

Searching for New Spacetimes:
Towards A Dynamics-First View of Topology

Daniel Grimmer

Reuben College



University of Oxford

A thesis presented for the degree of

Doctor of Philosophy

Trinity 2024

I dedicate this work to my lovely wife, Caroline.

I am so excited for the future we are building together.

Adventure awaits!

Copyright © 2024 by Daniel Grimmer

All Rights Reserved

Acknowledgements

I would like to thank everyone who helped make this thesis possible. Firstly, I would like to thank my research supervisors, James Read and Oliver Pooley, for their guidance and support throughout my DPhil. I would also like to thank the wonderful people at Reuben College (and, before this, Pembroke College).

I would also like to thank my family for their continued love and support, especially light of the COVID-19 lockdowns which occurred at the start of my studies in Oxford. In this regard, my wife deserves special thanks as does my brother, Benjamin Grimmer.

Finally, I would like to thank the Clarendon Foundation for their financial support. Their generosity has allowed me to focus wholeheartedly on my studies.

Table of Contents

I From Unofficially Humean Laws to an Unofficially Kantian Spacetime 1

1 Introduction 2

2 Four Debates about Foundational Metaphysics Set Against an Epistemic Backdrop 14

2.1 Step 1: Organizing the Pre-Epistemic World into Phenomenal Experience 15

2.2 Step 2: Codifying Our Experiences into Formalized Theories of Physics 28

3 Four Benefits of Mastering our Capacities for Formal Redescription 41

3.1 Systematic Reformulation and Epistemic Simulation 42

3.2 Articulating Dynamics-First Views and Metaphysical Simulation 57

4 Spacetime Representation Theory and the ISE Equivalence Theorem 61

4.1 A First Example of Topological Redescription: Möbius-Euclid Duality 62

4.2 More Topological Redescriptions (a.k.a., Inequivalent Spacetime Representations) 74

4.3 The ISE-Equivalence Theorem (Comparison with Spacetime Algebraicism) . . . 86

5 Towards a Dynamics-First View of Spacetime Topology 91

5.1 An Unofficially Kantian Best Systems Account of Spacetime Topology 92

5.2 World as Structure and Representation (Comparison with Structural Realism) . . 99

II The ISE Method of Topological Redescription 103

6 An Overview of the ISE Method: Internalize, Search, Externalize 104

7 Demonstrating the ISE Method in some Familiar Settings 112

7.1 Surviving Internalization: Dynamics, Diffeomorphisms, Topology 113

7.2 Internalization in a Familiar Setting: Quantum Harmonic Oscillator 123

7.3 Externalization in a Familiar Setting: Implementing a Fourier Redescription . . . 125

7.4 Can we poplulate Fourier space other than via a Fourier transform? 134

8 Two More Example Applications of the ISE Method 142

8.1 Dimensional Reduction (and Expansion) à la Kaluza-Klein 142

8.2 From a Discrete to a Continuous Spacetime (and back again) à la Kempf 149

8.3 Conventionalism about Spacetime Topology? 158

9 The ISE Method in General: Internalization 164

9.1 Internalizing a Generic Spacetime Theory 166

9.2 Internalization given Spacetime-Kinematic Compatibility 173

9.3 The Naturalness of Spacetime-Kinematic Compatibility 178

10 The ISE Method in General: Externalization 185

10.1 Defining Pre-Spacetime Translation Operations 186

10.2 Building a New Spacetime Setting from PSTOs 191

10.3 Externalization Guarantees Spacetime-Kinematic Compatibility 196

11 Characterizing ISE-Relatedness via the ISE-Equivalence Theorem 200

11.1 Proof that ISE-Relatedness is an Equivalence Relation 201

11.2 Exact Characterization of ISE-Equivalence Classes 205

11.3 Applying The ISE Equivalence Theorem to the Möbius-Euclid Duality 208

12 Conclusion and Outlook 211

Appendix A Key Theorems Regarding Smooth Homogeneous Manifolds 230

A.1 Constructing and Characterizing Smooth Homogeneous Manifolds 232

A.2 Reconstructing the Sphere as a Quotient of its Diffeomorphisms 234

A.3 Reconstructing the Möbius Strip as a Quotient of its Diffeomorphisms 239

A.4 Proof of the Homogeneous Manifold Construction Theorem 245

A.5 Proof of the Homogeneous Manifold Characterization Theorem 247

A.6 Proof of Quotient Of Images Lemma 250

Appendix B Several Proofs in Support of the ISE Method **252**

B.1 Exact Conditions for Faithful and Robust Manifold Survival 252

B.2 Internalizing a Theory with Spacetime-Kinematic Compatibility Yields PSTOs . 257

Abstract

What if we could remove and replace the topological underpinnings of our spacetime theories just as easily as we can switch between different coordinate systems (or different axiomatizations, or different geometric framings)? This thesis will show that this is, in fact, possible by developing some powerful new topological redescription techniques, namely the ISE Method. For example, I will show how a theory about one type of particle floating around on the Euclidean plane can be topologically redescribed as instead being about a different type of particle floating around on a Möbius strip, and vice versa. Beyond this and other examples, I will prove that the ISE Method can be applied to a very wide scope of spacetime theories and gives us access to *effectively every possible topological redescription thereof*; Its range is limited only by a weak spacetime-kinematic compatibility constraint.

But what philosophical benefits can we expect to follow from this newfound capacity for topological redescription? I will argue that we can expect it to yield similar benefits to those that have already resulted from our existing capacities to reframe our theories using different axiom-like laws, coordinate systems, and geometric structures. In particular, I will highlight four kinds of benefits. Ranked from least to most metaphysically ambitious, these are as follows. Firstly, it might help physicists with theory development; Secondly, it might help philosophers (i.e., us) to better understand the epistemic process of theory development; Thirdly, it might help us to articulate various "dynamics-first vs dynamics-second debates" regarding how we should interpret our theories; Finally, it might help us characterize the common structure which all of these alternative descriptions are jointly descriptions of. After discussing how these four benefits might apply in the context of spacetime topology, the latter half of this thesis will then be spent discussing the ISE Method of topological redescription in some mathematical detail.

Part I

From Unofficially Humean Laws to an Unofficially Kantian Spacetime

1 | Introduction

How does the world hang itself together? That is, what (if anything) does nature use to help it facilitate the dynamical behavior of matter? In this thesis, I will approach these questions with a realist methodology:[1] Namely, I take it that the formal/mathematical structure of our best theories of physics is a relevant line of evidence when trying to address such foundational metaphysical questions. In this spirit, let's consider some candidate facilitators by examining four dynamical phenomena, each with a physics-inspired explanation (whose merits I will later challenge).[2] Q1: Why did that apple fall? A1: Because it was governed to do so by the laws of gravity. Q2: Why does Foucault's pendulum rotate? A2: Because it notices that the Earth's lines of latitude and longitude form a non-inertial coordinate system. Q3: Why do all kinds of clocks exhibit time dilation regardless of what material they are made of? A3: Because the spacetime manifold has certain geometric features carved into it, namely locally Lorentzian spacetime intervals. Moreover, for every kind of matter, its dynamical evolution happens by making reference to these spacetime intervals rather than any absolute time.

Our final example question is a bit more complicated. Consider a counterfactual world in which our leading theory of physics is best stated in terms of a 2+1-dimensional space-

[1] As I will discuss later on, adopting this realist *methodology* may or may not lead one to realist *conclusions* (e.g., about the laws of nature or about the geometric/topological features of spacetime). In fact, it is possible for one to follow along with my realist methodology without saying anything substantial (be it positive or negative) about the pre-epistemic world (a.k.a. the mind-independent/noumenal world). All that adopting this realist methodology entails is believing that a certain physics-inspired line of evidence is relevant for debates about foundational metaphysics; It does not prescribe in advance the kind of arguments/conclusions which one should then draw from this evidence.

[2] In addition to being physics-inspired, the four explanations given in the main text each have a distinctly dynamics-second character. That is, they seek to explain matter's dynamics by invoking some dynamics-facilitating counterparts for our theories' laws, coordinates, geometry, and/or topology. It is also possible, however, to give physics-inspired explanations of these dynamical phenomena in a dynamics-first way, i.e., without claiming that our theories have latched onto any such hard-working laws, coordinates, geometry, and/or topology. In particular, one might try to give a metaphysically supported explanation of these dynamical phenomena by appealing to further facts about the dynamical behavior of matter itself rather than any such dynamics-facilitating laws, geometry, or topology.

time manifold, $\mathcal{M}_{2+1}$, foliated into space-like surfaces, $M_2(t)$, each of which is homeomorphic to a 2-sphere, $M_2(t) \cong S^2$. Moreover, suppose that after spreading measuring rods across the entire universe, we determine that the total curvature on each of these surfaces is strictly positive, $\text{curv}(M_2(t)) > 0$, regardless of the gravitational field, g_{ab}. Q4: In such a world, why would this be the case? A4: First note that, according to the Gauss-Bonnet theorem, the total curvatures which we have measured are exactly the Euler characteristics of these surfaces, namely $\text{curv}(M_2(t)) = 2\pi\,\chi(M_2(t))$. Next note that given $M_2(t) \cong S^2$ we have that $\chi(M_2(t)) = \chi(S^2) = 2 > 0$. Hence, our measuring rods behave as they do (yielding positive total curvatures) because they exist within a spacetime manifold which has a certain topology.

From these four example explanations we can identify four candidate for that which helps nature to facilitate matter's dynamics: the laws of nature and special coordinate systems, as well as the geometry and topology of spacetime. As I will discuss in the next chapter, these four kinds of candidates regularly appear as formal structures within our best theories of physics. But how did our theories end up featuring the particular laws, coordinates, geometry and/or topology that they do, rather than some mathematically equivalent alternatives? This is an interesting epistemological question in its own right. Moreover, we can ask: What selection criteria have led us to pick out these rather than those theoretical structures? Are these selection criteria objective, or is there some element of conventionality to them? In Ch. 2 I will provide a sketch of a two-step epistemic story which everyone should find agreeable. Step 1: However the world is structured in itself, we unconsciously organize it into our phenomenal experiences of the world. Step 2: After centuries of hard scientific work, the scientific community has codified their observations of matter's dynamics into our current best theories of physics; This step includes everything from experimentalists calibrating their lab equipment to the theoreticians' for-

mal wizardry. I will give the name systematization to this whole process (organize + codify).

After sketching this epistemic story, Ch. 2 will then define a series of four debates about foundational metaphysics each of which are set against this epistemic backdrop. I shall call these the dynamics-first vs dynamics-second debates (about our theories' laws, coordinates, geometry, and/or topology). Each of these debates considers one of these four theoretical structures and asks: Where along the above-sketched epistemic story does it first gain its particular kind of metaphysical significance? Dynamics-second views claim that the theoretical structure under consideration acquires its particular kind of metaphysical significance "at step zero", i.e., by latching onto some dynamics-facilitating counterpart which exists out there in the pre-epistemic world. As such, dynamics-second views can then invoke these hard-working counterparts to give metaphysically supported explanations of various dynamical phenomena (e.g., as in answers A1 - A4). For instance, recall how answer A1 invoked governing laws (i.e., the dynamics-facilitating counterparts of our theories' laws) in order to explain the dynamical behavior of matter.

While my ultimate goal in Part I of this thesis is to put forward a dynamics-first view of spacetime topology, it is best to approach this topic through the dynamics-first views of laws and geometry. In opposition to governing views of laws, a dynamics-first approach would try to account for the nomological significance of our theories' laws without invoking any governing laws. Importantly, adopting such a dynamics-first view is compatible with (but does not require) believing that the laws of nature are *real* in the sense of being objective and mind-independent. Namely, the selection criteria which we use to pick out a given world's non-governing laws might be objective. For instance, Lewis (1994, pg. 479) argues that his law-selection criteria (i.e., best balancing simplicity and strength) will be objective "if nature is kind" to us. Moreover, each such non-governing law (e.g., "all Fs are Gs") might have a mind-independent counterpart (e.g., the brute regularity that all

Fs are Gs). Regardless of whether or not the laws of nature are real in this sense, the defining claim of any dynamics-first view of laws is as follows: Our theories' laws do not gain their nomological significance "at step zero" by latching onto any dynamics-facilitating law-like counterpart out there in the pre-epistemic world; Instead, they acquire whatever nomological significance they have only "along the way", i.e., by being a particularly nice way of systematizing the dynamical behavior of matter.

Of course, the question of whether the laws of nature govern the world's patterns or merely summarize them is a part of the well-established metaphysics of laws debate.[3] In this context, Hall (2015, pg. 265) has identified that aligning one's epistemic and metaphysical accounts of laws is the "unofficial guiding idea" behind Humeanism about laws; Namely, it consists in Humeans "taking standards that both sides [of the debate] endorse — but that [their anti-Humean] opponent views solely as epistemic standards — and elevating them to the status of standards constitutive of the laws of nature." Importantly, this unofficial best-systematization aspect of Humeanism is logically independent from its well-known ban on necessary connections between distinct existences. Indeed, it should be stressed that one's view can qualify as dynamics-first without putting forward any doctrine about the fundamental ontological structure of the pre-epistemic world (e.g., that it consists of a Humean mosaic). This is an important point of departure from Hall's (2015) framing of the metaphysics of laws debate. Indeed, as I will now discuss, dropping this reductionist thesis allows us to better connect Humeanism about laws with its historical roots; not à la David Lewis, but rather à la David Hume.

[3] For an overview of this debate see the SEP article by Carroll (2020) as well as Ott's (2022) recent book. Dynamics-second views of laws are roughly what Ott (2022) calls the "top-down" views in the metaphysics of laws debate: e.g., Cartesian occasionalism, see Ott (2009, Ch. 9); primitivism, see Maudlin (2007, Ch. 1), Schaffer (2016), and Carroll (1994); and the Dretske–Tooley–Armstrong conception of laws as relations between universals, see Dretske (1977), Tooley (1977), and Armstrong (1983). Dynamics-first views of laws are roughly what Ott (2022) calls the "bottom-up" views in the metaphysics of laws debate: e.g., causal powers, see Demarest (2017), and Bird (2007); and Humeanism, see Lewis (1983), Hall (2015), and Bhogal (2020) among many others.

On an epistemic reading, Hume's project is simply not concerned with establishing whether or not there are causally efficacious relationships at work out there in the pre-epistemic world. Instead, one can understand his project as aiming to identify how our phenomenal notion of causation has come to acquire its causative significance. Given Hume's belief that all of our ideas ultimately come from experience, he is led to conclude that our knowledge of causation only amounts to repeated experiences of constant conjunctions and so is not reflective of any causal efficacy out there in the world. In terms of the above-sketched epistemic story, one can gloss Hume's claim as follows: Our phenomenal notion of causation does not gain its causative significance "at step zero" by latching onto any causal efficacy out there in the pre-epistemic world; Instead, it acquires its causative significance only "along the way" by being a particularly nice way of organizing the pre-epistemic world. Notice that what makes cause/effect relationships "particularly nice" for Hume is that they are instances of certain psychologically compelling and exceptionless regularities of experience.

There are, of course, other readings of Hume according to which his project actually is concerned with whether or not there is any causal efficacy in the pre-epistemic world; Namely, he finds that there is not.[4] Importantly, even on such a metaphysical reading, Hume would still be in support of the claim which I have just attributed to him above. Our phenomenal notion of causation does not gain its causative significance "at step zero" simply because there are no causally efficacious relationships out there for it to latch onto. Hence, it must acquire its causative significance only "along the way" simply because there is no other way. These two readings of Hume show that one can apply Hall's (2015, pg. 265) "unofficial guiding idea" to the subject of causation either with or

[4] There are, of course, other possible readings of Hume beyond what I have here called the epistemic and metaphysical readings. For instance, a third possible reading of Hume (e.g., by Smith (2005) or Strawson (2014)) takes him to believe that there are in fact causally efficacious relationships out there in the world and, moreover, that they are in principle knowable. See Lorkowski (2023) for an overview of these three readings of Hume.

without making any substantive claims about the pre-epistemic world. Indeed, Hume's position on causation could be true for either epistemic or metaphysical reasons: Causally efficacious relationships might fail to exist or, even if they do exist, we might be epistemically alienated from them. Either way, Hume ends ups treating causation *as if* it is merely an organizational principle which we unconsciously project onto the world — given that this fully accounts for how it has earned its causative significance. Taking this Humean position as our blueprint, dynamics-first views (about laws, coordinates, geometry, and topology) will also come in epistemic and metaphysical flavors.

Returning to the metaphysics of laws debate, one may be surprised to discover that — despite its Humean origins — the umbrella of dynamics-first views of laws is actually wide enough to include causal powers views. This is because such views take our theories' laws to gain their nomological significance by latching onto causal powers, not governing laws. Ott (2022, pg. 4) puts it well: "[According to causal powers views,] if there are any laws at all, they merely describe the powers that are responsible for the events we see around us. Such a view is bottom-up in the sense that what determines the course of nature is Nature itself." It is in this sense that proponents of causal powers views take the laws of nature to acquire their nomological significance "along the way" rather than "at step zero". (Although they would then go on to disagree with Hume by saying that our phenomenal notion of causation earns its causative significance at step zero; This is what makes them only unofficially Humean).[5] For an example of a causal powers view which enthusiastically embraces these parallels with canonical Humeanism about laws, see the

[5] My term "unofficially Humean" can also be applied to Loewer's (2020) and Albert's (2015) Mentaculus taken together with Loewer's (2021) Package Deal Account (PDA) of laws. Loewer (2020, pg. 6) advertises the Mentaculus as a "Humean account along the lines of the best systems account of laws and chances developed by David Lewis" in which the past hypothesis and the statistical postulate are conceived of as non-governing laws. The PDA is similarly advertised as a Humean account of laws even though on this view the best systematization may (or may not) pick out dispositions and/or potencies as being among the world's fundamental properties. Loewer (2021, pg. 1084) argues that "Necessary connections introduced in this way don't compromise the PDA's Humean credentials since it is still the case that laws systematize regularities and don't enforcing [sic] them." In sum, the Mentaculus + PDA view is Humean in the best-systematization sense and is compatible with (but does not require) Humeanism in the modality-denying sense. Hence, in my terminology, this combination of views qualifies as unofficially Humean.

generalization of Lewis' (1983) Best System Account (BSA) by Demarest (2017) which allows for dispositional properties to appear on the now-not-so-Humean mosaic.[6] One good reason for dynamics-first views to include causal powers is that it enables them to give metaphysically-supported explanations of the dynamical behavior of matter (a well-noted inability of canonical Humeanism).

Let us next identify the dynamics-first vs dynamics-second debate about geometry. This debate is concerned with the metaphysical status of the geometric structures which appear in our best theories of physics (e.g., the Lorentzian metric, η_{ab}, in special relativity). In terms of the above-sketched epistemic story, the question is: Have these geometric structures earned their geometric significance "along the way" or "at step zero"? Indeed, this is a very near match to the topic of the well-established dynamical vs geometrical spacetime debate.[7] According to dynamics-second views of geometry, the geometric structures which appear in our best theories have earned their geometric significance by latching onto some geometric counterpart out there in the pre-epistemic world which helps to facilitate matter's dynamics. This then allows dynamics-second views of geometry to invoke these hard-working geometric structures in order to give metaphysically supported explanations of matter's dynamics. For instance, recall how answer A3 suggested that the dynamical evolution of clocks is aided by there being geometric features built into the spacetime manifold, e.g., locally Lorentzian spacetime intervals.

Just as in the laws' context, however, one might instead opt for a dynamics-first interpretation of our theories' geometric structures.[8] Namely, one might follow Brown (2005) and others in claiming that the geometric structures which appear in our spacetime the-

[6] The concept of a Humean mosaic will be introduced in Ch. 3.

[7] Dynamics-second views of geometry are on the geometrical side of the geometrical vs dynamical spacetime debate. This largely orthodox view is implicit in Friedman's textbook (1983, Ch. 6 Sec. 4). It can be seen more explicitly in Maudlin (2012); Janssen (2009); Norton (2008); Dorato (2007).

[8] The dynamics-first views of geometry are, roughly, the dynamical side of the geometrical vs dynamical spacetime debate. For a recent overview of the dynamics-first side of this debate see Brown and Read (2022) and references therein, especially Brown (2005); Brown and Pooley (2006); O. Pooley (1999); Menon (2019); Huggett (2006); Stevens (2014).

ories acquire their (chrono-)geometric significance only because they are a particularly nice way of systematizing the dynamical behavior of matter (perhaps conceived of non-geometrically). N.B.: This view is compatible with there being real geometric facts about the world just as there might be objective and mind-independent facts about the laws of nature even if they don't latch onto any governing laws. The world might have a definite geometric structure, albeit one which is metaphysically unemployed, so to speak. Unfortunately, a detailed discussion of the dynamical vs geometric spacetime debate is beyond the scope of this thesis.

While these two debates about the metaphysical status of laws and geometry are already well-established individually, a detailed discussion of their similarities is conspicuously missing from the existing literature.[9] In the next chapter I will fill this gap while also pointing out the existence of an analogous debate about the metaphysical status of coordinate systems. Beyond merely rehashing and comparing these well-established debates, my main goal in Part I of this thesis is to extend these three debates beyond their original contexts of laws, coordinates, and geometry into the context of spacetime topology (in answer to Norton (2008)). To this end, Chs. 2 and 3 will first do some substantive philosophical work in setting up these four metaphysical debates against a common epistemic backdrop. This will set the stage for Chs. 4 and 5 to then put forward a dynamics-first view of spacetime topology which is closely analogous to the other three dynamics-first views.

Said differently, my goal in this thesis is to take the above-discussed unofficially Humean approach to the metaphysics of laws and causation and apply it to the metaphysics of space and time. The prospect of extending certain aspects of Hume's work on causation to the metaphysics of space and time ought to ring some Kantian bells. In-

[9] One notable exception is an implementation of the dynamical approach to spacetime geometry by Huggett (2006) which enthusiastically embraces the above-noted parallels with Humeanism about laws.

deed, one of Kant's key philosophical contributions was to extend Hume's epistemology-aligned style of metaphysics from causation to the metaphysics of space and time.[10] Compare the above-discussed epistemic reading of Hume with the following epistemic reading of Kant. Kant's project is simply not concerned with establishing whether or not there are such things as space and time hard at work out there in the pre-epistemic world. Instead, one can understand Kant's project as aiming to identify how our phenomenal notions of space and time have come to acquire their spatiotemporal significance.

In terms of the above-sketched epistemic story (which I will flesh out in Ch. 2), one can gloss Kant's claim as follows: Our phenomenal notions of space and time do not gain their spatiotemporal significance "at step zero" by latching onto any dynamics-facilitating spacetime points or relations out there in the pre-epistemic world (contra Newton and Leibniz); Instead, they acquire their spatiotemporal significance only "along the way" by being a particularly nice way of organizing the pre-epistemic world.[11] Notice that what makes our phenomenal notions of space and time "particularly nice" for Kant is that they are necessary preconditions for the very possibility of human experience. While Kant clearly differs from Hume regarding how they are spelling out what "particular nice" means (i.e., a priori vs a posteriori), he has nonetheless followed Hume in giving an "along the way"-type answer rather than an "at step zero"-type answer. Given this historical connection between Hume and Kant, it is unsurprising that a dynamics-first view of spacetime will end up having an "unofficially Kantian" character.[12]

[10] Kant (2004, pg. 10).

[11] More will be said in Ch. 2 about what I take "organizing" to mean and how this compares with Kant's understanding of our spatiotemporal intuitions.

[12] Recall from above that the term "unofficially Humean" applies to views which accept one part of Hume's philosophy (i.e., the paralleling of metaphysics and epistemology, see Sec. 2.1) independent of whether or not it accepts some other aspects of his philosophy (e.g., his ban on necessary connections between distinct existences). Similarly, I will call a view "unofficially Kantian" if it accepts one part of Kant's philosophy (i.e., treating space and time as organizational principles, see Sec. 2.1) independent of whether or not it accepts some other parts (e.g., the a priori vs a posteriori character of space and time). Like other Neo-Kantian views, this unofficial Kantianism pins all of space and time's metaphysical significance upon the epistemological route via which they have entered into our experiences and our theories. See Ch. 5 for further discussion.

Indeed, as I will discuss in Ch. 5, the ultimate claim of my dynamics-first view of spacetime topology will be as follows. The spacetime manifold which appears in our best physical theories does not gain its spatiotemporal significance by latching onto some spatiotemporal stage upon which the world's drama plays out (contra Newton); Nor does it do so by latching onto any fundamental spatiotemporal relationships which help to facilitate interactions between the drama's players (contra Leibniz); Instead, it acquires its spatiotemporal significance only because it is a particularly nice way of systematizing the dynamical behavior of matter (perhaps conceived of non-spatiotemporally). N.B.: Just as there may be objective and mind-independent facts about non-governing laws, so too might there be real spatiotemporal facts on this view, so long as they are likewise metaphysically unemployed, so to speak.

In addition to setting up these four metaphysical debates (and highlighting their historical roots in Hume and Kant), this thesis will also discuss their epistemic underpinnings at length. Indeed, recall that these four debates are explicitly defined against an epistemic backdrop. Setting up this epistemic backdrop requires at least partially answering the following question: By what process have these particular Z = axiom-like laws, coordinate systems, geometry, and/or topology come to feature in our best physical theories, rather than some mathematically equivalent alternatives? Only then can we hope to account for how these theoretical structures have come to appear in our physical theories *while having the particular kind of metaphysical significance that they do*. Both sides of the dynamics-first vs dynamics-second debates ought to care about these underlying epistemic issues, albeit with different motivations. Of course, one might also be interested in our Z's epistemology for purely epistemic reasons (not picking sides in the the dynamics-first vs. dynamics-second debate about Z). For example, one might be interested in whether our Z-selection criteria are objective or conventional.

In any case, correctly identifying our Z-selection criteria will require some hard epistemological work. A full account of the *actual epistemic process* would likely involve: 0) some assumptions about the pre-epistemic world-in-itself, 1) an account of how we make organizational sense of the world, 2) an account of how we codify our phenomenal experiences into our theories, and 3) an account of the formal wizardry which is involved in the Z-reformulation process. Such an account would likely be very complicated requiring input from metaphysics, phenomenology, sociology, and mathematics among other fields of study. Fortunately, however, if we accept the realist methodology which I have proposed in this thesis, then we can leverage the formal structure of our best theories to shed some light on Z's epistemology (and thereby on the dynamics-first vs dynamics-second debate about Z).

As I will argue in Ch. 3, there is much to be gained from mastering our capacity for formal redescription (e.g., of our theories Z = axiom-like laws, coordinates, geometry, and topology). For instance, our ability to identify and articulate our Z-selection criteria would be enhanced if we could first get all of the options on the table, so to speak (even those which are horrendously complicated). Fortunately, we have good mathematical control over the process of reformulating our theories with respect to different axiom-like laws. Indeed, we have a similarly strong capacity for coordinate and geometric redescription as well. Part II of this thesis will deliver a similarly strong capacity for topological redescription, namely the ISE Method.

These new techniques will allow us to remove and replace the topological underpinnings of our spacetime theories just as easily as we can switch between different coordinate systems (or different axiomatizations, or different geometric framings). For example, in Ch. 4 I will show how a theory about a certain kind of particle floating around on the Euclidean plane can be topologically redescribed as instead being about a different type of particle floating around on a Möbius strip, and vice versa. This and other examples

of topological redescription will be considered throughout the latter half of this thesis. Ultimately, Part II of this thesis will culminate in Ch. 11 with a proof that the ISE Method can be applied to a very wide scope of spacetime theories and gives us access to effectively *every possible topological redescription thereof*; Its range is limited only by a weak spacetime-kinematic compatibility constraint.

2 | Four Debates about Foundational Metaphysics Set Against an Epistemic Backdrop

As I noted in Ch. 1, the four dynamics-first vs dynamics-second debates will be concerned with the metaphysical status of certain features of our best physical theories, i.e., their laws, coordinates, geometry, and topology. The central question in each of these debates will be about how the theoretical structure under consideration has come to acquire its particular kind of metaphysical significance. Should we take them to latch onto some worldly counterpart which helps to facilitate matter's dynamics (e.g., governing laws)? Or should we deny them any such metaphysical weight (e.g., non-governing laws), instead explaining their metaphysical significance solely in terms of how they help us systematize the dynamical behavior of matter? While one is free to choose sides in each of these four debates independently, this chapter will introduce and motivate the dynamics-first and dynamics-second perspectives across all four contexts in unison.

The goal of this chapter is to sharpen the contrast between the dynamics-first and dynamics-second approaches to metaphysics by staging them against an agreed-upon epistemic backdrop. The key question is as follows: By what process have these four kinds of structures come to show up in our best physical theories *while having the particular kind of metaphysical significance that they do*? I take it that both sides of the dynamics-first vs dynamics-second debates accept the two-step epistemic story outlined below. In short, we arrive at our physical theories by first unconsciously organizing the dynamical behavior of matter into our phenomenal experiences and then by painstakingly codifying these experiences into our formalized theories of physics. Hence, there is an

epistemic arrow which points from matter's dynamical behavior to the laws, coordinates, geometry, and topology which appear in our best physical theories.

Having agreed about the direction of this epistemic arrow, each dynamics-first vs dynamics-second debate can then be characterized in three equivalent ways: Firstly, as a disagreement regarding whether the metaphysical relationship between matter's dynamics and our theories' laws, coordinates, geometry, and/or topology ought to run parallel or anti-parallel to this epistemic arrow; Secondly, as a disagreement regarding where along this epistemic story these theoretical structures first acquire their particular kind of metaphysical significance (either before or after the dynamical behavior of matter, so to speak); Thirdly, as a disagreement regarding whether or not these theoretical structures have latched onto anything out there in the pre-epistemic world which helps to facilitate matter's dynamics.

2.1 Step 1: Organizing the Pre-Epistemic World into Phenomenal Experience

The first step in our epistemic story proceeds as follows. Step 1: However the world is structured in itself, we unconsciously organize it into our phenomenal experiences of the world. Let us call this the arrow of organization, see Fig. 2.1. Furthermore, let us adopt the term "the pre-epistemic world" for whatever *actual mind-independent stuff/goings-on/structure* our experiences are experiences of. (My usage of this term is comparable to Kant's term "the noumenal world" when understood as referring to the pre-epistemic world-in-itself and not to a limit concept accessible from within experience.) The foundational task of metaphysics is to disentangle (insofar as we can) the pre-epistemic world-in-itself from whatever organizational principles we are unconsciously bringing to the table ourselves. That is, our goal is to sort out to the best of our abilities what is going on in

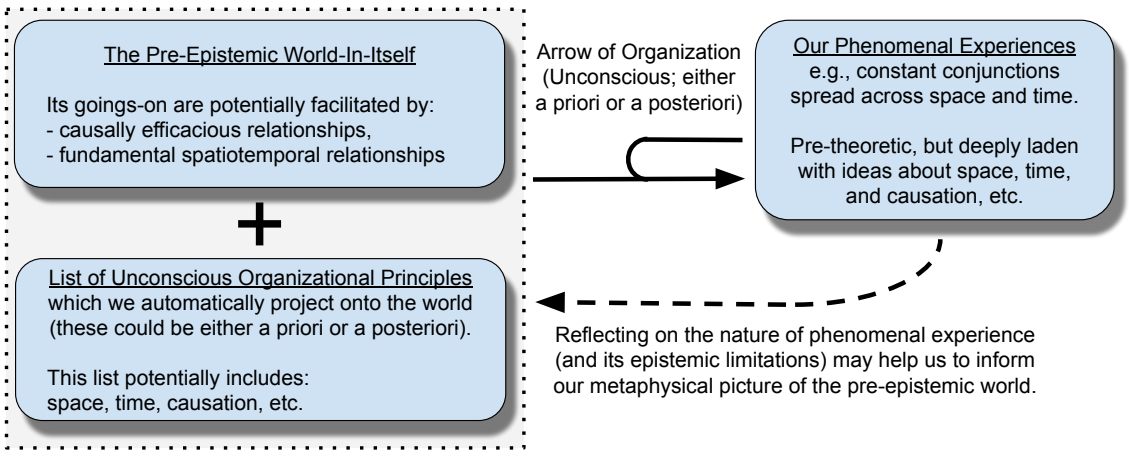

Figure 2.1: This figure sketches the first step in our two-step epistemic story: We arrive at our phenomenal experiences by applying some unconscious organizational principles to the pre-epistemic world's goings-on. The resulting experiences are deeply laden with various pre-theoretic ideas about the world including space, time, causation, etc. The debates for and against an unofficially Humean style of metaphysics ask: Where along this epistemic story do certain phenomenal notions (e.g., causation, space, and time, etc.) first acquire their particular kind of metaphysical significance? The unofficially Humean camps answer that they only gain their metaphysical significance "along the way" by being a particularly nice way of organizing the pre-epistemic world's goings-on. By contrast, the other camp (broadly anti-Humean) answers that they do so "at step zero" by latching onto some pre-epistemic counterpart which helps to facilitate these goings-on.

the left box of Fig. 2.1. Further metaphysics might then attempt to investigate the internal working mechanisms of the pre-epistemic world and/or the operation of our unconscious organizational principles (e.g., how they relate to each other and whether their epistemic character is a priori or a posteriori).

As an historically important illustration of this distinction, consider Hume's understanding of causation. Hume believed that all of our ideas come from experience. In particular, he argued that our knowledge of causation only amounts to repeated experiences of constant conjunctions and so is not reflective of any causal efficacy out there in the world. This, however, is not to say that nothing in Hume's metaphysics answers to the name "causation"; Hume identifies cause/effect relationships as being the instances of certain exceptionless regularities of experience, especially those which are psychologically compelling.[13] Namely, he argues that our judgements of causation are ultimately

[13] Hume actually provides us with two definitions of causation which are arguably inequivalent. Roughly speaking, the two definitions either omit or include the psychological criteria. According to Blackburn (1990, pg. 247), the first definition is concerned with "the contribution of the world to our apprehension" whereas the second definition is concerned with "the functional difference in the mind that apprehends the regularity." For illustrative purposes, I am here adopting the second definition.

just an organizational principle which we reflexively project onto the world as a matter of custom or habit.[14] According to Hume, we shouldn't think of cause/effect relationships as doing any metaphysical work to help facilitate the pre-epistemic world's goings-on; Instead, we ought to think that they acquire their causative significance only because of how they help us organize these goings-on.

It is a matter of lively debate whether Hume's critique of causation was meant to have metaphysical implications or whether it is merely epistemic.[15] On an epistemic reading, Hume's view is consistent with there nonetheless being causally efficacious relationships out there in the pre-epistemic world. On such a reading, Hume is not telling us about the existence or non-existence of causal efficacy; Instead, one can understand his project as aiming to identify how our phenomenal notion of causation has come to acquire its causative significance. Ultimately, however, the epistemic and metaphysical readings of Hume amount to the same thing in practice. Even if Hume is officially open to the existence of causal efficacy in the pre-epistemic world, he ends ups treating causation *as if* it is merely an organizational principle which we unconsciously project onto the world — given that this fully accounts for how it has earned its causative significance. Beyond this, any additional metaphysical significance which we might want to grant our phenomenal notion of causation would be unnecessarily speculative.

As another historically significant example, consider how Kant's metaphysics builds upon Hume's. Firstly (and more importantly for my purposes), Kant extended Hume's epistemology-aligned approach to causation to the metaphysics of space and time.[16] In particular, Kant denies that our phenomenal experiences give us any reason to believe that

[14] Hume (2020, Sec. 8 Part 1 pg. 82).

[15] There is, of course, a third possible reading of Hume (e.g., by Smith (2005) or Strawson (2014)) which takes him to believe that there are in fact causally efficacious relationships out there in the world and, moreover, that they are in principle knowable. See Lorkowski (2023) for an overview of these three readings of Hume.

[16] Kant (2004, pg. 10). See also, Weinberger et al. (2024) and Dewar et al. (2022) for two more recent calls to do epistemology-focused metaphysics regarding causation, space, and time.

there are such things as space and time out there helping to facilitate the pre-epistemic world's goings-on. Instead, it ought to be sufficient to think of our phenomenal notions of space and time as having acquired their spatiotemporal significance only because of how they help us organize these goings-on. Secondly, whereas Hume took our phenomenal notion of causation (and ostensibly space and time) to be derived from experience (making them *a posteriori*), Kant took them to be *a priori*. In particular, Kant took our phenomenal notions of causation, space, and time to play a significant role in grounding the very possibility of human experience.[17] There are hence two key ways in which Kant builds upon Hume's critique of speculative metaphysics: It should be sufficient to grant metaphysical significance to our phenomenal notions of causation *and space and time* only insofar as we can know about them through experience *as well as its a priori prerequisites*.

As with Hume, one ought to distinguish between two possible readings of Kant. A metaphysical reading would understand Kant as flatly rejecting that there are things such as space, time, and causation out there helping to facilitate goings-on in the pre-epistemic world. On an epistemic reading, however, Kant is largely silent about what is actually going on in the pre-epistemic world (besides acknowledging that it has a mind-independent existence and that it somehow affects us, e.g., by activating our senses).[18] Indeed, on such a reading Kant is simply not concerned with whether or not our phenomenal notions of causation, space, and time have any hard-working pre-epistemic counterparts; Instead, he is only concerned with how they have come to bear their particular kind of metaphysical significance. Given that Kant shares in Hume's critique of speculative metaphysics, however, these two readings of Kant amount to the same thing in practice; We ought to treat space and time (like causation) *as if* they are merely organizational principles which we

[17] Kant (2004).
[18] See Note II of Kant (2004, pg. 40) for Kant's rejection of idealism.

unconsciously project onto the world — given that this fully accounts for how they have earned their spatiotemporal significance. Beyond this, any additional metaphysical significance which we might want to grant our phenomenal notions of space and time would be unnecessarily speculative.

Of course, one is free to disagree with both Hume and Kant by arguing against their ban on speculative metaphysics. In fact, I will later argue that we may be justified in speculating about the pre-epistemic world based on the formal structure of our best physical theories. (Importantly, this claim goes beyond my above-discussed realist methodology, see Ch. 3.) There are, of course, many ways in which one might use physics to speculate about the pre-epistemic world. For instance, one could (although I won't) try to launch a physics-based argument that causation (or space, or time) are, in fact, hard-working aspects of the pre-epistemic world. Moreover, one might try to give a compelling story about how our phenomenal notions of causation (or space, or time) have more-or-less faithfully latched onto their hard-working pre-epistemic counterparts. The more similar that the pre-epistemic world is to our phenomenal experience thereof, the less epistemic work there is for any organizational principles to do. At a minimum, our brains might do very little organizational work for us before we consciously experience the world.

Acknowledging all of this room for disagreement, I take it to be agreed that there is some unconscious organizational gap (whatever its size) between the pre-epistemic world and our phenomenal experiences thereof. Hence, we have a consensus on the first step in our two-step epistemic story: The pre-epistemic world and our unconscious organizational principles somehow combine to yield our phenomenal experiences. Moreover, I take it to be agreed that the resulting experiences are deeply laden with various pre-theoretic ideas about the world including: space, time, causation, etc.[19]

[19] Us having such pre-theoretic ideas baked into our thinking does not preclude us from also having post-theoretic ideas similarly baked into our thinking. Indeed, reflecting on our theories often helps us to frame our phenomenal experiences of the world. Such a theoretical framing may even be necessary to get the scientific enterprise off the ground; There may be some unavoidable theory-ladeness to observation.

Before moving on to the next step of our epistemic story (in which our phenomenal experiences will be codified into formalized theories), allow me to first point out some features of the above discussion which will become important later in the dynamics-first vs dynamics-second debates. I should first note that accepting this epistemic story has some metaphysical consequences; Namely, it means believing that this story accounts for how causation, space, and time have shown up in our experiences (and, later, in our formalized theories) *while having the particular kind of metaphysical significance that they do*. In particular, accepting this epistemic story means the following.

> **Accepting this epistemic story**: Any aspect, Z, of our phenomenal experience (or, later, of our formalized theories) must acquire its particular kind of metaphysical significance somewhere along this epistemic story. The first place which this happens could be either:
>
> - "at step zero", i.e., by latching onto some pre-epistemic Z-like counterpart which helps to facilitate goings-on in the pre-epistemic world in a Z-like way,
> - or else, "along the way", i.e., by being a particularly nice way of organizing the pre-epistemic world (or, later, of codifying our experiences thereof).

Taking the "at step zero" option with Z = causation, one must claim that some cause/effect relationships really exist out there in the world. It is not enough, however, for these cause/effect relationships to merely exist (e.g., as a brute regularity: Cs always followed by Es). They must additionally do some metaphysical work at step zero (i.e., pre-epistemically) which could account for the causative significance of their phenomenal/theoretical counterparts. For instance, these pre-epistemic cause/effect relationships could help to facilitate the world's events by causally necessitating them. In this way, the cause/effect relationships would do some significant metaphysical work right at the beginning of our epistemic story, even before the organization step. Note that, so far, everything that we have said is still consistent with the epistemic reading of Hume. The final missing piece of our "at step zero" account is some bridging between these causally

efficacious relationships and our phenomenal notion of causation; This is what I mean by "latching onto". All of this is required if we are to say that our phenomenal notion of causation acquires its causative significance by latching onto some hard-working pre-epistemic counterpart.

To anticipate our later discussion, let's now take Z to be an aspect of our formalized theories. Namely, consider taking Z to be the gravitational field, g_{ab}, which appears in general relativity (GR). It is well known that g_{ab} has some (chrono-)geometric significance; Indeed, this is why it is often simply called "the metric field". But how has it come to have this geometric significance? When deciding which part of one's theory might deserve this title, it is insufficient to just pick any old object which a mathematician would be comfortable calling a Lorentzian metric. There may be several of these available to us: Consider the Einstein and Fokker (1914) reformulation of Nordström's (1913) second theory of gravity. Among all of the possible metrics (e.g., related by a Weyl transformation), not all of them will have geometric significance (i.e., they can't all relate appropriately to the dynamical behavior of rods and clocks). Some selection criteria are needed to determine which metric have (chrono-)geometric significance, see Duerr (2020).

Going beyond epistemics, however, what is required in order to take the "at step zero" route when accounting for g_{ab}'s geometric significance? Consider the following claim: "GR's g_{ab} field has latched onto a pre-epistemic g_{ab} field which is causally efficacious and thereby helps to facilitate the world's goings-on." This claim falls short of what we are looking for; It only establishes that GR's g_{ab} field acquires its causative significance at step zero whereas it might still pick up its geometric significance along the way. In order to give an "at step zero" accounts of the *specifically geometric* significance of the g_{ab} field in GR, we would need to make a few changes to this claim. Namely, we would need to say that the pre-epistemic g_{ab} field has its geometric character mind-independently and,

moreover, that it does its pre-epistemic work *in a geometric-way* such that it can (and, in fact, does) account for the geometric significance of g_{ab} in GR.

Alternatively, however, one could reject that Z (= causation, space, time, etc.) acquires its particular kind of metaphysical significance at step zero. This is one way to characterize what I called "unofficial Humeanism" earlier in Ch. 1,

> **Unofficial Humeanism about Z:** Acceptance of our epistemic story plus a *rejection* of the claim that some aspect, Z, of our phenomenal experience (or, later, of our formalized theories) gains its particular kind of metaphysical significance at step zero, i.e., by latching onto some pre-epistemic Z-like counterpart which helps to facilitate the pre-epistemic world's goings-on in a Z-like way.

For instance, Hume and Kant would hold this kind of view on either a metaphysical or an epistemic reading. Indeed, unofficial Humeanism about Z could be true for either metaphysical or epistemic reasons; Namely, such a hard-working pre-epistemic counterpart for Z might fail to exist or, even if it does exist, we might be epistemically alienated from it. Correspondingly, one could argue for unofficially Humean views on either metaphysical or epistemic grounds. Note that Hume's modality-denying credo (i.e., "there are no necessary connections between distinct existences") results from taking unofficial Humeanism about Z = causation to be true for metaphysical reasons. Importantly, however, these two senses of Humeanism (i.e., unofficial Humeanism and modality-denying Humeanism) will come apart in other contexts; For instance, as I noted in Ch. 1, one can reject an at-step-zero account of the laws of nature while accepting an at-step-zero account of causation.[20] Taking Z = space and time in this schematic gives us a view which we might call "unofficial Kantianism" (more will be said about this in Ch. 5).

But if our phenomenal notion of Z = causation (or space, or time) doesn't latch onto any hard-working pre-epistemic counterpart, then at what point in our epistemic story does it first gain its particular kind of metaphysical significance? For instance, what if our

[20] This is most clearly exemplified by the works of Demarest (2017) and of Loewer (2021).

phenomenal notion of causation has a pre-epistemic counterpart, but not a hard-working one (e.g., just a brute regularity: Cs always followed by Es)? Given what it means to accept the above epistemic story (i.e., an exclusive disjunction between the "at step zero" and "along the way" routes) Z must pick up its particular kind of metaphysical significance along the way, so to speak. This has the effect of paralleling our metaphysical account of Z with the epistemic route via which Z enters into our phenomenal experiences (or, later, our formalized theories). In particular, it leads to the following logically equivalent characterization of unofficial Humeanism:[21]

> **Unofficial Humeanism about Z:** Acceptance of our epistemic story plus the claim that some aspect, Z, of our phenomenal experience (or, later, of our formalized theories) gains its particular kind of metaphysical significance only "along the way", i.e., by being a particularly nice way of organizing the pre-epistemic world (or, later, of codifying our experiences thereof).

But what does "particularly nice" mean here exactly? The precise meaning of this term has been intentionally left open here, since there are ultimately many possible many ways of giving "along the way"-types accounts. For instance, we can see some examples of what "particularly nice" might mean by trying to approximate Hume's and Kant's positions. For Hume what makes cause/effect relationships "particularly nice" is that they are instances of certain psychologically compelling and exceptionless regularities of experience. For Kant, what makes our phenomenal notions of causation, space, and time "particularly nice" is that they are necessary preconditions for the very possibility of human experience. While Kant clearly differs from Hume regarding how they are spelling out what "particular nice" means (i.e., a priori vs a posteriori), he has nonetheless followed Hume in giving an "along the way"-type answer rather than an "at step zero"-type answer.

[21] As I noted in Ch. 1, my usage of this term originates with Hall's (2015, pg. 265) identification of "the unofficial guiding idea" behind Humeanism about laws as consisting in them aligning their epistemic and metaphysical account of laws.

The above-sketched debates (for and against an unofficially Humean style of metaphysics) are an important precursor to the coming dynamics-first vs dynamics-second debates. The above debates have been aimed at understanding to the best of our abilities what is going on in the left box of Fig. 2.1. Above, I called this the foundational task of metaphysics: To disentangle (insofar as we can) the hidden mind-independent structure of the pre-epistemic world from whatever organizational principles we are unconsciously bringing to the table ourselves (regardless of whether they are a priori or a posteriori).[22] So far, we have attempted to do this by reflecting upon the nature of phenomenal experience and its epistemic limitation. The opponents of unofficial Humeanism might argue that the prominence of causation, space, and time in our phenomenal experiences (and their explanatory merits) are good evidence for them having latched onto some hard-working pre-epistemic counterparts. By contrast, Hume and Kant would argue that this line of evidence is inconclusive at best.

Indeed, as I have noted above, according to their epistemic readings, Hume and Kant are pushed by the inconclusiveness of this kind of evidence towards a non-speculative style of metaphysics. Namely, they are led to treats causation (and, for Kant, space and time) *as if* they are merely organizational principles which we project onto the world — given that this is how they have earned their metaphysical significance. But must we stop here? With inconclusive evidence and epistemic modesty? One might hope that further reflection upon the nature of phenomenal experience would support unofficially Humean positions with some more conclusive evidence (be it on either epistemic or metaphysical grounds). For instance, one might try to build upon the above-mentioned metaphysical readings of Hume and Kant. Alternatively, one could try to follow Schopenhauer through his "narrow gateway to the truth".[23] While this might be a fruitful way to proceed, I am here proposing that we adopt a different approach. Namely, in accordance with my realist

[22] One need not be committed to the epistemological significance of this distinction, see Williamson (2013).
[23] Schopenhauer (2018, pg. 207).

methodology, I recommend that we go beyond such phenomenological considerations by bringing in a second physics-inspired line of evidence.

Indeed, as I noted in Ch. 1, this thesis will adopt a realist methodology: I take it that we can leverage the formal structure of our best physical theories in our pursuit of foundational metaphysics. The opponents of unofficial Humeanism ought to gladly invite this new line of evidence since, at first glance, it would seem to support their position. Nobody could have guessed from phenomenological considerations alone the lessons which quantum field theory and general relativity have taught us about causation, space, and time. Indeed, as I will discuss in the next section, our phenomenal notions of causation, space, and time can be seen to crystallize in our formalized theories becoming certain laws of nature and special coordinate systems as well as certain geometric and topological structures. Opponents of unofficial Humeanism might argue that the prominence of these theoretical structures in the physics discourse (and the quality of explanations they seem to give us, recall A1 - A4 from Ch. 1) are good evidence for them having latched onto some hard-working pre-epistemic counterparts.

But why should a proponent of unofficial Humeanism accept this turn towards a physics-inspired metaphysics? Later on, I will give some motivation for those who believe that unofficially Humean positions can be supported on metaphysical grounds. Let us start, however, with the more difficult task of dealing with the hard-line anti-speculativist who thinks that unofficially Humean positions are true on epistemic grounds but rejects any attempt to support them on metaphysical grounds. As I will now discuss, my realist methodology is actually compatible with this kind of view. Regarding the prospect of physics-inspired metaphysics one can imagine a hard-line anti-speculativist saying: "Regardless of what phenomenal notions or theoretical structures we are dealing with, it should be sufficient to concern ourselves with these notions/structures only insofar as we can justifiably know about them through experience (and perhaps its organizational

prerequisites). In fact, we ought to dismiss as unnecessarily speculative any approach to metaphysics which seeks to grant these notions/structures any metaphysical significance which goes beyond what they have earned via the epistemic route by which they have entered into our experiences and our theories. If we apply this non-speculative principle universally, then we are effectively cut off from saying anything substantial (either positive or negative) about the pre-epistemic world-in-itself.[24] There is simply no way to know using non-speculative methods whether or not the phenomenal notions or theoretical structures under consideration have hard-working pre-epistemic counterparts. Even if such counterparts do happen to exist, we would still be epistemically alienated from them."

According to this anti-speculative line of thought, adopting my realist methodology cannot grant us any non-speculative insights into the pre-epistemic world. But does this constitute a rejection of my realist methodology by the anti-speculativist? That is, does this mean that (granting their position) the formal structure of our best physical theories is irrelevant to debates about foundational metaphysics? No. Their complaint cannot be that: "Adopting this realist methodology yields a view which cannot provide non-speculative insights into the pre-epistemic world" since this also describes their own position. Instead, their complaint might be that: "Adopting this realist methodology yields a view which tries to provide such pre-epistemic insights but fails to do so". This complaint, however, misses the mark since my realist methodology only requires admitting the *relevance* of certain physics-inspired line of evidence and does not prescribe in advance what kind of arguments/conclusions one should draw from this evidence. In fact, as I will argue later in Ch. 3, this new line of evidence can be used to provide *additional epistemic support* for unofficially Humean positions. The anti-speculativist ought to find this situation acceptable and even desirable.

[24] Besides perhaps acknowledging that it has a mind-independent existence and that it somehow affects us, e.g., by activating our senses. See Note II of Kant (2004, pg. 40) for Kant's rejection of idealism.

Now, let's reintroduce the proponent of unofficial Humeanism who is comfortable supporting their position on metaphysical grounds. They ought to accept my realist methodology for much the same reason; It might provide them with a second line of support for their unofficially Humean positions. In fact, as I will argue in Ch. 3, this new line of evidence can be used to provide *additional metaphysical support* for unofficially Humean positions (despite the anti-speculativist's protests). Regardless of what kind of support this new line of evidence brings, it should go without saying that it is always beneficial to have support for one's position from multiple lines of argumentation (e.g., based on both phenomenological considerations and on physics). Indeed, this second line of argumentation is likely to be rhetorically effective given that, at first glance, taking this realist turn would seem to favor the opponents of unofficial Humeanism. There is here an element of "beating them on their own turf". That is, there are rhetorical benefits to arguing against one's opponent's conclusions (or for one's own conclusions) based on premises which they are especially sympathetic with.

Past these points, however, there are good philosophical reasons for any proponent of unofficial Humeanism to accept this new physics-inspired line of evidence. Recall that the task at hand is to determine (insofar as we can) whether certain structural aspects of our experiences/theories are reflective of hard-working aspects of the pre-epistemic world, or whether they are only reflective of the representational process itself. Such questions of structure and representation can be addressed with great precision when asked in mathematical terms. In Ch. 4 I will define what I call spacetime representation theory in close analogy with the mathematician's group representation theory. Indeed, there is a sense in which mathematics *simply is* the study of all coherent structures and their possible representations. Before discussing this, however, let us proceed to the second step in our epistemic story.

2.2 Step 2: Codifying Our Experiences into Formalized Theories of Physics

Before continuing on to the second step in our epistemic story, we first need to make a quick update to the terminology appearing in Fig. 2.1 as we move on to Fig. 2.2. First, recall that we are already committed (as is Kant himself) to the pre-epistemic world-in-itself (a.k.a. the noumenal world) having a mind-independent existence which somehow affects our experiences;[25] There is some (perhaps unknowable) pre-epistemic stuff/goings-on/structure out there which our experiences are experiences of. In light of our recent turn towards a physics-inspired metaphysics, I will now adopt a new name for this pre-epistemic stuff/goings-on/structure. The broadest possible term for the subject matter of physics is "the dynamical behavior of matter" (plus its potential facilitators: e.g., laws, coordinates, geometry, and/or topology). Hence, going forward I will adopt this terminology for "that which our experiences are experiences of".[26]

[25] This, of course, raises some issues for Kant if he is understood as a hard-line anti-speculativist whose methodology forbids him from saying anything substantial (either positive or negative) about the pre-epistemic world-in-itself (a.k.a. the noumenal world). One might then wonder, following Stang (2024, Sec. 3.4), how Kant can then claim that things-in-themselves: 1) exist mind-independently, 2) affect our experiences, and 3) are fundamentally non-spatiotemporal. While there are many ways to respond to this charge, it suffices to say that this kind of problem does not even arise for dynamics-first views which reject this sort of hard-line anti-speculativism.

[26] One might here complain that we are leaving out something important when we identify the pre-epistemic world's goings-on with the subject matter of physics. In order to get this two-step epistemic story off the ground, don't we also need to account for the fact that people have *conscious experiences of* the dynamical behavior of matter? One might feel the need at this point to invoke some hard-working pre-epistemic consciousness (e.g., as a property or as a substance) in order to facilitate our conscious experiences of matter's dynamics. This impulse is in some ways analogous to the urge which dynamics-second views feel to invoke the existence of some hard-working pre-epistemic laws, coordinates, geometry, and/or topology in order to explain matter's dynamics. Just as in these contexts, there is a "dynamics-first" option regarding consciousness: It could be the case that our conscious experiences arise automatically as our brains do their work organizing the pre-epistemic world. Talk of consciousness could then gain its metaphysical significance via this organizational route rather than by latching onto some hard-working pre-epistemic substance/property. Let us set this fifth debate aside, however, since it has a notably different character than (and is largely independent of) our four dynamics-first vs dynamics-second debates.

In what follows it is important to bear in mind the pre-epistemic character of the term "the dynamical behavior of matter";[27] Given that our experiences of matter's dynamics are ubiquitously staged in spatiotemporal/causal terms, it *does not* immediately follow that the dynamical behavior of matter itself is so staged. It is, of course, psychologically difficult to even think about dynamics without causation or without time. Similarly, we may struggle to think about matter without mentally placing it in some spatial context. Moreover, one could argue that it would be difficult to bring our theories into contact with experimental practice without relying on ideas about causation, space, and time.[28] There are, however, two possible explanations for our having these difficulties: 1) Causation, space, and time are organizational principles which we unconsciously project into our every experience of matter's dynamics; or, 2) Causation, space, and time are hard-working aspects of the pre-epistemic world which help to facilitate matter's dynamics and which our corresponding phenomenal notions have latched onto. But this is nothing new, it is just to rehearse the above-discussed debates in our new terminology.

Accepting this change in terminology, the second step of our epistemic story proceeds as follows. Step 2: Scientists have carefully studied various observable dynamical phenomena (e.g., the motion of needles in laboratories). After centuries of hard scientific work, the scientific community has codified their observations of matter's dynamics into our current best theories of physics. This step includes everything from experimentalists calibrating their lab equipment to the theoreticians' formal wizardry. Let us call this the arrow of codification, see Fig. 2.2. Furthermore, let us use the term "systematization" to refer to the joint process which connects the pre-epistemic world to our formalized the-

[27] This pre-epistemic usage of the term "the dynamical behavior of matter" may seem odd to those for whom the word "behavior" has certain psychological associations. Namely, behaviorism is the doctrine that the subject matter of psychology should be the (external, visible) behavior of humans/animals rather than their (internal, hidden) states of mind. Despite its association with this anti-speculative doctrine, my usage of the term "behavior" should be thought of as referring to matter's actual dynamics with no concern for what is epistemically accessible to us.

[28] Huggett and Wüthrich (2013).

ories through our phenomenal experiences. Notably, our physical theories near ubiquitously refer to matter and dynamics beyond those which were directly observed, e.g., they may refer to electrons, the electromagnetic field, and/or the gravitational field. Moreover, it is common for these theories to feature additional mathematical structures which do not refer to either matter or its dynamical behavior. To see what these additional structures might be like, I will now introduce a concrete example theory.

Consider a counterfactual world in which after centuries of carefully studying matter's dynamics, the scientific community eventually arrives at the following theory.

> **Scalar Electromagnetism (SEM) -** Consider a theory written in a specific global coordinate system, (t, x, y, z) about a scalar field, $\varphi(t, x, y, z)$, and a point-like particle with some trajectory, $X(\tau) = (t(\tau), x(\tau), y(\tau), z(\tau))$. (Note that I am working in units with $c = \hbar = 1$.) The only kinematic constraint on this theory's states is that φ and X must both be second-differentiable functions. In this coordinate system, the dynamical equations for φ and X are as follows:
>
> $$(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 - M^2)\,\varphi(t, x, y, z) = -\rho_Q/\epsilon_0, \tag{2.1}$$
> $$-(\partial_\tau t)^2 + (\partial_\tau x)^2 + (\partial_\tau y)^2 + (\partial_\tau z)^2 = -1, \tag{2.2}$$
> $$(\partial_\tau^2 t,\ \partial_\tau^2 x,\ \partial_\tau^2 y,\ \partial_\tau^2 z) = F_q[\varphi]/m, \tag{2.3}$$
>
> where $M \geq 0$ is the mass of the field and ρ_Q is the particle's charge density which sources the field. The particle has a rest mass of $m \geq 0$ and a charge of q. Here $F_q[\varphi]$ is the 4-force which is applied to the particle by the field. (The specific force law will be given later on.)[29]

Note that Eq. (2.1) is the inhomogeneous Klein-Gordon equation for a massive scalar field. Eq. (2.2) does two things. Firstly, it states that the particle is always moving slower than the speed of light (since the right-hand side is negative). Secondly, it establishes a convention for normalizing the τ-parameter in $X(\tau)$ by fixing the right-hand side to be -1. (N.B.: Our physics should be invariant under τ-reparameterization). Eq. (2.3) is the special-relativistic version of Newton's second law, $F = ma$, for a massive charged particle undergoing a certain 4-force, $F_q[\varphi]$, due to φ. Note that in the $F = 0$ case, Eq. (2.3) states that force-free particles travel on straight lines in the (t, x, y, z)-coordinate

[29] See Eq. (2.7) for a coordinate-free expression for $F_q[\varphi]$.

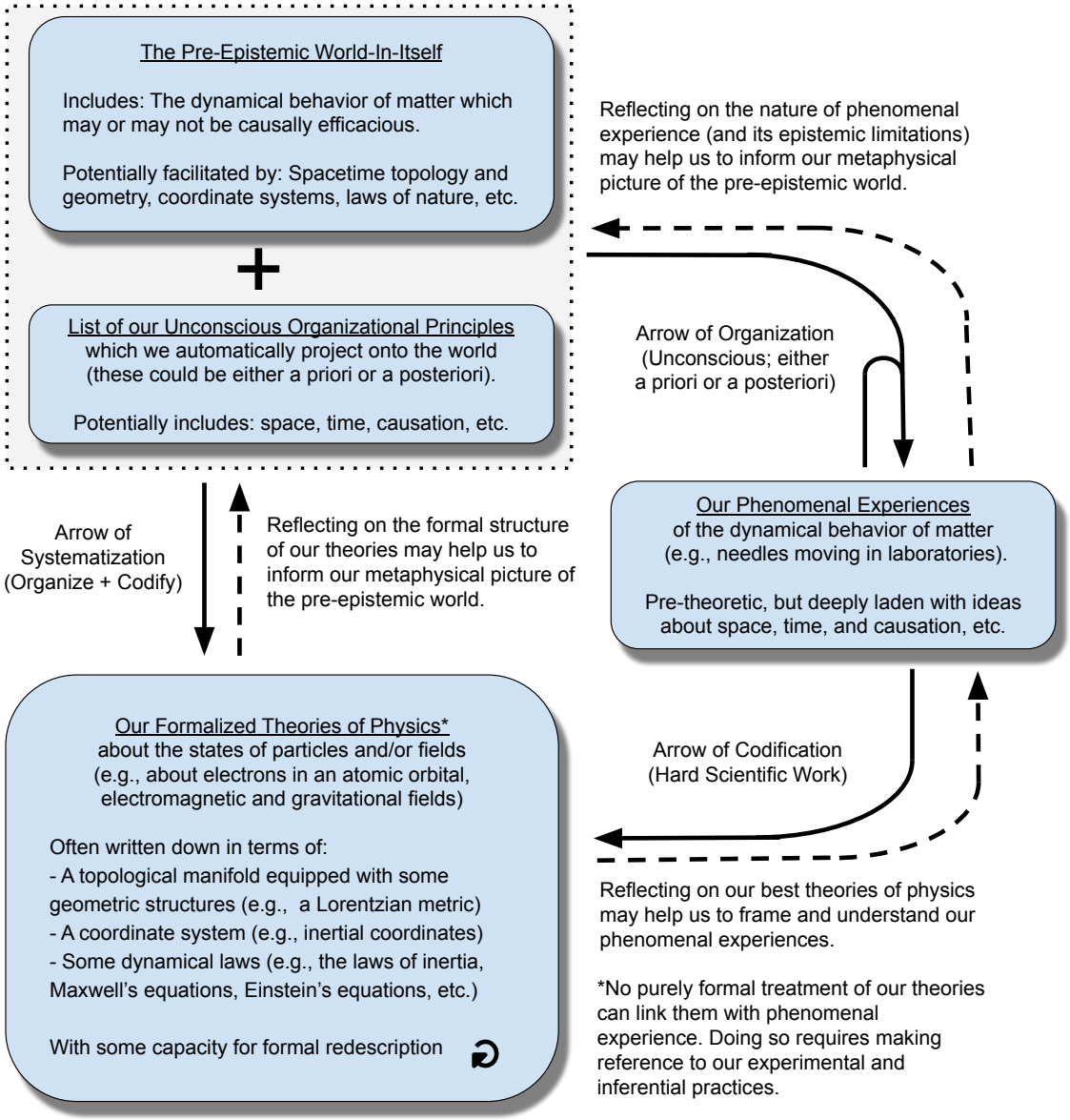

Figure 2.2: This figure sketches a two-step epistemic story showing how we have arrived at our best theories of physics by systematizing the dynamical behavior of matter. In the first step, we unconsciously organize the pre-epistemic world's goings-on into our phenomenal experiences. The resulting experiences are deeply laden with various pre-theoretic ideas about the world including: space, time, matter, dynamics, causation, etc. In the second step, we painstakingly codify our phenomenal experiences into our best physical theories. The resulting formalized theories of physics include various theoretical structures such as laws of nature and coordinate systems as well as a spacetime manifold with some geometric and topological structure. See, for instance, Eqs. (2.1) - (2.6). The four dynamics-first vs dynamics-second debates then ask: Where in this epistemic story do these theoretical structures first acquire their particular kind of metaphysical significance? The dynamics-second camps answer that they do so "at step zero" by latching onto some dynamics-facilitating pre-epistemic counterpart. By contrast, the dynamics-first camps answer that these theoretical structures only gain their metaphysical significance "along the way" by being a particularly nice way of systematizing the dynamical behavior of matter.

system. One can easily imagine this theory as part of a more complicated theory about multiple point-like particles with different charge-to-mass ratios, q/m.

In addition to describing the dynamical behavior of two types of matter (a scalar field, φ, and a point-like particle, X), the above theory also seems to talk about certain axiom-like laws of nature, e.g., Eqs. (2.1) - (2.3) themselves. This theory's nomological structure could be understood according to any of the three major viewpoints in the metaphysics of laws debate. On a governing view (which will count as a dynamics-second view of laws), Eqs. (2.1) - (2.3) reflect something over and above the world's non-nomic facts which somehow governs matter's dynamics from outside. This stands in stark contrast with either a causal powers view or (what I will call canonical) Humeanism both of which will count as dynamics-first views of laws.[30] For instance, we could think of these laws as systematizing the causal powers which φ and X have to influence each other; They somehow "force" and "source" each other in opposition to their respective powers of inertia. Alternatively, however, we could reject the existence of such causal powers instead opting for a non-modal understanding of this theory's dynamical content. For instance, we could spell out $F_q[\varphi]$ and ρ_Q in terms of φ and X thereby reducing Eqs. (2.1) - (2.3) to a set of differential equations which simply describe how φ and X sit on some Humean mosaic.

In addition to its nomological structure, we can also attribute to this theory a certain geometric and topological structure. These structures, however, are presently being obscured by our implicit use of one specific coordinate system, (t, x, y, z), to set our standards of distance and straightness. We can fix this issue by rewriting our theory in the coordinate-free language of differential geometry.[31] (Readers unfamiliar with differential

[30] As I noted in Ch. 1, one can adopt an unofficially Humean approach to laws without being a Humean in the modality-denying sense. This is most clearly exemplified by the works of Demarest (2017) and of Loewer (2021).

[31] Throughout this text, I adopt Einstein's summation convention in which matching pairs of upper (contravariant) and lower (covariant) indices are automatically contracted. Moreover, following Wald (1984) I also make the usual distinction between coordinate indices (greek: $\alpha, \beta, \gamma, \dots$) and abstract indices (roman: $\mathrm{a}, \mathrm{b}, \mathrm{c}, \dots$).

geometry need not worry, the core messages in this chapter do not require any technical know-how.) Rewritten in a coordinate-free way, our theory of scalar electromagnetism becomes:[32]

> **Coordinate-Free SEM -** Consider a theory set on a spacetime manifold of $\mathcal{M} \cong \mathbb{R}^4$ about a scalar field, $\varphi : \mathcal{M} \to \mathbb{R}$, and a particle trajectory, $X : \mathbb{R} \to \mathcal{M}$. The only kinematic constraint on this theory's states is that φ and X must both be second-differentiable functions. Let $\dot{X}^{\mathrm{a}} : \mathbb{R} \to \mathrm{T}^1\mathcal{M}$ be the tangent vector of this trajectory. This theory's dynamical equations can be written down with respect to a fixed flat Lorentzian metric, $\eta_{\mathrm{ab}} : \mathcal{M} \to \mathrm{T}_2\mathcal{M}$, with signature $(-,+,+,+)$. Let ∇_{c} be the unique covariant derivative compatible with η_{ab}, i.e., with $\nabla_{\mathrm{c}}\eta_{\mathrm{ab}} = 0$. In these terms, the dynamical equations for φ and X are as follows:
>
> $$(\eta^{\mathrm{ab}}\nabla_{\mathrm{a}}\nabla_{\mathrm{b}} - M^2)\,\varphi = -\rho_Q/\epsilon_0, \tag{2.4}$$
> $$\eta_{\mathrm{ab}}\dot{X}^{\mathrm{a}}\dot{X}^{\mathrm{b}} = -1, \tag{2.5}$$
> $$\dot{X}^{\mathrm{b}}\nabla_{\mathrm{b}}\dot{X}^{\mathrm{a}} = F_q^{\mathrm{a}}[\varphi]/m, \tag{2.6}$$
>
> The 4-force which is being applied to the particle by the field is given by,[33]
>
> $$F_q^{\mathrm{a}}[\varphi] \coloneqq -q\,(\eta^{\mathrm{ab}} + \dot{X}^{\mathrm{a}}\dot{X}^{\mathrm{b}})\,\nabla_{\mathrm{b}}\,\varphi. \tag{2.7}$$

On this coordinate-free formulation of our theory, we are still describing the dynamical behavior of φ and X in terms of certain axiom-like laws of nature, i.e., Eqs. (2.4) - (2.6). Indeed, these are the exact same laws as before, but re-written in a different mathematical language. Additionally, however, three new geometric and topological structures have also appeared: a Lorentzian metric, η_{ab}, a covariant derivative, ∇_{c}, and a smooth manifold, $\mathcal{M}$. In these terms, Eq. (2.6) still says that force-free particles travel in "straight lines". Now, however, rather than checking the straightness of the trajectory $X(\tau)$ against

[32] It should be noted that different interpretations of SEM would likely introduce this theory in slightly different ways. For instance, as I noted above, a canonical Humean would likely remove any reference to "forces" and "sources" by spelling out $F_q^{\mathrm{a}}[\varphi]$ and ρ_Q in terms of φ and X.

[33] It is impossible for a scalar field, φ, to impart a 4-force, F^{a}, which is simply proportional to its gradient, $\nabla^{\mathrm{a}}\varphi \coloneqq \eta^{\mathrm{ab}}\nabla_{\mathrm{b}}\varphi$. The issue here is the same as with Abraham's (1912) force law: It disrespects τ-reparameterization invariance. Recall our ability to normalize the particle's 4-velocity, $\dot{X}^{\mathrm{b}}$, as we did in Eq. (2.5), namely with $\eta_{\mathrm{ab}}\dot{X}^{\mathrm{a}}\dot{X}^{\mathrm{b}} = -1$. Maintaining this choice requires that F^{a} be orthogonal to every $\dot{X}^{\mathrm{b}}$ as $\eta_{\mathrm{ab}}F^{\mathrm{a}}\dot{X}^{\mathrm{b}} = 0$. The simple force law, $F^{\mathrm{a}} = -q\nabla^{\mathrm{a}}\varphi$, fails this test. We can fix this issue (as did Nordström (1912, 1913)) by adopting a velocity-dependent force law, $F_q^{\mathrm{a}}[\Phi] = -q\,(\eta^{\mathrm{ab}} + \dot{X}^{\mathrm{a}}\dot{X}^{\mathrm{b}})\,\nabla_{\mathrm{b}}\,\Phi$ where $\Phi = \Phi(\varphi)$ is a generic function of φ. Indeed, with some minor modifications, we can convert our theory of scalar electromagnetism into Nordström's (1912) first and Nordström's (1913) second theory of gravity with φ now being the scalar gravitational field. To get his first theory, NG-1, we simply need to take $M = 0$, $q = m$, and $\Phi = \varphi$ as well as to change the coefficient on the right-hand side of Eq. (2.4) from $-1/\epsilon_0$ to $4\pi G$. To get his second theory, NG-2, we need to instead take $\Phi = \ln(\varphi)$ and to additionally multiply the left-hand side of Eq. (2.4) by φ.

the (t, x, y, z)-coordinate system, we instead evaluate its straightness with respect to a piece of fixed geometric structure, namely ∇_c (which happens to be uniquely determined by η_{ab}). We can use η_{ab} and ∇_c to set our standards of distance and straightness for SEM in a coordinate-free way.

Notice that the four theoretical structures which we have highlighted in this example theory (i.e., laws, coordinates, geometry, and topology) are exactly the four kinds of candidates which were invoked by answers A1 - A4 in Ch. 1 as potentially helping to facilitate matter's dynamics. Hence, we have completed our two-step epistemic story by sketching how it is that these four kinds of structures have shown up in our best physical theories. Ultimately, it's as simple as this: We have systematized (= organized + codified) the dynamical behavior of matter. I take it that both sides of the dynamics-first vs dynamics-second debates accept at least the outline of this two-step epistemic story (although they will likely disagree about many of the details).

Moreover, I take it that both sides of the debate accept my above-discussed realist methodology; They are both interested in trying to leverage the formal structure of our best physical theories as evidence in these four debates about foundational metaphysics. The above discussion has thus identified the topic of the dynamics-first vs dynamics-second debates (the left-most box in Fig. 2.2) and the evidence to be considered (the right-most box in Fig. 2.2) as well as a sketch of an epistemic story connecting the former to the latter. We are now in a position to explicitly characterize the metaphysical disagreement at the heart of these debates. As I noted in Sec. 2.1, accepting the above-discussed epistemic story means believing that this story can account for how certain laws, coordinates, geometry, and topology have shown up in our formalized theories *while having the particular kind of metaphysical significance that they do*. As I am now defining them, dynamics-second views (about laws, coordinates, geometry, and/or topology) take these theoretical structures to get their particular kind of metaphysical significance "at step

zero", i.e., by having latched onto some dynamics-facilitating pre-epistemic counterparts. Hence, systematizing matter's dynamics has resulted in theories which have revealed to us the internal working mechanisms of the pre-epistemic world. We can then turn around and use these hard-working pre-epistemic counterparts to help give a metaphysically supported explanation of the very dynamical phenomena which we began by investigating (e.g., answers A1 - A4 in Ch. 1). In this way, dynamics-second views orient the arrow of metaphysical explanation anti-parallel to the epistemic arrow discussed above, see Fig. 2.3a).

Now, one of these four theoretical structures is clearly not like the others; Nobody seriously believes that there are coordinate labels out there in the pre-epistemic world doing dynamics-facilitating work. Believing in such hard-working coordinate labels would be similar to believing that objects and/or persons have "true names" which are doing some significant metaphysical work to help facilitate goings-on in the pre-epistemic world. But this seems to be a category error; Names and coordinate labels just aren't the kind of things which could do that kind of work. Dynamics-second views of coordinates are obviously wrong! Despite this, let us keep this bad apple in the batch and continue motivating the four dynamics-second views in unison. The line of thought presented in the following two paragraphs provides some motivation for the dynamics-second approach to metaphysics, to the extent to which it applies in each context.

As I noted above, our phenomenal experiences of the world are deeply laden with pre-theoretic ideas about causation, space, and time. Indeed, it is difficult to think about the dynamical behavior of matter without invoking such notions. These sorts of difficulties can be seen to persist into the formal structure of our best theories. Indeed, as we have seen in Eqs. (2.1) - (2.6), our pre-theoretic notions of causation, space, and time seem to crystallize in our formalized theories becoming certain laws of nature and special coordinate systems as well as certain geometric and topological structures. Moreover,

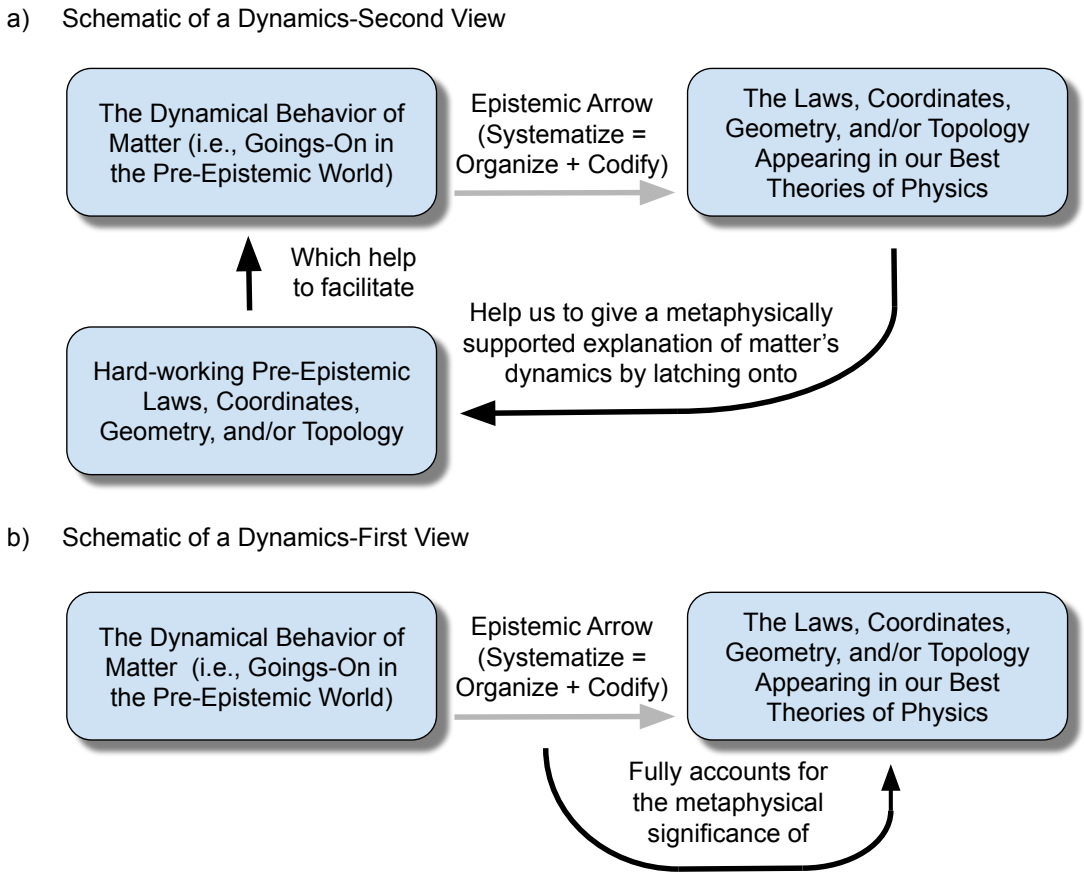

Figure 2.3: This figure shows schematics of the dynamics-second and dynamics-first approaches to foundational metaphysics. Past agreeing on a certain epistemic story (here summarized by the epistemic arrow, see Fig. 2.2) they subsequently have some substantial metaphysical disagreements. In particular, they disagree about whether a certain metaphysical arrow (in black) runs parallel or anti-parallel to this epistemic arrow (in gray).

a) According to dynamics-second views, the laws, coordinates, geometry, and/or topology which appear in our best physical theories gain their particular kind of metaphysical significance by latching onto some dynamics-facilitating pre-epistemic counterpart. Hence, systematizing matter's dynamics results in theories which reveal to us the internal working mechanisms of the pre-epistemic world; We can then turn around and use these hard-working pre-epistemic counterparts to help give a metaphysically supported explanation of the very dynamical phenomena which we began by investigating (e.g., answers A1 - A4 in Ch. 1).

b) According to dynamics-first views, the laws, coordinates, geometry, and/or topology which appear in our best theories of physics do not gain their particular kind of metaphysical significance by latching onto any such hard-working pre-epistemic counterparts. Instead, them having this metaphysical significance is fully accounted for by them being a particularly nice way of systematizing the dynamical behavior of matter. In this way, our metaphysical account of our theories' laws, coordinates, geometry, and/or topology respectively runs parallel to the epistemic route via which these theoretical structures have entered into our experience of and theories about matter's dynamics.

they crystallize in counter-intuitive ways which we simply could not have predicted beforehand. These unanticipated theoretical structures seem to be as necessary for doing physics as notions of causation, space, and time were for thinking about our experiences; Namely, we seem to need to write down a lot of topological, geometrical, coordinate, and nomological staging before we can even state our theory's dynamical equations.

