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SURVEY ARTICLE

Arithmetic progressions at the Journal of the LMS

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Abstract

We discuss the papers P. Erdős and P. Turán, *On some sequences of integers*, J. London Math. Soc. (1) **11** (1936), 261–264 and K. F. Roth, *On certain sets of integers*, J. London Math. Soc. (1) **28** (1953), 104–109, both foundational papers in the study of arithmetic progressions in sets of integers, and their subsequent influence.

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1 | THE PAPER OF ERDŐS AND TURÁN

The first of our featured papers, by Erdős and Turán [13], was probably the second (in chronological order) important work on the combinatorics of arithmetic progressions in the integers.

The first was the 1927 work of van der Waerden [53], where the following was shown.

This article forms part of the "100 years of the Journal of the London Mathematical Society" special issue celebrating the founding of the Journal of the London Mathematical Society in 1926.

Theorem 1.1 (van der Waerden). *Let $k \geqslant 3$ be an integer. Suppose that the set* $\mathbf{N}$ *is coloured with finitely many colours. Then, there is a monochromatic k-term arithmetic progression.*

The proof is via an elaborate inductive argument. One interpretation of the paper of Erdős and Turán is that they ask the following question: could it be the case that the 'largest' colour class already contains a k-term arithmetic progression?

Erdős and Turán both have very many important papers, including many substantial joint papers, so hopefully they would not be offended by my remarking that [13] sits a little below the marginal note of Fermat, the letter of Goldbach, and the statement of Waring in a ranking of works in number theory by the ratio of influence to content. Indeed the entire content of [13] can be summarised as follows. Here, and throughout the survey, $[N]$ denotes $\{1, \dots, N\}$ when N is a positive integer.

(1) Define the quantity $r_k(N)$, defined to be the maximum cardinality of a subset $A \subset [N]$ without k distinct elements in arithmetic progression;
(2) We have $r_k(N + N') \leqslant r_k(N) + r_k(N')$ (this is immediate);
(3) We have $r_3(2N) \leqslant N$ for $N \geqslant 8$, and $r_3(N) \leqslant (\frac{3}{8} + \varepsilon)N$ for $N \geqslant N_0(\varepsilon)$ (in fact details are only given for the weaker bound $r_3(N) \leqslant (\frac{4}{9} + \varepsilon)N$; the proof of the $\frac{3}{8}$ bound involves some 'very much longer' (omitted) case-checking);
(4) They state that it is 'probably' the case that $r_3(N) = o(N)$ (for an explanation of this notation, see below);
(5) Two conjectures of Szekeres are reported on the values of $r_k(N)$, for certain values of k and n, and it is observed that these conjectures would imply both van der Waerden's theorem and that there are arbitrarily long progressions of primes.

Unfortunately the two conjectures of Szekeres later turned out to be badly wrong. This leaves (1) and (4) as the most important contributions of the paper. The conjecture of Szekeres, which Erdős and Turán do not disbelieve, would imply that $r_k(N) = o(N)$. As a consequence of this and (4), it is conventionally asserted that the legacy of [13] is to define $r_k(N)$ and to conjecture[†] the following.

Conjecture 1.2. $r_k(N) = o(N)$. *That is, the density of the largest subset of* $[N]$*, free from k-term arithmetic progressions, tends to zero as $N \to \infty$.*

This conjecture turned out to be among the most profound and influential in 20th century mathematics.

Over the 15–20 years following the publication of [13], the only progress on these issues were some very influential constructions of Salem-Spencer [37] and Behrend [1], which show that conjectures of Szekeres reported in [13] are too optimistic. Behrend's result from 1946 is the following.

[†] This is not a completely unproblematic statement, since the conjecture is not actually written down in [13]. With help from https://www.erdosproblems.com/139 one can check that the 1961 open problems paper [10] appears to be the first place this occurs in print (it is described as 'plausible'). In various places, Erdős comments on the origin of the conjecture in ways similar to the following, which may be found in Erdős [12, p. 94]: "I only make a historical remark which cannot entirely be documented." E. Rothe in 1944 told me that his wife Dr. Hildegard Ille was given the problem of estimating $r_k(n)$ by I. Schur sometime in the 1930s. Thus perhaps Schur conjectured $r_k(n) = o(n)$ before Turán and myself'. For an extensive and interesting (and occasionally speculative) discussion of these points, see [44, Chapter 38].

