemma. □

Theorem 5.1.6. *Assume that $\overline{\rho}|_{G_{F_v}}$ is torsion semistable with Hodge–Tate weights contained in $[-r, r]$. Then there is a subfunctor $\mathcal{D}_v^{ss,r}$ of $\mathcal{D}_v^{\square}$ consisting of lifts which are torsion semistable with Hodge–Tate weights contained in $[-r, r]$. Moreover, $\mathcal{D}_v^{ss,r}$ is represented by a quotient $R_v^{\square} \to R_v^{ss,r}$ and two strictly equivalent lifts are torsion semistable if and only if one of them is. The same works with "crystalline" in place of "semistable". Hence if $S_{cris} = \{v \mid p : \overline{\rho}|_{G_{F_v}}$ is torsion crystalline.$\}$ and $S_{ss} = S_p \setminus S_{cris}$, then*

$$\mathcal{S} = (\overline{\rho}, S, \{\mathcal{D}_v^{cris,r}\}_{v\in S_{cris}} \cup \{\mathcal{D}_v^{ss,r}\}_{v\in S_{ss}} \cup \{\mathcal{D}_v^{\square}\}_{v\in S\setminus S_p})$$

is a well-defined global deformation problem.

Proof. See the main theorem of [Liu07]. □

Theorem 5.1.7. *Let $\mathcal{S} = (\overline{\rho}, S, \{\mathcal{D}_v^{cris,r}\}_{v\in S_{cris}} \cup \{\mathcal{D}_v^{ss,r}\}_{v\in S_{ss}} \cup \{\mathcal{D}_v^{\square}\}_{v\in S\setminus S_p})$ be as in the previous proposition. Let $x : R_{\mathcal{S}} \to E$ be a homomorphism and let ρ_x be the corresponding representation. Then $\widehat{(R_{\mathcal{S}})_x}$ pro-represents the functor $\mathcal{D}_{\mathcal{S},\rho_x}$ which sends an Artinian local E-algebra with residue field E to the set of deformations of ρ_x which are unramified outside S, semistable with Hodge–Tate weights equal to those of ρ_x and crystalline at the places in S_{cris}.*

The tangent space of $\widehat{(R_{\mathcal{S}})_x}$ is isomorphic to $H^1_{fg}(G_{F,S}, \mathrm{ad}\,\rho_x)$ (see section 1.5 for the definition of this space).

Proof. The interpretation of $\widehat{(R_{\mathcal{S}})_x}$ follows from the main theorem of [Liu07]. If $S_{cris} = \emptyset$, then the isomorphism with the tangent space is proven in [All16, Proposition 1.3.12]. For non-empty S_{cris}, the proof is similar and relies on the fact that an element of $H^1(G_{F,S}, \mathrm{ad}\,\rho_x)$ gives rise to an extension $0 \to \rho_x \to \eta \to \rho_x \to 0$ which is crystalline at v if and only if it is mapped to 0 in $H^1(G_{F_v}, \mathrm{ad}\,\rho_x \otimes B_{cris})$. □

Lemma 5.1.8. *For each $v \nmid p$, we have $\dim R_v^{\square}[1/p] \leq n^2$. Moreover, for each $v \mid p$ we have $\dim R_v^{ss,r}[1/p] \leq n^2 + \frac{n(n-1)}{2}[F_v : \mathbb{Q}_p]$ and $\dim R_v^{cris,r}[1/p] \leq n^2 + \frac{n(n-1)}{2}[F_v : \mathbb{Q}_p]$.*

Proof. See [All16, Proposition 1.2.2 and Theorem 1.2.4] and [Kis08, Theorem 3.3.8]. □

Proposition 5.1.9. *Let $\rho : G_F \to \mathrm{GL}_n(E)$ be a continuous representation, unramified outside S such that ρ is generic [All16, Definition 1.1.2] at all places $v \in S \setminus S_{cris}$. Then $(R_v)_\mathfrak{p}$ is regular for all $v \in S$, where R_v are the local framed deformation rings of*

$$\mathcal{S} = (\overline{\rho}, S, \{\mathcal{D}_v^{cris,r}\}_{v\in S_{cris}} \cup \{\mathcal{D}_v^{ss,r}\}_{v\in S_{ss}} \cup \{\mathcal{D}_v^{\square}\}_{v\in S\setminus S_p})$$

and $\mathfrak{p}$ is the kernel of the homomorphism $R_v \to E$ induced by ρ.

Proof. This follows from theorem 5.1.7 and [All16, Proposition 1.2.2 and Theorem 1.2.7] together with [Kis08, Theorem 3.3.8]. □

Definition 5.1.10. Let $\mathcal{S}$ be a global deformation datum, then a *Taylor–Wiles datum* for $\mathcal{S}$ is a tuple $(Q, (\alpha_{v,1}, \dots, \alpha_{v,n})_{v\in Q})$, where Q is a finite set of places of F disjoint from S such that

- for all $v \in Q$, $q_v \equiv 1 \pmod p$;
- $(\alpha_{v,1}, \dots, \alpha_{v,n})$ are the eigenvalues of $\overline{\rho}(\mathrm{Frob}_v)$, which are assumed to be pairwise distinct and k-rational.

