The bounded $\mathcal{H}^{\infty}$-calculus for sectorial, strip-type and half-plane operators

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Musa
Isa
Yusra

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Abstract

The main study of this thesis is the holomorphic functional calculi for three classes of unbounded operators: sectorial, strip-type and half-plane. The functional calculus for sectorial operators was introduced by McIntosh as an extension of the Riesz-Dunford model for bounded operators. More recently Haase has developed an abstract framework which incorporates analogous constructions for strip-type and half-plane operators. These operators are of interest since they arise naturally as generators of C_0-(semi)groups.

The theory of bounded $\mathcal{H}^\infty$-calculus for sectorial operators is well established and it has been found to have many applications in operator theory and parabolic evolution equations. We survey these known results, first on Hilbert space and then on general Banach space.

Our main goal is to fill the gaps in the parallel theory for strip-type operators. Whilst some of this can be deduced by taking exponentials and applying known results for sectorial operators, in general this is insufficient to obtain our desired results and so we pursue an independent approach. Starting on Hilbert space, we broaden known characterisations of the bounded $\mathcal{H}^\infty$-calculus for strip-type operators by introducing a notion of absolute calculus which is an analogue to the established notion for the sectorial case. Moving to general Banach space, we build on the work of Vörös, broadening his characterisation for strip-type operators in terms of weak integral estimates by introducing a new, but equivalent, notion of the bounded $\mathcal{H}^\infty$-calculus, which we call the m-bounded calculus.

We also demonstrate that these characterisations fail for half-plane operators and instead present a weaker form of the bounded $\mathcal{H}^\infty$-calculus which is more natural for these operators. This allows us to obtain new and simple proofs of well known generation theorems due to Gomilko and Shi-Feng, with extensions to polynomially bounded semigroups. The connection between the bounded $\mathcal{H}^\infty$-calculus of semigroup generators and polynomial boundedness of their associated Cayley Transforms is also explored.

Finally we present a series of results on sums of operators, in connection with maximal regularity. We also establish stability results for the bounded $\mathcal{H}^\infty$-calculus for strip-type operators by showing it is preserved under suitable bounded perturbations, which at time requires further assumptions on the underlying Banach space. This relies heavily on intermediate characterisations of the bounded $\mathcal{H}^\infty$-calculus due to Kalton and Weis.

Contents

1 Introduction

1.1 Background

The main topic of this thesis is the functional calculus for suitable classes of operators acting on Banach space. The idea behind the functional calculus is to assign to an operator A and scalar function f the operator $f(A)$. There is no obvious way to do this in complete generality, which leads one to impose certain assumptions on the operator A, scalar function f and Banach space X. The basic objective has been subject to study for a long time. For example, the Fourier transform presented an early example of a method by which one can reasonably define $f(A)$, where $A = \Delta$ is the Laplacian on $X = L^2(\mathbb{R})$ and f is any measurable function on $\mathbb{R}$. By a reasonable definition, we mean that the properties of $f(A)$ correlate in a natural way to the original function f. The Spectral Theorem allows us to generalise this to self-adjoint (or even normal) operators on Hilbert space. Moreover, the Hilbert-Schmidt Theorem allows one to assign an evaluation $f(A)$ to any compact, normal operator A and complex function f.

Leaving Hilbert space requires a different approach. With Cauchy's Integral Formulae guiding our intuition, it is possible to define $f(A)$ for any bounded operator acting on general Banach space, for functions f holomorphic in some neighbourhood of the spectrum. In other words, we can define

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda) R(\lambda, A) \, \mathrm{d}\lambda,$$

where the contour Γ lies inside a suitable region Ω containing the spectrum on which f is holomorphic. This idea goes back to the work of Riesz and Dunford [24] and can even be extended to functions which have singularities at some points of the boundary on the spectrum.

However, our motivation is largely derived from partial differential equations, so that the operators in question are usually unbounded. Of primary interest are the exponential functions, since they arise as solutions of the simplest partial differential equations. The evaluation of exponential functions has been studied in many frameworks, since as early as the 1930s [68]. In 1948, Hille and Yosida (see for example [39] [78]) characterised those operators which have a meaningful exponential evaluation. The Hille-Yosida theorem is somewhat restrictive in that it requires norm estimates on all powers of resolvents of the operator, which are difficult to obtain in practice. Gomilko [30] and Shi-Feng [66] independently obtained a sufficient (and necessary on Hilbert space) condition involving an integral estimate on a single power of the resolvent.

The exponential evaluations $\exp(tA)$, $t \geq 0$, for a fixed operator A, form a semigroup of bounded operators. Operator semigroups have undergone much study in their own right. Heuristically, starting with a semigroup of operators $(T(t))$, one ought to be able to extract an operator for which the bounded semigroup operators correspond to evaluations of exponential functions. One needs to impose conditions such as strong continuity on the semigroup to ensure that this so-called *generator* is well-defined. Generators of strongly continuous semigroups have nice spectral properties. In particular, they belong to the much studied class of *sectorial* operators. The Riesz-Dunford calculus described above has been generalised to a holomorphic functional calculus for unbounded sectorial operators by McIntosh [53]. Given the generator of a strongly continuous semigroup, one can then retrieve the semigroup from this functional calculus as the evaluation of exponential functions.

The functional calculus for sectorial operators also allows us to extract various other associated families of operators, such as fractional and imaginary powers. In particular, the logarithm function has received much coverage in the literature. Nollau [56] was the first to define the logarithm of a sectorial operator. Haase [34] has since successfully incorporated a consistent definition into the wider framework of functional calculus for sectorial operators. With either approach, one can quite easily show that the logarithm of a sectorial operator is a *strip-type* operator, in the sense that its spectrum lies in some (horizontal) strip, with appropriate resolvent estimates outside. Strip-type operators also arise naturally as generators of (strongly continuous) groups on the whole of $\mathbb{R}$. Analogously to the sectorial case, in the 1950s Bade [7] introduced a holomorphic functional calculus for strip-type operators, which has more recently been developed by Haase [34]. These operators are actually special types of operators belonging to the larger class of half-plane operators, where the spectrum is now assumed to lie in a half-plane rather than a strip, with the same resolvent estimates on one side. Every (strongly continuous) semigroup generator is a half-plane operator and Haase [35] has similarly constructed a functional calculus for these operators. The calculus is consistent with the one for sectorial operators, and many similarities arise due to the fact that every half-plane operator is, up to shifting, sectorial of angle $\frac{\pi}{2}$.

All of these constructions can be unified in an abstract framework which is laid out in [34, Chapter 1]. However, functional calculus is not merely a self-contained theory. Haase has reformulated ideas and theorems from operator theory into a functional calculus setting. In many cases he has simplified and even obtained new proofs of classical theorems, such as those of Hille-Yosida, Trotter-Kato, Dore-Venni, Monniaux, and Fattorini's Theorem for cosine functions. Fundamental to much of this is an approximation technique known as the Convergence Lemma, which holds under suitable assumptions for each of the classes of operators described above.

With the functional calculus in place, it is of great interest to consider a special class of operators which enjoy so-called *bounded $\mathcal{H}^\infty$-calculus* on appropriate regions. Namely, any bounded holomorphic function f on a neighbourhood of the spectrum $\sigma(A)$ evaluates as a bounded operator $f(A)$. This notion was first considered for sectorial operators by McIntosh [53] and has natural extensions to the strip and half-plane cases. Establishing boundedness of the calculus directly is difficult for concrete operators and so it is useful to have characterisations which only demand an estimate on a more narrow class of functions. On Hilbert space, McIntosh [53] achieved such a characterisation by considering *square function* estimates of the form

$$\int_0^\infty \|\psi(tA)x\|^2 \, \mathrm{d}t + \int_0^\infty \|\psi(tA^*)x^*\|^2 \, \mathrm{d}t < \infty \quad \forall x \in X \; \forall x^* \in X^* \; ,$$

for scalar functions ψ belonging to a special subclass of $\mathcal{H}^\infty$. We say A admits an *absolute* (*functional*) *calculus* if it satisfies this condition. By counterexamples due to Kalton [41], it is known that absolute calculus is stronger than boundedness of the $\mathcal{H}^\infty$-calculus in general Banach space. Thus we look towards a weaker form of these square function estimates, known as *Rademacher square function* estimates which take the form, for suitable $\mathcal{H}^\infty$-functions ψ,

$$\sup_{t>0} \Big\|\sum_{k\in\mathbb{Z}} r_k \psi(t2^k A)x\Big\|_{L^2(X)} + \sup_{t>0} \Big\|\sum_{k\in\mathbb{Z}} r_k \psi(t2^k A)^* x^*\Big\|_{L^2(X^*)} < \infty \quad \forall x \in X \; \forall x^* \in X^* \; .$$

A characterisation involving such estimates can then be obtained by passing through *weak square function estimates*:

$$\int_0^\infty |\langle \psi(tA)x, x^*\rangle| \, \frac{\mathrm{d}t}{t} < \infty \quad \forall x \in X \; \forall x^* \in X^* \; .$$

The argument for weak integral estimates was first presented in [16], and places a growth condition on the Fourier transform of the underlying scalar function ψ. Rademacher estimates were incorporated in [47, Chapter 12], although they were first shown to characterise the bounded $\mathcal{H}^\infty$-calculus in [44]. More recently, Weis [77] derived a special case of this characterisation through different means, which is most useful in applications. By modifying the harmonic analysis of the boundary of a sector to a strip, Vörös [76] was able to adapt the characterisation of [16] to strip-type operators. In particular, his weak integral estimates involved differences of first order resolvent operators, but not single resolvent powers which are easier to verify. Furthermore, the natural Rademacher estimates that arise for strip-type operators do not obviously characterise the boundedness of the $\mathcal{H}^\infty$-calculus on general Banach space. Assuming more geometry on the Banach space, namely that it has finite cotype, an intermediate characterisation due to an unpublished work of Kalton and Weis [45] is possible on strips and sectors.

Many of these arguments break down with half-plane operators, as the boundary loses the scope for necessary cancellation in these estimates. It is known that on Hilbert space, strip-type operators with a bounded $\mathcal{H}^\infty$-calculus are precisely generators of strongly continuous groups. However, the analogue for half-plane operators does not hold. Namely, there are semigroup generators acting on Hilbert space which do not admit a bounded $\mathcal{H}^\infty$-calculus on any half-plane. Thus we seek a characterisation of semigroup generators involving some weaker form of the bounded $\mathcal{H}^\infty$-calculus. This gives rise to the notion of an m-bounded calculus, in which only m^{th} derivatives of $\mathcal{H}^\infty$-functions are assumed to have bounded evaluation. As a by-product, we can establish simple proofs of the well known generation results of Gomilko and Shi-Feng, as well as more general results for polynomially bounded semigroups. On strips, m-boundedness is actually equivalent to the usual bounded calculus and an argument which builds on that of Vörös allows us to widen the class of integral estimates that characterise the bounded calculus of strip-type operators. This will also enable us to incorporate Rademacher estimates in our characterisation for strip-type operators. Furthermore, by exploiting the exp-log relationship between sectorial and strip-type operators, we can then broaden the weak integral estimates that arise in the characterisation of [16] for the bounded $\mathcal{H}^\infty$-calculus of sectorial operators.

On half-planes, we have another tool by which to explore properties of the functional calculus on Hilbert space. Working with the right half-plane $\mathbb{C}_+$, we can use the Cayley Transform

$$\delta(z) = \frac{z-1}{z+1}$$

to associate with a suitable half-plane operator the operator $\delta(A) = (A-I)(A+I)^{-1}$ which is bounded with spectrum contained in the closed unit disk. Since much is known about such operators on Hilbert space, we can reconcile properties of A with those of $\delta(A)$. In particular, we can characterise boundedness of the $\mathcal{H}^\infty$-calculus of a half-plane operator through estimates on appropriate classes of rational functions on that half-plane. This is in natural correspondence with the property of polynomial boundedness for bounded operators. An operator $T \in B(X)$, where X is a Hilbert space, is said to be *polynomially bounded* if there exists a constant C such that for all polynomials p,

$$\|p(T)\| \leq C \sup\{|p(z)| : \ |z| \leq 1\}.$$

This property has been studied in great depth in the context of the similarity problem for bounded operators. Von Neumann's classical inequality characterises contractions on Hilbert space as operators which are polynomially bounded with $C = 1$. A famous question posed by Halmos [38], which came to be known as the Similarity Problem, asks whether or not polynomial boundedness is equivalent to the operator being similar to a contraction. Eventually Pisier [61] answered the question in the negative, but a result due to Paulsen [57] asserts that operators similar to a contraction are precisely those which are completely polynomially bounded.

There is also a continuous version of the similarity problem for semigroups. A semigroup $(T(t))_{t\geq 0}$ is said to be *similar to a contraction semigroup* if there exists an invertible operator $S \in B(X)$ such that for every $t \geq 0$,

$$\|S^{-1}T(t)S\| \leq 1 \ .$$

An alternative form of von Neumann's inequality asserts that generators of contractive semigroups are precisely those which have a bounded $\mathcal{H}^\infty$-calculus on appropriate half-planes with constant 1. This is a special case of the correspondence between bounded $\mathcal{H}^\infty$-calculus of semigroup generators and polynomial boundedness of their Cayley Transform. So far a characterisation of semigroups similar to contractions in terms of functional calculus has not been precisely formulated. We introduce a natural analogue for the functional calculus to Paulsen's notion of complete polynomial boundedness, which has been used in special cases by Le Merdy [50].

In view of this correspondence, it is of interest to know when the Cayley Transform of the generator of a (bounded) strongly continuous semigroup is polynomially bounded. As mentioned above, we know this does not hold in general. In fact, so far it remains unkown if such an operator is even power bounded, although there are many partial results in this direction.

The functional calculus has powerful applications in the theory of maximal regularity, a thorough survey of which can be found in [47]. To establish maximal regularity it is sufficient to find conditions on a pair of (given) commuting sectorial operators A and B to ensure that their sum $A + B$, with domain $D(A) \cap D(B)$, is closed. This problem has been studied for a long time, starting with the work of Da

Prato and Grisvard [18], who showed that the sum is always closable. In their seminal paper, Dore and Venni [23] were the first to find conditions to ensure closedness of the sum. The Banach space was assumed to satisfy the UMD property and the operators A and B were assumed to be invertible and subject to admitting bounded imaginary powers, with further assumptions on the growth of the operators A^{is} and B^{is}. In fact, their proof showed that the operator $A + B$ is even invertible. Prüss and Sohr [64] later weakened the invertibility assumptions on A and B to injectivity. These ideas are crucial in the theory of maximal regularity and solutions of abstract evolution equations. Usually, one of the operators is known to be a differential operator, which is sectorial with bounded $\mathcal{H}^\infty$-calculus on an appropriate space. Kalton and Weis [44] proved that on any Banach space one obtains that the sum is closed if A admits a bounded $\mathcal{H}^\infty$-calculus and B is R-sectorial. Their arguments rely upon the theory of a joint functional calculus associated to a pair of sectorial operators (A, B). Lancien et al. [48] had already proven that the sum is closed whenever the joint calculus is bounded. Results have even been obtained for sums and products of non-commuting operators [55] [63] [36].

In the commuting case, one can similarly formulate the sum problem for strip-type operators. In particular, if one has a pair of strip-type operators that arise as logarithms of a pair of injective sectorial operators (A, B), one may expect the identity $\log A + \log B = \log AB$ to hold. It is possible to obtain such a result, provided the assumptions on A and B are such that the product AB is sectorial and $\log A, \log B$ do not cancel. Clark [12] proved a precise version of this result, exploiting the Kalton-Weis assumptions.

In practice, many operators arise as perturbed forms of known operators. Thus it is useful to have stability criteria for the bounded $\mathcal{H}^\infty$-calculus. This has been studied for special classes of (relatively) bounded perturbations, namely rank one and triangular perturbations, first for sectorial operators by Arendt and Batty [3] and Kalton [42] and then strip-type operators by Vörös [76].

1.2 Overview of thesis

Chapter 2 is a summary of the main objects of study in this thesis. We devote the first part of Chapter 2 to an overview of the basic theory of one-parameter strongly continuous semigroups, since much of our work is motivated by applications to these operator families. This will lead to definitions of the three classes of operators we are interested in: sectorial, strip-type and half-plane.

In Chapter 3 we present an in-depth discussion of the holomorphic functional calculi associated with these operators, with Haase's book [34] as our main reference point. Starting with an outline of the abstract framework, we explain in general terms how to construct a functional calculus, by starting with an algebra homorphism on a small class and using a regularisation procedure to extend the definition. We then apply this model to our classes of operators. As a by-product, we will define the logarithm of a sectorial operator and discuss the correspondence between sectorial and strip-type operators. Having defined precisely the notion of a bounded (functional) calculus, we will exploit the Convergence Lemma to deduce basic approximation results. With this background material in place, we will introduce the new notion of an m-bounded calculus, which will be employed throughout the thesis, and show that it is no different from the usual boundedness of the $\mathcal{H}^\infty$-calculus for sectorial and strip-type operators. We will also give some details on the well established natural extensions to the joint functional calculus and operator-valued functional calculus.

In Chapter 4, we explore various characterisations of the bounded $\mathcal{H}^\infty$-calculus for sectorial and strip-type operators. Starting on Hilbert space, we summarise the results of McIntosh for sectorial operators in relation to the absolute calculus. We then introduce this notion for strip-type operators, which is so far absent in the literature, and show that the two correspond in a natural way. This allows us to establish new characterisations of the bounded $\mathcal{H}^\infty$-calculus for strip-type operators on Hilbert space. Moving to general Banach space, we begin by surveying the known characterisations from [16] and [47] for sectorial operators, as well as providing a brief summary of the work of Vörös [76] for strip-type operators. Employing the m-bounded calculus, we will provide new characterisations of the bounded calculus in terms of both Rademacher estimates, which we introduce for strip-type operators, and weak integral estimates.

In Chapter 5 our attention turns to half-plane operators. First we use the m-bounded calculus to characterise semigroup generators, which yields new proofs of the Gomilko-Shi-Feng Theorem as well as extensions to polynomially bounded semigroups. Our focus will switch to the similarity problem for semigroups on Hilbert space. After discussing known results on the similarity problem for bounded operators, we will establish the expected correspondence between bounded calculus of a half-plane operator and polynomial boundedness of its Cayley Transform. The key to this will be a fundamental approximation technique for half-plane operators which we extract from known results on sectorial operators, in which estimates on rational functions are suffficient to obtain bounded calculus on half-planes. We will explore what is known about the Cayley Transform of semigroup generators and refine known conditions to guarantee power boundedness in special cases. We will also be interested in generation: namely, the question of when a bounded operator arises as the Cayley Transform of a (bounded) semigroup generator. The methods are easy but instructive, using the theorems of Hille-Yosida and Gomilko-Shi-Feng. We will also explore some connections between our functional calculus considerations and the inverse generator problem.

In Chapter 6 we consider sums and perturbations for a pair of sectorial and strip-type operators. The problem of sums is entwined with the concept of maximal regularity and we present a brief summary of known results in this area before filling in some gaps by employing the absolute calculus as an underlying assumption on one operator. We will provide a range of perturbation results to establish stability criteria for the bounded $\mathcal{H}^\infty$-calculus. Using the work of Kalton and Weis [45] we can provide more sophisticated results on spaces with property (α).

1.3 Notation

Banach spaces will be denoted by X, Y, etc. with possible deviation to H when working on Hilbert space. We denote the dual space of a Banach space X by X^*. The duality between $x \in X, y \in X^*$ will be denoted by one of

$$y(x) = \langle x, y \rangle = \langle y, x \rangle.$$

The unit balls of X and X^* are denoted by X_1 and X_1^* respectively.

The space of bounded linear operators from X to Y will be denoted $B(X, Y)$, or just $B(X)$ when $X = Y$. Bounded operators will usually be denoted by S, T, etc. and unbounded operators A, B, etc.

The Banach adjoint of an operator A is denoted A^* and its Hilbert adjoint is denoted A'.

The resolvent set of an operator A is denoted by $\rho(A)$, with $R(\lambda, A) := (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$. The spectrum is denoted by $\sigma(A)$.

Two operators A and B are said to be commuting if for every $\lambda \in \rho(A)$ and $\mu \in \rho(B)$ we have $R(\lambda, A)R(\mu, B) = R(\mu, B)R(\lambda, A)$.

We denote

$$\begin{aligned}
\Sigma_\theta &= \{z \in \mathbb{C} : \ |\arg z| < \theta\} \\
H_v &= \{z \in \mathbb{C} : \ |\operatorname{Im} z| < v\} \\
R_v &= \{z \in \mathbb{C} : \ \operatorname{Re} z > v\} \\
L_v &= \{z \in \mathbb{C} : \ \operatorname{Re} z < v\}
\end{aligned}$$

We define the function spaces

$$\begin{aligned}
&\mathcal{H}^\infty(\Omega) = \{f : \Omega \to \mathbb{C} : \ f \text{ is bounded and holomorphic}\}, \text{where } \Omega \text{ is any of the regions above.} \\
&\mathcal{H}_0^\infty(\Sigma_\theta) = \{f \in \mathcal{H}^\infty(\Sigma_\theta) : \ \text{there exists } C, \delta > 0 \text{ such that } |f(z)| \le C \frac{|z|^\delta}{1 + |z|^{2\delta}} \ \forall z \in \Sigma_\theta\}. \\
&\mathcal{H}_1^\infty(H_v) = \{f \in \mathcal{H}^\infty(H_v) : \ \text{there exists } C \ge 0, \delta > 1 \text{ such that } |f(z)| \le \frac{C}{1 + |\operatorname{Re} z|^\delta} \ \forall z \in H_v\}. \\
&\mathcal{H}_1^\infty(R_v) = \{f \in \mathcal{H}^\infty(R_v) : \ \text{there exists } \delta > 1 \text{ such that } |f(z)| \le \frac{C}{1 + |\operatorname{Im} z|^\delta} \ \forall z \in R_v\}.
\end{aligned}$$

For a suitable space Ω we denote

$$\begin{aligned}
\mathbf{C_0}(\overline{\Omega}) &= \{f : \overline{\Omega} \to \mathbb{C} : \ f \text{ is continuous, vanishes at infinity}\}. \\
C^{(k)}(\Omega) &= \{f : \Omega \to \mathbb{C} : \ f \text{ is } k\text{-times differentiable, } f^{(k)} \text{ is continuous}\}. \\
L^p(\Omega) &= \{f : \Omega \to \mathbb{C} : \ f \text{ is } p\text{-integrable}\}.
\end{aligned}$$

For a scalar function f, $f^{(m)}$ is the m^{th} derivative of f. Furthermore $\hat{f}$ denotes the Fourier transform of an integrable function f, given by

$$\hat{f}(\lambda) = \int_{\mathbb{R}} e^{-i\lambda x} f(x) \, \mathrm{d}x \ .$$

We denote by $\check{f}$ the inverse Fourier transform of f.

We will use C to indicate any constant that does not depend on the parameters involved. Often, C will be used repeatedly to denote different constants within the same equation. When C may depend on parameters, this will be made clear, e.g. C_v indicates that C depends on the paramater v.

2 Preliminaries

2.1 C_0-Semigroups

For a systematic treatment of semigroups we refer to [28] [4] [58], where much of our discussion is taken from. We begin with the definition of a one-parameter semigroup.

Definition 2.1.1. A family of bounded operators $(T(t))_{t\geq 0}$ on a Banach space X is said to be a (one-parameter) *semigroup* if for all $s, t \geq 0$

$$\begin{aligned} T(s+t) &= T(s)T(t) \\ T(0) &= I \end{aligned}$$

If for all $x \in X$ the map $t \mapsto T(t)x$, $\mathbb{R}_+ \to X$ is continuous we say the semigroup is *strongly continuous*, or a C_0-semigroup.

Observe that by the semigroup property, strong continuity is only required at $t = 0$.

The theory of one-parameter semigroups is motivated by the problem of assigning to an appropriate operator A and $t \in \mathbb{R}$, the evaluation e^{tA}, which we would expect to satisfy the conditions of Definition 2.1.1. This can be achieved with relative simplicity for special classes of operators. Indeed, if A is bounded we can define e^{tA} as the exponential series $\sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$, where the convergence is understood to be in the operator norm (as is done for matrices, for example). Furthermore, in the 1930s Stone [68] and von Neumann [74] gave precise definitions of e^{tA} for normal operators A on Hilbert space. In each case, the family of operators $(e^{tA})_{t\geq 0}$ can be shown to form a C_0-semigroup. In fact, Phillips [59] showed that the strong continuity assumption on the orbit maps $t \to T(t)x$ is equivalent to strong measurability and weak continuity (though not weak measurability).

Conversely, starting with a C_0-semigroup we can associate an operator A for which we can then define e^{tA} as the bounded operator $T(t)$.

Definition 2.1.2. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup. We define the *generator* of this semigroup as the operator A defined by

$$\begin{aligned} D(A) &= \{x \in X : \lim_{h\to 0+} \frac{T(h)x - x}{h} \text{ exists}\}\,, \\ Ax &= \lim_{h\to 0+} \frac{T(h)x - x}{h}\,. \end{aligned}$$

Then A exists as a closed, densely defined linear operator. Moreover, for each $x \in X$

we have

$$\lim_{n\to\infty} \left(\frac{n}{t} R\big(\frac{n}{t}, A\big)^n\right) x = T(t)x \ ,$$

so that the semigroup is uniquely determined by A. We now collect some properties of A, which can be found in [28, II.1.3].

Lemma 2.1.3. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Then the following properties hold.*

(i) If $x \in D(A)$ then $T(t)x \in D(A)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} T(t)x = T(t)Ax = AT(t)x \quad \text{for all } t \geq 0 \ .$$

(ii) For $t \geq 0$ and $x \in X$ we have

$$\int_0^t T(s)x \, \mathrm{d}s \in D(A) \ .$$

(iii) For every $t \geq 0$ we have

$$\begin{aligned} T(t)x - x &= A \int_0^t T(s)x \, \mathrm{d}s \quad (x \in X) \\ &= \int_0^t T(s)Ax \, \mathrm{d}s \quad (x \in D(A)) \ . \end{aligned}$$

In particular, properties (*ii*) and (*iii*) show that for each $x \in X$ $u(t) = T(t)x$ is a weak solution of the abstract Cauchy problem

$$\begin{aligned} \dot{u} &= Au \\ u(0) &= x \end{aligned} \tag{ACP}$$

In this sense, it seems natural to define e^{tA} as the operator $T(t)$. Furthermore, if A is bounded this definition is consistent with the usual one (see [28, II.1.5]).

Proposition 2.1.4. *Let A generate the C_0-semigroup $(T(t))_{t\geq 0}$. Then the following are equivalent.*

(i) A is a bounded operator.

(ii) $D(A) = X$.

(iii) $(T(t))_{t\geq 0}$ is norm continuous.

(iv) $\overline{\lim}_{\lambda\to\infty}\|\lambda A R(\lambda, A)\| < \infty$.

In this case the semigroup is given by

$$T(t) = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \ .$$

The natural question arises as to when an operator A is the generator of a C_0-semigroup. Before addressing this, we introduce some definitions. Note that for a C_0-semigroup $(T(t))_{t\geq 0}$, there exist constants $w \in \mathbb{R}$ and $M \geq 1$ such that

$$\|T(t)\| \leq M e^{wt} \quad \text{for all } t \geq 0 \ . \tag{2.1.1}$$

Definition 2.1.5. The infimum of all w for which there exists M such that (2.1.1) holds is called the *growth bound* of the semigroup, denoted w_0. The semigroup is said to be *bounded* if the choice $w = 0$ is possible and *contractive* if furthermore $M = 1$ is possible. If $\|T(t)x\| = \|x\|$ for all $x \in X$ and $t \geq 0$ then the semigroup is said to be *isometric*. We call the semigroup *uniformly exponentially stable* when $w_0 < 0$, which is equivalent to the property $\lim_{t\to\infty}\|T(t)\| = 0$ [28, V.1.7].

Furthermore, there exist semigroups for which

1. $w_0 = -\infty$,
2. w_0 is finite but not attained,
3. constants $M > 1$ are necessary.

(see [28, I.5.7])

Returning to the issue of generation, the celebrated Hille-Yosida theorem [28, II.3.8] provides necessary and sufficient conditions for a linear operator on a Banach space to generate a C_0-semigroup.

Theorem 2.1.6. (**Hille – Yosida**) *Let $(A, D(A))$ be a linear operator on a Banach space X, $w \in \mathbb{R}$, $M \geq 1$. Then the following are equivalent.*

(i) A generates a C_0-semigroup $(T(t))_{t\geq 0}$ such that

$$\|T(t)\| \leq M e^{wt} \quad \textit{for all}\, t \geq 0.$$

(ii) $(A, D(A))$ is closed, densely defined, and for every $\lambda > w$ we have $\lambda \in \rho(A)$ with

$$\big\|\big[(\lambda - w)R(\lambda, A)\big]^n\big\| \leq M \quad \textit{for all}\, n \in \mathbb{N}.$$

(iii) $(A, D(A))$ *is closed, densely defined, and for every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > w$ *we have* $\lambda \in \rho(A)$ *with*

$$\left\| \left[(\operatorname{Re}\lambda - w)R(\lambda, A)\right]^n \right\| \leq M \quad \textit{for all}\, n \in \mathbb{N}.$$

We collect two important properties of C_0-semigroups which are employed in the proof of Theorem 2.1.6 and will be useful later on. The first is a convergence condition which applies to any closed, densely defined operator and which is often used to refute the possibility that a given operator is the generator of a C_0-semigroup.

Lemma 2.1.7. *Suppose* A *is a closed, densely defined operator and there exist* $w \in \mathbb{R}$*,* $M \geq 1$ *such that* $[w, \infty) \subseteq \rho(A)$ *and* $\|\lambda R(\lambda, A)\| \leq M$ *for* $\lambda \geq w$*. Then for all* $x \in X$*,* $\lambda R(\lambda, A)x \to x$ *as* $\lambda \to \infty$*.*

The second result gives an expected relationship between the resolvent of a generator and the semigroup.

Lemma 2.1.8. *For the generator* A *of a* C_0*-semigroup* $(T(t))_{t\geq 0}$*,* $\operatorname{Re}\lambda > w_0$ *and any* $n \in \mathbb{N}$ *we have for every* $x \in X$*,*

$$R(\lambda, A)^n x = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-\lambda t} T(t)x \, \mathrm{d}t \,.$$

Remark 2.1.9. The proof of Lemma 2.1.8 for $n > 1$ follows by differentiating with respect to λ, once it is known for $n = 1$. Moreover, it is even possible to define the generator of a semigroup through this integral representation of the resolvent, in a manner which is consistent with Definition 2.1.2. This approach was employed in [4].

The Hille-Yosida theorem, whilst offering a complete characterisation, is limited in its use for applications since it requires estimates on all powers of resolvents, which are often difficult to obtain. Ideally, we would like a condition on a single resolvent power. Gomilko [30] and Shi and Feng [66] were able to provide a sufficient condition on general Banach space, which is also necessary on Hilbert space.

Theorem 2.1.10. *Let* A *be a densely defined operator on a Banach space* X *such that*

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0\} \,.$$

Consider the following conditions.

(i) For every $x \in X$ and $x^* \in X^*$

$$\sup_{w>0} w \int_{\mathbb{R}} \|R(w+it, A)x\|^2 \,\mathrm{d}t < \infty,$$

$$\sup_{w>0} w \int_{\mathbb{R}} \|R(w+it, A^*)x^*\|^2 \,\mathrm{d}t < \infty.$$

(ii) For every $x \in X$ and $x^* \in X^*$

$$\sup_{w>0} w \int_{\mathbb{R}} |\langle R(w+it, A)^2 x, x^* \rangle| \,\mathrm{d}t < \infty.$$

(iii) A generates a bounded C_0-semigroup.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). If X is a Hilbert space then (iii) $\Rightarrow$ (i) also holds.

We will also be interested in generators of semigroups which extend to the whole real line.

Definition 2.1.11. A family of operators $(T(t))_{t\in\mathbb{R}}$ is said to be a C_0-*group* if it is strongly continuous and the properties of Definition 2.1.1 hold for all $s, t \in \mathbb{R}$.

It is immediately clear that an operator $(A, D(A))$ generates a C_0-group if and only if A and $-A$ generate semigroups $T_+(t)$ and $T_-(t)$ respectively, in which case

$$T(t) = \begin{cases} T_+(t) & t \geq 0, \\ T_-(-t) & t \leq 0\ . \end{cases}$$

In view of this, it still makes sense to adopt the notation $T(t) = e^{tA}$. We call $(T_+(t))_{t\geq 0}$ and $(T_-(t))_{t\geq 0}$ the *forward* and *backward* semigroups corresponding to $(T(t))_{t\in\mathbb{R}}$, respectively.

Much of our discussion for C_0-semigroups then extends in a natural way to C_0-groups. It is clear that given a C_0-group $(T(t))_{t\in\mathbb{R}}$, there exist $w \in \mathbb{R}$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{|t|w}$ for all $t \in \mathbb{R}$. We define the *group type* w_0 as the infimum of all w for which such an M exists, which is easily seen to equal the minimum of the group types of the semigroups $(T_+(t))_{t\geq 0}$ and $(T_-(t))_{t\geq 0}$ (see Definition 2.1.5). We can also obtain versions of the Hille-Yosida and Gomilko-Shi Feng theorems for groups.

Theorem 2.1.12. *Let $(A, D(A))$ be a linear operator on a Banach space X, $w \in \mathbb{R}$, $M \geq 1$. Then the following are equivalent.*

(i) *A generates a C_0-group $(T(t))_{t\in\mathbb{R}}$ such that*

$$\|T(t)\| \le M e^{w|t|} \quad \textit{for all}\, t \in \mathbb{R}.$$

(ii) *$(A, D(A))$ is closed, densely defined, and for every $\lambda \in (-\infty, -w) \cup (w, \infty)$ we have $\lambda \in \rho(A)$ with*

$$\left\| \left[(|\lambda| - w) R(\lambda, A)\right]^n \right\| \le M \quad \textit{for all}\, n \in \mathbb{N}.$$

(iii) *$(A, D(A))$ is closed, densely defined, and for every $\lambda \in \mathbb{C}$ with $|\operatorname{Re}\lambda| > w$ we have $\lambda \in \rho(A)$ with*

$$\left\| \left[(|\operatorname{Re}\lambda| - w) R(\lambda, A)\right]^n \right\| \le M \quad \textit{for all}\, n \in \mathbb{N}.$$

Theorem 2.1.13. *Let A be a densely defined operator on a Banach space X such that for some $v \ge 0$,*

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \, |\operatorname{Re}\lambda| \le v\}\,.$$

Consider the following conditions.

(i) *For every $x \in X$ and $x^* \in X^*$,*

$$\sup_{w>v}\, w \int_{\mathbb{R}} \|R(\pm w + it, A)x\|^2 \,\mathrm{d}t < \infty,$$
$$\sup_{w>v}\, w \int_{\mathbb{R}} \|R(\pm w + it, A^*)x^*\|^2 \,\mathrm{d}t < \infty.$$

(ii) *For every $x \in X$ and $x^* \in X^*$*

$$\sup_{w>v}\, w \int_{\mathbb{R}} |\langle R(\pm w + it, A)^2 x, x^*\rangle| \,\mathrm{d}t < \infty.$$

(iii) *A generates a C_0-group $(T(t))_{t\in\mathbb{R}}$ for which there exists a constant C such that $\|T(t)\| \le C e^{v|t|}$ for every $t \in \mathbb{R}$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). If X is a Hilbert space then (iii) $\Rightarrow$ (i) also holds.

We conclude this section with some important examples of concrete semigroups, as well as some basic constructions of new semigroups from old.

Example 2.1.14. (i) Let Ω be a locally compact space (usually an interval in $\mathbb{R}$) and consider the space $C_0(\Omega)$ of all continuous, complex-valued functions that vanish at infinity. For a continuous function $q : \Omega \to \mathbb{C}$ define the *multiplication operator* M_q by

$$\begin{aligned} D(M_q) &= \{f \in C_0(\Omega) : \ q \cdot f \in C_0(\Omega)\}, \\ M_q f &= q \cdot f \quad (f \in D(M_q)) \ . \end{aligned}$$

If we suppose that $\sup_{s\in\Omega} \operatorname{Re} q(s) < \infty$ then M_q is the generator of the strongly continuous multiplication semigroup $(T_q(t))_{t\geq 0}$ given by

$$T_q(t)f = e^{tq} f \qquad (t \geq 0, f \in C_0(\Omega)) \ .$$

We can similarly define multiplication semigroups on the space $L^p(\Omega)$ where the function q is now subject to the condition $\operatorname{ess\,sup}_{s\in\Omega} \operatorname{Re} q(s) < \infty$. In either case, the generators that arise from these semigroups are precisely multiplication operators whose resolvents contain some right half plane R_w. Furthermore, every normal operator on a separable Hilbert space is unitarily equivalent to a multiplication operator on $L^2(\Omega, \mu)$ for some σ-finite measure space (Σ, Ω, μ). We refer to [28, I.4, II.2.9] for these details on multiplication semigroups.

(ii) The (left) translation operators defined by

$$T_l f(t) f(s) = f(s+t) \quad (s, t \in \mathbb{R})$$

form a C_0-semigroup on the spaces $C_{ub}(\mathbb{R})$ (endowed with the sup norm) and $L^p(\mathbb{R})$, $1 \leq p < \infty$. In each case, the generator $(A, D(A))$ is the differentiation operator, whose domain depends on the underlying space:

$$D(A) = \big\{f \in C_{ub}(\mathbb{R}) : \ f \text{ differentiable}, f' \in C_{ub}(\mathbb{R})\big\}$$

when $X = C_{ub}(\mathbb{R})$ and

$$\big\{f \in L^p(\mathbb{R}) : \ f \text{ absolutely continuous}, f' \in L^p(\mathbb{R})\big\}$$

when $X = L^p(\mathbb{R})$. In both spaces and for $\operatorname{Re} \lambda > 0$, we can explicitly compute the resolvent as

$$\big(R(\lambda, A) f\big)(s) = \int_s^\infty e^{-\lambda(\tau - s)} f(\tau) \, \mathrm{d}\tau \quad (f \in X, s \in \mathbb{R}) \ .$$

Notice that each operator T_l is an isometry on these spaces with inverse the right translation operator T_r defined by $(T_r(t)f(s) = f(s-t)$. Thus, we actually have a C_0-group of translation operators on these spaces. We refer to [28, I.4, II.2.10] for more details.

(iii) Our next example, from [40, 12.4], is of a semigroup which fails strong continuity. Let X be the set of sequence pairs $\{(x_n), (y_n)\}$ for which $\sum_{n=1}^{\infty}[|x_n|^2 + n|y_n|^2]^{\frac{1}{2}}$ exists, equipped with the norm $\|\{(x_n), (y_n)\}\| = \sum_{n=1}^{\infty}[|x_n|^2+n|y_n|^2]^{\frac{1}{2}}$. For $t > 0$ define $T(t)\{(x_n), (y_n)\} = \{(x_n^{'}), (y_n^{'})\}$, where

$$\begin{aligned} x_n^{'} &= e^{-(n+1)t}(x_n \cos nt - y_n \sin nt), \\ y_n^{'} &= e^{-(n+1)t}(x_n \sin nt + y_n \cos nt). \end{aligned}$$

Then $(T(t))_{t\geq 0}$ is a semigroup on X, strongly continuous for $t > 0$, satisfying

$$\|T(t)\| \leq Ct^{-\frac{1}{2}} \quad (t > 0) \ .$$

However, if the semigroup was strongly continuous, the generator A would satisfy

$$\|R(\lambda, A)\| \geq \frac{C}{(\lambda+1)^{\frac{1}{2}}} \quad (\lambda > 0) \ ,$$

which contradicts Lemma 2.1.7. Thus the semigroup is not strongly continuous at $t = 0$.

(iv) Similarity will be a key theme of Chapter 5. For the moment, we observe that if V is an isomorphism from a Banach space Y onto X and $(T(t))_{t\geq 0}$ is a C_0-semigroup on X with generator A, then the family of operators given by $S(t) = V^{-1}T(t)V$ forms a C_0-semigroup on Y with generator B given by

$$B = V^{-1}AV \quad \text{with domain} \quad D(B) = \{y \in Y : \ Vy \in D(A)\} \ .$$

It is clear that $\sigma(A) = \sigma(B)$ and $R(\lambda, B) = V^{-1}R(\lambda, A)V$ for every $\lambda \in \rho(A)$.

2.2 Sectorial operators

Having established the basic theory of C_0-semigroups in the previous section, we now turn our attention to properties of the generator. The Hille-Yosida theorem reveals the location of the spectrum of such an operator, as well as control on the growth of its resolvent. Having such information about an operator is useful from the point of view of developing a functional calculus, as we will outline in Chapter

3. First we consider special classes of operators for which the spectrum is assumed to lie in some special region of $\mathbb{C}$, with appropriate resolvent behaviour outside of this region. We begin with the well known class of sectorial operators.

For $\theta \in (0, \pi)$ we denote

$$\Sigma_\theta = \begin{cases} \{z \in \mathbb{C} : \ z \neq 0, |\arg z| < \theta\} & \theta \in (0, \pi), \\ (0, \infty) & \theta = 0 \ , \end{cases}$$

which, for $\theta > 0$, is the open sector symmetric about the positive real axis with opening angle 2θ.

Definition 2.2.1. The operator $(A, D(A))$ is said to be *sectorial* of angle ϕ if

(i) $\sigma(A) \subseteq \overline{\Sigma_\phi}$ and

(ii) for every $\phi' \in (\phi, \pi]$ there exists $C(A, \phi')$ such that

$$\|\lambda R(\lambda, A)\| \leq C(A, \phi') \quad \text{for all } \lambda \in \mathbb{C} \setminus \overline{\Sigma_{\phi'}} \ . \tag{2.2.1}$$

In this case we shall say $A \in \operatorname{Sect}(\phi)$. Furthermore, we denote by ϕ_A the minimum of all such ϕ - the *angle of sectoriality* of A. We will also denote $C(A, \pi)$ by $C(A)$. Often it will be convenient to further assume that

$$\overline{D(A)} = \overline{R(A)} = X,$$

which in particular implies that A is injective and that $D(A) \cap R(A)$ is dense in X (see [34, Proposition 2.1.1]).

Remark 2.2.2. It is without essential loss of generality that one can make these domain and range assumptions on a sectorial operator A. Indeed, if A is any sectorial operator on a Banach space X then taking the part of A in $\overline{D(A)} \cap \overline{R(A)}$ yields a sectorial operator with dense domain and range. Moreover if A already has dense domain and range then the dual A^* is a well-defined injective sectorial operator. Taking the part of A^* in $X_0 = \overline{D(A^*)} \cap \overline{R(A^*)}$ gives the so-called *moon dual* of A. The space X_0 is norming for X in the sense that the norm

$$\|x\|_0 = \sup\{|\langle x, x_0\rangle| : \ x_0 \in X_0, \|x_0\| \leq 1\}$$

is equivalent to the original norm on X.

We now collect some important properties of sectorial operators, the proofs of which can be found in [34, Chapter 2].

Proposition 2.2.3. *Let A be a closed operator acting on the Banach space X.*

(i) If A satisfies (2.2.1) for $\lambda < 0$ then $A \in \operatorname{Sect}(\pi - \arcsin \frac{1}{C(A)})$.

(ii) If $A \in \operatorname{Sect}(\phi)$ is injective then $A^{-1} \in \operatorname{Sect}(\phi)$. Indeed, the identity

$$\lambda R\big(\lambda, A^{-1}\big) = I - \frac{1}{\lambda} R\big(\frac{1}{\lambda}, A\big)$$

implies $C(A^{-1}, \phi) \leq 1 + C(A, \phi)$.

(iii) If A is sectorial with dense domain and range then for every $x \in X$ we have

$$\begin{aligned} &\lim_{t \to -\infty} tR(t, A)x = x && \lim_{t \to 0} tR(t, A)x = 0 \\ &\lim_{t \to -\infty} AR(t, A)x = 0 && \lim_{t \to 0} AR(t, A)x = x \quad . \end{aligned}$$

If A is the generator of a C_0-semigroup satisfying $\|T(t)\| \leq Ce^{wt}$, then $A - w$ generates the bounded C_0-semigroup $e^{-tw}T(t)$. By the Hille-Yosida theorem and Proposition 2.2.3 we have that $-(A - w)$ is sectorial with $\phi_A \leq \frac{\pi}{2}$. In fact, we can obtain an even stronger correspondence between semigroup generators and sectorial operators. To do so, we first define another special class of semigroups.

Definition 2.2.4. Let $\delta \in (0, \frac{\pi}{2}]$. A family of bounded operators $(T(t))_{t \in \Sigma_\delta \cup \{0\}}$ is said to be an *analytic semigroup* of angle δ if

- $T(0) = I$,
- $T(z + w) = T(z)T(w) \quad \text{for each} \ \ z, w \in \Sigma_\delta$,
- $$\lim_{z \to 0, z \in \Sigma_\epsilon} T(z)x = x \quad \text{for any } \epsilon \in (0, \delta) \ .$$

If furthermore $\|T(z)\|$ is bounded on Σ_ϵ for each $\epsilon \in (0, \delta)$ then the semigroup is said to be (sectorially) bounded, analytic. Note that this boundedness assumption is stronger than having a bounded C_0-semigroup which has analytic continuation to a sector.

Proposition 2.2.5. *Let $(-A, D(A))$ be sectorial of angle $0 < \delta < \frac{\pi}{2}$. Define $T(0) = I$ and for $z \in \Sigma_{\frac{\pi}{2} - \delta}$,*

$$T(z) = \frac{1}{2\pi i} \int_\Gamma e^{z\mu} R(\mu, A) \, \mathrm{d}\mu \ ,$$

where $\Gamma = [\rho, \infty)e^{\pm i\delta'} \cup \rho e^{i[-\delta',\delta']}$*, oriented counterclockwise,* $\rho > 0$ *and* $\frac{\pi}{2} + |\arg z| < \delta' < \pi - \delta$*. Then* $(T(z))$ *is a bounded, analytic semigroup of angle* $\frac{\pi}{2} - \delta$ *with generator* A*.*

In fact, the converse holds too and we have the following characterisation of generators of bounded, analytic semigroups [28, II.4.6].

Theorem 2.2.6. *Let* $(A, D(A))$ *be a densely defined operator on a Banach space* X*. Then the following are equivalent.*

(i) $-A$ *is sectorial of angle* $\delta < \frac{\pi}{2}$*.*

(ii) A *generates a bounded, analytic semigroup on* X*.*

(iii) There exists $\theta \in (0, \frac{\pi}{2})$ *such that the operators* $e^{\pm i\theta}A$ *generate bounded* C_0*-semigroups on* X*.*

(iv) A *generates a bounded* C_0*-semigroup on* X *such that for any* $t > 0$

$$R(T(t)) \subseteq D(A) \quad \text{and} \quad \sup_{t\geq 0} \|tAT(t)\| < \infty \, .$$

(v) A *generates a bounded* C_0*-semigroup and there exists a constant* C *such that for all* $\operatorname{Re}\lambda > 0$ *and* $\operatorname{Im}\lambda \neq 0$*,*

$$\|R(\lambda, A)\| \leq \frac{C}{|\operatorname{Im}\lambda|} \, .$$

2.3 Strip-type and half-plane operators

The next most important class of operators are those whose spectrum lie in a strip. For $w > 0$ define

$$H_w = \begin{cases} \{z \in \mathbb{C} : \ |\operatorname{Im} z| < w\} & w > 0, \\ \mathbb{R} & w = 0 \, . \end{cases}$$

Definition 2.3.1. The operator $(B, D(B))$ is said to be *strip-type* of height w, $B \in \operatorname{Strip}(w)$, if

(i) $\sigma(B) \subseteq \overline{H_w}$ and

(ii) for every $w' > w$ there exists $C(B, w')$ such that

$$\|R(\lambda, B)\| \leq C(B, w') \quad \text{for all } \lambda \in \mathbb{C} \setminus \overline{H_{w'}} \, . \tag{2.3.1}$$

If we have the stronger condition that for each $w' > w$ there exists $C(B, w')$ such that for every $\lambda \in \mathbb{C} \setminus \overline{H_{w'}}$,

$$\|(|\operatorname{Im}\lambda| - w')|R(\lambda, B)\| \leq C(B, w') \ , \tag{2.3.2}$$

then B is said to be of strong strip-type, $B \in \operatorname{SStrip}(w)$. As in the sectorial case, we define the minimum height w for which $B \in \operatorname{Strip}(w)$ ($\operatorname{SStrip}(w)$) by w_B^{st} (w_B^{sst}), or often just w_B.