Recall that we want to leverage the formal structures of our best physical theories as a guide in our pursuit of foundational metaphysics. On a dynamics-second account, the prominence of these structures in our best physical theories' dynamical equations guides us to believe that our theory's topological, geometrical, coordinate, and nomological structures have real counterparts out there in the pre-epistemic world. Moreover, the pre-dynamical necessity of these sorts of structures in both cognizing experience and in writing down our theories guides us to believe that they are pre-dynamically necessary in the pre-epistemic world as well;[34] Namely, this suggests that some kind of topological, geometrical, coordinate, and nomological staging must be present in the pre-epistemic world to help facilitate the dynamical behavior of matter. Given this, it is then no surprise that ideas about causation, space, and time show up so prominently in our phenomenal experiences and in our physical theories. Beyond this, dynamics-second views can also claim some explanatory merits. The hard-working pre-epistemic counterparts which they posit provide us with ample material for giving metaphysically supported explanations of various dynamical phenomena (recall answers A1- A4 in Ch. 1). We have come full circle.

As I will now argue, this strong capacity for coordinate redescription gives us good reasons to reject a dynamics-second view of coordinates. Firstly, it undercuts the premise in the above line of thought which (obviously wrongly) suggested that coordinate systems are necessary for either thinking about or for theorizing about matter's dynamics.

[34] It is, of course, this leap from "necessity in our minds/theories " to "necessity out there in the pre-epistemic world" which Hume and Kant would object to.

This then undercuts the (again, obviously wrong) conclusion that there must be some dynamics-facilitating coordinate labels out there in the pre-epistemic world. In place of this dynamics-second argument, reflecting upon our strong capacity for coordinate redescription leads us in a very different direction. Namely, the way in which individual coordinate systems exist as formal structures within our theories suggests that they are reflective of the representational process itself rather than any underlying coordinate-like structure; That is, it suggests that they gain whatever metaphysical significance they have, not by latching onto some dynamics-facilitating pre-epistemic counterpart, but rather only by helping us to systematize the dynamical behavior of matter (e.g., the inertial behavior of matter).[35]

Hence, we have been led to reject the dynamics-second view of coordinates by investigating the formal structure of our physical theories and discovering that we have a strong capacity for coordinate redescription. (Importantly, our strong capacity for coordinate redescription would point us in this direction even for a theory which has exactly one good coordinate description.) Notice how we have just reversed the arrow of metaphysical explanation. Individual coordinate systems don't latch onto any hard-working pre-epistemic counterparts which could then help us give a metaphysically supported explanation of matter's dynamics; Instead, it is the way in which we organize and codify matter's dy-

[35] One might try to split hairs here over the dynamics-first vs dynamics-second debate about geometry. We can all agree that inertial coordinate systems get their metaphysical significance via their associations with the inertial behavior of matter. However, on a dynamics-second view of geometry, the inertial behavior of matter itself is facilitated by the spacetime manifold itself having certain geometric features carved into it. On this basis, one could argue that inertial coordinate systems do, in fact, gain their metaphysical significance via its association with *some* dynamics-facilitating aspect of the pre-epistemic world. Importantly, however, this would be the pre-epistemic counterpart of our theory's geometric structure rather than the counterpart of any individual coordinate system. Hence, the dynamics-second view of coordinates is still false; Individual coordinate systems still only earn their metaphysical significance along the way by being a particularly nice way of systematizing the pre-epistemic world's goings-on. This is (with one slight correction) the dynamics-first view of coordinates which is stated in the main text. The slight correction is that the pre-epistemic stuff which coordinate systems are helping us to systematize now includes both the dynamical behavior of matter as well as its alleged facilitators (here, the spacetime manifold's inbuilt geometric features). An analogous correction could be made to a generic dynamics-first view of Z supposing that one holds a dynamics-second view of some $Z' \neq Z$.

namics which explains the metaphysical significance of certain coordinate systems. This reversal yields a dynamics-first view of coordinate systems, see Fig. 2.3b).

We are now in a position to officially define the other three dynamics-first views and to connect them with our above discussion of Hume and Kant. Dynamics-first views (about laws, coordinates, geometry, and/or topology) endorse the negation of the corresponding dynamics-second view. That is, each dynamics-first view denies that the theoretical structure under consideration gains its particular kind of metaphysical significance by latching onto some dynamics-facilitating pre-epistemic counterpart. But from whence then do our theory's laws, coordinates, geometry, and/or topology get their metaphysical significance? Given what it means to accept the above epistemic story, there is only one other possible route. In fact, the dynamics-first answer to this question is just an extension of the unofficially Humean approach to metaphysics which we have already discussed in Sec. 2.1. These theoretical structures must have acquired their particular kind of metaphysical significance "along the way" so to speak, by being particularly nice ways of systematizing (= organizing + codifying) the dynamical behavior of matter.[36]

This is the sense in which dynamics-first views align the arrow of metaphysical explanation with the two epistemic arrows discussed above, see Fig. 2.3b). It should be noted that (just like unofficial Humean views) dynamics-first views could be true for either metaphysical or epistemic reasons. For instance, our theories' laws, coordinates, geometry, and/or topology might simply have no hard-working pre-epistemic counterparts to latch onto; That is, such counterparts might simply fail to exist. Alternatively, even if such counterparts do happen to exist, we might be epistemically alienated from

[36] What exactly is being systematized (= organized + codified) here will depend upon whether or not one accepts any dynamics-second views. For instance, as I discussed in footnote 35 one might hold a dynamics-first view of coordinates as well as a dynamics-second view of geometry. In this case, coordinate systems would help us to systematize both the dynamical behavior of matter *as well as its alleged facilitators* (in this case, some hard-working pre-epistemic geometry). This minor clarification has been left out of the main text because the four dynamics-first views are here being motivated and articulated in unison.

them. This would also force us onto the "along the way" route. Hence, we can argue for dynamics-first positions on either metaphysical or epistemic grounds.

Indeed, recalling my discussion from the end of Sec. 2.1, one can imagine a hard-line anti-speculativist who thinks that dynamics-first views are true on epistemic grounds but rejects any attempt to support them on metaphysical grounds. Importantly, however, this does not constitute a rejection of my realist methodology; One can admit the relevance of this new physics-inspired line of evidence without using it for speculative ends. Indeed, as I will argue in the next chapter, this new line of evidence can be used to provide additional epistemic support for their dynamics-first positions. Indeed, it may even give us good metaphysical reasons for believing in dynamics-first views (despite the anti-speculativist's protests).

3 | Four Benefits of Mastering our Capacities for Formal Redescription

What if we could remove and replace the topological underpinnings of our spacetime theories just as easily as we can switch between different coordinate systems (or different axiomatizations, or geometric framings)? In Part II of this thesis I will establish that one can, in fact, do this by using the ISE Method of topological redescription. Before this, however, let us first ask: What benefits should we expect to follow from mastering our capacity for topological redescription? It is reasonable to expect that the benefits will be similar to those which have already come from our existing capacities for axiomatic, coordinate, and geometric redescription.

This chapter will outline four kinds of expected benefits: The first two benefits are epistemic (Sec. 3.1) whereas the latter two are metaphysical (Sec. 3.2). To overview, mastering our capacity for various types of formal redescription can:

1. Help physicists with theory development by giving them new (and possibly better) ways of formulating their theories. (This may also help philosophers of physics to better understand our existing theories);
2. Help philosophers (i.e., us) to better understand the epistemic process of theory development;
3. Help us to articulate various "dynamics-first vs dynamics-second debates" regarding how we should interpret our theories;
4. Help us characterize the common structure which all of these alternative descriptions are jointly descriptions of.

Turning to the epistemic benefits first, let us continue discussing the epistemic story which we sketched in Ch. 2.

3.1 Systematic Reformulation and Epistemic Simulation

Recall the two-step epistemic story which I sketched in Fig. 2.2 in Ch. 2. Step 1: However the world is structured in itself, we unconsciously organize it into our phenomenal experiences of the world. Step 2: After centuries of hard scientific work, the scientific community has codified their observations of matter's dynamics into our current best theories of physics. Recall that I have called this whole process systematization (= organize + codify). While there is much more that could be said about each step and each substep of this epistemic process, let us focus in on the codification step. The codification process doesn't simply end once we have some empirically adequate theory on the table. Indeed, the initial formulation of our theories (in terms of certain Z = axiom-like laws, coordinates, geometry, and/or topology) is often not their best formulation, whatever "best" ultimately means. After we have arrived at our initial Z-framing of the theory in question, some amount Z-reformulation will be required before we settle on a particularly nice Z-framing. Let us next discuss some historical examples which illustrate this point.

For instance, the initial coordinate framing of the Schwarzchild solution to general relativity (GR) had a coordinate singularity at the event horizon. Although these coordinates are still useful for some purposes, they obscure what is going on at the event horizon; This is analogous to how working in terms of latitude and longitude obscures what is going on at the north and south poles. As another historical example, consider how our capacity for geometric redescription was very helpful to physicists in their development of the Einstein and Fokker (1914) formulation of Nordström's (1913) second theory of gravity. This was an important stepping stone on the way to GR.

Given the usefulness of our capacity for formal redescription, it will undoubtedly play a role in the epistemic process via which we have arrived at our best theories of physics. In particular, it will pay a role after we have made organizational sense of the world and

after we have codified our experiences into an initial theory. After the initial codification step, there will generally be some process of redescription wherein we reframe this theory in terms of different, Z = axiom-like laws and coordinate systems as well geometric and topological structures. Adding this reformulation step to Fig. 2.2 we have Fig. 3.1.

Beyond helping physicists with theory development, engaging with our capacity for formal redescription also has clear benefits for the philosophers of physics who, by contrast, tend to be more focused on understanding our existing theories rather than on discovering new ones. As a concrete example of this, consider the ongoing efforts to re-interpret Newton's theory of gravity in light of GR. Moving directly from a fixed flat Newtonian spacetime to a dynamical curved Einsteinian spacetime requires us to make a significant jump both conceptually and mathematically. One can, however, minimize this distance by first reworking Newton's theory of gravity so that it more closely resembles GR. Namely, one can mathematically reformulate Newtonian gravity as a curved spacetime theory: i.e., Newton-Cartan theory.[37] In doing so, what seemed like a huge conceptual jump (from Newton to Einstein) becomes a much smaller jump (from Newton-Cartan to Einstein).

One can frame Newton's "failure" to anticipate either GR (or even just Newton-Cartan theory) as an instance of the problem of unconceived alternatives (à la Stanford (2006)). Newton simply lacked the necessary mathematical and conceptual tools to conceive of GR as a rival theory (or of Newton-Cartan as a better geometric reformulation of his own theory). Suppose counterfactually that Newton were magically granted our current level of mastery regarding geometric redescription, i.e., a mastery of differential geometry. He would likely still reject the possibility of a curved spacetime on metaphysical grounds. Importantly, however, this would no longer be the problem of unconceived alternatives but rather the problem of incorrectly rejected alternatives. As this example shows, the

[37] For more information on Newton-Cartan theory, see Malament (2012, Ch.4). See also the efforts originating from Saunders (2013) to reframe Newtonian gravity on a Maxwell spacetime.

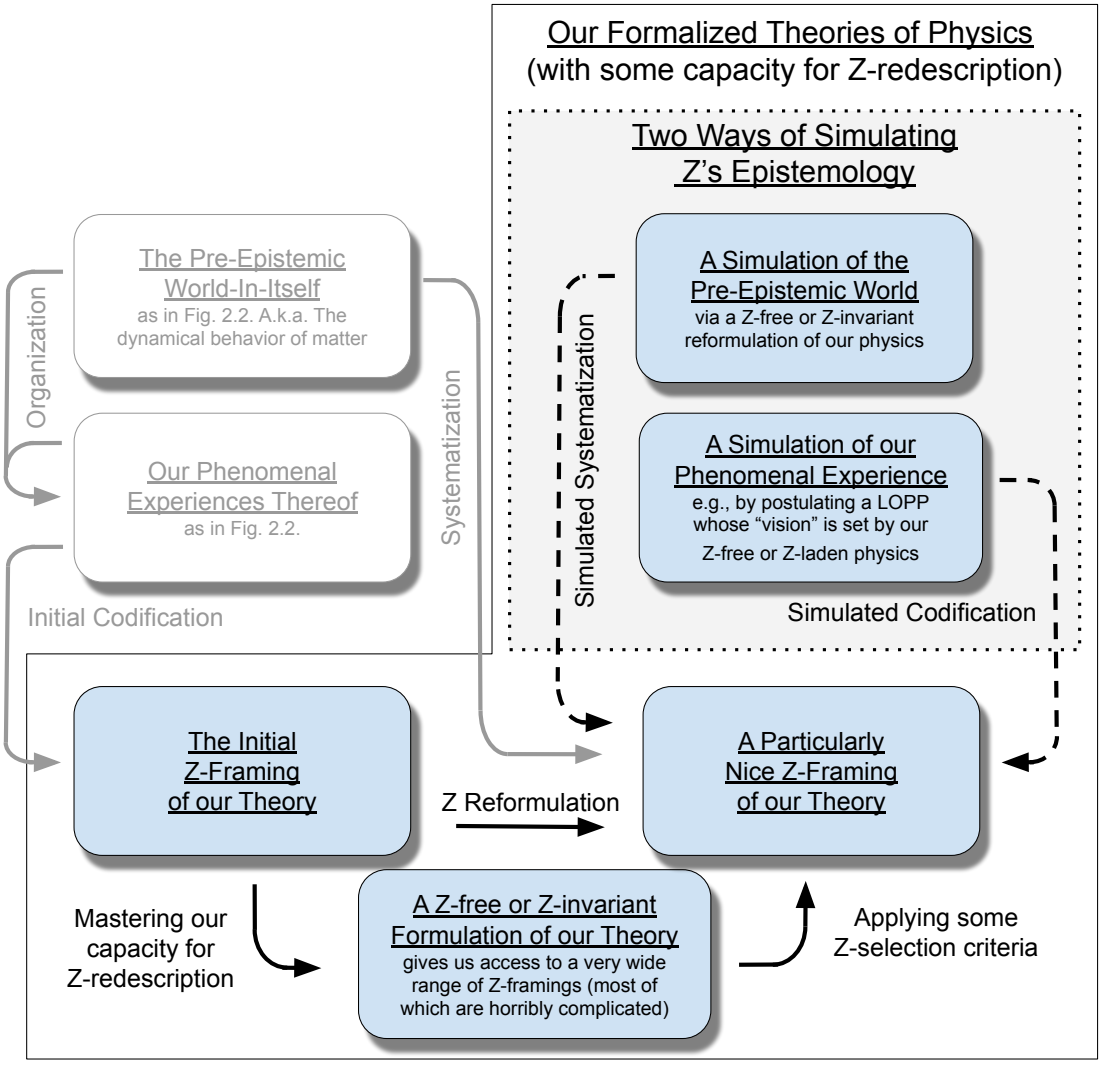

Figure 3.1: This figure shows different stages of the epistemic story regarding how our best physical theories have come to feature the particular Z = laws, coordinates, geometry, and topology that they do. We first make organizational sense of the world and then codified our phenomenal experience into an initial formulation of our theory. After this, some amount of Z-reformulation may be needed to find a particularly nice Z-framing of our theory. We can perform this last step systematically if we have mastered our capacity for Z-redescription. Namely, we can adopt a Z-free or Z-invariant formulation of our theory and then apply some Z-selection criteria. Using this Z-free or Z-invariant formulation of our theory we might also try to simulate either the pre-epistemic world or our phenomenal experiences thereof.

shortest path to discovering new physics is sometimes to first mathematically reformulate our existing physical theories, potentially in ways we find deeply counterintuitive.

As an aside, I claim that Newton's above-described situation regarding GR and curved spacetimes is in some ways analogous to that of present-day physicists regarding quantum gravity and spacetime non-fundamentality. Our situation, however, is more hopeful; We (unlike Newton) do have at least some anticipation of what is to come. Namely, in light of quantum gravity, many anticipate that (at least certain aspects of) our usual spacetime settings are emergent and hence non-fundamental.[38] Given this anticipation, we have good reasons to begin trying to reformulate and reinterpret our well-established physics with spacetime non-fundamentality in mind. This should hopefully make a viable theory of quantum gravity easier to discover (i.e., it should make it a smaller jump both conceptually and mathematically). Concretely, it should be easier to see spacetime as emergent if we can already see it as non-fundamental in our well-established physics. Like with Newton, however, we won't be able to succeed in this endeavour until we have developed the necessary mathematical and conceptual tools. It is towards this end that I am developing the ISE Method and the dynamics-first view of spacetime topology; They may help us to see spacetime non-fundamentality even in our well-established physics, as a bridge towards quantum gravity.

To recap the first potential benefit, mastering our capacity for formal redescription might help physicists to develop new theories and it might help philosophers of physics to better understand our existing theories. Suppose, however, that this first benefit never materializes. That is, suppose that we have already found the best way of framing our

[38] One ought to distinguish spacetime non-fundamentality from spacetime emergence. One can think of spacetime emergence as something that happens between theories, e.g., in the sense of Nagelian reduction; One begins from a non-spatiotemporal theory (e.g., some theory of quantum gravity) and end up with some well-established theory wherein spacetime has traditionally been understood as fundamental. By contrast, spacetime non-fundamentality is an issue of interpretation involving only a single theory. Unlike emergence, developing a view of spacetime non-fundamentality requires no limits and no regime changes. Importantly, one can adopt an attitude of spacetime non-fundamentality for our well-established physics without invoking any speculative physics.

theories in terms of some Z = axiom-like laws, coordinates, geometry, and topology. This brings us to the second potential benefit: Gaining a mastery of Z-redescription might nonetheless be helpful in understanding the epistemic process of theory development. Indeed, in light of the abundance of Z-redescriptions available to us (even if most are horrendously complicated) the question looms: Why did our theories end up with this particular Z-framing rather than some mathematically equivalent alternative? That is, what are our Z-selection criteria?

While one might be interested in understanding Z's epistemology for its own sake, this may also be helpful in adjudicating the dynamics-first vs dynamics-second debates about Z which I introduced in Ch. 2. As I there framed them, the debate about Z concerns where along our two-step epistemic story it has acquired its particular kind of metaphysical significance: either "at step zero" or "along the way". In order to help navigate this debate, it would be beneficial (although extremely complicated) to provide a full treatment of the *actual epistemic process* which has led to our best theories featuring the particular Z that they do. As Fig. 3.1 illustrates, such account would likely involve: 0) possibly some assumptions about the pre-epistemic world-in-itself, 1) an account of how we make organizational sense of the world, 2) an account of how we codify our phenomenal experiences into our theories, and 3) an account of whatever kind of formal wizardry is involved in the Z-reformulation process. Such an account would likely be very complicated requiring input from metaphysics, phenomenology, sociology, and mathematics among other fields of study.

Fortunately, however, if we accept the realist methodology which I have proposed in this thesis, then we can leverage the formal structure of our best theories to shed some light on Z's epistemology. In particular, we may be able to use our strong capacity for Z-redescription to *theoretically simulate* the actual epistemic story of Z. No matter why one

is interested in Z's epistemology, the first step in simulating it theoretically is to decide which part of the above-discussed epistemic process we will try to simulate. As Fig. 3.1 indicates, our three options are to simulate either the process of: systematizing the pre-epistemic world; codifying our phenomenal experiences; or theoretically reformulating our theory in search of its best Z-framing.

Let us begin by trying to simulate the reformulation step. I say "simulate" to contrast what follows with the actual process of reformulating our theories. In practice, the reformulation step has often been an unstructured and ad hoc search through all of the possible Z-framing of our theory. However, once we have mastered our capacity for these various kinds of redescription we can then (in hindsight) conduct this search more systematically. For instance, given any spacetime theory, we can easily access every possible coordinate description of that theory and, moreover, we can even describe it in coordinate-invariant and coordinate-free ways. In fact, our mastery of coordinate redescription (and with it, a mastery of tensor calculus) proved itself invaluable in the development of GR.[39] Even once we have written our theory in the coordinate-free language of differential geometry, one is then still free to geometrically redescribe it using different metrics and connections. For instance, consider the Weyl transformation, $g_{ab} \mapsto \bar{g}_{ab} := \Omega^2 g_{ab}$, which carries one from Nordström's (1913) second theory of gravity to the Einstein and Fokker (1914) reformulation thereof.[40]

We have a similarly well-equipped toolbox for implementing nomological redescription, i.e., we can easily describe our theory according to different axiom-like laws. My goal in the latter half of this thesis will be to develop a similarly strong capacity for topological redescription. In particular, Ch. 4 will develop a general framework for thinking about topological redescriptions, which I shall call spacetime representation theory. Then

[39] See Norton (1993).
[40] See Duerr (2020).

in Part II of this thesis I will develop the ISE Method of topological redescription. This will allow us to access every spacetime representation of our theories which has a sufficient level of spacetime-kinematic compatibility.

In any of these four contexts, once we have mastered our capacity for Z-redescription, we can then comprehensively explore the full range of possibilities (even those which are terribly complicated or counter-intuitive). We can then propose some (simulated) selection criteria to account for how we have narrowed in on a particularly nice Z-framing of our theories, see Fig. 3.1. But why should we aim to put *all* of the options on the table? One might rightly point out that most axiomatizations and most coordinate systems are horrendously complicated (as are most geometric and topological framings). Wouldn't it be better to develop our capacity for Z-redescription with a focus on accessing only those Z-framings which are "simple" or "nice" in some sense? I will now offer two counterpoints to this way of thinking. Firstly, consider how impoverished the philosophy of space and time would be if we hadn't gained a mastery of our capacity for *generic* coordinate redescription. In particular, suppose that we were satisfied with our ability to switch between a few nice coordinate systems: inertial coordinates, polar coordinates, cylindrical coordinates, etc.

Secondly, suppose that we have provided some account of our Z-selection criteria and are then asked "So why didn't we end up formulating our theory in this terribly complicated way?". It would be unfortunate if our answer was as follows: "This possibility was not considered because it fell outside of the scope of our redescription techniques". A better answer would be "This possibility was considered but rejected because it violates this particular Z-selection criteria (e.g., one related to simplicity)". Ultimately, we want methods of formal redescription which —rather than having our preferred method of representation build into them—can instead help us see how we could have discovered these

preferred methods of representation in the first place. Building in our discoveries would only make it harder to see how the discovery process can be done in the first place.[41]

Let us now return to the other two parts of Z's epistemology which we might simulate, i.e., either systematizing the pre-epistemic world or codifying our phenomenal experiences thereof. In particular, I will next consider these two options in turn, albeit focused only on the laws' context since this is where this topic is more widely researched. Let us keep in mind, however, that analogous points can be made in the contexts of geometry and topology. To illustrate, let us take as our working example the theory of scalar electromagnetism (SEM) which I introduced in Sec. 2.2. One's natural impulse may be to pick out some part of SEM to be our simulated stand-in for the pre-epistemic world. This could mean, for instance, construing φ and X in non-modal terms and taking them to be the contents of a (simulated) Humean mosaic which we can then begin systematizing à la Lewis (1983). Allow me to briefly introduce the idea of a Humean mosaic before discussing what would make our SEM-inspired mosaic a *simulated* mosaic.

Lewis' (1986) account of laws begins "at step zero" by putting forward a concrete picture of the pre-epistemic world. In particular, he describes what is often called a Humean mosaic as follows:

> Humean supervenience is named in honor of the great denier of necessary connections. It is the doctrine that all there is to the world is a vast mosaic of local matters of particular fact, just one little thing and then another ... We have geometry: a system of external relations of spatiotemporal distances between points. ... And at those points we have local qualities: perfectly natural intrinsic properties which need nothing bigger than a point at which to be instantiated. For short: we have an arrangement of qualities. And that is

[41] This is a paraphrase of a quote taken from Sutton's (2019) "The Bitter Lesson" regarding the folly of trying to hard-code AI agents such that they think in the way that we think that we think. The full quote is as follows: "The second general point to be learned from the bitter lesson is that the actual contents of minds are tremendously, irredeemably complex; we should stop trying to find simple ways to think about the contents of minds, such as simple ways to think about space, objects, multiple agents, or symmetries. All these are part of the arbitrary, intrinsically-complex, outside world. They are not what should be built in, as their complexity is endless; instead we should build in only the meta-methods that can find and capture this arbitrary complexity. Essential to these methods is that they can find good approximations, but the search for them should be by our methods, not by us. We want AI agents that can discover like we can, not which contain what we have discovered. Building in our discoveries only makes it harder to see how the discovering process can be done."

> all. There is no difference without difference in the arrangement of qualities. All else supervenes on that.[42]

In particular, Lewis (1983) takes the laws of nature to be a part of the "all else" which supervenes on this Humean mosaic. In his own words, Lewis' Best System Account (BSA) of laws is as follows:[43]

> [A]n adequate analysis must be collective. It must treat regularities not one at a time, but rather as candidates to enter into integrated systems. ... I take a suitable system to be one that has the virtues we aspire to in our own theory-building, and that has them to the greatest extent possible given the way the world is. It must be entirely true; it must be closed under strict implication; it must be as simple in axiomatisation as it can be without sacrificing too much information content; and it must have as much information content as it can have without sacrificing too much simplicity.

Elsewhere, Lewis (1994, pg. 479) responds to anticipated concerns that BSA-selected lawhood will "come from us" in a problematic sense due to the conventionality of our "standards of simplicity and strength and balance." Against this "ratbag idealism" he claims that we can reasonably hope that BSA-selected lawhood will be objective nonetheless: "If nature is kind, the best system will be *robustly* best — so far ahead of its rivals that it will come out first under any standards of simplicity and strength and balance."

In sum, Lewis' BSA gives us a pre-epistemic picture of the world (his Humean mosaic) as well as a sketch of the epistemic process by which the laws of nature can be (hopefully objectively) read off of this mosaic. What makes this a "simulated" Humean mosaic rather than an actual one is the fact that it is located at the epistemically later stages of Fig. 3.1, i.e., within the white L-shaped box. Namely, the only reason that we would think to populate our Humean mosaic with a scalar field, φ, and point-like particles, X, is because these are the types of matter which appear in our best theory (here, SEM). In particular, these are the kinds of states which appear in its kinematically or dynamically possible models (KPMs or DPMs). Hence, while we may read the laws off of the Humean

[42] Lewis (1986, pg. ix).
[43] Lewis (1983, pg. 41).

mosaic, we read the contents of the simulated Humean mosaic off of our theory. For a certain brand of realist, the bridge between the simulated Humean mosaic and the actual Humean mosaic is easily crossed. However, others may want to make use of a simulated Humean mosaic for purely epistemic reasons, i.e., without claiming that it is an accurate simulation of the pre-epistemic world. Our present interest is in epistemic simulation, whereas the possibility of metaphysical simulation will be discussed in Sec. 3.2.

As Hall (2015, pg. 265) notes, the Humean position results from taking the above-discussed epistemic standards (which both sides endorse) and "elevating them to the status of standards constitutive of the laws of nature." One might rightly complain, however, that the other side of this debate are unlikely to accept "step zero" of Lewis' epistemic story since it depicts our world as one which gets by on its own without the governance of any axiom-like laws. Namely, it would bias our epistemic simulation with respect to the dynamics-first vs dynamics-second debate about axiom-like laws. Moreover, depending on how exactly one identifies SEM's subject matter, it might unnecessarily bias our epistemic simulation with respect to the other debates we are interested in elsewhere in this thesis. Does our epistemic simulation take φ and X to have causal powers, allowing them to force and source each other? Does our epistemic simulation take the topological and geometric structures (e.g., $\mathcal{M}$, η_{ab}, and ∇_c) which appear in Eqs. (2.4) - (2.6) to have latched onto dynamics-facilitating pre-epistemic counterparts? To avoid biasing ourselves in all of these debates, we ought to avoid making any unnecessary assumptions about the pre-epistemic world. As I will discuss later, Hall (2015) manages to (mostly) dodge my "biased starting point" complaint by instead simulating the codification step.

Before discussing the codification step, however, let us first see what Kant has to say about the epistemic process by which our theories of physics come to feature the specific laws of nature that they do. What is notable about Kant's approach is that it allows us to begin telling an epistemic story about the laws of nature while making few (if any)

assumptions about the pre-epistemic world.[44] In particular, Kant (1783, pg. 26) directs our attention to the a priori organizational pre-requisites which must be in place before an investigation of the empirical laws of nature can even begin:

> How can we know a priori that things (considered as objects of experience) must conform to law? ...The laws that govern our ways of knowing also govern the objects that we know, as long as these are considered as objects of experience and not as they are in themselves.

Importantly, one can accept this general point without accepting the feasibility of using such a priori considerations to place non-trivial constraints on the empirical laws of nature. For instance, one can safely reject Kant's (1783, Sec.37 - 38) attempted derivation of the inverse-square law from a priori geometric considerations, as well as his claim that we can know a priori that space and time have a Euclidean character.[45]

With our attention turned towards the agent who discovers the laws of nature, let us now move on to our next option: simulating the codification process. Doing so would require that we identify a theoretical stand-in — not for the pre-epistemic world — but for our phenomenally limited selves. Indeed, as I will now discuss, Hall (2015) makes a move in this direction with the introduction of his Limited Oracular Perfect Physicist (LOPP). As I noted above, Hall (2015, pg. 265) identifies the "unofficial guiding idea" (although he sometimes calls it the "second guiding idea") behind Humeanism about laws as consisting in them taking certain agreed-upon epistemic standards for identifying the laws of nature and promoting these standards to a full metaphysical account of what the laws of nature are. To help articulate this idea, Hall (2015, pg. 265) introduces his LOPP as a literal embodiment of these epistemic standards; In his own words,

> To help flesh this idea out more fully, let us imagine someone whom I will call a Limited Oracular Perfect Physicist. What makes our LOPP a perfect

[44] It is worth comparing Kant's approach to some more recent work on the epistemology of laws and causation. Kant seeks to identify the minimal organizational prerequisites which enable our search for the empirical laws of nature. By contrast, Weinberger et al. (2024) seek to identify the minimal "worldly infrastructure" which is necessary to support modern approaches to the epistemology of causation.

[45] See the Transcendental Aesthetic in Kant (1999).

> physicist is that, given as evidence any information about the world, she is perfectly able to judge what hypotheses about the fundamental physical laws are most strongly supported by that evidence. What makes her oracular is that she has, as evidence, quite a lot of information about the world. ... Specifically, we will suppose her evidence consists of all the facts about the distribution of perfectly natural magnitudes. Hence, in our Newtonian particle world: perfect information about the motions, masses, and charges of every particle, together with the further information that the world contains nothing else. The second guiding idea, then, is that the laws are whatever she says they are.

In contrast to Lewis' (1994) worries about "ratbag idealism" (i.e., that our law-selection criteria "comes from us"), Hall is more comfortable with this possibility. In fact, he says that,[46]

> there is plenty of room for the view that it is in part facts about us — *idiosyncratically* about us, and our peculiar human psychologies — that play a role in determining the most appropriate way for us to structure our investigation of the world [e.g., determining the laws of nature]. How could the details of our peculiar human situation not be relevant to this matter?

While one can invoke an LOPP without leaning too hard on her human-like qualities, Ott (2022, pg. 229) calls those views that do "Humeanism with a human face" and associates them with the following claim:

> Human LOPP-relative Nomic [Humean Supervenience]: the laws of a given world supervene on the mosaic (the spatio-temporal arrangement of properties), and are relative to the epistemic goals and cognitive limitations of a recognizably human LOPP.

In sum, Hall's framing of the metaphysics of laws debate is as follows: He posits his LOPP as embodying our epistemic standards for determining what the laws of nature are given some body of evidence. Following this agreed-upon epistemic story, the subsequent metaphysical debate is then about whether or not the nomological significance of these laws is fully accounted for by them being a particularly nice (i.e., LOPP-selected) way of codifying this body of evidence.

While Hall's framing of this debate does shift our attention away from systematizing a Humean mosaic and towards codifying a body of evidence, his (simulated) epistemology

[46] Hall (2015, pg. 268)

of laws does not go far enough in this direction for my present purposes. Fortunately, however, with two minor adjustments (one addition and one subtraction) Hall's LOPP will be exactly what we need to begin simulating the codification process. The addition (compatible with Hall's LOPP being idiosyncratically human) is the condition that her oracular access to the world is experienced through a recognizably human phenomenology. In particular, I demand that my LOPP applies the same unconscious organizational principles to the world as we do. Hence, the content of her experiences are events spread across space and time and she finds certain constant conjunctions to be psychologically compelling towards thoughts of necessity. Recall from Sec. 2.1 that this characterization of human phenomenology is meant to be uncontentious. Indeed, this is the agreed-upon epistemic backdrop against which a later metaphysical debate is to be set, i.e., the debate for and against unofficially Humean metaphysics. This epistemic backdrop alone says nothing about whether or not causation, space, and time are or are not hard-working aspects of the pre-epistemic world.

After adding this human-phenomenology condition to Hall's LOPP, I then need to substract the following condition from his view. Whereas I would prefer to focus on the wider class of views which I call unofficial Humeanism about laws (a.k.a. dynamics-first views of laws), Hall's (2015) focus is on a subset thereof which he calls *Humean reductionism*. In particular, his discussion of the metaphysics of laws debate starts as follows:

> Start with the assumption, widespread among reductionists and anti-reductionists alike, that there is such a thing as the fundamental ontological structure of the world: facts about the world to which all other facts reduce. The Humean reductionist needs a doctrine about the nature of this fundamental ontological structure. Lewis was admirably clear on this point.

But this seems unneccessarily restrictive given my above gloss of Hall's epistemology of laws: The laws of nature are what his LOPP says they are given some body of evidence (i.e., her phenomenal experiences of the world). One might then argue that the nomo-

logical significance of these laws is fully accounted for by them being a particularly nice (i.e., LOPP-selected) way of codifying this body of evidence. This would be sufficient to establish a dynamics-first view of laws (which is my aim, not Hall's). Indeed, recall the epistemic reading of Hume which I discussed in Ch. 1. One can hold dynamics-first views without putting forward any doctrine about the fundamental ontological structure of the pre-epistemic world (e.g., that it consists of a Humean mosaic).

If we now drop Hall's (arguably auxiliary) demand that both sides of the debate come equipped with a fully articulated reduction basis, then we get much nearer to my dynamics-first vs dynamics-second framing of the metaphysics of laws debate. Indeed, by converting talk of "the Humean mosaic" into talk of "a LOPP's phenomenal experiences", even a hard-line anti-speculativist can now safely enter into the metaphysics of laws debate on the dynamics-first side. With these minor alterations, we can take a Hall-esque LOPP as a theoretical stand-in for our phenomenally limited selves as we simulate the codification process, see Fig. 3.1. Making as few assumptions about the pre-epistemic world as possible, my LOPP's evidence regarding an SEM-like world will consist of her having oracular access to:

1. the strength (i.e., the voltage) of the electromagnetic field, φ, across all of spacetime,
2. the motions of every point-like particle, X, across all of spacetime (from which she can determine their masses and charges),
3. together with the further information that there is nothing else to be experienced in the realm of fundamental physics.

Importantly, we must here think of LOPP having access to X and φ as objects of experience and not as they are in themselves. But this is just to repeat the stipulation that my LOPP experiences the world with a recognizably human phenomenology; Her oracular access to φ and X comes from her "feeling" and "seeing" them in recognizably human

ways, e.g., in the way that we feel voltages and see specks of dust. This says nothing about the pre-epistemic world in itself.

Following Hall (2015), the laws of nature are what she says they are (trivially since she has the right epistemic standards built into her). But what can we say about the LOPP's epistemic standards? Given that she operates like a perfect physicist (per definition) we can apply our above discussion of the actual codification step. Namely, we can imagine that she first makes some initial codification of her experiences (using some axiom-like laws) and then reformulates it to find its best axiomatization. Given that our capacity for re-axiomatizing out theories is so great (as would be her's), it seems as though this initial codification step does not matter. However it is that she initially happened to formulated her theory, she would ultimately end up at the same place. Hence, specifying a LOPP just involves fixing her body of evidence (e.g., the nature and scope of her phenomenal experiences) and her theoretical law-selection criteria. Supposing that we can reach an agreement about the nature and scope of her evidence, then any questions regarding the objectivity vs conventionality of the laws of nature can be reduced to a question of her selection criteria.

While the above discussion has been focused on the laws' context, one could, of course invoke a LOPP in the other three contexts as well. A key difference, however, in the cases of geometry and topology is that these (like causation but unlike laws of nature) seem to be built into our phenomenal experience of the world. This, however, is not contentious and indeed was assumed in my above discussion. We can, as before, specify our LOPP by fixing her body of evidence and her theoretical selection criteria but now for geometry and topology. Having discussed the first two epistemic benefits let us now turn our attention to the next two metaphysical benefits.

3.2 Articulating Dynamics-First Views and Metaphysical Simulation

The two benefits which we have discussed so far were both epistemic. Mastering our capacity for Z-redescription (with Z = axiom-like laws, coordinates, geometry, and/or topology) can help physicists with theory development and it can help philosophers to simulate the epistemic process of theory development. We were, however, led by metaphysical aspirations to focus our attention on simulating the codification and reformulation steps. Namely, I have suggested above that we can use a Hall-esque LOPP as a helpful epistemic backdrop which is neutral with respect to the dynamics-first vs dynamics-second debates. This is our third benefit: Mastering our capacity for Z-redescription can help us to articulate various dynamics-first vs dynamics-second debates regarding how we should interpret our theories.

Generalizing Hall (2015), both sides of these debates should be able to agree about some epistemic standards and build them into a LOPP such that the laws/geometry/topology are what she says they are (subject to some conventions if these cannot be picked out objectively). With this epistemic standard set, the dynamics-first vs dynamics-second debates are then about whether or not this epistemic story is sufficient to account for the metaphysical significance of these laws, geometry and/or topology.

One of the key benefits to framing these debates in terms of an agreed-upon LOPP, is how it helps us to highlight the exact amount of epistemic agreement which underlies these debates about foundational metaphysics. Namely, as I have stressed above neither side of this debate needs to come equipped with a fully articulated doctrine about the fundamental ontological structure of the pre-epistemic world.

Beyond merely helping us to articulate these debates, however, I claim that our capacity for Z-redescription can be brought to weigh in on one side or the other. Indeed, as I noted above, this thesis will adopt a realist methodology: I take it that we are justified in

using the formal/mathematical structure of our best physical theories as evidence in these debates about foundational metaphysics.

For instance, a close investigation of our Z-selection criteria might lead one to be a conventionalist about Z. This would give us good *epistemic grounds* for believing in dynamics-first views. Suppose, for instance, that multiple axiomatizations of our theory can reasonably qualify as best, then some tie-breaking convention may be needed to decide between them. While this tie breaking may be as arbitrary as flipping a coin, it also might be based upon our human (as opposed to Martian) idiosyncrasies (recall Humeanism with a face). In any case, dynamics-second views of laws would be hard-pressed to account for why our favorite axiom-like laws (rather than the Martians) have latched onto the world's governing laws. One might follow Lewis (1994) (quoted above) in hoping that "nature is kind" to us such that this kind of conventionalism can be avoided. At this point we can bring in Pitts's (2016, pg. 75) point (albeit in the context of geometry) that "debates on conventionalism . . . are seen to depend crucially on what one takes the relevant modal scope of the discussion to be." In particular Pitts (2016, pg. 84) notes that, "perhaps the conventionalists did not emphasize enough (though the themes can be found) that (1) the most interesting question is the modally broad one that considers a variety of theories rather than a single theory (even 'our best'), and (2) there exists a moderately interesting physical theory (or many of them) containing within itself 'rival' geometries, that is, more than one example of a given type of geometric object." Unfortunately, further discussion of conventionalism (regarding laws and geometry) is beyond the scope of this thesis. More will be said about conventionalism regarding spacetime topology in Ch. 8.

It is also possible that our making this realist turn could provide us with novel *metaphysical grounds* for believing in dynamics-first views. The keen reader may be able to anticipate how such an argument might go. Indeed, we have already seen such an argument in the context of coordinate systems at the end of Sec. 2.2. We were above convinced

that there are no coordinate labels (nor any "true names") out there in the pre-epistemic world which help to facilitate the dynamical behavior of matter. Given the mastery that we have for redescribing the world using different names/coordinates, it is clear that these aspects of our theories are reflective of the representational process itself rather than any underlying coordinate-like structure. Furthermore, it would be very strange if an aspect of our theories which is so obviously one of *our* representational tools to have a dynamics-facilitating pre-epistemic counterpart. Our mastery of coordinate redescription seems to give us good metaphysical grounds for a dynamics-first view of coordinates. In principle, an equally strong capacity for nomological, geometric, and/or topological redescriptions could give us equally solid metaphysical grounds for the other three dynamics-first views.

For instance, I claimed in Sec. 2.2 that insights about the metaphysical status of coordinate systems can be leveraged from our strong capacity for coordinate redescription; Given a theory described in one coordinate system, we have the mathematical tools to easily redescribe it in a wide variety of alternative coordinate systems (or even using no coordinates whatsoever). As I have argued above, this strong capacity for coordinate redescription gives us good reasons to favor dynamics-first views of coordinates over dynamics-second views.

In addition to shedding light on Z's metaphysical status, a strong capacity for Z-redescription also has the potential to help inform our overall picture of the pre-epistemic world. This is the fourth potential benefit of mastering our capacity for formal redescription. Suppose that one has become convinced of a dynamics-first view about Z. Namely suppose that one belives that Z's presence in our theories is only reflective of the representation process itself rather than any dynamics-facilitating pre-epistemic counterpart. The question then looms: What exactly is Z helping us to represent? This question calls out for us to take our Z-free or Z-invariant reformulation of our theories as a simulated stand-in for the pre-epistemic world. Indeed, defending any sort of realist position would

require that we put forward a concrete metaphysical picture of the mind-independent (i.e, the pre-epistemic) world.

In the context of coordinate systems, one might be here reminded of Einstein's principle of general covariance.[47] Applied to scalar electromagnetism (SEM) this means that we shouldn't begin analyzing SEM from its coordinate-specific formulation, Eqs. (2.1) - (2.3). Instead, we should begin our (meta-)physical investigations of SEM from either a coordinate-free formulation of SEM (i.e., Eqs. (2.4) - (2.6)). Namely, we can take this theory's axiom-like laws, and its geometric and topological structures (e.g., $\mathcal{M}$, η_{ab}, ∇_c) to be the subjects of the next rounds of the dynamics-first vs dynamics-second debates (i.e., about Z = laws, geometry, and topology). We might then adopt dynamics-first views in these contexts as well. This in turn would motivate us to seek out Z-free formulations of our theories. By repeatedly applying this process of elimination, we might eventually hone in on a better understanding of the world-in-itself.

[47] The terms "general covariance", "diffeomorphism invariance", and "background independence" are sometimes used interchangeably in the physics literature. In this thesis, I understand these terms as distinct as laid out in Pooley (2017).

4 | Spacetime Representation Theory and the ISE Equivalence Theorem

The previous chapter has motivated the need to develop some strong topological redescription techniques in order to aid our understanding of the epistemology and metaphysics of space and time. Such tools (namely, the ISE Method) will be developed in Part II of this thesis. Before this, however, this chapter will first develop a general framework for thinking about the topological redescriptions of our spacetime theories. To begin with, Sec. 4.1 will give our first example of topological redescription; This is a novel kind of spacetime duality which I shall call the Möbius-Euclid duality. This duality allows us to relate a certain spacetime theory set on the Möbius strip to another spacetime theory set on the Euclidean planes and vice versa. Despite their stark differences in topological staging, I will show that there is nonetheless a robust sense in which these two theories are equivalent: They have exactly the same kinematic and dynamical contents. Importantly, the Möbius and Euclid theories are both theories about particle trajectories and therefore have no algebraic structure (i.e., no notion of addition or multiplication).

Sec. 4.2 will then discuss some more algebraic examples of topological redescription. These include the well-known position-momentum duality in non-relativistic quantum mechanics (although I prefer to call it Fourier duality). Discussing these examples will lead us to adopt a very broad definition for the terms "spacetime theory" and "pre-spacetime theory". In these terms, the Möbius and Euclid theories are *inequivalent as spacetime theories* (because they disagree about the topological staging of their states). They are, however, *equivalent as pre-spacetime theories* because they are isomorphic at

the level of kinematics in a way which preserves their dynamics. Sec. 4.2 will end by laying the groundwork for what I will call "spacetime representation theory" which is analogous to the mathematician's representation theory for groups. In particular, I will define a sense in which the Möbius and Euclid theories are different spacetime representations of the same pre-spacetime theoretic structure.

Finally, in Sec. 4.3 I will state the main result of this thesis: The ISE-Equivalence Theorem (proved in Ch. 11). In effect, this theorem states that the ISE Method is a general-purpose tool for exploring and implementing a wide range of topological redescriptions. Namely, the ISE Method can be applied to a wide scope of spacetime theories and gives us access to effectively every possible topological redescription thereof, its range being limited only by a weak spacetime-kinematic compatibility constraint.

4.1 A First Example of Topological Redescription: Möbius-Euclid Duality

As a first example of topological redescription, consider the following two spacetime theories. The first theory is about a point-like particle moving on a Möbius strip, $\mathcal{M}_{\mathrm{M}} \cong \mathbb{M}$, over time. Before formalizing this theory, let us first discuss its dynamics informally, see the top row of Fig. 4.1. The point-like particle (in red) has two dynamical modes: 1) Neglecting any motion around the strip, the particle would undergo harmonic oscillations across the dashed blue line as if trapped in a quadratic potential. 2) Sitting at the minimum of this potential, the particle would be free to undergo uniform motion around the strip, eventually returning to its original location. Let us call the period of this motion $x = \pi$. Note that if the particle were held at a fixed distance away from the dashed blue line, then the period of this motion would instead be $x = 2\pi$. That is, the particle would then have

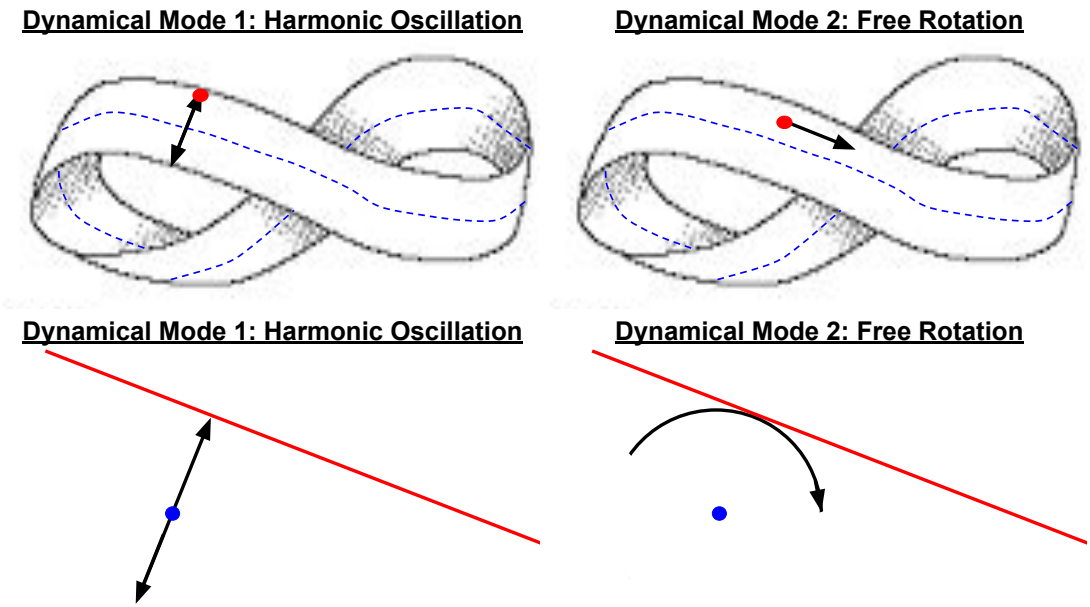

Figure 4.1: This figure shows the dynamics of the Möbius and Euclid theories. The two subfigures on the top row depict a point-like particle (the red point) moving on a Möbius strip over time. There is a special section of the Möbius strip (the blue dashed line) about which the particle oscillates harmonically as it moves freely in the other direction. The two subfigures on the bottom row depict a line-shaped particle (the red line) moving on the Euclidean plane over time. There is a special point (the blue point) about which the particle oscillates harmonically as it rotates freely in the other direction. Perhaps surprisingly, despite having clear differences in the topological staging of their states, these theories are nonetheless equivalent in a robust sense. Namely, they are isomorphic at the level of kinematics in a way which preserves their dynamics. This is what makes them topological redescriptions of each other. Moreover, as I will discuss in Sec. 4.2, we can think of these two theories as being different spacetime representations of the same pre-spacetime theory.

to travel twice around the Möbius strip to return to where it started. Combining these two types of motion, one can imagine the particle oscillating up and down as it travels around the Möbius strip.

The next spacetime theory which we will consider is (perhaps surprisingly) very closely related to the previous one. The key differences are as follows. Whereas the previous theory was about a particle moving on a Möbius strip, $\mathcal{M}_{\mathrm{M}} \cong \mathbb{M}$, the next theory considers a particle moving on a Euclidean plane, $\mathcal{M}_{\mathrm{E}} \cong \mathbb{R}^2$. Moreover, whereas the previous theory considered a point-like particle, the next theory considers an extended *line-shaped particle*. (That is, in spacetime this particle would be picked out by a world-sheet rather than a worldline.) Let us again start by getting an informal grasp of this

theory's dynamics, see the bottom row of Fig. 4.1. As before, the particle under consideration (in red) has two dynamical modes: 1) Neglecting any rotation, the particle would undergo harmonic oscillations across the special blue point as if trapped in a quadratic potential. 2) Sitting at the minimum of this potential, the particle would be free to undergo uniform rotational motion, eventually returning to its original location. Note that because the line is unoriented the period of this motion is $\theta = \pi$. Moreover, notice that if the particle were to be held at a fixed distance away from the special blue point, then the period of this motion would instead be $\theta = 2\pi$. Combining these two types of motion, one can imagine the line-shaped particle oscillating back and forth as it rotates around the Euclidean plane.

As strange as this second spacetime theory may seem, the previous paragraph ought to sound very familiar. In fact, it is nearly identical to the above description of the Möbius theory despite the fact that these two theories have radically different topological underpinnings ($\mathcal{M}_{\mathrm{M}} \cong \mathbb{M}$ vs $\mathcal{M}_{\mathrm{E}} \cong \mathbb{R}^2$) and are about radically different kinds of particles (point-like vs line-shaped). Indeed, these two theories disagree about seemingly fundamental metaphysical facts: e.g., whether or not the particles in question are mereologically simple. Despite their significant differences, I will demonstrate below that (once formalized) these two theories are nonetheless kinematically and dynamically equivalent. Indeed, shockingly, there is a robust sense in which these two theories are also *geometrically* equivalent, despite being topologically inequivalent.

Let us now formalize these two theories. The topological stage for the first theory includes only space and will be modeled as a Möbius strip, $\mathcal{M}_{\mathrm{M}} \cong \mathbb{M}$. Since the particle described by this theory is point-like, its trajectory will be given by a function $X_{\mathrm{M}} : \mathbb{R} \to \mathcal{M}_{\mathrm{M}}$. If we were to have included time in $\mathcal{M}_{\mathrm{M}}$, then X_{M} would have picked out the particle's worldline. Before stating the dynamics of this particle, it will be convenient to first introduce a fixed global coordinate system for its spatial stage, $\mathcal{M}_{\mathrm{M}} \cong \mathbb{M}$. For this

theory, there exists a convenient coordinate map, $C : \mathbb{R}^2 \to \mathcal{M}_{\mathrm{M}}$, which assigns to each point, $p \in \mathcal{M}_{\mathrm{M}}$, a set of coordinates,

$$C^{-1}(p) = \{(x + n\,\pi, (-1)^n\, y)\}_{n\in\mathbb{Z}} \subset \mathbb{R}^2. \tag{4.1}$$

One can think of advancing the x-coordinate as carrying us around the base space of the Möbius strip with a period of π. Advancing along the y-coordinate can be thought of as carrying us along the fibers of the Möbius strip. Note that the y-coordinate goes to infinity in both directions. In Fig. 4.1, the y-coordinate has been cropped for visualizability. Given this coordinate system, we can now officially state the Möbius theory as follows:

> **A Point-like Particle on a Möbius Strip (Möbius Theory) -** Consider a theory set on a spatial manifold of $\mathcal{M}_{\mathrm{M}} \cong \mathbb{M}$ about the trajectory of a particle, $X_{\mathrm{M}} : \mathcal{V}_{\mathrm{M}} \to \mathcal{M}_{\mathrm{M}}$, with a parameter space of $\mathcal{V}_{\mathrm{M}} \cong \mathbb{R}$. The only kinematic constraint on this theory's states is that X_{M} is a smooth map. For this theory there exists a map, $C : \mathbb{R}^2 \to \mathcal{M}_{\mathrm{M}}$, which assigns to each point, $p \in \mathcal{M}_{\mathrm{M}}$, a set of coordinates as in Eq. (4.1). In this coordinate system, the dynamical equation for the particle's trajectory is as follows:
>
> $$\partial_\tau^2\, x(\tau) = 0, \qquad \partial_\tau^2\, y(\tau) = -y(\tau). \tag{4.2}$$

To connect with Fig. 4.1, the $y = 0$ coordinate on each fiber picks out the special dashed blue line in Fig. 4.1.

Let us now officially introduce the Euclid theory. Whereas the previous theory considered a point-like particle, $X_{\mathrm{M}} : \mathbb{R} \to \mathcal{M}_{\mathrm{M}}$, the next theory considers an extended line-shaped particle, $X_{\mathrm{E}} : \mathbb{R}^2 \to \mathcal{M}_{\mathrm{E}}$. If we would have included time in $\mathcal{M}_{\mathrm{E}}$, then X_{E} would have picked out the particle's worldsheet. Before I formalize this theory's dynamics, the following fact should be recalled: The equation for any straight line in the (q_1, q_2)-plane can be written in the form $q_1 \cos(\theta) + q_2 \sin(\theta) = r$ for some parameters θ and r.

> **A Line-Shaped Particle in the Euclidean Plane (Euclid theory) -** Consider a theory set on a spatial manifold of $\mathcal{M}_{\mathrm{E}} \cong \mathbb{R}^2$ about the trajectory of a particle, $X_{\mathrm{E}} : \mathcal{V}_{\mathrm{E}} \to \mathcal{M}_{\mathrm{E}}$, with a parameter space of $\mathcal{V}_{\mathrm{E}} \cong \mathbb{R}^2$. The theory's states are subject to the following kinematic constraint: In some fixed global

coordinate system, (q_1, q_2), the particle's trajectory must be parameterized such that its two coordinate functions, $q_1(\tau, s)$ and $q_2(\tau, s)$, satisfy,

$$q_1(\tau, s)\cos(\theta(\tau)) + q_2(\tau, s)\sin(\theta(\tau)) = r(\tau) \tag{4.3}$$

for some smooth functions, $r(\tau)$ and $\theta(\tau)$. That is, at each fixed τ-parameter the extended particle must occupy a straight line in the (q_1, q_2)-plane. Here $r(\tau)$ and $\theta(\tau)$ are the polar coordinates for the point on this line which is nearest to the point $(q_1, q_2) = (0, 0)$. In terms of $r(\tau)$ and $\theta(\tau)$, the dynamical equation for the particle's trajectory is as follows:

$$\partial_\tau^2\,\theta(\tau) = 0, \qquad \partial_\tau^2\,r(\tau) = -r(\tau). \tag{4.4}$$

Note that $r(\tau)$ is here allowed to be negative with $(\theta, -r) \sim (\theta + \pi, r)$.

To connect with Fig. 4.1, the point with coordinates $(q_1, q_2) = (0, 0)$ picks out the special blue point in Fig. 4.1.

In order to see the sense in which the Möbius and Euclid theories are equivalent, let us first set aside their dynamics and focus on a time-slice of each theory. As I will now discuss, there is a robust geometric equivalence between the Möbius strip, $\mathcal{M}_{\rm M} \cong \mathbb{M}$, and the Euclidean plane, $\mathcal{M}_{\rm E} \cong \mathbb{R}^2$, despite their clear topological differences. In particular, there is a well-known isometry between the set of all lines in the Euclidean plane and the set of all points on the Möbius strip.[48] I have constructed the above two theories such that this isometry gives a one-to-one correspondence between their dynamics. To sketch this isometry:

- Every family of co-parallel lines in the Euclidean plane corresponds to one fiber on the Möbius strip. In particular, any two parallel lines will correspond to two points on the Möbius strip which are on the same fiber. The distance between these parallel lines corresponds to the distance-along-their-common-fiber between their corresponding points.
- Intersecting lines are in different co-parallel families and hence correspond to points on different fibers. Their acute intersection angles correspond to the angular distances between these two fibers.[49]

[48] The isometry which I discuss below is well-known to mathematicians. See, for instance, Beem and Parker (1991). My contribution is to build a novel spacetime duality on top of this isometry.

[49] Recall that we have defined the period of $\mathbb{M}$'s base space to be $x = \pi$ rather than 2π. If one measures the angular distance between fibers using the latter standard, then a factor of 2 is needed in this claim.

But where is the twist?[50] The Möbius strip famously has a topological twist to it; Where can this be seen in the above-sketched isometry? To demonstrate where the twist comes in, allow me to first state some obvious facts about the Euclidean plane before stating the isometric facts about the Möbius strip. Pick any point in the Euclidean plane to color blue (as in Fig. 4.1) and consider rotating any line in the Euclidean plane around this blue point by an angle of $\theta = \pi$. Doing so, this line would return to the same co-parallel family that it began in, but it would have moved to the opposite side of the blue point. For most lines, you have to rotate by $\theta = \pi$ twice to get home. The exceptional cases are the lines which happen to start on the special blue point (note, the lines are unoriented). Let us now use our isometry to state some equivalent facts about the Möbius strip. Pick any point on the Möbius strip and rotate it once around the base-space (which has period $x = \pi$ by definition).[51] Doing so, this point would end up on its original fiber but would move to the opposite side of the dashed blue line. For most points, you have to rotate by $x = \pi$ twice to get home. The exceptional cases are the points which happen to start on the blue dashed section.

Let us now approach this isometry more formally. First note that Euclidean geometry can be formalized equally well in terms of either points or lines (i.e., by defining points in

[50] There is another way to see where the twist comes from. I claim below that the space of all *unoriented* straight lines in the Euclidean plane is isomorphic to the Möbius strip; But what about oriented straight lines? One can again associate every family of co-parallel lines with an $\mathbb{R}$-fiber over S^1. This time, however, there is no twist. That is, the space of all *oriented* straight lines in the Euclidean plane is isomorphic to the cylinder, $S^1 \times \mathbb{R}$. Let us parameterize the cylinder as (θ, r). We can get from here to the Möbius strip by topologically identifying every line with its orientation-reversed counterpart. This amounts to taking a certain $\mathbb{Z}_2$ quotient of the cylinder, namely $(\theta, r) \sim (\theta + \pi, -r)$. One can check for oneself that $S^1 \times \mathbb{R}/\sim$ is the Möbius strip, $\mathbb{M}$, as desired.

[51] In the Euclidean case, we got to choose the special blue point which we then rotated about. Where is this choice in the Möbius case? To see this, let us first use the above-sketched isometry to find the correlate of an individual point in the Euclidean plane. Given any blue point, $p \in \mathcal{M}_{\mathrm{E}} \cong \mathbb{R}^2$, we can uniquely identify it with the set of all straight lines through p. Our isometry then maps this collection of straight lines to a section of the Möbius strip, e.g., the blue dashed section. Hence, individual point p on $\mathcal{M}_{\mathrm{E}} \cong \mathbb{R}^2$ correspond to certain sections s of $\mathcal{M}_{\mathrm{M}} \cong \mathbb{M}$. Given any such point p on $\mathcal{M}_{\mathrm{E}}$ we can define rotation around p as the unique transformation which: 1) preserves the distances between any pair of parallel lines, and 2) has that lines through p remain line through p. Similarly, given any such section s of $\mathcal{M}_{\mathrm{M}}$ we can uniquely define a way of "rotating" the Möbius strip such that: 1) it preserves the along-the-fiber distances between any pair of co-fibered points, and 2) it has that points on s remain points on s. In the main text, I am implicitly choosing to rotate the Möbius strip with respect to the section which corresponds to whichever blue point was chosen in the Euclidean case.

terms of the intersections of lines).[52] Formalizing Euclidean geometry in terms of lines, it is exhausted by the set of all straight lines, $L_{\text{Euc}} := \text{Lines}_{\text{Euclid}}$, together with their relative distances and angles. In particular, Euclidean geometry is fully captured by the quadruple, $\langle L_{\text{Euc}}, P_{\text{Euc}}, D_{\text{Euc}}, A_{\text{Euc}} \rangle$, where,

- $P_{\text{Euc}} : L_{\text{Euc}} \times L_{\text{Euc}} \to \{\text{T}, \text{F}\}$ reports whether or not two lines $\ell_1 \in L_{\text{Euc}}$ and $\ell_2 \in L_{\text{Euc}}$ are parallel to each other;
- $D_{\text{Euc}} : P^{-1}_{\text{Euc}}(\text{T}) \to \mathbb{R}_{\geq 0}$ takes as input two lines satisfying $P_{\text{Euc}}(\ell_1, \ell_2) = \text{T}$. That is, the two lines ℓ_1 and ℓ_2 are parallel. The function D_{Euc} then reports their linear separation;
- $A_{\text{Euc}} : P^{-1}_{\text{Euc}}(\text{F}) \to [0, 90°]$ takes as input two lines satisfying $P_{\text{Euc}}(\ell_1, \ell_2) = \text{F}$. That is, the two lines ℓ_1 and ℓ_2 are not parallel. The function $P^{-1}_{\text{Euc}}(\text{F})$ then reports their acute angle of intersection.

These three functions, P_{Euc}, D_{Euc}, and A_{Euc}, give us a robust sense of the distance (linear or angular) between any pair of lines in the Euclidean plane.

One can construct an isomorphic quadruple in terms of the set of all points on the Möbius strip, $O_{\text{Möb}} := \text{pOints}_{\text{Möbius}}$, as follows. Let us call two points on the Möbius strip "parallel" when they sit on the same fiber.[53] We can then define the following three functions,

- $P_{\text{Möb}} : O_{\text{Möb}} \times O_{\text{Möb}} \to \{\text{T}, \text{F}\}$ reports whether or not two points $o_1 \in O_{\text{Möb}}$ and $o_2 \in O_{\text{Möb}}$ are "parallel" to each other (i.e., whether or not they are on the same fiber);
- $D_{\text{Möb}} : P^{-1}_{\text{Möb}}(\text{T}) \to \mathbb{R}_{\geq 0}$ takes as input two points satisfying $P_{\text{Möb}}(o_1, o_2) = \text{T}$. That is, the two points o_1 and o_2 are parallel (i.e., on the same fiber). The function $D_{\text{Möb}}$ then reports the along-their-common-fiber distance between these two points;
- $A_{\text{Möb}} : P^{-1}_{\text{Möb}}(\text{F}) \to [0, 90°]$ takes as input two points satisfying $P_{\text{Möb}}(o_1, o_2) = \text{F}$. That is, the two points o_1 and o_2 are not parallel (i.e., they are on different fibers). The function $P^{-1}_{\text{Möb}}(\text{F})$ then reports the angular separation of their fibers. Recall that we have defined the period of $\mathbb{M}$'s base space to be $x = \pi$ rather than 2π. Hence, the largest possible angular separation is $\Delta x_{\text{max}} = \pi/2 = 90°$.

These three functions, $P_{\text{Möb}}$, $D_{\text{Möb}}$, and $A_{\text{Möb}}$, give us a robust sense of the distance between any pair of points on the Möbius strip (measured either along or between its fibers).

[52] See Barrett and Halvorson (2016, Sec.4) for a discussion of this fact in the context of Morita equivalence.

[53] Note that every point, $o \in O_{\text{Möb}}$, is "parallel" to itself since it sits on the same fiber as itself, $P_{\text{Möb}}(o, o) = \text{T}$. For consistency, this means that we need to consider every line in the plane, $\ell \in L_{\text{Euc}}$, to be parallel to itself, $P_{\text{Euc}}(\ell, \ell) = \text{T}$.

Above, I claimed that the Möbius strip and the Euclidean plane are *geometrically equivalent* despite their clear topological differences. Spelling this claim out formally, I meant that we have an isomorphism between the space of lines in the Euclidean plane and the space of points on the Möbius strip,

$$\langle L_{\text{Euc}}, P_{\text{Euc}}, D_{\text{Euc}}, A_{\text{Euc}} \rangle \cong \langle O_{\text{Möb}}, P_{\text{Möb}}, D_{\text{Möb}}, A_{\text{Möb}} \rangle. \tag{4.5}$$

I invite the reader to check that this isomorphism holds: The above-discussed bijection between the sets L_{Euc} and $O_{\text{Möb}}$ preserves all judgements of parallel-ness, distances, and angles. This justifies calling this isomorphism an isometry. Indeed, the entirety of Euclidean geometry can be faithfully redescribed in terms of points on a Möbius strip. Any figure, any construction, and any proof which one can produce in Euclidean geometry can be equivalently demonstrated using a Möbius strip.

Let us next use this isometry to see the sense in which the Möbius and the Euclid theories are kinematically and dynamically equivalent. To begin, note that both of these theories make a distinction between kinematically allowed and dynamically allowed states (that, is KPMs and DPMs). Let us begin by identifying the modal structure of the Möbius theory. At its broadest level, this theory is about all point-like particle trajectories on $\mathcal{M}_{\text{M}} \cong \mathbb{M}$ (even those which are non-smooth and/or discontinuous). That is, all $\mathbb{R}$-parameterized trajectories, $S^{\text{all}}_{\text{M}} = \{X_{\text{M}} : \mathcal{V}_{\text{M}} \to \mathcal{M}_{\text{M}}\}$, where $\mathcal{V}_{\text{M}} \cong \mathbb{R}$. Enforcing the theory's kinematic constraints, we can identify within $S^{\text{all}}_{\text{M}}$ a subset of kinematically allowed states, $S^{\text{kin}}_{\text{M}} \subset S^{\text{all}}_{\text{M}}$. This is the set of all smooth $\mathbb{R}$-parameterized trajectories on $\mathbb{M}$. Within this kinematically allowed set, we can then enforce the theory's dynamics to find its dynamically allowed states, $S^{\text{dyn}}_{\text{M}} \subset S^{\text{kin}}_{\text{M}} \subset S^{\text{all}}_{\text{M}}$.

Similarly, we can identify the modal structure of the Euclid theory as follows. At its broadest level, this theory is about all $\mathbb{R}^2$-parameterized particle trajectories on $\mathcal{M}_{\text{E}} \cong \mathbb{R}^2$ (even those which are non-smooth and/or discontinuous). Hence, we have that $S^{\text{all}}_{\text{E}} =$

$\{X_{\mathrm{E}} : \mathcal{V}_{\mathrm{E}} \to \mathcal{M}_{\mathrm{E}}\}$, where $\mathcal{V}_{\mathrm{E}} \cong \mathbb{R}^2$. Enforcing the theory's kinematic constraints, we have that $S^{\text{kin}}_{\mathrm{E}} \subset S^{\text{all}}_{\mathrm{E}}$ is the set of all smooth $\mathbb{R}^2$-parameterized trajectories on $\mathbb{R}^2$ *whose time-slices are all straight lines*. Within this kinematically allowed set, we can then enforce the theory's dynamics to find its dynamically allowed states, $S^{\text{dyn}}_{\mathrm{E}} \subset S^{\text{kin}}_{\mathrm{E}} \subset S^{\text{all}}_{\mathrm{E}}$.

For either theory, we can think of the theory's set of kinematically allowed states, $S^{\text{kin}}_{\mathrm{M/E}}$, as coming equipped with enough structure that we can identify within it the theory's dynamically allowed states. Indeed, as I will show momentarily, we can write down their dynamics equations in terms of some functions on $S^{\text{kin}}_{\mathrm{M/E}}$ (namely, $P_{\mathrm{M/E}}$, $D_{\mathrm{M/E}}$, $A_{\mathrm{M/E}}$, $\dots$) which are analogous to those discussed above. The details of how to do this are a bit thorny, but allow me to skip to the punchline. When equipped with the appropriate functions, we will have that,

$$\mathcal{S}^{\text{kin}}_{\mathrm{M}} = \langle S^{\text{kin}}_{\mathrm{M}}, P_{\mathrm{M}}, D_{\mathrm{M}}, A_{\mathrm{M}}, \dots \rangle \quad \text{and} \quad \mathcal{S}^{\text{kin}}_{\mathrm{E}} = \langle S^{\text{kin}}_{\mathrm{E}}, P_{\mathrm{E}}, D_{\mathrm{E}}, A_{\mathrm{E}}, \dots \rangle, \tag{4.6}$$

are sufficiently structured to pick out their dynamically allowed subsets, $S^{\text{dyn}}_{\mathrm{M}} \subset S^{\text{kin}}_{\mathrm{M}}$ and $S^{\text{dyn}}_{\mathrm{E}} \subset S^{\text{kin}}_{\mathrm{E}}$.

Now that we have granted these theories their respective kinematic structures, we can see the sense in which they are formally equivalent to each other. Namely, the Möbius and Euclid are isomorphic at the level of kinematics in a way which preserves their dynamics,

$$\text{Kinematic Isomorphism:} \quad \langle S^{\text{kin}}_{\mathrm{M}}, P_{\mathrm{M}}, D_{\mathrm{M}}, A_{\mathrm{M}}, \dots \rangle \cong \langle S^{\text{kin}}_{\mathrm{E}}, P_{\mathrm{E}}, D_{\mathrm{E}}, A_{\mathrm{E}}, \dots \rangle \tag{4.7}$$

$$\text{which preserves dynamics} \qquad S^{\text{dyn}}_{\mathrm{M}} \leftrightarrow S^{\text{dyn}}_{\mathrm{E}}$$

Hence, the Möbius and Euclid theories have the same kinematic and dynamical content despite them displaying this content on two radically different topological settings. Indeed, as I noted above, these two theories differ not only in their topological stages ($\mathcal{M}_{\mathrm{M}} \cong \mathbb{M}$ vs $\mathcal{M}_{\mathrm{E}} \cong \mathbb{R}^2$) but also in how they display their states upon these stages:

point-like vs line-shaped, i.e., $X_{\mathrm{M}} : \mathbb{R} \to \mathbb{M}$ vs $X_{\mathrm{E}} : \mathbb{R}^2 \to \mathbb{R}^2$. This leads us to the following definition:

> **Definition:**[54] Two spacetime theories are *topological redescriptions* of each other whenever they have the same kinematic and dynamical contents ($\mathcal{S}^{\text{kin}}_{\text{new}} \cong \mathcal{S}^{\text{kin}}_{\text{old}}$ with $S^{\text{dyn}}_{\text{new}} \leftrightarrow S^{\text{dyn}}_{\text{old}}$) despite radically disagreeing about the topological staging of their respective states.

Hence, the Möbius-Euclid duality is our first example of topological redescription.

The rest of this section will be spent filling in the kinematic details of the Möbius and Euclid theories, , $\mathcal{S}^{\text{kin}}_{\mathrm{M}}$ and $\mathcal{S}^{\text{kin}}_{\mathrm{E}}$, which I have gestured at above. The reader is invited to skip ahead to Sec. 4.2 on their first reading.

Let us begin with the Möbius theory. Whereas my above discussion has focused on a time-slice of this theory, in order to capture the theory's dynamics we must begin by re-introducing a space, $T_{\mathrm{M}} \cong \mathbb{R}$, for marking off times. Importantly, this auxiliary space, $T_{\mathrm{M}} \cong \mathbb{R}$, is distinct from the parameter space, $\mathcal{V}_{\mathrm{M}} \cong \mathbb{R}$, which appears in the old theory's states, $X_{\mathrm{M}} : \mathcal{V}_{\mathrm{M}} \to \mathcal{M}_{\mathrm{M}}$. For the Möbius theory these two spaces are isomorphic, $T_{\mathrm{M}} \cong \mathcal{V}_{\mathrm{M}}$. (This will not be the case, however in the Euclid theory.)

Using this auxiliary space, $T_{\mathrm{M}} \cong \mathbb{R}$, we can define three functions analogous to the $P_{\text{Möb}}$, $D_{\text{Möb}}$, and $A_{\text{Möb}}$ functions defined above

- $P_{\mathrm{M}} : S^{\text{kin}}_{\mathrm{M}} \times S^{\text{kin}}_{\mathrm{M}} \times T_{\mathrm{M}} \to \{\mathrm{T}, \mathrm{F}\}$ takes as input two kinematically allowed states $X_1, X_2 \in S^{\text{kin}}_{\mathrm{M}}$ and a parameter $\tau \in T_{\mathrm{M}}$ and reports whether or not the two points $X_1(\tau)$ and $X_2(\tau) \in \mathcal{M}_{\mathrm{M}}$ are on the same fiber. That is, it reports whether or not the two states, X_1 and X_2, are "parallel" in the above-defined sense at parameter τ.
- $D_{\mathrm{M}} : P^{-1}_{\mathrm{M}}(\mathrm{T}) \to \mathbb{R}_{\geq 0}$ takes as input two states $X_1, X_2 \in S^{\text{kin}}_{\mathrm{M}}$ and a parameter $\tau \in T_{\mathrm{M}}$ which satisfy $P_{\mathrm{M}}(X_1, X_2, \tau) = \mathrm{T}$. That is, X_1 and X_2 are on the same fiber at parameter τ. The function D_{M} then reports the along-their-common-fiber distance between $X_1(\tau)$ and $X_2(\tau)$,
- $A_{\mathrm{M}} : P^{-1}_{\mathrm{M}}(\mathrm{F}) \to [0, 90°]$ takes as input two states $X_1, X_2 \in S^{\text{kin}}_{\mathrm{M}}$ and a parameter $\tau \in T_{\mathrm{M}}$ which satisfy $P_{\mathrm{M}}(X_1, X_2, \tau) = \mathrm{F}$. That is, X_1 and X_2 are on different fibers at parameter τ. The function A_{M} then reports the angular separation between $X_1(\tau)$ fibers and $X_2(\tau)$ fiber.[55]

[54] This definition will be revised and made more formal at the end of Sec. 4.2,

[55] Recall that we have defined the period of $\mathbb{M}$'s base space to be $x = \pi$ rather than 2π. Hence, the largest possible angular separation is $\Delta x_{\text{max}} = \pi/2 = 90°$.

These three functions, however, are not quite enough to allow us to define the dynamics of the Möbius theory. Using just these functions, one cannot specify that a particle which sits on the Möbius strip's mid-line (i.e., the blue dashed section in Fig. 4.1) should remain stationary as we advance $\tau \in T_{\mathrm{M}}$.

From Eq. (4.2), the particle's future motion can be determined given its current angular velocity, its linear velocity, and its instantaneous distance from one of these special stationary states. We can already determine the state's angular and linear velocity using A_{M} and D_{M}. Hence, the final piece of structure which one needs in order to define this theory's dynamics is a sense of distance between a generic state and one of these special stationary states. The D_{M} function already gives us the relevant notion of distance between these states. Hence, all that we need to add to our theory is a list of these special stationary states, $F_{\mathrm{M}} \subset S^{\mathrm{kin}}_{\mathrm{M}}$, which will remain fixed under the theory's dynamics. In sum, we have:

$$\mathcal{S}^{\mathrm{kin}}_{\mathrm{M}} \coloneqq \langle S^{\mathrm{kin}}_{\mathrm{M}}, T_{\mathrm{M}}, P_{\mathrm{M}}, D_{\mathrm{M}}, A_{\mathrm{M}}, F_{\mathrm{M}} \rangle. \tag{4.8}$$

By equipping $S^{\mathrm{kin}}_{\mathrm{M}}$ with the above-discussed structures one can pick out the Möbius theory's dynamically allowed states, $S^{\mathrm{dyn}}_{\mathrm{M}} \subset S^{\mathrm{kin}}_{\mathrm{M}}$.

Let us now check that we can pick out the dynamically allowed states of the Euclid theory by defining analogous structures on its set of kinematically allowed states,

$$\mathcal{S}^{\mathrm{kin}}_{\mathrm{E}} \coloneqq \langle S^{\mathrm{kin}}_{\mathrm{E}}, T_{\mathrm{E}}, P_{\mathrm{E}}, D_{\mathrm{E}}, A_{\mathrm{E}}, F_{\mathrm{E}} \rangle. \tag{4.9}$$

Recall that $S^{\mathrm{kin}}_{\mathrm{E}}$ is the set of all smooth $\mathbb{R}^2$-parameterized trajectories, $X_{\mathrm{E}} : \mathcal{V}_{\mathrm{E}} \to \mathcal{M}_{\mathrm{E}}$, whose time-slices are all straight lines in the Euclidean plane, $\mathcal{M}_{\mathrm{E}} \cong \mathbb{R}^2$. As in the previous theory, we have an auxiliary space, $T_{\mathrm{E}} \cong \mathbb{R}$, for marking off times. Note that unlike before we have $T_{\mathrm{E}} \ncong \mathcal{V}_{\mathrm{E}}$. Despite this we can still use T_{E} to define the following three functions:

- $P_{\mathrm{E}} : S^{\mathrm{kin}}_{\mathrm{E}} \times S^{\mathrm{kin}}_{\mathrm{E}} \times T_{\mathrm{E}} \to \{\mathrm{T}, \mathrm{F}\}$ takes as input two kinematically allowed states $X_1, X_2 \in S^{\mathrm{kin}}_{\mathrm{E}}$ and a parameter $\tau \in T_{\mathrm{E}}$ and reports whether or not the two s-parameterized lines $X_1(\tau, s)$ and $X_2(\tau, s)$ are parallel.
- $D_{\mathrm{E}} : P^{-1}_{\mathrm{E}}(\mathrm{T}) \to \mathbb{R}_{\geq 0}$ takes as input two states $X_1, X_2 \in S^{\mathrm{kin}}_{\mathrm{E}}$ and a parameter $\tau \in T_{\mathrm{E}}$ which satisfy $P_{\mathrm{E}}(X_1, X_2, \tau) = \mathrm{T}$. That is, the two s-parameterized lines, $X_1(\tau, s)$ and $X_2(\tau, s)$, are parallel. The function D_{E} then reports their linear separation.
- $A_{\mathrm{E}} : P^{-1}_{\mathrm{E}}(\mathrm{F}) \to [0, 90°]$ takes as input two states $X_1, X_2 \in S^{\mathrm{kin}}_{\mathrm{E}}$ and a parameter $\tau \in T_{\mathrm{E}}$ which satisfy $P_{\mathrm{E}}(X_1, X_2, \tau) = \mathrm{F}$. That is, the two s-parameterized lines, $X_1(\tau, s)$ and $X_2(\tau, s)$, are not parallel. The function A_{E} then reports their acute angle of intersection.

