

Tent-Maps, Two-Point Sets, and the Self-Tietze Property



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Abstract

This thesis discusses three distinct topics.

A topological space X is said to be *self-Tietze* if for every closed $C \subset X$, every continuous $f: C \rightarrow X$ admits a continuous extension $F: X \rightarrow X$. We show that every disconnected, self-Tietze space is ultranormal. The Tychonoff Plank is an example of a compact self-Tietze space which is not completely normal, and we establish that a completely normal, zero-dimensional, homogeneous space need not be self-Tietze.

A subset of the plane is a *two-point set* if it meets every straight line in exactly two points. We show that a two-point set cannot contain a dense G_δ subset of an arc. We also show that the complement of a two-point set is necessarily path-connected. Finally, we construct a zero-dimensional subset of the plane of which the complement is simply-connected.

For $\lambda \in \mathbb{R}$, the *tent-map with slope λ* is the function $f: [0, 1] \rightarrow \mathbb{R}$ such that $f(x) = \lambda x$ for $x \leq \frac{1}{2}$ and $f(x) = \lambda(1 - x)$ for $x \geq \frac{1}{2}$. Properties of ω -limit sets of tent-maps, i.e. sets of the form

$$\bigcap_{n \in \mathbb{N}} \overline{\{f^{n+k}(x) \mid k \in \mathbb{N}\}}$$

for $x \in [0, 1]$, are examined, and an example of a tent-map and a closed, invariant, nonempty, internally chain transitive subset of $[0, 1]$ which is not an ω -limit set is given.

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A Comment on Definitions and Notation

In general, terms and notation are as defined in [7].

In contrast to [7], we define a topological space to be compact if and only if every open cover admits a finite subcover, the additional assumption that the space is Hausdorff not being made.

Terms and notation not defined in [7] are introduced when they are used. Some definitions from [7] are repeated where it is considered helpful to do so.

All that remains to mention here is the interval notation, $[a, b]$, which, throughout this thesis, is best thought of as the “convex hull” of the two points a and b . For an ordered space X we define,

$$[a, b]_X = \{x \in X \mid a \leq x \leq b \text{ or } b \leq x \leq a\}.$$

For $n \geq 2$ we define

$$[a, b]_{\mathbb{R}^n} = \{a + t(b - a) \mid t \in [0, 1]\}$$

We will omit the subscript where there is no ambiguity. Analogous definitions are made for (a, b) , $(a, b]$, and $[a, b)$.

Introduction

On the surface, the three chapters of this thesis will seem quite distinct and, indeed, I am not able to justify their relationship from a purely mathematical perspective. I am more fundamentally motivated by the methods and ideas employed in the proofs of mathematical results than in the results themselves and this has allowed me to pass quite freely between different topics in general topology.

In the first chapter we introduce and motivate a new topological property, “self-Tietze”, and investigate some of its properties.

Definition. A topological space X is said to be *self-Tietze* if for any closed $C \subset X$ and any continuous $f: C \rightarrow X$ there is a continuous $F: X \rightarrow X$ such that $F|_C = f$.

This is a basic and natural property in the setting of self maps and its study should create tools that will allow us to manipulate self maps with the same grace that existing machinery can handle real-valued maps.

We focus in particular on products of ordinals and discover that the Tychonoff Plank is self-Tietze, a result which will appear surprising given the theory established up to that point.

Toward the end of the chapter we look at ordered spaces and find an example of a completely normal, zero-dimensional, homogeneous space which is not self-Tietze.

In the second chapter we consider two-point sets.

Definition. A two-point set is a subset of the plane which meets every line in exactly two points.

The chief open problem concerning these sets is whether or not there exists one which is, in addition, Borel. With this problem in mind we take a constructive approach to two-point sets and prove that the closure of a two-point set cannot contain an arc and that the complement of a two-point set is necessarily path connected.

Inspired by the difficulty of strengthening the result about path-connectedness we proceed to construct a zero-dimensional subset of the plane of which the complement is simply-connected.

In the final chapter we look at tent-maps. These maps are central to the study of discrete dynamical systems in one dimension and have impressively rich dynamical properties for being so simply described.

Definition. For $\lambda \in \mathbb{R}$, the *tent-map of slope λ* is the real-valued function f on $[0, 1]$ such that $f(x) = \lambda x$ for $x \leq \frac{1}{2}$ and $f(x) = \lambda(1 - x)$ for $x \geq \frac{1}{2}$.

The aim of the chapter is to find those tent-maps with slopes in $[\sqrt{2}, 2]$ for which the closed, invariant, non-empty, internally chain transitive subsets are precisely the ω -limit sets. We show that a tent-map has this property whenever the critical point is periodic and we give an example of a tent-map which does not have this property. Although some details hold us back, much of the theory developed for handling tent-maps with periodic critical point can be generalised to work with a much larger subset of $[\sqrt{2}, 2]$.

Chapter 1

Self-Tietze Spaces

1.1 Basic Results

Definition 1.1. A topological space X is said to be *self-Tietze* if for any closed $C \subset X$ and any continuous $f: C \rightarrow X$ there is a continuous $F: X \rightarrow X$ such that $F|_C = f$.

Given the importance of continuity in general topology and of self-maps in so many areas of mathematics, the study of this property needs little motivation. Indeed, there is a substantial body of work on what is known as the extension problem: given two spaces X and Y , a closed subset A of X , and a continuous $f: A \rightarrow Y$, is there a continuous extension of f over X ? see [9, Section c-12: Extensors]. Oddly enough, self-maps have received little attention in this context.

The name *self-Tietze* comes from *Tietze's Extension Theorem*, see [7, 2.1.8]. It is this result which suggests that it would be natural to consider extending functions defined on *closed* subsets of a space. Tietze's Extension Theorem applies to give us that both $[0, 1]$ and $\mathbb{R}$ are self-Tietze.

Proposition 1.2. *Every Hausdorff self-Tietze space X is normal.*

Proof. Let C and D be closed, disjoint, non-empty subsets of X , and fix $c \in C$ and $d \in D$. Define $f: C \cup D \rightarrow X$ by mapping every element of C to c and every element of D to d . Since C and D are both closed in $C \cup D$ and f is constant on each of these regions we see that f is continuous. As X is self-Tietze and $C \cup D$ is closed in X , we obtain a continuous extension $F: X \rightarrow X$ of f . Now, because X is Hausdorff there are disjoint open U and V which contain c and d respectively. $F^{-1}(U)$ and $F^{-1}(V)$ are then disjoint open sets containing C and D respectively and hence X is normal. $\square$

We can use the same approach to show that a disconnected, self-Tietze space, X say, is zero-dimensional. In fact, we naturally obtain that such a space is ultranormal, a property equivalent to having large inductive dimension 0.

Definition 1.3. A space is said to be *ultranormal* if for any pair of disjoint, closed subsets, C and D , there exists a clopen set containing C and disjoint from D .

Proposition 1.4. *If a self-Tietze space X is disconnected then it is ultranormal.*

Proof. Let C and D be any pair of disjoint, closed subsets of X . Because X is disconnected there is a non-trivial clopen $E \subset X$; fix $a \in E$ and $b \in X \setminus E$. Define $f: C \cup D \rightarrow X$ such that $f(C) \subset \{a\}$ and $f(D) \subset \{b\}$. Since C and D are both closed in $C \cup D$ and f is constant on each of these regions, we see that f is continuous. As X is self-Tietze and $C \cup D$ is closed in X , we obtain a continuous $F: X \rightarrow X$ such that $F(C) \subset \{a\}$ and $F(D) \subset \{b\}$. Then $F^{-1}(E)$ is a clopen subset of X which contains C and is disjoint from D . Thus, X is ultranormal. $\square$

Definition 1.5. A subset A of a space X is a *retract* if there is a continuous $f: X \rightarrow A$ such that $f|_A = \text{id}_A$.

Proposition 1.6. *A closed retract D of a self-Tietze space X is self-Tietze.*

Proof. Let C be a closed subset of D and let $f: C \rightarrow D$ be continuous. Then C is closed in X and so there is a continuous function $F: X \rightarrow X$ such that $F|_C = f$. D being a retract of X means that there is a continuous $r: X \rightarrow D$ such that $r|_D = \text{id}_D$. Then for every $c \in C$,

$$(r \circ F)|_D(c) = r(F(c)) = r(f(c)) = f(c).$$

Thus $(r \circ F)|_D: D \rightarrow D$ is a continuous extension of f to D and so D is self-Tietze. $\square$

Definition 1.7. A space is said to be *retractifiable* if every closed subset is a retract.

Remark. Every retractifiable space is self-Tietze.

It is known that every retractifiable space is ultranormal so both $[0, 1]$ and $\mathbb{R}$ serve as examples of self-Tietze spaces which are not retractifiable. One natural question that arises is whether or not every disconnected, self-Tietze space is retractifiable. We answer this negatively later on when we show that the Tychonoff plank¹ is self-Tietze. Note that the Tychonoff plank cannot be retractifiable because every retractifiable space is hereditarily normal.

¹The Tychonoff plank is the space $(\omega_1 + 1) \times (\omega + 1)$ and is an example of a normal space which is not hereditarily normal.

Proposition 1.8. *Every retractsifiable space is hereditarily normal*

Proof. Let X be a retractsifiable space, let Y be a subspace of X , and let C and D be disjoint, closed subsets of Y . Let $r: X \rightarrow \overline{C \cup D}^X$ be a retraction. Then $U = \overline{C \cup D}^X \setminus \overline{D}^X$ and $V = \overline{C \cup D}^X \setminus \overline{C}^X$ are disjoint open subsets of $\overline{C \cup D}^X$ containing C and D respectively. It follows that $r^{-1}(U)$ and $r^{-1}(V)$ are disjoint open subsets of X separating C and D . Thus, X is hereditarily normal. $\square$

The following proposition is very simple, yet it is extremely valuable and will be used time and again.

Proposition 1.9 (The Self-Tietze Projection Characterisation). *A topological space $X = \prod_{i \in I} X_i$ is self-Tietze if and only if for every $i \in I$, every closed $C \subset X$, and every continuous $g: C \rightarrow X_i$, there is a continuous extension $G: X \rightarrow X_i$ of g .*

Proposition 1.10. *For every set I , 2^I is self-Tietze.*

Proof. Let C be any closed subset of 2^I . By Proposition 1.9 it is sufficient to show that every continuous $f: C \rightarrow 2$ has a continuous extension to 2^I .

Let $f: C \rightarrow 2$ be continuous and let $E = f^{-1}(\{1\})$. Then E is clopen in C and so is compact. Let $U = 2^I \setminus f^{-1}(\{0\})$ and note that $E \subset U$ and that U is open in 2^I . Because E is compact and U is open, there exists $V \subset U$, containing E , which is a finite union of the standard basic open sets. It follows that V is clopen because the standard basic open sets of 2^I are clopen.

Let $F: 2^I \rightarrow 2$ be such that $F^{-1}(\{1\}) = V$. Because V is clopen, F is continuous. Also, for every $c \in C$,

$$F(c) = 0 \implies c \notin V \implies c \notin U \implies f(c) = 0,$$

and

$$F(c) = 1 \implies c \in V \implies c \in E \implies f(c) = 1.$$

Thus, F is a continuous extension of f as required. $\square$

1.2 Ordinals and Products of Ordinals

We have introduced the property of being self-Tietze and have presented some basic results. Naturally, we want to know which topological spaces are self-Tietze and, just as naturally, we begin by considering some of the simplest topological spaces.

The Introduction of [14] comments that ordinals are retractsifiable but omits a proof. Such a result will be quite useful in the next two sections so we provide a proof here. It follows that ordinals are self-Tietze.

Proposition 1.11. *Ordinals are retractifiable.*

Proof. Let γ be any ordinal equipped with the order topology and let C be any closed subset of γ .

Let

$$A = \{\alpha \in \gamma \mid C \cap [\alpha, \gamma) = \emptyset\}$$

and let $\delta = \sup C$. If $A \neq \emptyset$ then $\delta < \gamma$ and $f: \gamma \rightarrow \delta$ such that $f|_\delta = \text{id}_\delta$ and $f(A) = \{\delta\}$ is a retraction. Thus, without loss of generality we may assume that $A = \emptyset$.

Define $r: \gamma \rightarrow C$ such that for all $\alpha \in \gamma$,

$$f(\alpha) = \inf (C \cap [\alpha, \gamma)).$$

We will show that r is continuous by showing that the preimage of every subbasic open subset of γ is open.

Fix $\beta \leq \gamma$. Because r is monotonically increasing, $r^{-1}([0, \beta))$ is downwards closed and therefore is open in γ . If $C \cap [0, \beta) = \emptyset$ then $r^{-1}((\beta, \gamma)) = \gamma$. If $C \cap [0, \beta) \neq \emptyset$ then

$$r^{-1}((\beta, \gamma)) = (\sup (C \cap [0, \beta]), \gamma).$$

In either case, $r^{-1}((\beta, \gamma))$ is open.

Thus, r is continuous and therefore C is a retract. □

Corollary 1.12. *Ordinals are self-Tietze.*

There are many paths we can take from here. We will proceed to consider products of ordinals because, as it will transpire, much of what is interesting about the property of being self-Tietze can be exhibited by products of ordinals.

We start with a simple result with a somewhat technical proof. We will show that any product of countable successor ordinals is necessarily self-Tietze. This is a generalisation of Proposition 1.10.

Lemma 1.13. *Every countable successor ordinal $\gamma + 1$ is homeomorphic to a closed retract of 2^γ .*

Proof. Throughout this proof we write $\chi_A \in 2^\gamma$ for the characteristic function of $A \subset \gamma$. Additionally, for each $\beta \in \gamma$, $\pi_\beta: 2^\gamma \rightarrow 2$ denotes the natural projection.

Define $\phi: \gamma + 1 \rightarrow 2^\gamma$ such that for all $\alpha \in \gamma + 1$,

$$\phi(\alpha) = \chi_{[0, \alpha)}$$

and fix $\beta \in \gamma$. For each $\alpha \in \gamma + 1$,

$$\phi(\alpha) \in \pi_\beta^{-1}(\{1\}) \iff \phi(\alpha)(\beta) = 1 \iff \beta \in [0, \alpha] \iff \alpha > \beta,$$

and equally,

$$\phi(\alpha) \in \pi_\beta^{-1}(\{0\}) \iff \alpha \leq \beta.$$

Thus, $(\pi_\beta \circ \phi)^{-1}(\{1\}) = (\beta, \gamma]$ and $(\pi_\beta \circ \phi)^{-1}(\{0\}) = [0, \beta]$. Because $(\beta, \gamma]$ and $[0, \beta]$ are both open in $\gamma + 1$, and $\beta \in \gamma$ was chosen arbitrarily, we see that ϕ is continuous.

Now we define $\psi: 2^\gamma \rightarrow \gamma + 1$ such that for all $f \in 2^\gamma$, $\psi(f) = \min f^{-1}(\{0\})$ (in this case $\min \emptyset$ is taken to be γ) and fix an open subset U of $\gamma + 1$. For each $f \in 2^\gamma$,

$$\begin{aligned} f &\in \psi^{-1}(U) \\ \iff \min f^{-1}(\{0\}) &\in U \\ \iff \text{there exists } \beta \in U &\text{ such that } f(\beta) = 0 \text{ and for all } \alpha < \beta, f(\alpha) = 1 \\ \iff \text{there exists } \beta \in U &\text{ such that } f \in \pi_\beta^{-1}(\{0\}) \cup \bigcup_{\alpha < \beta} \pi_\alpha^{-1}(\{1\}) \\ \iff f \in \bigcup_{\beta \in U} \left(\pi_\beta^{-1}(\{0\}) \cup \bigcup_{\alpha < \beta} \pi_\alpha^{-1}(\{1\}) \right). \end{aligned}$$

Thus $\psi^{-1}(U)$ is open and hence ψ is continuous.

Let C be the image of ϕ . C is closed because it is the continuous image of a compact space in a Hausdorff space.

Now, for each $\alpha \in \gamma + 1$,

$$(\psi \circ \phi)(\alpha) = \psi(\phi(\alpha)) = \psi(\chi_{[0, \alpha)}) = \min \chi_{[0, \alpha)}^{-1}(\{0\}) = \min [\alpha, \gamma] = \alpha.$$

Also, for each $\phi(\alpha) \in C$,

$$(\phi \circ \psi)(\phi(\alpha)) = (\phi \circ (\psi \circ \phi))(\alpha) = \phi(\alpha).$$

Thus, ϕ and $\psi|_C$ are mutual inverses and hence $\gamma + 1 \cong C$.

Finally, to see that C is a retract of 2^γ , note that $(\phi \circ \psi): 2^\gamma \rightarrow C$ is continuous and that $(\phi \circ \psi)|_C = \text{id}_C$. $\square$

Lemma 1.14. For any collection $\langle \langle C_i, X_i \rangle \mid i \in I \rangle$, where for each $i \in I$, C_i is a closed retract of the compact space X_i , $C = \prod_{i \in I} C_i$ is a closed retract of $X = \prod_{i \in I} X_i$.

Proof. For each $i \in I$, C_i is a closed subspace of a compact space and so is compact. By Tychonoff's theorem, both X and C are compact. Then C is a compact subset of the Hausdorff space X and hence is closed.

It remains to show that C is a retract of X . For each $i \in I$, let $f_i: X_i \rightarrow C_i$ be a retraction of X_i onto C_i . Define $f: X \rightarrow C$ such that $\pi_i \circ f = f_i \circ \pi_i$ for every $i \in I$. Then f is continuous because $\pi_i \circ f$ is continuous for every $i \in I$. Also, for every $i \in I$ and every $c \in C$,

$$(\pi_i \circ f|_C)(c) = (\pi_i \circ f)(c) = (f_i \circ \pi_i)(c) = f_i(\pi_i(c)),$$

and $\pi_i(c) \in C_i$ so $f_i(\pi_i(c)) = \pi_i(c)$. Thus, for each $i \in I$, $\pi_i \circ f|_C = \pi_i$ and hence $f|_C = \text{id}_C$. Therefore, f is a retraction of X onto C as required. $\square$

Proposition 1.15. *A product of countable successor ordinals is self-Tietze.*

Proof. Let $\langle \gamma_\alpha \mid \alpha \in \kappa \rangle$ be any sequence of countable ordinals. For each $\alpha \in \kappa$, Lemma 1.13 gives us a closed retract C_α of 2^{γ_α} which is homeomorphic to $\gamma_\alpha + 1$. For each $\alpha \in \kappa$, 2^{γ_α} is compact and so by Lemma 1.14, $\prod_{\alpha \in \kappa} C_\alpha$ is a closed retract of $\prod_{\alpha \in \kappa} 2^{\gamma_\alpha}$. $\prod_{\alpha \in \kappa} 2^{\gamma_\alpha}$ is homeomorphic to 2^λ for some cardinal λ , and hence by Proposition 1.10, $\prod_{\alpha \in \kappa} 2^{\gamma_\alpha}$ is self-Tietze. By Proposition 1.6, $\prod_{\alpha \in \kappa} C_\alpha \cong \prod_{\alpha \in \kappa} (\gamma_\alpha + 1)$ is self-Tietze. $\square$

We have shown that all ordinals are self-Tietze and that products of countable successor ordinals are self-Tietze. This naturally brings us to consider taking products of ordinals which are not countable successors.

