SUPPLEMENTARY MATERIALS: Multi-hump Collapsing Solutions in the Nonlinear Schrödinger Problem: Existence, Stability and Dynamics [*]

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SM1. Comparison with finite element computations. In this section, we compare numerical computations presented in the main text of our manuscript with those obtained using the open-source, finite-element software, `FreeFEM` [SM3]. Figure SM1 shows a comparison of two-humped solutions computed numerically with finite differences and P3 finite elements. The blow-up rate is $G = 0.01$ and the computational domain: $\xi \in [-300, 300]$. Both solutions appear visually indistinguishable. For the finite-difference computation, the domain is discretized using equidistant nodes with a nodal spacing of $\Delta\xi = 0.06$ (i.e., $10,000$ points) whereas for the finite-element, $10,000$ nodes were considered on $[-300, 300]$. In addition to the solution comparison, we also present the spectra computed using MATLAB's `eigs` function and `FreeFEM`'s eigensolver (ARPACK). This comparison demonstrates that both eigensolvers yield nearly identical eigenvalues.

SM2. Full domain dynamical simulations. In this section, we present dynamical simulations performed on the full domain, i.e., $\xi \in [-K, K]$. When simulations are performed on $[-K, K]$, convergence of an unstable multi-hump state to a steady-state corresponding to a single-humped solution is not observed. The initial data evolves toward the corresponding single-humped self-similar state; however the $\lambda = G$ associated eigenmode drives the solution away from this final state. Figure SM2 illustrates the $\xi - \tau$ spatiotemporal evolution of an initial two-humped self-similar solution computed at $\sigma = 2.01$, slightly perturbed along the symmetric eigenmode corresponding to the second largest eigenvalue. After an initial τ period spanning approximately $[0, 6]$, where the solution visibly restructures toward a single-humped state, then at $\tau \approx 40$, v starts shifting from the origin to positive ξ values. This shift is attributed to the $\lambda = G$ eigenvalue of the single-humped self-similar solution. In Fig-

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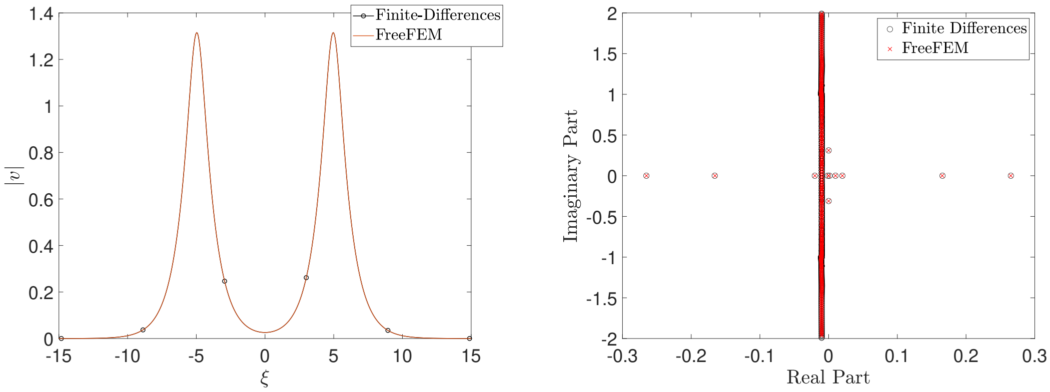
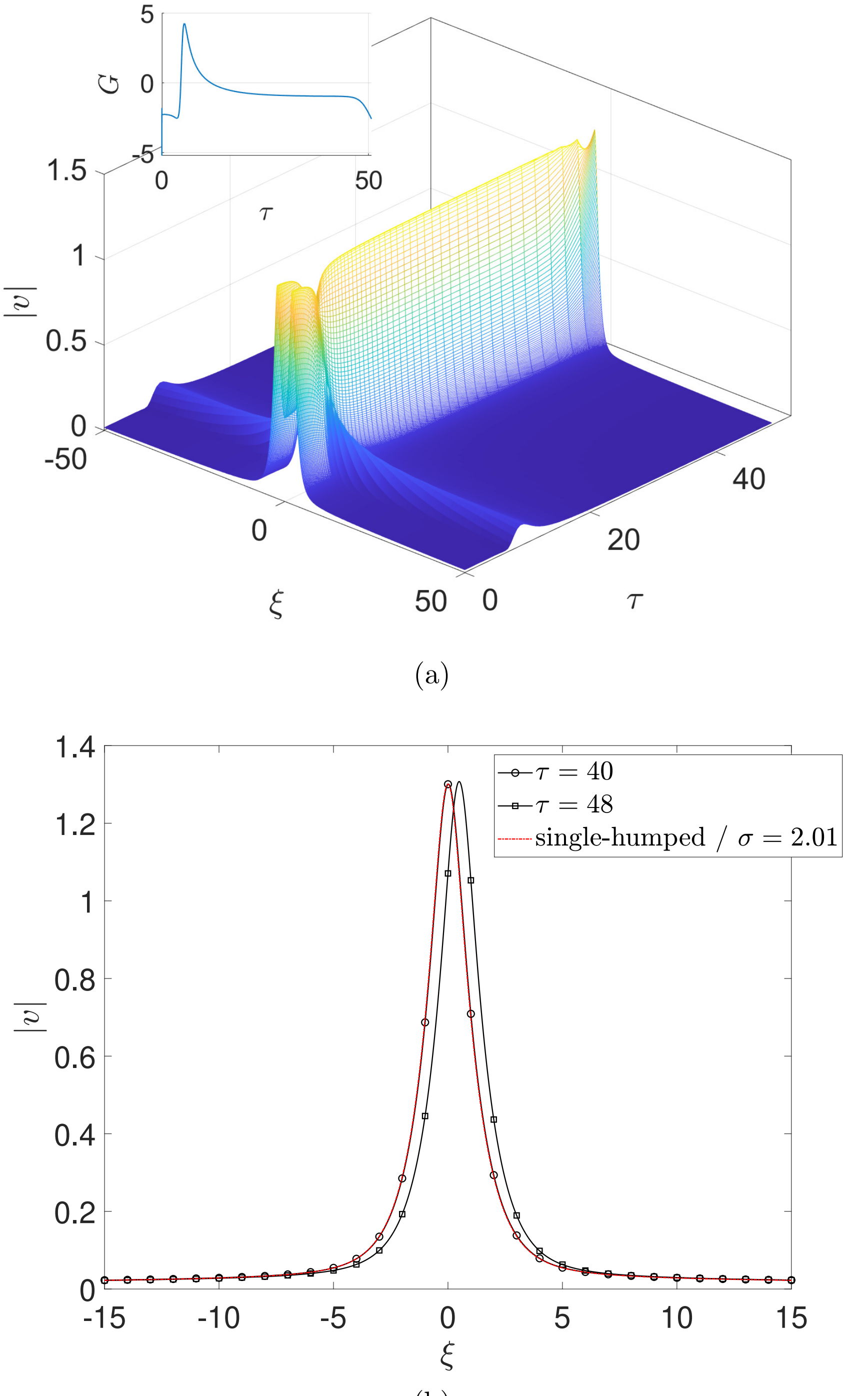


(a) (b)

Figure SM1: (a) Finite-difference (black line with open circles) and finite-element (red line) two-humped solutions with a blow-up rate of $G = 0.01$. The two solutions are visually indistinguishable. The computational domain $\xi \in [-300, 300]$ is discretized using equidistant nodes with a spacing of $\Delta\xi = 0.06$ (10,000 nodes). (b) Spectra comparison between the finite-difference discretization and MATLAB's `eigs` function (black circles) and `FreeFEM`'s ARPACK eigensolver (red crosses).

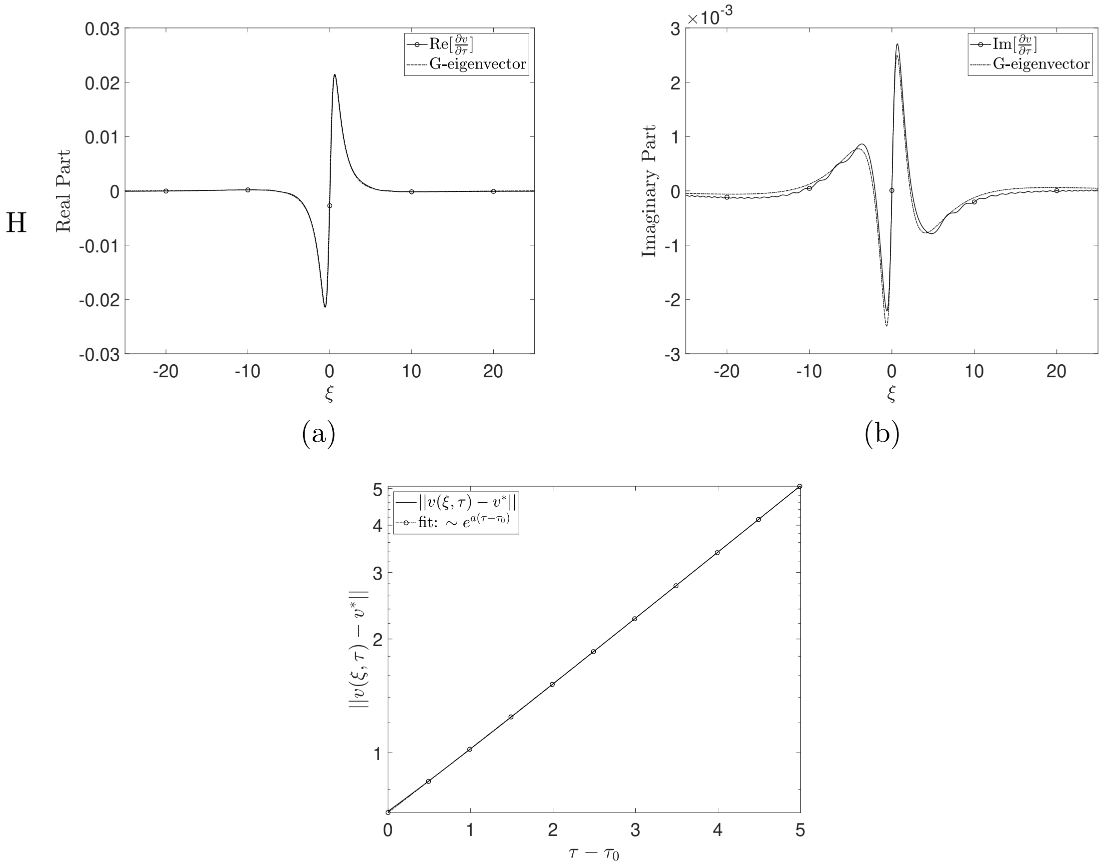
ure SM3(a)-(b), we compare the numerically approximated value of $\frac{\partial v}{\partial \tau}$ at $\tau = 48$ with the $\lambda = G$-associated eigenmode of the single-humped self-similar solution ((a) real part and (b) imaginary part). The comparison shows good agreement, indicating that v shifts in the direction of the $\lambda = G$-associated eigenmode. The blow-up rate for a single-humped solution at $\sigma = 2.01$ is $G = 0.403$. In Figure SM3(c), we plot the evolution of the deviation $||v(\xi,\tau) - v^*||$, with v^* the single-humped self-similar solution at $\sigma = 2.01$. Our starting point for measuring the deviation is $\tau_0 = 40$. This deviation is expected to grow exponentially in rescaled time, following: $||v(\xi,\tau) - v^*|| \sim e^{a(\tau-\tau_0)}$, where $a \approx G$. Indeed, fitting yields $a \approx 0.4085$ which is close to the expected value of the blow-up rate for a single-humped solution at $\sigma = 2.01$ (which, as mentioned above, is $G = 0.403$).



(a)

(b)

Figure SM2: (a) Dynamics in the renormalized/rescaled spatio-temporal framework $\xi - \tau$: Starting from a two-humped solution at $\sigma = 2.01$ and perturbing along the eigenfunction that corresponds to the second largest real eigenvalue the solution evolves toward the corresponding single-humped solution after an initial "restructuring" τ-period. Convergence is lost at around $\tau = 50$. The inset illustrates the evolution of blow-up rate, G. (b) To enhance visualization of the shift from the origin at later stages of the simulation, we plot two snapshots: one at $\tau = 40$ (solid line with open circles), where the solution is visually indistinguishable from the single-humped self-similar solution (red dashed line), and one at $\tau = 48$, where the solution has visibly shifted from the origin. $\xi \in [-50, 50]$, nodal distance: $\Delta x = 0.001$, and $\Delta\tau = 0.01$.



(a) (b)

(c)

Figure SM3: Comparison of solution's (a) real part and (b) imaginary part, partial τ derivative (computed at $\tau = 42$) with the real and imaginary part of the G-eigenvector indicating that the instability presented in Figure SM2 leads to growth along the direction of the G-eigenvector. (c) Exponential evolution of solution's v deviation from the single-humped solution, v^*, $||v(\xi,\tau) - v^*||$. The starting time to measure deviation is $\tau_0 = 40$. Here, $a \approx 0.4085$, which is close to the blow-up rate of the single-humped solution: $G = 0.403$.

In Figure SM4(a), we present a second simulation initialized with the two-humped solution for $\sigma = 2.01$. This time, no perturbation is directly applied to the initial condition, and we perform a direct numerical simulation with the roundoff error providing the initial perturbation. The unstable self-similar solution retains its initial shape for a time interval $\tau \in [0, 18]$ before gradually losing its symmetry in the later stages of the simulation. At this point, we observe an asymmetric shape for v and the blow-up rate that trends toward large negative values. The solution begins to deviate in the direction of the largest real eigenvalue of the two-humped solution. This behavior is illustrated in Figure SM4(b)-(c), where we plot the time-derivative of v (real and imaginary part) computed at $\tau = 18$ and compare against the eigenmode which is associated with the largest real eigenvalue of the two-humped solution. If we fit the exponential growth of the deviation from the two-humped solution v^*_{2h}, $E \equiv ||v(\xi, \tau - v^*_{2h}|| \sim e^{a(\tau-\tau_0)}$, then a is expected to be close to λ_1. This fitting is shown in Figure SM4(d), where $a = 1.2606$ close enough to $\lambda_1 = 1.245$. We start measuring the deviation at $\tau_0 = 18$.

SM3. Higher order terms in the bifurcation diagram.

To get a good quantitative analysis in Figure 4of the main manuscript when G is not so small we need to include higher-order terms in Eq. (4.23).We can do so by following a similar procedure to that by which we determined the higher-order terms for the one-hump solution in [SM2]. Many of the algebraic manipulations in this and the following sections were aided by Wolfram Mathematica.

SM3.1. Higher-order terms in the far field.

We proceed to calculate more terms in the expansion (4.16) of subsection 4.4. The calculation is similar to that in [SM2]. The equation for A_1 is

$$2A_1'\phi' + A_0'' + A_1\phi'' = 0,$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}\rho}A_1(-\phi')^{1/2} = \frac{A_0''}{2(-\phi')^{1/2}} = \frac{a_0(8+3\rho^2)}{4(4-\rho^2)^{5/2}}.$$

Thus,

$$A_1 = \frac{a_0\rho(24-\rho^2)}{24\sqrt{2}\,(4-\rho^2)^{7/4}} + \frac{2^{1/2}a_1}{(4-\rho^2)^{1/4}}.$$

Matching the near field gives $a_1 = 0$. At the next order

$$2A_2'\phi' + A_1'' + A_2\phi'' = 0,$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}\rho}A_2(-\phi')^{1/2} = \frac{A_1''}{2(-\phi')^{1/2}} = \frac{a_0\rho(3648+640\rho^2-3\rho^4)}{192(\rho^2-4)^4}.$$

Thus

$$A_2 = \frac{a_0(2320-996\rho^2+9\rho^4)}{576\sqrt{2}\,(4-\rho^2)^{13/4}} + \frac{2^{1/2}a_2}{(4-\rho^2)^{1/4}}.$$

As $\rho \to 0$,

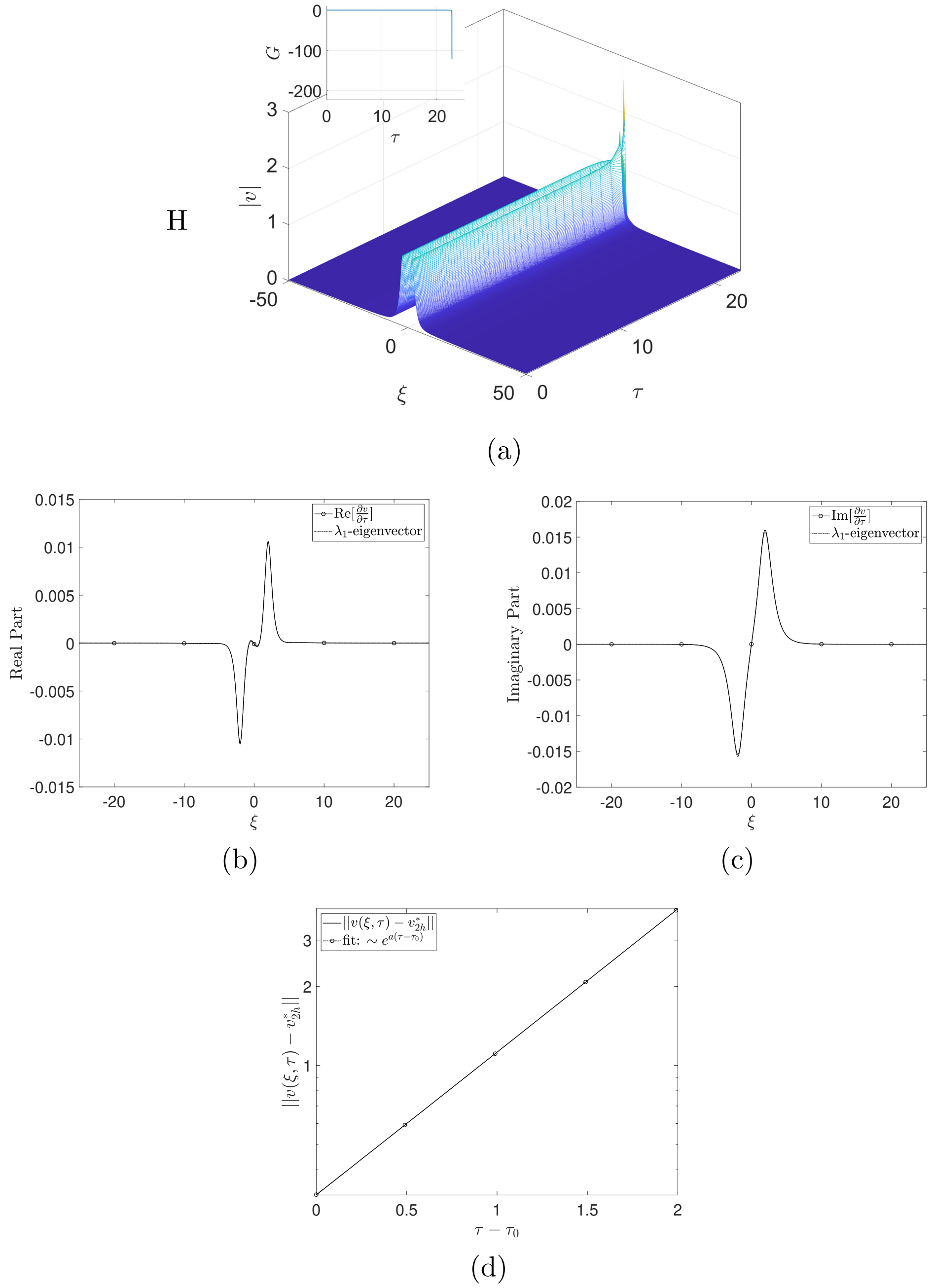
$$A_2 \to \frac{145a_0}{4608} + a_2. \tag{SM3.1}$$

SM3.2. Higher-order terms in outer limit of the inner.