Theorem 1.3. *We have $r_3(N) \geqslant Ne^{-c\sqrt{\log N}}$ for all sufficiently large N and some absolute constant $c > 0$.*

A careful analysis of Behrend's construction shows that one may take any $c < 2\sqrt{2\log 2} \approx 2.355$, for N sufficiently large. Until very recently, this was only improved in lower-order terms, and personally I might have speculated that this represents the truth. It was therefore quite surprising when, in 2024, Elsholtz, Hunter, Proske, and Sauermann [9] obtained the smaller value $2\sqrt{\log(24/7)} \approx 2.218$ via a somewhat elaborate variation on Behrend's construction. Rankin [32] obtained stronger lower bounds on $r_k(N)$ for larger k, though these are still of shape $Ne^{-(\log N)^{1-c_k}}$.

Notation. The letters c, C are reserved for absolute constants with $0 < c < 1 < C$, which may be different constants on different occasions. We use the standard asymptotic notation $X = O(Y)$ or equivalently $X \ll Y$ to mean that there is some absolute constant C such that $X \leqslant CY$. Similarly, $X \gg Y$ means $X \geqslant cY$ (and *not* that X is significantly bigger than Y). $f(N) = o(N)$ means $\lim_{N\to\infty} f(N)/N = 0$. Where dates are given these are publication dates, except for preprints where the date of first appearance on the arXiv is given. In tables, papers are listed by order of priority.

2 | ROTH's THEOREM

We now turn to the second of our featured papers, Roth's work [34] on three-term progressions. In this paper, Roth proved the first case of Conjecture 1.2, namely the case $k = 3$. His theorem is the following.

Theorem 2.1. *We have $r_3(N) \ll N/\log\log N$.*

This result is usually known as 'Roth's theorem' although sometimes[†] it needs to be remarked that there is an arguably more famous Roth's theorem on diophantine approximation of algebraic numbers.

We now sketch Roth's method of proof, which is at least as influential as the result itself. The basic scheme of the proof is the so-called *density increment argument*. One establishes the following dichotomy.

Proposition 2.2. *Let N be sufficiently large. Suppose that $0 < \alpha < 1$ and that $N \geqslant (8/\alpha)^{10}$. Suppose that $P \subset \mathbf{Z}$ is an arithmetic progression of length N and that $A \subset P$ is a set with cardinality at least αN. Then, one of the following two alternatives holds:*

(1) *A contains a nontrivial 3-term progression;*
(2) *A has a density increment: more precisely there is an arithmetic progression P' of length $N' \geqslant N^{1/5}$ such that, writing $A' := A \cap P'$ and $\alpha' := |A'|/|P'|$, we have $\alpha' \geqslant \alpha + \frac{\alpha^2}{112}$.*

[†] One such occasion was the ICM in 1958 where Davenport was describing the work leading to the award of the Fields Medal to Roth. After describing the famous result on diophantine approximation he turns to 3-term progressions and states "I turn now to another achievement of Dr Roth, which seems to me to be also of the first magnitude, though the problem to which it relates is perhaps of less universal interest ..."

The exact numbers and exponents here are not important; these are the ones obtained in my lecture notes [18], which may be consulted for more details.

Theorem 2.1 follows by iterating Proposition 2.2. The crucial observation is that we can only see case (2) a number of times bounded in terms of α before we would have $\alpha' > 1$, which is clearly impossible. In fact, the density of α doubles after $O(1/\alpha)$ steps, then doubles again after $O(1/2\alpha)$ steps, and so on, so we can only arrive at case (2) a total of $O(1/\alpha)$ times. For full details, see [18, Section 1].

The main tool for proving Proposition 2.2 is the Fourier transform, specifically the observation that this can be used to count 3-term arithmetic progressions. Indeed if $f_1, f_2, f_3 : \mathbf{Z} \to \mathbf{C}$ are three finitely-supported functions then we may consider the operator

$$T_3(f_1, f_2, f_3) := \sum_{x,d} f_1(x) f_2(x+d) f_3(x+2d). \tag{2.1}$$

This counts the number of 3-term progressions weighted by the functions f_i. The number of 3-term progressions in A (including the trivial ones (x, x, x)) is then $T_3(1_A, 1_A, 1_A)$, where $1_A(x) = 1$ if $x \in A$ and 0 otherwise. We have the formula

$$T_3(f_1, f_2, f_3) = \int_0^1 \widehat{f_1}(\theta)\widehat{f_2}(-2\theta)\widehat{f_3}(\theta) d\theta, \tag{2.2}$$

where the Fourier transform is given by

$$\widehat{f}(\theta) := \sum_n f(n) \mathrm{e}^{-2\pi i n\theta}.$$

Once written down, formula (2.2) is readily verified using the orthogonality relations for characters

$$\int_0^1 \mathrm{e}^{2\pi i m\theta} d\theta = \begin{cases} 1 & m = 0 \\ 0 & m \in \mathbf{Z} \setminus \{0\}. \end{cases} \tag{2.3}$$

