Clocks, Correlations and Interactions in Timeless Quantum Theory



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Time present and time past
Are both perhaps present in time future,
And time future contained in time past.
If all time is eternally present
All time is unredeemable.
What might have been is an abstraction
Remaining a perpetual possibility
Only in a world of speculation.
What might have been and what has been
Point to one end, which is always present.
Footfalls echo in the memory
Down the passage which we did not take
Towards the door we never opened
Into the rose-garden.

T. S. Eliot, Four Quartets

Abstract

The notion of time as an external parameter regulating the dynamics of physical systems is unsatisfactory from an explanatory point of view. It also clashes with the background-independent character of general relativity. These issues, which constitute part of the so-called "problem of time", can be solved via a relational approach, that is, by recovering a notion of time from the relations between different physical systems. Quantum relational approaches to time are the most promising as they would be compatible with a hypothetical quantum theory of gravity. These approaches share many similarities and find a natural explanation if quantum theory is understood as a universal physical theory, so we gather them under the term "timeless quantum theory". Timeless quantum theory recovers an emergent notion of time from the correlations between different subsystems of a stationary quantum Universe. Some of these systems play a special role by acting as clocks. Clocks are usually entangled with the rest of the Universe but do not interact with it. In this thesis, we study the role of clocks, correlations and interactions in timeless quantum theory. We clarify the interpretation of the time instants of the theory and show that the clocks can interact with the rest of the Universe. We also discuss how different correlations with the clocks affect the theory. Finally, we examine the role of interactions and their effect on the relational dynamics and the timekeeping qualities of the clocks. Our results elucidate crucial explanatory aspects of timeless

quantum theory and improve its robustness. They also point to several exciting avenues for future research, such as the generalisation to more realistic clocks, the prediction of novel physical effects, and the design of new experiments that could test quantum aspects of time. These lines of inquiry promise to advance our understanding of the interface between quantum theory and relativity.

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Chapter 1

Introduction

In this chapter, we briefly discuss the so-called "problem of time" in physics and outline some of its potential solutions, which generally adopt a relational interpretation of time. We then focus on one of these relational approaches in quantum theory, which we call "timeless quantum theory". This approach is based on a seminal work by Page and Wootters [1] and its subsequent refinements. We delve into the main features of timeless quantum theory, outlining the points that are still unclear and will be later developed in this thesis. This chapter reviews and summarises the main works in the literature of timeless quantum theory.

1.1 The Problem of Time

The concept of time is central to several physical theories. It usually appears as a parameter that regulates the evolution of other physical quantities in dynamical laws. In Newtonian mechanics, for instance, time is treated as an *external* entity that "flows" uniformly in any frame of reference (absolute time) [2]. It is external in the sense that its properties (e.g. its flowing) are not affected by ordinary matter, which seems to pertain to a different physical domain. The same holds in non-relativistic

quantum theory. In special relativity and quantum field theory, the situation changes as observers in different frames of reference will, in general, disagree about the "flow rate" of time. Nevertheless, time, or more generally spacetime, is still an external, fixed structure. As such, its status is unsatisfactory from an explanatory point of view. Why do such external structures exist?

In general relativity (GR), this conception of spacetime is subverted. The spacetime of GR is dynamic: it is affected by and affects ordinary matter. This implies that spacetime, and thus time, is not an external structure but something that is able to interact with the physical domain of ordinary matter. This property of spacetime contributes to the *background-independent* character of GR, loosely defined as the independence from any external structures (the precise definition is the subject of active debate both in physics and philosophy [3, 4, 5, 6, 7, 8]). As a result, GR forces us to abandon the external notion of time used from Newtonian mechanics onwards. For classical theories, this is not problematic since we should take GR as our best explanation of classical physics. The same does not hold true for quantum theory, whose nature is expected to clash, at some level, with the classical nature of GR. According to the domain of applicability of quantum theory, the clash could either happen only at scales where quantum relativistic effects are expected to be dominant (Planck scale) or at any scale. Nevertheless, since GR is one of the best explanations for how matter and spacetime interact, it is reasonable to expect that its background-independent nature should also be a feature of quantum theory.

Still, the abolition of external structures entailed by GR leaves some questions unanswered. What is spacetime time made of? How can it interact with ordinary matter without being ordinary matter? These questions become even more pressing when considering quantum theory. Is spacetime quantum? If not, how should we couple ordinary quantum systems to a classical spacetime? These issues, and many others which arise from the mismatch between background-dependent concepts of

time and the background-independent nature of GR, are generally gathered under the umbrella term *the problem of time* [9, 10, 11, 12, 13].

1.1.1 Time as an Emergent Property

A promising solution to the problem of time is to fully remove any appeal to an external time parameter by postulating that the Universe is stationary (i.e. not changing with respect to any external parameter) and that our common experience of the passing of time must be explained in terms of *relations* between different physical systems. According to this idea, time would not be a fundamental, external concept but rather an *emergent* feature of the physical reality. This would agree with the intuition that an "observer" can tell that time is flowing only because something else is changing with respect to it. Here, the term observer should be intended in the ideal sense of a physical system that can record external events but cannot evolve spontaneously. It should not be understood as related to a living or conscious being.

Within this relational approach, one can assume that there is a class of preferred physical systems (the clocks) which order the different instances of time by assigning a timestamp to each of them. These systems might be selected because they possess specific properties. They could be naturally occurring systems, such as planets, or man-made ones, such as atomic clocks. By synchronising these clocks and agreeing on a time scale, we could think of "mapping" the Universe with timestamps (and do similarly with rulers) and, therefore, reconstruct the spacetime of GR. However, the real meaning of saying that it is a *certain time* would only derive from one of these clocks telling *that time*.

Ideas similar to the ones we have vaguely described here have a long history in physics and philosophy, some of their most famous advocates being Parmenides [14], Leibniz [15] and Mach [16].

1.1.2 Relational Approaches to Time

The approach described above is made more precise in different physical theories that attempt to explain the emergence of time in a relational way. These theories have been proposed both at the classical and the quantum level [1, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. For instance, Barbour [33] explains why the external time of Newtonian mechanics is redundant as a fundamental concept and shows how duration and the behaviour of clocks emerge from a timeless law.

The removal of any external time parameter, or at least its unobservability, is central to all relational approaches to time. This fact arises explicitly when quantising the canonical formulation of GR, where usually one is confronted with the so-called Wheeler-DeWitt equation [34]:

$$\hat{H}_U|\Psi_U\rangle = 0. \tag{1.1}$$

Here, $\hat{H}_U$ and $|\Psi_U\rangle$ are the Hamiltonian and the state vector of the Universe, respectively. The Wheeler-DeWitt equation states that the Universe is in an eigenstate of its Hamiltonian and is thus stationary. Equations of this kind are usually termed as *Hamiltonian constraints* and are characteristic of background-independent theories, both at the classical and at the quantum level [35]. The Wheeler-DeWitt equation is often assumed to be the main aspect of the problem of time and, thus, the main motivation for adopting a relational approach to time. However, in the previous sections we have argued that the problem of time stems from deeper and broader issues, of which the Wheeler-DeWitt equation is a specific manifestation. This view avoids the potential criticism that canonical quantum gravity may not be the correct way to approach quantum gravity.

The stationarity problem highlighted by Eq. (1.1) invokes a relational solution. One of the most promising is the approach proposed by Page and Wootters [1].

Starting from similar assumptions, Page and Wootters showed how a notion of time can be recovered in a stationary Universe from the correlations between subsystems thereof. Due to its equivalence with some other quantum relational approaches to time [36], the construction and its subsequent refinements are usually referred to as "relational quantum dynamics". In this work, we will instead adopt the name "timeless quantum theory" because, as we will later explain, the application of quantum theory to a stationary Universe already contains an explanation for the relative states, so the term "relational" becomes superfluous.

What to make of the classical relational approaches to time? If one believes that quantum theory applies to any scale, it is reasonable to assume that time should also be a quantum concept, especially in light of the interaction between matter and spacetime in GR. However, there are additional reasons to prefer a quantum explanation of time. The main one is that the "existence" of different time instances is difficult to explain in classical relational approaches to time. In what sense do there "exist" different moments in the overall timeless Universe? Are we to consider each of those instants as real from an ontological point of view?

Quantum theory, interpreted as a universal physical theory [37], provides a satisfactory solution: all the time instances can coexist in the global state of the Universe [38, 39]. In this sense, classical relational approaches to time could be thought of as approximations of quantum ones in the same way that classical mechanics can be regarded as an approximation of quantum mechanics in some regimes. If, on the other hand, quantum theory were to have a limited domain of applicability, e.g. by breaking down at some mesoscopic scale, then classical relational approaches would have a prominent role on their own as explanations of time at the macroscopic scale. In this thesis, we will assume that quantum theory is a universal physical theory. Consequently, we will consider only quantum relational approaches to time.

1.2 Timeless Quantum Theory

In this section, we review a relational approach to time based on the Page-Wootters construction and its subsequent refinements. The starting point is the Wheeler-DeWitt equation (Eq. (1.1)), which can either be assumed as deriving from the canonical quantisation of GR, or can be justified as a condition to remove any appeal to an external time parameter in an Universe described a pure state, as discussed in Sec. 1.1.1. The approach described here is *non-relativistic*. However, relativistic effects can be included in the construction [40], and we will study them in Chapter 4.

1.2.1 The Minimal Model

Here, we present the Page-Wootters approach to time in a simple model along the lines of [1, 41]. We consider a Universe consisting of two systems, $\mathcal{A}$ and $\mathcal{C}$. For reasons that will be clear soon, we call the system $\mathcal{C}$ "the clock". We assume that the Universe is described by states on a Hilbert space $\mathscr{H}_U$ which has the following tensor product structure $\mathscr{H}_U = \mathscr{H}_A \otimes \mathscr{H}_C$. For now, we consider the standard scalar product on this Hilbert space, which is usually referred to as the "kinematical" scalar product ($\mathscr{H}_U$ is called, accordingly, the kinematical Hilbert space) [21]. We also assume that the state of the Universe is *pure*.

The Page-Wootters construction relies on the following assumptions [1, 42]:

1. $\mathcal{A}$ and $\mathcal{C}$ are *non-interacting*. This is described by the following Hamiltonian of the Universe:

$$\hat{H}_U = \hat{H}_\mathcal{A} \otimes \hat{\mathbb{1}}_\mathcal{C} + \hat{\mathbb{1}}_\mathcal{C} \otimes \hat{H}_\mathcal{C}, \tag{1.2}$$

   where $\hat{H}_\mathcal{A}$ ($\hat{H}_\mathcal{C}$) is the Hamiltonian of $\mathcal{A}$ ($\mathcal{C}$), and $\hat{\mathbb{1}}_\mathcal{A}$ ($\hat{\mathbb{1}}_\mathcal{C}$) is the identity operator on $\mathscr{H}_A$ ($\mathscr{H}_C$). In the following, we will sometimes omit the tensor products with the identity operators when their presence or absence is clear

from the context.

2. The state of the Universe $|\Psi_U\rangle\rangle$ is an eigenstate of the Hamiltonian of the Universe:

$$\hat{H}_U \, |\Psi_U\rangle\rangle = 0, \tag{1.3}$$

where we have used the double-ket notation as a visual aid to remind that this state belongs to the full Hilbert space of the Universe $\mathscr{H}_U$. This is the Wheeler-DeWitt equation discussed in Sec. 1.1.2, and it implies that the Universe is in a stationary state.

Before showing how we can recover the usual notion of time in a stationary Universe, let us consider more carefully the system $\mathcal{C}$. The main purpose of $\mathcal{C}$ is to serve as a clock. We do not have to think of it necessarily as a real macroscopic clock, although any real clock is admitted within this description because we are assuming that quantum theory holds at any scale. Whatever system is chosen, let us consider its state $|\phi_{\mathcal{C}}(0)\rangle$, which we can associate with the clock showing the time 0. At this stage, there is a lot of freedom in the choice of this state. We will only require that $|\phi_{\mathcal{C}}(0)\rangle$ is normalised and is not an eigenstate of $\hat{H}_{\mathcal{C}}$. We can also consider the family of states

$$|\phi_{\mathcal{C}}(t)\rangle \coloneqq e^{-i\hat{H}_{\mathcal{C}}t}|\phi_{\mathcal{C}}(0)\rangle \quad \forall t \in \mathbb{R}, \tag{1.4}$$

corresponding to the clock showing different times t. For the moment, let us assume that the clock is *ideal*, meaning that every $|\phi_{\mathcal{C}}(t)\rangle$ corresponds to a measurable state of the clock, so the latter has an infinite resolution. Let us also assume, for the moment, that the $|\phi_{\mathcal{C}}(t)\rangle$ are normalisable.

The core idea of the Page-Wootters construction is the following. The meaning of saying that the Universe is at a certain time t is that the clock is showing that

time. That is, we can think of the state of the Universe at time t as being given by [1]

$$|\Psi_U(t)\rangle\rangle \coloneqq \frac{\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle\,|\phi_{\mathcal{C}}(t)\rangle}{\|\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle\,|\phi_{\mathcal{C}}(t)\rangle\|}, \tag{1.5}$$

where $\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle$ is the *partial inner product* between $|\phi_{\mathcal{C}}(t)\rangle$ and $|\Psi_U\rangle\rangle$.[1] These states are slightly different from [1], as in the latter the authors focus on the state of the system $\mathcal{A}$ only. Also, note that we are renormalising the state as we would usually do after a selective measurement, while most subsequent literature on timeless quantum theory does not use this prescription [43].

The state $|\Psi_U(t)\rangle\rangle$ is what one would get after "measuring" the state $|\phi_{\mathcal{C}}(t)\rangle$ of the clock. However, $|\Psi_U(t)\rangle\rangle$ should not be interpreted as the state to which $|\Psi_U\rangle\rangle$ has collapsed after the measurement, but rather as an Everettian relative state [44]. This is for two reasons. First, the collapse would transform the stationary state of the Universe into a non-stationary state, which is not allowed by the construction. Second, there is no indication as to which $|\phi_{\mathcal{C}}(t)\rangle$ should be measured since there is no preferred time in the construction. The Everettian interpretation of quantum theory naturally accommodates different relative states without changing the overall state $|\Psi_U\rangle\rangle$.

Now, using Eqs. (1.2), (1.3) and (1.4) we find that

$$\begin{aligned}|\Psi_U(t)\rangle\rangle =& \frac{\langle\phi_{\mathcal{C}}(0)|e^{i\hat{H}_{\mathcal{C}}t}\,|\Psi_U\rangle\rangle\,e^{-i\hat{H}_{\mathcal{C}}t}|\phi_{\mathcal{C}}(0)\rangle}{\left\|\langle\phi_{\mathcal{C}}(0)|e^{i\hat{H}_{\mathcal{C}}t}\,|\Psi_U\rangle\rangle\,|\phi_{\mathcal{C}}(t)\rangle\right\|} = \frac{\langle\phi_{\mathcal{C}}(0)|e^{-i\hat{H}_{\mathcal{A}}t}\,|\Psi_U\rangle\rangle\,e^{-i\hat{H}_{\mathcal{C}}t}|\phi_{\mathcal{C}}(0)\rangle}{\left\|\langle\phi_{\mathcal{C}}(0)|e^{-i\hat{H}_{\mathcal{A}}t}\,|\Psi_U\rangle\rangle\,e^{-i\hat{H}_{\mathcal{C}}t}|\phi_{\mathcal{C}}(0)\rangle\right\|}\\ =& \frac{e^{-i\hat{H}_{\mathcal{A}}t}\,\langle\phi_{\mathcal{C}}(0)\,|\Psi_U\rangle\rangle\,e^{-i\hat{H}_{\mathcal{C}}t}|\phi_{\mathcal{C}}(0)\rangle}{\|\langle\phi_{\mathcal{C}}(0)\,|\Psi_U\rangle\rangle\,|\phi_{\mathcal{C}}(0)\rangle\|} = e^{-i\hat{H}_U t}\,|\psi_U(0)\rangle\rangle\,,\end{aligned} \tag{1.6}$$

which says that the state of the Universe relative to the clock showing the time t is given by the state relative to the clock showing the time 0 evolved with the

[1] The partial inner product is defined as map from $\mathscr{H}_U$ to $\mathscr{H}_{\mathcal{A}}$ such that, given two orthonormal basis $\{|\chi_j\rangle\}_j \in \mathscr{H}_{\mathcal{A}}$ and $\{|\theta_k\rangle\}_k \in \mathscr{H}_{\mathcal{C}}$, $\langle\phi\,|\Psi\rangle\rangle = \sum_{j,k}\langle\phi|\theta_k\rangle\,(\langle\chi_j|\langle\theta_k|\cdot|\Psi\rangle\rangle)\,|\chi_j\rangle$, for every $|\phi\rangle \in \mathscr{H}_{\mathcal{C}}$, with $\cdot$ the standard (kinematical) inner product on $\mathscr{H}_U$.

usual unitary evolution operator of quantum theory, as if $|\Psi_U(0)\rangle$ had evolved spontaneously for a time t. Therefore, we have recovered the usual time evolution of quantum theory from an overall stationary Universe.

We can reach the same conclusion by taking the derivative of Eq. (1.5) with respect to t. It is instructive to see this calculation in more detail, so let us focus first on the derivative of the numerator:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle\,|\phi_{\mathcal{C}}(t)\rangle\right) =& i\langle\phi_{\mathcal{C}}(t)|\hat{H}_{\mathcal{C}}\,|\Psi_U\rangle\rangle\,|\phi_{\mathcal{C}}(t)\rangle - i\,\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle\,\hat{H}_{\mathcal{C}}|\phi_{\mathcal{C}}(t)\rangle \\ =& -i\left(\hat{H}_{\mathcal{A}}+\hat{H}_{\mathcal{C}}\right)\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle\,|\phi_{\mathcal{C}}(t)\rangle, \end{aligned} \tag{1.7}$$

where we have used Eqs. (1.2) and (1.4). Using the equation above, we can easily find that the derivative of the denominator $\|\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle\,|\phi_{\mathcal{C}}(t)\rangle\| = \|\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle\|$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle\| = \frac{i\,\langle\langle\Psi_U|\left[\hat{H}_{\mathcal{A}},|\phi_{\mathcal{C}}(t)\rangle\langle\phi_{\mathcal{C}}(t)|\right]|\Psi_U\rangle\rangle}{2\|\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle\|} = 0. \tag{1.8}$$

Therefore, the derivative of Eq. (1.5) multiplied by i is

$$i\frac{\mathrm{d}\,|\psi_U(t)\rangle\rangle}{\mathrm{d}t} = \left(\hat{H}_{\mathcal{A}}+\hat{H}_{\mathcal{C}}\right)|\psi_U(t)\rangle\rangle\,, \tag{1.9}$$

which is the usual **Schrödinger equation**. Note that Eq. (1.8) guarantees that if $\|\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle\|$ is not zero for one value of t, then it will be so for any other value of t. Therefore, if we choose an initial state of the clock that is contained in $|\Psi_U\rangle\rangle$, the states of Eqs. (1.5) will be well defined for all values of t.

Instead of considering the states $|\Psi_U(t)\rangle$, the original paper from Page and Wootters [1] and the following literature focuses on the evolution of the system $\mathcal{A}$ alone, whose state relative to the clock showing the time t is given by:

$$|\psi_{\mathcal{A}}(t)\rangle \coloneqq \frac{\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle}{\|\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle\|}. \tag{1.10}$$

Similarly to Eq. (1.6), we can use Eqs. (1.2), (1.3) and (1.4) to get

$$|\psi_{\mathcal{A}}(t)\rangle = \frac{e^{-i\hat{H}_{\mathcal{A}}t}\,\langle\phi_{\mathcal{C}}(0)\,|\Psi_U\rangle\rangle}{\|\langle\phi_{\mathcal{C}}(0)\,|\Psi_U\rangle\rangle\|} = e^{-i\hat{H}_{\mathcal{A}}t}|\psi_{\mathcal{A}}(0)\rangle. \tag{1.11}$$

which is the usual unitary evolution of the state $|\psi_{\mathcal{A}}(0)\rangle$ for a time t. Moreover, similarly to Eq. (1.9), we find that

$$i\frac{\mathrm{d}\,|\psi_{\mathcal{A}}(t)\rangle\rangle}{\mathrm{d}t} = \hat{H}_{\mathcal{A}}\,|\psi_{\mathcal{A}}(t)\rangle\rangle\,, \tag{1.12}$$

which is the usual **Schrödinger equation** for the states of $\mathcal{A}$ *relative* to $\mathcal{C}$.

To summarise, we have shown that it is possible to recover the usual time evolution of unitary quantum theory (and so the usual notion of time) from a stationary quantum Universe. This was achieved by interpreting the notion of time in a relational way: the Universe (or the system $\mathcal{A}$) is at time t if the clock $\mathcal{C}$ measures that time.

1.2.1.1 A Circular Argument?

At first impression, it might seem that the argument above is circular as we decided to consider the states of the clock related by the unitary operator $e^{-i\hat{H}_{\mathcal{C}}t}$ as in Eq. (1.4) to recover the unitary evolution operator of the Universe $e^{-i\hat{H}_U t}$. This impression is misguided for different reasons. First, the Page-Wootters approach does not claim to explain *why* the Hamiltonian is related to the generation of time translations in physical systems. This fact is something we already know from the explanations we developed in ordinary physics (i.e., physics where time is an external parameter). Specifically, we already *know* that the different states of an isolated clock reading different times are related by Eq. (1.4) (assuming that the clock is a fundamentally quantum system). The main advance of the Page-Wootters approach is to show that this relation between the clock states is enough to recover a

good notion of dynamics at the level of the Universe, even if the latter is stationary. The second reason for why the approach is not circular is that it is not always the case that by assuming the unitary relation $e^{-i\hat{H}_C t}$ between the clock states we recover a similar relation for the states of the Universe. As we will see in Chapter 4, for more general Hamiltonians of the Universe the relational dynamics is no longer regulated by the Hamiltonian, but by more complicated operators. In some cases, the dynamics is even non-unitary. This dispels the circularity argument since we start with a unitary relation for states of the clock and we end up with a non-unitary one for the states of the Universe.

Some works in the literature [45, 40, 36] do not start from Eq. (1.4), but rather propose that the time readings are given by a Positive Operator Valued Measurement (POVM), a generalisation of the projective measurements in quantum theory, whose elements $E(t)$—corresponding to a measure of the time t on the clock—are related by [46]

$$e^{-i\hat{H}_C t} E(t) e^{i\hat{H}_C t} = E(t+\tau). \tag{1.13}$$

This condition is usually referred to as the *covariance* condition and characterises a good *time observable* [46] (we will further discuss POVMs in Sec. 1.2.3.4). It is then possible to use such a POVM to recover the dynamics of the Universe similarly to Eq. (1.6). Such an approach might seem to provide a better justification for the choice of clock states, grounding it in the theory of quantum measurements through Eq. (1.13). However, the motivation behind the covariance condition is to choose a POVM that is a good estimator of the parameter t that controls the evolution of quantum systems through the unitary $e^{-i\hat{H}_C t}$ [46, 47]. Therefore, this approach also starts from the assumption that the (relational) evolution of isolated quantum systems (in this case, the clocks) is regulated by $e^{-i\hat{H}_C t}$. This is conceptually equivalent to starting from the clock states of Eq. (1.4).

1.2.2 Generalised States of the Universe

Page and Wootters [1] proposed another argument for considering only stationary states of the Universe. They argued that, in any closed system, time is an inaccessible parameter. Therefore, all the physics of the system is equivalently described by the state obtained by averaging over all the possible time instants, which can be shown to be a stationary state [1]. This stationarity condition does not require the state of the system to be a zero-energy eigenstate or even a pure state, thus differing from the constraint approach of the Wheeler-DeWitt equation. In this section, we will consider such generalised stationary states of the Universe. While the results of this section lead to the same predictions one would get starting from the Wheeler-DeWitt equation, in Chapter 4 we will consider scenarios where the resulting dynamics is different.

1.2.2.1 Non-zero Energy Eigenstates

First, we can consider any energy eigenstate $|\Psi_U^{\varepsilon}\rangle\rangle$ of $\hat{H}_U$ [42] which is still as stationary state of the Universe:

$$\hat{H}_U \left|\Psi_U^{\mathcal{E}}\right\rangle\rangle = \mathcal{E} \left|\Psi_U^{\mathcal{E}}\right\rangle\rangle . \tag{1.14}$$

We can then apply the same approach outlined in the previous section and obtain

$$\left|\psi_{\mathcal{A}}^{\mathcal{E}}(t)\right\rangle := \frac{\left\langle\phi_{\mathcal{C}}(t)\left|\Psi_U^{\mathcal{E}}\right\rangle\right\rangle}{\|\langle\phi_{\mathcal{C}}(t)\left|\Psi_U^{\mathcal{E}}\right\rangle\rangle\|} = e^{-i(\hat{H}_{\mathcal{A}}-\mathcal{E})t}\left|\psi_{\mathcal{A}}^{\mathcal{E}}(0)\right\rangle, \tag{1.15}$$

which differs from Eq. (1.11) only for an unobservable phase factor. Therefore, any eigenstate of $\hat{H}_U$ will lead to the same observable relational evolution.

1.2.2.2 Mixed States

Let us now consider mixed states of the Universe. What is the meaning of such states? Usually, one would think that the Universe, being a closed system, must be in a pure state. However, some authors have argued that the Universe might be described by a mixed state [48]. Alternatively, we could think of being in a region of the Universe that is entangled with another region inaccessible to us. As a result, all our observations would depend on the reduced state of our region of the Universe, which must be a mixed state. Another reason to consider mixed states is that they will be useful to shed light on the types of correlations required by the timeless quantum theory.

The original paper from Page and Wootters already considers mixed states of the Universe [1]. The construction outlined above requires minimal modifications. Given a general state of the Universe $\hat{\mathcal{R}}_U$, the stationarity condition of Eq. (1.3) becomes

$$\left[\hat{\mathcal{R}}_U, \hat{H}_U\right] = 0. \tag{1.16}$$

The state of $\mathcal{A}$ relative to the clock showing the time t is given by [1]

$$\hat{\rho}_{\mathcal{A}}(t) \coloneqq \frac{\underset{\mathcal{C}}{\mathrm{Tr}}\left[\hat{\mathcal{R}}_U|\phi_{\mathcal{C}}(t)\rangle\langle\phi_{\mathcal{C}}(t)|\right]}{\mathrm{Tr}\left[\hat{\mathcal{R}}_U|\phi_{\mathcal{C}}(t)\rangle\langle\phi_{\mathcal{C}}(t)|\right]}, \tag{1.17}$$

where $\underset{\mathcal{C}}{\mathrm{Tr}}$ denotes the partial trace over $\mathcal{C}$.
Here, we are assuming that $\mathrm{Tr}\left[\hat{\mathcal{R}}_U|\phi(0)\rangle\langle\phi(0)|\right] \neq 0$, which means that the state of the clock at time 0 is part of the state of the Universe. Similarly to the previous section, this guarantees that the denominator of Eq. (1.17) is not 0 for any value of t. Using Eqs. (1.2) and (1.16) we find that

$$\hat{\rho}_{\mathcal{A}}(t) = e^{-i\hat{H}_{\mathcal{A}}t}\hat{\rho}_{\mathcal{A}}(0)e^{i\hat{H}_{\mathcal{A}}t}, \tag{1.18}$$

which is the usual unitary evolution of states in quantum theory.

Alternatively, we could have taken the derivative with respect to t of (1.18) and found

$$\frac{\mathrm{d}\hat{\rho}_{\mathcal{A}}(t)}{\mathrm{d}\hat{t}} = i\left[\hat{\rho}_{\mathcal{A}}(t), \hat{H}_{\mathcal{A}}\right], \tag{1.19}$$

which is the usual Liouville-von Neumann equation.

1.2.3 The Clock

1.2.3.1 Measuring the Clock

Up to now, we have not specified the structure of the clock $\mathcal{C}$, apart from assuming that we can extract the value of t from any $|\phi_{\mathcal{C}}(t)\rangle$. We have also been vague about the meaning of "measuring the clock". We have said that the relative states of Eq. (1.5) suggest that a measurement has been performed on $\mathcal{C}$. This measurement could not have happened at any "time" since the Universe is stationary and the Hamiltonian of the Universe has no interaction terms between $\mathcal{A}$ and $\mathcal{C}$. Rather, it seems that the Universe is in a state that is the result of a measurement process, and the states $|\Psi_U(t)\rangle\rangle$ are the relative states obtained after this measurement, although real measurements would impose some conditions on the actual $|\phi_{\mathcal{C}}(t)\rangle$ that can be measured. Moreover, the absence of interactions between $\mathcal{A}$ and $\mathcal{C}$ may cast some doubts about the fact that $\mathcal{C}$ can be measured by physical systems within the Universe while preserving its role as a clock. This is because actual measurements of clocks require other systems to interact with them.

We will discuss these issues in Chapter 2. For now, let us think of the relative states as emerging from an idealized projective measurement, although this measurement has not occurred at any point in time. This imposes some conditions on the states $|\phi_{\mathcal{C}}(t)\rangle$ that can be measured. When referring to measurements, it is common to use the projectors $\hat{\Pi}_t \coloneqq |\phi_{\mathcal{C}}(t)\rangle\langle\phi_{\mathcal{C}}(t)|$. We will do so from now on.

First, the projectors in $\mathcal{T}_{\mathcal{C}} \coloneqq \{\hat{\Pi}_t\}_t$ are not, in general, all mutually orthogonal. Therefore, only some of these states can be distinguished [49]. Since we are dealing with projective measurements, the best we can do is to choose a subset of mutually orthogonal states. The number and distribution of such states depend on the clock's resolution. A clock with a higher resolution can be used to resolve smaller time intervals.

We can denote with $\mathcal{T}_{\mathcal{C}}^M \coloneqq \{\hat{\Pi}_{t_j}\}_j$ a possible subset of $\mathcal{T}_{\mathcal{C}}$ that we decide to measure (the t_j can be discrete or continuous). $\mathcal{T}_{\mathcal{C}}^M$ is such that all the $\hat{\Pi}_{t_j}$ are mutually orthogonal. $\mathcal{T}_{\mathcal{C}}^M$ does not yet define a projective measurement because the $\hat{\Pi}_{t_j}$ may not form a complete set of projectors. There are two possible ways to approach this point. The first one is to require that $\mathcal{T}_{\mathcal{C}}^M$ is complete, that is $\sum_{j=1}^N \hat{\Pi}_{t_j} = \hat{\mathbb{1}}_{\mathcal{C}}$, where we assumed that $\dim(\mathscr{H}_{\mathcal{C}}) = N$. This completeness condition greatly restricts the possible $|\phi_{\mathcal{C}}(0)\rangle$ that can be used to generate $\mathcal{T}_{\mathcal{C}}$ and the spectrum that $\hat{H}_{\mathcal{C}}$ can have [36, 50]. The second way is to complete $\mathcal{T}_{\mathcal{C}}^M$ with other external projectors and explain why these are not relevant for the construction. We will further explain this approach in Chapter 2.

There is a final joint restriction on the states of $\mathcal{A}$ and $\mathcal{C}$. To guarantee that $|\Psi_U\rangle\rangle$ is stationary as in Eq. (1.3), only eigenstates of $\hat{H}_{\mathcal{A}}$ and $\hat{H}_{\mathcal{C}}$ with opposite eigenvalues can appear in $|\Psi_U\rangle\rangle$. That is, $|\Psi_U\rangle\rangle$ must be a superposition of terms like $|E_{\mathcal{A}}\rangle|-E_{\mathcal{C}}\rangle$ with $|E_{\mathcal{A}}\rangle$ ($|E_{\mathcal{C}}\rangle$) eigenstate of $\hat{H}_{\mathcal{A}}$ ($\hat{H}_{\mathcal{C}}$) with eigenvalue E. This means, for instance, that energy eigenstates of $\mathcal{A}$ that do not have a counterpart in $\mathcal{C}$ with opposite eigenvalue must be excluded. The same holds for the energy eigenstate of $\mathcal{C}$. We could select different energy eigenstates of $\mathcal{A}/\mathcal{C}$ by using different energy eigenstates of $\hat{H}_U$ as in Sec. 1.2.2.1. If instead we considered a mixed state of the Universe as in Sec. 1.2.2.2, all the eigenstates of $\hat{H}_{\mathcal{A}}$ and $\hat{H}_{\mathcal{C}}$ could be included in it.

Before considering different types of clocks, we can make a major distinction between them based on the structure of $\hat{H}_{\mathcal{C}}$. Let us consider the group of unitaries

$\mathbf{G} \coloneqq \left\{e^{-i\hat{H}_{\mathcal{C}}t}\right\}_{t\in G}$ with $G \subset \mathbb{R}$ the set of values necessary to parametrise the group. $\mathbf{G}$ can be either compact or non-compact. If $\mathbf{G}$ is compact, then there is a $T \in G$ such that [36]

$$e^{-i\hat{H}_{\mathcal{C}}T} = e^{i\varphi}\mathbb{1}_{\mathcal{C}}, \tag{1.20}$$

for some $\varphi \in [0, 2\pi)$. Clocks of such kind are periodic, and this implies that the state of $\mathcal{A}$ must also be periodic since, using Eq. (1.11), we get

$$|\psi_{\mathcal{A}}(t+T)\rangle = \frac{\langle\phi_{\mathcal{C}}(0)|e^{i\hat{H}_{\mathcal{C}}T}e^{i\hat{H}_{\mathcal{C}}t}\,|\Psi_U\rangle\rangle}{\|\langle\phi_{\mathcal{C}}(0)\,|\Psi_U\rangle\rangle\|} = e^{-i\varphi}e^{-i\hat{H}_{\mathcal{A}}t}|\psi_{\mathcal{A}}(0)\rangle = e^{-i\varphi}|\psi_{\mathcal{A}}(t)\rangle, \tag{1.21}$$

and so $|\psi_{\mathcal{A}}(t)\rangle$ is periodic with period T, apart from an unobservable phase.

1.2.3.2 Discrete-Spectrum Periodic Clocks

Let us now consider a periodic clock whose Hamiltonian has a discrete non-degenerate spectrum:

$$\hat{H}_{\mathcal{C}} = \sum_k \varepsilon_k |\varepsilon_k\rangle\langle\varepsilon_k|. \tag{1.22}$$

The periodicity condition of Eq. (1.20) implies that the eigenvalues ε_k of $\hat{H}_{\mathcal{C}}$ must be of the form [36]:

$$\varepsilon_k = \frac{2\pi n_k - \varphi}{T}, \tag{1.23}$$

with $n_k \in \mathbb{Z}$, that is, the spectrum of $\hat{H}_{\mathcal{C}}$ must be rational. Note that here we have set $\hbar = 1$. We will do so for the rest of the thesis. The simplest choice of such a clock is represented by N *equally spaced* eigenvalues: $\varepsilon_k = \varepsilon_1 + k\Delta\varepsilon$ with $\Delta\varepsilon = \frac{2\pi}{T}$. For simplicity, let us assume that $\varepsilon_1 = 0$ so that $\varphi = 0$. With this choice of spectrum, the only good time states of the clock are of the form [36, 50]

$$|\phi(t_j)\rangle = \frac{1}{\sqrt{N}}\sum_{k=1}^{N} e^{ig(\varepsilon_k)}e^{-i\varepsilon_k t_j}|\varepsilon_k\rangle, \tag{1.24}$$

with $t_j = t_1 + \frac{2\pi j}{N\Delta\varepsilon} = t_1 + j\frac{T}{N}$ and $g(\varepsilon_k)$ an arbitary real-valued phase. These states are orthonormal

$$\langle\phi(t_{j'})|\phi(t_j)\rangle = \frac{1}{N}\sum_{k=1}^{N} e^{-i\frac{2\pi k}{N}(j-j')} = \delta_{jj'}, \tag{1.25}$$

and thus $\mathcal{T}_{\mathcal{C}}^{M}$ forms a complete basis (because $\mathcal{C}$ is N-dimensional). Note that the time instants are also equally spaced: $t_j = t_1 + j\Delta t$ with $\Delta t = \frac{T}{N}$.

We can use these states to construct the following *time operator* [36, 50]

$$\hat{T}_{\mathcal{C}} \coloneqq \sum_{j=1}^{N} t_j \hat{\Pi}_{t_j}, \tag{1.26}$$

where $\hat{\Pi}_{t_j} = |\phi_{\mathcal{C}}(t_j)\rangle\langle\phi_{\mathcal{C}}(t_j)|$. This operator is self-adjoint with eigenstates $|\phi(t_j)\rangle$: $\hat{T}_{\mathcal{C}}|\phi(t_j)\rangle = t_j|\phi(t_j)\rangle$.[2]

Using the completeness of $\mathcal{T}_{\mathcal{C}}^{M}$ and Eq. (1.5), we can write the state of the Universe as

$$|\Psi_U\rangle\rangle = \sum_j \hat{\Pi}_{t_j}|\Psi_U\rangle\rangle = \left\|\hat{\Pi}_{t_1}|\Psi_U\rangle\rangle\right\| \cdot \sum_j |\Psi_U(t_j)\rangle\rangle\,, \tag{1.27}$$

which shows that $|\Psi_U\rangle\rangle$ is the superposition of all the relative states at the different times t_j. This state is *normalisable* and its norm can be made equal to 1. However, the exact value of the global norm is irrelevant since it is not an observable property of the theory. Eq. (1.6) now becomes

$$|\Psi_U(t_{j+1})\rangle\rangle = e^{-i\hat{H}_U j\Delta t}|\Psi_U(t_1)\rangle\rangle\,, \quad \forall j = 1, \dots N-1, \tag{1.28}$$

with the periodicity condition

$$e^{-i\hat{H}_U N\Delta t}|\Psi_U(t_1)\rangle\rangle = |\Psi_U(t_1)\rangle\rangle\,. \tag{1.29}$$

[2]This clock is known in the literature as the Salecker-Wigner-Peres clock [51, 52].

We can also write Eq. (1.27) in terms of the relative states of $\mathcal{A}$:

$$\begin{aligned}|\Psi_U\rangle\rangle =& \left\| \hat{\Pi}_{t_1} |\Psi_U\rangle\rangle \right\| \cdot \sum_j |\psi_{\mathcal{A}}(t_j)\rangle |\phi_{\mathcal{C}}(t_j)\rangle \\ =& \left\| \hat{\Pi}_{t_1} |\Psi_U\rangle\rangle \right\| \cdot \sum_j e^{-i\hat{H}_{\mathcal{A}} j\Delta t} |\psi_{\mathcal{A}}(t_1)\rangle |\phi_{\mathcal{C}}(t_j)\rangle,\end{aligned} \tag{1.30}$$

where we have used Eqs. (1.10) and (1.11).

The result is that we have recovered a unitary time evolution of U relative to $\mathcal{C}$, but we have lost the time-continuity of Eq. (1.6). Are all the other time instants lost? While any realistic clock is likely to have a finite resolution [53], we may not want to give up the continuous aspect of time. An intuitive way to solve this problem is to increase the number N of eigenstates of $\hat{H}_{\mathcal{C}}$. This will increase the resolution of the clock since. In the limit $N \to \infty$ we have that $\Delta t \to 0$, reaching a perfect resolution.

What if we start with an unbounded discrete spectrum? Let us consider, for instance, $\varepsilon_k = k\Delta\varepsilon$ with $\Delta\varepsilon = \frac{2\pi}{T}$ and $k \in \mathbb{Z}$. Note that this spectrum is unbounded from below, which is usually considered unphysical. This problem is related to the so-called "Pauli objection", which we will mention later in the context of clocks with a continuous spectrum. With this choice of spectrum the time states of Eq. (1.24) are no longer well defined as N is not finite. We can instead consider the following time states:

$$|\phi_{\mathcal{C}}(t)\rangle = \frac{1}{\sqrt{T}} \sum_{k=-\infty}^{+\infty} e^{ig(\varepsilon_k)} e^{-i\frac{2\pi}{T}kt} |\varepsilon_k\rangle, \tag{1.31}$$

which are periodic with period T. These states are orthogonal in the following sense

$$\langle \phi_{\mathcal{C}}(t')|\phi_{\mathcal{C}}(t)\rangle = \frac{1}{T} \sum_{k=-\infty}^{+\infty} e^{-i\frac{2\pi k}{T}(t-t')} = \sum_{j=-\infty}^{+\infty} \delta(t - t' - jT), \tag{1.32}$$

where in the last equation we have used the Fourier series representation of the Dirac

comb. Since the time states are periodic, we can restrict, for simplicity, to values of $t \in [0, T)$ so that

$$\langle \phi_{\mathcal{C}}(t')|\phi_{\mathcal{C}}(t)\rangle = \frac{1}{T}\sum_{k=-\infty}^{+\infty} e^{-i\frac{2\pi k}{T}(t-t')} = \delta(t-t'), \quad t, t' \in [0, T). \tag{1.33}$$

The time states are orthogonal but evidently non-normalisable. This means that they do not belong to $\mathscr{H}_{\mathcal{C}}$ and, therefore, should not be regarded as physical. Nevertheless, these time states are useful mathematical abstractions and can be thought of as approximating the behaviour of ever-more-precise realistic clocks that can distinguish ever-smaller time intervals. The rigorous way to treat these idealised states mathematically employs the so-called *rigged Hilbert space* formalism [54]. Within this formalism, we can work with these non-normalisable states and give a precise definition of states like $\int \mathrm{d}t\, \lambda(t)|\phi_{\mathcal{C}}(t)\rangle$. We discuss this structure in Appendix A.

Within the rigged Hilbert space formalism, the projectors $\hat{\Pi}_t$ form a complete set in the following sense

$$\int_{t=0}^{T} \mathrm{d}t\, \hat{\Pi}_t = \frac{1}{T}\int_{t=0}^{T} \mathrm{d}t \sum_{k,l=-\infty}^{+\infty} e^{-i\frac{2\pi}{T}(k-l)t}|\varepsilon_k\rangle\langle\varepsilon_l| = \sum_{k=-\infty}^{+\infty} |\varepsilon_k\rangle\langle\varepsilon_k| = \mathbb{1}_{\mathcal{C}}, \tag{1.34}$$

and we can build the corresponding time operator

$$\hat{T}_{\mathcal{C}} \coloneqq \int_0^T \mathrm{d}t\, t\, \hat{\Pi}_t. \tag{1.35}$$

The time states $|\phi_{\mathcal{C}}(t)\rangle$ are the so-called "generalised eigenstates" of $\hat{T}_{\mathcal{C}}$ [54].