By the Hille-Yosida theorem for groups, it immediately follows that if iB generates a C_0-group, then B is a strong strip-type operator with w_B^{sst} at most the group type.

Remark 2.3.2. The choice to work on horizontal strips rather than vertical strips is motivated by the fact that many strip-type operators can be considered as the logarithm of sectorial operators. This notion will be made more precise in Chapter 3 once we have developed a holomorphic functional calculus for sectorial operators. Indeed, much of our work for strip-type operators in this thesis will be approached from this viewpoint. Since all notions for semigroups and functional calculus are preserved modulo multiplication by $\pm i$, it is without loss of generality that we choose this convention.

Remark 2.3.3. Often we will assume that a strip-type operator B on a Banach space X is densely defined. Analogously to the sectorial case, there is no essential loss of generality (see Remark 2.2.2). This time, one takes the part of B in $\overline{D(B)}$. Furthermore one can pass to the *sun dual*, which is the part of B^* in $\overline{D(B^*)}$, to ensure the dual has dense domain. Again, this space is norming for X.

The following result is elementary but useful.

Proposition 2.3.4. *Let B be a densely defined strip-type operator. Then for $a > w_B^{st}$ and $x \in X$, $R(z, B)x \to 0$ as $z \to \pm\infty + ia$.*

Example 2.3.5. We conclude our preliminary remarks on strip-type operators with a presentation of a concrete example. Consider the second derivative operator $B = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $X = L^p(\mathbb{R})$, $1 \leq p < \infty$, with domain $W^{2,p}(\mathbb{R})$. Then A is (strong) strip-type but iA is a group generator only if $p = 2$. This result, due to Hörmander, can be found in [4, Theorem 3.9.4].

Our next class of operators are those whose spectrum lie in some half-plane. We have already seen that sectorial operators arise naturally as semigroup generators. The resolvent estimates provided by the Hille-Yosida theorem also allow us to classify generators as half-plane operators. The virtue of doing so will be made clear in

Chapter 3, where we will develop a half-plane functional calculus which is independent of, but consistent with, the treatment for sectorial operators. As convention, we will work with right half planes. For $w \in \mathbb{R}$ we denote

$$R_w = \{z \in \mathbb{C} : \operatorname{Re} z > w\},$$
$$L_w = \{z \in \mathbb{C} : \operatorname{Re} z < w\}.$$

For $w = \infty$ we set $R_w = \emptyset$, $L_w = \mathbb{C}$ and for $w = -\infty$ we set $R_w = \mathbb{C}$, $L_w = \emptyset$.

Definition 2.3.6. An operator $(B, D(B))$ is said to be *half-plane* of type $w \in \mathbb{R} \cup \{\infty\}$ if

(i) $\sigma(B) \subseteq \overline{R_w}$ and

(ii) for every $w' < w$ there exists $C(B, w')$ such that

$$\|R(\lambda, B)\| \leq C(B, w') \quad \text{for all } \lambda \in L_{w'} \ . \tag{2.3.3}$$

We define $s_0(B)$ to be the maximum of all such w. Furthermore, as with strip-type operators, we can define the slightly stronger notion of strong half-plane operators by replacing (2.3.3) with the condition

$$\sup\{\|(w' - \operatorname{Re} \lambda) R(\lambda, B)\| : \ \lambda \in L_{w'}\} < \infty \ .$$

It is clear that if B is a strong half-plane operator with $s_0(B) \geq 0$ then B is sectorial with $\phi_B \leq \frac{\pi}{2}$. At times we will not need this stronger notion but note that by the Hille-Yosida theorem, for every semigroup with generator $-A$ and growth bound w_0, the operator A is a strong half-plane operator with $s_0(A) \geq w_0$.

The following result will be used frequently later on.

Proposition 2.3.7. *Let A be a densely defined operator on X with non-empty resolvent set. Then A^2 is also densely defined.*

Proof. Let $\mu \in \rho(A)$. Then $R(R(\mu, A)) = D(A)$ is dense in X. Since the product of two bounded operators with dense range also has dense range, we have

$$\overline{D(A^2)} = \overline{D((\mu - A)^2)} = \overline{R(R(\mu, A)^2)} = X \ .$$

□

3 Functional Calculus

The idea behind the functional calculus for an operator A is to assign a meaningful valuation $f(A)$ for scalar functions f belonging to a suitable class. For specific functions this can be done through various means. For example, in Chapter 2 we defined e^{tA} for semigroup and group generators. Other examples of operators associated to A that are of interest and have been studied in their own right include

- Powers A^{α}, for suitable $\alpha \in \mathbb{C}$, when A is sectorial;
- The logarithm $\log A$ when A is sectorial;
- Resolvent powers $(AR(\lambda, A))^n$ and, more generally, $r(A)$ for a rational function r.

The functional calculus is a powerful tool in that it unifies these separate methods into a single framework by giving meaning to these operators in a manner that is consistent with other definitions. One of the earliest examples of functional calculus was the Riesz-Dunford calculus for bounded operators [24, VII.3], which is a special case of a more general functional calculus on Banach Algebras [15, VII.4]. The classes of operators of interest in this thesis are unbounded operators whose spectrum lies in some special region of the complex plane. The holomorphic functional calculus for sectorial operators was introduced by McIntosh in [53] and developed by Cowling et al. in [16]. The functional calculus for strip-type operators was introduced in the 1950s by Bade [7] and recently developed by Haase [34, Chapter 4], who has also extended this framework to include half-plane operators [35]. All of these approaches can be unified by Haase's abstract considerations [34, Chapter 2].

With the exception of Section 3.5, the material in this chapter is taken from the literature, with [34] as the main source of reference. In Section 3.1 we will begin our overview of the functional calculus with an outline of this abstract approach. In Section 3.2 we develop a functional calculus for the concrete familes of operators that will be studied later on, beginning with the standard construction for sectorial operators and moving onto strip-type and half-plane operators. We will also define the notion of a bounded $\mathcal{H}^{\infty}$-calculus. In relation to this, approximation will be a key theme throughout this thesis and basic techniques via the so-called Convergence Lemmas are introduced in Section 3.3. Using the functional calculus, in Section 3.4 we will define the logarithm of a sectorial operator, which is strip-type under suitable conditions, and consider the correspondence between the functional calculi of these two classes of operators. In Section 3.5 we introduce the new notion of an m-bounded $\mathcal{H}^\infty$-calculus for sectorial and strip-type operators, which involves estimates on derivatives of scalar functions only. In fact, the definition will be shown to be equivalent to the usual boundedness of the $\mathcal{H}^\infty$-calculus, but it serves as a useful tool for broadening known results as well as establishing new ones in Chapter 4. In Section 3.6 we review well known extensions of the functional calculus, first replacing scalar valued functions with operator-valued ones and then considering a calculus for a pair of (commuting) operators. Finally in Section 3.7 we discuss the notion of R-boundedness, which in recent years has emerged as an important property in the theory of functional calculus for unbounded operators in the context of maximal regularity.

3.1 Abstract framework

The modus operandi behind the functional calculus is to fix an operator A and consider a primary class of scalar functions f for which $f(A)$ can be reasonably defined. An extension procedure then allows us to widen this class of functions. Most of the material up to Section 3.5 is taken from [34] and as such we adopt much of the notation used there.

Definition 3.1.1. The triple $(\mathcal{E}, \mathcal{F}, \Lambda)$ is said to be an *abstract functional calculus* over the Banach space X if

1. $\mathcal{F}$ is a commutative unital algebra of functions with unit $\mathbf{1}$,
2. $\mathcal{E}$ is a subalgebra of $\mathcal{F}$, and
3. $\Lambda : \mathcal{E} \to B(X)$ is an algebra homomorphism.

Thus $\mathcal{E}$ represents the initial class of functions for which $f(A) := \Lambda(f)$ is defined as a bounded operator. We say $f \in \mathcal{F}$ is *regularisable* with *regulariser* e if there exists $e \in \mathcal{E}$ such that $ef \in \mathcal{E}$ and $e(A)$ is injective. In this case we define

$$\Lambda(f) := \Lambda(e)^{-1}\Lambda(ef), \quad \text{with domain} \quad D(\Lambda(f)) = \{x \in X : \Lambda(ef)x \in R(\Lambda(e))\} .$$

Proposition 3.1.2. *If $f \in \mathcal{F}$ is regularisable then $\Lambda(f)$ is closed and independent of the choice of regulariser e.*

Proof. The closedness of $\Lambda(f)$ is clear since $\Lambda(e)^{-1}$ is closed and $\Lambda(ef)$ is bounded. Suppose that e_1 and e_2 regularise f. Then $e_1e_2 = e_2e_1$ is also a regulariser and

$$\Lambda(e_1)^{-1}\Lambda(e_1f) = \Lambda(e_1)^{-1}\Lambda(e_2)^{-1}\Lambda(e_2)\Lambda(e_1f) = \Lambda(e_2e_1)^{-1}\Lambda(e_2e_1f) .$$

The result follows by symmetry. □

We say the functional calculus is *proper* if

$$\{e \in \mathcal{E} : \Lambda(e) \text{ is injective}\}$$

is non-empty, which holds if and only if the function $\mathbf{1}$ is regularisable. In this case the space of regularisable functions $\mathcal{F}_r$ is a subalgebra of $\mathcal{F}$ containing $\mathcal{E}$. Thus the regularisation process extends the map Λ from $\mathcal{E}$ to $\mathcal{F}_r$. Define $\mathcal{F}_b$ to be the space of all regularisable functions f for which $\Lambda(f) \in B(X)$. The following lemma collects some fundamental properties of the abstract functional calculus (see [34, Chapter 1]).

Lemma 3.1.3. *Let $(\mathcal{E}, \mathcal{F}, \Lambda)$ be a proper abstract functional calculus over a Banach space X, $f \in \mathcal{F}_r$. Then the following properties hold.*

(i) *If $T \in B(X)$ commutes with $\Lambda(e)$ for every $e \in \mathcal{E}$ then T commutes with $\Lambda(f)$.*

(ii) $\Lambda(\mathbf{1}) = I$.

(iii) *If $g \in \mathcal{F}_r$ then*

$$\Lambda(f) + \Lambda(g) \subseteq \Lambda(f+g) \quad \textit{and} \quad \Lambda(f)\Lambda(g) \subseteq \Lambda(fg) \ .$$

Furthermore, $D(\Lambda(f)\Lambda(g)) = D(\Lambda(fg)) \cap D(\Lambda(g))$, and equality holds in the above inclusions if $g \in \mathcal{F}_b$.

(iv) *The map $\Lambda : \mathcal{F}_b \to B(X)$ is a homomorphism of algebras.*

(v) *If $g \in \mathcal{F}_b$ is such that $\Lambda(g)$ is injective then $\Lambda(f) = \Lambda(g)^{-1}\Lambda(f)\Lambda(g)$.*

(vi) *If $g \in \mathcal{F}$ is such that $fg = \mathbf{1}$, then $g \in \mathcal{F}_r$ if and only if $\Lambda(f)$ is injective, and then $\Lambda(g) = \Lambda(f)^{-1}$.*

(vii) *Let F be a subspace of $D(\Lambda(f))$. Suppose there exists a sequence $(e_n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $\Lambda(e_n) \to I$ strongly as $n \to \infty$ and $R(\Lambda(e_n)) \subseteq F$ for all $n \in \mathbb{N}$. Then F is a core for $\Lambda(f)$.*

This preliminary discussion on the abstract functional calculus is sufficient for our purposes. Let us now turn our attention to concrete families of operators. In each case, the location of the spectrum is known and the resolvent is assumed to satisfy certain growth estimates outside of this region. The primary functional calculus is defined via contour integrals surrounding the spectrum, on a class of scalar functions which are holomorphic on some open set containing the spectrum and satisfy appropriate decay estimates to allow the integrals to converge absolutely. These constructions are essentially the same and the approach can be unified by Haase's notion of a meromorphic functional calculus. Given an open subset Ω of $\mathbb{C}$, define $\mathcal{H}(\Omega)$ and $\mathcal{M}(\Omega)$ to be the algebras of holomorphic and meromorphic functions on Ω, respectively. Furthermore $\mathcal{H}^\infty(\Omega)$ will denote the space of bounded, holomorphic functions on Ω. Suppose that $\mathcal{E}(\Omega)$ is a subalgebra of $\mathcal{M}(\Omega)$ and $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Lambda)$ is an abstract functional calculus over the Banach space X such that the following hold:

1. The function $\mathbf{z} : \Omega \to \mathbb{C}$, $z \to z$, is regularisable, so that $A := \Lambda(\mathbf{z})$ is well-defined.
2. An operator $T \in B(X)$ which commutes with A also commutes with $\Lambda(e)$ for every $e \in \mathcal{E}$.

We call such a triple a *meromorphic functional calculus* (MFC) for A. Define $f(A) := \Lambda(f)$ for every $f \in \mathcal{M}(\Omega)_r$. Then the properties of Lemma 3.1.3 can be translated as follows.

Lemma 3.1.4. *Let $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Lambda)$ be a meromorphic functional calculus for the closed operator A acting a Banach space X and let $f \in \mathcal{M}(\Omega)_r$. Then the following properties hold.*

(i) *If $T \in B(X)$ commutes with A then it also commutes with $f(A)$. Moreover, if $f(A) \in B(X)$ then $f(A)$ commutes with A.*

(ii) $\mathbf{1}(A) = I$ *and* $\mathbf{z}(A) = A$.

(iii) *If $g \in \mathcal{M}(\Omega)_r$ then*

$$f(A) + g(A) \subseteq (f+g)(A) \quad \textit{and} \quad (fg)(A) \subseteq f(A)g(A)\ .$$

Furthermore, $D(f(A)g(A)) = D((fg)(A)) \cap D(g(A))$, and equality holds in the above inclusions if $g \in \mathcal{M}(\Omega)_b$.

(iv) *The map $f \to f(A) : \mathcal{M}(\Omega)_b \to B(X)$ is a homomorphism of algebras.*

(v) *If $g \in \mathcal{M}(\Omega)_b$ is such that $g(A)$ is injective then $f(A) = g(A)^{-1}f(A)g(A)$.*

(vi) For $\lambda \in \mathbb{C}$,

$$(\lambda - f)^{-1} \in \mathcal{M}(\Omega)_r \Leftrightarrow \lambda - f(A) \textit{ is injective} .$$

In this case, $(\lambda - f)^{-1}(A) = (\lambda - f(A))^{-1}$. *In particular,* $\lambda \in \rho(f(A))$ *if and only if* $(\lambda - f)^{-1} \in \mathcal{M}(\Omega)_b$.

3.2 Examples of meromorphic functional calculus

In this section we develop the functional calculus for the classes of operators introduced in Chapter 2. In each case the functional calculus is meromorphic. We begin with sectorial operators and follow the approach of [34, Chapter 2], to which we refer for more details. Let Φ be the function given by

$$\Phi(\lambda) = \frac{\lambda}{(1+\lambda)^2}, \qquad \lambda \neq -1. \tag{3.2.1}$$

Then for $\phi \in (0, \pi)$, $\mathcal{H}_0^\infty(\Sigma_\phi)$ denotes the subspace of $\mathcal{H}^\infty(\Sigma_\phi)$ of scalar functions f for which there exist $C, \epsilon > 0$ such that

$$|f(\lambda)| \leq C|\Phi(\lambda)|^\epsilon \quad \text{for all } \lambda \in \Sigma_\phi.$$

Consider a sectorial operator A and a function $f \in \mathcal{H}_0^\infty(\Sigma_\phi)$, where $\phi > \phi_A$. Then the primary functional calculus is given by

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} f(\lambda) R(\lambda, A) \, \mathrm{d}\lambda \ \in B(X), \tag{3.2.2}$$

where $\theta \in (\phi_A, \phi)$ and Γ_θ denotes the positively oriented contour

$$\Gamma_\theta = e^{i\theta}(\infty, 0] \cup e^{-i\theta}[0, \infty).$$

By Cauchy's Theorem, this definition does not depend on the choice of $\theta \in (\phi_A, \phi)$. In order to include the constant functions and resolvents, we consider the extended class

$$\mathcal{E}(\Sigma_\phi) = \mathcal{H}_0^\infty(\Sigma_\phi) \ \oplus \ \langle (1+z)^{-1} \rangle \ \oplus \ \langle \mathbf{1} \rangle \ .$$

Given a function $f = g + c(1+z)^{-1} + d$ in this class, where $g \in \mathcal{H}_0^\infty(\Sigma_\phi)$ and $c, d \in \mathbb{R}$, we define

$$f(A) = g(A) + c(I+A)^{-1} + dI \ .$$

The triple $(\mathcal{E}(\Sigma_\phi), \mathcal{M}(\Sigma_\phi), \Lambda)$ is an MFC for A [34, Section 2.3.2]. The function Φ lies in $\mathcal{H}_0^\infty(\Sigma_\phi)$ and $\Phi(A) = A(I+A)^{-2}$ is injective when A is injective. In particular,

Φ regularises every function in $\mathcal{H}^\infty(\Sigma_\phi)$.

Definition 3.2.1. A sectorial operator A is said to admit a *bounded* $H^\infty(\Sigma_\phi)$-*functional calculus* if $f(A)$ defines a bounded operator for every $f \in \mathcal{H}^\infty(\Sigma_\phi)$ and there exists a constant C such that for every such f,

$$\|f(A)\| \leq C\|f\|_{\mathcal{H}^\infty(\Sigma_\phi)} \ .$$

We define the H^∞-*angle* of A by

$$\phi_A^H = \inf\{\phi > \phi_A : \text{A has a bounded } H^\infty(\Sigma_\phi)\text{-functional calculus}\}.$$

The following result confirms an expected consequence of the functional calculus.

Proposition 3.2.2. *Define the function* f *by*

$$f(\lambda) = \frac{\lambda}{(\lambda - \mu_1)(\lambda - \mu_2)}, \qquad \lambda \in \Sigma_\phi,$$

where $\mu_1, \mu_2 \notin \overline{\Sigma_\phi}$. *Then* $f \in \mathcal{H}_0^\infty(\Sigma_\phi)$ *and*

$$f(A) = AR(\mu_1, A)R(\mu_2, A).$$

By means of the functional calculus, we can also define fractional powers of a sectorial operator. For every $\alpha \in \mathbb{C}$, the function $\mathbf{z}^\alpha : z \to z^\alpha$ lies in the subalgebra

$$\mathcal{B}(\Sigma_\phi) = \{f \in \mathcal{H}(\Sigma_\phi) : \ \exists k \in \mathbb{N} \text{ such that } f\Phi^k \in \mathcal{H}_0^\infty(\Sigma_\phi)\}$$

of $\mathcal{M}(\Sigma_\phi)_r$. Thus when A is injective we can define

$$A^\alpha = \mathbf{z}^\alpha(A), \quad \alpha \in \mathbb{C}\ .$$

Definition 3.2.3. A sectorial operator A on a Banach space X is said to have *bounded imaginary powers*, *BIP*, if $D(A) \cap R(A)$ is dense in X and $A^{is} \in B(X)$ for each $s \in \mathbb{R}$.

It follows that the family $(A^{is})_{s\in\mathbb{R}}$ forms a C_0-group [34, Corollary 3.5.7]. We define the *power angle* θ_A of A to be the type of this group, i.e

$$\theta_A = \inf\{w \in \mathbb{R} : \exists\ M_w \text{ such that } \|A^{is}\| \leq M_w e^{w|s|}\ \forall s \in \mathbb{R}\}.$$

Suppose A has dense domain and range. It is clear that A has *BIP* whenever it admits a bounded $\mathcal{H}^\infty$-calculus. Moreover we have

$$\phi_A^H \geq \theta_A \geq \phi_A.$$

The proof of the second inequality is somewhat involved and we refer to [64] for the details. It has been shown [16, Theorem 5.4] that, remarkably, we actually have $\theta_A = \phi_A^H$ whenever A admits a bounded $\mathcal{H}^\infty$-calculus. Furthermore, it was also shown in [16] that BIP implies H^∞-sectoriality on Hilbert space, but that this is not true in general Banach space. The authors found an operator A on $L^p(\mathbb{R})$, $p \neq 2$, which has BIP with $\theta_A = \phi_A = 0$ but which does not have a bounded $H^\infty(\Sigma_\phi)$-calculus for any angle ϕ. We refer to [8] for examples of operators on Hilbert space that do not have BIP (and therefore do not have a bounded $H^\infty(\Sigma_\phi)$-calculus for any $\phi \in (0, \pi)$).

For an invertible, sectorial operator, Haase [34, Section 2.5] has refined the primary functional calculus by considering functions which only have the appropriate decay at ∞, with the contours avoiding the origin. This extension is consistent with the scheme described above in the sense that whenever f is regularisable in the usual functional calculus, it is also regularisable in the functional calculus for invertible sectorial operators and the operator $f(A)$ is the same in both cases. In fact, it is also possible to consider functions which are not even defined in a neighbourhood of the origin.

Let us now turn our attention to strip-type operators. The functional calculus can be constructed in an analogous framework to sectorial operators and we refer to [34, Chapter 3] for full details. For the primary functional calculus, consider for $w > 0$ the space

$$\mathcal{H}_1^\infty(H_w) = \{f \in \mathcal{H}(H_w) : \ f(z) = O(|\operatorname{Re} z|^{-\alpha}) \ \ (|z| \to \infty) \text{ for some } \alpha > 1\}.$$

Given $B \in \operatorname{Strip}(w)$, $v > w$ and $f \in \mathcal{H}_1^\infty(H_v)$, we define

$$f(B) = \frac{1}{2\pi i} \int_{\Gamma_{w'}} f(\lambda) R(\lambda, A) \, \mathrm{d}\lambda,$$

where $w' \in (w, v)$ and $\Gamma_{w'} = (iw' + \mathbb{R}) \cup (-iw' + \mathbb{R})$, positively oriented. Again, the definition does not depend on the choice of $w' \in (w, v)$. We have the following (see [34, Proposition 4.2.1, Lemma 4.2.2]).

Proposition 3.2.4. *(i) The mapping $f \mapsto f(B) : \mathcal{H}_1^\infty(H_v) \to B(X)$ is a homomorphism of algebras.*

(ii) *For* $\lambda, \mu \notin \overline{H_v}$, $((\lambda - z)(\mu - z))^{-1}(B) = R(\lambda, B)R(\mu, B)$.

(iii) $(f(z)(\lambda - z)^{-1})(B) = R(\lambda, B)f(B) = f(B)R(\lambda, B)$ *for* $\lambda \notin \overline{H_v}$.

(iv) *If* C *is a closed operator commuting with resolvents of* B *then* C *also commutes with* $f(B)$.

(v) *The triple* $(\mathcal{H}_1^\infty(H_w), \mathcal{M}(H_w), \Lambda)$ *is a meromorphic functional calculus for* B.

Given $B \in \mathrm{Strip}(w)$, $v > w$ and $\mu > v$, define

$$\psi(z) = \frac{1}{\mu^2 + z^2}\ .$$

Then $\psi \in \mathcal{H}_1^\infty(H_v)$ and $\psi(B) = R(i\mu, B)R(-i\mu, B)$ is injective. Thus, ψ regularises every function in $\mathcal{H}^\infty(H_v)$. We say that B admits a bounded $\mathcal{H}^\infty$-calculus on H_v if $f(B) \in B(X)$ for every $f \in \mathcal{H}^\infty(H_v)$ and there exists a constant C such that for every such f,

$$\|f(B)\| \leq C\|f\|_{\mathcal{H}^\infty(H_v)}.$$

We will say B admits a bounded $\mathcal{H}^\infty$-calculus of type v if it admits a bounded $\mathcal{H}^\infty$-calculus on $H_{v'}$ for each $v' > v$. The infimum of all such v is called the $\mathcal{H}^\infty$-*type* of B and will be denoted w_B^H. On Hilbert space, it turns out that strip-type operators with a bounded $\mathcal{H}^\infty$-calculus are precisely the generators of C_0-groups (see [34, Theorem 7.2.12]).

Theorem 3.2.5. *Let* $A \in \mathrm{Strip}(w)$ *be densely defined on a Hilbert space* X. *Then the following are equivalent.*

(i) A *admits a bounded* $\mathcal{H}^\infty$*-calculus of type* w.

(ii) iA *generates a* C_0*-group.*

(iii) iA *generates a* C_0*-semigroup.*

(iv) *For each* $|v| > w$ *there exists* $C = C(A, v)$ *such that for every* $x \in X$ *and* $y \in X^*$,

$$\int_{\mathbb{R}} \|R(t + iv, A)x\|^2 \,\mathrm{d}t \leq C\|x\|^2 \quad \textit{and}$$
$$\int_{\mathbb{R}} \|R(t + iv, A^*)y\|^2 \,\mathrm{d}t \leq C\|y\|^2\ .$$

In fact, the result is given when A^* is the Hilbert adjoint of A, but an identical proof works when A^* is the Banach adjoint. We omit the proof, other than to note that the implication (*iii*) ⇒ (*iv*) is a combination of Lemma 2.1.8 and Plancherel's Theorem.

Finally in this section we develop a functional calculus for (right) half-plane operators, following the work of Haase [35]. The construction is completely analogous to that for strip-type operators. Indeed, the functional calculus for a strip-type operator B is an extension of the (left or right) half-plane calculus for iB outlined below. Let B be a (right) half-plane operator and for fixed $w < s_0(B)$ consider the class

$$\mathcal{H}_1^\infty(R_w) = \{f \in \mathcal{H}(R_w) : \ f(z) = O(|z|^{-\alpha}) \ \ (|z| \to \infty) \text{ for some } \alpha > 1\} \ .$$

For such an f we define

$$f(B) = \frac{1}{2\pi i} \int_{\Gamma_\delta} f(\lambda) R(\lambda, B) \, \mathrm{d}\lambda,$$

where $\delta \in (w, s_0(B))$ and $\Gamma_\delta = \delta + i\mathbb{R}$, oriented from top to bottom. As usual, the definition does not depend on the choice of $\delta \in (w, s_0(B))$. As expected, we have the following (see [35, Proposition 2.1]).

Proposition 3.2.6. *(i) The mapping* $f \mapsto f(B) : \mathcal{H}_1^\infty(R_w) \to B(X)$ *is a homomorphism of algebras.*

(ii) For $\operatorname{Re}\lambda, \operatorname{Re}\mu < w$, $((\lambda - z)(\mu - z))^{-1}(B) = R(\lambda, B)R(\mu, B)$.

(iii) $(f(z)(\lambda - z)^{-1})(B) = R(\lambda, B)f(B) = f(B)R(\lambda, B)$ *for* $\operatorname{Re}\lambda < w$.

(iv) If C *is a closed operator commuting with resolvents of* B *then* C *also commutes with* $f(B)$.

(v) The triple $(\mathcal{H}_1^\infty(R_w), \mathcal{M}(R_w), \Lambda)$ *is a meromorphic functional calculus for* B.

Every $\mathcal{H}^\infty(R_w)$-function is regularisable by $e(z) = \frac{1}{(\lambda - z)^2}$, where $\lambda \in L_w$ is fixed. We say B admits a bounded $\mathcal{H}^\infty$-calculus on R_w if $f(B) \in B(X)$ for all $f \in \mathcal{H}^\infty(R_w)$ and there exists C such that

$$\|f(B)\| \leq C \|f\|_{\mathcal{H}^\infty(R_w)}.$$

Staying with convention, we will say B admits a bounded $\mathcal{H}^\infty$-calculus of type w if it admits a bounded $\mathcal{H}^\infty$-calculus on $R_{w'}$ for each $w' < w$.

The following criterion for a half-plane operator to be a C_0-semigroup generator will be needed later on. We refer to [35, Proposition 2.4] for the proof.

Proposition 3.2.7. *Let B be a half-plane operator with $s_0(B) \geq 0$. Then B is the generator of a C_0-semigroup T if and only if B is densely defined and $e^{tB} := e^{tz}(B)$ is bounded for every $t \geq 0$ with $\sup_{t\in[0,1]} \|e^{tB}\| < \infty$. In this case, $T(t) = e^{tB}$, $t \geq 0$.*

Remark 3.2.8. We conclude this section with a note on adjoints. If A is a densely defined sectorial operator (respectively strip-type, half-plane) then its Banach adjoint A^* is a well-defined sectorial operator (respectively strip-type, half-plane). If A has dense range then A^* is injective. Whenever $f(A)$ is defined via the primary functional calculus of A, we have $f(A^*) = f(A)^*$ since, for a suitable contour Γ, $f(A) = \int_\Gamma f(\lambda)R(\lambda, A)\,\mathrm{d}\lambda$.

3.3 Convergence Lemma and approximation

A useful tool in establishing the boundedness of the $\mathcal{H}^\infty$-calculus of an operator A (sectorial, strip-type or half-plane) is to approximate $\mathcal{H}^\infty$-functions by functions f_n in a smaller class for which $f_n(A)$ is better understood. This result, known as the Convergence Lemma, holds for the three classes of operators under consideration, under the appropriate assumptions. Furthermore, Haase [35] has employed the Convergence Lemma to obtain simpler proofs of well known theorems such as the Hille-Yosida theorem and Trotter-Kato theorem.

A key ingredient of the proof is the following classical result due to Vitali (see [4, Theorem A.5]).

Proposition 3.3.1. *Let $\Omega \subseteq \mathbb{C}$ be open and connected and let $(f_\alpha)_\alpha$ be a locally bounded net of holomorphic functions on Ω such that the set $\{z \in \Omega : (f_\alpha(z))_\alpha \text{ converges}\}$ has a limit point in Ω. Then (f_α) converges to a holomorphic function uniformly on compact subsets of Ω.*

An immediate consequence of Proposition 3.3.1 and the Dominated Convergence Theorem is the following.

Lemma 3.3.2. *Let $A \in \mathrm{Sect}(\theta)$, $\phi \in (\theta, \pi)$. Let $(f_\alpha)_\alpha \subseteq \mathcal{H}_0^\infty(\Sigma_\phi)$ be a net of functions converging pointwise to a function f. Suppose there exist $C, s > 0$ such that for every α*

$$|f_\alpha(z)| \leq C \min\left\{|z|^s, |z|^{-s}\right\} \quad (z \in \Sigma_\phi).$$

Then $f \in \mathcal{H}_0^\infty(\Sigma_\phi)$ and $\|f_\alpha(A) - f(A)\| \to 0$.

The Convergence Lemma for sectorial operators, which can be found in [34, Proposition 5.1.4] can now be stated as follows.

Proposition 3.3.3. *Let $A \in \mathrm{Sect}(\theta)$, $\phi \in (\theta, \pi)$, $(f_\alpha)_\alpha \subseteq \mathcal{H}^\infty(\Sigma_\phi)$. Suppose that $\sup_\alpha \|f_\alpha\|_{\mathcal{H}^\infty(\Sigma_\phi)} < \infty$ and that $f_\alpha(z)$ converges pointwise on Σ_ϕ to some function f. Suppose further that $f_\alpha(A)$ and $f(A)$ are defined by the natural functional calculus for sectorial operators. Then*

$$f_\alpha(A)x \to f(A)x$$

for all $x \in D(A) \cap R(A)$. Moreover:

(i) If A is injective and $f_\alpha(A) \in B(X)$ for each α and $f_\alpha(A) \to T \in B(X)$ strongly, then $T = f(A)$.

(ii) If A is densely defined with dense range and $\sup_\alpha \|f_\alpha(A)\| < \infty$ then $f(A) \in B(X)$ and $f_\alpha(A) \to f(A)$ strongly.

Proof. By Vitali's theorem we have $f \in \mathcal{H}^\infty(\Sigma_\phi)$. Define $\psi(z) = z(1+z)^{-2}$ and $g = f\psi$, $g_\alpha = f_\alpha\psi$. Then by Lemma 3.3.2 we have

$$f_\alpha(A)\psi(A) = g_\alpha(A) \to g(A) = f(A)\psi(A)$$

in operator norm. Thus $f_\alpha(A)x \to f(A)x$ for all $x \in R(\psi(A)) = D(A) \cap R(A)$.

To prove (i), fix $x \in X$ and let $y_\alpha = g_\alpha(A)x \in D((\psi(A))^{-1})$. Then $y_\alpha \to g(A)x$. However, $(\psi(A))^{-1}y_\alpha = f_\alpha(A)x \to Tx$ by assumption. By the closedness of $(\psi(A))^{-1}$, we have $y = g(A)x \in D(\psi(A)^{-1})$ with $(\psi(A))^{-1}y = Tx$. Thus $x \in D(f(A))$ and $f(A)x = Tx$.

Assertion (ii) follows immediately. □

The statement and proof of the Convergence Lemma for strip-type and half-plane operators is analogous, but now the restriction on the operator B is only that B, and hence B^2 by Proposition 2.3.7, is densely defined. We shall require the result in the following forms (see [34, Proposition 5.1.7], [35, Proposition 3.1]).

Proposition 3.3.4. *(***Convergence Lemma for strip-type operators***) Let $B \in \mathrm{Strip}(w)$ be densely defined, $v > w$. Let (f_α) be a net of functions in $\mathcal{H}^\infty(H_v)$ satisfying the following conditions.*

(i) $\sup_\alpha \|f_\alpha\|_{\mathcal{H}^\infty(H_v)} < \infty$,

(ii) $f_\alpha(A) \in B(X)$ *for each* α *and* $\sup_\alpha \|f_\alpha(A)\| < \infty$,

(iii) $f(z) := \lim_\alpha f_\alpha(z)$ *exists for each* $z \in H_v$.

Then $f \in \mathcal{H}^\infty(H_v)$*,* $f(A) \in B(X)$*,* $f_\alpha(A) \to f(A)$ *strongly and* $\|f(A)\| \leq \limsup \|f_\alpha(A)\|$.

Proposition 3.3.5. *(***Convergence Lemma for half − plane operators***) Let* B *be a densely defined (right) half-plane operator of type* w*,* $v < w$*. Let* (f_α) *be a net of functions in* $\mathcal{H}^\infty(R_v)$ *satisfying the following conditions.*

(i) $\sup_\alpha \|f_\alpha\|_{\mathcal{H}^\infty(R_v)} < \infty$*,*

(ii) $f_\alpha(A) \in B(X)$ *for each* α *and* $\sup_\alpha \|f_\alpha(A)\| < \infty$*,*

(iii) $f(z) := \lim_\alpha f_\alpha(z)$ *exists for each* $z \in R_v$*.*

Then $f \in \mathcal{H}^\infty(R_v)$*,* $f(A) \in B(X)$*,* $f_\alpha(A) \to f(A)$ *strongly and* $\|f(A)\| \leq \limsup \|f_\alpha(A)\|$.

The following is an easy consequence of Proposition 3.3.3 combined with the Closed Graph Theorem. Analogous results hold for strip-type and half-plane operators.

Proposition 3.3.6. *Let* A *be a densely defined sectorial operator with dense range,* $\phi > \phi_A$*. Then* $f(A) \in B(X)$ *for every* $f \in \mathcal{H}^\infty(\Sigma_\phi)$ *if and only if* A *admits a bounded* $\mathcal{H}^\infty$*-calculus on* Σ_ϕ*. In this case the map* $f \mapsto f(A)$ *is a (bounded) algebra homomorphism* $\mathcal{H}^\infty(\Sigma_\phi) \to B(X)$*.*

We can also employ the Convergence Lemma to reduce the bounded $\mathcal{H}^\infty$-calculus condition to an estimate on a smaller class of scalar functions. For a given $A \in \mathrm{Sect}(\theta)$, $\phi \in (\theta, \pi)$ and subalgebra $\mathcal{F} \subseteq \mathcal{H}^\infty(\Sigma_\phi)$, we say the (natural) $\mathcal{F}$-calculus for A is bounded if $f(A) \in B(X)$ for every $f \in \mathcal{F}$ and there exists C such that for every $f \in \mathcal{F}$,

$$\|f(A)\| \leq C\|f\|_{\mathcal{H}^\infty(\Sigma_\phi)}.$$

For $\phi \in (0, \pi)$ we define the classes

$$\mathcal{R}^\infty(\Sigma_\phi) = \{\frac{p}{q} :\ p, q \in \mathbb{C}[z],\ \text{the zeroes of } q \text{ lie outside } \overline{\Sigma_\phi},\ \deg p \leq \deg q\}$$

and

$$\mathcal{R}_0^\infty(\Sigma_\phi) = \{\frac{p}{q} :\ p, q \in \mathbb{C}[z],\ \text{the zeroes of } q \text{ lie outside } \overline{\Sigma_\phi},\ \deg p < \deg q\}\ .$$

Let $\mathbf{C_0}(\overline{\Sigma_\phi})$ denote the space of continuous functions on $\overline{\Sigma_\phi}$ vanishing at infinity. The following result is taken from [34, Proposition 5.3.4].

Proposition 3.3.7. *Let* $A \in \mathrm{Sect}(\theta)$*,* $\phi \in (\theta, \pi)$ *and suppose* A *has dense domain and range. The following are equivalent.*

(i) The natural $\mathcal{H}_0^\infty(\Sigma_\phi)$-calculus for A is bounded.

(ii) The natural $\mathcal{H}^\infty(\Sigma_\phi)$-calculus for A is bounded.

(iii) The natural $\mathcal{H}^\infty(\Sigma_\phi) \cap \mathbf{C_0}(\overline{\Sigma_\phi})$-calculus for A is bounded.

(iv) The natural $\mathcal{R}_0^\infty(\Sigma_\phi)$-calculus for A is bounded.

(v) The natural $\mathcal{R}^\infty(\Sigma_\phi)$-calculus for A is bounded.

In each case, the underlying constant C is the same.

Proof. The implications $(ii) \Rightarrow (i)$ and $(ii) \Rightarrow (v) \Rightarrow (iv)$ are immediate. Assuming (iv), we first observe that $\mathcal{R}_0^\infty(\Sigma_\phi)$ is uniformly dense in $\mathcal{H}^\infty(\Sigma_\phi) \cap \mathbf{C_0}(\overline{\Sigma_\phi})$ (see [34, Proposition F.3]). Thus we arrive at (iii) by the Convergence Lemma. Note that only injectivity of A is needed here.

Now assume (iii) holds and for $f \in \mathcal{H}_0^\infty(\Sigma_\phi)$, define the approximant $f_n(z) = f(z + \frac{1}{n})$. Then $f_n \in \mathcal{H}^\infty(\Sigma_\phi) \cap \mathbf{C_0}(\overline{\Sigma_\phi})$ and $f_n \to f$ pointwise on Σ_ϕ. The Convergence Lemma now yields (i).

It remains to prove $(i) \Rightarrow (ii)$. Assuming (i) holds, fix $f \in \mathcal{H}^\infty(\Sigma_\phi)$ and define $\psi_s(z) = \frac{z^s}{(1+z)^{2s}}$, for $s > 0$. Then $\psi_s, f\psi_s \in \mathcal{H}_0^\infty(\Sigma_\phi)$ and

$$\|f\psi_s\|_{\mathcal{H}^\infty(\Sigma_\phi)} \leq K^s \|f\|_{\mathcal{H}^\infty(\Sigma_\phi)},$$

where $K = \|\psi_1\|_{\mathcal{H}^\infty(\Sigma_\phi)}$. Thus $\|(f\psi_s)(A)\| \leq C\|f\|_{\mathcal{H}^\infty(\Sigma_\phi)} K^s$. The Convergence Lemma is again employed to deduce that $f(A) \in B(X)$ and $(f\psi_s)(A) \to f(A)$ strongly as $s \to 0$. This yields $\|f(A)\| \leq C\|f\|_{\mathcal{H}^\infty(\Sigma_\phi)}$, as required. $\square$

Using a similar argument, we can establish suitable versions of $(i) \Leftrightarrow (ii)$ for strip-type and half-plane operators, with the class $\mathcal{H}_0^\infty$ replaced by the class of scalar functions on which the primary functional calculus for the operator is defined.

By a shifting argument, we can formulate a version of Proposition 3.3.7 for half-plane operators which includes estimates on rational functions. This will be discussed in more detail when it is needed in Chapter 5.

3.4 The logarithm of a sectorial operator

So far our study of strip-type operators has been from the viewpoint of C_0-group generators. However, there is another way in which they arise, which allows one to reconcile this class of operators with the more familiar family of sectorial operators. Starting with an injective sectorial operator A, note that the function $\log z$ has subpolynomial growth at 0 and ∞. In particular, it can be regularised by $e(z) = \frac{z}{(1+z)^2}$ and so we define

$$\log A := (\log z)(A).$$

The logarithm of an injective sectorial operator was first introduced by Nollau [56], in which he proved the following fundamental identity. An adapted proof can be found in [34, Lemma 3.5.1].

Lemma 3.4.1. (**Nollau representation**) *Let $A \in \operatorname{Sect}(w)$ be injective. If $|\operatorname{Im}\lambda| > \pi$ then $\lambda \in \rho(\log A)$ and*

$$R(\lambda, \log A) = \int_0^\infty \frac{-1}{(\lambda - \log t)^2 + \pi^2}\,(t + A)^{-1}\,\mathrm{d}t. \tag{3.4.1}$$

Hence $\|R(\lambda, \log A)\| \le \pi\, C_A\,(|\operatorname{Im}\lambda| - \pi)^{-1}$.

A simple scaling argument [34, Proposition 3.5.2] confirms that the logarithm of an injective sectorial operator yields an operator of (strong) strip-type. By a suitable version of the composition rule (see, for example, [34, Theorem 4.2.4]) we have $\mathrm{e}^{\log A} = A$. The converse does not hold in general, as the following example illustrates.

Example 3.4.2. Consider the operator $B = -i\frac{\mathrm{d}}{\mathrm{d}t}$ acting on $X = L^1(\mathbb{R})$, with domain $W^{1,1}(\mathbb{R})$. Then B is a strong-strip type operator of type zero since iB generates a strongly continuous group of isometries. However, e^B has empty resolvent set and so is not sectorial (see [34, Corollary 8.4.6]).

The *inversion problem* deals with seeking conditions on a strip-type operator B to ensure that e^B is sectorial. Results due to Monniaux [54] and Clark [12] offer partial results in this direction. It turns out that boundedness of the $\mathcal{H}^\infty$-calculus is sufficient. Indeed, a simple argument involving the composition rule again delivers the following two way correspondence between sectorial and strip-type operators.

Proposition 3.4.3. *Let $A \in \operatorname{Sect}(w)$ be injective, $B = \log A$, $\theta \in (w, \pi]$. If A admits a bounded $\mathcal{H}^\infty$-calculus on Σ_θ, then B admits a bounded $\mathcal{H}^\infty(H_\theta)$-calculus. Conversely, if $B \in \operatorname{Strip}(w)$ admits a bounded $\mathcal{H}^\infty$-calculus on H_θ then $A = e^B$ is sectorial and admits a bounded $\mathcal{H}^\infty$-calculus on Σ_θ. In each case, for every $f \in \mathcal{H}^\infty(H_\theta)$ we have $f(B) = (f \circ \log)(A)$.*

We will usually work with sectorial operators which have dense domain and range. Moreover, strip-type operators will usually be densely defined. The following result shows that we can pass between the two via logarithms and exponentials.

Proposition 3.4.4. *If A is a sectorial operator with dense domain and range then* $\log A$*, which is well-defined since A is injective, is densely defined. Conversely if B is a densely defined strip-type operator and $B = \log A$ for some sectorial operator A, then A is densely defined with dense range.*

Proof. Let A be sectorial with dense domain and range. Then A^* is a well-defined injective sectorial operator (see Remark 3.2.8) and by (3.4.1) we have

$$R(\lambda, \log A^*) = R(\lambda, \log A)^* \ .$$

Thus if $\log A$ is not densely defined then there is a non-zero $x^* \in X^*$ such that $0 = x^* \circ R(\lambda, \log A) = R(\lambda, \log A^*)x^*$. But $R(\lambda, \log A^*)$ is injective which contradicts the fact that x^* is non-zero. Hence $\log A$ has dense domain.

To prove the converse, note that by (3.4.1), $D(\log A) \subseteq \overline{D(A)}$. Since $-\log A = \log A^{-1}$, we also have $D(\log A) \subseteq \overline{R(A)}$. Thus A has dense domain and range. □

3.5 m-bounded $\mathcal{H}^\infty$-calculus

In this section we introduce the notion of an m-bounded $\mathcal{H}^\infty$-calculus for strip-type operators, which requires estimates on derivatives of scalar functions only. This definition turns out to be equivalent to the usual notion of bounded $\mathcal{H}^\infty$-calculus, but provides a broader framework with which to view the condition. We can establish a corresponding notion for sectorial operators, which again is equivalent to boundedness of the $\mathcal{H}^\infty$-calculus. Although this may seem to make the notion trivial, there are two reasons why it is interesting. Firstly, it is m-boundedness that is naturally related to certain resolvent estimates (see Section 4.3). Secondly, the situation for half-plane operators is somewhat different and m-boundedness appears more naturally in the characterisations for these operators - we refer to Section 5.2 for more details.

In the sequel, for a scalar function f and integer $m \geq 0$, $f^{(m)}$ denotes the m^{th} derivative of f. Furthermore throughout this section we will assume that strip-type operators are densely defined and that sectorial operators have dense domain and range (see Remarks 2.2.2 and 2.3.3).

Definition 3.5.1. Let $A \in \mathrm{Strip}(w)$, $v > w$. Then A is said to admit an *m-bounded* $\mathcal{H}^\infty$-calculus on H_v if $f^{(m)}(A) \in B(X)$ for all $f \in \mathcal{H}^\infty(H_v)$ and there is a constant C such that

$$\|f^{(m)}(A)\| \leq C\|f\|_{\mathcal{H}^\infty(H_v)} \ . \tag{3.5.1}$$

We say A admits an m-bounded calculus of type v if it admits an m-bounded calculus on $H_{\tilde{v}}$ for every $\tilde{v} > v$.

It is an immediate consequence of Cauchy's Integral Formula that if $f \in \mathcal{H}^\infty(H_v)$ then $f^{(m)} \in \mathcal{H}^\infty(H_u)$ for any $w < u < v$ and $m \geq 0$. Thus, $f^{(m)}(A)$ exists as a closed, densely defined operator via the natural functional calculus for A. Moreover, if A admits an m-bounded $\mathcal{H}^\infty$-calculus then it admits an $(m+1)$-bounded $\mathcal{H}^\infty$-calculus on any larger strip. We have the following useful approximation lemma.

Lemma 3.5.2. *With A as above and any m, it is sufficient for A to admit an m-bounded $\mathcal{H}^\infty$-calculus that the estimate* (3.5.1) *holds for each $f \in \mathcal{H}_1^\infty(H_v)$.*

Proof. Suppose (3.5.1) holds for the class $\mathcal{H}_1^\infty(H_v)$ and let $f \in \mathcal{H}^\infty(H_v)$. Define $f_n = f\phi_n$, where $\phi_n(z) = (\frac{in}{in+z})^2$. Let $w < u < v$. Then for suitably large n and a constant C independent of n,

- $f_n \in \mathcal{H}_1^\infty(H_u)$
- $f_n^{(m)} \to f^{(m)}$ pointwise on H_u
- $\|f_n\|_{\mathcal{H}^\infty(H_u)} \leq C\|f\|_{\mathcal{H}^\infty(H_u)}$

By the Convergence Lemma for strip-type operators (Proposition 3.3.4), we deduce $f^{(m)}(A)$ is bounded with

$$\|f^{(m)}(A)\| \leq C\|f\|_{\mathcal{H}^\infty(H_u)} \leq C\|f\|_{\mathcal{H}^\infty(H_v)}.$$

□

Of course, for $m = 0$ we are reduced to the usual notion of bounded $\mathcal{H}^\infty$-calculus. In fact, the property is independent of m.