Finally, $F_{\mathrm{E}} \subset S^{\mathrm{kin}}_{\mathrm{E}}$ contains all of the kinematically allowed trajectories which pass through the blue point at every time-slice. We can write down this theory's dynamical equation in terms of the particle's instantaneous velocity (determined from A_{E} and D_{E}), and its instantaneous distance from one of these F_{E} states (determined from D_{E}).

Having formalized both of these spacetime theories, we are now in a position to see the sense in which they are formally equivalent: Namely, they have exactly the same kinematic and dynamical contents. Filling in the ellipses from Eq. (4.7) we have:

$$\begin{aligned} &\text{Kinematic Isomorphism:} \quad \langle S^{\mathrm{kin}}_{\mathrm{M}}, T_{\mathrm{M}}, P_{\mathrm{M}}, D_{\mathrm{M}}, A_{\mathrm{M}}, F_{\mathrm{M}} \rangle \cong \langle S^{\mathrm{kin}}_{\mathrm{E}}, T_{\mathrm{M}}, P_{\mathrm{E}}, D_{\mathrm{E}}, A_{\mathrm{E}}, F_{\mathrm{M}} \rangle \\ &\text{which preserves dynamics} \qquad\qquad S^{\mathrm{dyn}}_{\mathrm{M}} \leftrightarrow S^{\mathrm{dyn}}_{\mathrm{E}} \end{aligned} \tag{4.10}$$

As one can check, these two theories are isomorphic at the level of kinematics and that this isomorphism preserves the theories' dynamics. That is, there exists a bijection between $S^{\mathrm{kin}}_{\mathrm{M}}$ and $S^{\mathrm{kin}}_{\mathrm{E}}$ (and trivially one between the auxiliary spaces, $T_{\mathrm{M}} \cong \mathbb{R}$ and $T_{\mathrm{E}} \cong \mathbb{R}$) which preserves all judgements of parallel-ness, distances, and angles. Moreover, this bijection also maps the theories' special stationary states onto each other as $F_{\mathrm{M}} \leftrightarrow F_{\mathrm{E}}$. It is in this sense that these two theories are isomorphic at the level of kinematics, $\mathcal{S}^{\mathrm{kin}}_{\mathrm{M}} \cong \mathcal{S}^{\mathrm{kin}}_{\mathrm{E}}$. Furthermore, this bijection maps the theories' dynamically allowed states onto each other as $S^{\mathrm{dyn}}_{\mathrm{M}} \leftrightarrow S^{\mathrm{dyn}}_{\mathrm{E}}$. It is in this sense that the above-discussed isomorphism also preserves the theories' dynamics.

4.2 More Topological Redescriptions (a.k.a., Inequivalent Spacetime Representations)

This section will be spent working our way towards a definition of the terms "spacetime theories" and "pre-spacetime theories" with respect to which we can say that the Möbius and Euclid theories are different spacetime representations of the same pre-spacetime theoretic structure. To help demonstrate the breadth of the coming definitions, allow me to first introduce several more examples of topological redescription which I will discuss throughout this thesis. As with the Möbius-Euclid duality, the following pairs of theories will have the same kinematic and dynamical contents, $\mathcal{S}^{\text{kin}}_{\text{new}} \cong \mathcal{S}^{\text{kin}}_{\text{old}}$ with $S^{\text{dyn}}_{\text{new}} \leftrightarrow S^{\text{dyn}}_{\text{old}}$.

To begin, let us consider the following spacetime theory about a modified quantum harmonic oscillator (QHO):

> **Non-Local QHO -** Consider a theory set on a spacetime manifold, $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2$, about a complex-valued scalar field, $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to \mathcal{V}_{\text{old}}$, with a value space of $\mathcal{V}_{\text{old}} \cong \mathbb{C}$. This theory's states are subject to the following kinematic constraint: In some fixed global coordinate system, (t, x), at each fixed t-coordinate the state must be within the usual rigged Hilbert space of planewaves and Dirac deltas. At each t-coordinate, the metaphysically relevant way to judge the relative size and similarity of this theory's states is with an L^2 inner product. In this coordinate system, the dynamical equation for the field is as follows:
>
> $$\mathrm{i}\partial_t\varphi_{\text{old}}(t,x) = (-\partial_x^2 + x^2)\varphi_{\text{old}}(t,x) + \frac{\lambda}{2}\varphi_{\text{old}}(t,x-a) + \frac{\lambda}{2}\varphi_{\text{old}}(t,x+a), \tag{4.11}$$
>
> for some fixed $a > 0$ and $\lambda \in \mathbb{R}$.

It should be noted that by using the identity, $h(x + a) = \exp(a\partial_x)h(x)$, this theory's dynamical equation can be rewritten as:

$$\mathrm{i}\partial_t\varphi_{\text{old}}(t,x) = (-\partial_x^2 + x^2 + \lambda\cosh(a\partial_x))\varphi_{\text{old}}(t,x). \tag{4.12}$$

We can recognize this dynamical equation as the Schrödinger equation with a (rather

bizarre) Hamiltonian of $H_{\text{old}} = -\partial_x^2 + x^2 + \lambda\cosh(a\partial_x)$. This theory describes the dynamics of a standard QHO plus a non-local self-coupling.

To complete our second example of topological redescription, consider also the following closely related spacetime theory about a differently modified quantum harmonic oscillator (QHO):

> **Cosine QHO -** Consider a theory set on a spacetime manifold, $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$, about a complex-valued scalar field, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathcal{V}_{\text{new}}$, with a value space, $\mathcal{V}_{\text{new}} \cong \mathbb{C}$. The theory's states are subject to the following kinematic constraint: In some fixed global coordinate system, (τ, q), at each fixed τ-coordinated the state must be within the usual rigged Hilbert space of planewaves and Dirac deltas. At each τ-coordinate, the metaphysically relevant way to judge the relative size and similarity of this theory's states is with an L^2 inner product. In this coordinate system, the dynamical equation for the field is as follows:
>
> $$\mathrm{i}\partial_\tau \varphi_{\text{new}}(\tau, q) = (-\partial_q^2 + q^2 + \lambda\cos(a\,q))\,\varphi_{\text{new}}(\tau, q), \tag{4.13}$$
>
> for some $a > 0$ and $\lambda \in \mathbb{R}$.

We can again recognize this as the Schrödinger equation but this time with a Hamiltonian of $H_{\text{new}} = -\partial_q^2 + q^2 + \lambda\cos(a\,q)$. This Hamiltonian describes the dynamics of a standard QHO plus an extra cosine-shaped potential.

The keen reader may have noticed that the above two theories are related to each other via a Fourier transform.[56] Indeed, they are an instance of the position-momentum duality of non-relativistic quantum mechanics. One might complain that this example shouldn't count as an instance of topological redescription since these QHO theories ultimately have isomorphic spacetime manifolds, $\mathcal{M}_{\text{old}} \cong \mathcal{M}_{\text{new}} \cong \mathbb{R}^2$. Importantly, however, the Fourier transform which relates these two theories does not respect this isomorphism (more on this momentarily); Indeed, it radically reshapes how the theory's states sit on their respective spacetime manifolds. The old theory's planewaves become Dirac deltas in the new theory and vice versa. More generally, what was non-local becomes local and vice-versa. This is evidenced by the fact that the old theory's non-local dynamics has

[56] In particular they are related by the replacements $x \mapsto \mathrm{i}\partial_q$ and $\partial_x \mapsto \mathrm{i}q$ noting that $\cosh(\mathrm{i}az) = \cos(az)$.

become local dynamics in the new theory. As this example of topological redescription shows, it is possible to dramatically change the topological staging of a theory's states while preserving the topological character of the respective stages, $\mathcal{M}_{\text{old}} \cong \mathcal{M}_{\text{new}}$.

Despite the stark differences which these two QHO theories have in the topological staging of their states and dynamics, there is nonetheless a robust sense in which these theories are equivalent. Namely, as I will now discuss, we can see them as being kinematically and dynamically equivalent *in the exact same sense* as the Möbius and Euclid theories are. Namely, these two theories are isomorphic at the level of kinematics, and this isomorphism preserves the theories' dynamics. To see this explicitly, we will need to first identify the modal structure of the above two theories (i.e., their DPMs and KPMs, $S^{\text{dyn}}_{\text{old}} \subset S^{\text{kin}}_{\text{old}}$ and $S^{\text{dyn}}_{\text{new}} \subset S^{\text{kin}}_{\text{new}}$). We will then need to equip $\mathcal{S}^{\text{kin}}_{\text{old}} = \langle S^{\text{kin}}_{\text{old}}, \dots \rangle$ and $\mathcal{S}^{\text{kin}}_{\text{new}} = \langle S^{\text{kin}}_{\text{new}}, \dots \rangle$ with at least enough structure so that we can identify within them their dynamically allowed states, $S^{\text{dyn}}_{\text{old}} \subset S^{\text{kin}}_{\text{old}}$ and $S^{\text{dyn}}_{\text{new}} \subset S^{\text{kin}}_{\text{new}}$. For notational convenience, I will now drop the new/old subscripts whenever either could apply.

Regarding their modal structure, in the broadest sense, the two QHO theories are both about the set, S^{all}, of all $\mathbb{C}$-valued functions definable on $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2$ and $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$ respectively (even those which are non-smooth and/or discontinuous). Enforcing either theory's kinematic constraints, we can identify within S^{all} a set of kinematically allowed states, $S^{\text{kin}} \subset S^{\text{all}}$. For both QHO theories, this is the set of all $\mathbb{C}$-valued functions within "the usual rigged Hilbert space of planewaves and Dirac deltas". Within this set, we can then enforce the theory's dynamics to find its dynamically allowed states, $S^{\text{dyn}} \subset S^{\text{kin}} \subset S^{\text{all}}$.

Let us now identify what kinematic structure is required to state the dynamical equations of the two QHO theories. Given that their dynamical equations are both linear, we are led to equip S^{kin} with at least the structure of a vector space. Namely, we can give it an addition operation, $+ : S^{\text{kin}} \times S^{\text{kin}} \to S^{\text{kin}}$, and a scalar multiplication operation,

$\cdot : \mathbb{C} \times S^{\text{kin}} \to S^{\text{kin}}$. If the dynamics of these theories had been non-linear, then we would have additionally required a product operation, $\times : S^{\text{kin}} \times S^{\text{kin}} \to S^{\text{kin}}$. Using these structures we can state the theory's dynamical equations and define its dynamically allowed states, $S^{\text{dyn}} \subset S^{\text{kin}}$.

It is important to note that a theory's kinematic structures might go beyond the bare minimum which are required to state the theory's dynamics. Note the significant role that Hilbert space structure plays in interpreting non-relativistic quantum mechanics despite the fact that no inner products appear in the Schrödinger equation. Moreover, such an inner product might nonetheless be crucial in grounding whatever metaphysical story one wants to tell about this theory. For these reasons, the two QHO theories also come equipped with an inner product which has been flagged as metaphysically relevant. To define this inner product let us introduce an auxiliary space, $T_{\text{old}} \cong T_{\text{new}} \cong \mathbb{R}$, which parameterizes time for each theory. Using this space we can define at each parameter, $s \in T$, an inner product,

$$I(\varphi, \phi, s) \coloneqq \langle \varphi, \phi \rangle(s) \coloneqq \int_{-\infty}^{\infty} \varphi(s, y)^* \phi(s, y) \, \mathrm{d}y. \tag{4.14}$$

Namely, for either the new or old theory we have a function $I : S^{\text{kin}} \times S^{\text{kin}} \times T \to \mathbb{C}$ for judging the relative size and similarity of any two kinematically allowed states, $\varphi, \phi \in S^{\text{kin}}$ at a parameter, $s \in T$.

In total, for the two QHO theories we have identified the following kinematic structures as being dynamically and/or metaphysically relevant,

$$\mathcal{S}^{\text{kin}}_{\text{old}} \coloneqq \langle S^{\text{kin}}_{\text{old}}, +_{\text{old}}, \cdot_{\text{old}}, T_{\text{old}}, I_{\text{old}} \rangle, \tag{4.15}$$

$$\mathcal{S}^{\text{kin}}_{\text{new}} \coloneqq \langle S^{\text{kin}}_{\text{new}}, +_{\text{new}}, \cdot_{\text{new}}, T_{\text{new}}, I_{\text{new}} \rangle. \tag{4.16}$$

That is, at a kinematic level, both of these theories have the structure of a Hilbert space (at each parameter $s \in T$). For both theories, these structures are sufficient to define the

theory's dynamically allowed states, $S^{\text{dyn}} \subset S^{\text{kin}}$. As with the Möbius and Euclid theories, there is a clear sense in which these two QHO theories are kinematically and dynamically equivalent.

$$\text{Kinematic Isomorphism:} \quad \langle S^{\text{kin}}_{\text{old}}, +_{\text{old}}, \cdot_{\text{old}}, T_{\text{old}}, I_{\text{old}} \rangle \cong \langle S^{\text{kin}}_{\text{new}}, +_{\text{new}}, \cdot_{\text{new}}, T_{\text{new}}, I_{\text{new}} \rangle$$

$$\text{which preserves dynamics} \qquad S^{\text{dyn}}_{\text{old}} \leftrightarrow S^{\text{dyn}}_{\text{new}} \tag{4.17}$$

Namely, despite their differences in topological staging, we nonetheless have $\mathcal{S}^{\text{kin}}_{\text{old}} \cong \mathcal{S}^{\text{kin}}_{\text{new}}$ with $S^{\text{dyn}}_{\text{old}} \leftrightarrow S^{\text{dyn}}_{\text{new}}$. Hence, these two QHO theories qualify as our second example of topological redescription.

But what is it exactly which these two QHO theories have in common which is being topologically redescribed? The obvious (and correct) answer is that they are two representations of the $\langle S, +, \cdot, T, I \rangle$-structure which we have just identified. One might, however, try to go beyond this by pointing out that there is a third spatiotemporal description of the above-identified structure,

$$\mathcal{S}_{\text{Hil}} \coloneqq \langle S^{\text{kin}}_{\text{Hil}}, +_{\text{Hil}}, \cdot_{\text{Hil}}, T_{\text{Hil}}, I_{\text{Hil}} \rangle \tag{4.18}$$

where $S^{\text{kin}}_{\text{Hil}} = \{|\psi\rangle : \mathcal{M}_{\text{Hil}} \to \mathcal{H}_{\text{Hil}}\}$ is the set of all smooth functions from a manifold of times, $\mathcal{M}_{\text{Hil}} \cong \mathbb{R}$, to $\mathcal{H}_{\text{Hil}}$, "the usual rigged Hilbert space of planewaves and Dirac deltas". The operations $+_{\text{Hil}}$, $\cdot_{\text{Hil}}$, and I_{Hil} are then the Hilbert space structure of $\mathcal{H}_{\text{Hil}}$ and $T_{\text{Hil}} = \mathcal{M}_{\text{Hil}}$ just is the manifold of times at which the states are defined. We hence have three topological redescriptions of this Hilbert space dynamics.

Who knows what other topological redescriptions of this theory might be waiting around the corner? Indeed, two other possibilities are worth mentioning. Suppose that QHO theory had instead been about multiple particles (say n of them) in $3 + 1$ dimensions. Then it would also be possible to represent this theory's dynamics by staging it on the configuration space of these particles, $\mathcal{M}_{\text{Conf}} = \mathbb{R}^{1+3n}$, as being about a field

$\varphi_{\text{Conf}}(t, x_1, x_2, x_3, \ldots, z_1, z_2, z_3)$. Alternatively, it could be staged on the usual spacetime, $\mathcal{M}_{3+1} = \mathbb{R} \times \mathbb{R}^3$, as being about a multi-field, $\varphi_{3+1}(t; x_1, x_2, x_3; \ldots; z_1, z_2, z_3)$, which is valued over one time, $t \in \mathbb{R}$, and n points $(y_1, y_2, y_3) \in \mathbb{R}^3$. There is a well-established debate between these two perspectives on quantum mechanics.[57]

Given all of these possible topological representations of this theory, one might feel some pressure to pick one of them as being the canonical/best/right representation. For the sake of concreteness let us suppose everyone agrees that the best representation of this theory turns out to be $\mathcal{S}_{\text{Hil}}$. One may be tempted to understand all of the other representations (e.g., $\mathcal{S}_{\text{old}}$, $\mathcal{S}_{\text{new}}$, etc.) as variants of this canonical representation. Namely, one might be tempted say that $\mathcal{S}_{\text{old}}$ and $\mathcal{S}_{\text{new}}$ are just two different representations of $\mathcal{S}_{\text{H}}$ using the $|x\rangle$ and $|q\rangle$ bases for $\mathcal{H}_{\text{Hil}}$ respectively. I would like to resist this analysis for two reasons. Firstly, it would be terminologically confusing to say that $\mathcal{S}_{\text{old}}$ and $\mathcal{S}_{\text{new}}$ are different representations of $\mathcal{S}_{\text{Hil}}$, given that this too is a representation of the above-defined $\langle S, +, \cdot, T, I\rangle$-structure. Even if the $\mathcal{S}_{\text{Hil}}$ representation of this theory is somehow objectively best, it is nonetheless a representation of $\langle S, +, \cdot, T, I\rangle$ in just the same way that $\mathcal{S}_{\text{old}}$ and $\mathcal{S}_{\text{new}}$ are.

Secondly, while there is nothing technically wrong with arriving at the $\mathcal{S}_{\text{old}}$ and $\mathcal{S}_{\text{new}}$ representations by picking a basis for $\mathcal{H}_{\text{Hil}}$ in the $\mathcal{S}_{\text{Hil}}$ representation, in my view this is an unnecessarily algebra-heavy way of understanding the relationship between these three spacetime theories. Importantly, it does not follow from the fact that these theories are algebraically structured that the redescription techniques which we use to move between them need to invoke this algebraic structure. Indeed, if this were true then it would not be possible to develop topological redescription techniques which simultaneously cover the above-discussed Fourier-duality as well as the notably non-algebraic Möbius-Euclid duality. To the contrary, however, the ISE Method will do exactly this. In order to see that

[57] For an overview of this debate, see Ney (2021) and references therein.

the QHO theories are topological redescriptions of each other one just needs to note that they are structurally isomorphic. From the perspective of the ISE Method, the fact that their common structure is algebraic in nature is incidental.

Before formalizing the sense in which cases of topological redescription amount to different spacetime representations of some underlying pre-spacetime theoretic structure, allow me to first introduce a few more examples of topological redescription (more briefly). These examples will be discussed in further detail throughout this thesis. The two examples of topological redescription explored in Ch. 8 both start with a theory about a $\mathbb{R}$-valued scalar field, $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to \mathbb{R}$ on some spacetime manifold, $\mathcal{M}_{\text{old}}$. In each case, I will topologically redescribe the theory in question as being about some other field, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathcal{V}_{\text{new}}$, valued elsewhere, $\mathcal{M}_{\text{new}}$, and potentially valued differently, $\mathcal{V}_{\text{new}}$. Concretely, in Sec. 8.2 I will begin from a lattice theory (i.e., a theory set on a discrete spacetime, $\mathcal{M}_{\text{old}} \cong \mathbb{R} \times \mathbb{Z}$) and end up on a continuous spacetime manifold, $\mathcal{M}_{\text{new}} \cong \mathbb{R} \times \mathbb{R}$.[58] By contrast, in Sec. 8.1 I will begin from a theory with some compactified dimensions, $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2 \times S^2$, and end up on lower dimensional spacetime, $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$. The cost, however, is that our old $\mathbb{R}$-valued field, $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to \mathbb{R}$, will become an $\mathbb{R}^\infty$-valued field, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathbb{R}^\infty$, whose components, $\varphi_{\text{new}}^{(\ell m)} : \mathcal{M}_{\text{new}} \to \mathbb{R}$, correspond to the harmonic mode of the old theory's compactified dimensions.

As all of these examples of topological redescription demonstrate, the types of kinematic structures, $\mathcal{S}^{\text{kin}} := \langle S^{\text{kin}}, \dots \rangle$, which we associate with our theories can vary significantly from case to case. For the Möbius and Euclid theories discussed above, their kinematic structures had a notable geometric flavor to them ("parallel", "distances", "angles", etc.). By contrast, for the two QHO theories these structures had a more algebraic

[58] It was through attempting to generalize an example of this kind that I discovered the ISE Method. This inspiring example was itself inspired by the work of the brilliant mathematical physicist Kempf (2010) entitled, "Spacetime could be simultaneously continuous and discrete, in the same way that information can be". For an overview of Kempf's work on this topic, see Kempf (2018). In fact, it was Kempf's work which first motivated me to look into the philosophical implications of topological redescription.

flavor to them ("addition", "scalar multiplication", "inner products", etc.). In general, the theory's kinematically allowed states, $\mathcal{S}^{\text{kin}}$, are allowed to be equipped with effectively any kind of structures one wishes. The only requirement placed upon these structures is that they contain all of the resources needed to state the theory's dynamics and to make sense of its metaphysics. However they are structured kinematically, our examples of topological redescription each realize this structure via very different topological stagings of the theory's states

Without further ado, let us now begin defining the terms "pre-spacetime theory" and "spacetime theory" with enough generality to capture all of the above examples. Let us first try to capture the sense in which the Möbius and Euclid theories (as well as our other examples) are nonetheless inequivalent, despite their similarities. Indeed, as I will now define they are inequivalent *as spacetime theories*.

> **Definition:** *Spacetime Theory* - In order to qualify as a spacetime theory, the theory under consideration must satisfy the following four assumptions:
>
> 1. *Modal Assumption:* The theory must distinguish between dynamically allowed states and kinematically allowed states as $S^{\text{dyn}} \subset S^{\text{kin}}$. That is, it must have DPMs and KPMs.
> 2. *Kinematic Structure Assumption:* At the level of kinematics, the theory must come equipped with all of its dynamically and metaphysically relevant structures, $\mathcal{S}^{\text{kin}} = \langle S^{\text{kin}}, \dots \rangle$. These structures must allow us to make sense of its metaphysics and to identify the theory's dynamically allowed states as S^{dyn} as a subset of S^{kin}.
> 3. *Declared Topological Staging Assumption:*[59] The theory must be formalized with a declared topological staging for its states. It must specify a specific topological stage for its states, Σ, and declare that it is about a certain kind of states, $S^{\text{all}} \supset S^{\text{kin}} \supset S^{\text{dyn}}$.
> 4. *Lifting Assumption:* There must be a canonical way to lift Σ's automorphisms, $d : \Sigma \to \Sigma$, to act on the theory's states, $d^* : S^{\text{all}} \to S^{\text{all}}$. More generally, we might have two copies of Σ and thereby two copies of S^{all}. We should then be able to lift any isomorphism $d : \Sigma_{\text{old}} \to \Sigma_{\text{new}}$, to act as a bijection on the theories' states as $d^* : S^{\text{all}}_{\text{old}} \to S^{\text{all}}_{\text{new}}$. Moreover, this $*_{\text{lift}} : d \mapsto d^*$ map must be a group homomorphism, i.e., we must have $d_1^* \, d_2^* = (d_1 \, d_2)^*$.

[59] It should be noted that — per this definition — general relativity (GR) is not a spacetime theory; Instead one can think of it as a schema for spacetime theories. For instance, GR with $\mathcal{M} \cong \mathbb{R}^4$ is a spacetime theory. So too is GR with $\mathcal{M} \cong \mathbb{R}^2 \times S^2$.

Regarding the declared topological staging assumption, we have a wide range of possibilities available to us. For instance, if the theory declares itself to be set on a smooth manifold, $\Sigma = \mathcal{M}$, then it could be about a particle trajectory, $X : \mathcal{V} \to \mathcal{M}$, or a scalar field, $\varphi : \mathcal{M} \to \mathcal{V}$, or a tangent vector field, $\vec{\varphi} : \mathcal{M} \to \mathrm{T}\mathcal{M}$. Alternatively, the theory might be about the global sections, $\varphi : B \to E$, of some fiber bundle, $\Sigma = \langle E, B, F, \pi : E \to B \rangle$. More exotically, the theory could be set on a supermanifold (à la supersymmetry) or on a topological space (e.g., a non-Hausdorff manifold).

As I will discuss momentarily, the lifting assumption is necessary to define a sense of equivalence between spacetime theories. Before seeing this, however, let me quickly confirm that it satisfied for a wide range of theories. For theories set on a smooth manifold $\Sigma = \mathcal{M}$ about a scalar field, $\varphi : \mathcal{M} \to \mathcal{V}$, we have $d^*\varphi \coloneqq \varphi \circ d^{-1}$. Alternatively, for theories about a particle trajectories, $X : \mathcal{V} \to \mathcal{M}$, we can simply take $d^*X \coloneqq d \circ X$. Our theory could even be about a tangent vector field, $\vec{\varphi} : \mathcal{M} \to \mathrm{T}\mathcal{M}$, in which case we would have that $d \in \mathrm{Diff}(\mathcal{M})$ lifts to act on $\vec{\varphi}$ as $d^*\vec{\varphi}$ via the push-forward.

Part II of this thesis will be focused on spacetime theories which are set on smooth manifolds, $\Sigma = \mathcal{M}$. In this context, the lifting assumption requires that our the theory under consideration be "natural" in the sense of Fatibene and Francaviglia (2003). It is important to note, however, that the lifting assumption is met by generic theories about sections $\varphi : B \to E$ of a fiber bundle $\Sigma = \langle E, B, F, \pi : E \to B \rangle$. The relevant automorphisms $d \in \mathrm{Auto}(E) \subset \mathrm{Diff}(E)$ are those diffeomorphisms which map fibers to fibers. Note that these are exactly the diffeomorphisms on E which have a well-defined image on the base space, $\pi[d] : B \to B$. Using this $\pi[d]$ map, we can lift any $d \in \mathrm{Auto}(E)$ to act on any global section, $\varphi : B \to E$, as $d^*\varphi \coloneqq d \circ \varphi \circ \pi[d^{-1}]$.

Given the lifting assumption, we can make the following definition: Two spacetime theories are equivalent if there is an isomorphism between their stages $d : \Sigma_{\text{old}} \to \Sigma_{\text{new}}$ which lifts to an isomorphism at the level of kinematics, $d^*|_{\text{kin}} : \mathcal{S}^{\text{kin}}_{\text{old}} \leftrightarrow \mathcal{S}^{\text{kin}}_{\text{new}}$, which

preserves the theories' dynamics, $d^*|_{\text{dyn}} : S^{\text{dyn}}_{\text{old}} \leftrightarrow S^{\text{dyn}}_{\text{new}}$. Hence, the Möbius and Euclid theories are inequivalent as spacetime theories because their stages are non-isomorphic, $\mathcal{M}_{\text{old}} \ncong \mathcal{M}_{\text{new}}$. For the two QHO theories their stages are isomorphic, $\mathcal{M}_{\text{old}} \cong \mathcal{M}_{\text{new}}$, but they are nonetheless inequivalent since their kinematic isomorphism is not the lift of any diffeomorphism relating $\mathcal{M}_{\text{old}}$ to $\mathcal{M}_{\text{new}}$.

There is also a sense in which these pairs of theories are nonetheless equivalent despite their differences. Indeed, as I will now define they are equivalent *as pre-spacetime theories*.

> **Definition:** *Pre-Spacetime Theory* - In order to qualify as a pre-spacetime theory, the theory under consideration must satisfy the following three assumptions:
>
> 1. *Modal Assumption:* Same as above.
> 2. *Kinematic Structure Assumption:* Same as above.
> 3. *No Declared Topological Staging Assumption:* The theory must be formalized without a declared topological staging for its states. That is $\mathcal{S}^{\text{kin}} = \langle S^{\text{kin}}, \dots \rangle$ must be formalized with S^{kin} being a set of ur-elements with no internal structure and *not as a set of maps* e.g., $\{X : \mathcal{V} \to \mathcal{M}\}$ or $\{\varphi : \mathcal{M} \to \mathcal{V}\}$ or $\{\vec{\varphi} : \mathcal{M} \to \mathrm{T}\mathcal{M}\}$ or $\{\varphi : B \to E\}$.

The two QHO theories are equivalent as pre-spacetime theories (because they have $\mathcal{S}^{\text{kin}}_{\text{old}} \cong \mathcal{S}^{\text{kin}}_{\text{new}}$ with $S^{\text{dyn}}_{\text{old}} \leftrightarrow S^{\text{dyn}}_{\text{new}}$) despite being inequivalent as spacetime theories. The same is true for the Möbius and Euclid theories. Indeed, it was precisely this fact about the Möbius-Euclid duality which motivated our definition of topological redescriptions at the end of Sec. 4.1. Hence, we can now rephrase this definition as follows:

> **Re-Definition:** Two spacetime theories are *topological redescriptions* of each other whenever they are equivalent as pre-spacetime theories but inequivalent as spacetime theories.

The sensibility of this definition turns upon our ability to alternatively conceive of a given theory as either a spacetime theory or as a pre-spacetime theory. To demonstrate how this is possible, a brief comparison with the representation theory of groups will be helpful. Recall that group representation theory is about exploring the different ways in

which various group structures (e.g., $G = \mathrm{SO}(3)$) can be realized as sets of linear maps, $\{g : V \to V\}$, on various vector spaces, V. For instance, we might have

$$G_5 = \{g : V_5 \to V_5\} \quad \text{or alternatively} \quad G_7 = \{g : V_7 \to V_7\} \tag{4.19}$$

where $\dim(V_5) = 5$ and $\dim(V_7) = 7$. If we think of G_5 and G_7 as group representations (treating $g : V \to V$ as a map from V to itself) then we have $G_5 \ncong G_7$ because we have $V_5 \ncong V_7$. We can, however, still treat G_5 and G_7 as groups, by simply ignoring the internal structure of their elements, (i.e., by treating g as an ur-element). Doing so we would find that we have $G_5 \cong G_7$ as groups. In sum, G_5 and G_7 are equivalent as groups but inequivalent as group representations.

This motivates us to adopt the same terminology for spacetime theories and pre-spacetime theories. For instance, we can analogously think of the Möbius and Euclid theories as being different spacetime representations of the same pre-spacetime theory. The Möbius theory realized their common pre-spacetime theoretic structure as maps, $S^{\text{kin}}_{\text{M}} = \{X_{\text{M}} : \mathbb{R} \to \mathbb{M}\}$. By contrast, the Euclid theory realizes this same pre-spacetime theoretic structure as a different kind of maps, $S^{\text{kin}}_{\text{E}} = \{X_{\text{E}} : \mathbb{R}^2 \to \mathbb{R}^2\}$. More generally, we can think of a *spacetime representation theory* which explores the various ways in which various pre-spacetime theoretic structures, $\mathcal{S}^{\text{kin}} = \langle S^{\text{kin}}, \dots \rangle$, can be realized as various kinds of states on various kinds of topological stages.

Having sketched an outline of spacetime representation theory, we can now restate the definition of topological redescription as follows:

> **Re-Definition:** Two spacetime theories are *topological redescriptions* of each other whenever they are inequivalent spacetime representations of the same pre-spacetime theory.

To make this more concrete, suppose that we begin with a spacetime theory (e.g., about $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to \mathbb{C}$) with some kinematic/dynamical content, namely $\mathcal{S}^{\text{kin}}_{\text{old}} = \langle S^{\text{kin}}_{\text{old}}, \dots \rangle$ and $S^{\text{dyn}}_{\text{old}} \subset S^{\text{kin}}_{\text{old}}$. Thinking of this as a pre-spacetime theory, we can then explore different

spacetime representations of this theory, $\mathcal{S}^{\text{kin}}_{\text{new}} \cong \mathcal{S}^{\text{kin}}_{\text{old}}$ with $S^{\text{dyn}}_{\text{new}} \leftrightarrow S^{\text{dyn}}_{\text{old}}$. For instance, we might try to represent its states as

- $\mathbb{R}^\infty$-valued scalar field , $S^{\text{kin}}_{\text{new}} = \{\varphi : \mathcal{M}_1 \to \mathbb{R}^\infty\}$, on spacetime $\mathcal{M}_1$ or
- point-like particles, $S^{\text{kin}}_{\text{new}} = \{X : \mathbb{R} \to \mathcal{M}_2\}$, on spacetime $\mathcal{M}_1$ or
- line-shaped particles, $S^{\text{kin}}_{\text{new}} = \{X : \mathbb{R}^2 \to \mathcal{M}_3\}$, on spacetime $\mathcal{M}_2$ or
- tangent vector fields, $S^{\text{kin}}_{\text{new}} = \{\vec{\varphi} : \mathcal{M}_4 \to \mathrm{T}\mathcal{M}_4\}$, on spacetime $\mathcal{M}_4$, or
- global sections, $S^{\text{kin}}_{\text{new}} = \{\vec{\varphi} : B_5 \to E_5\}$, of a fiber bundle $\langle E_5, B_5, F_5, \pi : E_5 \to B_5\rangle$, or
- among many other options.

Motivated by our discussion in the previous chapter, it would be nice to get all of the options on the table, so to speak, no matter how complicated. If we could gain some mathematical control over the range of all possible spacetime representations, then we could begin exploring these possibilities systematically and investigating which kinds of selection criteria narrow in on the "particularly nice" representations. As I will now discuss, the ISE Method allows us to do exactly this.

Before this, however, my approach should be briefly compared with De Haro and Butterfield (2021) who also employ an analogy with group representation theory in relation to what they call "dualities". In particular, they think of dualities as a "giant symmetry" which operates one level higher than we are used to; Namely, whereas symmetries are a certain kind of dynamics-preserving maps between a theory's models, dualities are a certain kind of dynamics-preserving maps *between two theories*. But how do their dualities compare to my topological redescriptions? The chief point of contrast is that whereas their dualities are meant to be symmetry-like, my topological redescriptions are meant to be more like coordinate redescriptions (or nomological redescriptions, or geometric redescriptions). It should be noted that dynamical symmetries (understood as lifted diffeomorphisms) are essentially a subspecies coordinate transformations. It is therefore

conceivable that De Haro and Butterfield's (2021) dualities are a subspecies of my topological redescriptions. Further investigation of this claim, however, is outside of the scope of this thesis.

To stimulate further reflection, I will close here with two points of contrast between symmetries and coordinate redescription. At first glance, both of the following two points seem to be reflected in the difference between dualities and topological redescriptions. Firstly, symmetries (unlike coordinate transformations) are typically required to preserve the *form* of the theory's dynamics. Secondly, a dynamical symmetry might suddenly disappear if one changes the theory's dynamical equations. By contrast, the existence of coordinate redescriptions is unaffected by changes to the theory's dynamics. It should be noted, however, that while a theory's dynamics plays essentially no role in the process of coordinate redescription, it nonetheless plays a significant role in informing our coordinate selection criteria.

4.3 The ISE-Equivalence Theorem (Comparison with Spacetime Algebraicism)

Ideally we would have some strong and widely applicable topological redescription techniques to help us explore the range of all possible spacetime representations. Our above discussion has already divided this space up into pre-spacetime equivalence classes, see the large blue boxes in Fig. 4.2. Two spacetime theories are in the same blue box if and only if their kinematic and dynamical contents are isomorphic, $\mathcal{S}^{\text{kin}}_{\text{new}} \cong \mathcal{S}^{\text{kin}}_{\text{old}}$ with $S^{\text{dyn}}_{\text{new}} \leftrightarrow S^{\text{dyn}}_{\text{old}}$.

The ISE Method applies to the subset of spacetime theories with spacetime-kinematic compatibility (SKC). I will define this constraint in Sec. 9.2 and argue that it is very weak in Sec. 9.3. In Sec. 11.1, I will prove that ISE-relatedness is an equivalence relation over

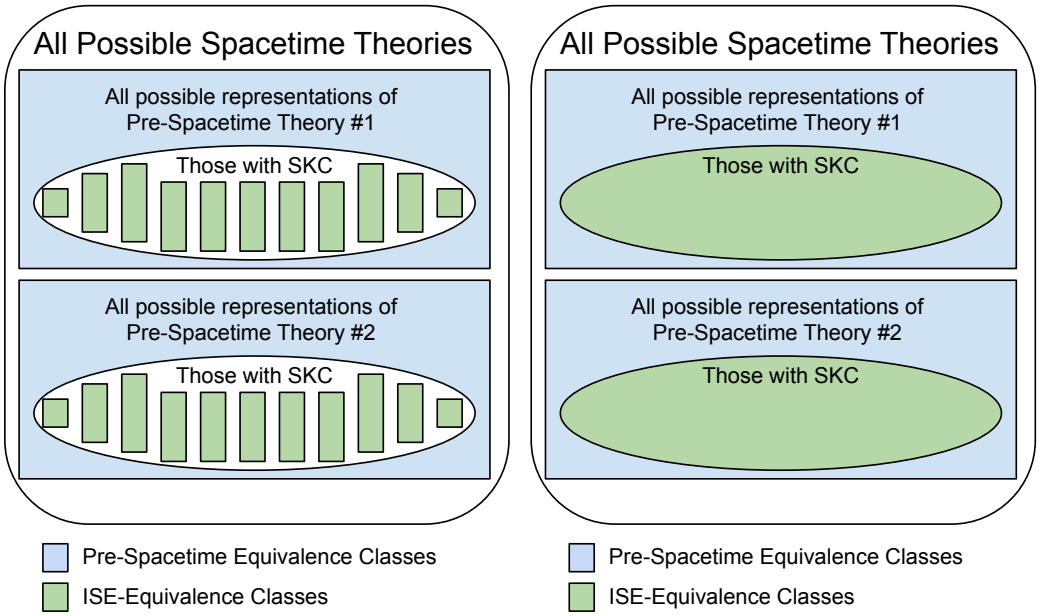

Figure 4.2: Each side of this figure shows all possible spacetime theories being divided up into pre-spacetime equivalence classes (the large blue boxes). Two spacetime theories are in the same blue box if and only if their kinematic and dynamical contents are isomorphic, $\mathcal{S}^{\text{kin}}_{\text{new}} \cong \mathcal{S}^{\text{kin}}_{\text{old}}$ with $S^{\text{dyn}}_{\text{new}} \leftrightarrow S^{\text{dyn}}_{\text{old}}$. We can say that any two such theories are different spacetime representations of the same pre-spacetime theoretic structure. The ISE Method applies to the subset of spacetime theories with spacetime-kinematic compatibility (SKC). I will define this in Sec. 9.2 and I will argue that it is a weak constraint in Sec. 9.3. In Sec. 11.1, I will prove that ISE-interrelatedness divides the theories with SKC up into ISE-Equivalence classes (the small green boxes, left). Two spacetime theories are in the same green box if and only if they can be redescribed as one another using the ISE Method of topological redescription. Moreover, in Sec. 11.2 I will prove that these ISE-equivalence classes are as large as possible (the green oval, right). That is, the ISE Method offers us every possible topological description of a given theory, bounded only by a weak spacetime-kinematic compatibility constraint.

all spacetime theories with spacetime-kinematic compatibility. Hence, it divides these theories up into ISE-Equivalence classes. See the small green boxes on the left half of Fig. 4.2. Two spacetime theories are in the same green box if and only if they can be redescribed as one another using the ISE Method of topological redescription. We want these ISE-Equivalence classes to be as large as possible such that our one topological redescription technique (the ISE Method) can cover a very wide range of topological redescriptions.

In Sec. 11.2 I will prove that these ISE-equivalence classes are as large as they can possibly be. Indeed, the small green boxes on the left half of Fig. 4.2 are misleading. Instead, a theory's ISE-Equivalence class includes all possible topological redescriptions

of that theory which have spacetime-kinematic compatibility. See the green ovals in the right half of Fig. 4.2. Thus, the ISE Method applies to a wide range of spacetime theories and gives us access to effectively every possible topological redescription thereof; It is limited only by a weak spacetime-kinematic compatibility constraint.

To better contextualize the wide scope of topological redescriptions which the ISE Method offers us, allow me to compare it with well-known results from the literature on spacetime algebraicism.[60] Consider a spacetime theory with spacetime-kinematic compatibility about a real-scalar field, $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to \mathbb{R}$, with some kinematic/dynamical content, namely $\mathcal{S}^{\text{kin}}_{\text{old}} = \langle S^{\text{kin}}_{\text{old}}, \dots \rangle$ and $S^{\text{dyn}}_{\text{old}} \subset S^{\text{kin}}_{\text{old}}$. In particular, suppose that its kinematic structure is the C^∞-algebra of smooth functions,

$$\mathcal{S}^{\text{kin}}_{\text{old}} = C^\infty(\mathcal{M}_{\text{old}}) = \langle S^{\text{kin}}_{\text{old}}, +_{\text{old}}, \cdot_{\text{old}}, \times_{\text{old}}, \Phi_{\text{old,f}} \rangle, \tag{4.20}$$

where $S^{\text{kin}}_{\text{old}}$ (abbreviated S) is the set of all smooth functions $c : \mathcal{M}_{\text{old}} \to \mathcal{M}_{\text{old}}$. Moreover, the three operations $+_{\text{old}} : S \times S \to S$, and $\cdot_{\text{old}} : \mathbb{R} \times S \to S$, and $\times_{\text{old}} : S \times S \to S$ are the standard point-wise additions, scalar multiplication, and product operations on $\mathcal{M}_{\text{old}}$. Finally, for every $n \geq 1$ and every smooth function $f : \mathbb{R}^n \to \mathbb{R}$ we have a function $\Phi_{\text{old,f}} : S^n \to S$ defined by $\Phi_f(c_1, \dots, c_n)(x) = f(c_1(x), \dots c_n(x))$ where $x \in \mathcal{M}_{\text{old}}$ is a free variable. Given such a C^∞-style theory, what topological redescriptions are available to us?

In principle, the ISE Method could allow us to topologically redescribe this theory as being about a tangent vector field, $\vec{\varphi}_{\text{new}} : \mathcal{M}_{\text{new}} \to \text{T}\mathcal{M}_{\text{new}}$, or as being about a particle trajectory, $X_{\text{new}} : \mathcal{V}_{\text{new}} \to \mathcal{M}_{\text{new}}$. If these kinds of spacetime representations are possible, however, they are not likely to be "nice" representations. Particle trajectories, X_{new} :

[60] Roughly put, spacetime algebraicism is the view that the dynamical behavior of matter (specifically, field states) can exist without any topological underpinnings (specifically, because they can be thought of algebraically). For further discussion of spacetime algebraicism, see Chen and Fritz (2021); Menon (2019); Linnemann and Salimkhani (2021); Rynasiewicz (1992); Earman (1989); Rosenstock et al. (2015). Notably, Geroch (1972) uses Einstein Algebras to reconstruct general relativity in this way.

$\mathcal{V}_{\text{new}} \to \mathcal{M}_{\text{new}}$, have no natural notion of addition, $+_{\text{new}}$. Tangent vector fields, $\vec{\varphi}_{\text{new}} : \mathcal{M}_{\text{new}} \to \text{T}\mathcal{M}_{\text{new}}$, have no natural notion of product, $\times_{\text{new}}$. But these are not the only options on the table. Indeed, recall the Kalutza-Klein-like dimensional reduction example which we discussed earlier in Sec. 4.2 (and which will be discussed further in Sec. 8.1). Suppose that we begin with a theory about a real-valued field, $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to \mathbb{R}$, set on a spacetime with some compactified dimensions, e.g., $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2 \times S^2$. It is possible to topologically redescribe this theory as being set on a spacetime without compactified dimensions, e.g., $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$. The cost of doing so that the new theory will be about an $\mathbb{R}^\infty$-valued field, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathbb{R}^\infty$, whose components, $\varphi_{\text{new}}^{(\ell m)} : \mathcal{M}_{\text{new}} \to \mathbb{R}$, correspond to the harmonic mode of the old theory's compactified dimensions. In addition to this, who knows what other possible spacetime representations might be available to us?

In order to continue our comparison with spacetime algebraicism, however, we need to set such possibilities aside and limit our search for spacetime representations of our old theory which are about a real-valued field, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathbb{R}$. Moreover, let us limit our search to spacetime representations of $C^\infty(\mathcal{M}_{\text{old}})$ which realize its C^∞-algebraic structure via natural operations on φ_{new}. Namely, let us assume that the new theory's $+_{\text{new}}$, $\cdot_{\text{new}}$, $\times_{\text{new}}$, and $\Phi_{\text{new,f}}$ operations will be those of its natural C^∞-algebra, that is $C^\infty(\mathcal{M}_{\text{new}})$. This is a non-trivial restriction which the ISE Method does not make automatically. Indeed, recall the Fourier redescription example which we discussed earlier in Sec. 4.2 (and which will be discussed later in Ch. 7). If either of these QHO theories had been equipped with a point-wise field product, $\times_{\text{old}}$, then the Fourier-related theory would be equipped with a non-point-wise convolutional product, $*_{\text{new}}$.

Having noted the non-triviality of these restrictions, let us continue our search for new spacetime representations of our old theory which are about a real-valued field, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathbb{R}$, with $\mathcal{S}^{\text{kin}}_{\text{new}} = C^\infty(\mathcal{M}_{\text{new}})$. Given that our new and old theories must be

equivalent as pre-spacetime theories we must have, $C^\infty(\mathcal{M}_{\text{new}}) \cong C^\infty(\mathcal{M}_{\text{old}})$. Hence, applying the (Generalized) Stone Representation Theorem yields the following result:[61] We must have $\mathcal{M}_{\text{new}} \cong \mathcal{M}_{\text{old}}$ with the map between our two theories, $d^*|_{\text{kin}} : S^{\text{kin}}_{\text{old}} \leftrightarrow S^{\text{kin}}_{\text{new}}$, being the lift of some diffeomorphism, $d : \mathcal{M}_{\text{old}} \leftrightarrow \mathcal{M}_{\text{new}}$. In sum, we have a unique representation result for C^∞-style pre-spacetime theories; There is no possibility for topological redescription — after we restrict our search to theories about scalar fields, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathbb{R}$, which represent this algebraic structure via its natural C^∞-algebra, namely $C^\infty(\mathcal{M}_{\text{new}})$.

Let us pause to contrast this algebra-heavy example, $\mathcal{S}^{\text{kin}}_{\text{old}} = C^\infty(\mathcal{M}_{\text{old}})$, with our earlier example of the Möbius-Euclid duality which lacks any algebraic structure. Recall that we had,

$$\mathcal{S}^{\text{kin}}_{\text{M}} = \langle S^{\text{kin}}_{\text{M}}, P_{\text{M}}, D_{\text{M}}, A_{\text{M}}, \dots \rangle, \quad \text{and} \quad \mathcal{S}^{\text{kin}}_{\text{E}} = \langle S^{\text{kin}}_{\text{E}}, P_{\text{E}}, D_{\text{E}}, A_{\text{E}}, \dots \rangle, \tag{4.21}$$

with no sums or products in sight. Despite the clear differences between these algebraic and non-algebraic examples, the ISE Method applies to both of these cases *in exactly the same way.* Indeed, the ISE Method applies to all of the above examples of topological redescription in exactly the same way. Hence, each of these topological redescriptions is to be accomplished *without invoking any algebraic structure whatsoever*. In particular, this means I will not be identifying spacetime points with the maximal ideals of certain rings. Moreover, I will be able to implement the position-momentum duality in QM without invoking a change of basis in Hilbert space. As I noted above, it does not follow from the fact that these two QHO theories are algebraically structured that the redescription techniques which we use to move between them need to invoke this algebraic structure. The fact that the kinematic structures of these theories are algebraic in nature is incidental to their equivalence as pre-spacetime theories.

[61] See Rosenstock et al. (2015).

5 | Towards a Dynamics-First View of Spacetime Topology

The ultimate goal of this thesis is to develop some powerful new tools for topological redescription and to then use these novel techniques to inform various debates about the epistemology and metaphysics of space and time. Whereas the second half of this thesis will focus on developing a strong capacity for topological redescription (namely, the ISE Method), Part I has focused on completing the following conditional statement: "If we could develop a strong capacity for topological redescription, then the philosophical consequences of this would likely be ...". In particular, I have argued in Ch. 3 that we can expect similar benefits to those which have already come about from our existing capacities for nomological, coordinate, and geometric redescription. I there discussed the impact which our existing capacities for formal redescription have had on various debates about the epistemology and metaphysics of axiom-like laws, coordinate systems, and spacetime geometry.

Ultimately, we can expect that mastering our capacity for topological redescription will teach us important lessons about *both* the epistemology and the metaphysics of space and time. A detailed discussion of its epistemic benefits, however, is best saved for a later chapter (namely, Ch. 12) after we have seen my new topological redescription techniques in action. Hence, this chapter will focus solely on the metaphysical benefits which follow from our newfound capacity for topological redescription. Namely, Sec. 5.1 will discuss how it can help us to articulate the new dynamics-first vs dynamics-second debates about spacetime topology. More speculatively, Sec. 5.2 will discuss how spacetime representation theory might help us characterize the common structure which all of these alternative descriptions are jointly descriptions of.

5.1 An Unofficially Kantian Best Systems Account of Spacetime Topology

Of particular interest throughout Part I has been a series of four debates about foundational metaphysics (i.e., the dynamics-first vs dynamics-second debates). As I noted in Ch. 1, an important prototype for the dynamics-first style of metaphysics is Hume's position on causation. In terms of the epistemic story which I have sketched in Ch. 2, I take Hume's view to be as follows: Our phenomenal notion of causation does not gain its causative significance "at step zero" by latching onto any causal efficacy out there in the pre-epistemic world; Instead, it acquires its causative significance only "along the way" by being a particularly nice way of organizing the pre-epistemic world. In particular, what makes cause/effect relationships "particularly nice" for Hume is that they are instances of certain psychologically compelling and exceptionless regularities of experience.

Importantly, I take Hume to subscribe to this claim on either an epistemic reading or a metaphysical reading of his project.[62] Indeed, Hume's position on causation (or at least my gloss of it) could be true for either metaphysical or epistemic reasons. Namely, such causally efficacious relationships might fail to exist or, even if they do exist, we might be epistemically alienated from them. Either way, our phenomenal notion of causation would pick up its causative significance "along the way" rather than "at step zero". For this reason, the metaphysical and epistemic readings of Hume amount to the same thing in practice: Either way, one is led to treat causation *as if* it is merely an organizational principle which we unconsciously project onto the world — given that this is how it has earned its causative significance.

[62] There are several possible ways of understanding the aim of Hume's (2020) project. See Lorkowski (2023) for an overview of several possible readings of Hume.

Whereas this Hume-inspired style of metaphysics has been most thoroughly explored in contexts of laws and causation, this thesis has sought to show how it can also be applied to the metaphysics of space and time. The prospect of extending certain aspects of Hume's work on causation to the metaphysics of space and time ought to ring some Kantian bells. Indeed, one of Kant's key philosophical contributions was to extend Hume's epistemology-aligned style of metaphysics from causation to the metaphysics of space and time.[63] Consider the following epistemic reading of Kant. Kant's project is simply not concerned with establishing whether or not there are such things as space and time hard at work out there in the pre-epistemic world. Instead, one can understand Kant's project as aiming to identify how our phenomenal notion of space and time have come to acquire their spatiotemporal significance.

In terms of the epistemic story which I have sketched in Ch. 2, one can gloss Kant's claim as follows: Our phenomenal notions of space and time do not gain their spatiotemporal significance "at step zero" by latching onto any dynamics-facilitating spacetime points or relations out there in the pre-epistemic world; Instead, they acquire their spatiotemporal significance only "along the way" by being a particularly nice way of organizing the pre-epistemic world. In particular, what makes our phenomenal notions of space and time "particularly nice" for Kant is that they are necessary preconditions for the very possibility of human experience. While Kant clearly differs from Hume regarding how they spell out what "particular nice" means (i.e., a priori vs a posteriori), he has nonetheless followed Hume in giving an "along the way"-type answer rather than an "at step zero"-type answer. Given this historical connection between Hume and Kant, it is unsurprising that a dynamics-first view of spacetime will end up having an "unofficially Kantian" character.

[63] Kant (2004, pg. 10).

But what changes as we move from these historical debates for and against an (unofficially) Humean style of metaphysics to the dynamics-first vs dynamics-second debates? As I will now review, it is primarily the subject matter of the debates which changes and, correspondingly, a new line of evidence becomes relevant. The four dynamics-first vs dynamics-second debates are concerned with how certain aspects of our formalized theories (as opposed to our phenomenal experiences) have come to acquire their particular kind of metaphysical significance. That is, regarding our theories' axiom-like laws and coordinate systems as well as their geometric and topological structures, the dynamics-first vs dynamics-second debates ask: Where along our two-step epistemic story (see Fig. 2.2) have these theoretical structures acquired their particular kind of metaphysical significance? The two opposing sides still advance "along the way" and "at step zero" styles of metaphysics (now called dynamics-first vs dynamics-second, respectively). However, due to this shift in subject matter (from phenomenal to theoretical) a new line of evidence becomes relevant in these debates (or so I have argued). Indeed, this is my realist methodology: I take it that we can leverage the formal/mathematical structure of our best theories of physics as evidence in these debates about foundational metaphysics.

Returning to the metaphysics of laws debate, the defining claim of any dynamics-first view of laws is as follows: Our theories' laws do not gain their nomological significance "at step zero" by latching onto any dynamics-facilitating law-like counterpart out there in the world (i.e., governing laws); Instead, they acquire whatever nomological significance they have only "along the way" by being a particularly nice way of systematizing the dynamical behavior of matter. Perhaps surprisingly, despite its above-traced Humean origins, the umbrella of dynamics-first views of laws is wide enough to include causal powers views. This is because such views take the laws of nature to gain their nomological significance by latching onto causal powers, not governing laws. Ott (2022, pg. 4) puts it well: "[According to causal powers views,] if there are any laws at all, they merely

describe the powers that are responsible for the events we see around us. Such a view is bottom-up in the sense that what determines the course of nature is Nature itself." It is in this sense that proponents of causal powers views take the laws of nature to acquire their nomological significance "along the way" rather than "at step zero". (Although they would then go on to disagree with Hume by saying that our phenomenal notion of causation earns its causative significance at step zero). For an example of a causal powers view which enthusiastically embraces these parallels with canonical Humeanism about laws, see the generalization of Lewis's (1983) Best System Account (BSA) by Demarest (2017) which allows for dispositional properties to appear on the now-not-so-Humean mosaic.

In Ch. 2 I have argued that it is this best-systematization aspect of Humeanism (moreso than its ban on necessary connections) which can be profitably extended beyond the debates about the metaphysics of laws and causation. Defending any kind of "best systematization" view (be it about laws or, soon, about spacetime topology) will require some discussion of the epistemic standards which we use to associate certain theoretical structures to the world's goings-on. Given that our subject matter is a certain aspect of our formalized theories of physics (i.e., their axiom-like laws) there is always some capacity for formal redescription. That is, we are always free to formally redescribe our theories as if they are subject to radically different (but mathematically equivalent) axiom-like laws. As I have discussed in Ch. 3 this is actually a very important step in the process of theory development.

But given all of these possibilities, the question looms: Why didn't we use some other mathematically equivalent axiom-like laws? In order to separate the wheat from the chaff, some selection criteria must be employed. Identifying these selection criteria will require some hard epistemological work. For this it would be helpful to first get all of the options

on the table, so to speak, even those which are horrendously complicated (see Ch. 3). Beyond this strictly epistemic motivation, we also ought to care about these selection criteria if we want to defend and articulate dynamics-first views of laws. Indeed, it is ultimately these epistemic standards which (if we follow Hall's (2015) unofficial guiding idea) are to be promoted to an account of the nomological significance of our theory's laws. Fortunately, we have good mathematical control over the process of reaxiomatization, i.e., of reformulating our theories with respect to different axiom-like laws.

Part II of this thesis will deliver a similarly strong capacity for topological redescription. By analogy, we can expect this to help us investigate our topology selection criteria (for further discussion, see Ch. 12). Moreover, we can also expect this newly gained mastery of topological redescription to help us articulate and defend dynamics-first views of spacetime topology. Indeed, in light of the above discussion, one way to gloss such a view would be to call it a best system account of spacetime topology.[64] In close analogy with the above discussion of laws, the defining claim of any dynamics-first view of topology is as follows. The spacetime manifold which appears in our best physical theories does not gain its spatiotemporal significance by latching onto some spatiotemporal stage upon which the world's drama plays out (contra Newton); Nor does it do so by latching onto any fundamental spatiotemporal relationships which help to facilitate interactions between the drama's players (contra Leibniz); Instead, it acquires its spatiotemporal significance only because it is a particularly nice way of systematizing the dynamical behavior of matter (perhaps conceived of non-spatiotemporally).

Given the historical connection between Hume and Kant, it should not be surprising that this best system account of spacetime topology has ended up resembling Kant's view

[64] Similarly, one can give a similar gloss to the dynamical approach to spacetime geometry founded by Brown (2005) and others. In particular, see Huggett (2006) for an implementation of the dynamical approach which enthusiastically embraces these parallels with Humeanism about laws. One might call this a best system account of spacetime geometry.

of space and time in some ways. Indeed, compare the dynamics-first view of topology with the following quote from Kant regarding the metaphysical nature of space:[65]

> Space is not something objective and real, nor a substance, nor an accident, nor a relation; instead, it is subjective and ideal, and originates from the mind's nature in accord with a stable law as a scheme, as it were, for co-ordinating everything sensed externally.

That is, according to Kant space is not a substance (denying Newton) nor a relation (denying Leibniz) but rather something which our minds project onto the world as an organizing principle. While Kant's view is *an important example* of a dynamics-first view of topology, it should be stressed that the following several aspects of his view are optional. As I will soon discuss, dynamics-first views of spacetime topology may or may not agree with Kant on: the a priority necessity of spatiotemporal intuitions; its unique Euclidean character; its objectivity; and his overall non-speculative attitude towards metaphysics. Allowing for all of these points of departure is what makes dynamics-first views of topology only "unofficially" Kantian.

Recall that Kant's position arose from him reflecting upon the a priori necessity of spatiotemporal intuitions for cognizing experience. By contrast, dynamics-first views of topology can take spatiotemporal intuitions to be non-necessary for cognizing experience, and even to arise only a posteriori.[66] Kant (at least in some places) insinuates that it is conceptually necessary that our spatiotemporal intuitions be Euclidean.[67] By contrast, our strong capacities for geometric and topological redescription seem to indicate that we can spatiotemporally frame the world in a wide variety of ways (even though most would be horrendously complicated). We are thus led to something more like the relativized a priori discussed by Reichenbach and Reichenbach (1965) or like Friedman (2001). Alternatively, we could adopt a view paralleling Poincaré's (2018) conventionalism about

[65] While this quote is taken from Kant's (1992, pg. 397) Inaugural Dissertation, he holds effectively the same position post-critique.
[66] One need not be committed to the epistemological significance of this distinction, see Williamson (2013).
[67] See the Transcendental Aesthetic in Kant (1999).

geometry.[68] Despite this overabundance of spacetime framings, it may nonetheless turn out that one of these options is far and away the best, in a way which is insensitive to the details of our topology selection criteria; To take Lewis (1994, pg. 479) out of context it may turn out that "nature is kind" to us in this way. Alternatively, however, it may turn out that our choice of topological staging turns out to be somewhat conventional, see Sec. 8.3. Finally, whereas Kant is famously reluctant to engage in speculative metaphysics, dynamics-first views of topology are allowed (but not required) to make substantive claims about the pre-epistemic world.

One might complain that if one allows for all of these points of depature, then almost nothing of Kant's position remains. Against this, one should note the following sense in which I stand to Kant as Lewis stands to Hume. Namely, as I have discussed above the key part of Hume's position which remains in Lewis' unofficial Humeanism is its epistemology-aligned style of metaphysics. Similarly, it is the same part of Kant's position which remains in my unofficial Kantianism. Like other Neo-Kantian views, this unofficial Kantianism pins all of space and time's metaphysical significance upon the epistemological route via which they have entered into our experiences and our theories. This is, in fact, what sets a broadly Kantian metaphysics for space and time apart from its rivals: substantivalism, and relationalism (à la Newton and à la Leibniz, respectively). Lastly, it should be noted that much of Kant's position remains still on the table; It is conceivable that further argumentation could re-establish the pillars of Kant's view which are considered optional within the wider umbrella of dynamics-first views of topology.

[68] See Ivanova (2015) for some discussion of where Poincaré sits on the spectrum from Neo-Kantians to conventionalists.

5.2 World as Structure and Representation (Comparison with Structural Realism)

Suppose that one has adopted a dynamics-first view of spacetime topology. That is, suppose one has become convinced that the spacetime manifold which appears in our physical theories is reflective only of the representational process itself rather than any dynamics-facilitating pre-epistemic structure. What claim (if any) then follows about the fundamental ontological structure of the world? Nothing. As I stressed in my above discussion of Hume, one can adopt dynamics-first views for purely epistemic reasons, i.e., without making any substantial posits about the pre-epistemic world. Despite this, one is nonetheless led to do a certain kind of metaphysics in practice. Recall from Ch. 1 that on either an epistemic or metaphysical reading, Hume was ultimately led to treats causation *as if* it is merely an organizational principle which we unconsciously project onto the world — given that this is how it has earned its causative significance.

Let us now see how this kind of "as if" metaphysics works in the laws context. The following view qualifies as dynamics-first view of laws: "Even if there were laws of nature out there governing the world (and for all we know there might be), we would still be epistemically alienated from them. Hence, given this epistemic limitation, our theories' laws must acquire their nomological significance 'along the way', even if unbeknownst to us they do happen to have dynamics-facilitating pre-epistemic counterparts." On such a view we would be led to treat the laws of nature *as if* they are merely particularly nice ways of systematizing the dynamical behavior of matter — given that this is how they have earned their nomological significance.

We can imagine an analogous epistemically motivated dynamics-first view of topology as follows: "Even if there are fundamental spacetime points/relations out there facili-

tating matter's dynamics (and for all we know there might be), we would still be epistemically alienated from them. Hence, given this epistemic limitation, our theories' spacetime manifold must acquire spatiotemporal significance 'along the way', even if unbeknownst to us it does happen to have a dynamics-facilitating pre-epistemic counterpart." On such a view we would be led to treat the spacetime manifold *as if* it is merely a particularly nice ways of systematizing the dynamical behavior of matter — given that this is how it has earned its spatiotemporal significance.

It should be stressed that this kind of epistemically motivated dynamics-first view is compatible with realism. Allow me to demonstrate this in the laws' context. Dynamics-first views of laws can claim that non-governing laws are nonetheless *real* in the sense of being objective and mind-independent. Namely, the selection criteria which we use to pick out a given world's non-governing laws might be objective, e.g., "if nature is kind" to us;[69] Moreover, each such non-governing law (e.g., "all Fs are Gs") might have a mind-independent counterpart (e.g., the brute regularity that all Fs are Gs). I claim that a similar kind of realism is possible for dynamics-first views of spacetime topology. It is possible that our topology selection criteria will uniquely favor one possible spacetime representation of the world over all others. Recall from Ch. 4 that every C^∞-style pre-spacetime theory has exactly one nice real-scalar-field representation (up to diffeomorphism). Moreover, we might be able to identify the uniqueness of this best spacetime framing with some mind-independent aspect of the pre-epistemic world (e.g., its C^∞-algebaic structure).

This brings us to an important point of contrast between dynamics-first views of laws and of spacetime topology. Whereas one can take different axiom-like laws to be descriptions of the same underlying "stuff" (e.g., a Humean mosaic), it is not as straightforward to characterize the common structure which all of our topological descriptions are jointly descriptions of. Recall from Ch. 4 that topological redescriptions can radically change

[69] Lewis (1994, pg. 479).

not only our theory's topological backdrop but also the way that its matter content is displayed thereon. For instance, the matter content of the Möbius and Euclid theories from Sec. 4.1 appeared as a point-like particle in one spacetime setting but a line-shaped particle another. What is the common core here? Looking to our discussion of spacetime representation theory, it would seem they are different spacetime representations of a single underlying pre-spacetime theory. Recall, however, that pre-spacetime theories are specified in purely structural terms; They are explicitly not about this or that kind of matter content on this or that kind of spacetime. Hence, realism about pre-spacetime theories must be structural realism.

Notably, structural realism comes in epistemic and ontological varieties.[70] In particular, adding realism to the above-discussed epistemically motivated dynamics-first views would lead us to epistemic structural realism. As I noted above, however, dynamics-first views are allowed to (but are not required to) make substantive claims about the pre-epistemic world. Indeed, dynamics-first views can be true for either epistemic or metaphysical reasons; As I noted in Ch. 1, this aligns well with the epistemic and metaphysical readings of Hume and Kant. In particular, one might be led to adopt a dynamics-first view of laws, not out of epistemic modesty, but rather because one is convinced that there simply are no governing laws for our theory's laws to latch onto. Such a metaphysically-motivated view would claim that the laws of nature are merely a particularly nice way of systematizing the dynamical behavior of matter. We can make an analogous move to strengthen the above-discussed epistemic structural realism into ontic structural realism. Namely, we can take it that some pre-spacetime theory accurately captures the pre-epistemic world at an ontological level (not just "as if" out of epistemic modesty).

This brings us back to the two opening questions of this thesis: How does the world hang itself together? That is, what (if anything) does nature use to help facilitate the dy-

[70] I am here following Ladyman (1998) as well as Ladyman et al. (2014) and French (2016).

namical behavior of matter? We can now answer these questions from the point of view of someone who holds dynamics-first views across the board. There are no dynamics-facilitating laws to tell matter what to do; There are no dynamics-facilitating geometric structures to tell matter which way is straight ahead; There are no dynamics-facilitating topological structures against which matter can be identified as this or that kind of particle or field. In sum, the dynamical behavior of matter gets by on its own without the need to invoke any nomological, geometric, and/or topological substructure. In sum, on such a thorough-going dynamics-first view, the world hangs itself together as a seamless structured unity. (One can, of course, add in all of the epistemic hedging and "as if" clauses which we are by now used to.)

Importantly, this thorough-going dynamics-first view is not claiming that there are no objective and mind-independent facts about the world's laws, geometry, and topology; Such facts might exist even if dynamics-first views are true for solely metaphysical reasons. Importantly, however, these sorts of facts would then have to be metaphysically unemployed so to speak, they don't help facilitate the dynamical behavior of matter (e.g., as non-governing laws don't). From the metaphysics of laws debate (e.g., Lewis (1983) and Demarest (2017)), we have some insights into how the laws of nature can be read off of something like a Humean mosaic. Similarly, Huggett (2006) and others have shown us how certain geometric structures can be read off of something like a Humean mosaic (sans its assumed geometric underpinnings). This thesis seeks to go beyond this by showing how certain topological structures can be read off of an even less structured version of the Humean mosaic (i.e., sans its assumed topological underpinnings). To this end, the second half of this thesis will now introduce the ISE Method of topological redescription.

Part II

The ISE Method of Topological Redescription

6 | An Overview of the ISE Method: Internalize, Search, Externalize

So far in Part I of this thesis I have introduced the idea of topological redescription via spacetime representation theory (see Ch. 4) and given several examples thereof. Moreover, I have claimed that the ISE Method of topological redescription can handle these examples, as well as many more. The ultimate scope of the ISE Method is set by the ISE Equivalence theorem which I have already stated in Sec. 4.3 and will prove in Ch. 11. In effect, it says that the ISE Method can be used to implement every case of topological redescription for which new theory has a sufficient amount of spacetime-kinematic compatibility. Importantly, the ISE Method handles all of these cases *in the same way* no matter whether or not the pre-spacetime theoretic structure in question is algebraic or non-algebraic (e.g., the Möbius-Euclid duality from Sec. 4.1).

Part II of this thesis will introduce the ISE Method of topological redescription. In this chapter, I will overview the ISE Method before discussing it in more technical detail in the coming chapters. In particular, Chs. 7 and 8 will demonstrate the ISE Method by applying it to some of the above-discussed examples of topological redescription. Then Chs. 9 and 10 will discuss how the ISE Method works in general when applied to a wide range of spacetime theories and pre-spacetime theories. Part II will then conclude with a proof of the ISE Equivalence theorem in Ch. 11. Without further ado, allow me to introduce the ISE Method of topological redescription.

The ISE Method is a three-step process: Internalize, Search, Externalize (hence the initials, ISE). The internalization step takes as input a spacetime theory and de-emphasizes its given topological staging (e.g., $\mathcal{M}_{\text{old}} \cong \mathbb{M}$, a Möbius strip or, $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2$ the Eu-

clidean plane). In particular, it does so by converting the input spacetime theory into a pre-spacetime theory (as defined in Ch. 4). That is, internalization results in a pre-spacetime theoretic characterization of the theory in question. Recall from Sec. 4.2 that spacetime theories have enough structure at the level of kinematics, $\mathcal{S}^{\text{kin}}_{\text{old}} = \langle S^{\text{kin}}_{\text{old}}, \dots \rangle$, to pick out its dynamically allowed states, $S^{\text{dyn}}_{\text{old}} \subset S^{\text{kin}}_{\text{old}}$. From this, it follows that, in a certain sense, the theory's dynamics faithfully survives internalization. One might expect, however, that we will lose topological information about $\mathcal{M}_{\text{old}}$ in the internalization process. Indeed, in order to divorce the dynamical behavior of matter from its topological underpinnings we simply "forget" that our theory is about a certain kind of states (e.g., a certain kind of maps, $X_{\text{old}} : \mathbb{R} \to \mathcal{M}_{\text{old}}$, or, $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to \mathbb{R}$) by treating its states as ur-elements. Perhaps surprisingly, however, there is a sense in which our theory's spacetime will actually survive internalization faithfully if it has a sufficient level of spacetime-kinematic compatibility (defined in Sec. 9.2).

In the second step of the ISE Method, one next searches for the building blocks of a new spacetime framing of the theory. I call these building blocks pre-spacetime translation operations, abbreviated PSTOs. Ultimately, the new and old framings of our theory will differ regarding their choice of which PSTOs should qualify as spatiotemporal. While a technical definition of PSTOs can be found in Sec. 7.3 and Sec. 10.1, the following characterization is sufficient for our current purposes. PSTOs are a Lie group action on the theory's states which is, in a certain sense, structurally indistinguishable from spacetime translations. (Technically, PSTOs are *a pair of* Lie group actions, but let us simplify

to ease this introduction.)[71] A relatively trivial example of PSTOs would be any large enough group of translation operations (i.e., lifted diffeomorphisms) with respect to the theory's old spacetime, $\mathcal{M}_{\text{old}}$.[72] A non-trivial but still intuitive example of PSTOs would be translations, not in the old theoy's spacetime, but rather in old theory's Fourier space. To see why these Fourier shifts should qualify as PSTOs, consider the theory in question from a pre-spacetime theoretic perspective; That is, suppose we don't know what is space and what is Fourier space. Our Fourier shifts are structurally indistinguishable from spacetime translations when viewed as a Lie group action on the theory's states; Indeed they are translations somewhere, namely *in Fourier space*. More examples of PSTOs will be discussed in Chs. 7 and 8, see also Fig. 6.1 below.

Finally, in the externalization step, one uses whichever PSTOs one has chosen to construct a new topological setting for the theory in question. One does this simply by taking these PSTOs seriously as spacetime translations. In particular, a unique spacetime manifold is picked out by declaring that one's hand-picked PSTOs will become honest-to-goodness spacetime translations in the new spacetime setting. For instance, if I demand that the above-discussed Fourier shifts are to be regarded as spacetime translations, then I will construct a new spacetime manifold, $\mathcal{M}_{\text{new}}$, which is effectively a copy of the old theory's Fourier space. More will be said in Ch. 10 about how this construction works in general.

To complete the externalization process, one must next map the old theory's states and dynamics onto $\mathcal{M}_{\text{new}}$. As I will discuss in Sec. 7.4, this final stage of the externaliza-

[71] As I will discuss in Sec. 7.3, a choice of PSTOs consists of a pair of Lie groups $G_{\text{fix}} \subset G_{\text{trans}}$ which act on the theory's kinematically allowed states, S^{kin}. One of the compatibility requirements between these two groups is that G_{fix} is a core-free subgroup of G_{trans}. More plainly, G_{fix} is required to be in "lowest terms" with G_{trans} such that nothing immediately cancels in their quotient $G_{\text{trans}}/G_{\text{fix}}$. This means that G_{fix} is not allowed to contain any non-trivial normal subgroups of G_{trans}. In cases where G_{trans} is an abelian group, then all of its subgroups are normal subgroups. Hence, in such cases, we must have that $G_{\text{fix}} = \{\mathbb{1}\}$ is the trivial group. Therefore, when G_{trans} is abelian one can talk about PSTOs as if they consist of only one Lie group action (just G_{trans}) instead of two (both G_{fix} and G_{trans}). To simplify our discussion in this chapter, I will henceforth adopt this simplified one-group mode of speech regarding PSTOs.

[72] This fact is proved in Appendix B.2.

tion process is tightly constrained by our choice of PSTOs. For instance, in the Fourier example, from minor assumptions it follows that the new theory's states and dynamics are required to be the Fourier transform of the old theory's states and dynamics. Ch. 7 will discuss this Fourier redescription example at length in the context of non-relativistic quantum mechanics where it amounts to a position-momentum duality.

Let us now see some more explicit examples of PSTOs. As a first trivial example, consider smoothly shifting the theory's states with respect to the theory's old spacetime manifold. To make things concrete, consider a theory about a complex scalar field, $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to \mathbb{C}$, with $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2$. In some fixed global coordinate system, (t, x), we might smoothly shift the theory's states as,

$$\varphi_{\text{old}}(t, x) \mapsto \varphi_{\text{old}}(t - \Delta t, x - \Delta x), \tag{6.1}$$

for any $\Delta t, \Delta x \in \mathbb{R}$. The action of this Lie group on $\varphi_{\text{old}}(t, x)$ is shown in the first column of Fig. 6.1. In general, the old theory's spacetime translations trivially qualify as PSTOs. They are structurally indistinguishable from spacetime translations because they are spacetime translations in the theory's old spacetime setting. As I will discuss in Ch. 10, by externalizing these sorts of PSTOs one can easily return back to the theory's old spacetime framing, up to diffeomorphism.[73] This, however, is not compulsory. We can choose to externalize different PSTOs.

As a non-trivial example of PSTOs, consider smoothly increasing the frequency of every wave state described by the theory in question. Returning to the above-discussed, complex scalar field theory, we might smoothly transform the theory's states as,

$$\varphi_{\text{old}}(t, x) \mapsto \exp(\mathrm{i}\,\Delta k\, x)\, \varphi_{\text{old}}(t - \Delta t, x), \tag{6.2}$$

[73] I prove this fact in Appendix B.2.

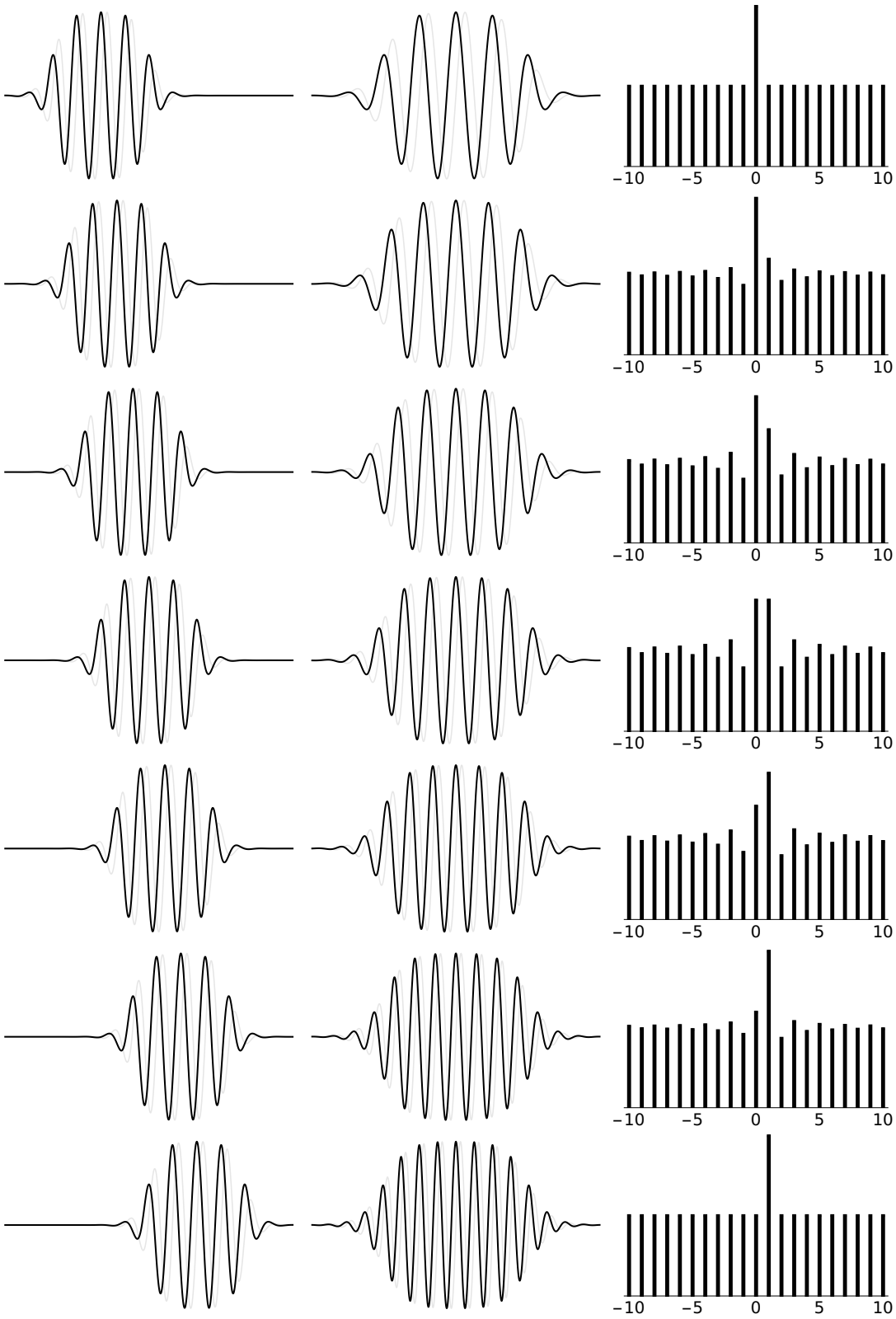

Figure 6.1: Three examples of pre-spacetime translation operations (PSTOs) are shown. The first column shows a wave packet being smoothly translated with respect to the old theory's spacetime manifold. The second column shows a wave packet being smoothly increased in frequency. The third column shows one way of smoothly implementing a discrete translation operation on a lattice theory. In all three cases, when viewed as a Lie group action on the theory's states, these smooth transformations are structurally indistinguishable from spacetime translations. Hence, one can use the ISE Method to build a new spacetime setting for the theory in question such that these PSTOs become honest-to-goodness spacetime translations. *Reproduced with permission from Grimmer (2023).*

for any $\Delta t, \Delta k \in \mathbb{R}$. The action of this Lie group on $\varphi_{\text{old}}(t, x)$ is shown in the second column of Fig. 6.1. Why do these pitch-shifting transformations qualify as PSTOs? While they are not translations in spacetime, they are nonetheless translation operations somewhere, namely in Fourier space. Viewed as a Lie group action on the theory's states, these Fourier shift operations are structurally indistinguishable from spacetime translations. Hence, these smooth transformations also qualify as PSTOs.