To prove Proposition 2.2, we proceed as follows. Suppose that $P = [N]$ (by affine rescaling we may assume this is the case) and that $|A| = \alpha N$. Then, we decompose 1_A as $\alpha 1_{[N]} + f_A$ where, by definition, the average value of f_A is zero. We may then expand

$$T_3(1_A, 1_A, 1_A) = \alpha^3 T_3(1_{[N]}, 1_{[N]}, 1_{[N]}) + \cdots + T_3(f_A, f_A, f_A), \tag{2.4}$$

where the dots denote 6 unwritten terms, each involving at least one copy of f_A. The quantity $T_3(1_{[N]}, 1_{[N]}, 1_{[N]})$ can be computed; the exact answer depends on the parity of N, but it is roughly $N^2/4$. Thus, unless α is rather small, the term $\alpha^3 T_3(1_{[N]}, 1_{[N]}, 1_{[N]})$ is of significant size and in particular is much bigger than the number of trivial progressions in A. Therefore, if A has no non-trivial 3-term progressions, then at least one of the seven terms involving f_A on the right in (2.4) is $\gg \alpha^3 N^2$. The seven cases can be analysed with one argument, and the case that $T_3(f_A, f_A, f_A) \gg \alpha^3 N^2$ is representative.

Appealing to (2.2) and the Parseval identity

$$\int_0^1 |\widehat{f}_A(\theta)|^2 d\theta = \alpha(1-\alpha)N,$$

TABLE 1 Progression of bounds for progressions (length 3).

Author	Publication date	Reference	Bound ($\ll \dots$)
Roth	1953	[34]	$N/\log\log N$
Szemerédi	2007	[48]	$Ne^{-c\sqrt{\log\log N}}$
Heath–Brown, Szemerédi[1]	1987, 1990	[23, 47]	$N(\log N)^{-c}$
Bourgain	1999	[5]	$N(\log N)^{-1/2+o(1)}$
Bourgain	2008	[6]	$N(\log N)^{-2/3+o(1)}$
Sanders	2012	[39]	$N(\log N)^{-3/4+o(1)}$
Sanders	2011	[38]	$N(\log N)^{-1}(\log\log N)^6$
Bloom	2016	[2]	$N(\log N)^{-1}(\log\log N)^4$
Schoen	2021	[42]	$N(\log N)^{-1}(\log\log N)^3$
Bloom and Sisask	2020	[3]	$N(\log N)^{-1-c}$
Kelley and Meka	2023	[27]	$Ne^{-(\log N)^{1/12}}$

[1]For a discussion of the relationship between these two works and [48] (whose ideas date to 'the 1980s'), see [48] and the MathSciNet review of [47].

one may conclude that

$$\sup_{\theta\in[0,1]} |\widehat{f_A}(\theta)| \gg \alpha^2 N. \tag{2.5}$$

To complete the proof of Proposition 2.2, and hence of Roth's theorem, it suffices to show that a set $A \subset [N]$ satisfying (2.5) has a suitable density increment on a subprogression $P \subset [N]$. Before sketching the argument, let us remark on the nature of the condition (2.5). This statement asserts that A has a significant 'bias', very different to what one expects for a random set of similar size. This is easiest to think about in the case $\alpha = \frac{1}{2}$, with A generated by flipping a fair coin for each $n \in [N]$ and including n in A if coin n comes up heads†. Then, $\widehat{f}_A(\theta) = \frac{1}{2}\sum_{n\in[N]} X_n e^{-2\pi i n\theta}$ where $X_n = 2(1_A(n) - \frac{1}{2})$ is a Rademacher random variable, that is to say takes the values ± 1 with equal probability. A sum like this will typically have magnitude on the order $\sqrt{N}$, rather than N.

To actually convert (2.5) to a density increment one uses a small amount of diophantine analysis. If $|\widehat{f}_A(\theta)| \gg \alpha^2 N$, then by Dirichlet's theorem one may select a moderate-sized d for which θd is nearly an integer. Using the fact that $e^{-2\pi i n\theta}$ is almost constant as n ranges over progressions with common difference d, one can easily show that the average of f_A on one of these progressions P is $\gg \alpha^2$, which immediately translates to A having density at least $\alpha + c\alpha^2$ on P. This concludes the proof.