Proposition 3.5.3. *Let $A \in \mathrm{Strip}(w)$ and suppose A admits an m-bounded $\mathcal{H}^\infty$-calculus on H_v, $v > w$. Then A also admits a bounded $\mathcal{H}^\infty$-calculus on H_v.*

Proof. Fix $f \in \mathcal{H}^\infty(H_v)$ and for fixed $s \in \mathbb{R}$ define $g(z) = e^{izs}$ and observe that by the m-boundedness assumption, $g(A) \in B(X)$. Then $g \in \mathcal{H}^\infty(H_v)$ and we have

$$\begin{aligned}(fg)^{(m)} &= \sum_{j=0}^{m-1} \binom{m}{j} f^{(j)} g^{(m-j)} + f^{(m)} g \\ &= \sum_{j=0}^{m-1} (is)^{m-j} \binom{m}{j} f^{(j)} g + f^{(m)} g\ .\end{aligned}$$

By the m-boundedness assumption and Lemma 3.1.4 (iii), we have that $(fg)^{(m)}(A)$ and $(f^{(m)}g)(A)$ are bounded. Furthermore $g(A)$ is invertible with inverse e^{-isA}. Hence by Lemma 3.1.4 (iii) again, we deduce that for every $s \in \mathbb{R}$,

$$\big[\sum_{j=0}^{m-1} (is)^{m-j} \binom{m}{j} f^{(j)}\big](A) \in B(X)\ . \tag{1}$$

Replacing s with $2s$ gives

$$\big[\sum_{j=0}^{m-1} 2^{m-j} (is)^{m-j} \binom{m}{j} f^{(j)}\big](A) \in B(X)\ . \tag{2}$$

Combining (1) and (2) and using Lemma 3.1.4 again, we arrive at

$$\big[\sum_{j=0}^{m-2} (2^{m-j}-2)(is)^{m-j} \binom{m}{j} f^{(j)}\big](A) \in B(X)\ .$$

Continuing in this way a further $(m-2)$ times, we eventually obtain

$$\big[\prod_{j=1}^{m-1} (is)^{m}(2^m - 2^j) f\big](A) \in B(X).$$

Thus the operator $f(A)$ is bounded and the result is proven. $\square$

Note that we did not need the estimate (3.5.1) in this proof, but only the assumption that the operator $f^{(m)}(A)$ is bounded for $f \in \mathcal{H}^\infty(H_v)$. In particular, this shows that (3.5.1) is automatic once we know that the operators $f^{(m)}(A)$ are bounded.

We now seek to extend the notion of an m-bounded $\mathcal{H}^\infty$-calculus to sectorial operators, which in view of Proposition 3.5.3 should be equivalent to boundedness of the $\mathcal{H}^\infty$-calculus. In fact we can extract a natural definition of m-bounded $\mathcal{H}^\infty$-calculus for sectorial operators via the one already in place for strip-type operators.

Definition 3.5.4. Let $A \in \mathrm{Sect}(w)$, $\theta > w$. Then A is said to admit an m-bounded $\mathcal{H}^\infty$-calculus on Σ_θ if the operator $\log A$ admits an m-bounded $\mathcal{H}^\infty$-calculus on H_θ.

By Proposition 3.5.3 it follows that this definition is independent of m and therefore equivalent to boundedness of the $\mathcal{H}^\infty$-calculus for sectorial operators. Consider the case $m = 1$. Given a function $f \in \mathcal{H}^\infty(\Sigma_\theta)$, define $g \in \mathcal{H}^\infty(H_\theta)$ by $g(z) = f(e^z)$. Then $g'(z) = e^z f(e^z)$. Conversely, from $g \in \mathcal{H}^\infty(H_\theta)$ define $f \in \mathcal{H}^\infty(\Sigma_\theta)$ by $f(z) = g(\log z)$, then $zf'(z) = g'(\log z)$. Thus the following result is immediate.

Corollary 3.5.5. *Let $A \in \mathrm{Sect}(w)$, $\theta > w$. Then A admits a 1-bounded $\mathcal{H}^\infty$-calculus on Σ_θ if and only if there exists a constant C such that for each $f \in \mathcal{H}^\infty(\Sigma_\theta)$, $Af'(A) \in B(X)$ and*

$$\|Af'(A)\| \le C\|f\|_{\mathcal{H}^\infty(\Sigma_\theta)} \,. \tag{3.5.2}$$

We remark that since derivatives of $\mathcal{H}^\infty$-functions on sectors need not be bounded (consider for example, $f(z) = z^{is}$, $s \in \mathbb{R}$), it would have been naive to merely import the definition of m-boundedness for sectorial operators from the one for strip-type operators. This is reinforced by the next two results. By $\tilde{\mathcal{H}^\infty}(\Sigma_\theta)$ we denote the space of functions $f \in \mathcal{H}^\infty(\Sigma_\theta)$ for which $f' \in \mathcal{H}^\infty(\Sigma_\theta)$.

Proposition 3.5.6. *Let A be an invertible sectorial operator, $\theta > \phi_A$. Then for every $f \in \tilde{\mathcal{H}^\infty}(\Sigma_\theta)$, $f'(A)$ is bounded and there exists C such that for every such f,*

$$\|f'(A)\| \le C\|f\|_{\mathcal{H}^\infty(\Sigma_\theta)} \,.$$

Proof. We write for every $f \in \tilde{\mathcal{H}^\infty}(\Sigma_\theta)$ and $z \in \Sigma_\theta$,

$$f'(z) = \left(1 + \frac{1}{z}\right) g(z),$$

where $g(z) = \left(\frac{z}{z+1} f'(z)\right) \in \mathcal{H}^\infty_0(\Sigma_\theta)$. By the extended functional calculus for invertible sectorial operators, $\left(1 + \frac{1}{z}\right)(A) = 1 + A^{-1}$ is bounded and $f'(A) \in B(X)$ with

$$\begin{aligned}
\|f'(A)\| &\le \|(1 + A^{-1})\| \left\| \int_{\Gamma_{r,\theta}} \frac{\lambda}{\lambda+1} f'(\lambda) R(\lambda, A) \,\mathrm{d}\lambda \right\| \\
&\le C\|f\|_{\mathcal{H}^\infty(\Sigma_\theta)} \int_{\Gamma_{r,\theta}} \frac{1}{|\lambda+1|} \frac{|\mathrm{d}\lambda|}{|\lambda|} \\
&\le C\|f\|_{\mathcal{H}^\infty(\Sigma_\theta)},
\end{aligned}$$

where $\Gamma_{r,\theta}$ is the contour $(-\infty, r]e^{-i\theta} \cup re^{i[-\theta,\theta]} \cup [r,\infty)e^{i\theta}$ and r is sufficiently small.

□

Proposition 3.5.7. *Let A be a sectorial operator, $\theta \in (\phi_A, \pi)$, and suppose there exists a constant C such that for every $f \in \mathcal{H}^\infty(\Sigma_\theta)$, $f'(A) \in B(X)$ with*

$$\|f'(A)\| \le C\|f\|_{\mathcal{H}^\infty(\Sigma_\theta)} \,. \tag{3.5.3}$$

Then A is invertible.

Proof. For $\epsilon > 0$, define the function f_ϵ by $f_\epsilon(z) = (\epsilon + z)^{\frac{1}{2}}(1+z)^{-1}$. Then $f_\epsilon \in \mathcal{H}^\infty(\Sigma_\theta)$ and $f_\epsilon'(z) = \frac{1}{2}(\epsilon + z)^{-\frac{1}{2}}(1+z)^{-1} - (\epsilon+z)^{\frac{1}{2}}(1+z)^{-2}$. Thus by (3.5.3) there exists a constant C independent of $\epsilon \in (0,1)$ such that

$$\|\frac{1}{2}(\epsilon + A)^{-\frac{1}{2}}(1+A)^{-1} - (\epsilon + A)^{\frac{1}{2}}(1+A)^{-2}\| \leq C\|f_\epsilon\|_{\mathcal{H}^\infty(\Sigma_\theta)} \leq C \; . \tag{3.5.4}$$

Moreover, $(\epsilon + A)^{\frac{1}{2}}(1+A)^{-2}$ is a closed operator whose domain contains the domain of the bounded operator $A^{\frac{1}{2}}(1+A)^{-2}$. Thus, for $\epsilon \in (0,1)$ the operator $(\epsilon+A)^{\frac{1}{2}}(1+A)^{-2}$ is bounded. By [4, Proposition 3.8.2] these operators converge strongly to $A^{\frac{1}{2}}(1+A)^{-2}$ as $\epsilon \to 0$, thus $\sup_{\epsilon\in(0,1)} \|(\epsilon + A)^{\frac{1}{2}}(1+A)^{-2}\| < \infty$. Combining this with (3.5.4), the operators $T_\epsilon := (\epsilon + A)^{-\frac{1}{2}}(1+A)^{-1}$ are bounded with norm uniformly bounded in $\epsilon \in (0,1)$. Let $c_\epsilon = \frac{1}{1-\epsilon}$, then

$$\left[(\epsilon + A)^{-\frac{1}{2}}(1+A)^{-1}\right]^2 = \frac{c_\epsilon}{\epsilon + A} - \frac{{c_\epsilon}^2}{(1+A)^2} - \frac{c_\epsilon}{1+A} \; .$$

Thus it follows that

$$C := \sup_{\epsilon \in (0, \frac{1}{2}} \|(\epsilon + A)^{-1}\| < \infty \; .$$

The invertibility of A now follows from the identity $I = \epsilon(\epsilon + A)^{-1} + A(\epsilon + A)^{-1}$ and the density of the range of A, i.e. we have for every $x \in D(A)$,

$$\|x\| \leq C(\epsilon\|x\| + \|Ax\|)$$

and so, letting $\epsilon \to 0$,

$$\|x\| \leq C\|Ax\|$$

for every $x \in D(A)$. □

3.6 Operator-valued and joint functional calculus

There are two main ways in which the functional calculus can be extended, which are both well established. The first is to consider operator-valued functions. There are many approaches to this; the simplest is to extend the construction in the scalar-valued situation described already. Our discussion will be largely confined to sectorial operators, but the notions have natural extensions to strip-type and half-plane operators.

Given a sectorial operator A we define the commutant of A in $B(X)$ by

$$E_A = \{T \in B(X) : \; TR(\lambda, A) = R(\lambda, A)T \;\; \forall \lambda \in \rho(A)\}.$$

This is clearly a closed subalgebra of $B(X)$. Next, for $\phi > \phi_A$, we define $\mathcal{H}^\infty(\Sigma_\phi; E_A)$ to be the space of all bounded holomorphic functions $F : \Sigma_\phi \to E_A$ and the corresponding subspace $\mathcal{H}_0^\infty(\Sigma_\phi; E_A)$ of all functions F for which there exist $C, \epsilon > 0$ such that

$$\|F(\lambda)\|_{B(X)} \leq C|\Phi(\lambda)|^\epsilon \qquad \text{for all } \lambda \in \Sigma_\phi,$$

where Φ is as defined in (3.2.1). Analogously to the scalar case, for $F \in \mathcal{H}_0^\infty(\Sigma_\phi; E_A)$ we define $F(A) \in B(X)$ by

$$F(A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} F(\lambda) R(\lambda, A) \, \mathrm{d}\lambda,$$

where $\theta \in (\phi_A, \phi)$. Again our aim is to extend this definition to functions in $\mathcal{H}^\infty(\Sigma_\phi; E_A)$. To do so, note that we can identify $\mathcal{H}^\infty(\Sigma_\phi)$ with a subspace of $\mathcal{H}_0^\infty(\Sigma_\phi; E_A)$ via the map

$$f \to F, \qquad F(\lambda) = f(\lambda)I.$$

Then certainly $\Phi \in \mathcal{H}_0^\infty(\Sigma_\phi; E_A)$ with $\Phi(A) = A(I + A)^{-2}$. Moreover, this operator is injective when A is injective and it has range $D(A) \cap R(A)$. Hence we regularise as before, i.e. for $F \in \mathcal{H}_0^\infty(\Sigma_\phi; E_A)$ we define the operator $F(A)$ by

$$F(A) = \Phi(A)^{-1}(F\Phi)(A),$$

with natural domain

$$D(F(A)) = \{x \in X : (F\Phi)(A)x \in D(A) \cap R(A)\}.$$

Definition 3.6.1. A is said to admit a bounded $\mathcal{H}^\infty(\Sigma_\phi; E_A)$-functional calculus if for every $F \in \mathcal{H}^\infty(\Sigma_\phi; E_A)$ the operator $F(A)$ is bounded and there exists a constant C such that for every such F,

$$\|F(A)\| \leq C \ \sup\{\|F(\lambda)\| : \ \lambda \in \Sigma_\phi\} \ .$$

The other extension of the functional calculus is to a finite family of sectorial operators which are pairwise resolvent commutative. We follow the approach introduced in [48] as it is analogous to the functional calculus we have developed already for a single operator. Moreover since our interest lies with pairs of operators (A, B), we will consider the case $n = 2$ only. The following considerations have natural generalisations to the case $n > 2$, which can be found in [44].

Let A and B be two sectorial operators with spectral angles ϕ_A and ϕ_B respectively. For $\theta, \theta' \in (0, \pi)$ we denote by $\mathcal{H}^\infty(\Sigma_\theta \times \Sigma_{\theta'})$ the Banach algebra of all bounded holomorphic scalar-valued functions on $\Sigma_\theta \times \Sigma_{\theta'}$ and define Ψ by

$$\Psi(z, w) = \Phi(z)\Phi(w) = \frac{zw}{(1+z)^2(1+w)^2}.$$

Then we define $\mathcal{H}_0^\infty(\Sigma_\theta \times \Sigma_{\theta'})$ to be the subspace of functions f for which there exist $C, \epsilon > 0$ such that

$$|f(\lambda_1, \lambda_2)| \leq C|\Psi(\lambda_1, \lambda_2)|^\epsilon \qquad \text{for all } (\lambda_1, \lambda_2) \in \Sigma_\theta \times \Sigma_{\theta'}.$$

Then for $\phi > \phi_A, \phi' > \phi_B$ and $F \in \mathcal{H}_0^\infty(\Sigma_\phi \times \Sigma_{\phi'})$ we define

$$F(A, B) = -\frac{1}{4\pi^2} \int_{\Gamma_\theta \times \Gamma'_\theta} F(\lambda, \lambda') R(\lambda, A) R(\lambda', B) \, \mathrm{d}\lambda \, \mathrm{d}\lambda',$$

with $(\theta, \theta') \in (\phi_A, \phi) \times (\phi_B, \phi')$. This integral converges in $B(X)$ and is independent of the choice of (θ, θ'). To define $F(A, B)$ for general $F \in \mathcal{H}^\infty(\Sigma_\phi \times \Sigma_{\phi'})$ we proceed as in the one-variable case. The operator $\Psi(A, B) = A(A+I)^{-2}B(B+I)^{-2}$ is injective when A and B are injective. Moreover, for $F \in \mathcal{H}^\infty(\Sigma_\phi \times \Sigma_{\phi'})$, the function $F\Psi$ belongs to $\mathcal{H}_0^\infty(\Sigma_\theta \times \Sigma_{\theta'})$. Thus Ψ regularises any such F so that we may define

$$F(A, B) = \Psi(A, B)^{-1}(F\Psi)(A, B).$$

We say that (A, B) admits a *bounded* $\mathcal{H}^\infty(\Sigma_\phi \times \Sigma_{\phi'})$*-joint functional calculus* if $F(A, B)$ is a bounded operator for every $F \in \mathcal{H}^\infty(\Sigma_\phi \times \Sigma_{\phi'})$. Analogously to the single operator case (see Propositions 3.3.6 and 3.3.7), we have the following.

Proposition 3.6.2. *Let A and B be sectorial operators and let ϕ, ϕ' be such that $\phi > \phi_A$ and $\phi' > \phi_B$. Then the following assertions are equivalent.*

1. *(A, B) admits a bounded $\mathcal{H}^\infty(\Sigma_\phi \times \Sigma_{\phi'})$-joint functional calculus;*
2. *there exists a constant $C > 0$ such that for all $F \in \mathcal{H}_0^\infty(\Sigma_\phi \times \Sigma_{\phi'})$, $\|F(A, B)\| \leq C\|F\|_{\mathcal{H}^\infty(\Sigma_\phi \times \Sigma_{\phi'})}$;*
3. *there exists a constant $C > 0$ such that for all $F \in \mathcal{H}^\infty(\Sigma_\phi \times \Sigma_{\phi'})$, $F(A, B) \in B(X)$ and $\|F(A, B)\| \leq C\|F\|_{\mathcal{H}^\infty(\Sigma_\phi \times \Sigma_{\phi'})}$.*

In this case the map $F \to F(A, B)$ is a (bounded) algebra homomorphism $\mathcal{H}^\infty(\Sigma_\phi \times \Sigma_{\phi'}) \to B(X)$.

If f is regularisable in the functional calculus for the single operator A, then we can identify it with a function regularisable in the joint functional calculus for the pair of operators (A, B). More precisely, on appropriate sectors define $F(z, w) = f(z)$. If $e(z)$ is a regulariser for f, it is easy to show that $E(z, w) = e(z)\Phi(w)$ is a regulariser for F. Moreover, $F(A, B) = f(A)$. In particular, this shows that if (A, B) admits a bounded $\mathcal{H}^\infty$-joint functional calculus on $\Sigma_\phi \times \Sigma_{\phi'}$, then A and B admit bounded $\mathcal{H}^\infty$-calculi on Σ_ϕ, $\Sigma_{\phi'}$ respectively. The converse does not hold in general.

The triple $(\mathcal{H}_0^\infty(\Sigma_\phi \times \Sigma_{\phi'}), \mathcal{M}(\Sigma_\phi \times \Sigma_{\phi'}), \Lambda)$, where Λ denotes the map $f \mapsto f(A, B)$, is an abstract functional calculus. Using Proposition 3.6.2, we can obtain the following version of Lemma 3.1.4.

Lemma 3.6.3. *Let (A, B) be a pair of commuting sectorial operators, $\phi \in (\phi_A, \pi), \phi' \in (\phi_B, \pi)$. Let $f \in \mathcal{M}(\Sigma_\phi \times \Sigma_{\phi'})_r$. Then the following properties hold.*

(i) If $T \in B(X)$ commutes with A and B (in the sense of resolvents) then it also commutes with $f(A, B)$. Moreover, if $f(A, B) \in B(X)$ then $f(A, B)$ commutes with A and B.

(ii) For $\lambda \notin \overline{\Sigma_\phi}, \mu \notin \overline{\Sigma_{\phi'}}$, the functions

$$\mathbf{1} : (z, w) \to 1, \quad \mathbf{z} : (z, w) \to z, \quad \mathbf{w} : (z, w) \to w$$
$$f_\lambda : (z, w) \to (\lambda - z)^{-1}, \quad g_\mu : (z, w) \to (\mu - w)^{-1}$$

are all regularisable with

$$\mathbf{1}(A, B) = I, \quad \mathbf{z}(A, B) = A, \quad \mathbf{w}(A, B) = B,$$
$$f_\lambda(A, B) = R(\lambda, A), \quad g_\mu(A, B) = R(\mu, B).$$

(iii) If $g \in \mathcal{M}(\Sigma_\phi \times \Sigma_{\phi'})_r$ then

$$f(A, B) + g(A, B) \subseteq (f + g)(A, B) \quad \text{and} \quad (fg)(A, B) \subseteq f(A, B)g(A, B)\ .$$

Furthermore, $D(f(A, B)g(A, B) = D((fg)(A, B)) \cap D(g(A, B)))$, and equality holds in the above inclusions if $g(A, B)$ is bounded.

(iv) The map $f \to f(A, B) : \mathcal{M}(\Sigma_\phi \times \Sigma_{\phi'})_b \to B(X)$ is a homomorphism of algebras.

(v) If $g \in \mathcal{M}(\Sigma_\phi \times \Sigma_{\phi'})_b$ is such that $g(A, B)$ is injective then

$$f(A, B) = g(A, B)^{-1} f(A, B) g(A, B) \,.$$

(vi) For $\lambda \in \mathbb{C}$,

$$(\lambda - f)^{-1} \in \mathcal{M}(\Sigma_\phi \times \Sigma_{\phi'})_r \Leftrightarrow \lambda - f(A, B) \textit{ is injective} \,.$$

In this case, $(\lambda - f)^{-1}(A, B) = (\lambda - f(A, B))^{-1}$. In particular, $\lambda \in \rho(f(A, B))$ if and only if $(\lambda - f)^{-1} \in \mathcal{M}(\Sigma_\phi \times \Sigma_{\phi'})_b$.

In a similar manner we can define the joint functional calculus for a pair of commuting strip-type operators. The following definition and result are then a natural extension of the single operator case.

Definition 3.6.4. Let A, B be a pair of densely defined commuting strip-type operators, $k, l \in \mathbb{N}$. (A, B) is said to admit a (k, l)-bounded $\mathcal{H}^\infty$-calculus on $H_v \times H_w$, $v > w_A^{st}$, $w > w_B^{st}$, if for every $f \in \mathcal{H}^\infty(H_v \times H_w)$, $[\frac{\partial^{k+l}}{\partial z^k \partial w^l} f](A, B) \in B(X)$.

Proposition 3.6.5. *Let A, B be as above and suppose (A, B) admits a (k, l)-bounded $\mathcal{H}^\infty$-calculus on $H_v \times H_w$. Then (A, B) admits a bounded joint functional calculus on $H_v \times H_w$.*

Proof. Given $f \in \mathcal{H}^\infty(H_v \times H_w)$ and $s \in \mathbb{R}$, define $g \in \mathcal{H}^\infty(H_v \times H_w)$ by $g(z, w) = e^{izs} e^{isw} f(z, w)$. Then

$$\frac{\partial^{k+l}}{\partial z^k \partial w^l} g = e^{izs} e^{isw} \left[\sum_{r=0}^{l} \sum_{j=0}^{k} (is)^{l-r+k-j} l \binom{l}{r} \binom{k}{j} \frac{\partial^{j+r}}{\partial z^j \partial w^r} f \right] .$$

By our assumption, for $s \neq 0$ $[e^{isz} e^{isw}](A, B) = \left[(is)^{-k+l} \frac{\partial^{k+l}}{\partial z^k \partial w^l} e^{isz} e^{isw}\right](A, B) \in B(X)$ and so we have

$$\left[\sum_{r=0}^{l} \sum_{j=0}^{k} (is)^{l-r+k-j} \binom{l}{r} \binom{k}{j} \frac{\partial^{j+r}}{\partial z^j \partial w^r} f(A, B) \right] \in B(X) \,. \tag{1}$$

Replacing s by $2s$ gives

$$\left[\sum_{p=0}^{k+l} \sum_{r+j=p} 2^{l+k-p} (is)^{l+k-p} \binom{l}{r} \binom{k}{j} \frac{\partial^{j+r}}{\partial z^j \partial w^r} f(A, B) \right] \in B(X) \,. \tag{2}$$

Combining (1) and (2) gives

$$\Big[\sum_{p=0}^{k+l-2}\sum_{r+j=p}(2^{l+k-p}-2)(is)^{l+k-p}\binom{l}{r}\binom{k}{j}\frac{\partial^{j+r}}{\partial z^j\partial w^r}f(A,B)\Big]\in B(X)\ .$$

Continuing in this way we eventually arrive at $f(A,B)\in B(X)$, as in the proof of Proposition 3.5.3 .

□

Analogously to the single operator case (see Definition 3.5.4), we say that a commuting pair (A,B) of sectorial operators has a joint (k,l)-bounded joint functional calculus if $(\log A,\log B)$ does so. By Proposition 3.6.5 this is equivalent to (A,B) having a bounded joint functional calculus. For example, a $(1,1)$-bounded joint functional calculus corresponds to the condition that for every $f\in\mathcal{H}^\infty$,

$$AB\frac{\partial^2}{\partial z\partial w}f(A,B)\in B(X)\ .$$

3.7 R-boundedness

As mentioned in the introduction, the property of R-boundedness is important in the theory of operator sums. In this section we provide a brief overview of the wide reaching and well established theory behind this concept. We refer to [47, Chapter 2] and [20], from which much of the material in this section is taken, for more details.

We denote by $(r_j)_{j=1}^\infty$ a Rademacher sequence on $[0,1]$, that is, a sequence of independent random variables on $[0,1]$ such that $\mathbb{P}(r_j=1)=\mathbb{P}(r_j=-1)=1/2$ for all $j\geq 1$ (where $\mathbb{P}$ denotes the Lebesgue measure on $\mathbb{R}$). For example, we can define

$$r_j(t)=\operatorname{sgn}\sin(2^j\pi t)\qquad\text{for all } j\geq 1 \text{ and } t\in[0,1].$$

For a finite subset $\{x_1,\dots,x_n\}$ of X we set

$$\begin{aligned}\Big\|\sum_{j=1}^n r_jx_j\Big\|_{Rad_p(X)} &= \Big(\mathbb{E}\Big\|\sum_{j=1}^n r_jx_j\Big\|^p\Big)^{1/p}\\ &= \Big(\int_0^1\Big\|\sum_{j=1}^n r_j(t)x_j\Big\|_X^p\,\mathrm{d}t\Big)^{1/p},\end{aligned}$$

where $p\in[1,\infty)$.

Definition 3.7.1. A subset $\tau\subseteq B(X)$ is said to be *R-bounded* if there exists a constant C such that

$$\Big\|\sum_{j=1}^{n} r_j T_j x_j\Big\|_{Rad_2(X)} \le C\Big\|\sum_{j=1}^{n} r_j x_j\Big\|_{Rad_2(X)} \tag{3.7.1}$$

for all $T_1, \dots, T_n \in \tau$ and $x_1, \dots, x_n \in X$. The *R-bound* of τ is the smallest constant C for which (3.7.1) holds, denoted by $\mathcal{R}(\tau)$.

The following well known result tells us that we can replace the $Rad_2(X)$ norm in Definition 3.7.1 by the $Rad_p(X)$ norm for any $p \in [1, \infty)$. We refer to [47, Chapter 2] for a proof.

Theorem 3.7.2. *(Kahane's Inequality): Let $p \in [1, \infty)$. Then there exists a constant C_p such that*

$$\frac{1}{C_p}\Big\|\sum_{j=1}^{n} r_j x_j\Big\|_{Rad_2(X)} \le \Big\|\sum_{j=1}^{n} r_j x_j\Big\|_{Rad_p(X)} \le C_p\Big\|\sum_{j=1}^{n} r_j x_j\Big\|_{Rad_2(X)}$$

for all $x_1, \dots, x_n \in X$.

We mention another famous inequality for the case $X = \mathbb{C}$ which enables us to estimate L^p-norms of sums of Rademacher functions (see [47, Chapter 2]).

Theorem 3.7.3. *(Khinchine's Inequality): Let $p \in [1, \infty)$. Then there exists a constant C_p such that*

$$\frac{1}{C_p}\Big(\sum_{j=1}^{n} |a_j|^2\Big)^{1/2} \le \Big\|\sum_{j=1}^{n} a_j r_j\Big\|_{Rad_p(\mathbb{C})} \le C_p\Big(\sum_{j=1}^{n} |a_j|^2\Big)^{1/2}$$

for all $a_1, \dots, a_n \in \mathbb{C}$.

Here we gather some important properties of R-boundedness. For the proofs and more results on R-boundedness we refer to [47, Chapter 2] and [20].

Proposition 3.7.4. *Let σ and τ be R-bounded sets.*

(i) The sets

$$\sigma + \tau = \{S + T : \, S \in \sigma, T \in \tau\} \quad \textit{and} \qquad \sigma \circ \tau = \{ST : S \in \sigma, T \in \tau\}$$

are R-bounded with $\mathcal{R}(\sigma + \tau) \le \mathcal{R}(\sigma) + \mathcal{R}(\tau)$ and $\mathcal{R}(\sigma \circ \tau) \le \mathcal{R}(\sigma)\mathcal{R}(\tau)$.

(ii) The sets $\sigma \cup \tau$ and $\sigma \cup \{0\}$ are R-bounded with $\mathcal{R}(\sigma \cup \tau) \le \mathcal{R}(\sigma) + \mathcal{R}(\tau)$ and $\mathcal{R}(\sigma \cup \{0\}) = \mathcal{R}(\sigma)$.

(iii) *If* $(\sigma_i)_{i\in\mathbb{N}}$ *is a sequence of R-bounded sets with* $\sigma_i \subseteq \sigma_{i+1}$ *and* $\sup_i \mathcal{R}(\sigma_i) < \infty$ *then* $\bigcup_i \sigma_i$ *is R-bounded with* $\mathcal{R}(\bigcup_i \sigma_i) \leq \sup_i \mathcal{R}(\sigma_i)$.

(iv) *The convex hull, absolute convex hull* $\operatorname{absco}(\sigma) = \{\sum_{k=1}^n \lambda_k S_k : \ n \in \mathbb{N}, S_k \in \sigma, \lambda_k \in \mathbb{C}, \sum_{k=1}^n |\lambda_k| = 1\}$ *and their closures in the strong operator topology are R-bounded with*

$$\mathcal{R}(\overline{\operatorname{co}(\sigma)}^s) \leq \mathcal{R}(\sigma) \quad \textit{and} \qquad \mathcal{R}(\overline{\operatorname{absco}(\sigma)}^s) \leq 2\mathcal{R}(\sigma)\ .$$

(v) *If* X *has non-trivial type (see Section 6.3 for a definition) then* $\sigma^* = \{S^* : \ S \in \sigma\}$ *is R-bounded with* $\mathcal{R}(\sigma^*) \leq \mathcal{R}(\sigma)$.

Using these basic properties, we can establish the following useful result which will be employed in the proof of Theorem 6.2.5.

Proposition 3.7.5. *Let* $(\sigma_i)_{i\in\mathbb{N}}$ *be a sequence of R-bounded sets such that* $\sum_i \mathcal{R}(\sigma_i) < \infty$. *Then* $\sum_i \sigma_i$ *is R-bounded with* $\mathcal{R}(\sum_i \sigma_i) \leq \sum_i \mathcal{R}(\sigma_i)$.

Proof. Define $\tau_n = \sum_{i=1}^n \{\sigma_i \cup \{0\}\}$, then $\tau_n \subseteq \tau_{n+1}$ for each n. By Proposition 3.7.4 $(i)-(iii)$ we have that $\bigcup_n \tau_n$ is R-bounded with R-bound at most $\sum_i \mathcal{R}(\sigma_i)$. Taking the strong closure of $\bigcup_n \tau_n$ and applying (iv) now yields the result, since this set contains $\sum_i \sigma_i$. ☐

We will also need the following important R-boundedness result for integrals [47, Corollary 2.14]:

Proposition 3.7.6. *Let* τ *be an R-bounded set. For every strongly measurable function* $N : \Omega \to \tau$ *on a* σ*-finite measure space* Ω *and every* $h \in L^1(\Omega, \mu)$, *define* $T_{N,h} \in B(X)$ *by*

$$T_{N,h}x = \int_\Omega h(w)N(w)x \, \mathrm{d}\mu(w), \qquad x \in X\ .$$

Then $\sigma = \{T_{N,h} : \ \|h\|_{L^1} \leq 1, N \textit{ as above}\}$ *is R-bounded with* $\mathcal{R}(\sigma) \leq 2\mathcal{R}(\tau)$.

Recall that a key component of the definition of a sectorial operator A is the condition that the set

$$\{t(t+A)^{-1} : t > 0\}$$

is uniformly bounded. With our updated catalogue of definitions we can modify this notion.

Definition 3.7.7. A sectorial operator A on a Banach space X is said to be *R-sectorial* if the set $\{t(t+A)^{-1} : t > 0\}$ is R-bounded. In this case we define ϕ_A^R by

$$\phi_A^R = \inf\{\theta \in (\phi_A, \pi) : \mathcal{R}(\{\lambda(\lambda + A)^{-1} : |\arg\lambda| \leq \pi - \theta\}) < \infty\}.$$

We can also reconcile these notions to strengthen the definition of a bounded H^∞-calculus.

Definition 3.7.8. A sectorial operator A on a Banach space X is said to admit a bounded RH^∞-calculus if there exists $\sigma \in (0, \pi)$ such that the set $\{f(A) : \|f\|_{H^\infty(\Sigma_\sigma)} \leq 1\}$ is R-bounded. In this case we define ϕ_A^{RH} to be the infimum of all such σ.

It is known (see, for example, [47, Chapter 2]) that on Hilbert space, R-boundedness is actually equivalent to uniform boundedness, so that these notions are no different to those already considered. However, this is not true on general Banach space, as the following example illustrates.

Example 3.7.9. Let $X = L^p(\mathbb{R})$, with $p \in [1, 2)$. For $n = 0, 1, 2, \ldots$, define T_n to be the right-shift operator given by

$$T_n f(t) = f(t - n)$$

for all $f \in X, t \in \mathbb{R}$. Then certainly $T_n \in B(X)$ for all n with $\|T_n\| = 1$. So the set $\tau = \{T_n : n = 0, 1, 2, \ldots, \}$ is uniformly bounded. We claim the set τ fails to be R-bounded. To see this, consider the characteristic functions defined by $f_n = \chi_{[0,1]}$ for each n. Then for each n we have

$$\Big\|\Big(\sum_{j=0}^{n-1} |T_j f_j|^2\Big)^{1/2}\Big\|_X = \|\mathcal{X}_{[0,n]}\|_X = n^{1/p}$$

and

$$\Big\|\Big(\sum_{j=1}^{n-1} |f_j|^2\Big)^{1/2}\Big\|_X = \|n^{1/2}\mathcal{X}_{[0,1]}\|_X = n^{1/2}.$$

Clearly then there is no constant C such that (3.7.1) holds and so τ is not R-bounded.

We can similarly use R-boundedness to define R-strip-type and R-half-plane operators, with extensions to the functional calculus as described above. We omit the details.

4 Characterisations of the bounded $\mathcal{H}^\infty$-calculus for sectorial and strip-type operators

A powerful tool in the theory of the bounded $\mathcal{H}^\infty$-calculus is to reduce the property to an estimate involving a more specific family of operators, such as resolvent powers. In the Hilbert space setting, McIntosh [53] obtained a characterisation for sectorial operators in terms of these so-called square function (or *quadratic*) estimates. This condition, which we call *absolute calculus*, is too strong in general and so needs to be weakened in general Banach space. This gives rise to Rademacher square function estimates, which were shown by Kalton and Weis [44] to characterise the bounded $\mathcal{H}^\infty$-calculus in general Banach space. These estimates also appear in a characterisation from [47, Chapter 12], where the argument incorporates an equivalance of bounded $\mathcal{H}^\infty$-calculus with weak integral estimates, which is actually a special case of an earlier, more general characterisation due to Cowling et al [16].

In principle, one can then deduce the theory for strip-type operators by taking exponentials. In general, however, the exponential of a strip-type operator is not easily identified and so it is more useful to develop a parallel theory on strips. Our aim in this chapter, therefore, is to obtain an analogous but independent characterisation for strip-type operators. As a starter, we consider what the appropriate square function estimates are when working on strips, rather than sectors. In Section 4.1 we introduce a notion of absolute calculus for strip-type operators in a manner that corresponds in a reasonable way with the definition for sectorial operators. By this we mean that if A is a sectorial operator with absolute calculus then $\log A$ has absolute calculus as a strip-type operator and, conversely, if B is a strip-type operator with absolute calculus then e^B is sectorial with absolute calculus. By exploiting this connection, we will show that the absolute calculus is equivalent to boundedness of the $\mathcal{H}^\infty$-calculus on Hilbert space.

Moving to the more general Banach space setting, in Section 4.2 we explore the known characterisations of the bounded $\mathcal{H}^\infty$-calculus in terms of Rademacher square function estimates. First, we summarise the results of [44], [47] and [16] for sectorial operators. Moving on to strips, we present a modification of this theory by introducing Rademacher square function estimates for strip-type operators. With the help of [76] this allows for a characterisisation of the bounded $\mathcal{H}^\infty$-calculus for strip-type operators in terms of these estimates, by passing through weak integral estimates in a manner similar to [16]. The class of functions allowed in the arguments from [76] is restrictive; in particular, they do not allow for estimates on a single resolvent power. Employing the notion of an m-bounded $\mathcal{H}^\infty$-calculus from Section 3.5, and its equivalence to the boundedness of the $\mathcal{H}^\infty$-calculus for sectorial and strip-type operators, in Section 4.3 we will refine the arguments of [16] and [76] to broaden the class of functions involved in these characterisations. This will allow us to incorporate weak integral estimates and, in turn, Rademacher estimates, in our characterisation for strip-type operators.

Finally in Section 4.4 we extend these considerations to a pair of (commuting) operators, by first obtaining analogous square function characterisations and then refining these conditions via an adapted form of the m-bounded calculus for two operators.

Throughout this chapter, sectorial operators will be assumed to have dense domain and range and strip-type operators will be assumed to have dense domain (see Remark 2.2.2, Remark 2.3.3 and Proposition 3.4.4).

4.1 Absolute calculus

The notion of an absolute calculus has received much coverage in the literature, though in different guises. Here we collect some important results that highlight the relative strength of this property over the usual notion of bounded $\mathcal{H}^\infty$-calculus. For a sectorial operator A on a Banach space X and scalar function $\psi \in \mathcal{H}_0^\infty(\Sigma_\phi)$, $\phi > \phi_A$, we define the *quadratic norm* associated to ψ by

$$\|x\|_{A,\psi} = \left\{ \int_0^\infty \|\psi(tA)x\|^2 \, \frac{\mathrm{d}t}{t} \right\}^{\frac{1}{2}},$$

which indeed defines a norm on the space $X_{A,\psi}$ of all elements x for which this integral is finite.

Definition 4.1.1. A is said to have *absolute calculus* (on Σ_θ) if for some $\psi \in \mathcal{H}_0^\infty(\Sigma_\theta)$

$$X_{A,\psi} = X \ \text{ and } \ \|x\| \sim \|x\|_{A,\psi}. \tag{4.1.1}$$

In fact, the space $X_{A,\psi}$ is independent of the choice of $\psi \in \mathcal{H}_0^\infty(\Sigma_\phi)$ and $\theta \in (\phi_A, \pi)$, due to the following result from [1, Section 3].

Proposition 4.1.2. *Given (non-zero) functions $\phi \in \mathcal{H}_0^\infty(\Sigma_{\theta_1})$ and $\psi \in \mathcal{H}_0^\infty(\Sigma_{\theta_2})$, with $\theta_1, \theta_2 > \phi_A$, there exists a constant C such that for every $f \in \mathcal{H}^\infty(\Sigma_{\theta_3})$, $\theta_3 \in (\phi_A, \pi)$ and every $x \in X_{A,\phi} \cap D(f(A))$, we have $f(A)x \in X_{A,\psi}$ and*

$$\|f(A)x\|_{A,\psi} \leq c\|f\|_{\mathcal{H}^\infty} \|x\|_{A,\phi} \ .$$

The choice $f = 1$ then yields the equivalence of the quadratic norms on the space $X_{A,\psi}$, which we can denote X_A without ambiguity. Thus, when speaking of absolute calculus, it is understood that $X = X_A$ and that all of these norms are equivalent to the original norm on X. Furthermore, the minimal angle of absolute calculus is always ϕ_A. The proof of Proposition 4.1.2 relies upon the following useful auxiliary lemma from [1]. Here, ψ_t is the function defined by $\psi_t(z) = \psi(tz)$.

Lemma 4.1.3. *Let A, ψ, ϕ and θ be as above. Then there exists a constant C such that*

(i) $\|(f\psi_t)(A)\| \leq C\ \|f\|_{\mathcal{H}^\infty(\Sigma_\theta)}$ *for all* $f \in \mathcal{H}^\infty(\Sigma_\theta), t > 0$ *and*

(ii) $$\left\{ \int_0^\infty \left\| \int_\alpha^\beta \psi_\tau(A)\phi_t(A)g(\tau)\ \frac{\mathrm{d}\tau}{\tau} \right\|^2 \frac{\mathrm{d}t}{t} \right\}^{\frac{1}{2}} \leq C \left\{ \int_\alpha^\beta \|g(\tau)\|^2\ \frac{\mathrm{d}\tau}{\tau} \right\}^{\frac{1}{2}}$$

for all $0 < \alpha < \beta < \infty$ *and continuous functions* $g : [\alpha, \beta] \to X$.

We omit the proof here, as the core ingredients will be present in the analogous result for strip-type operators (see Lemma 4.1.10). Lemma 4.1.3 can also be employed to describe the absolute calculus property in terms of a dual condition.

Proposition 4.1.4. *A admits an absolute calculus (on Σ_θ) if and only if for some/all non-zero $\psi \in \mathcal{H}_0^\infty(\Sigma_\theta)$ there exists a constant C such that for every $x \in X$,*

$$\int_0^\infty \|\psi(tA)x\|^2\ \frac{\mathrm{d}t}{t} \leq C\ \|x\|^2 \tag{4.1.2}$$

and for every $y \in X^*$,

$$\int_0^\infty \|\psi(tA^*)y\|^2\ \frac{\mathrm{d}t}{t} \leq C\ \|y\|^2\ . \tag{4.1.3}$$

In this case we say that the pairs $\{A, \psi\}$ and $\{A^*, \psi\}$ satisfy *quadratic estimates*. Note that, by the Closed Graph Theorem, (4.1.2) and (4.1.3) are equivalent to having, for every $x \in X$ and $y \in X^*$,

$$\int_0^\infty \|\psi(tA)x\|^2\ \frac{\mathrm{d}t}{t} < \infty \quad \text{and}$$
$$\int_0^\infty \|\psi(tA^*)y\|^2\ \frac{\mathrm{d}t}{t} < \infty\ .$$

We will now verify that absolute calculus is indeed a strengthening of the usual notion of bounded $\mathcal{H}^\infty$-calculus. To this end, let us first note that our definition of absolute calculus implies the following one given by Kucherenko in [46], which we shall refer to as K-absolute calculus for clarity. Here $\psi_\delta(z) = \frac{z^\delta}{(1+z)^{2\delta}}$.

Definition 4.1.5. Let A be sectorial, $\theta \in (\phi_A, \pi)$. Then A is said to have K-absolute calculus on Σ_θ if there exist $g \in \mathcal{H}_0^\infty(\Sigma_\theta)$ and $C, \delta > 0$ such that

$$\|\psi_\delta(tA)g(tA)x\| \leq \|g(tA)y\| \quad (0 < t < \infty) \Rightarrow \|x\| \leq C\,\|y\|.$$

Proposition 4.1.6. *Let A be a sectorial operator with absolute calculus. Then A has K-absolute calculus on Σ_θ for every $\theta > \phi_A$.*

Proof. Assuming that A has absolute calculus, we can choose (any) $g \in \mathcal{H}_0^\infty(\Sigma_\theta)$ for which $\|x\| \sim \|x\|_{A,g}$. Then whenever $\|\psi_\delta(tA)g(tA)x\| \leq \|g(tA)y\|$, it follows that

$$\begin{aligned}\|x\| &\leq C\Big\{\int_0^\infty \|(\psi_\delta g)(tA)x\|^2 \,\frac{\mathrm{d}t}{t}\Big\}^{\frac{1}{2}}\\ &\leq C\Big\{\int_0^\infty \|g(tA)y\|^2 \,\frac{\mathrm{d}t}{t}\Big\}^{\frac{1}{2}}\\ &\leq C\|y\|,\end{aligned}$$

where C depends only on g and ψ_δ.

□

We can now employ a result of Kucherenko [46, Lemma 4.6.2] to show that bounded $\mathcal{H}^\infty$-calculus follows from absolute calculus.

Corollary 4.1.7. *Suppose A has absolute calculus on Σ_ϕ. Then A also admits a bounded $\mathcal{H}^\infty$-calculus on Σ_ϕ.*

Once again, we postpone the proof to our discussion of strip-type operators (see Corollary 4.1.15). An immediate consequence is that $\phi_A^H = \phi_A$ when A has absolute calculus. Kalton [41] found, for each $\theta \in (0, \pi)$, examples of sectorial operators for which $\phi_A = 0$ and $\phi_A^H = \theta$. An earlier example was presented in [16, Example 5.5], although the operator there does not have dense range. Thus these operators do not have absolute calculus. One can extract further examples of operators without absolute calculus from known results on sums in connection with maximal regularity (see Remark 6.1.7).

As mentioned, on Hilbert space the two notions are equivalent. The following was proven by McIntosh [53] (see also [34, Theorem 7.3.1]).

Theorem 4.1.8. *Let A be a sectorial operator on a Hilbert space X. Then the following are equivalent.*

(i) A admits a bounded $\mathcal{H}^\infty$-calculus (on some sector Σ_θ).

(ii) A has bounded imaginary powers.

(iii) A has absolute calculus.

In this case, $\phi_A^H = \phi_A$.

Our immediate aim is to develop a notion of absolute calculus for strip-type operators in an analogous framework. To reconcile the two notions for sectorial operators and strip-type operators, we would hope to have the following correspondence with Proposition 3.4.3:

- If A is a sectorial operator with absolute calculus, then $\log A$ has absolute calculus as a strip-type operator.
- If B is a strip-type operator with absolute calculus then e^B is sectorial and has absolute calculus.

One may hope to adapt the notion of absolute calculus for sectorial operators to the strip-type case by modifying the harmonic analysis of the boundary of the region. With this and Proposition 4.1.4 in mind, the following seems a reasonable starting point.

Definition 4.1.9. Let $A \in \mathrm{Strip}(w)$, $v > w$. Then A is said to have absolute calculus (on H_v) if for some non-zero $\psi \in \mathcal{H}_1^\infty(H_v)$ there exists a constant C such that the following estimates hold:

$$\forall x \in X \int_{\mathbb{R}} \|\psi(t-A)x\|^2 \, \mathrm{d}t \le C \, \|x\|^2, \tag{4.1.4}$$

$$\forall y \in X^* \int_{\mathbb{R}} \|\psi(t-A^*)y\|^2 \, \mathrm{d}t \le C \, \|y\|^2 \, . \tag{4.1.5}$$

We note that, similarly to the sectorial case, for (4.1.4) and (4.1.5) to hold it is sufficient to have that the integrals are finite for each $x \in X, y \in X^*$. Again, we say A has absolute calculus of type v if it has absolute calculus on $H_{\tilde{v}}$ for every $\tilde{v} > v$. Our first goal is to establish that the conditions in Definition 4.1.9 are independent of the choice of ϕ in $\mathcal{H}_1^\infty(H_v)$. We proceed as in the sectorial case, beginning with the following auxiliary lemma. The proof is in the same vein as that of Lemma 4.1.3.