From section SM5 as $x \to \infty$, the asymptotic behaviour of $V_1 = V_o + V_e$ is given by

$$V_o \sim 3^{1/4}\frac{X_i}{16}\sqrt{2}\,\mathrm{e}^{-x}\left(2x^2+2x-2\log 2+1\right), \tag{SM3.2}$$

$$V_e \sim 12^{1/4}(\alpha_0+\alpha_1 x+\alpha_2 x^2+\alpha_3 x^3)\mathrm{e}^{-x} - \frac{X_i^2}{16}3^{1/4}\sqrt{2}\,(1-2x)\,\mathrm{e}^{-x}, \tag{SM3.3}$$



(a)

(b)

(c)

(d)

Figure SM4: (a) Dynamics in the renormalized/rescaled spatio-temporal framework $\xi - \tau$: Starting from a two-humped solution at $\sigma = 2.01$, we observe that the dynamics for the time interval $[0, 18]$ are slow. The accumulation of numerical errors drives the solution to asymmetric profiles for $\tau > 20$, while the blow-up rate, G (shown in the inset) shifts towards large negative values, and convergence is lost at around $\tau \approx 23$ [in line with the blow-up of such solutions also reported in [SM1]]. (b)-(c) Comparison of solution's (rescaled-τ) time derivative ((b) real part and (c) imaginary part) computed at $\tau = 18$ with the real and imaginary part of the eigenvector associated with the largest real eigenvalue of the two-humped solution, $\lambda_1 = 1.245$. The agreement indicates that the deviation grows along the direction of the λ_1 eigenvector. (d) Exponential evolution of solution's v deviation from the two-humped solution, v^*_{2h}, $||v(\xi,\tau) - v^*_{2h}||$. The starting time to measure deviation is $\tau_0 = 18$. Here, $a \approx 1.26$, which is close to the largest real eigenvalue, λ_1 of the two-humped solution.

with α_i, $i = 0, 1, 2, 3$ given by (SM5.9). With $x = \xi - X_i$, The coefficient of $\mathrm{e}^{-\xi}$ is

$$\left(3^{1/4}\frac{X_i}{16}\sqrt{2}\,(2X_i^2 - 2X_i - 2\log 2 + 1) + 12^{1/4}(\alpha_0 - \alpha_1 X_i + \alpha_2 X_i^2 - \alpha_3 X_i^3)\right.$$
$$\left. - \frac{X_i^2}{16}3^{1/4}\sqrt{2}(1 + 2X_i)\right)\mathrm{e}^{X_i}.$$

Matching with (SM3.1) gives

$$a_2 = \left(3^{1/4}\frac{X_i}{16}\sqrt{2}\,(2X_i^2 - 2X_i - 2\log 2 + 1) + 12^{1/4}(\alpha_0 - \alpha_1 X_i + \alpha_2 X_i^2 - \alpha_3 X_i^3)\right.$$
$$\left. - \frac{X_i^2}{16}3^{1/4}\sqrt{2}(1 + 2X_i)\right)\mathrm{e}^{X_i} - \frac{145a_0}{4608}.$$

SM3.3. Higher-order terms in the integral of V_s^2. The final step in finding higher-order terms in (4.21) is to get a better approximation to the right-hand side, for which we need the integral of V_s^2. If we split the integral we have

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{X_1 - R} + \int_{X_1 - R}^{X_1 + R} + \int_{X_1 + R}^{X_2 - R} + \cdots + \int_{X_n + R}^{\infty}.$$

Now, using the inner expansion,

$$\begin{aligned}
\int_{X_i - R}^{X_i + R} V_s^2 \,\mathrm{d}\xi &= \int_{-R}^{R} (V_0^2 + 2G^2 V_0 V_1 + \cdots)\,\mathrm{d}x \\
&\sim \frac{\sqrt{3}\pi}{2} + 2G^2 \int_{-R}^{R} V_{\text{even}} V_0 \,\mathrm{d}x \\
&\sim \frac{\sqrt{3}\pi}{2} + 2G^2 \int_{-R}^{R} \left(\frac{3^{1/4}}{2}\left(\frac{\mathrm{e}^{X_i - X_{i+1}}}{G^2} + \frac{\mathrm{e}^{X_{i-1} - X_i}}{G^2}\right) v_2(x)\right. \\
&\quad \left. + \hat{V}_1(x) - \frac{X_i^2}{4}\left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x}\right)\right) V_0 \,\mathrm{d}x \\
&= \frac{\sqrt{3}\pi}{2} + 2G^2 \frac{\sqrt{3}\pi^3}{256} + 3^{1/4}\left(\mathrm{e}^{X_i - X_{i+1}} + \mathrm{e}^{X_{i-1} - X_i}\right) \int_{-R}^{R} v_2 V_0 \,\mathrm{d}x \\
&= \frac{\sqrt{3}\pi}{2} + 2G^2 \frac{\sqrt{3}\pi^3}{256} + 3^{1/4}\left(\mathrm{e}^{X_i - X_{i+1}} + \mathrm{e}^{X_{i-1} - X_i}\right) \left[2 \times 3^{1/4}(x - \tanh 2x)\right]_{-R}^{R} \\
&= \frac{\sqrt{3}\pi}{2} + G^2 \frac{\sqrt{3}\pi^3}{128} + 4\sqrt{3}\left(\mathrm{e}^{X_i - X_{i+1}} + \mathrm{e}^{X_{i-1} - X_i}\right)(R - 1).
\end{aligned}$$

Using the outer (between humps) expansion,

$$\begin{aligned}
\int_{X_i + R}^{X_{i+1} - R} V_s^2 \,\mathrm{d}\xi &= \int_{X_i + R}^{X_{i+1} - R} (V_0^2 + 2G^2 V_1 V_0 + \cdots)\,\mathrm{d}\xi \\
&= \int_{X_i + R}^{X_{i+1} - R} 2\sqrt{3}(\mathrm{e}^{-\xi + X_i} + \mathrm{e}^{\xi - X_{i+1}})^2 + 2G^2 12^{1/4} V_1 (\mathrm{e}^{-\xi + X_i} + \mathrm{e}^{\xi - X_{i+1}}) + \cdots)\,\mathrm{d}\xi \\
&\sim -8\sqrt{3}\,R\mathrm{e}^{X_i - X_{i+1}} + 4\sqrt{3}(X_{i+1} - X_i)\mathrm{e}^{X_i - X_{i+1}} + \cdots.
\end{aligned}$$

Thus, together

$$\begin{aligned}
\int_{-\infty}^{\infty} V_s^2\,\mathrm{d}x \sim{}& \frac{\sqrt{3}\pi}{2}\left(1+\frac{G^2\pi^2}{64}\right)n + 4\sqrt{3}(R-1)\sum_{i=1}^{n}\left(\mathrm{e}^{X_i-X_{i+1}}+\mathrm{e}^{X_{i-1}-X_i}\right)\\
&- 8\sqrt{3}\,R\sum_{i=1}^{n-1}\mathrm{e}^{X_i-X_{i+1}} + 4\sqrt{3}\sum_{i=1}^{n-1}(X_{i+1}-X_i)\mathrm{e}^{X_i-X_{i+1}}\\
\sim{}& \frac{\sqrt{3}\pi}{2}\left(1+\frac{G^2\pi^2}{64}\right)n + 4\sqrt{3}(R-1)\sum_{i=1}^{n-1}\mathrm{e}^{X_i-X_{i+1}}\\
&+ 4\sqrt{3}(R-1)\sum_{i=1}^{n-1}\mathrm{e}^{X_i-X_{i+1}} - 8\sqrt{3}\,R\sum_{i=1}^{n-1}\mathrm{e}^{X_i-X_{i+1}}\\
&+ 4\sqrt{3}\sum_{i=1}^{n-1}(X_{i+1}-X_i)\mathrm{e}^{X_i-X_{i+1}}\\
\sim{}& \frac{\sqrt{3}\pi}{2}\left(1+\frac{G^2\pi^2}{64}\right)n + 4\sqrt{3}\sum_{i=1}^{n-1}(X_{i+1}-X_i-2)\mathrm{e}^{X_i-X_{i+1}}.
\end{aligned}$$

SM3.4. Normal form.

With $X_1=-X_n$ (4.21) now becomes

$$\begin{aligned}
4\sqrt{3}\mathrm{e}^{2X_n}\mathrm{e}^{-\pi/G}\Bigg(1+G^2\Bigg(&\frac{X_i}{16}(2X_i^2-2X_i-2\log 2+1)\\
&+(\alpha_0-\alpha_1X_i+\alpha_2X_i^2-\alpha_3X_i^3)-\frac{X_i^2}{16}(1+2X_i)\Bigg)\mathrm{e}^{X_i}-\frac{145G^2}{4608}\Bigg)^2\\
&= \frac{(\sigma-2)G}{2\sigma}\left(\frac{\sqrt{3}\pi}{2}\left(1+\frac{G^2\pi^2}{64}\right)n+4\sqrt{3}\sum_{i=1}^{n-1}(X_{i+1}-X_i-2)\mathrm{e}^{X_i-X_{i+1}}\right).
\end{aligned} \tag{SM3.4}$$

This is what is plotted in Figure 4.

SM4. Detailed asymptotic analysis of the eigenvalues $\lambda = O(G)$.

With the expansions (5.34)-(5.38), equating coefficients at $O(G)$ we have

$$f_1 = a_i^{(1)}U_0 + b_i^{(0)}W_1,$$

where

$$\frac{\mathrm{d}^2W_1}{\mathrm{d}x^2}+2V_0^4W_1^*+3V_0^4W_1-W_1=\lambda_1V_0.$$

The solvability conditions on W_1 are satisfied automatically, and

$$W_1=\lambda_1\left(\frac{V_0}{4}+\frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x}\right).$$

Equating coefficients at $O(G^2)$ we have

$$f_2 = a_i^{(1)}U_1 + b_i^{(2)}W_0 + b_i^{(0)}W_2,$$

where

$$\frac{\mathrm{d}^2 U_1}{\mathrm{d}x^2} + 2V_0^4 U_1^* + 3V_0^4 U_1 - U_1 = -\mathrm{i}\lambda_1 \frac{\mathrm{d}V_0}{\mathrm{d}x},$$

$$\frac{\mathrm{d}^2 W_2}{\mathrm{d}x^2} + V_0^4(2W_2^* + 3W_2) - W_2 = -\mathrm{i}\lambda_1^2\left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x}\right) + \lambda_2 V_0 - \mathrm{i}\frac{(x+X_i)^2}{4}V_0 - 4\mathrm{i}V_1V_0^4.$$

The solvability conditions on U_1 are automatically satisfied, while those on W_2 are

$$\operatorname{Im}\left[\frac{\mathrm{d}W_2}{\mathrm{d}x}V_0 - W_2\frac{\mathrm{d}V_0}{\mathrm{d}x}\right]_{-R}^{R} = \left[\frac{\mathrm{d}V_1}{\mathrm{d}x}V_0 - V_1\frac{\mathrm{d}V_0}{\mathrm{d}x}\right]_{-R}^{R}, \tag{SM4.1}$$

$$\operatorname{Re}\left[\frac{\mathrm{d}W_2}{\mathrm{d}x}\frac{\mathrm{d}V_0}{\mathrm{d}x} - W_2\frac{\mathrm{d}^2V_0}{\mathrm{d}x^2}\right]_{-R}^{R} = 0. \tag{SM4.2}$$

The solutions are

$$U_1 = -\frac{\mathrm{i}\lambda_1 x}{2}V_0, \qquad W_2 = -\frac{\mathrm{i}\lambda_1^2 x^2 V_0}{8} + \mathrm{i}V_1 + \lambda_2\left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x}\right). \tag{SM4.3}$$

Note that W_2 grows at infinity, as does V_1, so that to evaluate (SM4.1) we need to match with the region between humps. In preparation for matching we list the limiting behaviour of the near-hump solutions as $x \to \pm\infty$:

$$U_0 \sim \mp 12^{1/4}\mathrm{e}^{\mp x} \quad \text{as } x \to \pm\infty, \qquad W_0 \sim 12^{1/4}\mathrm{i}\mathrm{e}^{\mp x} \quad \text{as } x \to \pm\infty,$$

$$U_1 \sim -\frac{12^{1/4}\mathrm{i}}{2}x\mathrm{e}^{\mp x} \quad \text{as } x \to \pm\infty, \qquad W_1 \sim 12^{1/4}\left(\frac{1}{4} \mp \frac{x}{2}\right)\mathrm{e}^{\mp x} \quad \text{as } x \to \mp\infty.$$

Thus,

$$f_0 \sim 12^{1/4}\mathrm{i}b_i^{(0)}\mathrm{e}^{\mp x} \quad \text{as } x \to \pm\infty, \qquad f_1 \sim 12^{1/4}b_i^{(0)}\left(\frac{1}{4} \mp \frac{x}{2}\right)\mathrm{e}^{\mp x} \mp 12^{1/4}a_i^{(1)}\mathrm{e}^{\mp x} \quad \text{as } x \to \pm\infty.$$

SM4.1. In between the humps.