But what happens when we externalize these Fourier shifts? As I noted above, externalization takes seriously whichever PSTOs you give it as spacetime translations; It builds a new spacetime setting for the theory such that these PSTOs become honest-to-goodness spacetime translations in the new theory. Hence, by construction what were translations in Fourier space in the old theory will become translations in spacetime in the new theory. The new theory's spacetime is effectively a copy of the old theory's Fourier space. In this example, the ISE Method effectively implements a Fourier transform, see Ch. 7. Perhaps surprisingly, this kind of topological redescription can sometimes give the theory's dynamics a more local expression than it had in its original spacetime setting.

The exact same topological redescription techniques can also be used to redescribe a lattice theory (i.e., a theory set on a discrete spacetime, $\mathcal{M}_{\text{old}} \cong \mathbb{R} \times \mathbb{Z}$) as existing on a continuous spacetime manifold, $\mathcal{M}_{\text{new}} \cong \mathbb{R} \times \mathbb{R}$. But how is this possible? Discrete spacetime theories (i.e., lattice theories) obviously admit discrete spacetime translations, e.g., every lattice site moves over one place. Our goal, however, is to build a continuous spacetime setting for these discrete theories. Hence, we need to find within this discrete theory some continuous pre-spacetime translation operations. Perhaps surprisingly, such continuous PSTOs exist and are shown in the third column of Fig. 6.1. As Sec. 8.2 will discuss in detail, these continuous PSTOs act on the theory's discrete states so as to smoothly interpolate between the theory's discrete spacetime translations. Viewed as a Lie group action on the theory's states, these transformations are structurally indistinguishable from

spacetime translations. (Namely, they are structurally indistinguishable from spacetime translations in exactly the same sense as the above-discussed Fourier shifts were). Hence, they too will qualify as PSTOs.

But what prompts us to take these continuous PSTOs seriously as spacetime translations? As Sec. 8.2 will discuss, the transformations shown in Fig. 6.1 are not only PSTOs but also dynamical symmetries (i.e., they map solutions to solutions). We have thus discovered that these theories have a hidden dynamical symmetry. The ISE Method allows us to redescribe these lattice theories in such a way that their hidden dynamical symmetries are re-expressed as spacetime symmetries in the new theory. Arguably, this new continuous spacetime description is a better fit for the theory's dynamics than its original spacetime setting was (i.e., better in terms of the expression of its dynamical symmetries).

The process for constructing this new continuous spacetime is identical to the Fourier example discussed above. Given this choice of PSTOs, the old theory's discrete field states *must* appear on the new continuous spacetime setting as bandlimited functions (see Sec. 8.2 for details). We have as little freedom in mapping the old theory's states onto the new theory's spacetime as we did in the Fourier example; There is simply no good way to populate the new theory's spacetime (i.e., a copy of the old theory's Fourier space) other than with the Fourier transform of the old theory's states. See Sec. 7.4 for details.

As a third example application, Sec. 8.1 will use the ISE Method to implement a Kaluza-Klein dimensional reduction (and to then undo this dimensional reduction). Chs. 9 and 10 will also discuss how the ISE Method can be applied outside of the scalar field case. Indeed, it can also be applied to theories about a tangent vector field, $\vec{\varphi} : \mathcal{M} \to \mathrm{T}\mathcal{M}$, or even a parameterized particle trajectory, $X : \mathcal{V} \to \mathcal{M}$. For instance, recall the Möbius-Euclid duality from Sec. 4.1. The Möbius theory considered a point-like particle, $X_{\text{old}} : \mathbb{R} \to \mathcal{M}_{\text{old}}$, floating around on the Möbius strip, $\mathcal{M}_{\text{old}} \cong \mathbb{M}$, over time. As I will discuss in Sec. 11.3 one can use the ISE Method to redescribe the Möbius theory as the Euclid theory which is

about a line-shaped particle, $X_{\text{new}} : \mathbb{R}^2 \to \mathcal{M}_{\text{new}}$, floating around on the Euclidean plane, $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$, over time.

Despite their striking differences, the ISE Method applies to all of these examples in exactly the same way. Indeed, moving beyond these example applications, Ch. 11 will prove the ISE Equivalence Theorem which states the following. The ISE Method gives us access to effectively *every possible* spacetime framing of a given theory's kinematical and dynamical content, bounded only by a weak spacetime-kinematic compatibility condition. This proves that the ISE Method is a completely general tool for topological redescription (at least, within the realm of spacetime theories which satisfy this spacetime-kinematic compatibility condition). Every instance of topological redescription between any two such theories can be implemented by the ISE Method. As Sec. 9.3 will discuss, spacetime-kinematic compatibility is an extremely weak assumption. Hence, the scope of topological redescriptions offered to us by the ISE Method is extremely broad.

7 | Demonstrating the ISE Method in some Familiar Settings

This chapter will demonstrate the ISE Method of topological redescription by applying it to some example theories. In particular, Sec. 7.1 and Sec. 7.2 will provide two demonstrations of the internalization process. Sec 7.3 will then provide one demonstration of the externalization process. For the sake of pedagogy, the example of topological redescription demonstrated in this chapter will be rather tame. Indeed, it is just the Fourier duality of non-relativistic quantum mechanics applied to the two quantum harmonic oscillator theories which we have already discussed in Sec. 4.2. While this topological redescription ultimately just amounts to taking a Fourier transform, it will nonetheless be helpful to first see the ISE Method implemented in a familiar setting.

For the sake of concreteness, the example applications considered here (and in Ch. 8) will be focused on theories about a scalar field, $\varphi : \mathcal{M} \to \mathcal{V}$. The real power of the ISE Method, however, is in its scope and generality. Chs. 9 and 10 will show how the internalization and externalization processes work more generally. Indeed, beyond scalar fields, it can also be applied to theories about tangent vector fields, $\vec{\varphi} : \mathcal{M} \to \mathrm{T}\mathcal{M}$, and even theories about particle trajectories, $X : \mathcal{V} \to \mathcal{M}$. Two examples of this latter kind would be the Möbius and Euclid theories discussed in Sec. 4.1. Sec. 11.3 will demonstrate how the ISE Method can be applied to these two theories in particular. For now, however, let us see the ISE Method of topological redescription in action.

7.1 Surviving Internalization: Dynamics, Diffeomorphisms, Topology

Consider the following non-linear theory set on a curved spacetime:

> **A Quartic Klein-Gordon Theory (3+1D QKG) -** Consider a theory set on a spacetime manifold, $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2 \times S^2$, about a scalar field, $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to \mathcal{V}_{\text{old}}$, with value space, $\mathcal{V}_{\text{old}} \cong \mathbb{R}$. For this theory, there exists an injective map, $C : \mathcal{M}_{\text{old}} \to \mathbb{R}^5$, which assigns to each point, $p \in \mathcal{M}_{\text{old}}$, some coordinates, $C(p) = (t, q, x, y, z) \in \mathbb{R}^5$, with $x^2 + y^2 + z^2 = R^2$ for some fixed $R > 0$. This theory's states are subject to the following kinematic constraint: In this fixed coordinate system the field, φ_{old}, must be smooth. In this coordinate system, the metaphysically relevant way to judge the relative size and similarity of this theory's states is with an L^2 inner product. In this coordinate system, the dynamical equation for φ_{old} is as follows:
>
> $$\left(\partial_t^2 - \partial_q^2 - \frac{L_x^2}{R^2} - \frac{L_y^2}{R^2} - \frac{L_z^2}{R^2} + M^2\right) \varphi_{\text{old}} + \lambda\, \varphi_{\text{old}}^3 = 0, \tag{7.1}$$
>
> where L_x, L_y, and L_z are the generators of rotations about the x, y, and z-axis respectively. $\lambda \in \mathbb{R}$ is some fixed self-coupling constant and $M \geq 0$ is some fixed mass parameter.

A crucial first step in the internalization process is to frame this theory as a spacetime theory in the sense defined in Sec. 4.2. To do so, we must first identify the modal structure of this theory. At its broadest level, this theory is about the set, $S_{\text{old}}^{\text{all}}$, of all $\mathbb{R}$-valued functions definable on $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2 \times S^2$ (even those which are non-smooth and/or discontinuous). Enforcing the theory's kinematic constraints, we can identify within $S_{\text{old}}^{\text{all}}$ the theory's set of kinematically allowed states, $S_{\text{old}}^{\text{kin}} \subset S_{\text{old}}^{\text{all}}$. This is the set of all smooth $\mathbb{R}$-valued functions on $\mathcal{M}_{\text{old}}$. Within this set, we can then enforce the theory's dynamics to find its dynamically allowed states, $S_{\text{old}}^{\text{dyn}} \subset S_{\text{old}}^{\text{kin}} \subset S_{\text{old}}^{\text{all}}$. Elsewhere these are called the theory's kinematically possible models (KPMs) and its dynamically possible models (DPMs).

Next, we need to identify which structural features of this theory are dynamically and metaphysically relevant. In particular, the theory's kinematically allowed states, $S_{\text{old}}^{\text{kin}}$, must come equipped with all of the resources needed to pick out the theory's dynamically allowed states and to make sense of its metaphysics. For instance, if we are going to use

the dynamical equation, Eq. (7.1), to pick out $S^{\text{dyn}}_{\text{old}}$, then we are required to equip $S^{\text{kin}}_{\text{old}}$ with some addition and multiplication operations. Namely, it requires an addition operation, $+_{\text{old}} : S^{\text{kin}}_{\text{old}} \times S^{\text{kin}}_{\text{old}} \to S^{\text{kin}}_{\text{old}}$, and a scalar multiplication operation, $\cdot_{\text{old}} : \mathbb{R} \times S^{\text{kin}}_{\text{old}} \to S^{\text{kin}}_{\text{old}}$, and finally a product operation, $\times_{\text{old}} : S^{\text{kin}}_{\text{old}} \times S^{\text{kin}}_{\text{old}} \to S^{\text{kin}}_{\text{old}}$ each of which are pointwise operations on $\mathcal{M}_{\text{old}}$. Using these we can state the theory's dynamical equations and thereby define its dynamically allowed states, $S^{\text{dyn}}_{\text{old}} \subset S^{\text{kin}}_{\text{old}}$.

Past its dynamically-relevant structure, this theory also comes equipped with a metaphysically relevant inner product, $I_{\text{old}}(\varphi, \phi) = \langle \varphi, \phi \rangle_{\text{old}}$. Namely, we can use this inner product to judge the relative size and similarity of any two kinematically allowed states, $\varphi \in S^{\text{kin}}_{\text{old}}$ and $\phi \in S^{\text{kin}}_{\text{old}}$. Note that while no such inner product appears in the theory's dynamical equation, it might nonetheless be crucial in grounding whatever metaphysical story one wants to tell about this theory. Recall from Sec. 4.2 the role that Hilbert space structure plays in non-relativistic quantum mechanics despite the fact that no inner products appear in the Schrödinger equation. Collecting together all of the structures which we have associated with the old theory's kinematically allowed states, we have:

$$\mathcal{S}^{\text{kin}}_{\text{old}} := \langle S^{\text{kin}}_{\text{old}}, +_{\text{old}}, \cdot_{\text{old}}, \times_{\text{old}}, I_{\text{old}} \rangle. \tag{7.2}$$

While this theory has the structure of an algebra, not all theories are like this; Recall that from Sec. 4.1 that the Möbius and Euclid theories have no algebraic structure.

Having satisfied the modal assumption and the kinematic structure assumption, let us next identify this theory's declared topological staging. Indeed, one should recall from Ch. 6 that the goal of the internalization process is to divorce the dynamical behavior of matter from the theory's assumed topological background structure (i.e., its spacetime manifold). But where do the dynamical fields, $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to \mathcal{V}_{\text{old}}$, make contact with the spacetime manifold? Note that each of the modal spaces discussed above, $S^{\text{dyn}}_{\text{old}} \subset S^{\text{kin}}_{\text{old}} \subset S^{\text{all}}_{\text{old}}$, is populated by *functions on the spacetime manifold*, $\mathcal{M}_{\text{old}}$. We can associate to this

declared topological staging a special operation, eval : $S^{\text{kin}}_{\text{old}} \times \mathcal{M}_{\text{old}} \to \mathcal{V}_{\text{old}}$, which allows us to evaluate the field states at spacetime points, $\text{eval}(\varphi_{\text{old}}, p) = \varphi_{\text{old}}(p)$.

It is ultimately this declared topological staging which connects φ_{old} to the theory's assumed topological underpinnings, $\mathcal{M}_{\text{old}}$. While I have above introduced $+_{\text{old}}$, $\cdot_{\text{old}}$, and $\times_{\text{old}}$ as being carried out point-wise with respect to $\mathcal{M}_{\text{old}}$, it is not compulsory that we think of them in this way. We can instead think of them primarily as binary operations on $S^{\text{kin}}_{\text{old}}$ which only qualify as being "point-wise" operations on $\mathcal{M}_{\text{old}}$ in light of their relationship to this special eval operation. In this way, the field-to-spacetime connection can be isolated to within this special eval function and the function-like role which it allows $\varphi_{\text{old}} \in S^{\text{kin}}_{\text{old}}$ to play. It is this connection which the internalization process aims to sever.

Before we can begin internalizing this theory, one final piece of mathematical structure needs to be pointed out. Note that the Quartic Klein-Gordon theory, admits a map $*_{\text{lift}} : d \mapsto d^*$, which can lift the theory's diffeomorphisms, $d \in \text{Diff}(\mathcal{M}_{\text{old}})$, to act on $S^{\text{all}}_{\text{old}}$ as a bijection, $d^* : S^{\text{all}}_{\text{old}} \to S^{\text{all}}_{\text{old}}$. Concretely, we have $d^*\varphi_{\text{old}} \coloneqq \varphi_{\text{old}} \circ d^{-1}$. One important property of this $*_{\text{lift}}$ map is that it is a group homomorphism, i.e., we have $(d_1\, d_2)^* = d_1^*\, d_2^*$. Analogous lifting maps can be defined for theories about particle trajectories, $X_{\text{old}} : \mathcal{V}_{\text{old}} \to \mathcal{M}_{\text{old}}$, or even tangent vector fields, $\vec{\varphi}_{\text{old}} : \mathcal{M}_{\text{old}} \to \text{T}\mathcal{M}_{\text{old}}$. In any case, we can restrict these lifted diffeomorphisms to act only on the theory's kinematically allowed states as $d^*|_{\text{kin}} : S^{\text{kin}}_{\text{old}} \to S^{\text{all}}_{\text{old}}$. For the Quartic Klein-Gordon theory in particular, we have that every lifted diffeomorphism, d^*, maps kinematically allowed states to kinematically allowed states. Hence, we actually have $d^*|_{\text{kin}} : S^{\text{kin}}_{\text{old}} \to S^{\text{kin}}_{\text{old}}$.[74]

[74] We can hence say that for the Quartic Klein-Gordon theory every diffeomorphism, $d \in \text{Diff}(\mathcal{M}_{\text{old}})$, is kinematically allowed. This fact, however, is not generic. As Ch. 9 will discuss, in generally only a subset $d \in \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}}) \subset \text{Diff}(\mathcal{M}_{\text{old}})$ of diffeomorphisms will be kinematically allowed. The fortunate coincidence $\text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}}) = \text{Diff}(\mathcal{M}_{\text{old}})$ which we have in the present case will become relevant in two places below, namely Eqs. (7.7) and (7.10). Another notable property of this theory is that it is kinematically reduced: The only diffeomorphism which has $d^*|_{\text{kin}} = \mathbb{1}$ acting as the identity on $S^{\text{kin}}_{\text{old}}$ is $d = \mathbb{1}$. This fact will become important following Eq. (7.11).

We are now ready to begin the internalization process. The way in which the ISE Method divorces the theory's states from the old spacetime setting is similar in spirit to the algebraic approach to spacetime. Let us first discuss what these approaches have in common before seeing how the ISE Method goes further. On either approach, we begin by noting that a large portion of the theory's structure can be preserved while breaking the field-to-spacetime connection. In order to break this connection, all we need to do is forget about $\mathcal{M}_{\text{old}}$ (and relatedly its special eval and $*_{\text{lift}}$ operations). We can do this via some forgetful operation, $\mathcal{F}_{\text{alg}}$, which remembers only algebraic structure as follows:

$$\mathcal{F}_{\text{alg}} : \mathcal{S}^{\text{kin}}_{\text{old}} \to \mathcal{S}^{\text{kin}}_{\text{pst}} \coloneqq \langle S^{\text{kin}}_{\text{pst}}, +_{\text{pst}}, \cdot_{\text{pst}}, \times_{\text{pst}}, I_{\text{pst}} \rangle. \tag{7.3}$$

Two things have changed here. Firstly, the set, $S^{\text{kin}}_{\text{old}}$, has been replaced with an equal-sized set, $S^{\text{kin}}_{\text{pst}}$, whose elements have no connection with $\mathcal{M}_{\text{old}}$. Namely, whereas $\varphi_{\text{old}} \in S^{\text{kin}}_{\text{old}}$ is a functions on $\mathcal{M}_{\text{old}}$, its corresponding element is not, $\varphi_{\text{pst}} \coloneqq \mathcal{F}_{\text{alg}}(\varphi_{\text{old}}) \in S^{\text{kin}}_{\text{pst}}$. Secondly, the algebraic operations appearing in the above expression (e.g., $+$, $\cdot$, $\times$, and I) are now defined on $S^{\text{kin}}_{\text{pst}}$ instead of $S^{\text{kin}}_{\text{old}}$ such that we have an isomorphism, $\mathcal{S}^{\text{kin}}_{\text{pst}} \cong \mathcal{S}^{\text{kin}}_{\text{old}}$, as algebras. Concretely, we have,

$$\begin{aligned}
\varphi_{\text{pst}} \ +_{\text{pst}} \ \phi_{\text{pst}} &\coloneqq \mathcal{F}_{\text{alg}}(\mathcal{F}^{-1}_{\text{alg}}(\varphi_{\text{pst}}) +_{\text{old}} \mathcal{F}^{-1}_{\text{alg}}(\phi_{\text{pst}})), \\
\alpha \ \cdot_{\text{pst}} \ \varphi_{\text{pst}} &\coloneqq \mathcal{F}_{\text{alg}}(\alpha \ \cdot_{\text{old}} \ \mathcal{F}^{-1}_{\text{alg}}(\varphi_{\text{pst}})), \\
\varphi_{\text{pst}} \ \times_{\text{pst}} \ \phi_{\text{pst}} &\coloneqq \mathcal{F}_{\text{alg}}(\mathcal{F}^{-1}_{\text{alg}}(\varphi_{\text{pst}}) \times_{\text{old}} \mathcal{F}^{-1}_{\text{alg}}(\phi_{\text{pst}})), \\
I_{\text{pst}}(\varphi_{\text{pst}}, \phi_{\text{pst}}) &\coloneqq I_{\text{old}}(\mathcal{F}^{-1}_{\text{alg}}(\varphi_{\text{pst}}), \mathcal{F}^{-1}_{\text{alg}}(\phi_{\text{pst}})),
\end{aligned} \tag{7.4}$$

where $\varphi_{\text{pst}}, \phi_{\text{pst}} \in S^{\text{kin}}_{\text{pst}}$ are generic pre-spacetime states and $\alpha \in \mathbb{R}$. In total, $\mathcal{F}_{\text{alg}}$ forgets everything about the Quartic Klein-Gordon theory except for the algebraic structure of its kinematically allowed states.

It is important to note that exactly how forgetful the $\mathcal{F}$ operation ought to be will vary from theory to theory. Ultimately, the correct level of forgetfulness for $\mathcal{F}$ will be set

by the dynamical and metaphysical details of the theory in question. Recall the Möbius and Euclid theories from Sec. 4.1; Their metaphysically relevant structures were all non-algebraic. The internalization of these two theories will be discussed in Sec. 11.3. As this and other examples will show, the ISE Method allows for a wide variability in the dynamically and metaphysically relevant structures which are at play. Importantly, however, no specific kind of structure is required.

If one comes from a spacetime algebraicist's perspective, one might have expected that some degree of algebraic structure (e.g., $+$, $\cdot$, $\times$, etc.) will be required later on in constructing a new spacetime framing of this theory (e.g., if one seeks to identify spacetime points with certain maximal ideals of some ring). It is at this point, however, that the ISE Method diverges substantially from spacetime algebraicism. Namely, unlike the algebraic approach to spacetime, the ISE Method does not make use of any algebraic structure in recovering the theory's old spacetime setting (or in generating new spacetime settings). As I noted in Ch. 6, the new spacetime settings offered to us by the ISE Method are to be built solely from our ability to smoothly vary the theory's states (recall Fig. 6.1); By this, I mean that $S^{\text{kin}}_{\text{old}}$ can be the subject of a Lie group action (no algebra required). This is how the ISE Method is applicable to non-algebraic theories such as the Möbius-Euclid duality discussed in Sec. 4.1.

To be clear, there may be dynamical or metaphysical reasons for preserving other aspects of the theory in question through the internalization process (e.g., its algebraic structure if it has any). These structures may play a substantial role in stating the theory's dynamics, in grounding a satisfying metaphysical story, and even in choosing the best spacetime setting for the theory. Nonetheless, any such structures are *mechanically irrelevant* throughout the ISE Method. They merely come along for the ride so to speak.

Let us return to internalizing the Quartic Klein-Gordon theory, which incidentally does require us to preserve its algebraic structure for both dynamical and metaphysi-

cal reasons. What else survives internalization? The old theory's kinematically allowed states, $\varphi_{\text{old}} \in S^{\text{kin}}_{\text{old}}$, survive internalization as follows,

$$\varphi_{\text{old}} \xrightarrow[\text{int. as}]{\text{survives}} \varphi_{\text{pst}} \coloneqq \mathcal{F}_{\text{alg}}(\varphi_{\text{old}}). \tag{7.5}$$

Note that it is only the theory's kinematically allowed states, $S^{\text{kin}}_{\text{old}} \subset S^{\text{all}}_{\text{old}}$, which survive internalization, not all of $S^{\text{all}}_{\text{old}}$.

Moreover, given that $\mathcal{S}^{\text{kin}}_{\text{old}}$'s algebraic structure has faithfully survived internalization, we ought to have the resources within $\mathcal{S}^{\text{kin}}_{\text{pst}}$ to state the theory's dynamics. Let us see how this happens. To begin, recall the theory's dynamical equation given by Eq. (7.1),

$$\left(\partial_t^2 - \partial_q^2 - \frac{L_x^2}{R^2} - \frac{L_y^2}{R^2} - \frac{L_z^2}{R^2} + M^2\right) \varphi_{\text{old}} + \lambda\, \varphi_{\text{old}}^3 = 0, \tag{7.6}$$

Each part of this dynamical equation survives internalization as follows:[75]

$$\begin{aligned} \partial_t &\xrightarrow[\text{int. as}]{\text{survives}} & D_0 &\coloneqq \mathcal{F}_{\text{alg}} \circ \partial_t \circ \mathcal{F}_{\text{alg}}^{-1} \\ \partial_q &\xrightarrow[\text{int. as}]{\text{survives}} & D_1 &\coloneqq \mathcal{F}_{\text{alg}} \circ \partial_q \circ \mathcal{F}_{\text{alg}}^{-1} \\ L_{x,y,z} &\xrightarrow[\text{int. as}]{\text{survives}} & D_{2,3,4} &\coloneqq \mathcal{F}_{\text{alg}} \circ L_{x,y,z} \circ \mathcal{F}_{\text{alg}}^{-1} \end{aligned} \tag{7.7}$$

Hence, we have that the dynamical equation survives internalization as follows:

$$\begin{aligned} &\left(\partial_t^2 - \partial_q^2 - \frac{L_x^2}{R^2} - \frac{L_y^2}{R^2} - \frac{L_z^2}{R^2} + M^2\right) \varphi_{\text{old}} + \lambda\, \varphi_{\text{old}}^3 = 0 \\ &\xrightarrow[\text{int. as}]{\text{survives}} \left(D_0^2 - D_1^2 - \frac{D_2^2}{R^2} - \frac{D_3^2}{R^2} - \frac{D_4^2}{R^2} + M^2\right) \varphi_{\text{pst}} + \lambda \cdot \varphi_{\text{pst}}^3 = 0 \end{aligned} \tag{7.8}$$

where the $\cdot$, $+$, and $\times$ operations in the pre-spacetime theory are those induced by the forgetful operation, $\mathcal{F}_{\text{alg}}$, see Eq. (7.4). Hence, we can we can pick out the theory's dynamically allowed states in a pre-spacetime theoretic way,

$$S^{\text{dyn}}_{\text{old}} \subset \mathcal{S}^{\text{kin}}_{\text{old}} \xrightarrow[\text{int. as}]{\text{survives}} S^{\text{dyn}}_{\text{pst}} \subset \mathcal{S}^{\text{kin}}_{\text{pst}}. \tag{7.9}$$

[75] In order for these partial derivatives to survive internalization, it is necessary that they are kinematically allowed, see footnote 74. That is, they must map kinematically allowed states to kinematically allowed states. Fortunately, for the Quartic Klein-Gordon theory, all of these partial derivatives are kinematically allowed.

We have so far seen how a theory's states and dynamics can survive being divorced from the old theory's declared topological background structure. Importantly, however, internalization does not simply delete all of the theory's topological information. Surprisingly, a great deal of topological information about the old spacetime setting can survive internalization. The key to seeing how any topological information survives internalization is the $*_{\text{lift}} : d \mapsto d^*$ map discussed above. Recall that this map lifts diffeomorphisms, $d : \mathcal{M}_{\text{old}} \to \mathcal{M}_{\text{old}}$, to act on the theory's states as $d^* : S^{\text{all}}_{\text{old}} \to S^{\text{all}}_{\text{old}}$. Once we restrict the action of d^* to kinematically allowed states (e.g., $d^*|_{\text{kin}} : S^{\text{kin}}_{\text{old}} \to S^{\text{kin}}_{\text{old}}$), it becomes possible to see it surviving internalization. That is, we can see them as maps on $S^{\text{kin}}_{\text{pst}}$, namely $G(d) \coloneqq \mathcal{F}_{\text{alg}} \circ d^*|_{\text{kin}} \circ \mathcal{F}^{-1}_{\text{alg}}$.

For the Quartic Klein-Gordon theory, a notable family of diffeomorphism on $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2 \times S^2$ survive internalization in this way. Let $d_t(\epsilon) \in \text{Diff}(\mathcal{M}_{\text{old}})$ be the diffeomorphism on $\mathcal{M}_{\text{old}}$ which act in the C-coordinate system as $d_t(\epsilon) : (t, q, x, y, z) \mapsto (t - \epsilon, q, x, y, z)$. Similarly, let $d_q(\epsilon) \in \text{Diff}(\mathcal{M}_{\text{old}})$ act as $d_q(\epsilon) : (t, q, x, y, z) \mapsto (t, q - \epsilon, x, y, z)$. Let $d_x(\theta), d_y(\theta), d_z(\theta) \in \text{Diff}(\mathcal{M}_{\text{old}})$, be the diffeomorphisms which rigidly rotate $\mathcal{M}_{\text{old}}$ around the x, y, and z-axis in the C-coordinate system. Let $H_{\text{trans}} \cong (\mathbb{R}^2, +) \times \text{SO}(3)$ be the Lie group formed by these diffeomorphisms. Each of these diffeomorphisms, $d \in H_{\text{trans}}$, survives internalization as follows:[76]

$$
\begin{aligned}
d^*_t(\epsilon)|_{\text{kin}} = \exp(-\epsilon\, \partial_t) &\xrightarrow[\text{int. as}]{\text{survives}} G(d_t(\epsilon)) = \exp(-\epsilon\, D_0), \\
d^*_q(\epsilon)|_{\text{kin}} = \exp(-\epsilon\, \partial_q) &\xrightarrow[\text{int. as}]{\text{survives}} G(d_q(\epsilon)) = \exp(-\epsilon\, D_1), \\
d^*_x(\theta)|_{\text{kin}} = \exp(-\theta L_x) &\xrightarrow[\text{int. as}]{\text{survives}} G(d_x(\theta)) = \exp(-\theta D_2), \\
d^*_y(\theta)|_{\text{kin}} = \exp(-\theta L_y) &\xrightarrow[\text{int. as}]{\text{survives}} G(d_y(\theta)) = \exp(-\theta D_3), \\
d^*_z(\theta)|_{\text{kin}} = \exp(-\theta L_z) &\xrightarrow[\text{int. as}]{\text{survives}} G(d_z(\theta)) = \exp(-\theta D_4).
\end{aligned}
\tag{7.10}
$$

[76] In order for these diffeomorphisms to survive internalization, it is necessary that they are kinematically allowed, see footnote 74. Fortunately, for the Quartic Klein-Gordon theory, every diffeomorphism is kinematically allowed.

Indeed, these diffeomorphisms survive internalization not only individually, but also as a Lie group (i.e., the G map is a Lie group homomorphism, see Sec. 9.2 for proof). In total, therefore, we have that the Lie group $H_{\text{trans}} \subset \text{Diff}(\mathcal{M}_{\text{old}})$ survives internalization as follows:

$$H_{\text{trans}} \xrightarrow[\text{int. as}]{\text{survives}} G_{\text{trans}} \coloneqq G(H_{\text{trans}}). \tag{7.11}$$

Moreover, H_{trans} survives internalization faithfully (i.e., we have $G_{\text{trans}} \cong H_{\text{trans}} \cong (\mathbb{R}^2, +) \times \text{SO}(3)$) since the kernel of the G map is trivial, $\ker(G) = \{\mathbb{1}\}$.[77] Thus, at least some topological information regarding the old spacetime setting, $\mathcal{M}_{\text{old}}$, has survived internalization.

Perhaps surprisingly, not just partial but complete topological information about $\mathcal{M}_{\text{old}}$ has faithfully survived internalization. Seeing this, however, will require that we first understand a bit of mathematics regarding the quotients of Lie groups, namely, smooth homogeneous manifolds. A summary of this topic along with proofs of some key results can be found in Appendix A. For our purposes, some key results should be quickly noted.

A smooth manifold, $\mathcal{M}$, is said to be homogeneous if there exists a (finite-dimensional) Lie group, $H_{\text{trans}} \subset \text{Diff}(\mathcal{M})$, which acts smoothly and transitively on it. A group action is said to be transitive if any point $p \in \mathcal{M}$ can be mapped to any other point $q \in \mathcal{M}$ by some group action $h(p) = q$. That is, $\forall\, p, q \in \mathcal{M}\ \ \exists h \in H_{\text{trans}}\ \ h(p) = q$. Some examples of homogeneous smooth manifolds are as follows. The 2-sphere, S^2, is homogeneous with respect to $\text{SO}(3)$. The plane, $\mathbb{R}^2$, is homogeneous with respect to $(\mathbb{R}^2, +)$. The plane, $\mathbb{R}^2$, is also homogeneous with respect to $\text{SE}(2)$. Perhaps surprisingly, the Möbius strip, $\mathbb{M}$, is also homogeneous with respect to $\text{SE}(2)$. For a non-example, the double-torus, $\mathcal{M} \cong \mathbb{T}^2 \# \mathbb{T}^2$, is homogeneous with respect to $\text{Diff}(\mathcal{M})$ but not with respect to

[77] This follows immediately from the Quartic Klein Gordon theory being kinematically reduced, see footnote 74.

any finite-dimensional Lie subgroup thereof.[78] Importantly, this notion of homogeneity is entirely topological as $\mathcal{M}$ here is not equipped with any kind of metric structure.

Such smooth manifolds are homogeneous in the sense that from the perspective of the Lie group, G, all points are interchangeable. Said differently, there is only one G-orbit, namely the whole manifold $\mathcal{M}$. This point can also be made in the contrapositive direction. If G's action on $\mathcal{M}$ wasn't transitive, then it would divide $\mathcal{M}$ into several distinct pieces (i.e., the several G-orbits). Hence, from G's perspective $\mathcal{M}$ would then look inhomogeneous (i.e., it would have several different kinds of points, namely these distinct G-orbits).

It is well known that the quotient of two Lie groups is generally a smooth homogeneous manifold. For instance: The quotient of $\mathrm{SO}(3)$ by $\mathrm{SO}(2)$ is $\mathrm{SO}(3)/\mathrm{SO}(2) \cong S^2$, the 2-sphere; The quotient of $(\mathbb{R}^2, +)$ by the trivial group $\{\mathbb{1}\}$ is the plane, $\mathbb{R}^2$; The quotient of $\mathrm{SE}(2)$ by its rotation subgroup $\mathrm{SO}(2)$ is also $\mathbb{R}^2$; Perhaps surprisingly, the quotient of $\mathrm{SE}(2)$ by its $\mathrm{E}(1)$ subgroup is $\mathbb{M}$, the Möbius strip. Concrete demonstrations of these quotients can be found in Appendix A.2 and A.3. In general, given any finite-dimensional Lie group, G_1, and any closed Lie subgroup thereof, G_2, their quotient $\mathcal{M}_{\mathrm{G}} \coloneqq G_1/G_2$ is a smooth homogeneous manifold with G_1 acting smoothly and transitively over it. This is the Homogeneous Manifold Construction Theorem discussed in Appendix A.

But are there any homogeneous manifolds which cannot be built by taking the quotient of Lie groups, G_1/G_2? No there are not! Recall that homogeneous smooth manifolds, $\mathcal{M}$, comes along with a transitively acting Lie group $H_{\text{trans}} \subset \mathrm{Diff}(\mathcal{M})$. This will be the group in the numerator. Now let $p_0 \in \mathcal{M}$ and let H_{fix} be the stabilizer subgroup of H_{trans} at the point p_0. Namely, $\forall h \in H_{\text{fix}} \;\; h(p_0) = p_0$. This will be the group in the denominator. By the Homogeneous Manifold Construction Theorem, the quotient of these two groups is a homogeneous manifold, $\mathcal{M}_{\mathrm{H}} := H_{\text{trans}}/H_{\text{fix}}$. It turns out, this quotient manifold is always

[78] For proof see Mostow (2005).

diffeomorphic to the smooth manifold which we started with, $\mathcal{M}_{\text{H}} \cong \mathcal{M}$. This is the Homogeneous Manifold Characterization Theorem discussed in Appendix A. This result is robust in the sense that it does not matter which transitively acting Lie group, H_{trans}, and which stabilizer subgroup, H_{fix}, we use; The result is always $\mathcal{M}_{\text{H}} \cong \mathcal{M}$. Given these two results, the class of all smooth homogeneous manifolds are, up to diffeomorphism, exactly the class of all such Lie group quotients.

We are now in a position to see how $\mathcal{M}_{\text{old}}$ can faithfully survive internalization. To do so we will first reconstruct it as a quotient of its diffeomorphisms. To do this, we need to pick out a finite-dimensional Lie group, $H \subset \text{Diff}(\mathcal{M}_{\text{old}})$, which acts transitively on $\mathcal{M}_{\text{old}}$. The Lie group $H_{\text{trans}} \cong (\mathbb{R}^2, +) \times \text{SO}(3)$ discussed above is just such a Lie group (hence the subscript "trans"). One can easily check that $H_{\text{fix}} \cong \{\mathbb{1}\} \times \text{SO}(2)$ is one of H_{trans}'s stabilizer subgroups (e.g., the subgroup of H_{trans} which fixes the north pole at $t = 0$ and $q = 0$, hence the subscript "fix"). By using the above-discussed Homogeneous Manifold Characterization Theorem we can reconstruct $\mathcal{M}_{\text{old}}$ up to diffeomorphism as,

$$\mathcal{M}_{\text{old}} \cong \mathcal{M}_{\text{H}} \coloneqq \frac{H_{\text{trans}}}{H_{\text{fix}}} \cong \frac{(\mathbb{R}^2, +) \times \text{SO}(3)}{\{\mathbb{1}\} \times \text{SO}(2)} \cong \mathbb{R}^2 \times S^2. \tag{7.12}$$

A concrete demonstration of the non-trivial part of this quotient, $\text{SO}(3)/\text{SO}(2) \cong S^2$, can be found in Appendix A.2. Moreover, a reconstruction of the Möbius strip, $\mathcal{M} \cong \text{SE}(2)/\text{E}(1)$, in terms of a Lie group quotient can be found in Appendix A.3.

Having reconstructed the old theory's spacetime manifold in this way, we can now see it as faithfully surviving internalization. As noted above, the Lie group H_{trans} (and, consequently, its subgroup $H_{\text{fix}} \subset H_{\text{trans}}$) faithfully survives internalization:

$$\begin{aligned} H_{\text{trans}} &\xrightarrow[\text{int. as}]{\text{survives}} G_{\text{trans}} \coloneqq G(H_{\text{trans}}) \cong H_{\text{trans}}, \\ H_{\text{fix}} &\xrightarrow[\text{int. as}]{\text{survives}} G_{\text{fix}} \coloneqq G(H_{\text{fix}}) \cong H_{\text{fix}}, \end{aligned} \tag{7.13}$$

becoming some Lie groups, $G_{\text{fix}} \subset G_{\text{trans}}$, acting on $S^{\text{kin}}_{\text{pst}}$. By taking the quotient of these

surviving Lie groups we can faithfully reconstruct $\mathcal{M}_{\text{H}}$ (and, consequently, $\mathcal{M}_{\text{old}} \cong \mathcal{M}_{\text{H}}$) post-internalization as follows:

$$\mathcal{M}_{\text{old}} \cong \mathcal{M}_{\text{H}} \coloneqq \frac{H_{\text{trans}}}{H_{\text{fix}}} \xrightarrow[\text{int. as}]{\text{survives}} \mathcal{M}_{\text{G}} \coloneqq \frac{G_{\text{trans}}}{G_{\text{fix}}} \cong \frac{H_{\text{trans}}}{H_{\text{fix}}} \cong \mathcal{M}_{\text{old}}. \tag{7.14}$$

Thus, the Quartic Klein-Gordon theory's spacetime manifold has faithfully survived internalization.

This is a surprising conclusion. Recall that the stated purpose of internalization was to conceptually divorce the theory's states, φ_{old}, from any assumed topological background structure. Despite this divorce, it seems that complete topological information about $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2 \times S^2$ has survived internalization. As Ch. 10 will discuss (and as Appendix B.2 will prove), for any theory with a sufficient level of spacetime-kinematic compatibility, there is enough topological information contained in the surviving Lie groups G_{trans} and G_{fix} to perfectly reconstruct the theory's original spacetime setting. Namely, by applying the yet-to-be-discussed externalization process to these Lie groups, one can perfectly reconstruct the old theory's spacetime setting, up to diffeomorphism.

7.2 Internalization in a Familiar Setting: Quantum Harmonic Oscillator

Before discussing the externalization process, however, allow me to first quickly discuss how the internalization process applies to the non-local quantum harmonic oscillator theory discussed in Sec. 4.2. We have already seen there how to frame this theory as a spacetime theory. In particular, recall from Eq. (4.15) that we have already identified that this theory's dynamically and metaphysically relevant features constitute a Hilbert space (at each parameter $t \in T_{\text{old}}$),

$$\mathcal{S}^{\text{kin}}_{\text{old}} \coloneqq \langle S^{\text{kin}}_{\text{old}}, +_{\text{old}}, \cdot_{\text{old}}, T_{\text{old}}, I_{\text{old}} \rangle. \tag{7.15}$$

Hence, as with the Quartic Klein-Gordon theory, we can internalize this theory by applying a forgetful operation, $\mathcal{F}_{\text{vec}}$, at the level of kinematics which only remembers $\mathcal{S}^{\text{kin}}_{\text{old}}$'s algebraic structure as follows:

$$\mathcal{F}_{\text{vec}} : \mathcal{S}^{\text{kin}}_{\text{old}} \to \mathcal{S}^{\text{kin}}_{\text{pst}} \coloneqq \langle S^{\text{kin}}_{\text{pst}}, +_{\text{pst}}, \cdot_{\text{pst}}, T_{\text{pst}}, I_{\text{pst}} \rangle. \tag{7.16}$$

Ultimately, this means that we can state this theory's dynamical equation in a pre-spacetime theoretic way and hence identify $S^{\text{dyn}}_{\text{pst}} \subset \mathcal{S}^{\text{kin}}_{\text{pst}}$.

To finish internalizing this theory, let us see how its spacetime manifold faithfully survives internalization. If $\mathcal{M}_{\text{old}}$ is to survive internalization we must first reconstruct it as a quotient of its diffeomorphisms. To do this, we need a pick out a finite-dimensional Lie group, $H_{\text{trans}} \subset \text{Diff}(\mathcal{M}_{\text{old}})$, which acts transitively $\mathcal{M}_{\text{old}}$. According to the above-discussed Homogeneous Manifold Characterization Theorem it actually does not matter which such Lie group, H_{trans}, we pick. Namely, we can choose H_{trans} to be any Lie group which acts transitively on $\mathcal{M}$ and we can then choose H_{fix} to be any of H_{trans}'s stabilizer subgroups. The simplest Lie group which satisfies these criteria is the group, $H_{\text{trans}} \cong (\mathbb{R}^2, +)$, which implements rigid translations in the old theory's (t, x) coordinate system. Note that the stabilizer subgroups for this choice of H_{trans} are all identical, $H_{\text{fix}} = \{\mathbb{1}\}$.

Whichever Lie groups we use to reconstruct the old theory's spacetime manifold, $\mathcal{M}_{\text{old}} \cong H_{\text{trans}}/H_{\text{fix}}$, we can subsequently see it surviving internalization as follows. The Lie groups in the numerator and denominator of this quotient each faithfully survive internalization becoming a pair of Lie groups, $G_{\text{trans}} \coloneqq G(H_{\text{trans}}) \cong H_{\text{trans}}$ and $G_{\text{fix}} \coloneqq G(H_{\text{fix}}) \cong H_{\text{fix}}$, which act on $S^{\text{kin}}_{\text{pst}}$, the space of kinematically allowed states in the pre-spacetime theory.[79] Hence, the theory's old spacetime manifold, $\mathcal{M}_{\text{old}} \cong H_{\text{trans}}/H_{\text{fix}}$, also faithfully survives internalization becoming $\mathcal{M}_{\text{G}} \coloneqq G_{\text{trans}}/G_{\text{fix}} \cong \mathcal{M}_{\text{old}}$.

[79] We are here benefiting from the fortunate coincidence that (like in the Quartic Klein-Gordon theory) every diffeomorphism is kinematically allowed $\text{Diff}_{\text{kin}}(\mathcal{M}) = \text{Diff}(\mathcal{M})$. Moreover, this theory is kinematically reduced. See footnote 74.

Let's recap: So far we have applied the internalization process to two example theories. In each case, we have divorced the dynamical behavior of matter from its assumed topological underpinnings (in the sense that $\varphi_{\text{old}} \mapsto \varphi_{\text{pst}}$ is no longer a function of spacetime points). As we have seen, all of the theory's dynamically and metaphysically relevant structures can be made to survive internalization. Surprisingly, however, despite this attempted topological divorce we have seen that a great deal of topological information actually survives internalization. In fact, in each of these examples, it seems that complete topological information about the old theory's spacetime manifold, $\mathcal{M}_{\text{old}}$, has survived internalization.

7.3 Externalization in a Familiar Setting: Implementing a Fourier Redescription

The previous section has discussed how a theory's original spacetime manifold, $\mathcal{M}_{\text{old}}$, can faithfully survive internalization. Namely, it can survive as a quotient of two Lie groups, $G^{(\text{int})}_{\text{trans}}$ and $G^{(\text{int})}_{\text{fix}}$, which act on $S^{\text{kin}}_{\text{pst}}$. The superscript (int) here indicates that these Lie groups were produced by the internalization process. Of course, knowing how a theory's spacetime manifold survives internalization is very suggestive of how externalization might be able to generate new spacetime settings for this theory.

Up front, a brief comment regarding notation is needed. Externalization is, roughly, the inverse process of internalization. Hence, many of the mathematical objects used in the externalization process are closely analogous to the objects used in the internalization process. To highlight these structural similarities, I have chosen to denote the analogous objects with the same symbols (e.g., $G^{(\text{int})}_{\text{trans}}$ vs. $G^{(\text{ext})}_{\text{trans}}$ and $H^{(\text{int})}_{\text{trans}}$ vs. $H^{(\text{ext})}_{\text{trans}}$). Where necessary, superscripts (ext) and (int) will be adopted to distinguish the objects used in externalization and internalization respectively. Most of the time, however, these superscripts will not be necessary.

This section will introduce the externalization process by using it to build a new spacetime setting for the non-local QHO theory discussed above. In particular, the new theory's spacetime will be effectively a copy of the old theory's Fourier space. Moreover, its states will be related to the old theory's states via a Fourier transform. While it may seem like overkill to use the ISE Method to achieve this Fourier redescription, the techniques demonstrated here are applicable much more generally. Other applications of the externalization process can be found in Ch. 8. Beyond these examples, I will prove in Ch. 11 that the ISE Method is a very general tool for topological redescription; Using it, one can access effectively every possible spacetime framing of a given theory, bounded only by a certain spacetime-kinematic compatibility condition.

Without further ado, let us see the externalization process in action. Externalization builds new spacetime settings from certain building blocks, namely, pre-spacetime translation operations (PSTOs). Concretely, these are a pair of Lie groups, $G^{(\text{ext})}_{\text{fix}} \subset G^{(\text{ext})}_{\text{trans}}$, which act on $S^{\text{kin}}_{\text{pst}}$. However, not just any pair of Lie group actions will qualify as PSTOs, they must be structurally indistinguishable from spacetime translations per the following definition.[80]

> **Definition: Pre-Spacetime Translation Operations (PSTOs) -** Let $G_2 \subset G_1$ be a pair of Lie groups which act on some space $\mathcal{S}^{\text{kin}} = \langle S^{\text{kin}}, \dots \rangle$. This pair of Lie group actions will be called *pre-spacetime translation operations* when they have both faithful construction compatibility and injection compatibility.

The technical definitions of faithful construction compatibility and injection compatibility will be given throughout this section. (Additionally, the various pieces of this definition are collected together and generalized past the scalar field case in Sec. 10.1.) Externalization takes some hand-picked PSTOs, $G^{(\text{ext})}_{\text{fix}} \subset G^{(\text{ext})}_{\text{trans}}$, and builds a new spacetime from them, $\mathcal{M}_{\text{new}}$, in such a way that our PSTOs become honest-to-goodness spacetime

[80] N.B.: The term "translation" is here being used very broadly, such that it includes rotations (e.g., the action of SO(3) on S^2) as well as generic diffeomorphisms.

translations, $H^{(\text{ext})}_{\text{fix}} \subset H^{(\text{ext})}_{\text{trans}} \subset \text{Diff}(\mathcal{M}_{\text{new}})$. Our requirement that PSTOs have faithful construction compatibility will guarantee that, $G^{(\text{ext})}_{\text{trans}}$, ends up faithfully represented as diffeomorphisms on the new spacetime setting, namely, $H^{(\text{ext})}_{\text{trans}} \cong G^{(\text{ext})}_{\text{trans}}$ and $H^{(\text{ext})}_{\text{fix}} \cong G^{(\text{ext})}_{\text{fix}}$. Injection compatibility will then guarantee that the Lie group action of $G^{(\text{ext})}_{\text{trans}}$ on $S^{\text{kin}}_{\text{pst}}$ is faithfully represented by the lifted actions these diffeomorphisms, $H^{(\text{ext})*}_{\text{trans}}$, on some yet-to-be-defined states, $S^{\text{kin}}_{\text{new}}$, defined on $\mathcal{M}_{\text{new}}$. This justifies the above gloss of the externalization process.

Let us begin by briefly revisiting the example PSTOs discussed in Ch. 6 (and displayed in Fig. 6.1). I there claimed that the old theory's spacetime translation operations will always qualify as PSTOs (see the first column of Fig. 6.1). More technically, this claim is about the Lie group actions which have faithfully survived internalization; Namely, it is about $G^{(\text{int})}_{\text{trans}}$ and $G^{(\text{int})}_{\text{fix}}$ which act on $S^{\text{kin}}_{\text{pst}}$. In Appendix B.2 I prove that this pair of Lie group actions will always qualify as PSTOs whenever the theory in question has a sufficient level of spacetime-kinematic compatibility. Moreover, it is there proved that by externalizing these PSTOs one can easily return to the theory's original spacetime setting.

Importantly, however, we are not forced to go back the way we came. We are free to search for other pre-spacetime translation operations to externalize, i.e., ones with $G^{(\text{ext})}_{\text{trans}} \neq G^{(\text{int})}_{\text{trans}}$ and/or $G^{(\text{ext})}_{\text{fix}} \neq G^{(\text{int})}_{\text{fix}}$. To have a concrete alternative in mind, consider the non-local QHO theory discussed in Sec. 7.2. Let the maps,

$$f(\Delta t, \Delta k)\, \varphi_{\text{old}}(t, x) = e^{-\mathrm{i}\Delta k x}\, \varphi_{\text{old}}(t - \Delta t, x), \tag{7.17}$$

form a Lie group $f(\Delta t, \Delta k) \in F_{\text{Fourier}}$ which acts on $S^{\text{kin}}_{\text{old}}$. Some Lie group actions of this kind are depicted in the second column of Fig. 6.1. Note that these F_{Fourier} transformations are kinematically allowed and hence faithfully survive internalization becoming a Lie group, $G_{\text{Fourier}} = \mathcal{F}_{\text{vec}} \circ F_{\text{Fourier}} \circ \mathcal{F}^{-1}_{\text{vec}}$, which acts on $S^{\text{kin}}_{\text{pst}}$.

There is an intuitive sense in which G_{Fourier}'s Lie group action on $S^{\text{kin}}_{\text{pst}}$ should qualify as PSTOs; Namely, when viewed as a Lie group action on the theory's states, there is a sense in which these Fourier shift operations are structurally identical to spacetime translations. Indeed, while these aren't translations in the old theory's spacetime, they are nonetheless translations *somewhere*, namely in the old theory's Fourier space. Intuitively, we should expect that externalizing these PSTOs ought to produce a new theory which is, roughly speaking, set on a copy of the old theory's Fourier space. With this example of PSTOs in mind, let us next see how the externalization process works more generally.

Recall from Sec. 7.1 that a theory's spacetime manifold can faithfully survive internalization as a quotient of two Lie groups, $G^{(\text{int})}_{\text{trans}}$ and $G^{(\text{int})}_{\text{fix}}$, which act on $S^{\text{kin}}_{\text{pst}}$. Analogously, externalization begins from a pair of Lie groups $G^{(\text{ext})}_{\text{trans}}$ and $G^{(\text{ext})}_{\text{fix}}$, which act on $S^{\text{kin}}_{\text{pst}}$. Externalization puts forward the quotient manifold of these Lie groups as the theory's new spacetime manifold,

$$\mathcal{M}_{\text{new}} \cong \mathcal{M}_{\text{G}} \coloneqq G^{(\text{ext})}_{\text{trans}}/G^{(\text{ext})}_{\text{fix}}, \tag{7.18}$$

at least up to diffeomorphism. I say here "up to diffeomorphism" because the quotient manifold $\mathcal{M}_{\text{G}}$ contains excess structure; Its elements are sets of maps on $S^{\text{kin}}_{\text{pst}}$ as well as manifold points on $\mathcal{M}_{\text{G}}$. We can remove this excess structure by applying a forgetful operation $\mathcal{F}_{\text{man}}$ to $\mathcal{M}_{\text{G}}$. Namely, this is a diffeomorphism, $\mathcal{F}_{\text{man}} : \mathcal{M}_{\text{G}} \to \mathcal{M}_{\text{new}}$, where $\mathcal{M}_{\text{new}}$ is some smooth manifold which is isomorphic to $\mathcal{M}_{\text{G}}$, i.e., with $\mathcal{M}_{\text{new}} \cong \mathcal{M}_{\text{G}}$. In terms of this forgetful operation, the new spacetime manifold is defined as,

$$\mathcal{M}_{\text{new}} \coloneqq \mathcal{F}_{\text{man}}(G^{(\text{ext})}_{\text{trans}}/G^{(\text{ext})}_{\text{fix}}). \tag{7.19}$$

For notational convenience, let us now drop the (ext) superscripts.

It is worth reflecting momentarily on where the new spacetime, $\mathcal{M}_{\text{new}}$, gets its smooth topological structure from. As the above definition indicates, $\mathcal{M}_{\text{new}}$ gets its smooth struc-

ture from the quotient manifold, $\mathcal{M}_{\text{G}} = G_{\text{trans}}/G_{\text{fix}}$. But where does $\mathcal{M}_{\text{G}}$ get its smooth structure from? Ultimately, it gets it from the smooth structure of its numerator, G_{trans}, via the quotient with G_{fix}.[81] But where does G_{trans} get its smooth structure from? G_{trans} itself has a naturally motivated smooth structure coming from it being a Lie group. Namely, for every Lie group there is a unique smooth structure which is compatible with its algebraic and topological structure. Hence, the ultimate source of the new spacetime's smooth structure is the smoothness of the Lie group G_{trans} which acts on $S^{\text{kin}}_{\text{pst}}$.[82] In contrast with the algebraic approach to spacetime, the theory's states, $\varphi_{\text{pst}} \in \mathcal{S}^{\text{kin}}_{\text{pst}}$, do not need to have any algebraic structure for this construction to proceed. Instead, this construction proceeds based solely upon our ability to subject the theory's states to a Lie group action.

In addition to giving $\mathcal{M}_{\text{new}}$ its smooth structure, the Lie group G_{trans} is also naturally represented as diffeomorphisms on $\mathcal{M}_{\text{new}}$. To see this, note that the left action of G_{trans} on itself can be seen as smoothly permuting its G_{fix}-cosets (that is, smoothly permuting the points $[g_0] \in \mathcal{M}_{\text{G}}$). Concretely, we have that $g \in G_{\text{trans}}$ acts smoothly on $\mathcal{M}_{\text{G}}$ as $\bar{g}([g_0]) \coloneqq [g\, g_0]$. These maps, $\bar{g} : \mathcal{M}_{\text{G}} \to \mathcal{M}_{\text{G}}$, are diffeomorphisms and form a Lie group $\overline{G}_{\text{trans}}$. These diffeomorphisms, $\bar{g} \in \text{Diff}(\mathcal{M}_{\text{G}})$, can be transferred over into diffeomorphisms on $\mathcal{M}_{\text{new}}$ as,

$$H(g) \coloneqq \mathcal{F}_{\text{man}} \circ \bar{g} \circ \mathcal{F}^{-1}_{\text{man}}. \tag{7.20}$$

In total therefore we have a Lie group homomorphism, $H : G_{\text{trans}} \to \text{Diff}(\mathcal{M}_{\text{new}})$. The image of this map, $H^{(\text{ext})}_{\text{trans}} \coloneqq H(G_{\text{trans}})$, is a representation of G_{trans} as diffeomorphisms on $\mathcal{M}_{\text{new}}$.

[81] This transference of smooth structure from G_{trans} to $\mathcal{M}_{\text{G}}$ (and hence to $\mathcal{M}_{\text{new}}$) requires that G_{fix} is a closed subgroup of G_{trans}. See the Quotient Manifold Theorem discussed in Appendix A. This is guaranteed by our assumption that G_{trans} and G_{fix} have faithful construction compatibility.

[82] One may wish to grant $\mathcal{S}^{\text{kin}}_{\text{pst}}$ either a topological structure or a smooth structure and then demand that G_{trans}'s action on $\mathcal{S}^{\text{kin}}_{\text{pst}}$ is either continuous or smooth. In this case, one could then attribute the new spacetime's continuity/smoothness to the continuity/smoothness of $\mathcal{S}^{\text{kin}}_{\text{pst}}$. As natural as this extra demand and re-attribution are, it is not required by the ISE Method of topological redescription. Hence, it should be thought of as a part of the selection criteria which we use to evaluate possible spacetime framings of our theories.

Recall that our goal is to build a new spacetime setting for the theory in question such that our PSTOs end up as honest-to-goodness spacetime translations on $\mathcal{M}_{\text{new}}$. To achieve this, we need some assurance that the Lie group, $H^{(\text{ext})}_{\text{trans}}$, will be a faithful representation of our PSTOs, namely that $H^{(\text{ext})}_{\text{trans}} \cong G_{\text{trans}}$. As I prove in Appendix A, this will happen if and only if G_{trans} and G_{fix} have faithful construction compatibility.

> **Definition: Faithful Construction Compatibility (for a smooth manifold)**
> - Let G_1 and G_2 be two Lie groups. These two Lie groups have *faithful construction compatibility* if and only if G_2 is both a closed subgroup of G_1 as well as a core-free subgroup of G_1.

The demand that G_2 be a core-free subgroup of G_1 is equivalent to demanding that the quotient G_1/G_2 is already in "lowest terms". Namely, it is equivalent to demanding that G_2 does not contain any non-trivial normal subgroups of G_1 (any such subgroups would immediately cancel in the quotient). For later reference, a fact regarding abelian groups should be noted. If G_1 is abelian, then all of its subgroups, G_2, are normal, $G_2 \lhd G_1$. Hence, if G_1 and G_2 are supposed to have faithful construction compatibility and G_1 is abelian then we must have $G_2 = \{\mathbb{1}\}$. In such cases, I will sometimes talk about PSTOs as if they consist of one Lie group (only G_{trans}) instead of two (G_{trans} and G_{fix}).

Let us now return our attention to externalizing the non-local QHO theory with the following choice of PSTOs, $G_{\text{trans}} = G_{\text{Fourier}}$. In order to complete our choice of PSTOs, however, we also need to pick out a Lie subgroup, $G_{\text{fix}} \subset G_{\text{trans}}$, which has faithful construction compatibility with G_{trans}. What are our options? As I have just discussed, it follows from $G_{\text{Fourier}} \cong (\mathbb{R}^2, +)$ being abelian that we are forced to pick $G_{\text{fix}} = \{\mathbb{1}\}$. Given these PSTOs, the new theory's spacetime manifold is,

$$\mathcal{M}_{\text{new}} \cong G_{\text{trans}}/G_{\text{fix}} \cong \mathbb{R}^2. \tag{7.21}$$

Note that while this new spacetime is diffeomorphic to the old one, $\mathcal{M}_{\text{new}} \cong \mathcal{M}_{\text{old}} \cong \mathbb{R}^2$ it is nonetheless distinct, $\mathcal{M}_{\text{new}} \neq \mathcal{M}_{\text{old}}$. To see the distinction, recall that Fourier

shifts in the old theory, $f(\Delta t, \Delta k) \in F_{\text{Fourier}}$, have faithfully survived internalization becoming $g(\Delta t, \Delta k) \in G_{\text{Fourier}} = G^{(\text{ext})}_{\text{trans}}$ acting on $S^{\text{kin}}_{\text{pst}}$. But this is the Lie group which we have just externalized. Thus, post-externalization we will have $g(\Delta t, \Delta k)$ becoming a diffeomorphism, $h(\Delta t, \Delta k) \coloneqq H(g(\Delta t, \Delta k)) \in H^{(\text{ext})}_{\text{trans}} \subset \text{Diff}(\mathcal{M}_{\text{new}})$, on our new two-dimensional manifold. Let (τ, q) be a coordinate system for $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$ such that $h(\Delta t, \Delta k)(\tau, q) = (\tau - \Delta t, q - \Delta k)$. Thus, shifts in Fourier space in the old theory, F_{Fourier}, become shifts in spacetime in the new theory, $H^{(\text{ext})}_{\text{trans}}$. As promised, externalization has turned our choice of PSTOs into honest-to-goodness spacetime translations on $\mathcal{M}_{\text{new}}$.

The connection between $\mathcal{M}_{\text{new}}$ and the old theory's Fourier space will only become stronger as we proceed to the next step in the externalization process, namely injecting the old theory's states and dynamics into the new spacetime setting. Externalization must somehow map the theory's pre-spacetime states, $\varphi_{\text{pst}} \in \mathcal{S}^{\text{kin}}_{\text{pst}}$, and their dynamics onto $\mathcal{M}_{\text{new}}$. Let us assume that the new theory will be about a scalar field, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathcal{V}_{\text{new}}$ for some value space, $\mathcal{V}_{\text{new}}$.[83]

In order to begin identifying the new theory's states we must first build $S^{\text{all}}_{\text{new}}$, the set of all $\mathcal{V}_{\text{new}}$-valued fields definable on $\mathcal{M}_{\text{new}}$ (even those which are non-smooth and/or discontinuous). Within this wide scope we must next pick out the subset of states which are kinematically allowed in the new theory, $S^{\text{kin}}_{\text{new}} \subset S^{\text{all}}_{\text{new}}$. Recall that we already have a pre-spacetime theoretic characterization of the theory's kinematically allowed states, namely, $S^{\text{kin}}_{\text{pst}}$. All we need to do is to map these states onto the new spacetime setting via some map $J : \varphi_{\text{pst}} \mapsto \varphi_{\text{new}}$. Demanding that distinct states, $\varphi_{\text{pst}} \in S^{\text{kin}}_{\text{pst}}$, are mapped onto distinct states, $\varphi_{\text{new}} \in S^{\text{all}}_{\text{new}}$, we are looking for an injective map, $J : S^{\text{kin}}_{\text{pst}} \to S^{\text{all}}_{\text{new}}$. Fixing this injector, J, will fix the new theory's kinematically allowed states as $S^{\text{kin}}_{\text{new}} \coloneqq J(S^{\text{kin}}_{\text{pst}})$.

[83] It should be noted that we are not forced into having the new theory be about a scalar field, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathcal{V}_{\text{new}}$. For instance, Ch. 10 will discuss how to externalize into theories about particle trajectories, $X_{\text{new}} : \mathcal{V}_{\text{new}} \to \mathcal{M}_{\text{new}}$ and tangent vector fields, $\vec{\varphi}_{\text{new}} : \mathcal{M}_{\text{new}} \to \text{T}\mathcal{M}_{\text{new}}$.

Given the above-discussed connection between the new theory's spacetime and the old theory's Fourier space, an intuitive choice of J would be the one which implements a Fourier transform. Namely, we should be able to externalize via an injective map J which (together with $\mathcal{F}_{\text{vec}}$) defines φ_{new} as follows:

$$J \circ \mathcal{F}_{\text{vec}} : \varphi_{\text{old}} \mapsto \varphi_{\text{new}}(\tau, q) = \tilde{\varphi}_{\text{old}}(\tau, q). \tag{7.22}$$

That is, the value of φ_{new} at coordinate (τ, q) has the same value that $\tilde{\varphi}_{\text{old}}(t, k)$ does (namely, $\varphi_{\text{old}}(t, x)$'s Fourier transform) at $(t, k) = (\tau, q)$. At this point one may rightly wonder, are we somehow forced into this choice of J or are there other ways in which we could inject this theory's states onto $\mathcal{M}_{\text{new}}$? I will return to this question in Sec. 7.4. For now, however, let us proceed with this intuitive choice of J and see where it leads us.

With this choice of injector $J : S^{\text{kin}}_{\text{pst}} \to S^{\text{all}}_{\text{new}}$, one can straight-forwardly translate the old theory's dynamics and kinematics onto its new spacetime setting. The result is exactly the Cosine QHO theory discussed in Sec. 4.2. Recall that our two QHO theories have the following dynamical equations:

$$\begin{aligned} \mathrm{i}\partial_t \varphi_{\text{old}}(t, x) &= (-\partial_x^2 + x^2 + \lambda \cosh(a\, \partial_x))\, \varphi_{\text{old}}(t, x) \\ \mathrm{i}\partial_\tau \varphi_{\text{new}}(\tau, q) &= (-\partial_q^2 + q^2 + \lambda \cos(a\, q))\, \varphi_{\text{new}}(\tau, q). \end{aligned} \tag{7.23}$$

Note that the new theory is about a standard QHO plus a cosine-shaped potential. Moreover, note that whereas the old theory's dynamics was non-local, this new theory has local dynamics. In this way, this new spacetime setting is a better fit for this theory's dynamics than its old setting was.

To finalize the externalization process, let us next see how the old theory's dynamically and metaphysically relevant structures have survived into the new theory. Recall from Sec. 7.2 (specifically, Eq. (7.16)) that the following features of the non-local QHO theory survived internalization: an addition operation (which was point-wise on $\mathcal{M}_{\text{old}}$),

an scalar multiplication operation (which was also point-wise on $\mathcal{M}_{\text{old}}$), an inner product (specifically the L^2 inner product at each parameter $t \in T_{\text{old}}$). Note that since the externalization map, $J : S^{\text{kin}}_{\text{pst}} \to S^{\text{all}}_{\text{new}}$, is injective we have a bijection, $J : S^{\text{kin}}_{\text{new}} \leftrightarrow S^{\text{kin}}_{\text{pst}}$ between our two spacetime framing of this theory's kinematically allowed states. We can use this bijection to transfer over all of the old theory's dynamically and metaphysically relevant structure onto $S^{\text{kin}}_{\text{new}}$ as follows:[84]

$$J : \mathcal{S}^{\text{kin}}_{\text{pst}} \to \mathcal{S}^{\text{kin}}_{\text{new}} \coloneqq \langle S^{\text{kin}}_{\text{new}}, +_{\text{new}}, \cdot_{\text{new}}, T_{\text{new}}, I_{\text{new}} \rangle. \tag{7.24}$$

Note that these new structures on $S^{\text{kin}}_{\text{new}}$ are exactly those induced by the J map (i.e., they are defined analogously to Eq. (7.4)).

But what are these induced structures exactly? It follows from familiar properties of the Fourier transform that $+_{\text{new}}$ and $\cdot_{\text{new}}$ are the standard point-wise addition and scalar multiplications operation on $\mathcal{M}_{\text{new}}$. Moreover, it follows that I_{new} is the L^2 inner product at each parameter $\tau \in T_{\text{new}}$. Thus, given our above choice of J the new theory's induced structures all seem to fit very well on its new spacetime manifold. It should be stressed, however, that this is not guaranteed to happen in general. For instance, a different choice of J might induce a different inner product, I_{new}, than the standard L^2 inner product. Alternatively, suppose that (like the Quartic Klein-Gordon theory) the non-local QHO theory had come equipped with a pointwise product operation, $\times_{\text{old}}$, on $\mathcal{M}_{\text{old}}$. In this case, the induced product operation in the new theory, $\times_{\text{new}}$, would have been a convolutional product on $\mathcal{M}_{\text{new}}$. It is important to note that the new theory's induced structures follow solely from our choice of injective map, $J : S^{\text{kin}}_{\text{pst}} \to S^{\text{all}}_{\text{new}}$. They are not arrived at by identifying which operations are natural (e.g., pointwise or local) on $\mathcal{M}_{\text{new}}$. Although, of course, our choice of PSTOs might be guided by whether or not a nice J map will be available to us.

[84] In addition to using this bijection, we also need to make a copy of the old theory's auxiliary space as $T_{\text{new}} \cong T_{\text{old}}$.

7.4 Can we poplulate Fourier space other than via a Fourier transform?

The remainder of this section will discuss the latter half of the externalization process more generally (i.e., how to pick an injector $J : S^{\text{kin}}_{\text{pst}} \to S^{\text{all}}_{\text{new}}$ given some PSTOs). The following questions will be answered in turn: What are the minimal constraints which our choice of J must satisfy? For a generic set of PSTOs, how do we know that such a J map exists? Given that such a J map exists, to what extent is it uniquely determined by our PSTOs?

Let us begin by placing some constraints on our choice of J. Recall that our goal is to build a new spacetime setting for the theory in question such that every pre-spacetime translation operation, $g \in G_{\text{trans}}$, becomes an honest-to-goodness spacetime translation in the new theory. We have so far seen how for every $g \in G_{\text{trans}}$ the externalization process generates a corresponding diffeomorphism, $H(g) \in \text{Diff}(\mathcal{M}_{\text{new}})$. In order for this to truly count as a spacetime translation, however, it needs to lift to act in the right way on the new theory's states, $J(\varphi_{\text{pst}})$. Establishing this will require that we demand some compatibility between our choice of injector J and our choice of PSTOs.

For any Lie group element, $g \in G_{\text{trans}}$, we already know how it acts on any pre-spacetime state, $\varphi_{\text{pst}} \in S^{\text{kin}}_{\text{pst}}$. Namely, it acts as $\varphi_{\text{pst}} \mapsto g\,\varphi_{\text{pst}}$. In addition to converting every $g \in G_{\text{trans}}$ into a diffeomorphism, $H(g) \in H_{\text{trans}} \subset \text{Diff}(\mathcal{M}_{\text{new}})$, externalization also converts every pre-spacetime state, $\varphi_{\text{pst}} \in S^{\text{kin}}_{\text{pst}}$, into a state on the new spacetime, $J(\varphi_{\text{pst}}) \in S^{\text{kin}}_{\text{new}}$. Our choice of J must be such that these two externalization maps, $g \mapsto H(g)$ and $\varphi_{\text{pst}} \mapsto J(\varphi_{\text{pst}})$, are compatible:

$$\forall g\, \forall \varphi_{\text{pst}} \;\; (J \circ g)\, \varphi_{\text{pst}} = (H(g)^* \circ J)\, \varphi_{\text{pst}}. \tag{7.25}$$

with g and φ_{pst} ranging over G_{trans} and $S^{\text{kin}}_{\text{pst}}$ respectively. That is, enacting any $g \in G_{\text{trans}}$

on any $\varphi_{\text{pst}} \in S^{\text{kin}}_{\text{pst}}$ then externalizing via J ought to be equivalent to first externalizing φ_{pst} with J and then enacting the corresponding lifted diffeomorphism, $H(g)^*$. Demanding Eq. (7.25) of our injector J ensures that externalization preserves not only the Lie group structure of G_{trans} but also its structure as a Lie group action.

We must, however, demand slightly more than Eq. (7.25) in order to ensure our desired result, i.e., that the action of $g \in G_{\text{trans}}$ on $S^{\text{kin}}_{\text{pst}}$ is faithfully represented by the action of $H(g)^* \in {H_{\text{trans}}}^*$ on $S^{\text{kin}}_{\text{pst}}$. Given only Eq. (7.25) we could conceivably have distinct $g_1 \neq g_2$ which correspond to identical actions on $S^{\text{kin}}_{\text{pst}}$, namely $H(g_1)^*|_{\text{kin}} = H(g_2)^*|_{\text{kin}}$. If this were to happen, then $H(g)^*$'s action on $S^{\text{kin}}_{\text{new}}$ would not be a faithful representation of g's action on $S^{\text{kin}}_{\text{pst}}$. In order to avoid this we must demand that for all diffeomorphisms, $h \in H_{\text{trans}} \subset \text{Diff}(\mathcal{M}_{\text{new}})$, we have,

$$\Big(\forall \varphi_{\text{pst}} \;\; h^* J(\varphi_{\text{pst}}) = J(\varphi_{\text{pst}})\Big) \iff h = \mathbb{1}, \tag{7.26}$$

with φ_{pst} ranging over $S^{\text{kin}}_{\text{pst}}$. One can easily check that our above-discussed choice of J (i.e., the one which implements a Fourier transform) satisfies these two conditions.

To review, externalization builds a new spacetime setting for our theory from the following materials: A pair of Lie groups, G_{trans} and G_{fix}, with faithful construction compatibility which act on $S^{\text{kin}}_{\text{pst}}$; a value space, $\mathcal{V}_{\text{new}}$, for the new theory; and an injective map, $J : S^{\text{kin}}_{\text{pst}} \to S^{\text{all}}_{\text{new}}$. This construction is considered successful if Eq. (7.25) and Eq. (7.26) are satisfied. But given some pair of Lie group actions, G_{trans} and G_{fix}, how do we know that such value spaces, $\mathcal{V}_{\text{new}}$, and injective maps, J, exist? Their existence is guaranteed by demanding that G_{trans} and G_{fix} have injection compatibility for some kind of scalar field state.

> **Definition: Injection Compatibility (for a $\mathcal{V}_{\mathbf{G}}$-valued scalar field)**[85] - Let G_1 be a Lie group which acts on some space $\mathcal{S}^{\text{kin}} = \langle S^{\text{kin}}, \dots \rangle$. Let $G_2 \subset G_1$

[85] In Ch. 10 I will define injection compatibility *for a particle trajectory* as well as *injection compatibility for a tangent vector field*. A set of PSTOs has injection compatibility simpliciter just in case it has injection compatibility for some kind of state.

be a closed subgroup thereof. Let $\mathcal{M}_{\mathrm{G}} \coloneqq G_1/G_2$ be the quotient manifold of G_2 and G_1. Let $S^{\mathrm{all}}_{\mathrm{G}}$ be the space of all $\mathcal{V}_{\mathrm{G}}$-valued functions definable on $\mathcal{M}_{\mathrm{G}}$. Note that generic diffeomorphisms, $d \in \mathrm{Diff}(\mathcal{M}_{\mathrm{G}})$, lift to act on a generic scalar fields, $\varphi : \mathcal{M}_{\mathrm{G}} \to \mathcal{V}_{\mathrm{G}}$, as $d^*\varphi = \varphi \circ d^{-1}$.

This pair of Lie groups, $G_2 \subset G_1$, acting on S^{kin} have *injection compatibility for a $\mathcal{V}_G$-valued scalar field* if and only if the following condition holds. There exists an injective map, $J_{\mathrm{G}} : S^{\mathrm{kin}} \to S^{\mathrm{all}}_{\mathrm{G}}$, with the following two properties: Firstly, we must have,

$$\forall g \, \forall \sigma \;\; \Big((J_{\mathrm{G}} \circ g)\, \sigma = (\bar{g}^* \circ J_{\mathrm{G}})\, \sigma \Big), \tag{7.27}$$

and, secondly, for all diffeomorphisms, $d \in \mathrm{Diff}(\mathcal{M}_{\mathrm{G}})$, we must have,[86]

$$\Big(\forall \sigma \;\; d^* J_{\mathrm{G}}(\sigma) = J_{\mathrm{G}}(\sigma) \Big) \iff d = \mathbb{1}. \tag{7.28}$$

In both of these expressions g ranges over G_1 and σ ranges over S^{kin}.

Together with faithful construction compatibility, we are now demanding the following from our PSTOs. From G_{trans} and G_{fix} we can build a quotient manifold $\mathcal{M}_{\mathrm{new}} \cong G_{\mathrm{trans}}/G_{\mathrm{fix}}$ such that: 1) the acted-upon space, $S^{\mathrm{kin}}_{\mathrm{pst}}$, can be mapped onto some kind of states on $\mathcal{M}_{\mathrm{new}}$, which we have here chosen to be scalar field states, $\varphi_{\mathrm{new}} : \mathcal{M}_{\mathrm{new}} \to \mathcal{V}_{\mathrm{new}}$; and 2) The Lie group, G_{trans}, is faithfully represented as diffeomorphisms on $\mathcal{M}_{\mathrm{new}}$ whose lifted action on $S^{\mathrm{kin}}_{\mathrm{new}}$ is a faithful representation of G_{trans}'s action on $S^{\mathrm{kin}}_{\mathrm{pst}}$. This justifies the following gloss of the externalization process: Externalization builds a new spacetime by taking some hand-picked pre-spacetime translation operations, $G_{\mathrm{fix}} \subset G_{\mathrm{trans}}$, acting on $S^{\mathrm{kin}}_{\mathrm{pst}}$, and letting them be honest-to-goodness spacetime translations, $H_{\mathrm{fix}} \subset H_{\mathrm{trans}} \subset \mathrm{Diff}(\mathcal{M}_{\mathrm{new}})$, which lift to act on $S^{\mathrm{kin}}_{\mathrm{new}}$.

Now that we have identified the general conditions which our choice of injective map, $J : S^{\mathrm{kin}}_{\mathrm{pst}} \to S^{\mathrm{all}}_{\mathrm{new}}$, must satisfy (namely, Eq. (7.25) and Eq. (7.26)), we are in a position to ask: Were there any other options for J available to us in externalizing the non-local QHO theory? Recall that prior to making our intuitive choice of J in Sec. 7.3 we had already constructed $\mathcal{M}_{\mathrm{new}}$ from our Fourier-shift PSTOs and noted its close relationship with the

[86] A keen reader may note that what is demanded in Eq. (7.28) is slight more than what is needed to satisfy Eq. (7.26). The former quantifies over more diffeomorphisms than the latter does. As I will discuss in Ch. 10, making this only very slightly stronger demand greatly simplifies our analysis of the ISE Method.

old theory's Fourier space. Hence, we are really asking the following: Is there any good way to populate the new theory's spacetime (i.e., a copy of the old theory's Fourier space) other than with the Fourier transform of the old theory's states?

The remainder of this section will be spent showing that the above-discussed constraints come very close to forcing the intuitive choice of injector upon us. In particular, I will now show that there is only one choice of J which satisfies Eq. (7.25) as well as the following three assumptions about how the theory's states are to be represented on the new spacetime setting:

1. The new theory must be about a complex scalar field, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathcal{V}_{\text{new}}$, with $\mathcal{V}_{\text{new}} \cong \mathbb{C}$,
2. The new theory's addition and scalar multiplication operations, $+_{\text{new}}$ and $\cdot_{\text{new}}$ must be the standard pointwise operations on, $\mathcal{M}_{\text{new}}$. Let us call these $+_p$ and $\cdot_p$,
3. The new theory's inner product, I_{new}, must be the standard L^2 inner product on $\mathcal{M}_{\text{new}}$.

To see this, let us define the following state in the old theory,

$$\varphi_{\text{old}}^{(\lambda_1,\lambda_2)}(t,x) \coloneqq \exp(\mathrm{i}\,\lambda_1\,t)\,\delta(x-\lambda_2), \tag{7.29}$$

and note that the Fourier transform of this state,

$$\tilde{\varphi}_{\text{old}}^{(\lambda_1,\lambda_2)}(t,k) = \exp(\mathrm{i}\,\lambda_1\,t)\exp(\mathrm{i}\,\lambda_2\,k) \tag{7.30}$$

is a planewave *in the old theory's Fourier space*. Namely, $\tilde{\varphi}_{\text{old}}^{(\lambda_1,\lambda_2)}(t,k)$ is a simultaneous eigenvector of both ∂_t and ∂_k with eigenvalues $\mathrm{i}\lambda_1$ and $\mathrm{i}\lambda_2$ respectively. These differential operators are, of course, the generators of translations in the old theory's Fourier space, $f(\Delta t,\Delta k) \in F_{\text{Fourier}}$. Namely, we have $f(\mathrm{d}t,0) = \mathbb{1} + \partial_t$ and $f(0,\mathrm{d}k) = \mathbb{1} + \partial_k$. This means that $\varphi_{\text{old}}^{(\lambda_1,\lambda_2)}(t,x)$ is a simultaneous eigenvector of $f(\mathrm{d}t,0)$ and $f(0,\mathrm{d}k)$ with eigenvalues $1+\mathrm{i}\lambda_1$ and $1+\mathrm{i}\lambda_2$ respectively.

Per the above discussion, for every Fourier shift in the old theory, $f(\Delta t,\Delta k) \in F_{\text{Fourier}}$, there is a corresponding diffeomorphism on $\mathcal{M}_{\text{new}}$, namely $h(\Delta t,\Delta k) \in H_{\text{trans}}^{(\text{ext})}$.