Now, due to the divergent norm of the clock states $|\phi_{\mathcal{C}}(t)\rangle$, we need to modify the definition of relative states in Eq. (1.5), which has a divergent term in the denominator. The simplest solution is to use the following definition:

$$|\Psi_U(t)\rangle\rangle \coloneqq \frac{\hat{\Pi}_t\,|\Psi_U\rangle\rangle}{\langle\langle\Psi_U|\,\hat{\Pi}_t\,|\Psi_U\rangle\rangle}, \tag{1.36}$$

which differs from Eq. (1.5) only in the denominator (now not divergent). The term $\langle\langle\Psi_U|\,\hat{\Pi}_t\,|\Psi_U\rangle\rangle$ is called the "physical" inner product [21, 43] and does not depend, in the examples we are considering in this chapter, on the value of t, so we can denote it with the symbol $\langle\langle\Psi_U\,|\Psi_U\rangle\rangle_{\,phy} = \|\,|\Psi_U\rangle\rangle\|^2_{phy} \coloneqq \langle\langle\Psi_U|\,\hat{\Pi}_t\,|\Psi_U\rangle\rangle$ [43]. Using Eqs. (1.34) and (1.36), Eq. (1.27) becomes

$$|\Psi_U\rangle\rangle = \int_{t=0}^{T} \mathrm{d}t\,\hat{\Pi}_t\,|\Psi_U\rangle\rangle = \|\,|\Psi_U\rangle\rangle\|_{phy} \cdot \int_{t=0}^{T} \mathrm{d}t\,|\Psi_U(t)\rangle\rangle\,. \tag{1.37}$$

It is possible to choose $|\Psi_U\rangle\rangle$ so that it is still *normalisable* according to the *kinematical* inner product $\langle\langle\Psi_U\,|\Psi_U\rangle\rangle < \infty$ and its norm can be set equal to 1. We will later see that this is not possible when the clock has a continuous spectrum. In that case, the kinematical inner product will have to be replaced by the physical inner product. Also, note that we have recovered the continuous aspect of time we are familiar with, although time is still periodic in this scenario.

To conclude the section, let us consider what happens when the spectrum of $\hat{H}_{\mathcal{C}}$ is degenerate (and finite). In this case, the completeness condition

$$\sum_j |\phi(t_j)\rangle\langle\phi(t_j)| = \hat{\mathbb{1}}_{\mathcal{C}}, \tag{1.38}$$

cannot be satisfied. To see this, consider the generic expression for the time states $|\phi(t_j)\rangle = \sum_k \lambda_{k,d_k} e^{-i\varepsilon_k t_j}|\varepsilon_k, d_k\rangle$, where d_k labels the degeneracy of the ε_k-energy level. The completeness condition then becomes

$$\sum_j |\phi(t_j)\rangle\langle\phi(t_j)| = \sum_{j,k,k',d_k,d'_{k'}} \lambda_{k,d_k}\lambda^*_{k',d'_{k'}} e^{-i(\varepsilon_k-\varepsilon_{k'})t_j}|\varepsilon_k, d_k\rangle\langle\varepsilon_{k'}, d'_{k'}| = \hat{\mathbb{1}}_{\mathcal{C}}, \tag{1.39}$$

but we can see that this does not hold because

$$\langle \varepsilon_k, d_k | \left(\sum_j |\phi(t_j)\rangle\langle\phi(t_j)| \right) |\varepsilon_k, d'_k\rangle \propto \lambda_{k,d_k} \lambda^*_{k,d'_k}, \tag{1.40}$$

and the latter cannot be proportional to δ_{d_k,d'_k} as required by Eq. (1.39) unless there is no degeneracy in the energy levels. Similar considerations hold for unbounded energy spectra.

1.2.3.3 Continuous-Spectrum Clocks

In the previous section, we dealt with clocks with a discrete energy spectrum. Now, we turn our attention to clocks with a continuous spectrum. These clocks have an infinite resolution because any two different clock states—labelled by the continuous parameter t—are orthogonal so that any time difference, however small, can be distinguished. They also provide a convenient tool for removing the periodicity condition from the relative states of the Universe.

So let us consider a Hamiltonian $\hat{H}_\mathcal{C}$ with a continuous, non-degenerate spectrum. Since the spectrum is continuous, it includes irrational numbers and the group $\mathbf{G}$ is thus non-compact (i.e. non-periodic). The Hamiltonian admits a set $\{|\varepsilon_\mathcal{C}\rangle\}_\varepsilon$ of *generalised eigenstates*: $\hat{H}_\mathcal{C}|\varepsilon_\mathcal{C}\rangle = \varepsilon|\varepsilon_\mathcal{C}\rangle$. As mentioned in the previous section, these states are not normalisable and belong to an extended mathematical structure called the *rigged Hilbert space*. For the clock states to be orthogonal, the spectrum of $\hat{H}_\mathcal{C}$ must be unbounded from above and below [36]. We will call clocks with such a spectrum *ideal clocks*.

The clock states are of the form $|\phi_\mathcal{C}(t)\rangle = e^{-i\hat{H}_\mathcal{C}t}|\phi_\mathcal{C}(0)\rangle$ generated from [36]

$$|\phi_\mathcal{C}(0)\rangle \coloneqq \int_{-\infty}^{+\infty} \mathrm{d}\varepsilon \, \frac{e^{ig(\varepsilon)}}{\sqrt{2\pi}} |\varepsilon_\mathcal{C}\rangle, \tag{1.41}$$

where $g(\varepsilon)$ is a real-valued function of ε that reflects some freedom in the choice of

the clock states. The $|\phi_{\mathcal{C}}(t)\rangle$ also belong to the rigged Hilbert space as they are not normalisable. Moreover, they are orthogonal in the sense

$$\langle\phi_{\mathcal{C}}(t')|\phi_{\mathcal{C}}(t)\rangle = \delta(t-t'). \tag{1.42}$$

These ideal clocks are unrealistic because $\hat{H}_{\mathcal{C}}$ is unbounded from below. This constitutes the core of the so-called "Pauli objection" to the existence of a time operator in quantum theory [55]. However, the objection does not apply to realistic clocks, whose spectrum would necessarily be bounded from below.

Another issue with these clocks is that the states $|\phi_{\mathcal{C}}(t)\rangle$ are non-normalisable and therefore unphysical. As discussed in the previous section, this is not an issue if we think of these clocks as mathematical objects approximating the behaviour of an increasingly precise real clock and work within the rigged Hilbert space formalism (Appendix A). We can then consider the *improper* projectors $\hat{\Pi}_t = |\phi_{\mathcal{C}}(t)\rangle\langle\phi_{\mathcal{C}}(t)|$ which form a resolution of the identity in the following sense [54]

$$\int_{-\infty}^{+\infty} \mathrm{d}t\, \hat{\Pi}_t = \hat{\mathbb{1}}_{\mathcal{C}}, \tag{1.43}$$

and can be used to construct the corresponding time operator

$$\hat{T}_{\mathcal{C}} \coloneqq \int_{-\infty}^{+\infty} \mathrm{d}t\, t\, \hat{\Pi}_t, \tag{1.44}$$

which is conjugate to $\hat{H}_{\mathcal{C}}$:

$$\left[\hat{T}_{\mathcal{C}}, \hat{H}_{\mathcal{C}}\right] = i. \tag{1.45}$$

Now, using Eqs. (1.43) and (1.36), we can write $|\Psi_U\rangle\rangle$ as

$$|\Psi_U\rangle\rangle = \||\Psi_U\rangle\rangle\|_{phy} \cdot \int_{-\infty}^{+\infty} \mathrm{d}t\ |\Psi_U(t)\rangle\rangle\,, \tag{1.46}$$

where we have used the *physical* inner product defined in the previous section. We can choose $|\Psi_U\rangle\rangle$ so that $\||\Psi_U\rangle\rangle\|_{phy} = 1$ [43] and thus

$$|\Psi_U\rangle\rangle = \int_{-\infty}^{+\infty} \mathrm{d}t\, |\psi_{\mathcal{A}}(t)\rangle|\phi_{\mathcal{C}}(t)\rangle = \int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-i\hat{H}_{\mathcal{A}}t}|\psi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(t)\rangle, \tag{1.47}$$

where we have used Eqs. (1.10) and (1.11). However, unlike in the previous section, the state of the Universe is no longer normalisable according to the kinematical inner product. In this case, we need to adopt the *physical* inner product defined in the previous section [21, 43]. According to this inner product, $|\Psi_U\rangle\rangle$ is normalised. This is necessary whenever the zero eigenvalue of $\hat{H}_U$ is in the continuous part of the energy spectrum [21]. In the rest of this thesis, we will almost always use ideal clocks, so the state of the Universe must always be understood as normalised according to the physical inner product.

Finally, from Eqs. (1.46) and (1.47) we see that we have recovered the continuity and unboundedness of time: all the possible time instants are contained in $|\Psi_U\rangle\rangle$.

1.2.3.4 POVM

It is possible to include all the clock states $|\phi_{\mathcal{C}}(t)\rangle$ in the construction without using Hamiltonians that are unbounded from below by relaxing the requirement that measurements of the clock must be projective and considering instead a more general class of measurements: positive operator-valued measurements (POVM) [46, 45, 40, 36].

In a simplified definition, a POVM is a set of positive semi-definite self-adjoint operators $\{\hat{E}_i\}_i$ such that they are normalised, i.e. $\sum_i \hat{E}_i = \mathbb{1}_{\mathcal{C}}$ [49]. For a more general definition in the context of timeless quantum theory, see [45, 40, 36]. The set of projectors $\mathcal{T}_{\mathcal{C}} \coloneqq \{|\phi_{\mathcal{C}}(t)\rangle\langle\phi_{\mathcal{C}}(t)|\}_t$ is a special case of rank 1 POVM, provided

that they are normalised.[3] The normalisation imposes some conditions on the state $|\phi_{\mathcal{C}}(0)\rangle$ (we refer the reader to [36] for a thorough discussion). If we call $G \subseteq \mathbb{R}$ the set of values that parametrise the group $\mathbf{G} = \left\{e^{-i\hat{H}_{\mathcal{C}}t}\right\}_t$ up to a phase, then there is a suitable choice of initial state $|\phi'_{\mathcal{C}}(0)\rangle$ such that [36]

$$\int_G \mathrm{d}t\, \frac{1}{\mu} |\phi'_{\mathcal{C}}(t)\rangle\langle\phi'_{\mathcal{C}}(t)| = \mathbb{1}_{\mathcal{C}}, \tag{1.48}$$

with μ a constant, so that the POVM is normalised.[4]

Note that these POVMs allow for a continuity of time instants to be measured. Moreover, we can have $G = \mathbb{R}$ while having the spectrum of $\hat{H}_{\mathcal{C}}$ bounded from below [36], thus solving Pauli's objection without renouncing the continuity of time.

When $\mathbf{G}$ is compact and the spectrum of $\hat{H}_{\mathcal{C}}$ is bounded and discrete as, for instance, in Eq. (1.23), we are faced with the choice between the discrete projective measurement $\mathcal{T}_{\mathcal{C}}^{M} := \{|\phi_{\mathcal{C}}(t_j)\rangle\langle\phi_{\mathcal{C}}(t_j)|\}_j$ and the continuous POVM $\mathcal{T}_{\mathcal{C}} := \{|\phi'_{\mathcal{C}}(t)\rangle\langle\phi'_{\mathcal{C}}(t)|\}_{t\in G}$.[5] We can thus write the state of the Universe either as

$$|\Psi_U\rangle\rangle = \|\langle\phi_{\mathcal{C}}(t_1)\,|\Psi_U\rangle\rangle\,|\phi_{\mathcal{C}}(t_1)\rangle\|\cdot\sum_j |\Psi_U(t_j)\rangle\rangle\,, \tag{1.49}$$

or

$$|\Psi_U\rangle\rangle = \frac{1}{T}\|\langle\phi'_{\mathcal{C}}(0)\,|\Psi_U\rangle\rangle\,|\phi'_{\mathcal{C}}(0)\rangle\|\cdot\int_G \mathrm{d}t\;|\Psi_U(t)\rangle\rangle\,. \tag{1.50}$$

Whilst mathematically equivalent, the interpretation of the two equations differs depending on what quantities of the clock we consider measurable. We will further discuss this point in Chapter 2.

[3] This POVM is rank 1 because its elements are the rank 1 projectors $\hat{\Pi}_t$. It is also called "covariant" because its elements are such that $\hat{\Pi}_{t+\theta} = e^{-i\hat{H}_{\mathcal{C}}\theta}\hat{\Pi}_t e^{i\hat{H}_{\mathcal{C}}\theta}$ [45, 40, 36].

[4] It is not possible to define the POVM when $\hat{H}_{\mathcal{C}}$ has a discrete spectrum and $\mathbf{G}$ is non-compact [36].

[5] In this case, $|\phi_{\mathcal{C}}(t)\rangle$ and $|\phi'_{\mathcal{C}}(t)\rangle$ differ only by an overall real constant T/N.

1.2.4 Normalisation of the Relative States

When considering the states of $U/\mathcal{A}$ relative to the clock showing the time t, we have prescribed that these states should be normalised as in Eqs. (1.5) and (1.10) (or Eq. (1.36) when the clock states are not normalisable). This is in agreement with the original paper from Page and Wootters [1]. However, subsequent literature [41, 42, 43] uses a different prescription for the relative states of $\mathcal{A}$:

$$|\psi_{\mathcal{A}}(t)\rangle \coloneqq \langle \phi_{\mathcal{C}}(t)\, |\Psi_U\rangle\rangle\,, \tag{1.51}$$

together with the condition $\langle\psi_{\mathcal{A}}(t_0)|\psi_{\mathcal{A}}(t_0)\rangle = 1$ for some value of t_0. Similarly, we could define the relative states of U as

$$|\Psi_U(t)\rangle\rangle \coloneqq \langle \phi_{\mathcal{C}}(t)\, |\Psi_U\rangle\rangle\, |\phi_{\mathcal{C}}(t)\rangle = \hat{\Pi}_t\, |\Psi_U\rangle\rangle\,, \tag{1.52}$$

with the analogous condition $\langle\langle \Psi_U(t_0)\, |\Psi_U(t_0)\rangle\rangle = 1$ for some value of t_0. At the moment, the two prescriptions lead to the same result since, due to Eq. (1.8), $\left\|\hat{\Pi}_t\, |\Psi_U\rangle\rangle\right\| = \left\|\hat{\Pi}_{t_0}\, |\Psi_U\rangle\rangle\right\|, \forall t$. So Eq. (1.5) and Eqs. (1.52) differ just on whether the same normalisation is applied to the state of the Universe or at the moment of measuring the time of the clock. In fact, note that, using Eqs. (1.52) , Eq. (1.27) becomes

$$|\Psi_U\rangle\rangle = \sum_j |\Psi_U(t_j)\rangle\rangle\,, \tag{1.53}$$

which differs from Eq. (1.27) only by a global normalisation factor. Note that this $|\Psi_U\rangle\rangle$ is not normalised but this is not problematic because $|\Psi_U\rangle\rangle$ is stationary [43]. Also, note that in the case of ideal clocks $|\Psi_U\rangle\rangle$ is normalisable according to the physical inner product, as discussed in Sec. 1.2.3.3.

While the two prescriptions here lead to the same result, this is not always the case. For instance, in later sections we will see that they lead to different results

when there are interactions between $\mathcal{A}$ and $\mathcal{C}$.

1.2.5 Correlations

The reason why $\mathcal{A}$ evolves with respect to $\mathcal{C}$ is that $\mathcal{A}$ and $\mathcal{C}$ are *correlated*: they "tick" in the same way. This is apparent from Eqs. (1.30) and (1.47), which show that $\mathcal{A}$ and $\mathcal{C}$ are correlated in a uniquely quantum way: they are *entangled*. $|\Psi_U\rangle\rangle$ is separable only if $|\psi_{\mathcal{A}}(t)\rangle = |\psi_{\mathcal{A}}(t')\rangle$ for all the values of t and t' appearing in $|\Psi_U\rangle\rangle$. This means that $|\Psi_U\rangle\rangle$ is separable only when $|\psi_{\mathcal{A}}(t)\rangle$ is stationary, that is when the evolution of $\mathcal{A}$ relative to $\mathcal{C}$ is *trivial*. In other words, only entangled states of the Universe result in a *non-trivial* relational evolution. This fact is sometimes expressed in the literature with the slogan "time from entanglement" [41, 42, 56].

However, it is not clear that the mixed states of the Universe discussed in Sec. 1.2.2.2 should also be entangled. Moreover, some authors have claimed that entanglement is not necessary [57]. We will discuss the role of correlations and entanglement in Chapter 3.

1.2.6 Interactions with the Clock

The original construction due to Page and Wootters [1] does not include any interactions between $\mathcal{A}$ and $\mathcal{C}$. This is reflected in the absence of interaction terms in the Hamiltonian of the Universe of Eq. (1.2). However, it is crucial to study what happens when $\mathcal{A}$ and $\mathcal{C}$ are interacting. This is for multiple reasons. First, the absence of interactions between $\mathcal{A}$ and $\mathcal{C}$ precludes the accessibility of the clock, which, therefore, cannot be considered a physical system. For instance, an observer included in $\mathcal{A}$ is not able to access the time readings of $\mathcal{C}$. Second, it may not be possible to shield all the interactions between $\mathcal{A}$ and $\mathcal{C}$ (for instance, gravitational interactions). Third, interaction terms appear in relativistic approaches to timeless

quantum theory [40]. Therefore, it is essential to study how interactions between $\mathcal{A}$ and $\mathcal{C}$ affect the relational approach to time outlined above.

The discussion of the problem depends on whether the relative states are defined as in Eq. (1.5)/(1.10) or as in Eq. (1.52)/(1.51). In both cases, let us consider a Hamiltonian of the Universe of the form [43]

$$\hat{H}_U = \hat{H}_\mathcal{A} \otimes \hat{\mathbb{1}}_\mathcal{C} + \hat{\mathbb{1}}_\mathcal{C} \otimes \hat{H}_\mathcal{C} + \hat{V}, \tag{1.54}$$

where $\hat{V}$ is an interaction term between $\mathcal{A}$ and $\mathcal{C}$. Let us also assume that $\mathcal{C}$ is an ideal clock. The state of the Universe $|\Psi_U\rangle\rangle$ will now be a (generalised) eigenstate of this new Hamiltonian with eigenvalue 0.

1.2.6.1 Unnormalised Relative States

Let us define the relative states of U as in Eq. (1.5) but without the normalisation term. In the presence of interactions, is not easy to find the new relational evolution of the Universe in the *integral* form as in Eq. (1.6). It is easier, instead, to consider the *differential* form:

$$\begin{aligned} i\frac{\mathrm{d}}{\mathrm{d}t}|\Psi_U(t)\rangle\rangle &= -\langle\phi_\mathcal{C}(t)|\hat{H}_\mathcal{C}|\Psi_U\rangle\rangle\,|\phi_\mathcal{C}(t)\rangle + \langle\phi_\mathcal{C}(t)\,|\Psi_U\rangle\rangle\,\hat{H}_\mathcal{C}|\phi_\mathcal{C}(t)\rangle \\ &= \left(\hat{H}_\mathcal{A} + \hat{H}_\mathcal{C}\right)|\Psi_U(t)\rangle\rangle + \langle\phi_\mathcal{C}(t)|\hat{V}\,|\Psi_U\rangle\rangle\,|\phi_\mathcal{C}(t)\rangle, \end{aligned} \tag{1.55}$$

where we have used Eqs. (1.3), (1.4) (1.54). This equation represents a *modified* Schrödinger equation. We can find a similar equation for the relative states of $\mathcal{A}$ [43]:

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\psi_\mathcal{A}(t)\rangle = -\langle\phi_\mathcal{C}(t)|\hat{H}_\mathcal{C}\,|\Psi_U\rangle\rangle = \hat{H}_\mathcal{A}|\psi_\mathcal{A}(t)\rangle + \langle\phi_\mathcal{C}(t)|\hat{V}\,|\Psi_U\rangle\rangle\,, \tag{1.56}$$

which we can rewrite as [43]

$$\begin{aligned} i\frac{\mathrm{d}}{\mathrm{d}t}|\psi_{\mathcal{A}}(t)\rangle =& \hat{H}_{\mathcal{A}}|\psi_{\mathcal{A}}(t)\rangle + \int_{-\infty}^{+\infty} \mathrm{d}t' \, \langle \phi_{\mathcal{C}}(t)|\hat{V}\,\hat{\Pi}_{t'}\,|\Psi_U\rangle\rangle \\ =& \hat{H}_{\mathcal{A}}|\psi_{\mathcal{A}}(t)\rangle + \int_{-\infty}^{+\infty} \mathrm{d}t' \, \hat{K}\,(t,t')\,|\psi_{\mathcal{A}}(t')\rangle, \end{aligned} \tag{1.57}$$

where we have used the resolution of the identity in terms of the $\hat{\Pi}_{t'}$ and defined $\hat{K}\,(t,t') \coloneqq \langle \phi_{\mathcal{C}}(t)|\hat{V}|\phi_{\mathcal{C}}(t')\rangle$ [43]. Eq. (1.57) also represents a modified Schrödinger equation. The equation is *linear* but, unlike the usual Schrödinger equation, it is *time non-local* [43], meaning that the solution of the equation at a certain time t may depend on the solution at other times.

This equation has been solved for some specific types of interactions. Depending on the form of $\hat{V}$, Eq. (1.57) can either lead to a unitary evolution with an effective or time-dependent Hamiltonian [43], or to a non-unitary evolution [58]. We can see a signal of non-unitarity if we calculate [43]

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle \psi_{\mathcal{A}}(t)|\psi_{\mathcal{A}}(t)\rangle = i\,\langle\langle \Psi_U|\left[\hat{V},\hat{\Pi}_t\right]|\Psi_U\rangle\rangle\,, \tag{1.58}$$

which is, in general, not zero. If this is the case, then the norm of $|\psi_{\mathcal{A}}(t)\rangle$ is not constant, which is a sign that $|\psi_{\mathcal{A}}(t)\rangle$ is not evolving unitarily. The same holds for $|\Psi_U(t)\rangle\rangle$.

1.2.6.2 Normalised Relative States

If we define the relative states of U as in Eq. (1.5), we force the relative states to be normalised at all times. However, the evolution of these states becomes even more complicated. The derivative of the numerator of Eq. (1.5) is the same as in Eq.

(1.55), while the derivative of the denominator is [6]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\hat{\Pi}_t\left|\Psi_U\right\rangle\right\rangle\right\| = \frac{i\left\langle\left\langle\Psi_U\right|\left[\hat{V},\hat{\Pi}_t\right]\left|\Psi_U\right\rangle\right\rangle}{2\|\langle\phi_{\mathcal{C}}(t)\left|\Psi_U\right\rangle\rangle\|}. \tag{1.59}$$

Combining these results, we find

$$\begin{aligned} i\frac{\mathrm{d}}{\mathrm{d}t}\left|\Psi_U(t)\right\rangle\rangle = \left(\hat{H}_{\mathcal{A}}+\hat{H}_{\mathcal{C}}\right)\left|\Psi_U(t)\right\rangle\rangle + \frac{\hat{\Pi}_t\hat{V}\left|\Psi_U\right\rangle\rangle}{\|\langle\phi_{\mathcal{C}}(t)\left|\Psi_U\right\rangle\rangle\|} \\ + \frac{\left\langle\left\langle\Psi_U\right|\left[\hat{V},\hat{\Pi}_t\right]\left|\Psi_U\right\rangle\right\rangle}{2\|\langle\phi_{\mathcal{C}}(t)\left|\Psi_U\right\rangle\rangle\|^2}\left|\Psi_U(t)\right\rangle\rangle, \end{aligned} \tag{1.60}$$

where an additional term appears with respect to Eq. (1.55). The analogous equation for the relative states of $\mathcal{A}$ is

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\psi_{\mathcal{A}}(t)\rangle = \hat{H}_{\mathcal{A}}|\psi_{\mathcal{A}}(t)\rangle + \frac{\hat{\Pi}_t\hat{V}\left|\Psi_U\right\rangle\rangle}{\|\langle\phi_{\mathcal{C}}(t)\left|\Psi_U\right\rangle\rangle\|} + \frac{\left\langle\left\langle\Psi_U\right|\left[\hat{V},\hat{\Pi}_t\right]\left|\Psi_U\right\rangle\right\rangle}{2\|\langle\phi_{\mathcal{C}}(t)\left|\Psi_U\right\rangle\rangle\|^2}|\psi_{\mathcal{A}}(t)\rangle. \tag{1.61}$$

We can further manipulate this expression by noting that [59]

$$\frac{\langle\phi_{\mathcal{C}}(t)|\hat{V}\left|\Psi_U\right\rangle\rangle}{\|\langle\phi_{\mathcal{C}}(t)\left|\Psi_U\right\rangle\rangle\|} = \left[\hat{V}_{\mathcal{A}}(t) - \frac{\left\langle\left\langle\Psi_U\right|\hat{V}\,\hat{\Pi}_t\left|\Psi_U\right\rangle\right\rangle}{\|\langle\phi_{\mathcal{C}}(t)\left|\Psi_U\right\rangle\rangle\|^2}\right]|\psi_{\mathcal{A}}(t)\rangle, \tag{1.62}$$

where we have defined

$$\hat{V}_{\mathcal{A}}(t) \coloneqq \frac{\langle\phi_{\mathcal{C}}(t)|\left\{\hat{V},\left|\Psi_U\right\rangle\rangle\left\langle\left\langle\Psi_U\right|\right\}\right|\phi_{\mathcal{C}}(t)\rangle}{\|\langle\phi_{\mathcal{C}}(t)\left|\Psi_U\right\rangle\rangle\|^2}, \tag{1.63}$$

which is self-adjoint. Then, Eq. (1.61) can be rewritten as

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\psi_{\mathcal{A}}(t)\rangle = \left(\hat{H}_{\mathcal{A}}+\hat{V}_{\mathcal{A}}(t)-\frac{v(t)}{2}\right)|\psi_{\mathcal{A}}(t)\rangle, \tag{1.64}$$

[6] If the clock states are not normalised, the denominator should be defined using the physical inner product as in Eq. (1.36) and (1.46). The same results hold even in this case.

where $v(t)$ is the *real number*

$$v(t) \coloneqq \frac{\langle\langle \Psi_U | \left\{ \hat{V}, \hat{\Pi}_t \right\} | \Psi_U \rangle\rangle}{\| \langle \phi_{\mathcal{C}}(t) | \Psi_U \rangle\rangle \|^2}. \tag{1.65}$$

Eq. (1.64) is *similar* to a Schrödinger equation with a time-dependent, self-adjoint potential. This may suggest that $|\psi_{\mathcal{A}}(t)\rangle$ evolves unitarily.

Since $-\frac{v(t)}{2}$ in Eq. (1.64) is just an energy shift, we can absorb it in $|\psi_{\mathcal{A}}(t)\rangle$ by defining $|\psi'_{\mathcal{A}}(t)\rangle \coloneqq e^{-i\int_0^t \mathrm{d}t' \, v(t')/2} |\psi_{\mathcal{A}}(t)\rangle$ to finally get

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\psi'_{\mathcal{A}}(t)\rangle = \left(\hat{H}_{\mathcal{A}} + \hat{V}_{\mathcal{A}}(t)\right) |\psi'_{\mathcal{A}}(t)\rangle, \tag{1.66}$$

This equation was derived in a slightly different way in [59]. To the best of our knowledge, the connection between the results of [59] and the treatment of normalised states outlined in this section does not appear in the literature.

We can do similar manipulations to transform Eq. (1.60) into

$$i\frac{\mathrm{d}}{\mathrm{d}t} |\Psi'_U(t)\rangle\rangle = \left(\hat{H}_{\mathcal{A}} + \hat{H}_{\mathcal{C}} + \hat{V}_{\mathcal{A}}(t)\right) |\Psi'_U(t)\rangle\rangle \,, \tag{1.67}$$

with $|\Psi'_U(t)\rangle\rangle \coloneqq e^{-i\int_0^t \mathrm{d}t' \, v(t')/2} \, |\Psi_U(t)\rangle\rangle$. This equation is also similar to a Schrödinger equation with a self-adjoint Hamiltonian.

Is the evolution given by Eqs. (1.66)-(1.67) unitary? While the equations preserve the norm of the relative states and the Hamiltonian is self-adjoint, we observe that these equations are *non-linear*.[7] This is because $\hat{V}_{\mathcal{A}}(t)$ depends on $|\Psi_U\rangle\rangle$. Therefore, we cannot say that the evolution described by these equations is unitary because there might not be a unitary operator describing such evolution. For instance, we could have, in general, two different initial states $|\psi_{\mathcal{A}}(0)\rangle$ and $|\chi_{\mathcal{A}}(0)\rangle$

[7] This makes the equations difficult to solve despite their familiar appearance and is in stark contrast to the linear nature of quantum theory.

such that $\langle \psi_{\mathcal{A}}(t) \, | \chi_{\mathcal{A}}(t) \rangle \rangle$ is not constant in time. Note that Eq. (1.57) is, instead, linear. This is one of the reasons why we will opt for the unnormalised relative states in later chapters of the thesis.

1.3 Further Developments

The interest in the seminal papers by Page and Wootters [1, 41] was later revived by some subsequent refinements [60, 42] and an experimental illustration of the mechanism [56]. The approach has since been developed and generalised in multiple directions.

Part of the literature focuses on understanding the role of interactions between the system and the clock. After a seminal work on the topic [43], interactions were explored in different scenarios [61, 40, 62, 63, 58, 64, 65, 59, 66]. While some interactions lead to interesting relativistic effects [61, 40, 64, 66], other can lead to a non-unitary relational dynamics [58]. The role of interactions in timeless quantum theory thus appears fundamental, but it is not fully clear. Moreover, we lack a clear understanding of why non-unitarity appears in the relational dynamics, and it is not known how to characterise the types of interactions that lead to such an effect.

A consistent part of the literature explores alternative formulations of timeless quantum theory. Some of the works recover a relational Heisenberg picture [45, 36, 67], others are based on the notion of Dirac observable [68, 69, 36], and others still are based on a clock-neutral approach [70, 71, 72, 73]. Most of these approaches have been proven to be equivalent [36] and have been considered in relativistic [74] and quantum field theoretic contexts [75]. Moreover, some of these approaches turned out to be closely related to the so-called "quantum reference frames" proposed in [76] (see [77] for an overview of the link) and were further explored for different types of symmetry groups [78, 79, 80, 81, 82].

Some works on timeless quantum theory studied other features of the construction, such as the choice of tensor product structure [83], relativistic generalisations [84, 85], the role of coherence [57] and asymmetry [86, 87], classical limits [88], the role of energy measurements [89] and modular operators [90], the notion of subsystems [91], extensions to spatial degrees of freedom [92, 93, 94], and the role of periodic clocks [95]. The approach was also applied in other scenarios, such as in the study of causal structures [96], cosmology [97], classical relational approaches to time [32], and time travel [98].

1.4 Outline of the Thesis

The body of this thesis consists of three chapters that analyse three different aspects of timeless quantum theory. In Chapter 2, we will explore the role of clocks. The core of the chapter will be devoted to dealing with some issues around clocks and relative states in timeless quantum theory. We will argue for an Everettian view of the timeless approach and clarify the interpretation of time. We will also show why the construction allows the clock to be accessed, thus granting it the status of a proper physical system.

In Chapter 3, we will examine the role of correlation in the construction. We will first show how to cast timeless quantum theory in the Heisenberg picture by transferring crucial correlations from the state vector of the Universe to its observables. We will do this for both pure and mixed states of the Universe. We will then discuss the role of entanglement and other types of correlations in the construction. Finally, we will introduce a differential notation that will later simplify the study of interactions.

Chapter 4 will focus on the role of interaction in timeless quantum theory. After considering some explicit examples of interactions, we will discuss how these affect

the time readings of the clock. We will explain why the timekeeping qualities of the clock are not affected when considering macroscopic systems and propose a potential criterion to characterise interactions that lead to a unitary relational evolution. We will then show how another type of non-unitarity arises when considering mixed states of the Universe that are not entangled. Finally, we will show how to recover the Heisenberg picture version of timeless quantum theory even in the presence of some types of interactions.

In Chapter 5, we will summarise our results and outline directions for future research.

Chapter 2

The Role of Clocks

In the previous chapter, the clock $\mathcal{C}$ was instrumental in defining the relative states of the Universe associated with different times. We associated these relative states with covariant PVMs or POVMs of the clock, which considerably restricted the allowed types of Hamiltonian and clock states. However, the meaning of these relative states is not fully clear. Why does the clock $\mathcal{C}$ and not the system $\mathcal{A}$ define the relative states? Should we adopt PVMs or POVMs? Do realistic clocks conform to the requirements discussed in Chapter 1?

In this chapter, we clarify the role of clocks in timeless quantum theory. We start by showing how to construct a stationary state of the Universe from any initial state. This provides some insights into the content of timeless quantum theory, and the results will be useful in later chapters. We then discuss the interpretation of time in timeless quantum theory. The interpretation is based on the Everettian approach to quantum theory and sheds light on the role of clocks and the meaning of their relative states. In the last section, we show why timeless quantum theory allows the clock to be accessed by the other subsystems of the Universe. This is crucial to guarantee that the clock is a physical system.

2.1 Construction of the State of the Universe

As we discussed in the introduction, the state of the Universe in timeless quantum theory must be an eigenstate of the Hamiltonian of the Universe. This ensures the removal of any external time parameter but can greatly restrict the states of $\mathcal{A}$ and $\mathcal{C}$ that are available. For example, in the trivial case in which the 0-eigenspace of $\hat{H}_U$ is empty, there is no solution to the Wheeler-DeWitt equation (1.3).[1] So, ideally, the 0-eigenspace of $\hat{H}_U$ would have a high degeneracy to guarantee the availability of enough different states of $\mathcal{A}$ and $\mathcal{C}$ [83]. Finding an eigenstate of $\hat{H}_U$ could be complicated, so here we will show how to construct a suitable one. This section is slightly more technical but the results obtained here will be useful in the rest of the thesis and provide the first indications that the structure of the POVMs/PVMs considered in the previous chapters can be relaxed.

First, let us consider a generic 0-eigenstate of the non-interacting Hamiltonian $\hat{H}_U$ of Eq. (1.2). Assuming, for the moment, that the spectra of $\hat{H}_\mathcal{A}$ and $\hat{H}_\mathcal{C}$ are discrete, we can write it as

$$|\Psi_U\rangle\rangle = \sum_{E,d_E} \lambda_{E,d_E} |E_\mathcal{A}, d_E\rangle |-E_\mathcal{C}\rangle, \tag{2.1}$$

where $|E_\mathcal{A}, d_E\rangle$ is an eigenstate of $\hat{H}_\mathcal{A}$ with eigenvalue E and degeneracy label d_E, and $|-E_\mathcal{C}\rangle$ is an eigenstate of $\hat{H}_\mathcal{C}$ with eigenvalue $-E$.[2] $|\Psi_U\rangle\rangle$ can be normalised, although its norm may or may not be equal to 1 depending on how we define the relative states (see Sec. 1.2.4).

There is another way to construct the 0-eigenstates of $\hat{H}_U$ *independently from*

[1] Of course there are still eigenstates of $\hat{H}_U$ with different eigenvalues, or mixed states that commute with $\hat{H}_U$. We will examine these states in later chapters.

[2] We remind the reader that the clock cannot have degenerate energy eigenvalues as discussed in Sec. 1.2.3.2. Also, note that the eigenstates of the clock will be improper if $\hat{H}_\mathcal{C}$ has a continuous spectrum.

the form of the Hamiltonian using the following *improper* projector [21]:[3]

$$\delta(\hat{H}_U) \coloneqq \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-i\hat{H}_U t}. \tag{2.2}$$

We can easily see that this operator is Hermitian

$$\delta(\hat{H}_U)^\dagger = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}t\, e^{i\hat{H}_U t} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}t'\, e^{-i\hat{H}_U t'} = \delta(\hat{H}_U), \tag{2.3}$$

but it is improper since

$$\begin{aligned}\delta(\hat{H}_U)^2 &= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \mathrm{d}t\mathrm{d}t'\, e^{-i\hat{H}_U(t+t')} \\ &= \frac{1}{8\pi^2}\left(\int_{-\infty}^{+\infty} \mathrm{d}s'\right)\int_{-\infty}^{+\infty} \mathrm{d}s\, e^{-i\hat{H}_U(s)} \sim \delta(0)\delta(\hat{H}_U),\end{aligned} \tag{2.4}$$

where we performed the change of variables $s = t + t'$ and $s' = t - t'$.

If we take *any* initial states $|\chi_{\mathcal{A}}(0)\rangle$ of $\mathcal{A}$ (we do not call it $|\psi_{\mathcal{A}}(0)\rangle$ because it may be different from $\langle\phi_{\mathcal{C}}(0)\,|\Psi_U\rangle\rangle$) and $|\phi_{\mathcal{C}}(0)\rangle$ of the clock, and act on them with this projector, we obtain an *improper* 0-eigenstate of $\hat{H}_U$ [21]

$$\left|\tilde{\Psi}_U\right\rangle\!\!\right\rangle = \delta(\hat{H}_U)|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-i\hat{H}_U t}|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle, \tag{2.5}$$

where the tilde indicates that this state may differ from the eigenstates of $\hat{H}_U$ of Eq. (2.1). We have that, for any $\theta \in \mathbb{R}$,

$$\begin{aligned}e^{-i\hat{H}_U\theta}\left|\tilde{\Psi}_U\right\rangle\!\!\right\rangle &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-i\hat{H}_U(t+\theta)}|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}t'\, e^{-i\hat{H}_U t'}|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle = \left|\tilde{\Psi}_U\right\rangle\!\!\right\rangle,\end{aligned} \tag{2.6}$$

[3] We are extending the meaning of "improper" to denote states and projectors that lead to divergencies even if $\hat{H}_U$ has a finite, discrete spectrum. Compare this with the discussions in Sections 1.2.3.2-1.2.3.3.

which means that $\left|\tilde{\Psi}_U\right\rangle\!\!\right\rangle$ is an (improper) eigenstate of $\hat{H}_U$ with eigenvalue 0.

We can also write $|\chi_{\mathcal{A}}(0)\rangle = \sum_{E,d_E} \mu_{E,d_E}|E_{\mathcal{A}}, d_E\rangle$ and $|\phi_{\mathcal{C}}(0)\rangle = \sum_{\varepsilon} \nu_{\varepsilon}|\varepsilon_{\mathcal{C}}\rangle$, and observe that, if $\hat{H}_U$ is non-interacting, $\delta(\hat{H}_U)$ acts on $|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle$ in the following way

$$\begin{aligned}\left|\tilde{\Psi}_U\right\rangle\!\!\right\rangle &= \delta(\hat{H}_U) \sum_{E,d_E,\varepsilon} \mu_{E,d_E}\nu_{\varepsilon}|E_{\mathcal{A}}, d_E\rangle|\varepsilon_{\mathcal{C}}\rangle \\ &= \sum_{E,d_E,\varepsilon} \mu_{E,d_E}\nu_{\varepsilon} \int_{-\infty}^{+\infty} \mathrm{d}t\, \frac{e^{-i(E+\varepsilon)t}}{2\pi}|E_{\mathcal{A}}, d_E\rangle|\varepsilon_{\mathcal{C}}\rangle \\ &= \sum_{E,d_E,\varepsilon} \mu_{E,d_E}\nu_{\varepsilon}\, \delta(E+\varepsilon)|E_{\mathcal{A}}, d_E\rangle|\varepsilon_{\mathcal{C}}\rangle, \end{aligned} \tag{2.7}$$

thus selecting only the $|E_{\mathcal{A}}, d_E\rangle|-E_{\mathcal{C}}\rangle$ states, which are the 0-eigenstates of $\hat{H}_U$. If we consider a Hamiltonian of the clock with a continuous spectrum, we can write $|\phi_{\mathcal{C}}(0)\rangle = \int \mathrm{d}\varepsilon\, \nu_{\varepsilon}|\varepsilon_{\mathcal{C}}\rangle$, and Eq. (2.7) becomes

$$\begin{aligned}\left|\tilde{\Psi}_U\right\rangle\!\!\right\rangle &= \sum_{E,d_E} \int \mathrm{d}\varepsilon\, \mu_{E,d_E}\nu_{\varepsilon}\, \delta(E+\varepsilon)|E_{\mathcal{A}}, d_E\rangle|\varepsilon_{\mathcal{C}}\rangle \\ &= \sum_{E,d_E} \mu_{E,d_E}\nu_{-E}|E_{\mathcal{A}}, d_E\rangle|-E_{\mathcal{C}}\rangle, \end{aligned} \tag{2.8}$$

which is of the form of Eq. (2.1). The Dirac deltas have disappeared, although the state is still unnormalisable due to the eigenstates of the clock $|-E_{\mathcal{C}}\rangle$ being improper. Nevertheless, this problem can be solved by adopting the physical inner product, as discussed in Sec. 1.2.3.

What we have achieved so far is to construct a 0-eigenstate of $\hat{H}_U$ starting from any state $|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle$. Eq. (2.5) shows that the state of the Universe consists of the superposition of all the time-translated states of the initially chosen state. This resembles a block-Universe picture where all the time instants exist and are included in a big superposition. However, this interpretation is not always correct because

the states $e^{-i\hat{H}_U t}|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle$ might not coincide with the state of the Universe relative to the clock showing a certain time $\hat{\Pi}_t \left|\tilde{\Psi}_U\right\rangle\!\right\rangle$. It could even be the case that $\hat{\Pi}_0 \left|\tilde{\Psi}_U\right\rangle\!\right\rangle \neq |\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle$, meaning that the state of the Universe at time 0 is not the state we have used to construct $\left|\tilde{\Psi}_U\right\rangle\!\right\rangle$. As we will see in the next section, this is because we need to impose further requirements on the clock states.

2.1.1 Further Requirements

Let us consider the state of Eq. (2.8) relative to the clock showing the time 0

$$\left\langle \phi_{\mathcal{C}}(0) \left|\tilde{\Psi}_U\right\rangle\!\right\rangle = \sum_{E,d_E} \mu_{E,d_E} \left|\nu_{-E}\right|^2 |E_{\mathcal{A}}, d_E\rangle. \tag{2.9}$$

This is equal to $|\chi_{\mathcal{A}}(0)\rangle$ only if $\left|\nu_{-E}\right|^2 = 1$ for all the values of E such that $\mu_{E,d_E} \neq 0$. We can relax this requirement with impunity to $\left|\nu_{-E}\right|^2 = M \in \mathbb{R}$ because the precise value of the normalisation of the state of the Universe is irrelevant. This means that if we want the state $|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle$ that we used to construct the stationary state of the Universe to actually be the state of the Universe at time 0, then we have to choose a suitable state of the clock at time 0. It is easy to show that this requirement is equivalent to

$$\left(\int_{-\infty}^{+\infty} \mathrm{d}t\, |\phi_{\mathcal{C}}(t)\rangle\langle\phi_{\mathcal{C}}(t)|\right) \left|\tilde{\Psi}_U\right\rangle\!\right\rangle = 2\pi M \left|\tilde{\Psi}_U\right\rangle\!\right\rangle, \tag{2.10}$$

which is similar to the completeness condition on PVMs/POVMs discussed in Sec. 1.2.3.4. However, this condition is weaker and is dependent on $\left|\tilde{\Psi}_U\right\rangle\!\right\rangle$. If this condition holds, then we have that, in the non-interacting case,

$$\hat{\Pi}_t \left|\tilde{\Psi}_U\right\rangle\!\right\rangle = e^{-i\hat{H}_U t}|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle, \tag{2.11}$$

that is, the state of the Universe of Eq. (2.5) is indeed the superposition of all its relative states. Finally, if the condition of Eq. (2.10) holds for any state $\left|\tilde{\Psi}_U\right\rangle\!\!\rangle$, then the clock state must satisfy exactly the completeness condition of Eq. (1.48).