As usual, away from the humps,

$$\mathrm{i}\lambda f + \frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} - f + \frac{G^2\xi^2}{4}f = O(G^8 \log G^4 f).$$

Consider the gap between X_i and X_{i+1}. At leading order

$$\frac{\mathrm{d}^2 f_0}{\mathrm{d}\xi^2} - f_0 = 0.$$

Matching requires

$$f_0 \sim 12^{1/4}\mathrm{i}b_{i+1}^{(0)}\mathrm{e}^{\xi - X_{i+1}} \quad \text{as } \xi \to X_{i+1}, \qquad f_0 \sim 12^{1/4}\mathrm{i}b_i^{(0)}\mathrm{e}^{-\xi + X_i} \quad \text{as } \xi \to X_i.$$

Thus,

$$f_0 = 12^{1/4}\mathrm{i}b_{i+1}^{(0)}\mathrm{e}^{\xi - X_{i+1}} + 12^{1/4}\mathrm{i}b_i^{(0)}\mathrm{e}^{-\xi + X_i}.$$

In terms of $\xi = X_{i+1} + x$ this is

$$f_0 = 12^{1/4}\mathrm{i}b_{i+1}^{(0)}\mathrm{e}^{x} + 12^{1/4}\mathrm{i}b_i^{(0)}\mathrm{e}^{-x+X_i-X_{i+1}},$$

giving the matching condition (translating to $i+1 \to i$)

$$G^2 f_2 \sim 12^{1/4}\mathrm{i}b_{i-1}^{(0)}\mathrm{e}^{-x+X_{i-1}-X_i} \qquad \text{as } x \to -\infty \tag{SM4.4}$$

on the local solution near the ith hump. Similarly we find

$$G^2 f_2 \sim 12^{1/4}\mathrm{i}b_{i+1}^{(0)}\mathrm{e}^{x+X_i-X_{i+1}} \qquad \text{as } x \to \infty \tag{SM4.5}$$

on the local solution near the ith hump. Since

$$f_2 = a_i^{(1)}U_1 + b_i^{(2)}W_0 + b_i^{(0)}W_2,$$

and the local hump solutions for U_1 and W_0 decay at infinity, matching gives

$$\mathrm{Im}\left[b_i^{(0)}\frac{\mathrm{d}W_2}{\mathrm{d}x}V_0 - b_i^{(0)}W_2\frac{\mathrm{d}V_0}{\mathrm{d}x}\right]_{-R}^{R} = \frac{4\sqrt{3}\,b_{i+1}^{(0)}}{G^2}\mathrm{e}^{X_i-X_{i+1}} + \frac{4\sqrt{3}\,b_{i-1}^{(0)}}{G^2}\mathrm{e}^{X_{i-1}-X_i}, \tag{SM4.6}$$

$$\mathrm{Re}\left[b_i^{(0)}\frac{\mathrm{d}W_2}{\mathrm{d}x}\frac{\mathrm{d}V_0}{\mathrm{d}x} - b_i^{(0)}W_2\frac{\mathrm{d}^2V_0}{\mathrm{d}x^2}\right]_{-R}^{R} = 0. \tag{SM4.7}$$

Using (SM4.6), (SM4.7) and (5.17) in (SM4.1)-(SM4.2) gives

$$b_i^{(0)}\left(\frac{4\sqrt{3}}{G^2}\mathrm{e}^{-X_{i+1}+X_i} + \frac{4\sqrt{3}}{G^2}\mathrm{e}^{-X_i+X_{i-1}}\right) = \frac{4\sqrt{3}\,b_{i+1}^{(0)}}{G^2}\mathrm{e}^{X_i-X_{i+1}} + \frac{4\sqrt{3}\,b_{i-1}^{(0)}}{G^2}\mathrm{e}^{X_{i-1}-X_i}. \tag{SM4.8}$$

This is equation (5.39) from the main text. Equation (SM4.8) fixes $b_i^{(0)} = b^{(0)}$ for all i.

SM4.2. The outer limit of f_2. We will need to know the outer limit of f_2, for which we need to know the behaviour of W_2 as $x \to \pm\infty$. We have just evaluated the exponentially growing component by matching, but we will find that unfortunately we need also the exponentially decaying component. This requires the asumptotic behaviour of V_1. Note that there are also higher-order terms arising from the expansion of f_0 and f_1 as $x \to \pm\infty$, but these don't contribute until $O(\mathrm{e}^{-5x}) = O(G^{10})$ so we will ignore them.

We show in section SM5 that, as $x \to -\infty$,

$$V_1 \sim 3^{1/4}\frac{X_i}{16}\sqrt{2}\,\mathrm{e}^{x}\left(-2x^2 + 2x + 2\log 2 - 1\right) + 3^{1/4}\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\sqrt{2}\,\mathrm{e}^{-x} + 12^{1/4}(\alpha_0 - \alpha_1 x + \alpha_2 x^2 - \alpha_3 x^3)\mathrm{e}^{x} - \frac{X_i^2}{16}3^{1/4}\sqrt{2}\,(1+2x)\,\mathrm{e}^{x}, \tag{SM4.9}$$

while, as $x \to \infty$,

$$\begin{aligned} V_1 \sim{}& 3^{1/4}\frac{X_i}{16}\sqrt{2}\,\mathrm{e}^{-x}\left(2x^2+2x-2\log 2+1\right)+3^{1/4}\sqrt{2}\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}\mathrm{e}^{x} \\ &+12^{1/4}(\alpha_0+\alpha_1x+\alpha_2x^2+\alpha_3x^3)\mathrm{e}^{-x}-\frac{X_i^2}{16}3^{1/4}\sqrt{2}\,(1-2x)\,\mathrm{e}^{-x}. \end{aligned} \tag{SM4.10}$$

Thus

$$\begin{aligned} W_2 \sim{}& -\frac{\mathrm{i}\lambda_1^2x^2}{8}12^{1/4}\mathrm{e}^{-x}+12^{1/4}\mathrm{i}\frac{X_i}{16}\,\mathrm{e}^{-x}\left(2x^2+2x-2\log 2+1\right)+12^{1/4}\mathrm{i}\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}\mathrm{e}^{x} \\ &+12^{1/4}\mathrm{i}(\alpha_0+\alpha_1x+\alpha_2x^2+\alpha_3x^3)\mathrm{e}^{-x} \\ &-\mathrm{i}\frac{X_i^2}{16}12^{1/4}\,(1-2x)\,\mathrm{e}^{-x}+\frac{\lambda_2}{4}12^{1/4}\,(1-2x)\,\mathrm{e}^{-x} \qquad \text{as } x\to\infty, \\ W_2 \sim{}& -\frac{\mathrm{i}\lambda_1^2x^2}{8}12^{1/4}\mathrm{e}^{x}+12^{1/4}\mathrm{i}\frac{X_i}{16}\,\mathrm{e}^{x}\left(-2x^2+2x+2\log 2-1\right)+12^{1/4}\mathrm{i}\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\mathrm{e}^{-x} \\ &+12^{1/4}\mathrm{i}(\alpha_0-\alpha_1x+\alpha_2x^2-\alpha_3x^3)\mathrm{e}^{x} \\ &-\mathrm{i}\frac{X_i^2}{16}12^{1/4}\,(1+2x)\,\mathrm{e}^{x}+\frac{\lambda_2}{4}12^{1/4}\,(1+2x)\,\mathrm{e}^{x} \qquad \text{as } x\to-\infty. \end{aligned}$$

Thus

$$\begin{aligned} f_2 \sim{}& 12^{1/4}b_i^{(2)}\mathrm{i}\mathrm{e}^{-x}-12^{1/4}a_i^{(1)}\frac{\mathrm{i}\lambda_1x}{2}\mathrm{e}^{-x}-b^{(0)}\frac{\mathrm{i}\lambda_1^2x^2}{8}12^{1/4}\mathrm{e}^{-x} \\ &+12^{1/4}\mathrm{i}b^{(0)}\frac{X_i}{16}\,\mathrm{e}^{-x}\left(2x^2+2x-2\log 2+1\right) \\ &+12^{1/4}\mathrm{i}b^{(0)}\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}\mathrm{e}^{x}+12^{1/4}\mathrm{i}b^{(0)}(\alpha_0+\alpha_1x+\alpha_2x^2+\alpha_3x^3)\mathrm{e}^{-x} \\ &-\mathrm{i}\frac{X_i^2}{16}12^{1/4}b^{(0)}\,(1-2x)\,\mathrm{e}^{-x}+\frac{\lambda_2}{4}12^{1/4}b^{(0)}\,(1-2x)\,\mathrm{e}^{-x} \qquad \text{as } x\to\infty, \end{aligned} \tag{SM4.11}$$

$$\begin{aligned} f_2 \sim{}& 12^{1/4}b_i^{(2)}\mathrm{i}\mathrm{e}^{x}-12^{1/4}a_i^{(1)}\frac{\mathrm{i}\lambda_1x}{2}\mathrm{e}^{x}-\frac{\mathrm{i}\lambda_1^2x^2}{8}12^{1/4}b^{(0)}\mathrm{e}^{x} \\ &+12^{1/4}b^{(0)}\mathrm{i}\frac{X_i}{16}\,\mathrm{e}^{x}\left(-2x^2+2x+2\log 2-1\right) \\ &+12^{1/4}b^{(0)}\mathrm{i}\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\mathrm{e}^{-x}+12^{1/4}b^{(0)}\mathrm{i}(\alpha_0-\alpha_1x+\alpha_2x^2-\alpha_3x^3)\mathrm{e}^{x} \\ &-b^{(0)}\mathrm{i}\frac{X_i^2}{16}12^{1/4}\,(1+2x)\,\mathrm{e}^{x}+\frac{\lambda_2}{4}12^{1/4}b^{(0)}\,(1+2x)\,\mathrm{e}^{x} \qquad \text{as } x\to-\infty. \end{aligned} \tag{SM4.12}$$

SM4.3. $O(G^3)$ terms in the inner expansion.

At $O(G^3)$ we find

$$f_3 = a_i^{(3)}U_0+a_i^{(1)}U_2+b_i^{(2)}W_1+b_i^{(0)}W_3,$$

where

$$\frac{\mathrm{d}^2U_2}{\mathrm{d}x^2} + V_0^4(2U_2^* + 3U_2) - U_2 = -\frac{\lambda_1^2 x}{2}V_0 - \mathrm{i}\lambda_2\frac{\mathrm{d}V_0}{\mathrm{d}x} - \frac{(x+X_i)^2}{4}\frac{\mathrm{d}V_0}{\mathrm{d}x} - 20V_1V_0^3\frac{\mathrm{d}V_0}{\mathrm{d}x},$$

$$\begin{aligned}\frac{\mathrm{d}^2W_3}{\mathrm{d}x^2} + V_0^4(2W_3^* + 3W_3) - W_3 = -\frac{\lambda_1^3x^2V_0}{8} + \lambda_1V_1 - 2\mathrm{i}\lambda_2\lambda_1\left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x}\right) + \lambda_3V_0\\ - \frac{(x+X_i)^2}{4}\lambda_1\left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x}\right) - 20V_1V_0^3\lambda_1\left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x}\right).\end{aligned}$$

The solvability conditions are

$$\mathrm{Im}\left[\frac{\mathrm{d}U_2}{\mathrm{d}x}V_0 - U_2\frac{\mathrm{d}V_0}{\mathrm{d}x}\right]_{-R}^{R} = 0, \tag{SM4.13}$$

$$\mathrm{Re}\left[\frac{\mathrm{d}U_2}{\mathrm{d}x}\frac{\mathrm{d}V_0}{\mathrm{d}x} - U_2\frac{\mathrm{d}^2V_0}{\mathrm{d}x^2}\right]_{-R}^{R} = \frac{\sqrt{3}\,\pi(\lambda_1^2-1)}{8} + \left[\frac{\mathrm{d}^2V_1}{\mathrm{d}x^2}\frac{\mathrm{d}V_0}{\mathrm{d}x} - \frac{\mathrm{d}V_1}{\mathrm{d}x}\frac{\mathrm{d}^2V_0}{\mathrm{d}x^2}\right]_{-R}^{R}, \tag{SM4.14}$$

$$\mathrm{Im}\left[\frac{\mathrm{d}W_3}{\mathrm{d}x}V_0 - W_3\frac{\mathrm{d}V_0}{\mathrm{d}x}\right]_{-R}^{R} = 0, \tag{SM4.15}$$

$$\begin{aligned}\mathrm{Re}\left[\frac{\mathrm{d}W_3}{\mathrm{d}x}\frac{\mathrm{d}V_0}{\mathrm{d}x} - W_3\frac{\mathrm{d}^2V_0}{\mathrm{d}x^2}\right]_{-R}^{R} &= \\ = \lambda_1\int_{-R}^{R}\left(V_1\frac{\mathrm{d}V_0}{\mathrm{d}x} - \left(\frac{xX_i}{2} + 20V_1V_0^3\right)\left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x}\right)\frac{\mathrm{d}V_0}{\mathrm{d}x}\right)\mathrm{d}x,\end{aligned} \tag{SM4.16}$$

where we have taken advantage of the evenness of V_0 to simplify. The terms forced by λ_2 and λ_3 can be evaluated, so that we can write

$$\begin{aligned}U_2 &= U_2^R - \frac{\mathrm{i}\lambda_2 x}{2}V_0,\\ W_3 &= W_3^R - \mathrm{i}\lambda_2\lambda_1\frac{x^2V_0}{4} + \lambda_3\left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x}\right),\end{aligned}$$

where

$$\frac{\mathrm{d}^2U_2^R}{\mathrm{d}x^2} + 5V_0^4U_2^R - U_2^R = -\frac{\lambda_1^2x}{2}V_0 - \frac{(x+X_i)^2}{4}\frac{\mathrm{d}V_0}{\mathrm{d}x} - 20V_1V_0^3\frac{\mathrm{d}V_0}{\mathrm{d}x}, \tag{SM4.17}$$

$$\begin{aligned}\frac{\mathrm{d}^2W_3^R}{\mathrm{d}x^2} + 5V_0^4W_3^R - W_3^R = &-\frac{\lambda_1^3x^2V_0}{8}\\ &+ \lambda_1V_1 - \frac{(x+X_i)^2}{4}\lambda_1\left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x}\right)\\ &- 20V_1V_0^3\lambda_1\left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x}\right).\end{aligned} \tag{SM4.18}$$

SM4.4. $O(G)$ terms in the outer expansion.

At $O(G)$ in the outer we find

$$\mathrm{i}\lambda_1 f_0 + \frac{\mathrm{d}^2 f_1}{\mathrm{d}\xi^2} - f_1 = 0,$$

with

$$f_0 = 12^{1/4}\mathrm{i}b^{(0)}\mathrm{e}^{\xi - X_{i+1}} + 12^{1/4}\mathrm{i}b^{(0)}\mathrm{e}^{-\xi+X_i}.$$

Matching requires

$$f_1 \sim 12^{1/4}a_{i+1}^{(1)}\mathrm{e}^{\xi-X_{i+1}} + 12^{1/4}b^{(0)}\left(\frac{\lambda_1(\xi - X_{i+1})}{2} + \frac{\lambda_1}{4}\right)\mathrm{e}^{\xi - X_{i+1}} \qquad \text{as } \xi \to X_{i+1},$$

$$f_1 \sim -12^{1/4}a_{i}^{(1)}\mathrm{e}^{-\xi+X_{i}} + 12^{1/4}b^{(0)}\left(-\frac{\lambda_1(\xi - X_{i})}{2} + \frac{\lambda_1}{4}\right)\mathrm{e}^{-\xi + X_{i}} \qquad \text{as } \xi \to X_{i}.$$

Thus

$$f_1 = 12^{1/4}\left(a_{i+1}^{(1)} + \frac{\lambda_1 b^{(0)}(\xi - X_{i+1})}{2} + \frac{\lambda_1 b^{(0)}}{4}\right)\mathrm{e}^{\xi - X_{i+1}} + 12^{1/4}\left(-a_i^{(1)} - \frac{\lambda_1 b^{(0)}(\xi - X_i)}{2} + \frac{\lambda_1 b^{(0)}}{4}\right)\mathrm{e}^{-\xi+X_i}.$$

In terms of $\xi = X_{i+1} + x$ this is

$$f_1 = 12^{1/4}\left(a_{i+1}^{(1)} + \frac{\lambda_1 x}{2}b^{(0)} + \frac{\lambda_1}{4}b^{(0)}\right)\mathrm{e}^{x} + 12^{1/4}\left(-a_i^{(1)} - \frac{\lambda_1(x + X_{i+1} - X_i)}{2}b^{(0)} + \frac{\lambda_1}{4}b^{(0)}\right)\mathrm{e}^{-x-X_{i+1}+X_i},$$

giving the matching condition

$$f_3 G^2 \sim 12^{1/4}\left(-a_{i-1}^{(1)} - \frac{\lambda_1(x + X_i - X_{i-1})}{2}b^{(0)} + \frac{\lambda_1}{4}b^{(0)}\right)\mathrm{e}^{-x-X_i+X_{i-1}}, \tag{SM4.19}$$

as $x \to -\infty$ on the inner solution. In terms of $\xi = X_i + x$ we have

$$f_1 = 12^{1/4}\left(a_{i+1}^{(1)} + \frac{\lambda_1(x + X_i - X_{i+1})}{2}b^{(0)} + \frac{\lambda_1}{4}b^{(0)}\right)\mathrm{e}^{x+X_i-X_{i+1}} + 12^{1/4}\left(-a_i^{(1)} - \frac{\lambda_1 x}{2}b^{(0)} + \frac{\lambda_1}{4}b^{(0)}\right)\mathrm{e}^{-x},$$

giving the matching condition

$$f_3 G^2 \sim 12^{1/4}\left(a_{i+1}^{(1)} + \frac{\lambda_1(x + X_i - X_{i+1})}{2}b^{(0)} + \frac{\lambda_1}{4}b^{(0)}\right)\mathrm{e}^{x+X_i-X_{i+1}}, \tag{SM4.20}$$

as $x \to \infty$ on the inner solution.

SM4.5. Solvability condition on f_3.