If we drop the ‘successor’ condition in the statement of Proposition 1.15, which is equivalent to compactness, we will quickly run into difficulties. In particular, by [10, Theorem 3] we know that ω^{ω_1} is not normal and it follows from Proposition 1.2 that ω^{ω_1} is not self-Tietze. We will revisit this approach in the next section.

Things become more interesting if we instead drop the ‘countable’ condition and leave compactness in place. To handle uncountable ordinals we will make light use of some set theoretic notions and machinery. See [12, Chapter II, Section 6] for a thorough treatment. In contrast with this source, we use ‘club’ instead of ‘c.u.b.’ to refer to a closed unbounded subset of an ordinal.

We begin with an example which establishes that the ‘countable’ condition of 1.15 is required.

Example 1.16. $(\omega_1 + 1)^2$ is not self-Tietze.

Proof. Write $X = (\omega_1 + 1)^2$, let

$$C = \{\langle \alpha, \beta \rangle \in X \mid \beta \in \{\alpha, \omega_1\}\},$$

and note that this is a closed subset of X .

Let $f: C \rightarrow \omega_1$ be such that for all $\alpha \in \omega_1$, $f(\alpha, \alpha) = \omega_1$ and $f(\alpha, \omega_1) = \alpha$. f is continuous because it is the union of two compatible continuous functions defined on closed subsets of X .

If X were self-Tietze then the self-Tietze projection characterisation would give us the existence of a continuous extension of f to X . We will show that f admits no such extension and so establish that X is not self-Tietze. Suppose for a contradiction that f admits a continuous extension $F: X \rightarrow \omega_1$. For each $\alpha \in \omega_1$, $F|_{[0, \alpha] \times \{\alpha\}}$ is continuous and takes the value ω_1 at $\langle \alpha, \alpha \rangle$. Because ω_1 has uncountable cofinality we deduce that there exists $g(\alpha) < \alpha$ such that $F|_{[g(\alpha), \alpha] \times \{\alpha\}}$ is constant. By the Pressing Down Lemma (see [12, Lemma 6.15]) there exists $\gamma \in \omega_1$ such that $g^{-1}(\{\gamma\})$ is unbounded. But then for each $\beta \in g^{-1}(\{\gamma\})$,

$$F(\gamma, \beta) = F(g(\beta), \beta) = F(\beta, \beta) = f(\beta, \beta) = \omega_1.$$

Because $g^{-1}(\{\gamma\})$ is unbounded, continuity of F forces $F(\gamma, \omega_1)$ to be ω_1 . But then $F(\gamma, \omega_1) = f(\gamma, \omega_1) = \gamma$, a contradiction. $\square$

This example is able to do a lot of work for us. By Proposition 1.6 we have that any product of compact ordinals, at least two of which are uncountable, must fail to be self-Tietze. Given the results of this section so far, we are practically forced to consider the Tychonoff plank. Intuition suggests that it too will fail to be self-Tietze, yet a technical but nevertheless satisfying proof establishes the opposite.

Before we begin, we need some technical lemmas. The proofs are fairly long but there is nothing very deep going on here.

Lemma 1.17. *Let X be any space and let Y be a self-Tietze space. Let C and D be closed subsets of X and suppose D is homeomorphic to a closed subset of Y . Then for every continuous $f: C \rightarrow Y$ there is a continuous $F: C \cup D \rightarrow Y$ such that $F|_C = f$.*

Proof. Let D_Y be a closed subset of Y which is homeomorphic to D and let h be a homeomorphism from D onto D_Y . Then $h(C \cap D)$ is a closed subset of Y and $f \circ h^{-1}: h(C \cap D) \rightarrow Y$ is continuous. Because Y is self-Tietze, there is a continuous $F_Y: Y \rightarrow Y$ such that $F_Y(y) = (f \circ h^{-1})(y)$ for every $y \in h(C \cap D)$. Note that $F_Y|_{D_Y} \circ h: D \rightarrow Y$ is continuous, and that for every $c \in C \cap D$,

$$(F_Y|_{D_Y} \circ h)(c) = F_Y(h(c)) = (f \circ h^{-1})(h(c)) = f(c);$$

i.e. $F_Y|_{D_Y} \circ h$ and f agree on $C \cap D$. This allows us to define $F: C \cup D \rightarrow Y$ by declaring that $F|_C = f$ and that $F|_D = F_Y|_{D_Y} \circ h$. Finally, because C and D are

closed in $C \cup D$, and because $F|_C$ and $F|_D$ are continuous, we see that the preimage of any closed subset of Y is the union of two closed subsets of $C \cup D$ and hence that F is continuous. $\square$

Lemma 1.18. *For every continuous $f: \omega_1 \rightarrow \omega_1$ which is not eventually constant, there exists a club subset C of ω_1 such that*

$$f([c_0, c_1]) \subset [c_0, c_1]$$

for every $c_0 \leq c_1 \in C$.

Proof. Firstly, we define

$$D = \{\alpha \in \omega_1 \mid f([0, \alpha]) \subset [0, \alpha]\}$$

and show that this set is a club in ω_1 .

To see that D is closed let A be any bounded, non-empty subset of D . It suffices to show that $\sup A \in D$. For each $\beta < \sup A$ we have

$$\begin{aligned} \beta &< a \text{ for some } a \in A \\ \implies f(\beta) &\leq a \text{ for some } a \in A \\ \implies f(\beta) &\leq \sup A. \end{aligned}$$

Also, $f(\sup A) \leq \sup \{f(a) \mid a \in A\}$ since f is continuous, and hence, because $f(a) \leq a$ for every $a \in A$, we have that $f(\sup A) \leq \sup A$. We have shown that for every $\beta \leq \sup A$, $f(\beta) \leq \sup A$, and hence we see that $\sup A \in D$, as required.

Now, suppose for a contradiction that D is bounded and let γ be an upper bound for D . Recursively define the sequence $\langle \alpha_n \mid n \in \omega \rangle$ such that $\alpha_0 = \gamma + 1$ and for all $n \in \omega$, $\alpha_{n+1} = \sup f([0, \alpha_n])$. Certainly, $\alpha_0 > \gamma$. Also, if $\alpha_n > \gamma$ for some $n \in \omega$ then

$$\begin{aligned} \alpha_n &\notin D \\ \implies &\text{there exists } \beta \leq \alpha_n \text{ such that } f(\beta) > \alpha_n \\ \implies &\sup f([0, \alpha_n]) > \alpha_n \\ \text{i.e. } &\alpha_{n+1} > \alpha_n. \end{aligned}$$

This gives us that $\alpha_{n+1} > \alpha_n > \gamma$ for every $n \in \omega$. Let $\alpha_\omega = \sup_{n \in \omega} \alpha_n$ and note that $\alpha_\omega > \gamma$ and so cannot be an element of D .

By continuity,

$$f(\alpha_\omega) = f\left(\sup_{n \in \omega} \alpha_n\right) \leq \sup_{n \in \omega} f(\alpha_n).$$

For each $n \in \omega$, $f(\alpha_n) \in f([0, \alpha_n])$ which gives us that $f(\alpha_n) \leq \alpha_{n+1}$. Combining these two observations gives us that $f(\alpha_\omega) \leq \alpha_\omega$. Also, for each $\beta < \alpha_\omega$ there exists an $n \in \omega$ such that,

$$\begin{aligned} \beta &< \alpha_n \\ \implies \beta &\in [0, \alpha_n] \\ \implies f(\beta) &\in f([0, \alpha_n]) \\ \implies f(\beta) &\leq \sup f([0, \alpha_n]) = \alpha_{n+1} \\ \implies f(\beta) &\leq \alpha_\omega. \end{aligned}$$

We have shown that for every $\beta \leq \alpha_\omega$, $f(\beta) \leq \alpha_\omega$, a contradiction to the fact that α_ω does not lie in D .

Thus, D is a club subset of ω_1 , as required.

Now define

$$E = \{\alpha \in \omega_1 \mid f([0, \alpha]) \subset [0, \alpha]\}.$$

We shall establish that E , too, is a club subset of ω_1 .

To see that E is closed, let A be any non-empty, bounded subset of E . If $\beta \in \omega_1$ is such that $\beta \geq \sup A$ then

$$\begin{aligned} \beta &\geq \alpha \text{ for each } \alpha \in A \\ \implies f(\beta) &\geq \alpha \text{ for each } \alpha \in A \\ \implies f(\beta) &\geq \sup A. \end{aligned}$$

Hence, $\sup A \in E$ and so E is closed.

Note that for every $\alpha \in \omega_1$, $\{f^{-1}(\{\beta\}) \mid \beta \leq \alpha\} \cup \{f^{-1}([0, \alpha])\}$ is a partition of ω_1 into closed sets. Because the intersection of any two clubs in ω_1 is a club, it follows that there can be at most one element of this partition which is unbounded. Because f is not eventually constant, $f^{-1}(\{\beta\})$ cannot be unbounded for any $\beta \leq \alpha$. This gives us that $f^{-1}([0, \alpha])$ is bounded.

Now, suppose for a contradiction that E is bounded. Let α be any upper bound for E . Then $\alpha + 1 \notin E$ which gives us that there exists a $\beta \in \omega_1$ such that $\beta \geq \alpha + 1$ and $f(\beta) < \alpha + 1$. Then $\beta \in f^{-1}([0, \alpha])$ and so $f^{-1}([0, \alpha]) \neq \emptyset$. Furthermore, because f is not eventually constant, $f^{-1}([0, \alpha])$ is bounded. This allows us to consider $\sup f^{-1}([0, \alpha])$ and we see immediately that $\alpha < \beta \leq \sup f^{-1}([0, \alpha])$. Fixing an upper bound $\gamma \in \omega_1$ for E , we may now recursively define a sequence $\langle \alpha_n \mid n \in \omega \rangle$ such that $\alpha_0 = \gamma + 1$ and for all $n \in \omega$, $\alpha_{n+1} = \sup f^{-1}([0, \alpha_n])$.

Let $\alpha_\omega = \sup_{n \in \omega} \alpha_n$. Because the sequence $\langle \alpha_n \mid n \in \omega \rangle$ is increasing, we have that $\alpha_\omega > \gamma$, and hence that α_ω is not an element of E . This means that there is a $\beta \in \omega_1$ such that $\beta \geq \alpha_\omega$ and $f(\beta) < \alpha_\omega$. But then $f(\beta) < \alpha_n$ for some $n \in \omega$, and hence $\beta \in f^{-1}([0, \alpha_n])$. This forces

$$\beta \leq \sup f^{-1}([0, \alpha_n]) = \alpha_{n+1} < \alpha_\omega,$$

a contradiction.

Now let $C = D \cap E$. Noting that the intersection of two clubs in ω_1 is a club, one quickly verifies that C has the required properties. $\square$

Corollary 1.19. *Suppose $f: \omega_1 + 1 \rightarrow \omega_1 + 1$ is continuous and such that $f|_{\omega_1}$ is not eventually constant. Then $f|_{\omega_1}^{-1}(\{\omega_1\})$ is bounded and there exists a club subset C of ω_1 such that*

$$f([c_0, c_1]) \subset [c_0, c_1]$$

for every $c_0 \leq c_1 \in C$.

Proof. Firstly we show that $f|_{\omega_1}^{-1}([0, \omega_1))$ is closed in ω_1 . To see this, let B be any bounded, non-empty subset of $f|_{\omega_1}^{-1}([0, \omega_1))$. Then $f(B)$ is a countable set of countable ordinals and so $\sup f(B) \in [0, \omega_1)$. By continuity of f , $f(\sup B) \leq \sup f(B)$ and so $f(\sup B) \in [0, \omega_1)$. But then $\sup B \in f|_{\omega_1}^{-1}([0, \omega_1))$ and therefore $f|_{\omega_1}^{-1}([0, \omega_1))$ is closed.

Because $f|_{\omega_1}$ is not eventually constant, $f|_{\omega_1}^{-1}([0, \omega_1))$ is unbounded. This gives us that $f|_{\omega_1}^{-1}(\{\omega_1\})$ is bounded because the intersect of any two clubs in ω_1 is non-empty. Hence, there exists $\gamma < \omega_1$ such that $f|_{[\gamma, \omega_1)}: [\gamma, \omega_1) \rightarrow \omega_1$. Because ω_1 is self-Tietze, there exists a continuous extension $F: \omega_1 \rightarrow \omega_1$ of $f|_{[\gamma, \omega_1)}$. Clearly, F is not eventually constant, and so by Lemma 1.18 there exists a club subset D of ω_1 such that

$$F([d_0, d_1]) \subset [d_0, d_1]$$

for every $d_0 \leq d_1 \in D$. But then $C = D \cap [\gamma, \omega_1)$ is a club and

$$f([c_0, c_1]) \subset [c_0, c_1]$$

for every $c_0 \leq c_1 \in C$. $\square$

We are now ready to prove that the Tychonoff Plank is self-Tietze. The proof is broken into several lemmas and we begin by establishing some common notation.

$X = (\omega_1 + 1) \times (\omega + 1)$ will denote the Tychonoff Plank itself. For each $\alpha_0 \leq \alpha_1 \in \omega_1 + 1$, let V_{α_0, α_1} denote $[\alpha_0, \alpha_1] \times [0, \omega]$. For each $\beta \in \omega + 1$, let

$$X_\beta = (\omega_1 + 1) \times \{\beta\},$$

and for each $\alpha \in \omega_1 + 1$, let

$$Y_\alpha = \{\alpha\} \times (\omega + 1).$$

Given a closed subset C of X and $\beta \in \omega + 1$, we define C_β such that

$$C_\beta \times \{\beta\} = C \cap (\omega_1 \times \{\beta\})$$

Lemma 1.20. *Let $f: C \rightarrow \omega_1 + 1$ be a continuous function on a closed subset C of X . Write A_γ for $(\gamma + 1) \times (\omega + 1)$ for each $\gamma \in \omega_1$. If there exists $\gamma \in \omega_1$ such that $f|_{C \setminus A_\gamma}$ is continuously extendable to $X \setminus A_\gamma$, then f is continuously extendable to X .*

Proof. Suppose $\gamma \in \omega_1$ is such that there exists a continuous extension of $f|_{C \setminus A_\gamma}$ to X . Let

$$\delta = \max \{ \gamma, \sup (f(C \cap A_\gamma) \setminus \{\omega_1\}) \}$$

and note that δ is a countable ordinal because A_γ is countable. Clearly,

$$((\delta + 1) \cup \{\omega_1\}) \times (\omega + 1)$$

is homeomorphic to a product of countable successor ordinals and hence is self-Tietze. By the self-Tietze projection characterisation, it follows that there is a continuous extension of $f|_{C \cap A_\gamma}$ to A_γ . Noting that A_γ is clopen in X and combining this result with our assumption, we see that there is a continuous extension of f to X . $\square$

On fixing a closed subset C of X containing X_ω and a continuous $f: C \rightarrow \omega_1 + 1$, we have two cases to consider:

- $f|_{X_\omega}$ is eventually constant,
- $f|_{X_\omega}$ is the identity on a club subset of X_ω .

These two cases require fundamentally different approaches. The next four lemmas build our knowledge about the first case.

Lemma 1.21. *Let C be a closed subset of X and let $f: C \rightarrow \omega_1 + 1$ be continuous. If $X_\omega \subset C$ and $f|_{X_\omega}$ is constant then f is continuously extendable to X .*

Proof. Let $\gamma \in \omega_1 + 1$ be the constant value taken by f on X_ω . Note that we can trivially extend f to take the value γ on all of

$$(\omega_1 + 1) \times \{\alpha \in \omega \mid C \cap X_\alpha = \emptyset\}$$

without compromising continuity. Thus, without loss of generality we assume that $C \cap X_\alpha \neq \emptyset$.

For each $\alpha \in \omega + 1$, X_α is homeomorphic to $\omega_1 + 1$, which is retractifiable. Hence, for each $\alpha \in \omega + 1$, there exists a retraction $r_\alpha: X_\alpha \rightarrow C \cap X_\alpha$.

Define $F: X \rightarrow \omega_1 + 1$ as $\bigcup_{\alpha \in \omega+1} (f|_{C \cap X_\alpha} \circ r_\alpha)$ and note that this is an extension of f to X . To see that F is continuous, consider an arbitrary subset B of X and fix an element $b_\infty \in \overline{B}$.

For each $\alpha < \omega$, X_α is open in X and so $\overline{B} \cap X_\alpha = \overline{B \cap X_\alpha}^{X_\alpha}$. Also, $F|_{X_\alpha} = f|_{C \cap X_\alpha} \circ r_\alpha$ which is continuous, and so we have

$$F|_{X_\alpha}(\overline{B} \cap X_\alpha) = F|_{X_\alpha}(\overline{B \cap X_\alpha}^{X_\alpha}) \subset \overline{F|_{X_\alpha}(B \cap X_\alpha)} \subset \overline{F(B)}.$$

Hence, if $b_\infty \in X_\alpha$ for some $\alpha < \omega$ then $F(b_\infty) \in \overline{F(B)}$.

If $b_\infty \in X_\omega$ then $F(b_\infty) = f(b_\infty) = \gamma$. Also, for each $\alpha \in \omega$ there exists $h(\alpha) \in \omega + 1$ and $b_\alpha \in B$ such that $h(\alpha) > \alpha$ and $b_\alpha \in X_{h(\alpha)}$; for otherwise, $\bigcup_{\alpha' > \alpha} X_{\alpha'}$ would be an open set containing $b_\infty \in \overline{B}$ but disjoint from B . This gives us a sequence $\langle r_{h(\alpha)}(b_\alpha) \mid \alpha \in \omega \rangle$ in C . Because C is compact Hausdorff, the sequence has a limit point, c_∞ say, in C . Because $r_{h(\alpha)}(b_\alpha) \in X_{h(\alpha)}$ and $h(\alpha) > \alpha$ for each $\alpha \in \omega$, we must have that $c_\infty \in X_\omega$. But then

$$\gamma = f(c_\infty) \in \overline{f(\{r_{h(\alpha)}(b_\alpha) \mid \alpha \in \omega\})} = \overline{F(\{b_\alpha \mid \alpha \in \omega\})} \subset \overline{F(B)}.$$

and hence $F(b_\infty) \in \overline{F(B)}$.