Roth's argument may be considered as somewhat related to the Hardy–Littlewood method, on which Roth had previously published—perhaps most notably giving in [33] an asymptotic for the number of representations of $n = x^2 + y^3 + z^4$. So far as I am aware, [34] was the first paper to use a version of this method to say something about an essentially arbitrary set A rather than a specific classical set such as the kth powers or the primes‡.

The problem of bounding $r_3(N)$ has received considerable attention over the years. See Table 1 below for the chronology.

† We ignore the minor point that such a set need not have size *exactly* $N/2$.

‡ Davenport [7] said essentially the same thing.

We remark that the paper [2] was also published in JLMS.

Bloom and Sisask [4] improved Kelley and Meka's exponent of $\frac{1}{12}$ to $\frac{1}{9}$.

All of these bounds use some variant of the density increment argument. Up to and including [3] these works replaced the relatively simple use of Fourier analysis in Roth's original by increasingly sophisticated and complex harmonic analysis arguments. The very recent work of Kelley and Meka is quite remarkable for how much it improves on what went before, and also because it uses Fourier analysis only somewhat minimally.

Much of this progress has been somewhat motivated by a celebrated conjecture of Erdős[§] [11] to the effect that any set $A \subset \mathbf{N}$ with $\sum_{a\in A} \frac{1}{a} = \infty$ should contain progressions of arbitrary length. This is roughly equivalent to proving that $r_k(N) \leqslant N(\log N)^{-1+o(1)}$. Thus, Bloom and Sisask [3] were the first to establish this conjecture of Erdős for $k = 3$.

A major reason for Erdős to state this conjecture, other than its intrinsic appeal, is that it implies that the primes contain arbitrarily long arithmetic progressions 'on density grounds alone'. However, from the point of view of arithmetic progressions alone, the threshold $N(\log N)^{-1+o(1)}$ is probably not a natural one.

3 | PROGRESSIONS OF LENGTH $k \geqslant 4$

After Roth's paper, it took another 16 years for further progress to be made on Conjecture 1.2. In 1969, Szemerédi [45] proved[†] that $r_4(N) = o(N)$, and then a few years later, in a masterpiece of combinatorial reasoning, proved the full conjecture that $r_k(N) = o(N)$. Shortly thereafter, in 1977, Furstenberg [14] gave a completely different proof of the result using ergodic theory. Both of these arguments have opened up many new directions, but the reader is referred to other accounts such as [49] for a fuller discussion.

Neither Szemerédi's work nor that of Furstenberg gave any bounds on $r_k(N)$. Szemerédi's work does give a bound in principle but it would be exceptionally weak due to the highly recursive nature of the argument. Furstenberg's work is highly infinitary and uses essentially a compactness argument; it does not give any bound, even in principle.

Given that Roth's theorem and subsequent works had given good quantitative information about $r_3(N)$, it was natural to ask whether the density increment method and Fourier analysis could give effective bounds for $r_k(N)$, $k \geqslant 4$. Roth [35, 36] carried out some investigations in this direction but they were inconclusive. Other very interesting variants of the density increment argument were given by Sárközy [40, 41]. In [40], Sárközy showed that any set $A \subset [N]$ not containing distinct elements a', a differing by a square has size $\ll N(\log N)^{-1/3+o(1)}$, and in [41] showed that any set $A \subset [N]$ not containing distinct elements a', a differing by $p - 1$ is $\ll N(\log\log N)^{-2+o(1)}$. In particular, both bounds are $o(N)$. We remark that, while an ergodic proof of the bound $|A| = o(N)$ for sets with no square difference was given by Furstenberg around the same time, it again gives no bounds, and there is no such proof for sets with no shifted prime difference.

In terms of applications to arithmetic progressions, the real breakthrough came with the work of Gowers in the late 1990s [15, 16].

[§] Erdős stated this question many times but with help from https://www.erdosproblems.com/3. It seems that [11] is the earliest mention, at least by publication date; note that this is nearly 40 years later than the Erdős-Turán paper. The problem is in fact stated as 4.33.6 in the problems section immediately following the cited paper.

[†] For an entertaining anecdote about how Szemerédi came to work on this problem, see [26, p. 3] (this reference is available at https://www.math.tugraz.at/kang/papers/Kang-Szemeredi.pdf).