What about the cases where $\hat{H}_{\mathcal{C}}$ has a discrete spectrum? We can write Eq. (2.7) as

$$\left|\tilde{\Psi}_U\right\rangle\!\!\rangle = \delta(0) \sum_{E,d_E,\varepsilon} \delta_{E,-\varepsilon}\, \mu_{E,d_E} \nu_\varepsilon \, |E_{\mathcal{A}}, d_E\rangle |\varepsilon_{\mathcal{C}}\rangle, \tag{2.12}$$

and Eq. (2.9) as

$$\left\langle \phi_{\mathcal{C}}(0) \left|\tilde{\Psi}_U\right\rangle\!\!\rangle = \delta(0) \sum_{E,d_E,\varepsilon} \delta_{E,-\varepsilon}\, \mu_{E,d_E} \left|\nu_\varepsilon\right|^2 |E_{\mathcal{A}}, d_E\rangle, \tag{2.13}$$

both manifestly non-normalisable even according to the physical inner product. At this point, motivated by the irrelevance of the overall normalisation of the state of the Universe, one might decide to *non-rigorously* renormalise the relative states to remove the divergent term $\delta(0)$. While we leave the questions about the validity of this procedure open for future investigation, we note that the result would be, in some cases, in agreement with what was discussed in Sec. 1.2.3.4. For instance, when the clock is periodic, the only eigenvalues surviving in Eq. (2.12) must have rational energy gaps and thus result in periodic relative states (see Appendix B for a concrete example). However, note that we can construct a state of the Universe and define the relative states even if the clock has a discrete, non-periodic spectrum. This was not possible with POVMs in Sec. 1.2.3.4.

In the rest of the thesis, we will mostly deal with ideal clocks, which have two main advantages. First, as we have discussed above, $\left|\tilde{\Psi}_U\right\rangle\!\!\rangle$ is normalised according to the physical inner product. Second, since the clock states are such that $\left|\nu_\varepsilon\right|^2 = (2\pi)^{-1}\ \forall \varepsilon$, we can choose any initial state of $|\chi_{\mathcal{A}}(0)\rangle$ and build a corresponding state of the Universe $\left|\tilde{\Psi}_U\right\rangle\!\!\rangle$ such that $\left\langle \phi_{\mathcal{C}}(0) \left|\tilde{\Psi}_U\right\rangle\!\!\rangle = |\chi_{\mathcal{A}}(0)\rangle$ (see Eq. (2.9)).

Finally, note that we can relax the requirement that $\hat{H}_{\mathcal{C}}$ must be non-degenerate.

For example, let us consider a generic clock state

$$|\phi_{\mathcal{C}}(0)\rangle = \int \mathrm{d}\varepsilon \sum_{l_\varepsilon} \nu_{\varepsilon,l_\varepsilon} |\varepsilon_{\mathcal{C}}, l_\varepsilon\rangle, \tag{2.14}$$

where we have assumed, for simplicity, that the degeneracies can be labelled by the discrete index l_ε. Then, Eq. (2.8) becomes

$$\left|\tilde{\Psi}_U\right\rangle\!\!\right\rangle = \sum_{E,d_E,l_{-E}} \mu_{E,d_E}\nu_{-E,l_{-E}} |E_{\mathcal{A}}, d_E\rangle|-E_{\mathcal{C}}, l_{-E}\rangle, \tag{2.15}$$

and Eq. (2.9)

$$\left\langle \phi_{\mathcal{C}}(0) \left|\tilde{\Psi}_U\right\rangle\!\!\right\rangle = \sum_{E,d_E} \mu_{E,d_E} \left(\sum_{l_{-E}} \left|\nu_{-E,l_{-E}}\right|^2 \right) |E_{\mathcal{A}}, d_E\rangle, \tag{2.16}$$

which is equal to $|\chi_{\mathcal{A}}(0)\rangle$ (up to a global factor) if $\sum_{l_{-E}} \left|\nu_{-E,l_{-E}}\right|^2 = M \in \mathbb{R}$ for all the values of E such that $\mu_{E,d_E} \neq 0$. This requirement is equivalent to a weak completeness condition as in Eq. (2.10). We thus recover the usual time evolution of the relative even if the clock has a degenerate spectrum. This was excluded with POVMs in Sec. 1.2.3.4. We will further comment on the required properties of the clock in Sec. 2.2.

2.1.2 Generalised States of the Universe

It is easy to extend the discussion above to eigenstates of $\hat{H}_U$ with energy ε by using the projector

$$\delta^{\varepsilon}(\hat{H}_U) \coloneqq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-i\left(\hat{H}_U - \varepsilon\right)t} \tag{2.17}$$

to construct $\left|\tilde{\Psi}_U^\varepsilon\right\rangle\!\!\right\rangle = \delta^\varepsilon(\hat{H}_U)|\chi_\mathcal{A}(0)\rangle|\phi_\mathcal{C}(0)\rangle$. Similarly to Eq. (2.6), we have that

$$e^{-i\left(\hat{H}_U-\varepsilon\right)\theta}\left|\tilde{\Psi}_U^\varepsilon\right\rangle\!\!\right\rangle = \left|\tilde{\Psi}_U^\varepsilon\right\rangle\!\!\right\rangle, \tag{2.18}$$

for every $\theta \in \mathbb{R}$, meaning that $\left|\tilde{\Psi}_U^\varepsilon\right\rangle\!\!\right\rangle$ is an eigenstate of $\hat{H}_U$ with eigenvalue ε.

In the case of mixed states of the Universe, we can build a stationary state $\hat{\mathcal{R}}_U$ in the following way

$$\hat{\mathcal{R}}_U = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-i\hat{H}_U t}\hat{\mathcal{R}}_U(0)e^{i\hat{H}_U t}, \tag{2.19}$$

for any initial state $\hat{\mathcal{R}}_U(0)$. Let us check that $\hat{\mathcal{R}}_U$ is stationary:

$$e^{-i\hat{H}_U\theta}\hat{\mathcal{R}}_U e^{i\hat{H}_U\theta} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-i\hat{H}_U(t+\theta)}\hat{\mathcal{R}}_U(0)e^{i\hat{H}_U(t+\theta)} = \hat{\mathcal{R}}_U, \tag{2.20}$$

for every $\theta \in \mathbb{R}$, which implies that $\left[\hat{\mathcal{R}}_U, \hat{H}_U\right] = 0$. We can also construct the following state

$$\hat{\mathcal{R}}_U^0 = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \mathrm{d}t\mathrm{d}t'\, e^{-i\hat{H}_U t}\hat{\mathcal{R}}_U(0)e^{i\hat{H}_U t'}. \tag{2.21}$$

Similarly to Eq. (2.20), one can check that $\hat{H}_U\hat{\mathcal{R}}_U = \hat{\mathcal{R}}_U\hat{H}_U = 0$, meaning that $\hat{\mathcal{R}}_U^0$ is a state living in the 0-eigenspace of $\hat{H}_U$.

2.1.3 Most General Stationary State

So far, we have built stationary states of the Universe starting from an initial state of the form $|\chi_\mathcal{A}(0)\rangle|\phi_\mathcal{C}(0)\rangle$. While this was convenient for the discussion, this initial state is clearly not the most general state from which we can start. The most general state will be of the form (assuming, for simplicity, $\hat{H}_\mathcal{C}$ has a continuous non-degenerate spectrum) $|\chi_U(0)\rangle\!\rangle := \sum_{E,d_E}\int \mathrm{d}\varepsilon\, \lambda_{E,d_E,\varepsilon}|E_\mathcal{A}, d_E\rangle|\varepsilon_\mathcal{C}\rangle$ from which we can

build the stationary state

$$\begin{aligned}\left|\tilde{\Psi}_U\right\rangle\!\right\rangle =& \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-i\hat{H}_U t}\,|\chi_U(0)\rangle\!\rangle \\ =& \sum_{E,d_E}\int \mathrm{d}\varepsilon\, \lambda_{E,d_E,\varepsilon}\,\delta(E+\varepsilon)|E_{\mathcal{A}},d_E\rangle|\varepsilon_{\mathcal{C}}\rangle \\ =& \sum_{E,d_E} \lambda_{E,d_E,-E}|E_{\mathcal{A}},d_E\rangle|-E_{\mathcal{C}}\rangle, \end{aligned} \tag{2.22}$$

which is of the form of Eq. (2.1). Does this add extra solutions? No, because we can always write $\left|\tilde{\Psi}_U\right\rangle\!\right\rangle$ as originating from a stationary state of the form $|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle$. For instance, given some clock state $|\phi_{\mathcal{C}}(0)\rangle = \int \mathrm{d}\varepsilon\, \nu_\varepsilon|\varepsilon_{\mathcal{C}}\rangle$ such that Eq. (2.10) holds, $\left|\tilde{\Psi}_U\right\rangle\!\right\rangle$ is equivalent to $\delta(\hat{H}_U)|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle$ with $|\chi_{\mathcal{A}}(0)\rangle = \sum_{E,d_E}\lambda_{E,d_E,-E}\nu^*_{-E}|E_{\mathcal{A}},d_E\rangle$. This equivalence may not hold in the presence of interactions.

2.2 Interpreting Time

The contents of this chapter are based, in part, on [99].

In the previous section, we saw that some of the requirements we imposed on the states of the clock in Chapter 1 can be relaxed. For instance, we saw that the clock states do not necessarily need to form a complete set, but they only need to satisfy Eq. (2.10). This adds to the confusion about the role of clocks and relative states in timeless quantum theory. In this section, we will shed some light on these points. We will start by arguing for an Everettian approach to timeless quantum theory. This will be the guiding principle for the rest of the section.

2.2.1 Measurements in Quantum Theory

In the introduction, we saw that there are two possible ways to define the relative states of $\mathcal{A}/U$ relative to the clock. One renormalises the relative states, the other does not. The renormalisation procedure of the state after a measurement is typical of interpretations of quantum theory that assume a collapse of the state vector. According to these interpretations, the state vector collapses to one of the possible outcomes after the measurement. For example, if we have a PVM $\{\hat{\Pi}_j\}_j$ on a quantum system S and we observe a specific outcome j^*, the state vector $|\psi_S\rangle$ of S collapses to $\left|\psi_S^{j^*}\right\rangle = \hat{\Pi}_{j^*}|\psi_S\rangle$ after the measurement. The resulting state vector must then be normalised to reflect the fact that if we repeat the measurement, there is no other outcome that we can obtain: $\left|\psi_S^{j^*}\right\rangle = \hat{\Pi}_{j^*}|\psi_S\rangle \cdot \left\|\hat{\Pi}_{j^*}|\psi_S\rangle\right\|^{-1}$. This account of measurements effectively skirts around the dynamics of the measurement process, and the normalisation procedure hides its non-unitarity.

In the Everettian interpretation of quantum theory [44, 100], the measurement process is described by a unitary transformation of the quantum system S and its measurement apparatus M. After the measurement, the state of $S + M$ is given by $\left|\Psi'_{S+M}\right\rangle = \hat{U}|\psi_S\rangle \otimes |0_M\rangle$, where $\hat{U}$ is the unitary describing the measurement process and $|0_M\rangle$ is the "ready" state of the measurement apparatus. In general, $\left|\Psi'_{S+M}\right\rangle$ will be in a superposition of different states ("relative states" or "branches") associated with a definite measurement outcome: $\left|\Psi'_{S+M}\right\rangle = \sum_j \lambda_j \left|\phi_S^j\right\rangle \otimes |j_M\rangle$, where $|j_M\rangle$ is the state of the apparatus displaying the outcome j. The different $|j_M\rangle$ must be orthogonal as they represent different definite outcomes. In the case of the projective (non-disturbing) measurement $\{\hat{\Pi}_j\}_j$, the final state of $S + M$ will be $\left|\Psi'_{S+M}\right\rangle = \sum_j \hat{\Pi}_j|\psi_S\rangle \otimes |j_M\rangle = \sum_j \left|\psi_S^j\right\rangle \otimes |j_M\rangle$, and by selecting a specific branch j^* and renormalising it we recover the "collapsed" state. If we consider, instead, a POVM $\{\hat{E}_j\}_j$ with $\hat{E}_j = \hat{M}_j^\dagger \hat{M}_j$ [49], the final state will be $\left|\Psi'_{S+M}\right\rangle = \sum_j \hat{M}_j|\psi_S\rangle \otimes |j_M\rangle = \sum_j \left|\chi_S^j\right\rangle \otimes |j_M\rangle$ where the $\left|\chi_S^j\right\rangle$ are not, in general, orthogonal.

In any case, note that the global state vector must not collapse (and thus be renormalised) after the measurement. Clearly, if we consider the expectation values of some observable in one of these relative states, then we need to condition these expectation values on the probability of being in one of these states: $\left\langle \hat{O} \right\rangle_j = \langle \psi_S^j | \hat{O} | \psi_S^j \rangle \cdot \langle \psi_S^j | \psi_S^j \rangle^{-1}$. This is equivalent to choosing a "collapsed" relative state and renormalising it. However, this can be done as long as the other relative states of the global state vector do not interfere with the chosen state. If they do interfere, then we cannot neglect them lest we provide a wrong description of the evolution of $S + M$.

The approach to timeless quantum theory discussed so far clearly resembles Everettian quantum theory. For instance, it is crucial that the state of the Universe does not collapse into one of the relative states. If this happened, then we would either be forever "stuck" in one of the relative states, or we would have transformed the stationary state of the Universe into a non-stationary state that could now evolve according to an external time parameter. Note that these are exactly two of the criticisms of the Page-Wootters construction raised by Kuchař [11]. We can thus understand them as the result of not adopting an Everettian view of quantum theory, or at least a no-collapse interpretation (see also [36, 74] for a resolution of these criticisms).

2.2.2 Interpreting the Relative States

We have previously shown that in timeless quantum theory one can recover the usual evolution of a system $\mathcal{A}$ by considering its state relative to a clock showing a certain time in an overall stationary Universe. One of the main advantages of an Everettian view of quantum theory is that we do not need to consider these states as necessarily deriving from a dynamical measurement process. In fact, while relative states are usually invoked in the context of measurements, they can be defined

meaningfully without appealing to a dynamical process. For instance, consider a (non-stationary) Universe that "started" in a state which is the same as what one would get after measuring some quantum system. Since the Universe started in this state, there could not have been any dynamical process leading to it. Still, it is meaningful to consider the relative states "as if a measurement had occurred". Thus, Everettian quantum theory already naturally contains the relational aspect that defines relational approaches to time.

However, there are still some unclear points:

(i) the clock $\mathcal{C}$ seems to have a privileged role in the construction. Why is it $\mathcal{C}$, and not $\mathcal{A}$, that determines the relative states?

(ii) The relative states can be defined in terms of either projective measurements or POVMs on the clock. It is unclear which is more fundamental in timeless quantum theory.

(iii) The notion of measurement required us to consider a complete set of measurement operators. This implied strong conditions on the Hamiltonian and the states of the clock as discussed in Sec. 1.2.3.

(iv) The absence of interactions in the Hamiltonian of the Universe prevents $\mathcal{C}$ from interacting with other physical systems. Since actual measurements require interactions, does this mean that $\mathcal{C}$ cannot be both accessed and still retain its role of clock in timeless quantum theory?

Almost all of these points can be clarified by considering more carefully the concept of Everettian relative state. While any kind of relative state can be mathematically defined, only some of them are useful in the description of physical phenomena. Usually, these are states that instantiate well-defined higher-level structures [100]. For example, the relative states after a measurement process are meaningful because

each of them instantiates a well-defined structure, i.e., the "classical" description of the world we are familiar with [101]. Usually, the subspaces (sometimes called "preferred basis") that instantiate these structures are thought to be selected by decoherence. Among other things, decoherence maintains these structures well-defined in time and space by preventing interference between the different relative states instantiating them (see [100] for an in-depth discussion of these concepts). However, these structures can sometimes be defined without appealing to decoherence but rather to information-theoretic concepts such as the capability of performing a classical computation [102, 103]. Regardless of how the type of structure is defined, a common point among all approaches is that relative states instantiating different structures must be orthogonal.

We propose that these ideas should be carried over into timeless quantum theory. This way, we can explain how time instants emerge naturally from the global state of the Universe as relative states that instantiate well-defined high-level structures. However, we soon encounter a problem: there are many more relative states than possible time instants. Which relative states should be associated with different time instants? A promising solution, which underlies many relational approaches to time, is to assume the existence of a class of systems (called "clocks") that play a special role in defining the relative states associated with different times.[4] These systems might have some special properties, such as being capable of performing certain tasks. While studying these systems is important, in this thesis we will assume that such systems exist and correspond to the clocks we currently use. As such, they select, among all the relative states that instantiate higher-level structures in the Universe, a subset thereof that we associate with different time instances. These states can be labelled by a parameter t and are associated with some projectors

[4]While clocks are central to all relational approaches to time, the properties that make certain physical systems good clocks are not always specified. For instance, covariant POVMs can be defined for vast classes of physical systems. So can all of these systems act as clocks?

$\hat{\Pi}_t$ of the clock. We also *know* that, when the clock is isolated, these projectors must be related by $\hat{\Pi}_{t'} = e^{-i\hat{H}_C(t'-t)}\hat{\Pi}_t e^{i\hat{H}_C(t'-t)}$, for any admissible value of t' and t. The assumption just described is equivalent to adopting the operational stance that "time is what the clock shows" without addressing what a "clock" is. We leave the latter question open for future investigations.

We can now see that this approach clarifies the points outlined above:

(i) We have proposed that certain systems in the Universe, called clocks, play a special role in determining the relative states of the Universe that are associated with different time instants.

(ii) While relative states can be defined in terms of POVMs even though they are not, in general, orthogonal, this is usually because the states of the apparatus corresponding to the different outcomes of the POVM are orthogonal (see Sec. 2.2.1). In timeless quantum theory, there is no external system measuring the clock, so the relative states associated with POVMs of the clock do not belong, in general, to orthogonal branches and thus cannot instantiate well-defined higher-level structures. As a result, projective measurements play a more fundamental role in timeless quantum theory. POVMs can still be seen as useful mathematical tools that include a continuity of time instants even with finite-dimensional clocks. Note that we are not saying that only the time instants associated with PVMs exist. The real, existing object is the state vector of the Universe; the states of the PVM are the only ones we can instantiate well-defined (orthogonal) higher-level structures.

(iii) By subordinating the relative states of the Universe to the *real* clocks existing in it, we can also relax some requirements on the PVM/POVM that we discussed in Sec. 1.2.3. To see this, let us describe a realistic (isolated) clock schematically. We do not know the Hamiltonian of this clock; however, to

be a good clock, it must possess some projectors $\left\{\hat{\Pi}_{t_j}\right\}_j$ that are associated with the clock showing the time t_j. These projectors will be, in general, of high rank, including all the microstates that correspond to the clock showing a specific time. They must also be such that $\hat{\Pi}_{t_{j'}} = e^{-i\hat{H}_{\mathcal{C}}(t_{j'}-t_j)}\hat{\Pi}_{t_j}e^{i\hat{H}_{\mathcal{C}}(t_{j'}-t_j)}$, for some values of $t_j, t_{j'}$ according to the range and resolution of the clock. Now, any realistic clock will perform its function in a certain regime. For instance, a pendulum will act as a good clock in some controlled temperature conditions, while if it becomes too hot, the string will dilate, the mass could melt, etc., and it will lose its timekeeping properties. Therefore, we cannot expect that the projectors $\left\{\hat{\Pi}_{t_j}\right\}_j$ sum up to the identity operator on the clock's Hilbert space. In general, there will be another projector $\hat{\overline{\Pi}}$ that includes all the configurations that are outside the operating regime of the clock. The completeness condition on the clock states thus reads

$$\sum_j \hat{\Pi}_{t_j} + \hat{\overline{\Pi}} = \hat{\mathbb{1}}_{\mathcal{C}}, \tag{2.23}$$

which is weaker than the completeness conditions discussed in Sec. 1.2.3. Also, note that $\hat{H}_{\mathcal{C}}$ can have degeneracies as these can either be included in the higher-dimensional $\hat{\Pi}_{t_j}$ or in $\hat{\overline{\Pi}}$. Finally, in our Universe we have access to many different types of clocks, each with its own operating regime. By synchronising and linking the time readings of all these clocks, we can "patch" all the time instants of the Universe (excluding cases where this might be, in principle, impossible).

Having clarified this point, in the rest of the thesis we will adopt the ideal clocks discussed in Sec. 1.2.3.3. While these clocks are unrealistic, they provide an infinite time resolution and operating range and can be used to highlight the main features of timeless quantum theory.

The last issue will be solved in Sec. 2.3.

As a final remark, note that Everettian quantum theory and the considerations above explain why we experience a single time instant instead of the global stationary Universe.[5] The explanation is the same for why we experience only one of the measurement outcomes in Everettian quantum theory: our "classical experience" could not perceive more than one. So, in any stationary quantum Universe, we would be "forced" to experience some time slice of it. The role of clocks is to "order" the different slices. We believe that this explanatory power is not achieved by classical relational approaches to time.

2.3 Physicality of the Clock

The contents of this section are based on the work [104], done in collaboration with Samuel Kuypers.

The last issue to be solved is whether the clocks in timeless quantum theory can be accessed. The original construction from Page and Wootters [1] and subsequent literature [83, 36] considers only clocks that are isolated from the rest of the Universe. As a result, while the clock plays an instrumental part in defining the time instants of the Universe, the clock readings cannot be accessed. This is in stark contrast with our everyday experience, where we access clock readings all the time. Moreover, the inaccessibility of the clock raises questions about its physicality: how can we consider it a physical system if it cannot interact with other physical systems?

In this section, we will introduce a simplified model in which the clock interacts with a qubit. We will show that this model can be thought of as describing a measurement of the clock by the qubit. Importantly, the type of interaction in-

[5] Here, we are overlooking deep philosophical questions such as whether it is possible to experience something in "one time" or why do we "jump" from one time to the other.

troduced does not lead to the non-unitarity problems discussed in Sec. 1.2.6 [43]. Our approach is based on a model from [105], where the author studies a measurement process in a stationary Universe. Although similar types of interactions were already discussed in the literature [43, 61], our approach shows that the clock can be measured and thus be considered a fully-fledged physical system.

2.3.1 A Qubit Measuring the Clock

Let us consider a stationary Universe consisting of a qubit $\mathcal{A}$ and an ideal clock $\mathcal{C}$. The Hamiltonian of the Universe is given by

$$\hat{H}_U = \hat{H}_{\mathcal{A}} \otimes \hat{P}_{[0,T]} + \hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}, \tag{2.24}$$

where

$$\hat{P}_{[0,T]} \coloneqq \int_0^T \mathrm{d}t\, \hat{\Pi}_t, \tag{2.25}$$

is the projector on the clock states with eigenvalues between 0 and T. Note that the term $\hat{H}_{\mathcal{A}} \otimes \hat{P}_{[0,T]}$ is an interaction term between $\mathcal{A}$ and $\mathcal{C}$.

The qubit $\mathcal{A}$ has four basic observables $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$, and the identity $\hat{1}_{\mathcal{A}}$. They satisfy the Pauli algebra $[\hat{\sigma}_i, \hat{\sigma}_j] = i2\epsilon_{ijk}\hat{\sigma}_k$, with $i, j, k = x, y, z$ and ϵ_{ijk} the Levi-Civita symbol, and $\hat{\sigma}_x^2 = \hat{\sigma}_y^2 = \hat{\sigma}_z^2 = \hat{1}_{\mathcal{A}}$. Let us denote with $|0_{\mathcal{A}}\rangle$ ($|1_{\mathcal{A}}\rangle$) the eigenstate of $\hat{\sigma}_z$ with eigenvalue 1 (-1). Let us also assume that $\hat{H}_{\mathcal{A}}$ is given by

$$\hat{H}_{\mathcal{A}} = -\frac{\pi}{2T}\hat{\sigma}_y. \tag{2.26}$$

We will shortly see that the qubit $\mathcal{A}$ functions as a simple timer. It stores exactly one bit of information about what time it is according to the time "shown" on the clock. This is thanks to the interaction term between $\mathcal{A}$ and $\mathcal{C}$ which causes the

qubit to evolve under the Hamiltonian $\hat{H}_{\mathcal{A}}$ during the time interval $[0, T]$. In this period, $\hat{H}_{\mathcal{A}}$ effectively acts as a NOT gate on the qubit. If $\mathcal{A}$ starts in the state $|0_{\mathcal{A}}\rangle$, the interaction will toggle its state to $|1\rangle$ after the time T, which is evidence that the clock has passed the time $t = 0$.

Let us see how this description of events can be implemented in timeless quantum theory. To do so, let us consider the state of the Universe

$$|\Psi_U\rangle\rangle \coloneqq \int_{-\infty}^{+\infty} \mathrm{d}t \, \Big(\theta(t<0)|0_{\mathcal{A}}\rangle + \theta(0\leq t\leq T)e^{-i\hat{H}_{\mathcal{A}}t}|0_{\mathcal{A}}\rangle + \theta(t>T)|1_{\mathcal{A}}\rangle\Big)\,|\phi_{\mathcal{C}}(t)\rangle, \quad (2.27)$$

where the function θ equals 1 for the interval shown in its argument and zero otherwise. This state coincides with the description of events provided above. To see this, consider, for instance, the state of $\mathcal{A}$ relative to the clock showing the time t

$$\langle\phi_{\mathcal{C}}(t)\,|\Psi\rangle\rangle = \begin{cases} |0_{\mathcal{A}}\rangle, & t<0, \\ e^{-i\hat{H}_{\mathcal{A}}t}|0_{\mathcal{A}}\rangle, & 0\leq t\leq T, \\ |1_{\mathcal{A}}\rangle, & t>T. \end{cases} \quad (2.28)$$

To show that this state is compatible with timeless quantum theory, we need to prove that it is an eigenstate of $\hat{H}_U$. Let us first consider how the interaction term acts on $|\Psi_U\rangle\rangle$

$$\Big(\hat{H}_{\mathcal{A}}\otimes\hat{P}_{[0,T]}\Big)\,|\Psi_U\rangle\rangle = \int_0^T \mathrm{d}t\,\hat{H}_{\mathcal{A}}e^{-i\hat{H}_{\mathcal{A}}t}|0_{\mathcal{A}}\rangle|\phi_{\mathcal{C}}(t)\rangle, \quad (2.29)$$

which can alternatively be expressed as

$$\left(\hat{H}_{\mathcal{A}} \otimes \hat{P}_{[0,T]}\right) |\Psi_U\rangle\rangle = i \int_{-\infty}^{+\infty} \mathrm{d}t \, \frac{\mathrm{d}}{\mathrm{d}t} \Big(\theta(t<0)|0_{\mathcal{A}}\rangle \\ + \theta(0 \leq t \leq T) e^{-i\hat{H}_{\mathcal{A}} t}|0_{\mathcal{A}}\rangle + \theta(t>T)|1_{\mathcal{A}}\rangle \Big) |\phi_{\mathcal{C}}(t)\rangle, \quad (2.30)$$

where we have used the fact that the derivative of θ with respect to t is a sum of Dirac delta functions; these all cancel out in the expression above. We can combine Eq. (2.30) with

$$\hat{\mathbb{1}}_{\mathcal{C}} \otimes \hat{H}_{\mathcal{C}} \, |\Psi_U\rangle\rangle = i \int_{-\infty}^{+\infty} \mathrm{d}t \, \Big(\theta(t<0)|0_{\mathcal{A}}\rangle \\ + \theta(0 \leq t \leq T) e^{-i\hat{H}_{\mathcal{A}} t}|0_{\mathcal{A}}\rangle + \theta(t>T)|1_{\mathcal{A}}\rangle \Big) \frac{\mathrm{d}}{\mathrm{d}t} |\phi_{\mathcal{C}}(t)\rangle, \quad (2.31)$$

to get

$$\hat{H}_U \, |\Psi_U\rangle\rangle = i \int_{-\infty}^{+\infty} \mathrm{d}t \, \frac{\mathrm{d}}{\mathrm{d}t} \Big[\Big(\theta(t<0)|0_{\mathcal{A}}\rangle \\ + \theta(0 \leq t \leq T) e^{-i\hat{H}_{\mathcal{A}} t}|0_{\mathcal{A}}\rangle + \theta(t>T)|1_{\mathcal{A}}\rangle \Big) |\phi_{\mathcal{C}}(t)\rangle \Big]. \quad (2.32)$$

As discussed in Appendix A in the context of Rigged Hilbert Spaces, this expression is equivalent to

$$\hat{H}_U \, |\Psi_U\rangle\rangle = 0, \quad (2.33)$$

due to the boundary terms vanishing at infinity. Eq. (2.33) shows that $|\Psi_U\rangle\rangle$ is an energy eigenstate of $\hat{H}_U$ with eigenvalue 0, and is thus compatible with timeless quantum theory. Therefore, we have shown that timeless quantum theory allows for measurements of the clock. This is important because it guarantees that the clock can be accessed and is thus a physical system.

2.3.1.1 Information Content of the Measurement

In the simple model we have just discussed, the qubit acts as an extremely simple timer for the ideal clock $\mathcal{C}$. This is because $\mathcal{A}$ contains one bit of information about the state of the clock. To see this, consider the following expectation values

$$\langle\langle\Psi|\,|0_{\mathcal{A}}\rangle\langle 0_{\mathcal{A}}|\otimes\hat{\Pi}_t\,|\Psi_U\rangle\rangle = 0 \quad \text{for all} \quad t \geq T, \tag{2.34}$$

$$\langle\langle\Psi|\,|0_{\mathcal{A}}\rangle\langle 0_{\mathcal{A}}|\otimes\hat{\Pi}_t\,|\Psi_U\rangle\rangle \neq 0 \quad \text{for all} \quad t < T. \tag{2.35}$$

These show that when the qubit is in the state $|0_{\mathcal{A}}\rangle$, it is certain that the time of the clock is $t < T$. Similarly, when the qubit is in the state $|1_{\mathcal{A}}\rangle$, it is certain that the time of the clock is $t > 0$ because

$$\langle\langle\Psi|\,|1_{\mathcal{A}}\rangle\langle 1_{\mathcal{A}}|\otimes\hat{\Pi}_t\,|\Psi_U\rangle\rangle = 0 \quad \text{for all} \quad t \leq 0, \tag{2.36}$$

$$\langle\langle\Psi|\,|1_{\mathcal{A}}\rangle\langle 1_{\mathcal{A}}|\otimes\hat{\Pi}_t\,|\Psi_U\rangle\rangle \neq 0 \quad \text{for all} \quad t > 0. \tag{2.37}$$

Notably, there is a slight ambiguity in the timer since, during the period $0 < t < T$, the qubit is in an intermediate state. This means that if there were another system that measured $\mathcal{A}$ (in the z-basis) during that period, then the outcome could be either $|0_{\mathcal{A}}\rangle$ or $|1_{\mathcal{A}}\rangle$. This must be a general feature of any real timer because the state of any real system must change continuously in time. For the qubit to be a good timer, the period T must be small relative to the relevant time scales. In principle, T can always be made small enough to be irrelevant because there is no bound on how fast a gate, such as the NOT gate, can be applied [106].

As we have seen, the qubit $\mathcal{A}$ contains an answer to the binary question: "Does the clock show it is earlier than $t = T$ or later than $t = 0$?". Therefore, it contains precisely 1 bit of information about the clock's time. By extending the system $\mathcal{A}$, we can devise a more sophisticated timer that can measure the clock time arbitrarily

well. We will discuss more general examples in the following sections.

2.3.1.2 Comment on the Type of Interaction

The interaction term in Eq. (2.24) allowed $\mathcal{A}$ to measure the clock. Importantly, this interaction did not result in a non-unitary evolution of the relative states. This is because it is a type of interaction that leads to a Hermitian, time-dependent effective Hamiltonian for the system $\mathcal{A}$ [43]. In fact, looking at Eq. (2.28), we can see that the relative states of $\mathcal{A}$ evolve according to the following Hamiltonian

$$\hat{H}_{\mathcal{A}}^{(eff)}(t) \coloneqq \theta(0 \leq t \leq T)\hat{H}_{\mathcal{A}}. \tag{2.38}$$

Such a Hamiltonian leads to a unitary time evolution for the relative states of $\mathcal{A}$. This is crucial to guarantee that the clock can be accessed while remaining within the boundaries of unitary quantum theory.

While one may object to the use of time-dependent Hamiltonians due to their less fundamental status, they are a convenient tool to represent the main dynamical features of quantum systems without modelling precisely the underlying physics. This is why they are widely used, for instance, in quantum circuits [49].

2.3.2 A More General Timer

The timer described in Sec. 2.3.1 is a simple device consisting of a single qubit. It can be made more sophisticated by including additional qubits, which allow it to measure clock time more accurately. Here we describe one way to do so. Let us assume that $\mathcal{A}$ is a composite system with internal degrees of freedom $\mathcal{A}_1$, which we are not interested in, and a part $\mathcal{A}_2$ consisting of N qubits that acts as a timer. We can factorise its Hilbert space accordingly: $\mathscr{H}_{\mathcal{A}} = \mathscr{H}_{\mathcal{A}_1} \otimes \mathscr{H}_{\mathcal{A}_2}$. Let us also assume

that the Hamiltonian of the Universe is given by

$$\hat{H}_U = \hat{H}_{\mathcal{A}_1} \otimes \hat{\mathbb{1}}_{\mathcal{A}_2} \otimes \hat{\mathbb{1}}_{\mathcal{C}} + \sum_{j=1}^{N-1} \hat{\mathbb{1}}_{\mathcal{A}_1} \otimes \hat{H}^{(j)}_{\mathcal{A}_2} \otimes \hat{P}_{[(j-1)\Delta t,(j-1)\Delta t+T]} + \hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}, \tag{2.39}$$

with $\hat{H}_{\mathcal{A}_1}$ the Hamiltonian of the internal degrees of freedom and $\Delta t \geq T$ is the timestep we decide to measure. $\mathcal{A}_2$ is initially in the state $|00\ldots01_{\mathcal{A}_1}\rangle$, a -1-eigenstate of $\hat{\sigma}^{(1)}_z$, and

$$\hat{H}^{(j)}_{\mathcal{A}_2} \coloneqq \frac{\pi}{2T}\left(\hat{\sigma}^{(j+1)}_x - \hat{\sigma}^{(j)}_x\right), \tag{2.40}$$

where $\hat{\sigma}^{(j)}_x$ is the Pauli operator of the j-th qubit of $\mathcal{A}_2$. With this definition, the effect of the interaction terms in Eq. (2.39) will be to "switch on" the j-th bit by the time $j\Delta t + T$ on the clock, signalling that the clock has passed the timestamp $j\Delta t$. The state of the Universe is given by

$$\begin{aligned}|\Psi_U\rangle\rangle \coloneqq \int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-i\hat{H}_{\mathcal{A}_1}t}|\psi_{\mathcal{A}_1}(0)\rangle \Big(&\theta(t<0)|0\ldots01_{\mathcal{A}_2}\rangle \\ &+\theta(0\leq t\leq T)e^{-i\hat{H}^{(1)}_{\mathcal{A}_2}t}|0\ldots01_{\mathcal{A}_2}\rangle + \theta(T<t<\Delta t)|0\ldots010_{\mathcal{A}_2}\rangle \\ +\theta(\Delta t\leq t\leq \Delta t+T)e^{-i\hat{H}^{(2)}_{\mathcal{A}_2}t}|0\ldots010_{\mathcal{A}_2}\rangle &+ \theta(\Delta t+T<t<2\Delta t)|0\ldots0100_{\mathcal{A}_2}\rangle \\ &+\cdots+\theta\left(t>(N-1)\Delta t+T\right)|10\ldots0_{\mathcal{A}_2}\rangle\Big)\,|\phi_{\mathcal{C}}(t)\rangle,\end{aligned} \tag{2.41}$$

which can be checked to be stationary following the procedure outlined in the previous section.

Effectively, we have constructed a timer which can record N timestamps between 0 and $(N-1)\Delta t$ with a maximum resolution of T. All these parameters can be changed at will by refining the timer. Note that our encoding is not efficient because we are using N qubits to measure N timestamps, while we could devise an interaction that uses all the 2^N orthogonal states of $\mathcal{A}_2$.

2.3.3 An Ideal Clock as a Timer

In this section, we show how an ideal clock can measure the time of another ideal clock. To that end, let us consider a Universe consisting of two ideal clocks, with Hamiltonian $\hat{H}_{\mathcal{C}_1}$ and $\hat{H}_{\mathcal{C}_2}$, respectively. Let us assume that the Hamiltonian of the model Universe is

$$\hat{H}_U = \hat{H}_{\mathcal{C}_1} \otimes \hat{P}_{[t_0,\infty]} + \hat{\mathbb{1}}_{\mathcal{C}_1} \otimes \hat{H}_{\mathcal{C}_2} \tag{2.42}$$

with

$$\hat{P}_{[t_0,\infty]} = \int_{t_0}^{\infty} \mathrm{d}t\, \hat{\Pi}_t, \tag{2.43}$$

the projector on the clock states of $\mathcal{C}_2$ with eigenvalues greater than or equal to t_0. Effectively, the interaction term in Eq. (2.42) "turns on" the Hamiltonian $\hat{H}_{\mathcal{C}_1}$. The result is that the two clocks will be ticking in lockstep from t_0 onwards. The following energy eigenstate of $\hat{H}_U$ reflects a history in which the two ideal clocks become synchronised:

$$|\Psi_U\rangle\rangle \coloneqq \int_{-\infty}^{+\infty} \mathrm{d}t\, \Big(\theta(t < t_0)|\phi_{\mathcal{C}_1}(0)\rangle + \theta(t \geq t_0)|\phi_{\mathcal{C}_1}(t)\rangle\Big)\, |\phi_{\mathcal{C}_2}(t)\rangle. \tag{2.44}$$

It can be easily verified, similarly to the previous sections, that $|\Psi_U\rangle\rangle$ is a 0-eigenstate of $\hat{H}_U$ since

$$\hat{H}_U\, |\Psi_U\rangle\rangle = i \int_{-\infty}^{+\infty} \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t} \left[\left(\theta(t < t_0)|\phi_{\mathcal{C}_1}(0)\rangle + \theta(t \geq t_0)|\phi_{\mathcal{C}_1}(t)\rangle\right) |\phi_{\mathcal{C}_2}(t)\rangle\right] = 0. \tag{2.45}$$

This scenario represents an idealisation of the previously analysed timers since, in this case, $\mathcal{C}_1$ and $\mathcal{C}_2$ can be synchronised instantly. Moreover, $\mathcal{C}_1$ has infinitely many distinguishable states $\{|\phi_{\mathcal{C}_1}(t)\rangle\}_t$, so $\mathcal{C}_1$ can record time arbitrarily well. In contrast,

the timers discussed above could only store a finite number of bits of information.

We can now think of letting $t_0 \to -\infty$ so that the synchronisation procedure occurs in the infinite past. In this case, the Hamiltonian of the Universe reduces to one without interactions since

$$\lim_{t_0 \to -\infty} \hat{H}_U = \hat{H}_{\mathcal{C}_1} \otimes \hat{\mathbb{1}}_{\mathcal{C}_2} + \hat{\mathbb{1}}_{\mathcal{C}_1} \otimes \hat{H}_{\mathcal{C}_2}, \tag{2.46}$$

due to

$$\hat{P}_{[-\infty,\infty]} = \hat{\mathbb{1}}_{\mathcal{C}_2}. \tag{2.47}$$

In this limit, the state of the Universe becomes

$$\lim_{t_0 \to -\infty} |\Psi_U\rangle\rangle = \int_{-\infty}^{+\infty} \mathrm{d}t \, |\phi_{\mathcal{C}_1}(t)\rangle |\phi_{\mathcal{C}_2}(t)\rangle. \tag{2.48}$$

We have thus recovered the standard non-interacting scenario as the limiting case of an interacting one where the clocks become synchronised in the infinite past. This should not be surprising because "turning on" the Hamiltonian of $\mathcal{C}_2$ in the infinite past is equivalent to $\mathcal{C}_2$ having a time-independent Hamiltonian. However, this is a consequence of the simple type of interactions discussed in these sections. A more realistic modelling of the measuring process would likely leave a trace of the interaction between the two clocks at the level of the observables.

As a final remark, note that, in the scenario just discussed, the two clocks appear synchronised after the time t_0. Usually, the synchronisation process consists of setting the counter of two clocks (that intrinsically tick with the same speed in the same frame of reference) to the same value. After this, the two clocks will remain synchronised as long as they are in the same reference frame. Here, the situation is slightly different as the first clock perpetually records the time on the

second one rather than interacting with it only once. We leave the study of how the synchronisation process can be implemented in timeless quantum theory open for future investigation.

2.3.4 Instantaneous Measurements

In Sec. 2.3.1.1, we said that there are no limits on how fast the clock can be measured. As a result, the measurement can indefinitely *approach* an ideal instantaneous measurement. Here, we show that we can also mathematically implement truly instantaneous measurements in timeless quantum theory.

Let us consider a qubit $\mathcal{A}$, a clock $\mathcal{C}$ and a (non-Hermitian) Hamiltonian of the Universe

$$\hat{H}_U = i\left(\hat{1}_{\mathcal{A}} - \hat{G}^{\dagger}_{\mathcal{A}}\right) \otimes \hat{\Pi}_{t^*} + \hat{1}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}, \tag{2.49}$$

where $\hat{1}_{\mathcal{A}}$ is the identity on $\mathcal{A}$, $\hat{G}_{\mathcal{A}}$ is a gate (unitary) on the qubit, and t^* is the time at which the gate is to be applied. The zero-eigenstate of $\hat{H}_U$ that describes the application of the gate $\hat{G}$ at time t^* is

$$|\Psi_U\rangle\rangle = \int_{-\infty}^{+\infty} \mathrm{d}t \left(\theta(t < t^*)|0_{\mathcal{A}}\rangle + \theta(t \geq t^*)\hat{G}_{\mathcal{A}}|0_{\mathcal{A}}\rangle\right)|\phi_{\mathcal{C}}(t)\rangle. \tag{2.50}$$

If $\hat{G}_{\mathcal{A}}$ is, for example, a NOT gate, the qubit has effectively measured the state $|\phi_{\mathcal{C}}(t^*)\rangle$ of the clock instantaneously.