Given a Taylor–Wiles datum we define the augmented global deformation problem

$$\mathcal{S}_Q = (\overline{\rho}, S \cup Q, \{\mathcal{D}_v^{\square}\}_{v\in Q} \cup \{\mathcal{D}_v\}_{v\in S})$$

Lemma 5.1.11. *For a subset $T \subset S$ and Taylor–Wiles datum $(Q, (\alpha_{v,1}, \dots, \alpha_{v,n})_{v\in Q})$, the deformation ring $R^T_{\mathcal{S}_Q}$ obtains a natural local $\mathcal{O}[\Delta_Q^n]$-algebra structure, where Δ_Q is the maximal p-quotient of $\prod_{v\in Q} k(v)^\times$ and the maximal ideal of $\mathcal{O}[\Delta_Q^n]$ is the kernel of $\mathcal{O}[\Delta_Q^n] \to k : g \mapsto 1$. Moreover, the kernel of the natural map $R^T_{\mathcal{S}_Q} \to R^T_{\mathcal{S}}$, is generated by the augmentation ideal of $\mathcal{O}[\Delta_Q^n]$.*

Proof. This follows from [ACC+23, Lemma 6.2.19] and the discussion afterwards and goes back to [DDT97, Lemma 2.44]. □

5.2 Base Change

Ultimately, we wish to show that the Bloch–Kato Selmer group of a representation of G_F vanishes. It turns out that to do so one can first restrict to G_L for any finite extension L/F.

Lemma 5.2.1. *Let L/F be a finite extension of number fields unramified outside a finite set of places S of F containing all places above p and Σ be the set of places of L above S. Let $S_{cris} \cup S_{ss}$ be a partition of the places of F above p and $\Sigma_{cris} \cup \Sigma_{ss}$ the corresponding partition of places of L above p. If $\rho : G_{F,S} \to \mathrm{GL}_n(E)$ is a continuous Galois representation, then $H^1_{fg}(G_{L,\Sigma}, \mathrm{ad}\,\rho) = 0$ implies that $H^1_{fg}(G_{F,S}, \mathrm{ad}\,\rho) = 0$.*

Proof. There is a commutative square

$$\begin{array}{ccc} H^1(G_{F,S}, \mathrm{ad}\,\rho) & \longrightarrow & \prod_{v \in S_{cris}} H^1(F_v, B_{cris} \otimes_{\mathbb{Q}_p} \mathrm{ad}\,\rho) \times \prod_{v \in S_{ss}} H^1(F_v, B_{dR} \otimes_{\mathbb{Q}_p} \mathrm{ad}\,\rho) \\ \downarrow & & \downarrow \\ H^1(G_{L,\Sigma}, \mathrm{ad}\,\rho) & \longrightarrow & \prod_{v' \in \Sigma_{cris}} H^1(L_{v'}, B_{cris} \otimes_{\mathbb{Q}_p} \mathrm{ad}\,\rho) \times \prod_{v' \in \Sigma_{ss}} H^1(L_{v'}, B_{dR} \otimes_{\mathbb{Q}_p} \mathrm{ad}\,\rho) \end{array}$$

with injective vertical arrows by [Ser02, Chapter I, Proposition 9]. The right vertical arrow is defined as

$$(\sigma_v)_{v|p} \mapsto (\sigma_w|_{G_{L_{v'}}})_{v'|p},$$

where w denotes the unique place of F lying below v'. The other maps in the diagram are restriction maps. Taking kernels of the horizontal arrows, we find an injective map

$$H^1_{fg}(G_{F,S}, \mathrm{ad}\,\rho) \hookrightarrow H^1_{fg}(G_{L,\Sigma}, \mathrm{ad}\,\rho)$$

and the claim follows. □

This lemma allows us to liberally use automorphic base change [AC89] as a reduction step in the proof of the main theorem. Paired with class field theory [AT09] it becomes a very powerful method. Here we state a few such results which we will use below. Recall that an extension L/F is called solvable if there exists a chain of subfields $F = L_0 \subset L_1 \subset \ldots L_n = L$ such that L_i/L_{i-1} is cyclic for each i.

Proposition 5.2.2. *Let F be a CM number field, L/F a solvable extension and π be a cohomological cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. If the Galois representation $r_\iota(\pi)|_{G_L}$ is absolutely irreducible, then there exists a unique cohomological cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_L)$ such that*

$$r_\iota(\Pi) \cong r_\iota(\pi)|_{G_L}.$$

Proof. By induction it suffices to treat the case when L/F is cyclic of prime degree l. By class field theory there is a finite order Hecke character $\eta : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ which cuts out L, i.e. satisfies $\ker(\eta \circ \mathrm{Art}_F^{-1}) = G_L$. If we can show that $\pi \otimes (\eta \circ \det) \not\cong \pi$, then [AC89, Chapter 3, Theorem 4.2 (a)] will imply the proposition by the Chebotarev density theorem and the explicit description of Hecke eigenvalues in unramified local base change [AC89, Chapter 1, §4.2].