The first[†] key realisation in Gowers's work (which was anticipated, albeit in a rather different context, in the ergodic literature, particularly in various works of Conze–Lesigne and Furstenberg–Weiss and later of Host–Kra and Ziegler) is that a Fourier-based density increment argument analogous to that of Roth cannot work. This can be explained by the existence of a relatively simple example of a set $A \subset [N]$ of size αN for which

$$T_4(1_A, 1_A, 1_A, 1_A) \not\approx \alpha^4 T_4(1_{[N]}, 1_{[N]}, 1_{[N]}, 1_{[N]})$$

but for which f_A has no large Fourier coefficient. Here, we define

$$T_4(f_1, f_2, f_3, f_4) := \sum_{x,d} f_1(x) f_2(x+d) f_3(x+2d) f_4(x+3d). \tag{3.1}$$

which is an obvious generalisation of T_3 to 4-term progressions.

Such an example can be described, at least heuristically, as follows. Let θ be a suitably irrational number ($\theta = \sqrt{2}$ is fine) and set $A := \{n \in [N] : \|\theta n^2\|_{\mathbf{R}/\mathbf{Z}} \leqslant \alpha/2\}$, where $\|x\|_{\mathbf{R}/\mathbf{Z}}$ denotes the distance from x to the nearest integer. This set has size $|A| \approx \alpha N$. Now, let us look at the possible 4-term progressions $x, x+d, x+2d, x+3d$ with $d > 0$ and contained in $[N]$; the number of such progressions is easily checked to be roughly $N^2/6$. Selecting such a progression at random, we first look at the probability that all three of $x, x+d, x+2d$ lie in A. The probability that each fixed $x + id$ ($i \in \{0,1,2\}$) lies in A is $\approx \alpha$, and it is reasonable to suppose that these three events are roughly independent, so $\mathbb{P}(x, x+d, x+2d \in A) \approx \alpha^3$.

However, the further event $x + 3d \in A$ is *not* independent of the first three on account of the relationship

$$(x+3d)^2 = x^2 - 3(x+d)^2 + 3(x+2d)^2.$$

This suggests that

$$\mathbb{P}(x+3d \in A \mid x, x+d, x+2d \in A) \approx \mathbb{P}\left(X_0 - 3X_1 + 3X_2 \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right) = \frac{8}{27},$$

where X_0, X_1, X_2 are i.i.d. $U[-\frac{1}{2}, \frac{1}{2}]$ random variables. Therefore, we expect

$$\begin{aligned}\mathbb{P}(x, x+d,&x+2d, x+3d \in A) \\ &= \mathbb{P}(x+3d \in A \mid x, x+d, x+2d \in A) \cdot \mathbb{P}(x, x+d, x+2d \in A) \\ &\approx \frac{8}{27}\alpha^3,\end{aligned}$$

and so $T_4(1_A, 1_A, 1_A, 1_A) \approx \frac{8}{27}\alpha^3 \cdot \frac{N^2}{6}$ whilst $\alpha^4 T_4(1_{[N]}, 1_{[N]}, 1_{[N]}, 1_{[N]}) \approx \alpha^4 \cdot \frac{N^2}{6}$. These quantities are significantly different when α is small.

[†] For a more historically accurate description of the thought process involved, the reader may refer to Gowers's own account in minutes 1:40–3:00 of his lecture 'The afterlife of Szemerédi's Theorem' available at https://www.youtube.com/watch?v=8W_YDHNNeBU

The above heuristics may be turned into a proof using Fourier analysis and standard exponential sum estimates, notably Weyl's inequality, and with the same tools one may confirm that $f_A = 1_A - \alpha 1_{[N]}$ has no large Fourier coefficient.

Remark. Although the set A we have just described in fact has *more* 4-term arithmetic progressions than a random set of density α, more elaborate examples can be constructed with fewer [17].

The example above shows that any naïve attempt to generalise Roth's argument to longer progressions is doomed to fail. Gowers's way around this first makes the observation that quadratic examples such as the above do not actually defeat a density increment strategy, because quadratic phases such as $e^{-2\pi i\theta n^2}$ are still constant on long progressions. Indeed, the set A in the example actually *contains* progressions of length $\gg_\alpha N^{1/8}$. This follows using well-known results about diophantine approximation using squares: for instance by [54] there is some $q \leqslant N^{1/2}$ such that $\|\theta q^2\|_{\mathbf{R}/\mathbf{Z}} \leqslant N^{-1/4}$, and then all the elements $q, 2q, 3q, \ldots, mq$ lie in A for $m \leqslant (\alpha N^{1/4}/2)^{1/2}$.