Let us check that $|\Psi_U\rangle\rangle$ is actually an eigenstate of $\hat{H}_U$. The interaction term acts on $|\Psi_U\rangle\rangle$ in the following way:

$$\begin{aligned} i\left[\left(\hat{1}_{\mathcal{A}} - \hat{G}^{\dagger}_{\mathcal{A}}\right) \otimes \hat{\Pi}_{t^*}\right]|\Psi_U\rangle\rangle &= i\left(\hat{G}_{\mathcal{A}} - \mathbb{1}_{\mathcal{A}}\right)|0_{\mathcal{A}}\rangle|\phi_{\mathcal{C}}(t^*)\rangle \\ = i\int_{-\infty}^{+\infty} \mathrm{d}t \,\frac{\mathrm{d}}{\mathrm{d}t}\left(\theta(t < t^*)|0\rangle + \theta(t \geq t^*)\hat{G}_{\mathcal{A}}|0_{\mathcal{A}}\rangle\right)|\phi_{\mathcal{C}}(t)\rangle, \end{aligned} \tag{2.51}$$

and, therefore,

$$\hat{H}_U \left|\Psi\right\rangle\rangle = i \int_{-\infty}^{+\infty} \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t} \left[\left(\theta(t < t^*) |0_{\mathcal{A}}\rangle + \theta(t \geq t^*) \hat{G}_{\mathcal{A}} |0_{\mathcal{A}}\rangle \right) |\phi_{\mathcal{C}}(t)\rangle \right] = 0, \quad (2.52)$$

where the last equality follows from the discussion on rigged Hilbert spaces in Appendix A

We have thus implemented an instantaneous measurement in timeless quantum theory. The price we had to pay is having introduced a *non-Hermitian* interaction term in the Hamiltonian of the Universe:

$$i \left(\hat{1}_{\mathcal{A}} - \hat{G}_{\mathcal{A}}^{\dagger} \right) \neq \left[i \left(\hat{1}_{\mathcal{A}} - \hat{G}_{\mathcal{A}}^{\dagger} \right) \right]^{\dagger}, \quad (2.53)$$

unless $(\hat{G}_{\mathcal{A}} + \hat{G}_{\mathcal{A}}^{\dagger})/2 = \hat{1}_{\mathcal{A}}$ which implies that $(\hat{G}_{\mathcal{A}} - \hat{1}_{\mathcal{A}})^2 = 0$ and thus $\hat{G}_{\mathcal{A}} = \hat{1}_{\mathcal{A}}$ (i.e., no gate is applied). The non-Hermiticity is linked to the fact that an instantaneous change cannot be described as a continuous unitary evolution with a Hermitian Hamiltonian.

Finally, if instead of the qubit $\mathcal{A}$ we consider another ideal clock $\mathcal{C}_2$ with a pair of canonically conjugate observables $(\hat{T}_{\mathcal{C}_2}, \hat{H}_{\mathcal{C}_2})$ and starting in the state $|\phi_{\mathcal{C}_2}(t')\rangle$, any change under the unitary $\hat{G}_{\mathcal{C}_2} = e^{-i\hat{H}_{\mathcal{C}_2}\delta t}$ will result in a distinguishable state $|\phi_{\mathcal{C}_2}(t' + \delta t)\rangle = e^{-i\hat{H}_{\mathcal{C}_2}\delta t} |\phi_{\mathcal{C}_2}(t')\rangle$. If $\delta t \ll 1$, the interaction term is well approximated by

$$i \left(\hat{1}_{\mathcal{C}_2} - \hat{G}_{\mathcal{C}_2}^{\dagger} \right) = \hat{H}_{\mathcal{C}_2} \delta t + O(\delta t^2), \quad (2.54)$$

which is, to first order in δt, self-adjoint. Within this approximation, an ideal clock can thus measure the time of another ideal clock instantaneously via self-adjoint Hamiltonians.

Chapter 3

The Role of Correlations

The contents of this chapter are based on [107].

In Sec. 1.2.5, we have seen that the correlations between $\mathcal{A}$ and $\mathcal{C}$ are crucial for recovering a non-trivial relational evolution. In this chapter, we further explore the nature and the role of these correlations. We start by studying a closely related problem: how do we formulate the relational approach to timeless quantum theory in the Heisenberg picture? We show that the solution to this problem involves "transferring" the $\mathcal{A}-\mathcal{C}$ correlations from the state vector of the Universe to its observables. In turn, this will help us characterise the states of the Universe that result in non-trivial relational time evolutions. Finally, we discuss a formalism that condenses the results of this chapter.

3.1 Heisenberg Picture

The construction outlined in Chapter 1 was cast in the Schrödinger picture since it was concerned only with the relational evolution of the state vectors of U or $\mathcal{A}$. Although the Schrödinger picture (SP) and the Heisenberg picture (HP) are empirically equivalent, their description of physical reality is different, especially

with regard to locality. In fact, the HP provides a local description of quantum theory [108, 109, 110, 111].[1] Moreover, in some cases, the HP description may be the only one available [112, 113]. It is thus desirable to be able to cast timeless quantum theory in the HP.

One could think of recovering the Heisenberg picture relational evolution of U ($\mathcal{A}$) by simply shifting the action of the unitary in Eq. (1.6) ((1.11)) from the state vector to the observables. However, this makes the HP subordinate to the SP since one has to first find the SP relational evolution in order to move the HP. Therefore, we need to find the HP of timeless quantum theory at the level of the Universe before considering the evolution relative to the clock. We are now faced with another problem. The Universe is stationary, so there is no apparent distinction between SP and HP at this level. Does this mean that the HP does not exist at the level of the Universe?

Some works in the literature shed light on the problem through different approaches [45, 70, 68, 69, 72, 36, 74, 114]. In this section, we show how to formulate timeless quantum theory in the HP at the level of the Universe by performing a *unitary* transformation on both the state vector and the observables of the Universe, which transfers crucial system-clock correlations from the former to the latter. The approach and interpretation we outline here differ from [45, 36, 74], where the Heisenberg observables are recovered from other axioms and the choice of Heisenberg state is motivated differently. They also differ from [68, 69, 70, 72], which generally start from invariant observables (i.e. observables that commute with a constraint operator), and from [114], which postulates that the HP is defined by observables of the system that obey a generalised Heisenberg equation.[2]

[1] Since the Universe in timeless quantum theory is stationary, here locality would refer to a local description *relative to the time of the clock*.

[2] In Ref. [114] the author requires that the Hamiltonian of the Universe be equal to the Hamiltonian of the clock to prevent a real-valued time parameter from appearing in the equations of motion for the system's observables. Taken at face value, this requirement implies that the Hamiltonian of the clock must be identically zero. A correct interpretation of this statement,

Here, starting from the assumption that the Universe is in a stationary state, we recover the HP description of the whole Universe via a unitary transformation on both the observables and the state vector. The observables and Heisenberg state of [45, 36] are recovered in a simple and well-motivated way; the assumptions of [114] find a natural explanation; and the invariant observables appearing in [45, 68, 69, 36, 70, 72, 74] can be recovered via a group-average operation on the HP-observables of the system.[3] Moreover, we extend the analysis to include mixed states of the Universe and, in a later chapter, interactions with the clock.

In the following sections, we will consider $\mathcal{C}$ to be an ideal clock with an associated time operator as in Eq. (1.35).

3.1.1 Pure States of the Universe

Let us consider again the eigenstate of the non-interacting Hamiltonian of Eq. (1.2) given by Eq. (1.47). We can rewrite this state in the following way [42]:

$$|\Psi_U\rangle\rangle = \int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-i\hat{H}_{\mathcal{A}}t}|\psi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(t)\rangle = e^{-i\hat{H}_{\mathcal{A}}\otimes\hat{T}_{\mathcal{C}}}|\psi_{\mathcal{A}}(0)\rangle|\mathrm{TL}_{\mathcal{C}}\rangle, \tag{3.1}$$

where $\hat{T}_{\mathcal{C}}$ is the time operator of the clock as in Eq. (1.35) and the "Time Line" state $|\mathrm{TL}_{\mathcal{C}}\rangle$ is defined as $|\mathrm{TL}_{\mathcal{C}}\rangle \coloneqq \int_{-\infty}^{+\infty} \mathrm{d}t\, |\phi_{\mathcal{C}}(t)\rangle = \sqrt{2\pi}|0_{\mathcal{C}}\rangle$, with $|0_{\mathcal{C}}\rangle$ the 0-eigenstate of $\hat{H}_{\mathcal{C}}$ [42].

Now, let us consider the unitary

$$\hat{\mathbb{U}} \coloneqq e^{-i\hat{H}_{\mathcal{A}}\otimes\hat{T}_{\mathcal{C}}}, \tag{3.2}$$

which is difficult to formulate in the framework of [114], is that the Hamiltonian of the Universe in the HP must be equal to the Hamiltonian of the clock in the SP. This property finds a natural explanation in the approach described below.

[3] Note that the time measurements in [45, 36] are represented, in general, by a POVM while we restrict to projective measurements as discussed in the previous section. My results are in agreement with [45, 36] for this specific type of time measurements.

appearing in Eq. (3.1). This unitary is, apart from a phase and a constant, the same as the "trivialisation map" of [36], and similar to a "quantum reference frame transformation" as in [76]. We propose that we can recover the HP formulation of timeless quantum theory using this unitary. To do so, we transform the SP observables of the Universe $\hat{O}$ in the following way

$$\hat{O}^{(\mathbb{H})} = e^{i\hat{H}_{\mathcal{A}}\otimes\hat{T}_{\mathcal{C}}}\hat{O}e^{-i\hat{H}_{\mathcal{A}}\otimes\hat{T}_{\mathcal{C}}}. \tag{3.3}$$

In the specific case of the observables $\hat{O}_{\mathcal{A}}$ of $\mathcal{A}$, the equation above reads

$$\hat{O}^{(\mathbb{H})}_{\mathcal{A}} = e^{i\hat{H}_{\mathcal{A}}\otimes\hat{T}_{\mathcal{C}}}\left(\hat{O}_{\mathcal{A}}\otimes\hat{\mathbb{1}}_{\mathcal{C}}\right)e^{-i\hat{H}_{\mathcal{A}}\otimes\hat{T}_{\mathcal{C}}}, \tag{3.4}$$

which is an observable depending on the clock's time operator $\hat{T}_{\mathcal{C}}$ and has support on $\mathscr{H}_U$. The observables $\hat{H}_{\mathcal{A}}$, $\hat{H}_{\mathcal{C}}$ and $\hat{T}_{\mathcal{C}}$ transform in the following way

$$\hat{H}^{(\mathbb{H})}_{\mathcal{A}} = \hat{\mathbb{U}}^{\dagger}\left(\hat{H}_{\mathcal{A}}\otimes\hat{\mathbb{1}}_{\mathcal{C}}\right)\hat{\mathbb{U}} = \hat{H}_{\mathcal{A}}\otimes\hat{\mathbb{1}}_{\mathcal{C}}, \tag{3.5}$$

$$\hat{H}^{(\mathbb{H})}_{\mathcal{C}} = \hat{\mathbb{U}}^{\dagger}\left(\hat{\mathbb{1}}_{\mathcal{A}}\otimes\hat{H}_{\mathcal{C}}\right)\hat{\mathbb{U}} = \hat{\mathbb{1}}_{\mathcal{A}}\otimes\hat{H}_{\mathcal{C}} - \hat{H}_{\mathcal{A}}\otimes\hat{\mathbb{1}}_{\mathcal{C}}, \tag{3.6}$$

$$\hat{T}^{(\mathbb{H})}_{\mathcal{C}} = \hat{\mathbb{U}}^{\dagger}\left(\hat{\mathbb{1}}_{\mathcal{A}}\otimes\hat{T}_{\mathcal{C}}\right)\hat{\mathbb{U}} = \hat{\mathbb{1}}_{\mathcal{A}}\otimes\hat{T}_{\mathcal{C}}, \tag{3.7}$$

which show that the Hamiltonian of $\mathcal{A}$ and the time operator are invariant under $\hat{\mathbb{U}}$, while the Hamiltonian of the clock is not.[4] However, the Hamiltonian of the Universe is not invariant:

$$\hat{H}^{(\mathbb{H})}_{U} = \hat{\mathbb{U}}^{\dagger}\hat{H}_{U}\hat{\mathbb{U}} = \hat{\mathbb{1}}_{\mathcal{A}}\otimes\hat{H}_{\mathcal{C}}. \tag{3.8}$$

[4]Since the transformation between the SP and the HP is unitary, the relation $\left[\hat{H}^{(\mathbb{H})}_{\mathcal{C}}, \hat{O}^{(\mathbb{H})}_{\mathcal{A}}\right] = \left[\hat{H}_{\mathcal{C}} - \hat{H}_{\mathcal{A}}, \hat{O}^{(\mathbb{H})}_{\mathcal{A}}\right] = 0$ holds for every observable of the system $\hat{O}^{(\mathbb{H})}_{\mathcal{A}}$. This commutation relation is the initial assumption of [114].

This is because here $\hat{\mathbb{U}}$ is a *formal* transformation between the two pictures and not a time translation generated by $\hat{H}_U$ (which usually defines the transformation between the two pictures).

For consistency with the SP, the Heisenberg state of the Universe must be

$$\left|\Psi_U^{(\mathbb{H})}\right\rangle\!\!\right\rangle := \hat{\mathbb{U}}^\dagger \left|\Psi_U\right\rangle\rangle = |\psi_{\mathcal{A}}(0)\rangle|\mathrm{TL}_{\mathcal{C}}\rangle, \tag{3.9}$$

which is a *separable* state with respect to $\mathcal{A}$ and $\mathcal{C}$. The state $\left|\Psi_U^{(\mathbb{H})}\right\rangle\!\!\right\rangle$ still satisfies the stationarity constraint $\hat{H}_U^{(\mathbb{H})}\left|\Psi_U^{(\mathbb{H})}\right\rangle\!\!\right\rangle = 0$ but this is now entirely due to the part of the state living in the clock's Hilbert space: $\hat{H}_{\mathcal{C}}|\mathrm{TL}_{\mathcal{C}}\rangle = 0$. Incidentally, this also shows that one is free to choose any initial state of $\mathcal{A}$, $|\psi_{\mathcal{A}}(0)\rangle$, while still preserving the constraint $\hat{H}_U^{(\mathbb{H})}\left|\Psi_U^{(\mathbb{H})}\right\rangle\!\!\right\rangle = 0$. This is similar to the freedom in the choice of the initial state of the Universe discussed in Sec. 2.1.

If we used the state $\left|\Psi_U^{(\mathbb{H})}\right\rangle\!\!\right\rangle$ to recover the relational evolution of $\mathcal{A}$ in the Schrödinger picture, we would find that it is trivial, i.e., the relative state of $\mathcal{A}$ is stationary. Therefore, the unitary transformation $\hat{\mathbb{U}}$ has transferred the time evolution "encoded" in the SP stationary state of the Universe to the HP observables by transferring the correlations (entanglement) from the former to the latter.

The Heisenberg operators $\hat{O}^{(\mathbb{H})}$ and the Heisenberg state $\left|\Psi_U^{(\mathbb{H})}\right\rangle\!\!\right\rangle$ are all one needs to recover the relational time evolution of the observables of $\mathcal{A}$ and their expectation values. First, note that $\hat{O}_{\mathcal{A}}^{(\mathbb{H})}$ contains the "whole history" of $\hat{O}_{\mathcal{A}}$:

$$\left\langle\!\!\left\langle \hat{O}_{\mathcal{A}}^{(\mathbb{H})} \right\rangle\!\!\right\rangle = \int_{-\infty}^{+\infty} \mathrm{d}t\, \langle\psi_{\mathcal{A}}(0)|e^{i\hat{H}_{\mathcal{A}}t}\,\hat{O}_{\mathcal{A}}\,e^{i\hat{H}_{\mathcal{A}}t}|\psi_{\mathcal{A}}(0)\rangle, \tag{3.10}$$

where we have defined $\langle\langle\cdot\rangle\rangle := \left\langle\!\!\left\langle \Psi_U^{(\mathbb{H})}\right| \cdot \left|\Psi_U^{(\mathbb{H})}\right\rangle\!\!\right\rangle$. In order to get one time instance of $\hat{O}_{\mathcal{A}}^{(\mathbb{H})}$, let us consider the *improper* projectors $\hat{\Pi}_t(\hat{T}_{\mathcal{C}}) = \hat{\mathbb{1}}_{\mathcal{A}} \otimes |\phi_{\mathcal{C}}(t)\rangle\langle\phi_{\mathcal{C}}(t)|$ on the eigenspace of $\hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{T}_{\mathcal{C}}$ with eigenvalue t. We have made explicit the dependence of the projectors on the time operator of the clock. Now consider the following

"relative observables"

$$\hat{O}^{(\mathbb{H})}_{\mathcal{A}}(t) \coloneqq \hat{O}^{(\mathbb{H})}_{\mathcal{A}}\hat{\Pi}^{(\mathbb{H})}_{t}(\hat{T}_{\mathcal{C}}), \tag{3.11}$$

where $\hat{\Pi}^{(\mathbb{H})}_{t}(\hat{T}_{\mathcal{C}}) \coloneqq \hat{\mathbb{U}}^{\dagger}\hat{\Pi}_{t}(\hat{T}_{\mathcal{C}})\hat{\mathbb{U}} = \hat{\Pi}_{t}(\hat{T}^{(\mathbb{H})}_{\mathcal{C}})$. Also, note that Eq. (3.7) implies that $\hat{\Pi}^{(\mathbb{H})}_{t}(\hat{T}_{\mathcal{C}}) = \hat{\Pi}_{t}(\hat{T}_{\mathcal{C}})$. Eq. (3.11) describes the observables of $\mathcal{A}$ relative to the clock being in an eigenstate of $\hat{T}_{\mathcal{C}}$ with eigenvalue t. Similar relative observables were defined in [67] in the context of a measurement process described in the Heisenberg picture.

The relative observables form an "improper relative sub-algebra" at each time t:

$$\left(\hat{O}^{(\mathbb{H})}_{\mathcal{A}}\hat{\Pi}^{(\mathbb{H})}_{t}(\hat{T}_{\mathcal{C}})\right)^{2} = \left(\hat{O}^{(\mathbb{H})}_{\mathcal{A}}\right)^{2}\hat{\Pi}^{(\mathbb{H})}_{t}(\hat{T}_{\mathcal{C}})\delta(0), \tag{3.12}$$

$$\left[\hat{O}^{(\mathbb{H})}_{\mathcal{A}}\hat{\Pi}^{(\mathbb{H})}_{t}(\hat{T}_{\mathcal{C}}), \hat{Q}^{(\mathbb{H})}_{\mathcal{A}}\hat{\Pi}^{(\mathbb{H})}_{t}(\hat{T}_{\mathcal{C}})\right] = \left[\hat{O}^{(\mathbb{H})}_{\mathcal{A}}, \hat{Q}^{(\mathbb{H})}_{\mathcal{A}}\right]\hat{\Pi}^{(\mathbb{H})}_{t}(\hat{T}_{\mathcal{C}})\delta(0), \tag{3.13}$$

for all the Heisenberg-picture observables $\hat{O}^{(\mathbb{H})}_{\mathcal{A}}, \hat{Q}^{(\mathbb{H})}_{\mathcal{A}}$ of $\mathcal{A}$. We defined this sub-algebra *improper* because the projectors $\hat{\Pi}^{(\mathbb{H})}_{t}(\hat{T}_{\mathcal{C}})$ are improper since $\hat{\Pi}^{(\mathbb{H})}_{t}(\hat{T}_{\mathcal{C}})^{2} = \hat{\Pi}^{(\mathbb{H})}_{t}(\hat{T}_{\mathcal{C}})\delta(0)$.

Now, the expectation value of $\hat{O}^{(\mathbb{H})}_{\mathcal{A}}$ relative to the clock reading the time t is given by

$$\left\langle\left\langle \hat{O}^{(\mathbb{H})}_{\mathcal{A}}(t)\right\rangle\right\rangle = \langle\psi_{\mathcal{A}}(0)|e^{i\hat{H}_{\mathcal{A}}t}\hat{O}_{\mathcal{A}}e^{-i\hat{H}_{\mathcal{A}}t}|\psi_{\mathcal{A}}(0)\rangle, \tag{3.14}$$

in agreement with what one would have obtained by taking the expectation value of $\hat{O}_{\mathcal{A}}$ in the SP with the relative state of $\mathcal{A}$ at time t given in Eq. (1.11).

Since the Heisenberg state of the Universe is separable with respect to $\mathcal{A}$ and $\mathcal{C}$, one can find a "reduced description" of $\hat{O}^{(\mathbb{H})}_{\mathcal{A}}(t)$ on the Hilbert space of the system given by

$$\left[\hat{O}^{(\mathbb{H})}_{\mathcal{A}}(t)\right]_{\mathcal{A}} \coloneqq \langle \mathrm{TL}_{\mathcal{C}}|\hat{O}^{(\mathbb{H})}_{\mathcal{A}}(t)|\mathrm{TL}_{\mathcal{C}}\rangle = e^{i\hat{H}_{\mathcal{A}}t}\hat{O}_{\mathcal{A}}e^{-i\hat{H}_{\mathcal{A}}t}. \tag{3.15}$$

These are the usual HP observables of a system with Hamiltonian $\hat{H}_{\mathcal{A}}$. We have thus successfully recovered the usual HP evolution of $\mathcal{A}$'s observables from an overall

stationary Universe.

3.1.1.1 Uniqueness of the Transformation

The way to transform between the SP and the HP was suggested by how we wrote the state of the Universe in Eq. (3.1). While the unitary that we used led to recovering the correct HP evolution of $\mathcal{A}$'s observables, one may question whether this transformation is unique. To answer this question, consider a generic unitary transformation $\hat{\mathbb{U}}'$ between the two pictures and the associated Heisenberg state of the Universe $\left|\Psi'_U{}^{(\mathbb{H})}\right\rangle\!\rangle := \hat{\mathbb{U}}'^{\dagger}\left|\Psi_U\right\rangle\!\rangle$. We postulate that a *good* transformation between the SP and the HP is defined by the following properties:

1. $\hat{\mathbb{U}}'$ is such that

$$\hat{\mathbb{U}}'^{\dagger}\left(\hat{\mathbb{1}}_{\mathcal{A}}\otimes\hat{T}_{\mathcal{C}}\right)\hat{\mathbb{U}}'=\hat{\mathbb{1}}_{\mathcal{A}}\otimes\hat{T}'_{\mathcal{C}}, \tag{3.16}$$

   with $\hat{T}'_{\mathcal{C}}$ a self-adjoint operator of the clock, so that the notion of partial inner product and partial trace are still well-defined. In turn, this will be crucial to recover the reduced observables of $\mathcal{A}$ as in Eq. (3.15).

2. The transformation between the SP and the HP transfers all the "encoded time evolution" from the state of the Universe to the observables. Specifically, $\left|\Psi'^{(\mathbb{H})}_U\right\rangle\!\rangle$ is such that

$$\left\langle\!\left\langle\phi'_{\mathcal{C}}(t')\right|\Psi'_U{}^{(\mathbb{H})}\right\rangle\!\rangle=e^{i\phi(t,t')}\left\langle\!\left\langle\phi'_{\mathcal{C}}(t)\right|\Psi'_U{}^{(\mathbb{H})}\right\rangle\!\rangle\quad\forall\,t,t', \tag{3.17}$$

   where $|\phi'_{\mathcal{C}}(t)\rangle$ is an eigenstate of $\hat{T}'_{\mathcal{C}}$ with eigenvalue t and $\phi(t,t')$ is a (real) phase. This requirement implies that the state of the Universe will act as a Heisenberg state (that is, a state fixed in time) and thus guarantee that we are working in the HP.

We will now see that the unitary transformation between the SP and the HP satis-

fying these properties is unique apart from local unitary transformations.[5]

Theorem 3.1. *A unitary $\hat{\mathbb{U}}'$ satisfies Eqs. (3.16) and (3.17) if and only if it is related to $\hat{\mathbb{U}} = e^{-i\hat{H}_{\mathcal{A}}\otimes\hat{T}_{\mathcal{C}}}$ via local unitary transformations.*

Proof. It is trivial to show that any unitary related to $\hat{\mathbb{U}}$ via local unitary transformations satisfies the two requirements above.

To prove the other direction, let us define the unitary $\hat{\mathbb{W}} \coloneqq \hat{\mathbb{U}}'^{\dagger}\hat{\mathbb{U}}$, which is such that $\left|\Psi'^{(\mathbb{H})}_{U}\right\rangle\!\!\right\rangle \coloneqq \hat{\mathbb{W}}\left|\Psi^{(\mathbb{H})}_{U}\right\rangle\!\!\right\rangle$. For the moment, let us assume that $\hat{\mathbb{U}}'$ leaves $\hat{\mathbb{1}}_{\mathcal{A}}\otimes\hat{T}_{\mathcal{C}}$ invariant, that is $\left[\hat{\mathbb{U}}', \hat{\mathbb{1}}_{\mathcal{A}}\otimes\hat{T}_{\mathcal{C}}\right] = 0$. This means that $\hat{\mathbb{W}}$ does the same and thus shares a complete set of eigenstates with $\hat{\mathbb{1}}_{\mathcal{A}}\otimes\hat{T}_{\mathcal{C}}$. The eigenstates must be of the form $\{|w_j(t)\rangle|\phi_{\mathcal{C}}(t)\rangle\}_{j,t}$ with eigenvalues $w_j(t)$. For simplicity, let us write $\hat{\mathbb{W}} = \int_{-\infty}^{+\infty} \mathrm{d}t\, \hat{W}(t)\otimes|\phi_{\mathcal{C}}(t)\rangle\langle\phi_{\mathcal{C}}(t)|$, where $\hat{W}(t) \coloneqq \sum_j w_j(t)|w_j(t)\rangle\langle w_j(t)|$ is a t-dependent unitary. The condition of Eq. (3.17) implies that[6]

$$\hat{W}(t')\left\langle\phi_{\mathcal{C}}(t')\left|\Psi^{(\mathbb{H})}_{U}\right\rangle\!\!\right\rangle = e^{i\phi(t,t')}\hat{W}(t)\left\langle\phi_{\mathcal{C}}(t)\left|\Psi^{(\mathbb{H})}_{U}\right\rangle\!\!\right\rangle \quad \forall\, t, t', \tag{3.18}$$

where we have used $\left|\Psi'^{(\mathbb{H})}_{U}\right\rangle\!\!\right\rangle = \hat{\mathbb{W}}\left|\Psi^{(\mathbb{H})}_{U}\right\rangle\!\!\right\rangle$. Since $\left\langle\phi_{\mathcal{C}}(t')\left|\Psi^{(\mathbb{H})}_{U}\right\rangle\!\!\right\rangle = \left\langle\phi_{\mathcal{C}}(t)\left|\Psi^{(\mathbb{H})}_{U}\right\rangle\!\!\right\rangle = |\psi_{\mathcal{A}}(0)\rangle\ \forall\, t, t'$ (see Eq. (3.9)) we have that

$$\hat{W}(t')|\psi_{\mathcal{A}}(0)\rangle = e^{i\phi(t,t')}\hat{W}(t)|\psi_{\mathcal{A}}(0)\rangle \quad \forall\, t, t'. \tag{3.19}$$

If we also require this to hold for any $|\psi_{\mathcal{A}}(0)\rangle$, then $\hat{W}(t') = e^{i\phi(t,t')}\hat{W}(t)\ \forall\, t, t'$ and we can write

$$\hat{\mathbb{W}} = \hat{W}(0)\otimes\int \mathrm{d}t\, e^{i\phi(0,t)}|\phi_{\mathcal{C}}(t)\rangle\langle\phi_{\mathcal{C}}(t)|, \tag{3.20}$$

which is a local unitary operator.

What if, more in general, $\hat{\mathbb{U}}'^{\dagger}\left(\hat{\mathbb{1}}_{\mathcal{A}}\otimes\hat{T}_{\mathcal{C}}\right)\hat{\mathbb{U}}' = \hat{\mathbb{1}}_{\mathcal{A}}\otimes\hat{T}'_{\mathcal{C}}$? In this case, it is always

[5] With "local unitary" we mean any unitary of the form $\hat{W} = \hat{W}_{\mathcal{A}}\otimes\hat{W}_{\mathcal{C}}$.

[6] Note that we are using the states $|\phi_{\mathcal{C}}(t)\rangle$ because $\hat{\mathbb{1}}_{\mathcal{A}}\otimes\hat{T}_{\mathcal{C}}$ is invariant under $\hat{\mathbb{U}}'$.

possible to find a local unitary of the clock $\hat{V}$ such that $\hat{V}^\dagger \left(\hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{T}'_{\mathcal{C}}\right) \hat{V} = \hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{T}_{\mathcal{C}}$. Let us then consider the unitary $\hat{\mathbb{U}}'\hat{V}$ and the state $\left|\Psi''^{(\mathbb{H})}_U\right\rangle\!\rangle \coloneqq \hat{V}^\dagger \hat{\mathbb{U}}'^\dagger \left|\Psi_U\right\rangle\!\rangle$. We have that

$$\hat{V}^\dagger \hat{\mathbb{U}}'^\dagger \left(\hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{T}_{\mathcal{C}}\right) \hat{\mathbb{U}}'\hat{V} = \hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{T}_{\mathcal{C}}, \tag{3.21}$$

and, due to Eq. (3.17),

$$\left\langle\!\left\langle \phi_{\mathcal{C}}(t') \left| \Psi''^{(\mathbb{H})}_U \right\rangle\!\right\rangle = e^{i\phi(t,t')} \left\langle\!\left\langle \phi_{\mathcal{C}}(t) \left| \Psi''^{(\mathbb{H})}_U \right\rangle\!\right\rangle \quad \forall\, t, t', \tag{3.22}$$

where have used the fact that $\hat{V}^\dagger |\phi'_{\mathcal{C}}(t)\rangle = |\phi_{\mathcal{C}}(t)\rangle$. Therefore, applying the same reasoning as before, $\hat{\mathbb{U}}'\hat{V}$ must be a local unitary. This means that $\hat{\mathbb{U}}'$ must be a local unitary because it is the composition of local unitaries. □

3.1.1.2 Gauge Invariance

Let us now discuss some additional features of the construction. First, note that there is what one may call a gauge freedom. In the HP, the unitaries

$$\hat{\mathcal{U}}(\theta) \coloneqq \hat{\mathbb{1}}_{\mathcal{A}} \otimes e^{-i\hat{H}_{\mathcal{C}}\theta} \tag{3.23}$$

are gauge transformations since

$$\hat{\mathcal{U}}(\theta) \left|\Psi^{(\mathbb{H})}_U\right\rangle\!\rangle = \left|\Psi^{(\mathbb{H})}_U\right\rangle\!\rangle \quad \forall\, \theta \in \mathbb{R}. \tag{3.24}$$

Under $\hat{\mathcal{U}}(\theta)$, the time operator $\hat{T}_{\mathcal{C}}$ undergoes a time translation:

$$\hat{\mathcal{U}}^\dagger(\theta) \left(\hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{T}_{\mathcal{C}}\right) \hat{\mathcal{U}}(\theta) = \hat{\mathbb{1}}_{\mathcal{A}} \otimes \left(\hat{T}_{\mathcal{C}} + \theta \hat{\mathbb{1}}_{\mathcal{C}}\right). \tag{3.25}$$

All the HP observables of $\mathcal{A}$ related to $\hat{O}^{(\mathbb{H})}_{\mathcal{A}}$ by $\hat{\mathcal{U}}(\theta)$ can be written as

$$\hat{\mathcal{U}}^{\dagger}(\theta)\hat{O}^{(\mathbb{H})}_{\mathcal{A}}\hat{\mathcal{U}}(\theta) = e^{i\hat{H}_{\mathcal{A}}\otimes\left(\hat{T}_{\mathcal{C}}+\theta\hat{\mathbb{1}}_{\mathcal{C}}\right)}\hat{O}_{\mathcal{A}}e^{-i\hat{H}_{\mathcal{A}}\otimes\left(\hat{T}_{\mathcal{C}}+\theta\hat{\mathbb{1}}_{\mathcal{C}}\right)}. \tag{3.26}$$

They form an equivalence class because all their expectation values do not depend on θ.

For a general system, these new observables can be thought of as describing a different physical situation even if all the expectation values are the same [109]. The difference lies in the different dynamics described by the old and the new observables, and it can be detected by "starting" with another Heisenberg state that is not an eigenstate of $\hat{\mathcal{U}}(\theta)$. However, the PW construction does not allow states that do not satisfy the stationarity condition, so this difference is impossible to detect.

This impossibility may not apply to other unitary transformations that have $\left|\Psi^{(\mathbb{H})}\right\rangle\!\rangle$ as an eigenstate (for instance, a local transformation on the system). Still, it is unclear what it means to start with a different Heisenberg state of the Universe and, thus, whether it is possible to detect the difference between the new and the old observables, even in principle. We leave the questions regarding this possibility open for future discussion.

Now, if we consider the relative observables of Eq. (3.11), we can take their so-called "group average" [115]:

$$\hat{\mathcal{O}}^{(\mathbb{H})}_{\mathcal{A}}(t) \coloneqq \int_{-\infty}^{+\infty} \mathrm{d}\theta\, \hat{\mathcal{U}}^{\dagger}(\theta)\left[\hat{O}^{(\mathbb{H})}_{\mathcal{A}}\hat{\Pi}^{(\mathbb{H})}_{t}(\hat{T}_{\mathcal{C}})\right]\hat{\mathcal{U}}(\theta). \tag{3.27}$$

These observables commute with $\hat{\mathbb{1}}_{\mathcal{A}}\otimes\hat{H}_{\mathcal{C}} = \hat{H}^{(\mathbb{H})}_{U}$. They are the HP version of the so-called "Dirac observables" found in, for example, [36]. It is easy to see that

$$\hat{\mathcal{O}}^{(\mathbb{H})}_{\mathcal{A}}(t) = e^{i\hat{H}_{\mathcal{A}}t}\hat{O}_{\mathcal{A}}e^{-i\hat{H}_{\mathcal{A}}t}\otimes\hat{\mathbb{1}}_{\mathcal{C}}, \tag{3.28}$$

which are the usual HP observables of $\mathcal{A}$ at time t.

3.1.1.3 Non-zero Energy Eigenstates

If $\big|\Psi_U^{\mathcal{E}}\big\rangle\big\rangle$ is an eigenstates of $\hat{H}_U$ with eigenvalue $\mathcal{E}$, we can write it as

$$\big|\Psi_U^{\mathcal{E}}\big\rangle\big\rangle = \int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-i\left(\hat{H}_{\mathcal{A}}-\mathcal{E}\right)t}|\psi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(t)\rangle = e^{-i\hat{H}_{\mathcal{A}}\otimes\hat{T}_{\mathcal{C}}}|\psi_{\mathcal{A}}(0)\rangle\big|\mathrm{TL}_{\mathcal{C}}^{\mathcal{E}}\big\rangle, \tag{3.29}$$

where $\big|\mathrm{TL}_{\mathcal{C}}^{\mathcal{E}}\big\rangle \coloneqq \int_{-\infty}^{+\infty} \mathrm{d}t\, e^{i\mathcal{E}t}|\phi_{\mathcal{C}}(t)\rangle = \sqrt{2\pi}|\mathcal{E}_{\mathcal{C}}\rangle$. We can then apply the same unitary $\hat{\mathbb{U}}$ as in Eq. (3.2) to recover the HP. In this case, the Heisenberg states will be

$$\left|\Psi_U^{\mathcal{E}(\mathbb{H})}\right\rangle\big\rangle = |\psi_{\mathcal{A}}(0)\rangle\big|\mathrm{TL}_{\mathcal{C}}^{\mathcal{E}}\big\rangle. \tag{3.30}$$

As in the SP, the value of $\mathcal{E}$ does not lead to any observable difference in the construction.

Alternatively, one could keep the Heisenberg state $|\psi_{\mathcal{A}}(0)\rangle|\mathrm{TL}_{\mathcal{C}}\rangle$ and recover the HP observables with the unitary

$$\hat{\mathbb{U}}^{\mathcal{E}} \coloneqq e^{-i\left(\hat{H}_{\mathcal{A}}-\mathcal{E}\right)\otimes\hat{T}_{\mathcal{C}}}. \tag{3.31}$$

Both prescriptions lead to the same results.

3.1.1.4 The Role of Entanglement

We have seen that the Heisenberg state of the Universe $\left|\Psi_U^{(\mathbb{H})}\right\rangle\big\rangle$ is *separable* with respect to system $\mathcal{A}$ and the clock $\mathcal{C}$. As we have seen in Sec. 3.1.1.1, any other good Heisenberg state is related to $\left|\Psi_U^{(\mathbb{H})}\right\rangle\big\rangle$ by local unitary transformation and so must also be separable. Does this mean that entanglement does not play a fundamental role in the construction? The answer is negative. Entanglement is still essential in the construction. The SP-HP transformation simply "transfers"

the entanglement from the state of the Universe to the observables. As always, entanglement depends not only on the state vector but also on the subalgebras of observables considered [116]. In our case, the unitary transformation $\hat{\mathbb{U}}$ changes the subalgebras of observables that we assign to $\mathcal{A}$ and $\mathcal{C}$. Therefore, although the Heisenberg state is separable, $\mathcal{A}$ and $\mathcal{C}$ are still entangled.

Note that here we are talking about entanglement in the kinematical Hilbert space $\mathscr{H}_U$. This notion of entanglement is not gauge-invariant and cannot be probed within the so-called "physical Hilbert space" $\mathscr{H}_{phys}$ consisting of all the states of the Universe that satisfy the stationary constraint of Eq. (1.3) [36, 91]. Nevertheless, entanglement plays a crucial explanatory role in timeless quantum theory with pure states of the Universe. It is the reason why different states of $\mathcal{A}$ "appear" at different times of the clock. We will later see that entanglement is no longer strictly necessary when considering mixed states of the Universe.

3.1.2 Mixed states

The construction outlined above can also be applied to mixed states of the Universe. The stationarity constraint, in this case, reads as in Eq. (1.16), which we report here:

$$\left[\hat{\mathcal{R}}_U, \hat{H}_U\right] = 0. \tag{3.32}$$

The density operator of the Universe $\hat{\mathcal{R}}_U$ in the SP must then be of the form

$$\hat{\mathcal{R}}_U = \int_{-\infty}^{+\infty} \mathrm{d}\mathcal{E} \sum_{d_\mathcal{E}} \nu_{\mathcal{E},d_\mathcal{E}} \left|\mathcal{E}, d_\mathcal{E}\right\rangle\!\rangle \left\langle\!\left\langle \mathcal{E}, d_\mathcal{E}\right|, \tag{3.33}$$

where $\{|\mathcal{E}, d_\mathcal{E}\rangle\!\rangle\}_{\mathcal{E},d_\mathcal{E}}$ is a complete set of improper energy eigenstates $\hat{H}_U \left|\mathcal{E}, d_\mathcal{E}\right\rangle\!\rangle = \mathcal{E}\left|\mathcal{E}, d_\mathcal{E}\right\rangle\!\rangle \ \forall\, \mathcal{E}, d_\mathcal{E}$, with $d_\mathcal{E}$ the degeneracy label of the $\mathcal{E}$ eigenvalue, such that $\left\langle\!\left\langle \mathcal{E}', d'_{\mathcal{E}'} \left|\mathcal{E}, d_\mathcal{E}\right\rangle\!\right\rangle = \delta(\mathcal{E}-\mathcal{E}')\delta_{d'_{\mathcal{E}'},d_\mathcal{E}}$ and $\left\langle\!\left\langle \mathcal{E}, d_\mathcal{E} \left|\phi_\mathcal{C}(t)\right\rangle\!\right\rangle \left\langle\!\left\langle \phi_\mathcal{C}(t) \left|\mathcal{E}, d_\mathcal{E}\right\rangle\!\right\rangle = 1 \ \forall t$. We also

must have that $\nu_{\mathcal{E},d_\mathcal{E}} \geq 0 \; \forall \, \mathcal{E}, d_\mathcal{E}$ (positivity) and $\int_{-\infty}^{+\infty} \mathrm{d}\mathcal{E} \, \sum_{d_\mathcal{E}} \nu_{\mathcal{E},d_\mathcal{E}} = 1$ (normalisation according to the physical inner product). Note that we have considered $\mathcal{A}$ to be a finite-dimensional system. As a result, there is only a finite number of degenerate energy eigenstates for any energy level.

Now, using Eqs. (3.29) and (3.2), we can write each $|\mathcal{E}, d_\mathcal{E}\rangle\rangle \, \langle\langle \mathcal{E}, d_\mathcal{E}|$ as

$$|\mathcal{E}, d_\mathcal{E}\rangle\rangle \, \langle\langle \mathcal{E}, d_\mathcal{E}| = \hat{\mathbb{U}} \Big| \psi_\mathcal{A}^{\mathcal{E},d_\mathcal{E}}(0) \Big\rangle \Big\langle \psi_\mathcal{A}^{\mathcal{E},d_\mathcal{E}}(0) \Big| \otimes \big| \mathrm{TL}_\mathcal{C}^\mathcal{E} \big\rangle \big\langle \mathrm{TL}_\mathcal{C}^\mathcal{E} \big| \hat{\mathbb{U}}^\dagger, \tag{3.34}$$

where, due to the unitarity of $\hat{\mathbb{U}}$, we must have that $\Big\langle \psi_\mathcal{A}^{\mathcal{E},d_\mathcal{E}}(0) \Big| \psi_\mathcal{A}^{\mathcal{E},d'_\mathcal{E}}(0) \Big\rangle = \delta_{d_\mathcal{E},d'_\mathcal{E}}$. Therefore, $\hat{\mathcal{R}}_U$ can be written as $\hat{\mathcal{R}}_U = \hat{\mathbb{U}} \hat{\mathcal{R}}_U^{(\mathbb{H})} \hat{\mathbb{U}}^\dagger$ with

$$\hat{\mathcal{R}}_U^{(\mathbb{H})} = \int_{-\infty}^{+\infty} \mathrm{d}\mathcal{E} \sum_{d_\mathcal{E}} \nu_{\mathcal{E},d_\mathcal{E}} \Big| \psi_\mathcal{A}^{\mathcal{E},d_\mathcal{E}}(0) \Big\rangle \Big\langle \psi_\mathcal{A}^{\mathcal{E},d_\mathcal{E}}(0) \Big| \otimes \big| \mathrm{TL}_\mathcal{C}^\mathcal{E} \big\rangle \big\langle \mathrm{TL}_\mathcal{C}^\mathcal{E} \big|. \tag{3.35}$$

$\hat{\mathcal{R}}_U^{(\mathbb{H})}$ is a convex combination of product states, so it is separable. Working, for simplicity, with a discrete number of energy eigenvalues $\mathcal{E}_k$, we can also write

$$\hat{\mathcal{R}}_U^{(\mathbb{H})} = \sum_k p_k \, \hat{\rho}_k(0) \otimes \Big| \mathrm{TL}_\mathcal{C}^{\mathcal{E}_k} \Big\rangle \Big\langle \mathrm{TL}_\mathcal{C}^{\mathcal{E}_k} \Big|, \tag{3.36}$$

which shows that $\hat{\mathcal{R}}_U^{(\mathbb{H})}$ is a so-called *one-way classically correlated* or *quantum-classical* state [117, 118]. Due to Eqs. (3.32) and (3.8), this state satisfies

$$\Big[\hat{\mathcal{R}}_U^{(\mathbb{H})}, \hat{H}_\mathcal{C} \Big] = 0, \tag{3.37}$$

meaning that

$$\operatorname*{Tr}_\mathcal{C} \Big[\hat{\mathcal{R}}_U^{(\mathbb{H})} \hat{\Pi}_t(\hat{T}_\mathcal{C}) \Big] = \operatorname*{Tr}_\mathcal{C} \Big[\hat{\mathcal{R}}_U^{(\mathbb{H})} \hat{\Pi}_{t'}(\hat{T}_\mathcal{C}) \Big] \quad \forall \, t, t', \tag{3.38}$$

that is, no time evolution is encoded in $\hat{\mathcal{R}}^{(\mathbb{H})}$. Therefore, let us take $\hat{\mathcal{R}}_U^{(\mathbb{H})}$ as the Heisenberg state.