In particular, it also suffices to show that $r_\iota(\pi \otimes (\eta \circ \det)) \ncong r_\iota(\pi)$. By the Chebotarev density theorem we find that

$$r_\iota(\pi \otimes (\eta \circ \det)) \cong r_\iota(\pi) \otimes (\eta^{-1} \circ \mathrm{Art}_F^{-1})$$

But if $r_\iota(\pi) \otimes (\eta^{-1} \circ \mathrm{Art}_F^{-1}) \cong r_\iota(\pi)$, then the intertwining operator realising such an isomorphism is a non-scalar G_L-equivariant endomorphism of $r_\iota(\pi)|_{G_L}$. Otherwise, there would exist $\alpha \in \overline{\mathbb{Q}_p}^\times$ and $g \in G_F$ such that $\eta(\mathrm{Art}_F^{-1}(g)) \neq 1$ and $\alpha r(g)v = \eta(\mathrm{Art}_F^{-1}(g))r(g)\alpha v$ for all vectors v of $r_\iota(\pi)$. By Schur's lemma this contradicts the assumption that $r_\iota(\pi)|_{G_L}$ is absolutely irreducible. □

Proposition 5.2.3. *Let $F/\mathbb{Q}_\ell$ be a finite extension and π a irreducible smooth representation of $\mathrm{GL}_n(F)$. Then there exists a finite solvable extension F'/F such that the base change Π of π to $\mathrm{GL}_n(F')$ exists and has Iwahori fixed vectors.*

Proof. We use the functorial properties of the local Langlands correspondence [HT01]. The precise statement we need is conveniently stated in [Sch13]. Let $\pi \mapsto \sigma(\pi)$ be the normalized Weil group representation (without the monodromy) attached to a smooth representation π of $\mathrm{GL}_n(F)$. By construction of the topology on the Weil group there exists a finite Galois extension F'/F such that $\sigma(\pi)|_{W_{F'}}$ is unramified. Since G_F is pro-solvable, any such F'/F is solvable. By [BZ76, Proposition 3.19], π can be embedded into a normalized parabolic induction $i_P^{\mathrm{GL}_n(F)}(\pi')$ for some supercuspidal representation π' of M, where $P = MN$ is a standard parabolic subgroup of $\mathrm{GL}_n(F)$. The Weil representation $\sigma(\pi)$ is determined by π' [Sch13, Theorem 12.1 (ii)] and we can inductively apply [Sch13, Theorem 12.1 (iv)] to π', to obtain the base change Π of π satisfying

$$\sigma(\Pi) = \sigma(\pi)|_{W_{F'}}.$$

Now [Sch13, Theorem 12.1 (v)] shows that Π has Iwahori fixed vectors. □

Proposition 5.2.4. *Let F be a number field, p a prime such that $\zeta_p \notin F$, S a finite set of places of F and $\phi : G_F \to H$ a continuous homomorphism to a finite group H. Suppose that for each $v \in S$ we are given a finite extension F'_v/F_v, then there exists a finite solvable extension L/F such that*

- *$\zeta_p \notin L$;*
- *$\phi(G_{L(\zeta_p)}) = \phi(G_{F(\zeta_p)})$;*

- *$\phi(G_L) = \phi(G_F)$;*
- *For each $v \in S$ and place $w \mid v$ of L we have $L_w \cong F'_v$.*

Proof. By induction it suffices to prove the claim when all the F'_v/F_v are cyclic. So let $\chi_v : F_v^\times \to \mathbb{C}^\times$ be continuous local characters cutting out the F'_v, i.e. such that $\ker(\chi_v \circ \mathrm{Art}_{F_v}^{-1}) = G_{F_{v'}}$. Moreover, by the Chebotarev density theorem, there exists a finite set of places T' of F such that $\{\phi(\mathrm{Frob}_v) : v \text{ lies above } T'\}$ generates $\phi(G_{F(\zeta_p)})$. Moreover, there exists a place v_0 of F such that Frob_{v_0} generates $\mathrm{Gal}(F(\zeta_p)/F)$. Let $T = T' \cup \{v_0\}$ and $\chi_v = 1$ for $v \in T$. By [AT09, Chapter X, Theorem 5], there exists a global character $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ with local components equal to χ_v at all $v \in S \cup T$. Now the cyclic Galois extension L/F cut out by χ satisfies

- $\zeta_p \notin L$ since $\zeta_p \notin F_{v_0}$;
- Every place $v \in T$ is totally split in L;
- For each $v \in S$ and place $w \mid v$ of L we have $L_w \cong F'_v$.

In particular the second property implies that for $v \in T$, the conjugacy class $\mathrm{Frob}_v \subset G_F$ is equal to the union of the conjugacy classes $\mathrm{Frob}_w \subset G_L$, for $w \mid v$. Now by construction of T' we find that $\phi(G_{L(\zeta_p)}) = \phi(G_{F(\zeta_p)})$. Since $v_0 \in T$ is totally split and Frob_{v_0} generates $\mathrm{Gal}(F(\zeta_p)/F)$ we also find that $\phi(G_L) = \phi(G_F)$. □

5.3 Main Theorem

We collect some final preliminary results on the existence of Taylor–Wiles primes and the existence of Hecke algebra valued Galois representations before finally proving the main theorem.

Definition 5.3.1. A subgroup $H \subset \mathrm{GL}_n(k)$ is called *enormous* over k (the fixed finite residue field) if it satisfies the following

1. The representation of H acting on k^n, given by the inclusion $H \hookrightarrow \mathrm{GL}_n(k)$ is absolutely irreducible.
2. H has no non-trivial quotients of p-power order.
3. $H^0(H, \mathrm{ad}^0) = H^1(H, \mathrm{ad}^0) = 0$, where ad^0 is the vector space of traceless $n \times n$ matrices over k on which $\mathrm{GL}_n(k)$ acts by conjugation.

4. For any simple $k[H]$-submodule $M \subset \mathrm{ad}^0$, there is a regular semisimple $h \in H$ such that $M^h \neq 0$.

