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Equilibrium Reward for Liquidity Providers in Automated Market Makers

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ABSTRACT

We find the equilibrium contract that an automated market maker (AMM) offers to their strategic liquidity providers (LPs) in order to maximize the order flow that gets processed by the venue. Our model is formulated as a leader–follower stochastic game, where the venue is the leader and a representative LP is the follower. We derive approximate closed-form equilibrium solutions to the stochastic game and analyze the reward structure. The equilibrium strategies we find characterize the scenarios when LPs have incentives to add liquidity to the pool. The equilibrium contract depends on the external price, the pool reference price, and the pool reserves. Our framework offers insights into AMM design for maximizing order flow while ensuring LP profitability.

1 | Introduction

Automated market makers (AMMs) are one of the latest developments in financial technology. An AMM is a venue with predefined trading rules where liquidity takers (LTs) and liquidity providers (LPs) trade. These venues became popular after Uniswap V2 was released in May 2020. As of 2025, the fourth version of the protocol has already been announced; see Adams et al. (2021, 2023).

The academic literature on AMMs is still young but already quite rich. The first papers studying the mechanisms of AMMs are in Chiu and Koepl (2019), Angeris and Chitra (2020), Angeris et al. (2021), Capponi and Jia (2021), Lipton and Treccani (2021), the arbitrage between AMMs and external centralized venues is studied in Cartea et al. (2023a, 2023b), and price formation in AMMs is studied in Capponi et al. (2026). It is now well-known that, in the absence of fees, an agent providing liquidity in a standard constant function market (CFM)—where prices are only determined as a function of the reserves—are exposed to a

concave payoff that is inferior to that of holding coins outside of the pool, a phenomenon called impermanent loss (IL). In fact, empirical studies such as Loesch et al. (2021) show that currently, even in the presence of fees, LPs are on average incurring a loss. Recently, Fukasawa et al. (2023) study the hedging of this IL, and Milionis et al. (2022) introduce the notion of loss-versus-rebalancing—see also Milionis et al. (2023). Lastly, Cartea et al. (2024) consider a more general notion of predictable loss and derive optimal liquidity provision strategies.

In order to tackle the IL issue, another active area of research is that of AMM design. Bergault et al. (2024) propose an alternative mechanism in which the pricing function uses external information about the current market exchange rates, allowing the AMM to update its bid and offer prices not only after a trade but also after a price oracle has fired an update.¹ Cartea et al. (2024) introduce a new family of AMMs that they call decentralized liquidity pools (DLPs). In their paper, they optimize a range of performance criteria involving the profitability of LPs. The optimal policy of a given performance criterion gives rise to

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a new DLP design. Depending on the model, the resulting DLP could monitor external prices or even filter arrivals in search of fundamental values.² Recently, Cao et al. (2025) studied the problem of computing dynamic optimal fees in AMMs adjusting for instantaneous volatility. Lastly, Adams et al. (2024) introduce the so-called “auction-managed AMM,” that reduces losses to informed order flow, and maximizes revenue from uninformed order flow. With the exception of Adams et al. (2024), these designs do not aim to maximize the trading activity in the venue.

In this paper, we find the optimal contract that a CFM should offer to their strategic LPs in order to attract the maximum amount of order flow to the CFM. We formulate the design as the solution to a leader–follower stochastic game in which the venue is the leader and the representative LP is the follower.³ Closest to our paper is El Euch et al. (2021) who work in a centralized limit order book (LOB) and find the optimal make–take fees in such a setting;⁴ see also Baldacci et al. (2021) for a multiplayer version of the model, and Baldacci and Bergault (2024) for a different approach to make–take fees. In our paper, the entity designing the contract is the CFM. Although the derivation of the contract follows what is done in El Euch et al. (2021), there are additional steps that we need to take to make the derivations work; see, for example, Lemma 2, which we use to prove that the value function of the representative LP is strictly negative. Furthermore, in El Euch et al. (2021), the agent only controls the intensity of the jump processes, whereas in this work, the agent also controls the drift of the Brownian motion which they use to add or subtract liquidity from the pool. Here, we establish the viscosity properties of the value function associated with our problem. Another contribution of our work is that we find approximate closed-form solutions to the stochastic game, and we obtain insights into the reward structure that attracts, in equilibrium, the highest possible order flow to the venue. For this approximation, we establish the existence and uniqueness of the matrix Riccati differential equation that characterizes the closed-form solutions.

To the best of our knowledge, this is the first article to propose AMM designs as the solution to a leader–follower stochastic game. In our framework, LPs aim to maximize exponential utility of wealth and the venue wishes to maximize trading activity. As customary in leader–follower games, for a given reward structure, we compute the optimal liquidity providing strategy from the representative LP (follower), and using this, we compute the optimal reward structure to be provided by the venue (leader). The new design comes in the form of a “contract” that encodes the reward structure obtained in the second optimization. With the optimal reward structure, the venue compensates LPs for external price movements and adverse liquidity taking activity. We find that, even under the equilibrium contract, if higher liquidity (in the pool) does not attract noise trading, then LPs do not have incentives to add liquidity to the pool. This underlines the fact that the venue on its own cannot fully compensate LPs for the losses to arbitrageurs. On the other hand, if higher liquidity attracts noise trading, then, the optimal contract incentivizes LPs to add liquidity to the pool in order to benefit from the noise order flow. In such a scenario, the transaction costs in the external market scale the amount that the representative LP adds to the pool. In equilibrium, the optimal contract depends on the external price, the pool’s reference price, and the reserves in the pool. Our

equilibrium solutions provide insights into the design of these emerging venues.

The remainder of the paper proceeds as follows. Section 2 introduces the probabilistic framework of the model. In particular, Section 2.3 introduces the problem of the representative LP and Section 2.4 introduces the problem of the venue. The solutions to these two problems are derived in Sections 3 and 4, respectively. Finally, Section 5 presents the numerical results and we collect proofs in the appendix.

2 | The Model

2.1 | Probabilistic Framework

Let $T > 0$ be a trading horizon (e.g., one day), Ω_c the set of continuous functions from $\mathfrak{T} = [0, T]$ into \mathbb{R} , Ω_d the set of piecewise constant càdlàg functions from \mathfrak{T} into \mathbb{N} , and $\Omega = (\Omega_c)^2 \times (\Omega_d)^2$ with the corresponding Borel algebra \mathcal{F} . The observable state is the canonical process $(\chi_t)_{t \in \mathfrak{T}} = (W_t, B_t, \hat{N}_t^+, \hat{N}_t^-)_{t \in \mathfrak{T}}$ of the measurable space (Ω, \mathcal{F}) , with

$$\begin{aligned} W_t(\omega) &:= w(t), \quad B_t(\omega) := b(t), \quad \hat{N}_t^+(\omega) := \hat{n}^+(t), \quad \hat{N}_t^-(\omega) \\ &:= \hat{n}^-(t), \quad \text{for all } t \in \mathfrak{T}, \end{aligned}$$

where $\omega := (w, b, \hat{n}^+, \hat{n}^-) \in \Omega$.