We define the corresponding HP observables of $\mathcal{A}$ as in Eq. (3.4). The relative observables $\hat{O}^{(\mathbb{H})}_{\mathcal{A}}(t)$ are defined as in Eq. (3.11) and the expectation value of $\hat{O}^{(\mathbb{H})}$ relative to the clock being in the eigenstate with eigenvalue t is given by

$$\mathrm{Tr}\left[\hat{O}^{(\mathbb{H})}_{\mathcal{A}}(t)\hat{\mathcal{R}}^{(\mathbb{H})}_{U}\right] = \underset{\mathcal{A}}{\mathrm{Tr}}\left[e^{i\hat{H}_{\mathcal{A}}t}\hat{O}_{\mathcal{A}}e^{-i\hat{H}_{\mathcal{A}}t}\hat{\rho}^{(\mathbb{H})}_{\mathcal{A}}\right], \tag{3.39}$$

where $\hat{\rho}^{(\mathbb{H})}_{\mathcal{A}} = \mathrm{Tr}\left[\hat{\mathcal{R}}^{(\mathbb{H})}_{U}\hat{\Pi}_t(\hat{T}_{\mathcal{C}})\right] = \sum_k p_k\hat{\rho}_k(0)$ is the Heisenberg state of $\mathcal{A}$. Eq. (3.39) is equivalent to what one would have obtained from the PW construction in the SP at time t.

Finally, since the Heisenberg state of the Universe is separable and since $\left\langle \mathrm{TL}^{\mathcal{E}_j}_{\mathcal{C}}\middle|\hat{O}^{(\mathbb{H})}_{\mathcal{A}}(t)\middle|\mathrm{TL}^{\mathcal{E}_j}_{\mathcal{C}}\right\rangle = \left\langle \mathrm{TL}^{\mathcal{E}_k}_{\mathcal{C}}\middle|\hat{O}^{(\mathbb{H})}_{\mathcal{A}}(t)\middle|\mathrm{TL}^{\mathcal{E}_k}_{\mathcal{C}}\right\rangle \forall\, j,k$, one can define the reduced relative observables as

$$\left[\hat{O}^{(\mathbb{H})}_{\mathcal{A}}(t)\right]_{\mathcal{A}} \coloneqq \underset{\mathcal{C}}{\mathrm{Tr}}\left[\hat{O}^{(\mathbb{H})}_{\mathcal{A}}(t)\left(\hat{\mathbb{1}}_{\mathcal{A}}\otimes\hat{\rho}^{(\mathbb{H})}_{\mathcal{C}}\right)\right] = e^{i\hat{H}_{\mathcal{A}}t}\hat{O}_{\mathcal{A}}e^{-i\hat{H}_{\mathcal{A}}t}, \tag{3.40}$$

where $\hat{\rho}^{(\mathbb{H})}_{\mathcal{C}} \coloneqq \underset{\mathcal{A}}{\mathrm{Tr}}\left[\hat{\mathcal{R}}^{(\mathbb{H})}_{U}\right] = \sum_k p_k\left|\mathrm{TL}^{\mathcal{E}_k}_{\mathcal{C}}\right\rangle\left\langle \mathrm{TL}^{\mathcal{E}_k}_{\mathcal{C}}\right|$. These are the usual Heisenberg-evolved observables of the system.

In conclusion, the PW construction can be formulated in the HP also for mixed states of the Universe. Furthermore, the mixed state of Eq. (3.33) consists of an ensemble of eigenstates of $\hat{H}_U$ with different eigenvalues and is thus more general than the pure state of Eq. (3.1). If $\hat{\mathcal{R}}^{(\mathbb{H})}_{U}$ contains only states with the same eigenvalue $\mathcal{E}$, then the Heisenberg state of Eq. (3.35) becomes

$$\hat{\mathcal{R}}^{(\mathbb{H})} = \hat{\rho}_{\mathcal{A}}(0)\otimes|\mathrm{TL}_{\mathcal{E}}\rangle\langle\mathrm{TL}_{\mathcal{E}}|, \tag{3.41}$$

which is a product state. In general, however, the Heisenberg state, although separable, may have some other kinds of non-classical correlations, such as discord [119].

3.1.2.1 Uniqueness of the Transformation

Is the SP-HP transformation outlined above unique? Here, we show that—under the assumptions discussed in Sec. 3.1.1.1—all the other *unitary* transformations between the two pictures differ from $\hat{\mathbb{U}}$ only by local unitaries. To see this, consider another unitary $\hat{\mathbb{U}}'$ and the associated Heisenberg state $\hat{\mathcal{R}}'^{(\mathbb{H})}_U := \hat{\mathbb{U}}'^\dagger \hat{\mathcal{R}}_U \hat{\mathbb{U}}'$. The unitary $\hat{\mathbb{U}}'$ must be such that Eq. (3.16) holds and

$$\operatorname*{Tr}_{\mathcal{C}}\left[\hat{\mathcal{R}}'^{(\mathbb{H})}_U \hat{\Pi}_t(\hat{T}'_{\mathcal{C}})\right] = \operatorname*{Tr}_{\mathcal{C}}\left[\hat{\mathcal{R}}'^{(\mathbb{H})}_U \hat{\Pi}_t(\hat{T}'_{\mathcal{C}})\right] \quad \forall\, t, t'. \tag{3.42}$$

with $\hat{\Pi}_t(\hat{T}'_{\mathcal{C}}) := \hat{\mathbb{U}}'^\dagger \hat{\Pi}_t(\hat{T}_{\mathcal{C}}) \hat{\mathbb{U}}'$.

Theorem 3.2. *A unitary $\hat{\mathbb{U}}'$ satisfies Eqs. (3.16) and (3.42) if and only if it is related to $\hat{\mathbb{U}} = e^{-i\hat{H}_{\mathcal{A}} \otimes \hat{T}_{\mathcal{C}}}$ via local unitary transformations.*

Proof. The necessary condition is trivial to prove.

To prove the other direction, let us then define $\hat{\mathbb{W}} := \hat{\mathbb{U}}'^\dagger \hat{\mathbb{U}}$, which is such that $\hat{\mathcal{R}}'^{(\mathbb{H})}_U = \hat{\mathbb{W}} \hat{\mathcal{R}}^{(\mathbb{H})}_U \hat{\mathbb{W}}^\dagger$. For the moment, let us assume that $\hat{\mathbb{U}}'$, and thus $\hat{\mathbb{W}}$, leave $\hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{T}_{\mathcal{C}}$ invariant. As discussed in the proof of Theorem 3.1, this means that we can write $\hat{\mathbb{W}} = \int_{-\infty}^{+\infty} \mathrm{d}t\, \hat{W}(t) \otimes |\phi_{\mathcal{C}}(t)\rangle\langle\phi_{\mathcal{C}}(t)|$, with $\hat{W}(t)$ a t-dependent unitary. Eq. (3.42) then implies

$$\hat{W}(t) \operatorname*{Tr}_{\mathcal{C}}\left[\hat{\mathcal{R}}^{(\mathbb{H})}_U \hat{\Pi}_t(\hat{T}_{\mathcal{C}})\right] \hat{W}^\dagger(t) = \hat{W}(t') \operatorname*{Tr}_{\mathcal{C}}\left[\hat{\mathcal{R}}^{(\mathbb{H})}_U \hat{\Pi}_{t'}(\hat{T}_{\mathcal{C}})\right] \hat{W}^\dagger(t') \quad \forall\, t, t'. \tag{3.43}$$

We will further require that this equation holds for any choice of $\hat{\mathcal{R}}^{(\mathbb{H})}_U$ given by Eq. (3.35). For instance, let us choose $\hat{\mathcal{R}}^{(\mathbb{H})}_U = |\psi_{\mathcal{A}}\rangle\langle\psi_{\mathcal{A}}| \otimes \left|\mathrm{TL}^{\mathcal{E}}_{\mathcal{C}}\right\rangle\!\left\langle\mathrm{TL}^{\mathcal{E}}_{\mathcal{C}}\right|$ for any $|\psi_{\mathcal{A}}\rangle$ and $\mathcal{E}$. We can then rewrite Eq. (3.43) as

$$\hat{W}(t)|\psi_{\mathcal{A}}\rangle\langle\psi_{\mathcal{A}}|\hat{W}^\dagger(t) = \hat{W}(t')|\psi_{\mathcal{A}}\rangle\langle\psi_{\mathcal{A}}|\hat{W}^\dagger(t') \quad \forall\, t, t', \tag{3.44}$$

for any state $|\psi_{\mathcal{A}}\rangle$. This implies that $\hat{W}(t') = e^{i\phi(t,t')}\hat{W}(t)$ and thus $\hat{W} = \hat{W}(0) \otimes \int \mathrm{d}t\, e^{i\phi(t,0)}|\phi_{\mathcal{C}}(t)\rangle\langle\phi_{\mathcal{C}}(t)| = \hat{W}(0) \otimes e^{-i\hat{\mathbb{1}}_{\mathcal{A}}\otimes\phi(\hat{T}_{\mathcal{C}})}$ which is a local unitary. The demonstration for the case when $\hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{T}_{\mathcal{C}}$ is not invariant under $\hat{\mathbb{U}}'$ (but is still of the form $\hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{T}'_{\mathcal{C}}$) is similar to that in the proof of Theorem 3.1. □

If we restrict to transformations that leave $\hat{T}_{\mathcal{C}}$ invariant and do not change the basis of the $\mathcal{A}$'s Hilbert space, the class of unitaries and Heisenberg states is given by

$$\hat{\mathbb{U}}_{\phi} \coloneqq \hat{\mathbb{U}} e^{-i\hat{\mathbb{1}}_{\mathcal{A}}\otimes\phi(\hat{T}_{\mathcal{C}})}, \tag{3.45}$$

$$\hat{\mathcal{R}}^{(\mathbb{H})}_{U,\phi} \coloneqq e^{i\hat{\mathbb{1}}_{\mathcal{A}}\otimes\phi(\hat{T}_{\mathcal{C}})}\hat{\mathcal{R}}^{(\mathbb{H})}_{U} e^{-i\hat{\mathbb{1}}_{\mathcal{A}}\otimes\phi(\hat{T}_{\mathcal{C}})}, \tag{3.46}$$

with $\phi(\hat{T}_{\mathcal{C}})$ a self-adjoint function of $\hat{T}_{\mathcal{C}}$ only. Similarly to the previous section, the simplest choice is represented by $\hat{\mathbb{U}}$ and $\hat{\mathcal{R}}^{(\mathbb{H})}_{U}$.

3.1.2.2 Gauge Invariance, Entanglement and Discord

As for the pure Heisenberg state of the Universe $\left|\Psi^{(\mathbb{H})}_{U}\right\rangle\!\!\right\rangle$, also $\hat{\mathcal{R}}^{(\mathbb{H})}_{U}$ is invariant under the unitaries $\hat{\mathcal{U}}(\theta)$ of Eq. (3.23) because of Eq. (3.37):

$$\hat{\mathcal{U}}^{\dagger}(\theta)\hat{\mathcal{R}}^{(\mathbb{H})}_{U}\hat{\mathcal{U}}(\theta) = \hat{\mathcal{R}}^{(\mathbb{H})}_{U} \quad \forall\theta \in \mathbb{R}. \tag{3.47}$$

This means that there is a gauge freedom in the construction as discussed in Sec. 3.1.1.2.

The possible mixed Heisenberg states must also be separable with respect to $\mathcal{A}$ and $\mathcal{C}$. This is because $\hat{\mathcal{R}}^{(\mathbb{H})}_{U}$ in Eq. (3.35) is separable and any other Heisenberg state is related to it by local unitary transformations. The last point also implies that any Heisenberg state must commute with a local observable of the clock $\hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{H}'_{\mathcal{C}}$ related to $\hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}$ by a unitary transformation. In turn, this means that any Heisenberg

state must be quantum-classical [119], as we will now show for a finite-dimensional bipartite system.

Theorem 3.3. *A state $\hat{\mathcal{R}}$ of a bipartite system $\mathscr{H}_{\mathcal{A}} \otimes \mathscr{H}_{\mathcal{C}}$ is quantum-classical if and only if there exists a self-adjoint operator $\hat{O}_{\mathcal{C}}$ with non-degenerate eigenvalues such that $\left[\hat{\mathcal{R}}, \hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{O}_{\mathcal{C}}\right] = 0$.*

Proof. First, assume that $\hat{\mathcal{R}}$ is quantum-classical. This means that it can be written as $\hat{\mathcal{R}} = \sum_j p_j \hat{\rho}_{\mathcal{A}}^{(j)} \otimes |j\rangle\langle j|$ with $p_j \geq 0$, $\sum_j p_j = 1$, and $\{|j\rangle\}_j$ an orthonormal basis of $\mathcal{C}$ [119]. We can then define the observable $\hat{O}_{\mathcal{C}} \coloneqq \sum_j o_j |j\rangle\langle j|$, with $o_j \neq o_k \ \forall j, k$. We have that $\left[\hat{\mathcal{R}}, \hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{O}_{\mathcal{C}}\right] = 0$.

To prove the other direction, consider the observable $\hat{O}_{\mathcal{C}}$. Its eigenstate $|o_j\rangle$ has eigenvalue o_j, with $o_j \neq o_k \ \forall j \neq k$. The eigenstates of $\hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{O}_{\mathcal{C}}$ must then have the form $|\psi\rangle|o_j\rangle$ for any $|\psi\rangle \in \mathscr{H}_{\mathcal{A}}$. $\left[\hat{\mathcal{R}}, \hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{O}_{\mathcal{C}}\right] = 0$ implies that $\hat{\mathcal{R}}$ and $\hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{O}_{\mathcal{C}}$ share at least a common basis of eigenstates. This basis will be, in general, of the form $\{|\chi_{k,j}\rangle|o_j\rangle\}_{k,j}$ with $\langle\chi_{k',j}|\chi_{k,j}\rangle = \delta_{k',k} \ \forall k, k', j$, but with $|\chi_{k,j}\rangle$ not necessarily orthogonal to $|\chi_{k',j'}\rangle$ when $j \neq j'$. $\hat{\mathcal{R}}$ can then be written as

$$\hat{\mathcal{R}} = \sum_{k,j} \nu_{kj} |\chi_{k,j}\rangle\langle\chi_{k,j}| \otimes |o_j\rangle\langle o_j|, \tag{3.48}$$

which is a quantum-classical state. □

The theorem can be straightforwardly generalised to the case where $\mathcal{C}$ is an ideal clock. Since $\hat{H}_{\mathcal{C}}$ is self-adjoint and has non-degenerate eigenvalues, $\hat{H}'_{\mathcal{C}}$ has the same properties. Therefore, any Heisenberg state $\hat{\mathcal{R}}'^{(\mathbb{H})}_U$ must be quantum-classical. Although these states are separable, they are, in general, not classically correlated and contain other types of correlations that are not entanglement, such as discord [119].

3.1.3 Some Examples

3.1.3.1 Qubit and Ideal Clock

Let us consider a model where the system $\mathcal{A}$ is a qubit and the clock is an ideal clock with two conjugate observables $\hat{T}_\mathcal{C}$ and $\hat{H}_\mathcal{C}$. Let us assume that the Hamiltonian of $\mathcal{A}$+$\mathcal{C}$ is $\hat{H}_U = \hat{H}_\mathcal{A} \otimes \hat{\mathbb{1}}_\mathcal{C} + \hat{\mathbb{1}}_\mathcal{A} \otimes \hat{H}_\mathcal{C}$ with $\hat{H}_\mathcal{A} = \sigma_x/2$. Here $\sigma_{x,y,z}$ denote the Pauli matrices. Let us also assume that the state of system+clock is an eigenstate of $\hat{H}_U$ with eigenvalue 0 and that the system is in the state $|\psi_\mathcal{A}(0)\rangle$ at time $t = 0$. Since all the qubit's observables can be expressed in terms of two generators of its algebra, one can choose to keep only track of $\left(\sigma_x \otimes \hat{\mathbb{1}}_\mathcal{C}, \sigma_z \otimes \hat{\mathbb{1}}_\mathcal{C}\right)$ [120].

To move to the HP, one can choose the Heisenberg state $|\psi_\mathcal{A}(0)\rangle|\mathrm{TL}_\mathcal{C}\rangle$ and act with the unitary $\hat{\mathbb{U}} = e^{-i\hat{H}_\mathcal{A}\otimes\hat{T}_\mathcal{C}}$ on the generators obtaining

$$\left(\sigma_x \otimes \hat{\mathbb{1}}_\mathcal{C}, \sigma_z \otimes \hat{\mathbb{1}}_\mathcal{C}\right) \rightarrow \left(\sigma_x \otimes \hat{\mathbb{1}}_\mathcal{C}, \sigma_z \otimes \cos\left(\hat{T}_\mathcal{C}\right) + \sigma_y \otimes \sin\left(\hat{T}_\mathcal{C}\right)\right), \tag{3.49}$$

where the dependence of the system's observables on the time operator $\hat{T}_\mathcal{C}$ is explicit. Moreover, the qubit's generators relative to the clock being in the eigenstate with eigenvalue t are given by $\left(\sigma_x\hat{\Pi}_t(\hat{T}_\mathcal{C}), \left[\sigma_z\cos(t) + \sigma_y\sin(t)\right]\hat{\Pi}_t(\hat{T}_\mathcal{C})\right)$. Using equation (3.15) one can get the evolved generators of the system $(\sigma_x, \sigma_z\cos(t) + \sigma_y\sin(t))$ that one would have obtained through the Heisenberg equation.

3.1.3.2 Qubit System and Clock

As a further simplification, let us assume that the clock consists of a single qubit [56] and that the time operator $\hat{T}_\mathcal{C}$ is given by $\hat{T}_\mathcal{C} \coloneqq t_0|0_\mathcal{C}\rangle\langle 0_\mathcal{C}| + t_1|1_\mathcal{C}\rangle\langle 1_\mathcal{C}|$ where $|0_\mathcal{C}\rangle$ ($|1_\mathcal{C}\rangle$) is the +1 (-1) eigenstate of $\hat{\mathbb{1}}_\mathcal{A} \otimes \sigma_z$. In this case, to avoid a trivial evolution, one must consider only non-degenerate Hamiltonians of clock and system

such that their sum has a doubly degenerate zero eigenvalue.[7] For example, one can choose $\hat{H}_{\mathcal{A}} = \sigma_x/2$ and $\hat{H}_{\mathcal{C}} = -\sigma_x/2$ (when $\Delta t \coloneqq t_1 - t_0 = \pi$, this Hamiltonian evolves the time eigenstates correctly: $e^{-i\hat{H}_{\mathcal{C}}\Delta t}|0_{\mathcal{C}}\rangle = i|1_{\mathcal{C}}\rangle$). Since now the clock is finite dimensional, Eq. (1.45) does not hold. Instead, the commutator is now $\left[\hat{T}_{\mathcal{C}}, \hat{H}_{\mathcal{C}}\right] = -i\Delta t/2\,\sigma_y$.

A suitable Heisenberg state of system+clock is now $|0_{\mathcal{A}}\rangle\left(|0_{\mathcal{C}}\rangle + |1_{\mathcal{C}}\rangle\right)/\sqrt{2}$. Under the unitary $\hat{\mathbb{U}}(t_0) \coloneqq e^{-i\hat{H}\otimes(\hat{T}_{\mathcal{C}} - t_0\hat{\mathbb{1}}_{\mathcal{C}})}$, the generators $\left(\sigma_x \otimes \hat{\mathbb{1}}_{\mathcal{C}}, \sigma_z \otimes \hat{\mathbb{1}}_{\mathcal{C}}\right)$ become

$$\left(\sigma_x \otimes \hat{\mathbb{1}}_{\mathcal{C}}, \sigma_z \otimes \left[|0_{\mathcal{C}}\rangle\langle 0_{\mathcal{C}}| + \cos(\Delta t)\,|1_{\mathcal{C}}\rangle\langle 1_{\mathcal{C}}|\right] + \sin(\Delta t)\,\sigma_y \otimes |1_{\mathcal{C}}\rangle\langle 1_{\mathcal{C}}|\right). \tag{3.50}$$

This means that if the clock is in the state $|0\rangle_{\mathcal{C}}$, the reduced relative generators of the system will be (σ_x, σ_z); if the clock is in the state $|1\rangle_{\mathcal{C}}$ the generators will be $(\sigma_x, \cos(\Delta t)\,\sigma_z + \sin(\Delta t)\,\sigma_y)$, as expected.

Finally, let us check that these results are consistent with the state of the system in the SP. To do so, let us write the density matrix of the system as

$$\rho_{\mathcal{A}}(t) = \frac{1}{2}\left(\hat{\mathbb{1}}_{\mathcal{A}} + \sum_i \lambda_i(t)\sigma_i\right), \tag{3.51}$$

with $\lambda_i(t) \coloneqq \mathrm{Tr}\left[\rho_{\mathcal{A}}(t)\sigma_i\right] = \mathrm{Tr}\left[\rho_{\mathcal{A}}(t_0)\sigma_i(\Delta t)\right]$, where $\{\sigma_i(\Delta t)\}_i$ are the Heisenberg-evolved Pauli matrices after a time Δt and $\rho_{\mathcal{A}}(t_0) = |0_{\mathcal{A}}\rangle\langle 0_{\mathcal{A}}|$. As we have shown above, at time t_0 the Pauli matrices are unchanged and thus, trivially, $\rho_{\mathcal{A}}(t_0) = |0_{\mathcal{A}}\rangle\langle 0_{\mathcal{A}}|$. Using the Heisenberg-evolved generators above one finds at time t_1: $\{\lambda_i(t_1)\}_i = (0, -\sin(\Delta t), \cos(\Delta t))$. Plugging these values into Eq. (3.51) one gets

$$\rho_{\mathcal{A}}(t_1) = \frac{1}{2}\begin{pmatrix} 1+\cos(\Delta t) & i\sin(\Delta t) \\ -i\sin(\Delta t) & 1-\cos(\Delta t) \end{pmatrix}. \tag{3.52}$$

[7] In the previous example this was guaranteed for any non-degenerate Hamiltonian of the qubit since the spectrum of the clock Hamiltonian was $\mathbb{R}$.

One would have obtained this density matrix by evolving the system's state in the SP.

3.2 Correlations and Time Evolution

In this section, we explore what kind of correlations between $\mathcal{A}$ and $\mathcal{C}$ are necessary to recover a non-trivial relational dynamics. The situation is straightforward with pure states of the Universe but becomes more complicated when working with mixed states. The results obtained here will be useful in Chapter 4 when considering an interaction between $\mathcal{A}$ and $\mathcal{C}$.

3.2.1 Pure States

In Sec. 1.2.5, we showed that the state of the Universe $|\Psi_U\rangle\rangle$ must be entangled if we want to recover a *non-trivial* time evolution.[8] We arrived at this conclusion by looking at the form of $|\Psi_U\rangle\rangle$ in Eqs. (1.30) and (1.47), but we can easily prove it in a more general way. To do so, let us define a trivial relational dynamics as one such that $\langle\phi_{\mathcal{C}}(t')\,|\Psi_U\rangle\rangle = z(t,t')\,\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle\ \forall\, t,t'$, with $z(t,t')$ a generic complex number. We will assume $z(t,t') \neq 0$.

Theorem 3.4. *A stationary state $|\Psi_U\rangle\rangle$ results in a trivial relational dynamics if and only if it is separable with respect to $\mathcal{A}$ and $\mathcal{C}$.*

Proof. Let $|\Psi_U\rangle\rangle$ be a stationary state that encodes a trivial time evolution. This implies that

$$|\Psi_U\rangle\rangle = \int dt\,|\phi_{\mathcal{C}}(t)\rangle\langle\phi_{\mathcal{C}}(t)|\cdot|\Psi_U\rangle\rangle = \langle\phi_{\mathcal{C}}(0)\,|\Psi_U\rangle\rangle \int dt\,z(t,0)|\phi_{\mathcal{C}}(t)\rangle = \\ = |\psi_{\mathcal{A}}(0)\rangle \int dt\,z(t,0)|\phi_{\mathcal{C}}(t)\rangle, \tag{3.53}$$

[8] A trivial evolution would be one where $\mathcal{A}$ is stationary apart from an unobservable phase.

which is a separable state of $\mathcal{A}$ and $\mathcal{C}$.

Vice versa, if $|\Psi_U\rangle\rangle$ is separable, then it must be of the form $|\Psi_U\rangle\rangle = |X_\mathcal{A}\rangle|Y_\mathcal{C}\rangle$ which leads to a trivial relational dynamics because

$$\langle\phi_\mathcal{C}(t')\,|\Psi_U\rangle\rangle = z(t,t')|X_\mathcal{A}\rangle\,\langle\phi_\mathcal{C}(t)|Y_\mathcal{C}\rangle = z(t,t')\,\langle\phi_\mathcal{C}(t)\,|\Psi_U\rangle\rangle \quad \forall\, t,t', \tag{3.54}$$

where we have used the fact that $\langle\phi_\mathcal{C}(t')|Y_\mathcal{C}\rangle$ is a complex number and is thus proportional to $\langle\phi_\mathcal{C}(t)|Y_\mathcal{C}\rangle$. We have also assumed that $z(t,t') \neq 0$, meaning that all the time states are present in $|\Psi_U\rangle\rangle$. □

Note that this result *does not depend on the specific form of the Hamiltonian* (it could even be an interacting one). In conclusion, only **entangled** pure states of the Universe result in non-trivial (i.e. non-stationary) relational dynamics.

3.2.2 Mixed States

The situation becomes more complicated when dealing with mixed states of the Universe. We can prove the following.

Theorem 3.5. *If a stationary state $\hat{\mathcal{R}}$ is such that $\left[\hat{\mathcal{R}}, \hat{H}_\mathcal{C}\right] = 0$, then the state results in a trivial relational dynamics.*

Proof. Consider the state of $\mathcal{A}$ given that the clock shows the time t:
$\hat{\rho}_\mathcal{A}(t) = \underset{\mathcal{C}}{\mathrm{Tr}}\left[\hat{\mathcal{R}}\hat{\Pi}_t(\hat{T}_\mathcal{C})\right] = \underset{\mathcal{C}}{\mathrm{Tr}}\left[\hat{\mathcal{R}}e^{-i\hat{H}_\mathcal{C}t}\hat{\Pi}_0(\hat{T}_\mathcal{C})e^{i\hat{H}_\mathcal{C}t}\right] = \underset{\mathcal{C}}{\mathrm{Tr}}\left[e^{i\hat{H}_\mathcal{C}t}\hat{\mathcal{R}}e^{-i\hat{H}_\mathcal{C}t}\hat{\Pi}_0(\hat{T}_\mathcal{C})\right] =$
$\underset{\mathcal{C}}{\mathrm{Tr}}\left[\hat{\mathcal{R}}\hat{\Pi}_0(\hat{T}_\mathcal{C})\right] = \hat{\rho}_\mathcal{A}(0)\ \forall t$. Therefore, the evolution is trivial. □

Note that this is *independent* of the form of $\hat{H}_U$. Now, the set of states $\hat{\mathcal{R}}'_U$ such that $\left[\hat{\mathcal{R}}'_U, \hat{H}_\mathcal{C}\right] = 0$ is a subset of the set of all the possible separable states of the Universe. This means that other **separable** states can lead to a non-trivial system's

evolution in the SP. An example of such a state is

$$\hat{\mathcal{R}}'_U = \int \mathrm{d}t\, |\psi_{\mathcal{A}}(t)\rangle\langle\psi_{\mathcal{A}}(t)| \otimes |\phi_{\mathcal{C}}(t)\rangle\langle\phi_{\mathcal{C}}(t)|, \tag{3.55}$$

with $|\psi_{\mathcal{A}}(t)\rangle = e^{-i\hat{H}_{\mathcal{A}}t}|\psi_{\mathcal{A}}(0)\rangle$. It is easy to see that this state is stationary, i.e. $\left[\hat{\mathcal{R}}'_U, \hat{H}_U\right] = 0$, and separable. Nevertheless, this state leads to the usual relational evolution. This is because $\left[\hat{\mathcal{R}}'_U, \hat{H}_{\mathcal{C}}\right] \neq 0$.[9]

Therefore, when working with mixed states of the Universe, entanglement is no longer necessary to recover a non-trivial relational dynamics. There are separable states of the Universe that encode the relevant time evolution in correlations that are not entanglement. This was made possible by allowing $\hat{\mathcal{R}}_U$ to live on eigenspaces of $\hat{H}_U$ with different eigenvalues. To see this, let us prove the following theorem where we assume, for simplicity, that both the $\mathcal{A}$ and $\mathcal{C}$ are finite-dimensional.

Theorem 3.6. *If a separable bipartite state $\hat{\mathcal{R}}$ is such that $\hat{\mathcal{R}}\hat{H}_U = \hat{H}_U\hat{\mathcal{R}} = 0$ with $\hat{H}_U = \hat{H}_{\mathcal{A}} \otimes \hat{\mathbb{1}}_{\mathcal{C}} + \hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}$, then $\left[\hat{\mathcal{R}}, \hat{H}_{\mathcal{C}}\right] = 0$.*

Proof. Consider a generic separable state $\hat{\mathcal{R}} = \sum_j p_j \hat{\rho}^{(j)} \otimes \hat{\sigma}^{(j)}$. We can write it in terms of the eigenstates of $\hat{H}_{\mathcal{A}}$ and $\hat{H}_{\mathcal{C}}$ as

$$\hat{\mathcal{R}} = \sum_{j,E,E',\varepsilon,\varepsilon'} p_j \rho^{(j)}_{E,E'} \sigma^{(j)}_{\varepsilon,\varepsilon'} |E\rangle\langle E'| \otimes |\varepsilon\rangle\langle\varepsilon'|. \tag{3.56}$$

$\hat{\mathcal{R}}\hat{H}_U = \hat{H}_U\hat{\mathcal{R}} = 0$ implies that

$$\sum_j p_j \rho^{(j)}_{E,E'} \sigma^{(j)}_{\varepsilon,\varepsilon'} \,(E' + \varepsilon') = 0 = (E + \varepsilon) \sum_j p_j \rho^{(j)}_{E,E'} \sigma^{(j)}_{\varepsilon,\varepsilon'}, \tag{3.57}$$

for any $E, E', \varepsilon, \varepsilon'$. For any $\hat{\rho}^{(j)}$ and $\hat{\sigma}^{(j)}$ there must be at least one positive diagonal element in the $\{|E\rangle|\varepsilon\rangle\}_{E,\varepsilon}$ basis (because $\hat{\rho}^{(j)}$ and $\hat{\sigma}^{(j)}$ have trace 1). In

[9] Note that, instead, we have $\left[\hat{\mathcal{R}}'_U, \hat{T}_{\mathcal{C}}\right] = 0$.

generality, assume $\rho^{(j^*)}_{E_1,E_1} > 0$ for some values j^*. Then the only nonzero diagonal element of $\hat{\sigma}^{(j^*)}$ must be $\sigma^{(j^*)}_{-E_1,-E_1}$, otherwise $(E_1+\varepsilon)\sum_j p_j \rho^{(j)}_{E_1,E_1}\sigma^{(j)}_{\varepsilon,\varepsilon} = (E_1+\varepsilon)\sum_{j^*} p_{j^*}\rho^{(j^*)}_{E_1,E_1}\sigma^{(j^*)}_{\varepsilon,\varepsilon} > 0$ for $\varepsilon \neq -E_1$ (remember that $\sigma^{(j)}_{\varepsilon,\varepsilon}$ must be non-negative for every j). The same reasoning implies that there cannot be two values of E such that $\rho^{(j)}_{E,E} \neq 0$ for any fixed j. To summarise, for each value of j there is only one non-zero diagonal element of $\hat{\rho}^{(j)}$ and $\hat{\sigma}^{(j)}$: $\rho^{(j)}_{E_j,E_j} = 1$ and $\sigma^{(j)}_{-E_j,-E_j} = 1$. Now consider the off-diagonal elements $\sigma^{(j)}_{\varepsilon,\varepsilon'}$. Since $\hat{\sigma}^{(j)}$ has only one non-zero diagonal element, all the off-diagonals must be zero to guarantee that $\hat{\sigma}^{(j)}$ is positive semi-definite. The same reasoning applies to $\rho^{(j)}$. Therefore, $\hat{\mathcal{R}}$ must be of the form $\hat{\mathcal{R}} = \sum_j p_j |E_j\rangle\langle E_j| \otimes |-E_j\rangle\langle -E_j|$. This state commutes with $\hat{H}_{\mathcal{C}}$. □

This theorem straightforwardly extends to systems with continuous energy spectrum. It implies that if $\hat{\mathcal{R}}_U$ is a stationary separable state that leads to a non-trivial relational dynamics, i.e. $\left[\hat{\mathcal{R}}_U, \hat{H}_{\mathcal{C}}\right] \neq 0$, then $\hat{\mathcal{R}}_U \hat{H}_U \neq 0$, which means that $\hat{\mathcal{R}}_U$ lives on multiple eigenspaces of $\hat{H}_U$ with different eigenvalues. If one restricts to a single eigenspace, then $\hat{\mathcal{R}}_U$ must be **entangled** to encode a non-trivial time evolution. This discussion will become important in the presence of interaction between $\mathcal{A}$ and $\mathcal{C}$, as discussed in Chapter 4.

Finally, if one has access only to the system $\mathcal{A}$, then it is not possible to distinguish an entangled state of the Universe $\hat{\mathcal{R}}_U$ from the state

$$\hat{\mathcal{R}}_{\text{dec}} \coloneqq \int \mathrm{d}t\, \langle\phi_{\mathcal{C}}(t)|\hat{\mathcal{R}}_U|\phi_{\mathcal{C}}(t)\rangle \otimes |\phi_{\mathcal{C}}(t)\rangle\langle\phi_{\mathcal{C}}(t)|, \tag{3.58}$$

which is a **separable** stationary state of the Universe with no coherences between the different eigenstates of $\hat{T}_{\mathcal{C}}$.

3.3 Mixed States in a Tripartite Universe

In Sec. 1.2.2.2, we discussed some motivations behind the use of mixed states of the Universe. While the Universe, intended as a closed system with no systems external to it, is usually considered to be in a pure state (see [48] for an opposite argument), we could think of being in a region of the Universe that is entangled with another region inaccessible to us. As a result, all our observations would depend on the reduced state of our region of the Universe, which must be a mixed state.

In this section, we will discuss such an example. So let us consider a Universe consisting of a system $\mathcal{A}$ and two ideal clocks $\mathcal{B}$ and $\mathcal{C}$. Let us also consider a Hamiltonian

$$\hat{H}_U = \hat{H}_{\mathcal{AB}} \otimes \hat{\mathbb{1}}_{\mathcal{C}} + \hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{\mathbb{1}}_{\mathcal{B}} \otimes \hat{H}_{\mathcal{C}}, \tag{3.59}$$

which allows for interactions between $\mathcal{A}$ and the clock $\mathcal{B}$ (e.g. $\mathcal{A}$ reading the time of $\mathcal{B}$). There is some extra freedom in the choice of a stationary state of the Universe $|\Psi_U\rangle\rangle$ because the structure of the 0-eigenspace of $\hat{H}_U$ is more complex than in the case of a bipartite Universe. We will choose the following state

$$|\Psi_U\rangle\rangle = \int_{-\infty}^{+\infty} \mathrm{d}t \, \left(e^{-i\hat{H}_{\mathcal{AB}}t}|\psi_{\mathcal{A}}(0)\rangle \otimes |\phi_{\mathcal{B}}(0)\rangle\right) \otimes |\phi_{\mathcal{C}}(t)\rangle, \tag{3.60}$$

for generic states $|\psi_{\mathcal{A}}(0)\rangle$ of $\mathcal{A}$. If we take $\langle\phi_{\mathcal{B}}(t)\,|\Psi_U\rangle\rangle$ we recover the relational dynamics of $\mathcal{A}$ with respect to the clock $\mathcal{B}$. Now since the clock $\mathcal{C}$ is inaccessible to $\mathcal{A}$ and $\mathcal{B}$, we might decide to trace out its degrees of freedom. If we do this to $\hat{\mathcal{R}}_U \coloneqq |\Psi_U\rangle\rangle\,\langle\langle\Psi_U|$, we get

$$\hat{\mathcal{R}}_{\mathcal{AB}} = \int_{-\infty}^{+\infty} \mathrm{d}t \, e^{-i\hat{H}_{\mathcal{AB}}t}|\psi_{\mathcal{A}}(0)\rangle\langle\psi_{\mathcal{A}}(0)| \otimes |\phi_{\mathcal{B}}(0)\rangle\langle\phi_{\mathcal{B}}(0)|e^{i\hat{H}_{\mathcal{AB}}t}, \tag{3.61}$$

which is a mixed state of $\mathcal{A}$ and $\mathcal{B}$ such that $\left[\hat{\mathcal{R}}_{\mathcal{AB}}, \hat{H}_{\mathcal{AB}}\right] = 0$. We have thus recovered a mixed stationary state of a sub-region of the Universe from an overall

pure state. As usual, this is because $|\Psi_U\rangle\rangle$ is entangled with respect to $\mathcal{A}-\mathcal{B}$ and $\mathcal{C}$. This would not be the case if $\mathcal{A}-\mathcal{B}$ and $\mathcal{C}$ were not synchronised. For instance, consider the stationary state

$$|\Psi'_U\rangle\rangle = \int_{-\infty}^{+\infty} \mathrm{d}t \, \left(e^{-i\hat{H}_{\mathcal{AB}}t}|\psi_{\mathcal{A}}(0)\rangle \otimes |\phi_{\mathcal{B}}(0)\rangle\right) \otimes |0_{\mathcal{C}}\rangle, \tag{3.62}$$

where $\mathcal{C}$ is not synchronised with $\mathcal{A}$ and $\mathcal{B}$. We have that $\hat{\mathcal{R}}'_{\mathcal{AB}} \coloneqq \underset{\mathcal{C}}{\mathrm{Tr}}\left[|\Psi'_U\rangle\rangle\,\langle\langle\Psi'_U|\right]$ is given by

$$\hat{\mathcal{R}}'_{\mathcal{AB}} = \int_{-\infty}^{+\infty} \mathrm{d}t\,\mathrm{d}t' \, e^{-i\hat{H}_{\mathcal{AB}}t}|\psi_{\mathcal{A}}(0)\rangle\langle\psi_{\mathcal{A}}(0)| \otimes |\phi_{\mathcal{B}}(0)\rangle\langle\phi_{\mathcal{B}}(0)|e^{i\hat{H}_{\mathcal{AB}}t'}, \tag{3.63}$$

which is a pure stationary state of $\mathcal{A}$ and $\mathcal{B}$.

Finally, if $\mathcal{A}$ and $\mathcal{B}$ are non-interacting, then the relational dynamics of $\mathcal{A}$ with respect to $\mathcal{B}$ given by $\hat{\mathcal{R}}_{\mathcal{AB}}$ is the same as the one given by $|\Psi_U\rangle\rangle$. In the case of interactions, this may not always hold.

3.4 Differential Formulation

In this section, we introduce an alternative notation for timeless quantum theory in terms of derivatives with respect to the time operator of an ideal clock. Time-operator derivatives have already been discussed in this context in [61, 114]. Here, we expand on these works by showing how timeless quantum theory can be expressed in terms of time-operator derivatives both in the SP and the HP, as well as for both pure and mixed states of the Universe. We will later use this notation to explore system-clock interactions.

Extending the notion of derivative to include variations over operators gives rise to some subtleties linked to the non-commutativity of the operators and the choice of the variation [121]. Here, we will focus on a specific type of operator derivatives,

namely derivatives taken along the diagonal. This removes some of the subtleties mentioned above. It is also important to note that the results of this section hinge on the specific algebra of the ideal clock and the fact that the time operator is left unchanged when moving to the HP.

First, we introduce the derivative of a *pure* state of the Universe with respect to the time operator $\hat{T}_{\mathcal{C}}$ of the clock:

$$\frac{\mathrm{d}\,|\Psi_U\rangle\rangle}{\mathrm{d}\hat{T}_{\mathcal{C}}} \coloneqq \int \mathrm{d}t\, \frac{\mathrm{d}\,\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle}{\mathrm{d}t} |\phi_{\mathcal{C}}(t)\rangle, \tag{3.64}$$

where

$$\frac{\mathrm{d}\,\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle}{\mathrm{d}t} = \lim_{\delta\to 0} \frac{\langle\phi_{\mathcal{C}}(t+\delta)\,|\Psi_U\rangle\rangle - \langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle}{\delta}, \tag{3.65}$$

is the usual time derivative of a state vector. Eq. (3.64) resembles a directional derivative. It is easy to show that

$$\frac{\mathrm{d}\,|\Psi_U\rangle\rangle}{\mathrm{d}\hat{T}_{\mathcal{C}}} = i\left(\hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}\right)|\Psi_U\rangle\rangle\,, \tag{3.66}$$

which is similar to the action of momentum in single-particle quantum mechanics. Using this, the stationarity condition on $\left|\Psi_U^{\mathcal{E}}\right\rangle\rangle$ can be written as

$$i\frac{\mathrm{d}\left|\Psi_U^{\mathcal{E}}\right\rangle\rangle}{\mathrm{d}\hat{T}_{\mathcal{C}}} = \left(\hat{H}_{\mathcal{A}} \otimes \hat{\mathbb{1}}_{\mathcal{C}} - \mathcal{E}\right)\left|\Psi_U^{\mathcal{E}}\right\rangle\rangle\,, \tag{3.67}$$

which has the form of the Schrödinger equation but with time-operator derivatives instead of ordinary time derivatives. By taking the partial inner product of Eq. (3.67) with $|\phi_{\mathcal{C}}(t)\rangle$, and using the definition of time-operator derivative of Eq. (3.64), one gets, at any time t,

$$i\frac{\mathrm{d}\left|\psi_{\mathcal{A}}^{\mathcal{E}}(t)\right\rangle}{\mathrm{d}t} = \left(\hat{H}_{\mathcal{A}} - \mathcal{E}\right)\left|\psi_{\mathcal{A}}^{\mathcal{E}}(t)\right\rangle, \tag{3.68}$$

the usual Schrödinger equation on the system's Hilbert space (apart from an unobservable energy shift $\mathcal{E}$).