Lemma 4.1.10. *Let A, v be as above and fix ψ and ϕ in $\mathcal{H}_1^\infty(H_v)$. For a scalar function χ, denote $\chi_t(z) = \chi(t-z)$. Then for some constant C we have* *(i)* $\|(f\psi_t)(A)\| \leq C\,\|f\|_{\mathcal{H}^\infty(H_v)}$ *for all* $f \in \mathcal{H}^\infty(H_v), t \in \mathbb{R}$*, and*

(ii) $\left\{\int_{\mathbb{R}} \left\| \int_\alpha^\beta \psi_\tau(A)\phi_t(A)g(\tau)\,\mathrm{d}\tau \right\|^2 \mathrm{d}t \right\}^{\frac{1}{2}} \leq C \left\{ \int_\alpha^\beta \|g(\tau)\|^2\,\mathrm{d}\tau \right\}^{\frac{1}{2}}$

for all $-\infty < \alpha < \beta < \infty$ *and continuous functions* $g : [\alpha, \beta] \to X$*.*

Proof. The desired inequality in (i) can be obtained by a direct estimate of $\|(f\psi_t)(A)\|$. To this end, let Γ be the boundary of H_u, $u \in (w, v)$. Since $f\psi_t \in \mathcal{H}_1^\infty(H_v)$,

$$(f\psi_t)(A) = \frac{1}{2\pi i}\int_\Gamma f(z)\psi(t-z)R(z,A)\,\mathrm{d}z\,.$$

Thus we have, for some constant C and $s > 1$,

$$\begin{aligned}\|(f\psi_t)(A)\| &\leq C\int_\Gamma \frac{|f(z)|}{1+|\operatorname{Re} z - t|^s}\,|\mathrm{d}z| \\ &\leq C\|f\|_{\mathcal{H}^\infty(H_v)}\int_{\mathbb{R}}\frac{1}{1+|t|^s}\,\mathrm{d}t \\ &\leq C\|f\|_{\mathcal{H}^\infty(H_v)}.\end{aligned}$$

To show (ii), note

$$\begin{aligned}&\int_{\mathbb{R}}\left\|\int_\alpha^\beta \psi_\tau(A)\phi_t(A)g(\tau)\,\mathrm{d}\tau\right\|^2\mathrm{d}t \\ &\quad\leq \int_{\mathbb{R}}\left\{\int_\alpha^\beta \Big\|\psi_\tau(A)\phi_t(A)\Big\|^{\frac{1}{2}}\,\Big\|\psi_\tau(A)\phi_t(A)\Big\|^{\frac{1}{2}}\,\Big\|g(\tau)\Big\|\,\mathrm{d}\tau\right\}^2\mathrm{d}t \\ &\quad\leq \int_{\mathbb{R}}\left\{\int_\alpha^\beta \Big\|\psi_\tau(A)\phi_t(A)\Big\|\,\mathrm{d}\tau\int_\alpha^\beta\Big\|\psi_\tau(A)\phi_t(A)\Big\|\,\Big\|g(\tau)\Big\|^2\,\mathrm{d}\tau\right\}\mathrm{d}t \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(Cauchy-Schwarz)} \\ &\quad\leq \sup_t\left\{\int_\alpha^\beta\Big\|(\psi_\tau\phi_t)(A)\Big\|\,\mathrm{d}\tau\right\}\sup_\tau\left\{\int_{\mathbb{R}}\Big\|(\psi_\tau\phi_t)(A)\Big\|\,\mathrm{d}t\right\}\left\{\int_\alpha^\beta\Big\|g(\tau)\Big\|^2\,\mathrm{d}\tau\right\},\end{aligned}$$

by Fubini's theorem, provided the right hand side is finite. For some $C > 0$ and $s_1, s_2 > 1$ independent of α, β and g we have

$$\begin{aligned}\|(\psi_\tau\phi_t)(A)\| &\leq C\int_\Gamma\frac{|\mathrm{d}z|}{(1+|\operatorname{Re} z-\tau|^{s_1})(1+|\operatorname{Re} z - t|^{s_2})} \\ &\leq C\int_{\mathbb{R}}\frac{\mathrm{d}r}{(1+|r-\tau|^{s_1})(1+|r-t|^{s_2})} \\ &= C\int_{\mathbb{R}}\frac{\mathrm{d}s}{(1+|s+t-\tau|^{s_1})(1+|s|^{s_2})}\end{aligned}$$

where Γ is as above. Then

$$\begin{aligned}\int_\alpha^\beta \left\|(\psi_\tau\phi_t)(A)\right\| \,\mathrm{d}\tau &\leq C\int_\alpha^\beta\int_{\mathbb{R}} \frac{\mathrm{d}s\,\mathrm{d}\tau}{(1+|s+t-\tau|^{s_1})(1+|s|^{s_2})} \\ &= C\int_{\mathbb{R}}\int_\alpha^\beta \frac{\mathrm{d}\tau\,\mathrm{d}s}{(1+|s+t-\tau|^{s_1})(1+|s|^{s_2})} \\ &\leq C\int_{\mathbb{R}} \frac{\mathrm{d}\tau}{1+|\tau|^{s_1}}\int_{\mathbb{R}}\frac{\mathrm{d}s}{1+|s|^{s_2}},\end{aligned}$$

again by Fubini, since this quantity is finite. Moreover, it is independent of t, as required. We can similarly estimate

$$\sup_\tau\left\{\int_{\mathbb{R}}\left\|(\psi_\tau\phi_t)(A)\right\|\,\mathrm{d}t\right\},$$

and the result is proven.

□

Our next step, analogously to the sectorial case, is to employ this lemma with a view to showing that the spaces defined for $\psi \in \mathcal{H}_1^\infty(H_v)$ by $X_{A,\psi} = \{x \in X : \int_{\mathbb{R}} \|\psi(t-A)x\|^2 \,\mathrm{d}t < \infty\}$ are in fact all equal with equivalent norms.

Proposition 4.1.11. *Given $A \in \mathrm{Strip}(w)$ and $v > w$, let $\psi, \phi \in \mathcal{H}_1^\infty(H_v)$, ϕ non-zero. Then there exists a constant C such that for all $x \in X_{A,\phi}$*

$$\int_{\mathbb{R}} \|\psi(t-A)x\|^2 \,\mathrm{d}t \leq C\int_{\mathbb{R}} \|\phi(t-A)x\|^2 \,\mathrm{d}t\ .$$

Proof. First we observe that there exist functions $\theta, \underline{\theta} \in \mathcal{H}_1^\infty(H_v)$ satisfying

$$\int_{\mathbb{R}} \theta(t)\underline{\theta}(t)\phi(t)\,\mathrm{d}t = 1\ .$$

Next, for $-\infty < \alpha < \beta < \infty$ we define

$$\phi_{\alpha,\beta}(z) = \int_\alpha^\beta \theta_t(z)\underline{\theta}_t(z)\phi_t(z)\,\mathrm{d}t\ .$$

Then $\phi_{\alpha,\beta} \in \mathcal{H}_1^\infty(H_v)$ and by analytic continuation we have $\lim_{\alpha\to\infty,\beta\to-\infty}\phi_{\alpha,\beta}(z) = 1$ for every $z \in H_v$. Thus by the Convergence Lemma for strip-type operators (Proposition 3.3.4) we have

$$\lim_{\alpha\to-\infty,\beta\to\infty}\phi_{\alpha,\beta}(A)x = \lim_{\alpha\to-\infty,\beta\to\infty}\int_\alpha^\beta \theta_t(A)\underline{\theta}_t(A)\phi_t(A)x\,\mathrm{d}t = x$$

for every $x \in X$. On the other hand, for all $x \in X_{A,\phi}$ we have by Lemma 4.1.10,

$$\begin{aligned}
\int_{\mathbb{R}} \|\psi(t-A)\phi_{\alpha,\beta}(A)x\|^2 \,\mathrm{d}t &= \int_{\mathbb{R}} \left\| \int_{\alpha}^{\beta} \psi_t(A)\theta_\tau(A)\underline{\theta}_\tau(A)\phi_\tau(A)x \,\mathrm{d}\tau \right\|^2 \mathrm{d}t \\
&= \int_{\mathbb{R}} \left\| \int_{\alpha}^{\beta} \theta_\tau(A)\psi_t(A)\underline{\theta}_\tau(A)\phi_\tau(A)x \,\mathrm{d}\tau \right\|^2 \mathrm{d}t \\
&\le C \int_{\alpha}^{\beta} \|\underline{\theta}_\tau(A)\phi_\tau(A)x\|^2 \,\mathrm{d}\tau && \text{(by (ii))} \\
&\le C \int_{\alpha}^{\beta} \|\phi(\tau-A)x\|^2 \,\mathrm{d}\tau && \text{(by (i))} \\
&\le C \int_{\mathbb{R}} \|\phi(\tau-A)x\|^2 \,\mathrm{d}\tau \,.
\end{aligned}$$

Let $\gamma = (\alpha,\beta)$, $x_\gamma = \phi_\gamma(A)x$. Then by the above, $(x_\gamma) \to x$ as $\gamma \to (-\infty,\infty)$. Thus by Fatou's lemma and the above estimate,

$$\begin{aligned}
\int_{\mathbb{R}} \|\psi(t-A)x\|^2 \,\mathrm{d}t &= \int_{\mathbb{R}} \liminf_{\alpha\to\infty,\beta\to-\infty} \|\psi(t-A)\phi_{\alpha,\beta}(A)x\|^2 \,\mathrm{d}t \\
&\le \liminf_{\alpha\to\infty,\beta\to-\infty} \int_{\mathbb{R}} \|\psi(t-A)\phi_{\alpha,\beta}(A)x\|^2 \,\mathrm{d}t \\
&\le C \int_{\mathbb{R}} \|\phi(t-A)x\|^2 \,\mathrm{d}t \,.
\end{aligned}$$

Thus x belongs to $X_{A,\psi}$ and satisfies the required estimate.

□

Since we can apply an identical proof to A^*, we arrive at our desired result.

Corollary 4.1.12. *Let $A \in \mathrm{Strip}(w)$, $v > w$. If A satisfies* (4.1.4) *and* (4.1.5) *for some $\psi \in \mathcal{H}_1^\infty(H_v)$ then it does so for every $\phi \in \mathcal{H}_1^\infty(H_v)$.*

Note that the argument in Proposition 4.1.11 does not depend on the choice of $v > w$. Thus the minimal height of absolute calculus is always w_A^{st}. Our next goal is to establish that absolute calculus implies bounded $\mathcal{H}^\infty$-calculus for a strip-type operator A. We adapt our approach in the sectorial case, beginning with the following result.

Proposition 4.1.13. *Let $A \in \mathrm{Strip}(w)$, $v > w$. Define $\phi(z) = \frac{1}{(iu+z)^2}$, where $u > v$. Suppose A has absolute calculus on H_v. Then for any $g \in \mathcal{H}_1^\infty(H_v)$ there exists a constant C such that for all $x, y \in X$,*

$$\|\phi(t-A)g(tA)x\| \le \|g(t-A)y\| \quad (t \in \mathbb{R}) \Rightarrow \|x\| \le C\,\|y\|.$$

In fact, this result follows in much the same way as the sectorial case (Proposition 4.1.6) once we observe that absolute calculus of a strip-type operator can be described in terms of the equivalence of the quadratic norms with the original norm on X. The following result is analogous to Proposition 4.1.4.

Proposition 4.1.14. *Let $A \in \mathrm{Strip}(w)$, $v > w$. Then A has absolute calculus on H_v if and only if for some (hence all, by Proposition 4.1.11) non-zero $\psi \in \mathcal{H}_1^\infty(H_v)$ there exists a constant C such that for all $x \in X$,*

$$C^{-1}\|x\|^2 \leq \int_{\mathbb{R}} \|\psi(t-A)x\|^2 \,\mathrm{d}t \leq C\|x\|^2 \,. \tag{4.1.6}$$

Proof. Assume first that A has absolute calculus on H_v. Similarly to the proof of [1, Theorem E] (see also [33, Section 4]), choose $c > 0$ such that

$$\int_{\mathbb{R}} \frac{c^2}{(u^2+t^2)^2} \,\mathrm{d}t = 1,$$

where $u > v$ is fixed. Then, setting $\psi(z) = \frac{c}{u^2+z^2}$, we have by Vitali's theorem that $\int_{\mathbb{R}} \psi^2(t-z) \,\mathrm{d}t = 1$ for every $z \in H_v$. Thus for any $x \in X, y \in X^*$ we have by Cauchy-Schwarz,

$$\begin{aligned} |\langle x, y\rangle| &= \left|\Big\langle \int_{\mathbb{R}} \psi^2(t-A)x \,\mathrm{d}t, y\Big\rangle\right| = \left|\int_{\mathbb{R}} \langle \psi(t-A)x, \psi(t-A^*)y\rangle \,\mathrm{d}t\right| \\ &\leq \Big(\int_{\mathbb{R}} \|\psi(t-A)x\|^2 \,\mathrm{d}t\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} \|\psi(t-A^*)y\|^2 \,\mathrm{d}t\Big)^{\frac{1}{2}} \,. \end{aligned}$$

Hence with the assumption of (4.1.5) we have

$$\begin{aligned} \|x\| = \sup_{y\in X_1^*} |\langle x, y\rangle| &\leq \sup_{y\in X_1^*} \Big(\int_{\mathbb{R}} \|\psi(t-A)x\|^2 \,\mathrm{d}t\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} \|\psi(t-A^*)y\|^2 \,\mathrm{d}t\Big)^{\frac{1}{2}} \\ &\leq C\Big(\int_{\mathbb{R}} \|\psi(t-A)x\|^2 \,\mathrm{d}t\Big)^{\frac{1}{2}}. \end{aligned}$$

The reverse inequality is built into the assumption (4.1.4), so the proof is complete.

Conversely, it is enough to establish the one-sided dual estimate (4.1.3). Fix $x^* \in X^*$ and choose an interval I, depending on ψ and x^*, large enough such that

$$\int_I \|\psi(t-A^*)x^*\|^2 \,\mathrm{d}t \geq \frac{1}{2}\int_{\mathbb{R}} \|\psi(t-A^*)x^*\|^2 \,\mathrm{d}t \,.$$

For each $t \in I$ choose $x_t \in X_1$ such that $\langle \psi(t-A^*)x^*, x_t\rangle \geq \frac{1}{2}\|\psi(t-A^*)x^*\|$. Since $t \mapsto \psi(t-A^*)x^*$ is continuous, x_t can be chosen in a way that $t \mapsto x_t$ is a step function. Thus the X-valued function g, defined on I by $g(t) = \|\psi(t-A^*)x^*\|x_t$, is measurable. Then $\|\psi(t-A^*)x^*\|^2 \leq 2\langle \psi(t-A^*)x^*, g(t)\rangle$ and thus employing Lemma 4.1.10 we can estimate for $x = \int_I \psi(t-A)g(t)\,\mathrm{d}t$,

$$\begin{aligned}\int_{\mathbb{R}} \|\psi(t-A)x\|^2\,\mathrm{d}t &= \int_{\mathbb{R}} \left\| \int_I \psi(\tau-A)\psi(t-A)g(t)\,\mathrm{d}t \right\|^2 \mathrm{d}\tau \\ &\leq C\|g\|_{L^2}^2 \leq C\int_{\mathbb{R}} \|\psi(t-A^*)x^*\|^2\,\mathrm{d}t\,.\end{aligned}$$

Hence we have

$$\begin{aligned}\Big(\int_{\mathbb{R}} \|\psi(t-A)x\|^2\,\mathrm{d}t\Big)^{\frac{1}{2}} \|x^*\|\,\mathrm{d}t \geq C\|x\|\|x^*\| \geq C\langle x^*, x\rangle \\ &= C\int_I \langle \psi(t-A^*)x^*, g(t)\rangle\,\mathrm{d}t \\ &\geq C\Big(\int_{\mathbb{R}} \|\psi(t-A^*)x^*\|^2\,\mathrm{d}t\Big) \\ &\geq C\Big(\int_{\mathbb{R}} \|\psi(t-A^*)x^*\|^2\,\mathrm{d}t\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} \|\psi(t-A)x\|^2\,\mathrm{d}t\Big)^{\frac{1}{2}}\,.\end{aligned}$$

Thus the dual estimate (4.1.3) holds and A has absolute calculus.

□

Corollary 4.1.15. *Let $A \in \mathrm{Strip}(w)$ have absolute calculus on H_v, $v > w$. Then A admits a bounded $\mathcal{H}^\infty$-calculus on H_v with $w_A^H = w_A$.*

Proof. We adapt the proof of [46, Lemma 4.6.2]. Let $f \in \mathcal{H}^\infty(H_v)$ and, for $n > v$, define $\phi_n(z) = (\frac{in}{in+z})^2$. Set $\phi(z) = (\frac{1}{iu+z})^2$, $u > v$. Then for all such n and every $t \in \mathbb{R}$ and an appropriate contour Γ we have

$$\begin{aligned}\|\phi(t-A)(\phi_n f)(A)\| &\leq \int_\Gamma \|\phi(t-z)\phi_n(z)f(z)R(z,A)\|\,|\mathrm{d}z| \\ &\leq C\|f\|_{\mathcal{H}^\infty(H_v)} \int_\Gamma |\phi(t-z)|\,|\mathrm{d}z| \\ &\leq C\|f\|_{\mathcal{H}^\infty(H_v)},\end{aligned}$$

since the functions $\{\phi_n : n > v\}$ are uniformly bounded on H_v. Set

$$M = \sup_{t,n} \|\phi(t-A)(\phi_n f)(A)\|.$$

Given $y \in X$, define $x_n = \frac{1}{M}(\phi_n f)(A)y$. Then for any $\psi \in \mathcal{H}_1^\infty(H_v)$,

$$\begin{aligned}
\|\phi(t-A)\psi(t-A)x_n\| &= \|\phi(t-A)\psi(t-A)\frac{1}{M}(\phi_n f)(A)y\| \\
&\le \frac{1}{M}\|\phi(t-A)(\phi_n f)(A)\| \, \|\psi(t-A)y\| \\
&\le \|\psi(t-A)y\| \ .
\end{aligned}$$

Thus by Proposition 4.1.13 there exists a constant C such that $\frac{1}{M}\|(\phi_n f)(A)y\| = \|x_n\| \le C\|y\|$. Since $\phi_n f \to f$ on H_v, the result follows from the Convergence Lemma for strip-type operators (Proposition 3.3.4). Since absolute calculus is independent of the height of the strip (see the comment after Corollary 4.1.12), the second claim is immediate.

□

Next we seek a correspondence between the absolute calculus of sectorial operators and strip-type operators. Given that B is a strip-type operator with absolute calculus, combining Corollary 4.1.15 with Proposition 3.4.3 we deduce that e^B is sectorial and admits a bounded $\mathcal{H}^\infty$-calculus. We now show that in fact e^B has absolute calculus. The converse holds as well; given a sectorial operator A with absolute calculus, we have by Corollary 4.1.7 and Proposition 3.4.3 that $\log A$ is strip-type and admits a bounded $\mathcal{H}^\infty$-calculus. Again, we show that in fact this operator has absolute calculus. More precisely, we have the following.

Theorem 4.1.16. *Let $A \in \mathrm{Sect}(\theta)$, $B \in \mathrm{Strip}(w)$.*

(i) If A has absolute calculus on Σ_v, $v \in (\theta, \pi)$, then $\log A$ *has absolute calculus on H_v.*

(ii) If B has absolute calculus on H_v, $v \in (w, \pi)$, then e^B is sectorial and has absolute calculus on Σ_v.

Proof. Let $\psi(z) = \frac{z^{\frac{1}{2}}}{1+z} \in \mathcal{H}_0^\infty(\Sigma_v)$, $\phi(z) = \frac{e^{\frac{z}{2}}}{1+e^z} \in \mathcal{H}_1^\infty(H_v)$. By Proposition 4.1.4 A has absolute calculus on Σ_v if and only if there exists a constant C such that for each $x \in X_1$

$$\int_0^\infty \|\psi(tA)x\|^2 \, \frac{\mathrm{d}t}{t} \le C \ ,$$

and the corresponding dual condition (4.1.3). By making the substitutions $u = \frac{1}{t}$

and $q = \log u$ we see that this is equivalent to

$$\begin{aligned}\int_0^\infty \|t^{\frac{1}{2}}A^{\frac{1}{2}}(1+tA)^{-1}x\|^2\,\frac{\mathrm{d}t}{t}\ \le C &\Leftrightarrow \int_0^\infty \|A^{\frac{1}{2}}(u+A)^{-1}x\|^2\,\mathrm{d}u\ \le C\\ &\Leftrightarrow \int_0^\infty \|e^{\frac{1}{2}B}(u+e^B)^{-1}x\|^2\,\mathrm{d}u\ \le C\\ &\Leftrightarrow \int_0^\infty \|[e^{\frac{1}{2}z}(u+e^z)^{-1}](B)x\|^2\,\mathrm{d}u \le C\\ &\Leftrightarrow \int_{\mathbb{R}} \|[e^{\frac{1}{2}(z+q)}(e^q+e^z)^{-1}](B)x\|^2\,\mathrm{d}q\ \le C\\ &\Leftrightarrow \int_{\mathbb{R}} \|\phi(B-q)x\|^2\,\mathrm{d}q\ \le C\ ,\end{aligned}$$

where $B = \log A$. We argue similarly for the dual condition and by Corollary 4.1.12 the result follows.

□

With this correspondence, we can extend the class of scalar functions which are permitted in considerations of the absolute calculus of sectorial operators in Definition 4.1.1 and Proposition 4.1.4. Indeed, given a sectorial operator A and $\phi \in \mathcal{H}_1^\infty(H_v)$, define $\tilde{\phi}(z) = \phi(\log z) \in \mathcal{H}^\infty(\Sigma_v)$. Then by the composition rule we have

$$\begin{aligned}\int_0^\infty \|\tilde{\phi}(tA)x\|^2\,\frac{\mathrm{d}t}{t} &= \int_0^\infty \|\phi(\log tA)x\|^2\,\frac{\mathrm{d}t}{t}\\ &= \int_0^\infty \|\phi(\log t + \log A)x\|^2\,\frac{\mathrm{d}t}{t}\\ &= \int_{\mathbb{R}} \|\phi(\log A - u)x\|^2\,\mathrm{d}u\end{aligned}$$

We can establish a corresponding dual equality, since by [34, Proposition 2.6.3] we have $(\log A)^* = \log A^*$. Thus by Theorem 4.1.16 A has absolute calculus if and only if the pair $\{A, \tilde{\phi}\}$ satisfy the estimates (4.1.2) and (4.1.3). For example, fix $\theta \in (\phi_A, \pi)$, $v \in (\theta, \pi)$ and let $\phi(z) = \left(\frac{1}{iv+z}\right)^2$. Then $\tilde{\phi}(z) = \phi(z) = \left(\frac{1}{iv+\log z}\right)^2$ does not lie in $\mathcal{H}_0^\infty(\Sigma_\theta)$ but to show A has absolute calculus it is sufficient (and necessary) to establish quadratic estimates for this function. Further, by Proposition 4.1.14 this is also equivalent to showing that the quadratic norm associated to $\tilde{\phi}$ is equivalent to $\|\cdot\|_X$. We summarise this discussion in the following result.

Corollary 4.1.17. *Let $A \in \operatorname{Sect} w$, $v \in (w, \pi)$, $\phi \in \mathcal{H}_1^\infty(H_v)$, $\tilde{\phi}(z) = \phi(\log z)$. Then A has absolute calculus on Σ_v if and only if one of the following conditions holds.*

(i) $\{A, \tilde{\phi}\}$ and $\{A^*, \tilde{\phi}\}$ satisfy quadratic estimates.

(ii) There exists a constant C such that for every $x \in X$,

$$C^{-1}\|x\|^2 \le \int_0^\infty \|\tilde{\phi}(tA)x\|^2 \,\frac{\mathrm{d}t}{t} \le C\|x\|^2 \,.$$

As an immediate consequence of Theorem 4.1.16, Theorem 4.1.8 and Proposition 3.4.3, we can now establish that absolute calculus of a strip-type operator is equivalent to bounded $\mathcal{H}^\infty$-calculus on Hilbert space.

Corollary 4.1.18. *Let $A \in \mathrm{Strip}(w)$ on a Hilbert space X. Then A has absolute calculus on H_v, $v > w$, if and only if A admits a bounded $\mathcal{H}^\infty$-calculus on H_v.*

4.2 Weak integral estimates

Having characterised the bounded $\mathcal{H}^\infty$-calculus of sectorial and strip-type operators in terms of the absolute functional calculus in Hilbert space, it is now our aim to extend these considerations to general Banach space. As discussed in Section 4.1, absolute calculus is too strong a property in general; one cannot always expect equivalence of the quadratic norms when the operator has bounded $\mathcal{H}^\infty$-calculus. Thus we seek a weaker form of these square function estimates. We start by surveying known results for the sectorial case. Given a sectorial operator A on a Hilbert space X, $\theta \in (\phi_A, \pi)$ and $\psi \in \mathcal{H}_0^\infty(\Sigma_\theta)$, we have for any x in X, and a Rademacher sequence (r_k),

$$\begin{aligned}
\int_0^\infty \|\psi(tA)x\|^2 \,\frac{\mathrm{d}t}{t} &= \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \|\psi(tA)x\|^2 \,\frac{\mathrm{d}t}{t} \\
&= \int_1^2 \sum_{k\in\mathbb{Z}} \|\psi(2^k tA)x\|^2 \,\frac{\mathrm{d}t}{t} \\
&= \int_1^2 \left(\int_0^1 \Big\|\sum_{k\in\mathbb{Z}} r_k(u)\psi(2^k tA)x\Big\|^2 \mathrm{d}u \right) \frac{\mathrm{d}t}{t} \,.
\end{aligned}$$

Thus it seems sensible to consider so-called Rademacher norms on general Banach space, defined for such ψ by

$$\begin{aligned}
\|x\|_{R,\psi} &= \sup_{t>0} \Big\|\sum_{k\in\mathbb{Z}} r_k \psi(t2^k A)x\Big\|_{L^2(X)} \,, \qquad x \in X, \\
\|x^*\|_{R,\psi} &= \sup_{t>0} \Big\|\sum_{k\in\mathbb{Z}} r_k \psi(t2^k A)^* x^*\Big\|_{L^2(X^*)} \,, \quad x^* \in X^* \,.
\end{aligned}$$

Definition 4.2.1. We say A admits *Rademacher square function estimates* with respect to ψ if there exists a constant C such that for every $x \in X$ and $x^* \in X^*$,

$$\|x\|_{R,\psi} \leq C\|x\| \ ,$$
$$\|x^*\|_{R,\psi} \leq C\|x^*\| \ .$$

To establish that these estimates characterise the bounded $\mathcal{H}^\infty$-calculus of sectorial operators, we can employ a result due to Cowling et al. [16] which connects the two notions through weak integral estimates. One must impose certain growth estimates on the Fourier transform of the underlying scalar functions. To formulate their precise result, we must first introduce some notation. Given $\mu \in (0,\pi)$ and $\psi \in \mathcal{H}_0^\infty(\Sigma_\mu)$, denote by ψ_e the function $\psi \circ \exp$. Let $\hat{\cdot}$ denote the Fourier transform; it then follows that

$$\begin{aligned}\hat{\psi}_e(\lambda) &= \int_{\mathbb{R}} e^{-i\lambda x}\psi_e(x)\,\mathrm{d}x = \int_0^\infty \tau^{-i\lambda}\psi(\tau)\,\frac{\mathrm{d}\tau}{\tau} \\ &= e^{\theta\lambda}\int_0^\infty \tau^{-i\lambda}\psi(\tau e^{i\theta})\,\frac{\mathrm{d}\tau}{\tau} \qquad \forall\lambda \in \mathbb{R}\ \forall\theta \in (-\mu,\mu)\ .\end{aligned}$$

Hence

$$\left|\hat{\psi}_e(\lambda)\right| \leq Ce^{-\mu|\lambda|} \qquad \forall\lambda \in \mathbb{R}\ .$$

Similarly we have

$$\Big|\frac{\mathrm{d}}{\mathrm{d}\lambda}\hat{\psi}_e(\lambda)\Big| \leq Ce^{-\mu|\lambda|} \quad \forall\lambda \in \mathbb{R}.$$

For $0 < \mu < \nu < \pi$, define the classes

$$\mathcal{H}_{0,\nu}^\infty(\Sigma_\mu) = \left\{\psi \in \mathcal{H}_0^\infty(\Sigma_\mu) : \inf_{t\in\mathbb{R}} e^{\nu|t|}|\hat{\psi}_e(t)| > 0\right\}\ .$$

and

$$\mathcal{H}_{0,\nu}^\infty(\Sigma_\nu-) = \bigcap_{\mu\in(0,\nu)} \mathcal{H}_{0,\nu}^\infty(\Sigma_\mu)\ .$$

The characterisation in terms of weak integral estimates from [16, Theorem 4.4] can then be stated as follows.

Theorem 4.2.2. *Let A be a sectorial of type ω. Suppose $\omega < \mu < \nu < \pi$ and $2\nu - \mu < \eta < \pi$. Suppose that for some $\psi \in \mathcal{H}_{0,\nu}^\infty(\Sigma_\mu)$ there exists a constant C such that*

$$\int_0^\infty |\langle\psi(tA)x, y\rangle|\frac{\mathrm{d}t}{t} \leq C\|x\|\|y\| \ \ \forall x \in X\ \forall y \in X^*. \tag{4.2.1}$$

Then A admits a bounded $\mathcal{H}^\infty$-calculus on Σ_η.

Conversely, if A admits a bounded $\mathcal{H}^\infty$-calculus on Σ_μ and $\psi \in \mathcal{H}_0^\infty(\Sigma_\mu)$ then A

satisfies (4.2.1).

We note that, by the uniform boundedness theorem, the estimate (4.2.1) is automatic once the integrals $\int_0^\infty |\langle \psi(tA)x, y\rangle| \frac{\mathrm{d}t}{t}$ are known to be finite for each $x \in X, y \in X^*$. It is an immediate corollary that if, for A as above, (4.2.1) is satisfied for some $\psi \in \mathcal{H}^\infty_{0,\nu}(\Sigma_\nu -)$, $\nu \in (0,\pi)$, then A admits a bounded $\mathcal{H}^\infty$-calculus of type ν. This is restrictive in the sense that it does not allow us to establish boundedness of the $\mathcal{H}^\infty$-calculus of A on Σ_η for any $\eta \in (0,\nu)$. The next result [16, Theorem 4.6] circumvents this issue.

Theorem 4.2.3. *Let A be a sectorial operator of type ω in a Banach space X. Suppose that $\omega < \mu < \nu < \pi$, $0 \le \theta < \mu - \omega$ and $2\nu - \mu - \theta < \eta < \pi$. If*

$$\int_0^\infty |\langle \psi(te^{\pm i\theta}A)x, y\rangle| \frac{\mathrm{d}t}{t} \le C\|x\|\|y\| \ \ \forall x \in X \ \forall y \in X^*, \tag{4.2.2}$$

for some $\psi \in \mathcal{H}^\infty_{0,\nu}(\Sigma_\mu)$, then A admits a bounded $\mathcal{H}^\infty$-calculus of type η.

Example 4.2.4. It was also shown in [16] that the following functions ψ lie in the class $\mathcal{H}^\infty_{0,\nu}(\Sigma_\mu)$, for appropriate ν and μ, so that the results above can be applied.

(i) $\psi(z) = \frac{\alpha}{z-\alpha} - \frac{\overline{\alpha}}{z-\overline{\alpha}}$, where $\alpha \notin \Sigma_\mu$. Then $\psi \in \mathcal{H}^\infty_{0,\nu}(\Sigma_\nu -)$ for $\nu = |\arg \alpha|$. Thus if A satisfies (4.2.1) for this choice of ψ, A admits a bounded $\mathcal{H}^\infty$-calculus of type $|\arg \alpha|$.

(ii) $\psi(z) = ze^{-z}$, defined on $\Sigma_{\frac{\pi}{2}}$. Then $\psi \in \mathcal{H}^\infty_{0,\frac{\pi}{2}}(\Sigma_{\frac{\pi}{2}} -)$. Thus if A is sectorial of type $\omega < \frac{\pi}{2}$ and

$$\int_0^\infty |\langle Ae^{-tA}x, y\rangle| \,\mathrm{d}t \le C\|x\| \, \|y\| \quad \forall x \in X, y \in X^*,$$

then by (the corollary to) Theorem 4.2.2 we infer that A has a bounded $\mathcal{H}^\infty$-calculus of type $\frac{\pi}{2}$.

More generally, for $\theta \in (0, \frac{\pi}{2} - \omega)$ we can show by Theorem 4.2.3 that if

$$\int_0^\infty |\langle Ae^{-te^{\pm i\theta}A}x, y\rangle \,\mathrm{d}t \le C\|x\| \, \|y\| \quad \forall x \in X, y \in X^*,$$

then A admits a bounded $\mathcal{H}^\infty$-calculus of type $\frac{\pi}{2} - \theta$.

(iii) $\psi(z) = \frac{z}{(1+z)^2}$ lies in $\mathcal{H}^\infty_{0,\pi}(\Sigma_\pi -)$. Thus if A is sectorial of angle $\omega < \pi$ and $0 \le \theta < \pi - \omega$ and

$$\int_0^\infty |\langle A(t + e^{\pm i\theta}A)^{-2}x, y\rangle \,\mathrm{d}t \le C\|x\| \, \|y\| \quad \forall x \in X, y \in X^*,$$

by Theorem 4.2.3 A admits a bounded $\mathcal{H}^\infty$-calculus of type $\pi - \theta$.

The characterisation of bounded $\mathcal{H}^\infty$-calculus of sectorial operators in terms of Rademacher estimates can now be stated. The proofs of (a) and (b) can be found in [47, Chapter 12], whereas (c) and (d) are from [16, Theorem 4.6]. The original version of this result, which bypasses weak integral estimates, can be found in [44, Section 4].

Theorem 4.2.5. *Let A be a sectorial operator. For $\sigma \in (\phi_A, \pi)$ and a scalar function $\chi \in \mathcal{H}_0^\infty(\Sigma_{\sigma'})$, $\sigma' \in (\phi_A, \pi)$, consider the following conditions.*

(i) A admits a bounded $\mathcal{H}^\infty$-calculus on Σ_σ.

(ii)

$$\sup_N \sup_{t>0} \sup_{\epsilon_k = \pm 1} \| \sum_{k=-N}^{N} \epsilon_k \chi(2^k t A) \|_{B(X)} < \infty \, .$$

(iii) A admits Rademacher square function estimates with respect to χ.

(iv) There exists a constant C such that, for every $x \in X$, $y \in X^*$,

$$\int_0^\infty |\langle \chi(te^{\pm i\theta}A)x, y\rangle| \, \frac{\mathrm{d}t}{t} \leq C \|x\| \, \|y\| \, .$$

Then

(a) $(i) \Rightarrow (ii) \Rightarrow (iii)$ for every $\chi \in \mathcal{H}_0^\infty(\Sigma_\sigma)$.

(b) Let $\theta < \sigma - \phi_A$, $\psi \in \mathcal{H}_0^\infty(\Sigma_\sigma)$. If (iii) holds for the functions $\chi(z) = \psi(e^{\pm i\theta} z) \in \mathcal{H}_0^\infty(\Sigma_{\sigma-\theta})$ then (iv) holds for $\chi = \psi^2$.

(c) If (iv) holds for some $\chi \in \mathcal{H}_{0,\nu}^\infty(\Sigma_\mu)$, with $\phi_A < \mu < \nu < \pi$ and $0 \leq \theta < \mu - \phi_A$, then (i) holds for $\sigma > 2\nu - \mu - \theta$.

(d) $(i) \Rightarrow (iv)$ for every $\chi \in \mathcal{H}_0^\infty(\Sigma_\sigma)$.

We can gain more control on the angles involved via concrete functions, such as those in Example 4.2.4. Indeed, defining $\psi_{\pm\theta}(z) = z^{\frac{1}{2}}(e^{\pm i\theta} - z)^{-1}$, we have by Example 4.2.4 (iii) that for $\phi_A < \sigma < \theta$,

A admits a bounded $\mathcal{H}^\infty$-calculus on $\Sigma_\sigma \Rightarrow \{A, \psi_{\pm\theta}\}$ satisfy Rademacher estimates

$$\Rightarrow \int_0^\infty |\langle AR(te^{\pm i\theta}, A)^2 x, y\rangle| \mathrm{d}t \leq C\|x\| \, \|y\| \, .$$

Conversely for $\sigma > \theta > \phi_A$ we have

$$\int_0^\infty |\langle AR(te^{\pm i\theta}, A)^2 x, y\rangle| \mathrm{d}t \le C\|x\|\,\|y\| \Rightarrow A \text{ has a bounded } \mathcal{H}^\infty\text{-calculus on } \Sigma_\sigma.$$

Following this discussion for sectorial operators, we may hope for an analogous result for strip-type operators. This can be achieved by again adapting the natural harmonic analysis of the boundary of a sector to that of a strip. We start in an analogous way to the sectorial case (see the opening paragraph of this section) by making the following calculation in Hilbert space. Again, (r_k) represents a Rademacher sequence of random variables.

$$\begin{aligned}\int_0^\infty \|\psi(t-A)x\|^2 \,\mathrm{d}t &= \sum_{k\in\mathbb{Z}} \int_k^{k+1} \|\psi(t-A)x\|^2 \,\mathrm{d}t \\ &= \int_0^1 \sum_{k\in\mathbb{Z}} \|\psi(t+k-A)x\|^2 \,\mathrm{d}t \\ &= \int_0^1 \left(\int_0^1 \Big\|\sum_{k\in\mathbb{Z}} r_k(u)\psi(t+k-A)x\Big\|^2 \mathrm{d}u \right) \mathrm{d}t \,.\end{aligned}$$

With this in mind, it is natural to define, for $A \in \mathrm{Strip}(w)$, $v > w$ and $\psi \in \mathcal{H}_1^\infty(H_v)$,

$$\begin{aligned}\|x\|_{R,\psi} &= \sup_{t\in\mathbb{R}} \Big\|\sum_{k\in\mathbb{Z}} r_k \psi(t+k-A)x\Big\|_{L^2(X)}\,, & x \in X, \\ \|x^*\|_{R,\psi} &= \sup_{t\in\mathbb{R}} \Big\|\sum_{k\in\mathbb{Z}} r_k \psi(t+k-A)^* x^*\Big\|_{L^2(X^*)}\,, & x^* \in X^*\,.\end{aligned}$$

As in the sectorial case, we will say a strip-type operator A satisfies Rademacher (square function) estimates with respect to ψ if there exists a constant C such that for every $x \in X$ and $x^* \in X^*$,

$$\begin{aligned}\|x\|_{R,\psi} &\le C\|x\|, \\ \|x^*\|_{R,\psi} &\le C\|x^*\|\,.\end{aligned}$$

By Kahane's inequality (Theorem 3.7.2), we can replace the L^2-norm in this definition by any L^p-norm for $p \in [1,\infty)$.

Our aim is to obtain a version of Theorem 4.2.5 for strip-type operators by showing that the bounded $\mathcal{H}^\infty$-calculus for these operators is characterised by these square function estimates. Again, we will need to go through weak integral estimates to achieve this. We formulate the properties for strip-type operators for clarity:

(A) A admits a bounded $\mathcal{H}^\infty$-calculus on H_v.

(B)

$$\sup_n \sup_{t\in\mathbb{R}} \sup_{\epsilon_k=\pm 1} \|\sum_k \epsilon_k \psi(t+k-A)\|_{B(X)} < \infty\ .$$

(C) A admits Rademacher estimates with respect to ψ.

(D) There exists a constant C such that for every $x \in X, y \in X^*$,

$$\int_{\mathbb{R}} |\langle \psi(t-A)x, y\rangle|\ \mathrm{d}t \le C\|x\|\ \|y\|\ . \tag{4.2.3}$$

We note that, by the uniform boundedness theorem, the estimate (4.2.3) is automatic once the integrals are known to be finite for each $x \in X, y \in X^*$. As in the sectorial case, the difficult implication is (D) ⇒ (A). Vörös [76, Chapter 4] was able to adapt the arguments of [16] and obtain a class of functions for which conditions (A) and (D) are equivalent. The implication (A) ⇒ (D) follows much as in the sectorial case.

Proposition 4.2.6. *Let* $A \in \mathrm{Strip}(w)$*. If* A *admits a bounded* $\mathcal{H}^\infty$*-calculus on* H_v*,* $v > w$*, then for every* $\psi \in \mathcal{H}^\infty_1(H_v)$ *we have for some* C *and every* $x \in X, y \in X^*$*,*

$$\int_{\mathbb{R}} |\langle \psi(t-A)x, y\rangle|\ \mathrm{d}t \le C\|x\|\ \|y\|\ .$$

Proof. Fix $x \in X_1, y \in X_1^*$. For a suitable measurable sign function ϵ, we can write the integral as

$$\int_{\mathbb{R}} \langle \psi(A-t)x, y\rangle \epsilon(t,x,y)\ \mathrm{d}t = \langle f(A)x, y\rangle,$$

where $f(z) = \int_{\mathbb{R}} \psi(z-t)\epsilon(t,x,y)\ \mathrm{d}t$. Then for some $s > 1$ and any $z \in H_v$,

$$|f(z)| \le \int_{\mathbb{R}} |\psi(z-t)|\ \mathrm{d}t \le C \int_{\mathbb{R}} \frac{\mathrm{d}t}{1+|t|^s} \le C.$$

Thus $f \in \mathcal{H}^\infty(H_v)$ and $\int_{\mathbb{R}} |\langle \psi(A-t)x, y\rangle|\ \mathrm{d}t \le C\|f(A)\| \le C\|f\|_{\mathcal{H}^\infty(H_v)} \le C.$ □

The proof of the converse direction relies upon the underlying scalar function ψ satisfying appropriate decay estimates. We present the proof in the special case where

$$\psi(z) = \frac{1}{z+i\tilde{v}} - \frac{1}{z-i\tilde{v}},$$

since our motivating examples are resolvent operators. Note that if $\hat{\cdot}$ denotes the Fourier transform, we have

$$\hat{\psi}(z) = \frac{\pi}{\tilde{v}} e^{-\tilde{v}|z|}, \quad \hat{\psi}'(z) = \pi(\operatorname{sgn} z)e^{-\tilde{v}|z|}.$$

For a given $f \in \mathcal{H}^\infty_1(H_u)$, $u > v$, choose $\tilde{v} > u$ and ψ as above. It then makes sense to define

$$q = (\hat{f}\cosh(\alpha\,\cdot))\check{\ } * \Big(\frac{1}{\hat{\psi}\cosh(\alpha\,\cdot)}\Big)\check{\ },$$

where $u < \alpha < \tilde{v}$, so that $q = (\frac{\hat{f}}{\hat{\psi}})\check{\ }$, where $\check{\ }$ denotes the inverse Fourier transform. Since

$$\left|\frac{1}{\hat{\psi}(z)\cosh(\alpha z)}\right| + \left|\left(\frac{1}{\hat{\psi}\cosh(\alpha\,\cdot)}\right)'(z)\right| \le Ce^{(u-\alpha)|z|},$$

we have by Hölder's Inequality and Plancherel's Theorem that

$$\begin{aligned}
\left\|\left(\frac{1}{\hat{\psi}\cosh(\alpha\,\cdot)}\right)\check{\ }\right\|_{L^1(\mathbb{R})} &\le \left\|(1-i\,\cdot)\left(\frac{1}{\hat{\psi}\cosh(\alpha\,\cdot)}\right)\check{\ }\right\|_{L^2(\mathbb{R})}\left\|\frac{1}{1-i\,\cdot}\right\|_{L^2(\mathbb{R})} \\
&\le C\left\|\left(\frac{1}{\hat{\psi}\cosh(\alpha\,\cdot)}\right) + \left(\frac{1}{\hat{\psi}\cosh(\alpha\,\cdot)}\right)'\right\|_{L^2(\mathbb{R})} \\
&\le C\left(\int_{\mathbb{R}} e^{2(u-\alpha)|t|}\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\le C.
\end{aligned}$$

Thus

$$\|q\|_{L^\infty(\mathbb{R})} \le \left\|(\hat{f}\cosh(\alpha\,\cdot))\check{\ }\right\|_{L^\infty(\mathbb{R})}\left\|\left(\frac{1}{\hat{\psi}\cosh(\alpha\,\cdot)}\right)\check{\ }\right\|_{L^1(\mathbb{R})} \le C\|f\|_{\mathcal{H}^\infty(H_u)},$$

since

$$(\hat{f}\cosh(\alpha\,\cdot))\check{\ } = \frac{1}{2}\sum f(\cdot\ \pm i\alpha)\ .$$

Next we observe that, by the definition of q, we have $f = q * \psi$ on $\mathbb{R}$, and so on H_v by analytic continuation. Hence we have

$$\begin{aligned}
\|f(A)\| = \sup_{x\in X_1, y\in X_1^*}|\langle f(A)x, y\rangle| &= \sup_{x\in X_1, y\in X_1^*}|\lim_{n\to\infty}\int_{-n}^{n} q(t)\langle\psi(A-t)x, y\rangle\,\mathrm{d}t| \\
\le \|q\|_{L^\infty}\sup_{x\in X_1, y\in X_1^*}\int_{\mathbb{R}}|\langle\psi(A-t)x, y\rangle|\,\mathrm{d}t& \\
\le C\|f\|_{\mathcal{H}^\infty(H_u)}.&
\end{aligned}$$

Thus f admits a bounded $\mathcal{H}^\infty$-calculus on H_u, by Proposition 3.3.7.

This, along with an almost identical proof for more general ψ, yields the following (see also [76, Theorem 4.2.4]).

Theorem 4.2.7. *Let* $A \in \mathrm{Strip}(w)$*,* $w < \phi < v$ *and suppose there exists* $\psi \in \mathcal{H}_1^\infty(H_\phi)$ *such that*

- *(i)* $|\hat{\psi}(t)| \geq Ce^{-v|t|}$ *and*
- *(ii)* $|\hat{\psi}'(t)| \leq Ce^{-\phi|t|}$.

Suppose further that for some C *and every* $x \in X, y \in X^*$ *we have*

$$\int_{\mathbb{R}} |\langle \psi(t-A)x, y\rangle| \, \mathrm{d}t \leq C\|x\| \, \|y\| \, .$$

Then A *admits a bounded* $\mathcal{H}^\infty$*-calculus on* $H_{2v-\phi}$*. In particular, if*

$$\int_{\mathbb{R}} |\langle [R(iv+t, A) - R(-iv+t, A)]x, y\rangle| \, \mathrm{d}t \leq C\|x\| \, \|y\| \, , \tag{4.2.4}$$

then A *admits a bounded* $\mathcal{H}^\infty$*-calculus of type* v*.*

In a similar manner, we can adapt the proof of Theorem 4.2.3 in order to narrow the height of the strip. The proof is omitted, but for the main ideas we refer to the proof of Theorem 4.4.4, where a similar result is established for a pair of commuting sectorial operators.

Theorem 4.2.8. *Let* $A \in \mathrm{Strip}(w)$*,* $w < \phi < v$*,* $0 \leq \theta < \phi - w$*,* $2v - \phi - \theta < \pi$ *and suppose there exists* $\psi \in \mathcal{H}_1^\infty(H_\phi)$ *such that*

- *(i)* $|\hat{\psi}(t)| \geq Ce^{-v|t|}$ *and*
- *(ii)* $|\hat{\psi}'(t)| \leq Ce^{-\phi|t|}$.
- *(iii) For some* C *and every* $x \in X, y \in X^*$*,*

$$\int_{\mathbb{R}} |\langle \psi(\pm i\theta + t - A)x, y\rangle| \, \mathrm{d}t \leq C\|x\| \, \|y\| \, .$$

Then A *admits a bounded* $\mathcal{H}^\infty$*-calculus on* $H_{2v-\phi-\theta}$*.*

Integral estimates which involve a single resolvent term are preferable to (4.2.4). We can use the methods of Gomilko-Cojuhari [14, Lemma 3.4] and Shi-Feng [66] to arrive at such a criterion by imposing some control on the growth of the integral estimates. These estimates also appear in more general form in Theorem 4.3.1.

Corollary 4.2.9. *Let* $A \in \mathrm{Strip}(w)$*,* $v > w$*.*

(i) If A admits a bounded $\mathcal{H}^\infty$-calculus on H_v then there exists a constant C such that for every $x \in X$, $y \in X^*$,

$$\sup_{\tilde{v}>v} (\tilde{v} - v) \int_{\mathbb{R}} |\langle R(t \pm i\tilde{v}, A)^2 x, y\rangle| \, \mathrm{d}t \leq C\|x\| \, \|y\| \, . \tag{4.2.5}$$

(ii) If

$$\sup_{\tilde{v}>v} (\tilde{v} - v) \int_{\mathbb{R}} |\langle R(t \pm i\tilde{v}, A)^2 x, y\rangle| \, \mathrm{d}t \leq C\|x\| \, \|y\| \;\; \forall x \in X \; \forall y \in X^*, \tag{4.2.6}$$

then A admits a bounded $\mathcal{H}^\infty$-calculus of type v.