Proposition 5.3.2. *Let $\mathcal{S} = (\overline{\rho}, S, \{\mathcal{D}_v\}_{v \in S})$ be a global deformation datum. Assume that*

- *$p \nmid 2n$,*
- *$\overline{\rho}$ is absolutely irreducible,*
- *$F = F^+ F_0$ with F^+ totally real and F_0 an imaginary quadratic field,*
- *$\zeta_p \notin F$ and $\overline{\rho}(G_{F(\zeta_p)})$ is enormous.*

Then for a large enough integer q and any $N \geq 1$, there exists a Taylor–Wiles datum $(Q_N, (\alpha_{v,1}, \ldots, \alpha_{v,n})_{v \in Q_N})$ satisfying

1. *$\# Q_N = q$.*
2. *For each $v \in Q_N$, $q_v \equiv 1 \pmod{p^N}$ and the rational prime below v splits in F_0.*
3. *There is a local $\mathcal{O}$-algebra surjection $R_{\mathcal{S}}^{S,loc}[[X_1, \ldots, X_g]] \to R^S_{\mathcal{S}_{Q_N}}$, with*

$$g = qn - n^2[F^+ : \mathbb{Q}]$$

Proof. This is [ACC+23, Proposition 6.2.32]. □

Proposition 5.3.3. *Assume that $p > n$. Let $\mathfrak{m} < \mathbb{T}^S$ be a non-Eisenstein ideal and λ a dominant weight of GL_n / F. Suppose K is a good open compact subgroup $K < \mathrm{GL}_n(\mathbb{A}_F^\infty)$ such that $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ for all $v \notin S$ for some finite set of places S containing those above p. Let $(Q, (\alpha_{v,1}, \ldots, \alpha_{v,n})_{v \in Q})$ be a Taylor–Wiles datum for*

$$(\overline{\rho}_{\mathfrak{m}}, S, \{\mathcal{D}_v^{cris,r}\}_{v \in S_{cris}} \cup \{\mathcal{D}_v^{ss,r}\}_{v \in S_{ss}} \cup \{\mathcal{D}_v^{\square}\}_{v \in S \setminus S_p}),$$

where $S_{cris} \cup S_{ss}$ is a partition of the places of F above p. We define level subgroups $K_1(Q) = \prod_v K_1(Q)_v \subset K_0(Q) = \prod_v K_0(Q)_v \subset K$ by $K_1(Q) = K_0(Q)_v = K_v$ for $v \notin Q$ and for $v \in Q$ we set $K_0(Q) = \mathrm{Iw}_v$ and $K_1(Q)_v$ the kernel of the maximal abelian p-quotient of Iw_v for $v \in Q$. We have $K_0(Q)/K_1(Q) \cong \Delta_Q^n$.

Consider the commutative $\mathcal{O}$-subalgebra

$$\mathbb{T}_Q^{S \cup Q} := \mathbb{T}^{S \cup Q}[U_{v,1}, \ldots, U_{v,n}, v \in Q] \subset \mathcal{H}(\mathrm{GL}_n(\mathbb{A}_F^S), K_0(Q)) \otimes_{\mathbb{Z}} \mathcal{O},$$

where $U_{v,i}$ is the characteristic function of the double coset

$$\mathrm{Iw}_v \operatorname{diag}(\varpi_v, \dots, \varpi_v, 1, \dots, 1) \mathrm{Iw}_v,$$

where ϖ_v appears i times on the diagonal. Conjugation by elements of $K_0(Q)/K_1(Q)$ induces an embedding

$$\mathbb{T}_Q^{S\cup Q}[\Delta_Q^n] \to \mathcal{H}(\mathrm{GL}_n(\mathbb{A}_F^S), K_1(Q)) \otimes_{\mathbb{Z}} \mathcal{O}$$

and given an object M of $\mathbf{D}(\mathcal{O}[\Delta_Q^n])$ with Hecke action we write $\mathbb{T}_\Delta^{S\cup Q}(M)$ for the image of $\mathbb{T}_Q^{S\cup Q}[\Delta_Q^n]$ in $\mathrm{End}_{\mathbf{D}(\mathcal{O}[\Delta_Q^n])}(M)$. With these notations, there are maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_0$ and local $\mathcal{O}$-algebra homomorphisms ϕ, ψ

$$\mathbb{T}_\Delta^{S\cup Q}(R\Gamma(X_{\mathrm{GL}_n/F}^{K_1(Q)}, \mathcal{V}_\lambda))_{\mathfrak{m}_1} \xrightarrow{\phi} \mathbb{T}_Q^{S\cup Q}(R\Gamma(X_{\mathrm{GL}_n/F}^{K_0(Q)}, \mathcal{V}_\lambda))_{\mathfrak{m}_0} \xrightarrow{\psi} \mathbb{T}^S(R\Gamma(X_{\mathrm{GL}_n/F}^{K}, \mathcal{V}_\lambda))_{\mathfrak{m}}$$

such that ψ is an isomorphism and ϕ is surjective, containing the augmentation ideal of $\mathcal{O}[\Delta_Q^n]$ in its kernel. Moreover, there are corresponding maps

$$R\Gamma(X_{\mathrm{GL}_n/F}^{K_1(Q)}, \mathcal{V}_\lambda)_{\mathfrak{m}_1} \xrightarrow{f} R\Gamma(X_{\mathrm{GL}_n/F}^{K_0(Q)}, \mathcal{V}_\lambda)_{\mathfrak{m}_0} \xrightarrow{g} R\Gamma(X_{\mathrm{GL}_n/F}^{K}, \mathcal{V}_\lambda)_{\mathfrak{m}},$$

where g is a quasi-isomorphism and f induces a quasi-isomorphism

$$R\Gamma(\Delta_Q^n, R\Gamma(X_{\mathrm{GL}_n/F}^{K_1(Q)}, \mathcal{V}_\lambda)_{\mathfrak{m}_1}) \to R\Gamma(X_{\mathrm{GL}_n/F}^{K_0(Q)}, \mathcal{V}_\lambda)_{\mathfrak{m}_0}.$$