We introduce a probability measure \mathbb{P} such that W and B are Brownian motions and \hat{N}^-, \hat{N}^+ are Poisson processes with intensity $a_0 > 0$. In this probability measure, $W, B, \hat{N}^-,$ and \hat{N}^+ are independent.⁵

We study trading in a pair of assets X and Y (e.g., USDC and ETH) that takes place in a CFM.⁶ In addition to the CFM, there is an external limit order book (LOB) venue where trading in X and Y occurs. We let $S = (S_t)_{t \in \mathfrak{T}}$ be the external midprice of asset Y in terms of asset X . Within the CFM, we let X_t and Y_t be the quantities of assets X and Y in the pool at time $t \in \mathfrak{T}$. The CFM has trading function $f(x, y) = xy$; this is the most popular choice of f and is known as the constant product market (CPM). We let the function $\varphi_{c_t}(y)$ be the level function of f such that $\varphi_{c_t}(Y_t) = X_t$ with

$$c_t = f(X_{t-}, Y_{t-}) = X_{t-} Y_{t-}.$$

More precisely, we have that

$$\varphi_c(y) = \frac{c}{y}, \quad (\varphi_c)'(y) = -\frac{c}{y^2}.$$

The external midprice satisfies

$$dS_t = \sigma dW_t, \quad S_0 \in \mathbb{R}^+,$$

where $\sigma > 0$ is the volatility parameter. We think of the external price as that of a venue where price formation takes place (e.g., Binance, Kraken).

In the CFM, LTs arrive according to the counting processes N^- (LT buys) and N^+ (LT sells) that model the number of

trades of size $\xi > 0$ through time. We denote $Z = (Z_t)_{t \in \mathfrak{T}}$ as the marginal price of Y in terms of X in the venue. We follow the characterization of CFMs in Cartea et al. (2024) that describes the mechanics of the reserves $(X_t)_{t \in \mathfrak{T}}$, $(Y_t)_{t \in \mathfrak{T}}$, and the instantaneous rate $(Z_t)_{t \in \mathfrak{T}}$, according to the arrival of orders N^- and N^+ , and extend it to include stochastic pool depths (given by c_t) due to the activity of LPs. In our model, we write

$$\begin{aligned} dX_t &= \underbrace{Z_t \eta dB_t}_{\text{activity from LPs}} - \underbrace{[\varphi_{c_t}(Y_{t-}) - \varphi_{c_t}(Y_{t-} + \xi)] dN_t^+ + [\varphi_{c_t}(Y_{t-} - \xi) - \varphi_{c_t}(Y_{t-})] dN_t^-}_{\text{activity from LTs}}, \\ dY_t &= \underbrace{\eta dB_t}_{\text{activity from LPs}} + \underbrace{\xi dN_t^+ - \xi dN_t^-}_{\text{activity from LTs}}, \\ dZ_t &= \underbrace{[-(\varphi_{c_t})'(Y_{t-} + \xi) + (\varphi_{c_t})'(Y_{t-})] dN_t^+ + [-(\varphi_{c_t})'(Y_{t-} - \xi) + (\varphi_{c_t})'(Y_{t-})] dN_t^-}_{\text{activity from LTs}}, \\ dN_t^- &= \mathbb{1}_{Y_{t-} > 2\xi} d\hat{N}_t^-, \\ dN_t^+ &= \mathbb{1}_{Y_{t-} > 2\xi} d\hat{N}_t^+, \end{aligned}$$

with $\eta > 0$ representing the volatility in liquidity provision, $N_0^+ = N_0^- = 0$, $X_0 \in \mathbb{R}^+$, $Y_0 \in (\xi, \infty)$, and $Z_0 = X_0/Y_0$. The constraint $\mathbb{1}_{\{Y_{t-} > 2\xi\}}$ is to guarantee that there is a strong solution to the above dynamics; in practical applications, this indicator is always equal to one. The interpretation of the above equations is as follows, given a liquidity taking buy order (resp. sell order), the quantity Y_t goes down (resp. up) by ξ , the quantity in X_t increases (resp. decreases) to its new level $\varphi_{c_t}(Y_{t-} - \xi)$ (resp. $\varphi_{c_t}(Y_{t-} + \xi)$), and the instantaneous rate moves to its new level $\varphi'_{c_t}(Y_{t-} - \xi)$ (resp. $\varphi'_{c_t}(Y_{t-} + \xi)$).

Finally, we endow the space (Ω, \mathcal{F}) with the \mathbb{P} -augmented canonical filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in \mathfrak{T}}$ generated by $(\mathcal{X}_t)_{t \in \mathfrak{T}}$.

2.2 | The Weak Formulation

We introduce $(L_t)_{t \in \mathfrak{T}}$ as the Doléans–Dade exponential

$$L_t = \exp \left\{ \sum_{i=+,-} \int_0^t \log \left(\frac{\lambda_u^i}{a_0} \right) d\hat{N}_u^i - (\lambda_u^i - a_0) du \right\},$$

where

$$\lambda_t^\pm = \bar{\lambda}^\pm(Z_{t-}, Y_{t-}, S_t),$$

with

$$\begin{aligned} \bar{\lambda}^\pm(Z, Y, S) &= \max \{a_0, a_1 + a_2 Y \pm a_3 (Z - S)\}, \quad a_0 > 0, \\ a_1, a_2, a_3 &\geq 0. \end{aligned} \quad (2.1)$$

The linear model for the intensities (inspired by Stoikov and Sağlam 2009) tries to capture three stylized facts: (i) a_1 is the baseline intensity of order arrivals of liquidity motivated trades, (ii) a_2 models the relationship between arrivals and depth (locally, all else being the same, higher depth may imply more arrivals),⁷ and (iii) a_3 captures the impact of trading by the arbitrageurs that align the quotes between the external LOB venue and the CFM. The effect of $S - Z$ in the intensities is asymmetric, this is

because when $S > Z$ (resp. $S < Z$), all else being equal, we expect a higher (resp. lower) buying intensity and a lower (resp. higher) selling intensity given that the instantaneous rate in the pool is underpriced (resp. overpriced). The minimum intensity $a_0 > 0$ is a technical condition to keep the intensity away from zero. This is nonrestrictive and in practice we expect both intensities to be bounded away from zero. Another way to understand these intensities is to think of them as linear approximations to the exponential trading intensities introduced in Avellaneda and Stoikov (2008). In their paper, the exponential form is derived from empirical evidence of the distribution of the sizes of market orders and price impact. To the best of our knowledge, there is no analogous empirical study for AMMs; in Subsection 4.3, we carry out experiments with market data using our linear modeling choices; the coefficients we obtain are all statistically significant.

Let $\nu_\infty > 0$ be a bound for the speed at which the representative LP adds or subtracts liquidity from the pool. The set of admissible strategies of the representative LP is as follows:

$$\mathcal{A} := \{\nu = (\nu_t)_{t \in \mathfrak{T}} : \nu \text{ is } \mathbb{F}\text{-progressively measurable and } |\nu_t| \leq \nu_\infty \mathbb{P} \otimes dt \text{ a.e.}\},$$

and for each $\nu \in \mathcal{A}$ we introduce the Radon–Nikodym derivative $(K_t^\nu)_{t \in \mathfrak{T}}$ given by the following:

$$K_t^\nu := \exp \left(\int_0^t \frac{\nu_s}{\eta} dB_s - \frac{1}{2} \int_0^t \left(\frac{\nu_s}{\eta} \right)^2 ds \right).$$

Given that $\nu \in \mathcal{A}$, the process $(K_t^\nu)_{t \in \mathfrak{T}}$ is a martingale (by Novikov’s condition using that ν is bounded) and we define the probability measure \mathbb{P}^ν given by the following:

$$\frac{d\mathbb{P}^\nu}{d\mathbb{P}} := K_T^\nu L_T.$$

Given that all the probability measures $(\mathbb{P}^\nu)_{\nu \in \mathcal{A}}$ are equivalent, we use the notation *a.s.* for almost surely without ambiguity.