Proof. (*i*) Given $\tilde{v} > v$, $x \in X_1$ and $y \in X_1^*$, we proceed as in the proof of Proposition 4.2.6 on H_v with the function $\phi(z) = \frac{1}{(z \pm i\tilde{v})^2}$. This allows us to estimate

$$\int_{\mathbb{R}} |\langle R(t \pm i\tilde{v}, A)^2 x, y\rangle| \, \mathrm{d}t \leq \|f(A)\| \leq C_v \|f\|_{\mathcal{H}^\infty(H_v)},$$

where C_v is the $\mathcal{H}^\infty$-constant corresponding to A on H_v. This last term can further be estimated by

$$\sup_{z \in H_v} \int_{\mathbb{R}} \frac{\mathrm{d}t}{t^2 + |\tilde{v} - \operatorname{Im} z|^2} \leq \int_{\mathbb{R}} \frac{\mathrm{d}t}{t^2 + (\tilde{v} - v)^2} = \frac{C}{\tilde{v} - v}.$$

(*ii*) First observe that by (4.2.6) and Theorem 2.1.10, the operator $iA - v$ generates a uniformly bounded C_0-semigroup, so that iA generates a semigroup e^{isA} satisfying $\|e^{isA}\| \leq Ce^{vs} (s > 0)$. Fix $\tilde{v} > v$. Then for $s > 0$,

$$R(t - i\tilde{v}, A) = iR(it + \tilde{v}, iA) = -i \int_0^\infty e^{-\tilde{v}s} e^{-its} e^{isA} \mathrm{d}s, \qquad t > 0,$$

and setting $\lambda = t + i\tilde{v}$,

$$R(\lambda, A) - R(\overline{\lambda}, A) = 2i\tilde{v} R(\lambda, A)[R(\lambda, A) - 2ivR(\overline{\lambda}, A)].$$

Hence for $x \in X_1$ and $y \in X_1^*$ we have

$$\begin{aligned} \langle R(\lambda, A)x, y\rangle - &\langle R(\overline{\lambda}, A)x, y\rangle = \\ &2i\tilde{v}\langle R(\lambda, A)^2 x, y\rangle - 4i\tilde{v}^2 \int_0^\infty e^{-\tilde{v}s} e^{-its} \langle R(\lambda, A)^2 e^{isA} x, y\rangle \, \mathrm{d}s. \end{aligned}$$

Furthermore,

$$\begin{aligned}
\int_{\mathbb{R}} \left| \int_0^\infty e^{-\tilde{v}s} e^{ist} \langle R(t+i\tilde{v})^2 e^{isA} x, y\rangle \, \mathrm{d}s \right| \mathrm{d}t \\
&\leq \int_0^\infty e^{-\tilde{v}s} \int_{\mathbb{R}} \left| \langle R(t+i\tilde{v}, A)^2 e^{isA} x, y\rangle \right| \mathrm{d}t \, \mathrm{d}s \\
&\leq \frac{C}{(\tilde{v}-v)} \int_0^\infty e^{-\tilde{v}s} \| e^{isA} x\| \, \mathrm{d}s \qquad \text{(by (4.2.6))} \\
&\leq \frac{C}{(\tilde{v}-v)} \int_0^\infty e^{(-\tilde{v}+v)s} \, \mathrm{d}s = \frac{C}{(\tilde{v}-v)^2} \, .
\end{aligned}$$

Combining this with (4.2.6), we arrive at

$$\int_{\mathbb{R}} |\langle (R(t+i\tilde{v}, A) - R(t-i\tilde{v}, A))x, y\rangle| \, \mathrm{d}t \leq C_{\tilde{v}}.$$

The result now follows from Theorem 4.2.7.

□

Note that the proof of (*ii*) only requires the decay rate of $\int_{\mathbb{R}} |\langle R(t-i\tilde{v}, A)^2 x, y\rangle| \, \mathrm{d}t$, whereas the integrals $\int_{\mathbb{R}} |\langle R(t+i\tilde{v}, A)^2 x, y\rangle| \, \mathrm{d}t$ can just be assumed to be finite (by the Uniform Boundedness Theorem, these integrals are then bounded independently of $x \in X_1$, $y \in X_1^*$). A similar argument for higher order estimates is presented in Theorem 4.3.4.

Corollary 4.2.9 falls short of our aspirations in two respects. On the one hand, we would like to lose the growth bounds on the integrals involved. Further, to meet our original aim of characterising bounded $\mathcal{H}^\infty$-calculus in terms of Rademacher estimates, we will need to incorporate a broader class of functions into the possible choices of ψ in the implication (D) ⇒ (A) (see page 67).

4.3 Characterisation of the bounded $\mathcal{H}^\infty$-calculus for strip-type operators

Employing the m-bounded calculus from Section 3.5, we will now seek to derive characterisations of the bounded $\mathcal{H}^\infty$-calculus which are broader than those established so far in Theorem 4.2.7 and Corollary 4.2.9. As in the previous section, we restrict our attention initially to resolvents.

Theorem 4.3.1. *Let $A \in \mathrm{Strip}(w)$, $v > w$, $m \in \mathbb{N}$. Then the following are equivalent.*

(i) A admits a bounded $\mathcal{H}^\infty$-calculus of type v.

(ii) A admits an m-bounded $\mathcal{H}^\infty$-calculus of type v.

(iii) For each $\tilde{v} > v$ there exists $C_{\tilde{v},m}$ such that for every $x \in X$, $y \in X^*$,

$$\int_{\mathbb{R}} |\langle (R(t+i\tilde{v},A)^{m+1} - R(t-i\tilde{v},A)^{m+1})x, y\rangle| \,\mathrm{d}t \leq C_{\tilde{v},m}\|x\| \, \|y\| \, .$$

(iv) For each $\tilde{v} > v$ there exists $C_{\tilde{v},m}$ such that for every $x \in X$, $y \in X^*$,

$$\int_{\mathbb{R}} |\langle (R(t \pm i\tilde{v},A)^{m+2}x, y\rangle| \,\mathrm{d}t \leq C_{\tilde{v},m}\|x\| \, \|y\| \, .$$

Proof. The equivalence of (i) and (ii) has already been established in Proposition 3.5.3. Moreover, the implications $(i) \Rightarrow (iii)$ and $(i) \Rightarrow (iv)$ are a direct consequence of Proposition 4.2.6.

Assuming (iii), fix $u > v$ and consider $\psi(z) = \frac{1}{z+i\tilde{v}} - \frac{1}{z-i\tilde{v}}$, where $\tilde{v} > u$. We proceed as in the proof of Theorem 4.2.7 so that given $f \in \mathcal{H}_1^\infty(H_u)$, there exists a function $q \in L^\infty(\mathbb{R})$ for which

$$f = q * \psi \qquad \text{and} \qquad \|q\|_{L^\infty(\mathbb{R})} \leq C\|f\|_{\mathcal{H}^\infty(H_u)},$$

for some C independent of f. Then $f^{(m)} = q * \psi^{(m)}$ and we obtain

$$\begin{aligned}
\|f^{(m)}(A)\| &\leq \|q\|_{L^\infty} \sup_{x\in X_1, y \in X_1^*} \int_{\mathbb{R}} |\langle \psi^{(m)}(A-t)x, y\rangle| \,\mathrm{d}t \\
&= m! \, \|q\|_{L^\infty} \sup_{x\in X_1, y \in X_1^*} \int_{\mathbb{R}} |\langle (R(t+i\tilde{v},A)^{m+1} - R(t-i\tilde{v},A)^{m+1})x, y\rangle| \,\mathrm{d}t \\
&\leq C_m\|f\|_{\mathcal{H}^\infty(H_u)} \, .
\end{aligned}$$

By Lemma 3.5.2 we deduce that A has an m-bounded $\mathcal{H}^\infty$-calculus on H_u, giving (ii).

Finally, assuming (iv) holds, we obtain (iii) for the value $m+1$ and thus m, since (iii) has already been shown to be equivalent to (i). Thus the result is proven. ☐

The case $m = 0$ yields the following refinement of the Gomilko-Shi-Feng Theorem for groups (Theorem 2.1.13), in which the growth constraints on the integral estimates can be dropped. It also allows us to add weak integral estimates to the characterisation given by Theorem 3.2.5 on Hilbert space.

Theorem 4.3.2. *Let $A \in \mathrm{Strip}(v)$ on a Banach space X. Consider the following conditions.*

(i) For every $w > v$ there exists C_w such that for every $x \in X$ and $y \in X^*$,

$$\int_{\mathbb{R}} \|R(\pm iw + t, A)x\|^2 \,\mathrm{d}t \leq C_w \|x\|^2,$$
$$\int_{\mathbb{R}} \|R(\pm iw + t, A^*)y\|^2 \,\mathrm{d}t \leq C_w \|y\|^2.$$

(ii) For every $w > v$ there exists C_w such that for every $x \in X$ and $y \in X^*$,

$$\int_{\mathbb{R}} |\langle R(\pm iw + t, A)^2 x, y\rangle| \,\mathrm{d}t \leq C_w \|x\| \,\|y\| \,.$$

(iii) A admits a bounded $\mathcal{H}^\infty$-calculus of type v.

(iv) iA generates a C_0-group of type at most v.

(v) iA generates a C_0-semigroup of type at most v.

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). If X is a Hilbert space then (v) $\Rightarrow$ (i) also holds.

In fact, on Hilbert space we can combine the arguments used in the Gomilko-Shi-Feng Theorem with results from Section 4.1 to obtain the following characterisation of bounded $\mathcal{H}^\infty$-calculus. As a by-product, we obtain that for a strip-type operator A, the semigroup $(e^{isA})_{s\geq 0}$, defined via the natural functional calculus for A, enjoys strong continuity as soon as it satisfies certain decay estimates near zero (compare this with Example 2.1.14 (*iii*)).

Theorem 4.3.3. *Let $A \in \mathrm{Strip}(w)$ on a Hilbert space X, $v > w$, $m \geq 0$. Then the following are equivalent.*

(i) A admits a bounded $\mathcal{H}^\infty$-calculus of type v.

(ii) A admits an m-bounded $\mathcal{H}^\infty$-calculus of type v.

(iii) The family $(e^{isA})_{s\geq 0}$ form a semigroup of bounded operators such that for each $u > v$

$$\|e^{isA}\| \leq \frac{C_{m,u}}{s^m}\, e^{su} \quad (s > 0).$$

(iv) A has absolute calculus.

In this case, $w_A^H = w_A$.

Proof. $(i) \Rightarrow (ii)$ follows from the definitions. Assuming (ii), define for $s \geq 0$ the function $f_s(z) = e^{isz}$. Then for fixed $u > v$, $f \in \mathcal{H}^\infty(H_u)$ and so $s^m e^{isA}$ is bounded with

$$\|s^m e^{isA}\| = \|{f_s}^{(m)}(A)\| \leq C_{m,u}\|f_s\|_{\mathcal{H}^\infty(H_u)} = C_m e^{su}.$$

This gives (iii). Assuming (iii), we have by [35, Lemma 2.2] that for $\operatorname{Re}\lambda > v$ and $x \in D(A^2)$, the map $t \mapsto e^{-vt}e^{itA}x : [0,\infty) \to X$ is continuous and bounded with

$$R(\lambda, iA)x = \int_0^\infty e^{-\lambda t}e^{itA}x \,\mathrm{d}t \qquad (\operatorname{Re}\lambda > v),$$

for all $x \in D(A^2)$. Differentiating both sides $m-1$ times with respect to λ, we obtain for every $m > 0$ and $x \in D(A^2)$,

$$R(\lambda, iA)^{m+1}x = \frac{1}{m!}\int_0^\infty t^m e^{-\lambda t}e^{itA}x \,\mathrm{d}t \qquad (\operatorname{Re}\lambda > v)\ . \tag{4.3.1}$$

Since A is densely defined we have by Proposition 2.3.7 that A^2 is too. Moreover, the integral operator

$$x \mapsto \int_0^\infty t^m e^{-\lambda t}e^{itA}x \,\mathrm{d}t$$

is bounded by assumption (iii). Thus (4.3.1) holds for every $x \in X$. An application of Plancherel's Theorem then yields for $v < \tilde{u} < u$,

$$\begin{aligned}\int_{\mathbb{R}} \|R(t-iu, A)^{m+1}x\|^2 \,\mathrm{d}t &= \int_{\mathbb{R}} \|R(it+u, iA)^{m+1}x\|^2 \,\mathrm{d}t \\ &= \frac{2\pi}{m!}\int_0^\infty \|e^{-su}s^m e^{isA}x\|^2 \,\mathrm{d}s \\ &\leq C_{m,\tilde{u}}\int_0^\infty e^{-2s(u-\tilde{u})} \,\mathrm{d}s\ \|x\|^2 \\ &= C_{m,\tilde{u},u}\|x\|^2\ .\end{aligned}$$

Thus if $m = 0$ then the required estimate (4.1.4) holds for A on $H_{\tilde{u}}$ with the function $\psi(z) = \left(\frac{1}{z-iu}\right)^2 \in \mathcal{H}_1^\infty(H_{\tilde{u}})$ since the resolvent of A is uniformly bounded on horizontal lines. For $m > 0$, the estimate (4.1.4) holds for A on $H_{\tilde{u}}$ with $\psi(z) = \left(\frac{1}{z-iu}\right)^{m+1} \in \mathcal{H}_1^\infty(H_{\tilde{u}})$. Since also $iA^* = (iA)^*$ generates a semigroup satisfying the condition in (iii), we obtain the required dual estimate (4.1.5) for this same choice of ψ. We conclude that A has absolute calculus.

The claim $(iv) \Rightarrow (i)$ follows from Corollary 4.1.18. The final assertion is immediate from Corollary 4.1.15.

□

The implication $(iv) \Rightarrow (ii)$ in Theorem 4.3.1 can be arrived at by a more direct route, using the identity $f^{(m)}(A) = \frac{m!}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A)^{m+1}\,\mathrm{d}\lambda$. This idea is explored in the more general setting of half-plane operators in Chapter 5 (see Theorem 5.2.3 and the discussion thereafter).

We can bypass the functional calculus route and obtain a form of $(iv) \Rightarrow (iii)$ in Theorem 4.3.1 for a more general class of unbounded operators. As a compromise, we must impose growth conditions on the integral estimates involved. As preparation, we first need to generalise the result of Gomilko and Shi-Feng to higher order resolvents.

Theorem 4.3.4. *Let A be a closed, densely defined operator with $\sigma(A) \subseteq \{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$ and suppose that, for some $m \geq 2$, A satisfies*

$$\forall x \in X \ \forall y \in X^* \ \sup_{\delta > 0} \delta^{m-1} \int_{\delta - i\infty}^{\delta + i\infty} |\langle R(\lambda, A)^m x, y\rangle| \, |\mathrm{d}\lambda| < \infty. \tag{4.3.2}$$

Then A generates a uniformly bounded C_0-semigroup.

Proof. The argument is adapted from [66]. The first step is to express the resolvent power as an integral operator. Indeed, for any $\delta > 0$ the function $z \mapsto \langle R(A, z+\delta)^m x, y\rangle$ belongs to the Hardy space H^1 in the half-plane $\operatorname{Re} z > 0$. It follows that for any $x \in X$ and $y \in X^*$,

$$\langle R(t, A)^m x, y\rangle = -\frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty} \frac{\langle R(\lambda, A)^m x, y\rangle}{\lambda - t}\,\mathrm{d}\lambda \quad (0 < \delta < t).$$

For $k > m - 1$ we have

$$(-1)^{k-m+1} m(m+1)\ldots k R(t, A)^{k+1} = \frac{\mathrm{d}^{k-m+1}}{\mathrm{d}t^{k-m+1}} R(t, A)^m \quad (t > 0)\ .$$

Thus we get

$$\begin{aligned}\langle R(t, A)^{k+1} x, y\rangle &= -\frac{1}{2\pi i}\frac{(m-1)!}{k!}\int_{\delta - i\infty}^{\delta + i\infty} \frac{\mathrm{d}^{k-m+1}}{\mathrm{d}t^{k-m+1}}\Big(\frac{\langle R(\lambda, A)^m x, y\rangle}{\lambda - t}\Big)\,\mathrm{d}\lambda \\ &= \frac{(-1)^{k-m}}{2\pi i}\frac{(k-m+1)!\,(m-1)!}{k!}\int_{\delta - i\infty}^{\delta + i\infty}\frac{\langle R(\lambda, A)^m x, y\rangle}{(\lambda - t)^{k-m+2}}\,\mathrm{d}\lambda\ .\end{aligned}$$

Given our assumption (4.3.2), this implies

$$|\langle R(t, A)^{k+1} x, y\rangle| \leq \frac{C\,(k-m+1)!\,(m-1)!}{k!}\,\delta^{1-m}\,(t-\delta)^{m-k-2} \quad (0 < \delta < t).$$

Choosing $\delta = \frac{t}{k}$ gives

$$\begin{aligned} |\langle R(t,A)^{k+1}x, y\rangle| &\leq \frac{C\ (k-m+1)!\ (m-1)!}{k!}\ \big(\frac{t}{k}\big)^{1-m}\ \big(t - \frac{t}{k}\big)^{m-k-2} \\ &= C \cdot C_{k,m} \cdot t^{-(k+1)}, \end{aligned}$$

where

$$\begin{aligned} C_{k,m} &= \frac{(m-1)!\ (k-m+1)!}{k!}\ k^{m-1}\ \big(\frac{k}{k-1}\big)^{k-m+2} \\ &= (m-1)! \left(\frac{k}{k-m+2}\right) \left(\frac{k}{k-m+3}\right) \cdots \left(\frac{k}{k-m+m}\right) \left(\frac{k}{k-1}\right)^{k-1} \left(\frac{k}{k-1}\right)^{3-m}. \end{aligned}$$

For fixed m, the sequence $(C_{k,m})_k$ converges as $k \to \infty$ and so is bounded indepedently of k. Hence we have

$$\|R(t,A)^{k+1}\| \leq \frac{C_m}{t^{k+1}} \quad \forall k > m-1.$$

The rest now follows from Theorem 2.1.6 (ii), in which we only need the resolvent estimate to hold for sufficiently large powers.

□

We will show later that in fact (4.3.2) is independent of $m \in \mathbb{N}, m > 1$ (see Remark 5.2.4). The following result, and proof, generalise Corollary 4.2.9.

Theorem 4.3.5. *Let A be a closed, densely defined operator with $\sigma(A) \subseteq H_w$ and let $v > w$, $m \geq 0$. If*

$$\sup_{\tilde{v}>v} (\tilde{v}-v)^{m+1} \int_{\mathbb{R}} |\langle (R(t-i\tilde{v},A)^{m+2}x, y\rangle|\ \mathrm{d}t \leq C\|x\|\ \|y\| \ \ \forall x \in X\ \forall y \in X^*, \quad (4.3.3)$$

and for each $\tilde{v} > v$,

$$\int_{\mathbb{R}} |\langle (R(t+i\tilde{v},A)^{m+2}x, y\rangle|\ \mathrm{d}t \leq C\|x\|\ \|y\| \ \ \forall x \in X\ \forall y \in X^*, \quad (4.3.4)$$

then for each $\tilde{v} > v$

$$\int_{\mathbb{R}} |\langle (R(t+i\tilde{v},A)^{m+1} - R(t-i\tilde{v},A))^{m+1}x, y\rangle|\ \mathrm{d}t \leq C_{\tilde{v}}\|x\|\ \|y\| \ \ \forall x \in X\ \forall y \in X^*.$$

Proof. Fix $\tilde{v} > v$ and for $t \in \mathbb{R}$, set $\lambda = t + i\tilde{v}$. Then

$$\begin{aligned}
&R(\lambda, A)^{m+1} - R(\overline{\lambda}, A)^{m+1} \\
&\qquad = [R(\lambda, A) - R(\overline{\lambda}, A)] \sum_{k=0}^{m} R(\overline{\lambda}, A)^k R(\lambda, A)^{m-k} \\
&\qquad = -2iv \sum_{k=0}^{m} R(\lambda, A)^{m+1-k} R(\overline{\lambda}, A)^{k+1} \\
&\qquad = -2iv \sum_{k=0}^{m} R(\lambda, A)^{m+1-k} [R(\lambda, A) + 2ivR(\lambda, A)R(\overline{\lambda}, A)]^{k+1} \\
&\qquad = -2iv \sum_{k=0}^{m} \sum_{j=0}^{k+1} \binom{k+1}{j} R(\lambda, A)^{m+1-k} R(\lambda, A)^j [2ivR(\lambda, A)R(\overline{\lambda}, A)]^{k+1-j} \,.
\end{aligned}$$

Every term in this sum is of the form $C_{m,k,j,v} R(\lambda, A)^{m+2} R(\overline{\lambda}, A)^k$ for some $0 \leq k \leq m+1$. Moreover, by Theorem 4.3.4 and (4.3.3) the operator $iA - v$ generates a uniformly bounded C_0-semigroup, so iA generates a C_0-semigroup e^{isA} satisfying $\|e^{isA}\| \leq Ce^{sv}$, $s > 0$. Thus for $k \geq 1$ we can write

$$R(\overline{\lambda}, A)^k = R(t - i\tilde{v}, A)^k = i^k R(it + \tilde{v}, iA)^k = \frac{i^k}{(k-1)!} \int_0^\infty s^{k-1} e^{-\tilde{v}s} e^{-its} e^{isA} \mathrm{d}s.$$

Hence for $x \in X_1$ and $y \in X_1^*$ we have

$$\langle R(\lambda, A)^{m+2} R(\overline{\lambda}, A)^k x, y \rangle = \frac{i^k}{(k-1)!} \int_0^\infty s^{k-1} e^{-\tilde{v}s} e^{-its} \langle R(\lambda, A)^{m+2} e^{isA} x, y \rangle \, \mathrm{d}s.$$

Furthermore,

$$\begin{aligned}
&\int_{\mathbb{R}} \left| \int_0^\infty s^{k-1} e^{-\tilde{v}s} e^{ist} \langle R(t + i\tilde{v}, A)^{m+2} e^{isA} x, y \rangle \, \mathrm{d}s \right| \mathrm{d}t \\
&\qquad \leq \int_0^\infty s^{k-1} e^{-\tilde{v}s} \int_{\mathbb{R}} \left| \langle R(t + i\tilde{v}, A)^{m+2} e^{isA} x, y \rangle \right| \mathrm{d}t \, \mathrm{d}s \\
&\qquad \leq \frac{C}{(\tilde{v} - v)^{m+1}} \int_0^\infty s^{k-1} e^{-\tilde{v}s} \|e^{isA} x\| \, \mathrm{d}s \qquad \text{(by (4.3.4))} \\
&\qquad \leq \frac{C}{(\tilde{v} - v)^{m+1}} \int_0^\infty s^{k-1} e^{(-\tilde{v}+v)s} \, \mathrm{d}s = C_{\tilde{v}} \,.
\end{aligned}$$

Combining this with (4.3.4), we arrive at

$$\int_{\mathbb{R}} |\langle (R(t + i\tilde{v}, A)^{m+1} - R(t - i\tilde{v}, A))^{m+1} x, y \rangle| \, \mathrm{d}t \leq C_{\tilde{v}},$$

as required.

□

With the help of Theorem 4.3.1, we are almost ready to state our characterisation of bounded $\mathcal{H}^\infty$-calculus for strip-type operators in terms of Rademacher square function estimates. First we need an auxiliary lemma, adapted from [47, Lemma 12.6].

Lemma 4.3.6. *For strongly measurable, locally bounded, operator-valued functions ψ, ϕ and M on $\mathbb{R}$ for which $\{M(t) : t \in \mathbb{R}\}$ is R-bounded, $x \in X$ and $x^* \in X^*$ we have*

$$\lim_{R\to\infty} \int_{-R}^{R} |\langle \psi(t)M(t)\phi(t)x, x^*\rangle| \,\mathrm{d}t \leq$$
$$2R(\{M(t) : t \in \mathbb{R}\}) \sup_{t\in\mathbb{R}} \sup_{N} \big\| \sum_{|k|\leq N} r_k\phi(k+t)x\big\|_{L^2(X)} \big\| \sum_{|l|\leq N} r_k\psi(l+t)^*x^*\big\|_{L^2(X^*)} .$$

Proof. We have

$$\begin{aligned}
\int_{-N}^{N+1} |\langle \psi(t)M(t)\phi(t)x, x^*\rangle| \,\mathrm{d}t &= \sum_{k=-N}^{N} \int_{k}^{k+1} |\langle \psi(t)M(t)\phi(t)x, x^*\rangle| \,\mathrm{d}t \\
&\leq \sup_{t\in\mathbb{R}} \sum_{k=-N}^{N} |\langle M(k+t)\phi(k+t)x, \psi(k+t)^*x^*\rangle| \\
&\leq \sup_{t\in\mathbb{R}} \sup_{|\epsilon_k|=1} \big| \sum_{k=-N}^{N} \epsilon_k\langle M(k+t)\phi(k+t)x, \psi(k+t)^*x^*\rangle\big| \\
&= \sup_{t\in\mathbb{R}} \sup_{|\epsilon_k|=1} \big\| \sum_{k=-N}^{N} r_k^2\langle \epsilon_k M(k+t)\phi(k+t)x, \psi(k+t)^*x^*\rangle\big\|_{L^1(0,1)} \\
&\leq \sup_{t\in\mathbb{R}} \sup_{|\epsilon_k|=1} \big\|\sum_{k} r_k\epsilon_k M(k+t)\phi(k+t)x\big\|_{L^2(X)} \big\|\sum_{l} r_l\psi(l+t)^*x^*\big\|_{L^2(X^*)} .
\end{aligned}$$

The result now follows from the R-boundedness assumption on M.

□

For $v > \omega > 0$, we denote by $\mathcal{H}^\infty_{1,v}(H_\omega)$ the set of all functions $\psi \in \mathcal{H}^\infty_1(H_\omega)$ for which there exists constants $C > 0$ and $C' \geq 0$ such that $|\hat{\psi}(\eta)| \geq Ce^{-v|\eta|}$, $\eta \in \mathbb{R}$, and $|(\hat{\psi})'(\eta)| \leq C'e^{-\omega|\eta|}$, $\eta \neq 0$.

Theorem 4.3.7. *Let A be a strip-type operator. For $v, \sigma > w_A$ and a scalar function $\chi \in \mathcal{H}^\infty(H_{\sigma'})$, $\sigma' > w_A$, consider the following conditions.*

(i) A admits a bounded $\mathcal{H}^\infty$-calculus on H_σ.

(ii)

$$\sup_N \sup_{t\in\mathbb{R}} \sup_{\epsilon_k=\pm 1} \| \sum_{k=-N}^{N} \epsilon_k \chi(k+t-A)\|_{B(X)} < \infty \,.$$

(iii) A admits Rademacher square function estimates with respect to χ*.*

(iv) There exists a constant C *such that for every* $x \in X$*,* $y \in X^*$*,*

$$\int_{\mathbb{R}} |\langle \chi(\pm i\theta + t - A)x, y\rangle| \, \mathrm{d}t \le C\|x\| \, \|y\| \,.$$

Then

(a) $(i) \Rightarrow (ii) \Rightarrow (iii)$ *for every* $\chi \in \mathcal{H}_1^\infty(H_\sigma)$*.*

(b) Let $\theta < v - w_A$*,* $\psi \in \mathcal{H}_1^\infty(H_v)$*. If* (iii) *holds for the functions* $\chi(z) = \psi(\pm i\theta - z) \in \mathcal{H}_1^\infty(H_{v-\theta})$ *then* (iv) *holds for* $\chi = \psi^2$*.*

(c) If (iv) *holds for some* $\psi \in \mathcal{H}_{1,v}^\infty(H_u)$*, or for some* $\chi = \phi'$ *with* $\phi \in \mathcal{H}_{1,v}^\infty(H_u)$*, with* $w_A < u < v$*,* $0 \le \theta < u - w_A$ *and* $2v - u - \theta < \pi$*, then* (i) *holds for* $\sigma > 2v - u - \theta$*.*

(d) $(i) \Rightarrow (iv)$ *for every* $0 \le \theta < \sigma$ *and* $\chi \in \mathcal{H}_1^\infty(H_{\sigma+\theta})$*.*

In the particular case where $\chi_\pm(z) = \frac{1}{(z \pm iv)^2}$*,* $v > w_A$*, we have for* $\sigma < v$

$$\begin{aligned} A \text{ admits a bounded } \mathcal{H}^\infty\text{-calculus on } H_\sigma &\Rightarrow \{A, \chi_\pm\} \text{ satisfy Rademacher estimates} \\ &\Rightarrow \int_{\mathbb{R}} |\langle \chi_\pm(t-A)^2 x, y\rangle| \mathrm{d}t \le C\|x\| \, \|y\| \,. \end{aligned}$$

Conversely for $\sigma > v$ *we have*

$$\int_{\mathbb{R}} |\langle \chi_\pm(t-A)^2 x, y\rangle| \mathrm{d}t \le C\|x\| \, \|y\| \Rightarrow A \text{ has a bounded } \mathcal{H}^\infty\text{-calculus on } H_\sigma.$$

Proof. (a) Given that (i) holds, fix $\psi \in \mathcal{H}_1^\infty(H_\sigma)$. Note that for each $t \in \mathbb{R}$ the operator $t - A$ is strip-type with bounded $\mathcal{H}^\infty$-calculus on H_σ, with the $\mathcal{H}^\infty$-constant independent of t. Thus for some $s > 1$,

$$\begin{aligned}
&\Big\|\sum_{k=-N}^{N} \epsilon_k \psi(t+k-A)\Big\| = \Big\|\Big(\sum_{k=-N}^{N} \epsilon_k \psi(z+k)\Big)(t-A)\Big\| \le C \sup_{z\in H_\sigma} \sum_{k\in\mathbb{Z}} |\psi(z+k)| \\
&\le C \sup_{z\in H_\sigma} \left(\sum_{k=0}^{\infty} \frac{1}{(2+k+\lfloor \operatorname{Re} z\rfloor - \operatorname{Re} z)^s} + \sum_{k=0}^{\infty} \frac{1}{(2+k+\operatorname{Re} z - \lfloor \operatorname{Re} z\rfloor)^s} \right) \\
&\le C \sup_{z\in H_\sigma} \left(\int_0^\infty \frac{\mathrm{d}x}{(1+x+\lfloor \operatorname{Re} z\rfloor - \operatorname{Re} z)^s} + \int_0^\infty \frac{\mathrm{d}x}{(1+x+\operatorname{Re} z - \lfloor \operatorname{Re} z\rfloor)^s} \right) \\
&= C \sup_{z\in H_\sigma} \left(\frac{(1+\lfloor \operatorname{Re} z\rfloor - \operatorname{Re} z)^{1-s}}{s-1} + \frac{(1+\lfloor \operatorname{Re} z\rfloor - \operatorname{Re} z)^{1-s}}{s-1} \right) \le \frac{C}{s-1}\,.
\end{aligned}$$

This gives (ii). The implication $(ii) \Rightarrow (iii)$ is trivial since, for example,

$$\begin{aligned}
\int_0^1 \|\sum_k r_k(u)\psi(k+t-A)^* x^*\|^2 \,\mathrm{d}u &\le \sup_{\epsilon_k=\pm 1} \big\|\big[\sum_k \epsilon_k \psi(k+t-A)\big]^*\big\|^2 \,\|x^*\|^2 \\
&\le C\|x^*\|^2\,.
\end{aligned}$$

(b) is an immediate consequence of Lemma 4.3.6, with $M(t) = I$.

(c) The first case follows from Theorem 4.2.8. When $\psi = \phi'$, an almost identical proof to the implication $(iii) \Rightarrow (ii)$ of Theorem 4.3.1 yields that A admits a 1-bounded, and hence by Proposition 3.5.3 a bounded, $\mathcal{H}^\infty$-calculus.

(d) follows from Proposition 4.2.6.

□

We now show that in the case where A is R-strip-type, the Rademacher norms on X are independent of the choice of $\psi \in \mathcal{H}^\infty_1$, so that the implication $(ii) \Rightarrow (i)$ of Theorem 4.3.7 holds for any such ψ. Moreover, such a pair $\{A, \psi\}$ admit Rademacher estimates if and only if this norm is equivalent to the original norm on X. In particular, this equivalence of norms characterises the bounded H^∞-calculus for such operators. Analogous results hold in the sectorial case for the class H^∞_0, from which the proofs are adapted (see [47, Chapter 12]). We start with a preparatory lemma.

Lemma 4.3.8. *Let $M, N : \mathbb{R} \to B(X)$ be strongly measurable, bounded functions, $h \in L^1(\mathbb{R})$. Define*

$$M(t) = \int_{\mathbb{R}} h(t-s)N(s)\,\mathrm{d}s, \qquad t \in \mathbb{R}.$$

Then there exists a constant C such that for all $x \in X$,

$$\sup_{t\in\mathbb{R}} \|\sum_{k\in\mathbb{Z}} r_k M(k+t)x\|_{L^1(X)} \leq C \sup_{t\in\mathbb{R}} \|\sum_{k\in\mathbb{Z}} r_k N(k+t)x\|_{L^1(X)}.$$

Proof. For every $k \in \mathbb{Z}$ and $t \in \mathbb{R}$ we have

$$M(t+k)x = \sum_{j\in\mathbb{Z}} \int_0^1 h((t+k)-(s+j))N(s+j)x \,\mathrm{d}s \,.$$

Thus

$$\begin{aligned}\|\sum_k r_k(\cdot)M(t+k)x\| &\leq \int_0^1 \|\sum_j \sum_k r_k(\cdot)h(t+k-s-j)N(s+j)x\| \,\mathrm{d}s \\ &\leq \sum_l \int_0^1 \|\sum_j r_{j-l}(\cdot)h(t-s-l)N(s+j)x\| \,\mathrm{d}s \,.\end{aligned}$$

Then by Fubini we obtain

$$\begin{aligned}\|\sum_{k\in\mathbb{Z}} r_k M(k+t)x\|_{L^1(X)} &\leq 2\sum_l \int_0^1 |h(t-s-l)|\mathrm{d}s \left(\sup_{s\in[0,1]} \|\sum_j r_{j-l}N(s+j)x\|_{L^1(X)}\right) \\ &\leq 2\|h\|_{L^1(\mathbb{R})}\left(\sup_{s\in[0,1]} \|\sum_j r_j N(s+j)x\|_{L^1(X)}\right) \,.\end{aligned}$$

The result now follows.

□

Proposition 4.3.9. *Let $A \in R$-Strip(w), $v > w$, $\phi, \psi \in \mathcal{H}_1^\infty(H_v)$ non-zero. Then there exists a constant C such that for every $x \in X$,*

$$\frac{1}{C}\|x\|_{R,\phi} \leq \|x\|_{R,\psi} \leq C\|x\|_{R,\phi} \,.$$

Proof. Choose $g \in \mathcal{H}_1^\infty(H_v)$ such that

$$\int_{\mathbb{R}} g(t)\phi(t) \,\mathrm{d}t = 1 \,.$$

As in Proposition 4.1.14,

$$\int_{\mathbb{R}} g(t-A)\phi(t-A) \,\mathrm{d}t = I \,.$$

Then for $s \in \mathbb{R}$ and $w < u < v$ we have by Fubini,

$$\begin{aligned}
\psi(s-A) &= \int_{\mathbb{R}} [\psi(s-A)g(t-A)]\phi(t-A)\ \mathrm{d}t \\
&= \int_{\mathbb{R}} \left[\frac{1}{2\pi i}\int_{\partial H_u} \psi(s-\lambda)g(t-\lambda)R(\lambda,A)\ \mathrm{d}\lambda\right]\phi(t-A)\ \mathrm{d}t \\
&= \frac{1}{2\pi i}\int_{\partial H_u} \psi(s-\lambda)R(\lambda,A)\left(\int_{\mathbb{R}} g(t-\lambda)\phi(t-A)\ \mathrm{d}t\right)\ \mathrm{d}\lambda \\
&= \frac{1}{2\pi i}\int_{\mathbb{R}} \psi(s-iu-t)M(iu+t)\ \mathrm{d}t - \frac{1}{2\pi i}\int_{\mathbb{R}} \psi(s+iu-t)M(-iu+t)\ \mathrm{d}t\ ,
\end{aligned}$$

where $M(\lambda) = R(\lambda, A)N(\lambda)$, $N(\lambda) = \int_{\mathbb{R}} g(t-\lambda)\phi(t-A)\ \mathrm{d}t$. The R-boundedness of $\{R(\lambda, A) :\ \lambda \in H_v\}$, along with Lemma 4.3.8, gives

$$\begin{aligned}
\|x\|_{R,\psi} &\leq C \sup_{\pm}\sup_{s\in\mathbb{R}} \|\sum_j r_j M(\pm iu + s + j)x\|_{L^p(X)} \\
&\leq C \sup_{\pm}\sup_{s\in\mathbb{R}} \|\sum_j r_j N(\pm iu + s + j)x\|_{L^p(X)} \\
&\leq C\|x\|_{R,\phi}\ .
\end{aligned}$$

The result now follows by symmetry.

□

Theorem 4.3.10. *Let $A \in R$-Strip(w) on a Banach space.*

(i) If A has a bounded $\mathcal{H}^\infty$-calculus on H_v, $v > w$ then for every $u > v$ and $\phi \in \mathcal{H}_1^\infty(H_u)$ there exists a constant C such that for every $x \in X$,

$$\frac{1}{C}\|x\| \leq \|x\|_{R,\phi} \leq C\|x\|\ . \tag{4.3.5}$$

(ii) If (4.3.5) holds for some $\phi \in \mathcal{H}_1^\infty(H_v)$, $v > w$, then A admits a bounded $\mathcal{H}^\infty$-calculus on H_u for every $u > v$.

Proof. (*i*) Given that A has a bounded $\mathcal{H}^\infty(H_v)$-calculus, $u > v$ and $\phi \in \mathcal{H}_1^\infty(H_u)$, choose $g \in \mathcal{H}_1^\infty(H_u)$ such that

$$\int_{\mathbb{R}} \phi(t)g(t)\ \mathrm{d}t = 1\ .$$

As in Proposition 4.1.14, we have $\int_{\mathbb{R}} \phi(t-A)g(t-A)\ \mathrm{d}t = I$. Then by Lemma 4.3.6

and statement (a) of Theorem 4.3.7, we have for every $x \in X, y \in X^*$,

$$|\langle x, y\rangle| = \left|\left\langle \int_{\mathbb{R}} \phi(t-A)g(t-A)x \, \mathrm{d}t, y \right\rangle\right| \leq \int_{\mathbb{R}} |\langle \phi(t-A)x, g(t-A)^* y\rangle| \, \mathrm{d}t$$
$$\leq C\|x\|_{R,\phi} \, \|y\|_{R,g} \leq C\|x\|_{R,\phi} \, \|y\| \, .$$

Thus for every $x \in X$ we have $\|x\| \leq C\|x\|_{R,\phi}$. The reverse inequality of (4.3.5) is another consequence of Theorem 4.3.7 (a).

(ii) Given the assumptions, note that for $f \in \mathcal{H}^\infty(H_u)$, $t \in \mathbb{R}$ and $v' \in (w, v)$,

$$f(A)\phi(t-A) = \frac{1}{2\pi i} \int_{\partial H_{v'}} \phi(t-\lambda)[f(\lambda)R(\lambda, A)] \, \mathrm{d}\lambda \, .$$

Since $\{f(\lambda)R(\lambda, A) : \ \lambda \in \partial H_{v'}\}$ is R-bounded, by Proposition 3.7.6 we deduce that there is a constant C such that for every $f \in \mathcal{H}^\infty(H_u)$,

$$\mathcal{R}\{f(A)\phi(t-A) : \ t \in \mathbb{R}\} \leq C\|f\|_{\mathcal{H}^\infty(H_u)} \, .$$

Along with Proposition 4.3.9 this yields for every $f \in \mathcal{H}^\infty(H_u)$ and $x \in X$,

$$\begin{aligned}\|f(A)x\| &\leq C\|f(A)x\|_{R,\phi} \leq C\|f(A)x\|_{R,\phi^2} \\ &= \sup_{t\in\mathbb{R}} \|\sum_k r_k[f(A)\phi(t+k-A)]\phi(t+k-A)x\|_{L^p(X)} \\ &\leq C\|f\|_{\mathcal{H}^\infty(H_u)}\|x\|_{R,\phi} \leq C\|f\|_{\mathcal{H}^\infty(H_u)}\|x\| \, .\end{aligned}$$

The result now follows.

□

We can apply our results for strip-type operators to broaden the class of functions which appear in the characterisation of the bounded $\mathcal{H}^\infty$-calculus for sectorial operators (Theorem 4.2.5). Let A be a sectorial operator and set $B = \log A$. Given $\psi \in \mathcal{H}^\infty_{0,v}(\Sigma_\theta)$, $\theta \in (\phi_A, \pi)$, note that $\chi := \psi \circ \exp \in \mathcal{H}^\infty_{1,v}(H_\theta)$. Then Theorem 4.3.7 applied to the operator B and function χ, along with Proposition 3.4.3, yield the following result.

Theorem 4.3.11. *Let A be a sectorial operator of type ω.*

(i) Let $\omega < \mu < \nu < \pi$ and $2\nu - \mu < \eta < \pi$. Suppose that for some $\psi \in \mathcal{H}^\infty_{0,\nu}(\Sigma_\mu)$ there exists a constant C such that

$$\int_0^\infty |\langle A\psi'(tA)x, y\rangle| \mathrm{d}t \leq C\|x\|\|y\| \ \ \forall x \in X \ \forall y \in X^*. \qquad (4.3.6)$$

Then A admits a bounded $\mathcal{H}^\infty$-calculus on Σ_η.

(ii) Suppose that $\omega < \mu < \nu < \pi$, $0 \le \theta < \mu - \omega$ and $2\nu - \mu - \theta < \eta < \pi$. If

$$\int_0^\infty |\langle A\psi'(te^{\pm i\theta}A)x, y\rangle| \mathrm{d}t \le C\|x\|\|y\| \;\; \forall x \in X \; \forall y \in X^* \, . \tag{4.3.7}$$

for some $\psi \in \mathcal{H}^\infty_{0,\nu}(\Sigma_\mu)$ then A admits a bounded $\mathcal{H}^\infty$-calculus of type η.

We remark that Theorem 4.3.11 can be arrived at more directly by showing that the estimates (4.3.6) and (4.3.7) imply that $Af'(A) \in B(X)$ for each $f \in \mathcal{H}^\infty(\Sigma_\eta)$ with $\|Af'(A)\| \le C\|f\|_{\mathcal{H}^\infty(\Sigma_\eta)}$ (see Corollary 3.5.5).

Revisiting the examples of 4.2.4, we arrive at the following estimates.

(i) $\psi(z) = \frac{\alpha}{z-\alpha} - \frac{\overline{\alpha}}{z-\overline{\alpha}}$, $\alpha \notin \Sigma_\mu$: if A satisfies (4.3.6) for this choice of ψ, then by Theorem 4.3.11 A admits a bounded $\mathcal{H}^\infty$-calculus of type $|\arg\alpha|$.

(ii) $\psi(z) = ze^{-z}$, defined on $\Sigma_{\frac{\pi}{2}}$: if A is sectorial of type $\omega < \frac{\pi}{2}$ and

$$\int_0^\infty |\langle A(-tA + I)e^{-tA}x, y\rangle| \, \mathrm{d}t \le C\|x\| \, \|y\| \quad \forall x \in X, y \in X^*,$$

then A has a bounded $\mathcal{H}^\infty$-calculus of type $\frac{\pi}{2}$.

For $\theta \in (0, \frac{\pi}{2} - \omega)$, it follows from (4.3.7) that if

$$\int_0^\infty |\langle A(-te^{\pm i\theta}A + I)e^{-te^{\pm i\theta}A}x, y\rangle| \, \mathrm{d}t \le C\|x\| \, \|y\| \quad \forall x \in X, y \in X^*,$$

then by Theorem 4.3.11 A admits a bounded $\mathcal{H}^\infty$-calculus of type $\frac{\pi}{2} - \theta$.

(iii) $\psi(z) = \frac{z}{(1+z)^2}$: if A is sectorial of angle $\omega < \pi$, $0 \le \theta < \pi - \omega$ and

$$\int_0^\infty |\langle A(t - e^{\pm i\theta}A)(t + e^{\pm i\theta}A)^{-3}x, y\rangle| \, \mathrm{d}t \le C\|x\| \, \|y\| \quad \forall x \in X, y \in X^*,$$

then by Theorem 4.3.11 A admits a bounded $\mathcal{H}^\infty$-calculus of type $\pi - \theta$.

Similar estimates can also be obtained for higher order derivatives, using the fact that m-boundedness is equivalent to boundedness of the $\mathcal{H}^\infty$-calculus for any $m \ge 0$.

4.4 Square function estimates for a pair of commuting operators

In this section we characterise boundedness of the joint functional calculus of a pair of commuting sectorial operators in terms of weak integral estimates. The approach is a natural extension of the single operator case, where we now employ the Fourier transform in two dimensions, defined for $f \in L^1(\mathbb{R}^2)$ by

$$\hat{f}(\lambda,\mu) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\lambda x}e^{-i\mu y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y\ ,$$

with corresponding inverse. Here we gather some basic properties of the Fourier transform in two dimensions, which can be found in [67, Section 6.2]. Given a two-variable function $\psi \in \mathcal{H}_0^\infty(\Sigma_\theta \times \Sigma_{\theta'})$ and $x, y \in \mathbb{R}$, we denote $\psi_e(x,y) = \psi(e^x, e^y)$. Furthermore we define $(f * g)(s,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(u,v)g(s-u,t-v)\,\mathrm{d}u\,\mathrm{d}v$.

(a) Given $\psi \in \mathcal{H}_0^\infty(\Sigma_\theta \times \Sigma_{\theta'})$, there exists a constant C such that for all $\lambda, \mu \in \mathbb{R}$,

$$|\hat{\psi}_e(\lambda,\mu)| + \Big|\frac{\partial}{\partial\lambda}\hat{\psi}_e(\lambda,\mu)\Big| + \Big|\frac{\partial}{\partial\mu}\hat{\psi}_e(\lambda,\mu)\Big| + \Big|\frac{\partial^2}{\partial\lambda\partial\mu}\hat{\psi}_e(\lambda,\mu)\Big| \le Ce^{-\theta|\lambda|}e^{-\theta'|\mu|}\ .$$

since, for example, we have by Cauchy's Theorem, for every $\lambda, \mu \in \mathbb{R}$ and $\nu \in (-\lambda, \lambda)$, $\nu' \in (-\mu, \mu)$,

$$\begin{aligned}
\Big|\frac{\partial}{\partial\lambda}\hat{\psi}_e(\lambda,\mu)\Big| &= \Big|\int_{\mathbb{R}}\int_{\mathbb{R}} xe^{-i\lambda x}e^{-i\mu y}\psi_e(x,y)\,\mathrm{d}x\mathrm{d}y\Big| \\
&= \Big|\int_0^\infty\int_0^\infty (\log u)u^{-i\lambda}v^{-i\mu}\psi(u,v)\,\frac{\mathrm{d}u}{u}\frac{\mathrm{d}v}{v}\Big| \\
&= \Big|\int_0^\infty\int_0^\infty (\log se^{i\nu})(se^{i\nu})^{-i\lambda}(te^{i\nu'})^{-i\mu}\psi(se^{i\nu},te^{i\nu'})\,\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}\Big| \\
&\le e^{\nu\lambda}e^{\nu'\mu}\int_0^\infty\int_0^\infty (|\log s| + |\nu|)\psi(se^{i\nu},te^{i\nu'})\,\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}\Big| \\
&\le Ce^{\nu\lambda}e^{\nu'\mu}\ .
\end{aligned}$$

Letting $\nu \to \pm\theta$ and $\nu' \to \pm\theta'$ we arrive at $\big|\frac{\partial}{\partial\lambda}\hat{\psi}_e(\lambda,\mu)\big| \le Ce^{-\theta|\lambda|}e^{-\theta'|\mu|}$ for every $\lambda, \mu \in \mathbb{R}$.