In order to move to the HP, we need to introduce time-operator derivatives of operators. Let us define the derivative of an operator $\hat{A}$ (function of $\hat{T}_\mathcal{C}$) with respect to $\hat{T}_\mathcal{C}$ as [121]

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}\hat{T}_\mathcal{C}} := \lim_{\delta\to 0} \frac{\hat{A}(\hat{T}_\mathcal{C} + \delta\hat{\mathbb{1}}_\mathcal{C}) - \hat{A}(\hat{T}_\mathcal{C})}{\delta}, \tag{3.69}$$

where the variation has been taken along the diagonal. Since the algebra of observables of $\mathcal{C}$ is generated by $\left(\hat{T}_\mathcal{C}, \hat{H}_\mathcal{C}\right)$ and $\hat{T}_\mathcal{C}$ and $\hat{H}_\mathcal{C}$ are conjugate observables, we can write this derivative as

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}\hat{T}_\mathcal{C}} = \lim_{\delta\to 0} \frac{e^{i\hat{H}_\mathcal{C}\delta}\hat{A}e^{-i\hat{H}_\mathcal{C}\delta} - \hat{A}}{\delta} = i\left[\hat{\mathbb{1}}_\mathcal{A}\otimes\hat{H}_\mathcal{C}, \hat{A}\right]. \tag{3.70}$$

This equation implies that a chain rule holds for the time-operator derivatives:

$$\begin{aligned}\frac{\mathrm{d}(\hat{A}\hat{B})}{\mathrm{d}\hat{T}_\mathcal{C}} &= i\left[\hat{\mathbb{1}}_\mathcal{A}\otimes\hat{H}_\mathcal{C}, \hat{A}\hat{B}\right] = i\left[\hat{\mathbb{1}}_\mathcal{A}\otimes\hat{H}_\mathcal{C}, \hat{A}\right]\hat{B} + i\hat{A}\left[\hat{\mathbb{1}}_\mathcal{A}\otimes\hat{H}_\mathcal{C}, \hat{B}\right] \\ &= \frac{\mathrm{d}\hat{A}}{\mathrm{d}\hat{T}_\mathcal{C}}\hat{B} + \hat{A}\frac{\mathrm{d}\hat{B}}{\mathrm{d}\hat{T}_\mathcal{C}}.\end{aligned} \tag{3.71}$$

Now, if we insert two completeness relations in terms of the $|\phi_\mathcal{C}(t)\rangle$ in Eq. (3.70), we get

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}\hat{T}_\mathcal{C}} = \lim_{\delta\to 0} \frac{\int \mathrm{d}t\,\mathrm{d}t'\,|\phi_\mathcal{C}(t)\rangle\langle\phi_\mathcal{C}(t+\delta)|\hat{A}|\phi_\mathcal{C}(t'+\delta)\rangle\langle\phi_\mathcal{C}(t')| - \hat{A}}{\delta}. \tag{3.72}$$

Moreover, if we consider the operator $|\Psi_U\rangle\rangle\langle\langle\Psi_U|$, Eq. (3.72) implies that

$$\frac{\mathrm{d}\,|\Psi_U\rangle\rangle\langle\langle\Psi_U|}{\mathrm{d}\hat{T}_\mathcal{C}} = \frac{\mathrm{d}\,|\Psi_U\rangle\rangle}{\mathrm{d}\hat{T}_\mathcal{C}}\langle\langle\Psi_U| + |\Psi_U\rangle\rangle\frac{\mathrm{d}\,\langle\langle\Psi_U|}{\mathrm{d}\hat{T}_\mathcal{C}}, \tag{3.73}$$

with $\frac{\mathrm{d}|\Psi_U\rangle\rangle}{\mathrm{d}\hat{T}_\mathcal{C}}$ as in Eq. (3.64). This connects Eq. (3.64) and Eq. (3.69) through a

chain rule, which is why we use the same notation for both derivatives.

The notion of time-operator derivative of an operator and Eq. (3.70) allow us to recast the stationarity constraint for a mixed state of the Universe $\hat{\mathcal{R}}_U$ as

$$\frac{\mathrm{d}\hat{\mathcal{R}}_U}{\mathrm{d}\hat{T}_{\mathcal{C}}} = i\left[\hat{\mathcal{R}}_U, \hat{H}_{\mathcal{A}} \otimes \hat{\mathbb{1}}_{\mathcal{C}}\right], \tag{3.74}$$

which is in the form of the Liouville–von Neumann equation but with time-operator derivatives instead of ordinary time derivatives. Using Eq. (1.17) to get the relative states of Eq. (3.74) we get

$$\frac{\mathrm{d}\hat{\rho}_{\mathcal{A}}(t)}{\mathrm{d}t} = i\left[\hat{\rho}_{\mathcal{A}}(t), \hat{H}_{\mathcal{A}}\right], \tag{3.75}$$

which is the usual Liouville–von Neumann equation for the state of the system $\mathcal{A}$.

One can also use this notation for timeless quantum theory in the Heisenberg Picture. The time-operator derivative of $\hat{\mathbb{U}}$ in Eq. (3.2) is

$$\frac{\mathrm{d}\hat{\mathbb{U}}}{\mathrm{d}\hat{T}_{\mathcal{C}}} = -i\left(\hat{H}_{\mathcal{A}} \otimes \hat{\mathbb{1}}_{\mathcal{C}}\right)\hat{\mathbb{U}}, \tag{3.76}$$

and thus, using the chain rule on the HP observables of the $\mathcal{A}$,

$$\frac{\mathrm{d}\hat{O}^{(\mathbb{H})}_{\mathcal{A}}}{\mathrm{d}\hat{T}_{\mathcal{C}}} = i\left[\hat{H}_{\mathcal{A}} \otimes \hat{\mathbb{1}}_{\mathcal{C}}, \hat{O}^{(\mathbb{H})}_{\mathcal{A}}\right], \tag{3.77}$$

which is the usual equation of motion for the observables of $\mathcal{A}$ in the HP but with time-operator derivatives instead of usual time derivatives. The usual version of the equation at the level of the system's Hilbert space can be recovered via the procedure of Eq. (3.11). Finally, let us calculate the time-operator derivative of the

Heisenberg state of the Universe. For pure states, this is

$$\frac{\mathrm{d}\left|\Psi_U^{\mathcal{E}(\mathbb{H})}\right\rangle\!\right\rangle}{\mathrm{d}\hat{T}_{\mathcal{C}}} = \mathcal{E}\left|\Psi_U^{\mathcal{E}(\mathbb{H})}\right\rangle\!\right\rangle, \tag{3.78}$$

while for mixed states, we have

$$\frac{\mathrm{d}\hat{\mathcal{R}}^{(\mathbb{H})}}{\mathrm{d}\hat{T}_{\mathcal{C}}} = 0. \tag{3.79}$$

Both equations show that no time evolution is encoded in the Heisenberg state.

Chapter 4

The Role of Interactions

The contents of this chapter are based, in part, on [107, 99, 122].

In Chapter 1, we considered adding interaction terms between the system $\mathcal{A}$ and the clock $\mathcal{C}$ in the Hamiltonian of the Universe. While most formulations of timeless quantum theory do not consider such interactions, they are crucial for multiple reasons, as discussed in the introduction. They give the construction more generality since truly isolated quantum systems are not common in Nature. Moreover, interactions allow the clock to be accessed by the rest of the Universe. This is a desirable feature of timeless quantum theory that we discussed in Chapter 2.3.

In this chapter, we start by analysing the effect of different types of interactions in timeless quantum theory. We discuss their effect on the relational dynamics and on the timekeeping qualities of ideal clocks. Finally, we explore the interplay between interactions and correlation and show how the Heisenberg picture version of timeless quantum theory is affected.

4.1 Types of Interactions

Let us consider a Universe consisting of a system $\mathcal{A}$ and an ideal clock $\mathcal{C}$. The simplest way to introduce interactions between $\mathcal{A}$ and $\mathcal{C}$ is to consider a Hamiltonian of the Universe of the following type [43]:

$$\hat{H}_U = \hat{H}_{\mathcal{A}} \otimes \hat{\mathbb{1}}_{\mathcal{C}} + \hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}} + \hat{V}, \tag{4.1}$$

where $\hat{V}$ is the interaction term, that is, a term that cannot be written as $\hat{W}_{\mathcal{A}} \otimes \hat{\mathbb{1}}_{\mathcal{C}}$ or $\hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{W}_{\mathcal{C}}$.[1] Different interactions result in wildly different types of relational dynamics, some of which are non-unitarity [90]. It is crucial to understand how interactions affect the dynamics and to investigate the meaning of non-unitarity in this context.

First, let us discuss whether $\hat{V}$ must be self-adjoint. In Sec. 2.3.4, we saw that we could model an instantaneous measurement using a non-Hermitian interaction. Is this property of the interaction problematic? Usually, the answer would be positive since non-Hermitian operators can have non-real expectation values. In timeless quantum theory, however, the only allowed states of the Universe are the zero-eigenstates $|\Psi_U\rangle\rangle$ of the total Hamiltonian of the Universe, so that all the possible expectation values of $\hat{V}$ at the level of the Universe are given by

$$\langle\langle \Psi_j | \hat{V} | \Psi_j \rangle\rangle = - \langle\langle \Psi_j | \hat{H}_{\mathcal{A}} + \hat{H}_{\mathcal{C}} | \Psi_j \rangle\rangle \quad \forall j, \tag{4.2}$$

which are real since both $\hat{H}_{\mathcal{A}}$ and $\hat{H}_{\mathcal{C}}$ are self-adjoint. Therefore, even if $\hat{V}$ is non-Hermitian, all its possible expectation values at the level of the Universe are real. If instead one considers the average expectation value of $\hat{V}$ in a time interval Δt

[1] In the following, we will sometimes omit the tensor products with identity operators on the other systems for better visual clarity. It should be clear from the context whether an operator acts on a single or multiple Hilbert spaces.

around t, defined as

$$\overline{\langle\langle \hat{V} \rangle\rangle}_t = \frac{\int_{t-\Delta t/2}^{t+\Delta t/2} \mathrm{d}t' \ \langle\langle \Psi_j | \, \Pi_{t'}(\hat{T}_{\mathcal{C}}) \hat{V} \Pi_{t'}(\hat{T}_{\mathcal{C}}) \, | \Psi_j \rangle\rangle}{\int_{t-\Delta t/2}^{t+\Delta t/2} \mathrm{d}t' \ \langle\langle \Psi_j | \, \Pi_{t'}(\hat{T}_{\mathcal{C}}) \Pi_{t'}(\hat{T}_{\mathcal{C}}) \, | \Psi_j \rangle\rangle}, \tag{4.3}$$

one can get some non-real values. For example, using the interaction term of Eq. (2.49) and the state of Eq. (2.50) one gets

$$\overline{\langle\langle \hat{V} \rangle\rangle}_t = \begin{cases} i\frac{\left(1-\langle 0|\hat{G}^\dagger|0\rangle\right)}{\Delta t} & \text{if } t^* \in [t-\Delta t/2, t+\Delta t/2] \,, \\ 0 & \text{otherwise,} \end{cases} \tag{4.4}$$

which is non-real when $\langle 0|\hat{G}^\dagger|0\rangle \neq 1$. However, it is not clear whether these expectation values can be measured within the construction. Since instantaneous operations are unrealistic, we will not consider them in the rest of the chapter, where we will focus only on self-adjoint interaction terms.

Now, since the algebra of the ideal clock $\mathcal{C}$ is generated by $\left(\hat{T}_{\mathcal{C}}, \hat{H}_{\mathcal{C}}\right)$, we can characterise the types of interactions based on what observables of the clock appear in them. Let us consider each case in detail. While most of these interactions have been considered in the literature, here we show how the compact notation introduced in Sec. 3.4 can be used to easily find the equations governing the relational dynamics of the construction and understand their main features.

4.1.1 Time-Independent Interactions

Let us first consider an interaction $\hat{V}(\hat{H}_{\mathcal{C}})$ that is a function only of $\hat{H}_{\mathcal{C}}$ and some observables of $\mathcal{A}$. Using (3.66), we can rewrite the stationary constraint $\hat{H}_U \, |\Psi_U\rangle\rangle = 0$ as

$$\left[-\hat{V}\left(-i\frac{\mathrm{d}}{\mathrm{d}\hat{T}_{\mathcal{C}}}\right) + i\frac{\mathrm{d}}{\mathrm{d}\hat{T}_{\mathcal{C}}}\right] |\Psi_U\rangle\rangle = \hat{H}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{C}} \, |\Psi_U\rangle\rangle \,. \tag{4.5}$$

By taking the partial inner product with the time states of the clock and using the fact that $\langle\phi_{\mathcal{C}}(t)|\hat{V}(\hat{H}_{\mathcal{C}})\,|\Psi_U\rangle\rangle = \hat{V}(-i\frac{\mathrm{d}}{\mathrm{d}t})\,\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle$, we get

$$\left[-\hat{V}\left(-i\frac{\mathrm{d}}{\mathrm{d}t}\right)+i\frac{\mathrm{d}}{\mathrm{d}t}\right]|\psi_{\mathcal{A}}(t)\rangle = \hat{H}_{\mathcal{A}}|\psi_{\mathcal{A}}(t)\rangle, \tag{4.6}$$

which is not, in general, a type of Schrödinger equation. This can lead to a number of problems, including non-unitarity ones, depending on the specific form of $\hat{V}(\hat{H}_{\mathcal{C}})$. It is instructive to consider some explicit examples.

4.1.1.1 $\hat{V} = \frac{1}{\Lambda}\hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}$

Let us consider the interaction term

$$\hat{V} = \frac{1}{\Lambda}\hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}. \tag{4.7}$$

This interaction term was already considered in [123, 43, 61, 64] and is usually used to approximate a gravitational interaction. In fact, assume that $\mathcal{A}$ has mass $m_{\mathcal{A}}$. Then, we can describe a gravitational interaction between $\mathcal{A}$ and $\mathcal{C}$ in the low-energy regime in the following way [123]

$$\hat{V}_G \coloneqq -\frac{G}{dc^4}\hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}, \tag{4.8}$$

where d is the distance between $\mathcal{A}$ and $\mathcal{C}$.[2] Since gravitational interactions cannot be shielded, we expect interaction terms that are well-approximated by $\hat{V}_G$ in low-energy regimes to appear in any realistic scenario [123]. Crucially, this interaction term has been shown to lead to a gravitational time dilation effect [43, 61, 64].

[2]This potential is obtained from the usual Newtonian potential by performing the substitutions $m_{\mathcal{A}} \to m_{\mathcal{A}} + \hat{H}_{\mathcal{A}}/c^2$ and $m_{\mathcal{C}} \to m_{\mathcal{C}} + \hat{H}_{\mathcal{C}}/c^2$. Here, we are including the rest energy $m_{\mathcal{A}}c^2$ in $\hat{H}_{\mathcal{A}}$ and we are assuming that $\mathcal{C}$ is an ideal clock, meaning that $m_{\mathcal{C}} = 0$.

With the interaction term $\hat{V}$, Eq. (4.6) becomes

$$i\left(\hat{\mathbb{1}}_{\mathcal{A}}+\frac{1}{\Lambda}\hat{H}_{\mathcal{A}}\right)\frac{\mathrm{d}|\psi_{\mathcal{A}}(t)\rangle}{\mathrm{d}t}=\hat{H}_{\mathcal{A}}|\psi_{\mathcal{A}}(t)\rangle. \tag{4.9}$$

Now, if $\left(\hat{\mathbb{1}}_{\mathcal{A}}+\frac{1}{\Lambda}\hat{H}_{\mathcal{A}}\right)$ is invertible, we can rewrite this equation as [64]

$$i\frac{\mathrm{d}|\psi_{\mathcal{A}}(t)\rangle}{\mathrm{d}t}=\left(\hat{\mathbb{1}}_{\mathcal{A}}+\frac{1}{\Lambda}\hat{H}_{\mathcal{A}}\right)^{-1}\hat{H}_{\mathcal{A}}|\psi_{\mathcal{A}}(t)\rangle, \tag{4.10}$$

which is a type of Schrödinger equation with a *self-adjoint* effective Hamiltonian

$$\hat{H}^{eff}_{\mathcal{A}}\coloneqq\left(\hat{\mathbb{1}}_{\mathcal{A}}+\frac{1}{\Lambda}\hat{H}_{\mathcal{A}}\right)^{-1}\hat{H}_{\mathcal{A}}=\frac{\hat{H}_{\mathcal{A}}}{\hat{\mathbb{1}}_{\mathcal{A}}+\hat{H}_{\mathcal{A}}/\Lambda}. \tag{4.11}$$

Eq. (4.10) results in a unitary relational evolution. Its explicit solution, which was first found in [43], is

$$|\psi_{\mathcal{A}}(t)\rangle=e^{-i\hat{H}^{eff}_{\mathcal{A}}t}|\psi_{\mathcal{A}}(0)\rangle. \tag{4.12}$$

Alternatively, this solution can be found using Eq. (2.5). In any case, the state of the Universe can be written as

$$|\Psi_U\rangle\rangle=\int\mathrm{d}t\,e^{-i\hat{H}^{eff}_{\mathcal{A}}t}|\psi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(t)\rangle, \tag{4.13}$$

which is the superposition of all the different relative states of the Universe.

4.1.1.2 $\hat{V}=\hat{X}_{\mathcal{A}}\otimes\hat{H}_{\mathcal{C}}$

If the interaction term is given by

$$\hat{V}=\hat{X}_{\mathcal{A}}\otimes\hat{H}_{\mathcal{C}}, \tag{4.14}$$

with $\hat{X}_{\mathcal{A}}$ a generic observable of $\mathcal{A}$, the effective Hamiltonian of Eq. (4.11) becomes

$$\hat{H}_{\mathcal{A}}^{eff} = \left(\hat{\mathbb{1}}_{\mathcal{A}} + \hat{X}_{\mathcal{A}}\right)^{-1} \hat{H}_{\mathcal{A}}. \tag{4.15}$$

If $\left[\hat{X}_{\mathcal{A}}, \hat{H}_{\mathcal{A}}\right] \neq 0$, the effective Hamiltonian is non-Hermitian and thus does not lead to a unitary evolution. This case was studied in [58] and will be considered again later in this chapter.

4.1.1.3 $\hat{V} = \frac{1}{\Lambda}\hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}^2$

Let us consider the interaction term

$$\hat{V} = \frac{1}{\Lambda}\hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}^2. \tag{4.16}$$

Although this interaction is slightly different from the one in Eq. (4.7), the construction is dramatically affected. In fact, Eq. (4.9) becomes

$$\frac{1}{\Lambda}\hat{H}_{\mathcal{A}}\frac{\mathrm{d}^2|\psi_{\mathcal{A}}(t)\rangle}{\mathrm{d}t^2} + i\frac{\mathrm{d}|\psi_{\mathcal{A}}(t)\rangle}{\mathrm{d}t} = \hat{H}_{\mathcal{A}}|\psi_{\mathcal{A}}(t)\rangle. \tag{4.17}$$

We can find an explicit solution to this equation using Eq. (2.5). First, note that the states $|E_{\mathcal{A}}\rangle|\varepsilon_{\mathcal{C}}\rangle$ are eigenstates of $\hat{H}_U$ with eigenvalue $E + \varepsilon + E\varepsilon^2/\Lambda$. Therefore, using Eq. (2.5) we get

$$|\Psi_U\rangle\rangle = \sum_E \int \mathrm{d}E\, \mathrm{d}\varepsilon\, \mu_E \nu_\varepsilon \delta(E + \varepsilon + E\varepsilon^2/\Lambda)|E_{\mathcal{A}}\rangle|\varepsilon_{\mathcal{C}}\rangle, \tag{4.18}$$

which is non-zero only if $\frac{4E^2}{\Lambda} \leq 1$. Let us assume, for now, that $\Lambda < 0$ so this condition is always satisfied. We can use the properties of the Dirac delta to write

$$\delta(E + \varepsilon + E\varepsilon^2/\Lambda) = \frac{\delta(\varepsilon - \varepsilon_+)}{\sqrt{1 - 4E^2/\Lambda}} + \frac{\delta(\varepsilon - \varepsilon_-)}{\sqrt{1 - 4E^2/\Lambda}}, \tag{4.19}$$

with

$$\varepsilon_\pm \coloneqq \frac{-\Lambda \pm \sqrt{\Lambda^2 - 4E^2\Lambda}}{2E}. \tag{4.20}$$

Then, Eq. (4.18) becomes

$$|\Psi_U\rangle\rangle = \sum_E \frac{\mu_E \nu_{\varepsilon_+}}{\sqrt{1-4E^2/\Lambda}} |E_\mathcal{A}\rangle |\varepsilon_{+,\mathcal{C}}\rangle + \sum_E \frac{\mu_E \nu_{\varepsilon_-}}{\sqrt{1-4E^2/\Lambda}} |E_\mathcal{A}\rangle |\varepsilon_{-,\mathcal{C}}\rangle, \tag{4.21}$$

and taking the partial inner product with the time states of the clock we get

$$|\psi_\mathcal{A}(t)\rangle = e^{-i\hat{H}_+^{eff}t} |\psi_{+,\mathcal{A}}(0)\rangle + e^{-i\hat{H}_-^{eff}t} |\psi_{-,\mathcal{A}}(0)\rangle, \tag{4.22}$$

where we have defined the *self-adjoint* operators

$$\hat{H}_\pm^{eff} \coloneqq \frac{-\Lambda \pm \sqrt{\Lambda^2 - 4\hat{H}_\mathcal{A}^2\Lambda}}{2\hat{H}_\mathcal{A}}, \tag{4.23}$$

and the states

$$|\psi_{\pm,\mathcal{A}}(0)\rangle \coloneqq \sum_E \frac{\mu_E \nu_{\varepsilon_\pm}}{\sqrt{2\pi}\sqrt{1-4E^2/\Lambda}} |E_\mathcal{A}\rangle. \tag{4.24}$$

Eq. (4.21) does not describe a unitary evolution, although each of the two "branches" $|\psi_{\pm,\mathcal{A}}(0)\rangle$ undergoes a unitary evolution with a different effective Hamiltonian. If we focus only on one of these branches, for instance by setting $\nu_{\varepsilon_-} = 0$ for every E such that $\mu_E \neq 0$, then we recover a unitary evolution (see [74] for a similar result).

Finally, when $\Lambda > 0$ we get a similar solution but the only allowed eigenstates of $\hat{H}_\mathcal{A}$ in Eq. (4.18) must have eigenvalue $-\sqrt{\Lambda}/2 < E < \sqrt{\Lambda}/2$. If $E = \pm\sqrt{\Lambda}/2$, the zero of the Dirac delta's argument in Eq. (4.18) has multiplicity 2 and therefore it makes the integral diverge.

4.1.2 Time-Dependent Interactions

Let us now consider an interaction that depends on $\hat{T}_{\mathcal{C}}$. First, we will focus on an interaction $\hat{V}(\hat{T}_{\mathcal{C}})$ that is a function only of $\hat{T}_{\mathcal{C}}$ and the observables of $\mathcal{A}$. This is the type of interaction we used in Sec. 2.3 and does not suffer from non-unitary problems [43]. Given a stationary state of the Universe $|\Psi_U\rangle\rangle$, Eq. (3.66) becomes

$$i\frac{\mathrm{d}\,|\Psi_U\rangle\rangle}{\mathrm{d}\hat{T}_{\mathcal{C}}} = \left(\hat{H}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{C}} + \hat{V}(\hat{T}_{\mathcal{C}})\right)|\Psi_U\rangle\rangle\,. \tag{4.25}$$

By taking the partial inner product with the time states of the clock and using the fact that $\langle\phi_{\mathcal{C}}(t)|\hat{V}(\hat{T}_{\mathcal{C}})\,|\Psi_U\rangle\rangle = \hat{V}(t)\,\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle$, we get

$$i\frac{\mathrm{d}|\psi_{\mathcal{A}}(t)\rangle}{\mathrm{d}t} = \left(\hat{H}_{\mathcal{A}} + \hat{V}(t)\right)|\psi_{\mathcal{A}}(t)\rangle, \tag{4.26}$$

which is the standard Schrödinger equation with a time-dependent potential. This equation was first found in [43]. Since the effective Hamiltonian is self-adjoint, the evolution of $|\Psi_{\mathcal{A}}(t)\rangle\rangle$ is unitary. Given the form of $\hat{V}(\hat{T}_{\mathcal{C}})$ we can then solve Eq. (4.26). The solution is well-known to be [61]

$$|\psi_{\mathcal{A}}(t)\rangle = \mathcal{T}e^{-i\int_0^t \mathrm{d}s\,\left(\hat{H}_{\mathcal{A}}+\hat{V}(s)\right)}|\psi_{\mathcal{A}}(0)\rangle, \tag{4.27}$$

where $\mathcal{T}$ is the time ordering operator.

What if $\hat{V}$ depends also on $\hat{H}_{\mathcal{C}}$? In this case, we can use Eq. (1.45) to rearrange the $\hat{H}_{\mathcal{C}}$ and $\hat{T}_{\mathcal{C}}$ appearing in it so that all the powers of $\hat{T}_{\mathcal{C}}$ are on the left of the powers of $\hat{H}_{\mathcal{C}}$. Let us denote with $\hat{V}^{ord}(\hat{T}_{\mathcal{C}}, \hat{H}_{\mathcal{C}})$ this ordered interaction term. We can then rewrite the stationarity constraint as

$$\left[-\hat{V}^{ord}\left(\hat{T}_{\mathcal{C}}, -i\frac{\mathrm{d}}{\mathrm{d}\hat{T}_{\mathcal{C}}}\right) + i\frac{\mathrm{d}}{\mathrm{d}\hat{T}_{\mathcal{C}}}\right]|\Psi_U\rangle\rangle = \hat{H}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{C}}\,|\Psi_U\rangle\rangle\,, \tag{4.28}$$

and take the partial inner product with the time states of the clock to get

$$\left[-\hat{V}^{ord}\left(t,-i\frac{\mathrm{d}}{\mathrm{d}t}\right)+i\frac{\mathrm{d}}{\mathrm{d}t}\right]|\psi_{\mathcal{A}}(t)\rangle=\hat{H}_{\mathcal{A}}|\psi_{\mathcal{A}}(t)\rangle. \tag{4.29}$$

This equation will not be, in general, a type of Schrödinger equation.

4.1.3 Non-Unitarity

As we have just seen, interactions give rise to various types of relational dynamics, some of which are non-unitary. Characterising which interactions lead to non-unitarity is, in general, not trivial. However, we can restrict our attention to certain classes of interaction. For instance, most dynamical equations of physical interest in non-relativistic quantum theory are of first order in the time derivative. In fact, the state vector of any closed quantum system evolving unitarily in a sufficiently well-behaved manner must obey a Schrödinger-like equation with a self-adjoint Hamiltonian. Therefore, we can focus on interaction terms that contain at most first powers of $\hat{H}_C$. In general, such terms can be written as

$$\hat{V}=\hat{V}_1(\hat{T}_{\mathcal{C}})+\hat{V}_2(\hat{T}_{\mathcal{C}})\hat{H}_{\mathcal{C}}, \tag{4.30}$$

with $\hat{V}_2^{\dagger}=\hat{V}_2$ but $\hat{V}_1$ not, in general, self-adjoint. Eq. (4.29) then becomes

$$i\left[\hat{V}_2\left(t\right)+\hat{\mathbb{1}}_{\mathcal{A}}\right]\frac{\mathrm{d}|\psi_{\mathcal{A}}(t)\rangle}{\mathrm{d}t}=\left[\hat{H}_{\mathcal{A}}+\hat{V}_1(t)\right]|\psi_{\mathcal{A}}(t)\rangle. \tag{4.31}$$

If the *self-adjoint* operator $\left[\hat{V}_2\left(t\right)+\hat{\mathbb{1}}_{\mathcal{A}}\right]$ is invertible, we can further rewrite this equation as

$$i\frac{\mathrm{d}|\psi_{\mathcal{A}}(t)\rangle}{\mathrm{d}t}=\left[\hat{V}_2\left(t\right)+\hat{\mathbb{1}}_{\mathcal{A}}\right]^{-1}\left[\hat{H}_{\mathcal{A}}+\hat{V}_1(t)\right]|\psi_{\mathcal{A}}(t)\rangle, \tag{4.32}$$

which is a type of Schrödinger equation. Now, the unitarity of the evolution is guaranteed if the operator $\left[\hat{V}_2(t) + \hat{\mathbb{1}}_{\mathcal{A}}\right]^{-1}\left[\hat{H}_{\mathcal{A}} + \hat{V}_1(t)\right]$ is self-adjoint.

If $\left[\hat{V}_2(t) + \hat{\mathbb{1}}_{\mathcal{A}}\right]$ is not invertible, then it must have some eigenstates $\{|\chi_j\rangle\}_j$ with eigenvalue 0. By multiplying Eq. (4.31) on the left with $\langle\chi_j|$ we get

$$\langle\chi_j|\left[\hat{H}_{\mathcal{A}} + \hat{V}_1(t)\right]|\psi_{\mathcal{A}}(t)\rangle = 0, \tag{4.33}$$

for every j. This condition is not of immediate interpretation, and, as we will shortly see in an example, it can restrict the type of interaction terms or the allowed states of $\mathcal{A}$.

Another possible restriction on the types of interaction excludes time-dependent terms. While these terms were essential in modelling measurements of the clock in Chapter 2.3, time-dependent interactions are usually considered as effective interactions and thus less fundamental. We have studied time-independent interactions that are first order in $\hat{H}_{\mathcal{C}}$ in Sec. 4.1.1.1. Let us now consider the meaning of Eq. (4.33) in this case. Assuming the interaction is given by $\hat{V} = \frac{1}{\Lambda}\hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}$, if $\hat{H}_{\mathcal{A}}$ has an eigenstate $|-\Lambda_{\mathcal{A}}\rangle$ with eigenvalue $-\Lambda$, then $\left[\hat{H}_{\mathcal{A}}/\Lambda + \hat{\mathbb{1}}_{\mathcal{A}}\right]$ is not invertible because it has a null eigenvalue. The condition of Eq. (4.33) becomes

$$-\Lambda\,\langle-\Lambda_{\mathcal{A}}|\psi_{\mathcal{A}}(t)\rangle = 0, \tag{4.34}$$

which means that $\langle-\Lambda_{\mathcal{A}}|\psi_{\mathcal{A}}(t)\rangle = 0\ \forall t$. Therefore, the allowed states of $\mathcal{A}$ must not contain the state $|-\Lambda_{\mathcal{A}}\rangle$. With this restriction, we have that the state of $\mathcal{A}$ at time t is given by (neglecting degeneracies in $\hat{H}_{\mathcal{A}}$ for ease of notation)

$$|\psi_{\mathcal{A}}(t)\rangle = \sum_{E\neq-\Lambda} \mu_E e^{-i\frac{E}{1+E/\Lambda}}|E_{\mathcal{A}}\rangle, \tag{4.35}$$

which describes a unitary evolution.

In conclusion, although characterising the types of interaction that lead to a unitary relational dynamics is a difficult task, the differential notation we introduced in Chapter 3.4 can be used to quickly understand whether a specific interaction will lead to a unitary relational dynamics in many cases of physical interest.

4.1.4 Relativistic Interactions

Describing interactions as in Eq. (4.1) is quite general in non-relativistic quantum theory but is usually not appropriate in relativistic settings. Deriving suitable Hamiltonian constraints in a relativistic setting requires some care, and we refer the reader to [40, 74] for a discussion. Here, we review some of the constraints found in those works. This will be important for a broader discussion on the role of interactions in later sections. Instead of the more rigorous approach of [40], here we show how to derive the relevant constraint in a heuristic way similar to how the Klein-Gordon equation is derived in relativistic wave mechanics.

Let us start from the relativistic energy-momentum equation for a particle of mass m: $E^2 = \mathbf{p}^2c^2 + m^2c^4$. The Klein-Gordon equation is usually obtained by promoting $\mathbf{p}$ to the momentum operator of the particle and replacing $E \to i\hbar\frac{\partial}{\partial t}$. In timeless quantum theory, if we add an external ideal clock $\mathcal{C}$, it is easy to see that the Klein-Gordon equation for the particle $\mathcal{A}$ can be recovered from imposing the following constraint on the state vector of $\mathcal{A}+\mathcal{C}$ [74]

$$\hat{C}_U = \hat{H}_{\mathcal{C}}^2 - \hat{\mathbf{p}}_{\mathcal{A}}^2c^2 - m_{\mathcal{A}}^2c^4, \tag{4.36}$$

which can be thought of as deriving from the substitution $i\hbar\frac{\partial}{\partial t} \to -\hat{H}_{\mathcal{C}}$ in the Klein-Gordon equation.

Let us now add some internal structure to the particle $\mathcal{A}$ by enlarging its Hilbert space to $\mathscr{H}_{\mathcal{A}_{TOT}} = \mathscr{H}_{\mathcal{A}_{CM}} \otimes \mathscr{H}_{\mathcal{A}_{in}}$, where $\mathscr{H}_{CM} = L^2(\mathbb{R}^3)$ is the Hilbert space of

the centre of mass and $\mathscr{H}_{\mathcal{A}_{in}}$ is the Hilbert space of the internal degrees of freedom [40]. If the internal degrees of freedom are regulated by the Hamiltonian $\hat{H}_{\mathcal{A}_{in}}$, the constraint becomes [40]

$$\hat{C}_U = \hat{H}_{\mathcal{C}}^2 - \hat{\mathbf{p}}_{\mathcal{A}}^2 c^2 - \left(m_{\mathcal{A}} c^2 + \hat{H}_{\mathcal{A}_{in}}\right)^2 . \tag{4.37}$$

Now, we can "fix" the particle $\mathcal{A}$ by assuming that it is in an 0-eigenstate of $\hat{\mathbf{p}}_{\mathcal{A}}$. The meaning of this is that $\mathcal{A}$ is fixed with respect to the (inertial) clock $\mathcal{C}$. By tracing out the degrees of freedom associated with the centre of mass of $\mathcal{A}$, we obtain the following constraint on $\mathscr{H}_{\mathcal{A}_{in}} \otimes \mathscr{H}_{\mathcal{C}}$ [74]

$$\hat{C}_U = \left(m_{\mathcal{A}} c^2 + \hat{H}_{\mathcal{A}_{in}}\right)^2 - \hat{H}_{\mathcal{C}}^2, \tag{4.38}$$

which can be written as [74]

$$\hat{C}_U = \left(m_{\mathcal{A}} c^2 + \hat{H}_{\mathcal{A}_{in}} + \hat{H}_{\mathcal{C}}\right)\left(m_{\mathcal{A}} c^2 + \hat{H}_{\mathcal{A}_{in}} - \hat{H}_{\mathcal{C}}\right). \tag{4.39}$$

Each of the two terms in Eq. (4.39), considered independently, leads to a Schrödinger equation for the internal states of $\mathcal{A}$ relative to the clock $\mathcal{C}$. Since these terms commute, it is enough that the state of $\mathcal{A}_{in} + \mathcal{C}$ vanishes on one of the two. In this case, the relational dynamics will be unitary. However, more general solutions do not evolve unitarily, similarly to what we have seen in Sec. 4.1.1.3. Also, note that if we focus on only one of the two terms, the term mc^2 does not alter the relational dynamics of $\mathcal{A}_{in}$ as it represents just an energy shift.

If $\mathcal{A}$ is moving with respect to the clock $\mathcal{C}$, the constrain can be written as [40]

$$\hat{C}_U = \left(\hat{H}_{\mathcal{C}} - \sqrt{\hat{\mathbf{p}}_{\mathcal{A}}^2 c^2 + \left(m_{\mathcal{A}} c^2 + \hat{H}_{\mathcal{A}_{in}}\right)^2}\right)\left(\hat{H}_{\mathcal{C}} + \sqrt{\hat{\mathbf{p}}_{\mathcal{A}}^2 c^2 + \left(m_{\mathcal{A}} c^2 + \hat{H}_{\mathcal{A}_{in}}\right)^2}\right). \tag{4.40}$$

We can then focus on states of the Universe that vanish on the term [40]

$$\hat{C}_U^+ \coloneqq \hat{H}_{\mathcal{C}} + \sqrt{\hat{\mathbf{p}}_{\mathcal{A}}^2 c^2 + \left(m_{\mathcal{A}} c^2 + \hat{H}_{\mathcal{A}_{in}}\right)^2}. \tag{4.41}$$

The authors of [40] expand the square root term in series and find that the constraint can be written, to first order in relativistic contributions, as

$$\hat{C}_U^+ \approx \hat{H}_{\mathcal{C}} + \hat{H}_{\mathcal{A}_{CM}} + \hat{H}_{\mathcal{A}_{in}} + \hat{V} + m_{\mathcal{A}} c^2, \tag{4.42}$$

with $\hat{H}_{\mathcal{A}_{CM}} \coloneqq \frac{\hat{\mathbf{p}}_{\mathcal{A}}^2}{2m_{\mathcal{A}}}$ and

$$\hat{V} \coloneqq -\frac{1}{m_{\mathcal{A}} c^2}\left[\hat{H}_{\mathcal{A}_{CM}} \otimes \hat{H}_{\mathcal{A}_{in}} + \frac{1}{2}\left(\hat{H}_{\mathcal{A}_{CM}}\right)^2\right], \tag{4.43}$$

which is similar to the interaction term discussed in Sec. 4.1.1.1. The interpretation of such a constraint is that, in the frame of reference of $\mathcal{C}$, the centre of mass of $\mathcal{A}$ couples to its internal degrees of freedom. The authors of [40] then assume that $\mathcal{A}$ has an internal clock and, using time POVMs, show that the constraint of Eq. (4.42) leads to a time dilation effect that coincides with the usual relativistic one in some circumstances but deviates from it in others. Crucially, these deviations are in principle experimentally verifiable.

In conclusion, we have seen that accounting for relativistic effects usually leads to constraints that are more complex than the one of Eq. (4.1). Moreover, coupling terms between systems that are not interacting (in their frame of reference) seem to arise when moving to other reference frames. While the approach outlined so far suffers from the same problems of relativistic wave mechanics that were later solved by quantum field theory, it represents a first step towards a relativistic version of timeless quantum theory. It also shows that some types of interactions with the clock are inevitable and play a crucial role in accounting for relativistic effects such

as time dilation.

4.2 Interactions and Clocks

4.2.1 The Dynamical Effect of Interactions

Consider again the Hamiltonian of the Universe in Eq. (4.1). While $\hat{V}$ represents an interaction term between $\mathcal{A}$ and $\mathcal{C}$, it does not have any consequences at the dynamical level because the state of the Universe is stationary. To better understand the role of $\hat{V}$, consider the stationary state of Eq. (2.5) given an initial state $|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle$

$$\left|\tilde{\Psi}_U\right\rangle\!\!\right\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-i\left(\hat{H}_{\mathcal{A}}+\hat{H}_{\mathcal{C}}+\hat{V}\right)t}|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle. \tag{4.44}$$

As discussed in Sec. 2.1, we can interpret $\left|\tilde{\Psi}_U\right\rangle\!\!\right\rangle$ as the superposition of the states obtained from evolving $|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle$ with the unitary $e^{-i\left(\hat{H}_{\mathcal{A}}\otimes\hat{\mathbb{1}}_{\mathcal{C}}+\hat{\mathbb{1}}_{\mathcal{C}}\otimes\hat{H}_{\mathcal{C}}+\hat{V}\right)t}$. In this picture, $\hat{V}$ acts as a usual interaction term between $\mathcal{A}$ and $\mathcal{C}$ at the *relational* dynamical level.

4.2.2 The Effect of Interactions on Timekeeping

Eq. (4.44) also shows an important effect of interactions. While in a Universe without interactions the unitary $e^{-i\left(\hat{H}_{\mathcal{A}}+\hat{H}_{\mathcal{C}}\right)t}$ transforms the initial state $|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle$ into the state $e^{-i\hat{H}_{\mathcal{A}}t}|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(t)\rangle$ that has a well-defined clock time, the interaction term $\hat{V}$ can transform $|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle$ into a state without a well-defined time. This is because interactions usually entangle a separable state. For instance, assuming

$\hat{V} = \frac{1}{\Lambda}\hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}$, we have that

$$e^{-i\left(\hat{H}_{\mathcal{A}}+\hat{H}_{\mathcal{C}}+\hat{V}\right)t}|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle = \sum_{E} \mu_E e^{-iEt}|E_{\mathcal{A}}\rangle|\phi_{\mathcal{C}}(t(1+E/\Lambda))\rangle, \tag{4.45}$$

which is an entangled state unless $|\chi_{\mathcal{A}}(0)\rangle$ is an energy eigenstate of $\hat{H}_{\mathcal{A}}$. A notable exception is when $\hat{V} = \hat{V}(\hat{T}_{\mathcal{C}})$. In this case, $e^{-i\left(\hat{H}_{\mathcal{A}}+\hat{H}_{\mathcal{C}}+\hat{V}\right)t}|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle$ remains separable at all times.