The matching conditions (SM4.19), (SM4.20) give

$$
\begin{aligned}
G^2\left[\frac{\mathrm{d}f_3}{\mathrm{d}x}V_0 - f_3\frac{\mathrm{d}V_0}{\mathrm{d}x}\right]_{-R}^{R} &= 2\sqrt{3}\left(2a_{i+1}^{(1)} + \lambda_1(1+R+X_i-X_{i+1})b^{(0)}\right)\mathrm{e}^{X_i-X_{i+1}}\\
&\quad - 2\sqrt{3}\left(2a_{i-1}^{(1)} + \lambda_1(-1-R+X_i-X_{i-1})b^{(0)}\right)\mathrm{e}^{-X_i+X_{i-1}},\\
G^2\left[\frac{\mathrm{d}f_3}{\mathrm{d}x}\frac{\mathrm{d}V_0}{\mathrm{d}x} - f_3\frac{\mathrm{d}^2V_0}{\mathrm{d}x^2}\right]_{-R}^{R} &= -2\sqrt{3}\left(2a_{i+1}^{(1)} + \lambda_1(1+R+X_i-X_{i+1})b^{(0)}\right)\mathrm{e}^{X_i-X_{i+1}}\\
&\quad - 2\sqrt{3}\left(2a_{i-1}^{(1)} + \lambda_1(-1-R+X_i-X_{i-1})b^{(0)}\right)\mathrm{e}^{-X_i+X_{i-1}}.
\end{aligned}
$$

The solvability conditions (SM4.13), (SM4.14), (SM4.15), (SM4.16) then give

$$
\begin{aligned}
&\frac{\sqrt{3}\,\pi a_i^{(1)}(\lambda_1^2-1)}{8} + a_i^{(1)}\left[\frac{\mathrm{d}^2V_1}{\mathrm{d}x^2}\frac{\mathrm{d}V_0}{\mathrm{d}x} - \frac{\mathrm{d}V_1}{\mathrm{d}x}\frac{\mathrm{d}^2V_0}{\mathrm{d}x^2}\right]_{-R}^{R}\\
&\qquad + \lambda_1 b^{(0)}\int_{-R}^{R}\left(V_1\frac{\mathrm{d}V_0}{\mathrm{d}x} - \left(\frac{xX_i}{2} + 20V_1V_0^3\right)\left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x}\right)\frac{\mathrm{d}V_0}{\mathrm{d}x}\right)\mathrm{d}x\\
&\qquad\qquad = -2\sqrt{3}\left(2a_{i+1}^{(1)} + \lambda_1(1+R+X_i-X_{i+1})b^{(0)}\right)\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}\\
&\qquad\qquad\quad - 2\sqrt{3}\left(2a_{i-1}^{(1)} + \lambda_1(-1-R+X_i-X_{i-1})b^{(0)}\right)\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}.
\end{aligned}
\tag{SM4.21}
$$

We show in section SM7 that the terms proportional to R vanish as they should, and (SM4.21) simplifies to

$$
\begin{aligned}
&\frac{\pi a_i^{(1)}(\lambda_1^2-1)}{32} + (a_{i+1}^{(1)} - a_i^{(1)})\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2} + (a_{i-1}^{(1)} - a_i^{(1)})\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\\
&\qquad = \frac{3\pi}{64}\lambda_1 b^{(0)}X_i - \frac{\lambda_1 b^{(0)}}{2}(X_i-X_{i+1})\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2} - \frac{\lambda_1 b^{(0)}}{2}(X_i-X_{i-1})\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}.
\end{aligned}
\tag{SM4.22}
$$

Equation (SM4.22) is (5.40) from the main text. When $b^{(0)}$ is non-zero it gives $a_i^{(1)}$ in terms of $b^{(0)}$, with no restriction on λ_1. To determine λ_1 we need to go to one more order.

SM4.6. $O(G^4)$ terms in the inner expansion.

At $O(G^4)$ in the inner expansion near hump i we find

$$
f_4 = a_i^{(3)}U_1 + a_i^{(1)}U_3 + b_i^{(4)}W_0 + b_i^{(2)}W_2 + b_i^{(0)}W_4,
$$

where

$$
\frac{\mathrm{d}^2U_3}{\mathrm{d}x^2} + V_0^4(2U_3^* + 3U_3) - U_3 = -\mathrm{i}\lambda_1 U_2^R - \lambda_1\lambda_2 xV_0 - \mathrm{i}\lambda_3 V_0' + \frac{\mathrm{i}\lambda_1 x(x+X_i)^2}{8}V_0 + 2\mathrm{i}\lambda_1 xV_1V_0^4,
$$

$$
\begin{aligned}
\frac{\mathrm{d}^2W_4}{\mathrm{d}x^2} + V_0^4(2W_4^* + 3W_4) - W_4 = -\mathrm{i}\lambda_1 &\left(W_3^R - \mathrm{i}\lambda_1\lambda_2 \frac{x^2V_0}{4} \right) \\
&- \mathrm{i}\lambda_2 \left(-\frac{\mathrm{i}\lambda_1^2 x^2 V_0}{8} + \mathrm{i}V_1 + \lambda_2 \left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x} \right) \right) \\
&- \mathrm{i}\lambda_3\lambda_1 \left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x} \right) + \lambda_4 V_0 \\
&- \frac{(x+X_i)^2}{4} \left(-\frac{\mathrm{i}\lambda_1^2 x^2 V_0}{8} + \mathrm{i}V_1 + \lambda_2 \left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x} \right) \right) \\
&- 4V_1V_0^3 \left(-\frac{\mathrm{i}\lambda_1^2 x^2 V_0}{8} + \mathrm{i}V_1 + 5\lambda_2 \left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x} \right) \right) \\
&- 2V_0^2(3V_1^2 + 2V_0V_2)\mathrm{i}V_0.
\end{aligned}
$$

The relevant solvability conditions are

$$
\text{(SM4.23)} \qquad \mathrm{Im}\left[\frac{\mathrm{d}U_3}{\mathrm{d}x} V_0 - U_3 \frac{\mathrm{d}V_0}{\mathrm{d}x} \right]_{-R}^{R} = \int_{-R}^{R} \left(-\lambda_1 U_2^R V_0 + \frac{\lambda_1}{4} x^2 X_i V_0^2 + 2\lambda_1 x V_1 V_0^5 \right) \mathrm{d}x,
$$

$$
\begin{aligned}
\mathrm{Im}\left[\frac{\mathrm{d}W_4}{\mathrm{d}x} V_0 - W_4 \frac{\mathrm{d}V_0}{\mathrm{d}x} \right]_{-R}^{R} = -\int_{-R}^{R} \Bigg(\lambda_1 W_3^R V_0 + \left(\frac{(x+X_i)^2 V_0}{4} + 4V_1V_0^4 \right) &\left(V_1 - \frac{\lambda_1^2 x^2 V_0}{8} \right) \\
+ 2V_0^4(3V_1^2 + 2V_0V_2) &\Bigg) \mathrm{d}x. \qquad \text{(SM4.24)}
\end{aligned}
$$

SM4.7. $O(G^2)$ terms in the outer expansion. At $O(G^2)$ in the outer

$$
\mathrm{i}\lambda_1 f_1 + \mathrm{i}\lambda_2 f_0 + \frac{\mathrm{d}^2 f_2}{\mathrm{d}\xi^2} - f_2 = -\frac{\xi^2 f_0}{4}.
$$

Matching with (SM4.11), (SM4.12) gives the solution as

$$
\begin{aligned}
f_2 = {} & 12^{1/4} b_i^{(2)} \mathrm{i} \mathrm{e}^{-\xi+X_i} - 12^{1/4} a_i^{(1)} \frac{\mathrm{i}\lambda_1(\xi - X_i)}{2} \mathrm{e}^{-\xi+X_i} - b^{(0)} \frac{\mathrm{i}\lambda_1^2(\xi - X_i)^2}{8} 12^{1/4} \mathrm{e}^{-\xi+X_i} \\
& + 12^{1/4} \mathrm{i} b^{(0)} \frac{X_i}{16}\, \mathrm{e}^{-\xi+X_i} \left(2(\xi - X_i)^2 + 2(\xi - X_i) - 2\log 2 + 1\right) + 12^{1/4} \mathrm{i} b^{(0)} \frac{\mathrm{e}^{X_i - X_{i+1}}}{G^2} \mathrm{e}^{\xi - X_i} \\
& + 12^{1/4} \mathrm{i} b^{(0)} (\alpha_0 + \alpha_1(\xi - X_i) + \alpha_2(\xi - X_i)^2 + \alpha_3(\xi - X_i)^3) \mathrm{e}^{-\xi+X_i} \\
& - \mathrm{i} \frac{X_i^2}{16} 12^{1/4} b^{(0)} \left(1 - 2(\xi - X_i)\right) \mathrm{e}^{-\xi+X_i} + \frac{\lambda_2}{4} 12^{1/4} b^{(0)} \left(1 - 2(\xi - X_i)\right) \mathrm{e}^{-\xi+X_i} \\
& + 12^{1/4} b_{i+1}^{(2)} \mathrm{i} \mathrm{e}^{\xi - X_{i+1}} - 12^{1/4} a_{i+1}^{(1)} \frac{\mathrm{i}\lambda_1(\xi - X_{i+1})}{2} \mathrm{e}^{\xi - X_{i+1}} - \frac{\mathrm{i}\lambda_1^2(\xi - X_{i+1})^2}{8} 12^{1/4} b^{(0)} \mathrm{e}^{\xi - X_{i+1}} \\
& + 12^{1/4} b^{(0)} \mathrm{i} \frac{X_{i+1}}{16}\, \mathrm{e}^{\xi - X_{i+1}} \left(-2(\xi - X_{i+1})^2 + 2(\xi - X_{i+1}) + 2\log 2 - 1\right) \\
& + 12^{1/4} b^{(0)} \mathrm{i} \frac{\mathrm{e}^{X_{i-1} - X_i}}{G^2} \mathrm{e}^{-\xi + X_{i+1}} \\
& + 12^{1/4} b^{(0)} \mathrm{i} (\alpha_0 - \alpha_1(\xi - X_{i+1}) + \alpha_2(\xi - X_{i+1})^2 - \alpha_3(\xi - X_{i+1})^3) \mathrm{e}^{\xi - X_{i+1}} \\
& - b^{(0)} \mathrm{i} \frac{X_{i+1}^2}{16} 12^{1/4} \left(1 + 2(\xi - X_{i+1})\right) \mathrm{e}^{\xi - X_{i+1}} + \frac{\lambda_2}{4} 12^{1/4} b^{(0)} \left(1 + 2(\xi - X_{i+1})\right) \mathrm{e}^{\xi - X_{i+1}}.
\end{aligned}
\tag{SM4.25}
$$

This then gives the matching conditions

$$
\begin{aligned}
G^2 f_4 \sim {} & 12^{1/4} b_{i-1}^{(2)} \mathrm{i} \mathrm{e}^{-x - X_i + X_{i-1}} - 12^{1/4} a_{i-1}^{(1)} \frac{\mathrm{i}\lambda_1(x + X_i - X_{i-1})}{2} \mathrm{e}^{-x - X_i + X_{i-1}} \\
& - b^{(0)} \frac{\mathrm{i}\lambda_1^2(x + X_i - X_{i-1})^2}{8} 12^{1/4} \mathrm{e}^{-x - X_i + X_{i-1}} \\
& + 12^{1/4} \mathrm{i} b^{(0)} \frac{X_{i-1}}{16}\, \mathrm{e}^{-x - X_i + X_{i-1}} \left(2(x + X_i - X_{i-1})^2 + 2(x + X_i - X_{i-1}) - 2\log 2 + 1\right) \\
& + 12^{1/4} \mathrm{i} b^{(0)} (\alpha_0 + \alpha_1(x + X_i - X_{i-1}) + \alpha_2(x + X_i - X_{i-1})^2 + \alpha_3(x + X_i - X_{i-1})^3) \mathrm{e}^{-x - X_i + X_{i-1}} \\
& - \mathrm{i} \frac{X_{i-1}^2}{16} 12^{1/4} b^{(0)} \left(1 - 2(x + X_i - X_{i-1})\right) \mathrm{e}^{-x - X_i + X_{i-1}} \\
& + \frac{\lambda_2}{4} 12^{1/4} b^{(0)} \left(1 - 2(x + X_i - X_{i-1})\right) \mathrm{e}^{-x - X_i + X_{i-1}} \qquad \text{as } x \to -\infty,
\end{aligned}
\tag{SM4.26}
$$

$$\begin{aligned}
G^2 f_4 \sim{}& 12^{1/4} b_{i+1}^{(2)} \mathrm{i} \mathrm{e}^{x+X_i-X_{i+1}} - 12^{1/4} a_{i+1}^{(1)} \frac{\mathrm{i}\lambda_1(x+X_i-X_{i+1})}{2} \mathrm{e}^{x+X_i-X_{i+1}} \\
&- \frac{\mathrm{i}\lambda_1^2(x+X_i-X_{i+1})^2}{8} 12^{1/4} b^{(0)} \mathrm{e}^{x+X_i-X_{i+1}} \\
&+ 12^{1/4} b^{(0)} \mathrm{i} \frac{X_{i+1}}{16} \mathrm{e}^{x+X_i-X_{i+1}} \left(-2(x+X_i-X_{i+1})^2 + 2(x+X_i-X_{i+1}) + 2\log 2 - 1\right) \\
&+ 12^{1/4} b^{(0)} \mathrm{i} (\alpha_0 - \alpha_1(x+X_i-X_{i+1}) + \alpha_2(x+X_i-X_{i+1})^2 - \alpha_3(x+X_i-X_{i+1})^3) \mathrm{e}^{x+X_i-X_{i+1}} \\
&- b^{(0)} \mathrm{i} \frac{X_{i+1}^2}{16} 12^{1/4} \left(1 + 2(x+X_i-X_{i+1})\right) \mathrm{e}^{x+X_i-X_{i+1}} \\
&+ \frac{\lambda_2}{4} 12^{1/4} b^{(0)} \left(1 + 2(x+X_i-X_{i+1})\right) \mathrm{e}^{x+X_i-X_{i+1}} \qquad \text{as } x \to \infty
\end{aligned} \tag{SM4.27}$$

on the inner solution near hump i.

SM4.8. Final solvability condition.

We have, from (SM4.23), (SM4.24)

$$\begin{aligned}
\mathrm{Im}\left[\frac{\mathrm{d}f_4}{\mathrm{d}x} V_0 - f_4 \frac{\mathrm{d}V_0}{\mathrm{d}x}\right]_{-R}^{R} ={}& a_i^{(1)} \int_{-R}^{R} \left(-\lambda_1 U_2^R V_0 + \frac{\lambda_1}{4} x^2 X_i V_0^2 + 2\lambda_1 x V_1 V_0^5\right) \mathrm{d}x \\
&- b^{(0)} \int_{-R}^{R} \left(\lambda_1 W_3^R V_0 + \left(\frac{(x+X_i)^2 V_0}{4} + 4 V_1 V_0^4\right)\left(V_1 - \frac{\lambda_1^2 x^2 V_0}{8}\right) + 2V_0^4(3V_1^2 + 2V_0 V_2)\right) \mathrm{d}x \\
&+ b_i^{(2)} \left(\frac{4\sqrt{3}}{G^2} \mathrm{e}^{-X_{i+1}+X_i} + \frac{4\sqrt{3}}{G^2} \mathrm{e}^{-X_i+X_{i-1}}\right).
\end{aligned} \tag{SM4.28}$$

Evaluating the LHS by matching using (SM4.26), (SM4.27) gives

$$\begin{aligned}
\mathrm{Im}\left[\frac{\mathrm{d}f_4}{\mathrm{d}x} V_0 - f_4 \frac{\mathrm{d}V_0}{\mathrm{d}x}\right]_{-R}^{R} ={}& \frac{4\sqrt{3}}{G^2} \mathrm{e}^{X_i-X_{i+1}} b_{i+1}^{(2)} - \frac{\sqrt{3}}{G^2} \mathrm{e}^{X_i-X_{i+1}} \lambda_1 a_{i+1}^{(1)} (1 + 2R + 2X_i - 2X_{i+1}) \\
&- \frac{b^{(0)}}{32\sqrt{3}G^2} \mathrm{e}^{X_i-X_{i+1}} \big(\pi^2 + 48\lambda_1^2 R + 48\lambda^2 R^2 + 16R^3 + 48(\lambda_1^2 + R) X_i^2 \\
&\qquad +16X_i^3 - 48(\lambda_1^2 + 2\lambda_1^2 R + \log 2) X_{i+1} + 48(1+\lambda_1^2) X_{i+1}^2 - 16X_{i+1}^3 \\
&\qquad +48X_i(\lambda_1^2 + 2\lambda_1^2 R + R^2 - 2\lambda_1^2 X_{i+1})\big) \\
&+ \frac{4\sqrt{3}}{G^2} \mathrm{e}^{X_{i-1}-X_i} b_{i-1}^{(2)} - \frac{\sqrt{3}}{G^2} \mathrm{e}^{X_{i-1}-X_i} \lambda_1 a_{i-1}^{(1)} (-1 - 2R - 2X_{i-1} + 2X_i) \\
&- \frac{b^{(0)}}{32\sqrt{3}G^2} \mathrm{e}^{X_{i-1}-X_i} \big(\pi^2 + 48\lambda_1^2 R + 48\lambda_1^2 R^2 + 16R^3 + 48(1+\lambda_1^2) X_{i-1}^2 \\
&\qquad +16X_{i-1}^3 - 48(R^2 + \lambda_1^2(1+2R)) X_i + 48(\lambda_1^2 + R) X_i^2 - 16X_i^3 \\
&\qquad -48X_{i-1}(\lambda_1^2(-1-2R) - \log 2 + 2\lambda_1^2 X_i)\big).
\end{aligned}$$

We show in section SM8 that all the terms in R cancel as they should, and (SM4.28) simplifies to

$$\begin{aligned} &\frac{b^{(0)}\lambda_1^2(\lambda_1^2-4)\pi^3}{512} + \frac{3\pi\lambda_1 X_i}{64}(b^{(0)}\lambda_1 X_i - 2a_i^{(1)}) \\ &\quad + 4(b_{i+1}^{(2)}-b_i^{(2)})\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2} + 4(b_{i-1}^{(2)}-b_i^{(2)})\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2} \\ &\quad + 2a_{i+1}^{(1)}(X_{i+1}-X_i)\lambda_1\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2} + 2a_{i-1}^{(1)}(X_{i-1}-X_i)\lambda_1\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2} \\ &\quad - 3a_{i+1}^{(1)}\lambda_1\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2} + 3a_{i-1}^{(1)}\lambda_1\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2} \\ &\quad + \frac{\lambda_1^2}{2}b^{(0)}\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}\left(-X_i^2+3X_{i+1}+2X_iX_{i+1}-X_{i+1}^2\right) \\ &\quad + \frac{\lambda_1^2}{2}b^{(0)}\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\left(-X_i^2-3X_{i-1}+2X_iX_{i-1}-X_{i-1}^2\right) = 0. \end{aligned} \tag{SM4.29}$$

Equation (SM4.29) is (5.41) from the main text. The sum over i is the equation which determines λ_1.