Proof. This works in exactly the same way as [ACC+23, Lemma 6.5.9]. There, our $\mathfrak{m}_1$ is called $\mathfrak{n}_1^Q$ and $\mathfrak{m}_0$ is called $\mathfrak{n}_0^Q$. □

Theorem 5.3.4. *Let $\mathfrak{m} < \mathbb{T}^S$ be a non-Eisenstein ideal and λ a dominant weight of GL_n/F. Suppose K is a good open compact subgroup $K < \mathrm{GL}_n(\mathbb{A}_F^\infty)$ such that $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ for all $v \notin S \cup R$ for some finite set of places $S \cup R$ satisfying (Σ_p). Moreover, let $S_p = S_{cris} \cup S_{ss} \subset S$ be a partition, stable under complex conjugation such that $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ for $v \in S_{cris}$ and $K_v = Iw_v$ for $v \in S_{ss}$ and $K_v = Iw_{v,1}$ for $v \in R$. Consider the global deformation datum*

$$\mathcal{S} = (\overline{\rho}_{\mathfrak{m}}, S \cup R, \{\mathcal{D}_v^{cris,r}\}_{v\in S_{cris}} \cup \{\mathcal{D}_v^{ss,r}\}_{v\in S_{ss}} \cup \{\mathcal{D}_v^{\square}\}_{v\in R\cup S\setminus S_p})$$

and let $(Q, (\alpha_{v,1}, \dots, \alpha_{v,n})_{v\in Q})$ be a Taylor–Wiles datum. Suppose that the following are satisfied

(1) For each embedding $\tau : F \hookrightarrow E$ inducing a place above p, we have $-\lambda_{\tau c,1} - \lambda_{\tau,1} \geq 0$;

(2) For each place $v \mid p$ of F let $\overline{v}$ be the place of F^+ lying below v. Then there exists a place $\overline{v}' \neq \overline{v}$ of F^+ such that $\overline{v}' \mid p$ and

$$\sum_{\overline{v}'' \neq \overline{v}, \overline{v}'} [F^+_{\overline{v}''} : \mathbb{Q}_p] > \frac{1}{2}[F^+ : \mathbb{Q}];$$

(3) The residual representation $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible and decomposed generic [ACC+23, Definition 4.3.1],

then there is an integer $\delta \geq 1$ depending only on $[F : \mathbb{Q}]$ and n, a nilpotent ideal $I \subset \mathbb{T}^{S \cup R \cup Q}_{\Delta}(R\Gamma(X^{K_1(Q)}_{\mathrm{GL}_n/F}, \mathcal{V}_\lambda))_{\mathfrak{m}_1}$ such that $I^\delta = 0$ and a surjective local $\mathcal{O}[\Delta_Q]$-algebra homomorphism

$$x : R_{\mathcal{S}_Q} \to \mathbb{T}^{S \cup R \cup Q}_{\Delta}(R\Gamma(X^{K_1(Q)}_{\mathrm{GL}_n/F}, \mathcal{V}_\lambda))_{\mathfrak{m}_1}/I$$

such that

$$\det(z \cdot \mathrm{id} - \rho_x(\mathrm{Frob}_v)) = P_v(z) \in (\mathbb{T}^{S \cup R \cup Q}_{\Delta}(R\Gamma(X^{K_1(Q)}_{\mathrm{GL}_n/F}, \mathcal{V}_\lambda))_{\mathfrak{m}_1}/I)[z]$$

for $v \notin S \cup R \cup Q$.

Proof. When $Q = \emptyset$, this follows from theorem 4.3.1. In general the Δ_Q-equivariance follows from of [ACC+23, Theorem 3.1.1]. □

Theorem 5.3.5. *Let $F \subset L$ be CM fields and let $\rho : G_F \to \mathrm{GL}_n(E)$ be a continuous representation, where $E \subset \overline{\mathbb{Q}}_p$ is a finite extension of $\mathbb{Q}_p$ containing the images of all field homomorphisms $L \to \overline{\mathbb{Q}}_p$. Moreover, let Π be a cohomological cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_L)$ and let S be a finite set of places of F, stable under complex conjugation, containing the archimedean ones and those above p such that Π_v is unramified for all v not lying above a place of S. Let $S_p = S_{cris} \cup S_{ss}$ be a partition of the places of F lying above p which is stable under complex conjugation such that the following are satisfied.*

(a) $p > n$;

(b) There exists an isomorphism $\iota : \overline{\mathbb{Q}}_p \to \mathbb{C}$ such that $\rho|_{G_L} \otimes_E \overline{\mathbb{Q}}_p \cong r_\iota(\Pi)$;

(c) The residual representation $\overline{\rho}|_{G_L}$ is absolutely irreducible and decomposed generic [ACC+23, Definition 4.3.1]. Moreover, $\zeta_p \notin L$ and $\overline{\rho}|_{G_{L(\zeta_p)}}$ has enormous image [ACC+23, Definition 6.2.28], where $\zeta_p \in \overline{L}$ is a primitive pth root of unity;

(d) If v lies above a place in $S \setminus S_p$, then the Weil–Deligne representation $\mathrm{WD}(\rho|_{G_{L_v}})$ is generic [All16, Definition 1.1.2];

(e) If v lies above a place in S_{cris}, then Π_v is unramified. If v lies above a place in S_{ss}, then $\rho|_{G_{L_v}}$ is de Rham and for any finite extension L'_v/L_v, $\mathrm{WD}(\rho|_{G_{L'_v}})$ is generic.