In particular, corresponding to the infinitesimal Fourier shifts, $f(\mathrm{d}t, 0)$ and $f(0, \mathrm{d}k)$, there are corresponding infinitesimal diffeomorphisms on $\mathcal{M}_{\text{new}}$, namely $h(\mathrm{d}t, 0)$ and $h(0, \mathrm{d}k)$. Lifting these diffeomorphisms in the new spacetime setting we have $h(\mathrm{d}t, 0)^* = \mathbb{1} +_p \partial_\tau$ and $h(0, \mathrm{d}k)^* = \mathbb{1} +_p \partial_q$.Eq. (7.25) demands that these lifted diffeomorphisms act on the new theory's states in the same way that the above-discussed Fourier shifts acted on the old theory's states. Concretely, taking $\varphi_{\text{pst}} = \mathcal{F}_{\text{vec}}(\varphi_{\text{old}}^{(\lambda_1,\lambda_2)})$ and $g = g(\mathrm{d}t, 0)$ in Eq. (7.25) it follows that the new state,

$$\varphi_{\text{new}}^{(\lambda_1,\lambda_2)} \coloneqq J \circ \mathcal{F}_{\text{vec}}(\varphi_{\text{old}}^{(\lambda_1,\lambda_2)}), \tag{7.31}$$

must be related to the old state $\varphi_{\text{old}}^{(\lambda_1,\lambda_2)}$ as follows:

$$\begin{aligned}
J\left(g(\mathrm{d}t, 0)\, \mathcal{F}_{\text{vec}}(\varphi_{\text{old}}^{(\lambda_1,\lambda_2)})\right) &= h(\mathrm{d}t, 0)^*(J \circ \mathcal{F}_{\text{vec}})(\varphi_{\text{old}}^{(\lambda_1,\lambda_2)}), \qquad (7.32)\\
(J \circ \mathcal{F}_{\text{vec}})\left(f(\mathrm{d}t, 0)\varphi_{\text{old}}^{(\lambda_1,\lambda_2)}\right) &= (\mathbb{1} +_p \partial_\tau)\varphi_{\text{new}}^{(\lambda_1,\lambda_2)},\\
(J \circ \mathcal{F}_{\text{vec}})\left((1 + \mathrm{i}\lambda_1)\varphi_{\text{old}}^{(\lambda_1,\lambda_2)}\right) &= (\mathbb{1} +_p \partial_\tau)\varphi_{\text{new}}^{(\lambda_1,\lambda_2)},\\
(1 + \mathrm{i}\lambda_1) \cdot_{\text{new}} \varphi_{\text{new}}^{(\lambda_1,\lambda_2)} &= (\mathbb{1} +_p \partial_\tau)\varphi_{\text{new}}^{(\lambda_1,\lambda_2)},\\
\varphi_{\text{new}}^{(\lambda_1,\lambda_2)} +_{\text{new}} \mathrm{i}\lambda \cdot_{\text{new}} \varphi_{\text{new}}^{(\lambda_1,\lambda_2)} &= \varphi_{\text{new}}^{(\lambda_1,\lambda_2)} +_p \partial_\tau \varphi_{\text{new}}^{(\lambda_1,\lambda_2)}.
\end{aligned}$$

Note that on the left-hand-side we have the new theory's induced addition and scalar multiplication operations, $+_{\text{new}}$ and $\cdot_{\text{new}}$, whereas on the right-hand-side we have the standard point-wise addition operations on $\mathcal{M}_{\text{new}}$, namely $+_p$ and $\cdot_p$. Fortunately, however, we invoke our above assumption that these are the same. Applying the same logic to $g = g(0, \mathrm{d}k)$ then we have,

$$\partial_\tau\, \varphi_{\text{new}}^{(\lambda_1,\lambda_2)} = \mathrm{i}\lambda_1 \cdot_p \varphi_{\text{new}}^{(\lambda_1,\lambda_2)} \quad \text{and} \quad \partial_q\, \varphi_{\text{new}}^{(\lambda_1,\lambda_2)} = \mathrm{i}\lambda_2 \cdot_p \varphi_{\text{new}}^{(\lambda_1,\lambda_2)}, \tag{7.33}$$

That is, $\varphi_{\text{new}}^{(\lambda_1,\lambda_2)}$ must be a simultaneous eigenvector of ∂_τ and ∂_q with eigenvalues $\mathrm{i}\lambda_1$ and $\mathrm{i}\lambda_2$. Namely, $\varphi_{\text{new}}^{(\lambda_1,\lambda_2)}$ must be a planewave in the new theory's spacetime.

But what does this tell us about which injective maps J are allowed? Recall that $\varphi_{\text{old}}^{(\lambda_1,\lambda_2)}(t,x)$ is a planewave *in the old theory's Fourier space*. We thus have a planewave-to-planewave correspondence between the old theory's Fourier space and the new theory's spacetime. Concretely, Eq. (7.25) enforces that we have,

$$\begin{aligned}\varphi_{\text{new}}^{(\lambda_1,\lambda_2)}(\tau,q) &= a(\lambda_1,\lambda_2)\ \exp(\mathrm{i}\,\lambda_1\,\tau)\ \exp(\mathrm{i}\,\lambda_2\,q) \\ &= a(\lambda_1,\lambda_2)\ \tilde{\varphi}_{\text{old}}^{(\lambda_1,\lambda_2)}(\tau,q),\end{aligned} \tag{7.34}$$

for some scaling parameter $a(\lambda_1,\lambda_2)$.[87] A quick calculation shows that a generic state in the old theory, $\varphi_{\text{old}}(t,x) \in S_{\text{old}}^{\text{kin}}$, must be mapped to,

$$J \circ \mathcal{F}_{\text{vec}} : \varphi_{\text{old}} \mapsto \varphi_{\text{new}}(\tau,q) = A(\tau,q) *_{\text{conv}} \tilde{\varphi}_{\text{old}}(\tau,q), \tag{7.35}$$

for some profile $A(\tau,q)$. Recalling that convolution in space corresponds to point-wise multiplication in Fourier space, this $A(\tau,q)$ freedom captures our ability to freely rescale each of the new theory's planewaves via the above-discussed scaling parameter, $a(\lambda_1,\lambda_2)$. For later convenience, let us define $*_{\text{conv}}^{-1}(A)$ to be the profile associated with $1/a(\lambda_1,\lambda_2)$. We then have,

$$\tilde{\varphi}_{\text{old}}(\tau,q) = *_{\text{conv}}^{-1}(A) *_{\text{conv}} \varphi_{\text{new}}(\tau,q). \tag{7.36}$$

Note that our intuitive choice of J (namely, Eq. (7.22)) corresponds to picking a scaling of $a(\lambda_1,\lambda_2) = 1$ for each of the new theory's planewaves. This then corresponds to a profile $A(\tau,q) = \delta(\tau-0)\delta(q-0)$. To recap, having only invoked our first and second demands, we have found that, up to this rescaling freedom, there is only one way to map the old theory's states onto the new theory's spacetime (namely, via a Fourier transform).

But what should we make of this rescaling freedom? One might think that this planewave-rescaling freedom reflects a genuine ability to make certain planewaves to be

[87] Note that we must have $a(\lambda_1,\lambda_2) \neq 0$ otherwise, J would fail to be injective.

of larger or smaller magnitude on the new spacetime. As I will now argue, however, this choice of $A(\tau, q)$ is actually metaphysically irrelevant regarding the relative sizes and similarity of the new theories states. To see this, first recall from Sec. 4.2 that the non-local QHO theory came equipped with a metaphysically relevant inner product, I_{old}, at each t-coordinate. In particular, this was the L^2 product in the old theory's (t, x) coordinate system,

$$I_{\text{old}}(\varphi_{\text{old}}, \phi_{\text{old}}) = \int_{-\infty}^{\infty} \varphi_{\text{old}}(t, x)^* \phi_{\text{old}}(t, x)\, \mathrm{d}x = \int_{-\infty}^{\infty} \tilde{\varphi}_{\text{old}}(t, k)^* \tilde{\phi}_{\text{old}}(t, k)\, \mathrm{d}k, \tag{7.37}$$

which incidentally is also the L^2 inner product in Fourier space.

Post-internalization another inner product, I_{pst}, is induced on $\mathcal{S}^{\text{kin}}_{\text{pst}}$. Post-externalization yet another inner product, I_{new}, is induced on $\mathcal{S}^{\text{kin}}_{\text{new}}$. Throughout both of these processes, this inner product retains its status as the metaphysically relevant way to judge the size and similarity of states. Moreover, we have,

$$I_{\text{new}}(\varphi_{\text{new}}, \phi_{\text{new}}) = I_{\text{pst}}(\varphi_{\text{pst}}, \phi_{\text{pst}}) = I_{\text{old}}(\varphi_{\text{old}}, \phi_{\text{old}}), \tag{7.38}$$

such that the new planewave states, $\varphi^{(\lambda_1,\lambda_2)}_{\text{new}}(\tau, q)$, are normalized automatically *no matter what* scalings, $a(\lambda_1, \lambda_2)$, we choose. To see this more concretely, let us use Eq. (7.36) to write down I_{new} in terms of our choice of $A(\tau, q)$,

$$\begin{aligned} I_{\text{new}}(\varphi_{\text{new}}, \phi_{\text{new}}) &= I_{\text{old}}(\varphi_{\text{old}}, \phi_{\text{old}}) \\ &= \int_{-\infty}^{\infty} \tilde{\varphi}_{\text{old}}(t, k)^* \tilde{\phi}_{\text{old}}(t, k)\, \mathrm{d}k \\ &= \int_{-\infty}^{\infty} (*^{-1}_{\text{conv}}(A) *_{\text{conv}} \varphi_{\text{new}}(\tau, q))^* (*^{-1}_{\text{conv}}(A) *_{\text{conv}} \phi_{\text{new}}(\tau, q))\, \mathrm{d}q \end{aligned} \tag{7.39}$$

Hence, if we were to pick an injector J which has $a(\lambda_1, \lambda_2) = 2$ for some λ_1 and λ_2 then the new theory's induced inner product I_{new} will automatically compensate in order to maintain every state's metaphysically relevant norm. More generally, the new

theory's induced inner product will deviate from the new spacetime's L^2 inner product so as to keep φ_{new} normalized whenever φ_{old} was normalized (with respect to I_{new} and I_{old} respectively).

What does this say about our final criteria for singling out the intuitive choice of J discussed above? Demanding that the new theory's induced inner product is the standard L^2 inner product on $\mathcal{M}_{\text{new}}$, is not a substantive demand about how we ought to judge the relative size and similarity of the new theory's states. Rather it is a demand that our pre-existing standards (i.e., I_{old}) ought to be nicely represented in the new spacetime setting.

As the above discussion has shown, the externalization process is driven almost entirely by our choice of PSTOs. These determine not only the new theory's spacetime manifold (up to diffeomorphism) but also almost entirely determine how the old theory's states are to be mapped onto this new spacetime setting. The injection map is completely fixed if we then make some minor restrictions regarding how the metaphysically relevant notions of addition, scalar product, and inner product are to be represented in the new spacetime setting.

8 | Two More Example Applications of the ISE Method

This chapter will apply the ISE Method to two example theories. Firstly in Sec. 8.1 I will revisit the 3+1D Quartic Klein-Gordon theory which was introduced in Sec. 7.1. Following this, Sec. 8.2 will show how one can use the ISE Method to redescribe a lattice theory (i.e., a theory set on a discrete spacetime, $\mathcal{M}_{\text{old}} \cong \mathbb{R} \times \mathbb{Z}$) as existing on a continuous spacetime manifold, $\mathcal{M}_{\text{new}} \cong \mathbb{R} \times \mathbb{R}$. Sec. 8.3 will then use these two examples to discuss the possibility of conventionalism about spacetime topology.

8.1 Dimensional Reduction (and Expansion) à la Kaluza-Klein

Let us begin by briefly recalling that the internalization of the 3+1D Quartic Klein-Gordon theory has already been discussed in Sec. 7.1. Recall that the theory's old spacetime setting, $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2 \times S^2$, consists of two extended dimensions (spanned by t and q) and two compactified dimensions (spanned by x, y, and z with $x^2 + y^2 + z^2 = R^2$). Recall from Sec. 7.1 the Lie group which we used in the internalization process, $H^{(\text{int})}_{\text{trans}} \cong (\mathbb{R}^2, +) \times \text{SO}(3)$. Recall that this group $H^{(\text{int})}_{\text{trans}} \subset \text{Diff}(\mathcal{M}_{\text{old}})$ implemented rigid translations in the old theory's (t, q) coordinates and rigid rotations in the old theory's (x, y, z) coordinates. Its stabilizer subgroup was $H^{(\text{int})}_{\text{fix}} \cong \{\mathbb{1}\} \times \text{SO}(2)$. Recall from Eq. (7.13) that $H^{(\text{int})}_{\text{trans}}$ (and, consequently, its subgroup $H^{(\text{int})}_{\text{fix}} \subset H^{(\text{int})}_{\text{trans}}$) faithfully survives internalization:

$$\begin{aligned} H^{(\text{int})}_{\text{trans}} &\xrightarrow[\text{int. as}]{\text{survives}} G^{(\text{int})}_{\text{trans}} := G(H^{(\text{int})}_{\text{trans}}) \cong H^{(\text{int})}_{\text{trans}}, \\ H^{(\text{int})}_{\text{fix}} &\xrightarrow[\text{int. as}]{\text{survives}} G^{(\text{int})}_{\text{fix}} := G(H^{(\text{int})}_{\text{fix}}) \cong H^{(\text{int})}_{\text{fix}}, \end{aligned} \tag{8.1}$$

becoming some Lie groups, $G^{(\text{int})}_{\text{fix}} \subset G^{(\text{int})}_{\text{trans}}$, acting on $S^{\text{kin}}_{\text{pst}}$.

We can now begin to search for good PSTOs to externalize, let us call them $G^{(\text{ext})}_{\text{trans}}$ and $G^{(\text{ext})}_{\text{fix}}$. One pair of PSTOs which immediately come to mind are the Lie groups which were just produced by the internalization process, namely $G^{(\text{int})}_{\text{trans}} \cong (\mathbb{R}^2, +) \times \text{SO}(3)$ and $G^{(\text{int})}_{\text{fix}} \cong \{\mathbb{1}\} \times \text{SO}(2)$. As I prove in Appendix B.2, it follows from the 3+1D QKG theory having spacetime-kinematic compatibility, that this pair of Lie group actions are PSTOs. Moreover, as I will there discuss, one can easily externalize these PSTOs in order to return back to the theory's original spacetime setting. Fortunately, however, there are also more exciting PSTOs available to us.

Let us next search for a non-trivial set of PSTOs. Instead of re-externalizing all four of the old theory's dimensions, one can choose to leave the old theory's compactified dimensions internalized and only re-externalize the other two dimensions. Intuitively, this should result in a new spacetime framing of our theory which is $1 + 1$ dimensional, i.e., $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$. Let us embark with this destination in mind. In particular, let us focus on the Lie group which implements rigid (t, q)-translations, namely $F_{1+1} \subset H^{(\text{int})}_{\text{trans}}$. Concretely, let $f(\Delta t, \Delta q) \in F_{1+1} \cong (\mathbb{R}^2, +)$ act in the (t, q, x, y, z) coordinate system as $f(\Delta t, \Delta q) : (t, q, x, y, z) \mapsto (t - \Delta t, q - \Delta q, x, y, z)$. Since the lifted action of F_{1+1} is kinematically allowed, it faithfully survives internalization:

$$F_{1+1} \xrightarrow[\text{int. as}]{\text{survives}} G_{1+1} \coloneqq G(F_{1+1}) \cong F_{1+1}, \tag{8.2}$$

becoming a Lie group, $G_{1+1} \subset G^{(\text{int})}_{\text{trans}}$, acting on $S^{\text{kin}}_{\text{pst}}$.

As I will now discuss, we can choose $G^{(\text{ext})}_{\text{trans}} \coloneqq G_{1+1}$ to be our PSTOs. Intuitively, externalizing with these PSTOs should yield a new spacetime framing of our theory which is $1 + 1$ dimensional, i.e., $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$. Before seeing whether or not this happens, however, we need to complete our choice of PSTOs by picking out another Lie group, $G^{(\text{ext})}_{\text{fix}} \subset G^{(\text{ext})}_{\text{trans}}$, to externalize. What are our options? Recall that $G^{(\text{ext})}_{\text{fix}}$ must have faithful construction compatibility with $G^{(\text{ext})}_{\text{trans}}$. In particular, $G^{(\text{ext})}_{\text{fix}}$ must be a core-free subgroup

of $G^{(\text{ext})}_{\text{trans}} = G_{1+1}$ (i.e., it must contain no non-trivial normal subgroups of $G^{(\text{ext})}_{\text{trans}}$). Next, note that $G_{1+1} \cong (\mathbb{R}^2, +)$ is abelian and hence all of its subgroups are normal. This means that the only core-free subgroup of G_{1+1} is the trivial group. Hence, given our choice of $G^{(\text{ext})}_{\text{trans}} = G_{1+1}$ we are forced to take $G^{(\text{ext})}_{\text{fix}} = \{\mathbb{1}\}$.

Proceeding with this choice of Lie group actions, $G^{(\text{ext})}_{\text{trans}} = G_{1+1} \cong (\mathbb{R}^2, +)$ and $G^{(\text{ext})}_{\text{fix}} = \{\mathbb{1}\}$, externalization will build a new spacetime manifold from their quotient,

$$\mathcal{M}_{\text{new}} \cong G^{(\text{ext})}_{\text{trans}}/G^{(\text{ext})}_{\text{fix}} \cong \mathbb{R}^2. \tag{8.3}$$

As expected, we do end up with a new spacetime manifold consisting of only one dimension of space and one dimension of time. To see that these two dimensions deserve the names "space" and "time" respectively, recall that the old theory's rigid (t, q)-shifts, $f(\Delta t, \Delta q) \in F_{1+1} \subset \text{Diff}(\mathcal{M}_{\text{old}})$, have faithfully survived internalization becoming $g(\Delta t, \Delta q) \coloneqq G(f(\Delta t, \Delta q)) \in G_{1+1}$ acting on $S^{\text{kin}}_{\text{pst}}$. But this is the Lie group which we have just externalized. Thus, post-externalization we will have $g(\Delta t, \Delta q)$ becoming $h(\Delta t, \Delta q) \coloneqq H(g(\Delta t, \Delta q)) \in \text{Diff}(\mathcal{M}_{\text{new}})$, a diffeomorphism on our new 1+1-dimensional spacetime manifold, $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$. Let (τ, ξ) be a coordinate system for $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$ such that $h(\Delta t, \Delta q)(\tau, \xi) = (\tau - \Delta t, \xi - \Delta q)$. Thus, rigid (t, q)-shifts in the old theory become rigid (τ, ξ)-shifts in the new theory. Externalization has turned our choice of PSTOs, $G^{(\text{ext})}_{\text{trans}} = G_{1+1}$, into honest-to-goodness spacetime translations on $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$. Namely, $H^{(\text{ext})}_{\text{trans}} \coloneqq H(G^{(\text{ext})}_{\text{trans}})$.

After constructing this new spacetime manifold, $\mathcal{M}_{\text{new}}$, the next step in the externalization process is to inject the new theory's states and dynamics onto this new spacetime setting. Concretely, this is to be done by picking out an injective map, $J : S^{\text{kin}}_{\text{pst}} \to S^{\text{all}}_{\text{new}}$. Here $\varphi_{\text{new}} \in S^{\text{all}}_{\text{new}}$ is the set of all scalar fields states, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathcal{V}_{\text{new}}$, definable on the new spacetime setting for some yet-to-be fixed value space, $\mathcal{V}_{\text{new}}$. What constrains our choice of J? As Sec. 7.4 has stressed, the new diffeomorphisms, $h(\Delta t, \Delta q) \in H^{(\text{ext})}_{\text{trans}}$,

must faithfully represent our PSTOs not only as a Lie group but also must lift to a faithful representation of our PSTOs Lie group action on the theory's kinematically allowed states. In particular, it follows from Eq. (7.25) that infinitesimal (t, q)-shifts in the old theory must correspond to infinitesimal (τ, ξ)-shifts in the new theory. Moreover, this correspondence must hold even once we lift these diffeomorphisms to act on the new theory's states. As I will now discuss, it follows from this that a generic planewave in the old theory's spacetime must map onto a corresponding planewave in the new theory's spacetime.

Consider a generic planewave in the old theory's spacetime,

$$\varphi_{\text{old}}^{(\omega\, k\, \ell\, m)}(t, q, x, y, z) = Y_{\ell m}(x/R, y/R, z/R)\, \exp(\mathrm{i}\omega t + \mathrm{i}kq), \tag{8.4}$$

where $Y_{\ell m}$ are the spherical harmonics with $\ell = 0, \ldots, \infty$ and $-\ell \leq m \leq \ell$. Note that this state is a simultaneous eigenvector of ∂_t and ∂_q with eigenvalues $\mathrm{i}\omega$ and $\mathrm{i}k$ respectively. It follows from Eq. (7.25) that the corresponding state in the new theory, $\varphi_{\text{new}}^{(\omega\, k\, \ell\, m)}(\tau, \xi) \coloneqq J \circ \mathcal{F}_{\text{alg}}(\varphi_{\text{old}}^{(\omega\, k\, \ell\, m)})$, must be a simultaneous eigenvector of ∂_τ and ∂_ξ with eigenvalues $\mathrm{i}\omega$ and $\mathrm{i}k$ respectively. Namely, we must have,[88]

$$J \circ \mathcal{F}_{\text{alg}} : \varphi_{\text{old}}^{(\omega\, k\, \ell\, m)} \mapsto \varphi_{\text{new}}^{(\omega\, k\, \ell\, m)} = a(\omega, k, \ell, m)\, \exp(\mathrm{i}\omega\tau + \mathrm{i}k\xi), \tag{8.5}$$

for some scaling parameter, $a(\omega, k, \ell, m) \in \mathcal{V}_{\text{new}}$. Thus, up to some rescaling freedom, the externalization process must map planewaves in the old theory's spacetime onto a corresponding planewave in the new theory's spacetime.

But what should we make of this sort of planewave-rescaling freedom? Sec. 7.4 has discussed this issue at length. To summarize: This rescaling freedom is metaphysically irrelevant. In particular, it only affects how the theory's states are being *represented* in this

[88] As in Sec. 7.4 this part of the calculation requires that the new theory's induced addition and scalar multiplication operations, $+_{\text{new}}$ and $\cdot_{\text{new}}$, are the standard pointwise operations on $\mathcal{M}_{\text{new}}$, namely $+_p$ and $\cdot_p$.

new spacetime setting. It does not actually change their relative sizes in a metaphysically meaningful way. Recall that, by stipulation, the metaphysically relevant way to judge the relative size and similarity of the old theory's states, is with a certain inner product, I_{old}. The $\mathcal{F}_{\text{alg}}$ and J maps combine to induce a corresponding inner product, I_{new}, for the new theory's states. Just as in Sec. 7.4, the metaphysically relevant way to judge the relative size and similarity of the new theory's states is with this induced inner product (whether or not it is the natural inner product for the new spacetime setting, $\mathcal{M}_{\text{new}}$). If one investigates this induced inner product, one would find that normalized states in the old theory (with respect to I_{old}) automatically map onto normalized states in the new theory (with respect to I_{new}). See Eq. 7.38.

Given that our choice of $a(\omega, k, \ell, m)$ is merely a choice about how to represent the new theory's states on $\mathcal{M}_{\text{new}}$, we might as well choose a nice way of representing them. In particular, we can choose $a(\omega, k, \ell, m) = \vec{e}_{\ell m} \in \mathcal{V}_{\text{new}} \cong \mathbb{R}^{\infty}$ where the set $\{\vec{e}_{\ell m}\}$ form a basis for an infinite-dimensional vector space, $\mathcal{V}_{\text{new}} \cong \mathbb{R}^{\infty}$, where the new field is valued, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathcal{V}_{\text{new}}$. Note that for this choice of $a(\omega, k, \ell, m)$ the new theory's induced inner product, I_{new}, will be the L^2 inner product in the (τ, ξ)-coordinate system together with the native inner product on $\mathcal{V}_{\text{new}} \cong \mathbb{R}^{\infty}$. That is, whichever basis, $\{\vec{e}_{\ell m}\}$, we choose for $\mathcal{V}_{\text{new}}$ will end up being an orthonormal basis with respect to this inner product.

Having settled on a choice of scaling, $a(\omega, k, \ell, m)$, we have now fixed the injective map, $J : S^{\text{kin}}_{\text{pst}} \to S^{\text{all}}_{\text{new}}$, which defines the new theory's states and dynamics. Indeed, after some straightforward calculations, one finds that the new theory is defined as follows:[89]

> **A Quartic Klein-Gordon Theory (1+1D QKG) -** Consider a theory set on a spacetime manifold, $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$, about a scalar field, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathcal{V}_{\text{new}}$, with value space, $\mathcal{V}_{\text{new}} \cong \mathbb{R}^{\infty}$. For this theory, there exists an orthonormal basis $\vec{e}_{\ell m} \in \mathcal{V}_{\text{new}}$ with respect to which we can decompose the states as $\varphi_{\text{new}} = \sum_{\ell m} \varphi^{(\ell m)}_{\text{new}} \vec{e}_{\ell m}$ with $\ell = 0, \ldots, \infty$ and $-\ell \leq m \leq \ell$. Adopting a phrase from Grimmer et al. (2021), let us call each of these components a "subfield". This theory's states are subject to the following kinematic constraint: In some fixed

[89] See Grimmer et al. (2021) for details.

> global coordinate system, (τ, ξ), each subfield, $\varphi^{(\ell m)}_{\text{new}}(\tau, \xi)$, must be smooth. The metaphysically relevant way to judge the relative size and similarity of this theory's states is with an L^2 inner product in the (τ, ξ)-coordinate system together with the native inner product on $\mathcal{V}_{\text{new}} \cong \mathbb{R}^\infty$. In this coordinate system, the dynamical equations for each subfield, $\varphi^{(\ell m)}_{\text{new}}(\tau, \xi)$, is as follows:
>
> $$0 = (\partial^2_\tau - \partial^2_\xi + M^2_{\ell m})\, \varphi^{(\ell m)}_{\text{new}} + \lambda\, \mathcal{O}\big(\varphi^{(\ell m)}_{\text{new}}\, \varphi^{(\ell' m')}_{\text{new}}\, \varphi^{(\ell'' m'')}_{\text{new}}\big) \tag{8.6}$$
>
> where $\lambda \in \mathbb{R}$ is some fixed coupling constant and $M^2_{\ell m} \coloneqq M^2 + \ell(\ell+1)/R^2$ where $M \geq 0$ is some fixed mass parameter.

Note that in the non-interacting case (with $\lambda = 0$) each subfield evolves as a massive 1+1 dimensional Klein-Gordon field. From the perspective of the old theory, the new theory's subfields acquire their masses, $M_{\ell m} \coloneqq \sqrt{M^2 + \ell(\ell+1)/R^2}$, partly from the original field's mass, M and partly from their geometric confinement. Those familiar with Kaluza Klein dimensional reduction should recognize what has happened here. We started with a theory set on a spacetime, $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2 \times S^2$, about an $\mathbb{R}$-valued field, $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to \mathbb{R}$. We then removed the two compactified dimensions to find a new theory set on $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$ about $\mathbb{R}^\infty$-valued field, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathbb{R}^\infty$, whose components, $\varphi^{(\ell m)}_{\text{new}}$, correspond to the harmonic modes of the compactified dimensions, S^2. There is no approximation made in this dimensional reduction. Nor is there any restriction placed upon the radius R of the compactified dimensions. If R is small then almost all of the subfields, $\varphi^{(\ell m)}_{\text{new}}$, will be incredibly massive. This is a rigorous way to see why "when viewed from a sufficient distance" the scalar field on $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2 \times S^2$ can be well approximated by *a single scalar field* on $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$; The approximation is in neglecting all of the high-mass subfields. See Grimmer et al. (2021) for further discussion.

But what about the interacting case with $\lambda \neq 0$? In the old theory, the cubic self-interaction term was governed by the old theory's point-wise product operation, $\times_{\text{old}}$. One can trace this product operation through the internalization process to a pre-spacetime product operation, $\times_{\text{pst}}$. One can then follow this product operation through the externalization process to a new induced product operation, $\times_{\text{new}}$, in the new theory. Ultimately,

$\times_{\text{new}}$ ends up being a point-wise product on $\mathcal{M}_{\text{new}}$ complemented by a product operation in the new theory's value space, $\mathcal{V}_{\text{new}} \cong \mathbb{R}^{\infty}$. This induced product operation on $\mathcal{V}_{\text{new}}$ is related to the harmonic decomposition of the products of spherical harmonics. It is ultimately this induced product operation on $\mathcal{V}_{\text{new}}$ which defines the cubic couplings between the new theory's subfields.

In sum, this example has shown how the ISE Method can be used to build a new spacetime setting for the 3+1D Quartic Klein-Gordon theory. Beginning from a single scalar field φ_{old} on a large spacetime $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2 \times S^2$, we could choose to re-externalize just a part of it yielding an infinite collection of massive scalar fields $\{\varphi_{\text{new}}^{(\ell m)}\}$ on a smaller spacetime $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$. Namely, the result is the above-discussed 1+1D Quartic Klein-Gordon theory. As I will prove in Ch. 11, the ISE Method can be applied equally well to achieve the reverse redescription. Namely, suppose that we had started from the 1+1D QKG theory about an infinite collection of subfields living on $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$. One might then notice that this theory has an internal dynamical symmetry of $\text{SO}(3)$ acting on $\mathcal{V}_{\text{new}} \cong \mathbb{R}^{\infty}$. Despite being an internal symmetry, this dynamical symmetry is nonetheless very much like a continuous spacetime translation symmetry. Indeed, viewed as a Lie group acting on pre-spacetime theoretic states, they are structurally indistinguishable from spacetime translations (indeed, they are PSTOs). By externalizing them we can build a new (or, in this case, rebuild the old) spacetime setting for the QKG theory such that they become honest-to-goodness spacetime translations. In effect, we would reconstruct the spacetime $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2 \times S^2$ as a quotient of Lie groups and then inject the theory's states onto $\mathcal{M}_{\text{old}}$ as a single field living on this larger spacetime. Ultimately, the result would be the above-discussed 3+1D Quartic Klein-Gordon theory.

As the Fourier and QKG examples hopefully demonstrate, it can be fruitful to search for novel unexpected PSTOs beyond those given to us via the internalization process. These alternate PSTOs might lead us to new and fruitful topological redescriptions of our

theories. In the Fourier redescription example these alternative PSTOs were not dynamical symmetries but nonetheless externalizing them yielded a more local expression of the theory's dynamics. In the present example, these alternative PSTOs were dynamical symmetries and hence became spacetime symmetries in the new theory. The next section will continue in this theme. I will there identify some hidden dynamical symmetries which are present in a wide range of lattice theories. Despite being internal transformations, these hidden dynamical symmetries are nonetheless very much like continuous spacetime translation operations (indeed, they are PSTOs). By externalizing them we can redescribe these lattice theories as existing on a continuous spacetime manifold.

8.2 From a Discrete to a Continuous Spacetime (and back again) à la Kempf

This section will show how one can use the ISE Method to redescribe a lattice theory (i.e., a theory set on a discrete spacetime, $\mathcal{M}_{\text{old}} \cong \mathbb{R} \times \mathbb{Z}$) as existing on a continuous spacetime manifold, $\mathcal{M}_{\text{new}} \cong \mathbb{R} \times \mathbb{R}$. The applicability of (an early version of) the ISE Method to these discrete spacetime theories was first demonstrated in Grimmer (2022b) and Grimmer (2022a). What follows is a condensed and updated version of this treatment. My analysis of these discrete spacetime theories is largely inspired by the work of the brilliant mathematical physicist Achim Kempf. It was, in fact, Kempf's work which first motivated me to look into the philosophical implications of the existence of spacetime theories with multiple inequivalent spacetime settings. See, for instance, Kempf (2010) entitled, "Spacetime could be simultaneously continuous and discrete, in the same way that information can be". His paper explores this continuous-to-discrete correspondence by using Nyquist-Shannon sampling theory. For an overview of Kempf's work on these topics see Kempf (2018).

Before this, however, some mathematical definitions are needed relating to various discrete approximations of the derivative. Consider the nearest-neighbor and next-to-nearest-neighbor approximations of the second derivative,

$$\Delta^2_{(1)}/\epsilon^2 : \ \partial_x^2 f(x) \approx \frac{f(x+\epsilon) - 2f(x) + f(x-\epsilon)}{\epsilon^2}, \tag{8.7}$$
$$\Delta^2_{(2)}/\epsilon^2 : \ \partial_x^2 f(x) \approx \frac{-f(x+2\epsilon) + 16\, f(x+\epsilon) - 30\, f(x) + 16\, f(x-\epsilon) - f(x+2\epsilon)}{12\,\epsilon^2}.$$

Closely, associated with these derivative approximations are the following two infinite matrices,

$$\Delta^2_{(1)} \coloneqq \text{Toeplitz}(\,\ldots, 0, 0, 0, 0, 1, \mathbf{-2}, 1, 0, 0, 0, 0, \ldots\,) \tag{8.8}$$
$$\Delta^2_{(2)} \coloneqq \text{Toeplitz}(\,\ldots, 0, 0, 0, \frac{-1}{12}, \frac{16}{12}, \frac{\mathbf{-30}}{\mathbf{12}}, \frac{16}{12}, \frac{-1}{12}, 0, 0, 0, \ldots\,).$$

Toeplitz matrices are the so-called diagonal-constant matrices with $[A]_{i,j} = [A]_{i+1,j+1}$. Thus, the values listed in the above expression give the matrix's values on either side of the main diagonal. The value in the middle of the list (in bold) corresponds to the main diagonal. Past this, five values to either side of the main diagonal are given.

These two infinite matrices can be easily generalized to, $\Delta^2_{(n)}$, the Toeplitz matrix related to the (next-to)^n-nearest-neighbor approximation of the second derivative. Taking the limit $n \to \infty$ one can define,

$$D^2 \coloneqq \text{Toeplitz}(\,\ldots, \frac{2}{25}, \frac{-2}{16}, \frac{2}{9}, \frac{-2}{4}, \frac{2}{1}, \frac{\mathbf{-2\pi^2}}{\mathbf{6}}, \frac{2}{1}, \frac{-2}{4}, \frac{2}{9}, \frac{-2}{16}, \frac{2}{25}, \ldots\,), \tag{8.9}$$

as the infinite Toeplitz matrix related to the infinite-range discrete approximation of the second derivative. For details see Grimmer (2022b). One can also define an infinite Toeplitz matrix related to the infinite-range discrete approximation of the first derivative,

$$D \coloneqq \text{Toeplitz}(\,\ldots, \frac{-1}{5}, \frac{1}{4}, \frac{-1}{3}, \frac{1}{2}, -1, \mathbf{0}, 1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \frac{1}{5}, \ldots\,). \tag{8.10}$$

As it turns out, D^2 is the square of D.

For later reference, I should here note how a generic symmetric Toeplitz matrix acts on the discrete planewaves, $\varphi_{\text{old}}^{(\omega,k)}(t,n) \coloneqq \exp(\mathrm{i}\omega t + \mathrm{i}k\,n)$, with $\omega \in \mathbb{R}$ and $k \in [-\pi,\pi]$.[90] Given

$$T_{\text{sym}} \coloneqq \text{Toeplitz}(\,\ldots, c_5, c_4, c_3, c_2, c_1, \mathbf{c}_0, c_1, c_2, c_3, c_4, c_5, \ldots\,). \tag{8.11}$$

the (vectorized) discrete planewave,

$$\varphi_{\text{old}}^{(\omega,k)}(t) \coloneqq (\ldots, \varphi_{\text{old}}^{(\omega,k)}(t,-2), \varphi_{\text{old}}^{(\omega,k)}(t,-1), \varphi_{\text{old}}^{(\omega,k)}(t,0), \varphi_{\text{old}}^{(\omega,k)}(t,1), \varphi_{\text{old}}^{(\omega,k)}(t,2)\ldots) \tag{8.12}$$

turns out to be an eigenvector of T_{sym} as follows,

$$T_{\text{sym}}\,\varphi_{\text{old}}^{(\omega,k)}(t) = \left(c_0 + 2\sum_{n=1}^{\infty} c_n\ \cos(n\,k)\right)\ \varphi_{\text{old}}^{(\omega,k)}(t). \tag{8.13}$$

That is, taking the c_n values in T_{sym} as the coefficients of a cosine series gives the eigenvalue of $\varphi_{\text{old}}^{(\omega,k)}$ with respect to T_{sym}. A similar result holds for generic anti-symmetric Toeplitz matrices except with a sine series. For the four Toeplitz matrices discussed above we have,

$$\begin{aligned}
\Delta_{(1)}^2\,\varphi_{\text{old}}^{(\omega,k)}(t) &= (2\cos(k) - 2)\ \varphi_{\text{old}}^{(\omega,k)}(t), \\
\Delta_{(2)}^2\,\varphi_{\text{old}}^{(\omega,k)}(t) &= \frac{1}{12}(-2\cos(2k) + 32\cos(k) - 30)\ \varphi_{\text{old}}^{(\omega,k)}(t), \\
D^2\,\varphi_{\text{old}}^{(\omega,k)}(t) &= -k^2\ \varphi_{\text{old}}^{(\omega,k)}(t), \\
D\,\varphi_{\text{old}}^{(\omega,k)}(t) &= \mathrm{i}k\ \varphi_{\text{old}}^{(\omega,k)}(t).
\end{aligned} \tag{8.14}$$

Notably, the discrete planewave, $\varphi_{\text{old}}^{(\omega,k)}(t)$, is simultaneously an eigenvector of each of these four linear operators with different eigenvalues. Incredibly, the eigenvalues for D^2 and D are $-k^2$ and $\mathrm{i}k$ exactly matching the eigenvalues of the continuum derivatives, ∂_x^2 and ∂_x. This fact will become useful later: There exists a discrete approximation

[90] Note that these discrete planewaves are periodic in their wavenumber k with a period of 2π. Namely, it follows from Euler's identity that $\varphi_{\text{old}}^{(\omega,k+2\pi)}$ and $\varphi_{\text{old}}^{(\omega,k)}$ are literally the same functions of (t,n).

of the derivative (albeit one with infinite range) which makes *literally zero error* when applied to planewaves with a sufficiently low wavenumber, $k \in [-\pi, \pi]$. (By linearity, this derivative "approximation" is also perfect when applied to any function with a sufficiently low bandwidth.)

Given these definitions, we can now state the following three lattice theories:[91]

> **Heat Equation on a Discrete Spacetime** - Consider a theory set on a spacetime manifold, $\mathcal{M}_{\text{old}} \cong \mathbb{R} \times \mathbb{Z}$, about a scalar field, $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to \mathcal{V}_{\text{old}}$, with value space, $\mathcal{V}_{\text{old}} \cong \mathbb{R}$. This theory's states are subject to the following kinematic constraint: In some fixed global coordinate system, (t, n), each field must be smooth. At each t-coordinate, the metaphysically relevant way to judge the relative size and similarity of this theory's states is with an L^2 inner product. There are three variants of this theory with the following three dynamical equations respectively:
>
> $$\begin{aligned} \textbf{H1}: &\quad \frac{\mathrm{d}}{\mathrm{d}t}\varphi_{\text{old}}(t) = \alpha\, \Delta^2_{(1)}\, \varphi_{\text{old}}(t), \\ \textbf{H2}: &\quad \frac{\mathrm{d}}{\mathrm{d}t}\varphi_{\text{old}}(t) = \alpha\, \Delta^2_{(2)}\, \varphi_{\text{old}}(t), \\ \textbf{H3}: &\quad \frac{\mathrm{d}}{\mathrm{d}t}\varphi_{\text{old}}(t) = \alpha\, D^2\, \varphi_{\text{old}}(t), \end{aligned} \tag{8.15}$$
>
> for some fixed parameter $\alpha > 0$.

These are the nearest neighbor (H1), next-to-nearest neighbor (H2), and infinite-range (H3) discrete heat equations respectively.

As with the previous theories, internalization begins by identifying this theory's dynamically and metaphysically relevant structures. Since this theory's dynamical equations are linear it will be convenient to equip the space of kinematically allowed states, $\mathcal{S}^{\text{kin}}_{\text{old}}$, with at least a vector space structure. The space of dynamically allowed states is then a vector subspace thereof, $S^{\text{dyn}}_{\text{old}} \subset \mathcal{S}^{\text{kin}}_{\text{old}}$. Past its dynamically relevant linear structure, this theory also has a metaphysically-relevant inner product, $I_{\text{old}}(\varphi, \phi)$, defined for $\varphi, \phi \in \mathcal{S}^{\text{kin}}_{\text{old}}$. We can internalize this theory by applying a forgetful operation, $\mathcal{F}_{\text{vec}}$, at the

[91] In these theories, space is discrete whereas time is continuous. The ISE Method can also be applied when both space and time are discrete. For instance, in Grimmer (2022a) these techniques are applied to a discrete version of the Klein-Gordon equation.

level of kinematics which only remembers $\mathcal{S}^{\text{kin}}_{\text{old}}$'s vector space structure as follows:

$$\mathcal{F}_{\text{vec}} : \mathcal{S}^{\text{kin}}_{\text{old}} \to \mathcal{S}^{\text{kin}}_{\text{pst}} \coloneqq \langle S^{\text{kin}}_{\text{pst}}, +_{\text{pst}}, \cdot_{\text{pst}}, I_{\text{pst}} \rangle. \tag{8.16}$$

Moreover, this theory's spacetime manifold can also faithfully survive internalization. It can do so by means of the Lie group action $H_{\text{trans}} \cong (\mathbb{R} \times \mathbb{Z}, +)$ which implements rigid translations in the (t, n) coordinate system. Note that all of H_{trans}'s stabilizer subgroups are trivial, $H_{\text{fix}} = \{\mathbb{1}\}$. Hence, by the Homogeneous Manifold Characterization Theorem (see Appendix A.1) we can reconstruct $\mathcal{M}_{\text{old}}$ up to diffeomorphism as,

$$\mathcal{M}_{\text{old}} \cong \mathcal{M}_{\text{H}} \coloneqq \frac{H_{\text{trans}}}{H_{\text{fix}}} \cong \mathbb{R} \times \mathbb{Z}. \tag{8.17}$$

Having reconstructed $\mathcal{M}_{\text{old}} \cong \mathbb{R} \times \mathbb{Z}$ in this way, we can now see that it faithfully survives internalization as follows:

$$\mathcal{M}_{\text{old}} \cong \mathcal{M}_{\text{H}} \coloneqq \frac{H_{\text{trans}}}{H_{\text{fix}}} \xrightarrow[\text{int. as}]{\text{survives}} \mathcal{M}_{\text{G}} \coloneqq \frac{G^{(\text{int})}_{\text{trans}}}{G^{(\text{int})}_{\text{fix}}} \cong \frac{H_{\text{trans}}}{H_{\text{fix}}} \cong \mathcal{M}_{\text{old}}. \tag{8.18}$$

This follows from the fact that H_{trans} and H_{fix} both faithfully survive internalization as Lie groups, $G^{(\text{int})}_{\text{trans}} \coloneqq G(H_{\text{trans}}) \cong (\mathbb{R} \times \mathbb{Z}, +)$ and $G^{(\text{int})}_{\text{fix}} \coloneqq G(H_{\text{fix}}) = \{\mathbb{1}\}$, acting on $S^{\text{kin}}_{\text{pst}}$. Thus, we have a faithful survival of this theory's spacetime manifold.

Having finished internalization, we next search for good PSTOs to externalize. As I prove in Appendix B.2, the pair of Lie group actions produced via internalization (namely $G^{(\text{int})}_{\text{trans}}$, and $G^{(\text{int})}_{\text{fix}}$) here qualify as PSTOs. While we could externalize these PSTOs in order to get back to the theory's original spacetime setting, we are not compelled to do so. But what other PSTOs are available to us? Perhaps surprisingly, these lattice theories also admit continuous pre-spacetime translation operations. In particular, these are the smooth transformations sketched in the third column of Fig. 6.1. As I will now show, by externalizing these PSTOs we can find a new continuous spacetime setting for our formerly discrete theories.

But how is it that the above-discussed lattice theories can have both continuous and discrete spacetime representations? Consider first the discrete shift diffeomorphism, $d_m : (t, n) \mapsto (t, n - m)$ for $m \in \mathbb{Z}$, acting on the old theory's spacetime manifold, $\mathcal{M}_{\text{old}}$. Due to the discreteness of the old spacetime manifold, we cannot promote $m \in \mathbb{Z}$ to a continuous parameter, $\epsilon \in \mathbb{R}$. Next, consider lifting these discrete shift maps to act on $\varphi_{\text{old}} \in S^{\text{kin}}_{\text{old}}$ as $d^*_m : S^{\text{kin}}_{\text{old}} \to S^{\text{kin}}_{\text{old}}$. Unlike the old spacetime, $\mathcal{M}_{\text{old}}$, the state space, $S^{\text{kin}}_{\text{old}}$, is continuous; Hence, if we focus our attention on the lifted maps, d^*_m, then we might be able to promote $m \in \mathbb{Z}$ to a continuous parameter, $\epsilon \in \mathbb{R}$. This is how we will find our continuous PSTOs.

Recall that D (defined in Eq. (8.10)) is the best infinite-range discrete approximation of the derivative. Consider the Lie group action on $S^{\text{kin}}_{\text{old}}$ given by $\exp(-\epsilon\, D)$ for $\epsilon \in \mathbb{R}$. It turns out that for $\epsilon = m \in \mathbb{Z}$ we have $\exp(-m\, D) = d^*_m$. That is, the continuous family of smooth transformation, $\exp(-\epsilon D)$, nicely interpolates between the discrete shift operations.[92] Indeed, it was this Lie group action, $\exp(-\epsilon D)$, maps which is shown in the third column of Fig. 6.1.

Let us now construct a set of continuous PSTOs. Let the maps,

$$f(\Delta t, \epsilon)\, \varphi_{\text{old}}(t) \coloneqq \exp(-\epsilon D)\, \varphi_{\text{old}}(t - \Delta t), \tag{8.19}$$

form a Lie group $f(\Delta t, \epsilon) \in F_{\text{Cont}} \cong (\mathbb{R}^2, +)$ which acts on $S^{\text{kin}}_{\text{old}}$. These transformations are kinematically allowed and hence faithfully survive internalization becoming a Lie group $g(\Delta t, \epsilon) \in G_{\text{Cont}}$ acting on $S^{\text{kin}}_{\text{pst}}$. Let us take $G^{(\text{ext})}_{\text{trans}} = G_{\text{Cont}} \cong (\mathbb{R}^2, +)$ to be our PSTOs. As in the previous examples, it follows from $G^{(\text{ext})}_{\text{trans}}$ being abelian that we must pick $G^{(\text{ext})}_{\text{fix}} = \{\mathbb{1}\}$.

Constructing a new spacetime manifold from these PSTOs we have

$$\mathcal{M}_{\text{new}} \cong \frac{G^{(\text{ext})}_{\text{trans}}}{G^{(\text{ext})}_{\text{fix}}} = \frac{G_{\text{Cont}}}{\{\mathbb{1}\}} \cong \mathbb{R}^2. \tag{8.20}$$

[92] Indeed, another way to define D is as $D \coloneqq \ln(d_1)$.

By construction, the Lie group in the numerator, $g(\Delta t, \epsilon) \in G_{\text{Cont}}$, will be faithfully represented as a diffeomorphism, $h(\Delta t, \epsilon) \coloneqq H(g(\Delta t, \epsilon)) \in \text{Diff}(\mathcal{M}_{\text{new}})$, on the new spacetime. As in the previous examples, it will be helpful to introduce a coordinate system associated with these diffeomorphisms. Let (τ, q) be a coordinate system for $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$ such that $h(\Delta t, \epsilon)(\tau, q) = (\tau - \Delta t, q - a\,\epsilon)$. Note that I have defined the (τ, q)-coordinates such that a unit shift of ϵ in the old spacetime corresponds to a shift of the q-coordinate by a distance a in the new theory. Externalization has turned our choice of $G_{\text{trans}} = G_{\text{Cont}}$ into honest-to-goodness spacetime translations on $\mathcal{M}_{\text{new}}$.

Externalization does not stop here, however, we still need to map the old theory's states and dynamics onto this new spacetime setting. In particular, we need to pick out an injective map, $J : S^{\text{kin}}_{\text{pst}} \to S^{\text{all}}_{\text{new}}$, which obeys Eq. (7.25) and Eq. (7.26). Namely, the new diffeomorphisms, $h(\Delta t, \epsilon) \in \text{Diff}(\mathcal{M}_{\text{new}})$, must lift to act in the right way on the new theory's states, $\varphi_{\text{new}} = J(\varphi_{\text{pst}})$. As in the Fourier example, an eigenvector argument will fix our choice of J up to a metaphysically irrelevant rescaling of the new theory's Fourier modes.

In particular, it follows from Eq. (7.25) that infinitesimal action by $f(\Delta t, \epsilon)$ in the old theory must correspond to infinitesimal action by $h(\Delta t, \epsilon)^*$ in the new theory. It follows from this that a generic discrete planewave in the old theory's spacetime must map onto a corresponding continuous planewave in the new theory's spacetime. To see this, consider a generic discrete planewave in the old theory's spacetime,

$$\varphi^{(\omega,k)}_{\text{old}}(t, n) \coloneqq \exp(\mathrm{i}\omega t + \mathrm{i}k\, n), \tag{8.21}$$

Note that this state is a simultaneous eigenvector of ∂_t and D with eigenvalues $\mathrm{i}\omega$ and $\mathrm{i}k$ respectively. It follows from Eq. (7.25) that the corresponding state in the new theory, $\varphi^{(\omega\, k)}_{\text{new}}(\tau, q) \coloneqq J \circ \mathcal{F}_{\text{vec}}(\varphi^{(\omega\, k)}_{\text{old}})$, must be a simultaneous eigenvector of ∂_τ and ∂_q with

eigenvalues $\mathrm{i}\omega$ and $\mathrm{i}k/a$ respectively. Namely, we must have,[93]

$$J \circ \mathcal{F}_{\text{vec}} : \varphi_{\text{old}}^{(\omega\, k)} \mapsto \varphi_{\text{new}}^{(\omega\, k)}(\tau, q) = b(\omega, k)\, \exp(\mathrm{i}\omega\tau + \mathrm{i}kq/a), \tag{8.22}$$

for some scaling parameter $b(\omega, k)$. Thus, up to some rescaling freedom, the externalization process must map discrete planewaves in the old theory's spacetime onto a corresponding continuous planewave in the new theory's spacetime. Note that the old theory's discrete planewaves (which are exhausted by the wavenumbers $k \in [-\pi, \pi]$ and $\omega \in \mathbb{R}$) must become continuous planewaves with wavenumbers $k/a =: \kappa \in [-\pi/a, \pi/a]$. Hence, all of the new theory's states are guaranteed to be *bandlimited* with a bandwidth of $K \coloneqq \pi/a$: Namely, their support in Fourier space is limited to $\kappa \in [-K, K]$. Note that this conclusion holds independent of our choice of scalings, $b(\omega, k)$.

A quick calculation shows that a generic state in the old theory, $\varphi_{\text{old}}(t, n) \in S^{\text{kin}}_{\text{old}}$, must be mapped onto,

$$J \circ \mathcal{F}_{\text{vec}} : \varphi_{\text{old}} \mapsto \varphi_{\text{new}}(\tau, q) = B(\tau, q/a) *_{\text{conv}} \sum_{n \in \mathbb{Z}} \delta(q/a - n)\, \varphi_{\text{old}}(\tau, n), \tag{8.23}$$

for some profile $B(\tau, q/a)$ given by the Fourier transform of $b(\omega, k)$ extended to vanish for $k \notin [-\pi, \pi]$. This bandlimited profile is convolved against a direct embedding of the discrete field state, $\varphi_{\text{old}}(\tau, n)$, as Dirac deltas onto the new continuous spacetime.

Just as in the Fourier example, there is effectively only one way to map the old theory's states onto the new theory's spacetime. The only freedom we have in populating the new spacetime is in a free choice of scalings for each of the new theory's planewaves, $b(\omega, k)$ for $\omega \in \mathbb{R}$ and $k \in [-\pi, \pi]$. But what should we make of this planewave-rescaling freedom? Just as in the Fourier example, this rescaling freedom is metaphysically irrelevant if we judge the relative size and similarity of states using the new theory's induced

[93] As in Sec. 7.4 and Sec. 8.1, this part of the calculation requires that the new theory's induced addition and scalar multiplication operations, $+_{\text{new}}$ and $\cdot_{\text{new}}$, are the standard point-wise operations on $\mathcal{M}_{\text{new}}$, namely $+_p$ and $\cdot_p$.

inner product, I_{new}. The only difference this choice makes is in how the theory's states are being represented in this new spacetime setting. Hence, we are free to pick $b(\omega, k) = 1$ for all $\omega \in \mathbb{R}$ and all $k \in [-\pi, \pi]$. Noting that $b(\omega, k) = 0$ outside of this region we have a profile of $B(\tau, q/a) = \text{sinc}(q/a - n)$ and therefore,

$$J \circ \mathcal{F}_{\text{vec}} : \varphi_{\text{old}} \mapsto \varphi_{\text{new}}(\tau, q) = \sum_{n \in \mathbb{Z}} \text{sinc}(q/a - n)\, \varphi_{\text{old}}(\tau, n) \tag{8.24}$$

where $\text{sinc}(z) \coloneqq \sin(\pi\, z)/(\pi\, z)$.

Continuing with this choice of J one can map the old theory's dynamics into this new spacetime setting in order to define the following theories:[94]

> **Bandlimited Heat Equation on a Continuous Spacetime -** Consider a theory set on a spacetime manifold, $\mathcal{M}_{\text{new}} \cong \mathbb{R}^2$, about a scalar field, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathcal{V}_{\text{new}}$, with value space, $\mathcal{V}_{\text{new}} \cong \mathbb{R}$. This theory's states are subject to the following kinematic constraint: In some fixed global coordinate system, (τ, q), each field must be smooth in the τ direction. Moreover, at each τ-coordinate the field must have a well-defined Fourier transform, $\tilde{\varphi}_{\text{new}}(t, \kappa)$, with support only over $\kappa \in [-\pi/a, \pi/a]$. At each τ-coordinate, the metaphysically relevant way to judge the relative size and similarity of this theory's states is with an L^2 inner product. There are three variants of this theory with the following three dynamical equations respectively:
>
> $$\begin{aligned}
> \textbf{H1}: &\quad \partial_\tau \varphi_{\text{new}}(\tau, q) = \alpha\, [2\text{cosh}(a\partial_q) - 2]\, \varphi_{\text{new}}(\tau, q), \\
> \textbf{H2}: &\quad \partial_\tau \varphi_{\text{new}}(\tau, q) = \frac{\alpha}{12}[-2\text{cosh}(2a\partial_q) + 32\text{cosh}(a\partial_q) - 30]\varphi_{\text{new}}(\tau, q), \\
> \textbf{H3}: &\quad \partial_\tau \varphi_{\text{new}}(\tau, q) = \frac{\alpha}{a^2}\, \partial_q^2\, \varphi_{\text{new}}(\tau, q),
> \end{aligned} \tag{8.25}$$
>
> for some fixed parameter $\alpha > 0$.

Note that H3 is the standard continuum heat equation (albeit bandlimited). H1 and H2 are similar to the continuum heat equation but feature some non-locality at a length scale of order $\sim a$.

One remarkable feature of these bandlimited heat equations should be noted: They each have a continuous translation symmetry in their q-coordinate. In the old theory, this corresponds to the fact that action by $\exp(-\epsilon D)$ is a dynamical symmetry i.e., it maps solutions to solutions. To see this, note that each of the infinite-dimensional matrices

[94] See Grimmer (2022b) for details.

$\Delta^2_{(1)}$, $\Delta^2_{(2)}$, D^2, and D are all diagonal in the same basis (see Eq. (8.14)). In hindsight, one can understand this application of the ISE Method as follows: In Eq. (8.19), we identified a nice set of dynamical symmetries, $f(\Delta t, \epsilon) \in F_{\text{Cont}} \cong (\mathbb{R}^2, +)$, in our old lattice-based theory. Despite not being lifted diffeomorphisms these transformations on $S^{\text{kin}}_{\text{old}}$ are structurally indistinguishable from spacetime translations. Namely, they are PSTOs. The ISE Method allows us to build a new spacetime setting for our theory such that these PSTOs become honest-to-goodness spacetime translations. Indeed, since these PSTOs are dynamical symmetries, they turn into spacetime symmetries of the new theory.

Perhaps surprisingly, the reverse topological redescription is also possible using the ISE Method (see Ch. 11 for proof). Namely, suppose that we had started from the bandlimited theory discussed above. One can internalize this theory using the above-discussed $(\mathbb{R}^2, +)$ group. While one could then use this group to re-externalize the theory, it is actually sufficient to just use the discrete subgroup $(\mathbb{R} \times \mathbb{Z}, +)$. This is only possible because of this theory's bandlimiting kinematic constraint. An alternate way to understand this continuous-to-discrete redescription is via the Nyquist-Shannon sampling theory: Any bandlimited function can be *exactly reconstructed* by a sufficiently dense discrete sampling. This sampling could be uniform as above or (remarkably!) non-uniform so long as it remains sufficiently dense. Unfortunately, further discussion of Nyquist-Shannon sampling theory is beyond the scope of this thesis. I direct the interested reader to Jerri (1977) and Shannon (2001).

8.3 Conventionalism about Spacetime Topology?

I will now discuss how these two examples of topological redescription suggest the possibility of conventionalism about spacetime topology. In Sec. 8.1 we seem to have a choice between a spacetime with and without extra compactified dimensions. In Sec. 8.2

we seem to have a choice between a continuous or a discrete spacetime. This raises the question: What are our topology selection criteria? Moreover, are they objective or conventional? One might hope that "if nature is kind" to us (to apply Lewis (1994, pg. 479) in a new context) then our topology selection criteria will be objective. We have already seen, for instance, in Sec. 4.3 that certain C^∞-style pre-spacetime theories have exactly one nice real scalar field representation (up to diffeomorphism). As evidence against the kindness of nature, however, let's reflect on the above two examples of topological redescription.

It should first be noted that both kinds of topological redescription which I have demonstrated in this chapter are not tied to the specific details of my example theories. For instance, given any theory with compactified dimensions, we can get rid of them by expanding its fields in their harmonic modes. Similarly, any discrete spacetime theory can be recast as a bandlimited theory (and vice versa by the Nyquist-Shannon sampling theory). In both cases, these redescriptions are exact, i.e., not approximations. For narrative purposes, however, let's assume that the length scales appearing in these examples (i.e., R and a) are both relatively small such that they were at-first-unnoticed in the historical development of these theories.

Let us start with the discrete-to-continuous example. If a is relatively small (whether this ends up being interpreted as a lattice spacing a or a bandwidth $K = \pi/a$) the macroscopic creatures of this world are likely to first theorize in terms of fields on a continuous spacetime with no bandlimit. That is, they will likely first arrive at one of the above-discussed theories in the continuum limit, $a \to 0$, or, equivalently, $K \to \infty$. For either spacetime framing of the theory, taking this continuum limit amounts to the same thing; They will begin with a theory about non-bandlimited fields on a continuous spacetime. Now as they investigate the microscopic world, they will find that this kind of theory fails to give them the right predictions. They will say "Something funny is going on in the

microscopic world". But how to account for this funniness? One group of theorists has a long think about it and announces "The fundamental fields are bandlimited! Their bandwidth is $K = \pi/a$." The next day, another group of theorists announces "At a microscopic level space is discrete! Its lattice spacing is a."

As our above discussion has shown, the two theories which are being put forward here are topological redescriptions of each other. Namely, they are isomorphic at the level of kinematics in a way which aligns their dynamics one-to-one. Moreover, we can switch between these two theories using the ISE Method; Hence, what they ultimately disagree about is which Lie group actions on their states are to count as spatiotemporal. In particular, the bandlimited theory considers strictly more transformations as being spatiotemporal than the discrete theory does, $(\mathbb{R}^2, +)$ vs $(\mathbb{R} \times \mathbb{Z}, +)$. Namely, the continuous theory considers the transformations shown in the third column of Fig. 6.1 to be spacetime translations.

How are we to decide between these two formally equivalent pictures of the world? One's first thought might be that these two theories are empirically inequivalent. It should be recalled, however, that these two theories agree about all macroscopic predictions. Moreover, there is a robust dictionary relating how the two theories account for their microscopic predictions. What the continuous framing calls a derivative, $\partial_q \varphi_{\text{new}}(\tau, q)$, the discrete framing knows as $D\,\varphi_{\text{old}}(t)$. Where the continuous framing assigns field values between the lattice sites, the discrete framing sees linear combinations of field values at the lattice sites (this follows from Eq. (8.24)).

But what about non-empirical selection criteria? Two immediate considerations which come to mind are locality and symmetry. As I noted above, the discrete theories all have a hidden continuous symmetry which the bandlimited theory more naturally expresses as a spacetime symmetry. This points in favor of the bandlimited theory. Moreover, if the field's dynamics is like H3 then the continuous framing also has better locality than

the discrete theory. In this case, the continuous framing seems to win 2-0. However, if the dynamics is more like H1 or H2 then the discrete framing would seem to have better locality properties. We would then be tied 1-1. While there is much more to say about this topic, it is hopefully clear how investigating the world's microdynamics might leave us with a choice on our hands. We could either describe the microdynamics in terms of: 1) a bandlimited field on a continuous spacetime, or 2) a field on a discrete spacetime. Prima facie, it seems possible that competing considerations might pull us in opposite directions making this a hard choice. For further discussion see Grimmer (2022b) and Grimmer (2022a) both of which were inspired by Kempf's (2010) "Spacetime could be simultaneously continuous and discrete in the same way that information can be".

Our other example of topological redescription (i.e., dimensional reduction in Sec. 8.1) can also be framed as a case in which it is unclear how we should extend our macroscopic understanding of space and time into the microscopic world. If R is relatively small (whether this ends up being interpreted as a radius or as a parameter in the field masses $M_{\ell m} = \sqrt{M^2 + \ell(\ell+1)/R^2}$) the macroscopic creatures of this world are likely to first theorize in terms of a single real-valued field on a spacetime without compactified dimensions. In either case, at low energies all but the $\ell = 0$, $m = 0$ subfield are dynamically frozen. Whether this subfield is later understood as one harmonic mode of $\varphi_{\text{old}} : \mathbb{R}^2 \times S^2 \to \mathbb{R}$ or as one component of $\varphi_{\text{new}} : \mathbb{R}^2 \to \mathbb{R}^\infty$, it is the only dynamically relevant degree of freedom at low energies.

Now as they investigate the microscopic world, they will find that this kind of theory fails to give them the right predictions. They will say "Something funny is going on in the microscopic world". But how to account for this funniness? One group of theorists has a long think about it and announces "There are extra compactified spacetime dimensions! (But there is still just one real-valued field.)" The next day, another group of theorists announces "There are no extra dimensions! Instead, a series of increasingly

massive fields becomes relevant at high energies." As our above discussion has shown, the two theories which are being put forward here are topological redescriptions of each other. Namely, they are isomorphic at the level of kinematics in a way which aligns their dynamics one-to-one. Moreover, we can switch between these two theories using the ISE Method; Hence, what they ultimately disagree about is which Lie group actions on their states are to count as spatiotemporal. In particular, the theory with compactified dimensions considers strictly more transformations as being spatiotemporal than the other theory does, $(\mathbb{R}^2, +) \times \mathrm{SO}(3)$ vs just $(\mathbb{R}^2, +)$.

How are we to decide between these two formally equivalent pictures of the world? One's first thought might be that these two theories are empirically inequivalent. It should be recalled, however, that these two theories agree about all macroscopic predictions. Moreover, there is a robust dictionary relating how the two theories account for their microscopic predictions. As noted above, what the old theory calls a harmonic mode of $\varphi_{\text{old}} : \mathbb{R}^2 \times S^2 \to \mathbb{R}$, the other theory calls a subfield, i.e., a component of $\varphi_{\text{new}} : \mathbb{R}^2 \to \mathbb{R}^\infty$. But what about non-empirical selection criteria? Again, locality and symmetry are two considerations which immediately come to mind. Unlike in the previous case, both of these theories have local dynamics. Regarding symmetries, both framing of the theories recognize the theory's dynamics as having an $\mathrm{SO}(3)$ symmetry. The old theory regards this as a spacetime symmetry whereas the new theory regards this as an internal symmetry.

In the previous example, I argued in favor of the theory which expressed its hidden dynamical symmetries as spacetime symmetry. Can we make the same argument here? While there may be some benefit in this, it has the downside of introducing new spacetime dimensions with no macroscopic analog. In the standard course of physics, it feels very natural to add more kinds of particles/fields to our theories as we explore higher energies. By contrast, the addition of new spacetime dimensions would seem to require a higher threshold of evidence. But as the above example shows, the latter option is always ex-

changeable with the former. That is, one might can always choose to add new kinds of particles/fields instead of adding new compactified dimensions. But wouldn't this mean that we cannot discover compactified dimensions (even if they are there) simply because of our standards of scientific practice? In this case, perhaps more so than the previous one, we seem to have a hard choice on our hands.

9 | The ISE Method in General: Internalization

The next two chapters will overview the ISE Method in its full generality. Before this, however, allow me to introduce two more example theories to help demonstrate the scope of these topological redescription techniques. The following two closely related spacetime theories will also help to make parts of the following discussion more concrete.

> **Periodic Matter on a Non-Periodic Spacetime -** Consider a theory set on a spacetime manifold, $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2$, about a scalar field, $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to \mathcal{V}_{\text{old}}$, with value space, $\mathcal{V}_{\text{old}} \cong \mathbb{R}$. The theory's states are subject to the following kinematic constraint: In some fixed global coordinate system, (t, x), the field, φ_{old}, must be second-differentiable (and so potentially non-smooth). Moreover, this theory's states are kinematically constrained to be periodic in the x-coordinate with some fixed period $L > 0$ as,
>
> $$\varphi_{\text{old}}(t, x + L) = \varphi_{\text{old}}(t, x). \tag{9.1}$$
>
> In this coordinate system, the dynamical equation for φ_{old} is as follows:
>
> $$\partial_t \varphi_{\text{old}}(t, x) = \alpha\, \partial_x^2 \varphi_{\text{old}}(t, x), \tag{9.2}$$
>
> for some fixed $\alpha > 0$.

Note that this theory describes the dynamics of a periodic distribution of heat, $\varphi_{\text{old}}(t, x)$, evolving on a non-periodic spacetime, $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2$. For brevity, let us refer to this theory as the *periodic matter theory*. Contrast this with the following closely spacetime related theory.

> **Matter on a Periodic Spacetime -** Consider a theory set on a spacetime manifold, $\mathcal{M}_{\text{new}} \cong \mathbb{R} \times S^1$ about a scalar field, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathcal{V}_{\text{new}}$, with value space, $\mathcal{V}_{\text{new}} \cong \mathbb{R}$. For this theory there exists a map, $C : \mathbb{R}^2 \to \mathcal{M}_{\text{new}}$, which assigns to each point, $p \in \mathcal{M}_{\text{new}}$, a set of coordinates, $C^{-1}(p) = \{(\tau, q + n\, L)\}_{n \in \mathbb{Z}} \subset \mathbb{R}^2$. The theory's states are subject to the following kinematic constraint: In this fixed coordinate system the field, φ_{new}, must be second-differentiable (and so potentially non-smooth). In this coordinate system, the dynamical equation for φ_{new} is as follows:
>
> $$\partial_\tau \varphi_{\text{new}}(\tau, q) = \alpha\, \partial_q^2 \varphi_{\text{new}}(\tau, q), \tag{9.3}$$
>
> for some fixed $\alpha > 0$.

It should be noted that the q-coordinate in this theory is periodic with period L, unlike the x-coordinate of the previous theory. Hence, this theory describes the dynamics of a generic distribution of heat, $\varphi_{\text{new}}(\tau, q)$, evolving on a periodic spacetime, $\mathcal{M}_{\text{new}} \cong \mathbb{R} \times S^1$. For brevity, let us refer to this theory as the *periodic spacetime theory*.

The keen reader may have noted that these two theories are topological redescriptions of each other (as defined in Sec. 4.2). The metaphysically and dynamically relevant structures for these theories would include a vector space structure for their kinematically allowed states and potentially other algebraic structures such as an inner product. So equipped, we can then see these two theories as being different spacetime representations of the same pre-spacetime theory. They have the same pre-spacetime theoretic structure because there is an isomorphism between their kinematic structures, $\langle S^{\text{kin}}_{\text{new}}, \dots \rangle \cong \langle S^{\text{kin}}_{\text{old}}, \dots \rangle$, which preserves their dynamics, $S^{\text{dyn}}_{\text{new}} \leftrightarrow S^{\text{dyn}}_{\text{old}}$. They are nonetheless inequivalent as spacetime theories because they represent this pre-spacetime theoretic structure with different topological staging of their states (e.g., as scalar fields on $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2$ vs on $\mathcal{M}_{\text{new}} \cong \mathbb{R} \times S^1$ respectively). To connect with Fig. 4.2 in Sec. 4.3, these two spacetime theories are in the same pre-spacetime equivalence class, i.e., the same large box.

As Sec. 11.2 will discuss further, we can use the ISE Method to topologically redescribe the periodic matter theory as the periodic spacetime theory, *but not vice-versa*. Namely, for this pair of theories we can move from old to new but not backwards from new to old. This asymmetry is due to the periodic matter theory lacking a certain kind of spacetime-kinematic compatibility (defined in Sec. 9.2). Roughly, this theory's spacetime manifold is excessive given the theory's kinematics; It has distinct spacetime points which are valued identically in every kinematically allowed state. To connect with Fig. 4.2 in Sec. 4.3, while these two theories are in the same pre-spacetime equivalence class (i.e., the same large box) the periodic matter theory is outside of the SKC oval whereas the periodic spacetime theory is inside.

9.1 Internalizing a Generic Spacetime Theory

Hopefully, the variety of the spacetime theories which we have already discussed indicates the ISE Method is applicable to a very wide range of spacetime theories. But what exactly is its scope? More concretely, what kinds of spacetime theories can be taken as input by the ISE Method? As I will now discuss, the ISE Method can be applied to *any spacetime theory*. Let us briefly recall my definition of the term "spacetime theory" from Sec. 4.2. To earn this title: 1) our theory must distinguish between DPMs and KPMs, $S^{\text{dyn}}_{\text{old}} \subset S^{\text{kin}}_{\text{old}}$; 2) it must have a sufficient amount of structure at the level of kinematics, $\mathcal{S}^{\text{kin}} = \langle S^{\text{kin}}, \dots \rangle$; 3) it must have a declared topological staging, Σ, for its states, $S^{\text{all}}_{\text{old}} \supset S^{\text{kin}}_{\text{old}} \supset S^{\text{dyn}}_{\text{old}}$; and 4) it must have a canonical way to lift its automorphisms, $d \in \text{Auto}(\Sigma)$, to maps on its states, $d^* : S^{\text{all}}_{\text{old}} \to S^{\text{all}}_{\text{old}}$. Recall that this $*_{\text{lift}}$ map must be a group homomorphims, i.e., we must have $d_1^* \, d_2^* = (d_1 \, d_2)^*$.

We are now in a position to discuss how the internalization process applies to a generic spacetime theory. Recall from Ch. 6 that the purpose of internalization is to break any connection between the old theory's states and its assumed smooth topological background structure (i.e., the spacetime manifold). This divorce is to be carried out at the level of kinematics. Recall that by the declared topological staging assumption a spacetime theory specifies a specific topological stage for its states, Σ, and declare that it is about a certain kind of states. For instance, if the theory's declared stage is a smooth manifold $\Sigma = \mathcal{M}$ then it might be about a particle trajectory, $X : \mathcal{V} \to \mathcal{M}$, or a scalar field, $\varphi : \mathcal{M} \to \mathcal{V}$, or a tangent vector field, $\vec{\varphi} : \mathcal{M} \to \text{T}\mathcal{M}$. More generally, the theory in question could be about global sections, $\varphi : B \to E$, of some fiber bundle, $\Sigma = \langle E, B, F, \pi : E \to B \rangle$. To remain neutral between all of these kinds of states, I will henceforth refer to the old theory's states generically as $\sigma \in S^{\text{all}}$. In any case, it is this field-to-spacetime connection which the internalization process aims to sever.

Negating the declared topological staging assumption (and relatedly dropping the natural lifting assumption) yields what I called a *pre-spacetime theory* in Sec. 4.2. In effect, a pre-spacetime theory is just a spacetime theory which has lost its commitment to a declared topological staging. In sum: The internalization process takes as input a spacetime theory and outputs the corresponding pre-spacetime theory.

Formally, we can implement the internalization process by applying some forgetful bijection operation, $\mathcal{F}$, to $\mathcal{S}^{\text{kin}}_{\text{old}}$ as follows:

$$\mathcal{F} : \mathcal{S}^{\text{kin}}_{\text{old}} \rightarrow \mathcal{S}^{\text{kin}}_{\text{pst}} \coloneqq \langle S^{\text{kin}}_{\text{pst}}, \dots \rangle. \tag{9.4}$$

Two things have changed here. Firstly, the old theory's kinematically allowed states, $S^{\text{kin}}_{\text{old}}$, have been replaced via the bijection, $\mathcal{F} : S^{\text{kin}}_{\text{old}} \leftrightarrow S^{\text{kin}}_{\text{pst}}$, with an equal-sized set, $S^{\text{kin}}_{\text{pst}}$. The key difference here is that whereas $\sigma_{\text{old}} \in S^{\text{kin}}_{\text{old}}$ is a function valued on/in Σ_{old}, the corresponding state, $\sigma_{\text{pst}} = \mathcal{F}(\sigma_{\text{old}}) \in S^{\text{kin}}_{\text{pst}}$, is not. Secondly, isomorphic copies of each of the old theory's dynamically and metaphysically relevant structures have been induced in $S^{\text{kin}}_{\text{pst}}$. Namely, using the bijection, $\mathcal{F} : S^{\text{kin}}_{\text{old}} \leftrightarrow S^{\text{kin}}_{\text{pst}}$, we have defined the relevant structures in $\mathcal{S}^{\text{kin}}_{\text{pst}}$ such that we have the appropriate kind of isomorphism, $\mathcal{S}^{\text{kin}}_{\text{pst}} \cong \mathcal{S}^{\text{kin}}_{\text{old}}$. For an explicit example of how the relevant structures are induced in the pre-spacetime theory, see Eq. (7.4). Internalizing the theory in this way completely severs the connection between the old theory's states, σ_{old}, and the old theory's topological staging, Σ_{old}.

It is important to note that while the theory's dynamically and metaphysically relevant structures are guaranteed to faithfully survive internalization, the mathematical character of these structures is, in some sense, mechanically irrelevant; They merely come along for the ride and play no role in helping us get from A to B. Indeed, the ISE Method applies in exactly the same way to a wide variety of spacetime theories regardless of how they are structured, algebraically or otherwise. As I have discussed in Part I, this mechanism of topological redescription is in some ways analogous to the well-understood processes

of coordinate redescription and logical re-axiomization; In each case, the redescription mechanism is largely insensitive to the details of the theory in question. For instance, recall the Möbius-Euclid duality from Sec. 4.1 which lacked any algebraic structure. The ISE Method of topological redescription is to be carried out *without invoking any algebraic structure whatsoever*. This stands in stark contrast to the algebraic approach to spacetime, which identifies spacetime points with the maximal ideals of certain rings.

Despite being mechanically irrelevant, a theory's metaphysical and dynamical details are, of course, of great philosophical importance. Indeed, recalling the kinematic structure assumption we must first do the hard philosophical work of identifying which structural features of one's theory are dynamically and metaphysically relevant. Moreover, while these structures don't help us get from A to B, they will nonetheless play an essential role in helping us evaluate A and B as possible destinations. That is, the quality of any given spacetime representation (e.g., if it qualifies as "particularly nice") will depend in no small part upon how naturally these pre-spacetime theoretic structures are represented there.

But what exactly survives internalization? Each of the old theory's kinematically allowed states, $\sigma_{\text{old}} \in S^{\text{kin}}_{\text{old}}$, has a corresponding pre-spacetime state $\sigma_{\text{pst}} \coloneqq \mathcal{F}(\sigma_{\text{old}})$. Hence, we can see these states as having survived internalization as follows,

$$\sigma_{\text{old}} \xrightarrow[\text{int. as}]{\text{survives}} \sigma_{\text{pst}} \coloneqq \mathcal{F}(\sigma_{\text{old}}). \tag{9.5}$$

Just as some (but not all) of theory's states surviving internalization (e.g., only $S^{\text{kin}}_{\text{old}} \subset S^{\text{all}}_{\text{old}}$), we also have some (but not all) of the maps between these states survive internalization. Let us say that a map $f_{\text{old}} : S^{\text{all}}_{\text{old}} \to S^{\text{all}}_{\text{old}}$ is kinematically allowed when it preserves the old theory's kinematic constraints, $f_{\text{old}}(S^{\text{kin}}_{\text{old}}) \subset S^{\text{kin}}_{\text{old}}$. For any such map, we can restrict its domain to $S^{\text{kin}}_{\text{old}}$ as, $f_{\text{old}}|_{\text{kin}} : S^{\text{kin}}_{\text{old}} \to S^{\text{kin}}_{\text{old}}$. Note that each of the old theory's kinematically allowed maps has a corresponding pre-spacetime map given by $f_{\text{pst}} \coloneqq \mathcal{F} \circ f_{\text{old}}|_{\text{kin}} \circ \mathcal{F}^{-1}$.

Hence, we can see these maps as surviving internalization as follows,

$$f_{\text{old}}|_{\text{kin}} \xrightarrow[\text{int. as}]{\text{survives}} f_{\text{pst}} \coloneqq \mathcal{F} \circ f_{\text{old}}|_{\text{kin}} \circ \mathcal{F}^{-1}. \tag{9.6}$$

Maps, $f_{\text{old}} : S^{\text{all}}_{\text{old}} \to S^{\text{all}}_{\text{old}}$ which are not kinematically allowed do not survive internalization.

Next, notice that, by design, all of the theory's dynamically and metaphysically relevant structures will survive internalization. Hence, we ought to have the resources within $\mathcal{S}^{\text{kin}}_{\text{pst}}$ to state the theory's dynamics in a pre-spacetime theoretic way. In light of this, the theory's dynamically allowed subset survives internalization,

$$S^{\text{dyn}}_{\text{old}} \subset S^{\text{kin}}_{\text{old}} \xrightarrow[\text{int. as}]{\text{survives}} S^{\text{dyn}}_{\text{pst}} \subset S^{\text{kin}}_{\text{pst}}. \tag{9.7}$$

Indeed, we have $S^{\text{dyn}}_{\text{pst}} = \mathcal{F}(S^{\text{dyn}}_{\text{old}})$. For an explicit example of a theory's dynamics surviving internalization, see Sec. 7.1.