SM5. Analysis of V_1.

We have (4.11), which we restate for convenience

$$\frac{\mathrm{d}^2V_1}{\mathrm{d}x^2} + 5V_0^4V_1 - V_1 = -\frac{(x+X_i)^2V_0}{4}. \tag{SM5.1}$$

The exponential growth of V_1 at infinity arises due to the term proportional to xV_0 on the RHS, which is not orthogonal to $\mathrm{d}V_0/\mathrm{d}x$. We divide V_1 into terms forced by the odd and even components on the RHS by setting

$$V_1 = V_o + V_e,$$

where

$$\frac{\mathrm{d}^2V_o}{\mathrm{d}x^2} + 5V_0^4V_o - V_o = -\frac{xX_iV_0}{2}, \tag{SM5.2}$$

$$\frac{\mathrm{d}^2V_e}{\mathrm{d}x^2} + 5V_0^4V_e - V_e = -\frac{(x^2+X_i^2)V_0}{4}, \tag{SM5.3}$$

and $V_e \to 0$ as $x \to \pm\infty$. Note that V_e is even but this last boundary condition means that V_o is not odd but has an even component (which satisfies the homogeneous equation).

Since the homogeneous version of (SM5.1) has linearly independent solutions

$$v_1 = \frac{\sinh(2\xi)}{\cosh^{3/2}(2\xi)}, \qquad v_2 = \frac{\cosh(4\xi)-3}{\cosh^{3/2}(2\xi)}, \tag{SM5.4}$$

the general solution of (SM5.2) may be written as

$$V_o = \frac{X_i}{2}\frac{v_1(x)}{4}\int_a^x v_2(\bar{x})\bar{x}V_0(\bar{x})\,\mathrm{d}\bar{x} - \frac{X_i}{2}\frac{v_2(x)}{4}\int_b^x v_1(\bar{x})\bar{x}V_0(\bar{x})\,\mathrm{d}\bar{x},$$

for some constants a and b. Varying the constant a corresponds to translating the position of the inner region. To fix it we need to be more specific about how the X_i are defined. If we define X_i to be the position of the local maximum of the modulus of V, then we have

$\mathrm{d}V/\mathrm{d}\xi = 0$ at $\xi = X_i$, so that in the inner coordinate $\mathrm{d}V/\mathrm{d}x = 0$ at $x = 0$. This fixes $a = 0$. To determine b we use the matching conditions (SM4.9), (SM4.10), which give

$$V_o \sim 12^{1/4}\frac{\mathrm{e}^{x+X_i-X_{i+1}}}{G^2} \quad \text{as } x \to \infty, \qquad V_o \sim 12^{1/4}\frac{\mathrm{e}^{-x+X_{i-1}-X_i}}{G^2} \quad \text{as } x \to -\infty.$$

We can set $b = -\infty$ if we add on the multiple of $v_2'(x)$ we expect there. This gives

$$\begin{aligned} V_o &= \frac{X_i}{2}\frac{v_1(x)}{4}\int_0^x v_2(\bar{x})\bar{x}V_0(\bar{x})\,\mathrm{d}\bar{x} - \frac{X_i}{2}\frac{v_2(x)}{4}\int_{-\infty}^x v_1(\bar{x})\bar{x}V_0(\bar{x})\,\mathrm{d}\bar{x} + 3^{1/4}\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}v_2(x) \\ &= 3^{1/4}\frac{X_i}{2}\frac{v_1(x)}{4}\left(\log(1+\mathrm{e}^{4x}) + x^2 - 2x - 2x\tanh 2x - \log 2\right) \\ &\quad + \frac{3^{1/4}}{4}\frac{X_i}{2}\frac{v_2(x)}{4}\left(2x\operatorname{sech}2x - \sin^{-1}\operatorname{sech}2x\right) + 3^{1/4}\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}v_2(x). \end{aligned} \tag{SM5.5}$$

Note the branch of $\sin^{-1}$ is such that

$$\sin^{-1}\operatorname{sech}2x = \begin{cases} \sin^{-1}\operatorname{sech}2x & x < 0, \\ \pi - \sin^{-1}\operatorname{sech}2x & x > 0. \end{cases}$$

The odd part of (SM5.5) is

$$\begin{aligned} V_{\mathrm{odd}} = 3^{1/4}\frac{X_i}{2}\frac{v_1(x)}{4}&\left(\log(1+\mathrm{e}^{4x}) + x^2 - 2x - 2x\tanh 2x - \log 2\right) \\ &+ \frac{3^{1/4}}{4}\frac{X_i}{2}\frac{v_2(x)}{4}\left(2x\operatorname{sech}2x + \frac{\pi}{2} - \sin^{-1}\operatorname{sech}2x\right) \end{aligned} \tag{SM5.6}$$

leaving an even multiple of v_2,

$$V_o^e = 3^{1/4}\left(-\frac{X_i\pi}{64} + \frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\right)v_2(x) = \frac{3^{1/4}}{2}\left(\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2} + \frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\right)v_2(x).$$

As $x \to -\infty$,

$$V_o \sim \quad 3^{1/4}\frac{X_i}{16}\sqrt{2}\,\mathrm{e}^{x}\left(-2x^2 + 2x + 2\log 2 - 1\right) + 3^{1/4}\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\sqrt{2}\,\mathrm{e}^{-x}, \tag{SM5.7}$$

while as $x \to \infty$,

$$V_o \sim 3^{1/4}\frac{X_i}{16}\sqrt{2}\,\mathrm{e}^{-x}\left(2x^2 + 2x - 2\log 2 + 1\right) + 3^{1/4}\sqrt{2}\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}\mathrm{e}^{x}, \tag{SM5.8}$$

where we have used (4.15) to simplify. Let us now turn to (SM5.3). The solution may be written

$$V_e = \hat{V}_1 - \frac{X_i^2}{4}\left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x}\right),$$

where

$$\hat{V}_1(x) = v_1(x)\int_0^x \frac{v_2(\bar{x})\bar{x}^2V_0(\bar{x})\,\mathrm{d}\bar{x}}{16} + v_2(x)\int_{\bar{x}}^\infty \frac{v_1(\bar{x})\bar{x}^2V_0(\bar{x})\,\mathrm{d}\bar{x}}{16},$$

is the perturbation to the single hump solution (i.e., without the change of origin). As $x \to \infty$,

$$\hat{V}_1 \sim 12^{1/4}(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)\mathrm{e}^{-x},$$

with

$$\text{(SM5.9)} \qquad \alpha_0 = \frac{1}{32}\left(1 - \frac{\pi^2}{12}\right), \qquad \alpha_1 = \frac{1}{16}, \qquad \alpha_2 = \frac{1}{16}, \qquad \alpha_3 = \frac{1}{24}.$$

Since $\hat{V}_1$ is even, as $x \to -\infty$,

$$\hat{V}_1 \sim 12^{1/4}(\alpha_0 - \alpha_1 x + \alpha_2 x^2 - \alpha_3 x^3)\mathrm{e}^{x}.$$

Thus

(SM5.10)

$$V_e \sim 12^{1/4}(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)\mathrm{e}^{-x} - \frac{X_i^2}{16}3^{1/4}\sqrt{2}\,(1-2x)\,\mathrm{e}^{-x} \qquad \text{as } x \to \infty,$$

(SM5.11)

$$V_e \sim 12^{1/4}(\alpha_0 - \alpha_1 x + \alpha_2 x^2 - \alpha_3 x^3)\mathrm{e}^{x} - \frac{X_i^2}{16}3^{1/4}\sqrt{2}\,(1+2x)\,\mathrm{e}^{x} \qquad \text{as } x \to -\infty.$$

Together (SM5.7), (SM5.8), (SM5.10), and (SM5.11) give (SM4.9) and (SM4.10).

SM6. Analysis of V_2.

If we proceed to $O(G^4)$ in the local analysis of subsection 4.1, subsection 4.3 we find the equation for V_2 is

$$\frac{\mathrm{d}^2V_2}{\mathrm{d}x^2} + 5V_0^4V_2 - V_2 = -\frac{(x+X_i)^2V_1}{4} - 10V_0^3V_1^2.$$

Multiplying by V_0 and integrating gives

$$\begin{aligned} -\int_{-R}^R \frac{(x+X_i)^2V_1V_0}{4} + 10V_0^4V_1^2\,\mathrm{d}x &= \int_{-R}^R V_0\left(\frac{\mathrm{d}^2V_2}{\mathrm{d}x^2} + 5V_0^4V_2 - V_2\right)\mathrm{d}x \\ &= \int_{-R}^R \left(V_2\frac{\mathrm{d}^2V_0}{\mathrm{d}x^2} + 5V_0^5V_2 - V_0V_2\right)\mathrm{d}x \\ &\quad + \left[V_0\frac{\mathrm{d}V_2}{\mathrm{d}x} - V_2\frac{\mathrm{d}V_0}{\mathrm{d}x}\right]_{-R}^R \\ &= \int_{-R}^R 4V_0^5V_2\,\mathrm{d}x + \left[V_0\frac{\mathrm{d}V_2}{\mathrm{d}x} - V_2\frac{\mathrm{d}V_0}{\mathrm{d}x}\right]_{-R}^R. \end{aligned}$$

Thus

$$\int_{-R}^{R} 4V_0^5 V_2 + 10V_0^4 V_1^2 + \frac{(x+X_i)^2 V_1 V_0}{4}\,\mathrm{d}x = -\left[V_0\frac{\mathrm{d}V_2}{\mathrm{d}x} - V_2\frac{\mathrm{d}V_0}{\mathrm{d}x}\right]_{-R}^{R}. \tag{SM6.1}$$

To evaluate the right-hand side we need to go to one more term in the outer region between humps than we did in subsection 4.2. Equating coefficients of G^2 in the region between humps gives

$$\frac{\mathrm{d}^2 V_1}{\mathrm{d}\xi^2} - V_1 = -\frac{\xi^2}{4}V_0 = -\frac{12^{1/4}\xi^2}{4}\left(\mathrm{e}^{-\xi+X_i} + \mathrm{e}^{\xi-X_{i+1}}\right). \tag{SM6.2}$$

Thus

$$V_1 = 12^{1/4}\left(C_1 - \frac{\xi}{16} + \frac{\xi^2}{16} - \frac{\xi^3}{24}\right)\mathrm{e}^{\xi-X_{i+1}} + 12^{1/4}\left(C_2 + \frac{\xi}{16} + \frac{\xi^2}{16} + \frac{\xi^3}{24}\right)\mathrm{e}^{-\xi+X_i}. \tag{SM6.3}$$

The coefficients C_1 and C_2 are determined by matching with the subdominant exponentials in the inner solution V_1. As $x \to -\infty$ in the inner solution, from (SM5.7), (SM5.11),

$$\begin{aligned} V_1 \sim 3^{1/4}\frac{X_i}{16}\sqrt{2}\,\mathrm{e}^{x}\left(-2x^2 + 2x + 2\log 2 - 1\right) + 3^{1/4}\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\sqrt{2}\,\mathrm{e}^{-x} \\ + 12^{1/4}(\alpha_0 - \alpha_1 x + \alpha_2 x^2 - \alpha_3 x^3)\mathrm{e}^{x} - \frac{X_i^2}{16}3^{1/4}\sqrt{2}\,(1+2x)\,\mathrm{e}^{x}. \end{aligned}$$

With $x = \xi - X_{i+1}$ (and $i \to i+1$) this is

$$\begin{aligned} V_1 \sim {} & 3^{1/4}\frac{X_{i+1}}{16}\sqrt{2}\,\mathrm{e}^{\xi-X_{i+1}}\left(-2(\xi-X_{i+1})^2 + 2(\xi-X_{i+1}) + 2\log 2 - 1\right) \\ & + 3^{1/4}\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}\sqrt{2}\,\mathrm{e}^{-(\xi-X_{i+1})} \\ & + 12^{1/4}(\alpha_0 - \alpha_1(\xi-X_{i+1}) + \alpha_2(\xi-X_{i+1})^2 - \alpha_3(\xi-X_{i+1})^3)\mathrm{e}^{\xi-X_{i+1}} \\ & - \frac{X_{i+1}^2}{16}3^{1/4}\sqrt{2}\,(1+2(\xi-X_{i+1}))\,\mathrm{e}^{\xi-X_{i+1}}. \end{aligned}$$

Removing the terms which match with V_0 in the outer leaves

$$\begin{aligned} V_1 &\sim 12^{1/4}\mathrm{e}^{\xi-X_{i+1}}\left(\frac{X_{i+1}}{16}\left(-2(\xi-X_{i+1})^2 + 2(\xi-X_{i+1}) + 2\log 2 - 1\right)\right. \\ &\qquad \left. + (\alpha_0 - \alpha_1(\xi-X_{i+1}) + \alpha_2(\xi-X_{i+1})^2 - \alpha_3(\xi-X_{i+1})^3) - \frac{X_{i+1}^2}{16}\left(1 + 2(\xi-X_{i+1})\right)\right) \\ &\sim 12^{1/4}\mathrm{e}^{\xi-X_{i+1}}\left(\frac{1}{32}\left(1-\frac{\pi^2}{12}\right) - \frac{\xi}{16} + \frac{\xi^2}{16} - \frac{\xi^3}{24} - \frac{X_{i+1}^2}{8} + \frac{X_{i+1}^3}{24} + \frac{X_{i+1}\log 2}{8}\right). \end{aligned}$$

Thus

$$C_1 = \frac{1}{32}\left(1-\frac{\pi^2}{12}\right) - \frac{X_{i+1}^2}{8} + \frac{X_{i+1}^3}{24} + \frac{X_{i+1}\log 2}{8}.$$

As $x \to \infty$ in the inner solution, from (SM5.8), (SM5.10),

$$V_1 \sim 3^{1/4}\frac{X_i}{16}\sqrt{2}\,\mathrm{e}^{-x}\left(2x^2+2x-2\log 2+1\right)+3^{1/4}\sqrt{2}\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}\mathrm{e}^{x}$$
$$+12^{1/4}(\alpha_0+\alpha_1 x+\alpha_2 x^2+\alpha_3 x^3)\mathrm{e}^{-x}-\frac{X_i^2}{16}3^{1/4}\sqrt{2}\,(1-2x)\,\mathrm{e}^{-x}.$$

With $x=\xi-X_i$ this is

$$\begin{aligned}V_1 \sim{}& 3^{1/4}\frac{X_i}{16}\sqrt{2}\,\mathrm{e}^{-(\xi-X_i)}\left(2(\xi-X_i)^2+2(\xi-X_i)-2\log 2+1\right)\\ &+3^{1/4}\sqrt{2}\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}\mathrm{e}^{(\xi-X_i)}\\ &+12^{1/4}(\alpha_0+\alpha_1(\xi-X_i)+\alpha_2(\xi-X_i)^2+\alpha_3(\xi-X_i)^3)\mathrm{e}^{-(\xi-X_i)}\\ &-\frac{X_i^2}{16}3^{1/4}\sqrt{2}\,(1-2(\xi-X_i))\,\mathrm{e}^{-(\xi-X_i)}.\end{aligned}$$

Removing the terms which match with V_0 in the outer leaves

$$\begin{aligned}V_1 \sim{}& 12^{1/4}\,\mathrm{e}^{-(\xi-X_i)}\left(\frac{X_i}{16}\left(2(\xi-X_i)^2+2(\xi-X_i)-2\log 2+1\right)\right.\\ &\left.+\left(\alpha_0+\alpha_1(\xi-X_i)+\alpha_2(\xi-X_i)^2+\alpha_3(\xi-X_i)^3\right)-\frac{X_i^2}{16}\left(1-2(\xi-X_i)\right)\right)\\ \sim{}& 12^{1/4}\mathrm{e}^{-\xi+X_i}\left(\frac{1}{32}\left(1-\frac{\pi^2}{12}\right)+\frac{\xi}{16}+\frac{\xi^2}{16}+\frac{\xi^3}{24}-\frac{X_i^2}{8}-\frac{X_i^3}{24}-\frac{X_i\log 2}{8}\right).\end{aligned}$$

Thus

$$C_2=\frac{1}{32}\left(1-\frac{\pi^2}{12}\right)-\frac{X_i^2}{8}-\frac{X_i^3}{24}-\frac{X_i\log 2}{8}.$$

Thus the outer solution for V_1 when $X_i<\xi<X_{i+1}$ is

$$\begin{aligned}V_1 ={}& 12^{1/4}\mathrm{e}^{-\xi+X_i}\left(\frac{1}{32}\left(1-\frac{\pi^2}{12}\right)+\frac{\xi}{16}+\frac{\xi^2}{16}+\frac{\xi^3}{24}-\frac{X_i^2}{8}-\frac{X_i^3}{24}-\frac{X_i\log 2}{8}\right)\\ &+12^{1/4}\mathrm{e}^{\xi-X_{i+1}}\left(\frac{1}{32}\left(1-\frac{\pi^2}{12}\right)-\frac{\xi}{16}+\frac{\xi^2}{16}-\frac{\xi^3}{24}-\frac{X_{i+1}^2}{8}+\frac{X_{i+1}^3}{24}+\frac{X_{i+1}\log 2}{8}\right).\end{aligned}\tag{SM6.4}$$