In summary, under \mathbb{P}^ν , the processes \hat{N}^+ and \hat{N}^- have respective intensities $(\lambda_t^+)_{t \in \mathfrak{T}}$ and $(\lambda_t^-)_{t \in \mathfrak{T}}$, the process

$$B_t^\nu = B_t - \int_0^t \frac{\nu_s}{\eta} ds$$

is a standard Brownian motion, and W is a standard Brownian motion with $\langle W, B^\nu \rangle = 0$. Thus, under \mathbb{P}^ν , we write the following:

$$\begin{aligned} dS_t &= \sigma dW_t, \\ dX_t &= \underbrace{Z_t \nu_t dt}_{\text{representative LP}} + \underbrace{Z_t \eta dB_t^\nu}_{\text{other LPs}} - \underbrace{[\varphi_{c_t}(Y_{t-}) - \varphi_{c_t}(Y_{t-} + \xi)] dN_t^+ + [\varphi_{c_t}(Y_{t-} - \xi) - \varphi_{c_t}(Y_{t-})] dN_t^-}_{\text{activity from LTs}}, \\ dY_t &= \underbrace{\nu_t dt}_{\text{representative LP}} + \underbrace{\eta dB_t^\nu}_{\text{other LPs}} + \underbrace{\xi dN_t^+ - \xi dN_t^-}_{\text{activity from LTs}}, \\ dZ_t &= \underbrace{[-(\varphi_{c_t})'(Y_{t-} + \xi) + (\varphi_{c_t})'(Y_{t-})] dN_t^+ + [-(\varphi_{c_t})'(Y_{t-} - \xi) + (\varphi_{c_t})'(Y_{t-})] dN_t^-}_{\text{activity from LTs}}. \end{aligned}$$

2.3 | The Problem of the Follower: The Representative LP

The representative LP finances her stake in the pool by trading in the external market. That is, to add (resp. subtract) $v_t dt$ units of Y in (resp. from) the pool at time t , the representative LP purchases (resp. sells) $v_t dt$ units in the external market at price $(S_t + \mathbf{a} v_t) v_t dt$, where \mathbf{a} is the temporary price impact in the external venue. Thus, for $0 \leq s \leq t \leq T$, the performance criterion of the representative LP is defined as follows:

$$Q_{s,t}^v := \underbrace{(Z_t Y_t + Y_t S_t) - (Z_s Y_s + Y_s S_s)}_{(X_t + Y_t S_t) - (X_s + Y_s S_s)} - \int_s^t Z_{u-} v_{u-} du - \int_s^t (S_u + \mathbf{a} v_{u-}) v_{u-} du.$$

The first two terms account for the mark-to-market of the stake in the pool, with the quantity Y_t valued at the external price S_t . The term $-\int_s^t Z_{u-} v_{u-} du$ is the financing of the units added or subtracted in asset X .⁸ Lastly, the term $-\int_s^t (S_u + \mathbf{a} v_{u-}) v_{u-} du$ is the financing of the units added or subtracted in asset Y —the representative LP pays the costs of trading (e.g., walking the LOB and spread) through the quadratic term $\mathbf{a} v_t^2$.

The above can be written as follows:

$$Q_{s,t}^v = - \int_s^t \mathbf{a} v_{u-}^2 du + \eta \int_s^t (S_{u-} + Z_{u-}) dB_u^v + \sigma \int_s^t Y_{u-} dW_u + \int_s^t \Delta_u^- dN_u^- + \int_s^t \Delta_u^+ dN_u^+,$$

for $s < t$ where

$$\Delta_u^\pm := \pm \xi \left(S_u - \frac{Z_{u-} Y_{u-}}{Y_{u-} \pm \xi} \right) \mathbb{1}_{|Y_{t-}| > 2\xi}.$$

Thus, the control problem that the representative LP wishes to solve is given by the following:

$$V_t(\mathfrak{R}) = \sup_{v \in \mathcal{A}} \mathbb{E}_t^v \left[-\exp \left\{ -\gamma (\mathfrak{R} + Q_{t,T}^v) \right\} \right],$$

where $\mathfrak{R} \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}^v)$ is the reward offered by the venue to the representative LP, $\gamma > 0$ is the risk aversion parameter of the representative LP, and \mathbb{E}_t^v represents the conditional expectation with respect to the sigma-algebra \mathcal{F}_t under the probability measure \mathbb{P}^v . To ensure that the above expectation is well-defined, we need the following technical condition.

Condition 1. There exists $\gamma' > \gamma$ such that

$$\sup_{v \in \mathcal{A}} \mathbb{E}^v [\exp\{-\gamma' \mathfrak{R}\}] < +\infty.$$

In Section 3, for each reward \mathfrak{R} , we prove that there exists a unique optimal response $v^*(\mathfrak{R})$.

2.4 | The Problem of the Leader: The Venue

The venue provides a contract \mathfrak{R} to the representative LP and wishes to attract as much order flow as possible to the venue (i.e., $N_T^- + N_T^+$). Thus, the venue wishes to maximize the following performance criterion:

$$\mathbb{E}^{v^*(\mathfrak{R})} \left[-\exp \left\{ -\zeta (\mathbf{r} (N_T^- + N_T^+) - \mathfrak{R}) \right\} \right],$$

where $\zeta > 0$ is the venue's risk aversion parameter and $\mathbf{r} > 0$ is a constant transaction fee paid by LTs to the venue. For the above expectation to be well-defined, we need the following technical condition.

Condition 2. There exists $\zeta' > \zeta$ such that

$$\sup_{v \in \mathcal{A}} \mathbb{E}^v [\exp\{\zeta' \mathfrak{R}\}] < +\infty.$$

The set of admissible contracts is given by the following:

$$\mathcal{A}^R = \{\mathfrak{R}, \mathcal{F}_T\text{-measurable, such that } V_0(\mathfrak{R}) \geq R \text{ and Conditions 1 and 2 are satisfied}\},$$

where $R < 0$ is the reservation level of the LP, that is, the smallest acceptable utility level, in the sense that the LP refuses the contract if it doesn't allow them to reach this utility level; see Remark 1 below for a more detailed discussion.

Next, we characterize the solution to the leader-follower game between the venue and the representative LP. Section 3 studies the optimal response of the follower $v^*(\mathfrak{R})$, and Section 4 finds the optimal contract \mathfrak{R}^* .

3 | Solving the Problem of the LP

In this section, we look for the optimal response of an LP facing a given contract. We start by introducing a contract representation that allows us to simplify the LP's problem.