So far we have seen how the states and dynamics of a wide range of theories can survive being divorced from the old theory's spacetime manifold (i.e., its smooth topological background structure). As the examples from Ch. 7 and Ch. 8 have shown, internalization does not simply delete all of the theory's topological information. Indeed, in the three theories which were considered there, a great deal of topological information about the old spacetime setting survives internalization. As I will now show, for every spacetime theory we will have some topological information surviving internalization.

The key to seeing how topological information survives internalization is the lifting assumption. This guarantees that the theory's stage automorphisms, $d \in \text{Auto}(\Sigma_{\text{old}})$, can be lifted to act on the theory's states as $d^* : S^{\text{all}}_{\text{old}} \to S^{\text{all}}_{\text{old}}$. It should be noted that a subgroup of these lifted automorphisms, d^*, will preserve the theory's kinematic constraints, mapping $S^{\text{kin}}_{\text{old}}$ onto itself. Let us call this the theory's group of *kinematically allowed*

automorphisms,

$$\text{Auto}_{\text{kin}}(\Sigma_{\text{old}}) \coloneqq \{d \in \text{Auto}(\Sigma_{\text{old}}) \mid \ d^*(S^{\text{kin}}_{\text{old}}) \subset S^{\text{kin}}_{\text{old}}\}. \tag{9.8}$$

Note that every kinematically allowed automorphism, $d \in \text{Auto}_{\text{kin}}(\Sigma_{\text{old}})$, can be lifted to d^* and then restricted to act on $S^{\text{kin}}_{\text{old}}$ as $d^*|_{\text{kin}} : S^{\text{kin}}_{\text{old}} \to S^{\text{kin}}_{\text{old}}$. We can then see that these kinematically allowed automorphisms survive internalization in light of Eq. (9.6). Namely, we have that

$$d \in \text{Auto}_{\text{kin}}(\Sigma_{\text{old}}) \xrightarrow[\text{int. as}]{\text{survives}} G(d) \coloneqq \mathcal{F} \circ d^*|_{\text{kin}} \circ \mathcal{F}^{-1}. \tag{9.9}$$

Thus, we have that every kinematically allowed automorphisms survives internalization.

In fact, as I will now discuss these automorphisms survive not only individually but as a group (albeit potentially unfaithfully). In order for this to happen G must be a group homomorphism. This will happen exactly when the kernel of $\ker(G)$ is a normal subgroup of $\text{Auto}_{\text{kin}}(\Sigma_{\text{old}})$. Let us call this kernel the theory's group of *kinematically trivial automorphisms*,

$$\ker(G) = H_{\text{kin}} \coloneqq \{d \in \text{Auto}_{\text{kin}}(\Sigma_{\text{old}}) \mid \ d^*|_{\text{kin}} = \mathbb{1}\}, \tag{9.10}$$

These are the automorphisms which, when lifted, act trivially on every kinematically allowed state, i.e., they have $d^*\sigma_{\text{old}} = \sigma_{\text{old}}$ for all $\sigma_{\text{old}} \in S^{\text{kin}}_{\text{old}}$.

As I will now prove, the fact that H_{kin} is normal in $\text{Auto}_{\text{kin}}(\Sigma_{\text{old}})$ follows directly from $*_{\text{lift}}$ being a group homomorphism. The desired normality claim is that for all $d \in \text{Auto}_{\text{kin}}(\Sigma_{\text{old}})$ and all $h \in H_{\text{kin}}$ we have $d^{-1}\, h\, d \in H_{\text{kin}}$. To see this, note that for all $\sigma_{\text{old}} \in S^{\text{kin}}_{\text{old}}$ we have,

$$\begin{aligned}(d^{-1}\, h\, d)^*(\sigma_{\text{old}}) &= d^{*-1}\, h^*\, d^*(\sigma_{\text{old}}) \\ &= d^{*-1}\, d^*(\sigma_{\text{old}}) \\ &= \sigma_{\text{old}}.\end{aligned} \tag{9.11}$$

The first line follows from $*_{\text{lift}}$ being a group homomorphism. Namely, it follows from $(a\,b)^* = a^*\,b^*$ and $(a^{-1})^* = (a^*)^{-1}$. Next, note that since $d \in \text{Auto}_{\text{kin}}(\Sigma_{\text{old}})$ we know it preserves the theory's kinematically allowed states; Hence, we have that $d^*(\sigma_{\text{old}}) \in S^{\text{kin}}_{\text{old}}$ is a kinematically allowed state. The second line follows from the fact that $h \in H_{\text{kin}}$ acts trivially on all such states. Thus, we have that H_{kin} is a normal subgroup of $\text{Auto}_{\text{kin}}(\Sigma_{\text{old}})$. Therefore G is a group homomorphism.

Given that G is a group homomorphism, we therefore have that:

$$\text{Auto}_{\text{kin}}(\Sigma_{\text{old}}) \xrightarrow[\text{int. as}]{\text{survives}} G(\text{Auto}_{\text{kin}}(\Sigma_{\text{old}})) \cong \text{Auto}_{\text{kin}}(\Sigma_{\text{old}})/H_{\text{kin}}. \tag{9.12}$$

That is, up to isomorphism, $\text{Auto}_{\text{kin}}(\Sigma_{\text{old}})$ survives internalization as a group by being quotiented by the theory's kinematically trivial group, H_{kin}. The above discussion can be easily generalized to analyze the survival of an arbitrary subgroup of the kinematically allowed automorphisms, $H \subset \text{Auto}_{\text{kin}}(\Sigma_{\text{old}})$. Restricting the G map to this subgroup as $G|_H$ yields a group homomorphism if and only if restricting H_{kin} to H yields a normal subgroup of H. One can easily check that $\ker(G|_H) = H_{\text{kin}} \cap H$ is a normal and subgroup of H. In fact, the above arguments used in the $H = \text{Auto}_{\text{kin}}(\Sigma_{\text{old}})$ case apply unchanged. Thus, for every kinematically allowed group, $H \subset \text{Auto}_{\text{kin}}(\Sigma_{\text{old}})$, we have that,

$$H \xrightarrow[\text{int. as}]{\text{survives}} G(H) \cong H/(H_{\text{kin}} \cap H). \tag{9.13}$$

That is, up to isomorphism, $H \subset \text{Auto}_{\text{kin}}(\Sigma_{\text{old}})$ survives internalization as a group by being quotiented by $H_{\text{kin}} \cap H$, its kinematically trivial subgroup.

Thus we have seen that *some* automorphisms can survive internalization (albeit potentially unfaithfully). Let's see some examples. For all but one of the spacetime theories which we have discussed so far, we have $H_{\text{kin}} = \{\mathbb{1}\}$, only the trivial automorphism is kinematically trivial. (Later I shall call such theories kinematically reduced). Only the Periodic Matter on a Non-Periodic Spacetime Theory has $H_{\text{kin}} \neq \{\mathbb{1}\}$. To see this,

consider the diffeomorphisms, $d(\Delta t, \Delta x)$, which act in the (t, x) coordinate system as $d(\Delta t, \Delta x) : (t, x) \mapsto (t - \Delta t, x - \Delta x)$. Let these diffeomorphisms, $d(\Delta t, \Delta x)$, together form a group, $H_{\text{trans}} \cong (\mathbb{R}^2, +)$. Each of these diffeomorphisms is kinematically allowed, $d(\Delta t, \Delta x) \in \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$, since each of them maps kinematically allowed states to kinematically allowed states. Some of them, however, are kinematically trivial. Namely, when $\Delta t = 0$ and $\Delta x = n\,L$ for $n \in \mathbb{Z}$ these diffeomorphisms act as the identity on every kinematically allowed state,

$$\forall \varphi_{\text{old}} \in S^{\text{kin}}_{\text{old}} \quad d(0, n\,L)^* \varphi_{\text{old}} = \varphi_{\text{old}}. \tag{9.14}$$

This is due to the fact that all of the theory's kinematically allowed states are periodic in the x-coordinate with period L. Thus, we have $d(0, n\,L)^*|_{\text{kin}} = \mathbb{1}$ despite the fact that $d(0, n\,L) \neq \mathbb{1}$. The kinematically trivial subgroup of H_{trans} is $H_{\text{kin}} \cap H_{\text{trans}} \cong (\mathbb{Z}, +)$ containing these $d(0, n\,L)$ maps.

Let us now see how having $H_{\text{kin}} \cap H_{\text{trans}} \neq \{\mathbb{1}\}$ interferes with H_{trans} faithfully surviving internalization. Applying Eq. (9.13) we have that H_{trans} survives internalization albeit unfaithfully as follows:

$$H_{\text{trans}} \xrightarrow[\text{int. as}]{\text{survives}} G_{\text{trans}} := G(H_{\text{trans}}) \cong \frac{H_{\text{trans}}}{H_{\text{kin}} \cap H_{\text{trans}}} \cong \frac{(\mathbb{R}^2, +)}{(\mathbb{Z}, +)} \cong (\mathbb{R}, +) \times U(1). \tag{9.15}$$

This is the case in general: For kinematically unreduced theories, we will have that $\text{Auto}_{\text{kin}}(\Sigma_{\text{old}})$ and all of its subgroups survive internalization albeit filtered through the theory's kinematically allowed states.

To review: We can apply the internalization process to any spacetime theory (as defined in Sec. 4.2). For all such theories, the above discussion has demonstrated that at least some automorphisms (namely, $\text{Auto}_{\text{kin}}(\Sigma_{\text{old}}) \subset \text{Auto}(\Sigma_{\text{old}})$) will survive the internalization process as a group. Thus, under four very weak assumptions, at least some topological information regarding the old spacetime setting survives internalization. Next, I will

discuss the conditions under which *complete* topological information survives internalization.

9.2 Internalization given Spacetime-Kinematic Compatibility

This section will define a notion of spacetime-kinematic compatibility which is sufficient for a spacetime theory's topological stage to survive internalization both faithfully and robustly (definitions to follow). The remainder of this thesis will be focused solely on spacetime theories which are set on smooth manifolds, $\Sigma = \mathcal{M}$. I do so solely for reasons of brevity and ease of introduction. There exist similar mathematical techniques to those employed below for fiber bundles, $\Sigma = \langle E, B, F, \pi : E \to B \rangle$, as well as supermanifolds, and topological spaces. A full discussion of these, however, is beyond the scope of this thesis.

Given that I am only considering theories set on a smooth manifold, $\Sigma = \mathcal{M}$, the relevant automorphisms are $\mathrm{Auto}(\Sigma) = \mathrm{Diff}(\mathcal{M})$. In this context, the lifting assumption becomes a requirement that the theory under consideration is "natural" in the sense of Fatibene and Francaviglia (2003). Importantly, this does stop our theory from making use of non-trivial fiber bundles. The theory in question could still be about global sections, $\varphi : \mathcal{M} \to E$, of some fiber bundle, $\pi : E \to \mathcal{M}$, under the following condition: E must be a *natural bundle* over $\mathcal{M}$. This means that there must be a natural way to lift any given $d \in \mathrm{Diff}(\mathcal{M})$ to an automorphism of E, namely $d^\ell \in \mathrm{Auto}(E)$ with $\pi \circ d^\ell = d \circ \pi$. Using this automorphism one can define $d^*\varphi := d^\ell \circ \varphi \circ d^{-1}$. For instance, the tangent bundle, $\mathrm{T}\mathcal{M}$, is a natural bundle over $\mathcal{M}$ with $d^\ell : \mathrm{T}\mathcal{M} \to \mathrm{T}\mathcal{M}$ being the push-forward. By contrast, the Möbius strip, $\mathbb{M}$, is not a natural bundle over the circle. Relatedly, unlike a tangent bundle over $\mathcal{M}$, a spin bundle over $\mathcal{M}$ is not a natural bundle. Hence, some technical upgrades will be needed before the ISE Method can be applied to

spinor theories. To remain neutral between all of these kinds of states, I will henceforth refer to the old theory's states generically as $\sigma \in S^{\text{all}}$.

Applied to natural bundle theories, the key result of the previous section is as follows. For any spacetime any group of kinematically allowed diffeomorphisms, $H \subset \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$, will survive internalization as a group, $G(H)$, which acts on $S^{\text{kin}}_{\text{pst}}$. The following discussion will demonstrate that given a certain amount of spacetime-kinematic compatibility these automorphisms will survive internalization faithfully, i.e., with $G(H) \cong H$ as a Lie group. Relatedly, I will demonstrate that under these conditions the theory's spacetime manifold, $\mathcal{M}_{\text{old}}$, will also survive internalization faithfully and robustly. Without further ado, spacetime-kinematic compatibility is defined as follows:

> **Definition: Spacetime-Kinematic Compatibility (for theories set on a smooth manifold) -** A spacetime theory has spacetime-kinematic compatibility just in case:
>
> 1. it is kinematically reduced, i.e., $H_{\text{kin}} = \{\mathbb{1}\}$. N.B.: This implies that H_{kin} is a closed subgroup of $\text{Diff}_{\text{kin}}(\mathcal{M})$.[95]
> 2. it is kinematically homogeneous. That is, there exists a finite-dimensional Lie group of kinematically allowed diffeomorphisms, $H_{\text{trans}} \subset \text{Diff}_{\text{kin}}(\mathcal{M})$, which acts transitively on $\mathcal{M}$.

As I will discuss in Sec. 9.3, spacetime-kinematic compatibility is a remarkably weak assumption.

I will now discuss how the internalization process applies to spacetime theories with spacetime-kinematic compatiblity. We have already seen in Sec. 9.1, that the sequence of maps, $d \mapsto d^* \mapsto d^*|_{\text{kin}} \mapsto G(d)$, is a group homomorphism. For our later purposes, however, we want these maps to preserve not only group structure but also Lie group structure. For this to happen it is necessary that H_{kin} is not only a normal subgroup of $\text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$ but also a closed subgroup thereof. This, however, follows directly from our assumption that the theory in question is kinematically reduced (since the singleton

[95] Note that this implicitly requires that we are able to equip $\text{Diff}_{\text{kin}}(\mathcal{M})$ with some topological structure. While nothing crucially depends on our choice of topology, I have in mind the compact-open topology for $\text{Diff}(\mathcal{M})$.

$\{\mathbb{1}\}$ is a closed set). If we also want G to be a Lie group isomorphism then we must further demand that its kernel be trivial. We have already seen in Sec. 9.1, that a theory's kinematically allowed diffeomorphisms, $\mathrm{Diff}_{\mathrm{kin}}(\mathcal{M}_{\mathrm{old}})$, might survive internalization unfaithfully if $H_{\mathrm{kin}} \coloneqq \ker(G)$ is non-trivial. Given our assumption that the theory in question is kinematically reduced, however, we will have that $\mathrm{Diff}_{\mathrm{kin}}(\mathcal{M}_{\mathrm{old}})$ and all of its subgroups will survive internalization faithfully. For the rest of this section, let us assume that the theory in question is kinematically reduced.

We are halfway to our next goal of showing that not only $\mathrm{Diff}_{\mathrm{kin}}(\mathcal{M}_{\mathrm{old}})$ but also $\mathcal{M}_{\mathrm{old}}$ itself survives internalization faithfully. See this, however, will require us to introduce a bit of mathematics and to make another assumption relating to the homogeneity of the theory's spacetime manifold. Recall from Sec. 7.1 that in order for $\mathcal{M}_{\mathrm{old}}$ to survive internalization we must first reconstruct it as a quotient of two finite-dimensional Lie groups of its diffeomorphisms, $\mathrm{Diff}(\mathcal{M}_{\mathrm{old}})$. The set of smooth manifolds which can be reconstructed in this way are exactly the smooth homogeneous manifolds. A more detailed discussion of smooth homogeneous manifolds can be found in Appendix A. For now, it is sufficient to recall that a smooth manifold, $\mathcal{M}$, is *homogeneous* if and only if there exists a (finite-dimensional) Lie group of diffeomorphisms, $H_{\mathrm{trans}} \subset \mathrm{Diff}(\mathcal{M})$, which acts transitively on it.[96] Given such a Lie group we can reconstruct $\mathcal{M}$ up to diffeomorphism as

[96] Two slightly different definitions of homogeneity need to be distinguished: One might here either allow or disallow infinite-dimensional Lie groups. For an example where this distinction matters, consider the double-torus, $\mathcal{M} \cong \mathbb{T}^2 \# \mathbb{T}^2$. This smooth manifold is homogeneous with respect to the infinite-dimensional Lie group, $\mathrm{Diff}(\mathcal{M})$, but not with respect to any finite-dimensional subgroup thereof. For proof of the latter claim, see Mostow (2005). For what I expect are merely technical reasons, certain theorems which are central to the ISE Method have only been proven in the finite-dimensional case (infinite-dimensional Lie groups are very difficult to work with). Indeed, the theorems proved in Appendix A only deal with the finite-dimensional notion of homogeneity. Until analogous theorems are proved in the infinite-dimensional case, the ISE Method ought to be understood as employing the finite-dimensional definition of homogeneity. However, while it is difficult to prove theorems in the infinite-dimensional case, it is much easier to simply talk about $\mathrm{Diff}(\mathcal{M})$ rather than quantifying over all of its finite-dimensional Lie subgroups. Hence, for expositional convenience, I will sometimes make use of the infinite-dimensional definition in the main text and clarify in a footnote what changes are needed in the finite-dimensional case.

$\mathcal{M} \cong H_{\text{trans}}/H_{\text{fix}}$ where H_{fix} is one of H_{trans}'s stabilizer subgroups.[97] These homogeneous manifold reconstruction results motivate the definition of *kinematic homogeneity* given above. Let us assume that the theory in question is kinematically homogeneous. Once again, this is a very weak assumption, see Sec. 9.3.

Now that we have finished defining spacetime-kinematic compatibility, we are in a position to see how it facilitates the faithful survival of the old theory's spacetime manifold, $\mathcal{M}_{\text{old}}$. It follows from the theory in question being kinematically homogeneous that its spacetime manifold, $\mathcal{M}_{\text{old}}$, is homogeneous. Hence, we can use the above-discussed characterization of smooth homogeneous manifolds to reconstruct $\mathcal{M}_{\text{old}}$ as a quotient of its diffeomorphisms. Namely, there will be some transitively acting Lie group, $H_{\text{trans}} \subset \text{Diff}(\mathcal{M}_{\text{old}})$, such that $\mathcal{M}_{\text{old}} \cong H_{\text{trans}}/H_{\text{fix}}$ where H_{fix} is one of H_{trans}'s stabilizer subgroups.

Indeed, since the theory in question is not only homogeneous but kinematically homogeneous we can take these two Lie groups to be kinematically allowed, $H_{\text{fix}} \subset H_{\text{trans}} \subset \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$. Recall that it follows from a theory being kinematically reduced that every Lie group of kinematically allowed diffeomorphisms survives internalization faithfully. Hence, we have that

$$H_{\text{trans}} \xrightarrow[\text{int. as}]{\text{survives}} G_{\text{trans}} \coloneqq G(H_{\text{trans}}) \cong H_{\text{trans}} \tag{9.16}$$

$$H_{\text{fix}} \xrightarrow[\text{int. as}]{\text{survives}} G_{\text{fix}} \coloneqq G(H_{\text{fix}}) \cong H_{\text{fix}}.$$

[97] Similar reconstruction results can be achieved for topological spaces by taking the quotient of topological groups. Moreover, a wide range of supermanifolds can be reconstructed as the quotient of Lie supergroups. Finally, one can reconstruct many fiber bundles as the pair-wise quotients of three Lie groups. Further discussion of these reconstruction techniques is unfortunately beyond the scope of this thesis.

Given that the Lie groups in both the numerator and denominator of $\mathcal{M}_{\text{old}} \cong H_{\text{trans}}/H_{\text{fix}}$ have faithfully survived internalization we have that,

$$\mathcal{M}_{\text{old}} \cong \mathcal{M}_{\text{H}} \coloneqq \frac{H_{\text{trans}}}{H_{\text{fix}}} \xrightarrow[\text{int. as}]{\text{survives}} \mathcal{M}_{\text{G}} \coloneqq \frac{G_{\text{trans}}}{G_{\text{fix}}} \cong \frac{H_{\text{trans}}}{H_{\text{fix}}} \cong \mathcal{M}_{\text{old}}. \tag{9.17}$$

We have thus demonstrated the faithful survival of $\mathcal{M}_{\text{old}}$ for any theory with spacetime-kinematic compatibility.

In addition to surviving internalization faithfully, $\mathcal{M}_{\text{old}}$ also survives robustly in the following sense. The surviving quotient manifold, $\mathcal{M}_{\text{G}}$, is diffeomorphic to $\mathcal{M}_{\text{old}}$ independent of which Lie groups, H_{trans} and H_{fix}, we used to reconstruct $\mathcal{M}_{\text{old}}$. Namely, for H_{trans}, we could have picked any kinematically allowed, transitively acting Lie group. For H_{fix}, we could have picked any stabilizer subgroup of H_{trans}.

In sum, the above discussion has demonstrated how for any theory with spacetime-kinematic compatibility, its spacetime manifold, $\mathcal{M}_{\text{old}}$, survives internalization both faithfully and robustly. But what goes wrong when our theory lacks spacetime-kinematic compatibility? Recall that the Periodic Matter on a Non-Periodic Spacetime theory is not kinematically reduced. Let us see whether or not its spacetime manifold faithfully survives internalization. Note that the Lie group defined above, $H_{\text{trans}} \cong (\mathbb{R}^2, +)$, acts transitively on $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2$. Note that all of H_{trans}'s stabilizer subgroups are trivial, $H_{\text{fix}} = \{\mathbb{1}\}$. Using these Lie groups we can reconstruct $\mathcal{M}_{\text{old}}$ up to diffeomorphism as follows,

$$\mathcal{M}_{\text{old}} \cong \mathcal{M}_{\text{H}} \coloneqq \frac{H_{\text{trans}}}{H_{\text{fix}}} \cong \frac{(\mathbb{R}^2, +)}{\{\mathbb{1}\}} \cong \mathbb{R}^2. \tag{9.18}$$

We have already seen how the Lie group in the numerator survives internalization unfaithfully. Namely, $H_{\text{trans}} \cong (\mathbb{R}^2, +)$ becomes $G_{\text{trans}} \cong (\mathbb{R}, +) \times U(1)$. By contrast, the Lie group in the denominator, $H_{\text{fix}} = \{\mathbb{1}\}$ does survive internalization faithfully becoming $G_{\text{fix}} \coloneqq G(H_{\text{fix}}) = \{\mathbb{1}\}$. Hence, we have the theory's spacetime manifold surviving

internalization as follows:

$$\mathcal{M}_{\text{old}} \cong \mathcal{M}_{\text{H}} \coloneqq \frac{H_{\text{trans}}}{H_{\text{fix}}} \xrightarrow[\text{int. as}]{\text{survives}} \mathcal{M}_{\text{G}} \coloneqq \frac{G_{\text{trans}}}{G_{\text{fix}}} \cong \frac{(\mathbb{R},+)\times U(1)}{\{\mathbb{1}\}} \cong \mathbb{R}\times S^1. \tag{9.19}$$

Thus, while $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2$ has survived internalization, it has been done so unfaithfully becoming $\mathcal{M}_{\text{G}} \cong \mathbb{R}\times S^1$.[98]

Given spacetime-kinematic compatibility, however, a theory's spacetime manifold will always survive internalization both faithfully and robustly. This is a strong conclusion from what the next section will argue are very weak assumptions. Indeed, the spacetime-kinematic compatibility assumptions made in this section are, in fact, slightly stronger than necessary to achieve the desired result. As I prove in Appendix B.1, a slightly weakened notion of spacetime-kinematic compatibility is both necessary and sufficient for a theory's spacetime manifold to faithfully and robustly survive internalization in this way.

9.3 The Naturalness of Spacetime-Kinematic Compatibility

The remainder of this section will be spent arguing that the above-discussed spacetime-kinematic compatibility assumptions are extremely weak. There is a clear sense in which theories lacking spacetime-kinematic compatibility are unnatural. As I will now discuss, all such theories exhibit at least one of the following two flaws: 1) They contain an excessive amount of spacetime given their kinematics, or 2) they are kinematically inhomogeneous in the sense that the theory's kinematics makes a topological distinction between at least two kinds of spacetime points. As I will discuss, failing to have spacetime-kinematic compatibility in either of these ways requires our theories to be subject to rather severe kinematic constraints.

[98] One might recognize this as (up to diffeomorphism) the spacetime manifold for the periodic spacetime theory. Indeed, as Sec. 11.2 will discuss, these two theories are ISE-related.

The Weakness of the Kinematically Reduced Assumption

Consider a generic theory about a particle trajectory, $X_{\text{old}} : \mathcal{V}_{\text{old}} \to \mathcal{M}_{\text{old}}$. It follows from the definition of H_{kin} that a diffeomorphism, $k \in \text{Diff}(\mathcal{M}_{\text{old}})$, is kinematically trivial (i.e., $k \in H_{\text{kin}}$) if and only if,

$$\forall X_{\text{old}} \;\; \forall \nu \quad k \circ X_{\text{old}}(\nu) = X_{\text{old}}(\nu). \tag{9.20}$$

where X_{old} ranges over all kinematically allowed states, $S^{\text{kin}}_{\text{old}}$, and ν ranges over all parameter values, $\mathcal{V}_{\text{old}}$. Notice, however, that $X_{\text{old}}(\nu)$ ranges over all of the points on $\mathcal{M}_{\text{old}}$ which some kinematically allowed trajectory, $X_{\text{old}} \in S^{\text{kin}}_{\text{old}}$, visits. Let us call this the kinematically accessible part of the spacetime, $\mathcal{M}^{(\text{kin})}_{\text{old}} \coloneqq S^{\text{kin}}_{\text{old}}(\mathcal{V}_{\text{old}}) \subset \mathcal{M}_{\text{old}}$. In light of this definition, if follows from Eq. (9.20) that every kinematically trivial diffeomorphism, $k \in H_{\text{kin}}$, acts trivially, $k(p) = p$, on every kinematically accessible spacetime point, $p \in \mathcal{M}^{(\text{kin})}_{\text{old}}$.

Now suppose that the theory in question is kinematically unreduced, $H_{\text{kin}} \neq \{\mathbb{1}\}$, such that there exists some $k \neq \mathbb{1}$ with $k \in H_{\text{kin}}$. This $k \neq \mathbb{1}$ must act non-trivially somewhere, i.e., there must then exist some point, $q \in \mathcal{M}_{\text{old}}$, with $k(q) \neq q$. It then follows from (9.20) that no kinematically allowed trajectory, $X_{\text{old}} \in S^{\text{kin}}_{\text{old}}$, goes through this point. Namely $q \notin \mathcal{M}^{(\text{kin})}_{\text{old}}$ is kinematically inaccessible.

Hence, every kinematically unreduced theory about a particle trajectory, $X_{\text{old}} : \mathcal{V}_{\text{old}} \to \mathcal{M}_{\text{old}}$, has a spacetime manifold which is in excess of the theory's kinematics. Namely, all such theories contain kinematically redundant spacetime points (namely, the point $q \notin \mathcal{M}^{(\text{kin})}_{\text{old}}$) where no kinematically allowed trajectory goes. Taking the contrapositive, any theory which lacks such kinematically inaccessible spacetime points is guaranteed to be kinematically reduced. The only way which a spacetime theory might fail to be kinematically reduced is if it has some rather severe kinematic constraints which make

certain portions of its spacetime kinematically inaccessible.

Next, consider a generic spacetime theory about a scalar field, $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to \mathcal{V}_{\text{old}}$, which is not kinematically reduced (i.e., with $H_{\text{kin}} \neq \{\mathbb{1}\}$). It follows from the definition of H_{kin} that any spacetime points, $p_1, p_2 \in \mathcal{M}_{\text{old}}$, which are related by some $k \in H_{\text{kin}}$ take the identical values in all kinematically allowed states:

$$\big(\exists k \in H_{\text{kin}} \;\; p_2 = k(p_1)\big) \Longrightarrow \big(\forall \varphi_{\text{old}} \in S^{\text{kin}}_{\text{old}} \;\; \varphi_{\text{old}}(p_2) = \varphi_{\text{old}}(p_1)\big). \tag{9.21}$$

If $H_{\text{kin}} \neq \{\mathbb{1}\}$ is non-trivial, then it acts non-trivially somewhere. That is, there exist distinct points $p_1 \neq p_2$ related by some $k \in H_{\text{kin}}$. It then follows that there exist distinct but kinematically identical spacetime points.

Hence, every kinematically unreduced theory about a particle trajectory, $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to \mathcal{V}_{\text{old}}$, has a spacetime manifold which is in excess of the theory's kinematics. Namely, it contains kinematically identical spacetime points. Taking the contrapositive, any theory which lacks such kinematically redundant spacetime points is guaranteed to be kinematically reduced. Thus any theory which grants a sufficient level of independent variability to the field values at different spacetime points will automatically be kinematically reduced. The only way which a spacetime theory might fail to be kinematically reduced is if it has some rather severe kinematic constraints which make certain portions of its spacetime kinematically redundant.

Finally, let us consider a generic spacetime theory about the global sections, $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to E_{\text{old}}$, of some natural fiber bundle, $\pi_{\text{old}} : E_{\text{old}} \to \mathcal{M}_{\text{old}}$, over $\mathcal{M}_{\text{old}}$. Recall that E_{old} being a natural bundle over $\mathcal{M}_{\text{old}}$ means that there is a natural way to lift any given $d \in \text{Diff}(\mathcal{M}_{\text{old}})$ to an automorphism of E_{old}, namely $d^\ell \in \text{Auto}(E_{\text{old}})$ with $\pi_{\text{old}} \circ d^\ell = d \circ \pi_{\text{old}}$. Using this automorphism one can define $d^* \varphi_{\text{old}} \coloneqq d^\ell \circ \varphi_{\text{old}} \circ d^{-1}$. For instance, the tangent bundle, $\text{T}\mathcal{M}_{\text{old}}$, is a natural bundle over $\mathcal{M}_{\text{old}}$ with $d^\ell : \text{T}\mathcal{M}_{\text{old}} \to \text{T}\mathcal{M}_{\text{old}}$ being the push-forward of d.

It follows from the definition of H_{kin} that any pair of spacetime points, $p_1, p_2 \in \mathcal{M}_{\text{old}}$, which are related by some $k \in H_{\text{kin}}$ will take k^ℓ-related values in all kinematically allowed states:[99]

$$\big(\exists k \in H_{\text{kin}} \;\; p_2 = k(p_1)\big) \Longrightarrow \big(\forall \varphi_{\text{old}} \in S^{\text{kin}}_{\text{old}} \;\; \varphi_{\text{old}}(p_2) = k^\ell \circ \varphi_{\text{old}}(p_1)\big). \tag{9.22}$$

Note that if $H_{\text{kin}} \neq \{\mathbb{1}\}$ is non-trivial, then it acts non-trivially somewhere. That is, there exist distinct points $p_1 \neq p_2$ related by some $k \in H_{\text{kin}}$. It then follows that there exist distinct but kinematically redundant spacetime points. To see this, think of any kinematically allowed state, $\varphi_{\text{old}} \in S^{\text{kin}}_{\text{old}}$, and consider all of the possible values $\varphi_{\text{old}}(p_1)$ that it can take at $p_1 \in \mathcal{M}$. It follows from Eq. (9.22) that in each of these states, everything about φ_{old}'s behavior at p_2 is already fixed given $\varphi_{\text{old}}(p_1)$ and the k which relates p_1 to p_2.

Hence, every kinematically unreduced theory about a section of a natural bundle, $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to E_{\text{old}}$, has a spacetime manifold which is in excess of the theory's kinematics. Namely, it contains kinematically redundant spacetime points. Taking the contrapositive, any theory which lacks such kinematically redundant spacetime points is guaranteed to be kinematically reduced. Thus any theory which grants a sufficient level of independent variability to the field values at different spacetime points will automatically be kinematically reduced. The only way which a spacetime theory might fail to be kinematically reduced is if it has some rather severe kinematic constraints which make certain portions of its spacetime kinematically redundant.

In sum: Every kinematically unreduced theory (be it about X_{old} or φ_{old} or $\vec{\varphi}_{\text{old}}$) contains an excessive amount of spacetime given its kinematics. Let us next turn our attention to the assumption of kinematic homogeneity.

[99] Note that in the trivial bundle case where $E_{\text{old}} = \mathcal{M}_{\text{old}} \times \mathcal{V}_{\text{old}}$ we have k^ℓ, acting trivially on the value space. Hence, the following equation recovers Eq. (9.21).

The Weakness of the Kinematic Homogeneity Assumption

The assumption of kinematic homogeneity is also a very weak assumption. It should first be noted that whether or not a smooth manifold is homogeneous is solely a matter of topology. Indeed, a spacetime theory can fail to be kinematically homogeneous for solely topological reasons. Namely, it might be set on a spacetime manifold, $\mathcal{M}$, which is itself inhomogeneous. This would happen, for instance, if we had $\mathcal{M} \cong \mathbb{T}^2 \dot{\cup} \mathbb{R}^2$ being the disjoint union of a torus and a two-dimensional plane. This manifold is inhomogeneous because it contains two distinct kinds of points which no smooth transformation can relate (namely, torus-points and plane-points).

If one considers all diffeomorphisms, $d \in \text{Diff}(\mathcal{M})$, then this example of inhomogeneity is, in fact, generic:[100] The only way in which a smooth manifold can be inhomogeneous is by having dissimilar connected components. Otherwise, $\mathcal{M}$ is homogeneous with respect to the infinite-dimensional Lie group, $\text{Diff}(\mathcal{M})$. Thus, all spacetime manifolds commonly encountered in physics are homogeneous (at least with respect to $\text{Diff}(\mathcal{M})$).

Simply being set on a homogeneous manifold, however, is not sufficient for a spacetime theory to qualify as *kinematically* homogeneous. This would only guarantee homogeneity with respect to $\text{Diff}(\mathcal{M}_{\text{old}})$ and not with respect to $\text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$ as we actual

[100] Unfortunately, however, if one considers only homogeneity with respect to finite-dimensional Lie subgroups, $H \subset \text{Diff}(\mathcal{M})$, then things get slightly more complicated. For instance, consider the case of the double-torus, $\mathcal{M} \cong \mathbb{T}^2 \# \mathbb{T}^2$ discussed in footnote 96. This smooth manifold is homogeneous with respect to the infinite-dimensional Lie group, $\text{Diff}(\mathcal{M})$, but not with respect to any finite-dimensional subgroup thereof. Hence, the claim in the main text about inhomogeneity requiring dissimilar connected components needs some qualification if one is using the finite-dimensional notion of homogeneity (see footnote 96). Importantly, however, it is still the case that almost all spacetime manifolds commonly encountered in physics are homogeneous under this stricter definition.

require.[101] Often, however, it will be the case that every diffeomorphism is kinematically allowed, i.e., $\text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}}) = \text{Diff}(\mathcal{M}_{\text{old}})$. For instance, consider the Quartic Klein-Gordon theory where our only kinematic constraint was that φ_{old} must be smooth. Every diffeomorphism lifts to map smooth states to smooth states. Hence, every diffeomorphism is kinematically allowed. By contrast, consider the bandlimited theory from Sec. 8.2; Not every diffeomorphism respects the kinematically-enforced bandwidth constraint. In this case, however, the diffeomorphisms which are kinematically allowed are still sufficient to make the periodic matter theory kinematically homogeneous.

For an example of a kinematically inhomogeneous theory, consider a spacetime theory about a scalar field, $\varphi_{\text{old}} : \mathcal{M}_{\text{old}} \to \mathcal{V}_{\text{old}}$, set on $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2 \dot{\cup} \mathbb{R}^2$, i.e., the disjoint union of two planes. On one of these planes (but not the other) the theory's matter content is kinematically constrained to be periodic in both space and time. While this theory is set on a homogeneous manifold, it is not kinematically homogeneous. To see this, consider any point p_0 on the periodic copy of $\mathbb{R}^2$. Note that by using arbitrary diffeomorphisms, $d \in \text{Diff}(\mathcal{M}_{\text{old}})$, one can map p_0 to anywhere on either plane.[102] Note, however, that none of the kinematically allowed diffeomorphisms, $d \in \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$, can map p_0 onto the non-periodic plane (i.e., all d which do so are kinematically disallowed). Hence, at the level of kinematics, this theory makes a topological distinction between two kinds of spacetime points: those reachable from p_0 via $\text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$ (i.e., the periodic plane), and those not reachable from p_0 via $\text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$ (i.e., the non-periodic plane). Said differently, when one divides $\mathcal{M}_{\text{old}}$ into $\text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$-orbits one finds multiple pieces. This aspect of

[101] If one here adopts a finite-dimensional understanding of homogeneity (see footnote 96) then things get slightly more complicated. Under this stricter definition, in order for $\mathcal{M}_{\text{old}}$ to qualify as homogeneous it must be homogeneous not just with respect to the infinite-dimensional Lie group, $\text{Diff}(\mathcal{M}_{\text{old}})$, but also with respect to some finite-dimensional subgroup thereof, $H \subset \text{Diff}(\mathcal{M}_{\text{old}})$. Similarly for a finite-dimensional understanding of kinematic homogeneity. To simplify the exposition, however, I will continue to talk about $\text{Diff}(\mathcal{M}_{\text{old}})$ and $\text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$, rather than quantifying over the finite-dimensional Lie subgroups thereof.

[102] Indeed, one can map p_0 to anywhere on either plane by using only a finite-dimensional subgroup of $\text{Diff}(\mathcal{M}_{\text{old}})$. Hence, this manifold is also homogeneous on a finite-dimensional understanding of homogeneity (see footnote 96).

kinematic inhomogeneity is generic:[103] For any kinematically inhomogeneous spacetime theory, the theory's kinematical structure will divide the spacetime manifold, $\mathcal{M}_{\text{old}}$, into several distinct pieces (namely, into several distinct $\text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$-orbits).

In sum, consider any spacetime theory set on a homogeneous manifold, $\mathcal{M}_{\text{old}}$. In order for this theory to be kinematically inhomogeneous it must be subject to rather severe kinematic constraint which not only disallows some diffeomorphisms, $\text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}}) \neq \text{Diff}(\mathcal{M}_{\text{old}})$, but also cuts $\mathcal{M}_{\text{old}}$ into distinct $\text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$-orbits. As the above examples have shown, common kinematic constraints involving smoothness, second-differentiability, and even periodicity are not sufficient to do this. Hence, one can expect that most theories encountered in physics will be kinematically homogeneous.

[103] On a finite-dimensional understanding of homogeneity (see footnote 96) this claim should be qualified as follows. For any kinematically inhomogeneous spacetime theory, and for any finite-dimensional Lie subgroup, $H \subset \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$, the spacetime manifold $\mathcal{M}_{\text{old}}$ will be divided into several distinct H-orbits.

10 | The ISE Method in General: Externalization

The previous chapter has shown that, for a wide range of spacetime theories, the theory's original spacetime manifold, $\mathcal{M}_{\text{old}}$, survives internalization both faithfully and robustly. Namely, given spacetime-kinematic compatibility, it survives as a quotient of two Lie groups, $G^{(\text{int})}_{\text{trans}}$ and $G^{(\text{int})}_{\text{fix}}$ which act on $S^{\text{kin}}_{\text{pst}}$. The superscript (int) here indicates that these Lie groups were produced by the internalization process. Of course, knowing how a theory's spacetime manifold can survive internalization is very suggestive of how externalization might be able to generate a new spacetime setting for this theory.

As Ch. 7 and Ch. 8 have demonstrated, externalization builds new spacetime settings from certain building blocks, namely, pre-spacetime translation operations (PSTOs). These chapters, however, have only demonstrated the externalization process when the new theory is about a scalar field, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathcal{V}_{\text{new}}$. The present chapter will generalize this discussion to cover externalization into theories about sections of a natural bundle (e.g., a tangent vector field, $\vec{\varphi}_{\text{new}} : \mathcal{M}_{\text{new}} \to \text{T}\mathcal{M}_{\text{new}}$) as well as theories about particle trajectories, $X_{\text{new}} : \mathcal{V}_{\text{new}} \to \mathcal{M}_{\text{new}}$. Indeed, Ch. 11 will discuss how the ISE Method applies in the particle trajectory case, particularly to the Möbius and Euclid theories introduced in Sec. 4.1.

Before this, however, Sec. 10.1 will first recall the definition of pre-spacetime translation operations (PSTOs) from Ch. 7 while also generalizing it past the scalar field case. Next, Sec. 10.2 will discuss in complete generality how the externalization process builds a new spacetime setting given some PSTOs. Finally, in Sec. 10.3 I will prove that every spacetime theory produced by the externalization process is guaranteed to have spacetime-kinematic compatibility. In fact, I will prove that a spacetime theory is producible via externalization if and only if it has spacetime-kinematic compatibility.

10.1 Defining Pre-Spacetime Translation Operations

Before defining the term "pre-spacetime translation operations", let's first review what the externalization process takes as its input. The starting point for the externalization process is a *pre-spacetime theory* in the sense defined in Sec. 4.2. Such theories distinguish between dynamically allowed and kinematically allowed states (i.e., DPMs and KPMs, $S^{\text{dyn}}_{\text{pst}} \subset S^{\text{kin}}_{\text{pst}}$). Recall that the states of such theories are pre-spacetime theoretic in the sense that they have no declared topological staging; They are not functions on or into any smooth manifold. Moreover, recall that the set of kinematically allowed pre-spacetime states is allowed to be structured in effectively any way which one wishes so long as it contains all of the theory's dynamically and metaphysically relevant structures. Namely, $\mathcal{S}^{\text{kin}}_{\text{pst}} = \langle S^{\text{kin}}_{\text{pst}}, \dots \rangle$ needs to be structured enough that we can pick out its dynamically allowed subset, $S^{\text{dyn}}_{\text{pst}} \subset S^{\text{kin}}_{\text{pst}}$. Moreover, $\mathcal{S}^{\text{kin}}_{\text{pst}}$ needs to be structured enough to ground whatever metaphysical story one wants to tell about this pre-spacetime theory. For instance, in Sec. 8.1 we had that $\mathcal{S}^{\text{kin}}_{\text{pst}}$ was an algebra whereas in Sec. 8.2 we had that it was merely a vector space. It should be noted, however, that the common linear structure present in these two examples is incidental. Indeed, Ch. 11 will demonstrate how the externalization process can be applied to the Möbius and Euclid theories from Sec. 4.1. Recall that these theories lack any algebraic structure.

No matter how the pre-spacetime theory is structured (e.g., algebraically or otherwise) the only structural feature which is needed for the externalization process to begin is that $S^{\text{kin}}_{\text{pst}}$ can be the subject of a certain kind of Lie group action. Indeed, this brings us to the topic of pre-spacetime translation operations (PSTOs). While this term has already been defined in Ch. 7 its definition there was spread across two sections (Secs. 7.3 and 7.4) and was limited to cases in which the new theory is to be about a scalar field, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathcal{V}_{\text{new}}$. This section will collect together the relevant parts of the definition

while generalizing them past the scalar field case (e.g., so it can produce theories about particle trajectories, $X_{\text{new}} : \mathcal{V}_{\text{new}} \to \mathcal{M}_{\text{new}}$). What remains true in every context is that PSTOs are a pair of smooth Lie group actions on a given pre-spacetime theory's states which are, in a certain sense, structurally indistinguishable from spacetime translations. The externalization process will build a new spacetime setting, $\mathcal{M}_{\text{new}}$, from these hand-picked PSTOs such that they become honest-to-goodness spacetime translations on $\mathcal{M}_{\text{new}}$. (N.B.: The term translation is here being used broadly, such that it includes rotations as well as generic diffeomorphisms.)

> **Definition: Pre-Spacetime Translation Operations (PSTOs) -** Let $G_2 \subset G_1$ be a pair of Lie groups which act on some space S^{kin}. This pair of Lie group actions will be called *pre-spacetime translation operations* when they have both faithful construction compatibility and injection compatibility.

Before restating the definition of faithful construction compatibility, allow me to quickly provide some motivation. Consider the quotient $\mathcal{M}_{\text{G}} = G_1/G_2$ of a Lie group G_1 and one of its closed Lie subgroups, $G_2 \subset G_1$. As discussed in Sec. 7.3, the group in the numerator, G_1, will have a natural group action on the quotient space, $\mathcal{M}_{\text{G}}$. Namely, the left action of G_1 on itself can be seen as smoothly permuting its G_2-cosets (that is, smoothly permuting the points $[g_0] \in \mathcal{M}_{\text{G}}$). Concretely, we have that $g \in G_1$ acts smoothly on $\mathcal{M}_{\text{G}}$ as $\bar{g}([g_0]) \coloneqq [g\, g_0]$. These maps $\bar{g} : \mathcal{M}_{\text{G}} \to \mathcal{M}_{\text{G}}$ are diffeomorphisms on $\mathcal{M}_{\text{G}}$ and together form a group $\bar{g} \in \overline{G}_1 \subset \text{Diff}(\mathcal{M}_{\text{G}})$. Faithful construction compatibility is exactly the condition under which $\overline{G}_1 \cong G_1$ is a faithful representation of G_1 as diffeomorphisms on $\mathcal{M}_{\text{G}}$ (see Appendix A.4 for proof).

> **Definition: Faithful Construction Compatibility (for a smooth manifold)-** Let G_1 and G_2 be two Lie groups. These two Lie groups have *faithful construction compatibility* if and only if G_2 is both a closed subgroup of G_1 as well as a core-free subgroup of G_1. (N.B.: G_2 is a core-free subgroup of G_1 if and only if G_2 contains no non-trivial normal subgroups of G_1.)

Before stating the definition of injection compatibility, it should be noted that so far we have only guaranteed that the structure of G_1 *as a Lie group* is faithfully captured

by these diffeomorphisms, $\bar{g} \in \overline{G}_1$, on $\mathcal{M}_{\text{G}}$. Recall that we want G_1 to be structurally identical to spacetime translations. Demanding injection compatibility will additionally guarantee that the structure of G_1 *as a Lie group action* on S^{kin} is faithfully captured by the action of the lifted diffeomorphisms, $\bar{g}^* \in \overline{G}_1{}^*$, on some yet-to-be-specified states on $\mathcal{M}_{\text{G}}$. Sec. 7.4 has already defined injection compatibility for a theory about a scalar field, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathcal{V}_{\text{new}}$. But what can be said of injection compatibility *simpliciter*? This is, after all, the notion of injection compatibility which appears in the definition of pre-spacetime translation operations. The definition is as follows: A pair of Lie group actions on S^{kin} will be said to have injection compatibility *simpliciter* just in case they have injection compatibility for *some kind of state* definable over $\mathcal{M}_{\text{G}}$.

But what, in general, goes into defining a "kind of state" over $\mathcal{M}_{\text{G}}$? One firstly needs to construct $S^{\text{all}}_{\text{G}}$, the set of all such states definable over $\mathcal{M}_{\text{G}}$ (e.g., all $\mathcal{V}$-parameterized particle trajectories, all $\mathcal{V}$-valued scalar fields, all tangent vector fields, etc.). Once we have constructed $S^{\text{all}}_{\text{G}}$, we need a lift map, $*_{\text{lift}} : d \mapsto d^*$, which can lift the theory's diffeomorphisms, $d : \mathcal{M}_{\text{G}} \to \mathcal{M}_{\text{G}}$, to act as a bijection, $d^* : S^{\text{all}}_{\text{G}} \to S^{\text{all}}_{\text{G}}$, on the theory's states. This $*_{\text{lift}}$ map must be a group homomorphism, i.e., we must have $(d_1\, d_2)^* = d_1^*\, d_2^*$.

Given a way of constructing $S^{\text{all}}_{\text{G}}$, and such a lift map, $*_{\text{lift}} : d \mapsto d^*$, we can define injection compatibility (for that kind of state). To make this concrete, let us now see the definition of injection compatibility for a theory about a particle trajectory, $X_{\text{new}} : \mathcal{V}_{\text{new}} \to \mathcal{M}_{\text{new}}$.

> **Definition: Injection Compatibility (for a $\mathcal{V}_{\text{G}}$-parameterized particle trajectory)** - Let G_1 be a Lie group which acts on some space S^{kin}. Let $G_2 \subset G_1$ be a closed subgroup thereof. Let $\mathcal{M}_{\text{G}} \coloneqq G_1/G_2$ be the quotient manifold of G_2 and G_1. Let $S^{\text{all}}_{\text{G}}$ be the space of all $\mathcal{V}_{\text{G}}$-parameterized trajectories definable on $\mathcal{M}_{\text{G}}$. Note that a generic diffeomorphism, $d \in \text{Diff}(\mathcal{M}_{\text{G}})$, lifts to act on a generic particle trajectory, $X : \mathcal{V}_{\text{G}} \to \mathcal{M}_{\text{G}}$, as $d^* x = d \circ x$.
>
> This pair of Lie group actions, $G_2 \subset G_1$, on S^{kin} have *injection compatibility for a $\mathcal{V}_G$-parameterized particle trajectory* if and only if the following condition holds. There must exist an injective map, $J_{\text{G}} : S^{\text{kin}} \to S^{\text{all}}_{\text{G}}$, with the

following two properties: Firstly, we must have,

$$\forall g \; \forall \sigma_{\text{pst}} \;\; (J_{\text{G}} \circ g)\, \sigma_{\text{pst}} = (\bar{g}^* \circ J_{\text{G}})\, \sigma_{\text{pst}}, \tag{10.1}$$

and, secondly, for all diffeomorphisms, $d \in \text{Diff}(\mathcal{M}_{\text{G}})$, we must have,

$$\Big(\forall \sigma_{\text{pst}} \;\; d^* J_{\text{G}}(\sigma_{\text{pst}}) = J_{\text{G}}(\sigma_{\text{pst}})\Big) \iff d = \mathbb{1}. \tag{10.2}$$

In both of these expressions g ranges over G_1 and σ_{pst} ranges over S^{kin}.

Note that unlike in the definition of injection compatibility given in Sec. 7.4, we here have that $J_{\text{G}}(\sigma_{\text{pst}})$ is a $\mathcal{V}_{\text{G}}$-parameterized particle trajectory on $\mathcal{M}_{\text{G}}$ rather than a $\mathcal{V}_{\text{G}}$-valued scalar field. Correspondingly, the diffeomorphisms $\bar{g}$ lift differently in these two cases. Namely, we here have $\bar{g}^*\, J_{\text{G}}(\sigma_{\text{pst}}) = \bar{g} \circ J_{\text{G}}(\sigma_{\text{pst}})$ rather than $\bar{g}^*\, J_{\text{G}}(\sigma_{\text{pst}}) = J_{\text{G}}(\sigma_{\text{pst}}) \circ \bar{g}^{-1}$. Other than these minor differences, however, this definition of injection compatibility is identical to the one given in Sec. 7.4.

Here, as before, Eq. (10.1) says that enacting any $g \in G_1$ on any state, $\sigma_{\text{pst}} \in S^{\text{kin}}$, and then mapping the result onto $\mathcal{M}_{\text{G}}$ via J_{G} is equivalent to first mapping σ_{pst} onto $\mathcal{M}_{\text{G}}$ via J_{G} and then enacting the corresponding lifted diffeomorphism, $\bar{g}^*$. As was motivated in Ch. 7, this guarantees that $\overline{G}_1{}^*$ is a representation of G_1 not only as a Lie group but also as a Lie group action on $S^{\text{all}}_{\text{G}}$.

But what is the motivation behind the additional demand (i.e., Eq. (10.2)) that injection compatibility makes on G_1 and G_2? So far, we have represented the action of $g \in G_1$ on S^{kin} by the action of $\bar{g}^* \in \overline{G}_1{}^*$ on $S^{\text{all}}_{\text{G}}$. Demanding Eq. (10.2) guarantees that this representation is faithful even if we restrict the action of $\bar{g}^* \in \overline{G}_1{}^*$ to a smaller set $J_{\text{G}}(S^{\text{kin}}) \subset S^{\text{all}}_{\text{G}}$. As the worked examples in Ch. 7 and Ch. 8 have shown, the image of the J_{G} map (namely, $J_{\text{G}}(S^{\text{kin}}) \subset S^{\text{all}}_{\text{G}}$) is the analog in these definitions of the new theory's kinematically allowed states, $S^{\text{kin}}_{\text{new}} \subset S^{\text{all}}_{\text{new}}$. Given this, we would like to demand the following of our PSTOs. The action of $g \in G_1$ on S^{kin} ought to be faithfully represented by the action of $\bar{g}^* \in \overline{G}_1{}^*$ on $J_{\text{G}}(S^{\text{kin}}) \subset S^{\text{all}}_{\text{G}}$. To achieve this, we need not only that the map

$g \mapsto \bar{g}$ is injective for all $g \in G_1$ but also that the map $\bar{g}^* \mapsto \bar{g}^*|_{J_{\mathrm{G}}(S^{\mathrm{kin}})}$ is too. This final demand is exactly Eq. (10.2) except with $d = \bar{g}$ ranging over $\overline{G}_1$.

Of course, Eq. (10.2) actually makes a stronger demand with d ranging not only over $\overline{G}_1$ but over all of $\mathrm{Diff}(\mathcal{M}_{\mathrm{G}})$. Indeed, what Eq. (10.2) actually demands is that the map $d \mapsto d^*|_{J_{\mathrm{G}}(S^{\mathrm{kin}})}$ is injective for all $d \in \mathrm{Diff}(\mathcal{M}_{\mathrm{G}})$ not just $\overline{G}_1$. As Sec. 10.3 will discuss, this extra demand is equivalent to demanding that spacetime theory produced by externalizing G_1 and G_2 is kinematically reduced (i.e., that only the trivial diffeomorphism, $d = \mathbb{1}$, lifts to act trivially on every kinematically allowed state). The motivation for making this slightly stronger-than-required demand of our PSTOs can be found in Sec. 10.3. For now, it is sufficient to recall from Sec. 9.3 that being kinematically reduced is a very weak condition to demand of our new spacetime theory.

To close out this section, let's see the definition of injection compatibility for a tangent vector field.

> **Definition: Injection Compatibility (for a tangent vector field)** - Let G_1 be a Lie group which acts on some space S^{kin}. Let $G_2 \subset G_1$ be a closed subgroup thereof. Let $\mathcal{M}_{\mathrm{G}} \coloneqq G_1/G_2$ be the quotient manifold of G_2 and G_1. Let $S^{\mathrm{all}}_{\mathrm{G}}$ be the space of global sections, $\vec{\varphi} : \mathcal{M}_{\mathrm{G}} \to \mathrm{T}\mathcal{M}_{\mathrm{G}}$, of $\mathcal{M}_{\mathrm{G}}$'s tangent space. Note that a generic diffeomorphism, $d \in \mathrm{Diff}(\mathcal{M}_{\mathrm{G}})$, lift to act on a generic section, $\vec{\varphi}$, as $d^*\vec{\varphi} = d^{\ell} \circ \vec{\varphi} \circ d^{-1}$ where $d^{\ell} : \mathrm{T}\mathcal{M}_{\mathrm{G}} \to \mathrm{T}\mathcal{M}_{\mathrm{G}}$ is the push-forward of d.
>
> This pair of Lie group actions, $G_2 \subset G_1$, on S^{kin} have *injection compatibility for a tangent vector field* if and only if the following condition holds. There must exist an injective map, $J_{\mathrm{G}} : S^{\mathrm{kin}} \to S^{\mathrm{all}}_{\mathrm{G}}$, with the following two properties: Firstly, we must have,
>
> $$\forall g \; \forall \sigma_{\mathrm{pst}} \;\; (J_{\mathrm{G}} \circ g)\, \sigma_{\mathrm{pst}} = (\bar{g}^* \circ J_{\mathrm{G}})\, \sigma_{\mathrm{pst}}, \tag{10.3}$$
>
> and, secondly, for all diffeomorphisms, $d \in \mathrm{Diff}(\mathcal{M}_{\mathrm{G}})$, we must have,
>
> $$\Big(\forall \sigma_{\mathrm{pst}} \;\; d^* J_{\mathrm{G}}(\sigma_{\mathrm{pst}}) = J_{\mathrm{G}}(\sigma_{\mathrm{pst}})\Big) \iff d = \mathbb{1}. \tag{10.4}$$
>
> In both of these expressions g ranges over G_1 and σ_{pst} ranges over S^{kin}.

Note that across the three contexts which we have so far discussed, Eq. (10.3) and Eq. (10.4) do not change at all. Indeed, one can easily imagine an analogous definition for a generic

natural bundle theory, e.g., a rank $(2,0)$ metric field such as g_{ab}. Only two things will change from context to context: 1) the co-domain of the injective map, $J_G : S^{kin} \to S^{all}_G$, 2) the context-sensitive lift map, $*_{lift} : d \mapsto d^*$. Hence, we can easily adapt this definition to any kind of state.

10.2 Building a New Spacetime Setting from PSTOs

With these definitions in place, we are now in a position to discuss the externalization process in complete generality. Perhaps surprisingly, almost all of our prior discussion of the externalization process in Sec. 7.3 remains unchanged as we now move beyond the scalar field case.

Firstly, where can we find PSTOs to externalize? As Ch. 9 has demonstrated, internalizing any spacetime theory with spacetime-kinematic compatibility will yield a pair of Lie groups $G^{(int)}_{trans}$ and $G^{(int)}_{fix}$ acting on S^{kin}_{pst}. As I prove in Appendix B.2, the resulting pair of Lie group actions are always PSTOs, and externalizing them can take us back to the theory's original spacetime framing. Importantly, however, we are not forced to go back the way we came. We are free to search for other pre-spacetime translation operations to externalize, i.e., ones with $G^{(ext)}_{trans} \neq G^{(int)}_{trans}$ and/or $G^{(ext)}_{fix} \neq G^{(int)}_{fix}$. The various examples of topological redescription which are discussed throughout this thesis have shown that one can often find several interesting sets of PSTOs to externalize.

As discussed in Sec. 7.3, given some set of PSTOs, externalization puts forward their quotient manifold as the theory's new spacetime manifold,

$$\mathcal{M}_{new} \cong \mathcal{M}_G := G^{(ext)}_{trans}/G^{(ext)}_{fix}, \tag{10.5}$$

at least up to diffeomorphism. As before the quotient manifold $\mathcal{M}_G$ contains excess structure; Its elements are sets of maps on S^{kin}_{pst} as well as manifold points on $\mathcal{M}_G$. We

can remove this excess by applying a forgetful operation, $\mathcal{F}_{\text{man}}$, to $\mathcal{M}_{\text{G}}$. Namely, this is a diffeomorphism, $\mathcal{F}_{\text{man}} : \mathcal{M}_{\text{G}} \to \mathcal{M}_{\text{new}}$, where $\mathcal{M}_{\text{new}}$ is some smooth manifold which is isomorphic to $\mathcal{M}_{\text{G}}$, i.e., with $\mathcal{M}_{\text{new}} \cong \mathcal{M}_{\text{G}}$. In terms of this forgetful operation, the new spacetime manifold is defined as,

$$\mathcal{M}_{\text{new}} \coloneqq \mathcal{F}_{\text{man}}(G^{(\text{ext})}_{\text{trans}}/G^{(\text{ext})}_{\text{fix}}). \tag{10.6}$$

The quotient $\mathcal{M}_{\text{G}}$ (and hence, $\mathcal{M}_{\text{new}}$) inherits a unique smooth structure from the Lie group $G^{(\text{ext})}_{\text{trans}}$ via the quotient with $G^{(\text{ext})}_{\text{fix}}$ (see Appendix A for details). For notational convenience, the (ext) superscripts will now be suppressed where possible.

In addition to giving $\mathcal{M}_{\text{new}}$ its smooth structure, the Lie group G_{trans} is also faithfully represented as diffeomorphisms on $\mathcal{M}_{\text{new}}$. Recall from Sec. 10.1 that the faithful construction compatibility of G_{trans} and G_{fix} guarantees that each $g \in G_{\text{trans}}$ is faithfully represented as a diffeomorphism, $\bar{g} \in \overline{G_{\text{trans}}} \subset \text{Diff}(\mathcal{M}_{\text{G}})$. These diffeomorphisms can then be transformed into diffeomorphisms on $\mathcal{M}_{\text{new}}$ as,

$$H(g) \coloneqq \mathcal{F}_{\text{man}} \circ \bar{g} \circ \mathcal{F}^{-1}_{\text{man}}. \tag{10.7}$$

In total, therefore, we have an injective Lie group homomorphism, $H : G_{\text{trans}} \to \text{Diff}(\mathcal{M}_{\text{new}})$, which associates to every element, $g \in G_{\text{trans}}$, a diffeomorphism, $H(g) \in \text{Diff}(\mathcal{M}_{\text{new}})$. The image of this map, $H^{(\text{ext})}_{\text{trans}} \coloneqq H(G_{\text{trans}})$, is a faithful representation of G_{trans} as diffeomorphisms on $\mathcal{M}_{\text{new}}$. Similarly, G_{fix} is faithfully represented by $H^{(\text{ext})}_{\text{fix}} \coloneqq H(G_{\text{fix}}) \cong G_{\text{fix}}$. One can easily check that $H^{(\text{ext})}_{\text{fix}}$ is the stabilizer subgroup of $H^{(\text{ext})}_{\text{trans}}$ at $p_0 \coloneqq \mathcal{F}_{\text{man}}([\mathbb{1}])$.

In sum, this is how externalization builds a new spacetime manifold for the pre-spacetime theory in question: We pick two Lie groups G_{trans} and G_{fix} with faithful construction compatibility and take their quotient. This not only builds a new spacetime manifold, $\mathcal{M}_{\text{new}}$, but also faithfully represents both of these Lie groups as diffeomorphisms on $\mathcal{M}_{\text{new}}$. This justifies the following gloss of the externalization process: Externalization

builds a new spacetime by taking some hand-picked pre-spacetime translation operations, $G_{\text{fix}} \subset G_{\text{trans}}$, which act on $S^{\text{kin}}_{\text{old}}$ and letting them be honest-to-goodness spacetime translations, $H_{\text{fix}} \subset H_{\text{trans}} \subset \text{Diff}(\mathcal{M}_{\text{new}})$.

The externalization process cannot stop here, however. After constructing a new spacetime manifold, externalization must somehow map the theory's pre-spacetime states, $\sigma_{\text{pst}} \in S^{\text{kin}}_{\text{pst}}$, and their dynamics onto $\mathcal{M}_{\text{new}}$. In Sec. 7.3, in Ch. 8 we have already seen how this injection process works when the new theory is to be about a scalar field, $\varphi_{\text{new}} : \mathcal{M}_{\text{new}} \to \mathcal{V}_{\text{new}}$. Indeed, in each of these cases, we have seen that our choice of PSTOs nearly fixes how we are allowed to map the pre-spacetime theory's states onto the new spacetime setting, see Sec. 7.4.

For the sake of concreteness, let us suppose that the new theory's states are to be either particle trajectories, $\sigma_{\text{new}} = X_{\text{new}}$, or tangent vector fields, $\sigma_{\text{new}} = \vec{\varphi}_{\text{new}}$. In principle, the new theory could be about any kind of state one wishes so long as the following two conditions are met. Firstly, we must be able to construct the set, $\sigma_{\text{new}} \in S^{\text{all}}_{\text{new}}$, of all definable states of this kind on $\mathcal{M}_{\text{new}}$. This may require the introduction of some auxiliary space $\mathcal{V}_{\text{new}}$ e.g., as a value space or parameter space. Secondly, there must be a map, $*_{\text{lift}} : d \mapsto d^*$, which can lift its diffeomorphisms, d, to act on $S^{\text{all}}_{\text{new}}$ as a bijection, $d^* : S^{\text{all}}_{\text{new}} \to S^{\text{all}}_{\text{new}}$. For any kind of states meeting these criteria, the externalization process can proceed as follows.

Note, however, that $S^{\text{all}}_{\text{new}}$ is very large; In the particle trajectory case, it contains every non-smooth and discontinuous trajectory definable on $\mathcal{M}_{\text{new}}$. We next need to pick out the subset of states which are kinematically allowed in the new theory, $S^{\text{kin}}_{\text{new}} \subset S^{\text{all}}_{\text{new}}$. Recall that we already have a pre-spacetime theoretic characterization of the theory's kinematically allowed states, namely, $S^{\text{kin}}_{\text{pst}}$. All we need to do is to map these states onto the new spacetime setting via some map $J : S^{\text{kin}}_{\text{pst}} \to S^{\text{all}}_{\text{new}}$. If we demand that distinct states, $\sigma_{\text{pst}} \in S^{\text{kin}}_{\text{pst}}$, are mapped onto distinct states, $X_{\text{new}} \in S^{\text{all}}_{\text{new}}$, then we must have that J

is an injective map. Fixing this injector, J, will fix the new theory's kinematically allowed states as $S^{\text{kin}}_{\text{new}} \coloneqq J(S^{\text{kin}}_{\text{pst}})$.

Importantly, we are not completely free in our choice of injector, $J : S^{\text{kin}}_{\text{pst}} \to S^{\text{all}}_{\text{new}}$. Recall that our goal is to make every pre-spacetime translation operation, $g \in G_{\text{trans}}$, into an honest-to-goodness spacetime translation in the new theory. We have already seen how externalization generates a diffeomorphism on $\mathcal{M}_{\text{new}}$ from every $g \in G_{\text{trans}}$, namely $H(g) \in \text{Diff}(\mathcal{M}_{\text{new}})$. In order for this to qualify as a spacetime translation, however, it needs to act in the right way on the new theory's states, $\sigma_{\text{new}} \coloneqq J(\sigma_{\text{pst}})$. Establishing this will require that we demand some compatibility between our choice of J and our choice of G_{trans} and G_{fix}.

For any Lie group action $g \in G_{\text{trans}}$, we already know how it acts on any pre-spacetime state, $\sigma_{\text{pst}} \in S^{\text{kin}}_{\text{pst}}$. Namely, it acts as $\sigma_{\text{pst}} \mapsto g\,\sigma_{\text{pst}}$. As I discussed above, externalization converts every $g \in G_{\text{trans}}$ into a diffeomorphism, $H(g) \in H_{\text{trans}} \subset \text{Diff}(\mathcal{M}_{\text{new}})$. Moreover, externalization converts every pre-spacetime state, $\sigma_{\text{pst}} \in S^{\text{kin}}_{\text{pst}}$, into a state on the new spacetime, $J(\sigma_{\text{pst}}) \in S^{\text{kin}}_{\text{new}}$. Our choice of J must be such that these two externalization maps, $g \mapsto H(g)$ and $\sigma_{\text{pst}} \mapsto J(\sigma_{\text{pst}})$, are compatible:

$$\forall g\ \forall \sigma_{\text{pst}}\ \ (J \circ g)\,\sigma_{\text{pst}} = (H(g)^* \circ J)\,\sigma_{\text{pst}}. \tag{10.8}$$

with g and σ_{pst} ranging over G_{trans} and $S^{\text{kin}}_{\text{pst}}$ respectively. That is, enacting any $g \in G_{\text{trans}}$ on any $\sigma_{\text{pst}} \in S^{\text{kin}}_{\text{pst}}$ then externalizing via J ought to be equivalent to first externalizing σ_{pst} with J and then enacting the corresponding lifted diffeomorphism, $H(g)^*$. Demanding Eq. (10.8) of our injector J ensures that $\overline{H_{\text{trans}}}^*$ is a representation of G_{trans} not only as a Lie group but also as a Lie group action on $S^{\text{all}}_{\text{new}}$.

Given the close relationship between $S^{\text{kin}}_{\text{pst}}$ and $S^{\text{kin}}_{\text{new}} = J(S^{\text{kin}}_{\text{pst}})$, however, we would like to make a slightly stronger demand here. Namely, the action of $g \in G_{\text{trans}}$ on $S^{\text{kin}}_{\text{pst}}$ ought to be faithfully represented by the action of $H(g)^* \in {H_{\text{trans}}}^*$ on the new theory's kinemat-

ically allowed states, $S^{\text{kin}}_{\text{new}} \subset S^{\text{all}}_{\text{new}}$. At present, we are only guaranteed that distinct maps, $g_1 \neq g_2$, are mapped to distinct diffeomorphisms, $H(g_1) \neq H(g_2)$. Conceivably, however, it could presently be the case that these distinct maps $g_1 \neq g_2$ correspond to identical lifted-then-restricted actions; Namely, we could have that $H(g_1)^*|_{\text{kin}} = H(g_2)^*|_{\text{kin}}$ act in the same way on $S^{\text{kin}}_{\text{new}} \subset S^{\text{all}}_{\text{new}}$. If this were to happen, then the action of $H(g)^*$ on $S^{\text{kin}}_{\text{new}}$ would not be a faithful representation of the action of g on $S^{\text{kin}}_{\text{pst}}$. In order to avoid this, we must demand that the map $H(g) \mapsto H(g)^*|_{\text{kin}}$ is injective. Namely, we must demand that for all diffeomorphisms, $h \in H_{\text{trans}} \subset \text{Diff}(\mathcal{M}_{\text{new}})$, we have,

$$\Big(\forall \sigma_{\text{pst}} \;\; h^* J(\sigma_{\text{pst}}) = J(\sigma_{\text{pst}})\Big) \iff h = \mathbb{1}, \tag{10.9}$$

with σ_{pst} ranging over $S^{\text{kin}}_{\text{pst}}$. Under this condition, the way that $H(g)^*$ acts on $S^{\text{kin}}_{\text{new}}$ will be a faithful representation of the way that g acts on $S^{\text{kin}}_{\text{pst}}$.

We have nearly completed the externalization process. By construction, we so far have a bijection, $J : S^{\text{kin}}_{\text{pst}} \leftrightarrow S^{\text{kin}}_{\text{new}}$, between our two characterizations of the theory's kinematically allowed states. Recall from above that $\mathcal{S}^{\text{kin}}_{\text{pst}} \coloneqq \langle S^{\text{kin}}_{\text{pst}}, \dots \rangle$ comes equipped with all of the theory's dynamically and metaphysically relevant structures. These relevant structures can then be transferred from $\mathcal{S}^{\text{kin}}_{\text{pst}}$ to $S^{\text{kin}}_{\text{new}}$ via our choice of $J : S^{\text{kin}}_{\text{pst}} \to S^{\text{all}}_{\text{new}}$. Thus, by construction, all of the theory's dynamically and metaphysically relevant structures survive faithfully into the new theory. Explicit demonstrations of how this J map induces these structures in $\mathcal{S}^{\text{kin}}_{\text{new}}$ can be found in Sec. 7.3, Sec. 8.1, and Sec. 8.2. Another example will soon be given in Ch. 11 relating to the Möbius-Euclid duality. This completes the externalization process.

To review: Externalization builds a new spacetime setting for our theory from the following materials: A pair of Lie groups, G_{trans} and G_{fix}, which act on $S^{\text{kin}}_{\text{pst}}$ with faithful construction compatibility and injection compatibility (i.e., PSTOs); a choice of what kind of states the new theory will be about; and finally, an injective map, $J : S^{\text{kin}}_{\text{pst}} \to S^{\text{all}}_{\text{new}}$. This

construction is successful if Eq. (10.8) and Eq. (10.9) are satisfied. But given some pair of Lie group actions, G_{trans} and G_{fix}, how do we know that this construction can be carried out successfully? The existence of such choices is straightforwardly guaranteed by the injection compatibility of our PSTOs, see Sec. 10.1. Often a unique choice of J will be forced upon us (up to diffeomorphism) by demanding that $\mathcal{S}^{\text{kin}}_{\text{pst}}$ is nicely represented in terms of natural operations on $\mathcal{M}_{\text{new}}$, see Sec. 7.4.

10.3 Externalization Guarantees Spacetime-Kinematic Compatibility

This section will prove that every theory produced by the externalization process is guaranteed to be a spacetime theory (in the sense of Ch. 4) and, moreover, is guaranteed to have spacetime-kinematic compatibility. Indeed, I will further prove that a theory is producible via externalization *if and only if* it is a spacetime theory with spacetime-kinematic compatibility.

First, note that any theory produced by the externalization process will satisfy the modal assumption and the kinematic structure assumption. By construction, the new theory's kinematically allowed states, $S^{\text{kin}}_{\text{new}}$, are picked out as a subset of $S^{\text{all}}_{\text{new}}$ via our choice of $J : S^{\text{kin}}_{\text{pst}} \to S^{\text{all}}_{\text{new}}$. Next recall from above that our pre-spacetime theory comes equipped with all of its dynamically and metaphysically relevant structures. Hence, there is enough structure in $\mathcal{S}^{\text{kin}}_{\text{pst}} = \langle S^{\text{kin}}_{\text{pst}}, \dots \rangle$ to identify its dynamically allowed subspace, $S^{\text{dyn}}_{\text{pst}}$. Externalization induces isomorphic structures in $\mathcal{S}^{\text{kin}}_{\text{new}}$ via the bijection $J : S^{\text{kin}}_{\text{pst}} \leftrightarrow S^{\text{kin}}_{\text{new}}$. Using this we can identify the new theory's dynamically allowed subspace $S^{\text{dyn}}_{\text{new}}$. See Sec. 7.3, Chs. 8 and 11 for examples of how the J map induces this structure and thereby picks out the new theory's dynamics.

Next, note that any theory produced by the externalization process will satisfy the lifting assumption as well as the declared topological staging assumption. As discussed above, the externalization process guarantees that the new theory is about some kind of state (e.g., X_{new}, or φ_{new}, or $\vec{\varphi}_{\text{new}}$). Moreover, it guarantees that there will be a canonical way, $*_{\text{lift}} : d \mapsto d^*$, of lifting its diffeomorphisms, $d \in \text{Diff}(\mathcal{M}_{\text{new}})$, to act on these states as a bijection $d^* : S^{\text{all}}_{\text{new}} \to S^{\text{all}}_{\text{new}}$. In particular, $*_{\text{lift}} : d \mapsto d^*$ will be a group homomorphism.

The above discussion confirms that any spacetime theory produced via externalization will be a spacetime theory (in the sense defined in Ch. 4). Let us next confirm that any such theory will also be kinematically homogeneous. To do so, I will prove that $H^{(\text{ext})}_{\text{trans}} \coloneqq H(G^{(\text{ext})}_{\text{trans}})$ is a finite-dimensional Lie group which acts transitively and smoothly on $\mathcal{M}_{\text{new}}$ and, moreover, is kinematically allowed. All but the last of these claims follow directly from the fact that the H map defined in Eq. (10.7) is an injective Lie group homomorphism. This itself follows from $G^{(\text{ext})}_{\text{trans}}$ and $G^{(\text{ext})}_{\text{fix}}$ having faithful construction compatibility.

The final part of the new theory's kinematic homogeneity (namely, that $H^{(\text{ext})}_{\text{trans}}$ is kinematically allowed) follows from Eq. (10.8). To see this, note that for every $h \in H^{(\text{ext})}_{\text{trans}}$ we can find a corresponding $g \in G^{(\text{ext})}_{\text{trans}}$ with $h = H(g)$. By Eq. (10.8), enacting this lifted diffeomorphism, h^*, on any kinematically allowed state in the new theory is equivalent to enacting $g = H^{-1}(h)$ pre-externalization and then externalizing with J. But the J map always produces kinematically allowed states by definition. Hence, every $h^* \in H^{(\text{ext})}_{\text{trans}}{}^*$ maps the new theory's kinematically allowed states onto each other. Hence, every $h \in H^{(\text{ext})}_{\text{trans}}$ is kinematically allowed.

Finally, let us confirm that the new theory is kinematically reduced, i.e., that it has $H_{\text{kin}} = \{\mathbb{1}\}$ or equivalently that the map $d \mapsto d^*|_{\text{kin}}$ is injective. Recalling that $S^{\text{kin}}_{\text{new}} \coloneqq J(S^{\text{kin}}_{\text{pst}})$, the new theory being kinematically reduced follows immediately from the injection com-

patibility of $G^{(\text{ext})}_{\text{trans}}$ and $G^{(\text{ext})}_{\text{fix}}$. Namely, it follows directly from Eq. (10.2). This completes the proof that spacetime theories produced via externalization always have spacetime-kinematic compatibility.

A brief comment is here warranted about why injection compatibility demands Eq. (10.2) instead of something weaker. Indeed, as I noted in Sec. 10.1, a weaker demand would have sufficed to ensure that the action of G_{trans} on $S^{\text{kin}}_{\text{pst}}$ is faithfully captured by the action of H_{trans} on $S^{\text{kin}}_{\text{new}}$. In particular, if this were our aim then it would be sufficient to have d in Eq. (10.2) range over just $\overline{G}_{\text{trans}}$ instead of the much larger group $\text{Diff}(\mathcal{M}_{\text{G}})$. While making this lesser demand of our PSTOs would have been sufficient to guarantee that G_{trans} is faithfully represented as H_{trans}, it would not be sufficient to guarantee that the resulting theory has spacetime-kinematic compatibility. This is important because (as I have proved in Appendix B.1), a theory's spacetime manifold will faithfully *and robustly* survive internalization if and only if the theory has (a slightly weakened version of) spacetime-kinematic compatibility.[104] If one tries to internalize a theory which lacks this kind of spacetime-kinematic compatibility then whether or not $\mathcal{M}_{\text{old}}$ survives faithfully will depend upon which Lie groups, H_{trans} and H_{fix}, are used to first reconstruct $\mathcal{M}_{\text{old}}$. Hence, if we were to proceed with PSTOs satisfying this lesser demand, then the ISE Method would be greatly complicated by a kind of "path dependence"; It would suddenly matter which particular PSTOs one uses to implement a given topological redescription. We can avoid this headache and greatly simplify our analysis by making a slightly stronger demand of our PSTOs. Ultimately, this is a very small price to pay since spacetime-kinematic compatibility is such a weak assumption, see Sec. 9.3.

[104] Recall from Sec. 9.2 that spacetime-kinematic compatibility demands that our theory be kinematically reduced, $H_{\text{kin}} = \{\mathbb{1}\}$. This is equivalent to demanding that $H_{\text{kin}} \cap \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}}) = \{\mathbb{1}\}$ since $H_{\text{kin}} \subset \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$. As I discuss in Appendix B.1, the slightly weaker version of spacetime-kinematic compatibility makes a similar demand but with a slightly smaller group than $\text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$. Let $H_{\text{trans}} \subset \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$ range over all finite-dimensional and kinematically-allowed Lie groups which act smoothly and transitively on $\mathcal{M}_{\text{old}}$. Let $\text{Diff}^{\text{UFT}}_{\text{kin}}(\mathcal{M}_{\text{old}})$ be the union of all of these H_{trans} groups. The weakened demand is that $H_{\text{kin}} \cap \text{Diff}^{\text{UFT}}_{\text{kin}}(\mathcal{M}_{\text{old}}) = \{\mathbb{1}\}$. While it may be possible for this $\text{Diff}^{\text{UFT}}_{\text{kin}}(\mathcal{M}_{\text{old}})$ group to be strictly smaller than $\text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$, for every spacetime theory which I have been able to check, we actually have $\text{Diff}^{\text{UFT}}_{\text{kin}}(\mathcal{M}_{\text{old}}) = \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$.