As b_∞ was an arbitrary element of $\overline{B}$, we deduce that $F(\overline{B}) \subset \overline{F(B)}$. This holds for all $B \subset X$, and therefore, F is continuous. $\square$

Lemma 1.22. *Let C be a closed subset of X and let $f: C \rightarrow \omega_1 + 1$ be continuous. If $X_\omega \subset C$ and $f|_{X_\omega}$ is eventually constant then f is continuously extendable to X .*

Proof. Suppose $X_\omega \subset C$. If $f|_{X_\omega}$ is eventually constant then there exists $\alpha \in \omega_1$ and $\beta \in \omega_1 + 1$ such that $f|_{[\alpha, \omega_1] \times \{\omega\}}$ is constantly β . Let $\gamma = \max\{\alpha, \beta\}$ and let

$$g = f|_{C \setminus ((\gamma+1) \times (\omega+1))} \cup c_\gamma$$

where $c_\gamma: (\gamma + 1) \times \{\omega\} \rightarrow \omega_1 + 1$ maps every point to γ . $C \setminus ((\gamma + 1) \times (\omega + 1))$ and $(\gamma + 1) \times \{\omega\}$ are disjoint closed subsets of X , upon each of which g is continuous.

Because the union of these sets is the domain of g we have that g is continuous. Furthermore, it is clear that the domain of g is a closed subset of X , containing X_ω , and that g is constant on X_ω . Lemma 1.21 applies here and we obtain a continuous extension $G: X \rightarrow \omega_1 + 1$ of g . Then $G|_{X \setminus ((\gamma+1) \times (\omega+1))}$ is a continuous extension of $f|_{C \setminus ((\gamma+1) \times (\omega+1))}$ and we apply Lemma 1.20 for the result. $\square$

Lemma 1.23. *Let C be a closed subset of X and let $f: C \rightarrow \omega_1 + 1$ be continuous. If $C = (\omega_1 + 1) \times \{\omega\}$ then f is continuously extendable to X .*

Proof. Define $F: X \rightarrow \omega_1 + 1$ such that for all $\langle \alpha, \beta \rangle \in X$, $F(\alpha, \beta) = f(\alpha, \omega)$. Certainly, F is an extension of f . Also, for each open $U \subset \omega_1 + 1$, $F^{-1}(U) = f^{-1}(U) \times (\omega + 1)$, an open subset of X . Hence, F is continuous. $\square$

Lemma 1.24. *Let C be a closed subset of X and let $f: C \rightarrow \omega_1 + 1$ be continuous. If $X_\omega \subset C$ and C_n is bounded for each $n \in \omega$ then f is continuously extendable to F .*

Proof. For each $n \in \omega$ let α_n be an upper bound for C_n . Let $\alpha = \sup_{n \in \omega} \alpha_n$. Note that $f|_{X_\omega}$ admits a continuous extension G to X by Lemma 1.23. Then $G|_{[\alpha, \omega_1] \times (\omega+1)}$ is a continuous extension of $f|_{[\alpha, \omega_1] \times \{\omega\}}$ and so, by Lemma 1.20, there is a continuous extension of f to X . $\square$

We now tackle the second case with the following three lemmas.

Lemma 1.25. *Let C be a closed subset of X and let $f: C \rightarrow \omega_1 + 1$ be continuous. If there exists a club D such that $D \times (\omega + 1) \subset C$ and*

$$f(C \cap ([d_0, d_1] \times (\omega + 1))) \subset [d_0, d_1]$$

for all $d_0 \leq d_1 \in D$, then f is continuously extendable to X .

Proof. Fix a pair $d_0 < d_1 \in D$ such that $(d_0, d_1) \cap D = \emptyset$. Note that $[d_0, d_1] \times (\omega + 1)$ is homeomorphic to a product of countable successor ordinals and hence is self-Tietze. Note also that $f|_{C \cap ([d_0, d_1] \times (\omega+1))}$ only takes values in $[d_0, d_1]$. By the self-Tietze projection characterisation, there exists a continuous extension

$$F_{d_0, d_1}: [d_0, d_1] \times (\omega + 1) \rightarrow [d_0, d_1]$$

of $f|_{C \cap ([d_0, d_1] \times (\omega+1))}$. Define

$$F' = \bigcup \{F_{d_0, d_1} \mid d_0 < d_1 \in D \text{ and } (d_0, d_1) \cap D = \emptyset\} \cup f$$

and note that because $D \times (\omega + 1) \subset C$ that this is well-defined. Let

$$F = F'|_{(\min D, \omega_1 + 1] \times (\omega + 1)}.$$

F is an extension of $f|_{C \cap ((\min D, \omega_1 + 1] \times (\omega + 1))}$ and we aim to show that it is continuous. Let B be an arbitrary subset of $(\min D, \omega_1 + 1] \times (\omega + 1)$ and let $b \in \overline{B}$. For each $d_0 < d_1 \in D$ such that $(d_0, d_1) \cap D = \emptyset$, let $U_{d_0, d_1} = (d_0, d_1] \times (\omega + 1)$, an open subset of X .

If $b \in U_{d_0, d_1}$ for some $d_0, d_1 \in D$ then $b \in \overline{B \cap U_{d_0, d_1}}$. Because F_{d_0, d_1} is continuous,

$$F_{d_0, d_1}(\overline{B \cap U_{d_0, d_1}}) \subset \overline{F_{d_0, d_1}(B \cap U_{d_0, d_1})} \subset \overline{F(B)},$$

and hence $b \in \overline{F(B)}$.

Suppose therefore that $b = \langle \alpha, \beta \rangle \notin U_{d_0, d_1}$ for any $d_0, d_1 \in D$. Clearly, either $\alpha \in D$ or $\alpha = \omega_1$ and, in either case, we have that $F(b) = \alpha$. Fix $\delta \in [0, \alpha)$. One easily checks that α is not an isolated point in D and so we find $d \in D \cap (\delta, \alpha)$. Now, $(d, \alpha] \times (\omega + 1)$ is an open subset of X containing b and hence there exists $b' \in B \cap ((d, \alpha] \times (\omega + 1))$. As $d \leq \alpha \in D$, we have that

$$F([d, \alpha] \times (\omega + 1)) \subset [d, \alpha].$$

This gives us that $F(b') \in [d, \alpha] \subset (\delta, \alpha]$. As δ was arbitrary, we see that every open set containing α contains a point of $F(B)$ and hence that $F(b) = \alpha \in \overline{F(B)}$.

We have shown that $F(\overline{B}) \subset \overline{F(B)}$ and hence that F is continuous. The result now follows from an application of Lemma 1.20. $\square$

Lemma 1.26. *Let C be a closed subset of X and let $f: C \rightarrow \omega_1 + 1$ be continuous. If C_β is unbounded for each $n \in \omega$, $C_\omega = \omega_1$, and $f|_{X_\omega}$ is not eventually constant, then f is continuously extendable to X .*

Proof. For each $\beta \in \omega + 1$ let $f_\beta: C_\beta \rightarrow \omega_1 + 1$ be such that for all $\alpha \in \omega_1$, $f_\beta(\alpha) = f(\alpha, \beta)$. Let $D = \bigcap_{\beta \in \omega} C_\beta$ and note that D is a club subset of ω_1 . Because $\omega_1 + 1$ is self-Tietze, for each $\beta \in \omega + 1$ there exists a continuous extension $F_\beta: \omega_1 \rightarrow \omega_1 + 1$ of f_β . Suppose firstly that

$$S = \{\beta \in \omega \mid F_\beta \text{ is eventually constant}\}$$

is infinite. For each $\beta \in S$, let $\alpha_\beta \in \omega_1$ be such that $F_\beta(\alpha') = F_\beta(\alpha_\beta)$ for all $\alpha' \in (\alpha_\beta, \omega_1)$ and for all $\beta \in S$. Let $\alpha = \sup_{\beta \in S} \alpha_\beta$. For each $d \in D \cap (\alpha, \omega_1)$, and for each $\beta \in S$,

$$f_\beta(d) = F_\beta(d) = F_\beta(\alpha).$$

Therefore, because S is infinite, for each $d_0, d_1 \in D \cap (\alpha, \omega_1)$, $f_\omega(d_0) = f_\omega(d_1)$, and so f_ω is constant on $D \cap (\alpha, \omega_1)$, a club subset of ω_1 . Also, f_ω is not eventually constant and so, by Lemma 1.18, there is a club subset E of ω_1 such that $f_\omega|_E = \text{id}_E$. But then $D \cap E$ is a club and f_ω is required to be both the identity and a constant map on this club, a contradiction.

So S is finite. Let $T = (\omega + 1) \setminus S$. Then $(\omega_1 + 1) \times S$ and $(\omega_1 + 1) \times T$ is a partition of X into two clopen sets.

Because $\omega_1 + 1$ is self-Tietze, we can easily extend $f|_{C \cap ((\omega_1 + 1) \times S)}$ to $(\omega_1 + 1) \times S$.

For T , define $g: C \rightarrow \omega_1 + 1$ such that

$$g|_{C \cap ((\omega_1 + 1) \times T)} = f|_{C \cap ((\omega_1 + 1) \times T)}$$

and such that $g(\alpha, \beta) = \alpha$ for every $\beta \in S$. Then g satisfies the condition of Lemma 1.25 and hence, there is a continuous extension G of g to X . Then $G|_{(\omega_1 + 1) \times T}$ is a continuous extension of $f|_{C \cap ((\omega_1 + 1) \times T)}$.

Thus, f is continuously extendable to X . $\square$

Lemma 1.27. *Let C be a closed subset of X and let $f: C \rightarrow \omega_1 + 1$ be continuous. If $C_\omega = \omega_1$ and $f|_{X_\omega}$ is not eventually constant then f is continuously extendable to X .*

Proof. Let

$$S = \{\beta \in \omega \mid C_\beta \text{ is unbounded}\} \cup \{\omega\}$$

and let

$$T = \{\beta \in \omega \mid C_\beta \text{ is bounded}\} \cup \{\omega\}.$$

Let $g: X \rightarrow \omega_1 + 1$ be such that $g(\alpha, \beta) = f(\alpha, \omega)$ for all $\langle \alpha, \beta \rangle \in X$. Then

$$f|_{C \cap ((\omega_1 + 1) \times S)} \cup g|_{(\omega_1 + 1) \times T}$$

is a well-defined, continuous function. This function also satisfies the conditions of Lemma 1.26, and so admits a continuous extension, $F_0: X \rightarrow \omega_1 + 1$, say.

$f|_{C \cap ((\omega_1 + 1) \times T)}$ satisfies the conditions of Lemma 1.24, and so admits a continuous extension $F_1: X \rightarrow \omega_1 + 1$.

But then $F_0|_{(\omega_1 + 1) \times S} \cup F_1|_{(\omega_1 + 1) \times T}$ is a continuous extension of f to X , as required. $\square$

Finally, we bring everything together.

Proposition 1.28. *The Tychonoff Plank is self-Tietze*

Proof. Note that $X_\omega \cong \omega_1 + 1$ (a trivially closed subset of $\omega_1 + 1$) and hence, because $\omega_1 + 1$ is self-Tietze, we may apply Lemma 1.17 to obtain a continuous extension $F: C \cup X_\omega \rightarrow \omega_1 + 1$ of f . Therefore, without loss of generality we may assume that $X_\omega \subset C$.

The result now follows from Lemma 1.22, Lemma 1.27, and Proposition 1.9. $\square$

1.3 LOTSes and GO-Spaces

To further our investigation from the last section, we consider products of countable non-compact ordinals. The typical example being ω^ω which is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$, we find ourselves working with LOTSes.

Definition 1.29. A *linearly ordered topological space* (or *LOTS*) is a linearly ordered set equipped with the order topology.

Definition 1.30. A *generalized ordered space* (or *GO-space*) is a subspace of a linearly ordered space.

In [14], van Douwen shows that the rationals, the irrationals, and even the Sorgenfrey line, are retractifiable. Thus, we conclude that these spaces are self-Tietze. He establishes that every totally disconnected, locally compact GO-space is strongly retractifiable² so we know at once that the double-arrow space is also self-Tietze.

Writing $\mathbb{P}$ for the irrationals, $\mathbb{P} \times \{0, 1\}$ equipped with the lexicographical ordering and the order topology is the irrational double-arrow space. Van Douwen shows that this space is not retractifiable. Taking a different approach to the one he employed, we establish that the irrational double arrow space is not even self-Tietze. This is particularly surprising in light of the fact that the irrational double arrow space is homogeneous, suggesting that there is no significant shortage of continuous self-maps.

Let $P = \mathbb{P} \cap [0, 1]$ and let

$$X = P \times \{0, 1\}$$

with the topology given by the lexicographical order, (X is the irrational double-arrow space).

Let C' denote the cantor ternary set, i.e.

$$C' = \left\{ \sum_{n=1}^{\infty} a_n 3^{-n} \in [0, 1] \mid a_n \in \{0, 2\} \text{ for all } n \geq 1 \right\},$$

²A space X is *strongly retractifiable* if for every closed subset C there is a closed retraction $r: X \rightarrow C$.

and let $C = (P \cap C') \times \{0, 1\}$.

Define $f: P \cap C' \rightarrow [0, 1]$ such that for all $\langle a_n \mid n \geq 1 \rangle \in \{0, 2\}^{\mathbb{N}_1}$,

$$f\left(\sum_{n=1}^{\infty} a_n 3^{-n}\right) = \sum_{n=1}^{\infty} \frac{1}{2} a_n 2^{-n}.$$

f is the Cantor function restricted to $P \cap C'$. Note that f is a homeomorphism onto its image.

Firstly we prove a simple Lemma which helps form our picture of the function f .

Lemma 1.31. *For any dyadic rational³ $d \in (0, 1)$,*

$$(\sup f^{-1}([0, d)), \inf f^{-1}((d, 1])) \cap P \neq \emptyset.$$

Proof. Write d as $\sum_{k \leq n} d_k 2^{-k}$ where $d_k \in \{0, 1\}$ for each $k < n$ and $d_n = 1$. Let

$$\begin{aligned} a &= \sum_{k < n} 2d_k 3^{-k} + 3^{-n}, \\ b &= \sum_{k < n} 2d_k 3^{-k} + 2 \cdot 3^{-n}. \end{aligned}$$

The $a = \sup f^{-1}([0, d))$ and $b = \inf f^{-1}((d, 1])$. Certainly, $P \cap (a, b) \neq \emptyset$, as required. $\square$

Let $g: C \rightarrow X$ be such that for all $\langle x, \eta \rangle \in C$, $g(x, \eta) = \langle f(x), \eta \rangle$.

The rest of this section will establish that g cannot be continuously extended to the whole of X . All but the very end of the argument applies just as well to the function f as it does to g , and therefore, to avoid unnecessary complexity, we begin with a Lemma concerning continuous extensions of f .

Lemma 1.32. *If $F: P \rightarrow P$ is a continuous extension of f then there exists a sequence $\langle p_n \mid n \in \omega \rangle$ in P which converges to a point in $P \cap C'$ and is such that either:*

- $p_n < p$ and $F(p_n) > F(p)$ for all $n \in \omega$;
- $p_n > p$ and $F(p_n) < F(p)$ for all $n \in \omega$.

Proof. Suppose $F: P \rightarrow P$ is a continuous extension of f .

Let $D \subset (0, 1)$ denote the subset of dyadic rationals.

³A dyadic rational is a number of the form $\frac{q}{2^n}$ where $q \in \mathbb{Z}$ and $n \in \mathbb{N}$.

We recursively define non-empty, open intervals I_η , dyadic rationals d_η , and irrationals p_η , where $\eta \in \{0, 1\}^{<\omega}$, to satisfy the following requirements

$$I_\emptyset = (0, 1) \quad (1.1)$$

$$d_\eta \in I_\eta \cap D \quad (1.2)$$

$$p_\eta \in (\sup f^{-1}([0, d_\eta]), \inf f^{-1}((d_\eta, 1])) \cap P \quad (1.3)$$

$$\overline{I_{\eta \smallfrown \langle 0 \rangle}} \cup \overline{I_{\eta \smallfrown \langle 1 \rangle}} \subset I_\eta \cap (d_\eta, F(p_\eta)) \quad (1.4)$$

$$\overline{I_{\eta \smallfrown \langle 0 \rangle}} \cap \overline{I_{\eta \smallfrown \langle 1 \rangle}} = \emptyset \quad (1.5)$$

$$\text{len } I_\eta \leq 2^{-\text{len } \eta} \quad (1.6)$$

For sequences $\eta, \zeta \in \{0, 1\}^{<\omega}$, $\eta \smallfrown \zeta$ denotes the concatenation of η and ζ . Also, in condition (1.6), $\text{len } I_\eta$ is the distance between the two endpoints of the interval whereas $\text{len } \eta$ denotes the size of the domain of η .

To establish that this can be done we must show that for each $\eta \in \{0, 1\}^{<\omega}$ we can choose open intervals $I_{\eta \smallfrown \langle 0 \rangle}$ and $I_{\eta \smallfrown \langle 1 \rangle}$ consistent with conditions (1.4), (1.5), and (1.6). It suffices to show that

$$I_\eta \cap (d_\eta, F(p_\eta))$$

is a non-trivial open interval. Note that $(d_\eta, F(p_\eta))$ is an open interval, and is non-empty by virtue of the fact that $F(p_\eta)$ is irrational. Then we are done because I_η overlaps with $(d_\eta, F(p_\eta))$ by condition (1.2).

Also, we note that Lemma 1.31 allows us to choose p_η at each stage to conform with condition (1.3).

Conditions (1.4) and (1.6) give us that for each $\zeta \in \{0, 1\}^\omega$, $\bigcap_{\eta \subset \zeta} \overline{I_\eta}$ contains a single point, $\phi(\zeta)$, say. Furthermore, condition (1.5) ensures that $\phi: \{0, 1\}^\omega \rightarrow [0, 1]$ is an injection. Because $\{0, 1\}^\omega$ is uncountable and D is countable, we deduce that there exists $\zeta \in \{0, 1\}^\omega$ such that $\phi(\zeta) \notin D$. This allows us to define $p = f^{-1}(\{\phi(\zeta)\})$.