Thus the matching condition on V_2 is

$$\begin{aligned}G^2V_2 \sim{}& 12^{1/4}\mathrm{e}^{x+X_i-X_{i+1}}\left(\frac{1}{32}\left(1-\frac{\pi^2}{12}\right)-\frac{x+X_i}{16}+\frac{(x+X_i)^2}{16}-\frac{(x+X_i)^3}{24}\right.\\ &\left.-\frac{X_{i+1}^2}{8}+\frac{X_{i+1}^3}{24}+\frac{X_{i+1}\log 2}{8}\right)\\ \sim{}& 12^{1/4}\mathrm{e}^{x+X_i-X_{i+1}}\left(\frac{1}{32}\left(1-\frac{\pi^2}{12}\right)+\frac{x(-1+2X_i-2X_i^2)}{16}+\frac{x^2(1-2X_i)}{16}-\frac{x^3}{24}\right.\\ &\left.-\frac{X_i}{16}+\frac{X_i^2}{16}-\frac{X_i^3}{24}-\frac{X_{i+1}^2}{8}+\frac{X_{i+1}^3}{24}+\frac{X_{i+1}\log 2}{8}\right),\end{aligned}$$

as $x \to \infty$, and

$$
\begin{aligned}
G^2 V_2 &\sim 12^{1/4} \mathrm{e}^{-(x+X_i)+X_{i-1}} \left(\frac{1}{32}\left(1 - \frac{\pi^2}{12}\right) + \frac{(x+X_i)}{16} + \frac{(x+X_i)^2}{16} \right. \\
&\qquad \left. + \frac{(x+X_i)^3}{24} - \frac{X_{i-1}^2}{8} - \frac{X_{i-1}^3}{24} - \frac{X_{i-1}\log 2}{8} \right) \\
&\sim 12^{1/4} \mathrm{e}^{-x-X_i+X_{i-1}} \left(\frac{1}{32}\left(1 - \frac{\pi^2}{12}\right) + \frac{x(1+2X_i+2X_i^2)}{16} + \frac{x^2(1+2X_i)}{16} + \frac{x^3}{24} \right. \\
&\qquad \left. + \frac{X_i}{16} + \frac{X_i^2}{16} + \frac{X_i^3}{24} - \frac{X_{i-1}^2}{8} - \frac{X_{i-1}^3}{24} - \frac{X_{i-1}\log 2}{8} \right),
\end{aligned}
$$

as $x \to -\infty$. Thus, finally,

$$
\begin{aligned}
&\left[V_0 \frac{\mathrm{d}V_2}{\mathrm{d}x} - V_2 \frac{\mathrm{d}V_0}{\mathrm{d}x} \right]_{-R}^{R} = \qquad \text{(SM6.5)} \\
&= \frac{\sqrt{3}}{96} \mathrm{e}^{X_i - X_{i+1}} \left(-\pi^2 - 16R^3 - 48X_iR^2 - 48X_i^2R - 16X_i^3 - 48X_{i+1}^2 + 16X_{i+1}^3 + 48\log 2\, X_{i+1} \right) \\
&+ \frac{\sqrt{3}}{96} \mathrm{e}^{-X_i + X_{i-1}} \left(-\pi^2 - 16R^3 + 48X_iR^2 - 48X_i^2R + 16X_i^3 - 48X_{i-1}^2 - 16X_{i-1}^3 - 48\log 2\, X_{i-1} \right).
\end{aligned}
$$

SM7. Derivation of (SM4.22).

The solvability condition (SM4.21) involves the integral

$$
\begin{aligned}
\lim_{R\to\infty} \int_{-R}^{R} & \left(V_1 \frac{\mathrm{d}V_0}{\mathrm{d}x} - \left(\frac{xX_i}{2} + 20V_1V_0^3 \right) \left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x} \right) \frac{\mathrm{d}V_0}{\mathrm{d}x} \right) \mathrm{d}x \\
&= -\frac{\sqrt{3}\,\pi^3 X_i}{256} + \lim_{R\to\infty} \int_{-R}^{R} V_{\mathrm{odd}} \left(\frac{\mathrm{d}V_0}{\mathrm{d}x} - 20V_0^3 \left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x} \right) \frac{\mathrm{d}V_0}{\mathrm{d}x} \right) \mathrm{d}x,
\end{aligned}
$$

where we have used the fact that

$$
\int_{-\infty}^{\infty} \frac{x}{2} \left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x} \right) \frac{\mathrm{d}V_0}{\mathrm{d}x}\, \mathrm{d}x = \frac{\sqrt{3}\pi^3}{256},
$$

and that only the odd part of V_1 contributes to the integral. Using (SM5.6) we find numerically that

$$
\int_{-\infty}^{\infty} 20 V_{\mathrm{odd}} V_0^3 \left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x} \right) \frac{\mathrm{d}V_0}{\mathrm{d}x}\, \mathrm{d}x \approx -0.0731175 X_i.
$$

Thus we are left with evaluating

$$
\int_{-R}^{R} V_{\mathrm{odd}} \frac{\mathrm{d}V_0}{\mathrm{d}x}\, \mathrm{d}x.
$$

Now

$$
\int^{x} v_2(\bar{x}) V_0'(\bar{x})\, \mathrm{d}\bar{x} = -3^{1/4} (\log\cosh 2x + \operatorname{sech}^2 2x).
$$

Thus, integrating by parts,

$$\lim_{R\to\infty}\int_{-R}^{R} v_2(\bar{x})V_0'(\bar{x})3^{1/4}\frac{X_i}{32}\left(2\bar{x}\operatorname{sech}2\bar{x}+\frac{\pi}{2}-\sin^{-1}\operatorname{sech}2\bar{x}\right)\,\mathrm{d}\bar{x}$$

$$=\sqrt{3}\lim_{R\to\infty}\left[-(\log\cosh 2x+\operatorname{sech}^2 2x)\frac{X_i}{32}\left(2x\operatorname{sech}2x+\frac{\pi}{2}-\sin^{-1}\operatorname{sech}2x\right)\right]_{-R}^{R}$$

$$+\frac{\sqrt{3}X_i}{32}\int_{-\infty}^{\infty}(\log\cosh 2\bar{x}+\operatorname{sech}^2 2\bar{x})(-4\bar{x}\operatorname{sech}2\bar{x}\tanh 2\bar{x})\,\mathrm{d}\bar{x}$$

$$=\sqrt{3}(2R-\log 2)\frac{X_i\pi}{32}-0.31625X_i,$$

where we have used the fact that

$$\frac{\sqrt{3}}{32}\int_{-\infty}^{\infty}(\log\cosh 2\bar{x}+\operatorname{sech}^2 2\bar{x})(-4\bar{x}\operatorname{sech}2\bar{x}\tanh 2\bar{x})\,\mathrm{d}\bar{x}\approx -0.31625.$$

Since, numerically,

$$\int_{-\infty}^{\infty}\frac{3^{1/4}}{8}V_0'(x)v_1(x)\left(\log(1+\mathrm{e}^{4x})+x^2-2x-2x\tanh 2x-\log 2\right)\,\mathrm{d}x\approx -0.109394,$$

we have

$$\lim_{R\to\infty}\int_{-R}^{R}V_{\mathrm{odd}}\frac{\mathrm{d}V_0}{\mathrm{d}x}\,\mathrm{d}x=\sqrt{3}(2R-\log 2)\frac{X_i\pi}{32}-0.425644X_i. \tag{SM7.1}$$

Thus, finally,

$$\lim_{R\to\infty}\int_{-R}^{R}\left(V_1\frac{\mathrm{d}V_0}{\mathrm{d}x}-\left(\frac{xX_i}{2}+20V_1V_0^3\right)\left(\frac{V_0}{4}+\frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x}\right)\frac{\mathrm{d}V_0}{\mathrm{d}x}\right)\,\mathrm{d}x$$

$$=-\frac{\sqrt{3}\,\pi^3X_i}{256}+0.0731175X_i+\sqrt{3}(2R-\log 2)\frac{X_i\pi}{32}-0.425644X_i$$

$$=\frac{\sqrt{3}\,R\pi X_i}{16}-0.680174X_i. \tag{SM7.2}$$

Using (SM7.2) and (5.25) in (SM4.21) gives

$$\frac{\sqrt{3}\,\pi a_i^{(1)}(\lambda_1^2-1)}{8}+a_i^{(1)}\left(-\frac{4\sqrt{3}}{G^2}\mathrm{e}^{-X_{i+1}+X_i}-\frac{4\sqrt{3}}{G^2}\mathrm{e}^{-X_i+X_{i-1}}\right)$$

$$+\lambda_1 b^{(0)}\left(\frac{\sqrt{3}\,R\pi X_i}{16}-0.680174X_i\right)$$

$$=-2\sqrt{3}\left(2a_{i+1}^{(1)}+\lambda_1(1+R+X_i-X_{i+1})b^{(0)}\right)\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}$$

$$-2\sqrt{3}\left(2a_{i-1}^{(1)}+\lambda_1(-1-R+X_i-X_{i-1})b^{(0)}\right)\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}.$$

Using (4.15) the terms proportional to R cancel as they should, leaving

$$\begin{aligned}&\frac{\sqrt{3}\,\pi a_i^{(1)}(\lambda_1^2-1)}{8} + 4\sqrt{3}\,(a_{i+1}^{(1)} - a_i^{(1)})\frac{\mathrm{e}^{X_i - X_{i+1}}}{G^2} + 4\sqrt{3}\,(a_{i-1}^{(1)} - a_i^{(1)})\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\\ &\quad = 1.02026\,\lambda_1 b^{(0)} X_i - 2\sqrt{3}\,\lambda_1 b^{(0)}\left((X_i - X_{i+1})\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2} + (X_i - X_{i-1})\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\right).\end{aligned}$$

Dividing by $4\sqrt{3}$ gives

$$\begin{aligned}&\frac{\pi a_i^{(1)}(\lambda_1^2-1)}{32} + (a_{i+1}^{(1)} - a_i^{(1)})\frac{\mathrm{e}^{X_i - X_{i+1}}}{G^2} + (a_{i-1}^{(1)} - a_i^{(1)})\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\\ &\quad = 0.147262\,\lambda_1 b^{(0)} X_i - \frac{\lambda_1 b^{(0)}}{2}\left((X_i - X_{i+1})\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2} + (X_i - X_{i-1})\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\right).\end{aligned} \tag{SM7.3}$$

We can check our analysis, and identify the constant 0.147262, by considering the exact eigenfunction corresponding to $\lambda = 2G$. For this eigenfunction

$$f = \mathrm{i}V_s + G\left(\frac{V_s}{2} + \xi\frac{\mathrm{d}V_s}{\mathrm{d}\xi} - \frac{\mathrm{i}G\xi V_s}{2}\right),$$

so that with $\xi = x + X_i$,

$$f = \mathrm{i}V_0 + G\left(\frac{V_0}{2} + (x + X_i)\frac{\mathrm{d}V_0}{\mathrm{d}\xi}\right) + \mathrm{i}G^2\left(V_1 - \frac{(x+X_i)V_0}{2}\right) + G^3\left(\frac{V_1}{2} + \xi\frac{\mathrm{d}V_1}{\mathrm{d}\xi}\right) + \cdots .$$

Comparing to our expansion, with $b^{(0)} = 1$, and $\lambda_1 = 2$,

$$f = \mathrm{i}V_0 + G\left(a_i^{(1)}\frac{\mathrm{d}V_0}{\mathrm{d}x} + \frac{V_0}{2} + x\frac{\mathrm{d}V_0}{\mathrm{d}x}\right) + \mathrm{i}G^2\left(-a_i^{(1)}xV_0 + b_i^{(2)}V_0 - \frac{x^2V_0}{2} + V_1\right) + G^3\left(\cdots\right),$$

we have

$$a_i^{(1)} = X_i, \qquad b_i^{(2)} = -\frac{X_i^2}{2}.$$

Substituting this into (SM7.3) gives

$$\begin{aligned}&\frac{3\pi X_i}{32} + (X_{i+1} - X_i)\frac{\mathrm{e}^{X_i - X_{i+1}}}{G^2} + (X_{i-1} - X_i)\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\\ &\quad = 2 \times 0.147262\,X_i - \left((X_i - X_{i+1})\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2} + (X_i - X_{i-1})\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\right).\end{aligned} \tag{SM7.4}$$

This identifies the constant as

$$0.147262 = \frac{3\pi}{64}, \tag{SM7.5}$$

so that (SM7.3) becomes (SM4.22).

SM8. Derivation of (SM4.29).

Let us evaluate the integrals on the right-hand side of (SM4.28), namely

$$I_1 = \lim_{R\to\infty}\int_{-R}^{R}\left(-\lambda_1 U_2^R V_0 + \frac{\lambda_1}{4}x^2 X_i V_0^2 + 2\lambda_1 x V_1 V_0^5\right)\mathrm{d}x, \tag{SM8.1}$$

and

$$I_2 = -\lim_{R\to\infty}\int_{-R}^{R}\lambda_1 W_3^R V_0 + \left(\frac{(x+X_i)^2 V_0}{4} + 4V_1V_0^4\right)\left(V_1 - \frac{\lambda_1^2 x^2 V_0}{8}\right) + 2V_0^4(3V_1^2 + 2V_0V_2)\,\mathrm{d}x. \tag{SM8.2}$$

For (SM8.1) we require the odd part of V_1. Using (SM5.6) gives, numerically

$$\int_{-\infty}^{\infty}\left(\frac{\lambda_1}{4}x^2 X_i V_0^2 + 2\lambda_1 x V_1 V_0^5\right)\mathrm{d}x \approx 0.373466\lambda_1 X_i. \tag{SM8.3}$$

From (SM6.1) we have

$$\int_{-R}^{R} 4V_0^5 V_2 + 10V_0^4 V_1^2 + \frac{(x+X_i)^2 V_1 V_0}{4}\,\mathrm{d}x = -\left[V_0\frac{\mathrm{d}V_2}{\mathrm{d}x} - V_2\frac{\mathrm{d}V_0}{\mathrm{d}x}\right]_{-R}^{R}. \tag{SM8.4}$$

so that (SM8.2) is

$$I_2 = \left[V_0\frac{\mathrm{d}V_2}{\mathrm{d}x} - V_2\frac{\mathrm{d}V_0}{\mathrm{d}x}\right]_{-R}^{R} - \int_{-R}^{R}\lambda_1 W_3^R V_0 + \left(\frac{(x+X_i)^2 V_0}{4} + 4V_1V_0^4\right)\left(-\frac{\lambda_1^2 x^2 V_0}{8}\right)\mathrm{d}x. \tag{SM8.5}$$

Also

$$\int_{-\infty}^{\infty}\frac{(x+X_i)^2 V_0}{4}\left(-\frac{\lambda_1^2 x^2 V_0}{8}\right)\mathrm{d}x = -\lambda_1^2\frac{\sqrt{3}\,\pi^3(5\pi^2 + 16X_i^2)}{16384}, \tag{SM8.6}$$

$$\int_{-\infty}^{\infty} 4V_1V_0^4\left(-\frac{\lambda_1^2 x^2 V_0}{8}\right)\mathrm{d}x = -\frac{\lambda_1^2}{2}\int_{-\infty}^{\infty} x^2 V_{\mathrm{even}} V_0^5\,\mathrm{d}x, \tag{SM8.7}$$

where

$$V_{\mathrm{even}} = \frac{3^{1/4}}{2}\left(\frac{\mathrm{e}^{X_i - X_{i+1}}}{G^2} + \frac{\mathrm{e}^{X_{i-1} - X_i}}{G^2}\right)v_2(x) + \hat{V}_1(x) - \frac{X_i^2}{4}\left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x}\right). \tag{SM8.8}$$

Since

$$\int_{-\infty}^{\infty} v_2(x)x^2V_0(x)^5\,\mathrm{d}x = \frac{3^{1/4}(24-\pi^2)}{24},$$

$$\int_{-\infty}^{\infty}\hat{V}_1(x)x^2V_0(x)^5\,\mathrm{d}x \approx 0.0227007,$$

$$\int_{-\infty}^{\infty}\left(\frac{V_0}{4} + \frac{x}{2}\frac{\mathrm{d}V_0}{\mathrm{d}x}\right)x^2V_0(x)^5\,\mathrm{d}x = 0,$$

$$-\frac{\lambda_1^2}{2}\int_{-\infty}^{\infty} x^2 V_{\text{even}} V_0^5 \,\mathrm{d}x = -\lambda_1^2\left(\frac{\sqrt{3}(24-\pi^2)}{96}\left(\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}+\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\right)+0.0113504\right). \tag{SM8.9}$$

Collecting all this together gives

$$I_1 = -\lambda_1 \lim_{R\to\infty}\int_{-R}^{R} U_2^R V_0 \,\mathrm{d}x + 0.373466\lambda_1 X_i, \tag{SM8.10}$$

$$\begin{aligned} I_2 = {} & -\lambda_1\int_{-R}^{R} W_3^R V_0\,\mathrm{d}x + \left[V_0\frac{\mathrm{d}V_2}{\mathrm{d}x} - V_2\frac{\mathrm{d}V_0}{\mathrm{d}x}\right]_{-R}^{R} + \lambda_1^2\frac{\sqrt{3}\,\pi^3(5\pi^2+16X_i^2)}{16384} \\ & + \lambda_1^2\left(\frac{\sqrt{3}(24-\pi^2)}{96}\left(\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}+\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\right)+0.0113504\right). \end{aligned} \tag{SM8.11}$$

The term in square brackets involving V_0 and V_2 is evaluated in section SM6 and given by (SM6.5). We are left with the integrals involving U_2^R and W_3^R.