(b) For any $f, g \in L^1(\mathbb{R}^2)$, $(f * g)\hat{} = \hat{f}\hat{g}$.

(c) For any $f \in L^1(\mathbb{R}^2)$ such that $\hat{f} \in L^1(\mathbb{R}^2)$, $f = \frac{1}{4\pi^2}(\hat{f})\check{}$.

(d) For any $\psi \in \mathcal{H}_0^\infty(\Sigma_\theta \times \Sigma_{\theta'})$ and $\alpha, \beta \in \mathbb{R}$,

$$(\hat{\psi}_e(\lambda,\mu)e^{\alpha\lambda}e^{\beta\mu})\check{}(s,t) = \psi_e(s - i\alpha, t - i\beta)\ .$$

Our first result is the natural analogue of the converse direction of Theorem 4.2.2. The proof is almost identical and omitted.

Theorem 4.4.1. *Let (S,T) be a pair of commuting sectorial operators. Suppose further that (S,T) admit a joint bounded $\mathcal{H}^\infty$-calculus on $\Sigma_\lambda \times \Sigma_\mu$. Then for any $\Psi \in \mathcal{H}_0^\infty(\Sigma_\lambda \times \Sigma_\mu)$ there exists a constant C such that for every $x \in X, y \in X^*$,*

$$\int_0^\infty \int_0^\infty |\langle \Psi(sS,tT)x,y\rangle| \frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t} \leq C\|x\|\,\|y\| \,.$$

Looking to the converse, we begin with the following version of Theorem 4.2.2.

Theorem 4.4.2. *Let (S,T) be a pair of commuting sectorial operators. Suppose $\phi_S < \phi < \nu < \pi$, $\phi_T < \phi' < \omega < \pi$, $4\nu - 3\phi < \pi$ and $4\omega - 3\phi' < \pi$. Suppose there exists $\Psi \in \mathcal{H}_0^\infty(\Sigma_\phi \times \Sigma_{\phi'})$ such that*

(i) $|\hat{\Psi}_e(\lambda,\mu)| \geq Ce^{-\nu|\lambda|}e^{-\omega|\mu|} \quad \forall\, \lambda,\mu \in \mathbb{R}$ *and*

(ii)
$$\int_0^\infty \int_0^\infty |\langle \Psi(sS,tT)x,y\rangle| \frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t} \leq C\|x\|\,\|y\|$$
for every $x \in X, y \in X^*$.

Then (S,T) admits a bounded joint $\mathcal{H}^\infty$-calculus of type $(4\nu - 3\phi, 4\omega - 3\phi')$.

Proof. The proof is guided by [16]. We fix $4\nu - 3\phi < \eta < \pi$ and $4\omega - 3\phi' < \eta' < \pi$ and then choose α,β such that $4\nu - 3\phi < \alpha < \eta$ and $4\omega - 3\phi' < \beta < \eta'$. Define $\gamma : \mathbb{R}_+ \times \mathbb{R}_+$ by

$$\hat{\gamma_e}(\lambda,\mu) = \frac{1}{\hat{\Psi}_e(\lambda,\mu)\cosh(\alpha\lambda)\cosh(\beta\mu)} \,.$$

Assumption (i) on Ψ and property (a) (page 85) allows us to make the following estimates:

$$
\begin{aligned}
|\hat{\gamma_e}(\lambda,\mu)| &\le Ce^{(\nu-\alpha)|\lambda|}e^{(\omega-\beta)|\mu|},\\
\Big|\frac{\partial}{\partial\lambda}(\hat{\gamma_e}(\lambda,\mu))\Big| &= \Bigg|\frac{1}{\cosh(\beta\mu)}\Bigg(\frac{-\frac{\partial}{\partial\lambda}\hat{\Psi}_e(\lambda,\mu)\cosh(\alpha\lambda)-\alpha\sinh(\alpha\lambda)\hat{\Psi}_e(\lambda,\mu)}{\hat{\Psi}_e(\lambda,\mu)^2\cosh^2(\alpha\lambda)}\Bigg)\Bigg|\\
&\le Ce^{(2\nu-\alpha-\phi)|\lambda|}e^{(2\omega-\beta-\phi')|\mu|},\\
\Big|\frac{\partial}{\partial\mu}(\hat{\gamma_e}(\lambda,\mu))\Big| &\le Ce^{(2\nu-\alpha-\phi)|\lambda|}e^{(2\omega-\beta-\phi')|\mu|} \text{ (by symmetry)},\\
\Big|\frac{\partial^2}{\partial\mu\partial\lambda}(\hat{\gamma_e}(\lambda,\mu))\Big| &= \Bigg|\frac{1}{\cosh(\beta\mu)}\Bigg[\frac{1}{\cosh(\alpha\lambda)}\Bigg(\frac{\hat{\Psi}_e(\lambda,\mu)^2\frac{\partial^2}{\partial\lambda\partial\mu}\hat{\Psi}_e(\lambda,\mu)}{\hat{\Psi}_e(\lambda,\mu)^4}-\\
&\qquad\frac{2\frac{\partial}{\partial\lambda}\hat{\Psi}_e(\lambda,\mu)\frac{\partial}{\partial\mu}\Psi_e(\lambda,\mu)\Psi_e(\lambda,\mu)}{\hat{\Psi}_e(\lambda,\mu)^4}\Bigg)-\frac{\alpha\sinh(\alpha\lambda)\frac{\partial}{\partial\mu}\hat{\Psi}_e(\lambda,\mu)}{\cosh^2(\alpha\lambda)\hat{\Psi}_e(\lambda,\mu)^2}\Bigg]\\
&\qquad-\frac{\beta\sinh(\beta\mu)}{\cosh^2(\beta\mu)}\Bigg(\frac{\frac{\partial}{\partial\lambda}\hat{\Psi}_e(\lambda,\mu)}{\hat{\Psi}_e(\lambda,\mu)^2\cosh(\alpha\lambda)}+\frac{\alpha\sinh(\alpha\lambda)}{\hat{\Psi}_e(\lambda,\mu)\cosh^2(\alpha\lambda)}\Bigg)\Bigg|\\
&\le C\{e^{(4\nu-\alpha-3\phi)|\lambda|}e^{(4\omega-\beta-3\phi')|\mu|}+e^{(2\nu-\alpha-\phi)|\lambda|}e^{(2\omega-\beta-\phi')|\mu|}\\
&\qquad+e^{(\nu-\alpha)|\lambda|}e^{(\omega-\beta)|\mu|}\}\\
&\le C\{e^{(4\nu-\alpha-3\phi)|\lambda|}e^{(4\omega-\beta-3\phi')|\mu|}\}\ .
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&(1+ix)(1+iy)\gamma_e(x,y)=\\
&\frac{1}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{i\lambda x}e^{i\mu y}\big(\hat{\gamma_e}(\lambda,\mu)-\frac{\partial}{\partial\lambda}\hat{\gamma_e}(\lambda,\mu)-\frac{\partial}{\partial\mu}\hat{\gamma_e}(\lambda,\mu)+\frac{\partial^2}{\partial\lambda\partial\mu}\hat{\gamma_e}(\lambda,\mu)\big)\,\mathrm{d}\lambda\,\mathrm{d}\mu\ .
\end{aligned}
$$

Thus by Hölder's inequality, the Plancherel theorem and the estimates above we obtain

$$
\begin{aligned}
\int_0^\infty\int_0^\infty\big|\gamma(s,t)\big|\,\frac{\mathrm{d}s}{s}\,\frac{\mathrm{d}t}{t} &= \int_{\mathbb{R}}\int_{\mathbb{R}}\big|\gamma_e(x,y)\big|\,\mathrm{d}x\,\mathrm{d}y\\
&\le\Big(\int_{\mathbb{R}^2}\big|(1+ix)(1+iy)\gamma_e(x,y)\big|^2\mathrm{d}(x,y)\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^2}\Big|\frac{1}{(1+ix)(1+iy)}\Big|^2\mathrm{d}(x,y)\Big)^{\frac{1}{2}}\\
&\le C\Big(\int_{\mathbb{R}^2}|\hat{\gamma_e}(\lambda,\mu)-\frac{\partial}{\partial\lambda}\hat{\gamma_e}(\lambda,\mu)-\frac{\partial}{\partial\mu}\hat{\gamma_e}(\lambda,\mu)+\frac{\partial^2}{\partial\lambda\partial\mu}\hat{\gamma_e}(\lambda,\mu)|^2\,\mathrm{d}(\lambda,\mu)\Big)\\
&\le C\Big(\int_{\mathbb{R}^2}e^{2(4\nu-\alpha-3\phi)|\lambda|}e^{2(4\omega-\alpha-3\phi')|\mu|}\,\mathrm{d}(\lambda,\mu)\Big)^{\frac{1}{2}}\\
&<\infty\ .
\end{aligned}
$$

Given $f \in \mathcal{H}_0^\infty(\Sigma_\eta \times \Sigma_{\eta'})$, we define $\delta : \mathbb{R}_+ \times \mathbb{R}_+$ by

$$\hat{\delta}_e(\lambda, \mu) = \hat{f}_e(\lambda, \mu)\hat{\gamma}_e(\lambda, \mu) \cosh(\alpha\lambda) \cosh(\beta\mu) \ .$$

By (d) (page 85) we have

$$|(\hat{f}_e e^{\pm\alpha\cdot} e^{\pm\beta\cdot})(\tau, \tau')| = |f_e(\tau \mp i\alpha, \tau' \mp i\beta)| \leq \|f\|_{\mathcal{H}^\infty(\Sigma_\eta \times \Sigma_{\eta'})} \ .$$

Together with the fact that $\gamma_e \in L^1(\mathbb{R}^2)$, this shows that $\delta \in L^\infty(\mathbb{R}^2)$ with $\|\delta\|_\infty \leq \|f\|_{\mathcal{H}^\infty(\Sigma_\eta \times \Sigma_{\eta'})}$. We have

$$\hat{\delta}_e(\lambda, \mu)\hat{\Psi}_e(\lambda, \mu) = \hat{f}_e(\lambda, \mu)$$

and, inverting the Fourier transform and employing (b) (page 85), we have

$$f_e(z, z') = \int_{\mathbb{R}} \int_{\mathbb{R}} \delta_e(\tau, \tau')\Psi_e(z - \tau, z' - \tau') \, \mathrm{d}\tau \mathrm{d}\tau',$$

first for $(z, z') \in \mathbb{R}^2$ and then for $(z, z') \in H_\phi \times H_{\phi'}$ by analytic continuation. This yields

$$f(\zeta, \zeta') = \int_0^\infty \int_0^\infty \delta(s, t)\Psi(\frac{\zeta}{s}, \frac{\zeta'}{t}) \, \frac{\mathrm{d}s}{s} \frac{\mathrm{d}t}{t},$$

for $(\zeta, \zeta') \in \Sigma_\phi \times \Sigma_{\phi'}$. The rest follows as in the single operator case (see, for example, [16] or Theorem 4.2.7 for the case of strip-type operators). □

Corollary 4.4.3. *Let (S, T) be as above, $\phi_S < \nu < \pi, \phi_T < \omega < \pi$. Suppose there exists Ψ such that $\Psi \in \mathcal{H}_0^\infty(\Sigma_\phi \times \Sigma_{\phi'})$ for every $\phi \in (0, \nu)$ and $\phi' \in (0, \omega)$ and that $|\hat{\Psi}_e(\lambda, \mu)| \geq Ce^{-\nu|\lambda|}e^{-\omega|\mu|}$ for every $\lambda, \mu \in \mathbb{R}$. Suppose further that*

$$\int_0^\infty \int_0^\infty |\langle \Psi(sS, tT)x, y\rangle| \, \frac{\mathrm{d}s}{s} \, \frac{\mathrm{d}t}{t} \leq C\|x\| \, \|y\|$$

for every $x \in X, y \in X^*$. Then (S, T) admits a bounded joint $\mathcal{H}^\infty$-calculus of type (ν, ω).

Again, we can obtain bounded $\mathcal{H}^\infty$-calculus on smaller sectors by integrating along the boundary of appropriate sectors.

Theorem 4.4.4. *Let (S, T) be a pair of commuting sectorial operators. Suppose $\phi_S < \phi < \nu < \pi$, $\phi_T < \phi' < \omega < \pi$, $0 \leq \theta < \phi - \phi_S$, $0 \leq \theta' < \phi' - \phi_T$, $4\nu - 3\phi - \theta < \pi$ and $4\omega - 3\phi' - \theta' < \pi$. Suppose there exists $\Psi \in \mathcal{H}_0^\infty(\Sigma_\phi \times \Sigma_{\phi'})$ such that* *(i)* $|\hat{\Psi}_e(\lambda,\mu)| \geq Ce^{-\nu|\lambda|}e^{-\omega|\mu|} \quad \forall\, \lambda,\mu \in \mathbb{R}$ *and*

(ii)

$$\int_0^\infty \int_0^\infty |\langle \Psi(se^{\pm i\theta}S, te^{\pm i\theta'}T)x, y\rangle| \, \frac{\mathrm{d}s}{s}\, \frac{\mathrm{d}t}{t} \leq C\|x\|\,\|y\|$$

for every $x \in X, y \in X^*$.

Then (S,T) *admits a bounded joint* $\mathcal{H}^\infty$*-calculus of type* $(4\nu - 3\phi - \theta, 4\omega - 3\phi' - \theta')$.

Proof. We proceed as in the proof of Theorem 4.2.7, but now define γ by

$$\hat{\gamma_e}(\lambda,\mu) = \frac{1}{\hat{\Psi}_e(\lambda,\mu)\cosh((\alpha+\theta)\lambda)\cosh((\beta+\theta')\mu)},$$

where $4\nu - 3\phi - \theta < \alpha < \eta < \pi$ and $4\omega - 3\phi' - \theta' < \eta' < \pi$. In exactly the same way as before, we use estimates on γ and its derivatives, along with the assumptions on the underlying parameters, to establish $\gamma_e \in L^1(\mathbb{R}^2)$. Having fixed $f \in \mathcal{H}_0^\infty(\Sigma_\phi \times \Sigma_{\phi'})$, we define δ as in Theorem 4.4.2. Then

$$\hat{\delta_e}(\lambda,\mu) = \frac{1}{4}\sum_{i=1}^{4} \hat{\delta_e^i}(\lambda,\mu),$$

where each δ^i is defined by $\hat{\delta_e^i}(\lambda,\mu) = \hat{\gamma_e}(\lambda,\mu)\hat{f_e}(\lambda,\mu)e^{\pm\alpha\lambda}e^{\pm\beta\mu}$. Then by (d) (page 85) and the fact that $\gamma_e \in L^1(\mathbb{R}^2)$, we have that each $\delta^i \in L^\infty(\mathbb{R}^2)$ with $\|\delta^i\|_\infty \leq C\|f\|_{\mathcal{H}_0^\infty(\Sigma_\phi \times \Sigma_{\phi'})}$.

Furthermore, we have

$$\begin{aligned}\hat{f_e}(\lambda,\mu) &= \hat{\delta_e^1}(\lambda,\mu)\hat{\Phi}_e(\lambda,\mu)e^{\theta\lambda}e^{\theta'\mu} + \hat{\delta_e^2}(\lambda,\mu)\hat{\Phi}_e(\lambda,\mu)e^{\theta\lambda}e^{-\theta'\mu} \\ &\quad + \hat{\delta_e^3}(\lambda,\mu)\hat{\Phi}_e(\lambda,\mu)e^{-\theta\lambda}e^{\theta'\mu} + \hat{\delta_e^4}(\lambda,\mu)\hat{\Phi}_e(\lambda,\mu)e^{-\theta\lambda}e^{-\theta'\mu}\,.\end{aligned}$$

Inverting the Fourier transform gives

$$\begin{aligned}f_e(z,z') &= \int_{\mathbb{R}}\int_{\mathbb{R}} \delta_e^1(\tau,\tau')\Psi_e(z - i\theta - \tau, z' - i\theta' - \tau')\,\mathrm{d}\tau\,\mathrm{d}\tau' + \ldots \\ &\quad + \int_{\mathbb{R}}\int_{\mathbb{R}} \delta_e^4(\tau,\tau')\Psi_e(z + i\theta - \tau, z' + i\theta' - \tau')\,\mathrm{d}\tau\,\mathrm{d}\tau'\,,\end{aligned}$$

first for $z, z' \in \mathbb{R}$ and then $(z,z') \in \Sigma_\phi \times \Sigma_{\phi'}$. Then

$$\begin{aligned}f(\zeta,\zeta') &= \int_0^\infty\int_0^\infty \delta^1(s,t)\Psi\Big(\frac{\zeta e^{-i\theta}}{s}, \frac{\zeta' e^{-i\theta'}}{t}\Big)\,\frac{\mathrm{d}s}{s}\,\frac{\mathrm{d}t}{t} + \ldots \\ &\quad + \int_0^\infty\int_0^\infty \delta^4(s,t)\Psi\Big(\frac{\zeta e^{i\theta}}{s}, \frac{\zeta' e^{i\theta'}}{t}\Big)\,\frac{\mathrm{d}s}{s}\,\frac{\mathrm{d}t}{t},\end{aligned}$$

for $(\zeta, \zeta') \in \Sigma_\phi \times \Sigma_{\phi'}$. The rest now follows as in the proof of Theorem 4.4.2. $\square$

Remark 4.4.5. Suppose Ψ is of the form $\Psi(\lambda, \mu) = f(\lambda)g(\mu)$, with $f \in \mathcal{H}^\infty_{0,\nu}(\Sigma_\phi), g \in \mathcal{H}^\infty_{0,\omega}(\Sigma_{\phi'})$. Then Ψ satisfies the required Fourier estimate $|\hat{\Psi}(\lambda, \mu)| \geq Ce^{-\nu|\lambda|}e^{-\omega|\mu|}$. Thus, using the concrete examples in 4.2.4 we can derive sufficient (and necessary, by Theorem 4.4.1) conditions for the pair (S, T) to have a bounded joint $\mathcal{H}^\infty$-calculus.

The following result can similarly be obtained for a pair of (commuting) strip-type operators. We omit the details of the proof, which are analogous to those of Theorems 4.4.2 and 4.4.4.

Theorem 4.4.6. *Let (A, B) be a pair of commuting strip-type operators. Suppose $w_A < \phi < v$, $w_B < \phi' < w$, $0 \leq u < \phi - w_A$, $0 \leq \theta' < \phi' - w_B$, $4v - 3\phi - \theta$ and $4w - 3\phi' - \theta' < \pi$. Suppose there exists $\Psi \in \mathcal{H}^\infty_1(H_\phi \times H_{\phi'})$ such that*

(i) $|\hat{\Psi}(\lambda, \mu)| \geq Ce^{-v|\lambda|}e^{-w|\mu|} \quad \forall\, \lambda, \mu \in \mathbb{R}$ *and*

(ii)
$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\langle \Psi(\pm i\theta + s - A, \pm i\theta' + t - B)x, y\rangle| \, \mathrm{d}s\mathrm{d}t \leq C\|x\| \, \|y\|$$
for every $x \in X, y \in X^*$.

Then (A, B) admits a bounded joint $\mathcal{H}^\infty$-calculus of type $(4v - 3\phi - \theta, 4w - 3\phi' - \theta')$.

As in the single operator case (see Section 4.3), we can then employ (k, l)-boundedness (Proposition 3.6.5) to broaden the integral estimates in Theorems 4.4.4 and 4.4.6. For example, using the fact that $(1, 1)$-boundedness is equivalent to bounded $\mathcal{H}^\infty$-calculus, the following estimates are permissible in Theorem 4.4.4 for the same choice of Ψ:

$$\int_0^\infty \int_0^\infty |\langle ST \frac{\partial^2}{\partial z \partial w} \Psi(se^{\pm i\theta}S, te^{\pm i\theta'}T)x, y\rangle| \, \mathrm{d}s\mathrm{d}t \leq C\|x\| \, \|y\|$$

for every $x \in X, y \in X^*$.

5 Functional Calculus for half-plane operators

Much of our study of strip-type operators has been from the viewpoint of their connection with sectorial operators via the exponential and logarithm functions. In fact, on Hilbert space we obtain a complete characterisation of bounded $\mathcal{H}^\infty$-calculus for strip-type operators through Theorem 3.2.5, which tells us that strip-type operators with this property are precisely generators of a C_0-group, or even just those strip-type operators which are also generators of a C_0-semigroup. However, there exist C_0-semigroups on Hilbert space whose generators are not of strip-type, but only of half-plane type (i.e. those C_0-semigroups that are not groups). It is natural then to ask if this characterisation extends to these operators. Namely, does every generator of a C_0-semigroup on a Hilbert space enjoy a bounded $\mathcal{H}^\infty$-calculus in the half-plane sense outlined in Chapter 2? In the first instance we will work with bounded C_0-semigroups, so that for the generator $-A$, the operator A is a half-plane operator with $s_0(A) \geq 0$. We can then deduce more general results by shifting.

We will show in Section 5.1 that such operators need not admit a bounded $\mathcal{H}^\infty$-calculus on any half-plane. Thus we look to characterise semigroup generators on Hilbert space in terms of a property weaker than the boundedness of the $\mathcal{H}^\infty$-calculus. In Section 5.2 we show that this property is m-boundedness, provided we have some control on the underlying constants. In particular this shows that, in contrast to the case of strip-type operators, m-boundedness is a strict weakening of the bounded $\mathcal{H}^\infty$-calculus for half-plane operators. Through these considerations we also obtain new and simple proofs of the Gomilko-Shi-Feng theorem (Theorem 2.1.10), as well as a more general version for polynomially bounded semigroups.

We are interested then in identifying some class of semigroups for which the (negative) generator necessarily admits a bounded $\mathcal{H}^\infty$-calculus on appropriate half-planes. We attack the problem by studying the so-called Cayley transform of a semigroup generator (defined in Section 5.3). Since such an operator is bounded with spectrum in the closed unit disk, we can appeal to well known results on Hilbert space with a view to characterising bounded $\mathcal{H}^\infty$-calculus for semigroup generators. We will use von Neumann's classical inequality for contractions as a starting point and reconcile boundedness of the $\mathcal{H}^\infty$-calculus of a semigroup generator with polynomial boundedness of its Cayley transform. This is achieved by adapting the well known (and solved) similarity problem in Hilbert space. There are two considerations to note. Firstly, the boundedness of the $\mathcal{H}^\infty$-calculus will, in the first instance, be assumed to be uniform in the sense that the $\mathcal{H}^\infty$-constant does not depend on which half-plane we take. We can then apply these results to the more general case by a shifting argument. Secondly, given a bounded operator T with $\sigma(T) \subseteq \overline{D(0,1)}$ we must impose further spectral conditions on T to ensure that its inverse Cayley transform is well-defined. Even in this case, this may not give rise to a semigroup generator.

In Section 5.3 we introduce the Cayley transform and describe its role as a bridge between half-plane operators and bounded operators with spectrum in the closed unit disk. In Section 5.4 we proceed with an exposition of the well known discrete similarity problem on Hilbert space for bounded operators. We then describe an analogous similarity problem for a semigroup of bounded operators $(T(t))$, with a view to adapting the solution of the discrete problem. As a by-product, we introduce the notion of a completely bounded $\mathcal{H}^\infty$-calculus for half-plane operators (Section 5.5). In Section 5.6, we employ the familiar generation theorems, particularly those on Hilbert space, to obtain necessary and sufficient conditions for a bounded operator to arise as the Cayley transform of a semigroup generator. Finally, in Section 5.7 we apply our results in the more general setting where the $\mathcal{H}^\infty$-constant is assumed to depend on the half-plane.

5.1 A counterexample

In this section we present an example of a generator $-A$ of a bounded C_0-semigroup for which the half-plane operator A does not admit a bounded $\mathcal{H}^\infty$-calculus on any right half-plane $R_{-\epsilon}$, $\epsilon > 0$. The example is derived from constructions of invertible sectorial operators on Hilbert space which do not admit a bounded $\mathcal{H}^\infty$-calculus on any sector. On Hilbert space this is equivalent to the operator not having bounded imaginary powers (BIP). The construction is underpinned by the theory of conditional Schauder bases. Baillon and Clément [8] and Venni [73] were the first to use this approach to identify such an operator A for which A^{is} is bounded for some $s \in \mathbb{R}$ but not all. We omit the details as they are not in tune with our functional calculus motivations. Instead, we call upon the following similar example due to Le Merdy [49].

Example 5.1.1. Let X be a separable Hilbert space. Then there exists an operator A on X satisfying the following conditions.

(i) A is invertible and sectorial of angle 0.

(ii) A does not have BIP.

By Theorem 2.2.6, $-A$ is the generator of a bounded, analytic semigroup. We also have, by [34, Proposition 3.5.5], that for an invertible sectorial operator B, B

has BIP if and only if $B + \epsilon$ has BIP for every $\epsilon > 0$. Thus, given any $\epsilon > 0$, the function $f(z) = (z + \epsilon)^{is} \in \mathcal{H}^\infty(R_{-\epsilon})$ but $f(A) = (A + \epsilon)^{is}$ is not bounded for every $s \in \mathbb{R}$. Thus A fails to admit a bounded $\mathcal{H}^\infty$-calculus on any right half-plane.

Let us consider Example 5.1.1 in connection with the problem of inverse generators (Problem 5.1.3 below). First we establish the following easy result.

Theorem 5.1.2. *Let B be a densely defined half-plane operator of type $\epsilon > 0$ on a Banach space. Suppose B admits a bounded $\mathcal{H}^\infty$-calculus on $R_{\epsilon'}$ for some $0 < \epsilon' < \epsilon$. Then $-B$ and $-B^{-1}$ are generators of bounded C_0-semigroups.*

Proof. Given the assumption on B, define for each $t > 0$ the functions $f_t(z) = e^{-tz}$ and $g_t(z) = e^{-\frac{t}{z}}$. Then $f_t, g_t \in \mathcal{H}^\infty(R_\epsilon)$. Thus for each $t > 0$ the operators e^{-tB} and $e^{-tB^{-1}}$ are bounded. Moreover, there exists C such that for each $t > 0$

$$\|e^{-tB}\| + \|e^{-tB^{-1}}\| \leq C\big(\|e^{-tz}\|_{\mathcal{H}^\infty(R_\epsilon)} + \|e^{-\frac{t}{z}}\|_{\mathcal{H}^\infty(R_\epsilon)}\big) \leq C\ .$$

The result follows from Proposition 3.2.7. □

To determine whether or not the converse to Theorem 5.1.2 holds, namely if B admits a bounded half-plane calculus whenever $-B$ and $-B^{-1}$ are generators of C_0-semigroups, we consider the *inverse generator problem*, which is stated as follows.

Problem 5.1.3. Let A be the infinitesimal generator of a bounded C_0-semigroup on a Hilbert space X and suppose A^{-1} exists as a closed, densely defined operator. Does A^{-1} then necessarily generate a bounded C_0-semigroup?

The problem was originally posed on general Banach space by deLaubenfels in [19]. Zwart [79] constructed a counterexample on the function space $X = C_0([0, 1))$, equipped with the supremum norm, but as yet it is not known if such a counterexample exists on Hilbert space. It is known that if A^{-1} is necessarily the generator of a C_0-semigroup, then the boundedness of the semigroup is automatic (see [79]).

There are positive answers to Problem 5.1.3 for special classes of semigroups. We have by Theorem 2.2.6 that $-A$ generates a bounded analytic semigroup (that is, a semigroup satisfying $\|T(t)\| \leq C, |\arg t| < \theta$, some $\theta > 0$) if and only if A is sectorial of angle less than $\frac{\pi}{2}$. Furthermore, if A^{-1} exists as a closed, densely defined operator, then it is also sectorial of angle less than $\frac{\pi}{2}$, so $-A^{-1}$ also generates a bounded, analytic semigroup. On Hilbert space, we even have a positive result for the class of generators of contraction semigroups and, by extension, semigroups which are semi-contractive (see Definition 5.5.3). We refer to [79, Lemma 4.1] for the details.

Returning to the converse of Theorem 5.1.2, let A be as in Example 5.1.1 and note that by the considerations above A and A^{-1} are both generators of bounded analytic C_0-semigroups. However, as noted in Example 5.1.1, A does not admit a bounded $\mathcal{H}^\infty(R_\epsilon)$-calculus for any $0 < \epsilon < a$. Thus the converse to Theorem 5.1.2 fails on Hilbert space.

We conclude this section with a remark on the minimal type of bounded $\mathcal{H}^\infty$-calculus. On Hilbert space, we have seen in Section 4.1 that for sectorial and strip-type operators, boundedness of the $\mathcal{H}^\infty$-calculus is characterised by the absolute calculus, which in turn does not depend on the angle/height of the sector/strip, so that the $\mathcal{H}^\infty$-angle/$\mathcal{H}^\infty$-type coincides with the angle of sectoriality/strip-height. For half-plane operators, no such characterisation exists and it seems unclear if the $\mathcal{H}^\infty$-type is equal to the half-plane type of the operator. We can, however, offer a partial result in this direction.

Proposition 5.1.4. *Let A be a half-plane operator on a Hilbert space with $s_0(A) \geq 0$ and suppose there exist $\epsilon, \epsilon' > 0$ such that*

(i) A admits a bounded $\mathcal{H}^\infty$-calculus on $R_{-\epsilon}$ and

(ii) $A + \epsilon'$ is sectorial of angle less than $\frac{\pi}{2}$.

Then A admits a bounded $\mathcal{H}^\infty$-calculus on $R_{-\delta}$ for every $\delta > 0$.

Proof. Let $0 < \delta < \min\{\epsilon, \epsilon'\}$. We need to show that $A+\delta$ has bounded $\mathcal{H}^\infty$-calculus on $\Sigma_{\frac{\pi}{2}}$. By assumption (ii), it follows that $A + \delta$ is sectorial of angle less than $\frac{\pi}{2}$. Thus, by Theorem 4.1.8, $A+\delta$ has bounded $\mathcal{H}^\infty$-calculus on $\Sigma_{\frac{\pi}{2}}$ if and only if $A+\delta$ has BIP. Note that $A+\epsilon$ has BIP since it has bounded $\mathcal{H}^\infty$-calculus on $\Sigma_{\frac{\pi}{2}}$. Thus $A + \delta$ has BIP by [34, Proposition 3.5.5] and the result follows. □

5.2 m-bounded calculus for half-plane operators

In this section we consider m-boundedness of the $\mathcal{H}^\infty$-calculus for half-plane operators. Given $w \in \mathbb{R}$ and $f \in \mathcal{H}^\infty(R_w)$, it follows from Cauchy's Integral Formula that for any m, $f^{(m)} \in \mathcal{H}^\infty(R_{w'})$ for any $w' > w$. Thus the natural definition of m-bounded $\mathcal{H}^\infty$-calculus is essentially the same for half-plane operators as it is for strip-type operators.

Definition 5.2.1. Let A be a half-plane operator of type $w \in \mathbb{R}$, $u < w$. Then A is said to admit an m-bounded $\mathcal{H}^\infty$-calculus on R_u if $f^{(m)}(A) \in B(X)$ for every $f \in \mathcal{H}^\infty(R_u)$ and there exists a constant C such that for every such f,

$$\|f^{(m)}(A)\| \leq C\|f\|_{\mathcal{H}^\infty(R_u)} \ . \tag{5.2.1}$$

Our first aim is to obtain an appropriate version of Theorem 4.3.2 for half-plane operators. We saw in the previous section that not every C_0-semigroup generator on Hilbert space admits a half-plane bounded $\mathcal{H}^\infty$-calculus. Thus we expect to replace this with a (strictly) weaker condition. It turns out that 1-boundedness of the $\mathcal{H}^\infty$-calculus is necessary and sufficient, provided we have control on the constant C in (5.2.1) and on the integral estimates involved.

Theorem 5.2.2. *Let A be a densely defined half-plane operator of type $v \in \mathbb{R}$ on a Banach space X. Consider the following conditions.*

(i) There exists C such that for every $x \in X$ and $y \in X^*$ and every $w < v$,

$$(v-w)\int_{\mathbb{R}} \|R(w+it,A)x\|^2 \, \mathrm{d}t \le C\|x\|^2,$$
$$(v-w)\int_{\mathbb{R}} \|R(w+it,A^*)y\|^2 \, \mathrm{d}t \le C\|y\|^2.$$

(ii) There exists C such that for every $x \in X$ and $y \in X^*$ and every $w < v$,

$$\int_{\mathbb{R}} |\langle R(w+it,A)^2x, y\rangle| \, \mathrm{d}t \le \frac{C}{v-w}\|x\| \, \|y\| \, .$$

(iii) A admits a 1-bounded $\mathcal{H}^\infty$-calculus of type v and there exists C such that for every $w < v$ and $f \in \mathcal{H}^\infty(R_w)$,

$$\|f'(A)\| \le \frac{C}{v-w}\|f\|_{\mathcal{H}^\infty(R_w)} \, .$$

(iv) $-A$ generates a C_0-semigroup $(T(t))_{t\ge 0}$ for which there exists a constant C such that for every $t \ge 0$, $\|T(t)\| \le Ce^{-vt}$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$. If X is a Hilbert space then $(iv) \Rightarrow (i)$ also holds.

Proof. We will show $(ii) \Rightarrow (iii) \Rightarrow (iv)$. The other implications follow using the Cauchy-Schwarz inequality and Plancherel's theorem, in much the same way as in the Gomilko-Shi-Feng Theorem (Theorem 2.1.10). Without loss of generality we set $v = 0$. Assume (ii) holds, let $w < u < 0$ and consider the class

$$\mathcal{H}_1^\infty(R_w) = \{f \in \mathcal{H}^\infty(R_w) : \exists\, \epsilon > 0 \text{ such that } |f(z)| \le \frac{1}{|z|^{1+\epsilon}} \text{ for large } |z|, \, z \in R_w\}.$$

For such f and $w < w' < 0$, $|f'(z)| \le \frac{C_{w'}}{|z|^{1+\epsilon}}$, $z \in R_{w'}$, and so $f'(A)$ is defined via the natural functional calculus for A with

$$f'(A) = \frac{1}{2\pi i} \int_{u-i\infty}^{u+i\infty} f'(\lambda) R(\lambda, A) \, \mathrm{d}\lambda.$$

Integrating by parts gives

$$f'(A) = \lim_{n\to\infty} \left\{ \frac{1}{2\pi i} \big[f(\lambda) R(\lambda, A) \big]_{u-in}^{u+in} + \frac{1}{2\pi i} \int_{u-in}^{u+in} f(\lambda) R(\lambda, A)^2 \, \mathrm{d}\lambda \right\}.$$

The resolvent of A is uniformly bounded along vertical lines, so the norm of the first term can be estimated by $C_u(|f(u+in)| + |f(u-in)|)$, which converges to zero as $n \to \infty$ by the assumption on f. Denote the integral in the second term by T_n. We already have that T_n is bounded. Moreover, for any $x \in X$ and $y \in X^*$ we have

$$\begin{aligned} |\langle T_n x, y\rangle| &\le C \|f\|_{\mathcal{H}^\infty(R_w)} \int_{u-in}^{u+in} |\langle R(\lambda, A)^2 x, y\rangle| \, |\mathrm{d}\lambda| \\ &\le \frac{C}{|u|} \|f\|_{\mathcal{H}^\infty(R_w)} \|x\| \, \|y\| \, . \end{aligned}$$

Thus $\|T_n\| \le \frac{C}{|u|} \|f\|_{\mathcal{H}^\infty(R_w)}$. Letting $n \to \infty$, we deduce that $\|f'(A)\| \le \frac{C}{|u|} \|f\|_{\mathcal{H}^\infty(R_w)} = \frac{C'}{|w|} \|f\|_{\mathcal{H}^\infty(R_w)}$ for the choice $u = \frac{w}{2}$. For general $f \in \mathcal{H}^\infty(R_w)$, consider the approximating sequence $(f\phi_n)$, where $\phi_n(z) = (\frac{n}{n-w+z})^2$. Then we have

1. $(f\phi_n)' \to f'$ pointwise as $n \to \infty$,
2. $f\phi_n \in \mathcal{H}^\infty_*(R_w)$ for all n,
3. $\|f\phi_n\|_{\mathcal{H}^\infty(R_w)} \le \|f\|_{\mathcal{H}^\infty(R_w)}$ for each $n > 0$.

The result follows from the Convergence Lemma for half-plane operators (Proposition 3.3.5).

Now assume (*iii*) holds. For $w < 0$ and $t > 0$ define $f_{t,w}$ on R_w by $f_{t,w}(z) = e^{-tz}$. Then $f_{t,w} \in \mathcal{H}^\infty(R_w)$ with $f'_{t,w}(z) = -te^{-tz}$. Thus for any $t > 0$ we have

$$t\|e^{-tA}\| = \|f'_{t,w}(A)\| \le \frac{C}{|w|} \|f_{t,w}\|_{\mathcal{H}^\infty(R_w)} = \frac{C}{|w|} e^{-tw}.$$

In particular, choosing $w = -\frac{1}{t}$ we see $\|e^{-tA}\| \le C$ for $t \in [0, \infty)$. Thus $-A$ generates a bounded C_0-semigroup by Proposition 3.2.7, as required.

□

Observe that Theorem 5.2.2, along with Example 5.1.1, shows that 1-boundedness of the $\mathcal{H}^\infty$-calculus is strictly weaker than the usual boundedness for half-plane operators. We can also obtain a version of Theorem 5.2.2 for higher orders, as already indicated by Theorem 4.3.4.

Theorem 5.2.3. *Let A be a densely defined half-plane operator of type $v \in \mathbb{R}$ on a Banach space X. For $m \in \mathbb{N}$, consider the following conditions.*

(i) There exists C_m such that for every $x \in X$ and $y \in X^*$ and every $w < v$,

$$\int_{\mathbb{R}} |\langle R(w+it, A)^{m+1}x, y\rangle| \, \mathrm{d}t \leq \frac{C_m}{(v-w)^m} \|x\| \, \|y\| \, . \tag{5.2.2}$$

(ii) A admits an m-bounded $\mathcal{H}^\infty$-calculus of type v and there exists C_m such that for every $w < v$ and $f \in \mathcal{H}^\infty(R_w)$,

$$\|f^{(m)}(A)\| \leq \frac{C_m}{(v-w)^m} \|f\|_{\mathcal{H}^\infty(R_w)} \, .$$

(iii) $-A$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ for which there exists a constant C such that for every $t \geq 0$, $\|T(t)\| \leq Ce^{-vt}$.

Then conditions (i) and (ii) are independent of $m \geq 1$. Furthermore we have $(i) \Leftrightarrow (ii) \Rightarrow (iii)$. When X is a Hilbert space we have $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

Proof. Without loss of generality, set $v = 0$. First we show that for a fixed $m \geq 1$, (i) implies (ii). The proof is similar to the $m = 1$ case from Theorem 5.2.2. Let $w < u < 0$. This time we write

$$f^{(m)}(A) = \frac{m!}{2\pi i} \int_{u-i\infty}^{u+i\infty} f(\lambda) R(\lambda, A)^{m+1} \, \mathrm{d}\lambda,$$

for $f \in \mathcal{H}_1^\infty(R_w)$. Then using the assumption (i) we similarly arrive at

$$\|f^{(m)}(A)\| \leq \frac{C_m}{(v-w)^m} \|f\|_{\mathcal{H}^\infty(R_w)}$$

for this choice of f. The estimate for general $f \in \mathcal{H}^\infty(R_w)$ follows using a similar approximation argument with $f_n = f\phi_n$, ϕ_n as in the proof of Theorem 5.2.2.

To show $(ii) \Rightarrow (iii)$, we proceed as in Theorem 5.2.2 and define, for $w < 0$ and $t > 0$ the function $f_{t,w}$ on R_w by $f_{t,w}(z) = e^{-tz}$. Then for $t > 0$ we obtain

$$t^m \|e^{-tA}\| = \|f_{t,w}^{(m)}(A)\| \leq \frac{C_m}{|w|^m} \|f_{t,w}\|_{\mathcal{H}^\infty(R_w)} = \frac{C_m}{|w|^m} e^{-tw}.$$

Again we choose $w = -\frac{1}{t}$ to get $\|e^{-tA}\| \leq C_m$ for $t \in [0, \infty)$ so that $-A$ generates a bounded C_0-semigroup, by Proposition 3.2.7 .

Now suppose (ii) holds for some m. We will show that (i) holds for $m+1$, using a similar argument to the proof of Proposition 4.2.6. Fix $x \in X$ and $y \in X^*$. For a suitable measurable sign function ϵ, we can write

$$\begin{aligned}\int_{\mathbb{R}} |\langle R(w+it,A)^{m+2}x,y\rangle| \text{ d}t &= \int_{\mathbb{R}} \langle R(w+it,A)^{m+2}x,y\rangle \epsilon(t,x,y) \text{ d}t \\ &= \langle f^{(m)}(A)x,y\rangle,\end{aligned}$$

where $f(z) = \frac{1}{m!}\int_{\mathbb{R}}(w+it-z)^{-2}\epsilon(t,x,y) \text{ d}t$. Then for $z \in R_{\frac{w}{2}}$ we have

$$|f(z)| \le \int_{\mathbb{R}} \frac{\text{d}t}{(\operatorname{Re} z - w)^2 + t^2} \le \frac{C}{m!|w|}.$$

Thus $f \in \mathcal{H}^\infty(R_{\frac{w}{2}})$ and so by (ii), $\int_{\mathbb{R}} |\langle R(w+it,A)^{m+2}x,y\rangle| \text{ d}t \le \|f^{(m)}(A)\| \, \|x\| \, \|y\| \le \frac{C_m}{|w|^m}\|f\|_{\mathcal{H}^\infty(H_{\frac{w}{2}})}\|x\| \, \|y\| \le \frac{C_m}{|w|^{m+1}}\|x\| \, \|y\|$.

Now suppose (i) holds for some $m > 1$. We claim that (i) holds for for $m-1$. We have already shown that $-A$ generates a bounded C_0-semigroup, so that A is actually a strong half-plane operator with $s_0(A) \ge 0$. Thus for any $x \in D(A)$ and fixed $s \in \mathbb{R}$,

$$\lim_{u \to -\infty} R(u+is,A)x = 0 \; .$$

Since A is a densely defined half-plane operator we deduce $\lim_{u\to-\infty} R(u+is,A)^m x = 0$ for every $x \in X$. Thus the following identity holds for any $w < 0$:

$$R(w+is,A)^m = -m\int_{-\infty}^{w} R(u+is,A)^{m+1} \text{ d}u \; .$$

We can then invoke Fubini to obtain for any $x \in X, y \in X^*$,

$$\begin{aligned}\int_{\mathbb{R}} |\langle R(w+it,A)^m x,y\rangle| \text{ d}t &= \int_{\mathbb{R}} \Big|\Big\langle \int_{-\infty}^{w} R(u+it,A)^{m+1}x \text{ d}u, y\Big\rangle\Big| \text{ d}t \\ &\le \int_{\mathbb{R}}\int_{-\infty}^{w} |\langle R(u+it,A)^{m+1}x,y\rangle| \text{ d}u \text{ d}t \\ &= \int_{-\infty}^{w}\int_{\mathbb{R}} |\langle R(u+it,A)^{m+1}x,y\rangle| \text{ d}u \text{ d}t \\ &\le C\bigg(\int_{-\infty}^{w}\frac{\text{d}u}{|u|^m}\bigg)\|x\| \, \|y\| \\ &= \frac{C}{|w|^{m-1}}\|x\| \, \|y\| \; .\end{aligned}$$

Thus we obtain (i) for $m-1$.

Putting all this together, we have shown that (i) and (ii) are independent of m and equivalent to each other. When X is a Hilbert space, by Theorem 5.2.2 we have

$(iii) \Rightarrow (i)$ for $m = 1$. Thus the final claim holds.

$\square$

Remark 5.2.4. Suppose A is a closed operator with $\sigma(A) \subseteq \overline{\Sigma_{\frac{\pi}{2}}}$ which satisfies (5.2.2) with $v = 0$ for some $m > 1$. By Theorem 4.3.4, $-A$ generates a bounded C_0-semigroup so that A is a half-plane operator with $s_0(A) \geq 0$. Thus by Theorem 5.2.3 we deduce that A satisfies (5.2.2) with $v = 0$ for every $m \geq 1$. In fact slightly more is true. In [30] it is remarked that if for every $x \in X, y \in X^*$ there exists $m = m(x, y) \geq 1$ for which

$$\sup_{w<0} |w|^m \int_{\mathbb{R}} |\langle R(w + it, A)^{m+1}x, y\rangle| \,\mathrm{d}t < \infty ,$$

then (5.2.2), with $v = 0$, holds for $m = 1$. Since we do not need this version of the condition, we omit the details.

From the proof of $(i) \Rightarrow (ii)$ in Theorem 5.2.3, we see that if for every $u < 0$

$$\int_{\mathbb{R}} |\langle R(u + it, A)^{m+1}x, y\rangle| \,\mathrm{d}t \leq C_{m,u}\|x\| \, \|y\| \qquad \forall x \in X \; \forall y \in X^*, \tag{5.2.3}$$

then for $w < u < 0$ and suitable $f \in \mathcal{H}_1^\infty(R_w)$,

$$f^{(m)}(A) = \frac{m!}{2\pi i} \int_{u-i\infty}^{u+i\infty} f(\lambda)R(\lambda, A)^{m+1} \,\mathrm{d}\lambda , \tag{5.2.4}$$

where we note that the right hand integral is absolutely convergent in operator norm. We can establish a weaker form of this identity for general $f \in \mathcal{H}^\infty(R_w)$, by an approximation argument. More precisely, with $\phi_n(z) = (\frac{n}{n-w+z})^2$, consider the approximating sequence (f_n) defined by $f_n = f\phi_n$. Then f_n satisfies (5.2.4), $f_n \to f$ pointwise on R_w, $f_n^{(m)} \to f^{(m)}$ pointwise on R_w and $\|f_n^{(m)}(A)\| \leq C_{m,u}\|f\|_{\mathcal{H}^\infty(R_w)}$, so by the Convergence Lemma we have $f^{(m)}(A) \in B(X)$ and for every $x \in X$, $y \in X^*$,

$$\begin{aligned} \langle f^{(m)}(A)x, y\rangle &= \lim_{n\to\infty} \langle f_n^{(m)}(A)x, y\rangle = \lim_{n\to\infty} \frac{m!}{2\pi i} \int_{u-i\infty}^{u+i\infty} \langle f_n(\lambda)R(\lambda, A)^{m+1}x, y\rangle \,\mathrm{d}\lambda \\ &= \frac{m!}{2\pi i} \int_{u-i\infty}^{u+i\infty} \langle f(\lambda)R(\lambda, A)^{m+1}x, y\rangle \,\mathrm{d}\lambda , \end{aligned}$$

where this last term exists by the assumption (5.2.3).

Finally in this section we derive a known result due to Eisner [27] (see also her expanded work [25]) on the generation of polynomially bounded C_0-semigroups.

Definition 5.2.5. A C_0-semigroup $(T(t))_{t\geq 0}$ is said to be *polynomially bounded* if there exist constants $C, d \geq 0$ such that

$$\|T(t)\| \leq C(1+t^d), \qquad t \geq 0.$$

The following result is adapted from [27].

Proposition 5.2.6. *Let A be a densely defined operator with $\sigma(A) \subseteq \{z : \operatorname{Re} z \geq 0\}$ and suppose that for some $C, d \geq 0$ and every $w < 0$, A satisfies*

$$\int_{\mathbb{R}} |\langle R(w+it, A)^2 x, y\rangle| \,\mathrm{d}t \leq \frac{C}{|w|}(1+|w|^{-d})\|x\| \, \|y\| \, . \tag{5.2.5}$$

Then A is a half-plane operator with $s_0(A) \geq 0$.