For any $(v, A) \in [-v_\infty, v_\infty] \times \mathbb{R}^4$, with $A = (A^W, A^B, A^+, A^-)$, we define

$$h(v, A) = -\mathbf{a} v^2 + \frac{1}{\eta} A^B v.$$

It is easy to see that for any $A \in \mathbb{R}^4$, the maximizer of $h(\cdot, A)$ is reached at

$$\bar{v}(A) = \left(\frac{A^B}{2\mathbf{a}\eta} \vee -v_\infty \right) \wedge v_\infty.$$

We then define the following:

$$H(A) = \sup_{|v| \leq v_\infty} h(v, A) = h(\bar{v}(A), A).$$

For any constant $P_0 \in \mathbb{R}$ and predictable process $(A_t)_{t \in \mathfrak{Q}} = (A_t^W, A_t^B, A_t^+, A_t^-)_{t \in \mathfrak{Q}}$ satisfying that

$$\sup_{v \in \mathcal{A}} \mathbb{E}^v \left[\int_0^T (A_t^B)^2 + (A_t^W)^2 dt \right] < \infty,$$

and

$$\int_0^T \left| \frac{\gamma}{2} \left[(A_s^W + \sigma Y_{s-})^2 + (A_s^B + \eta(S_{s-} + Z_{s-}))^2 \right] - \sum_{i \in \{+, -\}} \frac{\lambda_s^i (1 - e^{-\gamma(A_s^i + \Delta_s^i)})}{\gamma} - H(A_s) \right| ds < \infty \quad \mathbb{P}\text{-a.s.},$$

we introduce the process

$$\begin{aligned} P_t^{P_0, A} &= P_0 + \sum_{i \in \{+, -\}} \int_0^t A_s^i d\hat{N}_s^i + \int_0^t A_s^W dW_s + \int_0^t A_s^B dB_s \\ &+ \int_0^t \left\{ \frac{\gamma}{2} \left[(A_s^W + \sigma Y_{s-})^2 + (A_s^B + \eta(S_{s-} + Z_{s-}))^2 \right] \right. \\ &\left. - \sum_{i \in \{+, -\}} \frac{\lambda_s^i (1 - e^{-\gamma(A_s^i + \Delta_s^i)})}{\gamma} - H(A_s) \right\} ds, \end{aligned}$$

and we denote by Λ the set of all processes $(A_t)_{t \in \mathfrak{T}}$ such that Conditions 1 and 2 are satisfied with reward $\mathfrak{R} = P_T^{P_0, A}$ and

$$\sup_{\nu \in \mathcal{A}} \sup_{t \in \mathfrak{T}} \mathbb{E}^\nu \left[\exp \left\{ -\gamma' P_t^{P_0, A} \right\} \right] < \infty \quad \text{for some } \gamma' > \gamma.$$

We denote by \mathcal{P} the set

$$\mathcal{P} = \left\{ P_T^{P_0, A} \text{ such that } P_0 \in \mathbb{R}, A \in \Lambda, \text{ and } V_0(P_T^{P_0, A}) \geq R \right\}.$$

It is clear that $\mathcal{P} \subset A^R$. The next two theorems are proved in Appendix A.

Theorem 1. For any $\mathfrak{R} \in A^R$, there exists a unique $(P_0, A) \in \mathbb{R} \times \Lambda$ such that $\mathfrak{R} = P_T^{P_0, A}$. In particular, $A^R = \mathcal{P}$.

Theorem 2. For any reward structure $\mathfrak{R} = P_T^{P_0, A}$ as stated in Theorem 1, the LP's value function is as follows:

$$V_0(P_T^{P_0, A}) = -\exp(-\gamma P_0),$$

with optimal liquidity providing speed

$$\nu_t^* = \bar{\nu}(A_t) = \left(\frac{A_t^B}{2\alpha\eta} \vee -\nu_\infty \right) \wedge \nu_\infty.$$

Remark 1. Theorem 2 shows that P_0 is the certainty equivalent of an LP that faces contract $P_T^{P_0, A}$ and trades optimally; it also shows that any admissible contract must satisfy $-\exp(-\gamma P_0) \geq R$. In papers such as El Euch et al. (2021), the reservation level R is chosen to be the utility level of the agent in the absence of a contract. This is reasonable in their framework since an agent behaving in an optimal way is already profitable even without a contract. This is not the case in our paper: it is well-known that, in the absence of fees, an agent providing liquidity for a CPM is exposed to a concave and negative payoff. Therefore, if the venue wants to attract LPs, it should redistribute some of the fees. More precisely, since the LP can simply choose not to participate in the pool, if the venue wants to retain them it should at least offer a contract that will allow the LP to have a non-negative certainty equivalent, that is, P_0 should be non-negative.

It is therefore reasonable to consider $-1 \leq R < 0$. Competition between venues in order to attract LPs should tend to increase this value. In the numerical examples below, we choose R so that the representative LP and the venue share the profits collected from fees.

4 | Solving the Problem of the Venue

Here, we formulate the control problem of the venue. First, we discuss the risk-neutral case in Subsection 4.1 and then the exponential utility case in Subsection 4.2.

4.1 | Risk-Neutral Venue

We first consider the case where $\zeta \downarrow 0$, in which case the venue's optimization problem amounts to the following:

$$\sup_{\mathfrak{R} \in A^R} \mathbb{E}^{\nu^*(\mathfrak{R})} \left[\mathfrak{r} (N_T^- + N_T^+) - \mathfrak{R} \right].$$

In view of Theorem 2, this is equivalent to the following:

$$\sup_{P_0 \geq \hat{P}_0} \sup_{A \in \Lambda} \mathbb{E}^{\nu^*(P_T^{P_0, A})} \left[\mathfrak{r} (N_T^- + N_T^+) - P_T^{P_0, A} \right],$$

where $\hat{P}_0 = -1/\gamma \log(-R)$. For any $A \in \Lambda$, the supremum over P_0 of the objective function is reached at $P_0 = \hat{P}_0$. The problem of the venue is therefore equivalent to the following:

$$\begin{aligned} \sup_{A \in \Lambda} \mathbb{E}^{\nu^*(P_T^{P_0, A})} \left[\int_0^T \left\{ -\frac{\gamma}{2} \left[(A_s^W + \sigma Y_{s-})^2 + (A_s^B + \eta(S_{s-} + Z_{s-}))^2 \right] + \sum_{i \in \{+, -\}} \frac{\lambda_s^i (1 - e^{-\gamma(A_s^i + \Delta_s^i)})}{\gamma} \right. \right. \\ \left. \left. + H(A_s) - \frac{A_s^B \nu_s^*}{\eta} \right\} ds + \sum_{i \in \{+, -\}} \left(\int_0^T (\mathfrak{r} - A_s^i) dN_s^i - \int_0^T A_s^i \mathbb{1}_{Y_{s-} \leq \xi} d\hat{N}_s^i \right) \right]. \end{aligned}$$

We define

$$\begin{aligned} v(t, z, y, s) &= \sup_{A \in \Lambda} \mathbb{E}^{\nu^*(P_T^{P_0, A})} \left[\int_t^T \left\{ -\frac{\gamma}{2} \left[(A_s^W + \sigma Y_{s-})^2 + (A_s^B + \eta(S_{s-} + Z_{s-}))^2 \right] \right. \right. \\ &+ \sum_{i \in \{+, -\}} \frac{\lambda_s^i (1 - e^{-\gamma(A_s^i + \Delta_s^i)})}{\gamma} + H(A_s) - \frac{A_s^B \nu_s^*}{\eta} \left. \right\} ds \\ &+ \sum_{i \in \{+, -\}} \left(\int_t^T (\mathfrak{r} - A_s^i) dN_s^i - \int_t^T A_s^i \mathbb{1}_{Y_{s-} \leq \xi} d\hat{N}_s^i \right). \end{aligned} \quad (4.1)$$