Then the Bloch–Kato Selmer group $H^1_{fg}(G_{F,S}, \mathrm{ad}\,\rho)$ vanishes, i.e. ρ is rigid. Moreover, $\rho|_{G_{L_v}}$ is crystalline for all v lying above a place in S_{cris}.

Proof. We immediately note that we can assume $L = F$ since it suffices to prove the Selmer vanishing over L by lemma 5.2.1. We now apply proposition 5.2.2 multiple times and enlarge F further, while keeping the places $v \in S \setminus S_{ss}$ totally split over the field L that we started with.

By proposition 5.2.3 we can find solvable local extensions F'_v/F_v for all $v \in S_{ss}$ such that the local base change of Π_v to $\mathrm{GL}_n(F'_v)$ has Iwahori fixed vectors. Using proposition 5.2.4 there is a solvable global extension which interpolates these, is totally split at all $v \in S \setminus S_{ss}$ and preserves condition *(c)*. Hence by proposition 5.2.2 and condition *(e)* we may assume that

(1) If $v \in S_{ss}$, then $\mathrm{WD}(\rho|_{G_{F_v}})$ is generic and $\Pi_v^{Iw_v} \neq 0$.

by choosing more local extensions at arbitrary places outside of S we can moreover assume that S satisfies (Σ_p) and

(2) For each place $v \mid p$ of F let $\overline{v}$ be the place of F^+ lying below v. Then there exists a place $\overline{v}' \neq \overline{v}$ of F^+ such that $\overline{v}' \mid p$ and

$$\sum_{\overline{v}'' \neq \overline{v}, \overline{v}'} [F^+_{\overline{v}''} : \mathbb{Q}_p] > \frac{1}{2}[F^+ : \mathbb{Q}].$$

Moreover, we can twist Π by a suitable power of $|\det|$ to arrange for the following condition to be satisfied.

(3) If c is the complex conjugation of F, then

$$-\lambda_{\tau c,1} - \lambda_{\tau,n} \geq 0$$

for all $\tau \in \mathrm{Hom}(F, \mathbb{C})$.

Then $r_\iota(\Pi)$ is twisted by a power of the cyclotomic character by which $\mathrm{ad}\, r_\iota(\Pi)$ is unaffected and by possibly enlarging the power of the character we can make sure that $\overline{r_\iota(\Pi)}$ remains the same. Hence all the above conditions are still satisfied.

We apply proposition 5.3.2 and let R be a large enough set of Taylor–Wiles primes for the deformation datum

$$\mathcal{S} = (\overline{\rho}_{\mathfrak{m}}, S, \{\mathcal{D}_v^{ss,r}\}_{v \in S_{ss}} \cup \{\mathcal{D}_v^{cris,r}\}_{v \in S_{cris}} \cup \{\mathcal{D}_v^{\square}\}_{v \in S \setminus S_p}).$$

In particular, for each $v \in R$, $\mathrm{WD}(\rho|_{F_v})$ is generic. We define the level subgroup $K = \prod_v K_v$ of $\mathrm{GL}_n(\mathbb{A}_F^\infty)$ by

- $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ for $v \notin R \cup S \setminus S_{cris}$,
- $K_v = Iw_v$ for $v \in S_{ss}$,
- K_v is any small enough open compact subgroup so that $\Pi_v^{K_v} \neq 0$ for $v \in S \setminus S_p$,
- $K_v = Iw_{v,1}$ is the group subgroup of matrices which are unipotent and upper triangular mod ϖ_v for $v \in R$.

Since R is sufficiently large, lemma 3.2.6 shows that K is neat.

It is proven in [ACC+23, Theorem 2.4.10] that there exists a maximal ideal $\mathfrak{m} \subset \mathbb{T}_{\mathcal{O}}^{S \cup R}(R\Gamma(X_{\mathrm{GL}_n/F}^K, \mathcal{V}_\lambda))$ such that $\overline{\rho}_{\mathfrak{m}} \cong \overline{r_\iota(\Pi)}$, where $\mathcal{O}$ is the ring of integers of E. Now theorem 4.3.1 shows that ρ is semistable at S_{ss} and crystalline at S_{cris} with Hodge–Tate weights bounded by some large enough r.