Proof. Since $\frac{\mathrm{d}}{\mathrm{d}z} R(z,A) = -R(z,A)^2$ we have for every $w < 0, x \in X, y \in X^*, r < s$,

$$\langle R(w+is, A)x, y\rangle = \langle R(w+ir, A)x, y\rangle - i\int_r^s \langle R(w+it, A)^2 x, y\rangle \,\mathrm{d}t \, . \tag{5.2.6}$$

By our assumption (5.2.5) this shows that $\|R(w+is, A)\|$ is uniformly bounded, by C_w say, on $w + i\mathbb{R}$. Let $z = w + i\tau$, $\tau \in \mathbb{R}$. Then for any $x \in D(A)$, $y \in X^*$,

$$|\langle R(z,A)x, y\rangle| = \frac{1}{|z|}|\langle x + R(z,A)Ax, y\rangle| \leq \frac{1}{|w+i\tau|}(\|x\| + C_w\|Ax\|)\|y\| \, .$$

Thus $\langle R(w+i\tau, A)x, y\rangle \to 0$ as $\tau \to \pm\infty$. By (5.2.5), (5.2.6) and the fact that A is densely defined, it follows that

$$\|R(w+is, A)\| \leq \frac{C}{|w|}(1+|w|^{-d}) \, .$$

□

As an immediate consequence, we see that in the case $d = 0$, the combined implications *(ii)* ⇒ *(iii)* ⇒ *(iv)* in Theorem 5.2.2 provide an alternative proof of the Gomilko-Shi-Feng Theorem (Theorem 2.1.10). For more general d, we can also establish an alternative proof of the following known result for polynomially bounded semigroups [27, Theorem 2.4].

Theorem 5.2.7. *Let A satisfy the assumptions of Proposition 5.2.6. Then $-A$ is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ satisfying*

$$\|T(t)\| \leq C(1+t^d), \qquad t \geq 0.$$

Proof. By Proposition 5.2.6 we know that A is a half-plane operator with type at least zero. Following the proof of *(ii)* $\Rightarrow$ *(iii)* in Theorem 5.2.2, there exists C such that for every $w < 0$ and $f \in \mathcal{H}^\infty(R_w)$,

$$\|f'(A)\| \leq \frac{C}{|w|}(1 + |w|^{-d}) \, \|f\|_{\mathcal{H}^\infty(R_w)} \, .$$

The result now follows in exactly the same way as *(iii)* $\Rightarrow$ *(iv)* in Theorem 5.2.2. $\square$

The (relative) simplicity of our proof, when compared with [27, Theorem 2.4], is in the employment of the functional calculus to derive the semigroup, rather than defining the semigroup explicitly. In fact, one can arrive at a weak form of this explicit representation of the semigroup via the discussion following Theorem 5.2.3. By the following result due to Malejki [52], we obtain a converse to Theorem 5.2.7 on Hilbert space. The proof is similar to the case $d = 0$, in that Plancherel's theorem is used in conjuction with the Laplace Transform representation of the resolvent of the generator (Lemma 2.1.8).

Theorem 5.2.8. *Let $-A$ be the generator of a C_0-semigroup on a Hilbert space satisfying*

$$\|T(t)\| \leq C(1 + t^d), \qquad t \geq 0 \, ,$$

for some $d \geq 0$. Then there exists a constant C such that for every $x \in X, y \in X^*$,

$$\int_{\mathbb{R}} \|R(w + it, A)x\|^2 \, \mathrm{d}t \leq \frac{C}{|w|}\big(1 + \frac{1}{|w|^{2d}}\big)\|x\|^2,$$
$$\int_{\mathbb{R}} \|R(w + it, A^*)y\|^2 \, \mathrm{d}t \leq \frac{C}{|w|}\big(1 + \frac{1}{|w|^{2d}}\big)\|y\|^2.$$

By the Cauchy-Schwarz inequality it follows that A satisfies (5.2.5) with $2d$ in place of d.

We now present a counterexample from [52] to highlight the failure of Theorems 5.2.2 and 5.2.8 on general Banach space. Let $X = \mathbf{C_0}(\mathbb{R})$, equipped with the supremum norm. Let A denote the operator of multiplication by the identity function on $\mathbb{R}$. Then iA generates a group of isometries. Suppose there exist constants $C, s \geq 0$ such that for every $x \in X$ and $w < 0$,

$$\int_{\mathbb{R}} \|R(w + it, A)x\|^2 \, \mathrm{d}t \leq \frac{C}{|w|}\big(1 + \frac{1}{|w|^s}\big)\|x\|^2 \, .$$

For $N \in \mathbb{N}$, choose $x_N \in X$ such that $x_N(s) = 1$ for $s \in [-N, N]$ and $\|x_N\|_\infty \leq 1$.

Then

$$\begin{aligned}\frac{C}{|w|}\big(1+\frac{1}{|w|^s}\big)\|x_N\|^2 \geq \int_{\mathbb{R}} \|R(w+it,A)^{-1}x_N\|^2 \,\mathrm{d}t &\geq \int_{-N}^{N} \|R(w+it,A)^{-1}x_N\|^2 \,\mathrm{d}t \\ &= \int_{-N}^{N} \sup_{s\in\mathbb{R}} \frac{|x_N(s)|^2}{(t-s)^2+w^2}\,\mathrm{d}t \\ &= \int_{-N}^{N} \frac{\mathrm{d}t}{w^2} = \frac{2N}{w^2}\|x_N\|^2 \,.\end{aligned}$$

This yields

$$C|w|\big(1+\frac{1}{|w|^s}\big) \geq 2N,$$

which can not hold for every $N \in \mathbb{N}, w < 0$. Thus A does not satisfy the estimates in Theorem 5.2.8. One can even find a bounded operator on a reflexive Banach space which is the generator of a bounded C_0-semigroup, but violates condition (ii) in Theorem 5.2.2. For more on this, we refer to the final paragraph of [30].

On general Banach space, one can also obtain the following version of the Hille-Yosida theorem for polynomially bounded semigroups. This was done by Malejki [52], but an alternative (and more simple) proof can be obtained by adapting Haase's proof of the Hille-Yosida theorem [35, Theorem 3.2]. The main difficulty here is in showing that A is a half-plane operator, which requires estimates on sums of the form

$$\sum_{n=0}^{\infty} n^s a^n, \qquad s>0,\ a\in(0,1).$$

Theorem 5.2.9. *Let $(A, D(A))$ be a closed, densely defined operator on a Banach space X, $d \geq 0$. Suppose that for some $M_1 \geq 0$ and $d \geq 0$ and every $n \in \mathbb{N}$,*

$$\|R(\lambda,A)^n\| \leq \frac{M_1}{\lambda^n}\left(1+\left(\frac{n}{\lambda}\right)^d\right), \qquad \lambda > 0\,.$$

Then A generates a C_0-semigroup $(T(t))_{t\geq 0}$ for which there exists $M_2 \geq 0$ such that

$$\|T(t)\| \leq M_2(1+t^d), \qquad t \geq 0\,.$$

5.3 The Cayley transform

The well known Cayley transform

$$\delta : z \mapsto \frac{z-1}{1+z}$$

is a conformal mapping from the right half plane $\overline{\Sigma_{\frac{\pi}{2}}}$ into the closed unit disk $\overline{D(0,1)}$. Thus, to a closed, densely defined operator A with $\sigma(A) \subseteq \overline{\Sigma_{\frac{\pi}{2}}}$ we can associate the bounded operator

$$\delta(A) = (A - I)(A + I)^{-1} = I - 2(A + I)^{-1}$$

which satisfies $\sigma(\delta(A)) \subseteq \overline{D(0,1)}$, by the spectral mapping theorem applied to $(I + A)^{-1}$. We call $\delta(A)$ the *Cayley transform* of A.

Conversely, given a bounded operator T for which $\sigma(T) \subseteq \overline{D(0,1)}$ and $I - T$ is injective with dense range, we can define the operator

$$A = (I + T)(I - T)^{-1}$$

which is closed, densely defined and satisfies $\sigma(A) \subseteq \overline{\Sigma_{\frac{\pi}{2}}}$. Moreover A is injective if $-1 \notin \sigma_p(T)$. We call A the *inverse Cayley transform* of T.

Our aim is to establish a correspondence, by means of the Cayley transform, between the half-plane operators we are interested in and bounded operators with spectrum in the closed unit disk. In particular, we will find an equivalence between boundedness of the $\mathcal{H}^\infty$-calculus for half-plane operators (although in a slightly stronger sense) and the so-called polynomial boundedness of their Cayley transform. This in turn will lead us to a solution of the similarity problem for semigroups (Theorem 5.5.12).

5.4 Polynomial boundedness

In this section we restrict our attention to bounded operators acting on Hilbert space with spectrum in the closed unit disk. Unless otherwise stated, throughout this section T will denote such an operator acting on a Hilbert space X. We begin with a classical result due to von Neumann, which characterises contractive operators acting on Hilbert space.

Theorem 5.4.1. *T is a contraction operator if and only if for all polynomials p*

$$\|p(T)\| \leq \|p\|_{\mathcal{H}^\infty(\overline{D(0,1)})} .$$

Since for an invertible operator S and polynomial p we have $p(S^{-1}TS) = S^{-1}p(T)S$, it follows that when T is similar to a contraction operator, T is polynomially bounded in the following sense:

Definition 5.4.2. T is said to be *polynomially bounded* if there exists a constant $C \geq 0$ such that for all polynomials p

$$\|p(T)\| \leq C\|p\|_{\mathcal{H}^\infty(\overline{D(0,1)})} \ . \tag{5.4.1}$$

Note that this is equivalent to having (5.4.1) hold for all rational functions with poles outside $\overline{D(0,1)}$, since the polynomials are dense in the disk algebra of continuous functions on $\overline{D(0,1)}$ which are holomorphic in $D(0,1)$, equipped with the supremum norm.

The similarity problem deals with the question of whether or not every polynomially bounded operator is similar to a contraction. The question was first posed for power bounded operators (T is power bounded if there exists C such that $\|T^n\| \leq C$ for all n) after Sz.-Nagy [70] proved the following.

Theorem 5.4.3. *Let T be power bounded and compact. Then T is similar to a contraction.*

Foguel [29] showed by means of a counterexample that power boundedness alone is not sufficient for an operator to be similar to a contraction. A refinement of his construction was yielded by Halmos [37] and is now outlined.

Example 5.4.4. Let H_0 be a Hilbert space with orthonormal basis $\{e_0, e_1, e_2, \cdots\}$ and let S denote the unilateral shift on this space, i.e.

$$Se_n = e_{n+1} \qquad n \geq 0 \ .$$

Now let Q be the projection of H_0 on $\mathrm{Sp}\{e_j : \ j \in J\}$, where $J = \{3^k : \ k \in \mathbb{N}\}$ (in fact, any set J such that $2i < j$ whenever $i, j \in J$ with $i < j$, will do). Set $H = H_0 \oplus H_0$ and define the operator A on H by

$$\begin{pmatrix} S^* & Q \\ 0 & S \end{pmatrix} .$$

To show A is power bounded, first note that (by induction)

$$A^n = \begin{pmatrix} S^{*n} & Q_n \\ 0 & S^n \end{pmatrix} ,$$

where $Q_{n+1} = \sum_{i=0}^{n} S^{*n-i}QS^i$, $n \geq 0$. Then it is sufficient to prove that the Q_n are uniformly bounded (in fact, they are partial isometries). To show A is not similar to a contraction, an argument of Foguel [29] is employed and it is sufficient to show that

$$Z(A) \cap (Z(A^*))^\perp \neq \{0\} \ ,$$

where $Z(A) = \{x \in X : \ A^n x \to 0 \text{ weakly}\}$.

It was later shown by Lebow [51] that the operator A is not polynomially bounded, giving rise to the similarity problem described after Definition 5.4.2 (first posed by Halmos in [38]). The problem was solved in two parts. First, Paulsen 5.4.6 provided a characterisation of operators similar to a contraction in terms of the extended notion of complete polynomial boundedness.

Definition 5.4.5. T is said to be *completely polynomially bounded* if there exists a constant $C \geq 0$ such that for any $n \geq 1$ and all polynomials $(P_{ij})_{1\leq i,j\leq n}$

$$\|[P_{ij}(T)]\| \leq C \sup \big\{ \|[P_{ij}(z)]\|_{M_n} : |z| \leq 1 \big\}.$$

In this context M_n denotes the space of $n \times n$ complex matrices acting as operators on the n-dimensional Hilbert space $\mathbb{C}^n$ and $[P_{ij}(T)]$ is an $n \times n$ matrix of operators acting on X^n in the usual way. For $n = 1$, the condition is that of polynomial boundedness.

Paulsen's result can be stated as follows.

Theorem 5.4.6. *T is similar to a contraction if and only if T is completely polynomially bounded.*

The question remained as to whether or not complete polynomial boundedness and polynomial boundedness are equivalent. A counterexample due to Pisier [61] answered this, and consequently the similarity problem, in the negative.

5.5 Bounded and completely bounded functional calculus for half-plane operators

We now return to half-plane operators on general Banach space, though many of the results in this section will be in the Hilbert space setting. In this subsection half-plane operators will be assumed to have dense domain. As a first step, we establish a result that corresponds to Theorem 5.4.1 for bounded operators.

Theorem 5.5.1. *Let A be a half-plane operator on a Hilbert space. Then $-A$ generates a contraction semigroup if and only if $s_0(A) \geq 0$ and for every $w < 0$, A admits a bounded $\mathcal{H}^\infty$-calculus on R_w with $\mathcal{H}^\infty$-constant* 1.

Proof. First assume $-A$ generates a contraction semigroup. Then A is $\Sigma_{\frac{\pi}{2}}$-accretive (meaning that its numerical range and spectrum are contained in $\overline{\Sigma_{\frac{\pi}{2}}}$). We can therefore invoke von Neumann's result that half-planes are spectral sets for these operators [75] (see also the more general result of Crouzeix and Délyon [17, Theorem 1]) to obtain that for any $w < 0$ and $f \in \mathcal{H}^\infty(R_w)$, we have $\|f(A)\| \leq \|f\|_{\mathcal{H}^\infty(\Sigma_{\frac{\pi}{2}})} \leq \|f\|_{\mathcal{H}^\infty(R_w)}$.

The converse direction is standard; we give the details for completeness. For each $t > 0$ consider the function f_t defined on R_w, $w < 0$, by $f_t(z) = e^{-tz}$. Then $f_t \in \mathcal{H}^\infty(R_w)$ and $\|e^{-tA}\| = \|f_t(A)\| \leq \|f_t\|_{\mathcal{H}^\infty(R_w)} = e^{-tw}$. Letting $w \to 0$, the result follows from Proposition 3.2.7. $\square$

Before we consider more general semigroup generators, we employ the following useful approximation result for half-plane operators. Here $\mathcal{R}^\infty(\Sigma_{\frac{\pi}{2}})$ denotes the space of all rational functions $\frac{p(z)}{q(z)}$ for which $\deg p \leq \deg q$ and the zeroes of q lie outside $\overline{\Sigma_{\frac{\pi}{2}}}$. Let A be a half-plane operator with $s_0(A) \geq 0$. Observe that for $r \in \mathcal{R}^\infty(\Sigma_{\frac{\pi}{2}})$ we have $r(A) \in B(X)$. We will say that A admits a *uniformly bounded $\mathcal{H}^\infty$-calculus* of type 0 if it admits a bounded $\mathcal{H}^\infty$-calculus on R_w for each $w < 0$ and the $\mathcal{H}^\infty(R_w)$-constant is independent of w, i.e. there exists a constant C such that for every $w < 0$ and $f \in \mathcal{H}^\infty(R_w)$, $f(A) \in B(X)$ with

$$\|f(A)\| \leq C\|f\|_{\mathcal{H}^\infty(R_w)} .$$

Using the known approximation result for sectorial operators (Proposition 3.3.7), we can obtain the following characterisation for half-plane operators.

Proposition 5.5.2. *Let A be a half-plane operator on a Banach space with $s_0(A) \geq 0$. Then the following are equivalent.*

(i) There exists a constant $C \geq 0$ such that for all $r \in \mathcal{R}^\infty(\Sigma_{\frac{\pi}{2}})$

$$\|r(A)\| \leq C\|r\|_{\mathcal{H}^\infty(\overline{\Sigma_{\frac{\pi}{2}}})}.$$

(ii) A admits a uniformly bounded $\mathcal{H}^\infty$-calculus of type 0.

Proof. The implication $(ii) \Rightarrow (i)$ is clear since, given $r \in \mathcal{R}^\infty(\Sigma_{\frac{\pi}{2}})$, we have for w sufficiently close to 0 that $r \in \mathcal{H}^\infty(R_w)$. Then for some constant C independent of w and r, we have

$$\|r(A)\| \leq C\|r\|_{\mathcal{H}^\infty(R_w)} .$$

We arrive at (i) by letting $w \to 0$, using the continuity of r.

Now assume (i) holds. Given $w < 0$, consider the operator $A - w$, which has spectrum in the closed right half-plane $\overline{R_{-w}}$. Let $r \in \mathcal{R}^\infty(\Sigma_{\frac{\pi}{2}})$ and define the function q by $q(z) = r(z - w)$, so that $q \in \mathcal{R}^\infty(\Sigma_{\frac{\pi}{2}})$. By the assumption on A we obtain that there exists a constant C independent of r such that

$$\|r(A - w)\| = \|q(A)\| \leq C\|q\|_{\mathcal{H}^\infty(\overline{\Sigma_{\frac{\pi}{2}}})} \leq C\|r\|_{\mathcal{H}^\infty(\Sigma_{\frac{\pi}{2}})}.$$

We note that $(v) \Rightarrow (ii)$ in Proposition 3.3.7 holds for the operator $A - w$ since we can now use the Convergence Lemma for half-plane operators. Thus

$$\|f(A - w)\| \leq C\|f\|_{\mathcal{H}^\infty(\Sigma_{\frac{\pi}{2}})}$$

for all $f \in \mathcal{H}^\infty(\Sigma_{\frac{\pi}{2}})$ (in particular, each such $f(A - w)$ is bounded). Then, given $g \in \mathcal{H}^\infty(R_w)$ we consider the function $f \in \mathcal{H}^\infty(\Sigma_{\frac{\pi}{2}})$ defined by $f(z) = g(z + w)$. So $g(A)$ is bounded and we arrive at

$$\|g(A)\| = \|f(A - w)\| \leq C\|f\|_{\mathcal{H}^\infty(\Sigma_{\frac{\pi}{2}})} = C\|g\|_{\mathcal{H}^\infty(R_w)},$$

as required. □

We can now proceed to formulate the similarity problem for semigroups, starting with the following definition.

Definition 5.5.3. A semigroup $(T(t))_{t\geq 0}$ is said to be *similar* to a contractive semigroup if there exists an invertible operator S such that

$$\|S^{-1}T(t)S\| \leq 1 \quad \forall t \geq 0.$$

For ease of terminology, we will henceforth call such semigroups *semi-contractive*. Note that this is stronger than the condition that each operator $T(t)$ is similar to a contraction. Furthermore, $(S^{-1}T(t)S)$ is a semigroup with generator $S^{-1}AS$. From Theorem 5.5.1 and Proposition 5.5.2 we immediately obtain the following result.

Theorem 5.5.4. *Let A be the generator of a semigroup on a Hilbert space which is similar to a contractive semigroup. Then A admits a uniformly bounded $\mathcal{H}^\infty$-calculus of type* 0.

Analogously to the single operator case, it is of interest then to characterise those semigroups on Hilbert space which are similar to a contractive semigroup. In view of Theorem 5.4.6 we will seek to develop a notion of completely bounded $\mathcal{H}^\infty$-calculus and show that it is equivalent to the semigroup being semi-contractive. Indeed, the following definition is a natural analogue of the discrete case.

Definition 5.5.5. An operator A on a Hilbert space X is said to have a *completely bounded* $\mathcal{H}^\infty$-calculus on the appropriate domain Ω (sector, strip or half plane) if $f(A) \in B(X)$ for all $f \in \mathcal{H}^\infty(\Omega)$ and there exists a constant C such that for any $n \geq 1$ and all bounded, holomorphic functions $(F_{ij})_{1 \leq i,j \leq n}$

$$\|[F_{ij}(A)]\|_{B(X^n)} \leq C \sup \big\{ \|[F_{ij}(z)]\|_{M_n} : \ z \in \Omega \big\}. \tag{5.5.1}$$

Note that we endow X^n with the canonical norm here. This property is clearly stronger than having bounded $\mathcal{H}^\infty$-calculus on Ω, since this is the condition for $n = 1$. Note that for each n the inequality (5.5.1) is automatic once A is known to have a bounded $\mathcal{H}^\infty$-calculus on Ω, but the constant C may depend on n. If A admits a completely bounded $\mathcal{H}^\infty(R_w)$-calculus for $w < 0$, with the constant C in (5.5.1) independent of w, then A is said to admit a *uniform completely bounded* $\mathcal{H}^\infty$-calculus of type 0.

When A is a half-plane operator with $s_0(A) \geq 0$, we can relate this property to that of complete polynomial boundedness for its Cayley transform. Before doing so, we establish the following analogue of Proposition 5.5.2.

Proposition 5.5.6. *Let A be a half-plane operator on a Hilbert space with $s_0(A) \geq 0$. Then the following are equivalent.*

(i) There exists a constant $C \geq 0$ such that for all $n \geq 1$ and all families $\{R_{i,j}\} \in \mathcal{R}^\infty(\Sigma_{\frac{\pi}{2}})$

$$\|[R_{ij}(A)]\|_{B(X^n)} \leq C \sup \left\{ \|[R_{ij}(z)]\|_{M_n} : \ z \in \overline{\Sigma_{\frac{\pi}{2}}} \right\} .$$

(ii) A admits a uniform completely bounded $\mathcal{H}^\infty$-calculus of type 0.

Proof. Assume (ii) holds. Then for a fixed $n \geq 1$ and family $\{R_{i,j}\} \in \mathcal{R}^\infty(\Sigma_{\frac{\pi}{2}})$, there exists $v < 0$ such that for all $w \in (v, 0)$ and every pair $\{i, j\}$ we have $R_{i,j} \in \mathcal{H}^\infty(R_w)$. Thus

$$\|[R_{ij}(A)]\| \leq C \sup \big\{ \|[R_{ij}(z)]\|_{M_n} : \ z \in R_w \big\},$$

for some constant C independent of w and $R_{i,j}$. As in Proposition 5.5.2, we let $w \to 0$ and employ the continuity of $[R_{ij}(z)]$ to arrive at (i).

Now assume (i) holds. Given $w < 0$ and $f \in \mathcal{H}^\infty(R_w)$, the case $n = 1$ and

Proposition 5.5.2 yield the boundedness of $f(A)$ and the estimate

$$\|f(A)\| \leq C\|f\|_{\mathcal{H}^\infty(R_w)} .$$

Given $n > 1$, consider a family of functions $f_{i,j} \subseteq \mathcal{H}^\infty(R_w)$. As in the proof of Proposition 5.5.2, without loss of generality we can set $w = 0$ and suppose A has spectrum in some strict right half-plane. We then observe that for each pair $\{i, j\}$ there exists a sequence $(r_{i,j,k})_k$ of rational functions with poles outside $\overline{\Sigma_{\frac{\pi}{2}}}$ converging uniformly to $f_{i,j}$ on $\Sigma_{\frac{\pi}{2}}$ [34, Proposition F.3]. Then the matrix $[r_{i,j,k}(z)]$ converges to $[f_{i,j}(z)]$ in M_n, uniformly for $z \in \Sigma_{\frac{\pi}{2}}$. Moreover, $[r_{i,j,n}(A)]$ converges in the strong operator topology to $[f_{i,j}(A)]$. The result now follows. □

We will also require the following approximation result for complete polynomial boundedness (see [57, Section 3]).

Proposition 5.5.7. *The operator T is completely polynomially bounded if and only if for every $n \geq 1$ and all families of rational functions $\{R_{i,j}\}$ with poles outside $\overline{D(0,1)}$,*

$$\|[R_{ij}(T)]\| \leq C \sup\big\{\|[R_{ij}(z)]\|_{M_n} : |z| \leq 1\big\}.$$

We can now obtain the desired link in one direction almost immediately.

Theorem 5.5.8. *Let A be a half-plane operator on a Banach space with $s_0(A) \geq 0$ and let $T = (A-I)(A+I)^{-1}$ denote its Cayley transform. If A admits a uniformly bounded $\mathcal{H}^\infty$-calculus of type 0, then T is polynomially bounded. If X is a Hilbert space and A admits a uniformly completely bounded $\mathcal{H}^\infty$-calculus of type 0, then T is completely polynomially bounded.*

Proof. By the definitions we have

$$\begin{aligned} T \text{ is polynomially bounded} &\Leftrightarrow \|p(T)\| \leq C\ \|p\|_{\mathcal{H}^\infty(\overline{D(0,1)})} \quad \text{for all polynomials } p \\ &\Leftrightarrow \|p((A-I)(A+I)^{-1})\| \leq C\ \|p\|_{\mathcal{H}^\infty(\overline{D(0,1)})} \\ &\qquad \text{for all polynomials } p. \end{aligned}$$

But for a polynomial p, $p((A-I)(A+I)^{-1}) = q(A)$, where $q(z) = p(\frac{z-1}{z+1}) \in \mathcal{R}^\infty(\Sigma_{\frac{\pi}{2}})$. Thus by Proposition 5.5.2,

$$\|p(T)\| = \|q(A)\| \leq C\|q\|_{\mathcal{H}^\infty(\overline{\Sigma_{\frac{\pi}{2}}})} = C\|p\|_{\mathcal{H}^\infty(\overline{D(0,1)})} ,$$

as required.

The completely bounded case on Hilbert space is similar. Given $n \geq 1$ and a family of polynomials $\{P_{i,j}\}$ we have for each pair $\{i,j\}$, $P_{i,j}(T) = Q_{i,j}(A)$, where $Q_{i,j}(z) = P_{i,j}(\frac{z-1}{z+1}) \in \mathcal{R}^\infty(\Sigma_{\frac{\pi}{2}})$. Then by Proposition 5.5.6 we have

$$\begin{aligned}\|[P_{ij}(T)]\| = \|[Q_{ij}(A)]\| &\leq C \sup\left\{\|[Q_{ij}(z)]\|_{M_n} :\ z \in \overline{\Sigma_{\frac{\pi}{2}}}\right\}\\ &= C \sup\left\{\|[P_{ij}(z)]\|_{M_n} :\ z \in \overline{D(0,1)}\right\}.\end{aligned}$$

□

We can approach the converse result via the work of Le Merdy [50]. In that paper, he solved the similarity problem for bounded, analytic semigroups via the following result. Note that under the assumptions of the following theorem, A is sectorial of angle less than $\frac{\pi}{2}$.

Theorem 5.5.9. *Let $(T(t))_{t\geq 0}$ be a bounded analytic semigroup on a Hilbert space with generator $-A$ and assume A is injective. Then the following are equivalent.*

(i) A admits bounded imaginary powers.

(ii) $(T(t))_{t\geq 0}$ is semi-contractive.

Furthermore, Le Merdy demonstrated that the analyticity assumption cannot be dropped. By taking the inverse Cayley transform of Pisier's counter-example to the similarity problem for single operators (which he showed to be well-defined), he found a generator of a bounded C_0-semigroup which has BIP but is not similar to a contraction semigroup. Using the methods of [50], which involves a fine-tuning of McIntosh's work on the functional calculus of A [53], we can offer the following converse to Theorem 5.5.8.

Theorem 5.5.10. *Let A be a half-plane operator on a Banach space with $s_0(A) \geq 0$ and let $T = (A - I)(A + I)^{-1}$ denote its Cayley transform. If T is polynomially bounded then A admits a uniformly bounded $\mathcal{H}^\infty$-calculus of type 0. If X is a Hilbert space and T is completely polynomially bounded then A admits a uniformly completely bounded $\mathcal{H}^\infty$-calculus of type 0.*

Proof. By Proposition 5.5.2, it is sufficient to prove that there exists a constant C such that for all $r \in \mathcal{R}^\infty(\Sigma_{\frac{\pi}{2}})$

$$\|r(A)\| \leq C\|r\|_{\mathcal{H}^\infty(\overline{\Sigma_{\frac{\pi}{2}}})}. \tag{5.5.2}$$

Given $r \in \mathcal{R}^\infty(\Sigma_{\frac{\pi}{2}})$, let $\phi : D(0,1) \to \mathbb{C}$ be defined by $\phi(z) = r(\frac{1+z}{1-z})$. Then ϕ has analytic continuation in a neighbourhood of $\overline{D(0,1)}$ (this can be verified first for r of the form $\frac{1}{z-a}$ with a outside of $\overline{\Sigma_{\frac{\pi}{2}}}$ and then by observing that these simple rational functions generate the algebra $\mathcal{R}^\infty(\Sigma_{\frac{\pi}{2}})$).

Clearly we have $\phi(T) = r(A)$ so that by the assumption that T is polynomially bounded we arrive at (5.5.2). The complete version on Hilbert space follows similarly from Propositions 5.5.6 and 5.5.7. □

We employ the following well known result from [69].

Theorem 5.5.11. *Let X be a Hilbert space, $T \in B(X)$. Then T is the Cayley transform of the (negative) generator of a contractive C_0-semigroup if and only if T is contractive and $T - I$ is injective.*

We then arrive at our desired result for semigroups.

Theorem 5.5.12. *Let A be a half-plane operator on a Hilbert space with $s_0(A) \geq 0$. Then A admits a uniform completely bounded $\mathcal{H}^\infty$-calculus of type 0 if and only if $-A$ is the generator of a semi-contractive semigroup.*

Proof. Denoting by T the Cayley Transform of A, we have by Theorems 5.5.8, 5.5.10, 5.4.6 and 5.5.11 that

$$\begin{aligned} A \text{ admits a uniform completely bounded } \mathcal{H}^\infty - \text{calculus of type } 0 &\iff \\ T \text{ is completely polynomially bounded} &\iff \\ T \text{ is similar to a contraction} &\iff \\ -A \text{ generates a semi-contractive semigroup.} & \end{aligned}$$

□

Moreover, by Theorems 5.5.8 and 5.5.10 we can reduce the problem of uniformly bounded $\mathcal{H}^\infty$-calculus of a semigroup generator to that of polynomial boundedness of its Cayley Transform:

Corollary 5.5.13. *Let A be a half-plane operator on a Banach space with $s_0(A) \geq 0$. Then A admits a uniformly bounded $\mathcal{H}^\infty$-calculus of type 0 if and only if its Cayley Transform T is polynomially bounded.*

We note that by combining Theorem 5.4.1 and 5.5.13, we obtain an alternative proof of the Crouzeix-Délyon result (Theorem 5.5.1).

Finally in this section we observe that by Theorem 5.5.12 and Corollary 5.5.13, Le Merdy's counterexample from [50] (see comment following Theorem 5.5.9) admits a uniformly bounded $\mathcal{H}^\infty$-calculus of type 0 but not a uniform completely bounded $\mathcal{H}^\infty$-calculus of type 0, so that the complete boundedness assumption cannot be removed in Theorem 5.5.12.

5.6 Properties of $\delta(A)$

The results of the previous section lead one to ask when the Cayley Transform of the generator of a bounded C_0-semigroup on a Hilbert space is polynomially bounded. So far, it is unknown if such an operator is always power bounded, although there are many results in this direction. We now survey these results. In what follows, $-A$ generates a bounded C_0-semigroup $(T(t))$ and $T = \delta(A)$ is the Cayley transform of A (often referred to in the literature as the *cogenerator* of $(T(t))$.

The first result shows that we cannot hope for power boundedness of T in general Banach space.

Theorem 5.6.1. *There exists a constant C such that for every n,*

$$\|T^n\| \leq C(1+\sqrt{n}) \ . \tag{5.6.1}$$

Moreover, this estimate is optimal.

The inequality (5.6.1) was proven by Brenner and Thomée [9], whereas optimality was established by Piskarev and Zwart [62] (in fact, even the weaker estimate $\|T^n(T-I)\| \leq C(1+\sqrt{n})$ was shown to be optimal). On Hilbert space, it is known that $(T(t))$ is contractive, normal, unitary, self-adjoint, isometric, completely non-unitary or strongly stable if and only if T is (see [69], [71]). Moreover, we have the following sufficient conditions for T to be power bounded.

Theorem 5.6.2. *Let $(T(t))$ and T be as above, X a Hilbert space. Then T is power bounded if one of the following holds:*

(i) $(T(t))$ is analytic.
(Gomilko [31])

(ii) A is injective and $-A^{-1}$ also generates a bounded C_0-semigroup.
(Guo, Zwart [32], Azizov et al. [6])

(iii)

$$\sup_{\rho>1} (\rho-1) \int_{|\lambda|=\rho} |\langle R^2(\lambda,T)x,y\rangle|\, |\mathrm{d}\lambda| < \infty \qquad \forall x \in X\ \forall y \in X^* \ . \tag{5.6.2}$$

(van Casteren [72])

(iv) *A is bounded.*
(Gomilko [31], Azizov et al. [6])

The following partial results have been established for general bounded semigroups on Hilbert space.

Theorem 5.6.3. *For a general bounded C_0-semigroup $(T(t))$ on Hilbert space we have*

(i) $\|T^n(T-I)\| \leq C$.
(Guo, Zwart [32])

(ii) $\|T^n\| \leq C(1+\log n)$.
(Gomilko [31])

It remains unknown if the estimate (*ii*) is optimal.

On the other hand, it is of interest to consider cogenerators which are not polynomially bounded, since then by Corollary 5.5.13 this will give rise to operators which do not admit a uniformly bounded $\mathcal{H}^\infty$-calculus of type 0. We have examples of operators which are power bounded but not polynomially bounded, such as Example 5.4.4, but we do not yet have conditions to determine whether or not these operators arise as cogenerators. Given an operator T for which $I-T$ is injective with dense range, we seek conditions for which $(I+T)(I-T)^{-1}$ is the generator of a bounded C_0-semigroup. We can apply the well known generation theorems to obtain such conditions on T. This was first done by Eisner and Zwart [26] on general Banach space via the Hille-Yosida theorem. Since the method is simple but instructive, we include the proof.

Theorem 5.6.4. *Let X be a Banach space, $T \in B(X)$, $M \geq 1$. The following are equivalent.*

(i) *T is the cogenerator of a C_0-semigroup $(T(t))$ on X satisfying $\|T(t)\| \leq M$ for all $t \geq 0$.*

(ii) *$T-I$ is injective with dense range, $(1,\infty) \subseteq \rho(T)$ and*

$$\|[(I-T)R(\lambda,T)]^n\| \leq \frac{2^n M}{(\lambda+1)^n} \text{ for all } \lambda > 1,\ n \in \mathbb{N}.$$

(iii) $T-I$ *is injective with dense range,* $(1,\lambda_0)\subseteq\rho(T)$ *for some* $\lambda_0>1$ *and*

$$\|[(I-T)R(\lambda,T)]^n\|\le\frac{2^nM}{(\lambda+1)^n}\ \textit{for all}\ \lambda\in(1,\lambda_0),\ n\in\mathbb{N}.$$

Proof. We consider the case $M=1$; the rest follows similarly. First note that for every cogenerator T, the operator $T-I$ is injective with dense range. Now suppose $T-I$ is injective with dense range. Then the operator $A=(T+I)(I-T)^{-1}$ is densely defined and by the Hille-Yosida theorem $-A$ generates a contractive semigroup if and only if there exists $\lambda_0\ge 0$ such that for all $\lambda\in(\lambda_0,\infty)$, $\|R(\lambda,-A)\|\le\frac{1}{\lambda}$. Set $\mu=\frac{\lambda+1}{\lambda-1}$. Then $\lambda>\lambda_0>1$ if and only if $1<\mu<\mu_0$ for $\mu_o=\frac{\lambda_0+1}{\lambda_0-1}$. Furthermore, by [26, Lemma 2.1] we have

$$\rho(-A)\backslash\{1\}=\left\{\frac{\mu+1}{\mu-1}:\ 1\neq\mu\in\rho(T)\right\}$$

and for $\lambda\in\rho(-A)\backslash\{1\}$,

$$R(\lambda,-A)=\frac{1}{\lambda-1}(I-T)R\biggl(\frac{\lambda+1}{\lambda-1},T\biggr)\ .$$

Thus for $1<\mu\in\rho(T)$ we have $1<\lambda=\frac{\mu+1}{\mu-1}\in\rho(-A)$ and

$$\lambda R(\lambda,-A)=\frac{\lambda}{\lambda-1}(I-T)R(\mu,T)=\frac{\mu+1}{2}(I-T)R(\mu,T).$$

This shows the equivalence of (i) and (iii). Similarly we can show $(i)\Leftrightarrow(ii)$.

$\square$

For applications it is desirable to have a cogeneration theorem for bounded semigroups involving a condition on a single resolvent power. In Hilbert space, we now obtain such a condition by employing the result of Gomilko-Shi-Feng (Theorem 2.1.10).

Theorem 5.6.5. *Let* X *be a Hilbert space,* $T\in B(X)$ *such that* $\sigma(T)\subseteq\overline{D(0,1)}$. *Suppose further that* $T-I$ *is injective with dense range. Then* T *is the cogenerator of a bounded* C_0*-semigroup if and only if for every* $x,y\in X$,

$$\sup_{w>0,w\neq1}\frac{w}{|1-w|}\int_0^{2\pi}\left|\left\langle\left[(I-T)R\left(\frac{e^{i\theta}}{|1-w|}+\frac{w}{w-1},T\right)\right]^2x,y\right\rangle\right|\,\mathrm{d}\theta\ <\infty\ .$$

Proof. The operator $A=(T+I)(T-I)^{-1}$ is closed and satisfies $\sigma(A)\subseteq\{z\in\mathbb{C}:\ \operatorname{Re}z\le0\}$. Thus by the Gomilko-Shi-Feng theorem, T is the cogenerator of a bounded semigroup if and only if

$$\sup_{w>0, w\neq 1} w \int_{w-i\infty}^{w+i\infty} |\langle R\big(\lambda, A\big)^2 x, y\rangle| \, |\mathrm{d}\lambda| < \infty \qquad \forall x, y \in X \, . \tag{5.6.3}$$

By [26, Lemma 2.1] we have for $\lambda \in \rho(A)\backslash\{1\}$,

$$R(\lambda, A) = \frac{\mu - 1}{2}(I - T)R(\mu, T), \qquad \mu = \frac{\lambda + 1}{\lambda - 1} \, .$$

Moreover, for $w > 0$ the function $z : z \to \frac{z+1}{z-1}$ maps the line $w + i\mathbb{R}$ to the circle $\Gamma_w = \{re^{i\theta} + \beta : \ \theta \in [0, 2\pi]\}$, where

$$r = \frac{1}{|1 - w|} \qquad \text{and} \qquad \beta = \frac{w}{w - 1} \, .$$

Hence with $\mu = \frac{\lambda+1}{\lambda-1}$ we have

$$\begin{aligned}
\int_{w-i\infty}^{w+i\infty} |\langle R(\lambda, A)^2 x, y\rangle| \, |\mathrm{d}\lambda| &= 2 \int_{\Gamma_w} \left|\left\langle \left[\frac{\mu - 1}{2}(I - T)R(\mu, T)\right]^2 x, y\right\rangle\right| \frac{|\mathrm{d}\mu|}{|\mu - 1|^2} \\
&= c \int_{\Gamma_w} \left|\left\langle \left[(I - T)R(\mu, T)\right]^2 x, y\right\rangle\right| |\mathrm{d}\mu| \\
&= c \int_0^{2\pi} r \left|\left\langle \left[(I - T)R(re^{i\theta} + \beta, T)\right]^2 x, y\right\rangle\right| \mathrm{d}\theta \\
&= \frac{c}{|1 - w|} \int_0^{2\pi} \left|\left\langle \left[(I - T)R\Big(\frac{e^{i\theta}}{|1 - w|} + \frac{w}{w - 1}, T\Big)\right]^2 x, y\right\rangle\right| \mathrm{d}\theta \, .
\end{aligned}$$

Thus the result follows from (5.6.3).

□

5.7 More general results

Let us now consider the usual notion of bounded $\mathcal{H}^\infty$-calculus for a (bounded) semigroup generator $-A$. For $\epsilon > 0$, A admits a bounded $\mathcal{H}^\infty(R_{-\epsilon})$-calculus if and only if $A+\epsilon$ admits a bounded $\mathcal{H}^\infty$-calculus on $\Sigma_{\frac{\pi}{2}}$. By our considerations in Section 5.5, we can show that this is equivalent to the operator $\delta(A + \epsilon)$ being polynomially bounded.

Theorem 5.7.1. *Let $-A$ be the generator of a bounded C_0-semigroup, $\epsilon > 0$. Then A admits a bounded $\mathcal{H}^\infty$-calculus on $R_{-\epsilon}$ if and only if $\delta(A + \epsilon)$ is polynomially bounded.*

Proof. By Theorems 5.5.8 and 5.5.10, $\delta(A + \epsilon)$ is polynomially bounded if and only if there exists a constant C such that for every $\delta > 0$ and $f \in \mathcal{H}^\infty(R_{-\delta})$,

$$\|f(A+\epsilon)\| \leq C\|f\|_{\mathcal{H}^\infty(R_{-\delta})} . \tag{5.7.1}$$

Clearly this holds if $A+\epsilon$ admits a bounded $\mathcal{H}^\infty$-calculus on $\Sigma_{\frac{\pi}{2}}$.

Conversely, suppose (5.7.1) holds and let $f \in \mathcal{H}^\infty(\Sigma_{\frac{\pi}{2}})$. For $\delta > 0$ define $f_\delta \in \mathcal{H}^\infty(R_{-\delta})$ by $f_\delta(z) = f(z+\delta)$. Then by the Convergence Lemma, letting $\delta \to 0$ gives $f(A+\epsilon) \in B(X)$ with

$$\|f(A+\epsilon) \leq C\|f\|_{\mathcal{H}^\infty(\Sigma_{\frac{\pi}{2}})} .$$

Thus $A+\epsilon$ has bounded $\mathcal{H}^\infty$-calculus on $\Sigma_{\frac{\pi}{2}}$, as required. $\square$

The operator $-(A+\epsilon)$ generates an exponentially stable semigroup, so $\sigma(\delta(A+\epsilon)) \cap \mathbb{T} = \{1\}$, where $\mathbb{T}$ is the closed unit circle. Thus we can refine van Casteren's condition (5.6.2) from Theorem 5.6.2 to obtain the following sufficient condition for power boundedness of $\delta(A+\epsilon)$.

Theorem 5.7.2. *Let $-A$ be the generator of a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space, $T = \delta(A+\epsilon)$. Then T is power bounded if for some $\theta > 0$,*

$$\sup_{\rho>1} (\rho - 1) \int_{\Gamma_{\rho,\theta}} |\langle R^2(\lambda, T)x, y\rangle| \, |\mathrm{d}\lambda| < \infty \qquad \forall x \in X \ \forall y \in X^* ,$$

where $\Gamma_{\rho,\theta} = \{\rho e^{i\psi} : \ |\arg\psi| \leq \theta\}$.

6 Sums and perturbations

In this chapter we consider sums and perturbations of strip-type operators in relation to the functional calculus.

6.1 Maximal regularity via sums

The theory of the functional calculus for sectorial operators bears fruitful applications to the problem of maximal regularity for evolution equations. Let us consider the following problem in a Banach space X:

$$\begin{aligned} u'(t) + Au(t) &= f(t), \qquad 0 \leq t \leq T, \\ u(0) &= 0, \end{aligned} \tag{6.1.1}$$

where $f \in L^p([0,T],X), p \in (1,\infty)$ and A is sectorial.

Definition 6.1.1. A is said to have *maximal L^p-regularity* if for every $f \in L^p([0,T],X)$, (6.1.1) has a unique solution $u \in W^{1,p}([0,T],X) \cap L^p([0,T],D(A))$ such that

$$\|u'\|_{L^p([0,T],X)} + \|Au\|_{L^p([0,T],X)} \leq C\|f\|_{L^p([0,T],X)},$$

for some constant $C > 0$.

This property is known to be independent of $p \in (1,\infty)$ and $T < \infty$ (see [22]), so we refer to maximal L^p-regularity simply as maximal regularity. Let us now tie this notion with operator sums. We define operators $\mathcal{A}$ and $\mathcal{B}$ on $L^p([0,T],X)$ as follows:

$$(\mathcal{A}u)(t) = A(u(t)) \qquad (\mathcal{B}u)(t) = u'(t),$$

for $t \in [0,T]$, with domains

$$\begin{aligned} D(\mathcal{A}) &= \Big\{u \in L^p([0,T],X) : u(t) \in D(A) \text{ for almost all t}, A(u(\cdot)) \in L^p([0,T],X)\Big\}, \\ D(\mathcal{B}) &= \{u \in W^{1,p}([0,T],X) : u(0) = 0\}. \end{aligned}$$

It is then clear that the abstract Cauchy problem (6.1.1) is equivalent to the operator equation

$$\mathcal{A}u + \mathcal{B}u = f$$

in the space $L^p([0,T],X)$. Thus maximal regularity is reduced to the problem of finding conditions under which $\mathcal{A}+\mathcal{B}$ is a well-defined, closed operator. In their seminal paper from 1987, Dore and Venni [23] were the first to provide such conditions when X is a UMD space.

Theorem 6.1.2. *Let X be a UMD space. Suppose that A and B are invertible sectorial operators whose resolvents commute and that they both have BIP. Suppose further that* $\theta_A + \theta_B < \pi$*. Then the sum* $(A + B, D(A) \cap D(B))$ *is closed and invertible.*

We note that the strong parabolicity condition $\theta_A + \theta_B < \pi$ is fundamental to results on operator sums of this kind. For example, trivial cases such as $B = -A$ (whence $A + B$ is just the zero operator) are avoided as a result. Soon after, Prüss and Sohr [64] removed the invertibility assumptions on A and B. As the theory of R-boundedness became more widely understood, Kalton and Weis [44] were able to derive the following conditions in connection with the functional calculus.

Theorem 6.1.3. *Let A and B be a pair of commuting sectorial operators. Suppose A admits a bounded* $\mathcal{H}^\infty$*-calculus and B is R-sectorial with* $\phi_A^H + \phi_B^R < \pi$*. Then* $(A + B, D(A) \cap D(B))$ *is closed and there exists a constant C such that*

$$\|Ax\| + \|Bx\| \leq C\|Ax + Bx\| \qquad \forall x \in D(A) \cap D(B) \ .$$

Furthermore, $A + B$ *is invertible if A or B is invertible.*

Note that on a UMD space, BIP is weaker than bounded $\mathcal{H}^\infty$-calculus but stronger than R-sectoriality [13]. L^p-spaces have UMD for $p > 1$ [10] and many classes of differential operators are known to have bounded $\mathcal{H}^\infty$-calculus on these spaces, making this result more useful in applications than the Dore-Venni theorem.

In fact, these considerations can be extended to the non-commuting case, under appropriate assumptions. Let A be an invertible sectorial operator, and $0 \leq \alpha < \beta < 1$. Consider the following condition on a sectorial operator B:

$$\begin{cases} \text{There exist constants } c = c_{A,\alpha,\beta}(B) > 0,\ \psi_A > \phi_A, \psi_B > \phi_B, \\ \text{such that } \psi_A + \psi_B < \pi \text{ and for all } \lambda \in \Sigma_{\pi-\psi_A}, \mu \in \Sigma_{\pi-\psi_B} \\ \|A(\lambda + A)^{-1}[A^{-1}(\mu + B)^{-1} - (\mu + B)^{-1}A^{-1}]\| \leq \frac{c}{(1+|\lambda|)^{1-\alpha}|\mu|^{1+\beta}}. \end{cases} \tag{6.1.2}$$

The following result is due to Prüss and Simonett [63].