We derive the Hamilton–Jacobi–Bellman (HJB) equation and substitute $H(A) = h(\bar{\nu}(A), A)$ to obtain the following:

$$\begin{aligned} 0 &= \bar{\lambda}^- [\mathfrak{r} + v(t, Z^-, Y - \xi, S) - v(t, Z, Y, S)] \mathbb{1}_{Y > 2\xi} + \bar{\lambda}^+ [\mathfrak{r} + v(t, Z^+, Y + \xi, S) - v(t, Z, Y, S)] \\ &+ \frac{1}{4} [-2\gamma\eta^2 S^2 - 4\gamma\eta^2 SZ - 2\gamma\eta^2 Z^2 - 2\gamma\sigma^2 Y^2 + 2\sigma^2 \partial_{SS} v + 2\eta^2 \partial_{YV} v + 4\partial_t v] \\ &+ \sup_A \left\{ \left[-\frac{\gamma}{2} (A^W)^2 - Y\gamma\sigma A^W \right] \right. \\ &+ \left[-\frac{\gamma}{2} (A^B)^2 + h(\bar{\nu}(A), A) - \frac{A^B h(A)}{\eta} - (S + Z)\gamma\eta A^B + \partial_Y v \bar{\nu}(A) \right] \\ &+ \bar{\lambda}^- (Z, Y, S) \left[-A^- + \frac{1 - e^{-\gamma(A^- + \Delta^-)}}{\gamma} \right] \mathbb{1}_{Y > 2\xi} + \bar{\lambda}^- \left[-A^- + \frac{1 - e^{-\gamma A^-}}{\gamma} \right] \mathbb{1}_{Y \leq 2\xi} \\ &+ \bar{\lambda}^+ (Z, Y, S) \left[-A^+ + \frac{1 - e^{-\gamma(A^+ + \Delta^+)}}{\gamma} \right] \mathbb{1}_{Y > 2\xi} + \bar{\lambda}^+ \left[-A^+ + \frac{1 - e^{-\gamma A^+}}{\gamma} \right] \mathbb{1}_{Y \leq 2\xi} \left. \right\}, \end{aligned} \quad (4.2)$$

where $Z^- = ZY^2/(Y - \xi)^2$, $Z^+ = ZY^2/(Y + \xi)^2$. We define a new function

$$\alpha(t, Z, Y, S) = \frac{\partial_Y v}{\alpha\eta} - \frac{2(S + Z)\gamma\eta}{2\gamma + \frac{1}{\alpha\eta^2}}.$$

Then the supremum in the HJB equation is attained at the following:

$$(A^W)^* = -Y \sigma, \quad (4.3)$$

$$(A^B)^* = \begin{cases} \alpha(t, Z, Y, S) & \text{if } \alpha(t, Z, Y, S) \in [-2a\eta v_\infty, 2a\eta v_\infty], \\ -(S+Z)\eta \vee 2a\eta v_\infty & \text{if } \alpha(t, Z, Y, S) > 2a\eta v_\infty, \\ -(S+Z)\eta \wedge (-2a\eta v_\infty) & \text{if } \alpha(t, Z, Y, S) < -2a\eta v_\infty, \end{cases} \quad (4.4)$$

$$(A^-)^* = -\Delta^- \mathbb{1}_{Y>2\xi}, \quad (4.5)$$

$$(A^+)^* = -\Delta^+ \mathbb{1}_{Y>2\xi}. \quad (4.6)$$

Observe that with this optimal reward structure, the venue compensates for any risk that the LPs face from the external price movement and liquidity taking. The PDE obtained from the HJB equation (4.2) and substituting (A^-, A^+, A^W, A^B) as in Equations (4.3)–(4.6) is difficult to solve, and we propose in Section 4.3 an approximation technique. However, we proved in Equation (B.2) that the value function v as defined in Equation (4.1) is the viscosity solution to the HJB equation (4.2).

4.2 | Risk-Averse Venue

Observe that under \mathbb{P}^ν , we have the following:

$$P_t = P_0 + \int_0^t \left\{ \frac{\gamma}{2} \left[(A_s^W + \sigma Y_{s-})^2 + (A_s^B + \eta(S_{s-} + Z_{s-}))^2 \right] - \sum_{i \in \{+, -\}} \frac{\lambda_s^i (1 - e^{-\gamma(A_s^i + \Delta_s^i)})}{\gamma} - H(A_s) + \frac{A_s^B \bar{v}(A_s)}{\eta} \right\} ds + \sum_{i \in \{+, -\}} \left(\int_0^t A_s^i dN_s^i + \int_0^t A_s^i \mathbb{1}_{Y_{s-} \leq 2\xi} d\hat{N}_s^i \right) + \int_0^t A_s^W dW_s + \int_0^t A_s^B dB_s^{\nu^*}.$$

We let V be the value function of the risk-averse venue, given by the following:

$$V(t, Z_t, Y_t, S_t, \hat{N}_t^-, \hat{N}_t^+, N_t^-, N_t^+, P_t^{\nu^*}) = \sup_{A \in \Lambda} \mathbb{E}_t^{\nu^*} \left[-\exp \left\{ -\zeta \left(\tau(N_t^- + N_t^+) - \mathfrak{R} \right) \right\} \right]. \quad (4.7)$$

The full HJB equation for the above control problem is

$$0 = \sup_{A^-, A^+, A^W, A^B} \left(\partial_t V + (\partial_Y V) \bar{v}(A) + (\partial_P V) \left\{ \frac{\gamma}{2} \left[(A^W + \sigma Y)^2 + (A^B + \eta(S+Z))^2 \right] - \sum_{i \in \{+, -\}} \frac{\bar{\lambda}^i(Z, Y, S)(1 - e^{-\gamma(A^i + \Delta^i)})}{\gamma} - H(A) + \frac{A^B \bar{v}(A)}{\eta} \right\} + (\partial_{Y^+} V) \frac{\eta^2}{2} + (\partial_{Y^-} V) A^B \eta + (\partial_{SS} V) \frac{\sigma^2}{2} + (\partial_{SP} V) A^W \sigma + (\partial_{PP} V) \frac{(A^W)^2 + (A^B)^2}{2} + \sum_{i \in \{+, -\}} \bar{\lambda}^i(Z, Y, S) [V^i - V] \right),$$

with

$$V^- = V(t, Z, Y^2/(Y - \xi \mathbb{1}_{Y>2\xi})^2, Y - \xi \mathbb{1}_{Y>2\xi}, S, \hat{N}^- + 1, \hat{N}^+, N^-, \mathbb{1}_{Y>2\xi}, N^+, P + A^-),$$

$$V^+ = V(t, Z, Y^2/(Y + \xi \mathbb{1}_{Y>2\xi})^2, Y + \xi \mathbb{1}_{Y>2\xi}, S, \hat{N}^-, \hat{N}^+ + 1, N^-, \mathbb{1}_{Y>2\xi}, P + A^+).$$

By taking the ansatz

$$V(t, Z, Y, S, \hat{N}^-, \hat{N}^+, N^-, N^+, P) = -e^{-\zeta[\tau(N^- + N^+) - P + v(t, Z, Y, S)]}$$

and dividing the HJB equation by $\zeta V(t, Z, Y, S, \hat{N}^-, \hat{N}^+, N^-, N^+, P)$, we obtain