This observation is only a small part of an actual proof. The remarkable advance of Gowers [15] was to show that any set $A \subset [N]$ with $|A| = \alpha N$ and not having $\approx \alpha^4 N^2/6$ arithmetic progressions must have some 'quadratic bias'. The argument uses the Freiman–Ruzsa structure theory of sets with small sumset coupled with the introduction of what are now called Gowers norms and the Balog–Szemerédi–Gowers theorem, both tools which have found very widespread application elsewhere.

4 | OTHER DIRECTIONS AND OPEN QUESTIONS

We mention here some other directions which have been influenced by the paper of Roth and the questions asked by Erdős and Turán.

(1) The whole subject of higher-order Fourier analysis has its roots in the work of Gowers mentioned above and the work of Host–Kra [24] and Ziegler [55] in ergodic theory, which itself can be traced back to Furstenberg's work on Szemerédi's theorem. Higher-order Fourier analysis has found many applications in number theory, for instance to asymptotics for linear equations in primes [21].

(2) Even more broadly, Roth's argument is perhaps the earliest instance of the 'Structure and randomness dichotomy', for further discussion of which we refer the reader to the elegant lectures of Tao [50, Section 2.1].

(3) The density increment argument has been applied in various settings concerning configurations more exotic than arithmetic progressions, and usually leads to the best known bounds where it can be used. For instance, Shkredov [43] used it to give effective bounds for the largest subset of $[N]^2$ not containing a *corner* $(x, y), (x + d, y), (x, y + d)$, and Shkredov's bounds were recently dramatically improved by Jaber, Liu, Lovett, Ostuni, and Sawhney [25], still using a variant of the density increment argument, to $N^2 e^{-(\log N)^c}$; that is, of comparable strength to the Kelley–Meka bounds for $r_3(N)$. We mention also bounds for polynomial progressions such as $(x, x + d, x + d^2)$, as may be found in the work of Peluse and Prendiville [30], as well as work on bounds in the density Hales–Jewett theorem [8, 31].

(4) Other variants of the density increment argument are energy-increment arguments, perhaps the most famous of which is the one used to prove Szemerédi's regularity lemma, a

TABLE 2 Progression of bounds for progressions (length 4).

Author	Date	Reference	Bound
Szemerédi	1969	[45]	$o(N)$
Gowers	1998	[15]	$N(\log\log\log N)^{-c}$
Gowers	2001	[16]	$N(\log\log N)^{-c}$
Green–Tao	2009	[20]	$Ne^{-c\sqrt{\log\log N}}$
Green–Tao	2017	[22]	$N(\log N)^{-c}$

TABLE 3 Progression of bounds for progressions (length 5 and higher).

Author	Date	Reference	Bound
Szemerédi	1975	[46]	$o(N)$
Gowers	2001	[16]	$N(\log\log N)^{-c_k}$
Leng–Sah–Sawhney	2024	[29]	$Ne^{-(\log\log N)^c}$ $(k=5)$
Leng–Sah–Sawhney	2024	[28]	$Ne^{-(\log\log N)^{c_k}}$ $(k \geqslant 6)$

fundamental tool in graph theory; and the entropy decrement argument of Tao [52] which was used to understand logarithmic two-point correlations of multiplicative functions such as $\sum_{n\leqslant X}\frac{1}{n}\mu(n)\mu(n+1)$ and hence to resolve the Erdős discrepancy problem [51].

Finally, we observe that this is a very active area of study, as can be seen by how recent some of the major advances mentioned above are. There remain many interesting questions. Perhaps the most fundamental is that of improving the bounds for $r_k(N)$ for $k \geqslant 4$ for progress on this question, see Tables 2 and 3. We still seem to be a long way from establishing the sum-of-reciprocals conjecture of Erdős [11] for progressions of length 4. Though our understanding of corners is now rather good, we still lack a good quantitative understanding of what density thresholds guarantee relatively basic higher-dimensional configurations such as squares $\{(x,y),(x,y+d),(x+d,y),(x+d,y+d)\}$. Finally, it is possible that any set $A\subset[N]$ with $|A|>N^{1-c}$ contains distinct elements differing by the square of an integer. Recently, it was shown using a density increment argument [19] that $|A|\geqslant Ne^{-c\sqrt{\log N}}$ does suffice, but it was also shown that a density increment argument cannot improve that bound significantly further. Substantially new ideas seem to be required for all of these problems, and one can hope that their study will lead to many further advances, as has been the case with the questions posed by Erdős and Turán.

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