Theorem 6.1.4. *Let A be an invertible sectorial operator with bounded* $\mathcal{H}^\infty$*-calculus, and* $0 \leq \alpha < \beta < 1$*. Consider R-sectorial operators B such that* $\phi_A^H + \phi_B^R < \pi$ *and* (6.1.2) *holds for some* $\psi_A > \phi_A^H$*,* $\psi_B > \phi_B^R$ *satisfying* $\psi_A + \psi_B < \pi$*. Then there exists* $c_0 = c_0(A, \alpha, \beta) > 0$ *such that the sum* $(A+B, D(A) \cap D(B))$ *is invertible and sectorial with* $\phi_{A+B} \leq \max\{\psi_A, \psi_B\}$ *whenever* $c_{A,\alpha,\beta}(B) < c_0$*.*

We remark that the smallness assumption on $c_{A,\alpha,\beta}(B)$ can be dropped at the expense of a shift. More precisely, suppose that (6.1.2) holds for an arbitrary $c_{A,\alpha,\beta}(B) > 0$. For $v > 0$ we can then replace the operator B by $v + B$ and estimate for $\alpha < \beta' < \beta$,

$$\|A(\lambda + A)^{-1}[A^{-1}(\mu + v + B)^{-1} - (\mu + v + B)^{-1}A^{-1}]\| \leq \frac{c}{(1 + |\lambda|)^{1-\alpha}|\mu + v|^{1+\beta}}$$
$$= \frac{c}{(1 + |\lambda|)^{1-\alpha}|\mu|^{1+\beta'}}\left(\frac{|\mu|^{1+\beta'}}{|\mu + v|^{1+\beta}}\right).$$

Since $\sup\{\frac{|\mu|^{1+\beta'}}{|\mu+v|^{1+\beta}} : \mu \in \Sigma_{\pi-\psi_B}\} \to 0$ as $v \to \infty$, we can choose $v > 0$ such that the constant in (6.1.2) corresponding to the pair of operators $\{A, v + B\}$ is less than $c_0(A, \alpha, \beta)$. Thus the conclusions of Theorem 6.1.4 apply to $(v+A+B, D(A)\cap D(B))$.

In Chapter 4 we noted that having absolute functional calculus is stronger than bounded $\mathcal{H}^\infty$-calculus in general, by considering minimal angles. Here we highlight the relative strength of absolute calculus by showing that under this assumption, we can remove the R-boundedness assumptions on the second operator in the results above. The commuting case is taken care of by the following result due to Kucherenko [46, Theorem 4.6.7], under her definition of K-absolute calculus (see Definition 4.1.5).

Theorem 6.1.5. *Suppose that A and B are a pair of commuting sectorial operators on a Banach space such that A has K-absolute calculus with $\phi_A + \phi_B < \pi$. Then the conclusions of Theorem 6.1.3 hold.*

The proof calls upon the following modification of a result from [44, Chapter 6], which can be found in [46, Chapter 4].

Proposition 6.1.6. *Let A be a sectorial operator of type ϕ_A with absolute calculus on Σ_ϕ for some/all $\phi > \phi_A$. Then A has a bounded $\mathcal{H}^\infty(\Sigma_\phi, E_A)$-calculus (in the sense of 3.6.1).*

Remark 6.1.7. Fix $\tau > 0$ and $1 < p < \infty$. The derivative operator $A_\tau = \frac{\mathrm{d}}{\mathrm{d}t}$ on $L^p((0,\tau), X)$, with domain $W^{1,p}((0,\tau), X)$, is sectorial of angle $\frac{\pi}{2}$ and $-A_\tau$ is a C_0-semigroup generator [34, Lemma 8.4.1]. Furthermore A_τ has bounded $\mathcal{H}^\infty$-calculus on $L^p((0,\tau), X)$ if and only if X is *UMD* [34, Theorem 8.5.8]. In particular, when $p = 2$ and X is a Hilbert space, $L^p((0,\tau), X)$ is also a Hilbert space and so A_τ has absolute calculus, since this is then equivalent to having a bounded $\mathcal{H}^\infty$-calculus. Furthermore if A_τ has absolute calculus on $L^p((0,\tau), X)$ then by Theorem 6.1.5 the (negative) generator of any analytic semigroup must have maximal regularity (since it is sectorial of angle less than $\frac{\pi}{2}$). Assume that X has an unconditional basis. Then by a result due to Kalton and Lancien [43] this implies that X is isomorphic to a Hilbert space. Thus A_τ does not have absolute calculus on $L^p((0,\tau),X)$ when X has an unconditional basis but is not a Hilbert space.

We can also adapt Theorem 6.1.4 to these assumptions to obtain the following new result, where now A is assumed to have absolute calculus in our usual sense.

Theorem 6.1.8. *Suppose that A and B are sectorial operators on a Banach space such that A has absolute calculus and $\phi_A + \phi_B < \pi$. Let $\psi_A > \phi_A$, $\psi_B > \phi_B$ be such that $\psi_A + \psi_B < \pi$ and* (6.1.2) *holds. Then the conclusions of Theorem 6.1.4 hold.*

The proof is mostly the same as [63, Theorem 3.1]. We give here the argument which replaces the step where the R-sectoriality of B was used in [63]. Let $\psi > \max\{\psi_A, \psi_B\}$ and let θ satisfy $\psi_A < \theta < \min\{\psi, \pi - \psi_B\}$. For $N \in \mathbb{N}$ define the contour

$$\Gamma_N = \{te^{i\theta} :\ 2^{-N} \le t \le 2^N\} \cup \{te^{-i\theta} :\ 2^{-N} \le t \le 2^N\}\ .$$

The R-sectoriality assumption in [63] was employed to estimate the operator norm of the integral

$$I_{\delta,\lambda} = \frac{1}{2\pi i}\int_{\Gamma_N} [A^\gamma(z-A)^{-1}]^a B(\lambda+z+B)^{-1}(1+\delta B)^{-1}[A^\gamma(z-A)^{-1}]^{1-a}\,\frac{\mathrm{d}z}{z^\gamma}$$

uniformly in $0 < \delta < 1$ and $\lambda \in \Sigma_{\pi-\psi}$ where $a, \gamma \in (0,1)$ are fixed. Using the assumptions of Theorem 6.1.8, we have for any $x \in X$ and $x \in X^*$,

$$\begin{aligned}
&|\langle I_{\delta,\lambda}x, x^*\rangle| \le \\
&C\sum_{\pm}\int_{2^{-N}}^{2^N} |\langle [A^\gamma(te^{\pm i\theta}-A)^{-1}]^a B(\lambda+te^{\pm i\theta}+B)^{-1}(1+\delta B)^{-1}[A^\gamma(te^{\pm i\theta}-A)^{-1}]^{1-a}x, x^*\rangle|\,\frac{\mathrm{d}t}{t^\gamma}\\
&\le C\sum_{\pm}\int_0^\infty |\langle [A^\gamma(te^{\pm i\theta}-A)^{-1}]^a B(\lambda+te^{\pm i\theta}+B)^{-1}(1+\delta B)^{-1}[A^\gamma(te^{\pm i\theta}-A)^{-1}]^{1-a}x, x^*\rangle|\,\frac{\mathrm{d}t}{t^\gamma}\\
&= C\sum_{\pm}\int_0^\infty |\langle [(\frac{A}{t})^\gamma(e^{\pm i\theta}-\frac{A}{t})^{-1}]^a B(\lambda+te^{\pm i\theta}+B)^{-1}(1+\delta B)^{-1}[(\frac{A}{t})^\gamma(e^{\pm i\theta}-\frac{A}{t})^{-1}]^{1-a}x, x^*\rangle|\,\frac{\mathrm{d}t}{t}\\
&\le C\sum_{\pm}\int_0^\infty |\langle \psi_{1,\pm}(\frac{A}{t})x, [B(\lambda+te^{\pm i\theta}+B)^{-1}(1+\delta B)^{-1}]^*\psi_{2,\pm}(\frac{A^*}{t})x^*\rangle|\,\frac{\mathrm{d}t}{t}\\
&\le C\sum_{\pm}\Big(\int_0^\infty \|\psi_{1,\pm}(\frac{A}{t})x\|^2\,\frac{\mathrm{d}t}{t}\Big)^{\frac12}\Big(\int_0^\infty \|[B(\lambda+te^{\pm i\theta}+B)^{-1}(1+\delta B)^{-1}]^*\psi_{2,\pm}(\frac{A^*}{t})x^*\|^2\,\frac{\mathrm{d}t}{t}\Big)^{\frac12}\\
&= C\sum_{\pm}\Big(\int_0^\infty \|\psi_{1,\pm}(tA)x\|^2\,\frac{\mathrm{d}t}{t}\Big)^{\frac12}\Big(\int_0^\infty \|[B(\lambda+\frac{1}{t}e^{\pm i\theta}+B)^{-1}(1+\delta B)^{-1}]^*\psi_{2,\pm}(tA^*)x^*\|^2\,\frac{\mathrm{d}t}{t}\Big)^{\frac12},
\end{aligned}$$

where $\psi_{1,\pm}(z) = \left(\frac{z^\gamma}{e^{\pm i\theta}-z}\right)^{1-a}$ and $\psi_{2,\pm}(z) = \left(\frac{z^\gamma}{e^{\pm i\theta}-z}\right)^{a}$. By the sectoriality assumption on B and the fact that A has absolute calculus, we can estimate this final term by $C\|x\|\|x^*\|$, with C independent of δ and λ. This gives the desired estimate.

In a similar manner we can adapt the proof in [36] for products of sectorial operators under these assumptions. Since the argument is essentially as above, we omit the details.

Theorem 6.1.9. *Let A and B be sectorial operators such that A has absolute calculus and $\phi_A + \phi_B < \pi$. Suppose further that the following set of conditions hold:*

$$\begin{cases} 0 \in \rho(A). \textit{ There exist constants } c = c(A,B) \geq 0, \alpha, \beta \geq 0, \alpha+\beta < 1, \\ \psi_A > \theta_A, \psi_B > \theta_B, \psi_A + \psi_B < \pi, \\ \textit{such that for all } \lambda \in \Sigma_{\pi-\psi_A}, \mu \in \Sigma_{\pi-\psi_B} \\ (\mu+B)^{-1}D(A) \subseteq D(A) \textit{ and} \\ \|[A(\mu+B)^{-1} - (\mu+B)^{-1}A](\lambda+A)^{-1}\| \leq \frac{c}{(1+|\lambda|)^{1-\alpha}|\mu|^{1+\beta}}. \end{cases} \tag{6.1.3}$$

Then there exists $v \geq 0$ such that $(v + AB, D(AB))$ is sectorial with $\phi_{v+AB} \leq \psi_A + \psi_B$.

We now turn our attention to sums of strip-type operators. Heuristically, for a pair of commuting injective sectorial operators A and B we may expect the identity

$$\log A + \log B = \log AB$$

to hold. However, we must impose suitable conditions to ensure that the operators $\log A$ and $\log B$ don't 'cancel' and that the product AB is sectorial. Clark [11] adopted Kalton-Weis type assumptions to achieve this with so called half strip-type operators. To formulate his result precisely, we employ the following notation. For $\theta \in (0,\pi)$ and $r > 0$ we say $A \in \mathrm{Sect}(\theta, r)$ if $A \in \mathrm{Sect}(\theta)$ and $\sigma(A) \subseteq \overline{\Sigma_{\theta,r}}$ where

$$\Sigma_{\theta,r} = \Sigma_\theta \cap \{\lambda \in \mathbb{C} : \ |\lambda| > r\} .$$

Furthermore for $w \geq 0$ and $\rho \in \mathbb{R}$ let

$$L_{w,\rho} = H_w \cap \{\mathrm{Re}\,\lambda < \rho\},$$

$$R_{w,\rho} = H_w \cap \{\mathrm{Re}\,\lambda > \rho\}.$$

We say B is a *half strip-type* operator, $B \in \mathrm{Strip}(w,\rho)$, if $B \in \mathrm{Strip}(w)$ and

(i) $\sigma(B) \subseteq \overline{R_{w,\rho}}$,

(ii) $\sup\{\|R(\lambda, B)\| : \operatorname{Re}\lambda \leq \rho'\} < \infty$ for each $\rho' < \rho$.

Clark's result can then be stated as follows.

Theorem 6.1.10. *Let $A \in$ Sect(w,r), $B \in$ Sect(w', r') be resolvent commuting with dense domain and range. Suppose A has a bounded $\mathcal{H}^\infty(\Sigma_\theta)$-calculus for some $\theta \in (w, \pi)$ and that B is R-sectorial. Suppose also that AB is sectorial (e.g. if $\phi_A^H + \phi_B^R < \pi$). Then* $\log A + \log B$ *is closed and*

$$\log A + \log B = \log AB \ .$$

Our aim is to drop the R-boundedness assumption on B by assuming that A has absolute calculus. A key step is to observe that the following appropriate modification of [11, Theorem 3.3.5] holds.

Theorem 6.1.11. *Let $A \in$ Sect(w,r), $B \in$ Sect(w', r') be resolvent commuting with dense domain and range. Suppose A has absolute calculus. Let $\alpha < \log r, \psi \in (\theta, \pi), \beta < \log r'$ and $\psi' \in (w', \pi)$. Suppose $f \in \mathcal{H}^\infty(R_{\psi,\alpha} \times R_{\psi',\beta})$ is such that $f(w, \log B)$ is bounded for every $w \in R_{\psi,\alpha}$ and that the set $\{f(w, \log B) : w \in R_{\psi,\alpha}\}$ is uniformly bounded. Then $f(\log A, \log B)$ is a bounded operator.*

The proof is very similar to that in [11, Theorem 3.3.5] and thus omitted. We observe the connection between the uniform boundedness assumption of $\{f(w, \log B) : w \in R_{\psi,\alpha}\}$ and Proposition 6.1.6 (which extends to operators in Sect(w, r)); whereas in [11, Theorem 3.3.5] it was R-boundedness that was needed. We will also need the following result for products.

Theorem 6.1.12. *Let A and B be commuting sectorial operators with dense domain and range of types ϕ_A and ϕ_B with $\phi_A + \phi_B < \pi$ and suppose A has absolute calculus. Then AB is sectorial with $\phi_{AB} \leq \phi_A + \phi_B$.*

Proof. Fix $\phi \in (\phi_A + \phi_B, \pi)$ and for $\lambda \in \Sigma_\phi$ define F_λ on Σ_ϕ by

$$F_\lambda(z) = \lambda(\lambda - zB)^{-1} = \frac{\lambda}{z}(\frac{\lambda}{z} - B)^{-1} \ .$$

Then F_λ satisfies the conditions in Proposition 6.1.6 and so, employing the sectoriality assumption on B, there is a constant C independent of λ such that $F_\lambda(A) \in B(X)$ with

$$\|F_\lambda(A)\| \leq C \ .$$

But $F_\lambda(A) = \lambda R(\lambda, AB)$ and so the result follows. $\square$

We can now state our desired result.

Theorem 6.1.13. *Let $A \in$ Sect(w,r), $B \in$ Sect(w', r') be resolvent commuting with dense domain and range. Suppose A has absolute calculus and that AB is sectorial (e.g. if $\phi_A + \phi_B < \pi$). Then* $\log A + \log B$ *is closed and*

$$\log A + \log B = \log AB \ .$$

Proof. As in [11], we can restrict our attention to the case $\log r + \log r' > 0$ and it is sufficient to prove closedness of $\log A + \log B$; the identity is then immediate under the additional assumption that AB is sectorial. In fact, we will show that $\log A + \log B$ is invertible. Observe that $\log A \in \mathrm{Strip}(w, \log r)$ and $\log B \in \mathrm{Strip}(w', \log r')$ (see [11, Proposition 3.2.3]). For $\psi \in (\theta, \pi), \rho \in (-\log r', \log r), \psi' \in (w', \pi)$ and $\rho' < \log r'$ define $f \in \mathcal{H}^\infty(R_{\psi,\rho} \times R_{\psi',\rho'}) \to \mathbb{C}$ by

$$f(w, z) = (w + z)^{-1} \ .$$

Then $\{f(w, \log B) : \ w \in R_{\psi,\rho}\} = \{(w + \log B)^{-1} : \ w \in R_{\psi,\rho}\} \subseteq B(X)$ and is uniformly bounded. By Theorem 6.1.11, $(\log A + \log B)^{-1} = f(\log A, \log B)$ is a bounded operator. This completes the proof. $\square$

We remark that there is a second proof of Theorem 6.1.10 also due to Clark [12] where Theorem 6.1.3 is applied to the operators $\log A$ and $\log B$, which can be shown to be sectorial of angle less than $\frac{\pi}{2}$ and satisfy the appropriate conditions. It seems unclear if one can adapt this argument to obtain an alternative proof of Theorem 6.1.13 since it is not obvious that $\log A$ has absolute calculus (in the sectorial sense).

6.2 $\mathcal{H}^\infty$-calculus for bounded perturbations

Given a strip-type operator A and a closed operator B, we will frequently make use of the following identity, whenever it makes sense:

$$R(\lambda, A + B) = R(\lambda, A)\sum_{n=0}^{\infty}(BR(\lambda, A))^n = \sum_{n=0}^{\infty}(R(\lambda, A)B)^n R(\lambda, A) \ . \qquad (6.2.1)$$

In particular, this expression is valid when $\|BR(\lambda, A)\| < 1$ or $\|R(\lambda, A)B\| < 1$.

Proposition 6.2.1. *If A is of strong strip-type and $B \in B(X)$, then $A + B$ is also of strong strip-type. If A is only strip-type, then $A + B$ is strip-type for any closed,* *linear operator B satisfying $D(A) \subseteq D(B)$ and $\|BR(\lambda, A)\| < 1$ for $|\operatorname{Im}\lambda|$ large enough (e.g. if B is bounded with sufficiently small norm).*

Proof. First suppose A is of strong strip-type, then for some $w > 0$ there exists a constant C such that

$$\|R(\lambda, A)\| \leq \frac{C}{|\operatorname{Im}\lambda| - w} \qquad (|\operatorname{Im}\lambda| \geq 2C\|B\| + w)$$

and so for such λ we have $\|BR(\lambda, A)\| < \frac{1}{2}$ with

$$\|R(\lambda, A+B)\| \leq \|2R(\lambda, A)\| \leq \frac{2C}{|\operatorname{Im}\lambda| - w} \leq \frac{C'}{|\operatorname{Im}\lambda| - (2C\|B\| + w)}\,.$$

The second claim follows similarly.

□

Our characterisation of the bounded $\mathcal{H}^\infty$-calculus for strip-type operators (Theorem 4.3.7) renders it sufficient to obtain estimates involving resolvent powers only. We may thus seek to exploit the identity (6.2.1) to establish perturbation results for the bounded $\mathcal{H}^\infty$-calculus. In the case where B is of finite rank or triangular, Vörös [76, Chapter 4] achieved this by computing weak integral estimates involving squares of resolvents. We refer to that thesis for details.

We seek to employ (6.2.1) with a view to deriving further sufficient conditions on A and B to ensure stability of the bounded $\mathcal{H}^\infty$-calculus. As a first step, we establish an easy perturbation result.

Proposition 6.2.2. *Let $A \in \operatorname{Strip}(w)$. Suppose A admits a bounded $\mathcal{H}^\infty$-calculus on H_u, $u > w$. Suppose further that B is a linear operator satisfying for some $w' \in (w, u)$,*

(i) $D(A) \subseteq D(B)$;

(ii) $\sup_{\lambda \in \partial H_{w'}} \|BR(\lambda, A)\| < 1$;

(iii) $\|R(\lambda, A)BR(\lambda, A)\|_{L^1(\partial H_{w'})} < \infty$.

Then $A + B$ admits a bounded $\mathcal{H}^\infty$-calculus on H_u.

Proof. By *(ii)* we can rewrite (6.2.1) for $|\operatorname{Im}\lambda| = w'$ as

$$R(\lambda, A+B) = R(\lambda, A) + S(\lambda),$$

where $S(\lambda) = R(\lambda, A)BR(\lambda, A)[1 - BR(\lambda, A)]^{-1}$. Then for $f \in \mathcal{H}_1^\infty(H_u)$ and $\Gamma = \partial H_{w'}$, positively oriented, we have

$$\begin{aligned} f(A+B) &= \frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A+B)\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A)\,\mathrm{d}\lambda + \frac{1}{2\pi i}\int_\Gamma f(\lambda)S(\lambda)\,\mathrm{d}\lambda\,. \end{aligned}$$

Since A has bounded $\mathcal{H}^\infty$-calculus and by assumption (*iii*), we can then estimate

$$\|f(A+B)\| \leq C\|f\|_{\mathcal{H}^\infty(H_u)}\,,$$

with C independent of $f \in \mathcal{H}_1^\infty(H_u)$. We deduce by Proposition 3.3.7 (see remark thereafter) that $A + B$ has bounded $\mathcal{H}^\infty$-calculus on H_u.

□

We now apply a more delicate argument in order to replace assumption (*iii*) in the theorem above with a commutator condition. The following perturbation result was formulated for BIP in [65].

Theorem 6.2.3. *Let A be a sectorial operator with bounded $\mathcal{H}^\infty$-calculus. Let B be a linear operator such that*

(i) $D(A) \subseteq D(B)$ and there exists $\beta > 0$ such that $\|Bx\| \leq \beta\|Ax\|$ for all $x \in D(A)$;

(ii) For some $\theta \in (\phi_A^H, \pi)$ and $\psi \in L^1(\mathbb{R}_+)$,

$$\|R(\lambda, A)Bx - BR(\lambda, A)x\| \leq \psi(|\lambda|)\|Ax\|, \quad x \in D(A), \lambda \notin \overline{\Sigma_\theta};$$

(iii) $\frac{1}{\beta} > \sup_{\lambda \notin \overline{\Sigma_\theta}} \|AR(\lambda, A)\|$.

Then $(A + B, D(A) \cap D(B))$ is closed, sectorial and admits a bounded $\mathcal{H}^\infty$-calculus.

The proof is a refinement of the one given for BIP in [65]. We present this refined argument in the next result, which provides an analogue on strips. Note that a strip-type operator with bounded $\mathcal{H}^\infty$-calculus is actually a strong strip-type operator.

Theorem 6.2.4. *Let A be a strip-type operator with bounded $\mathcal{H}^\infty$-calculus. Suppose B is a bounded operator, so $A+B$ is strip-type by Proposition 6.2.1. Suppose further that*

$$\|[R(\lambda, A), B]\|_{L^1(\partial H_w)} < \infty\,,$$

for some $w > \max\{w_A^H, w_{A+B}\}$*. Then for* B *of sufficiently small norm,* $A+B$ *admits a bounded* $\mathcal{H}^\infty$*-calculus on some strip.*

Proof. For $\lambda \in \rho(A)$, set $R_\lambda = R(\lambda, A)$ and $T_\lambda = [B, R_\lambda]$. We show by induction that for any n,

$$(BR_\lambda)^n = \sum_{l=1}^{n} \sum_{j=0}^{l-1} (BR_\lambda)^{n-l} R_\lambda^j T_\lambda R_\lambda^{l-1-j} B^{l-1} + R_\lambda^n B^n \ . \tag{6.2.2}$$

For $n = 1$ this is just the definition of T_λ. Given that (6.2.2) holds for n, we have

$$(BR_\lambda)^{n+1} = \sum_{l=1}^{n} \sum_{j=0}^{l-1} (BR_\lambda)^{n+1-l} R_\lambda^j T_\lambda R_\lambda^{l-1-j} B^{l-1} + BR_\lambda^{n+1} B^n,$$

with

$$BR_\lambda^{n+1} B^n = \sum_{j=0}^{n} R_\lambda^j T_\lambda R_\lambda^{n-j} B^n + R_\lambda^{n+1} B^{n+1} \ ,$$

so that (6.2.2) holds for $n+1$. Let $u > w$. For $\|B\|$ sufficiently small and $\lambda \in \partial H_w$, we can rewrite (6.2.1) as

$$\begin{aligned} R(\lambda, A+B) &= R(\lambda, A) \sum_{n=0}^{\infty} (BR(\lambda, A))^n \\ &= S(\lambda) + \sum_{n=0}^{\infty} R_\lambda^{n+1} B^n \ , \end{aligned}$$

where

$$S(\lambda) = R_\lambda \sum_{n=1}^{\infty} \sum_{l=1}^{n} \sum_{j=0}^{l-1} (BR_\lambda)^{n-l} R_\lambda^j T_\lambda R_\lambda^{l-1-j} B^{l-1} \ .$$

Fix $u > w + \|B\|$ and let $f \in \mathcal{H}_1^\infty(H_u)$. Then we can write, with $\Gamma = \partial H_w$,

$$\begin{aligned} f(A+B) &= \frac{1}{2\pi i} \int_\Gamma f(\lambda) S(\lambda) \, \mathrm{d}\lambda + \frac{1}{2\pi i} \int_\Gamma f(\lambda) \sum_{n=0}^{\infty} R(\lambda, A)^{n+1} B^n \, \mathrm{d}\lambda \\ &= I_1 + I_2 \ . \end{aligned}$$

By our commutator assumption and the fact that A is strip-type, we can estimate for $\lambda \in \partial H_w$,

$$\begin{aligned}
\|S(\lambda)\| &\le C \sum_{n=1}^{\infty} \sum_{l=1}^{n} \sum_{j=0}^{l-1} \|B\|^{n-1} \|R_\lambda\|^{n-1} \|T_\lambda\| \\
&\le C\|T_\lambda\| \sum_{n=1}^{\infty} M^{n-1} \frac{n(n+1)}{2} \\
&\le C\|T_\lambda\| \ ,
\end{aligned}$$

where $M = \|B\| \sup\{\|R(\lambda, A)\| : \ \lambda \in \partial H_w\} < 1$ for B of sufficiently small norm. This in turn gives the estimate $\|I_1\| \le C\|f\|_{\mathcal{H}^\infty(H_u)}$, by our commutator assumption.

Turning our attention to the second term, we write for $\Gamma' = \partial H_{u'}$, $w + \|B\| < u' < u$,

$$\begin{aligned}
I_2 &= \sum_{n=0}^{\infty} \biggl(\frac{1}{2\pi i} \int_\Gamma f(\lambda) R(\lambda, A)^{n+1} \, \mathrm{d}\lambda \biggr) B^n \\
&= f(A) + \sum_{n=1}^{\infty} \left[\frac{1}{2\pi i} \int_\Gamma f(\lambda) \biggl(\frac{1}{2\pi i} \int_{\Gamma'} \frac{1}{(\lambda - z)^{n+1}} R(z, A) \, \mathrm{d}z \biggr) \, \mathrm{d}\lambda \right] B^n \\
&= f(A) + \sum_{n=1}^{\infty} \left[\frac{1}{2\pi i} \int_{\Gamma'} \biggl(\frac{1}{2\pi i} \int_\Gamma f(\lambda) \frac{1}{(\lambda - z)^{n+1}} \, \mathrm{d}\lambda \biggr) R(z, A) \, \mathrm{d}z \right] B^n \\
&= f(A) + \sum_{n=1}^{\infty} \biggl(\frac{1}{2\pi i n!} \int_{\Gamma'} f^{(n)}(z) R(z, A) \, \mathrm{d}z \biggr) B^n \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(A)}{n!} B^n \ .
\end{aligned}$$

Since A has bounded $\mathcal{H}^\infty$-calculus, we can estimate

$$\|f^{(n)}(A)\| \le C\|f^{(n)}\|_{\mathcal{H}^\infty(H_w)} \ .$$

Furthermore, we have for any $z \in H_w$,

$$\begin{aligned}
|f^{(n)}(z)| = \left| \frac{n!}{2\pi i} \int_{\Gamma'} \frac{f(\lambda)}{(\lambda - z)^{n+1}} \, \mathrm{d}\lambda \right| &\le \frac{n! \|f\|_{\mathcal{H}^\infty(H_u)}}{2\pi} \int_{\Gamma'} \frac{1}{|\lambda - z|^{n+1}} \, |\mathrm{d}\lambda| \\
&\le C \frac{n!}{(u' - w)^n} \|f\|_{\mathcal{H}^\infty(H_u)} \ ,
\end{aligned}$$

so that

$$\|I_2\| \le C\|f\|_{\mathcal{H}^\infty(H_u)} \sum_{n=0}^{\infty} \left(\frac{\|B\|}{u' - w} \right)^n \le C\|f\|_{\mathcal{H}^\infty(H_u)} \ .$$

Putting together our estimates for I_1 and I_2 gives, for every $f \in \mathcal{H}^\infty_1(H_u)$,

$$\|f(A+B)\| \leq C\|f\|_{\mathcal{H}^\infty(H_u)} \ .$$

Thus $A+B$ has bounded $\mathcal{H}^\infty$-calculus by (the remark following) Proposition 3.3.7. $\square$

An alternative proof of Theorem 6.2.4, which uses weak integral estimate characterisations, can be found in [76, Theorem 4.53], in which a smallness requirement on the L^1-norm of the commutator is also required.

Next we employ Theorem 4.3.7 to derive a perturbation result with a commutator condition involving Rademacher norms.

Theorem 6.2.5. *Let A be an R-strip-type operator which admits a bounded $\mathcal{H}^\infty$-calculus. Suppose also that A^* is R-strip-type. Let B be a bounded operator with sufficiently small norm and suppose that for some $w > \max\{w_{A+B}, w^H_A, w^R_A\}$ there exists C such that for every $x \in X$,*

$$\sup_{t\in\mathbb{R}} \Big\|\sum_{k\in\mathbb{Z}} r_k[B, R(\pm iw + t - k, A)]x\Big\|_{L^2(X)} \leq C\|x\| \ ,$$

with the corresponding dual condition

$$\sup_{t\in\mathbb{R}} \Big\|\sum_{k\in\mathbb{Z}} r_k[B^*, R(\pm iw + t - k, A^*)]x\Big\|_{L^2(X^*)} \leq C\|x^*\| \ .$$

Then $A+B$ admits a bounded $\mathcal{H}^\infty$-calculus with $w^H_{A+B} \leq w$.

Proof. By Theorem 4.3.7 it is sufficient to show the estimate

$$\sup_{t\in\mathbb{R}} \Big\|\sum_{k\in\mathbb{Z}} r_k\psi(t + k - (A+B))x\Big\|_{L^2(X)} \leq C\|x\| \ ,$$

along with the corresponding dual condition, where $\psi(z) = \frac{1}{(\pm iw - z)^2}$. For $|\operatorname{Im}\lambda| = w$

and $\|B\| < \mathcal{R}\big(\{R(\lambda, A):\ |\operatorname{Im}\lambda| = w\}\big)^{-1}$ we have

$$\begin{aligned}
R(\lambda, A+B)^2 &= R(\lambda, A)^2 + R(\lambda, A)B[I - R(\lambda, A)B]^{-1}R(\lambda, A)^2 \\
&\quad + R(\lambda, A)[I - R(\lambda, A)B]^{-1}R(\lambda, A)BR(\lambda, A) \\
&\quad + R(\lambda, A)B[I - R(\lambda, A)B]^{-1}R(\lambda, A)[I - R(\lambda, A)B]^{-1}R(\lambda, A)BR(\lambda, A) \\
&= R(\lambda, A)^2 + R(\lambda, A)B[I - R(\lambda, A)B]^{-1}R(\lambda, A)^2 \\
&\quad + R(\lambda, A)[I - R(\lambda, A)B]^{-1}R(\lambda, A)[B, R(\lambda, A)] + R(\lambda, A)[I - R(\lambda, A)B]^{-1}R(\lambda, A)^2B \\
&\quad + R(\lambda, A)B[I - R(\lambda, A)B]^{-1}R(\lambda, A)[I - R(\lambda, A)B]^{-1}R(\lambda, A)[B, R(\lambda, A)] \\
&\quad + R(\lambda, A)B[I - R(\lambda, A)B]^{-1}R(\lambda, A)[I - R(\lambda, A)B]^{-1}R(\lambda, A)^2B\ . \qquad (6.2.3)
\end{aligned}$$

Furthermore, since

$$[I - R(\lambda, A)B]^{-1} = \sum_{n=0}^{\infty}(R(\lambda, A)B)^n,$$

we have by Propositions 3.7.4 and 3.7.5 that each of these terms is of the form $T_\lambda[B, R(\lambda, A)]$, $T_\lambda R(\lambda, A)^2$ or $T_\lambda R(\lambda, A)^2B$, with $\{T_\lambda :\ |\operatorname{Im}\lambda| = w\}$ R-bounded in each case. Furthermore, since A has bounded $\mathcal{H}^\infty$-calculus we can estimate by Theorem 4.3.7

$$\sup_{t\in\mathbb{R}} \|\sum_{k\in\mathbb{Z}} r_k\psi(t + k - A)B^ix\|_{L^2(X)} \le C\|B^ix\|\ ,$$

where $i \in \{0, 1\}$. Thus we can estimate

$$\begin{aligned}
&\sup_{t\in\mathbb{R}}\|\sum_{k\in\mathbb{Z}} r_kT_{\pm iw-t-k}R(\pm iw - t - k, -A)B^ix\|_{L^2(X)} \\
&\quad\le \mathcal{R}\{T_\lambda :\ |\operatorname{Im}\lambda| = w\} \sup_{t\in\mathbb{R}} \|\sum_{k\in\mathbb{Z}} r_kR(\pm iw - t - k, -A)B^ix\|_{L^2(X)} \\
&\quad\le C\|x\|\ .
\end{aligned}$$

Similarly, using our commutator assumption we can estimate

$$\begin{aligned}
&\sup_{t\in\mathbb{R}}\|\sum_{k\in\mathbb{Z}} r_kT_{\pm iw-t-k}[B, R(\pm iw - t - k, -A)]x\|_{L^2(X)} \\
&\quad\le C\|x\|\ .
\end{aligned}$$

The triangle inequality for the L^2-norm now gives us the required estimates. We can similarly establish the required dual condition.

$\square$

We remark that A^* is necessarily R-strip-type if A is R-strip-type and X has non-trivial type - see Section 6.3 for a definition - by Proposition 3.7.4. Note that, by a similar argument, the same conclusion of Theorem 6.2.5 holds (on the appropriate strip) if B is any bounded operator and A (along with A^*) is strong R-strip-type in the sense that

$$\mathcal{R}\{R(\lambda, A) : \; |\operatorname{Im} \lambda| > w\} \leq \frac{C}{|\operatorname{Im} \lambda| - w} \; .$$

Finally we observe that, by exploiting the identity (6.2.3) in the proof of Theorem 6.2.5 once more, we can establish a stability criterion for the absolute calculus.

Theorem 6.2.6. *Let A be a strip-type operator with absolute calculus (so A is actually strong strip-type) and let B be a bounded operator so that $A + B$ is also strip-type by Proposition 6.2.1. Suppose there exists a constant C such that for every $x \in X$,*

$$\int_{\mathbb{R}} \|[R(iw + t, A), B]x\|^2 \mathrm{d}t \leq C\|x\|^2 \; ,$$

with the corresponding dual condition, for some $w > w_{A+B}$. Then for B of sufficiently small norm, $A + B$ also has absolute calculus.

Proof. The proof unfolds as in Theorem 6.2.5, where we now observe that each term in (6.2.3) is of the form $T_\lambda[B, R(\lambda, A)]$, $T_\lambda R(\lambda, A)^2$ or $T_\lambda R(\lambda, A)^2 B$, with $\{T_\lambda : |\operatorname{Im} \lambda| = w\}$ uniformly bounded in each case. The required estimate, and the dual estimate, again follow from the triangle inequality for the L^2-norm. ☐

We conclude this section with a basic perturbation result for shifts of sectorial operators. The result also appears in [5] and [2].

Proposition 6.2.7. *Let A be a sectorial operator.*

(i) If A has bounded $\mathcal{H}^\infty$-calculus on Σ_θ then for any $c > 0$ the operator $A + c$ has bounded $\mathcal{H}^\infty$-calculus on Σ_θ.

(ii) Suppose that for some $c_0 > 0$ the operator $A + c_0$ has bounded $\mathcal{H}^\infty$-calculus on Σ_θ. Then $A + c$ has bounded $\mathcal{H}^\infty$-calculus on Σ_θ for every $c > 0$.

Proof. To prove (i), given $f \in \mathcal{H}^\infty(\Sigma_\theta)$ define $g \in \mathcal{H}^\infty(\Sigma_\theta)$ by $g(z) = f(z + c)$. Then $f(A + c) = g(A)$ is bounded and the result follows.

Turning to (ii), we may assume without loss of generality that A is invertible with $A + c_0$ having bounded $\mathcal{H}^\infty$-calculus on Σ_θ. We need to show that A has bounded $\mathcal{H}^\infty$-calculus on Σ_θ. We have for $f \in \mathcal{H}_0^\infty(\Sigma_\theta)$,

$$\begin{aligned} f(A) - f(A + c_0) &= \frac{1}{2\pi i} \int_\Gamma f(\lambda) R(\lambda, A) \, \mathrm{d}\lambda - \frac{1}{2\pi i} \int_\Gamma f(\lambda) R(\lambda, A + c_0) \, \mathrm{d}\lambda \\ &= \frac{c_0}{2\pi i} \int_\Gamma f(\lambda) R(\lambda, A) R(\lambda, A + c_0) \, \mathrm{d}\lambda, \end{aligned}$$

where Γ is a suitable contour avoiding zero. Thus for this choice of f we can estimate

$$\|f(A)\| \leq C \|f\|_{\mathcal{H}^\infty(\Sigma_\theta)}$$

and so A has bounded $\mathcal{H}^\infty$-calculus on Σ_θ, by Proposition 3.3.7. ☐

Turning our attention to half-plane operators, we see that an identical argument yields the corresponding version of (i) of Proposition 6.2.7. Namely, if A is a half-plane operator which admits a bounded $\mathcal{H}^\infty$-calculus on some half-plane R_w, then for any $c > 0$ the half-plane operator $A + c$ also admits a bounded $\mathcal{H}^\infty$-calculus on R_w. However, the argument in Proposition 6.2.7 (ii) relies on the resolvent decay and so does not extend naturally to half-plane operators. In fact, it is easy to see that the corresponding version of Proposition 6.2.7 (ii) for half-plane operators is equivalent to asking whether or not the $\mathcal{H}^\infty$-type of the operator (that is, the supremum of all w such that the operator has bounded $\mathcal{H}^\infty$-calculus on R_w) is equal to its half-plane type s_0. It is unknown if this holds in general, although a partial answer was presented in Proposition 5.1.4.

6.3 γ-norms on spaces of finite cotype

So far we have characterised the bounded $\mathcal{H}^\infty$-calculus for sectorial and strip-type operators on Hilbert space, through the absolute calculus, and on general Banach space, through Rademacher square function estimates. In this section we present an intermediate chacterisation due to Kalton and Weis and establish some immediate consequences for perturbations. To formulate their result precisely, we need some definitions.

We say a Banach space X has *type* $p \in (1, \infty)$ if there exists C such that for every sequence $n \in \mathbb{N}$ and every $x_1, \cdots, x_n \in X$,

$$\left(\mathbb{E}\|\sum_{i=1}^n r_i x_i\|^2\right)^{\frac{1}{2}} \leq C\left(\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}}.$$

Furthermore, X is said to have *cotype* $q \in (1, \infty]$ if there exists C such that for every sequence $n \in \mathbb{N}$ and every $x_1, \cdots, x_n \in X$,

$$\Big(\mathbb{E}\|\sum_{i=1}^{n} r_i x_i\|^2\Big)^{\frac{1}{2}} \geq C\Big(\sum_{i=1}^{n} \|x_i\|^q\Big)^{\frac{1}{q}} .$$

Every Banach space X has type 1 and cotype ∞. We say X has *non-trivial type* if it has type $p > 1$ and *finite cotype* if it has cotype $q < \infty$. X is a Hilbert space if and only if it has type 2 and cotype 2. For more on these concepts we refer to [20].

Let X be a Banach space and H a separable Hilbert space with orthonormal basis $(h_n)_{n\geq 1}$. Let $(\gamma_n)_{n\geq 1}$ be a sequence of independent standard Gaussian random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. An operator $R \in B(X, H)$ is said to be *γ-radonifying* if $\sum_{n\geq 1} \gamma_n R h_n$ converges to some $\xi \in L^2(\Omega, X)$. Moreover we define $\|R\|_{\gamma(H,X)} = \|\xi\|_{L^2(\Omega,X)}$. For an interval I in $\mathbb{R}$, let $\phi : I \to B(H, X)$ be strongly measurable such that for every $x^* \in X^*$, $\phi^* x^* \in L^2(I, H)$. Let $I_\phi : L^2(I, H) \to X$ be the integral defined by

$$I_\phi f = \int_I \phi(t) f(t)\ \mathrm{d}t .$$

We say $\phi \in \gamma(I, H, X)$ if $I_\phi : L^2(I, H) \to X$ is γ-radonifying. We write $\gamma(I, X) = \gamma(I, \mathbb{R}, X)$. In their elusive preprint, Kalton and Weis [45] were able to dualise this notion and define the norm $\|\phi\|_{\gamma^*(I,X^*)}$ for suitable strongly measurable functions $\phi : I \to X^*$. The details would require a prolonged detour and so are omitted. However, we call upon the following characterisation from that unpublished paper.

Theorem 6.3.1. *Let A be a strip-type operator on a Banach space X which has finite cotype. For $a, b > w_A$, consider the following conditions.*

(i) iA generates a C_0-group $(T(t))_{t\in\mathbb{R}}$ and there exists C such that for every $x \in X$, $x^* \in X^*$,

$$\|e^{-a|t|}T(t)x\|_{\gamma(\mathbb{R},X)} \leq C\|x\|,$$
$$\|e^{-a|t|}T(t)^* x^*\|_{\gamma^*(\mathbb{R},X^*)} \leq C\|x^*\|;$$

(ii) There exists C such that for every $x \in X$, $x^* \in X^*$,

$$\|R(\cdot \pm ia, A)x\|_{\gamma(\mathbb{R},X)} \leq C\|x\|,$$
$$\|R(\cdot \pm ia, A)^* x^*\|_{\gamma^*(\mathbb{R},X^*)} \leq C\|x^*\|;$$

(iii) There exists C such that for every $x \in X$,

$$C^{-1}\|x\| \leq \|R(\cdot \pm ia, A)x\|_{\gamma(\mathbb{R},X)} \leq C\|x\|;$$

(iv) A admits a bounded $\mathcal{H}^\infty$-calculus on H_b.

Then for $b > a$, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Moreover (iv) $\Rightarrow$ (i) for $b < a$.

When X is a Hilbert space, this reduces to our established characterisation Theorem 4.3.2, with the γ-norm now just the appropriate L^2-norm. We note that in this case the equivalence of condition *(iii)* and bounded $\mathcal{H}^\infty$-calculus is shown in [45, Theorem 2.2].

Indeed, the proof of Theorem 6.3.1 is a generalisation of this characterisation on Hilbert space. We wish not to repeat the details, except to note some fundamental properties of the L^2-norm that carry over to γ-norms. In particular, we have appropriate versions of Hölder's inequality and the Plancherel theorem. We state the following useful estimation lemma. For the proofs of these results we again refer to [45].

Lemma 6.3.2. *Let X have finite cotype and suppose $N : \Omega \to B(X)$ is strongly continuous. If $\{N(t) : t \in \Omega\}$ is R-bounded then there exists C such that for every $f \in \gamma(\Omega, X)$ and $g \in \gamma(\Omega, X^*)$,*

$$\|N(\cdot)[f(\cdot)]\|_{\gamma(\Omega,X)} \leq C\|f\|_{\gamma(\Omega,X)},$$
$$\|N(\cdot)^*[g(\cdot)]\|_{\gamma^*(\Omega,X^*)} \leq C\|g\|_{\gamma^*(\Omega,X^*)} .$$

We remark that on spaces of finite cotype, R-boundedness is equivalent to γ-boundedness (in which the Rademacher sequence (r_k) is replaced with a Gaussian sequence (γ_k)) - see [21].

The strength of Theorem 6.3.1 in applications is that it only requires an estimate on a single power of the resolvent. In particular, we can establish the following perturbation result.

Theorem 6.3.3. *Let A be an R-strip-type operator with bounded $\mathcal{H}^\infty$-calculus on a Banach space X of finite cotype. Let B be a bounded operator with sufficiently small norm. Then $A + B$ also has bounded $\mathcal{H}^\infty$-calculus.*

Proof. Note that for $\|B\|$ sufficiently small, $A + B$ is R-strip-type and for $|\operatorname{Im}\lambda|$ large enough,

$$R(\lambda, A + B) = \sum_{n=0}^{\infty}(R(\lambda, A)B)^n R(\lambda, A),$$

with $\{\sum_{n=0}^{\infty}(R(\lambda, A)B)^n : \ |\operatorname{Im}\lambda| > w\}$ R-bounded for some $w > w_A$ by Proposition 3.7.5. The result is now immediate from Theorem 6.3.1 and Lemma 6.3.2.

□

We remark that, as with Theorem 6.2.5, the smallness assumption on $\|B\|$ can be dropped provided A is of strong R-strip-type. If we impose tighter geometric restrictions on the underlying space, we can obtain a stronger characterisation in which we do not require an R-boundedness assumption on A or a small norm on B. A Banach space X is said to have property (α) if there is a constant C such that

$$\left(\mathbb{E}\left\|\sum_{j=1}^{n}\sum_{k=1}^{n}\alpha_{jk}r_j r_k^{'} x_{jk}\right\|^2\right)^{1/2} \leq C\max_{j,k}|\alpha_{jk}|\left(\mathbb{E}\left\|\sum_{j=1}^{n}\sum_{k=1}^{n}r_j r_k^{'} x_{jk}\right\|^2\right)^{1/2}$$

for any $(x_{jk})_{j,k=1}^n \subseteq X$ and $(\alpha_{jk})_{j,k=1}^n \subseteq \mathbb{C}$. Every such space already has finite cotype [60]. We will also need the following facts, from [47, Chapter 12] and [76, Chapter 4] respectively.

Theorem 6.3.4. *(i) Let A be a sectorial operator on a Banach space with property (α) and suppose A admits a bounded $\mathcal{H}^\infty$-calculus on some sector. Then A also admits an R-bounded $\mathcal{H}^\infty$-calculus on the same sector.*

(ii) Suppose A is the generator of a semigroup $(T(t))_{t\geq 0}$ which is R-exponentially bounded in the sense that there exists $a_1 > 0$ such that

$$\mathcal{R}\{e^{-|t|a_1}T(t)\}_{t\in\mathbb{R}} < \infty\ .$$

Then for any bounded operator B, the operator $A + B$ also generates an R-exponentially bounded semigroup.

Theorem 6.3.5. *Let A be a strip-type operator on a Banach space with property (α) and suppose A has bounded $\mathcal{H}^\infty$-calculus. Then for any bounded operator B, $A + B$ also has bounded $\mathcal{H}^\infty$-calculus.*

Proof. By Proposition 6.2.1, $A + B$ is strip-type. Indeed, this is clear from the fact that $i(A+B)$ generates a C_0-group. Since e^A is sectorial, we have by Theorem 6.3.4 (i) that the C_0-group generated by iA is R-exponentially bounded. By Theorem 6.3.4 (ii), the same is true of the C_0-group $(e^{it(A+B)})_{t\in\mathbb{R}}$, i.e. there exists $a > 0$ such that $\mathcal{R}\{e^{-|t|a}e^{it(A+B)}\}_{t\in\mathbb{R}} < \infty$. By Lemma 6.3.2 we then have for any $\epsilon > 0$ and every $x \in X$, $x^* \in X^*$,

$$\begin{aligned} \|e^{-(a+\epsilon)|t|} e^{it(A+B)} x\|_{\gamma(\mathbb{R},X)} &\leq C \|e^{-\epsilon|t|}\|_{L^2(\mathbb{R})} \|x\|, \\ \|e^{-(a+\epsilon)|t|} (e^{it(A+B)})^* x^*\|_{\gamma^*(\mathbb{R},X^*)} &\leq C \|e^{-\epsilon|t|}\|_{L^2(\mathbb{R})} \|x^*\| \ . \end{aligned}$$

The conclusion follows from Theorem 6.3.1. □

We remark that a weaker form of this theorem, in which X is also assumed to be UMD, can be found in [76, Chapter 4].

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