$$0 = \frac{1}{2} \inf_{A^B} \left\{ (\gamma + \zeta) (A^B)^2 + 2a \bar{v}(A)^2 + 2[\gamma \eta (S+Z) - \zeta \gamma \partial_Y v] A^B - 2(\partial_Y v) \bar{v}(A) \right\} + \frac{1}{4} \inf_{A^W} \left\{ 2(\gamma + \zeta) (A^W)^2 + 4\sigma (\gamma Y - \zeta \partial_S v) A^W \right\} - \sup_{A^-} \left\{ \bar{\lambda}^-(Z, Y, S) \left(\frac{1 - e^{-\zeta(v^- - v + \tau - A^-)}}{\zeta} \mathbb{1}_{Y>2\xi} + \frac{1 - e^{-\zeta A^-}}{\zeta} \mathbb{1}_{Y \leq 2\xi} + \frac{1 - e^{-\gamma(A^- + \Delta^-)}}{\gamma} \right) \right\} - \sup_{A^+} \left\{ \bar{\lambda}^+(Z, Y, S) \left(\frac{1 - e^{-\zeta(v^+ - v + \tau - A^+)}}{\zeta} \mathbb{1}_{Y>2\xi} + \frac{1 - e^{-\zeta A^+}}{\zeta} \mathbb{1}_{Y \leq 2\xi} + \frac{1 - e^{-\gamma(A^+ + \Delta^+)}}{\gamma} \right) \right\} + \frac{1}{2} \gamma \eta^2 S^2 + \gamma \eta^2 S Z + \frac{1}{2} \gamma \eta^2 Z^2 + \frac{1}{2} \gamma \sigma^2 Y^2 + \frac{1}{2} \zeta \sigma^2 (\partial_S v)^2 - \frac{1}{2} \sigma^2 \partial_{SS} v + \frac{1}{2} \zeta \eta^2 (\partial_Y v)^2 - \frac{1}{2} \eta^2 \partial_{YY} v - \partial_t v, \quad (4.8)$$

with

$$v^- = v(t, Z, Y^2/(Y - \xi \mathbb{1}_{Y>2\xi})^2, Y - \xi \mathbb{1}_{Y>2\xi}, S),$$

$$v^+ = v(t, Z, Y^2/(Y + \xi \mathbb{1}_{Y>2\xi})^2, Y + \xi \mathbb{1}_{Y>2\xi}, S)$$

and terminal condition $v(T, Z, Y, S) = 0$.

We define the following:

$$\alpha_\zeta(t, Z, Y, S) = \frac{\left(\frac{1}{a\eta} + 2\zeta\eta \right) \partial_Y v - 2\gamma\eta(S+Z)}{2(\gamma + \zeta) + \frac{1}{a\eta^2}}.$$

The optimal values for A^B , A^W , A^- , and A^+ are as follows:

$$(A^B)^* = \begin{cases} \alpha_\zeta(t, Z_{t-}, Y_{t-}, S_t) & \text{if } \alpha_\zeta(t, Z_{t-}, Y_{t-}, S_t) \in [-2a\eta v_\infty, 2a\eta v_\infty], \\ \frac{\xi \gamma \partial_Y v - \gamma \eta (S+Z)}{\gamma + \zeta} \vee 2a\eta v_\infty & \text{if } \alpha_\zeta(t, Z_{t-}, Y_{t-}, S_t) > 2a\eta v_\infty, \\ \frac{\xi \gamma \partial_Y v - \gamma \eta (S+Z)}{\gamma + \zeta} \wedge (-2a\eta v_\infty) & \text{if } \alpha_\zeta(t, Z_{t-}, Y_{t-}, S_t) < -2a\eta v_\infty, \end{cases} \\ (A^W)^* = \frac{\sigma(\zeta \partial_S v - \gamma Y)}{\gamma + \zeta}, \\ (A^-)^* = \frac{-\gamma \Delta^- + \zeta(v^- - v + \tau)}{\gamma + \zeta} \mathbb{1}_{Y_{t-} > 2\xi} \\ (A^+)^* = \frac{-\gamma \Delta^+ + \zeta(v^+ - v + \tau)}{\gamma + \zeta} \mathbb{1}_{Y_{t-} > 2\xi}.$$

The proof that the value function (4.7) is the viscosity solution to the HJB equation (4.8) is similar to the risk-neutral case and therefore omitted.

4.3 | Approximate Closed-Form Solutions

The HJB equations above are difficult to solve as they stand. Next, we carry out approximations that make the equations more tractable. The first set of approximations are minimal. For example, for the model parameters we calibrate from data, the probability that $\mathbb{1}_{Y_t > 2\xi}$ is equal to zero is negligible. Similarly, for the model parameters, we obtain from data we have that ξ/Y is of order 10^{-3} so that ignoring higher orders (order 10^{-6} or smaller) of the quotient seems reasonable in exchange for tractability. We refer the reader to Bergault et al. (2021) and Boyce et al. (2025) for examples that approximate the HJB equation characterizing the value function to obtain a closed-form solution, as opposed to approximating the solution via numerical methods. See Figures 8–11 in Bergault et al. (2021) or right panel of Figure 5 in Boyce et al. (2025) for a direct comparison between these two approaches in market making problems.

4.3.1 | Risk-Neutral Venue

First, we ignore the boundary imposed on the control, that is, we take $\nu_\infty \uparrow \infty$ so that $(A_t^B)^* = \alpha(t, Z_t, Y_t, S_t)$. Furthermore, we only consider the case where $Y > 2\xi$. Substituting this, we obtain

$$0 = \sum_{i \in \{+, -\}} \lambda^i \left[\mathbf{r} + h(t, Z^i, Y + \delta^i \xi, S) - h(t, Z, Y, S) \right] + \frac{1}{4} \left[-2\gamma \eta^2 S^2 - 4\gamma \eta^2 S Z - 2\gamma \eta^2 Z^2 + 2\sigma^2 \partial_{SS} h + 2\eta^2 \partial_{YY} h + 4\partial_t h \right] + \frac{1}{4} \left\{ \frac{\left(\frac{\partial_Y h}{a\eta} - 2(S+Z)\gamma\eta \right)^2}{2\gamma + \frac{1}{a\eta^2}} + 4 \sum_{i \in \{+, -\}} \bar{\lambda}^i(Z, Y, S) \left[\delta^i \xi \left(S - \frac{ZY}{Y + \delta^i \xi} \right) \right] \right\},$$

with $\delta^- = -1$, and $\delta^+ = 1$. Next, we remove the indicator $\mathbb{1}_{Y_t > 2\xi}$ from the stochastic intensities to obtain the following:

$$\lambda_t^- \approx \max\{a_0, a_1 + a_2 Y_t + a_3 (S_t - Z_t)\}, \quad \lambda_t^+ \approx \max\{a_0, a_1 + a_2 Y_t + a_3 (Z_t - S_t)\}.$$

We have four possibilities as follows: (i) $\lambda^-, \lambda^+ > a_0$, (ii) $\lambda^- = a_0, \lambda^+ > a_0$, (iii) $\lambda^- > a_0, \lambda^+ = a_0$, and (iv) $\lambda^-, \lambda^+ = a_0$. In what follows, we work under the assumption that model parameters are such that one is always in scenario (i), which helps us to obtain approximate closed-form solutions. First, we justify (with market data) that the above assumption is a good proxy for reality.

We employ market data from Uniswap V2 and from Binance for the pair ETH-USDC between January 1, 2022 and April 30, 2022. During these 4 months, we monitor the exchange rate in Binance (the midprice S_t), the instantaneous exchange rate in the Uniswap pool (Z_t), and the trading activity (liquidity taking orders) in the pool. We split the data into 10-min buckets, and within a given 10-min window, we compute the aggregate of the buy-initiated trades and sell-initiated trades.⁹ We think of these aggregates as a rough estimate for the intensities λ_t^\pm . Finally, we perform a linear regression that explains λ_t^\pm in terms of the average differences $\pm(S_t - Z_t)$; for now, we set $a_2 = 0$ for simplicity.¹⁰ We use the estimates \hat{a}_1, \hat{a}_3 to compute the boundaries $\pm d$ such that $0 = \hat{a}_1 + \hat{a}_3 d$. If $S_t - Z_t < d$ or $S_t - Z_t > -d$, then $\lambda_t^- = 0$ or $\lambda_t^+ = 0$, respectively. Figure 1 shows the boundaries $\pm d$ (in red dotted vertical lines) together with market data on $S_t - Z_t$.