Condition (1.2) tells us that $\langle d_\eta \mid \eta \subset \zeta \rangle$ converges to $\phi(\zeta)$. Consider a pair of points $a, b \in P$ such that $a < p < b$. Then

$$f(a) < f(p) = \phi(\zeta) < f(b)$$

and so there exists $\eta \subset \zeta$ such that $f(a) < d_\eta < f(b)$. Then, $a \leq \sup f^{-1}([0, d_\eta])$ and $b \geq \inf f^{-1}((d_\eta, 1])$, and so condition (1.3) gives us that $a < p_\eta < b$. Therefore, $\langle p_\eta \mid \eta \subset \zeta \rangle$ converges to p . Because F is continuous, we automatically have that $\langle F(p_\eta) \mid \eta \subset \zeta \rangle$ converges to $F(p)$.

Suppose $F(p_\eta) > F(p)$ for some $\eta \subset \zeta$. Note that

$$F(p) = f(p) = \phi(\zeta) \in \bigcap_{\theta \subset \zeta} \overline{I_\theta} \subset \overline{I_{\eta \smallfrown \langle 0 \rangle}} \cup \overline{I_{\eta \smallfrown \langle 1 \rangle}}. \quad (1.7)$$

Condition (1.4) gives us that

$$F(p) \in (d_\eta, F(p_\eta))$$

and as we are assuming that $F(p) < F(p_\eta)$ we must have that $d_\eta < F(p)$. Since $F(p) = f(p)$, this gives us that $p \in f^{-1}((d_\eta, 1])$. Thus, $p \geq \inf f^{-1}((d_\eta, 1])$ and so, by condition (1.3), $p_\eta < p$. A symmetric argument implies that, if $F(p_\eta) < F(p)$, then $p_\eta > p$.

Furthermore, by equation (1.7), $F(p_\eta) \neq F(p)$ for each $\eta \subset \zeta$, and therefore there exists a subsequence $\langle p_n \mid n \in \omega \rangle$ of $\langle p_\eta \mid \eta \subset \zeta \rangle$ such that either:

(1) $F(p_n) > F(p)$ for all $n \in \omega$ (in which case, $p_n < p$ for all $n \in \omega$);

or (2) $F(p_n) < F(p)$ for all $n \in \omega$ (in which case, $p_n > p$ for all $n \in \omega$).

□

Now we are ready for the main result.

Proposition 1.33. *The irrational double-arrow space is not self-Tietze.*

Proof. Suppose for a contradiction that X is self-Tietze and let $G: X \rightarrow X$ be a continuous extension of g . Let $\pi: [0, 1] \times \{0, 1\} \rightarrow [0, 1]$ denote the projection map and define $F: P \rightarrow P$ to be such that $F(p) = \pi(G(p, 0))$ for all $p \in P$. Then F is a continuous extension of f and hence, by Lemma 1.32, there exists a sequence $\langle p_n \mid n \in \omega \rangle$ in P which converges to a point $p \in P \cap C'$ and is such that either:

1. $p_n < p$ and $F(p_n) > F(p)$ for all $n \in \omega$;

2. $p_n > p$ and $F(p_n) < F(p)$ for all $n \in \omega$.

In case (1), we have that $\langle \langle p_n, 0 \rangle \mid n \in \omega \rangle$ is a sequence in X which converges to $\langle p, 0 \rangle$. Continuity of G gives us that $\langle G(p_n, 0) \mid n \in \omega \rangle$ converges to $G(p, 0)$. Because $p \in P \cap C'$, $G(p, 0) = g(p, 0) = \langle f(p), 0 \rangle$. Also, for each $n \in \omega$, there exists $\eta \in \{0, 1\}$ such that

$$G(p_n, 0) = \langle F(p_n), \eta \rangle > \langle F(p), 0 \rangle = \langle f(p), 0 \rangle.$$

But then $(-\infty, \langle f(p), 1 \rangle)$ is an open interval of X , containing $G(p, 0)$ and excluding $G(p_n, 0)$ for every $n \in \omega$. So $\langle G(p_n, 0) \mid n \in \omega \rangle$ does not converge to $G(p, 0)$, a contradiction.

In case (2), we note that the sequence $\langle \langle p_n, 0 \rangle \mid n \in \omega \rangle$ converges to $\langle p, 1 \rangle$ and that $G(p, 1) = g(p, 1) = \langle f(p), 1 \rangle$. Continuity of G gives us a similar contradiction to before. $\square$

1.4 Summary

We have an example of a compact, zero-dimensional, self-Tietze space, as well as an example of a completely normal, zero-dimensional, self-Tietze space. We also know that every separable, metric, zero-dimensional space is self-Tietze (see [7][6.2.B]). The natural plan of action of course is to close this gap.

Question 1.34. Is every compact, zero-dimensional, completely normal space self-Tietze?

We might also extend our work on products of ordinals by attempting to determine precisely which such spaces are self-Tietze.

Question 1.35. Is $(\omega + 1) \times (\omega_\omega + 1)$ self-Tietze?

It should not be difficult to prove that ω_1^2 is self-Tietze or to extend this to ω_1^n for $n < \omega$, and it is known that $\omega_1^{\omega_1}$ is not normal (see [10]) and therefore not self-Tietze.

Question 1.36. Is ω_1^ω self-Tietze?

Chapter 2

Two-Point Sets

2.1 Dense G_δ Subsets of Arcs

A two-point set is a subset of the plane which meets every line in exactly two points. In 1914, Mazurkiewicz proved the existence of such a set using the axiom of choice, see [13]. Attempts to build a two-point set without explicitly using the axiom of choice quickly ran into difficulties and ultimately led to the following classic open problem [15, Problem 1068].

Problem 2.1. Is there a Borel set M in $\mathbb{R}^2$ which meets each straight line in exactly two points? Can M be a G_δ set?

In this chapter we take a constructive approach to two-point sets. We start by making a small improvement to the following well known result for two-point sets, [11, Lemma 3].

Lemma 2.2. *A two-point set cannot contain an arc.*

We will show that a two-point set cannot even contain an dense G_δ subset of an arc.

Definition 2.3. An *arc* in a space X is a function $f: [0, 1] \rightarrow \mathbb{R}^2$ which is a homeomorphism onto its image.

Lemma 2.4. *Given an arc $f: [0, 1] \rightarrow \mathbb{R}^2$ which is not a straight line segment, there exists a line $L \subset \mathbb{R}^2$ such that:*

- *there exists a unique $x \in (0, 1)$ such that $f(x) \in L$;*
- *$\text{ran}(f)$ is contained in the closure of a connected component of $\mathbb{R}^2 \setminus L$.*

Proof. For any $A \subset \mathbb{R}^2$ and any $p \in \mathbb{R}^2 \setminus A$ let $D(p, A)$ denote the connected component of p in $\mathbb{R}^2 \setminus A$.

Let f be as described and let C be the convex hull of $\text{ran}(f)$. As f is not a line segment we must have $\text{int}(C) \neq \emptyset$; fix $p \in \text{int}(C)$.

Let $\mathcal{L}$ be the set of lines L in the plane with the following properties:

1. $f(0), f(1), p \notin L$;
2. $L \cap \text{ran}(f) \neq \emptyset$;
3. $C \subset \overline{D(p, A)}$.

For each $L \in \mathcal{L}$, let

$$\mathcal{R}_L = \{R \subset \mathbb{R}^2 \mid R \text{ is an infinite ray from } p \text{ and } R \cap \text{int}_L(C \cap L) \neq \emptyset\},$$

$$U_L = \bigcup \mathcal{R}_L \setminus \{p\}.$$

We first establish some properties of these sets.

For each $L \in \mathcal{L}$, because $p \notin L$, $\mathcal{R}_L$ is open in the space of all rays from p with the natural topology, and therefore U_L is open in $\mathbb{R}^2$.

Now, suppose $L_0, L_1 \in \mathcal{L}$ are such that there exists $R \in \mathcal{R}_{L_0} \cap \mathcal{R}_{L_1}$. For $i = 0, 1$, Let $p_i \in R \cap L_i \cap C$. If $p_0 \neq p_1$ w.l.o.g. we may assume $p_0 \in [p, p_1]$. L_0 passes through p_0 and not p and so separates p from p_1 . Thus $p_1 \notin \overline{D(p, L_0)}$; contradicting $L_0 \in \mathcal{L}$. Therefore $p_0 = p_1 \in R \cap L_0 \cap L_1$. Because $\mathcal{R}_{L_0}$ and $\mathcal{R}_{L_1}$ are open in the space of all rays from p , a space which has no isolated points, we must have that $|\mathcal{R}_{L_0} \cap \mathcal{R}_{L_1}| > 1$. Thus, there exists $R' \in \mathcal{R}_{L_0} \cap \mathcal{R}_{L_1}$ distinct from R . As with R , we can find a point $q \in R' \cap L_0 \cap L_1$. Because R and R' are distinct rays from p , we must have $p_0 \neq q$ and hence that $L_0 = L_1$. Taking the contrapositive, if $L_0 \neq L_1 \in \mathcal{L}$ then $\mathcal{R}_{L_0} \cap \mathcal{R}_{L_1} = \emptyset$, and hence $U_{L_0} \cap U_{L_1} = \emptyset$.

We now establish that $|\mathcal{L}| = 2^{\aleph_0}$. For each point $q \in (f(y), f(1))$, let L_q be the line through $f(0)$ and q . Let $z \in D(f(y), L_q) \cap \text{ran}(f)$ be maximally distant from L_q . Let M_q be the line through z parallel to L_q . Then $M_q \cap \text{ran}(f) \neq \emptyset$ because $z \in M_q \cap \text{ran}(f)$. Also, $M_q \subset D(f(y), L_q)$ and $f(0), f(1) \notin D(f(y), L_q)$ so $f(0), f(1) \notin M_q$. Maximality gives us that $\text{ran}(f) \subset \overline{D(f(1), M_q)}$ and therefore that $C \subset \overline{D(f(1), M_q)}$ because $\overline{D(f(1), M_q)}$ is convex. Then $\text{int}(C) \subset D(f(1), M_q)$ because $D(f(1), M_q)$ is open. Finally note that $p \notin M_q$ because $p \in \text{int}(C)$, and so $C \subset \overline{D(p, M_q)}$. Thus, $M_q \in \mathcal{L}$. If $q_0 \neq q_1 \in (f(y), f(1))$ then L_{q_0} is not parallel to L_{q_1} and hence M_{q_0} is distinct from M_{q_1} . Therefore $|\mathcal{L}| = 2^{\aleph_0}$.

Because $\mathbb{R}^2$ is separable, there exists $L \in \mathcal{L}$ such that $U_L = \emptyset$. This implies that $\text{int}_L(C \cap L) = \emptyset$, and because C is closed and convex and $L \cap C \neq \emptyset$, we must have that C is a singleton, $\{q\}$ say. $L \cap \text{ran}(f) \neq \emptyset$ and so $q \in \text{ran}(f)$, $q = f(x)$ say. Also, because f is injective, $x \in [0, 1]$ is unique such that $f(x) \in L$. Finally, if $a, b \in \text{ran}(f) \setminus L$ then $a, b \in C \subset \overline{D(p, L)}$ because $L \in \mathcal{L}$. Then $a, b \in \overline{D(p, L)} \setminus L = D(p, L)$, a convex set. Thus $[a, b] \subset D(p, L)$ and therefore $[a, b] \cap L = \emptyset$. $\square$

We are ready for the main result.

Proposition 2.5. *A two-point set cannot contain a dense G_δ subset of an arc.*

Proof. Let X be a two-point set and suppose for a contradiction that there exists a subset A of X and an arc $f: [0, 1] \rightarrow \mathbb{R}^2$ such that A is a dense, G_δ subset of $\text{ran}(f)$. By Lemma 2.4 there exists a line $L \subset \mathbb{R}^2$ such that:

1. $f(0), f(1) \notin L$;
2. there exists a unique $x \in [0, 1]$ such that $f(x) \in L$;
3. if $a, b \in \text{ran}(f) \setminus L$ then $[a, b] \cap L = \emptyset$.

As X is a two-point set, there exists a point $p \in L \cap X \setminus \{f(x)\}$.

For $\alpha \in \{0, x, 1\}$, let L_α be the line through p and $f(\alpha)$. Because $L \cap [f(0), f(1)] = \emptyset$, we must have that either $L_0 \cap [f(x), f(1)] \neq \emptyset$ or $L_1 \cap [f(x), f(0)] \neq \emptyset$. Without loss of generality we may assume that $L_0 \cap [f(x), f(1)] \neq \emptyset$.

For each $y \in (f(0), f(x))$, let L_y be the line through p and y and notice that this line separates $f(x)$ from $f(0)$ and $f(1)$. Let $\mathcal{L} = \{L_y \mid y \in (f(0), f(x))\}$, and let $U = \bigcup \mathcal{L} \setminus \{p\}$, an open set. Define $\psi: U \rightarrow (f(0), f(x))$ such that for all $u \in U$, $u \in L_{\psi(u)}$, and note that this is a well-defined continuous map.

Let $B_0 = \psi(f([0, x]) \cap A \cap U)$. $f([0, x])$ is a G_δ in $\mathbb{R}^2$, so $f([0, x]) \cap A \cap U$ is G_δ , and hence B_0 is a G_δ in $(f(0), f(x))$. Now let V be a non-empty open subset of $(f(0), f(x))$. for each $v \in V$, $\psi^{-1}(\{v\}) = L_v \setminus \{p\}$. L_v separates $f(0)$ from $f(x)$ so $f([0, x]) \cap L_v \neq \emptyset$. As $p \notin \text{ran}(f)$, we deduce that $f([0, x]) \cap \psi^{-1}(\{v\}) \neq \emptyset$ and hence that $f([0, x]) \cap \psi^{-1}(V) \neq \emptyset$. By continuity of ψ , we have that $f([0, x]) \cap \psi^{-1}(V)$ is a non-empty open subset of $f([0, x])$. Because A is dense in $\text{ran}(f)$ there exists $a \in A$ such that $\psi(a) \in V \cap B_0$. Thus B_0 is dense in $(f(0), f(x))$.

Similarly, we may set $B_1 = \psi(f([x, 1]) \cap A \cap U)$, and show that it is a dense G_δ in $(f(0), f(x))$.

But $(f(0), f(x))$ is Baire and so $B_0 \cap B_1 \neq \emptyset$. Let $y \in B_0 \cap B_1$. Then $L_y \cap f([0, x]) \cap A \neq \emptyset$ so there exists $b_0 \in [0, x]$ such that $f(b_0) \in L_y \cap A$. Similarly, there

exists $b_1 \in [x, 1]$ such that $f(b_1) \in L_y \cap A$. $b_0 \neq b_1$ because $f(x) \notin U$, and therefore $f(b_0) \neq f(b_1)$ because f is injective. Thus L_y meets $A \subset X$ in two points. But $p \in X \cap L_y \setminus \text{ran}(f)$ and so L_y meets X in at least 3 points; contradiction. $\square$

From this we have the following delightful corollary.

Corollary 2.6. *The closure of a G_δ two-point set cannot contain an arc.*

This seems to place some significant restrictions on G_δ two-point sets, particularly in view of the following result which follows from work in [8].

Definition 2.7. Given a metric space $\langle X, d \rangle$ and a subset A of X the diameter of A is

$$\text{diam } A = \sup \{d(x, y) \mid x, y \in A\}.$$

Definition 2.8. Given a metric space $\langle X, d \rangle$ and a subset A of X , the Hausdorff 1-measure of A is

$$\liminf_{\delta \rightarrow 0} \left\{ \sum_{n \in \mathbb{N}} \text{diam } B_n \mid A \subset \bigcup_{n \in \mathbb{N}} B_n \text{ and for all } n \in \mathbb{N}, \text{diam } B_n < \delta \right\}$$

Proposition 2.9. *The closure of a two-point set must have infinite Hausdorff 1-measure.*

2.2 Connection Properties of TPS Complements

In this section we will show that the complement of a two-point set is necessarily path-connected. It is known that every two-point set is zero-dimensional [11] but path-connectedness of the complement does not follow from this directly. The result itself does not tell us much about the constructability of two-point sets but the proof itself is quite constructive, providing a fresh way to handle these sets.

The fact that path-connectedness of the complement does not follow from zero-dimensionality is subtle and we look at this in more detail in the following section.

Definition 2.10. Given $A \subset Y \subset \mathbb{R}^2$, A is said to be convex with respect to Y if for every pair of points in A , the line segment between these points is a subset of $A \cup (\mathbb{R}^2 \setminus Y)$.

The following lemma is what really allows us to prove that the complement of a two-point set is necessarily path-connected. It tells us that if we consider any path-component of the complement of a two-point set then it must be either convex or it must contain a co-bounded subset of a line.

Lemma 2.11. *Let X be a two-point set and write Y for $\mathbb{R}^2 \setminus X$. If P is a path-component of Y and P is not convex with respect to Y then P contains a co-bounded subset of some line l .*

Proof. Because P is not convex with respect to Y , there exists a line l and points a , b , and c on l (with b between a and c) such that $a, c \in P$ and $b \in Y \setminus P$. Let R_a (resp. R_c) denote the infinite ray along l from a (resp. c). Because P is path connected, there must exist points x_a and x_c in X such that x_a is between a and b and x_c is between c and b . Then, because X is a two-point set, R_a and R_c are subsets of P . Thus, P contains a co-bounded subset of l . $\square$

Lemma 2.12. *Let X be a two-point set and write Y for $\mathbb{R}^2 \setminus X$. If Y is not path-connected then Y has a path-component which is convex with respect to Y .*

Proof. Suppose that Y is not path-connected and let P_0 and P_1 be distinct path-components of Y . Suppose for a contradiction that neither P_0 nor P_1 is convex with respect to Y . By Lemma 2.11, P_0 and P_1 contain co-bounded subsets C_0 and C_1 of some lines l_0 and l_1 respectively. By assumption P_0 and P_1 are disjoint and so l_0 and l_1 must be parallel. Let $f_0, f_1: \mathbb{R} \rightarrow \mathbb{R}^2$ be such that $C_i \subset \text{ran}(f_i) \subset P_i$ for $i = 0, 1$. One can find distinct lines l_2 and l_3 (parallel to l_0 and l_1) which do not intersect $\text{ran}(f_0) \cup \text{ran}(f_1)$ and which do not lie between l_0 and l_1 . As X is a two-point set, there exist points x_2 and x_3 in X such that $x_2 \in l_2$ and $x_3 \in l_3$. Let l be the line through x_2 and x_3 . l is not parallel to l_0 or l_1 so we must have that $l \cap \text{ran}(f_0) \neq \emptyset$ and $l \cap \text{ran}(f_1) \neq \emptyset$; let $x_0 \in l \cap \text{ran}(f_0)$ and let $x_1 \in l \cap \text{ran}(f_1)$. Because l_2 is not between l_0 and l_1 and because l_2 does not intersect $\text{ran}(f_0) \cup \text{ran}(f_1)$, it follows that x_2 is not between x_0 and x_1 . Similarly x_3 does not lie between x_0 and x_1 . As X is a two-point set, we must have that the straight line segment between x_0 and x_1 is completely contained in Y . This is a contradiction as x_0 and x_1 are in distinct path-components of Y . $\square$

An easy lemma with its proof omitted. If P is a convex path-connected subset of the plane which is also unbounded in many directions then it must cover the whole plane. Considering P convex w.r.t. a subset Y of the plane changes nothing.