SM8.1. Analysis of U_2^R and W_3^R. Differentiating (4.11) gives

$$\frac{\mathrm{d}^3V_1}{\mathrm{d}x^3} + 5V_0^4\frac{\mathrm{d}V_1}{\mathrm{d}x} - \frac{\mathrm{d}V_1}{\mathrm{d}x} = -20V_0^3V_1\frac{\mathrm{d}V_0}{\mathrm{d}x} - \frac{(x+X_i)V_0}{2} - \frac{(x+X_i)^2}{4}\frac{\mathrm{d}V_0}{\mathrm{d}x}.$$

Thus, recalling (SM4.17), (SM4.18) we may write

$$U_2^R = \frac{\mathrm{d}V_1}{\mathrm{d}x} + U, \tag{SM8.12}$$

$$W_3^R = \lambda_1\left(\frac{V_1}{4} + \frac{x}{2}\frac{\mathrm{d}V_1}{\mathrm{d}x}\right) + W, \tag{SM8.13}$$

where

$$\begin{aligned} \frac{\mathrm{d}^2U}{\mathrm{d}x^2} + 5V_0^4U - U &= \frac{(1-\lambda_1^2)}{2}xV_0 + \frac{X_i}{2}V_0, \\ \frac{\mathrm{d}^2W}{\mathrm{d}x^2} + 5V_0^4W - W &= \frac{\lambda_1(2x^2+3xX_i+X_i^2)}{4}V_0 - \frac{\lambda_1^3x^2}{8}V_0. \end{aligned}$$

Integrating by parts

$$\int_{-R}^{R}\frac{\mathrm{d}V_1}{\mathrm{d}x}V_0\,\mathrm{d}x = [V_1V_0]_{-R}^{R} - \int_{-R}^{R} V_1\frac{\mathrm{d}V_0}{\mathrm{d}x}\,\mathrm{d}x = [V_1V_0]_{-R}^{R} - \int_{-R}^{R} V_{\text{odd}}\frac{\mathrm{d}V_0}{\mathrm{d}x}\,\mathrm{d}x, \tag{SM8.14}$$

where we evaluated the final integral already in (SM7.1). Note that

$$[V_1V_0]_{-R}^{R} = 2\sqrt{3}\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2} - 2\sqrt{3}\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2} = -\frac{\sqrt{3}\,\pi X_i}{16}. \tag{SM8.15}$$

Also

$$\int_{-R}^{R}\left(\frac{V_1}{4}+\frac{x}{2}\frac{\mathrm{d}V_1}{\mathrm{d}x}\right)V_0\,\mathrm{d}x=\int_{-R}^{R}\left(\frac{V_{\mathrm{even}}}{4}+\frac{x}{2}\frac{\mathrm{d}V_{\mathrm{even}}}{\mathrm{d}x}\right)V_0\,\mathrm{d}x,$$

where V_{even} is given by (SM8.8). Since

$$\lim_{R\to\infty}\int_{-R}^{R}\left(\frac{v_2}{4}+\frac{x}{2}\frac{\mathrm{d}v_2}{\mathrm{d}x}\right)V_0\,\mathrm{d}x=3^{1/4}\left(\frac{1}{2}-\frac{\pi^2}{48}+R^2+R\right),$$

$$\int_{-\infty}^{\infty}\left(\frac{\hat{V}_1}{4}+\frac{x}{2}\frac{\mathrm{d}\hat{V}_1}{\mathrm{d}x}\right)V_0\,\mathrm{d}x\approx 0.173106,$$

$$\int_{-\infty}^{\infty}V_0\left(\frac{1}{4}+\frac{x}{2}\frac{\mathrm{d}}{\mathrm{d}x}\right)^2V_0\,\mathrm{d}x=-\frac{\sqrt{3}\,\pi^3}{256},$$

we find

$$\lim_{R\to\infty}\int_{-R}^{R}\left(\frac{V_1}{4}+\frac{x}{2}\frac{\mathrm{d}V_1}{\mathrm{d}x}\right)V_0\,\mathrm{d}x=\frac{\sqrt{3}}{2}\left(\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}+\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\right)\left(\frac{1}{2}-\frac{\pi^2}{48}+R^2+R\right)$$

$$+0.173106+\frac{\sqrt{3}\,\pi^3}{1024}X_i^2. \tag{SM8.16}$$

This leaves us just with the integrals involving U and W. To determine U and W we will need to match with the solution in between the humps, which means we have to reintroduce the constant multipliers. The actual term in the integral (SM4.28) is

$$I_3=\lim_{R\to\infty}\int_{-R}^{R}a_i^{(1)}U+b^{(0)}W\,\mathrm{d}x. \tag{SM8.17}$$

The general solution to

$$\frac{\mathrm{d}^2f}{\mathrm{d}x^2}+5V_0^4f-f=g$$

is

$$f=-\frac{v_1(x)}{4}\int_a^x v_2(\bar{x})g(\bar{x})\,\mathrm{d}\bar{x}+\frac{v_2(x)}{4}\int_b^x v_1(\bar{x})g(\bar{x})\,\mathrm{d}\bar{x},$$

where v_1 and v_2 are the homogeneous solutions given by (SM5.4). With $f=U$, we have

$$g=g_U=\frac{(1-\lambda_1^2)}{2}xV_0+\frac{X_i}{2}V_0,$$

while with $f=W$, we have

$$g=g_W=\frac{\lambda_1(2x^2+3xX_i+X_i^2)}{4}V_0-\frac{\lambda_1^3x^2}{8}V_0.$$

Note that

$$\int^x V_0(\bar{x})v_1(\bar{x})\,\mathrm{d}\bar{x}=-\frac{V_0(x)^2}{2\times 3^{1/4}},$$

$$\int^x V_0(\bar{x})v_2(\bar{x})\,\mathrm{d}\bar{x}=2\times 3^{1/4}(x-\tanh 2x).$$

Then, integrating by parts,

$$\begin{aligned}
\lim_{R\to\infty}\int_{-R}^{R} fV_0\,\mathrm{d}x &= \frac{1}{4\times 3^{1/4}}\lim_{R\to\infty}\int_{-R}^{R} V_0(x)V_0'(x)\int_a^x v_2(\bar{x})g(\bar{x})\,\mathrm{d}\bar{x}\,\mathrm{d}x \\
&\quad - \frac{1}{4\times 3^{1/4}}\lim_{R\to\infty}\int_{-R}^{R} V_0(x)v_2(x)\int_b^x V_0'(\bar{x})g(\bar{x})\,\mathrm{d}\bar{x}\,\mathrm{d}x \\
&= \frac{1}{4\times 3^{1/4}}\lim_{R\to\infty}\left[\frac{V_0(x)^2}{2}\int_a^x v_2(\bar{x})g(\bar{x})\,\mathrm{d}\bar{x}\right]_{-R}^{R} - \frac{1}{4\times 3^{1/4}}\lim_{R\to\infty}\int_{-R}^{R}\frac{V_0(x)^2}{2}v_2(x)g(x)\,\mathrm{d}x \\
&\quad - \frac{1}{2}\lim_{R\to\infty}\left[(x-\tanh 2x)\int_b^x V_0'(\bar{x})g(\bar{x})\,\mathrm{d}\bar{x}\right]_{-R}^{R} + \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}(x-\tanh 2x)V_0'(x)g(x)\,\mathrm{d}x \\
&= -\frac{1}{4\times 3^{1/4}}\int_{-\infty}^{\infty}\frac{V_0(x)^2}{2}v_2(x)g(x)\,\mathrm{d}x - \frac{1}{2}(R-1)\int_b^{\infty} V_0'(\bar{x})g(\bar{x})\,\mathrm{d}\bar{x} \\
&\quad - \frac{1}{2}(R-1)\int_b^{-\infty} V_0'(\bar{x})g(\bar{x})\,\mathrm{d}\bar{x} + \frac{1}{2}\int_{-\infty}^{\infty}(x-\tanh 2x)V_0'(x)g(x)\,\mathrm{d}x.
\end{aligned}$$

Evaluating the full range integrals we find

$$\begin{aligned}
\int_{-\infty}^{\infty} V_0(x)^2 v_2(x)g_U(x)\,\mathrm{d}x &= 0, \\
\int_{-\infty}^{\infty}(x-\tanh 2x)V_0'(x)g_U(x)\,\mathrm{d}x &= 0, \\
\int_{-\infty}^{\infty} V_0(x)^2 v_2(x)g_W(x)\,\mathrm{d}x &= \frac{\lambda_1(4-\lambda_1^2)}{8}\,\frac{3^{3/4}\pi}{2}, \\
\int_{-\infty}^{\infty}(x-\tanh 2x)V_0'(x)g_W(x)\,\mathrm{d}x &= \frac{\lambda_1(4-\lambda_1^2)}{8}\,\frac{\sqrt{3}\,\pi(4-\pi^2)}{32}.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{R\to\infty}\int_{-R}^{R} UV_0 &= -\frac{1}{2}(R-1)\int_b^{\infty} V_0'(\bar{x})g_U(\bar{x})\,\mathrm{d}\bar{x} \\
&\quad - \frac{1}{2}(R-1)\int_b^{-\infty} V_0'(\bar{x})g_U(\bar{x})\,\mathrm{d}\bar{x},
\end{aligned} \tag{SM8.18}$$

$$\begin{aligned}
\lim_{R\to\infty}\int_{-R}^{R} WV_0 &= -\frac{1}{2}(R-1)\int_b^{\infty} V_0'(\bar{x})g_W(\bar{x})\,\mathrm{d}\bar{x} \\
&\quad - \frac{1}{2}(R-1)\int_b^{-\infty} V_0'(\bar{x})g_W(\bar{x})\,\mathrm{d}\bar{x} - \frac{\sqrt{3}\,\pi^3\lambda_1(4-\lambda_1^2)}{512}.
\end{aligned} \tag{SM8.19}$$

To evaluate the remaining integrals we need to determine b using the matching conditions (SM4.19), (SM4.20) on f_3, which give

$$\begin{aligned}
(a_i^{(1)}U_2 + b^{(0)}W_3)G^2 &\sim 12^{1/4}\left(-a_{i-1}^{(1)} - \frac{\lambda_1(x+X_i-X_{i-1})}{2}b^{(0)} + \frac{\lambda_1}{4}b^{(0)}\right)\mathrm{e}^{-x-X_i+X_{i-1}}\\
&\qquad \text{as } x \to -\infty,\\
(a_i^{(1)}U_2 + b^{(0)}W_3)G^2 &\sim 12^{1/4}\left(a_{i+1}^{(1)} + \frac{\lambda_1(x+X_i-X_{i+1})}{2}b^{(0)} + \frac{\lambda_1}{4}b^{(0)}\right)\mathrm{e}^{x+X_i-X_{i+1}}\\
&\qquad \text{as } x \to \infty.
\end{aligned}$$

Using (SM4.10), (SM4.9), (SM8.12), (SM8.13), gives the matching conditions

$$\begin{aligned}
a_i^{(1)}U + b^{(0)}W &\sim 12^{1/4}\left(a_i^{(1)} - a_{i-1}^{(1)} - \frac{\lambda_1(X_i-X_{i-1})}{2}b^{(0)}\right)\frac{\mathrm{e}^{-x-X_i+X_{i-1}}}{G^2} \qquad \text{as } x \to -\infty,\\
a_i^{(1)}U + b^{(0)}W &\sim 12^{1/4}\left(-a_i^{(1)} + a_{i+1}^{(1)} + \frac{\lambda_1(X_i-X_{i+1})}{2}b^{(0)}\right)\frac{\mathrm{e}^{x+X_i-X_{i+1}}}{G^2} \qquad \text{as } x \to \infty.
\end{aligned}$$

We can satisfy the conditions at $-\infty$ by setting $b = -\infty$ in both U and W and adding an appropriate multiple of v_2 corresponding to the matching condition. Since

$$v_2 \sim \sqrt{2}\,\mathrm{e}^{-x} \text{ as } x \to -\infty,$$

this multiple is

$$3^{1/4}\left(a_i^{(1)} - a_{i-1}^{(1)} - \frac{\lambda_1(X_i-X_{i-1})}{2}b^{(0)}\right)\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}v_2(x). \tag{SM8.20}$$

As a check, at infinity then

$$\begin{aligned}
a_i^{(1)}U + b^{(0)}W &\sim 3^{1/4}\left(a_i^{(1)} - a_{i-1}^{(1)} - \frac{\lambda_1(X_i-X_{i-1})}{2}b^{(0)}\right)\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}v_2(x)\\
&\qquad + \frac{v_2}{4}\int_{-\infty}^{\infty} v_1(\bar{x})\left(a_i^{(1)}g_U + b^{(0)}g_W\right)\mathrm{d}\bar{x}\\
&= 3^{1/4}\left(a_i^{(1)} - a_{i-1}^{(1)} - \frac{\lambda_1(X_i-X_{i-1})}{2}b^{(0)}\right)\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}v_2(x)\\
&\qquad - \frac{v_2}{4\times 3^{1/4}}\left(a_i^{(1)}\frac{(1-\lambda_1^2)}{2} + b^{(0)}\frac{\lambda_1 3X_i}{4}\right)\int_{-\infty}^{\infty} V_0'(\bar{x})\bar{x}V_0(\bar{x})\,\mathrm{d}\bar{x}\\
&= 12^{1/4}\left(a_i^{(1)} - a_{i-1}^{(1)} - \frac{\lambda_1(X_i-X_{i-1})}{2}b^{(0)}\right)\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\mathrm{e}^{x}\\
&\qquad + 12^{1/4}\pi\left(a_i^{(1)}\frac{(1-\lambda_1^2)}{32} + b^{(0)}\frac{\lambda_1 3X_i}{64}\right)\mathrm{e}^{x}\\
&= 12^{1/4}\left(a_{i+1}^{(1)} - a_i^{(1)} + \frac{\lambda_1(X_i-X_{i+1})}{2}b^{(0)}\right)\frac{\mathrm{e}^{x+X_i-X_{i+1}}}{G^2},
\end{aligned}$$

as required, using (SM4.22) (which also provides proof of (SM7.5)). With $b=-\infty$ we have two further integrals to evaluate. We have

$$\begin{aligned}
-\frac{1}{2}(R-1)\int_{-\infty}^{\infty} V_0'(a_i^{(1)}g_U+b^{(0)}g_W)\,\mathrm{d}x &= -\frac{1}{2}(R-1)\left(a_i^{(1)}\frac{(1-\lambda_1^2)}{2}+b^{(0)}\frac{3\lambda_1 X_i}{4}\right)\int_{-\infty}^{\infty} V_0' x V_0\,\mathrm{d}x\\
&= \frac{1}{2}(R-1)\left(a_i^{(1)}\frac{(1-\lambda_1^2)}{2}+b^{(0)}\frac{3\lambda_1 X_i}{4}\right)\frac{\sqrt{3}\,\pi}{4},
\end{aligned}\tag{SM8.21}$$

and we also need the integral of the extra term (SM8.20),

$$\begin{aligned}
&3^{1/4}\left(a_i^{(1)}-a_{i-1}^{(1)}-\frac{\lambda_1(X_i-X_{i-1})}{2}b^{(0)}\right)\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\int_{-R}^{R} v_2 V_0\,\mathrm{d}x\\
&\qquad = 4\sqrt{3}\left(a_i^{(1)}-a_{i-1}^{(1)}-\frac{\lambda_1(X_i-X_{i-1})}{2}b^{(0)}\right)\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}(R-1).
\end{aligned}\tag{SM8.22}$$

Collecting together the results of this section we have, using (SM8.12), (SM8.15), and (SM7.1),

$$\lim_{R\to\infty}\int_{-R}^{R} U_2^R V_0\,\mathrm{d}x = -\frac{\sqrt{3}\,R X_i\pi}{16}+0.203422X_i+\lim_{R\to\infty}\int_{-R}^{R} U V_0\,\mathrm{d}x,\tag{SM8.23}$$

since

$$-\sqrt{3}\,\frac{X_i\pi}{16}+\frac{\sqrt{3}\,\pi\log 2}{32}+0.425644\approx 0.203422.$$

Using (SM8.13), (SM8.16),

$$\begin{aligned}
\lim_{R\to\infty}\int_{-R}^{R} W_3^R V_0\,\mathrm{d}x &= \frac{\sqrt{3}\,\lambda_1}{2}\left(\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}+\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\right)\left(\frac{1}{2}-\frac{\pi^2}{48}+R^2+R\right)\\
&\quad +0.173106\lambda_1+\frac{\sqrt{3}\,\pi^3\lambda_1}{1024}X_i^2+\lim_{R\to\infty}\int_{-R}^{R} W V_0\,\mathrm{d}x.
\end{aligned}\tag{SM8.24}$$

Using (SM8.18), (SM8.19), (SM8.21), (SM8.22),

$$\begin{aligned}
\int_{-R}^{R}(a_i^{(1)}U+b^{(0)}W)V_0\,\mathrm{d}x &= -b^{(0)}\lambda_1(4-\lambda_1^2)\frac{\sqrt{3}\,\pi^3}{512}\\
&\quad +(R-1)\left(a_i^{(1)}\frac{(1-\lambda_1^2)}{16}+b^{(0)}\frac{3\lambda_1 X_i}{32}\right)\sqrt{3}\,\pi\\
&\quad +4\sqrt{3}\left(a_i^{(1)}-a_{i-1}^{(1)}-\frac{\lambda_1(X_i-X_{i-1})}{2}b^{(0)}\right)\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}(R-1)\\
&= -b^{(0)}\lambda_1(4-\lambda_1^2)\frac{\sqrt{3}\,\pi^3}{512}\\
&\quad +\sqrt{3}\,(R-1)\left(\lambda_1 b^{(0)}(X_i-X_{i+1})\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}\right.\\
&\qquad\qquad \left.-\lambda_1 b^{(0)}(X_i-X_{i-1})\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\right)\\
&\quad +\sqrt{3}\,(R-1)\left(2(a_{i+1}^{(1)}-a_i^{(1)})\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}-2(a_{i-1}^{(1)}-a_i^{(1)})\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\right),
\end{aligned}\tag{SM8.25}$$

using (SM4.22).