What is the meaning of Eq. (4.45)? We can interpret the equation as telling us that the clock cannot keep a precise time in the presence of interactions. This becomes clearer when we examine a Universe consisting of a system $\mathcal{A}$ and two ideal clocks $\mathcal{B}$ and $\mathcal{C}$. Let us assume the Hamiltonian

$$\hat{H}_U = \hat{H}_{\mathcal{A}} + \hat{H}_{\mathcal{B}} + \hat{H}_{\mathcal{C}} + \hat{V}_{\mathcal{AB}}, \tag{4.46}$$

where $\hat{V}_{\mathcal{AB}} = \frac{1}{\Lambda}\hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{B}}$ is an interaction term between $\mathcal{A}$ and $\mathcal{B}$ only. Let us consider the following stationary state of the Universe

$$\left|\tilde{\Psi}_U\right\rangle\!\!\right\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-i\left(\hat{H}_{\mathcal{A}}+\hat{H}_{\mathcal{B}}+\hat{V}_{\mathcal{AB}}\right)t}|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{B}}(0)\rangle|\phi_{\mathcal{C}}(t)\rangle. \tag{4.47}$$

The state of $\mathcal{AB}$ relative to the clock $\mathcal{C}$ showing the time t is

$$|\psi_{\mathcal{AB}}(t)\rangle\rangle = e^{-i\left(\hat{H}_{\mathcal{A}}+\hat{H}_{\mathcal{B}}+\hat{V}_{\mathcal{AB}}\right)t}|\chi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{B}}(0)\rangle, \tag{4.48}$$

which is the same as in Eq. (4.45), meaning that the clock $\mathcal{B}$ is in a superposition of different times for each time t of $\mathcal{C}$. The "spread" in the clock readings depends on the coupling strength Λ. The greater the coupling strength (i.e., the smaller the value of Λ), the larger the spread.[3]

[3] A consequence of this spread is that if some time reading of $\mathcal{B}$ "triggers" a certain event of $\mathcal{A}$, then the time of this event is not well-defined from the point of view of $\mathcal{C}$ [61].

Eqs. (4.48) and (4.45) suggest that interactions affect the clock's timekeeping capability. When the time is t according to the ideal clock $\mathcal{C}$, $\mathcal{B}$ is in a superposition of different time readings, meaning that $\mathcal{C}$ cannot be "certain" about the time read by $\mathcal{B}$. Moreover, each of these times is different from t by a factor $1 + E/\Lambda$. Note that this problem was less marked in a similar scenario in [40] because the authors considered POVMs. These are already characterised by some uncertainty on the time readings, and interactions simply change the probability distribution of the readings [40]. Since in Chapter 2 we argued for using projective measurements and for clock readings instantiating well-defined higher-level structures, the issue becomes more pressing here. Is an observer moving relative to us automatically in a superposition of different time states as in Eq. (4.45)?

To guarantee that ideal clocks maintain their timekeeping qualities, we could postulate that they must not interact with other systems. However, we have seen that some interactions are likely inevitable (Sec. 4.1.1.1), while others appear in relativistic settings when changing reference frames (Sec. 4.1.4). Fortunately, these interactions lead to a unitary relational dynamics and predict time dilation effects that agree with the relativistic ones in some regimes. Therefore, these types of interactions should be considered as a good feature of timeless quantum theory.

So what should we make of the uncertainty of the clock readings caused by these interactions? We can observe that the problem seems to vanish when we consider macroscopic systems. To see this, consider again the Hamiltonian of Eq. (4.46) with $\hat{V}_{\mathcal{AB}} = \frac{1}{\Lambda}\hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{B}}$. As we have seen in Sections 4.1.1.1-4.1.4, this interaction can model both a gravitational interaction and a "relativistic interaction" due to a change of reference frame. $\hat{H}_U$ can then be interpreted either as describing the clock $\mathcal{B}$ close to a gravitational source $\mathcal{A}$ and the clock $\mathcal{C}$ far from it, or $\mathcal{C}$ being at rest and $\mathcal{A} - \mathcal{B}$ moving with respect to it. In both cases, Λ will have negative values. Now, we can calculate the average time reading $\overline{\tau_{\mathcal{B}}}$ of the clock $\mathcal{B}$ given that the

clock $\mathcal{C}$ reads the time t in the following way

$$\overline{\tau_{\mathcal{B}}}(t) \coloneqq \frac{\langle\langle\psi_{\mathcal{AB}}(t)|\,\hat{T}_{\mathcal{B}}\,|\psi_{\mathcal{AB}}(t)\rangle\rangle}{\langle\langle\psi_{\mathcal{AB}}(t)\,|\psi_{\mathcal{AB}}(t)\rangle\rangle} = t\left(1+\frac{\overline{E_{\mathcal{A}}}}{\Lambda}\right), \tag{4.49}$$

where we have used Eqs. (4.48) and (4.45) and defined $\overline{E_{\mathcal{A}}} \coloneqq \langle\chi_{\mathcal{A}}(0)|\hat{H}_{\mathcal{A}}|\chi_{\mathcal{A}}(0)\rangle$, the average energy of $\mathcal{A}$ (which is constant in time due to the form of $\hat{H}_U$).[4] It is also easy to show that the variance of these readings is given by

$$(\Delta\tau_{\mathcal{B}}(t))^2 = \frac{t^2}{\Lambda^2}\,(\Delta E_{\mathcal{A}})^2\,, \tag{4.50}$$

with $(\Delta E_{\mathcal{A}})^2 \coloneqq \langle\chi_{\mathcal{A}}(0)|\hat{H}^2_{\mathcal{A}}|\chi_{\mathcal{A}}(0)\rangle - \langle\chi_{\mathcal{A}}(0)|\hat{H}_{\mathcal{A}}|\chi_{\mathcal{A}}(0)\rangle^2$. Therefore, the relative spread of the time readings (in absolute value) is given by

$$\left|\frac{\Delta\tau_{\mathcal{B}}}{\overline{\tau_{\mathcal{B}}}}\right| = \left|\frac{\Delta E_{\mathcal{A}}}{\Lambda+\overline{E_{\mathcal{A}}}}\right|. \tag{4.51}$$

Considering $|\Lambda| \gg \left|\overline{E_{\mathcal{A}}}\right|$, we have $|\Delta\tau_{\mathcal{B}}/\overline{\tau_{\mathcal{B}}}| \approx |\Delta E_{\mathcal{A}}/\Lambda|$. Now, if $\mathcal{A}$ is a macroscopic system, we usually have $\Delta E_{\mathcal{A}}/\overline{E_{\mathcal{A}}} \approx 0$, so that $|\Delta\tau_{\mathcal{B}}/\overline{\tau_{\mathcal{B}}}|$ is negligible compared to the relativistic correction to the time readings $\overline{E_{\mathcal{A}}}/\Lambda$. Therefore, even in these gravitational/relativistic scenarios, a macroscopic system will read a single time on its clock according to any other clock.

The case $|\Lambda| \approx \left|\overline{E_{\mathcal{A}}}\right|$ requires more care. This regime is achieved for values of the total mass (rest+dynamical) $M_{\mathcal{A}} \approx dc^2/G$ when considering the gravitational interaction of Eq. (4.8), or for $\langle\hat{p}_{\mathcal{A}}\rangle \approx m_{\mathcal{A}}c$ when considering the relativistic interaction of Eq. (4.43). These are strongly relativistic regimes where we do not expect our simplified description to hold. Nevertheless, we explore some simple cases in Appendix C.

[4] Note that we have non-rigorously assumed that the divergent terms $\langle\phi_{\mathcal{B}}(0)|\phi_{\mathcal{B}}(0)\rangle$ at the numerator and denominator cancel out. This would be the case for the states of any realistic clock.

Finally, let us consider a microscopic particle with an internal clock moving relative to a clock at rest. According to the clock at rest, the particle is in a superposition of states corresponding to different time readings of the internal clock. Is this superposition problematic? No, for the same reason why we do not have a problem with particles being in a superposition of different states every day in the laboratory. The spread in the time readings is also not problematic because it simply indicates that the microscopic clock of the particle, when taken alone, is not a good clock—after all, no realistic clock uses a single particle to tell time. In this case, the most relevant aspect is the average time reading of the clock given by Eq. (4.49).

To summarise, while clocks should *ideally* not interact with other systems in their "frame of reference", some types of interactions are a crucial aspect of timeless quantum theory since they account for some relativistic effects to which we know all clocks must be subject. Moreover, these types of interactions preserve the unitarity of the relational dynamics and, apart from extremely strong relativistic regimes, alter the timekeeping properties of realistic clocks in a negligible way. Other kinds of interactions, instead, can spoil the properties of the clock and usually lead to a non-unitary relational dynamics. For example, some authors have shown that if the clock is subject to an interaction term of the type $\hat{V} = \hat{X}_{\mathcal{A}}\hat{H}_{\mathcal{C}}$, with $\left[\hat{X}_{\mathcal{A}}, \hat{H}_{\mathcal{A}}\right] \neq 0$, then the relational dynamics is non-unitary [58]. In that context, the authors link this interaction term to the clock's acceleration, concluding that non-inertial quantum clocks lead to a non-unitary dynamics [58].

The discussion of this section fits within the more general study of clocks in special and general relativity. In fact, it is well-known that accelerations and other effects can affect the timekeeping qualities of real clocks [124]. In relativity, it is common to assume that there are clocks affected only by time dilation due to their instantaneous speed with respect to an inertial frame and not by their acceleration. The validity of this assumption, sometimes called "the clock hypothesis" [125], has

been the subject of debate.[5] Some authors have argued for the existence of clocks that satisfy the clock hypothesis in general relativity [126], while others have shown inconsistencies in quantum theory [127, 128, 129, 130]. The results obtained in [58] seem to corroborate these inconsistencies within timeless quantum theory (see the next section), but we believe this aspect requires further attention. It would also be important to extend these considerations to other types of clocks, such as the one discussed in [126]. We leave these points open for future research.

4.2.3 Further Remarks on Non-unitarity

Let us further analyse in this context the non-unitarity effects that we have discussed in Sec. 4.1.3. First, note that while we have seen that, in general, an interaction between a system $\mathcal{A}$ and a clock $\mathcal{B}$ can lead to a non-unitary dynamics from $\mathcal{B}$'s perspective, there is no such problem from the perspective of another clock $\mathcal{C}$ not interacting with $\mathcal{A}$ or $\mathcal{B}$. This was already discussed in [58] and can be easily seen by looking at the states of $\mathcal{AB}$ relative to $\mathcal{C}$ in Eq. (4.48): clearly $|\psi_{\mathcal{AB}}(t)\rangle\rangle$ evolves unitarily. The non-unitarity problem arises when we look at the state of $\mathcal{AC}$ from $\mathcal{B}$'s perspective [58]. For example, consider the interaction $\hat{V}_{\mathcal{AB}} = \hat{X}_{\mathcal{A}}\hat{H}_{\mathcal{B}}$, which we know is problematic when $\left[\hat{X}_{\mathcal{A}}, \hat{H}_{\mathcal{A}}\right] \neq 0$. Then, the stationarity constraint deriving from the Hamiltonian of Eq. (4.46) can be written as

$$i\frac{\mathrm{d}\,|\Psi_U\rangle\rangle}{\mathrm{d}\hat{T}_{\mathcal{B}}} = \left(\hat{H}^{eff}_{\mathcal{A}} + \hat{H}^{eff}_{\mathcal{C}}\right)|\Psi_U\rangle\rangle\,, \tag{4.52}$$

with $\hat{H}^{eff}_{\mathcal{A}} = \left(1 + \hat{X}_{\mathcal{A}}/\Lambda\right)^{-1}\hat{H}_{\mathcal{A}}$ and $\hat{H}^{eff}_{\mathcal{C}} = \left(1 + \hat{X}_{\mathcal{A}}/\Lambda\right)^{-1}\hat{H}_{\mathcal{C}}$.
Since $\left(\hat{H}^{eff}_{\mathcal{A}}\right)^{\dagger} \neq \hat{H}^{eff}_{\mathcal{A}}$, the evolution of $\mathcal{AC}$ from $\mathcal{B}$'s perspective is non-unitary. Also, note that $\left(\hat{H}^{eff}_{\mathcal{C}}\right)^{\dagger} = \hat{H}^{eff}_{\mathcal{C}}$ and so it may be possible to separate the evolution

[5] More precisely, the clock hypothesis postulates that there exist clocks that tick at the same rate as a momentarily co-moving ideal inertial clock, regardless of their acceleration.

of the state of $\mathcal{C}$, which should be unitary.

Now, consider the following quantity $\hat{\gamma}^{-1}$ (the exponent here is just a notation that will be useful later, it does not indicate the inverse of the operator) defined in [58]

$$\hat{\gamma}^{-1} \coloneqq i\left[\hat{H}_{\mathcal{AB}}, \hat{T}_{\mathcal{B}}\right], \tag{4.53}$$

with $\hat{H}_{\mathcal{AB}} \coloneqq \hat{H}_{\mathcal{A}} + \hat{H}_{\mathcal{B}} + \hat{V}_{\mathcal{AB}}$. The authors of [58] call this quantity the "rate of the clock" $\mathcal{B}$ from the perspective of the $\mathcal{C}$, but we will call it the "rate operator" as it is not a c-number. We can check that $\hat{\gamma}^{-1}$ is related to the usual rate of $\mathcal{B}$ from $\mathcal{C}$'s perspective by calculating how the time readings of $\mathcal{B}$ (which we approximate with $\overline{\tau_{\mathcal{B}}}(t)$) change for small differences in the time readings t of $\mathcal{C}$, that is

$$\begin{aligned}
\frac{1}{\gamma(t)} \coloneqq \frac{\mathrm{d}\overline{\tau_{\mathcal{B}}}(t)}{\mathrm{d}t} &= \lim_{\mathrm{d}t\to 0} \frac{\langle\langle \psi_{\mathcal{AB}}(t+\mathrm{d}t)|\, \hat{T}_{\mathcal{B}}\, |\psi_{\mathcal{AB}}(t+\mathrm{d}t)\rangle\rangle - \langle\langle \psi_{\mathcal{AB}}(t)|\, \hat{T}_{\mathcal{B}}\, |\psi_{\mathcal{AB}}(t)\rangle\rangle}{\langle\langle \psi_{\mathcal{AB}}(t)\,|\psi_{\mathcal{AB}}(t)\rangle\rangle\, \mathrm{d}t} \\
&= \lim_{\mathrm{d}t\to 0} \frac{\langle\langle \psi_{\mathcal{AB}}(t)|\, e^{i\hat{H}_{\mathcal{AB}}\mathrm{d}t}\hat{T}_{\mathcal{B}}e^{-i\hat{H}_{\mathcal{AB}}\mathrm{d}t} - \hat{T}_{\mathcal{B}}\, |\psi_{\mathcal{AB}}(t)\rangle\rangle}{\langle\langle \psi_{\mathcal{AB}}(t)\,|\psi_{\mathcal{AB}}(t)\rangle\rangle\, \mathrm{d}t} \\
&= i\frac{\langle\langle \psi_{\mathcal{AB}}(t)|\left[\hat{H}_{\mathcal{AB}}, \hat{T}_{\mathcal{B}}\right]|\psi_{\mathcal{AB}}(t)\rangle\rangle}{\langle\langle \psi_{\mathcal{AB}}(t)\,|\psi_{\mathcal{AB}}(t)\rangle\rangle} = \frac{\langle\langle \psi_{\mathcal{AB}}(t)|\, \hat{\gamma}^{-1}\, |\psi_{\mathcal{AB}}(t)\rangle\rangle}{\langle\langle \psi_{\mathcal{AB}}(t)\,|\psi_{\mathcal{AB}}(t)\rangle\rangle},
\end{aligned} \tag{4.54}$$

where we have used the fact that

$$e^{i\hat{H}_{\mathcal{AB}}\mathrm{d}t}\hat{T}_{\mathcal{B}}e^{-i\hat{H}_{\mathcal{AB}}\mathrm{d}t} = \hat{T}_{\mathcal{B}} + i\mathrm{d}t\left[\hat{H}_{\mathcal{AB}}, \hat{T}_{\mathcal{B}}\right] + \frac{i(\mathrm{d}t)^2}{2!}\left[\hat{H}_{\mathcal{AB}}, \left[\hat{H}_{\mathcal{AB}}, \hat{T}_{\mathcal{B}}\right]\right] + \dots. \tag{4.55}$$

For the interaction term discussed above, assuming that $\mathcal{B}$ is an ideal clock, we have that [58]

$$\hat{\gamma}^{-1} = 1 + \hat{X}_{\mathcal{A}}. \tag{4.56}$$

The authors of [58] then relate $\hat{X}_{\mathcal{A}}$ to an acceleration of the clock, arguing that the clock rate is therefore affected by it (and thus the clock would likely violate the clock hypothesis as mentioned in the previous section). Here, we are not concerned with the clock hypothesis but rather with how $\hat{\gamma}^{-1}$ is linked to the non-unitarity of the

relational dynamics. In fact, we observe that the interaction just considered changes the clock rate in time, while the gravitational/relativistic interactions previously considered—which lead to a unitary relational dynamics—have a constant clock rate. It is not immediate to see this from $\hat{\gamma}^{-1}$. Rather, consider the following expression for $\gamma(t)^{-1}$

$$\begin{aligned}\frac{1}{\gamma(t)} &= \frac{\langle\langle\psi_{\mathcal{AB}}(t)|\,\hat{\gamma}^{-1}\,|\psi_{\mathcal{AB}}(t)\rangle\rangle}{\langle\langle\psi_{\mathcal{AB}}(t)\,|\psi_{\mathcal{AB}}(t)\rangle\rangle} = \frac{\langle\langle\psi_{\mathcal{AB}}(0)|\,e^{i\hat{H}_{\mathcal{AB}}t}\hat{\gamma}^{-1}e^{-i\hat{H}_{\mathcal{AB}}t}\,|\psi_{\mathcal{AB}}(0)\rangle\rangle}{\langle\langle\psi_{\mathcal{AB}}(0)\,|\psi_{\mathcal{AB}}(0)\rangle\rangle} \\ &= \frac{\langle\langle\psi_{\mathcal{AB}}(0)|\,\hat{\gamma}^{-1} + it\left[\hat{H}_{\mathcal{AB}},\hat{\gamma}^{-1}\right] + \ldots\,|\psi_{\mathcal{AB}}(0)\rangle\rangle}{\langle\langle\psi_{\mathcal{AB}}(0)\,|\psi_{\mathcal{AB}}(0)\rangle\rangle} \\ &= \frac{1}{\gamma(0)} + it\frac{\langle\langle\psi_{\mathcal{AB}}(0)|\left[\hat{H}_{\mathcal{AB}},\hat{\gamma}^{-1}\right]|\psi_{\mathcal{AB}}(0)\rangle\rangle}{\langle\langle\psi_{\mathcal{AB}}(0)\,|\psi_{\mathcal{AB}}(0)\rangle\rangle} + \ldots,\end{aligned} \tag{4.57}$$

where we have used again Eq. (4.55). For $\hat{\gamma}^{-1}$ as in Eq. (4.56), we have that $\left[\hat{H}_{\mathcal{AB}},\hat{\gamma}^{-1}\right] = \left[\hat{H}_{\mathcal{A}},\hat{X}_{\mathcal{A}}\right] \neq 0$ leading to $\frac{1}{\gamma(t)} \neq \frac{1}{\gamma(0)}\ \ \forall t \neq 0$.[6] Interestingly, the condition $\left[\hat{H}_{\mathcal{A}},\hat{X}_{\mathcal{A}}\right] \neq 0$ was exactly the source of non-unitarity in Sec. 4.1.1.2. Instead, we can easily verify that the interactions leading to a unitary relational dynamics discussed so far are such that $\gamma(t) = \gamma(0)$.

Therefore, one could investigate whether the condition $\gamma(t) = \gamma(0)\ \ \forall t$ guarantees that the relational dynamics is unitary. If we want the condition to hold for any initial state $|\psi_{\mathcal{AB}}(0)\rangle\rangle$, we must have

$$\left[\hat{H}_{\mathcal{AB}},\hat{\gamma}^{-1}\right] = i\left[\hat{H}_{\mathcal{AB}},\left[\hat{H}_{\mathcal{AB}},\hat{T}_{\mathcal{B}}\right]\right] = 0, \tag{4.58}$$

which, assuming that $\mathcal{B}$ is an ideal clock, we can also write as

$$\left[\hat{H}_{\mathcal{AB}},\frac{\mathrm{d}\hat{H}_{\mathcal{AB}}}{\mathrm{d}\hat{H}_{\mathcal{B}}}\right] = 0, \tag{4.59}$$

[6]The only exception is when $\mathcal{A}$ starts in an eigenstate of $\hat{H}_{\mathcal{A}}$. In this case, $\frac{1}{\gamma(t)} = \frac{1}{\gamma(0)}$, but note that in this case the relational evolution of $\mathcal{A}$ is unitary.

where we have defined the operator derivative

$$\frac{\mathrm{d}\cdot}{\mathrm{d}\hat{H}_{\mathcal{B}}} \coloneqq -i\left[\hat{T}_{\mathcal{B}}, \cdot\right], \tag{4.60}$$

similarly to Eq. (3.70).

However, note that this condition is not strictly valid when $\hat{V}_{AB} = \frac{1}{\Lambda}\hat{H}_{\mathcal{A}} \otimes \hat{H}^2_{\mathcal{B}}$, which satisfies Eq. (4.59) but generally leads to a non-unitary relational dynamics (although there are two "branches" that evolve unitarily as discussed in Sec. 4.1.1.3). Therefore, additional requirements might be necessary. For example, one could ask that the rate of $\gamma(t)^{-1}$ must be independent of the specific structure of the clock $\mathcal{B}$, meaning that the interaction $\hat{V}_{AB}$ would affect the rate of any clock that replaced $\mathcal{B}$ in the same way it affects $\mathcal{B}$.[7] We could express this requirement as

$$\frac{\mathrm{d}\hat{\gamma}^{-1}}{\mathrm{d}\hat{H}_{\mathcal{B}}} = \frac{\mathrm{d}^2\hat{H}_{\mathcal{AB}}}{\mathrm{d}\hat{H}^2_{\mathcal{B}}} = 0, \tag{4.61}$$

meaning that $\hat{H}_{\mathcal{AB}}$ can be at most of order 1 in $\hat{H}_{\mathcal{B}}$. While we leave the problem regarding the validity of these conditions open for future research, the considerations made here could be a first step towards a better characterisation of the non-unitarity effects appearing in timeless quantum theory.

4.2.4 Clock and System Around a Massive Object

Let us now show that the dynamics of the system $\mathcal{A}$ according to the clock $\mathcal{C}$ is unchanged if both $\mathcal{A}$ and $\mathcal{C}$ interact gravitationally in the same way with a third system $\mathcal{B}$ (see [66] for a similar scenario). This could describe, for example, a static description of system $\mathcal{A}$ and its clock $\mathcal{C}$ on Earth's surface. The Hamiltonian of the

[7] This is the case, for example, of the gravitational time dilation, which affects the rate of any clock in the same way (at the same point).

Universe is

$$\hat{H}_U = \hat{H}_{\mathcal{A}} + \hat{H}_{\mathcal{B}} + \hat{H}_{\mathcal{C}} + \frac{1}{\Lambda}\hat{H}_{\mathcal{A}}\hat{H}_{\mathcal{B}} + \frac{1}{\Lambda}\hat{H}_{\mathcal{B}}\hat{H}_{\mathcal{C}}, \tag{4.62}$$

with $\frac{1}{\Lambda} = -\frac{G}{dc^4}$. Assuming that $1+\hat{H}_{\mathcal{B}}/\Lambda$ is invertible, we can rewrite the stationarity constraint $\hat{H}_U |\Psi_U\rangle\rangle = 0$ on the state of the Universe $|\Psi_U\rangle\rangle$ as

$$i\frac{\mathrm{d}\,|\Psi_U\rangle\rangle}{\mathrm{d}\hat{T}_{\mathcal{C}}} = \left(\hat{H}_{\mathcal{A}} + \hat{H}^{eff}_{\mathcal{B}}\right)|\Psi_U\rangle\rangle\,, \tag{4.63}$$

with $\hat{H}^{eff}_{\mathcal{B}} \coloneqq \hat{H}_{\mathcal{B}}/(1+\hat{H}_{\mathcal{B}}/\Lambda)$. Taking the partial inner product with the time states of $\mathcal{C}$ we get

$$i\frac{\mathrm{d}\,|\psi_{\mathcal{AB}}(t)\rangle\rangle}{\mathrm{d}t} = \left(\hat{H}_{\mathcal{A}} + \hat{H}^{eff}_{\mathcal{B}}\right)|\psi_{\mathcal{AB}}(t)\rangle\rangle\,, \tag{4.64}$$

showing that the dynamics of $\mathcal{A}$ from $\mathcal{C}$'s perspective is not altered by their interaction with $\mathcal{B}$. Explicitly, given $|\psi_{\mathcal{AB}}(t)\rangle\rangle = |\psi_{\mathcal{A}}(0)\rangle|\psi_{\mathcal{B}}(0)\rangle$, we have that $|\psi_{\mathcal{AB}}(t)\rangle\rangle = e^{-i\hat{H}_{\mathcal{A}}t}|\psi_{\mathcal{A}}(0)\rangle e^{-i\hat{H}^{eff}_{\mathcal{B}}t}|\psi_{\mathcal{B}}(0)\rangle$.

If $\mathcal{A}$ and $\mathcal{C}$ interact differently with $\mathcal{B}$, for example via the interaction terms $\frac{1}{\Lambda_{\mathcal{A}}}\hat{H}_{\mathcal{A}}\hat{H}_{\mathcal{B}}$ and $\frac{1}{\Lambda_{\mathcal{C}}}\hat{H}_{\mathcal{B}}\hat{H}_{\mathcal{C}}$, respectively, then we get

$$i\frac{\mathrm{d}\,|\Psi_U\rangle\rangle}{\mathrm{d}\hat{T}_{\mathcal{C}}} = \left(\hat{H}^{eff}_{\mathcal{A}} + \hat{H}^{eff}_{\mathcal{B}}\right)|\Psi_U\rangle\rangle\,, \tag{4.65}$$

with

$$\hat{H}^{eff}_{\mathcal{A}} \coloneqq \left(1+\frac{\hat{H}_{\mathcal{B}}}{\Lambda_{\mathcal{C}}}\right)^{-1}\cdot\left(1+\frac{\hat{H}_{\mathcal{B}}}{\Lambda_{\mathcal{A}}}\right)\hat{H}_{\mathcal{A}} \approx \left(1+\Delta_\Lambda\hat{H}_{\mathcal{B}}\right)\hat{H}_{\mathcal{A}}, \tag{4.66}$$

where in the last step we have expanded $\left(1+\hat{H}_{\mathcal{B}}/\Lambda_{\mathcal{C}}\right)^{-1}$ to first order in $1/\Lambda_{\mathcal{C}}$ and defined $\Delta_\Lambda \coloneqq \frac{1}{\Lambda_{\mathcal{A}}} - \frac{1}{\Lambda_{\mathcal{C}}}$. This shows that if $\mathcal{A}$ and $\mathcal{C}$ interact differently with $\mathcal{B}$, then the dynamics of $\mathcal{A}$ from $\mathcal{C}$'s perspective is affected by $\mathcal{B}$. For instance, assuming that $\mathcal{B}$ is a macroscopic body of mass $\mathcal{M}_{\mathcal{B}}$—so that we can work in the approximation $\hat{H}_{\mathcal{B}} \to M_{\mathcal{B}}c^2$—and that $\mathcal{A}$ ($\mathcal{C}$) is at a distance d ($d+h$) from $\mathcal{B}$ (with $h \ll d$), then

we have that the evolution of $\mathcal{A}$ from $\mathcal{C}$'s perspective is slowed down by a factor

$$1 + \Delta_\Lambda M_\mathcal{B} c^2 = 1 - \frac{GM_\mathcal{B}}{dc^2} + \frac{GM_\mathcal{B}}{(d+h)c^2} \approx 1 - \frac{GM_\mathcal{B} h}{d^2 c^2}, \tag{4.67}$$

in agreement with the relativistic effect that one would get from the gravitational time dilation outside a massive non-rotating object with spherical symmetry.

4.3 Interactions and Correlations

In the introduction, we saw that there is some freedom regarding the choice of stationary state of the Universe in timeless quantum theory. For instance, if we choose eigenstates of the Hamiltonian of the Universe with non-zero eigenvalue or mixed states that commute with the Hamiltonian, then the relational dynamics is unaltered apart from a possible global phase. We will now discuss why this is generally no longer true in the presence of interactions: different stationary states lead to a different relational dynamics. The difference is especially important in the case of mixed states of the Universe, where the dynamics can become non-unitary even with the unproblematic interactions discussed earlier. The results we will obtain will then be used to impose a condition on the types of correlations that are compatible with a unitary relational dynamics.

4.3.1 Generalised States of the Universe

4.3.1.1 Non-zero Energy Eigenstates

Consider Hamiltonian of the Universe as in Eq. (4.1) and consider a state $\left|\Psi_U^\mathcal{E}\right\rangle\!\rangle$ such that $\hat{H}_U \left|\Psi_U^\mathcal{E}\right\rangle\!\rangle = \mathcal{E} \left|\Psi_U^\mathcal{E}\right\rangle\!\rangle$. We can rewrite such a constraint similarly to Eq. (4.29) as

$$\left[-\hat{V}^{ord}\left(t, -i\frac{\mathrm{d}}{\mathrm{d}t}\right) + i\frac{\mathrm{d}}{\mathrm{d}t}\right] |\psi_\mathcal{A}(t)\rangle = \left(\hat{H}_\mathcal{A} - \mathcal{E}\right) |\psi_\mathcal{A}(t)\rangle. \tag{4.68}$$

While $\mathcal{E}$ might seem to contribute only to a global energy shift in the relational dynamics (which is usually unobservable) this is not always the case. For example, consider the interaction term $\hat{V} = \frac{1}{\Lambda}\hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}$. With this interaction, Eq. (4.68) becomes

$$i\frac{\mathrm{d}|\psi_{\mathcal{A}}(t)\rangle}{\mathrm{d}t} = \hat{H}^{eff}_{\mathcal{A}}(\mathcal{E})|\psi_{\mathcal{A}}(t)\rangle, \tag{4.69}$$

with

$$\hat{H}^{eff}_{\mathcal{A}}(\mathcal{E}) \coloneqq \frac{\hat{H}_{\mathcal{A}} - \mathcal{E}}{\hat{\mathbb{1}}_{\mathcal{A}} + \hat{H}_{\mathcal{A}}/\Lambda}, \tag{4.70}$$

where, similarly to Sec. 4.1.1.1, we have assumed that $\hat{\mathbb{1}}_{\mathcal{A}} + \hat{H}_{\mathcal{A}}/\Lambda$ is invertible. Comparing this effective Hamiltonian with that of Eq. (4.10) we see that

$$\hat{H}^{eff}_{\mathcal{A}}(\mathcal{E}) \approx \hat{H}^{eff}_{\mathcal{A}}(0) - \mathcal{E} + \frac{\mathcal{E}\hat{H}_{\mathcal{A}}}{\Lambda}, \tag{4.71}$$

with the approximation holding as long as Λ is much bigger than the typical energy of the system. This leads to a non-trivial modification in the dynamics of the system. The difference in the dynamics between two eigenstates of the Universe with eigenvalues $\mathcal{E}$ and $\mathcal{E}'$ can be quantified in the following way:

$$\left|\left\langle \psi^{\mathcal{E}'}_{\mathcal{A}}(t)\middle|\psi^{\mathcal{E}}_{\mathcal{A}}(t)\right\rangle\right| \approx \left|\langle\psi_{\mathcal{A}}(0)|e^{i(\mathcal{E}'-\mathcal{E})\hat{H}_{\mathcal{A}}t/\Lambda}|\psi_{\mathcal{A}}(0)\rangle\right|, \tag{4.72}$$

where we have assumed that the initial state of $\mathcal{A}$ is the same in both cases.

Despite the difference in the dynamics, this is still unitary. In fact, Eq. (4.68) suggests that whenever the relational dynamics is unitary for $\mathcal{E} = 0$, it will be so for any value of $\mathcal{E}$. This is the case for the interactions studied so far.

4.3.1.2 Mixed States

When we consider stationary mixed states of the Universe the discussion becomes more complicated. This is because, unlike with pure states, it is more difficult to

deal with the generalisations of the operatorial equations of motions for mixed states (Eq. (3.74)). Nevertheless, we will now show how to find an explicit solution for interactions of the type $\hat{V} = \frac{1}{\Lambda}\hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}$.

The general form of state of the Universe $\hat{\mathcal{R}}_U$ such that $\left[\hat{\mathcal{R}}_U, \hat{H}_U\right] = 0$ is given by Eq. (3.33), which write again here

$$\hat{\mathcal{R}}_U = \int_{-\infty}^{+\infty} \mathrm{d}\mathcal{E} \sum_{d_\mathcal{E}} \nu_{\mathcal{E},d_\mathcal{E}} \left|\mathcal{E}, d_\mathcal{E}\right\rangle\!\right\rangle \left\langle\!\left\langle \mathcal{E}, d_\mathcal{E}\right| . \tag{4.73}$$

Each of the $|\mathcal{E}, d_\mathcal{E}\rangle\!\rangle$, eigenstates of $\hat{H}_U$ with eigenvalues $\mathcal{E}$, satisfies Eq. (4.69) and so can be written as

$$|\mathcal{E}, d_\mathcal{E}\rangle\!\rangle = \int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-i\hat{H}_\mathcal{A}^{eff}(\mathcal{E})t} \left|\psi_\mathcal{A}^{d_\mathcal{E}}(0)\right\rangle |\phi_\mathcal{C}(t)\rangle. \tag{4.74}$$

Therefore, assuming for simplicity that we have a finite number of different energy eigenvalues, we have that

$$\hat{\mathcal{R}}_U = \int \mathrm{d}t\mathrm{d}t' \sum_k p_k e^{-i\hat{H}_\mathcal{A}^{eff}(\mathcal{E}_k)t} \hat{\rho}_k(0) \otimes |\phi_\mathcal{C}(t)\rangle\langle\phi_\mathcal{C}(t')| e^{i\hat{H}_\mathcal{A}^{eff}(\mathcal{E}_k)t'}, \tag{4.75}$$

where we have defined $\hat{\rho}_k(0) \coloneqq \frac{1}{p_k} \sum_{d_{\mathcal{E}_k}} \nu_{\mathcal{E}_k,d_{\mathcal{E}_k}} \left|\psi_\mathcal{A}^{d_{\mathcal{E}_k}}(0)\right\rangle \left\langle\psi_\mathcal{A}^{d_{\mathcal{E}_k}}(0)\right|$ with p_k such that $\mathrm{Tr}\left[\hat{\rho}_k(0)\right] = 1$ ($p_k \geq 0 \;\; \forall k$ and $\sum_k p_k = 1$).

The state of $\mathcal{A}$ at time t reads

$$\hat{\rho}_\mathcal{A}(t) = \sum_k p_k e^{-i\hat{H}_\mathcal{A}^{eff}(\mathcal{E}_k)t} \hat{\rho}_k(0) e^{i\hat{H}_\mathcal{A}^{eff}(\mathcal{E}_k)t}. \tag{4.76}$$

We see that the map $\hat{\rho}_\mathcal{A}(0) \to \hat{\rho}_\mathcal{A}(t)$ preserves the positivity and the trace of $\hat{\rho}_\mathcal{A}(0)$. However, if the state of the Universe consists of eigenstates of $\hat{H}_U$ with different eigenvalues, then the evolution of the system is no longer unitary overall. Instead, states of the system associated with different eigensectors of $\hat{H}_U$ evolve with Hamil-

tonians which, due to Eq. (4.70), differ by more than just a trivial energy shift. In the special case where all the $\hat{\rho}_k(0)$ are equal, the evolution of $\hat{\rho}_{\mathcal{A}}$ is given by the CPTP map

$$\hat{\rho}_{\mathcal{A}}(t) = \sum_k \hat{B}_k(t)\hat{\rho}_{\mathcal{A}}(0)\hat{B}_k^\dagger(t), \tag{4.77}$$

with Kraus operators $\hat{B}_k(t) \coloneqq \sqrt{p_k}e^{-i\hat{H}_{\mathcal{A}}^{eff}(\mathcal{E}_k)t}$. The map can be unitary if and only if the state of the Universe belongs to a single eigensector of $\hat{H}_U$.

4.3.1.3 Quantifying the Effect of Non-Unitarity

To quantify the effect of non-unitarity due to mixed states of the Universe, let us consider a simple model consisting of a qubit $\mathcal{A}$ interacting with an ideal clock $\mathcal{C}$. Let us assume that the qubit has a Hamiltonian $\hat{H} = \sigma_x/2$ and the interaction is given $\hat{V} = \frac{1}{\Lambda}\hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}$. Let us also assume, for simplicity, that $\hat{\mathcal{R}}$ is an equal mixture of two eigensectors of $\hat{H}_U$, one with eigenvalue 0 and one with eigenvalue $\mathcal{E}$, and that the state of the qubit at time $t = 0$ is $|0_{\mathcal{A}}\rangle\langle 0_{\mathcal{A}}|$. Using Eq. (4.70) we find that the effective Hamiltonian of the qubit for the $\mathcal{E}$-eigenstates of $\hat{H}_U$ is

$$\hat{H}_{\mathcal{A}}^{eff}(\mathcal{E}) = \frac{\sigma_x/2 - \mathcal{E}}{\mathbb{1} + \sigma_x/2\Lambda} = \phi(\Lambda, \mathcal{E})\mathbb{1} + \omega(\Lambda, \mathcal{E})\frac{\sigma_x}{2}, \tag{4.78}$$

where

$$\phi(\Lambda, \mathcal{E}) = -\frac{1}{\Delta}\left(\mathcal{E} + \frac{1}{4\Lambda}\right), \tag{4.79}$$

$$\omega(\Lambda, \mathcal{E}) = \frac{1}{\Delta}\left(1 + \frac{\mathcal{E}}{\Lambda}\right), \tag{4.80}$$

with $\Delta \coloneqq \det(\mathbb{1} + \sigma_x/2\Lambda) = 1 - 1/4\Lambda^2$. This means that the state of the qubit at time t given by Eq. (4.76) is

$$\hat{\rho}_{\mathcal{A}}(t) = \frac{1}{2}\left(e^{-i\hat{H}_{\mathcal{A}}^{eff}(0)t}|0\rangle\langle 0|e^{i\hat{H}_{\mathcal{A}}^{eff}(0)t} + e^{-i\hat{H}_{\mathcal{A}}^{eff}(\mathcal{E})t}|0\rangle\langle 0|e^{i\hat{H}_{\mathcal{A}}^{eff}(\mathcal{E})t}\right). \tag{4.81}$$

Substituting the explicit expression for $\hat{H}_{\mathcal{A}}^{eff}(\mathcal{E})$ one gets

$$\hat{\rho}_{\mathcal{A}}(t) = \frac{1}{2}e^{-i\frac{t}{2\Delta}\sigma_x}\left(|0_{\mathcal{A}}\rangle\langle 0_{\mathcal{A}}| + e^{-i\frac{\mathcal{E}t}{2\Delta\Lambda}\sigma_x}|0_{\mathcal{A}}\rangle\langle 0_{\mathcal{A}}|e^{i\frac{\mathcal{E}t}{2\Delta\Lambda}\sigma_x}\right)e^{i\frac{t}{2\Delta}\sigma_x}, \tag{4.82}$$

which is manifestly non-unitary. To quantify the effect of this non-unitarity, consider the time it takes $\hat{\rho}_{\mathcal{A}}(t)$ to reach the maximally mixed state

$$\tau_D = \frac{\pi\hbar\Delta\Lambda}{\mathcal{E}} = \frac{\pi\hbar\Lambda}{\mathcal{E}}\left[1 - \frac{1}{4\Lambda^2}\right], \tag{4.83}$$

where we have reinstated $\hbar$. τ_D has the meaning of a coherence time.[8] This becomes small for small values of Λ (i.e. strong couplings) and for big differences between the energy of the eigensectors appearing in $\hat{R}_U$.

Another meaningful quantity is the number of full rotations that the system can undergo before it reaches the maximally mixed state:

$$\frac{\tau_D}{\tau} = \frac{\Lambda}{2\mathcal{E}}, \tag{4.84}$$

where $\tau = 2\pi\hbar\Delta$ is the time it takes for the qubit to go back to the initial state in the case of no interactions with the clock. For the gravitational interaction discussed in the previous sections we have $\Lambda = dc^4/G$, so that

$$\frac{\tau_D}{\tau} = \frac{dc^4}{2G\mathcal{E}}. \tag{4.85}$$

[8]However, note that the qubit goes back to its initial state after reaching the maximally mixed state.

To grasp the magnitude of this effect, let us consider an atomic-scale system. Assuming that the clock is gravitationally coupled to the system at a distance $d = 10^{-10}\,\mathrm{m}$, $\tau_D/\tau \approx 10^{51}$ for $\mathcal{E} = 10\,\mathrm{eV}$, while $\tau_D/\tau \approx 10^{41}$ for $\mathcal{E} = m_p c^2$, where m_p is the mass of the proton. Both these relative coherence times are very long. These results show that even if the system's evolution is non-unitary, this effect may not be easy to detect for weak gravitational system-clock interactions. Instead, the effect becomes significant for large values of $\mathcal{E}$ at scales where one would expect strong relativistic effects, e.g. for $\mathcal{E} \approx mc^2$ and $d \approx \frac{2Gm}{c^2}$. Alternatively, the effect could increase with the number of different energy eigenvalues appearing in the state of the Universe.

4.3.2 Unitarity Requires Entanglement

We have seen that the relational evolution deriving from a mixed stationary state of the Universe with interactions of the type $\hat{V} = \frac{1}{\Lambda}\hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}$ is, in general, non-unitary. This happens even though the same interactions lead to a unitary relational dynamics for pure stationary states of the Universe. Such an effect is (in principle) observable and might be interpreted as an argument against the use of mixed states in timeless quantum theory. While we think this requires further study, in this section we point out an interesting consequence for the type of system-clock correlations.

Similarly to Sec. 3.2.2, we can prove the following theorem, where we assume, for simplicity, that both the $\mathcal{A}$ and $\mathcal{C}$ are finite-dimensional.

Theorem 4.1. *If a separable bipartite state $\hat{\mathcal{R}}$ is such that $\hat{\mathcal{R}}\hat{H}_U = \hat{H}_U\hat{\mathcal{R}} = 0$ with $\hat{H}_U = \hat{H}_{\mathcal{A}} \otimes \hat{\mathbb{1}}_{\mathcal{C}} + \hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}} + \frac{1}{\Lambda}\hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}$, then $\left[\hat{\mathcal{R}}, \hat{H}_{\mathcal{C}}\right] = 0$.*

Proof. The proof is analogous to the one for the Theorem 3.6 but now $\hat{\mathcal{R}}$ must be of the form $\hat{\mathcal{R}} = \sum_j p_j |E_j\rangle\langle E_j| \otimes |-E_j/(1+E_j/\Lambda)\rangle\langle -E_j/(1+E_j/\Lambda)|$. This state still commutes with $\hat{H}_{\mathcal{C}}$. □

This theorem can also be extended to systems with continuous energy spectrum. We can combine it with Theorem 3.5 to conclude that if $\hat{\mathcal{R}}_U$ is a stationary separable state that leads to a non-trivial relational dynamics, i.e. $\left[\hat{\mathcal{R}}_U, \hat{H}_{\mathcal{C}}\right] \neq 0$, then $\hat{\mathcal{R}}_U \hat{H}_U \neq 0$, meaning that $\hat{\mathcal{R}}_U$ lives on multiple eigenspaces of $\hat{H}_U$ with different eigenvalues. In turn, this implies that $\hat{\mathcal{R}}_U$ leads to a *non-unitary* relational dynamics due to Eq. (4.70). If, instead, we want to preserve the unitarity of the relational dynamics by restricting to a single eigensector of $\hat{H}_U$, then $\hat{\mathcal{R}}_U$ must be entangled to encode a non-trivial time evolution. In other words, we have the following.