Lemma 2.13. *Let $Y \subset \mathbb{R}^2$ and suppose $P \subset Y$ is path-connected and is convex with respect to Y . Fix a point $p \in P$ and suppose further that for every $\theta_0 < \theta_1 \in [0, 2\pi)$ and every $r \in [0, \infty)$, there exist $\theta' \in (\theta_0, \theta_1)$ and $r' > r$ such that $p + r' \exp(i\theta') \in P$. Then $P = Y$.*

Let Y be the complement of a two-point set X . In the following proposition we show that non-trivial¹ path-components cannot be convex w.r.t. Y . To do this we consider a non-trivial path-component P which is convex w.r.t. Y and use this convexity to find a continuous boundary to P (or at least to some part of P). Of course, this boundary cannot lie wholly within our two-point set X and we achieve a contradiction by considering a point on this boundary that does not lie in X .

Proposition 2.14. *The complement of a two-point set is path-connected.*

Proof. Let X be any two-point set and write Y for $\mathbb{R}^2 \setminus X$. Suppose for a contradiction that Y is not path-connected. By Lemma 2.12, there exists a path component P of Y which is convex with respect to Y . Fix a point $p \in P$. As Y is not path-connected, $P \neq Y$, and so by Lemma 2.13, there exists $\theta_0 < \theta_1 \in [0, 2\pi)$ and there exists $r \in [0, \infty)$ such that

$$\{p + r' \exp(i\theta') \mid r' > r \text{ and } \theta' \in [\theta_0, \theta_1]\}$$

is disjoint from P . For each $\theta' \in (\theta_0, \theta_1)$, the line through p at angle θ' meets X in only a finite number of points, and so there exists $r' > 0$ such that $p + r' \exp(i\theta') \in P$. Thus, for each $\theta' \in (\theta_0, \theta_1)$,

$$\{r' \mid p + r' \exp(i\theta') \in P\}$$

is non-empty and bounded above.

Let $f: (\theta_0, \theta_1) \rightarrow (0, \infty)$ be the map

$$(\theta' \mapsto \sup \{r' \in (0, \infty) \mid p + r' \exp(i\theta') \in P\}).$$

For each $\theta \in (\theta_0, \theta_1)$, let l_θ denote the line through p at angle θ , and let $g(\theta) = p + f(\theta) \exp(i\theta)$. We will show that g is a continuous injection.

Fix $\theta \in (\theta_0, \theta_1)$ and find $\theta_a, \theta_b \in (\theta_0, \theta_1)$ such that $\theta_a < \theta < \theta_b$ and such that $\theta_b - \theta_a < \pi$. Let m_a (resp. m_b) denote the line that passes through $g(\theta)$ and $g(\theta_a)$ (resp. $g(\theta_b)$). Note that $f(\theta_a) > 0$ so m_a cannot be parallel to l_θ . Fix $\theta' \in (\theta, \theta_b)$ sufficiently close to θ that $l_{\theta'}$ is not parallel to m_a ; let w_a denote the point of intersection. $\theta_b - \theta < \pi$ and so $l_{\theta'}$ is not parallel to m_b ; let w_b be this point of intersection. Let $r_a = |w_a - p|$ and let $r_b = |w_b - p|$. We will show that $r_b \leq f(\theta') \leq r_a$ and this will allow us to deduce that g is continuous from the left.

¹Of course, if Y is path-connected then $P = Y$ is a trivial path-component which is convex w.r.t. Y

Firstly note that for each $r' < r_b$, there exist $p_0, p_1 \in P$ such that $p_0 \in l_\theta$, $p_1 \in l_{\theta_b}$ and such that the line segment between p_0 and p_1 crosses $l_{\theta'}$ at a point, w say, between $p + r' \exp(i\theta')$ and w_b . Furthermore, because X meets $l_{\theta'}$ at only two points, we can arrange that $w \notin X$. By convexity of P , w must lie in P , and so $f(\theta') \geq |w - p| > r'$. This holds for all $r' < r_b$ and so we deduce that $f(\theta') \geq r_b$.

Now suppose for a contradiction that $f(\theta') > r_a$. Then there exist $p_0, p_1 \in P$ such that $p_0 \in l_{\theta'}$, $p_1 \in l_{\theta_a}$ and such that the line segment between p_0 and p_1 crosses l_θ at a point, w say, where $|w - p| > f(\theta)$. As before, we can arrange that $w \notin X$. By convexity of P , w must lie in P , and this violates the definition of $f(\theta)$.

Now, for any $\epsilon > 0$, there exists a $\delta > 0$ sufficiently small that for each $\theta' \in (\theta, \theta + \delta)$, $l_{\theta'}$ is not parallel to m_a . We can also ensure that δ is sufficiently small that the points of intersection of $l_{\theta+\delta}$ with the lines m_a and m_b are both within the ϵ -ball about $g(\theta)$. With this, for each $\theta' \in (\theta, \theta + \delta)$, $g(\theta')$ lies in the ϵ -ball about $g(\theta)$. Thus, g is continuous from the left.

A symmetric argument gives us that g is continuous from the right, and so we have that g is continuous. Also, because $(\theta_0, \theta_1) \subset [0, 2\pi)$, we have that g is injective.

Hence, $\text{ran}(g)$ is a nondegenerate continuum and so, because X is a two-point set, $\text{ran}(g) \not\subset X$. Let $\theta \in (\theta_0, \theta_1)$ be such that $g(\theta) \notin X$ and consider the line l_θ . X meets l_θ at only two points and so there exist points $p_0, p_1 \in l_\theta \cap Y$ such that $g(\theta)$ is between p_0 and p_1 and such that every point on l_θ between p_0 and p_1 lies in Y . By definition of $f(\theta)$ and by convexity of P , we have that $p_0 \in P$ and that $p_1 \notin P$. But p_0 and p_1 are connected by a line segment entirely within Y , violating the fact that P is a path-component; a contradiction. $\square$

So, between any two points in the complement of a two-point set we can find a path completely contained within the complement. Given the fact that a two-point set X is necessarily zero-dimensional and that there is an abundance of straight line segments in the complement of X , we could expect to be able to do better than this.

Conjecture 2.15. *Given a two-point set X , for every $x \in X$ and every open neighbourhood U of x there exists a continuous function $f: S^1 \rightarrow U \setminus X$ such that the connected component of x in $\mathbb{R}^2 \setminus \text{ran}(f)$ is bounded.*

If this is true then the fact that the complement of a two-point set is path-connected would follow easily. One might try using the fact that a two-point set is zero-dimensional to get a base of clopen sets and then try to construct loops along the boundaries of these sets in the plane but this approach runs into problems.

2.3 A Zero-Dimensional Subset of $\mathbb{R}^2$ with Simply-Connected Complement.

Here we construct a non-empty, zero-dimensional subset of $\mathbb{R}^2$ with simply-connected complement. The problem of constructing such a set was inspired by the work on two-point sets in the previous section.

Aside from being a rather pretty result, the main interest is in the proof itself. The initial proof was so long and tedious that it is not even worth including for comparison's sake. Substantial refinement made the result much more accessible and, importantly, gave birth to a “mini-theory” which looks worthy of study in its own right. We begin with this mini-theory.

2.3.1 Shallow Paths and Deep Paths

Definition 2.16. Given two spaces X and Y , continuous functions $f, g: X \rightarrow Y$ are said to be *homotopic* if there exists a continuous $h: X \times [0, 1] \rightarrow Y$ such that for all $x \in X$, $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. f and g are said to be *homotopic relative to* $A \subset X$ if in addition, for all $x \in A$ and all $t \in [0, 1]$, $f(x) = h(x, t) = g(x)$.

Throughout this section we care about when two maps defined on $[a, b]$ (for $a \leq b \in \mathbb{R}$) are homotopic. In all cases we mean homotopic relative to $\{a, b\}$.

Lemma 2.17. *If $\{(a_n, b_n) \mid n \in \omega\}$ is a descending chain of non-empty, open intervals in $\mathbb{R}$ with empty intersection then one of $\langle a_n \mid n \in \omega \rangle$, $\langle b_n \mid n \in \omega \rangle$ is eventually constant.*

Proof. If not then for all $n \in \omega$ there exists $m > n$ such that

$$a_n < a_m < b_m < b_n.$$

Then

$$\overline{(a_m, b_m)} \subset (a_n, b_n)$$

which gives us that

$$\bigcap_{n \in \omega} \overline{(a_n, b_n)} = \bigcap_{n \in \omega} (a_n, b_n) = \emptyset.$$

But $\{\overline{(a_n, b_n)} \mid n \in \omega\}$ is a chain of compact Hausdorff spaces and so must have a non-empty intersection, a contradiction. $\square$

From this point we will fix a Hausdorff space X and suppose there exists an increasing chain $\mathcal{C} = \langle C_n \mid n \in \omega \rangle$ of simply-connected closed sets in X such that $\bigcup \mathcal{C} = X$.

Definition 2.18. For $a \leq b \in \mathbb{R}$, a continuous function $f: [a, b] \rightarrow X$ is said to be *shallow (with respect to $\mathcal{C}$)* if there exists continuous $g: [a, b] \rightarrow X$ homotopic to f and $n \in \omega$ such that $\text{ran}(g) \subset C_n$. Otherwise we say f is *deep (with respect to $\mathcal{C}$)*.

Fix a deep function $f: [0, 1] \rightarrow X$ and let $M \in \omega$ be such that $f(\{0, 1\}) \subset C_M$. Then for each $n \geq M$ we have that $f^{-1}(X \setminus C_n)$ is open in $(0, 1)$ and so admits a partition $\mathcal{P}_n$ into pairwise disjoint, maximal, open intervals. For each $n \geq M$ let

$$\mathcal{D}_n = \{U \in \mathcal{P}_n \mid f|_{\overline{U}} \text{ is deep}\}.$$

Lemma 2.19. For all $m \geq M$, all $U \in \mathcal{D}_m$, and all $n \geq m$, there exists $V \in \mathcal{D}_n$ such that $V \subset U$.

Proof. Suppose not and let $m \geq M$, $U \in \mathcal{D}_m$, and $n \geq m$ be such that whenever $V \in \mathcal{P}_n$ and $V \subset U$ then $V \notin \mathcal{D}_n$. Clearly each element of $\mathcal{P}_n$ is either a subset of U or is disjoint from U . It follows that

$$\overline{U} \subset f^{-1}(X \setminus C_n) \cup \bigcup (\mathcal{P}_n \setminus \mathcal{D}_n)$$

and we can then construct a homotopy between $f|_{\overline{U}}$ and a map from $\overline{U}$ into C_n , contradicting the fact that $f|_{\overline{U}}$ is deep. $\square$

Lemma 2.20. There exists $m \geq M$ and $U \in \mathcal{D}_m$ such that for all $n \geq m$ there exists a unique $V \in \mathcal{D}_n$ such that $V \subset U$.

Proof. If not then we would have a bifurcating tree of descending open intervals. At least one of these branches would have to fail the conclusion of Lemma 2.17 and consequently the intersection of that chain would have to be nonempty. This tells us that $\bigcup_{n \in \omega} C_n \neq X$, a contradiction. $\square$

Theorem 2.21. If $f: [0, 1] \rightarrow X$ is deep then there exists $a \neq b \in [0, 1]$ such that for all $t \in (a, b)$, $f|_{[a, t]}$ is shallow and $f|_{[t, b]}$ is deep.

Proof. By Lemma 2.20 we can find a descending chain of intervals on the closures of each of which f is deep. The intersection of this chain must be empty and so, by Lemma 2.17, either the left endpoints or the right endpoints form an eventually constant sequence, w.l.o.g. the right endpoints. Let (a, b) be an interval in this chain such that all later elements of the chain have right endpoint b . $\square$

Proposition 2.22. *If $f: [0, 1] \rightarrow X$ is such that for all $t \in (0, 1)$, $f|_{[0, t]}$ is shallow then there exists an arc $g: [0, 1] \rightarrow \text{ran}(f)$ homotopic to f such that for all $t \in (0, 1)$, there exists an $n \in \omega$ such that $g([0, t]) \subset C_n$.*

Proof. For each $k \in \omega$,

$$f|_{[1-2^{-k}, 1-2^{-(k+1)}]}$$

is homotopic to some map $g_k: [1-2^{-k}, 1-2^{-(k+1)}] \rightarrow C_n$ for some $n \in \omega$. Then $g = \bigcup_{k \in \omega} g_k \cup (1 \mapsto f(1))$ has the required properties. $\square$

This follows easily from the fact that every path-connected Hausdorff space is arc-connected (see [7]).

2.3.2 The Construction

With our mini-theory in place, we are ready to construct our strange subset of $\mathbb{R}^2$. We take the approach of constructing a simply-connected set which is a suitably “busy” subset of the unit sphere and then prove that the complement is zero-dimensional. To move from the sphere to the plane we simply remove a point from the complement.

Let $Z = (0, 1)^2$ and let $Z^* = [0, 1]^2$. Let $q: Z^* \rightarrow S^2$ be a quotient map which maps $Z^* \setminus Z$ to a single point $\star \in S^2$ and is otherwise injective. Let $\pi_0, \pi_1: Z^* \rightarrow [0, 1]$ be the projection maps and let $R: Z^* \rightarrow Z^*$ be the map

$$(\langle x, y \rangle \mapsto \langle y, x \rangle).$$

If a subset A of Z^* is declared to be *horizontal* then we automatically declare that $R(A)$ is *vertical* and we write $o_A = 0$ and $o_{R(A)} = 1$ (o is for *orientation*).

For each $n > 0$ let

$$D_n = \left\{ \frac{2k+1}{2^n} \in [0, 1] \mid k \in \omega \right\}.$$

Let $D = \bigcup_{n > 0} D_n$, the dyadic rationals in $(0, 1)$. Let $D_0 = \{0, 1\}$ and let $D^* = D_0 \cup D$. Note that $\{D_n \mid n \in \omega\}$ is a partition of D^* and so for each $d \in D^*$ we may uniquely define $n_d \in \omega$ such that $d \in D_{n_d}$. Equip D with the preorder order $\blacktriangleleft$ such that for all $d, d' \in D^*$, $d \blacktriangleleft d'$ if and only if $n_d < n_{d'}$. For $d < d' \in D^*$ let $\theta(d, d')$ be the $\blacktriangleleft$ -least element of $(d, d') \cap D^*$ ². Let $\triangleleft$ be a well-order on D such that for all $d, d' \in D$, if $d \blacktriangleleft d'$ then $d \triangleleft d'$.

Let

$$\mathcal{L}^{\text{horiz}} = \{[0, 1] \times \{d\} \mid d \in D\}.$$

² $\theta(d, d') = \{d \mid d'\}$ in the notation for surreal numbers developed in [4].

Declare the elements of $\mathcal{L}^{\text{horiz}}$ to be horizontal and for each $L = [0, 1] \times \{d\} \in \mathcal{L}^{\text{horiz}}$ let $n_L = n_d$. Let

$$\mathcal{L}^{\text{vert}} = \{R(L) \mid L \in \mathcal{L}^{\text{horiz}}\}$$

and for each $L \in \mathcal{L}^{\text{horiz}}$ let $n_{R(L)} = n_L$. Let

$$\mathcal{L} = \mathcal{L}^{\text{horiz}} \cup \mathcal{L}^{\text{vert}}.$$

For $L = [0, 1] \times \{d\} \in \mathcal{L}^{\text{horiz}}$ we define $\Lambda_L: [0, 1] \rightarrow L$ to be the map $(x \mapsto \langle x, d \rangle)$; we define Λ_L for $L \in \mathcal{L}^{\text{vert}}$ analogously. Equip $\mathcal{L}$ with the preorder $\blacktriangleleft$ such that for all $L, M \in \mathcal{L}$, $L \blacktriangleleft M$ if and only if $\langle n_L, o_L \rangle < \langle n_M, o_M \rangle$ in the lexicographical ordering, and extend this to a well-order $\triangleleft$.

Note that both $\langle D, \triangleleft \rangle$ and $\langle \mathcal{L}, \triangleleft \rangle$ have order-type ω .