SM8.2. Putting it all together.

Using (SM8.10), (SM8.11), (SM6.5), (SM8.23), (SM8.24) and (SM8.25) in (SM4.28) gives, after some simplification,

$$\begin{aligned}
\operatorname{Im}\left[\frac{\mathrm{d}f_4}{\mathrm{d}x}V_0 - f_4\frac{\mathrm{d}V_0}{\mathrm{d}x}\right]_{-R}^{R} &= 0.170044\, a_i^{(1)}\lambda_1 X_i + \frac{\sqrt{3}\, a_i^{(1)} R\lambda_1 X_i\pi}{16} + b^{(0)}\lambda_1^2(4-\lambda_1^2)\frac{\sqrt{3}\,\pi^3}{512}\\
&\quad - \sqrt{3}\,(R-1)\lambda_1^2\left(b^{(0)}(X_i - X_{i+1})\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}\right.\\
&\qquad\qquad\qquad\qquad \left. - b^{(0)}(X_i - X_{i-1})\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\right)\\
&\quad - 2\sqrt{3}\,(R-1)\lambda_1\left((a_{i+1}^{(1)} - a_i^{(1)})\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2} - (a_{i-1}^{(1)} - a_i^{(1)})\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\right)\\
&\quad - b^{(0)}\lambda_1^2\frac{\sqrt{3}}{2}\left(\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2} + \frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\right)(R^2+R)\\
&\quad + b^{(0)}\frac{\sqrt{3}}{96}\mathrm{e}^{X_i-X_{i+1}}\left(-\pi^2 - 16R^3 - 48X_iR^2\right.\\
&\qquad\qquad \left. -48X_i^2R - 16X_i^3 - 48X_{i+1}^2 + 16X_{i+1}^3 + 48\log 2\, X_{i+1}\right)\\
&\quad + b^{(0)}\frac{\sqrt{3}}{96}\mathrm{e}^{-X_i+X_{i-1}}\left(-\pi^2 - 16R^3 + 48X_iR^2 - 48X_i^2R + 16X_i^3\right.\\
&\qquad\qquad \left. -48X_{i-1}^2 - 16X_{i-1}^3 - 48\log 2\, X_{i-1}\right)\\
&\quad + b_i^{(2)}\left(\frac{4\sqrt{3}}{G^2}\mathrm{e}^{-X_{i+1}+X_i} + \frac{4\sqrt{3}}{G^2}\mathrm{e}^{-X_i+X_{i-1}}\right),
\end{aligned}$$

where we have set

$$0.173106 - 0.0113504 = \frac{5\sqrt{3}\,\pi^5}{16384}.$$

Thus the final solvability condition is

$$
\begin{aligned}
&\frac{4\sqrt{3}}{G^2}\mathrm{e}^{X_i-X_{i+1}}b_{i+1}^{(2)} - \frac{\sqrt{3}}{G^2}\mathrm{e}^{X_i-X_{i+1}}\lambda_1 a_{i+1}^{(1)}(1+2R+2X_i-2X_{i+1})\\
&\qquad - \frac{b^{(0)}}{32\sqrt{3}G^2}\mathrm{e}^{X_i-X_{i+1}}\big(\pi^2+48\lambda_1^2R+48\lambda_1^2R^2+16R^3+48(\lambda_1^2+R)X_i^2+16X_i^3\\
&\qquad\qquad -48(\lambda_1^2+2\lambda_1^2R+\log 2)X_{i+1}+48(1+\lambda_1^2)X_{i+1}^2\\
&\qquad\qquad -16X_{i+1}^3+48X_i(\lambda_1^2+2\lambda_1^2R+R^2-2\lambda_1^2X_{i+1})\big)\\
&\qquad + \frac{4\sqrt{3}}{G^2}\mathrm{e}^{X_{i-1}-X_i}b_{i-1}^{(2)} - \frac{\sqrt{3}}{G^2}\mathrm{e}^{X_{i-1}-X_i}\lambda_1 a_{i-1}^{(1)}(-1-2R-2X_{i-1}+2X_i)\\
&\qquad - \frac{b^{(0)}}{32\sqrt{3}G^2}\mathrm{e}^{X_{i-1}-X_i}\big(\pi^2+48\lambda_1^2R+48\lambda_1^2R^2+16R^3+48(1+\lambda_1^2)X_{i-1}^2\\
&\qquad\qquad +16X_{i-1}^3-48(R^2+\lambda_1^2(1+2R))X_i+48(\lambda_1^2+R)X_i^2\\
&\qquad\qquad -16X_i^3-48X_{i-1}(\lambda_1^2(-1-2R)-\log 2+2\lambda_1^2X_i)\big)\\
&= 0.170044\, a_i^{(1)}\lambda_1 X_i + \frac{\sqrt{3}\, a_i^{(1)}R\lambda_1X_i\pi}{16} + b^{(0)}\lambda_1^2(4-\lambda_1^2)\frac{\sqrt{3}\,\pi^3}{512}\\
&\qquad - \sqrt{3}\,(R-1)\lambda_1^2\left(b^{(0)}(X_i-X_{i+1})\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2} - b^{(0)}(X_i-X_{i-1})\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\right)\\
&\qquad - 2\sqrt{3}\,(R-1)\lambda_1\left((a_{i+1}^{(1)}-a_i^{(1)})\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2} - (a_{i-1}^{(1)}-a_i^{(1)})\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\right)\\
&\qquad - b^{(0)}\lambda_1^2\frac{\sqrt{3}}{2}\left(\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}+\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\right)(R^2+R)\\
&\qquad + b^{(0)}\frac{\sqrt{3}}{96}\mathrm{e}^{X_i-X_{i+1}}\big(-\pi^2-16R^3-48X_iR^2-48X_i^2R-16X_i^3\\
&\qquad\qquad -48X_{i+1}^2+16X_{i+1}^3+48\log 2\, X_{i+1}\big)\\
&\qquad + b^{(0)}\frac{\sqrt{3}}{96}\mathrm{e}^{-X_i+X_{i-1}}\big(-\pi^2-16R^3+48X_iR^2-48X_i^2R+16X_i^3\\
&\qquad\qquad -48X_{i-1}^2-16X_{i-1}^3-48\log 2\, X_{i-1}\big)\\
&\qquad + b_i^{(2)}\left(\frac{4\sqrt{3}}{G^2}\mathrm{e}^{-X_{i+1}+X_i}+\frac{4\sqrt{3}}{G^2}\mathrm{e}^{-X_i+X_{i-1}}\right).
\end{aligned}
$$

The terms proportional to R^3 are

$$
\begin{aligned}
-\frac{b^{(0)}}{32\sqrt{3}}\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}16R^3 - \frac{b^{(0)}}{32\sqrt{3}}\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}16R^3 &= -\frac{b^{(0)}\sqrt{3}}{96}\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}16R^3\\
&\quad -\frac{b^{(0)}\sqrt{3}}{96}\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}16R^3.
\end{aligned}
$$

These cancel as they should. The terms proportional to R^2 are

$$\begin{aligned}
&-\frac{b^{(0)}}{32\sqrt{3}}\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}(48\lambda_1^2R^2+48X_iR^2)-\frac{b^{(0)}}{32\sqrt{3}}\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}(48\lambda_1^2R^2-48X_iR^2)\\
&\qquad=-b^{(0)}\frac{\sqrt{3}\lambda_1^2}{2}\left(\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}+\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\right)R^2\\
&\qquad\qquad+\frac{b^{(0)}\sqrt{3}}{96}\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}(-48X_iR^2)+\frac{b^{(0)}\sqrt{3}}{96}\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}(48X_iR^2).
\end{aligned}$$

These cancel as they should. The terms proportional to R are

$$\begin{aligned}
&-\frac{\sqrt{3}}{G^2}\mathrm{e}^{X_i-X_{i+1}}\lambda_1a_{i+1}^{(1)}2R-\frac{b^{(0)}}{32\sqrt{3}}\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}(48\lambda_1^2R+48X_i^2R-96\lambda_1^2X_{i+1}R+96X_i\lambda_1^2R)\\
&-\frac{\sqrt{3}}{G^2}\mathrm{e}^{X_{i-1}-X_i}\lambda_1a_{i-1}^{(1)}(-2R)-\frac{b^{(0)}}{32\sqrt{3}}\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}(48\lambda_1^2R-96\lambda_1^2X_iR+48X_i^2R+96X_{i-1}\lambda_1^2R)\\
&\qquad=\lambda_1a_i^{(1)}\frac{2\sqrt{3}\,R\pi X_i}{32}-b^{(0)}\frac{\sqrt{3}\lambda_1^2}{2}\left(\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}+\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\right)R\\
&\qquad-\sqrt{3}\,R\lambda_1^2b^{(0)}\left((X_i-X_{i+1})\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}-(X_i-X_{i-1})\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\right)\\
&\qquad-2\sqrt{3}\,R\lambda_1\left((a_{i+1}^{(1)}-a_i^{(1)})\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}-(a_{i-1}^{(1)}-a_i^{(1)})\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\right)\\
&\qquad\qquad+\frac{b^{(0)}\sqrt{3}}{96}\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}(-48X_i^2R)+\frac{b^{(0)}\sqrt{3}}{96}\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}(-48X_i^2R).
\end{aligned}$$

These cancel as they should. Thus, finally,

$$\begin{aligned}
&\frac{4\sqrt{3}}{G^2}\mathrm{e}^{X_i-X_{i+1}}b^{(2)}_{i+1}-\frac{\sqrt{3}}{G^2}\mathrm{e}^{X_i-X_{i+1}}\lambda_1 a^{(1)}_{i+1}(1+2X_i-2X_{i+1})\\
&\quad-\frac{b^{(0)}}{32\sqrt{3}G^2}\mathrm{e}^{X_i-X_{i+1}}\left(\pi^2+48\lambda_1^2X_i^2+16X_i^3-48(\lambda_1^2+\log 2)X_{i+1}+48(1+\lambda_1^2)X_{i+1}^2\right.\\
&\qquad\left.-16X_{i+1}^3+48X_i(\lambda_1^2-2\lambda_1^2X_{i+1})\right)\\
&\quad+\frac{4\sqrt{3}}{G^2}\mathrm{e}^{X_{i-1}-X_i}b^{(2)}_{i-1}-\frac{\sqrt{3}}{G^2}\mathrm{e}^{X_{i-1}-X_i}\lambda_1 a^{(1)}_{i-1}(-1-2X_{i-1}+2X_i)\\
&\quad-\frac{b^{(0)}}{32\sqrt{3}G^2}\mathrm{e}^{X_{i-1}-X_i}\left(\pi^2+48(1+\lambda_1^2)X_{i-1}^2+16X_{i-1}^3-48\lambda_1^2X_i+48\lambda_1^2X_i^2\right.\\
&\qquad\left.-16X_i^3-48X_{i-1}(-\lambda_1^2-\log 2+2\lambda_1^2X_i)\right)\\
&=0.170044\,a^{(1)}_i\lambda_1X_i+b^{(0)}\lambda_1^2(4-\lambda_1^2)\frac{\sqrt{3}\,\pi^3}{512}\\
&\quad+\sqrt{3}\,\lambda_1^2\left(b^{(0)}(X_i-X_{i+1})\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}-b^{(0)}(X_i-X_{i-1})\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\right)\\
&\quad+2\sqrt{3}\,\lambda_1\left((a^{(1)}_{i+1}-a^{(1)}_i)\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}-(a^{(1)}_{i-1}-a^{(1)}_i)\frac{\mathrm{e}^{-X_i+X_{i-1}}}{G^2}\right)\\
&\quad+b^{(0)}\frac{\sqrt{3}}{96}\mathrm{e}^{X_i-X_{i+1}}\left(-\pi^2-16X_i^3-48X_{i+1}^2+16X_{i+1}^3+48\log 2\,X_{i+1}\right)\\
&\quad+b^{(0)}\frac{\sqrt{3}}{96}\mathrm{e}^{-X_i+X_{i-1}}\left(-\pi^2+16X_i^3-48X_{i-1}^2-16X_{i-1}^3-48\log 2\,X_{i-1}\right)\\
&\qquad+b^{(2)}_i\left(\frac{4\sqrt{3}}{G^2}\mathrm{e}^{-X_{i+1}+X_i}+\frac{4\sqrt{3}}{G^2}\mathrm{e}^{-X_i+X_{i-1}}\right).
\end{aligned}$$

Simplifying gives

$$\begin{aligned}
&-0.170044a^{(1)}_i\lambda_1X_i+b^{(0)}\lambda_1^2(\lambda_1^2-4)\frac{\sqrt{3}\,\pi^3}{512}\\
&\quad+4\sqrt{3}(b^{(2)}_{i+1}-b^{(2)}_i)\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}+4\sqrt{3}(b^{(2)}_{i-1}-b^{(2)}_i)\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\\
&\quad+2\sqrt{3}a^{(1)}_{i+1}(X_{i+1}-X_i)\lambda_1\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}+2\sqrt{3}a^{(1)}_{i-1}(X_{i-1}-X_i)\lambda_1\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\\
&\quad+\sqrt{3}(2a^{(1)}_i-3a^{(1)}_{i+1})\lambda_1\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}+\sqrt{3}(-2a^{(1)}_i+3a^{(1)}_{i-1})\lambda_1\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\\
&\quad+\frac{\sqrt{3}}{2}b^{(0)}\frac{\mathrm{e}^{X_i-X_{i+1}}}{G^2}\left(-3\lambda_1^2X_i-\lambda_1^2X_i^2+3\lambda_1^2X_{i+1}+2\lambda_1^2X_iX_{i+1}-\lambda_1^2X_{i+1}^2\right)\\
&\quad+\frac{\sqrt{3}}{2}b^{(0)}\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\left(3\lambda_1^2X_i-\lambda_1^2X_i^2-3\lambda_1^2X_{i-1}+2\lambda_1^2X_iX_{i-1}-\lambda_1^2X_{i-1}^2\right)=0.
\end{aligned}$$

We can check our analysis and identify the constant 0.170044 using the exact eigenvalue $\lambda_1=2$, for which

$$b^{(0)}=1,\qquad a^{(1)}_i=X_i,\qquad b^{(2)}_i=-\frac{X_i^2}{2}.$$

Substituting this in and simplifying using (4.15) gives

$$-0.170044 \times 2X_i^2 + \sqrt{3}X_i^2\frac{\pi}{16} = 0,$$

so that

$$0.170044 = \frac{\sqrt{3}\pi}{32}.$$

Thus, using (4.15) again to simplify, the final solvability condition is

$$\begin{aligned}\text{(SM8.26)}\quad & \frac{b^{(0)}\lambda_1^2(\lambda_1^2-4)\pi^3}{512} + \frac{3\pi\lambda_1 X_i}{64}(b^{(0)}\lambda_1 X_i - 2a_i^{(1)})\\ &+ 4(b_{i+1}^{(2)} - b_i^{(2)})\frac{\mathrm{e}^{X_i - X_{i+1}}}{G^2} + 4(b_{i-1}^{(2)} - b_i^{(2)})\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\\ &+ 2a_{i+1}^{(1)}(X_{i+1}-X_i)\lambda_1\frac{\mathrm{e}^{X_i - X_{i+1}}}{G^2} + 2a_{i-1}^{(1)}(X_{i-1}-X_i)\lambda_1\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\\ &- 3a_{i+1}^{(1)}\lambda_1\frac{\mathrm{e}^{X_i - X_{i+1}}}{G^2} + 3a_{i-1}^{(1)}\lambda_1\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\\ &+ \frac{\lambda_1^2}{2}b^{(0)}\frac{\mathrm{e}^{X_i - X_{i+1}}}{G^2}\left(-X_i^2 + 3X_{i+1} + 2X_iX_{i+1} - X_{i+1}^2\right)\\ &+ \frac{\lambda_1^2}{2}b^{(0)}\frac{\mathrm{e}^{X_{i-1}-X_i}}{G^2}\left(-X_i^2 - 3X_{i-1} + 2X_iX_{i-1} - X_{i-1}^2\right) = 0.\end{aligned}$$

This is (SM4.29).

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