We find that of the 17,131 data points in the histogram above, the boundaries are breached 30 times to the left and 34 times to the right. Thus, 0.37% of the data points violate the assumption that in our model $\hat{a}_1 \pm \hat{a}_3 (S_t - Z_t) > 0$.

Next, we use Laurent series to approximate the following terms:

$$\begin{aligned} \frac{Y}{Y + \delta^\pm \xi} &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{\delta^\pm \xi}{Y} \right)^n = 1 + O\left(\frac{\xi}{Y}\right), \\ \frac{Y^2}{Y + \delta^\pm \xi} &= Y + \sum_{n=1}^{\infty} (-1)^n \left(\frac{\delta^\pm \xi}{Y^{n-1}} \right) = Y - \delta^\pm \xi + O\left(\frac{\xi^2}{Y}\right), \\ \frac{Y^2}{(Y + \delta^\pm \xi)^2} &= \sum_{n=0}^{\infty} (n+1)(-1)^n \left(\frac{\delta^\pm \xi}{Y} \right)^n = 1 + O\left(\frac{\xi}{Y}\right), \\ \frac{Y^3}{(Y + \delta^\pm \xi)^2} &= \sum_{n=0}^{\infty} Y(n+1)(-1)^n \left(\frac{\delta^\pm \xi}{Y} \right)^n = Y - 2\delta^\pm \xi + O\left(\frac{\xi^2}{Y}\right). \end{aligned}$$

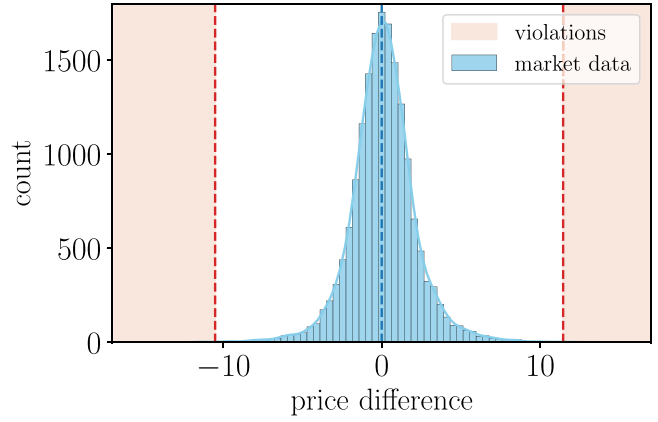


FIGURE 1 | Histogram of price differences between Binance and Uniswap V2 for ETH-USDC between January 1, 2022 and April 30, 2022. The red shaded area represents the region in which $a_1 \pm a_3(S_t - Z_t)$ becomes negative. We take $a_2 = 0$ for simplicity so that the violation boundary is fixed and does not depend on the number of ETH units in the pool. [Color figure can be viewed at wileyonlinelibrary.com]

Given that $Z^\pm = ZY^2/(Y + \delta^\pm \xi)^2$, and using the above approximations, we have that

$$\begin{aligned} h(t, Z^i, Y + \delta^\pm \xi, S) &= h(t, Z, Y + \delta^\pm \xi, S) + Z \left(\frac{Y^2}{(Y + \delta^\pm \xi)^2} - 1 \right) \partial_Z h(t, Z, Y + \delta^\pm \xi, S) + O\left(\frac{\xi^2}{Y^2}\right) \\ &\approx h(t, Z, Y + \delta^\pm \xi, S), \\ Y h(t, Z^i, Y + \delta^\pm \xi, S) &= Y h(t, Z, Y + \delta^\pm \xi, S) + Z \left(\frac{Y^3}{(Y + \delta^\pm \xi)^2} - Y \right) \partial_Z h(t, Z, Y + \delta^\pm \xi, S) + O\left(\frac{\xi^2}{Y}\right) \\ &\approx Y h(t, Z, Y + \delta^\pm \xi, S) - 2\delta^\pm \xi Z \partial_Z h(t, Z, Y + \delta^\pm \xi, S). \end{aligned}$$

Thus, the approximated HJB equation becomes the following:

$$0 \approx \partial_t h + \sum_{i \in \{+, -\}} \lambda^i \left[\mathbf{r} + h(t, Z, Y + \delta^i \xi, S) - h(t, Z, Y, S) \right] + \frac{1}{4} \left[-2\gamma \eta^2 S^2 - 4\gamma \eta^2 S Z - 2\gamma \eta^2 Z^2 + 2\sigma^2 \partial_{SS} h + 2\eta^2 \partial_{YY} h \right] + \frac{1}{4} \left\{ \frac{\left(\frac{\partial_Y h}{a\eta} - 2(S+Z)\gamma\eta \right)^2}{2\gamma + \frac{1}{a\eta^2}} + 4 \sum_{i \in \{+, -\}} \bar{\lambda}^i(Z, Y, S) \left[\delta^i \xi (S - Z) \right] \right\} + a_2 \sum_{i \in \{+, -\}} \left\{ \xi^2 Z - 2\delta^i \xi Z \partial_Z h(t, Z, Y + \delta^i \xi, S) \right\}. \quad (4.9)$$

Let \hat{h} be the approximation of h such that sign “ \approx ” in Equation (4.9) turns into an equality. By taking the ansatz

$$\hat{h}(t, Z, Y, S) = h_0(t, Y) + h_1(t, Y)Z + h_2(t, Y)S + h_3(t)Z^2 + h_4(t)ZS + h_5(t)S^2,$$

where

$$h_i(t, Y) = h_{i0}(t) + h_{i1}(t)Y, \quad i = 0, 1, 2,$$

we obtain the following:

$$\partial_Y \hat{h}(t, Z, Y, S) = h_{01}(t) + h_{11}(t)Z + h_{21}(t)S = 2a_2 \mathbf{r}(T - t).$$

Substituting this into $(A_t^B)^*$, we obtain the following:

$$(A_t^B)^* \approx \frac{1}{2\gamma + \frac{1}{a\eta^2}} \left(\frac{2a_2 \mathbf{r}(T - t)}{a\eta} - 2(S + Z)\gamma\eta \right).$$

Let \hat{v}^* be the approximation of v^* (where we replace h with \hat{h}); we obtain that

$$\hat{v}_t^* = \frac{1}{4\eta a\gamma + \frac{2}{\eta}} \left(\frac{2a_2 \mathbf{r}(T-t)}{a\eta} - 2(S+Z)\gamma\eta \right) = \frac{2a_2 \mathbf{r}(T-t) - 2(S+Z)\gamma a\eta^2}{4\eta^2 a^2 \gamma + 2a}.$$

Remark 2. For a risk-neutral venue, if the noise in the liquidity provision is small ($\eta \downarrow 0$), we have that

$$\hat{v}_t^* \rightarrow \frac{a_2 \mathbf{r}(T-t)}{a}.$$

That is, the provision of liquidity is inversely proportional to the transaction costs in the external venue a , and directly proportional to (i) the fees collected by the venue \mathbf{r} and (ii) the sensitivity of order flow to the depth of the pool a_2 . In particular, if $a_2 = 0$, the LP does not add nor remove liquidity from the pool in equilibrium.