Fix $\mu, \nu, \lambda, r \in D^*$ such that $\mu \neq \nu$ and $\lambda \pm r \in D^*$ and let $u = \langle \mu, \lambda \rangle$, $v = \langle \nu, \lambda \rangle$, and $\xi = \theta(\mu, \nu)$. Define

$$S_{u,v,r}^{\text{bulk}} = [u, \langle \tfrac{1}{2}(\mu + \xi), \lambda \rangle] \cup \bigcup_{n \in \omega} [\langle x_n, y_n \rangle, \langle x_n, y_{n+1} \rangle] \cup [\langle x_n, y_{n+1} \rangle, \langle x_{n+1}, y_{n+1} \rangle]$$

where

$$\begin{aligned} x_n &= \xi + (\mu - \xi) \left(\tfrac{1}{2} 2^{-n} \right) \quad \text{for all } n \in \omega, \\ y &= \langle \lambda + r, \lambda - r, \lambda + r, \lambda - r, \dots \rangle, \end{aligned}$$

and define

$$S_{u,v,r}^{\text{end}} = [\langle \xi, \lambda + r \rangle, \langle \xi, \lambda - r \rangle] \cup [\langle \xi, \lambda \rangle, v].$$

Let

$$S_{u,v,r} = S_{u,v,r}^{\text{bulk}} \cup S_{u,v,r}^{\text{end}}.$$

We refer to u and v as the *endpoints* of $S_{u,v,r}$. Because $\pi_1(u) = \pi_1(v)$ we declare $S_{u,v,r}$ to be horizontal. We also define

$$S_{R(u),R(v),r} = R(S_{u,v,r})$$

For each $L \in \mathcal{L}$ we will define $\kappa_L \in \omega$, and for each $L \in \mathcal{L}$ and each $k \in \omega$ we will define $u_k^L, v_k^L \in L$ and $r_k^L \in (0, \infty)$ below. For $L \in \mathcal{L}$ and $k \in \omega$ we define λ_k^L such that $\langle \lambda_k^L, \lambda_k^L \rangle \in L$ and write:

$$\begin{aligned} \mu_k^L &= \Lambda_L^{-1}(u_k^L), \\ \nu_k^L &= \Lambda_L^{-1}(v_k^L), \\ \xi_k^L &= \theta(\mu_k^L, \nu_k^L), \\ C_k^L &= S_{u_k^L, v_k^L, r_k^L}. \end{aligned}$$

We allow ourselves to extend this notation in natural ways by writing, for example,

$$C_{<k}^{\triangleleft L} = \bigcup \{C_j^M \mid M \triangleleft L \text{ and } j < k\}.$$

For all $L \in \mathcal{L}$ and all $k \in \omega$ let

$$U_k^L = Z \setminus \left(\overline{C_{\in \omega}^{\triangleleft L}} \cup C_{<k}^L \right).$$

and note that this is open.

For all $L \in \mathcal{L}$ we will ensure that:

- A1. $\overline{C_{<\omega}^L} \subset C_{<\omega}^L \cup L$;
- A2. For each $L' > L$ there exists $k \in \omega$ such that $L \cap L' \subset C_{<k}^{\triangleleft L} \cap L' = C_{<\omega}^{\triangleleft L} \cap L'$;
- A3. $q(C_{<\kappa_L}^{\triangleleft L})$ is simply-connected.

Furthermore, for all $L \in \mathcal{L}$ and all $k \in \omega$ we will ensure that:

- B1. $u_k^L, v_k^L \in L$;
- B2. $\|u_k^L - v_k^L\| \leq 2^{-(n_L-1)}$;
- B3. $r_k^L \leq \frac{1}{8} \|u_k^L - v_k^L\|$;
- B4. Taking the natural order on L , $u_k^L < v_k^L$ if and only if n_L is even;

We begin by explicitly defining the first few objects constructed by this induction for clarity.

The least element of $\mathcal{L}$ is $L = [0, 1] \times \{\frac{1}{2}\}$. $\kappa_L \in \omega$ is just some number sufficiently large that $q(C_{<\kappa_L}^{\triangleleft L})$ is simply-connected. Because L is the least element of $\mathcal{L}$, $\kappa_L = 0$ will do, and indeed κ_L is set to 0 in the general construction given later.

$U_0^L = Z = (0, 1)^2$ so we will take $(u_0^L, v_0^L) = (0, 1) \times \{\frac{1}{2}\}$. Since $n_L = n_{\frac{1}{2}} = 1$ is odd we will set $u_0^L = \langle 1, \frac{1}{2} \rangle$ and $v_0^L = \langle 0, \frac{1}{2} \rangle$. Analytic elements of the proof that the final construction is simply connected will require us to make r_0^L rather small. The construction does not specify a precise value for r_0^L but requires it to be in the range $(0, \frac{1}{16}]$. We set $r_0^L = \frac{1}{16}$ for simplicity.

Now we have u_0^L , v_0^L , and r_0^L we automatically obtain: $\mu_0^L = 1$, $\nu_0^L = 0$, $\lambda_0^L = \frac{1}{2}$, and $\xi_0^L = \theta(0, 1) = \frac{1}{2}$.

We now move on to $k = 1$. To find u_1^L and v_1^L we need to find the least $L' \triangleright L$ which meets $L \setminus C_0^L$. From the construction of C_0^L we see that $\langle \frac{t}{16}, \frac{1}{2} \rangle \in C_0^L$ for all $t = 0 \dots 16$ other than $t = 11$. Thus, $L' = \{\frac{11}{16}\} \times [0, 1]$ is the least element of $\mathcal{L}$

after L which meets $L \setminus C_0^L$. We must ensure that $L \cap L'$ is included in $C_{\infty}^{\triangleleft L}$ so we take $(u_1^L, v_1^L) = (\langle \frac{5}{8}, \frac{1}{2} \rangle, \langle \frac{3}{4}, \frac{1}{2} \rangle)$, the open line interval in $L \setminus C_0^L$ of maximal length containing $\langle \frac{11}{16}, \frac{1}{2} \rangle$. Again, since n_L is odd, we let u_1^L be the right endpoint, so $u_1^L = \frac{3}{4}$ and $v_1^L = \frac{5}{8}$. r_1^L is required to lie in the range $(0, \frac{1}{64}]$ and setting $r_1^L = \frac{1}{64}$ is sufficient here.

As before, we automatically generate $\mu_1^L = \frac{3}{4}$, $\nu_1^L = \frac{5}{8}$, $\lambda_1^L = \frac{1}{2}$, and $\xi_1^L = \theta(\frac{3}{4}, \frac{5}{8}) = \frac{11}{16}$ and construct C_1^L .

We continue like this until we have defined C_k^L for all $k \in \omega$ and then proceed to the next element of $\mathcal{L}$. The general construction is as follows.

Suppose $L \in \mathcal{L}$ is such that we have defined κ_M for all $M \triangleleft L$, and we have defined u_k^M , v_k^M , and r_k^M for all $M \triangleleft L$ and all $k < \omega$, consistent with the given properties.

We recursively define u_k^L , v_k^L , and r_k^L for $k \in \omega$. Suppose $k \in \omega$ is such that for all $j < k$, we have defined u_j^L , v_j^L , and r_j^L consistent with $(\text{Bn})_j^L$ for each n .

Because $D^2 \cap L$ is dense in L , there exists a least $L' \triangleright L$ such that $L' \cap L = \{x\}$ for some $x \in U_k^L$. Let (u_k^L, v_k^L) be the line interval in $U_k^L \cap L$ of maximal length containing x ; this establishes $(\text{B1})_k^L$.

Without loss of generality, assume that $u_k^L < v_k^L$ if and only if n_L is even; this establishes $(\text{B4})_k^L$.

Recall that for each $M \triangleleft L$, $M \cap L \subset C_{< j_M}^{\triangleleft M} \subset X \setminus U_k^L$. It follows from the definition of the order $\triangleleft$ on D that $\|u_k^L - v_k^L\| \leq 2^{-(n_L-1)}$. This establishes $(\text{B2})_k^L$.

Let $[u', v'] = (u_k^L, v_k^L) \setminus B_{\frac{1}{8}\|u_k^L - v_k^L\|}(\{u_k^L, v_k^L\})$. Then $[u', v']$ is a compact subset of U_k^L and so there exists $r_k^L \in (0, \leq 2^{-(n_L+3)}\|u_k^L - v_k^L\|]$ such that $B_{r_k^L}([u', v']) \subset U_k^L$. This establishes $(\text{B3})_k^L$.

We have established $(\text{Bn})_k^L$ for each n . Therefore, by induction, we can define u_k^L , v_k^L , and r_k^L for $k \in \omega$ satisfying all of the $(\text{Bn})_k^L$.

For each $M \triangleleft L$, by $(\text{A2})^M$ there exists $j_M \in \omega$ such that

$$M \cap L \subset C_{< j_M}^{\triangleleft M} \cap L = C_{< \omega}^{\triangleleft M} \cap L;$$

let $\kappa_L = \max \{j_M \mid M \triangleleft L\}$ (where $\max \emptyset$ is taken as 0). Then

$$C_{< \kappa_L}^{\triangleleft L} \cap L = C_{< \omega}^{\triangleleft L} \cap L.$$

Also, for each $M \triangleleft L$, by $(A1)^M$, $\overline{C_{<\omega}^M} \subset C_{<\omega}^M \cup M$. Thus

$$\begin{aligned} \overline{C_{<\omega}^M} \cap L &\subset (C_{<\omega}^M \cup M) \cap L \\ &\subset (C_{<\omega}^M \cap L) \cup (M \cap L) \\ &\subset (C_{<\omega}^{\triangleleft L} \cap L) \cup (C_{<j_M}^{\leq M} \cap L) \\ &\subset C_{<\kappa_L}^{\triangleleft L} \cap L. \end{aligned}$$

This gives us that

$$\overline{C_{<\omega}^{\triangleleft L}} \cap L = C_{<\kappa_L}^{\triangleleft L} \cap L.$$

If $L' \triangleright L$ is perpendicular to L then $L' \cap L = \{x\}$ for some $x \in X$. From the definition of U_k^L and by a simple induction there must exist $k \geq k_L$ such that $x \notin U_k^L$. Then $x \in (\overline{C_{<\omega}^{\triangleleft L}} \cup C_{<k}^L) \cap L \subset C_{<k}^{\triangleleft L}$. Thus, $x \in C_{<k}^{\triangleleft L} \cap L'$ and so

$$L \cap L' \subset C_{<k}^{\triangleleft L} \cap L'.$$

Also, as $x \notin U_k^L$, for any $k' > k$ we have that $x \notin (u_{k'}^L, v_{k'}^L)$, and we deduce that $C_{k'}^L \cap L' \subset \{x\}$. From this we deduce that $C_{<k}^L \cap L' = C_{<\omega}^L \cap L'$. Given that $k \geq \kappa_L$, $(A2)^L$ follows for $L' \triangleright L$ perpendicular to L .

If $L' \triangleright L$ is parallel to L then $L' \cap L = \emptyset$ and so trivially

$$L \cap L' \subset C_{<k}^{\triangleleft L} \cap L'.$$

for any $k \geq \kappa_L$. Also, thanks to the argument in the previous paragraph, $C_{<\omega}^L \cap L$ is dense in L . Obviously, $\|u_k^L - v_k^L\| \rightarrow 0$ as $k \rightarrow \omega$, and $(B3)_k^L$ gives us that $r_k^L \rightarrow 0$ as $k \rightarrow \omega$. Thus, for some $k \geq k_L$, $C_{k'}^L \cap L' = \emptyset$ for each $k' \geq k$. $(A2)^L$ follows for $L' \triangleright L$ parallel to L . Combining this with the previous paragraph gives us $(A2)^L$.

From the above, $r_k^L \rightarrow 0$ as $k \rightarrow \omega$, and from this we easily deduce $(A1)^L$.

$(A3)^L$ follows from $(A2)$, the fact that $\kappa_L \geq \kappa_M$ for $M \triangleleft L$, and because at each of the finitely many steps in building from $\star$ to $q(C_{<\kappa_L}^{\triangleleft L})$ we are extending a simply-connected set by essentially gluing a pair of path-connected sets at single points in such a way that no non-trivial loops are formed.

We have established $(An)^L$ for each n . Therefore, by induction, for each $L \in \mathcal{L}$ we can define $\kappa_L \in \omega$, and for each $L \in \mathcal{L}$ and $k \in \omega$ we can define $\langle u_k^L, v_k^L \rangle \in D^\star \times D^\star$ and $r_k^L > 0$, such that all $(An)^L$ and $(Bn)_k^L$ hold.

Let $C = C_{\in \omega}^{\in \mathcal{L}}$.

2.3.3 $S^2 \setminus q(C)$ is Zero-Dimensional

Note firstly that $S^2 \setminus q(C)$ is homeomorphic to $Z \setminus C$. We will show that $Z \setminus C$ is zero-dimensional.

Fix $z = \langle x, y \rangle \in Z \setminus C$ and find $\epsilon > 0$ such that

$$z \in B_\epsilon(x) \times B_\epsilon(y) \subset Z$$

For each $\sigma \in \{x, y\}$ and each $\eta \in \{-1, 1\}$, let $d_{\sigma, \eta} \in D \cap (\sigma, \sigma + \eta\epsilon)$ be minimal with respect to $\triangleleft$. Let

$$A = \{a_{\eta, \zeta} = \langle d_{x, \eta}, d_{y, \zeta} \rangle \mid \eta, \zeta \in \{-1, 1\}\}.$$

Let

$$\mathcal{L}' = \{\{d_{x, \eta}\} \times [0, 1], [0, 1] \times \{d_{y, \eta}\} \mid \eta \in \{-1, 1\}\}.$$

For each $L \in \mathcal{L}'$ let $R_L \subset L$ be the non-degenerate line interval with endpoints in A and let $R = \bigcup_{L \in \mathcal{L}'} R_L$. Let $\delta > 0$ be sufficiently small that

$$z \notin B_\delta(R) \subset B_\epsilon(x) \times B_\epsilon(y)$$

and then find $\kappa \in \omega$ sufficiently large that for each $L \in \mathcal{L}'$, $C_{\geq \kappa}^L \subset B_\delta(L)$.

Let $B \subset \mathcal{L}' \times [\kappa, \infty)$ be such that for every $\langle L, k \rangle \in B$,

$$[u_k^L, v_k^L] \cap C_{< \kappa}^{\triangleleft \max \mathcal{L}'} \cap R = \{u_k^L, v_k^L\}.$$

For each $L \in \mathcal{L}'$ let

$$E_L = \left(\bigcup_{\langle L, k \rangle \in B} C_k^L \right) \cup \left(C_{< \kappa}^{\triangleleft \max \mathcal{L}'} \cap R_L \right)$$

and let $E = \bigcup_{L \in \mathcal{L}'} E_L$.

B has been chosen to ensure that each E_L is closed and connected. With this we can see that E “encloses” z . Let U be the connected component of $Z \setminus E$ containing z . Because E is closed it follows that $Z \setminus U$ is a clopen neighbourhood of z in $Z \setminus C$ completely contained within $B_\epsilon(x) \times B_\epsilon(y)$.

Thus, $Z \setminus C$ is zero-dimensional.

2.3.4 $q(C)$ is Simply-Connected

With the construction done and a proof of the fact that the complement is zero-dimensional in place, we finish off by establishing that the constructed space really is simply-connected. In fact, this turns out to be the most difficult part of the proof. The complexities introduced in the construction which seemed to play no part in the proof of zero-dimensionality were made specifically for this section. Also, finally, our mini-theory comes into play.

Definition 2.23. A topological space X is said to be *simply-connected* if it is path-connected and every closed loop is null-homotopic (that is to say, every continuous $f: [0, 1] \rightarrow X$ such that $f(0) = f(1)$ is homotopic to a constant function).

For all $L \in \mathcal{L}$ and all $k \in \omega$ let $M_k^L \in \mathcal{L}$ be the least such that $L \leq M_k^L$ and $k < \kappa_{M_k^L}$. Define an order $\prec$ on $\mathcal{L} \times \omega$ such that for all $\langle L, k \rangle, \langle L', k' \rangle \in \mathcal{L} \times \omega$, $\langle L, k \rangle \prec \langle L', k' \rangle$ if and only if $\langle M_k^L, L, k \rangle < \langle M_{k'}^{L'}, L', k' \rangle$ in the lexicographical ordering.

Lemma 2.24. $\langle \mathcal{L} \times \omega, \prec \rangle$ has order-type ω .

Proof. This follows from the fact that for each $L \in \mathcal{L}$ and each $k \in \omega$,

$$\left\{ \langle L', k' \rangle \in \mathcal{L} \times \omega \mid M_{k'}^{L'} = M_k^L \right\}$$

is finite. □

Let $\langle \langle N_i, l_i \rangle \mid i \in \omega \rangle$ be the enumeration of $\mathcal{L} \times \omega$ such that for all $i < j \in \omega$, $\langle N_i, l_i \rangle \prec \langle N_j, l_j \rangle$. For each $i \in \omega$ let $E_i = C_{l_i}^{N_i}$ and let $E_{<i}$ denote $\bigcup_{j < i} E_j$.

The following three lemmas follow easily from the primary property of κ_L , $(A3)^L$, for $L \in \mathcal{L}$.

Lemma 2.25. For all $i \in \omega$, $E_{<i}$ is path-connected.

Lemma 2.26. For every $i \in \omega$, $q(E_{<i})$ is simply-connected.

Lemma 2.27. for all $i \in \omega$, S^1 does not embed into $q(E_{<i})$.

Proof. This follows from Lemma 2.26 and the fact that each $E_{<i}$ is a countable union of line segments. □

Lemma 2.28. $q(C)$ is path-connected.

Proof. The union of an increasing chain of path-connected sets is path connected so this follows from Lemma 2.25. □

Now we suppose, for a contradiction, that $q(C)$ is not simply-connected. We know that $q(C)$ is path-connected so there must then exist a continuous function $f_0: [0, 1] \rightarrow q(C)$ such that $f_0(0) = f_0(1)$ and f_0 is not null-homotopic. Lemma 2.26 tells us that f_0 is deep with respect to $\langle q(E_{<i}) \mid i \in \omega \rangle$. By Theorem 2.21 and Proposition 2.22, there must exist a deep arc $f_1: [0, 1] \rightarrow q(C)$ such that for all $t \in (0, 1)$ there exists $i \in \omega$ such that $f_1([0, t]) \subset q(E_{<i})$.

Lemma 2.29. *For all $i \in \omega$, $f_1^{-1}(q(E_{<i})) \cap [0, 1)$ is convex.*

Proof. Suppose not. Then there exists $a < b \in [0, 1)$ such that $f_1(\{a, b\}) \subset q(E_{<i})$ but $f_1((a, b)) \cap q(E_{<i}) = \emptyset$. By Lemma 2.25 we can find an arc from $f_1(a)$ to $f_1(b)$ contained within $q(E_{<i})$ and, with $f_1|_{[a, b]}$, we can form a simple closed curve contained entirely within $q(E_{<j})$ for some $j \in \omega$, contradicting Lemma 2.26. $\square$

Using this lemma, we can find a strictly increasing sequence $\langle i_\sigma \mid \sigma \in \omega \rangle$ in ω , a strictly increasing sequence $\langle t_\sigma \mid \sigma \in \omega \rangle$ starting at 0 and converging to 1, and a deep arc $f_2: [0, 1] \rightarrow q(C)$ such that for all $\sigma \in \omega$,

$$f_2^{-1}(q(E_{i_\sigma})) \cap [0, 1) = [t_\sigma, t_{\sigma+1}).$$

Additionally, we can easily arrange that $\star \notin f_2([0, 1))$. Because q is injective (and consequently a homeomorphic embedding) on Z we can find a continuous injective $g_0: [0, 1) \rightarrow Z$ such that $q \circ g_0 = f_2|_{[0, 1)}$.