To close off this section, I will now prove that the range of spacetime theories which can be output by externalization (i.e., the range of the ISE Method) are exactly those theories with spacetime-kinematic compatibility. The above discussion has proved one direction of this claim. To prove the other direction, we need to show that any theory with spacetime-kinematic compatibility can be produced via externalization. Given any such theory, let us take it as input into the ISE Method. As Ch. 9 has shown, internalizing any such theory yields a pair of Lie groups, $G^{(\text{int})}_{\text{trans}}$ and $G^{(\text{int})}_{\text{fix}}$, acting on $S^{\text{kin}}_{\text{pst}}$. In Appendix B.2, I have proved that this pair of Lie groups are always PSTOs. Moreover, I have proved that by externalizing them one can recover the theory's original spacetime framing, up to diffeomorphism. Hence, any spacetime theory with spacetime-kinematic compatibility can be output by the ISE Method (namely, by first internalizing it and then immediately re-externalizing it).

11 | Characterizing ISE-Relatedness via the ISE-Equivalence Theorem

Ch. 9 and Ch. 10 have discussed the internalization and externalization process in general. Three concrete applications of the internalization process have been discussed so far in Secs. 7.1, 7.2 and 8.2. After internalization, one is next supposed to search around for good pre-spacetime translation operations (PSTOs) to externalize. Three corresponding applications of the externalization process have been discussed in Secs. 7.3, 8.1, and 8.2. Beyond these example applications, however, what domain of spacetime theories can the ISE Method be applied to? And what range of spacetime theories does the ISE Method give us access to? How powerful exactly is the ISE Method as a tool for topological redescription?

This chapter will answer this question by proving the ISE-Equivalence Theorem: Two spacetime theories are ISE-related if and only if: 1) the output theory has spacetime-kinematic compatibility, and 2) they are topological redescriptions of each other in the sense of Sec. 4.2. (Recall that this means they have a certain kind of dynamics-preserving isomorphism between their kinematic structures.) Hence, by using the ISE Method one can access effectively every possible topological redescription of a given theory's dynamical and kinematic contents. Our capacity for topological redescription is bounded only by a spacetime-kinematic compatibility condition. Once Sec. 11.2 has proved this theorem, Sec. 11.3 will then apply it to the Möbius-Euclid duality which I first introduced in Sec. 4.1.

11.1 Proof that ISE-Relatedness is an Equivalence Relation

To set the scope of the ISE Method of topological redescription, we first need to know what kind of theories can be output by the externalization process. As Sec. 10.3 has proved, the spacetime theories which can be produced by the ISE Method are exactly those with spacetime-kinematic compatibility. Such theories constitute the *range* of the ISE Method. We must next ask what kind of spacetime theories can be input into the internalization process. As Ch. 9 has discussed, every spacetime theory can be input into the internalization process. Such theories constitute the *domain* of the ISE Method.

Having established both the domain and range of the ISE Method, we can now define ISE-relatedness. Let us say that spacetime theory (X) is ISE-related to spacetime theory (Y) when we can use the ISE Method to redescribe theory (X) as theory (Y), at least up to diffeomorphism. Three examples of ISE-related theories have been discussed so far in this thesis. Ch. 7 has shown that the non-local QHO theory is ISE-related to the cosine QHO theory. Sec. 8.1 has shown that the 3+1D QKG theory is ISE-related to the 1+1D QKG theory. Sec. 8.2 has shown that various theories about heat dissipating on a discrete spacetime are ISE-related to theories about bandlimited heat dissipating on a continuous spacetime. Later in Sec. 11.3 I will show that the Möbius and Euclid theories from Sec. 4.1 are ISE-related to each other. In each of these cases, I have claimed (but have not yet proved) that the reversed relation holds as well.

It is not guaranteed, however, that any given instance of ISE-relatedness is reversible. As I will discuss later, a concrete example of this ISE-irreversibility is the periodic matter theory and the periodic spacetime theory which were introduced in Ch. 9. As I will discuss below, the periodic matter theory is ISE-related to the periodic spacetime theory but the reversed relation *does not* hold. The general reason that the ISE Method can sometimes be irreversible is as follows; Strictly more theories can be input into the ISE Method than

can be output by it. The periodic matter theory is one such example, it qualifies as a spacetime theory but fails to have spacetime-kinematic compatibility. This theory cannot appear in the second place of an ISE-relation. In particular, neither the periodic spacetime theory nor the periodic matter theory itself are ISE-related to the periodic matter theory. Hence, ISE-relatedness is neither reflexive nor symmetric over its domain (i.e., over the space of all spacetime theories).

ISE-relatedness is, however, both reflexive and symmetric over its range (i.e., over the spacetime theories with spacetime-kinematic compatibility). In fact, as I will now prove, it is an equivalence relation over this range. To demonstrate its reflexiveness, recall the discussion at the end of Sec. 10.3. Consider a generic spacetime theory with spacetime-kinematic compatibility, and label it (A). As I prove in Appendix B.2, internalizing theory (A) yields a set of PSTOs, $G_{\text{trans}}(A)$ and $G_{\text{fix}}(A)$, acting on its pre-spacetime states, $S^{\text{kin}}_{\text{pst}}(A)$. We can then use these PSTOs to re-externalize theory (A). Doing so we end up back at the old theory's spacetime setting, up to diffeomorphism. Hence, the ISE-relatedness is reflexive over theories with spacetime-kinematic compatibility.

Let us next see that ISE-relatedness is symmetric over its range. Suppose that theories (A) and (B) have spacetime-kinematic compatibility and that we can get from theory (A) to theory (B) via the ISE Method. As I noted above, internalizing theory (A) yields a set of PSTOs, $G_{\text{trans}}(A)$ and $G_{\text{fix}}(A)$, which we could externalize to recover the theory's original (A)-framing. By assumption, however, we can also produce theory (B) by externalizing some other set of PSTOs, $G_{\text{trans}}(B)$ and $G_{\text{fix}}(B)$, which act on $S^{\text{kin}}_{\text{pst}}$. Since theory (B) has been produced via externalization it is a spacetime theory and hence can be re-internalized. After re-internalizing this theory, our original set of PSTOs, $G_{\text{trans}}(A)$ and $G_{\text{fix}}(A)$, are still available to us. By externalizing them we can return to theory (A). Hence, assuming that we can use the ISE Method to get from theory (A) to theory (B),

we can also use the ISE Method to get from theory (B) back to theory (A). Thus ISE-relatedness is symmetric over its range.

Finally, let us see that ISE-relatedness is transitive over its range. Consider three theories (A), (B), and (C) each with spacetime-kinematic compatibility and with (A) ISE-related to (B) and (B) ISE-related to (C). Suppose that we have just moved from theory (A) to a pre-spacetime theoretic description and then into theory (B). Since theory (B) has been produced via externalization, it qualifies as a spacetime theory (indeed, it has spacetime-kinematic compatibility). Hence, we can re-internalize it and then go in a different direction, say to theory (C). But re-internalizing theory (B) just takes us back to the pre-spacetime theoretic description between (A) and (B). Hence, we can also get directly from (A) to (C) using the ISE Method. This concludes the proof that ISE-relatedness is an equivalence relation over its range.

Having proved that ISE-relatedness is an equivalence relation over its range, the natural next step would be to try to characterize its equivalence classes. Indeed, this will be the subject of the next section. Before this, however, let us first connect this result with Fig. 4.2 in Sec. 4.3 (repeated here as Fig. 11.1 for the reader's convenience). This figure shows all possible spacetime theories being divided up into pre-spacetime equivalence classes (the large blue boxes). For each pre-spacetime theory let us focus on its spacetime representations which have spacetime-kinematic compatibility (the oval labeled SKC). The above discussion has proved that ISE-relatedness partitions these ovals into ISE-equivalence classes (the small green rectangles on the left side of Fig. 11.1).

Recall that the ISE Method was able to implement the three examples of topological redescriptions discussed in Ch. 7, Sec. 8.1, and Sec. 8.2. Recall also that all of these theories have spacetime-kinematic compatibility. Hence, considered pairwise, each of these theories fall into the same ISE-equivalence class. Since ISE-relatedness is an equivalence relation (and hence is symmetric) we have ISE-reversibility for these example applica-

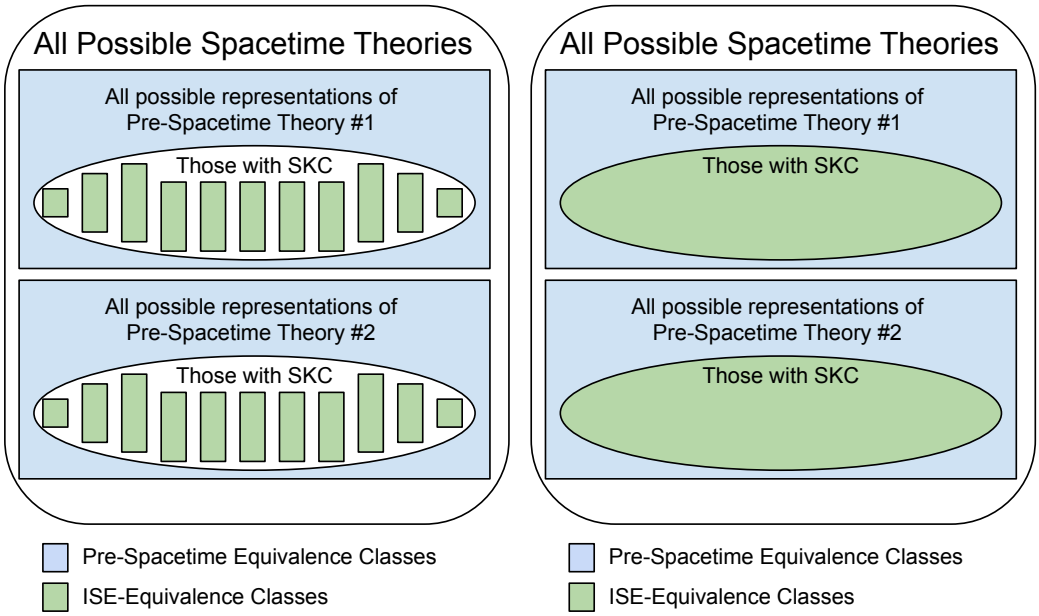

Figure 11.1: Each side of this figure shows all possible spacetime theories being divided up into pre-spacetime equivalence classes (the large blue boxes). Two spacetime theories are in the same blue box if and only if their kinematic and dynamical contents are isomorphic, $\mathcal{S}^{\text{kin}}_{\text{new}} \cong \mathcal{S}^{\text{kin}}_{\text{old}}$ with $S^{\text{dyn}}_{\text{new}} \leftrightarrow S^{\text{dyn}}_{\text{old}}$. We can say that any two such theories are different spacetime representations of the same pre-spacetime theoretic structure. The ISE Method applies to the subset of spacetime theories with spacetime-kinematic compatibility (SKC). I will define this in Sec. 9.2 and I will argue that it is a weak constraint in Sec. 9.3. In Sec. 11.1, I will prove that ISE-interrelatedness divides the theories with spacetime-kinematic compatibility up into ISE-equivalence classes (the small green boxes, left). Two spacetime theories are in the same green box if and only if they can be redescribed as one another using the ISE Method of topological redescription. Moreover, in Sec. 11.2 I will prove that these ISE-equivalence classes are as large as possible (the green oval, right). That is, the ISE Method offers us every possible topological description of a given theory, bounded only by a weak spacetime-kinematic compatibility constraint.

tions. Now recall the periodic matter theory and the periodic spacetime theory from Ch. 9. As I will discuss in Sec. 11.2, we can also use the ISE Method to topologically redescribe the former theory as the latter, *but not vice-versa*. Namely, for this pair of theories we can move from old to new but not backwards from new to old. This asymmetry is due to the periodic matter theory lacking spacetime-kinematic compatibility. While these two theories are in the same pre-spacetime equivalence class (i.e., the same large box) the periodic matter theory is outside of the SKC oval whereas the periodic spacetime theory is inside.

As the next section will prove, these ISE-equivalence classes are as large as possible. Indeed, the small green boxes on the left half of Fig. 11.1 are misleading. Instead, a theory's ISE-equivalence class includes all possible topological redescriptions of that theory

which have spacetime-kinematic compatibility. See the green ovals in the right half of Fig. 11.1. Thus, the ISE Method applies to a wide range of spacetime theories and gives us access to effectively every possible topological redescription thereof; It is limited only by a weak spacetime-kinematic compatibility constraint.

11.2 Exact Characterization of ISE-Equivalence Classes

Having proved that ISE-relatedness is an equivalence relation over its range, I will now characterize its equivalence classes. Indeed, the following theorem characterizes not only the ISE-equivalence classes but ISE-relatedness in general:

> **ISE-Equivalence Theorem -** One theory (A) is ISE-related to another theory (B) if and only if the following three conditions hold.
>
> 1) Firstly, theory (A) and (B) must both be spacetime theories (as defined in Sec. 4.2), and theory (B) must additionally have spacetime-kinematic compatibility (as defined in Sec. 9.2).
> 2) Secondly, there must exist an isomorphism $\mathcal{I} : \mathcal{S}^{\text{kin}}_{\text{old}}(A) \leftrightarrow \mathcal{S}^{\text{kin}}_{\text{old}}(B)$ between their kinematically allowed states which preserves all of their dynamically and metaphysically relevant structures.
> 3) Thirdly, this isomorphism, $\mathcal{I} : S^{\text{dyn}}_{\text{old}}(A) \leftrightarrow S^{\text{dyn}}_{\text{old}}(B)$, must also preserve the theory's dynamics. That is, it must map solutions to theory (A)'s dynamical equations onto solutions to theory (B)'s dynamical equations, and vice versa.
>
> Note that if theory (A) and theory (B) are to be ISE-inter-related, then the first condition must be strengthened to demand that both theories have spacetime-kinematic compatibility. Recall that, when restricted to such theories, ISE-relatedness is an equivalence relation. Hence, strengthening the first condition in this way gives an exact characterization of the ISE-equivalence classes.

The proof of this claim is relatively simple. Let us first demonstrate that any pair of ISE-related theories will have the above-listed properties.

1) In order for theory (A) to be an input to the ISE Method, it must be a spacetime theory. Likewise, for theory (B) to be producible by externalization it must be a spacetime theory with spacetime-kinematic compatibility, see Sec. 10.3.

2) Note that internalization and externalization both preserve all of the theory's dynamically and metaphysically relevant structures, $\mathcal{S}^{\text{kin}}_{\text{old}} \cong \mathcal{S}^{\text{kin}}_{\text{pst}} \cong \mathcal{S}^{\text{kin}}_{\text{new}}$. Concretely, these

structures are preserved by $\mathcal{I} \coloneqq J \circ \mathcal{F}$ where the bijection, $\mathcal{F} : S^{\text{kin}}_{\text{old}} \leftrightarrow S^{\text{kin}}_{\text{pst}}$, mediates internalization and the bijection, $J : S^{\text{kin}}_{\text{pst}} \leftrightarrow S^{\text{kin}}_{\text{new}}$, mediates externalization. N.B.: the kinematic structures appearing in the pre-spacetime theory and the new theory are *exactly* those induced by the $\mathcal{F}$ and J bijections respectively.

3) Finally, the isomorphism $\mathcal{I} = J \circ \mathcal{F}$ preserves the theory's dynamics since both $\mathcal{F}$ and J do. N.B.: the dynamics of the pre-spacetime theory and the new theory are *defined* by the $\mathcal{F}$ and J bijections, i.e., $S^{\text{dyn}}_{\text{pst}} \coloneqq \mathcal{F}(S^{\text{dyn}}_{\text{old}})$ and $S^{\text{dyn}}_{\text{new}} \coloneqq J(S^{\text{dyn}}_{\text{pst}})$.[105] This completes the proof that any pair of ISE-related theories will have the three above-listed properties.

To prove the converse result, let us consider two generic spacetime theories (A) and (B) which have the above-listed properties. Our goal is then to show that one can get from theory (A) to theory (B) via the ISE Method. Recall that internalization proceeds via a forgetful operation, $\mathcal{F}$, which preserves all of the theory's dynamically and metaphysically relevant structures. Hence, internalizing both theories we have two isomorphisms $\mathcal{F}_{\text{(A)}} : \mathcal{S}^{\text{kin}}_{\text{old}}(A) \leftrightarrow \mathcal{S}^{\text{kin}}_{\text{pst}}(A)$ and $\mathcal{F}_{\text{(B)}} : \mathcal{S}^{\text{kin}}_{\text{old}}(B) \leftrightarrow \mathcal{S}^{\text{kin}}_{\text{pst}}(B)$. Together with our assumed isomorphism, $\mathcal{I} : \mathcal{S}^{\text{kin}}_{\text{old}}(A) \leftrightarrow \mathcal{S}^{\text{kin}}_{\text{old}}(B)$, we can induce an isomorphism between $\mathcal{S}^{\text{kin}}_{\text{pst}}(B)$ and $\mathcal{S}^{\text{kin}}_{\text{pst}}(A)$, namely $\mathcal{I}_{\text{N}} : \mathcal{S}^{\text{kin}}_{\text{pst}}(B) \leftrightarrow \mathcal{S}^{\text{kin}}_{\text{pst}}(A)$ defined by $\mathcal{I}_{\text{N}} \coloneqq \mathcal{F}_{\text{(A)}} \circ \mathcal{I}^{-1} \circ \mathcal{F}^{-1}_{\text{(B)}}$.

Next note that since theory (B) has spacetime-kinematic compatibility, internalizing it yields pre-spacetime translation operations, $G_{\text{trans}}(B)$ and $G_{\text{fix}}(B)$, acting on $S^{\text{kin}}_{\text{pst}}(B)$. Using the isomorphism discussed above, $\mathcal{I}_{\text{N}} : \mathcal{S}^{\text{kin}}_{\text{pst}}(B) \leftrightarrow \mathcal{S}^{\text{kin}}_{\text{pst}}(A)$, we can convert these PSTOs acting on $S^{\text{kin}}_{\text{pst}}(B)$ into a pair of Lie groups acting on $S^{\text{kin}}_{\text{pst}}(A)$. Namely, we can faithfully convert $G_{\text{trans}}(B)$ and $G_{\text{fix}}(B)$ acting on $S^{\text{kin}}_{\text{pst}}(B)$ into another pair of Lie groups, $G_{\text{trans}}(A)$ and $G_{\text{fix}}(A)$, acting on $S^{\text{kin}}_{\text{pst}}(A)$. Note that these new Lie groups acting on $S^{\text{kin}}_{\text{pst}}(A)$ will be structurally identical to the old ones acting on $S^{\text{kin}}_{\text{pst}}(B)$. Hence, they will qualify as PSTOs. We are therefore free to pick them as our PSTOs when externalizing theory

[105] More elegantly, the dynamics of the pre-spacetime theory and the new theory would be picked out by the various kinematic structures induced in these theories. The end result is the same, however.

(A). Doing so, we find a topological redescription of theory (A) which is identical to our original description of theory (B), at least up to diffeomorphism. Hence, theory (A) is ISE-related to theory (B). This concludes the proof of the ISE-Equivalence theorem.

We can now apply this theorem to some of the example theories which we have considered throughout this thesis. As I noted at the end of Sec. 11.1, the following three theories are ISE-inter-related:

- From Ch. 7, the non-local QHO theory is ISE-related to the cosine QHO theory, and vice versa.
- From Sec. 8.1, the 3+1D QKG theory is ISE-related to the 1+1D QKG theory, and vice versa.
- From Sec. 8.2, various discrete heat theories are ISE-related to corresponding bandlimited heat theories, and vice versa.

This is all in accordance with the ISE-Equivalence Theorem since these pairs of theories have spacetime-kinematic compatibility while exhibiting an isomorphism at the level of kinematics which preserves the theory's dynamically allowed states.

Let us next consider the periodic matter and periodic spacetime theories introduced in Ch. 9. A quick inspection shows that there is an isomorphism between these two theories at the level of kinematics which preserves their dynamically allowed states. Hence, they meet the second and third conditions of the ISE-Equivalence Theorem. But do they satisfy the first condition? While both theories qualify as spacetime theories, the periodic matter theory lacks spacetime-kinematic compatibility. Hence, we have that the periodic matter theory is ISE-related to the periodic spacetime theory but not vice versa.

To spell this out in more detail, recall our discussion from Sec. 9.2 regarding the (unfaithful) internalization of the periodic matter theory. In particular, this theory's spacetime manifold $\mathcal{M}_{\text{old}} \cong H^{(\text{int})}_{\text{trans}}/H^{(\text{int})}_{\text{fix}} \cong \mathbb{R}^2$ survived internalization unfaithfully becoming $\mathcal{M}_{\text{G}} \cong G^{(\text{int})}_{\text{trans}}/G^{(\text{int})}_{\text{fix}} \cong \mathbb{R} \times S^1$. Since this theory does not have spacetime-kinematic compatibility we do not have a guarantee that the pair of Lie groups, $G^{(\text{int})}_{\text{trans}}$ and $G^{(\text{int})}_{\text{fix}}$, are

PSTOs; This theory falls outside of the scope of Appendix B.2. Fortunately, however, in this case, they are PSTOs. Externalizing these PSTOs, (i.e., taking $G^{(\text{ext})}_{\text{trans}} = G^{(\text{int})}_{\text{trans}}$ and $G^{(\text{ext})}_{\text{fix}} = G^{(\text{int})}_{\text{fix}}$) we can construct a new spacetime setting for the theory, namely, $\mathcal{M}_{\text{new}} \cong G^{(\text{ext})}_{\text{trans}}/G^{(\text{ext})}_{\text{fix}} \cong \mathbb{R} \times S^1$. This new spacetime manifold looks suspiciously like the spacetime from the periodic spacetime theory. Indeed, these are exactly the PSTOs which we would have found by internalizing the periodic spacetime theory. Hence, following the above proof, we can use these PSTOs to move from the periodic matter theory to the periodic spacetime theory.

11.3 Applying The ISE Equivalence Theorem to the Möbius-Euclid Duality

Finally, let us turn our attention to the Möbius-Euclid Duality introduced in Sec. 4.1. As I have discussed there, we have an isomorphism between these two theories at the level of kinematics which preserves their dynamically allowed states. Hence, they meet the second and third conditions of the ISE-Equivalence Theorem. All that remains is to check that these two theories both have spacetime-kinematic compatibility. Using the above theorem, this will then prove that these two theories are ISE-equivalent.

First let us note which diffeomorphisms are kinematically allowed for these two theories. For the Möbius theory, every lifted diffeomorphism is kinematically allowed, $\text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}}) = \text{Diff}(\mathcal{M}_{\text{old}})$. That is, every d^* maps kinematically allowed states to kinematically allowed states. This is not true, however, for the Euclid theory. Its kinematically allowed diffeomorphisms, $\text{Diff}_{\text{kin}}(\mathcal{M}_{\text{new}}) \subset \text{Diff}(\mathcal{M}_{\text{new}})$ must preserve the theory's non-trivial kinematic constraint: Namely, each time-slice of the particles world-sheet must be a straight line both before and after applying d^*.

We can now check whether or not these two theories have spacetime-kinematic compatibility. First note that both theories are kinematically reduced. That is, only $d = \mathbb{1}$ lifts to act trivially on all kinematically allowed states. Recalling Sec. 9.3, this follows immediately from the fact that neither theory has kinematically inaccessible spacetime points. That is, for both theories every point in $\mathcal{M}$ is accessible by some kinematically allowed state, $X \in S^{\text{kin}}$.

To see that the Euclid theory is kinematically homogeneous, note that the usual action of $\mathrm{SE}(2)$ on the plane is transitive and, moreover, kinematically allowed. The stabilizer subgroup of this action is $\mathrm{SO}(2)$. Hence, we can reconstruct the plane using the following quotient,

$$\mathbb{R}^2 \cong \frac{\mathrm{SE}(2)}{\mathrm{SO}(2)} \tag{11.1}$$

It is more difficult to see that the Möbius theory is kinematically homogeneous. Since this theory has $\mathrm{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}}) = \mathrm{Diff}(\mathcal{M}_{\text{old}})$, all we need to do is show that the Möbius strip is a homogeneous manifold. Perhaps surprisingly, $\mathcal{M}_{\text{old}} \cong \mathbb{M}$ is also homogeneous with respect to $\mathrm{SE}(2)$. The stabilizer subgroup of this action is $\mathrm{E}(1)$. I demonstrate these facts in Appendix A.3 and then use them to reconstruct the Möbius strip in terms of a Lie group quotient,

$$\mathbb{M} \cong \frac{\mathrm{SE}(2)}{\mathrm{E}(1)} \tag{11.2}$$

Hence, we can internalize the Möbius theory using $H_{\text{trans}} \cong \mathrm{SE}(2)$ and $H_{\text{fix}} \cong \mathrm{E}(1)$. This will result in some PSTOs, $G^{(\text{int})}_{\text{trans}} \cong \mathrm{SE}(2)$ and $G^{(\text{int})}_{\text{fix}} \cong \mathrm{E}(1)$, acting on the theory's pre-spacetime states, $S^{\text{kin}}_{\text{pst}}$. In order to externalize into the Euclid theory, we can actually keep the group in the numerator the same $G^{(\text{ext})}_{\text{trans}} = G^{(\text{int})}_{\text{trans}}$ and change only the group in the denominator to $G^{(\text{ext})}_{\text{fix}} \cong \mathrm{SO}(2)$. This example is therefore unlike all of the other theories discussed in this thesis. Usually, our topological redescriptions have come about

because the two representations of the theory choose different G_{trans} to externalize. Here, by contrast, the two theories differ only in their choice of G_{fix}.

12 | Conclusion and Outlook

The ultimate goal of this thesis has been to develop some powerful new tools for topological redescription (namely, the ISE Method) and to demonstrate how these new techniques can be leveraged in various debates about the epistemology and metaphysics of space and time. As we saw in Ch. 4, we can often represent the exact same kinematic and dynamical contents as occurring on two radically different topological stages. For instance, in the Möbius-Euclid Duality (see Sec. 4.1) a theory about a point-like particle floating around on the Möbius strip was redescribed as instead being about a line-shaped particle floating around on the Euclidean plane, and vice versa. Some more examples of topological redescription which we encountered throughout Part II are: the position-momentum duality in quantum mechanics (which I prefer to call Fourier duality); dimensional reduction à la Kaluza-Klein; and, discrete to continuous spacetime redescription à la Kempf (2010).

In order to help us survey this wide landscape of possible topological redescriptions, Ch. 4 introduced what I have called spacetime representation theory. There is a sense in which we can see the Möbius and Euclid theories as being two different spacetime representations of the same pre-spacetime theoretic structure (in analogy with how some group theoretic structure might admit multiple representations). Indeed, every instance of topological redescription can be thought of in terms of inequivalent spacetime representations. In Part II I then introduced the ISE Method as a powerful tool for exploring the wide range of alternative spacetime representations for our theories' pre-spacetime theoretic content. Indeed, the ISE-Equivalence theorem (proved in Ch. 11) gives an explicit characterization of the strength of these new topological redescription techniques. In effect, it says: For a

wide range of spacetime theories the ISE Method gives us access to *effectively every possible* topological redescription thereof; It is limited only by a weak spacetime-kinematic compatibility constraint. (I have argued for the weakness of this constraint in Sec. 9.3.)

To review, the ISE Method is a three-step process (Internalize, Search, Externalize). The first step, internalization, allows one to divorce a theory's dynamical and kinematical content from its declared topological staging (e.g., the spacetime manifold, $\mathcal{M}_{\text{old}}$). In the searching step, one then picks out a set of pre-spacetime translation operations (PSTOs). These are a pair of finite-dimensional Lie groups (called G_{trans} and G_{fix}) which act on the theory's kinematically allowed states and which are, in a certain sense, structurally indistinguishable from spacetime translations. An intuitive example explored in Sec. 7.3 are translations, not in the old theory's spacetime, but rather in the old theory's Fourier space. Finally, in the externalization step, one then builds a new spacetime setting, $\mathcal{M}_{\text{new}} \cong G_{\text{trans}}/G_{\text{fix}}$, from these hand-picked pre-spacetime translation operations such that they become honest-to-goodness spacetime translations on $\mathcal{M}_{\text{new}}$. In the Fourier example, the new theory's spacetime is effectively a copy of the old theory's Fourier space.

To complete the externalization process, one must then map the old theory's states and dynamics onto $\mathcal{M}_{\text{new}}$. As I have discussed in Sec. 7.4, this final stage of the externalization process is tightly constrained by our choice of PSTOs. For instance, in the Fourier example, under minor assumptions the new theory's states and dynamics are required to be the Fourier transform of the old theory's states and dynamics. Recall also from Ch. 4 that every C^∞-style pre-spacetime theory has exactly one nice real-scalar-field representation (up to diffeomorphism). In this case too, once we have chosen our PSTOs and constructed out new spacetime, $\mathcal{M}_{\text{new}} \cong G_{\text{trans}}/G_{\text{fix}}$, we are tightly constrained regarding how we can represent the old theory's states on this new spacetime setting.

There is a significant reversal at the heart of the ISE Method. Before seeing this, however, allow me to first describe the typical situation. Our spacetime theories typically

come with a declared-to-be-important piece of topological staging, i.e., the spacetime manifold, $\Sigma = \mathcal{M}$. In general, we will have access to some lifting map, $*_{\text{lift}} : d \mapsto d^*$, which lifts the diffeomorphisms of this stage, Diff($\mathcal{M}$), to act on the theory's states as a bijection, $d^* : S^{\text{all}} \to S^{\text{all}}$. These lifted automorphisms are a special kind of Lie group action on the theory's states. Indeed, it is common practice to grant some kind of metaphysical importance to certain group actions on the basis of them being lifted diffeomorphisms as well as having certain other properties. For instance, dynamical symmetries are often characterized as those Lie group actions which preserve the theory's dynamics and are lifted diffeomorphisms. In sum, the normal situation is as follows: A spacetime theory comes along with some declared topological staging, $\mathcal{M}$, for its states; This then fixes which group actions on S^{all} qualify as lifted diffeomorphisms, Diff$^*(\mathcal{M})$.

Here, now, is the ISE Method's reversal. We begin with a pre-spacetime theoretic understanding of our theory, i.e., with no declared topological staging of its states. We then pick out a certain Lie group action on its pre-spacetime states (namely, our PSTOs, $G_{\text{fix}} \subset G_{\text{trans}}$) and declare that these are to be the lifted diffeomorphisms of our new theory. This uniquely fixed the new spacetime manifold, $\mathcal{M}_{\text{new}} \cong G_{\text{trans}}/G_{\text{fix}}$, up to diffeomorphism. Moreover, it tightly constrains how we can represent the old theory's states on this new spacetime setting. Remarkably, on can access effectively every possible spacetime framing of one's theory in this way. As the ISE Equivalence theorem shows us, almost all instance of topological redescription boil down to a disagreement regarding which PSTOs are to be considered spatiotemporal.

The key piece of mathematics which allows for the ISE Method to do its work are smooth homogeneous manifolds (for an overview see Appendix A). For this reason, the ISE Method is (at present) limited to outputting spacetime theories on smooth homogeneous manifolds. To review: A smooth manifold, $\mathcal{M}$, is called homogeneous with respect to any (finite dimensional) Lie group $H_{\text{trans}} \subset$ Diff($\mathcal{M}$) which acts transitively on

it. Namely, any point $p \in \mathcal{M}$ can be mapped to any other point $q \in \mathcal{M}$ by some group action $h(p) = q$. For instance: The 2-sphere S^2 is homogeneous with respect to SO(3); the plane $\mathbb{R}^2$ is homogeneous with respect to $(\mathbb{R}^2, +)$; the plane $\mathbb{R}^2$ is also homogeneous with respect to SE(2); and, perhaps surprisingly, the Möbius strip $\mathbb{M}$ is also homogeneous with respect to SE(2). It should be noted that homogeneous does not mean constant curvature. Whether or not a smooth manifold is homogeneous is solely a matter of topology.

Since the ISE Method requires homogeneity, it cannot be applied (in its present state) to theories set on non-homogeneous manifolds. For instance, while the double-torus, $\mathcal{M} \cong \mathbb{T}^2 \# \mathbb{T}^2$ is homogeneous with respect to Diff$(\mathcal{M})$, it is not homogeneous with respect to any no finite-dimensional Lie subgroup thereof. For proof, see Mostow (2005). Fortunately, it is likely that the ISE Method can be improved so that it can be applied to theories set on $\mathcal{M} \cong \mathbb{T}^2 \# \mathbb{T}^2$. While I don't yet have proof, I am have good reasons to believe that all of the key theorems which support the ISE Method can be extended to work with infinite-dimensional Lie groups. If we can make such an allowance, then many previously-inhomogeneous manifold (such as $\mathcal{M} \cong \mathbb{T}^2 \# \mathbb{T}^2$) will become homogeneous. In particular, a smooth manifold, $\mathcal{M}$, is homogeneous with respect to the infinite-dimensional Lie group, Diff$(\mathcal{M})$, if and only if all of its connected components are pairwise diffeomorphic. While this broader definition still admits non-examples (such as the disjoint union of a torus and a two-dimensional plane, $\mathcal{M} \cong \mathbb{T}^2 \dot{\cup} \mathbb{R}^2$), its scope is vastly improved; Essentially every smooth manifold encountered in modern physics is homogeneous with respect to Diff$(\mathcal{M})$.

But what if one wants to topologically redescribe one's theory as taking place somewhere other than on a smooth manifold? For instance, one's desired topological stage may just be a topological space with no smooth structure. Alternatively, what if one is working with a supersymmetric theory and wants to set it in superspace? What if one is working with a theory set on a non-natural fiber bundle (such as a spinor theory)? Can the ISE

Method be used in these more complicated cases? Again, I have good reasons to believe that all of the key theorems which support the ISE Method can be adapted to such cases. While more technical work is needed to confirm these claims, the following is likely true. Just as one can build a wide range of smooth manifolds by taking the quotient of two Lie groups, one can also build a wide range of topological space by taking the quotient of two topological groups. Similarly, one can build a wide variety of super-manifolds out of the quotient two Lie super-groups. Finally, I have preliminary evidence that a wide variety of fiber bundles, $\langle E, B, F, \pi : E \to B \rangle$, can be constructed by taking the pair-wise quotients of three Lie groups, $G_3 \subset G_2 \subset G_1$, as follows: $E_{\mathrm{G}} \cong G_1/G_3$, $B_{\mathrm{G}} \cong G_1/G_2$, and $F_{\mathrm{G}} \cong G_2/G_3$. Note that we have a natural projector, $\pi_{\mathrm{G}} : E_{\mathrm{G}} \to B_{\mathrm{G}}$ which regroups the G_3-cosets of G_1 into larger G_2-cosets of G_1. If these upgrades can indeed be made, then the ISE Method will be applicable to essentially every spacetime theory encountered in modern physics.

Even without these upgrades, however, the ISE Method still allows us to remove and replace the topological underpinnings of a wide variety of spacetime theories just as easily as we can switch between different coordinate systems (or different axiom systems, or different geometric descriptions). But what benefits can the philosopher (and the physicist) expect to follow from mastering our capacity for topological redescription? In Ch. 3 I argued that we can expect similar benefits to those which have already come about from our existing capacities for nomological, coordinate, and geometric redescription. Hence, a significant portion of Part I was spent discussing the impact of these existing capacities for formal redescription on various debates about the epistemology and metaphysics of axiom-like laws, coordinate systems, and spacetime geometry. Ultimately, we can expect that mastering our capacity for topological redescription will teach us important lessons about *both* the epistemology and the metaphysics of space and time. Ch. 5 has already discussed some of the metaphysical benefits as they relate to a supporting a dynamics-first

of topology and as they relate to structural realism. I will return to these topics briefly at the end of this chapter. Before this, however, allow me to first discuss how mastering our capacity for topological redescription can help us understand the epistemology of space and time.

Allow me to first briefly review the epistemic story which I sketched in Ch. 2. Step 1: However the world is structured in itself, we unconsciously organize it into our phenomenal experiences of the world. Step 2: After centuries of hard scientific work, the scientific community has codified their observations of matter's dynamics into our current best theories of physics. Recall that I have called this whole process systematization (= organize + codify). While there is much more that could be said about each step and each substep of this epistemic process, let us focus in on the codification step. The codification process doesn't simply end once we have some empirically adequate theory on the table. Indeed, the initial formulation of our theories (in terms of certain Z = axiom-like laws, coordinates, geometry, and/or topology) is often not their best formulation, whatever "best" ultimately means. After we have arrived at our initial Z-framing of the theory in question, some amount Z-reformulation (see Fig. 3.1) will often be required before we find a particularly nice Z-framing.

This brings us to the first epistemic benefit of mastering our capacity for formal redescription: It gives physicists new (and possibly better) ways of reformulating their theories. Several historical examples of this have already been discussed in Ch. 3 regarding our theories axiom-like laws, coordinates, and geometry structures. It is possible that the same is true of spacetime topology. Indeed, just as Nordström (1913) was at first unaware of the Einstein and Fokker (1914) reformulation of his theory, we may be equally unaware of fruitful topological reframings of our current theories. We can make progress against this problem of unconceived alternatives by widening our search space. Fortunately, we can often do better than just searching around for better Z-reformulation in an ad hoc

and unsystematic way. Indeed, once we have mastered Z-redescription we can then begin to explore these possibilities more systematically. Regarding spacetime topology, the ISE Method offers us *effectively every possible* spacetime representations of our theories (limited only by a weak spacetime-kinematic compatibility constraint).

Suppose, however, that this first benefit never materializes. That is, suppose that we have already found the best spacetime framing of all of our theories. Gaining a mastery of topological redescription will nonetheless be helpful in understanding the epistemic process of theory development. Indeed, in light of the abundance of topological redescription available to us (even if most are horrendously complicated) the question looms: Why did our theories end up with the particular topological staging that they do, rather than some mathematically equivalent alternative? That is, what are our selection criteria for topological framings of our spacetime theories?

An easy answer for why our theories are set framed in terms of spacetime (e.g., rather than Fourier space) is that this aligns with our phenomenal experience of space and time. Indeed, connecting our theories with our spacetime-laden experience of the world as well as with experimental practice would seem to require access to a phenomenology-aligned spacetime framing of our theory (in at least some regime).[106] Such considerations might account for the codification step, but what about the organization step? Why did our spatiotemporal intuitions pick out this particular way of spatiotemporally framing the world rather than any other? Three possible answers are as follows: (approximate) locality, (approximate) symmetry, and (effective) low-dimensionality. Arguably, these are all features of our phenomenal experience of space and time which have helped us to evolve to by making it computational feasible to grapple with the world. Given our mastery of topological redescription, it is conceivable that one could prove the following kind of result: A certain kind of theory (e.g., C^∞-algebraic theories) have exactly one spacetime

[106] See Huggett and Wüthrich (2013).

representation with these evolutionarily useful properties.

Such a unique-representation result would seem to make our choice of spacetime framing reasonably objective (at least macroscopically). The notion of objectivity one could potentially achieve here is comparable to what Lewis reassures us is possible in the law's context regarding his Best Systems Account (BSA). Against "ratbag idealism" Lewis (1994, pg. 479) claims that we can reasonably hope that we can reasonably hope that our BSA-selected law will be objective:

> If nature is kind, the best system will be *robustly* best — so far ahead of its rivals that it will come out first under any standards of simplicity and strength and balance.

In contrast to Lewis's worries about "ratbag idealism", other Humeans are more comfortable with the possibility that our law-selection criteria might "comes from us". For instance, Hall (2015, pg. 268) says that,

> [T]here is plenty of room for the view that it is in part facts about us — *idiosyncratically* about us, and our peculiar human psychologies — that play a role in determining the most appropriate way for us to structure our investigation of the world [e.g., determining the laws of nature]. How could the details of our peculiar human situation not be relevant to this matter?

Ott (2022, pg. 229) gives the name "Humeanism with a human face" to those BSA-like view views which pick out the laws of nature relative to the "epistemic goals" and "cognitive limitations" of a recognizably human agent. Recall Beebee's (2000, pg. 547) suggestion that the laws of nature are like an efficient summary of "God's Big Book of Facts". As she notes, God himself has no need for this sort of summary, but we do because of our "feeble, finite minds".[107] The same ideas might be applied to spacetime topology; Namely, we might soften our notion of objectivity by making it relative to the epistemic goals and cognitive limitations of a recognizably human agent.

Additionally, however, there is another way in which we can soften worries that space and time may "comes from us" in a problematic sense. An analogy with mathematics is

[107] Beebee (2000, pg. 547).

here helpful. One can think of mathematics as the process of picking some axioms (often implicitly) and then discovering their richly structured consequences. Hence, mathematics features elements of both "invention" (of the axioms) and discovery (of their consequences). Similarly, one can think of physics as the process of discovering the richly structured consequences of our (implicit) choice of PSTOs. Recall from my above review of the ISE Method that once we have made our choice of PSTOs, the rest of the externalization process follows more-or-less automatically. Namely, our choice of PSTOs uniquely fix the topology of the new spacetime manifold up to diffeomorphism. Moreover, our choice of PSTOs tightly constrain what type of particles/fields the new theory can be about, e.g., point-like particles, line-shaped particles, a scalar field, etc. Finally, recall from Sec. 7.4 that the way the theory's states fit onto the new spacetime setting is also constrained. Hence, physics (like mathematics) can be thought of as having elements of both invention and discovery. However the world is structured in itself (e.g., like some pre-spacetime theory) we then organize it into phenomenal experiences (e.g., by implicitly picking some PSTOs). Taken together, these result in a certain spacetime representation of the world (e.g., a spacetime theory) whose matter content we then discover (e.g., certain particles and fields).

Let us proceed assuming that our choice how to stage the dynamical behavior of matter is reasonably objective, at least macroscopically. Should we expect this objectivity to be preserved as we extend our spacetime theories to cover micro-physics? Ch. 8 has already discussed two example scenarios in which there are multiple ways that we might extend our phenomenally-aligned spacetime to account for new physics. For instance, Sec. 8.1 has shown that we might have to choose between adding new compactified dimensions to our physics or, alternatively, adding an infinite number of massive fields. Similarly, Sec. 8.2 has shown that we might have to choose between staging our theories on a discrete spacetime or, alternatively, maintaining a continuous spacetime albeit

with bandlimited fields. As more exotic example, one might ask whether supersymmetric theories ought to be understood as occuring in superspace or in an ordinary relativistic spacetime.[108] Prima facie, it seems that in each of these examples there are competing considerations which pull us in opposite directions. Hence, I take it to be an open question whether or not our topology selection criteria are objective or to some extent conventional.

So far our discussion has been focused on the epistemology of space and time. Namely, we have asked: Why did our theories end up with the particular topological staging that they do, rather than some mathematically equivalent alternative? Moving beyond epistemics however, we can ask a metaphysical question: Where in our epistemic story, does our theory's topological staging (e.g., its spacetime manifold, $\mathcal{M}$) acquire its spatiotemporal significance? As I discussed in Ch. 2, dynamics-first views of topology adopt an unofficially Kantian approach to answering this question. The spacetime manifold which appears in our best physical theories does not gain its spatiotemporal significance by latching onto some spatiotemporal stage upon which the world's drama plays out (contra Newton); Nor does it do so by latching onto any fundamental spatiotemporal relationships which help to facilitate interactions between the drama's players (contra Leibniz); Instead, it acquires its spatiotemporal significance only because it is a particularly nice way of systematizing the dynamical behavior of matter (perhaps conceived of non-spatiotemporally).

In opposition to this "along the way"-type answer, dynamics-second views would rather say that our theory's spacetime manifold acquires its spatiotemporal significance "at step zero", i.e., by latching onto some dynamics-facilitating pre-epistemic counterpart (e.g., fundamental spacetime points or relations). To help articulate side of the debate, consider a case of AdS/CFT-like topological redescription in which we have a 3+1D bulk

[108] See Menon (2021) and Baker (2022) for some further discussion of philosophical questions about space and time raised by supersymmetric physics.

framing of the theory as well as a 2+1D boundary framing of the theory. Moreover, suppose that after much dynamical and empirical investigations it is discovered that the 3+1 dimensional bulk corresponds to our phenomenal notions of space and time (and, relatedly, to all of the dynamical phenomena that we expect from rods and clocks).

I will now introduce two characters both of which hold dynamics-second views. Consider Alice who believes that the bulk framing of the theory has latched onto some spacetime points and/or relations which provide the world's particles and fields a spatiotemporal context within which they can then carry out their fundamental dynamics. Alice holds a dynamics-second view of the bulk spacetime. By contrast, consider Bob who makes the following claim: Whereas our spatiotemporal intuitions have clearly latched onto the bulk framing, the boundary is nonetheless where the world's particles and fields actually "live" and carry out their fundamental dynamics. Bob holds a dynamics-second view of the boundary spacetime.

But what view does Bob hold of the bulk spacetime? Bob agrees with Alice that the bulk spacetime gets its spatiotemporal significance from its association with our phenomenal notions of space and time. Bob, however, disagrees with Alice because he doesn't believe that our phenomenal notions of space and time have any hard-working pre-epistemic counterparts. Hence, according to Bob, both the bulk spacetime and our phenomenal notions of space and time must have ultimately earned their spatiotemporal significance only by being a particularly nice way of organizing and systematizing the pre-epistemic world (which he understands in terms of the boundary spacetime). Ultimately, Bob holds a *dynamics-first view* of the bulk spacetime (while simultaneously holding a dynamics-second view of the boundary spacetime).

It should be stressed that Alice's and Bob's disagreement here is not merely semantic. We can imagine that Alice and Bob were raised in the same linguistic culture and use the same definitions for all sorts of geometric and topological terms when they are used

in a phenomenal context, e.g., when describing daily life and/or the results of physics experiments. Moreover, they both agree on the definition of the term "spacetime" when it is used to refer to the pre-epistemic world; Given that the world's material content is some array of fields and/or particles, they both agree that the term "the pre-epistemic spacetime" refers to whatever spatiotemporal context these fields/particles carry out their fundamental dynamics within. Hence, their disagreements are not semantic but rather stem from different beliefs about what is going on in the pre-epistemic world (and, relatedly, which topological framing of their common pre-spacetime theory accurately reflects these goings-on).

Indeed, it follows from Alice's and Bob's disagreements being purely about the pre-epistemic world that their interpretations of this theory are empirically equivalent. Bob can predict everything that Alice can by using what they both agree is the phenomenology-aligned formulation of the theory (here, the bulk-framing of the theory). In addition to this, Bob has access to a topological reformulation of this theory which he believes accurately describes the pre-epistemic world's goings-on (here, in terms of the 2+1 dimensional boundary spacetime). Bob feels inconvenienced by the fact that our spatiotemporal intuitions (including his own) have latched onto the a topological framing the world which has no dynamics-facilitating pre-epistemic counterpart. But what can you do?

It is important to note that one can (unlike Bob) hold a dynamics-first view of topology without making any substantial claims about the pre-epistemic world. Indeed, recall that dynamics-first views can be true for either metaphysical or epistemic reasons. Namely, dynamics-facilitating spacetime points/relations might fail to exist or, even if they do exist, we might be epistemically alienated from them. This aligns well with the epistemic and metaphysical readings of Hume and Kant which I discussed in Ch. 2. Earlier in Ch. 5

I connected these to the epistemic and ontic versions of structural realism.[109]

Adopting a structural realist approach may help us shed some light upon that which all of these alternate topological descriptions are jointly descriptions of. Namely, we can take it that some pre-spacetime theory is an accurate simulation of the pre-epistemic world (either in itself, or as far as we can know about it). This brings us back to the opening questions of this thesis: How does the world hang itself together? That is, what (if anything) does nature use to help facilitate the dynamical behavior of matter? According to a thorough-going dynamics-first view the laws, geometry and topology which appear in our best theories don't earn their metaphysical relevance by latching onto any dynamics-facilitating pre-epistemic counterpart. Instead, they gain their metaphysical relevance only along the way by being a particularly nice way of systematizing the dynamical behavior of matter. Hence, we are led to treat them *as if* they don't have dynamics-facilitating pre-epistemic counterpart — given that they don't feature in our account of the metaphysical signficance of these theoretical structures.

Sans any such facilitators, we can think of the dynamical behavior of matter "happening on its own" so to speak. Insofar as we can tell, the world hangs together as a seamless structured unity. This pre-spacetime theoretic structure can then be variously represented as different kinds of particles and fields depending on which transformations (i.e., PSTOs) are considered spatiotemporal. I propose that the question of which transformations we ought to consider to be spatiotemporal should be settled by something like a Lewisian Best Systems Account.

[109] For more on this epistemic vs ontic distinction see Ladyman (1998) as well as Ladyman et al. (2014) and French (2016).

Bibliography

Abraham, M. (1912). Zur Theorie der Gravitation, *Physikalische Zeitschrift* **13**: 1–4.

Albert, D. Z. (2015). *After Physics*, Harvard University Press, Cambridge, MA.

Armstrong, D. M. (1983). *What is a Law of Nature?*, Cambridge Studies in Philosophy, Cambridge University Press, Cambridge, UK.

Arvanitogeōrgos, A. (2003). *An Introduction to Lie Groups and the Geometry of Homogeneous Spaces*, Student Mathematical Library. v. 22, American Mathematical Society, Providence, RI.

Baker, D. J. (2022). Interpreting Supersymmetry, *Erkenntnis* **87**(5): 2375–2396.

Barrett, T. W. and Halvorson, H. (2016). Morita Equivalence, *The Review of Symbolic Logic* **9**(3): 556–582.

Beebee, H. (2000). The Non-Governing Conception of Laws of Nature, *Philosophy and Phenomenological Research* **61**(3): 571–594.

Beem, J. K. and Parker, P. E. (1991). The Space of Geodesics, *Geometriae Dedicata* **38**(1): 87–99.

Bhogal, H. (2020). Humeanism about Laws of Nature, *Philosophy Compass* **15**(8): e12696.

Bird, A. (2007). *Nature's Metaphysics: Laws and Properties*, Oxford Scholarship Online, Oxford University Press, Oxford, UK.

Blackburn, S. (1990). Hume and Thick Connexions, *Philosophy and Phenomenological Research* **50**: 237–250.

Brickell, F. and Clark, R. S. (1970). *Differentiable Manifolds: An Introduction*, New University Mathematics Series, Van Nostrand Reinhold, London, UK.

Brown, H. and Pooley, O. (2006). Minkowski Space-Time: A Glorious Non-Entity, *Philosophy and Foundations of Physics* **1**(C): 67–89.

Brown, H. R. (2005). *Physical Relativity: Space-Time Structure From a Dynamical Perspective*, Oxford University Press, Oxford, UK.

Brown, H. R. and Read, J. (2022). The Dynamical Approach to Spacetime Theories, *in* E. Knox and A. Wilson (eds), *The Routledge Companion to Philosophy of Physics*, 1st edn, Routledge, London, UK. 70-85.

Carroll, J. W. (1994). *Laws of Nature*, Cambridge University Press, Cambridge, UK.

Carroll, J. W. (2020). Laws of Nature, *in* E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*, Winter 2020 edn, Metaphysics Research Lab, Stanford University, Stanford, CA.

Chen, L. and Fritz, T. (2021). An Algebraic Approach to Physical Fields, *Studies in History and Philosophy of Science Part A* **89**: 188–201.

De Haro, S. and Butterfield, J. (2021). On Symmetry and Duality, *Synthese* **198**(4): 2973–3013.

Demarest, H. (2017). Powerful Properties, Powerless Laws, *in* J. D. Jacobs (ed.), *Causal Powers*, Oxford University Press, Oxford, UK. 38-53.

Dewar, N., Linnemann, N. and Read, J. (2022). The Epistemology of Spacetime, *Philosophy Compass* **17**(4): e12821.

Dorato, M. (2007). Relativity Theory between Structural and Dynamical Explanations, *International Studies in the Philosophy of Science* **21**(1): 95–102.

Dretske, F. I. (1977). Laws of Nature, *Philosophy of Science* **44**(2): 248–268.

Duerr, P. M. (2020). Unweyling Three Mysteries of Nordström Gravity, *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* **69**: 32–49.

Earman, J. (1989). *World Enough and Space-Time: Absolute versus Relational Theories of Space and Time*, MIT Press, Cambridge, MA.

Einstein, A. and Fokker, A. D. (1914). Die Nordströmsche Gravitationstheorie vom Standpunkt des absoluten Differentialkalküls, *Annalen der Physik* **349**(10): 321–328.

Fatibene, L. and Francaviglia, M. (2003). *Natural and Gauge Natural Formalism for Classical Field Theorie: A Geometric Perspective including Spinors and Gauge Theories*, Springer, Dordrecht, NL.

French, S. (2016). *The Structure of the World: Metaphysics and Representation*, Oxford University Press, Oxford, UK.

Friedman, M. (1983). *Foundations of Space-Time Theories: Relativistic Physics and Philosophy of Science*, Princeton University Press, Princeton, NJ.

Friedman, M. (2001). *Dynamics of Reason*, CSLI Publications, Stanford, CA.

Geroch, R. (1972). Einstein Algebras, *Communications in Mathematical Physics* **26**(4): 271–275.

Grimmer, D. (2022a). A Discrete Analog of General Covariance – Part 2: Despite What You've Heard, A Perfectly Lorentzian Lattice Theory, *arXiv preprint arXiv:2205.07701* .

Grimmer, D. (2022b). A Discrete Analog of General Covariance: Could the World be Fundamentally Set on a Lattice?, *arXiv preprint arXiv:2204.02276* .

Grimmer, D. (2023). Introducing the ISE Methodology: A Powerful New Tool for Topological Redescription, *arXiv preprint arXiv:2303.04130* .

Grimmer, D., Lopp, R. and Martín-Martínez, E. (2021). Dimensional Reduction of Cavities with Axial Symmetry: A Complete Analysis of when an Optical Fiber is Approximately One Dimensional, *Phys. Rev. A* **104**: 013723.

Hall, N. (2015). Humean Reductionism about Laws of Nature, *in* B. Loewer and J. Schaffer (eds), *A Companion to David Lewis*, John Wiley and Sons, Ltd, Chichester, UK. 262-277.

Huggett, N. (2006). The Regularity Account of Relational Spacetime, *Mind* **115**(457): 41–73.

Huggett, N. and Wüthrich, C. (2013). Emergent Spacetime and Empirical (In)coherence, *Studies in History and Philosophy of Modern Physics* **44**(3): 276–285.

Hume, D. (2020). *An Enquiry Concerning Human Understanding*, Oxford World's Classics, Oxford University Press, Oxford, UK.

Ivanova, M. (2015). Conventionalism, Structuralism and Neo-Kantianism in Poincaré's Philosophy of Science, *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* **52**: 114–122.

Janssen, M. (2009). Drawing the Line between Kinematics and Dynamics in Special Relativity, *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* **40**(1): 26–52.

Jerri, A. (1977). The Shannon Sampling Theorem — Its Various Extensions and Applications: A Tutorial Review, *Proceedings of the IEEE* **65**(11): 1565–1596.

Kant, I. (1783). *Prolegomena to Any Future Metaphysic*, in the version by Jonathan Bennett (2017), presented at www.earlymoderntexts.com.

Kant, I. (1992). On the Form And Principles of the Sensible and the Intelligable World [Inaugural Dissertation] (1770), *in* D. Walford and R. Meerbote (eds), *Theoretical Philosophy, 1755–1770*, The Cambridge Edition of the Works of Immanuel Kant, Cambridge University Press, Cambridge, UK. 373–416.

Kant, I. (1999). *The Critique of Pure Reason*, Cambridge Edition of the Works of Immanuel Kant, Cambridge University Press, Cambridge, UK.

Kant, I. (2004). *Prolegomena to Any Future Metaphysics that will be Able to Come Forward as Science: With Selections from the Critique of Pure Reason*, Cambridge Texts in the History of Philosophy, rev. edn, Cambridge University Press, Cambridge, UK.

Kempf, A. (2010). Spacetime Could Be Simultaneously Continuous And Discrete, In The Same Way That Information Can Be, *New Journal of Physics* **12**(11): e115001.

Kempf, A. (2018). Quantum Gravity, Information Theory and the CMB, *Foundations of Physics* **48**(10): 1191–1203.

Kobayashi, S. and Nomizu, K. (1996). *Foundations of Differential Geometry*, Wiley Classics Library, Wiley, New York City, NY.

Ladyman, J. (1998). What is Structural Realism?, *Studies in History and Philosophy of Science. Part A* **29**(3): 409–424.

Ladyman, J., Ross, D., Spurrett, D. and Collier, J. (2014). *Every Thing Must Go: Metaphysics Naturalized*, Oxford University Press, Oxford, UK.

Lewis, D. (1983). New Work for a Theory of Universals, *Australasian Journal of Philosophy* **61**(4): 343–377.

Lewis, D. (1986). *Philosophical Papers Volume II*, Oxford University Press, Oxford, UK.

Lewis, D. (1994). Humean Supervenience Debugged, *Mind* **103**(412): 473–490.

Linnemann, N. and Salimkhani, K. (2021). The Constructivist's Programme and the Problem of Pregeometry, *arXiv preprint arXiv:2112.09265* .

Loewer, B. (2020). The Mentaculus Vision, *Statistical Mechanics and Scientific Explanation*, World Scientific. 3-29.

Loewer, B. (2021). The Package Deal Account of Laws and Properties (PDA), *Synthese* **199**(1): 1065–1089.

Lorkowski, C. M. (2023). *David Hume: Causation*, The Internet Encyclopedia of Philosophy.

Malament, D. B. (2012). *Topics in the Foundations of General Relativity and Newtonian Gravitation Theory*, Chicago Lectures in Physics, University of Chicago Press, Chicago, IL.

Maudlin, T. (2007). *The Metaphysics Within Physics*, Oxford University Press, Oxford, UK.

Maudlin, T. (2012). *Philosophy of Physics: Space and Time*, Princeton Foundations of Contemporary Philosophy, Princeton University Press, Princeton, NJ.

Menon, T. (2019). Algebraic Fields and the Dynamical Approach to Physical Geometry, *Philosophy of Science* **86**(5): 1273–1283.

Menon, T. (2021). Taking Up Superspace: The Spacetime Setting for Supersymmetric Field Theory, *in* C. Wuthrich, B. Le Bihan and N. Huggett (eds), *Philosophy Beyond Spacetime: Implications from Quantum Gravity*, Oxford University Press, Oxford, UK. 103-128.

Mostow, G. D. (2005). A Structure Theorem for Homogeneous Spaces, *Geometriae Dedicata* **114**(1): 87–102.

Ney, A. (2021). *The World in the Wave Function: A Metaphysics for Quantum Physics*, Oxford Scholarship Online, Oxford University Press, Oxford, UK.

Nordström, G. (1912). Relativitätsprinzip und Gravitation, *Physikalische Zeitschrift* **13**: 1126–1129.

Nordström, G. (1913). Zur Theorie der Gravitation vom Standpunkt der Relativitätsprinzips, *Annalen der Physik* **42**: 533–554.

Norton, J. D. (1993). General Covariance and the Foundations of General Relativity: Eight Decades of Dispute, *Reports on Progress in Physics* **56**(7): 791–858.

Norton, J. D. (2008). Why Constructive Relativity Fails, *The British Journal for the Philosophy of Science* **59**(4): 821–834.

O. Pooley, H. B. (1999). The Origins of the Spacetime Metric: Bell's Lorentzian Pedagogy and its Significance in General Relativity, *in* C. Callender and N. Huggett (eds), *Physics Meets Philosophy at the Planck Scale*, Cambridge University Press, Cambridge, UK. 256–272.

Ott, W. (2009). *Causation and Laws of Nature in Early Modern Philosophy*, Oxford University Press, Oxford, UK.

Ott, W. (2022). *The Metaphysics of Laws of Nature: The Rules of the Game*, Oxford Academic, 1st edn, Oxford University Press, Oxford, UK.

Pitts, J. B. (2016). Space-Time Philosophy Reconstructed Via Massive Nordström Scalar Gravities? Laws Vs. Geometry, Conventionality, and Underdetermination, *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* **53**: 73–92.

Poincaré, H. (2018). *Science and Hypothesis: The Complete Text*, 1st edn, Bloomsbury Publishing Plc, London, UK.

Pooley, O. (2017). Background Independence, Diffeomorphism Invariance and the Meaning of Coordinates, *in* D. Lehmkuhl, G. Schiemann and E. Scholz (eds), *Towards a Theory of Spacetime Theories*, Springer, New York City, NY. 105-143.

Reichenbach, H. and Reichenbach, M. (1965). *The Theory of Relativity and A Priori Knowledge*, University of California Press, Berkeley, CA.

Rosenstock, S., Barrett, T. W. and Weatherall, J. O. (2015). On Einstein Algebras and Relativistic Spacetimes, *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* **52**: 309–316.

Rynasiewicz, R. (1992). Rings, Holes and Substantivalism: On the Program of Leibniz Algebras, *Philosophy of Science* **59**(4): 572–589.

Saunders, S. (2013). Rethinking Newton's Principia, *Philosophy of Science* **80**(1): 22–48.

Schaffer, J. (2016). It is the Business of Laws to Govern, *Dialectica* **70**(4): 577–588.

Schopenhauer, A. (2018). *The World as Will and Representation. Volume 2*, The Cambridge Edition of the Works of Schopenhauer, Cambridge University Press, Cambridge, UK.

Shannon, C. E. (2001). A Mathematical Theory of Communication, *SIGMOBILE Mob. Comput. Commun. Rev.* **5**(1): 3–55.

Smith, N. (2005). *The Philosophy of David Hume: A Critical Study of its Origins and Central Doctrines*, Palgrave Macmillan UK, London, UK.

Stanford, P. K. (2006). *Exceeding our Grasp Science, History, and the Problem of Unconceived Alternatives*, Oxford University Press, Oxford, UK.

Stang, N. F. (2024). Kant's Transcendental Idealism, *in* E. N. Zalta and U. Nodelman (eds), *The Stanford Encyclopedia of Philosophy*, Spring 2024 edn, Metaphysics Research Lab, Stanford University, Stanford, CA.

Stevens, S. (2014). *The Dynamical Approach to Relativity as a Form of Regularity Relationalism*, PhD thesis, University of Oxford, Oxford, UK.

Strawson, G. (2014). *The Secret Connexion: Causation, Realism, and David Hume: Revised Edition*, 1st edn, Oxford University Press, Oxford, UK.

Sutton, R. (2019). The Bitter Lesson, *http://www.incompleteideas.net/* .

Tooley, M. (1977). The Nature of Laws, *Canadian Journal of Philosophy* **7**(4): 667–698.

Wald, R. M. (1984). *General Relativity*, University of Chicago Press, Chicago, IL.

Warner, F. W. (1983). *Foundations of Differentiable Manifolds and Lie Groups*, Graduate Texts in Mathematics, Springer, New York City, NY.

Weinberger, N., Williams, P. and Woodward, J. (2024). The Worldly Infrastructure of Causation, *The British Journal for the Philosophy of Science* (appearing soon).

Williamson, T. (2013). How Deep is the Distinction Between A Priori and A Posteriori Knowledge?, *in* A. Casullo and J. C. Thurow (eds), *The A Priori in Philosophy*, Oxford University Press, Oxford, UK. 291–312.

Zikidis, M. (2016). *Homogeneous Spaces: Characterization, Construction and Important Examples (Lecture Notes)*, Heidelberg University, Heidelberg, DE.

A | Key Theorems Regarding Smooth Homogeneous Manifolds

A smooth manifold, $\mathcal{M}$, is said to be homogeneous *simpliciter* if there exists a finite-dimensional Lie group, G, which it is homogeneous with respect to. A smooth manifold, $\mathcal{M}$, is said to be homogeneous with respect to a finite-dimensional Lie group, G, if there exists a group action, $\theta : G \times \mathcal{M} \to \mathcal{M}$, which acts smoothly and transitively on $\mathcal{M}$. A group action, θ, is said to act transitively on $\mathcal{M}$ when for every pair of points, $p, q \in \mathcal{M}$, there is some $g \in G$ with $\theta(g, p) = q$. That is, a group action is transitive when it can map any point anywhere.

The intuition behind this definition of homogeneity is that all points on such a manifold, $\mathcal{M}$, are equivalent from the perspective of the Lie group, G. Concretely, any point $q \in \mathcal{M}$ can be replaced by any other point $p \in \mathcal{M}$ via action by some $g \in G$. Said differently, there is only one G-orbit, namely the whole manifold, $\mathcal{M}$. If G's action on $\mathcal{M}$ wasn't transitive, then it would divide $\mathcal{M}$ into several distinct pieces (i.e., the several G-orbits). This would mean that from G's perspective $\mathcal{M}$ would look inhomogeneous (i.e., with several different kinds of points, namely several G-orbits).

Some examples of homogeneous smooth manifolds are as follows. The 2-sphere, S^2, is homogeneous with respect to the group of rigid rotations, $\mathrm{SO}(3)$. The plane, $\mathbb{R}^2$, is homogeneous with respect to the group of rigid translations, $(\mathbb{R}^2, +)$. The plane, $\mathbb{R}^2$, is also homogeneous with respect to the group of rigid translations and rotations, $\mathrm{SE}(2)$. Perhaps surprisingly, the Möbius strip, $\mathbb{M}$, is also homogeneous with respect to $\mathrm{SE}(2)$. It should be noted that while some of these examples feature the geometrically loaded term

"rigid", the sense in which these manifolds are homogeneous has nothing to do with their geometry. In particular, homogeneous does not mean constant curvature. Whether or not a smooth manifold is homogeneous is solely a matter of topology.

A smooth manifold can fail to be homogeneous on purely topological grounds. For a simple example consider $\mathcal{M} \cong \mathbb{T}^2 \dot{\cup} \mathbb{R}^2$, the disjoint union of a torus and a two-dimensional plane. No smooth transformation on $\mathcal{M}$ can map a torus-point to a plane-point. For a less trivial example of a non-homogeneous manifold, consider the double torus, $\mathcal{M} \cong \mathbb{T}^2 \# \mathbb{T}^2$. The non-homogeneity of this manifold hinges crucially upon us restricting our attention to finite-dimensional Lie groups.[110] Namely, if we were to widen the definition of smooth homogeneous manifolds to allow for infinite-dimensional Lie groups, then the double torus would count as homogeneous. In fact, under this extended definition, a smooth manifold is homogeneous if and only if its connected components are pair-wise diffeomorphic. For any such manifold, the infinite-dimensional Lie group of diffeomorphisms, $\mathrm{Diff}(\mathcal{M})$, acts smoothly and transitively over $\mathcal{M}$. For mathematical simplicity, however, this thesis will use the more restrictive finite-dimensional definition of homogeneity.

The remainder of this appendix is organized as follows. Sec. A.1 will present several definitions and theorems regarding smooth homogeneous manifolds which are of central importance to the ISE Method.[111] To overview, it is well known that the quotient of two Lie groups is often a smooth homogeneous manifold. Moreover, it is well known that any homogeneous manifold is diffeomorphic to the quotient of some two Lie groups. Hence, up to diffeomorphism, the smooth homogeneous manifolds are exactly these Lie group

[110] All Lie groups which act smoothly and transitively over the double torus are infinite-dimensional. This follows from Mostow (2005)'s proof that all smooth connected manifolds which support smoothly and transitively-acting finite-dimensional Lie groups actions have Euler characteristic, $\chi \geq 0$. Note that the double torus has $\chi = -2$.

[111] For a general introduction to Lie groups and homogeneous spaces, see Arvanitogeōrgos (2003), particularly Chapter 4. For more technical details relating to the basic theorems of differential topology invoked below, see Warner (1983); Brickell and Clark (1970); Kobayashi and Nomizu (1996).

quotients. The theorems needed for the ISE Method, however, are slightly stronger. These improved theorems (stated below) show that, up to diffeomorphism, any smooth homogeneous manifold, $\mathcal{M}$, can be faithfully reconstructed as a quotient of *a certain kind* of Lie groups. Namely, we can take the Lie groups to be among $\mathcal{M}$'s own diffeomorphisms and, moreover, to have a certain kind of faithful construction compatibility (defined below). These slightly stronger theorems will be introduced in Sec. A.1, demonstrated in Secs. A.2 and A.3 and then proved in Sec. A.4 and A.5. Following this, Sec. A.6 will prove the quotient of images theorem for later use.

A.1 Constructing and Characterizing Smooth Homogeneous Manifolds

I should quickly rehearse two old definitions before making a new one. Firstly, let H be a group with some group action, $\theta : H \times \mathcal{M} \to \mathcal{M}$, on a smooth manifold, $\mathcal{M}$. The stabilizer subgroup of H at a point $p_0 \in \mathcal{M}$ is the subgroup of H whose group action on $\mathcal{M}$ maps p_0 to itself as $\theta(h, p_0) = p_0$. That is, the stabilizer subgroup of H at p_0 is the subgroup of H which maps p_0 onto itself.

Secondly, a group G_2 is said to be a core-free subgroup of G_1 if and only if G_2 contains no normal subgroups of G_1 (besides the trivial group, $\{\mathbb{1}\} \subset G_2 \subset G_1$). Note that this implies that G_2 itself is not a normal subgroup of G_1 (unless, of course, G_2 is itself trivial).

The following definition and theorem show how one can construct smooth homogeneous manifolds from a pair of Lie groups.

> **Definition: Faithful Construction Compatibility (for a smooth manifold)-** Let G_1 and G_2 be two finite-dimensional Lie groups. These two Lie groups will be said to have *faithful construction compatibility* if and only if G_2 is both a closed subgroup of G_1 as well as a core-free subgroup of G_1.

For later reference, a fact regarding abelian groups should be noted. If G_1 is abelian, then all of its subgroups, G_2, are normal, $G_2 \lhd G_1$. Hence, if G_1 and G_2 are supposed to have faithful construction compatibility and G_1 is abelian then we must have $G_2 = \{\mathbb{1}\}$.

> **Homogeneous Manifold Construction Theorem** - Let G_1 be a Lie group and $G_2 \subset G_1$ a closed subgroup thereof. The quotient of these groups, $\mathcal{M}_{\mathrm{G}} \coloneqq G_1/G_2$, is a smooth homogeneous manifold upon which G_1 acts smoothly and transitively via the following group action. Any $g \in G_1$ acts on any $[g_0] \in \mathcal{M}_{\mathrm{G}}$ as $\bar{g}$ with,
>
> $$\bar{g}([g_0]) = [g\, g_0], \tag{A.1}$$
>
> where $[g'] \in \mathcal{M}_{\mathrm{G}}$ is the G_2-coset in G_1 containing $g' \in G_1$. Together, these $\bar{g}$ maps form a Lie group of diffeomorphisms on $\mathcal{M}_{\mathrm{G}}$, namely, $\overline{G}_1 \subset \mathrm{Diff}(\mathcal{M}_{\mathrm{G}})$. These diffeomorphisms, $\overline{G}_1$, will be a faithful representation of G_1 (i.e., we will have $\overline{G}_1 \cong G_1$) if and only if G_1 and G_2 have faithful construction compatibility.

For proof see Sec. A.4. This construction theorem implies the well-known result that the quotient of a Lie group with one of its closed subgroups yields a smooth homogeneous manifold. The added content here is the condition under which the numerator group will be faithfully represented as diffeomorphisms on the resulting quotient space.

The following theorem gives the converse result.

> **Homogeneous Manifold Characterization Theorem** - Let $\mathcal{M}$ be a smooth manifold. $\mathcal{M}$ is homogeneous if and only if there exists a finite-dimensional Lie group of its own diffeomorphisms, $H_{\text{trans}} \subset \mathrm{Diff}(\mathcal{M})$, which acts transitively over $\mathcal{M}$. Let H_{fix} be any stabilizer subgroup of H_{trans}. Any such pair of Lie groups, H_{trans} and H_{fix}, will have faithful construction compatibility. Hence, by the Homogeneous Manifold Construction Theorem we can build a smooth manifold from them. The resulting manifold is diffeomorphic to the original,
>
> $$\mathcal{M} \cong \mathcal{M}_{\mathrm{H}} \coloneqq H_{\text{trans}}/H_{\text{fix}}. \tag{A.2}$$
>
> This reconstruction of $\mathcal{M}$ is robust in the sense that, up to diffeomorphism, it does not matter which H_{trans} and H_{fix} we use.

For proof see Sec. A.5. This characterization theorem implies the well-known result that every smooth homogeneous manifold, $\mathcal{M}$, can be reconstructed, up to diffeomorphism, as the quotient of two Lie groups. The added content here is that we can take these Lie groups to have faithful construction compatibility and, moreover, to be subgroups of $\mathcal{M}$'s

own diffeomorphisms, $H_{\text{fix}} \subset H_{\text{trans}} \subset \text{Diff}(\mathcal{M})$. The next two sections will demonstrate this reconstruction technique for the 2-sphere, $S^2 \cong \text{SO}(3)/\text{SO}(2)$ and the Möbius strip, $\mathbb{M} \cong \text{SE}(2)/\text{E}(1)$.

A.2 Reconstructing the Sphere as a Quotient of its Diffeomorphisms

To explicitly demonstrate how these theorems work, I will now use them to reconstruct the 2-sphere, $\mathcal{M} \cong S^2$. The reader is invited to think of $\mathcal{M} \cong S^2$ as a smaller version of the spacetime manifold, $\mathcal{M}_{\text{old}} \cong \mathbb{R}^2 \times S^2$, from the Quartic Klein-Gordon theory introduced in Sec. 7.1. Correspondingly, the relevant Lie group of diffeomorphisms for the following discussion, $H_{\text{trans}} \cong \text{SO}(3)$, is a smaller version of the $H_{\text{trans}} \cong (\mathbb{R}^2, +) \times \text{SO}(3)$ group discussed at the end of Sec. 7.1. The removal of the one time and one space dimension from this example is merely for pedagogical reasons; Removing these dimensions allows us to visualize the following discussion in terms of a nice three-dimensional picture, namely Fig. A.1.

In order to reconstruct $\mathcal{M} \cong S^2$ from H_{trans}, we must first identify H_{fix}, the stabilizer subgroup of H_{trans}. Recall that $H_{\text{trans}} \cong \text{SO}(3)$ is the group of rigid rotations of $\mathcal{M} \cong S^2$ in some coordinate system. If we take $p_0 = n$ to be the north pole, then we have that H_{fix} is the subgroup of rigid rotations which map the north pole to itself. That is, $H_{\text{fix}} \cong \text{SO}(2)$ is the group rotations about the z-axis. Plugging $H_{\text{trans}} \cong \text{SO}(3)$ and $H_{\text{fix}} \cong \text{SO}(2)$ into Eq. (A.2) we have,

$$\mathcal{M}_{\text{H}} \coloneqq H_{\text{trans}}/H_{\text{fix}} \cong SO(3)/SO(2). \tag{A.3}$$

According to the construction theorem, this quotient should be a smooth manifold. Moreover, according to the characterization theorem, it should be diffeomorphic to our original manifold, $\mathcal{M}_{\text{H}} \cong \mathcal{M} \cong S^2$. Let us now check this, paying careful attention to where the

quotient manifold gets its smooth structure from. See Fig. A.1 for an illustrated guide to how this quotient works.

In Eq. (A.3) we are supposed to regard the Lie group in the numerator, $H_{\text{trans}} \cong \text{SO}(3)$, as a smooth manifold: But what smooth manifold is $\text{SO}(3)$? To see this, consider first its Lie algebra, $\mathfrak{so}(3)$, and note that it is three-dimensional. The group exponential from $\mathfrak{so}(3)$ to $\text{SO}(3)$ is surjective such that each element of $\text{SO}(3)$ is represented (possibly multiple times) in the image of $\mathfrak{so}(3)$ under the group exponential. As I will now discuss, by identifying these redundancies we can see $\text{SO}(3)$ as a certain quotient space of $\mathbb{R}^3$.

Fig. A.1 illustrates how $\mathfrak{so}(3)$ redundantly represents $\text{SO}(3)$. In Fig. A.1 a point, $r \in \mathbb{R}^3$, represents a rotation of $\theta = |r|$ radians about the $\hat{r}$ axis (where $\hat{r}$ is the unit vector pointing at r). In this way, each rotation in $\text{SO}(3)$ can be represented within the region $|r| \leq \pi$ with the following redundancy: $\pi\,\hat{r}$ is identical to $-\pi\,\hat{r}$ for every $\hat{r}$. That is, a half-turn about any axis $\hat{r}$ is the same as a half-turn in the reverse direction.

For example, rotations of the sphere about the x-axis correspond to points on the x-axis in Fig. A.1 (e.g., the black points). Rotating about the x-axis through an angle just greater than π is equivalent to rotating in the reverse direction by an angle just less than π. Hence, the antipodal points on the x-axis in Fig. A.1 represent the same rotation and so should be identified. This holds not just for the x-axis but for every axis. Thus, as a smooth manifold $\text{SO}(3)$ is diffeomorphic to the unit ball in $\mathbb{R}^3$ with every pair of antipodal points on its surface identified. That is, as a smooth manifold we have $\text{SO}(3) \cong \text{RP}^3$, the real-projective space in three dimensions.

Note that RP^3 is not diffeomorphic to the manifold, $\mathcal{M} = S^2$, which we are hoping to recover. Indeed, according to Eq. (A.3) the desired manifold is a quotient of this manifold by $H_{\text{fix}} \cong \text{SO}(2)$. The colored lines in Fig. A.1 show some of these H_{fix}-cosets.[112]

[112] Note that only a subset of these H_{fix}-cosets (or equivalently, $\equiv_{p_0}$-equivalence classes) are shown in Fig. A.1. Namely, only equivalence classes which contain rotations about the x-axis with incrementally larger values of $\theta \in [-\pi, \pi]$ are shown.