Corollary 4.1.1. *Given a state $\hat{\mathcal{R}}_U$ such that $\left[\hat{\mathcal{R}}_U, \hat{H}_U\right] = 0$ with $\hat{H}_U = \hat{H}_{\mathcal{A}} \otimes \hat{\mathbb{1}}_{\mathcal{C}} + \hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}} + \frac{1}{\Lambda} \hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}$ that leads to a non-trivial relational dynamics, if the dynamics is unitary then $\hat{\mathcal{R}}_U$ must be entangled with respect to $\mathcal{A}$ and $\mathcal{C}$.*

This result is interesting because it shows the importance of entanglement in timeless quantum theory. Although we proved it for a specific kind of interaction, we expect this interaction to be ubiquitous in timeless quantum theory. Therefore, the non-unitarity deriving from a hypothetical mixed non-entangled state of the Universe could be (in principle) tested ubiquitously. Note that entanglement was not required in the absence of interactions, as discussed in Sec. 3.2. In some sense, interactions restore the role of entanglement in timeless quantum theory.

This result might also be used to explore the consistency of classical relational approaches to time for quantum systems, along the lines of the literature on quantum-classical dynamics [131, 132, 133, 134, 135, 136, 137, 138, 139, 140]. For instance, it might be possible to show inconsistency when coupling a classical clock with a quantum systems in a classical relational approach to time. We leave this open for future investigations.

4.4 Heisenberg Picture

We will now show how timeless quantum theory can be cast in the Heisenberg picture even in the presence of interactions. We do not expect to be able to do so for any type of interaction because we have seen that some of them lead to a non-unitary evolution in the Schrödinger picture, while the transformation between the two pictures should be unitary (we will discuss an exception later). In this section, we will show how to recover the HP for the interaction $\hat{V} = \frac{1}{\Lambda}\hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}$.

4.4.1 Pure States

We can easily see how to recover the HP for pure states of the Universe from Eq. (4.13) by noting that $|\Psi_U\rangle\rangle = \hat{\mathbb{U}}'|\psi_{\mathcal{A}}(0)\rangle|\mathrm{TL}_{\mathcal{C}}\rangle$, where $\hat{\mathbb{U}}'$ is the unitary

$$\hat{\mathbb{U}}' := e^{-i\hat{H}^{eff}_{\mathcal{A}} \otimes \hat{T}_{\mathcal{C}}}. \tag{4.86}$$

$\hat{\mathbb{U}}'$ can be used to move between the SP and the HP as described in Sec. 3.1.1. Under $\hat{\mathbb{U}}'$ Eqs. (3.5)-(3.7) become

$$\hat{H}^{(\mathbb{H})}_{\mathcal{A}} = \hat{H}_{\mathcal{A}} \otimes \hat{\mathbb{1}}_{\mathcal{A}}, \tag{4.87}$$

$$\hat{H}^{(\mathbb{H})}_{\mathcal{C}} = \hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}} - \hat{H}^{eff}_{\mathcal{A}} \otimes \hat{\mathbb{1}}_{\mathcal{A}}, \tag{4.88}$$

$$\hat{T}^{(\mathbb{H})}_{\mathcal{C}} = \hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{T}_{\mathcal{C}}. \tag{4.89}$$

Crucially, $\hat{T}_{\mathcal{C}}$ is invariant under $\hat{\mathbb{U}}'$ and $|\psi_{\mathcal{A}}(0)\rangle|\mathrm{TL}_{\mathcal{C}}\rangle$ encodes a trivial relational dynamics. Therefore, $\hat{\mathbb{U}}'$ represents a good SP-HP transformation according to the requirements of Section 3.1.1. The procedure outlined in Sec. 3.1.1 can then be straightforwardly applied to this case by replacing $\hat{H}_{\mathcal{A}}$ with $\hat{H}^{eff}_{\mathcal{A}}$.

We can prove more generally that *any interaction* that leads to a unitary relational evolution can be cast in the HP. This is because if the relational evolution is

unitary, i.e. $\langle \phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle = \hat{U}(t)|\psi_{\mathcal{A}}(0)\rangle$, then we can always write $|\Psi_U\rangle\rangle$ as

$$|\Psi_U\rangle\rangle = \int \mathrm{d}t\, \hat{U}(t)|\psi_{\mathcal{A}}(0)\rangle|\phi_{\mathcal{C}}(t)\rangle = \hat{U}(\hat{T}_{\mathcal{C}})|\psi_{\mathcal{A}}(0)\rangle|\mathrm{TL}_{\mathcal{C}}\rangle. \tag{4.90}$$

We can then use $\hat{U}(\hat{T}_{\mathcal{C}})$ to move to the HP because it is such that it leaves $\hat{T}_{\mathcal{C}}$ invariant.

Now, $\hat{H}^{eff}_{\mathcal{A}}$ and $\hat{V} = \frac{1}{\Lambda}\hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}$ transform in the following way

$$\hat{H}^{eff(\mathbb{H})}_{\mathcal{A}} = \hat{H}^{eff}_{\mathcal{A}} \otimes \hat{\mathbb{1}}_{\mathcal{C}}, \tag{4.91}$$

$$\hat{V}^{(\mathbb{H})} = \hat{V} - \hat{H}^{eff}_{\mathcal{A}} \cdot \frac{\hat{H}_{\mathcal{A}}}{\Lambda}. \tag{4.92}$$

and, finally, the Hamiltonian of the Universe in the HP becomes

$$\hat{H}^{(\mathbb{H})}_U = \hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}} + \hat{V}. \tag{4.93}$$

The last equation is important for two reasons. First, it shows that the stationarity constraint in the HP is satisfied for any initial state of $\mathcal{A}$ because $\hat{H}^{(\mathbb{H})}_U \left|\Psi^{(\mathbb{H})}_U\right\rangle\!\right\rangle = 0$ as long as $\left|\Psi^{(\mathbb{H})}_U\right\rangle\!\right\rangle$ is an eigenstate of $\hat{H}_{\mathcal{C}}$. In fact, from Eq. (4.90) we see that this must be the case for any interaction that leads to a unitary relational dynamics. Second, we see that the constraint is not "trivialised" in the same way as described in [36] because it contains extra terms in addition to $\hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}$. Therefore, we cannot recover the HP using the procedure of [36] in the presence of interactions. Nevertheless, note that the constraint is still trivialised in some sense because it is satisfied regardless of the Heisenberg state of $\mathcal{A}$.

4.4.2 Mixed States

Consider now a mixed state of the Universe $\hat{\mathcal{R}}_U$. We can rewrite Eq. (4.75) as

$$\hat{\mathcal{R}}_U = \sum_k p_k e^{-i\hat{H}^{eff}_{\mathcal{A}}(\mathcal{E}_k)\otimes\hat{T}_{\mathcal{C}}}\hat{\rho}_k(0)\otimes|\mathrm{TL}_{\mathcal{C}}\rangle\langle\mathrm{TL}_{\mathcal{C}}|e^{i\hat{H}^{eff}_{\mathcal{A}}(\mathcal{E}_k)\otimes\hat{T}_{\mathcal{C}}}. \tag{4.94}$$

If there is more than one term in the sum, then we have seen that there is no unitary to move between the SP and the HP. However, we can still recover the HP in the following way. Consider, for simplicity, the special case where all the $\hat{\rho}_k(0)$ are equal. We can then write $\hat{\mathcal{R}}_U$ as

$$\hat{\mathcal{R}}_U = \sum_k \hat{\mathbb{B}}_k\hat{\mathcal{R}}^{(\mathbb{H})}\hat{\mathbb{B}}_k^\dagger, \tag{4.95}$$

with $\hat{\mathcal{R}}^{(\mathbb{H})}_U \coloneqq \hat{\rho}_{\mathcal{A}}(0)\otimes|\mathrm{TL}_{\mathcal{C}}\rangle\langle\mathrm{TL}_{\mathcal{C}}|$ and Kraus operators $\hat{\mathbb{B}}_k \coloneqq \sqrt{p_k}e^{-i\hat{H}^{eff}_{\mathcal{A}}(\mathcal{E}_k)\otimes\hat{T}_{\mathcal{C}}}$. Now, we can recover the HP by acting with the adjoint map on the observables of the Universe and use the Heisenberg state $\hat{\mathcal{R}}^{(\mathbb{H})}_U$. For instance, the observables of $\mathcal{A}$ in the HP are given by

$$\hat{O}^{(\mathbb{H})}_{\mathcal{A}} = \sum_k \hat{\mathbb{B}}_k^\dagger\left(\hat{O}_{\mathcal{A}}\otimes\hat{\mathbb{1}}_{\mathcal{C}}\right)\hat{\mathbb{B}}_k. \tag{4.96}$$

These observables lead to expectation values in agreement with the SP ones. Moreover, we have that $\hat{T}^{(\mathbb{H})}_{\mathcal{C}} = \hat{T}_{\mathcal{C}}$, so the HP construction is the same as before. Also, note the Heisenberg state is such that $\left[\hat{\mathcal{R}}^{(\mathbb{H})}_U, \hat{H}_{\mathcal{C}}\right] = 0$, and so it does not encode any non-trivial time evolution.

For completeness, let us write how the main observables transform

$$\hat{H}_{\mathcal{A}}^{(\mathbb{H})} = \hat{H}_{\mathcal{A}} \otimes \hat{\mathbb{1}}_{\mathcal{A}}, \tag{4.97}$$

$$\hat{H}_{\mathcal{C}}^{(\mathbb{H})} = \hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}} - \hat{H}_{\mathcal{A}}^{eff}(\overline{E}) \otimes \hat{\mathbb{1}}_{\mathcal{A}}, \tag{4.98}$$

$$\hat{H}_{\mathcal{A}}^{eff(\mathbb{H})}(\mathcal{E}_k) = \hat{H}_{\mathcal{A}}^{eff}(\mathcal{E}_k) \otimes \hat{\mathbb{1}}_{\mathcal{C}}, \quad \forall k \tag{4.99}$$

$$\hat{V}^{(\mathbb{H})} = \hat{V} - \hat{H}_{\mathcal{A}}^{eff}(\overline{E}) \cdot \frac{\hat{H}_{\mathcal{A}}}{\Lambda}, \tag{4.100}$$

$$\hat{H}_{U}^{(\mathbb{H})} = \hat{\mathbb{1}}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}} + \hat{V} + \overline{E}, \tag{4.101}$$

where we have defined $\overline{E} \coloneqq \sum_k p_k \mathcal{E}_k$. Again, note that the constraint $\hat{H}_U^{(\mathbb{H})} \left|\Psi_U^{(\mathbb{H})}\right\rangle\!\rangle = 0$ is still satisfied for any initial state of $\mathcal{A}$.

Note that here we have recovered the HP formulation of timeless quantum theory despite the relational evolution being non-unitary. This shows that unitarity is not a necessary condition for recovering the HP. Still, we were able to do so in this case because the interaction we considered does not lead to a non-unitary dynamics for pure states of the Universe. Therefore, the non-unitarity effect discussed in this section and the one due to some types of interactions seen in previous sections are different. We expect that it is not possible to find the HP when the interaction leads to a non-unitarty dynamics, and we conjecture that this fact might be related to the invariance of $\hat{T}_{\mathcal{C}}$ under the transformation between the two pictures, as discussed in Secs. 3.1.1.1-3.1.2.1. We leave these problems for future work.

4.4.3 Interaction Picture

As a final comment, let us remark that the SP and the HP do not exhaust all the possibilities. There can be an infinity of pictures that differ on how the time dependence is split between the observables and the state of the Universe. For example, we can consider an "interaction picture" where the observables are evolved

with the "free" unitary $\hat{\mathbb{U}}$ of Eq. (3.2), while the state of the Universe $\left|\Psi_U^{(\mathbb{I})}\right\rangle\!\rangle := \hat{\mathbb{U}}^\dagger \left|\Psi\right\rangle\!\rangle$ obeys the following equation:

$$i\frac{\mathrm{d}\left|\Psi_U^{(\mathbb{I})}\right\rangle\!\rangle}{\mathrm{d}\hat{T}_{\mathcal{C}}} = \hat{V}^{(\mathbb{I})}\left|\Psi_U^{(\mathbb{I})}\right\rangle\!\rangle\,, \tag{4.102}$$

where $\hat{V}^{(\mathbb{I})} := \hat{\mathbb{U}}^\dagger \hat{V} \hat{\mathbb{U}}$ is the interaction term in the new picture. This shows that the state is still dependent on $\hat{T}_{\mathcal{C}}$, meaning that not all the time dependence has been transferred to the observables. In particular, $\left|\Psi_U^{(\mathbb{I})}\right\rangle\!\rangle$ is still entangled. This interaction picture may be more suitable for analysing some scenarios, as is the case in quantum field theory. We leave this open for future discussion.

Chapter 5

Conclusions

In this thesis, we explored some of the main features and consequences of timeless quantum theory. We focused on three central topics: clocks, correlations, and interactions.

After a review of the main features of timeless quantum theory in Chapter 1, we considered the role of clocks and the interpretation of time in Chapter 2. Here, we argued for adopting an Everettian view of the theory [39, 102]. In addition to solving some previously-raised criticism of timeless quantum theory [11, 36, 74], this view helped us clarify the interpretation of the relative states associated with different time instants. We saw that clocks play a special role in defining the time instants by ordering the multitude of possible relative states of the Universe. Moreover, we argued for the more fundamental role of projective measurements and showed that some assumptions related to the structure of the clocks can be relaxed.

Finally, we showed that timeless quantum theory allows for the clock readings to be measured by the rest of the Universe. This was achieved by adding an interaction between the clock and the rest of the Universe that does not alter the main features of the relational dynamics, such as unitarity. Although similar types of interactions were already discussed in the literature [105, 43, 61], our approach shows that the

clock can be measured and thus be considered a fully-fledged physical system. In this context, we also showed that these interactions can synchronise two ideal clocks, although in a different sense from what is commonly understood with the term "synchronisation".

In Chapter 3, we explored the role of correlation in timeless quantum theory. We showed how to recover the Heisenberg picture of the theory by transferring crucial system-clock correlations from the state vector of the Universe to its observables despite the Universe being stationary. This was achieved via a unitary transformation that we proved to be unique and applies to both pure and mixed states of the Universe.

Correlations also play a fundamental role in the relational dynamics of the theory, but their nature is not fully clear in the literature. Most works trace it back to entanglement [41, 56, 42, 83], while other argue that entanglement is not necessary [57] or show that it is a feature linked to the Schrödinger picture [45, 36, 91]. Here, we demonstrated that we can recover any kind of relational evolution starting from a separable (mixed) state of the Universe. The state still needs to contain correlations between the clock and the rest of the Universe, but these correlations are less strong than entanglement.

In the last part of the chapter, we introduced a compact notation for timeless quantum theory that uses derivatives with respect to operators. These types of derivatives were already used in the context of timeless quantum theory in [61, 114] and we extended them to study mixed states of the Universe and, later, system-clock interactions.

In Chapter 4, we considered the role of interactions between the clock and the rest of the Universe. These were first discussed in [43] and were later shown to sometimes lead to problems, such as the non-unitarity of the relational evolution [58]. We started by reviewing the main types of system-clock interactions, making extensive

use of the differential notation introduced in the previous chapter. We showed how this notation can be used to quickly find the equations governing the relational dynamics and understand their main features. We then examined the effect of some simple interactions on the clock's timekeeping abilities. While interactions can generally create some uncertainty in the clock's time readings, we saw that this effect is negligible for macroscopic systems. This observation, the arguments for the accessibility of the clock in Chapter 2, and the demonstration that some interactions lead to time dilation effects compatible with relativistic ones [40, 64, 66], show that interactions are an essential feature of timeless quantum theory. One of the main open problems about interactions is the non-unitarity of the relational dynamics due to some types of interactions. While this point requires further study, we proposed a potential criterion to characterise interactions that lead to a unitary relational evolution.

The presence of system-clock interactions also has consequences for the types of correlations in the state of the Universe. Differently from the results of Chapter 2, we saw that system-clock correlations less strong than entanglement lead to a non-unitary relational dynamics in a scenario with a simple interaction. This effect can be quantified and shows that system-clock entanglement is necessary for the relational dynamics to be unitary.

In the last part of Chapter 4, we demonstrated that the unitary transformation between the Schrödinger and the Heisenberg pictures discussed in Chapter 3 can also be applied in the presence of some types of interactions. To the best of our knowledge, this was not previously considered in the literature [45, 68, 36] and shows the versatility of our approach. We also discussed why the Schrödinger and the Heisenberg pictures do not exhaust all the possibilities: there is an infinite number of pictures that differ in how the time dependence is split between the observables and the state vector of the Universe. For instance, we showed that it is possible to

find an interaction picture where the observables of the system evolve freely while the state vector evolves under an interaction term.

Overall, the results of this thesis improve the robustness of timeless quantum theory and point to several avenues for future research. The main ones outlined throughout the thesis are the following:

- in Sec. 2.2, we saw that clocks play a crucial role in timeless quantum theory, but the properties that distinguish them from other physical systems should be better specified. This specification might not be fully achievable within quantum theory and could require other approaches. For instance, it may be possible to characterise clocks via tasks they can perform [141, 103].

- In Sec. 2.3.3, we saw how to effectively synchronise two ideal clocks. In that section, however, we discussed that this type of synchronisation differs from the usual interpretation of synchronisation because it requires the two clocks to interact continuously. It would be important to explore how a more realistic notion of synchronisation can be implemented in timeless quantum theory and how this could be extended to multiple clocks.

- In Chapter 4, we have seen that some interactions lead to a non-unitary relational dynamics [90]. While we have proposed a potential criterion to characterise interactions that lead to a unitary relational evolution, the problem is still open. Since unitarity is central to quantum theory, as exemplified by the no-cloning theorem [142], it is crucial to characterise and understand this effect. Unitarity is also crucial to guarantee that the physical inner product discussed in Sec. 1.2.3 is independent of the specific choice of time state. When evolution is non-unitary, the physical inner product is no longer globally well-defined. As mentioned in Sec. 4.4.2, we conjecture that the non-unitarity problem is related to the conditions for recovering the Heisenberg picture for-

mulation of timeless quantum theory.

- In Sec. 4.3.2, we have shown that entanglement is necessary to guarantee that the relational evolution is unitary in the presence of some types of interactions that are unproblematic when dealing with pure states of the Universe. The deviations from unitarity due to a correlated but not entangled state could be, in principle, detected. Alternatively, one might use this result to explore the consistency of classical relational approaches to time for quantum systems. For instance, it might be possible to show inconsistencies when coupling a classical clock with a quantum system in a classical relational approach to time. This bears some resemblance to recent works on entanglement as a witness of non-classicality [137, 138, 139, 140].

- In Chapter 4, we have also discussed some interactions that lead to time dilation effects consistent with relativistic predictions [40, 64, 66]. These interactions display some problems that also affect relativistic wave mechanics and were later solved in quantum field theory, so a quantum-field-theoretic approach might be essential [75]. In any case, these interactions represent a first step towards extending timeless quantum theory in a relativistic direction, one of the most exciting avenues for future research.

Solving these points would further strengthen the theory and could lead to the exclusion of different approaches to time or to further experimentally-verifiable predictions [40]. Moreover, as the universality of quantum theory plays a central explanatory role in timeless quantum theory, the latter stands out as one of the most promising approaches to time capable of encompassing a hypothetical quantum theory of gravity.

Appendix A

Rigged Hilbert Spaces

In this thesis, we often use ideal clocks, which are characterised by their continuous unbounded spectrum. It is well known that the eigenstates of the Hamiltonian and the "time operator" of these clocks (which are analogous to the position and momentum operators for a free particle in 1D) are not physical. Nevertheless, we have made extensive use of these states in the thesis with the understanding that they approximate the behaviour of ever more precise real clocks. Notably, these unphysical states can be mathematically defined in the *rigged Hilbert space* formalism. We briefly and non-rigorously review this formalism here, but we refer the reader to [54, 143] for an extensive discussion.

Let us consider the Hilbert space $\mathscr{H}_\mathcal{C} = L^2(\mathbb{R})$ and the algebra $\mathfrak{h}$ generated by $\hat{H}_\mathcal{C}$ and $\hat{T}_\mathcal{C}$ such that $\left[\hat{T}_\mathcal{C}, \hat{H}_\mathcal{C}\right] = i$ (in this case these operators are defined by their action on the functions in $L^2(\mathbb{R})$). We can consider the space of the so-called "test vectors" $|\chi_\mathcal{C}\rangle$ such that $\left\|\hat{O}_\mathcal{C}|\chi_\mathcal{C}\rangle\right\| < \infty$ for any $\hat{O}_\mathcal{C} \in \mathfrak{h}$. This will be the space of functions in $L^2(\mathbb{R})$ with partial derivatives rapidly decreasing at infinity (Schwartz space). If we call Φ the space of all such test vectors, we can consider its dual Φ^*, that is, the space of all the linear functionals on Φ. Φ^* is usually called the space of "distributions" and we have that $\Phi \subset \mathscr{H}_\mathcal{C} \subset \Phi^*$ [54]. We can denote the members

of Φ^* with the bra notation as $\langle\omega_{\mathcal{C}}|$, so that $\langle\omega_{\mathcal{C}}|\chi_{\mathcal{C}}\rangle \in \mathbb{C}$ for any test vector $|\chi_{\mathcal{C}}\rangle$.

Φ^* contains the so-called "generalised eigenvectors" of $\hat{H}_{\mathcal{C}}$ and $\hat{T}_{\mathcal{C}}$, defined as the bras $\langle\varepsilon_{\mathcal{C}}|$ and $\langle\phi_{\mathcal{C}}(t)|$ such that $\langle\varepsilon_{\mathcal{C}}|\hat{H}_{\mathcal{C}}|\chi_{\mathcal{C}}\rangle = \varepsilon\,\langle\varepsilon_{\mathcal{C}}|\chi_{\mathcal{C}}\rangle$ and $\langle\phi_{\mathcal{C}}(t)|\hat{T}_{\mathcal{C}}|\chi_{\mathcal{C}}\rangle = t\,\langle\phi_{\mathcal{C}}(t)|\chi_{\mathcal{C}}\rangle$ for every $|\chi_{\mathcal{C}}\rangle \in \Phi$, respectively. By taking their conjugate, we get the eigenstates $|\varepsilon_{\mathcal{C}}\rangle$ ($|\phi_{\mathcal{C}}(t)\rangle$) of $\hat{H}_{\mathcal{C}}$ ($\hat{T}_{\mathcal{C}}$) that we have used in the thesis. Within the rigged Hilbert space formalism, we can give precise meaning to expressions such as $\int \mathrm{d}t\,|\varepsilon_{\mathcal{C}}\rangle\langle\varepsilon_{\mathcal{C}}| = \hat{\mathbb{1}}_{\mathcal{C}}$ and $|\chi_{\mathcal{C}}\rangle = \int \mathrm{d}t\ \langle\varepsilon_{\mathcal{C}}|\chi_{\mathcal{C}}\rangle\ |\varepsilon_{\mathcal{C}}\rangle$. We can also explain why Eq. (2.32) in Sec. 2.3.1 vanishes despite an application of the integration-by-parts rule suggests that the equation can be written as

$$\hat{H}_U\,|\Psi_U\rangle\rangle = i\,|\psi_{\mathcal{A}}(t)\rangle|\phi_{\mathcal{C}}(t)\rangle\Big|_{-\infty}^{+\infty}, \tag{A.1}$$

which is not zero. This problem stems from the fact that $|\Psi_U\rangle\rangle$ belongs to the space of distributions, and to check if it is an eigenstate of $\hat{H}_U$, we need to calculate $\langle\langle\chi_U|\,\hat{H}_U\,|\Psi_U\rangle\rangle$ where $|\chi_U\rangle\rangle$ is a test vector. Assuming that the Hilbert space of the system $\mathcal{A}$ is finite, we can write $|\chi_U\rangle\rangle$ as

$$|\chi_U\rangle\rangle = \int_{-\infty}^{+\infty} \mathrm{d}t \sum_j \mu_j(t)|\chi_{\mathcal{A},j}\rangle|\phi_{\mathcal{C}}(t)\rangle, \tag{A.2}$$

where $\{|\chi_{\mathcal{A},j}\rangle\}_j$ is an orthonormal basis of $\mathscr{H}_{\mathcal{A}}$, and each $\mu_j(t)$ is a square-integrable function with partial derivatives rapidly decreasing at infinity. We have that

$$\langle\langle\chi_U|\,\hat{H}_U\,|\Psi_U\rangle\rangle = i\left(\textstyle\sum_j \mu_j^*(t)\,\langle\chi_{\mathcal{A},j}|\psi_{\mathcal{A}}(t)\rangle\right)\Big|_{-\infty}^{+\infty} = 0, \tag{A.3}$$

where we used Eq. (A.1) and the fact that $\mu_j^*(t) \to 0$ when $t \to \pm\infty$. Therefore, $|\Psi_U\rangle\rangle$ is indeed a generalised eigenstate of with $\hat{H}_U$ with eigenvalue zero.

What about the often-used conditioning procedure $\langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle$, where $|\phi_{\mathcal{C}}(t)\rangle$ is clearly not a test vector? This procedure should be intended as an idealised, non-

rigorous operation. Within timeless quantum theory, it is usually unproblematic because the state of the Universe is an energy eigenstate (and $\langle\phi_{\mathcal{C}}(t)|\varepsilon_{\mathcal{C}}\rangle = e^{i\varepsilon t}/\sqrt{2\pi}$ which is finite) and the boundary terms such as those in Eq. (A.1) can still be disregarded in this *non-rigorous* sense:

$$\langle\chi_{\mathcal{A},j}|\langle\phi_{\mathcal{C}}(t')|\hat{H}_U\,|\Psi_U\rangle\rangle = i\,\left(\sum_j \langle\chi_{\mathcal{A},j}|\psi_{\mathcal{A}}(t)\rangle\,\delta(t-t')\right)\Big|_{-\infty}^{+\infty} = 0, \tag{A.4}$$

for any $|\phi_j\rangle \in \mathscr{H}_{\mathcal{A}}$ and $t' \in \mathbb{R}$.

An alternative way to deal with the subtleties arising from Hamiltonians with a continuous spectrum is to introduce a "physical" inner product on the state vectors that annihilate the Hamiltonian constraint [43, 36]. These states form the "physical" Hilbert space (embedded in the "kinematical" Hilbert space $\mathscr{H}_{\mathcal{A}} \otimes \mathscr{H}_{\mathcal{C}}$) and are normalisable according to the physical inner product. It is easy to show that the boundary terms above do not appear in the physical inner product, similarly to Eq. (A.4).

Appendix B

Construction of a Stationary State with a Periodic Clock

Here, we show how to construct a stationary state using the projector discussed in Sec. 2.1 when the clock is periodic, that is, when the ratios of its energy gaps are rational. A consequence of the periodicity of the clock is that only a rational part of the spectrum of $\hat{H}_{\mathcal{S}}$ survives in $\left|\tilde{\Psi}_U\right\rangle\!\!\left\rangle$. The surviving part must then also be periodic with, at maximum, the same period of the clock. This can be observed in the following way

$$\begin{aligned}\delta(\hat{H}_U)|\chi_{\mathcal{S}}(0)\rangle|\phi_{\mathcal{C}}(0)\rangle &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-i\hat{H}_{\mathcal{S}}t}|\chi_{\mathcal{S}}(0)\rangle|\phi_{\mathcal{C}}(t)\rangle \\ &= \frac{1}{2\pi}\sum_{j=-\infty}^{+\infty}\int_0^T \mathrm{d}t\, e^{ij\varphi}e^{-i\hat{H}_{\mathcal{S}}(t+jT)}|\chi_{\mathcal{S}}(0)\rangle|\phi_{\mathcal{C}}(t)\rangle \\ &= \frac{1}{T}\int_0^T \mathrm{d}t\, e^{-i\hat{H}_{\mathcal{S}}t}|\chi'_{\mathcal{S}}(0)\rangle|\phi_{\mathcal{C}}(t)\rangle, \end{aligned} \tag{B.1}$$

where we have used the fact that $|\phi_{\mathcal{C}}(t+T)\rangle = e^{i\varphi}|\phi_{\mathcal{C}}(t)\rangle$, with T the period of the clock and φ a phase as discussed in Sec. 1.2.3. In the second equality, we have written the integral over $\mathbb{R}$ as a sum of integrals over $[0, T]$. Finally, in the last

equality, we have defined

$$
\begin{aligned}
|\chi'_{\mathcal{S}}(0)\rangle \coloneqq & \frac{T}{2\pi}\sum_{j=-\infty}^{+\infty} e^{ij\varphi}e^{-i\hat{H}_{\mathcal{S}}jT}|\chi_{\mathcal{S}}(0)\rangle = \frac{T}{2\pi}\sum_{j=-\infty}^{+\infty}\sum_{E}\mu_E\, e^{ij\varphi}e^{-ijET}|E_{\mathcal{S}}\rangle \\
= & T\sum_{k=-\infty}^{+\infty}\sum_{E}\mu_E\,\delta\left(\varphi - ET - 2\pi k\right)|E_{\mathcal{S}}\rangle. \qquad \text{(B.2)}
\end{aligned}
$$

Here, we have expressed $|\chi_{\mathcal{S}}(0)\rangle$ as $|\chi_{\mathcal{S}}(0)\rangle = \sum_E \mu_E|E_{\mathcal{S}}\rangle$ in the second equality and used the definition of the Dirac comb in the third one. In Eq. (B.2), only a rational part of the spectrum of $\hat{H}_{\mathcal{C}}$ is selected. The result is a non-normalisable, periodic state with period (at most) T. A non-rigorous way to get a normalisable state is to use the properties of the Dirac delta function in Eq. (B.2) to express it as $\delta\left(\varphi - ET - 2\pi k\right) = \delta\left(E - \frac{\varphi - 2\pi k}{T}\right)/T$ and "transform" it into a Kronecker delta:

$$
\left|\chi_{\mathcal{S}}^{\prime(norm)}(0)\right\rangle \coloneqq \sum_{k=-\infty}^{+\infty}\sum_{E}\mu_E\,\delta_{E,\frac{\varphi-2\pi k}{T}}|E_{\mathcal{S}}\rangle. \qquad \text{(B.3)}
$$

In this way, we get the state of the Universe

$$
|\Psi_U\rangle\rangle = \frac{1}{T}\int_0^T \mathrm{d}t\, e^{-i\hat{H}_{\mathcal{S}}t}\left|\chi_{\mathcal{S}}^{\prime(norm)}(0)\right\rangle|\phi_{\mathcal{C}}(t)\rangle. \qquad \text{(B.4)}
$$

If the $|\phi_{\mathcal{C}}(t)\rangle$ form a POVM, then this equation is in agreement with Eq. (1.50), and we have that $\left|\chi_{\mathcal{S}}^{\prime(norm)}(t)\right\rangle = \langle\phi_{\mathcal{C}}(t)\,|\Psi_U\rangle\rangle\ \forall t$.

Appendix C

Exploring Strong-Coupling Regimes

Here, we explore what happens when the coupling between $\mathcal{A}$ and $\mathcal{C}$ becomes strong. We again focus on the interaction $\hat{V} = \frac{1}{\Lambda}\hat{H}_{\mathcal{A}} \otimes \hat{H}_{\mathcal{C}}$, especially in its gravitational version. While we do not expect that such an interaction describes well gravitational effects in strongly relativistic regimes, our results could point to interesting effects that deserve further exploration.

So let us consider again the scenario discussed in Sec. 4.3.1.3. We have a qubit $\mathcal{A}$ with Hamiltonian $\hat{H}_{\mathcal{A}} = \sigma_x/2$ interacting with an ideal clock $\mathcal{C}$. If the state of the qubit at time $t = 0$ is $|0_{\mathcal{A}}\rangle$, then the state of the Universe with eigenvalue $\mathcal{E}$ is given by

$$\left|\Psi_U^{\mathcal{E}}\right\rangle\!\rangle = \int \mathrm{d}t\, e^{-i\frac{\hat{H}_{\mathcal{A}}-\mathcal{E}}{1+\hat{H}_{\mathcal{A}}/\Lambda}t}|0_{\mathcal{A}}\rangle|\phi_{\mathcal{C}}(t)\rangle. \tag{C.1}$$

The effective Hamiltonian is given by Eq. (4.78), which we write again here

$$\hat{H}_{\mathcal{A}}^{eff}(\mathcal{E}) = \frac{\sigma_x/2 - \mathcal{E}}{\mathbb{1} + \sigma_x/2\Lambda} = \phi(\Lambda, \mathcal{E})\mathbb{1} + \omega(\Lambda, \mathcal{E})\frac{\sigma_x}{2}, \tag{C.2}$$

where

$$\phi(\Lambda, \mathcal{E}) = -\frac{1}{\Delta}\left(\mathcal{E} + \frac{1}{4\Lambda}\right), \tag{C.3}$$

$$\omega(\Lambda, \mathcal{E}) = \frac{1}{\Delta}\left(1 + \frac{\mathcal{E}}{\Lambda}\right), \tag{C.4}$$

with $\Delta \coloneqq \det(\mathbb{1} + \sigma_x/2\Lambda) = 1 - 1/4\Lambda^2$. The effective Hamiltonian is well defined for $\Lambda \neq \pm 1/2$ and $\lambda \neq 0$ while for $\Lambda \to \pm 1/2$, ϕ and ω diverge. When $\Lambda = 0$ the interaction term is not defined, but for $\Lambda \to 0$, $\hat{H}^{eff}_{\mathcal{A}} \to 0$ and thus the solution approaches a stationary one. When $\Lambda \to \infty$ (no interaction), $\hat{H}^{eff}_{\mathcal{A}} \to \sigma_x/2$ for any value of $\mathcal{E}$, as expected. The unitary evolution of $\mathcal{A}$ is given by

$$\hat{U}^{eff}_{\mathcal{A}}(t) \coloneqq e^{-i\phi(\Lambda,\mathcal{E})t} e^{-i\omega(\Lambda,\mathcal{E})\sigma_x t/2}. \tag{C.5}$$

The first part induces an unobservable phase shift while the second rotates the state of the qubit. We can interpret the effect of the interaction as a change in the speed at which the qubit evolves. The observable part of the effect is determined by ω.

In Fig. C.1, we have plotted the value of $\omega(\Lambda, 0)$ and $\phi(\Lambda, 0)$ for different values of Λ. For $\Lambda \to 1/2$, the speed of the qubits's evolution diverges. Interestingly, $\omega(\Lambda, 0)$ is negative for values of Λ between 0 and 1/2, so the qubit evolves in the "opposite direction" with respect to the one in the limit $\Lambda \to \infty$. One might interpret this as an inversion in the direction of the time evolution. For $\Lambda < 0$, the situation is symmetrical for ω and antisymmetrical for ϕ. As can be seen in Eqs. (4.79)-(4.80), for $\mathcal{E} \neq 0$ there are additional zeroes for ϕ and ω. When $\omega = 0$ the qubit is stationary.

Let us now assume that the qubit has mass m so that its Hamiltonian is $\hat{H}_{\mathcal{A}} = mc^2 + E_I\sigma_x/2$, where we also introduce a scale E_I for the internal energy. The

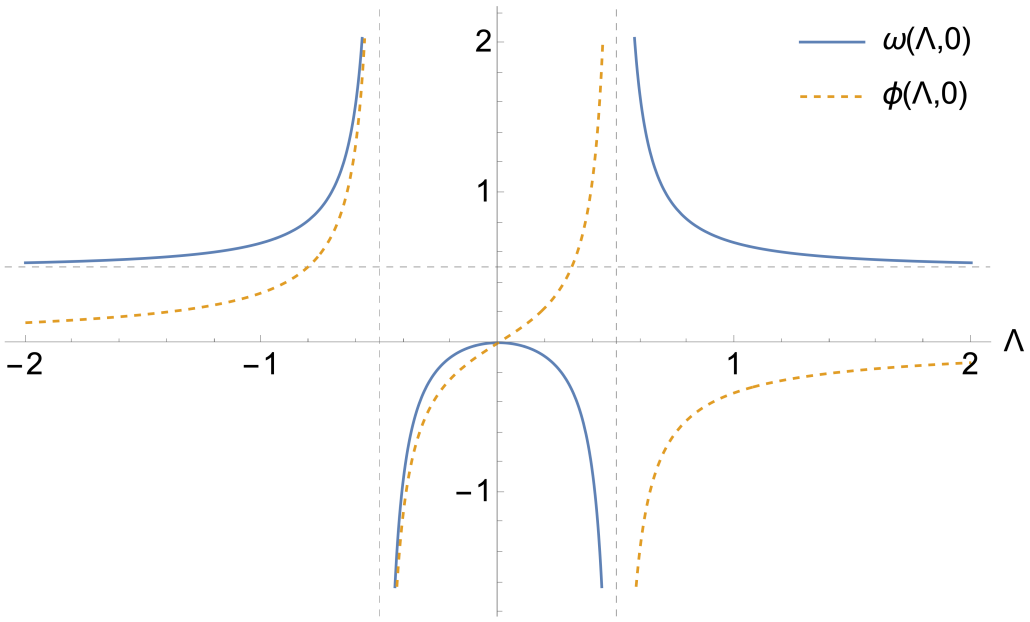


Figure C.1: Plot of the parameters governing the effective Hamiltonian of the two-level system discussed of Eq. (C.2). ω and ϕ are plotted as functions of the interaction strength Λ for $\mathcal{E} = 0$ (in arbitrary units). ϕ contributes to an unobservable global phase while ω represents a time-scale factor for the qubit's evolution. Note the inversion in the "direction" of the evolution for $-1/2 < \Lambda < 1/2$ and the divergences for $\Lambda = \pm 1/2$.

effective Hamiltonian is the same as in Eq. (C.2) but now with coefficients

$$\phi_m(\Lambda, \mathcal{E}) = \frac{mc^2}{\Delta}\left(1 + \frac{mc^2 - \mathcal{E}}{\Lambda} - \frac{\mathcal{E} + E_I^2/4\Lambda}{mc^2}\right), \tag{C.6}$$

$$\omega_m(\Lambda, \mathcal{E}) = \frac{E_I}{\Delta}\left(1 + \frac{\mathcal{E}}{\Lambda}\right), \tag{C.7}$$

with $\Delta = (1 + \frac{mc^2}{\Lambda})^2 - \frac{E_I^2}{4\Lambda^2}$. In the following, we will assume that the internal energy of the system is much smaller than its rest energy ($E_I \ll mc^2$). The observable part of the evolution is still governed by ω_m, which is plotted in Fig. C.2. The behaviour of ω_m is similar to the previous case, but now the divergences happen at $\Lambda = -mc^2 \pm E_I/2$.

If one assumes a Newtonian interaction $\Lambda = -dc^4/G$, with d the distance between

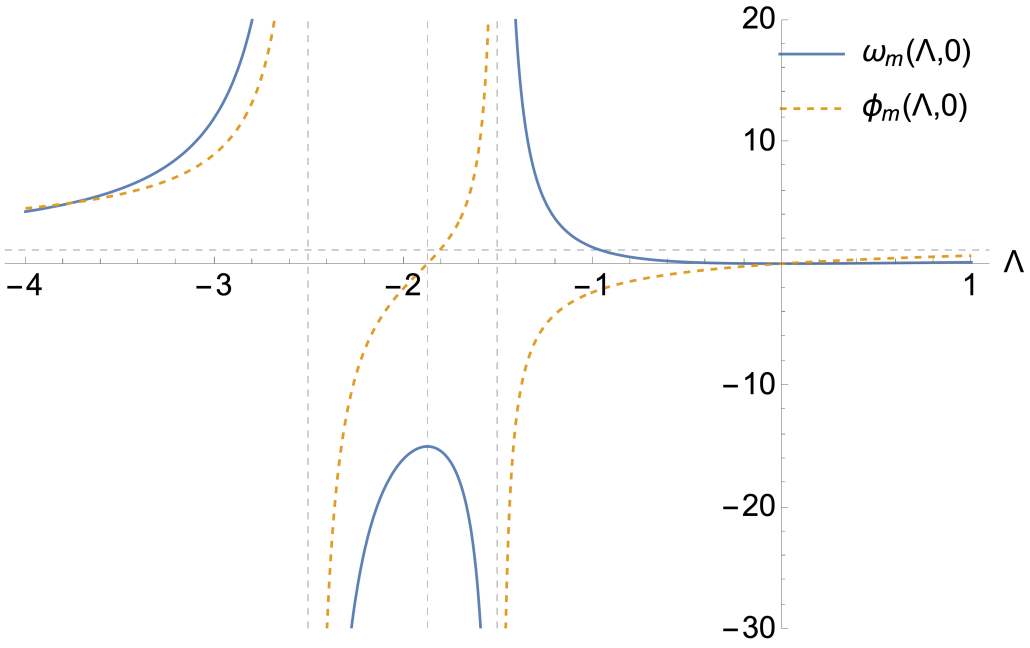


Figure C.2: Plot of the parameters in Eqs. (C.6)-(C.7) governing the effective Hamiltonian of a massive two-level system interacting with the clock. ω_m and ϕ_m are plotted as functions of the interaction strength Λ for $\mathcal{E} = 0$, $E_I = 1$ and $mc^2 = 2$ (in arbitrary units). ϕ_m contributes to an unobservable global phase while ω_m represents a time-scale factor for the system's evolution. Notice the inversion in the direction of the evolution and the divergences.

$\mathcal{A}$ and $\mathcal{C}$, then the divergences happen at the critical values of d

$$d_{\text{crit}} = \frac{R_S}{2} \mp \frac{E_I G}{2c^4}, \tag{C.8}$$

where $R_s = \frac{2Gm}{c^2}$ is the Schwarzschild radius of the system. Again, there is a time-inversion effect between these two values, as mentioned above. The factor 1/2 in Eq. (C.8) is common when working with approximations of the Schwarzschild metric away from the massive object and would disappear when considering the full metric. In that case, the time-inversion effect would happen in a small region around the "event horizon" of the massive qubit.

As we said earlier, we do not expect our scenario to describe such highly relativistic regimes well. Nevertheless, the effects we found deserve further exploration

and could lead to some interesting avenues of research.

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