To keep a handle on our notation, for $\sigma \in \omega$ we define $L_\sigma = N_{i_\sigma}$, $k_\sigma = l_{i_\sigma}$, and we will allow ourselves to write $@_\sigma$ for $@_{k_\sigma}^{L_\sigma}$ for $\sigma \in \omega$ where $@$ is any of the symbols: u , v , r , λ , μ , ν , ξ , C , U .

The following three lemmas are rather basic properties of g_0 and are only stated as lemmas so that they may be easily referred to.

Lemma 2.30. *For all $\sigma \in \omega$, $L_\sigma \trianglelefteq L_{\sigma+1}$.*

Lemma 2.31. *If $\sigma \in \omega$ is such that $L_\sigma \triangleleft L_{\sigma+1}$ then $L_\sigma \blacktriangleleft L_{\sigma+1}$.*

Proof. Suppose for a contradiction that there exists $\sigma \in \omega$ such that $L_\sigma \triangleleft L_\tau$ and $L_\sigma \not\blacktriangleleft L_\tau$ (where $\tau = \sigma + 1$). By Lemma 2.30 we have that $n_{L_\sigma} = n_{L_\tau}$ and $o_{L_\sigma} = o_{L_\tau}$. From (B2) and (B3) it follows that $\overline{B_{r_\sigma}(L_\sigma)} \cap L_\tau = \emptyset$. By construction of $C_\sigma = S_{u_\sigma, v_\sigma, r_\sigma}$ we deduce that $C_\sigma \cap L_\tau = \emptyset$. But $\{u_\tau, v_\tau\} \subset L_\tau$ and $\{u_\tau, v_\tau\} \cap C_\sigma \neq \emptyset$, a contradiction. $\square$

Lemma 2.32. *For all $L \in \mathcal{L}$ there exists $\sigma \in \omega$ such that $L_\sigma \triangleright L$.*

Lemma 2.33. Fix $L \in \mathcal{L}$ and suppose $\sigma \leq \tau \in \omega$ are such that $L_\sigma = L = L_\tau$. Fix $s \in \bigcup_{m < n_L} D_m$ such that

$$[u_\sigma, v_\sigma] \subset [g_0(t_\sigma), \Lambda_L(s)].$$

Then

$$|(\pi_{o_L} \circ g_0)(t_\sigma) - (\pi_{o_L} \circ g_0)(t_{\tau+1})| \leq \frac{1}{2}|(\pi_{o_L} \circ g_0)(t_\sigma) - s|$$

and

$$|(\pi_{1-o_L} \circ g_0)(t_\sigma) - (\pi_{1-o_L} \circ g_0)(t_{\tau+1})| \leq \frac{1}{8}2^{-n_L}.$$

Proof. Certainly, $(\pi_{o_L} \circ g_0)(t_\sigma)$ and ξ_σ are bounds for $(\pi_{o_L} \circ g_0)([t_\sigma, t_{\tau+1}])$. If

$$|(\pi_{o_L} \circ g_0)(t_\sigma) - (\pi_{o_L} \circ g_0)(t_{\tau+1})| > \frac{1}{2}|(\pi_{o_L} \circ g_0)(t_\sigma) - s|$$

then, defining $\xi = s + 2(\xi_\sigma - s)$, we have $\xi \in (\mu_\sigma, \nu_\sigma)$ and $n_\xi < n_{\xi_\sigma}$, contradicting the definition of ξ_σ as $\theta(\mu_\sigma, \nu_\sigma)$.

The second condition follows from (B1), (B2), and (B3). $\square$

By noting that the image under f_2 of any increasing sequence in $[0, 1]$ converging to 1 is Cauchy (thanks to Lemma 2.33 with suitable choices for s) we may deduce that there exists a continuous extension $g_1: [0, 1] \rightarrow X$ of g_0 such that $q \circ g_1 = f_2$.

Let $z = \langle x, y \rangle = g_1(1)$. At least one of x, y is dyadic. Suppose that x is dyadic. The case where y is dyadic is practically identical but there is a slight lack of symmetry which prevents us from saying “without loss of generality”, we omit this case. Let $L = \{x\} \times [0, 1]$.

From Lemmas 2.32 and 2.29, we may assume without loss of generality that $g_1([0, 1)) \cap L = \emptyset$.

Let $\rho: [0, 1] \rightarrow \mathbb{R}$ be the map $(t \mapsto |x - (\pi_0 \circ g_1)(t)|)$. For each $\sigma \in \omega$ we will say r is ρ -increasing if $\rho(t_\sigma) < \rho(t_{\sigma+1})$ and ρ -decreasing if $\rho(t_\sigma) > \rho(t_{\sigma+1})$.

Lemma 2.34. There do not exist ρ -decreasing $\sigma_0, \sigma_1, \sigma_2 \in \omega$ such that for $j \in \{0, 1, 2\}$, the $n_{L_{\sigma_j}}$ are consecutive and the $o_{L_{\sigma_j}}$ are equal.

Proof. Suppose for a contradiction that σ_0, σ_1 , and σ_2 are as above. By (B4) we have that for some $j \in \{0, 1\}$, $g_1(t_{\sigma_j}) = u_{\sigma_j}$. Write:

$$\sigma = \sigma_j,$$

$$\tau = \sigma_{j+1},$$

$$N = L_\sigma.$$

By (A2),

$$(u_\sigma, v_\sigma) \cap \bigcup \{M \in \mathcal{L} \mid n_M \leq n_N\} = \emptyset.$$

By (B3) and (B2) it follows that

$$(\mu_\sigma, \nu_\sigma) \times B_{r_\sigma}(\lambda_\sigma) \cap \bigcup \{M \in \mathcal{L} \mid n_M \leq n_N\} = \emptyset.$$

By the definition of θ and again with reference to (B3) and (B2) we have that

$$(\mu_\sigma, \xi_\sigma) \times B_{r_\sigma}(\lambda_\sigma) \cap \bigcup \{M \in \mathcal{L} \mid n_M \leq n_N + 1\} = \emptyset.$$

Thus,

$$S_{u_\sigma, v_\sigma, r_\sigma}^{\text{bulk}} \cap \bigcup \{M \in \mathcal{L} \mid n_M \leq n_N + 1\} = \emptyset.$$

But $n_{L_{i_\tau}} = n_N + 1$ and

$$u_\tau \in L_{i_\tau} \cap S_{u_\tau, v_\tau, r_\tau}^{\text{bulk}},$$

a contradiction. □

For all $n \in \omega$ and $o \in \{0, 1\}$ let

$$\sigma_{n,o} = \sup \{\sigma \in \omega \mid \langle n_{L_\sigma}, o_{L_\sigma} \rangle < \langle n, o \rangle\}$$

and let

$$\tau_{n,o} = \inf \{\tau \in \omega \mid \langle n_{L_\tau}, o_{L_\tau} \rangle > \langle n, o \rangle\}$$

For all $n \in \omega$ let $c_n = 2^{n-1} \rho(\sigma_{n,o})$.

Lemma 2.35. *There exists $\Gamma \in \omega$ such that for all $n > \Gamma$, $c_n \geq 1$.*

Proof. Fix $n \in \omega$. By setting $s = x$ in Lemma 2.33 we deduce that for $m \in \{n, n+1, n+2\}$:

$$\rho(\sigma_{m,1}) \geq \frac{1}{2} \rho(\sigma_{m,0}), \tag{2.1}$$

$$\rho(\sigma_{m+1,0}) \geq \frac{7}{8} \rho(\sigma_{m,1}). \tag{2.2}$$

Also, by Lemma 2.34, we must have that for some $m \in \{n, n+1, n+2\}$,

$$\rho(\sigma_{m,1}) \geq \rho(\sigma_{m,0})$$

Pulling all of these equations together we have that

$$\rho(\sigma_{n+3,0}) \geq \left(\frac{7}{8}\right)^3 \left(\frac{1}{2}\right)^2 \rho(\sigma_{n,0})$$

which gives us that

$$\frac{c_{n+3}}{c_n} \geq \frac{343}{256}$$

and the result follows. □

Now, for $n \geq \Gamma$ (as given in Lemma 2.35) and $\sigma > \sigma_{n,0}$ we apply Lemma 2.33 with $s \in \{u_\sigma, v_\sigma\}$ and deduce that

$$|(\pi_{o_{L_\sigma}} \circ g_0)(t_\sigma) - (\pi_{o_{L_\sigma}} \circ g_0)(t_{\tau+1})| \leq \frac{1}{2}|\mu_\sigma - \nu_\sigma|.$$

Using (B2) and taking the same approach as in the proof of Lemma 2.35 we show that

$$\begin{aligned} \rho(\sigma_{n,0}) &\leq \sum_{\substack{m \geq n \\ m \not\equiv n+2 \pmod{3}}} 2^{-m} + \sum_{m \geq n} \frac{1}{8} 2^{-m} \\ &\leq 2^{-n} \left(\frac{3}{2} \sum_{k \in \omega} 8^{-k} + \frac{1}{8} \sum_{k \in \omega} 2^{-k} \right) \\ &= 2^{-n+1} \frac{55}{56} \end{aligned}$$

But the definition of Γ forces

$$\rho(\sigma_{n,0}) \geq 2^{-n+1},$$

a contradiction.

Thus, finally, we deduce that $q(C)$ is simply-connected.

2.4 Summary

In light of the previous section, more than zero-dimensionality will be needed to improve the result that the complement of a two-point set is path-connected.

Conjecture 2.36. *Let X be a two-point set. If $U \subset \mathbb{R}^2$ is open and $U \cap X \neq \emptyset$ then $U \setminus X$ is not simply-connected.*

At first glance it seems somewhat strange that a non-empty zero-dimensional subset of the plane can have a simply-connected complement and so we are naturally motivated to investigate the space constructed in the previous section, X say. It seems unlikely that the complement of X is locally connected, but it's quite possible that it is “connected im kleinen” at some points.

Definition 2.37. A topological space X is said to be *connected im kleinen* at a point $x \in X$ if for every open neighbourhood U of x there exists a connected $K \subset X$ such that

$$x \in \text{int}(K) \subset K \subset U$$

Question 2.38. Is the complement of the X connected im kleinen at some point?

Symmetry of the space would suggest then that the space is connected im kleinen at a dense set of points but this is far from obvious at this stage.

Question 2.39. Is there a non-empty, zero-dimensional subset of the plane of which the complement is both simply-connected and locally connected?

Chapter 3

Tent-Maps

3.1 Introduction

In this chapter we study discrete dynamical systems in one dimension. Given a topological space X and a self-map $f: X \rightarrow X$ we apply f repeatedly and study the various associated limit objects; see [6] and [3] for more details. In this branch of mathematics, one of the central objects of study is the tent-map.

Definition 3.1. For $\lambda \in \mathbb{R}$, the *tent-map of slope λ* is the real-valued function f on $[0, 1]$ such that $f(x) = \lambda x$ for $x \leq \frac{1}{2}$ and $f(x) = \lambda(1 - x)$ for $x \geq \frac{1}{2}$. The tent-map with slope 2 is also called the *full tent-map*.

Note that for a tent-map f , $f(x) = f(1 - x)$ for all $x \in [0, 1]$.

At first glance the class of tent-maps may appear too specific and artificial to tell us much about dynamical systems in general and that we only study them because they are relatively easy to work with. One might prefer to consider an arbitrary continuous self-map on $\mathbb{R}$ or $[0, 1]$. However for many “real world” maps, the domain can be partitioned into finitely many pieces, on each of which the map is monotone. We naturally do away with unnecessary tedious complexities by considering a map with just two monotone regions and one “critical point”. The study of these “unimodal” maps tells us much directly about the dynamics of most continuous self-maps. We find that much of the dynamical behaviour exhibited by unimodal maps can also be demonstrated with tent-maps and we have simplified the context enormously from the family of all continuous unimodal maps to the class of tent-maps.

For tent-maps, much of the interesting dynamics occurs for slopes in the range $[\sqrt{2}, 2]$.

We will give some of the basic machinery used to study these maps and establish some new results.

3.2 Characterising ω -limit Sets

In investigating what happens to a single point as f is applied repeatedly we quickly encounter the following notion.

Definition 3.2. Given a tent-map f , for each $x \in [0, 1]$, the ω -limit set of x is the set

$$\omega(x) = \bigcap_{n \in \mathbb{N}} \overline{\{f^{n+k}(x) \mid k \in \mathbb{N}\}}.$$

A subset A of $[0, 1]$ is an ω -limit set if there exists $x \in [0, 1]$ such that $A = \omega(x)$.

One quickly realises that these sets are necessarily both closed and *invariant*.

Definition 3.3. Given a tent-map f , a subset A of $[0, 1]$ is *invariant* if $f(A) = A$. If, in addition, A is closed and non-empty, we may say A is CINE¹ for short.

We would like to characterise the ω -limit sets of tent-maps and so we ask, for a fixed tent-map, is every CINE set necessarily an ω -limit set for some point?

The answer is ‘no’. For the full tent-map, $\{0, \frac{2}{3}\}$ is an easy counter-example.

We need to add a property to ‘CINE’ which “binds the points together” with respect to f . The most promising property is one well studied in dynamics, that of *internal chain transitivity*. Before we define this, we will need the notion of a *pseudo-orbit*.

Definition 3.4. Let $\langle X, f \rangle$ be a dynamical system and fix $\epsilon > 0$. A sequence $\langle x_0, x_1, \dots, x_n \rangle$ is called an ϵ -pseudo-orbit if for each $k < n$, x_{k+1} is within ϵ of $f(x_k)$.

Definition 3.5. Let $\langle X, f \rangle$ be a dynamical system. A set $C \subset X$ is *internally chain transitive* (written ICT for short) if for all $x, y \in C$ and all $\epsilon > 0$ there exists an ϵ -pseudo-orbit, $\langle x_0, x_1, \dots, x_n \rangle$, contained within C , with $x_0 = x$ and $x_n = y$.

For a tent-map it is known that every ω -limit set is ICT.

Proposition 3.6. Fix a dynamical system $\langle X, f \rangle$. If X is compact then every ω -limit set is ICT.

¹CINE stands for “Closed, Invariant, Non-Empty”

Proof. Fix a compact metric space $\langle X, d \rangle$ and a continuous self-map $f: X \rightarrow X$. Suppose $C \subset X$ is an ω -limit set. Let $a \in X$ be such that $\omega(a) = C$. Fix $y, z \in C$ and $\epsilon > 0$. Compactness of X gives us that f is uniformly continuous so there exists $\delta \in (0, \frac{\epsilon}{2})$ such that for all $x \in X$, $f(B_\delta(x)) \subset B_{\frac{\epsilon}{2}}(f(x))$.

There must exist $N \in \mathbb{N}$ such that for all $n \geq N$, $d(f^n(a), C) < \delta$ for otherwise $\{f^n(a) \mid d(f^n(a), C) \geq \delta\}$ would be infinite and therefore have a point of accumulation outside of C , contradicting that C is an ω -limit set.

Let $n_y, n_z \in \mathbb{N}$ be such that $N \leq n_y < n_z$, $d(f^{n_y}(a), y) < \delta$, and $d(f^{n_z}(a), z) < \delta$. Let $x_0 = y$, $x_{n_z - n_y} = z$, and for all $1 < k < n_z - n_y$ pick $x_k \in B_\delta(f^{n_y + k}(a)) \cap C$. Then for all $k < n_z - n_y$,

$$d(f(x_k), x_{k+1}) \leq d(f(x_k), f^{n_y + k + 1}(a)) + d(f^{n_y + k + 1}(a), x_{k+1}) \leq \frac{\epsilon}{2} + \delta < \epsilon.$$

Thus, $\langle x_k \mid k \leq n_z - n_y \rangle$ is an ϵ -pseudo-orbit from y to z contained within C . Therefore C is ICT. $\square$

Question 3.7. Given a tent-map f , is every ICT CINE set A necessarily an ω -limit set?

It turns out that once again the answer is ‘no’. [1] gives further conditions that will guarantee that an ICT CINE set is also an ω -limit set but these conditions lack the simplicity of and motivation behind ICT and CINE. In an attempt to find a more satisfactory solution we ask ourselves for which slopes $\lambda \in [\sqrt{2}, 2]$ is it the case that every ICT CINE set is an ω -limit set.

To begin, we should at least find an example of one tent-map for which the implication holds and one for which it fails. As is shown in [2], for the full tent-map, the ω -limit sets are precisely the ICT CINE sets. We now explicitly identify a slope for which the implication fails.

Proposition 3.8. *There exists $\lambda \in [\sqrt{2}, 2]$ such that with respect to the tent-map with slope λ , there exists an ICT CINE set which is not an ω -limit set.*

Proof. We will define a tent-map with a pre-periodic but not periodic critical point. Any tent-map with slope in $[\sqrt{2}, 2]$ will have the property that $f(c) > c$ and $f^2(c) < c$. We additionally require that $f^3(c) < c$, $f^4(c) > c$, $f^5(c) > c$, and that $f^6(c) = f^3(c)$.

Unpacking, we have

$$\begin{aligned}
f(c) &= \frac{\lambda}{2} \\
f^2(c) &= \lambda\left(1 - \frac{\lambda}{2}\right) = \lambda - \frac{1}{2}\lambda^2 \\
f^3(c) &= \lambda\left(\lambda - \frac{1}{2}\lambda^2\right) = \lambda^2 - \frac{1}{2}\lambda^3 \\
f^4(c) &= \lambda\left(\lambda^2 - \frac{1}{2}\lambda^3\right) = \lambda^3 - \frac{1}{2}\lambda^4 \\
f^5(c) &= \lambda\left(1 - (\lambda^3 - \lambda^4)\right) = \lambda - \lambda^4 + \frac{1}{2}\lambda^5 \\
f^6(c) &= \lambda\left(1 - (\lambda - \lambda^4 + \lambda^5)\right) = \lambda - \lambda^2 + \lambda^5 - \frac{1}{2}\lambda^6
\end{aligned}$$

and applying the condition $f^6(c) = f^3(c)$ gives us

$$\begin{aligned}
\lambda - \lambda^2 + \lambda^5 - \frac{1}{2}\lambda^6 &= \lambda^2 - \frac{1}{2}\lambda^3 \\
\implies \lambda - 2\lambda^2 + \frac{1}{2}\lambda^3 + \lambda^5 - \frac{1}{2}\lambda^6 &= 0 \\
\implies \lambda(1 - \lambda)\left(1 - \lambda - \frac{1}{2}\lambda^2 - \frac{1}{2}\lambda^3 + \frac{1}{2}\lambda^4\right) &= 0
\end{aligned}$$

and this has solutions 0, 1, and

$$\frac{1}{4} + \frac{1}{2}b \pm \frac{1}{2}\left(\frac{11}{6} + \frac{21}{4b} - \frac{19}{18a} + 2a\right)^{\frac{1}{2}}$$

where

$$a = \sqrt[3]{\frac{161}{216} + \frac{\sqrt{2118}}{72}}$$

and

$$b = \pm \sqrt{\frac{11}{12} + \frac{19}{18a} + 2a}.$$

The particular map with gives a simple example of an ICT CINE set which is not an ω -limit set is We will use a slope of $\lambda = 1.82104947669432\dots$

This slope is fairly special in that repeatedly mapping the critical point quickly lands you in a loop with period 3, each of the 3 points being different from the critical point (so $c \notin \omega(c)$). Also, essentially, exactly 2 of these points are greater than the critical point.