Consider the global deformation problem

$$\mathcal{S} = (\overline{\rho}_{\mathfrak{m}}, S \cup R, \{\mathcal{D}_v^{ss,r}\}_{v \in S_{ss}} \cup \{\mathcal{D}_v^{cris,r}\}_{v \in S_{cris}} \cup \{\mathcal{D}_v^{\square}\}_{v \in R \cup S \setminus S_p})$$

The functor $\mathcal{D}_{\mathcal{S}}$ is represented by a complete local $\mathcal{O}$-algebra $R_{\mathcal{S}}$ by theorem 5.1.3. Let $\mathfrak{p}$ be the kernel of $R_{\mathcal{S}} \to E$ induced by $\rho = r_\iota(\Pi)$. Then theorem 5.1.7 shows that the tangent space of $(R_{\mathcal{S}})_{\mathfrak{p}}$ is isomorphic to $H^1_{fg}(G_{F,S}, \mathrm{ad}\,\rho)$. Set

$$C_0 = R\,\mathrm{Hom}_{\mathcal{O}}(R\Gamma(X_{\mathrm{GL}_n/F}^K, \mathcal{V}_\lambda)_{\mathfrak{m}}, \mathcal{O})$$

and $T_0 = \mathbb{T}_{\mathcal{O}}^S(C_0)$.

By theorem 5.3.4, there is a surjection $R_{\mathcal{S}} \to T_0/I_0$ of local $\mathcal{O}$-algebras, where $I_0 < T_0$ is the nilradical. Theorem 3.4.8 implies that $T_0[1/p]$ is a product of fields and thus the localisation at $\mathfrak{p}$ is a surjection $(R_{\mathcal{S}})_{\mathfrak{p}} \to E'$ for some field E'. If this map is also injective, then the tangent space $H^1_{fg}(G_{F,S}, \mathrm{ad}\,\rho)$ vanishes. Thus, we are left with applying lemma 2.6.4.

Let q be large enough and $g = qn - n^2[F^+ : \mathbb{Q}]$. For each $N \geq 1$, use proposition 5.3.2 to choose a Taylor–Wiles datum $(Q_N, (\alpha_{v,1}, \ldots, \alpha_{v,n})_{v \in Q_N})$ satisfying

(a) $\#Q_N = q$.

(b) For each $v \in Q_N$, $q_v \equiv 1 \pmod{p^N}$ and the rational prime below v splits in some imaginary quadratic subfield of F.

(c) There is a surjection of local $\mathcal{O}$-algebras $R_{\mathcal{S}}^{S\cup R,loc}[[X_1,\dots,X_g]] \to R_{\mathcal{S}_{Q_N}}^{S\cup R}$

Let $\mathcal{T} = \mathcal{O}[[X_1,\dots,X_{n^2|S\cup R|-1}]]$ and $S_\infty = \mathcal{T}[[Y_1,\dots,Y_{nq}]]$ with augmentation ideal $\mathfrak{a} < S_\infty$ given by

$$\mathfrak{a} := (X_1,\dots,X_{n^2|S\cup R|-1}, Y_1,\dots,Y_{nq}).$$

For $N \geq 1$ we define $\mathfrak{a}_N := ((X_i+1)^{p^N}-1, (Y_i+1)^{p^N}-1)$. To apply the main lemma (2.6.4) there are seven things to verify/define so we do the rounds.

(1) Let $R_\infty = R_{\mathcal{S}}^{S\cup R,loc}[[X_1,\dots,X_g]]$. It is a complete local $\mathcal{O}$-algebra since $R_{\mathcal{S}}^{S\cup R,loc}$ is.

(2) We put $R_0 = R_{\mathcal{S}}$ and for $N \geq 1$ we set $R_N = R_{\mathcal{S}_{Q_N}}^{S\cup R}/\mathfrak{a}_N$. By assumption on the Taylor–Wiles datum Q_N, each R_N is a quotient of R_∞. Using lemmas 5.1.11 and 5.1.5 we find that $R_N/\mathfrak{a} = R_0$ for all N.

(3) The complex C_0 is finitely generated since $X_{\mathrm{GL}_n/F}^K$ is homotopy equivalent to its Borel–Serre compactification (see proposition 3.2.4) and the cohomology of a compact manifold is finitely generated. Moreover, $R\Gamma(X_{\mathrm{GL}_n/F}^{K_1(Q)}, \mathcal{V}_\lambda)$ can be represented by a complex of finite free $\mathcal{O}[\Delta_Q^n]$-modules by choosing a Δ_Q^n-stable, finite Cech covering of $\overline{X}_{\mathrm{GL}_n/F}^{K_1(Q)}$. For $N \geq 1$, we let C_N be a minimal representative of

$$R\operatorname{Hom}_{\mathcal{O}}(R\Gamma(X_{\mathrm{GL}_n/F}^{K_1(Q)}, \mathcal{V}_\lambda)_{\mathfrak{m}_1}, \mathcal{T})/\mathfrak{a}_N.$$

Then $C_N/\mathfrak{a} = C_0$ by proposition 5.3.3.

(4) We let $q_0 = [F^+ : \mathbb{Q}]n(n-1)/2$ and $l_0 = n[F^+ : \mathbb{Q}] - 1$, then [ACC+23, Theorem 2.4.10] implies that $C_0[1/p]$ is not exact since it contains $(\Pi^\infty)^K$ in its cohomology. The same theorem also states that $C_0[1/p]$ is concentrated in degrees $[q_0, q_0 + l_0]$. On the other hand we have $\dim(S_\infty)_{\mathfrak{a}} = n^2|S\cup R| + nq - 1$ and lemma 5.1.8 implies

$$\begin{aligned}\dim R_\infty[1/p] &\leq g + n^2|S\cup R| + n(n-1)[F^+ : \mathbb{Q}] \\ &= qn + n^2|S\cup R| - n[F^+ : \mathbb{Q}] \\ &= \dim(S_\infty)_{\mathfrak{a}} - l_0\end{aligned}$$