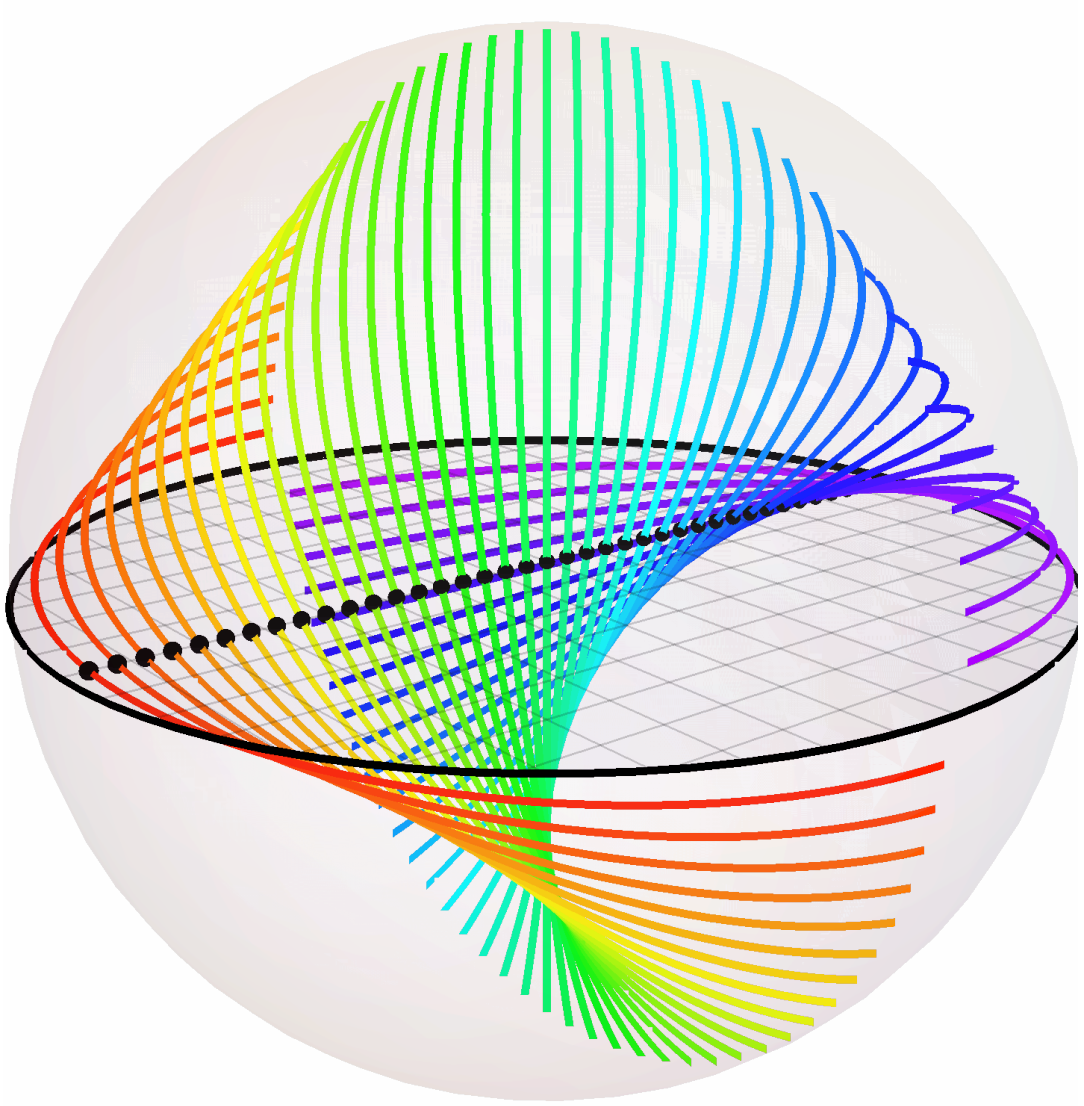

Figure A.1: The Lie group $\mathrm{SO}(3) \cong \mathrm{RP}^3$ is depicted here as a smooth manifold: Namely, as a ball with radius π in $\mathbb{R}^3$ with every pair of antipodal points on its surface identified. Concretely, each point $r \in \mathbb{R}^3$ in this figure represents a rotation of $\theta = |r|$ radians about the $\hat{r}$ axis (where $\hat{r}$ is the unit vector pointing at r). We can recover the sphere S^2 from $\mathrm{SO}(3)$ by taking its quotient with respect to a certain equivalence relation, $\equiv_{p_0}$. Two rotations are equivalent according to $\equiv_{p_0}$ if they map the north pole, $p_0 = n$, to the same place. The colored lines shown are some of the equivalence classes of $\mathrm{SO}(3)$ under $\equiv_{p_0}$. Note that the equator itself is an equivalence class. For every equivalence class except this one, we can pick out a unique representative on the x, y-plane (see the black points). Taking these points as our representatives the quotient space becomes a disk with its boundary compactified to a single point. Namely, this quotient of $\mathrm{SO}(3)$ by $\equiv_{p_0}$ yield the smooth manifold $\mathrm{SO}(3)/\equiv_{p_0} \cong S^2$ as desired.

At this point, it will be helpful to use the orbit-stabilizer theorem to rewrite the quotient manifold, $\mathcal{M}_{\text{H}} \coloneqq H_{\text{trans}}/H_{\text{fix}}$, in terms of an equivalence relation. This theorem tells us that H_{fix}-cosets in H_{trans} are identical to the equivalence classes of the following equivalence relation over H_{trans}:

$$h_2 \equiv_{p_0} h_1 \text{ iff } h_2(p_0) = h_1(p_0), \tag{A.4}$$

where $p_0 = n$ is the point at which H_{fix} is H_{trans}'s stabilizer subgroup. In other words, two diffeomorphisms, $h_1, h_2 \in H_{\text{trans}}$ are equivalent under $\equiv_{p_0}$ if and only if they map $p_0 = n$ to the same place. For any $h \in H_{\text{trans}}$ let $[h]_{\equiv(p_0)}$ denote its $\equiv_{p_0}$ equivalence class. The orbit-stabilizer theorem says that we have an exact matching between H_{fix}-cosets and $\equiv_{p_0}$ equivalence classes, namely $hH_{\text{fix}} = [h]_{\equiv(p_0)}$. Thus, the reconstructed manifold can be rewritten as,

$$\mathcal{M}_{\text{H}} = H_{\text{trans}}/\equiv_{p_0} . \tag{A.5}$$

We can understand Fig. A.1 in terms of this equivalence relation as follows: the colored lines in Fig. A.1 show which rotations in $H_{\text{trans}} \cong \text{SO}(3)$ map the north pole, $p_0 = n$ to the same place.

For later use, an alternate notation for $\equiv_{p_0}$ equivalence classes should be noted. As I will now discuss, we can efficiently label our cosets, $hH_{\text{fix}} = [h]_{\equiv(p_0)}$, not by which $h \in H_{\text{trans}}$ they contain but rather by which point $p \in \mathcal{M}$ they commonly send p_0 to. To see this, consider the map,

$$\begin{aligned} P : \mathcal{M}_{\text{H}} &\to \mathcal{M} \\ [h]_{\equiv(p_0)} &\mapsto h(p_0). \end{aligned} \tag{A.6}$$

This P map sends each equivalence class, $[h]_{\equiv(p_0)}$, onto the point, $h(p_0) \in \mathcal{M}$, which all of its elements jointly map p_0 onto. Note that no two equivalence classes will send p_0

to the same place (by definition). Hence, P is injective. Next note that P is surjective because H_{trans} acts transitively on $\mathcal{M}$ (i.e., for any p there is some h with $h(p_0) = p$). Hence, P is a bijection. Hence, for every $p \in \mathcal{M}$ we can define the set,

$$[h]_{p_0 \mapsto p} \coloneqq P^{-1}(p) = \{h' \in H_{\text{trans}} \mid h'(p_0) = p\}, \tag{A.7}$$

to be the set of maps $h' \in H_{\text{trans}}$ which map p_0 to p. In light of the orbit-stabilizer theorem, these $[h]_{p_0 \mapsto p}$ sets are in one-to-one correspondence with the H_{fix}-cosets in H_{trans}.

Let us quickly discuss two examples of equivalence classes. First, note that all rotations about the z-axis map the north pole to the same place (i.e., they leave it fixed). Hence, every z-axis rotation is in the same $\equiv_{p_0}$ equivalence class. Rotations about the z-axis are represented in Fig. A.1 by points on the z-axis. Hence, these points are connected by a vertical line in Fig. A.1. Second, note that all rotations about points on the equator through an angle of $\theta = \pi$ map the north pole to the same place (i.e., to the south pole). Hence, all of these rotations are in the same $\equiv_{p_0}$ equivalence class. In Fig. A.1, these rotations are represented by points on the sphere's equator. Hence, the equator is connected by a circular line in Fig. A.1. Besides these two equivalence classes, several others are also shown in Fig. A.1 which smoothly transition between them.

But how can we see the sphere S^2 arising from this collection of $\equiv_{p_0}$-equivalence class? For every equivalence class (except for the equatorial one) we can pick as its representative element its unique member on the x, y-plane (e.g., the black points in Fig. A.1). The exception to this claim is the equivalence class of rotations which map the north pole to the south pole (i.e., the black line around the equator). Just like every other equivalence class, we must treat the equator in Fig. A.1 like a single point in the quotient space. Thus, we can take the x, y-plane of Fig. A.1 to be our quotient space with one slight modification: We must treat the equator as being compactified into a single point in the quotient space. Compactifying the boundary of a disk yields the manifold $\mathcal{M}_{\text{H}} \cong S^2$ as desired.

A.3 Reconstructing the Möbius Strip as a Quotient of its Diffeomorphisms

In what follows the reader is invited to think of the Möbius strip, $\mathcal{M} \cong \mathbb{M}$, as the spacial stage from the Point-like Particle on a Möbius strip theory introduced in Sec. 4.1. Although it is not required for what follows, it will be helpful to think of the Möbius strip as a fiber bundle mostly so that we can talk about how a certain Lie group acts on its fibers. To set up my notation we have a projection $\pi : \mathcal{M} \to B$ from our total space, $\mathcal{M} \cong \mathbb{M}$, onto our base space, $B \cong S^1$, with fiber type, $F \cong \mathbb{R}$. It will also be helpful to recall from Sec. 4.1, a certain global coordinate system for the Möbius strip. Indeed, there exists a convenient coordinate map, $C : \mathbb{R}^2 \to \mathbb{M}$, which assigns to each point, $p \in \mathbb{M}$, a set of coordinates,

$$C^{-1}(p) = \{(x + n\,\pi, (-1)^n\, y)\}_{n\in\mathbb{Z}} \subset \mathbb{R}^2. \tag{A.8}$$

To connect with Fig. A.2, some point $p \in \mathbb{M}$ will receive many (x, y)-coordinates, four of which are shown in Fig. A.2a). One can think of advancing the y-coordinate of some point as carrying us along its fiber (e.g., in the direction of the black arrows). Advancing along the x-coordinate can be thought of as carrying us around the base space of the Möbius strip as follows:

$$d(\Delta x) : (x, y) \mapsto (x + \Delta x, y). \tag{A.9}$$

Note that the period of this motion is π on the base space (i.e., the dashed blue line with $y = 0$) but its period is 2π on the total space; Note that a rotation by $\Delta x = \pi$ simply flips all of the fibers over.

In order to reconstruct $\mathcal{M} \cong \mathbb{M}$ in terms of its diffeomorphisms, we must first find a finite-dimensional Lie group, H_{trans}, which acts smoothly and transitively on $\mathcal{M}$. That is, H_{trans} should be able to map any point to any point. Let us begin, however, with the easier task of finding a group which can map any fiber to any fiber. It is trivial to see that some Lie group action $d(\Delta x) \in \text{SO}(2)$ can map any fiber onto any other fiber. In order to get a Lie group, H_{trans}, which acts transitively over $\mathcal{M} \cong \mathbb{M}$ the following upgrade to our $\text{SO}(2)$ action would be sufficient: We need to expand it so that it acts transitively over at least one of $\mathcal{M}$'s fibers. To see that this would be sufficient, let us call the base point of this fiber $b_0 \in B \cong S^1$. Suppose that we want to map $p \in \mathcal{M}$ to any other point $q \in \mathcal{M}$. We can first use our above-discussed $\text{SO}(2) \subset H_{\text{trans}}$ action to map p and q onto b_0's fiber via some maps $h_1, h_3 \in H_{\text{trans}}$ with $h_1(p) \in \pi^{-1}(b_0)$ and $h_3(q) \in \pi^{-1}(b_0)$. If we manage to make the above-discussed upgrade, then we can find some map $h_2 \in H_{\text{trans}}$ which maps $h_1(p)$ to $h_3(q)$ as $h_2(h_1(p)) = h_3(q)$. In total we then have that $h_3^{-1} \circ h_2 \circ h_1(p) = q$ as desired.

Let us now make this upgrade to the above-discussed $\text{SO}(2)$ action. We would like to be able to move the $\pi^{-1}(b_0)$ along itself. Notice that one cannot simply shift all of the fibers uniformly in the same direction. That is, a uniform y-shift in Fig. A.2a) or Fig. A.2b) is disallowed because it disrespects the coordinate identification laid down in Eq. (A.8). If one of p's coordinate labels increases in y-value then its next coordinate label must decrease in y-value. Somewhere in between these our Lie group action must act trivially on some fiber. Let us take the fiber which must remain fixed, $\pi^{-1}(b_{\text{fix}})$, to be on opposite sides of the Möbius strip as the fiber we intend on moving. That is, let us take $b_{\text{fix}} = b_0 + \pi/2$.

To smoothly interpolate between the motion of b_0's fiber and the non-motion of b_{fix}'s fiber, we can take the relative "speed" at which each intermediate fibers, $\pi^{-1}(b_{\text{mid}})$, moves to be $\cos(\theta)$ where $-\pi/2 < \theta \leq \pi/2$ is the smallest angle between b_{mid} and b_0. Ul-

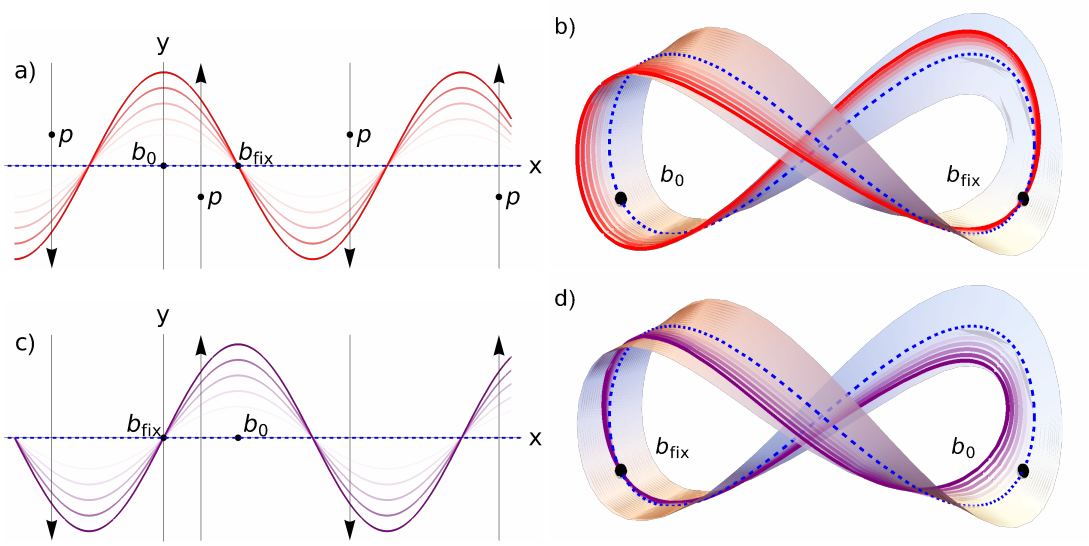

Figure A.2: This figure depicts the transitive action of SE(2) on the Möbius strip, $\mathcal{M} \cong \mathbb{M}$. We can decompose SE(2) into a rotation group SO(2) and two translation groups, $(\mathbb{R}, +)$, which I shall call A-shifts and B-shifts. The rotation group acts rigidly on the base space $B \cong S^1$ of $\mathcal{M} \cong \mathbb{M}$ and advances the x-coordinate in subfigures a) and c). In subfigures b) and d), the fibers are carried along by this rotation with the dashed blue line mapping onto itself. The A-shifts are shown in subfigures (a) and (b). Similarly, the B-shifts are shown in subfigures (c) and (d). Both of these transformations map each fiber onto itself. That is, each point changes only its y-coordinate in subfigures a) and c).

timately, we can define $d(\Delta x, A) \in \text{Diff}(\mathbb{M})$ be the diffeomorphism which acts in the (x, y)-coordinate system as follows:

$$d(\Delta x, A) : (x, y) \mapsto (x + \Delta x, y + A\cos(x)). \tag{A.10}$$

See Fig. A.2a) and note that this way of moving the fibers along themselves respects the coordinate identification laid down in Eq. (A.8).

Let us call the above-discussed cosine-shaped action on $\mathcal{M} \cong \mathbb{M}$ an A-shift, see Fig. A.2a) and b). These $d(\Delta x, A)$ transformation do not form a Lie group because they are not closed under composition. We can, however, take H_{trans} to be the closure of these A-shifts together with the SO(2) rotations discussed above. Fortunately, this closure is a finite-dimensional Lie group. To see this, note that the generator of our SO(2) rotations is ∂_θ. Hence, because $\partial_\theta \cos(\theta) = -\sin(\theta)$ forming this closure requires that we add to

H_{trans} a sine-shaped action which I shall call a B-shifts, see Fig. A.2c) and d). Concretely, we have,

$$d(\Delta x, A, B) : (x, y) \mapsto (x + \Delta x, y + A\sin(x) + B\cos(x)). \tag{A.11}$$

One can check that the A-shifts and B-shifts commute with each other. Moreover, B-shifts combine with infinitesimal rotations ∂_θ to yield A-shifts (this follows from $\partial_\theta \sin(\theta) = \cos(\theta)$). Hence, we have that H_{trans} is closed with only rotations, A-shifts, and B-shifts. Notice how these transformations interact with each other: rotation by $\theta = \pi/2$ converts A-shifts into B-shifts and B-shifts into negative A-shifts. It follows from this that these transformations together form a group, $H_{\text{trans}} \cong \text{SE}(2)$, namely the special Euclidean group. Hence, the Möbius strip, $\mathcal{M} \cong \mathbb{M}$, is homogeneous with respect to $\text{SE}(2)$.

In order to reconstruct $\mathcal{M} \cong \mathbb{M}$ from H_{trans} we must next identify H_{fix}, the stabilizer subgroup of H_{trans}'s action on $\mathcal{M}$. Looking at see Fig. A.2d), let us pick p_{fix} to be the point on b_{fix}'s fiber such that rotation by $\theta = \pi$ maps p_{fix} to p_{fix} (recall that rotation by $\theta = \pi$ flips every fiber). The transformations, $h \in H_{\text{trans}} \cong \text{SE}(2)$, which map p_{fix} to itself (i.e., $h \in H_{\text{fix}}$) are rotations by $\theta = \pi$ together with arbitrary B-shifts. Noting that rotation by $\theta = \pi$ inverts these B-shifts we have, $H_{\text{fix}} \cong (\mathbb{R}, +) \rtimes \mathbb{Z}_2 \cong \text{E}(1)$, the one-dimensional Euclidean group. Taking the quotient of $H_{\text{trans}} \cong \text{SE}(2)$ by $H_{\text{fix}} \cong \text{E}(1)$ we have,

$$\mathcal{M}_{\text{H}} \coloneqq \frac{H_{\text{trans}}}{H_{\text{fix}}} \cong \frac{\text{SE}(2)}{\text{E}(1)}. \tag{A.12}$$

According to the homogeneous manifold construction theorem, this quotient should be a smooth homogeneous manifold. Moreover, according to the homogeneous manifold characterization theorem, it should be diffeomorphic to our original manifold, $\mathcal{M}_{\text{H}} \cong \mathcal{M} \cong \mathbb{M}$. Let us now check this, paying careful attention to where the quotient manifold gets its smooth structure from. See the left side of Fig. A.3 for an illustrated guide to how this quotient works.

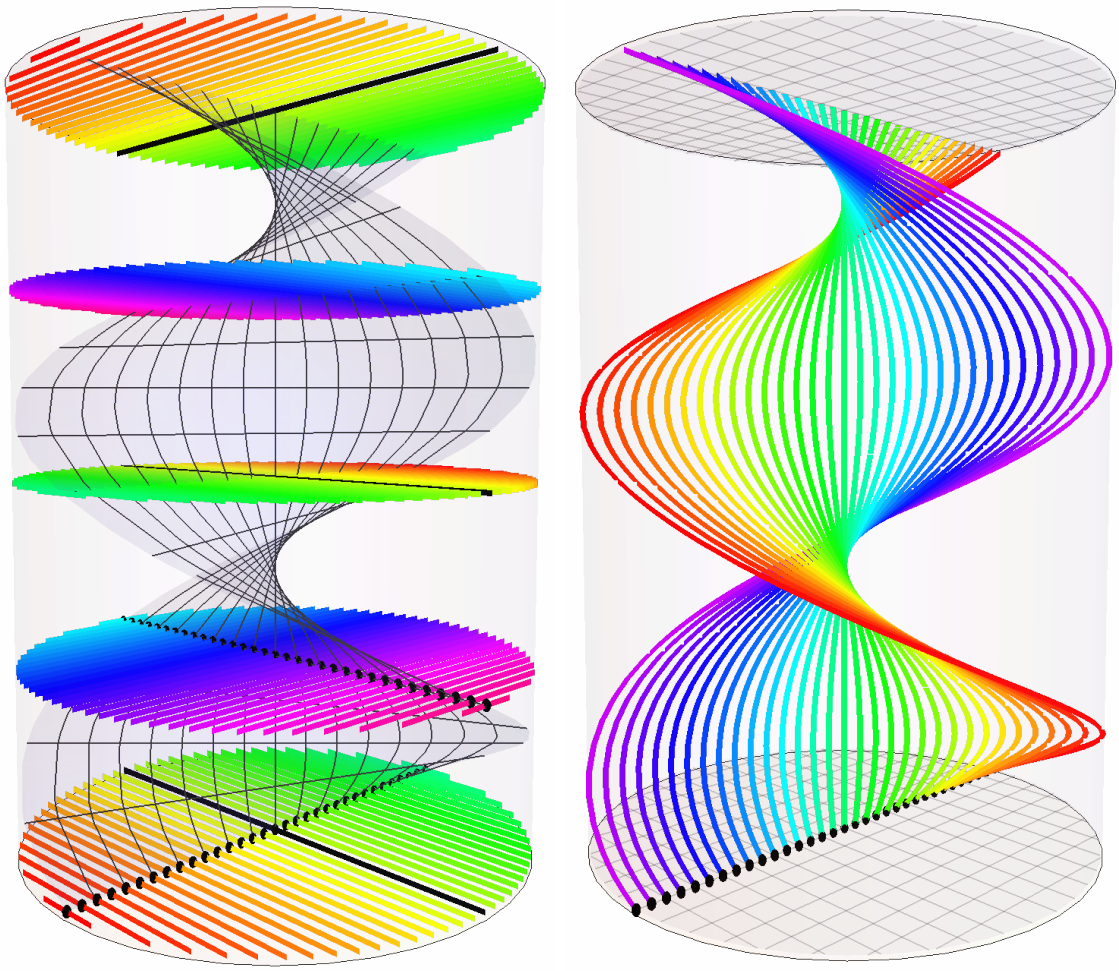

Figure A.3: The Lie group $H_{\text{trans}} \cong \text{SE}(2)$ is depicted here as a smooth manifold, $H_{\text{trans}} \cong \mathbb{R}^2 \times S^1$. The group's two translation degrees of freedom are represented in the horizontal x and y directions and its rotation degree of freedom is represented in the vertical z direction (which has a period of 2π; i.e., the top and bottom circles are to be identified). As noted in the text, $\text{SE}(2)$ acts transitively over both the Möbius strip, $\mathbb{M}$, and the plane $\mathbb{R}^2$. Hence, we should be able to realize either $\mathbb{M}$ or $\mathbb{R}^2$ as some quotient of $\text{SE}(2)$.
Left) Let $H_{\text{fix}}^{(\text{Mob})} \cong \text{E}(1)$ be the stabilizer subgroup of H_{trans}'s action on $\mathbb{M}$. The colored lines shown are some of the $H_{\text{fix}}^{(\text{Mob})}$-cosets in H_{trans}. For each $H_{\text{fix}}^{(\text{Mob})}$-cosets we can pick as its representative the point at which it intersects the twisting surface in the lower half of the figure. Some of their representatives are indicated in black. Identifying each $H_{\text{fix}}^{(\text{Mob})}$-cosets with its representative yield the smooth manifold $H_{\text{trans}}/H_{\text{fix}}^{(\text{Mob})} \cong \mathbb{M}$ as desired.
Right) Let $H_{\text{fix}}^{(\text{Euc})} \cong \text{SO}(2)$ be the stabilizer subgroup of H_{trans}'s action on $\mathbb{R}^2$. The colored lines shown are some of the $H_{\text{fix}}^{(\text{Euc})}$-cosets in H_{trans}. For each $H_{\text{fix}}^{(\text{Euc})}$-cosets we can pick as its representative the point at the bottom of the cylinder. Some of their representatives are indicated in black. Identifying each $H_{\text{fix}}^{(\text{Euc})}$-cosets with its representative yield the smooth manifold $H_{\text{trans}}/H_{\text{fix}}^{(\text{Euc})} \cong \mathbb{R}^2$ as desired.

To begin, note that as a smooth manifold, we have $H_{\text{trans}} \cong \text{SE}(2) \cong \mathbb{R}^2 \times S^1$. This space is depicted on both sides of Fig. A.3 with the A and B shifts corresponding to the x and y directions and the θ-rotations with $0 \leq \theta \leq 2\pi$ corresponding to the z-direction. Note that the z-direction in Fig. A.3 is periodic with the top circle identified with the bottom circle (without any twisting). The identity element $\mathbb{1} \in H_{\text{trans}}$ is marked by the black point in the center of the bottom circle.

In order to compute the quotient manifold, $\mathcal{M}_{\text{H}}$, we need to first find all of the H_{fix}-cosets in H_{trans}. The identity coset, $\mathbb{1}H_{\text{fix}}$, consists of arbitrary B-shifts and rotations by π. This coset is displayed as three black lines in Fig. A.3. It runs from left to right in the bottom, middle, and top circles. Similarly, every coloured line in Fig. A.3 represents an H_{fix}-coset in H_{trans} (with line segments of the same color belonging to the same coset). For instance, the A-shift H_{fix}-cosets (i.e., aH_{fix} where a is an A-shift) are shown as coloured lines in the bottom, middle, and top circles running from red to green. Next consider the rotated A-shift H_{fix}-cosets (i.e., RaH_{fix} where a is an A-shift and R is a rotation by $\theta = \pi/2$). These are shown in the intermediate circles with colors running from blue to purple. One can imagine how these H_{fix}-cosets twist as θ increases.

We can realize the above-discussed quotient, $\mathcal{M}_{\text{H}}$, by picking a representative for each H_{fix}-cosets in H_{trans}. For the cosets discussed above, we can take as representatives the black points in Fig. A.3. Note that we can take all of our representative points to lie in the range $0 \leq \theta < \pi$. For instance, we don't need a representative for the red line segment at the back of the middle circle, because we already have a representative for the red-coset (i.e., the black point on the red line in the bottom circle). For all other cosets, we can select representatives on the bottom half of the transparent twisting surface (i.e., with $0 \leq \theta < \pi$). Note that this surface completes a half turn as it moves from $\theta = 0$ to $\theta = \pi$. Concretely, note that in the bottom circle, the red-coset is nearest to us, but in the

middle circle it is farthest from us. After making the relevant identification, this surface of representatives is exactly the Möbius strip. Hence, we have $\mathcal{M}_{\mathrm{H}} \cong \mathbb{M}$ as desired.

The reader is invited to consider the quotient $\mathrm{SE}(2)$ by $\mathrm{SO}(2)$ which is depicted on the right side of Fig. A.3. The colored lines show $\mathrm{SO}(2)$ cosets. For each of them, we can pick as its representative the black points at the bottom of the figure. Hence, we have $\mathrm{SE}(2)/\mathrm{SO}(2) \cong \mathbb{R}^2$.

A.4 Proof of the Homogeneous Manifold Construction Theorem

This section will prove the Homogeneous Manifold Construction Theorem stated in Sec. A.1. Given any finite-dimensional Lie group, G_1, and closed subgroup thereof, $G_2 \subset G_1$, one can build a homogeneous manifold as follows. Taking the quotient of these Lie groups yields the following quotient space,

$$\mathcal{M}_{\mathrm{G}} \coloneqq G_1/G_2. \tag{A.13}$$

The equivalence classes under this quotient, $[g] = g\,G_2$, are G_2-cosets. Note that the quotient between these two groups is, in general, not itself a group. This is because the denominator, G_2, may not be a normal subgroup of the numerator, G_1. The quotient is, however, always a smooth manifold. In particular, this quotient space naturally adopts a smooth structure from the Lie group in the numerator, G_1, via its quotient with G_2. This follows from the quotient manifold theorem:[113]

> **Quotient Manifold Theorem -** Suppose G is a Lie group acting smoothly, freely, and properly on a smooth manifold M. Then the quotient space M/G has a unique smooth structure with the property that the quotient map $\pi : M \to M/G$ is a smooth submersion. (N.B.: If M is a Lie group and G is a closed subgroup thereof, then G acts smoothly, freely, and properly on M.)

In order to apply this to our current situation we take the replacements $G = G_2$ and $M = G_1$ such that $M/G = \mathcal{M}_{\mathrm{G}}$. The fact that G_2 acts smoothly, freely, and properly on

[113] See Theorem 1 of Zikidis (2016).

G_1 follows from G_2 being a closed subgroup of G_1. Hence, we can grant $\mathcal{M}_{\mathrm{G}}$ the smooth structure which is uniquely picked out by this theorem.

In terms of diffeomorphisms on $\mathcal{M}_{\mathrm{G}}$ this means the following: A generic transformation, $d_{\text{post}} : \mathcal{M}_{\mathrm{G}} \to \mathcal{M}_{\mathrm{G}}$, is smooth on $\mathcal{M}_{\mathrm{G}}$ if and only if there exists a smooth transformation, $d_{\text{pre}} : G_1 \to G_1$, which maps the equivalence classes onto each other in the same way,

$$\pi \circ d_{\text{pre}} = d_{\text{post}} \circ \pi. \tag{A.14}$$

That is, some permutation of the G_2 cosets, $[g] \mapsto d_{\text{post}}([g])$, will be smooth if and only if there is a way to implement it smoothly on G_1 prior to taking the quotient, i.e., to implement it as $[g] \mapsto [d_{\text{pre}}(g)]$.

This quotient space, $\mathcal{M}_{\mathrm{G}}$, is not only a smooth manifold, but is also homogeneous with G_1 acting smoothly and transitively over it. Namely, the following group action,

$$\begin{aligned} &\theta : G_1 \times \mathcal{M}_{\mathrm{G}} \to \mathcal{M}_{\mathrm{G}} \\ &\theta(g, [h]) = \bar{g}([h]) \coloneqq [g\, h]. \end{aligned} \tag{A.15}$$

acts smoothly and transitively over $\mathcal{M}_{\mathrm{G}}$. This follows straightforwardly from the left-action of G_1 on itself being both smooth and transitive.

Given that this group action is smooth, the above defined $\bar{g}$ maps are all diffeomorphisms on $\mathcal{M}_{\mathrm{G}}$. Collecting these together we have a Lie group, $\bar{g} \in \overline{G_1} \subset \mathrm{Diff}(\mathcal{M}_{\mathrm{G}})$. In some sense, the Lie group $\overline{G_1}$ is a representation of G_1 as diffeomorphisms on $\mathcal{M}_{\mathrm{G}}$. As I will now prove, this will be a faithful representation (i.e., $\overline{G_1} \cong G_1$) if and only G_2 is a core-free subgroup of G_1. (Recall that G_2 is a core-free subgroup of G_1 if and only if G_2 contains no non-trivial normal subgroups of G_1.)

The desired congruence result, $\overline{G_1} \cong G_1$, is equivalent to the map $g \mapsto \bar{g}$ having a trivial kernel. Namely, it is equivalent to the claim that $\bar{s}([g]) = [g]$ implies $s = \mathbb{1}$.

But $\bar{s}([g]) = [g]$ is equivalent to $[g^{-1}sg] = [e]$ which is itself equivalent to $g^{-1}sg \in G_2$. Hence, our desired congruence result, $\overline{G_1} \cong G_1$, is equivalent to the following:

$$\forall s \left((\forall g \;\; g^{-1} s \, g \in G_2) \implies s = \mathbb{1} \right) \tag{A.16}$$

with s and g both ranging over G_1. Let $N_{G_1}(G_2)$ denote the set of all $s \in G_1$ satisfying $\forall g \;\; g^{-1} \, s \, g \in G_2$. Note that $N_{G_1}(G_2) \subset G_2$ (consider taking $g = \mathbb{1}$). Note also that $N_{G_1}(G_2) \lhd G_1$ is a normal subgroup of G_1. In fact, $N_{G_1}(G_2)$ is the largest normal subgroup of G_1 which is contained within G_2. This is called the normal core of G_2 in G_1. Hence, our desired congruence result, $\overline{G_1} \cong G_1$, is equivalent to $N_{G_1}(G_2) = \{\mathbb{1}\}$. This happens exactly when G_2 contains no non-trivial normal subgroups of G_1. That is, when G_2 is a core-free subgroup of G_1.

A.5 Proof of the Homogeneous Manifold Characterization Theorem

This section will prove the Homogeneous Manifold Characterization Theorem stated in Sec. A.1. By definition, a smooth manifold, $\mathcal{M}$, is homogeneous if and only if there exists some finite-dimensional Lie group, G, which acts smoothly and transitively over it. Suppose that G acts smoothly on $\mathcal{M}$ with action $\theta : G \times \mathcal{M} \to \mathcal{M}$. Using this action we can associate to each $g \in G$ a diffeomorphism, $h \in \text{Diff}(\mathcal{M})$, with $h(p) = \theta(g, p)$. Collecting these diffeomorphisms together we have a finite-dimensional Lie group of diffeomorphisms, $H_{\text{trans}} \subset \text{Diff}(\mathcal{M})$. Note that H_{trans} acts transitively on $\mathcal{M}$ (because G does). Note also that H_{trans} automatically acts smoothly on $\mathcal{M}$ since it is a group of diffeomorphisms on $\mathcal{M}$. Hence, $\mathcal{M}$ is homogeneous if and only if there exists some finite-dimensional Lie group of its diffeomorphisms, $H_{\text{trans}} \subset \text{Diff}(\mathcal{M})$, which acts transitively over $\mathcal{M}$.

I will now prove that H_{trans} has faithful construction compatibility with any of its stabilizer subgroups H_{fix} (say the stabilizer subgroup at p_0). To begin, let us prove that H_{fix} is a closed subgroup of H_{trans}. Note that the function $f_{p_0} : H_{\text{trans}} \to \mathcal{M}$ given $f(h) = h(p_0)$ is continuous. Next note that $H_{\text{fix}} = f^{-1}(p_0)$. Since the singleton $\{p_0\}$ is a closed set and f is continuous, we have that H_{fix} is a closed set.

Next, let us prove that H_{fix} is a core-free subgroup of H_{trans}. As noted above, this claim is equivalent to the following:

$$\forall s \left((\forall h \;\; h^{-1} s \, h \in H_{\text{fix}}) \implies s = \mathbb{1}_{\mathcal{M}} \right) \tag{A.17}$$

with s and h both ranging over H_{trans}. This equivalent claim will now be proved by contrapositive. Note that if $s \neq \mathbb{1}_{\mathcal{M}}$ then there exists some $q \in \mathcal{M}$ with $s(q) \neq q$. Since H_{trans} acts transitively on $\mathcal{M}$ there is some $h \in H_{\text{trans}}$ which maps p_0 to q as $q = h(p_0)$. For this h we have $(s\,h)(p_0) \neq q$ whereas $h(p_0) = q$. Thus we have $(h^{-1} s\, h)(p_0) \neq p_0$ and therefore $h^{-1} s\, h \notin H_{\text{fix}}$ as desired. This completes the proof that there is faithful construction compatibility between any transitively acting Lie group $H_{\text{trans}} \subset \text{Diff}(\mathcal{M})$ and any of its stabilizer subgroups, H_{fix}.

Let us next use the above-proved construction theorem to build a homogeneous manifold from H_{trans} and H_{fix}, namely $\mathcal{M}_{\text{H}} \coloneqq H_{\text{trans}}/H_{\text{fix}}$. According to the characterization theorem, this should be diffeomorphic to the original manifold. But how do we know? To establish that $\mathcal{M}_{\text{H}}$ and $\mathcal{M}$ are diffeomorphic we need to find a bijection between them which is smooth in both directions (i.e., a diffeomorphism). As it turns out we have already met this bijection in Eq. (A.6), namely

$$\begin{aligned} P : \mathcal{M}_{\text{H}} &\to \mathcal{M} \\ [h]_{\equiv(p_0)} &\mapsto h(p_0). \end{aligned} \tag{A.18}$$

Recall that from the orbit stabilizer theorem that the H_{fix}-cosets in H_{trans} are exactly the

equivalence classes of a certain equivalence relation. Namely, we have that $hH_{\text{fix}} = [h]_{\equiv(p_0)}$ are the equivalence classes which group together those $h_1, h_2 \in H_{\text{trans}}$ which map p_0 to the same place as $h_2(p_0) = h_1(p_0)$. The above-defined P map sends each equivalence class, $[h]_{\equiv(p_0)}$, onto the point, $h(p_0) \in \mathcal{M}$, which all of its elements jointly map p_0 onto.

Note that no two equivalence classes will send p_0 to the same place (by definition). Hence, P is injective. Next note that P is surjective because H_{trans} acts transitively on $\mathcal{M}$ (i.e., for any q there is some h with $h(p_0) = q$). Hence, P is a bijection. Thus we have,

$$\mathcal{M} \leftrightarrow \mathcal{M}_{\text{H}} \coloneqq H_{\text{trans}}/H_{\text{fix}}. \tag{A.19}$$

But is this bijection P between $\mathcal{M}$ and $\mathcal{M}_{\text{H}}$ also a smooth map? It is, in fact, as can be seen from the following fundamental theorem of differential geometry:[114]

> **Unique Smoothness Theorem:** Let G, Q, and M be smooth manifolds and $\pi : G \to Q$ a surjective smooth submersion. If $F : G \to M$ is a smooth map that is constant across π (that is, $\pi(p) = \pi(q) \implies F(p) = F(q)$) then there exist a unique smooth map $R : Q \to M$ such that $R \circ \pi = F$

In order to apply this to our current situation we take the replacements $G = H_{\text{trans}}$, $Q = \mathcal{M}_{\text{H}}$, $M = \mathcal{M}$, and $F(h) = h(p_0)$. Note that each of $G = H_{\text{trans}}$, $Q = \mathcal{M}_{\text{H}}$, and $M = \mathcal{M}$ are all smooth manifolds either by definition or by construction. The map $\pi : H_{\text{trans}} \to \mathcal{M}_{\text{H}}$ is a smooth submersion by construction via the quotient manifold theorem. The F map is constant across π by construction.

Under these replacements, this theorem tells us that there exists a unique smooth map, R, such that $R([h]_{\equiv(p_0)}) = h(p_0)$. But this is exactly the P map defined in Eq. (A.18). Thus we have that $P = R$ is a smooth map as well as a bijection. To complete our proof that P is a diffeomorphism, we could next show that P^{-1} is smooth. Alternatively, we could show that P has a constant rank. This follows from P being equivariant under

[114] See Theorem 3 of Zikidis (2016).

H_{trans}'s joint action on $\mathcal{M}$ and $\mathcal{M}_{\text{H}}$,

$$P([g\,h]_{\equiv(p_0)}) = (g \circ h)(p_0) = g\,P([h]_{\equiv(p_0)}). \tag{A.20}$$

Hence, we have that P is a diffeomorphism and therefore that $\mathcal{M} \cong \mathcal{M}_{\text{H}} = H_{\text{trans}}/H_{\text{fix}}$ as smooth manifolds.

A.6 Proof of Quotient Of Images Lemma

This section will prove the following lemma for later use.

> **Quotient of Images Lemma:** Let $F \subset T \subset G$ be Lie groups with F a closed subgroup of T. Let $\phi : G \to H$ be a Lie group homomorphism. Let $K \coloneqq \ker(\phi) \cap T$. Note that since F is closed in T it follows that $\phi[F]$ is closed in $\phi[T]$. Hence, we can construct two smooth homogeneous manifolds, $\mathcal{M} \coloneqq T/F$ and $\mathcal{M}_\phi \coloneqq \phi[T]/\phi[F]$. Up to diffeomorphism, the quotient of the images, $\mathcal{M}_\phi$, is a certain quotient of $\mathcal{M}$, namely:
>
> $$\mathcal{M}_\phi \coloneqq \frac{\phi[T]}{\phi[F]} \cong \frac{T}{F\,K} \cong \frac{T/F}{K} = \frac{\mathcal{M}}{K} \tag{A.21}$$
>
> We will have $\mathcal{M} \cong \mathcal{M}_\phi$ if and only if $K \subset F$ (such that all of K's orbits on $\mathcal{M}$ are trivial).

Proof: Note that ϕ being a Lie group homomorphism implies that $\ker(\phi)$ is normal in G. This, in turn, implies that $K \coloneqq \ker(\phi) \cap T$ is normal in T. Similarly we have that $\ker(\phi) \cap F = K \cap F$ is normal in F. By the first isomorphism theorem of group theory, we then have,

$$\phi[T] \cong T/(\ker(\phi) \cap T) = T/K \tag{A.22}$$

$$\phi[F] \cong F/(\ker(\phi) \cap F) = F/(K \cap F).$$

Applying these results to the numerator and denominator of $\mathcal{M}_\phi$ we have,

$$\mathcal{M}_\phi \cong \frac{T/K}{F/(K \cap F)}. \tag{A.23}$$

The elements in this quotient space are $F/(K \cap F)$-cosets in the group T/K, namely they are elements of the form $tKF(K \cap F)$. Recall, however, that K is normal in T such that K can move past $F \subset T$. We thus have,

$$tKF(K \cap F) = tFK(K \cap F) = tFK, \tag{A.24}$$

since $K(K \cap F) = K$. Hence, we have $\mathcal{M}_\phi \cong T/(FK)$ as claimed in the lemma.[115]

To see the other diffeomorphic representations of $\mathcal{M}_\phi$ we will again use the fact that K is normal in T. As such we have,

$$tFK = tKF = KtF. \tag{A.25}$$

Hence, we can see the quotient of images, $\mathcal{M}_\phi$, as the quotient of $\mathcal{M} = T/F$ by the orbits of K acting on either the left or the right.

We will have $\mathcal{M} \cong \mathcal{M}_\phi$ if and only if every K-orbit on $\mathcal{M} = T/F$ is trivial, namely $KtF = tF$. Since we have K is normal in T we can pass the K through the t here. Canceling the factor of t on the left we then have $KF = F$ which is equivalent to $K \subset F$. This concludes the proof of the Quotient of Images Lemma.

[115] Note that it follows from K being closed and normal in T that the Frobenius product, $FK = KF$, is a closed Lie subgroup of T. Hence, the quotient $T/(FK) = T/(KF)$ is a smooth manifold.

B | Several Proofs in Support of the ISE Method

This appendix will prove several results in support of the main text. Firstly, Sec. B.1 will prove that in order for a theory's spacetime manifold to faithfully and robustly survive internalization requires a slightly weakened version of spacetime-kinematic compatibility. Indeed, I will there prove that this weakened condition is both necessary and sufficient. Next, Sec. B.2 will prove that given spacetime-kinematic compatibility the Lie group actions produced by the internalization process (namely, $G^{(\text{int})}_{\text{trans}}$ and $G^{(\text{int})}_{\text{fix}}$) are PSTOs. Moreover, I will prove that by externalizing these PSTOs one can immediately return to the theory's old spacetime setting, at least up to diffeomorphism.

B.1 Exact Conditions for Faithful and Robust Manifold Survival

Ch. 9 showed that spacetime-kinematic compatibility is a sufficient condition for a theory's spacetime manifold, $\mathcal{M}_{\text{old}}$, to faithfully and robustly survive internalization as quotient of its diffeomorphisms:

$$\mathcal{M}_{\text{old}} \cong \mathcal{M}_{\text{H}} \coloneqq \frac{H_{\text{trans}}}{H_{\text{fix}}} \xrightarrow[\text{int. as}]{\text{survives}} \mathcal{M}_{\text{G}} \coloneqq \frac{G_{\text{trans}}}{G_{\text{fix}}} \cong \frac{H_{\text{trans}}}{H_{\text{fix}}} \cong \mathcal{M}_{\text{old}}. \tag{B.1}$$

Surviving robustly here means that, up to diffeomorphism, the final result $\mathcal{M}_{\text{G}} \cong \mathcal{M}_{\text{old}}$ is independent of which H_{trans} and H_{fix} we have used to reconstruct $\mathcal{M}_{\text{old}}$. This section will prove a stronger result: The spacetime-kinematic compatibility condition can be weakened slightly while remaining sufficient; Moreover, I will prove that this weakened version of spacetime-kinematic compatibility is also a necessary condition. Hence, I will here provide a necessary and sufficient condition for $\mathcal{M}_{\text{old}}$ to survive internalization faithfully and robustly.

The weakened version of spacetime-kinematic compatibility is as follows.

> **Definition: (Weakened) Spacetime-Kinematic Compatibility -** A theory has the weakened version of spacetime-kinematic compatibility just in case:
>
> 1. it is a spacetime theory, as defined in Sec. 4.2,
> 2. it is kinematically homogeneous. That is, there exists a finite-dimensional and kinematically allowed Lie group, $H_{\text{trans}} \subset \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$, which acts transitively on $\mathcal{M}_{\text{old}}$,
> 3. it is kinematically reduced, $H_{\text{kin}} = \{\mathbb{1}\}$. Note that this is equivalent to demanding that $H_{\text{kin}} \cap \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}}) = \{\mathbb{1}\}$ since $H_{\text{kin}} \subset \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$. Or else failing this...
> 3′. it must have $H_{\text{kin}} \cap \text{Diff}^{\text{UFT}}_{\text{kin}}(\mathcal{M}_{\text{old}}) = \{\mathbb{1}\}$ where $\text{Diff}^{\text{UFT}}_{\text{kin}}(\mathcal{M}_{\text{old}})$ is the union of all of the finite-dimensional and kinematically allowed Lie groups, $H_{\text{trans}} \subset \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$, which acts transitively on $\mathcal{M}_{\text{old}}$.

Note that the definition of spacetime-kinematic compatibility appearing in Sec. 9.2 is simply $1. \wedge 2. \wedge 3.$ whereas the present definition is strictly weaker, namely $1. \wedge 2. \wedge (3. \vee 3'.)$. Note also that $3. \implies 3'.$ since $\text{Diff}^{\text{UFT}}_{\text{kin}}(\mathcal{M}_{\text{old}}) \subset \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$. To better see the difference between these two sets, consider the following possibility. There could exist a kinematically allowed map, $h_\infty \in \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$, such that for any finite-dimensional, kinematically allowed, and transitively acting Lie groups, H_{trans}, we have that closure$(h_\infty, H_{\text{trans}})$ is an infinite-dimensional Lie group. That is, every kinematically allowed and transitively acting Lie group which h_∞ participates in is infinite-dimensional. Such an h_∞ could exist, in principle, and if it does then $\text{Diff}^{\text{UFT}}_{\text{kin}}(\mathcal{M}_{\text{old}})$ is strictly smaller than $\text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$.

This section is organized as follows. Let P denotes the old theory's spacetime manifold, $\mathcal{M}_{\text{old}}$, faithfully and robustly surviving internalization. First I will argue that points $1.$ and $2.$ listed in the above definition are both necessary conditions for P. I will then argue that given points $1.$ and $2.$ we will have the faithful and robust survival of $\mathcal{M}_{\text{old}}$ if

and only if point $3. \vee 3'.$ is satisfied. The logic is then as follows:

$$\begin{aligned} P \implies (1. \wedge 2.) \\ \underline{(1. \wedge 2.) \implies (P \iff (3. \vee 3'.))} \\ \vdash \quad P \iff (1. \wedge 2. \wedge (3. \vee 3'.)). \end{aligned} \tag{B.2}$$

I will now re-trace the discussion in Ch. 9 regarding how topological information about $\mathcal{M}_{\text{old}}$ can survive internalization. The key difference here is that spacetime-kinematic compatibility is not assumed. To begin note that in order for $\mathcal{M}_{\text{old}}$ to survive the internalization process, we must be able to begin the internalization process. For this to happen the theory in question must be a spacetime theory in the sense defined in Sec. 4.2. This establishes that $P \implies 1.$

Given a spacetime theory, the discussion presented in Sec. 9.1 can proceed as before. Recall that *some* diffeomorphisms can survive internalization (albeit potentially unfaithfully). Namely, the theory's kinematically allowed diffeomorphisms survive via the map $G(d) \coloneqq \mathcal{F} \circ d^*|_{\text{kin}} \circ \mathcal{F}^{-1}$ where $\mathcal{F} : S^{\text{kin}}_{\text{old}} \leftrightarrow S^{\text{kin}}_{\text{pst}}$ is the map which internalizes the theory's states. In particular, recalling Eq. (9.13), we have that,

$$H \xrightarrow[\text{int. as}]{\text{survives}} G(H) \cong H/(H_{\text{kin}} \cap H), \tag{B.3}$$

for every kinematically allowed Lie group, $H \subset \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$. That is, up to isomorphism, all such H survive internalization as a Lie group by being quotiented by $H_{\text{kin}} \cap H$, its kinematically trivial subgroup.

Moreover, recall from Ch. 9 that *some* smooth manifolds can be reconstructed in terms of their diffeomorphisms (i.e., homogeneous manifolds can, $\mathcal{M}_{\text{old}} \cong H_{\text{trans}}/H_{\text{fix}}$, see Appendix A). Putting these possibilities together it seems conceivable that the spacetime manifolds of *some* theories might also survive internalization (potentially unfaithfully).

Concretely, the theory's spacetime manifold might survive internalization as a quotient of its surviving diffeomorphisms.

Let us next see that kinematic homogeneity is a necessary condition for the theory's spacetime to survive internalization simpliciter (i.e., faithfully or not). Firstly, in order for us to be able to reconstruct the old theory's spacetime manifold as $\mathcal{M}_{\text{old}} \cong H_{\text{trans}}/H_{\text{fix}}$ it is necessary that $\mathcal{M}_{\text{old}}$ is homogeneous. In particular, there must exist some finite-dimensional Lie group of diffeomorphisms, $H_{\text{trans}} \subset \text{Diff}(\mathcal{M}_{\text{old}})$, which acts smoothly transitively on $\mathcal{M}_{\text{old}}$. Secondly, in order for $\mathcal{M}_{\text{old}}$ to survive internalization *as a quotient of its surviving diffeomorphisms* the Lie group appearing in the numerator, H_{trans}, must itself survive internalization as a Lie group. That is, H_{trans} must lift to a Lie group of maps H^*_{trans} which are kinematically allowed. Putting these two requirements together, we have the following result: For a theory's spacetime to survive internalization (faithfully or not), the theory must be kinematically homogeneous. This establishes that $P \Longrightarrow 2.$ and hence that $P \Longrightarrow (1. \wedge 2.)$

Assuming points 1. and 2. we can continue the internalization process as follows. Kinematic homogeneity guarantees the existence of some finite-dimensional kinematically allowed transitively acting Lie group, $H_{\text{trans}} \subset \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$. Using this Lie group and one of its stabilizer subgroups H_{fix}, we can reconstruct $\mathcal{M}_{\text{old}} \cong H_{\text{trans}}/H_{\text{fix}}$ as a quotient and then internalize it,

$$\mathcal{M}_{\text{old}} \cong \mathcal{M}_{\text{H}} \coloneqq \frac{H_{\text{trans}}}{H_{\text{fix}}} \xrightarrow[\text{int. as}]{\text{survives}} \mathcal{M}_{\text{G}} \coloneqq \frac{G(H_{\text{trans}})}{G(H_{\text{fix}})} \cong \frac{\mathcal{M}_{\text{H}}}{H_{\text{kin}} \cap H_{\text{trans}}} \tag{B.4}$$

where the last step follows from the Quotient of Images Lemma proved in Appendix A.6. Thus, assuming only kinematic homogeneity the theory's old spacetime survives internalization, although it has been quotiented by the orbits of $H_{\text{kin}} \cap H_{\text{trans}}$. Note that $\mathcal{M}_{\text{old}}$ to will survive faithfully if and only if the orbits of $H_{\text{kin}} \cap H_{\text{trans}}$ in $\mathcal{M}_{\text{H}}$ are trivial. As noted in the Quotient of Images Lemma, this will happen exactly when we have $H_{\text{kin}} \cap H_{\text{trans}} \subset H_{\text{fix}}$.

This is a necessary and sufficient condition for faithful survival of $\mathcal{M}_{\text{old}}$ given our choice of H_{trans} and H_{fix} (and given points 1. and 2.).

If we moreover want faithful *and robust* survival, then we need to demand that we have $\mathcal{M}_{\text{G}} \cong \mathcal{M}_{\text{old}}$ no matter which H_{trans} and H_{fix} we used to reconstruct $\mathcal{M}_{\text{old}} \cong H_{\text{trans}}/H_{\text{fix}}$ as a quotient. Namely, we will have $\mathcal{M}_{\text{old}}$ surviving faithfully and robustly if and only if we have $H_{\text{kin}} \cap H_{\text{trans}} \subset H_{\text{fix}}$ for all H_{trans} and H_{fix}. Here H_{trans} ranges over all finite-dimensional, kinematically allowed, transitively acting Lie groups, $H_{\text{trans}} \subset \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$. Moreover, H_{fix} ranges over all of H_{trans}'s stabilizer subgroups (i.e., ranging over all $p_0 \in \mathcal{M}_{\text{old}}$). As I will now prove (given points 1. and 2.) this demand for faithful and robust survival will be satisfied if and only if the spacetime theory in question satisfies either points 3. or $3'$. Namely, $(1. \wedge 2.) \Longrightarrow (P \Longleftrightarrow (3. \vee 3'.))$.

Let us first show the sufficiency of $3. \vee 3'$. If we have point 3. (namely, $H_{\text{kin}} = \{\mathbb{1}\}$) then $H_{\text{kin}} \cap H_{\text{trans}} \subset H_{\text{fix}}$ follows immediately for all H_{trans} and H_{fix} since $\{\mathbb{1}\} \subset H_{\text{fix}}$. Moreover, if we have point $3'$. (namely, $H_{\text{kin}} \cap H_{\text{trans}} = \{\mathbb{1}\}$ for all such H_{trans}) than we automatically have $H_{\text{kin}} \cap H_{\text{trans}} \subset H_{\text{fix}}$ for all H_{trans} and H_{fix} for the same reason, $\{\mathbb{1}\} \subset H_{\text{fix}}$.

Let us next show the necessity of $3'.$, and hence the necessity of $3. \vee 3'$. Let $p_0 \in \mathcal{M}_{\text{old}}$ be the point at which H_{fix} is H_{trans}'s stabilizer subgroup. Note that $H_{\text{kin}} \cap H_{\text{trans}} \subset H_{\text{fix}}$ implies that every element of $H_{\text{kin}} \cap H_{\text{trans}}$ maps p_0 to itself. If this happens for all H_{fix} then it must happen for all p_0. Hence, if we want faithful and robust survival then we must have $H_{\text{kin}} \cap H_{\text{trans}}$ acting trivially everywhere. The only diffeomorphism which acts trivially everywhere is the identity. Hence, if we want faithful and robust survival then we must have $H_{\text{kin}} \cap H_{\text{trans}} = \{\mathbb{1}\}$ for all H_{trans}. Namely, H_{kin} has to have a trivial intersection with every finite-dimensional transitively acting Lie group, $H_{\text{trans}} \subset \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$. But this is just what point $3'$. demands. This completes the proof.

In sum: For a theory's spacetime manifold to faithfully and robustly survive internalization, the theory must be a spacetime theory (point 1.), and it must be kinematically homogeneous (point 2.). Given these two necessary conditions, the faithful and robust survival of $\mathcal{M}_{\text{old}}$ becomes equivalent to demanding something slightly weaker than that the theory in question is kinematically reduced (namely, point 3. vs. 3. $\vee$ 3.$'$). It follows from this that the weakened version of spacetime-kinematic compatibility defined at the top of this section is a necessary and sufficient condition for the faithful and robust survival of $\mathcal{M}_{\text{old}}$.

B.2 Internalizing a Theory with Spacetime-Kinematic Compatibility Yields PSTOs

As Ch. 9 has discussed, internalizing any theory with spacetime-kinematic compatibility yields a pair of Lie groups, $G^{(\text{int})}_{\text{trans}}$ and $G^{(\text{int})}_{\text{fix}}$, acting on the theory's pre-spacetime states, $S^{\text{kin}}_{\text{pst}}$. I will now prove that this pair of Lie group actions automatically qualify as PSTOs. Moreover, by externalizing these PSTOs one can always return to the theory's original spacetime setting up to diffeomorphism.

To begin, recall that internalization first reconstructs the old spacetime, $\mathcal{M}_{\text{old}}$, as a quotient of its kinematically allowed diffeomorphisms, $\mathcal{M}_{\text{old}} \cong H_{\text{trans}}/H_{\text{fix}}$. Specifically, $H_{\text{trans}} \subset \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$ is some Lie group which acts transitively on $\mathcal{M}_{\text{old}}$ with H_{fix} being one of its stabilizer subgroups. Since the theory in question has spacetime-kinematic compatibility, every kinematically allowed diffeomorphism, $d \in \text{Diff}_{\text{kin}}(\mathcal{M}_{\text{old}})$, survives internalization as $G(d) \coloneqq \mathcal{F} \circ d^*|_{\text{kin}} \circ \mathcal{F}^{-1}$. In particular, we have H_{trans} and H_{fix} faithfully surviving internalization becoming Lie groups acting on $S^{\text{kin}}_{\text{pst}}$, namely $G^{(\text{int})}_{\text{trans}} \coloneqq G(H_{\text{trans}}) \cong H_{\text{trans}}$ and $G^{(\text{int})}_{\text{fix}} \coloneqq G(H_{\text{fix}}) \cong H_{\text{fix}}$. Hence, the old theory's spacetime manifold faithfully survives internalization as $\mathcal{M}_{\text{old}} \cong \mathcal{M}_{\text{G}} \coloneqq G^{(\text{int})}_{\text{trans}}/G^{(\text{int})}_{\text{fix}}$.

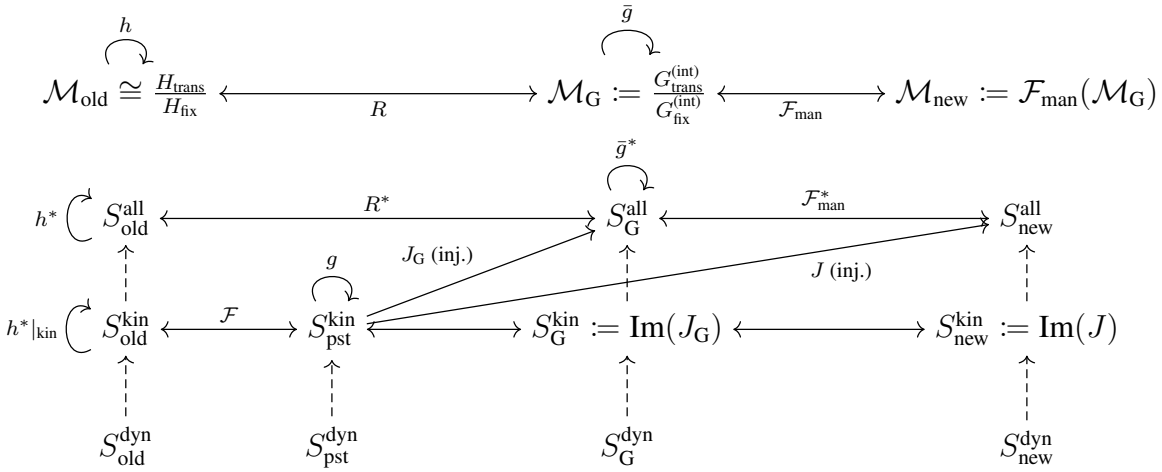

Figure B.1: A diagram to help the reader chase my arrows. The dashed arrows indicate inclusion as a subset, i.e., $S^{\text{dyn}} \subset S^{\text{kin}} \subset S^{\text{all}}$. The h maps are diffeomorphisms on $\mathcal{M}_{\text{old}}$. These lift to bijections, h^*, on $S^{\text{all}}_{\text{old}}$. Some of these are kinematically allowed and so restrict to bijections, $h^*|_{\text{kin}}$, on $S^{\text{kin}}_{\text{old}}$. It follows from the old theory having spacetime-kinematic compatibility that we can reconstruct its spacetime as $\mathcal{M}_{\text{old}} \cong H_{\text{trans}}/H_{\text{fix}}$ for some kinematically allowed Lie groups, $H_{\text{fix}} \subset H_{\text{trans}} \subset \text{Diff}(\mathcal{M}_{\text{old}})$. The $\mathcal{F}$ map implements the internalization process by giving us a bijection between $S^{\text{kin}}_{\text{old}}$ and $S^{\text{kin}}_{\text{pst}}$. The $h^*|_{\text{kin}}$ maps survive internalization becoming maps $g = \mathcal{F} \circ h^*|_{\text{kin}} \circ \mathcal{F}^{-1}$ which act on $S^{\text{kin}}_{\text{pst}}$. Collectively, we have that our pair of Lie groups, $H_{\text{fix}} \subset H_{\text{trans}}$, become a pair of Lie groups, $G^{(\text{int})}_{\text{fix}} \subset G^{(\text{int})}_{\text{trans}}$ acting on $S^{\text{kin}}_{\text{pst}}$. The quotient of these two Lie groups is also a reconstruction of the old theory's spacetime, $\mathcal{M}_{\text{old}} \cong \mathcal{M}_{\text{G}} := G^{(\text{int})}_{\text{trans}}/G^{(\text{int})}_{\text{fix}}$. Hence, there exists a diffeomorphism, $R : \mathcal{M}_{\text{old}} \leftrightarrow \mathcal{M}_{\text{G}}$, which we can lift to a bijection, $R^* : S^{\text{all}}_{\text{old}} \leftrightarrow S^{\text{all}}_{\text{G}}$. The fact this pair of Lie groups are PSTOs is witnessed by the existence of an injective map, $J_{\text{G}} : S^{\text{kin}}_{\text{pst}} \to S^{\text{all}}_{\text{G}}$, with certain properties. The main text proves that $J_{\text{G}} = R^* \circ \mathcal{F}^{-1}$ has the desired properties. It follows from J_{G} being injective that we have a bijection between its domain, $S^{\text{kin}}_{\text{pst}}$ and its image, $J_{\text{G}}(S^{\text{kin}}_{\text{pst}}) =: S^{\text{kin}}_{\text{G}}$. The externalization process is implemented by a closely related injective map, $J : S^{\text{kin}}_{\text{pst}} \to S^{\text{all}}_{\text{new}}$. The J_G and J maps are themselves related as $J = \mathcal{F}^*_{\text{man}} \circ J_{\text{G}}$ where $\mathcal{F}^*_{\text{man}} : S^{\text{all}}_{\text{G}} \leftrightarrow S^{\text{all}}_{\text{new}}$ is the lift of the diffeomorphism, $\mathcal{F}_{\text{man}} : \mathcal{M}_{\text{G}} \leftrightarrow \mathcal{M}_{\text{new}}$.

I will now prove that $G^{(\text{int})}_{\text{trans}}$ and $G^{(\text{int})}_{\text{fix}}$ are PSTOs. Firstly, note that it follows from H_{fix} being a stabilizer subgroup of $H_{\text{trans}} \subset \text{Diff}(\mathcal{M}_{\text{old}})$ that H_{trans} and H_{fix} have faithful construction compatibility (see the Homogeneous Manifold Characterization Theorem proved in Appendix A.5). It follows immediately that $G^{(\text{int})}_{\text{trans}}$ and $G^{(\text{int})}_{\text{fix}}$ have faithful construction compatibility too (because H_{trans} and H_{fix} do). For brevity, the (int) superscripts will now be dropped.

Thus, all that remains to show is that this pair of Lie group actions have Injection Compatibility, i.e., injection compatibility *for some kind of states*. In principle, one can use the ISE Method to switch between theories about different kinds of states (e.g., moving from a theory about a particle trajectory, X_{old}, to a theory about a tangent vector field, $\vec{\varphi}_{\text{new}}$). To demonstrate that $G^{(\text{int})}_{\text{trans}}$ and $G^{(\text{int})}_{\text{fix}}$ are PSTOs however, it will suffice to show that they have injection compatibility with respect to the kind of states that the old theory was about. In fact, I will now show how we can use them to return to the theory's original spacetime setting, up to diffeomorphism. Hence, the following discussion will implicitly assume that the new and old theories are about the same kind of states.

Demonstrating that $G^{(\text{int})}_{\text{trans}}$ and $G^{(\text{int})}_{\text{fix}}$ have injection compatibility requires that we construct an injective map, $J_{\text{G}} : S^{\text{kin}}_{\text{pst}} \rightarrow S^{\text{all}}_{\text{G}}$, with certain properties. Before this, however, we must construct the set $S^{\text{all}}_{\text{G}}$ of all states definable on $\mathcal{M}_{\text{G}}$. Ultimately, this must be done separately in the $\mathcal{V}_{\text{old}}$-paramterized particle trajectory case, X_{old}, the $\mathcal{V}_{\text{old}}$-valued scalar field case, φ_{old}, and the tangent vector field case, $\vec{\varphi}_{\text{old}}$. In the first two cases we can let $\mathcal{V}_{\text{G}} = \mathcal{V}_{\text{old}}$. We can then define $S^{\text{all}}_{\text{G}}$, the set of all $\mathcal{V}_{\text{G}}$-parameterized trajectories on $\mathcal{M}_{\text{G}}$. Similarly, in the scalar field case, $S^{\text{all}}_{\text{G}}$ is the set of all $\mathcal{V}_{\text{G}}$-valued functions definable over $\mathcal{M}_{\text{G}}$. In the tangent vector field case, $\sigma_{\text{old}} = \vec{\varphi}_{\text{old}}$ we do not need to pick out any space, $\mathcal{V}_{\text{G}}$. Instead, we can directly define $S^{\text{all}}_{\text{G}}$ to be the set of all definable sections on the tangent bundle, $\text{T}\mathcal{M}_{\text{G}} \cong \text{T}\mathcal{M}_{\text{old}}$.

In any case, once $S^{\text{all}}_{\text{G}}$ is defined, we then need to pick an injective map, $J_{\text{G}} : S^{\text{kin}}_{\text{pst}} \to S^{\text{all}}_{\text{G}}$, with the following two properties: Firstly, we must have,

$$\forall g \; \forall \sigma_{\text{pst}} \;\; \Big(J_{\text{G}}(g \, \sigma_{\text{pst}}) = \bar{g}^* J_{\text{G}}(\sigma_{\text{pst}}) \Big), \tag{B.5}$$

and, secondly, for all diffeomorphisms, $d_{\text{G}} \in \text{Diff}(\mathcal{M}_{\text{G}})$, we must have,

$$\Big(\forall g \; \forall \sigma_{\text{pst}} \;\; d^*_{\text{G}} \, J_{\text{G}}(\sigma_{\text{pst}}) = J_{\text{G}}(\sigma_{\text{pst}}) \Big) \Longleftrightarrow d_{\text{G}} = \mathbb{1}. \tag{B.6}$$

In both of these expressions (and throughout this section) the ranges of $g \in G_{\text{trans}}$ and $\sigma_{\text{pst}} \in S^{\text{kin}}_{\text{pst}}$ will be suppressed for notational convenience.

To find a map, J_{G}, satisfying the above two conditions, let us first identify the diffeomorphism, $R : \mathcal{M}_{\text{old}} \to \mathcal{M}_{\text{G}}$, which relates the old theory's spacetime to its post-internalization reconstruction. Recall that the internalization process first reconstructed $\mathcal{M}_{\text{old}}$ as $\mathcal{M}_{\text{H}} = H_{\text{trans}}/H_{\text{fix}}$ where H_{fix} is the stablizer subgroup of H_{trans} at $p_0 \in \mathcal{M}_{\text{old}}$. Then post-internalization this was then reconstructed as $\mathcal{M}_{\text{G}} = G_{\text{trans}}/G_{\text{fix}}$. Hence, we are looking for a map R which sends spacetime points in $p \in \mathcal{M}_{\text{old}}$ to its corresponding G_{fix}-cosets in G_{trans}. For notational convenience, let $[h]_{p_0 \mapsto p}$ denote the subset of H_{trans} which maps p_0 to p. It follows from the orbit-stabilizer theorem that $[h]_{p_0 \mapsto p} = h \, H_{\text{fix}}$ is an H_{fix}-coset in H_{trans} (see Appendix A.2). Hence, its image under $G(h) = \mathcal{F} \circ h^*|_{\text{kin}} \circ \mathcal{F}^{-1}$ will be a G_{fix}-coset in G_{trans}, i.e., a point in $\mathcal{M}_{\text{G}}$. The fact that this map

$$R(p) \coloneqq G([h]_{p_0 \mapsto p}) \tag{B.7}$$

is a diffeomorphism $R : \mathcal{M}_{\text{G}} \to \mathcal{M}_{\text{old}}$ follows from the Homogeneous Manifold Characterization Theorem proved in Appendix A.5. For later convenience, let us briefly discuss the inverse diffeomorphism, $R^{-1} : \mathcal{M}_{\text{G}} \to \mathcal{M}_{\text{old}}$, which acts on points $[g] \in \mathcal{M}_{\text{G}}$ as follows,

$$R^{-1}([g]) \coloneqq G^{-1}(g)(p_0) \tag{B.8}$$

where $p_0 \in \mathcal{M}_{\text{old}}$ is the point at which H_{fix} is the stabilizer subgroup of H_{trans}. One might worry that this definition of R^{-1} depends on which representative of $[g]$ we pick. Fortunately, it does not. Recall from above that the G map is the Lie group isomorphism relating H_{trans} and G_{trans}. The G^{-1} map thus relates each G_{fix}-coset in G_{trans} (i.e., each point $[g] \in \mathcal{M}_{\text{G}}$) to an H_{fix}-cosets in H_{trans} (i.e., to a point $[h] \in \mathcal{M}_{\text{H}}$). As I noted above, every element of an H_{fix}-coset maps p_0 to the same place. Hence, the above definition of the R^{-1} isomorphism is well-defined.

In addition to this isomorphism, R, relating $\mathcal{M}_{\text{old}}$ and $\mathcal{M}_{\text{G}}$ we also have another nice relationship between these smooth manifolds. Namely, we can think of G_{trans} as being faithfully represented as diffeomorphisms on both $\mathcal{M}_{\text{old}}$ and $\mathcal{M}_{\text{G}}$. Firstly, it follows from the old theory being kinematically reduced that we have that G_{trans} is faithfully represented as a group of diffeomorphisms on $\mathcal{M}_{\text{old}}$, namely $H_{\text{trans}} \subset \text{Diff}(\mathcal{M}_{\text{old}})$. Secondly, it follows from G_{trans}'s faithful construction compatibility with G_{fix} that it is faithfully represented as a group of diffeomorphisms on $\mathcal{M}_{\text{G}}$, namely $\overline{G_{\text{trans}}} \subset \text{Diff}(\mathcal{M}_{\text{G}})$. Recall from Appendix A that we have $\overline{G_{\text{trans}}} \cong G_{\text{trans}}$ where $\bar{g}([g']) \coloneqq [g\, g']$.

These two faithful representations of G_{trans} as diffeomorphisms (on $\mathcal{M}_{\text{old}}$ and $\mathcal{M}_{\text{G}}$) are naturally related by the $R : \mathcal{M}_{\text{old}} \to \mathcal{M}_{\text{G}}$ isomorphism introduced above. Namely, for each $g \in G_{\text{trans}}$ we have $h = G^{-1}(g) \in H_{\text{trans}}$ and $\bar{g} \in \overline{G_{\text{trans}}}$ related as follows:

$$\forall g \;\; h \circ R^{-1} = R^{-1} \circ \bar{g}. \tag{B.9}$$

This is easy to confirm. Evaluating the right-hand-side at a point $[g'] \in \mathcal{M}_{\text{G}}$ yields,

$$(R^{-1} \circ \bar{g})([g']) = G^{-1}(g\, g')(p_0). \tag{B.10}$$

Likewise, evaluating the left-hand-side at $[g'] \in \mathcal{M}_{\text{G}}$ yields,

$$(h \circ R^{-1})([g']) = (G^{-1}(g)\, G^{-1}(g'))(p_0). \tag{B.11}$$

But these are identical since G is a group isomorphism, $G^{-1}(g\,g') = G^{-1}(g)\,G^{-1}(g')$.

As I will now discuss, we can leverage this relationship between $\overline{G}_{\text{trans}} \subset \text{Diff}(\mathcal{M}_{\text{G}})$ and $H_{\text{trans}} \subset \text{Diff}(\mathcal{M}_{\text{old}})$ to find the desired J_{G} map. Perhaps surprisingly, we can actually handle all of the different kinds of states at the same time. In each case, our theory satisfies the lifting assumption. Therefore we can lift the $R : \mathcal{M}_{\text{old}} \to \mathcal{M}_{\text{G}}$ map to a map $R^* : S^{\text{all}}_{\text{old}} \to S^{\text{all}}_{\text{G}}$. For instance, $R^*\varphi_{\text{old}} \coloneqq \varphi_{\text{old}} \circ R^{-1}$ is a scalar field state on $\mathcal{M}_{\text{G}}$. Similarly, $R^*X_{\text{old}} \coloneqq R \circ X_{\text{old}}$ is a particle trajectory on $\mathcal{M}_{\text{G}}$. I leave it as an exercise for the reader to figure out how R^* should act in the tangent vector field case (and, more generally, the natural bundle case).

I will now prove that the injective map,

$$J_{\text{G}}(\sigma_{\text{pst}}) = R^*\mathcal{F}^{-1}(\sigma_{\text{pst}}), \tag{B.12}$$

has both of the desired properties, namely Eq. (B.5) and Eq. (B.6). First, note that we can rewrite Eq. (B.9) as,

$$\forall g \;\; R \circ h = \bar{g} \circ R. \tag{B.13}$$

For any kinematically allowed state, $\sigma_{\text{old}} \in S^{\text{kin}}_{\text{old}}$, we can apply these as lifted diffeomorphisms yielding,

$$\forall g\, \forall \sigma_{\text{old}} \;\; (R \circ h)^*\sigma_{\text{old}} = (\bar{g} \circ R)^*\sigma_{\text{old}}. \tag{B.14}$$

Recall that we have a bijection, $\mathcal{F} : S^{\text{kin}}_{\text{old}} \leftrightarrow S^{\text{kin}}_{\text{pst}}$, relating $\mathcal{F} : \sigma_{\text{old}} \mapsto \sigma_{\text{pst}}$. Hence, we have,

$$\forall g\, \forall \sigma_{\text{pst}} \;\; R^*h^*\mathcal{F}^{-1}(\sigma_{\text{pst}}) = \bar{g}^*R^*\mathcal{F}^{-1}(\sigma_{\text{pst}}). \tag{B.15}$$

Recalling that g and $h = G^{-1}(g)$ are related as $g = G(h) = \mathcal{F} \circ h^* \circ \mathcal{F}^{-1}$ we then have,

$$\forall g\, \forall \sigma_{\text{pst}} \;\; R^*\mathcal{F}^{-1}(g\,\sigma_{\text{pst}}) = \bar{g}^*(R^*\mathcal{F}^{-1}(\sigma_{\text{pst}})). \tag{B.16}$$

This establishes that $J_{\mathrm{G}} = R^* \circ \mathcal{F}^{-1}$ satisfies Eq. (B.5).

Similarly, one can prove Eq. (B.6) for this $J_{\mathrm{G}}(\sigma_{\mathrm{pst}})$ map as follows. Given that the old theory is kinematically reduced, we have $H_{\mathrm{kin}} = \{\mathbb{1}\}$. From this it follows that for all $d_{\mathrm{old}} \in \mathrm{Diff}(\mathcal{M}_{\mathrm{old}})$ we have,

$$\Big(\forall \sigma_{\mathrm{old}} \;\; d^*_{\mathrm{old}} \sigma_{\mathrm{old}} = \sigma_{\mathrm{old}}\Big) \iff d_{\mathrm{old}} = \mathbb{1}. \tag{B.17}$$

Taking $\sigma_{\mathrm{old}} = \mathcal{F}^{-1}(\sigma_{\mathrm{pst}}) = (R^{-1})^* J_{\mathrm{G}}(\sigma_{\mathrm{pst}})$ and applying R^* we then have,

$$\Big(\forall \sigma_{\mathrm{pst}} \;\; R^* \, d^*_{\mathrm{old}} \, (R^{-1})^* \, J_{\mathrm{G}}(\sigma_{\mathrm{pst}}) = J_{\mathrm{G}}(\sigma_{\mathrm{pst}})\Big) \iff d_{\mathrm{old}} = \mathbb{1}, \tag{B.18}$$

or equivalently,

$$\Big(\forall \sigma_{\mathrm{pst}} \;\; (R \circ d_{\mathrm{old}} \circ R^{-1})^* \, J_{\mathrm{G}}(\sigma_{\mathrm{pst}}) = J_{\mathrm{G}}(\sigma_{\mathrm{pst}})\Big) \iff d_{\mathrm{old}} = \mathbb{1}. \tag{B.19}$$

The diffeomorphisms, d_{old}, on $\mathcal{M}_{\mathrm{old}}$ are in one-to-one correspondence with diffeomorphisms, $d_{\mathrm{G}} = R \circ d_{\mathrm{old}} \circ R^{-1}$, on $\mathcal{M}_{\mathrm{G}}$. Consequently, we then have that for all $d_{\mathrm{G}} \in \mathrm{Diff}(\mathcal{M}_{\mathrm{G}})$,

$$\Big(\forall \sigma_{\mathrm{pst}} \;\; d^*_{\mathrm{G}} \, J_{\mathrm{G}}(\sigma_{\mathrm{pst}}) = J_{\mathrm{G}}(\sigma_{\mathrm{pst}})\Big) \iff d_{\mathrm{G}} = \mathbb{1}, \tag{B.20}$$

as desired. This completes the proof that $G^{(\mathrm{int})}_{\mathrm{trans}}$ and $G^{(\mathrm{int})}_{\mathrm{fix}}$ are PSTOs when the old theory has spacetime-kinematic compatibility.

Let us next see where externalizing these PSTOs can take us. The map which is used in externalization, $J : S^{\mathrm{kin}}_{\mathrm{pst}} \to S^{\mathrm{all}}_{\mathrm{new}}$ is related to the map $J_{\mathrm{G}} : S^{\mathrm{kin}}_{\mathrm{pst}} \to S^{\mathrm{all}}_{\mathrm{G}}$ discussed above as follows, $J(\sigma_{\mathrm{pst}}) = \mathcal{F}^*_{\mathrm{man}} J_{\mathrm{G}}(\sigma_{\mathrm{pst}})$. Recall from Ch. 10, that $\mathcal{F}_{\mathrm{man}} : \mathcal{M}_{\mathrm{G}} \to \mathcal{M}_{\mathrm{new}}$ is the diffeomorphism relating $\mathcal{M}_{\mathrm{G}}$ to $\mathcal{M}_{\mathrm{new}}$. Picking the map J_{G} map discussed above, we have the states of the new theory, σ_{new}, are related to the states of the old theory, σ_{old}, as follows:

$$\sigma_{\mathrm{new}} = J \circ \mathcal{F}\,(\sigma_{\mathrm{old}}) = (\mathcal{F}_{\mathrm{man}} \circ R)^* \sigma_{\mathrm{old}}, \tag{B.21}$$

where $\mathcal{F}_{\text{man}} \circ R : \mathcal{M}_{\text{old}} \to \mathcal{M}_{\text{new}}$ is an diffeomorphism between $\mathcal{M}_{\text{old}}$ and $\mathcal{M}_{\text{new}}$. Hence, this choice of J directly transplants the old theory's states (and their dynamics) onto the new spacetime manifold. We have thus recovered the old spacetime theory up to diffeomorphism.