The set I claim to be ICT CINE but not an ω -limit set is

$$\begin{aligned}
C = \{f^k(c) \mid 0 \leq k < 6\} &\cup \left\{ (h^2g)^k(c) \mid k \in \mathbb{N} \right\} \\
&\cup \left\{ \left(g(h^2g)^k\right)(c) \mid k \in \mathbb{N} \right\} \\
&\cup \left\{ \left(hg(h^2g)^k\right)(c) \mid k \in \mathbb{N} \right\}
\end{aligned}$$

where $g = (x \mapsto \frac{x}{s})$ and $h = (x \mapsto 1 - \frac{x}{s})$ (note: these are partial inverses to f , one for each side of c .)

Every point in C eventually maps into the loop of period 3 and so the ω -limit set of any member of C must be finite. Furthermore, if $x \in [0, 1]$ is such that $\omega(x) = C$ then there must exist $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$:

$$d(f^{n+k}(x), C) < 0.01$$

and there exists $k' \in \mathbb{N}$ such that

$$f^3(c) + 0.1 < d(f^{n+k'}(x), c) < f^3(c) + 0.1s^3$$

(this fact follows from the fact that x must have c in its ω -limit set.)

But $C \cap (f^3(c), f^3(c) + 0.01s^3) = \emptyset$, a contradiction. $\square$

It is not very difficult to extend this idea to show that there exists a countable dense set of slopes in $[\sqrt{2}, 2]$ for which there exists an ICT CINE set which is not an ω -limit set. We ask if the same might be said for slopes where ICT CINE implies ω -limit but the proof behind the full tent-map example does not translate to any other slopes.

3.3 Periodic tent-maps

In this section we show that there is in fact a countable dense set of slopes in $[\sqrt{2}, 2]$ for which every ICT CINE set is an ω -limit set. Indeed, every tent-map with a periodic critical point has this property.

Up until now we have allowed ourselves to tackle problems in a very hands-on way to help familiarise ourselves with the topic. From here we start to work hard to make our proofs modular so that the proof itself has some value to the study of dynamical systems. There are a few interesting ideas which make the following argument work and each idea will be presented more in mathematical terms and objects than in the supporting text.

To begin, fix $\lambda \in [\sqrt{2}, 2]$ and let $f: [0, 1] \rightarrow [0, 1]$ be the tent-map with slope λ . Let $c = \frac{1}{2}$, the critical point.

We will construct a new metric space which can be thought of as a merging of the interval with the symbol space. However, unlike these two spaces, the new metric space will not be independent of the tent-map. Different tent-maps will give rise to different metric spaces.

For each $n \in \mathbb{N}$, let $P_n = f^{-n}(\{c\})$ and let $P = \bigcup_{n \in \mathbb{N}} P_n$. For each $p \in P$, let $n_p \in \mathbb{N}$ be the least such that $p \in P_{n_p}$.

Let

$$X = ([0, 1] \setminus P) \times \{0\} \cup (P \times \{-1, 1\})$$

and equip it with the lexicographical order. For ease of notation we write x^η for $\langle x, \eta \rangle \in X$. Furthermore, we may write x^- , x^+ for $\langle x, -1 \rangle$, $\langle x, 1 \rangle$ respectively. Where appropriate we associate $x \in [0, 1] \setminus P$ with x^0 .

Define a metric d on X such that for all $x^\eta \leq y^\theta \in X$,

$$d(x^\eta, y^\theta) = y - x + \sum_{\substack{p \in P \\ x^\eta \leq p^- \leq p^+ \leq y^\theta}} 4^{-n_p}.$$

To see that this metric is well-defined notice that for all $n \in \mathbb{N}$, $|P_n| \leq 2^n$ and hence the distance between any two points of X cannot exceed

$$1 + \sum_{n \in \mathbb{N}} 2^n 4^{-n} = 2 < \infty.$$

This metric gives rise to the lexicographical order topology on X . Also, it is clear that X is a quotient of the double-arrow space and consequently must be compact.

Let $g: X \rightarrow X$ be such that for all $x^\eta \leq c^-$, $g(x^\eta) = (\lambda x)^\eta$ and for all $x^\eta \geq c^+$, $g(x^\eta) = (\lambda(1 - x))^{-\eta}$.

It can be shown that both $g|_{(-\infty, c^-]}$ and $g|_{[c^+, \infty)}$ are strictly monotone homeomorphic embeddings with clopen range. Because $(-\infty, c^-]$ and $[c^+, \infty)$ are a partition of X into clopen sets it follows that g is continuous and open.

The main value in $\langle X, g \rangle$ is in how well-behaved it is dynamically. We have already seen that g is open and we will now show that it is also *uniformly expanding on X^2* . It will then follow from [2, Theorem 5.5] that the ICT CINE sets of $\langle X, g \rangle$ are precisely the ω -limit sets. We can then use the close relationship between $\langle X, g \rangle$ and $\langle [0, 1], f \rangle$ to deduce that, with respect to f , the ICT CINE subsets of $[0, 1]$ are precisely the ω -limit sets.

Definition 3.9. Let $\langle Y, d_Y \rangle$ be a compact metric space, and $h: Y \rightarrow Y$ be continuous. If there are $\delta > 0$, $\mu > 1$ such that

$$d_Y(h(x), h(y)) \geq \mu d_Y(x, y)$$

provided that $x, y \in A \subset Y$ and $d_Y(x, y) < \delta$ then we say that h is *uniformly expanding on A* .

²The term and definition given here is taken directly from [2, Definition 4.9].

Proposition 3.10. *If c has periodic orbit then c^- and c^+ exist and g is uniformly expanding on X .*

Proof. Let $\mu = \lambda$ and let $\delta = 4^{-m}$ where $m = |\text{orb}(c)|$. Let $x^\eta, y^\theta \in X$ be such that $d(x^\eta, y^\theta) < \delta$. Let

$$P' = \{p \in P \mid p^- \text{ and } p^+ \text{ are between } x^\eta \text{ and } y^\theta \text{ (inclusive)}\}.$$

Now,

$$d(x^\eta, y^\theta) = |x - y| + \sum_{p \in P'} 4^{-n_p} < \delta.$$

By choice of δ we have that $P' \cap \text{orb}(c) = \emptyset$. In particular, either $x^\eta, y^\theta \leq c^-$ or $x^\eta, y^\theta \geq c^+$. Because $g|_{(-\infty, c^-]}$ and $g|_{[c^+, \infty)}$ are both monotone we can say that

$$P' = \{p \in P \mid g(p^-) \text{ and } g(p^+) \text{ are between } g(x^\eta) \text{ and } g(y^\theta) \text{ (inclusive)}\}.$$

Also, because $P' \cap \text{orb}(c) = \emptyset$ we must have that for all $p \in P'$, $n_{f(p)} = n_p - 1$. It follows that

$$\begin{aligned} & d(g(x^\eta), g(y^\theta)) \\ &= |f(x) - f(y)| + \sum_{p \in P'} 4^{-(n_p-1)} \\ &= \lambda|x - y| + 4 \sum_{p \in P'} 4^{-n_p} \\ &\geq \mu d(x^\eta, y^\theta). \end{aligned}$$

Thus, g is uniformly expanding on X , as required. $\square$

Now we can simply invoke the following direct corollary of [2, Theorem 5.5]

Corollary 3.11. *If $\langle Y, d_Y \rangle$ is a compact, metric space and $h: Y \rightarrow Y$ is continuous, open, and uniformly expanding on Y then the ICT CINE sets of $\langle Y, h \rangle$ are precisely the ω -limit sets.*

At this point we have that the ICT CINE sets of $\langle X, g \rangle$ are precisely the ω -limit sets. We now describe how to translate between $\langle X, g \rangle$ and $\langle [0, 1], f \rangle$.

Define the quotient map $\pi: X \rightarrow [0, 1]$ such that for all $x^\eta \in X$, $\pi(x^\eta) = x$. It is clear from the definition of g that

$$f \circ \pi = \pi \circ g.$$

Fix $x^\eta \in X$.

Firstly, fix $y \in \pi(\omega(x^\eta))$ and $\epsilon > 0$. Then there exists θ such that $y^\theta \in \omega(x^\eta)$. There must exist $n \in \mathbb{N}$ such that $g^n(x^\eta) \in B_\epsilon(y^\theta)$ and therefore that $|f^n(x) - y| < \epsilon$. It follows that $y \in \omega(x)$. Thus, $\pi(\omega(x^\eta)) \subset \omega(x)$.

Conversely, fix $y \in \omega(x)$ and $\epsilon > 0$. Find $n \in \mathbb{N}$ such that $2^{-n} < \frac{\epsilon}{2}$. $\bigcup_{k \leq n} P_k$ is finite so there exists $\delta < \frac{\epsilon}{2}$ such that $B_\delta(y) \setminus \{y\} \cap \bigcup_{k \leq n} P_k = \emptyset$. Note that for all $z^\zeta \in \pi^{-1}(B_\delta(y))$ there exists θ such that $y^\theta \in X$ and

$$\begin{aligned} d(y^\theta, z^\zeta) &= |y - z| + \sum_{\substack{p \in P \\ \pi^{-1}(\{p\}) \subset [y^\theta, z^\zeta]}} 4^{-n_p} \\ &\leq \delta + \sum_{\substack{p \in P \\ p \notin \bigcup_{k \leq n} P_k}} 4^{-n_p} \\ &\leq \delta + \sum_{k > n} 2^k \cdot 4^{-k} \\ &< \epsilon \end{aligned}$$

In the case where $y \in P_k$ for some $k \leq n$, $\theta \in \{-1, 1\}$ is chosen to minimise $d(y^\theta, z^\zeta)$ to avoid an extra 4^{-k} factor. Now, $y \in \omega(x)$ so we can find $m \in \mathbb{N}$ such that $|f^m(x) - y| < \delta$ and so there exists θ such that $y^\theta \in X$ and $d(g^m(x^\eta), y^\theta) < \epsilon$. θ takes one of a finite fixed set of values, namely $\{-1, 0, 1\}$, and so there must exist $\theta \in \{-1, 0, 1\}$ which works as above for an infinite sequence of values for ϵ converging to 0. It follows that for this θ , $y^\theta \in \omega(x^\eta)$ and consequently that $y \in \pi(\omega(x^\eta))$. Thus, $\omega(x) \subset \pi(\omega(x^\eta))$.

Therefore, the image of any ω -limit subset of X with respect to g is an ω -limit subset of $[0, 1]$ with respect to f .

Finally, we look at how to convert an ICT CINE set of $\langle [0, 1], f \rangle$ to an ICT CINE set of $\langle X, g \rangle$. To do this we need the following definition. Let $\zeta \in \{-1, 1\}$ be the unique value such that $c^\zeta \in \text{orb}(c^\zeta)$ and let

$$P_\zeta = \{p^\eta \in P \times \{-1, 1\} \mid g^{n_p}(p^\eta) = c^\zeta\}.$$

Suppose $D \subset [0, 1]$ is ICT CINE with respect to f . Then let

$$E' = ((D \setminus P) \times \{0\}) \cup (D \times \{-1, 1\}) \cap P_\zeta$$

and let $E = \overline{E'}$.

We will show that E' is ICT. It will follow automatically that E is ICT and then we can apply [2, Proposition 2.5] (given below) to deduce that E is ICT CINE.

Proposition. Let $\langle X, d \rangle$ be a compact metric space, and $f: X \rightarrow X$ be continuous. If A is a closed, internally chain transitive subset of X , then A is invariant.

After observing that $E' \subset \pi^{-1}(D)$ it is easy to establish that $\pi(E') = D$. From this we may deduce that the ω -limit sets of $\langle [0, 1], f \rangle$ are precisely the ICT CINE sets.

Lemma 3.12. *For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all*

$$x^\eta \in X \setminus \left(\bigcup_{n < N} P_n \times \{-1, 1\} \right)$$

there exists $\delta > 0$ such that

$$x^\eta \in \pi^{-1}(B_\delta(x)) \subset B_\epsilon(x^\eta)$$

Proof. Fix $\epsilon > 0$ and find $N \in \mathbb{N}$ such that $2^{-N+1} < \epsilon$. Fix

$$x^\eta \in X \setminus \left(\bigcup_{n \leq N} P_n \times \{-1, 1\} \right).$$

Find $\delta > 0$ such that $\delta < 2^{-N}$ and $B_\delta(x) \cap \bigcup_{n \leq N} P_n = \emptyset$. Then for all $y^\theta \in X$ such that $|x - y| < \delta$,

$$d(x^\eta, y^\theta) \leq |x - y| + \sum_{n > N} 2^{-n} < \delta + 2^{-N} < \epsilon \quad \square$$

Proposition 3.13. *E' is ICT.*

Proof. We consider three cases.

Firstly suppose $c \notin D$. Then $D \cap P = \emptyset$ because c is periodic and D is invariant. Let x^η, y^θ be two arbitrary elements of E' and fix $\epsilon > 0$. Because $x, y \notin P$ we can apply Lemma 3.12 to find $\delta > 0$ such that:

$$\begin{aligned} x^\eta &\in \pi^{-1}(B_\delta(x)) \subset B_\epsilon(x^\eta), \\ y^\theta &\in \pi^{-1}(B_\delta(y)) \subset B_\epsilon(y^\theta). \end{aligned}$$

f has periodic critical point and consequently, by [5][Theorem 4.2], has shadowing³. Combining this with the fact that D is ICT we can find a real orbit from a to b for some $a \in B_\delta(x)$ and $b \in B_\delta(y)$. We can easily find $\alpha, \beta \in \{-1, 0, 1\}$ such that $a^\alpha, b^\beta \in X$ and $b^\beta \in \text{orb}(a^\alpha)$. Then the orbit of a^α up to b^β is an ϵ -pseudo orbit from x^η to y^θ and hence E' is ICT.

³See the chapter summary for more details.

Now suppose that $c \in D$ and that c is isolated in D . By invariance we have that $\text{orb}(c) \subset D$ and that each of these finitely many points is isolated in D .

Let $\epsilon > 0$ be sufficiently small that $B_\epsilon(\text{orb}(c)) \cap D = \text{orb}(c)$. It is clear that there can be no ϵ -pseudo-orbit contained in D from a point of $\text{orb}(c)$ to a point of $D \setminus \text{orb}(c)$. Because D is ICT we must have that $D = \text{orb}(c)$.

$$E' = (\text{orb}(c) \times \{-1, 1\}) \cap P^\zeta$$

which is easily seen to be ICT.

Finally, suppose that $c \in D$ and that c is not isolated in D . It follows that for any $\epsilon > 0$ there exists a point z^ι within ϵ of c^ζ . Fix $x^\eta, y^\theta \in E'$ and $\epsilon > 0$. We consider two cases of x^η and y^θ .

If $x^\eta \in P_\zeta$ and $y^\theta \notin P_\zeta$ then simply take z^ι within $\frac{\epsilon}{2}$ of c^ζ and find an $\frac{\epsilon}{2}$ -pseudo-orbit from z^ι to y^θ as in the first case of this proof. By definition of P_ζ there is a real orbit from x^η to c^ζ and so we have an ϵ -pseudo-orbit from x^η to y^θ .

If $x^\eta \notin P_\zeta$ and $y^\theta \in P_\zeta$ then by invariance of D for any $N \in \mathbb{N}$ we can find

$$z^\iota \in E' \setminus \left(\bigcup_{n \leq N} P_n \times \{-1, 1\} \right)$$

such that $y^\theta \in \text{orb}(z^\iota)$. With this we can apply Lemma 3.12 to obtain $N \in \mathbb{N}$, $z^\iota \in E'$ and $\delta > 0$ such that:

$$\begin{aligned} x^\eta &\in \pi^{-1}(B_\delta(x)) \subset B_\epsilon(x^\eta); \\ z^\iota &\in \pi^{-1}(B_\delta(z)) \subset B_\epsilon(z^\iota); \\ y^\theta &\in \text{orb}(z^\iota). \end{aligned}$$

As before, we use shadowing of f to find an ϵ -pseudo-orbit from x^η to z^ι . Concatenate this with the real orbit from z^ι to y^θ to obtain an ϵ -pseudo-orbit from x^η to y^θ .

Of course, there are another two cases of x^η and y^θ to consider but these can be deduced from the first two cases. For example, given $x^\eta, y^\theta \in P_\zeta$, we fix an arbitrary $z^\iota \in E' \setminus P_\zeta$. Then, given $\epsilon > 0$, we can construct an ϵ -pseudo-orbit from x^η to y^θ from a pair of $\frac{\epsilon}{2}$ -pseudo-orbits from x^η to z^ι and from z^ι to y^θ .

Thus, we deduce that E' is ICT. □

3.4 Summary

Naturally we wish to extend the work of the previous section to handle more than just maps with periodic critical point. A special effort has been placed on keeping

the elements of the proof as general as possible but one or two details hold us firmly within the class of tent-maps with periodic critical point.

Problem 3.14. Find the set $A \subset [\sqrt{2}, 2]$ of tent-map slopes for which the ICT CINE sets are precisely the ω -limit sets. Characterise A using dynamical properties.

We are still a long way from this goal and a good starting point would be to investigate a concept which arises naturally in our work, that of *shadowing*.

Definition 3.15. A tent-map f has *shadowing* if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any δ -pseudo-orbit $\langle x_k \mid k \leq n \rangle$ there exists $y \in [0, 1]$ such that for all $k \leq n$, $|x_k - f^k(y)| < \epsilon$.

Question 3.16. For a tent-map with shadowing, are the ICT CINE sets precisely the ω -limit sets?

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