(5) For $N \geq 1$ we put $T_N = \mathbb{T}_\Delta^{S\cup R\cup Q_N}(C_N)_{\mathfrak{m}_1} \cdot \mathcal{T} \subset \mathrm{End}_{\mathbf{D}(S_N)}(C_N)_{\mathfrak{m}_1}$. Then it is clear that the image of T_N in $\mathrm{End}_{\mathbf{D}(S_0)}(C_0)$ equals

$$\mathbb{T}^{S\cup R\cup Q_N}(C_0) \subset \mathbb{T}^{S\cup R}(C_0) = T_0.$$

(6) By lemma 5.1.5 we can tensor the surjections from theorem 5.3.4 with $\mathcal{T}$ to obtain surjections $R_N \to T_N/I_N$, where I_N are ideals satisfying I_N^δ for some constant δ. The commutativity of the square follows from the Chebotarev density theorem and the computation of characteristic polynomials of Frobenius elements in terms of Hecke polynomials in the theorem 5.3.4.

(7) We already defined $\mathfrak{p}$ as the point corresponding to ρ. It is clear that this becomes a maximal ideal of $R_0[1/p]$. Since it corresponds to $\rho = r_\iota(\Pi)$ it is the pullback of the maximal ideal of $T_0[1/p]$ corresponding to the summand $(\Pi^\infty)^K$. Now $\widehat{(R_\infty)_\mathfrak{p}} = (\widehat{R_\mathcal{S}^{S\cup R,loc}})_\mathfrak{p}[[X_1,\ldots,X_g]]$ and $(\widehat{R_\mathcal{S}^{S\cup R,loc}})_\mathfrak{p}$ is regular by proposition 5.1.9. Hence $(R_\infty)_\mathfrak{p}$ is regular, too. □

Corollary 5.3.6. *Let A/F be an elliptic curve without complex multiplication over a CM field F and $p \geq 7$ a prime such that $\zeta_p \notin F$ and the image of G_F in $\mathrm{Aut}(A[p])$ contains $\mathrm{SL}_2(\mathbb{F}_p)$. Let*

$$V_pA = \left(\varprojlim A[p^n](\overline{F})\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

be the p-adic G_F-representation associated with A. Then V_pA is rigid, or equivalently any short exact sequence

$$0 \to V_pA \to V \to V_pA \to 0,$$

where V is a de Rham G_F-representation, splits. Furthermore, if $F/\mathbb{Q}$ is Galois and n is a positive integer, such that $p > 2n+3$, then the symmetric power $\mathrm{Sym}^n(V_pA)$ is rigid, too.

Proof. Let $\rho = \mathrm{Sym}^n(V_pA)$. Since the determinant of V_pA is the cyclotomic character, we find that in fact the image of $G_{F(\zeta_p)}$ in $\mathrm{Aut}(A[p])$ equals $\mathrm{SL}_2(\mathbb{F}_p)$. Hence [ACC+23, Lemma 7.1.4] implies that $\overline{\rho}(G_{F'(\zeta_p)})$ is enormous for all F'/F which are linearly disjoint from $\overline{F}^{\ker\overline{\rho}}$. Moreover, since the symmetric powers of the standard representation $\mathrm{Sym}^n(\overline{\mathbb{F}}_p^2)$ of $\mathrm{SL}_2(\mathbb{F}_p)$ are irreducible for $n < 2n+3 < p$, we find that $\overline{\rho}|_{G_{F'}}$ is absolutely irreducible for all F'/F which are linearly disjoint from $\overline{F}^{\ker\overline{\rho}}$.

If $n = 1$, then one can use the Chebotarev density theorem to show that $\overline{\rho}|_{G'_F}$ is decomposed generic for any F'/F disjoint from $\overline{F}^{\ker\overline{\rho}}$. If $n > 1$ and $F/\mathbb{Q}$ is Galois, then [ACC$^+$23, Lemma 7.1.6] implies that $\overline{\rho}|_{G_{F'}}$ is decomposed generic for any F'/F disjoint from $\overline{F}^{\ker \overline{\rho}}$.

Now [ACC$^+$23, Theorem 7.1.11] shows that there exists an F'/F, linearly disjoint from $\overline{F}^{\ker \overline{\rho}}$ such that $\rho|_{G_{F'}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \cong r_\iota(\Pi)$ for some cohomological cuspidal automorphic representation Π of $\mathrm{GL}_{n+1}(\mathbb{A}_{F'})$. Finally, to apply our main theorem it is enough to show that the Weil–Deligne representation $(r, N) = \mathrm{WD}(\rho|_{G_L})$ is generic for all finite extensions L/F'_v. If A has potentially good reduction at v, then by the Hasse bound, r and $r(1)$ share no common Frobenius eigenvalues, hence $\mathrm{Hom}((r, N), (r(1), N)) = 0$. If A has potentially multiplicative reduction at v, then N has maximal rank and any non-zero element of $\mathrm{Hom}((r, N), (r(1), N))$ would preserve the one-dimensional space $\ker N$. But $r(\mathrm{Frob}_v)$ acts on $\ker N$ by a root of unity and $r(1)(\mathrm{Frob}_v)$ acts on $\ker N$ by an algebraic number of complex absolute value q_v, implying that $\mathrm{Hom}((r, N), (r(1), N)) = 0$. □

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