4.3.2 | Risk-Averse Venue

Here we work out the approximate solution when the venue is risk-averse. We take $v_\infty \uparrow \infty$ such that $v^*(A_t) = A_t^B / (2a\eta)$. We also employ the approximation $e^{-x} \approx 1 - x$ and we only consider the case where $Y_{t-} > 2\xi$. We obtain that

$$\begin{aligned} 0 &\approx -\frac{1}{4} \frac{\left[\left(\frac{1}{a\eta} + 2\xi\eta \right) \partial_Y v - 2\gamma\eta(S+Z) \right]^2}{2(\gamma+\zeta) + \frac{1}{a\eta^2}} - \frac{1}{2} \frac{\sigma^2 (\zeta \partial_S v - \gamma Y)^2}{\gamma+\zeta} \\ &\quad - \sum_{i \in \{+, -\}} \bar{\lambda}^i(Z, Y, S) \left(v^i - v + \mathbf{r} + \delta^i \xi \left(S - \frac{ZY}{Y + \delta^i \xi} \right) \right) \\ &\quad + \frac{1}{2} \gamma \eta^2 S^2 + \gamma \eta^2 SZ + \frac{1}{2} \gamma \eta^2 Z^2 + \frac{1}{2} \gamma \sigma^2 Y^2 \\ &\quad + \frac{1}{2} \zeta \sigma^2 (\partial_S v)^2 - \frac{1}{2} \sigma^2 \partial_{SS} v + \frac{1}{2} \zeta \eta^2 (\partial_Y v)^2 - \frac{1}{2} \eta^2 \partial_{YY} v - \partial_t v, \end{aligned}$$

where $v^\pm = v(t, Z, Y + \delta^\pm \xi, S)$.

We focus on the regime where $\bar{\lambda}^\pm(t, Z_{t-}, Y_{t-}, S_t) > a_0$. Furthermore, we make the following approximations:

$$\begin{aligned} \frac{Y}{Y + \delta^\pm \xi} &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{\delta^\pm \xi}{Y} \right)^n = 1 + O\left(\frac{\xi}{Y}\right), \\ \frac{Y^2}{Y + \delta^\pm \xi} &= Y + \sum_{n=1}^{\infty} (-1)^n \left(\frac{\delta^\pm \xi}{Y} \right)^n = Y - \delta^\pm \xi + O\left(\frac{\xi^2}{Y}\right), \\ \frac{Y^2}{(Y + \delta^\pm \xi)^2} &= \sum_{n=0}^{\infty} (n+1) (-1)^n \left(\frac{\delta^\pm \xi}{Y} \right)^n = 1 + O\left(\frac{\xi}{Y}\right), \\ \frac{Y^3}{(Y + \delta^\pm \xi)^2} &= \sum_{n=0}^{\infty} Y(n+1) (-1)^n \left(\frac{\delta^\pm \xi}{Y} \right)^n = Y - 2\delta^\pm \xi + O\left(\frac{\xi^2}{Y}\right), \end{aligned}$$

which implies that

$$\begin{aligned} &v(t, Z, Y^2 / (Y + \delta^\pm \xi)^2, Y + \delta^\pm \xi, S) \\ &= v(t, Z, Y + \delta^\pm \xi, S) + Z \left(\frac{Y^2}{(Y + \delta^\pm \xi)^2} - 1 \right) \partial_Z v(t, Z, Y + \delta^\pm \xi, S) + O\left(\frac{\xi}{Y}\right) \\ &\approx v(t, Z, Y + \delta^\pm \xi, S), \end{aligned}$$

and

$$\begin{aligned} &Y v(t, Z^\pm, Y + \delta^\pm \xi, S) \\ &= Y v(t, Z, Y + \delta^\pm \xi, S) + Z \left(\frac{Y^3}{(Y + \delta^\pm \xi)^2} - Y \right) \partial_Z v(t, Z, Y + \delta^\pm \xi, S) + O\left(\frac{\xi}{Y}\right) \\ &\approx Y v(t, Z, Y + \delta^\pm \xi, S) - 2\delta^\pm \xi Z \partial_Z v(t, Z, Y + \delta^\pm \xi, S). \end{aligned}$$

We obtain that

$$\begin{aligned} 0 &\approx -\frac{1}{4} \frac{\left[\left(\frac{1}{a\eta} + 2\xi\eta \right) \partial_Y v - 2\gamma\eta(S+Z) \right]^2}{2(\gamma+\zeta) + \frac{1}{a\eta^2}} - \frac{1}{2} \frac{\sigma^2 (\zeta \partial_S v - \gamma Y)^2}{\gamma+\zeta} \\ &\quad - \sum_{i \in \{+, -\}} (a_1 + a_2 Y + a_3 \delta^i (Z-S)) (v^i - v) - 2a_1 \mathbf{r} - 2a_2 \mathbf{r} Y \\ &\quad - 2a_2 \xi^2 Z + 2a_3 \xi (S-Z)^2 + a_2 \sum_{i \in \{+, -\}} [2\delta^i \xi Z \partial_Z v(t, Z, Y + \delta^i \xi, S)] \\ &\quad + \frac{1}{2} \gamma \eta^2 S^2 + \gamma \eta^2 SZ + \frac{1}{2} \gamma \eta^2 Z^2 + \frac{1}{2} \gamma \sigma^2 Y^2 \\ &\quad + \frac{1}{2} \zeta \sigma^2 (\partial_S v)^2 - \frac{1}{2} \sigma^2 \partial_{SS} v + \frac{1}{2} \zeta \eta^2 (\partial_Y v)^2 - \frac{1}{2} \eta^2 \partial_{YY} v - \partial_t v. \end{aligned}$$

We let \hat{v} be the approximation of v such that the sign \approx above turns into an equality, and employ the ansatz

$$\hat{v}(t, Z, Y, S) = \mathbf{g}_{11}(t) + 2\boldsymbol{\eta}^\top G_1(t) + \boldsymbol{\eta}^\top G_2(t) \boldsymbol{\eta},$$

where $\boldsymbol{\eta} = (ZY S)^\top$ and

$$\begin{aligned} G_1(t) &= \begin{pmatrix} \mathbf{g}_{12}(t) & \mathbf{g}_{13}(t) & \mathbf{g}_{14}(t) \end{pmatrix}^\top, \\ G_2(t) &= \begin{pmatrix} \mathbf{g}_{22}(t) & \mathbf{g}_{23}(t) & \mathbf{g}_{24}(t) \\ \mathbf{g}_{23}(t) & \mathbf{g}_{33}(t) & \mathbf{g}_{34}(t) \\ \mathbf{g}_{24}(t) & \mathbf{g}_{34}(t) & \mathbf{g}_{44}(t) \end{pmatrix}, \end{aligned}$$

with terminal conditions $\mathbf{g}_{ij}(T) = 0$ for $i, j \in \{1, 2, 3, 4\}$.

We obtain the following system of differential equations:

$$\begin{aligned} 0 &= -G_2'(t) + G_2(t) U G_2(t) + V^\top G_2(t) + G_2(t) V + R, \\ 0 &= G_1'(t) + C(t) G_1(t)^\top + E(t), \\ 0 &= \mathbf{g}_{11}'(t) + 2a_1 \mathbf{r} + \frac{(1 + 2a\zeta\eta^2 - 4a^2\gamma\zeta\eta^4) \mathbf{g}_{13}(t)^2}{a(1 + 2a(\gamma + \zeta)\eta^2)} \\ &\quad - \frac{2\gamma\zeta\sigma^2 \mathbf{g}_{14}(t)^2}{\gamma + \zeta} + (2a_1 \xi^2 + \eta^2) \mathbf{g}_{33}(t) + \sigma^2 \mathbf{g}_{44}(t), \end{aligned}$$

where for the first equation, U, V , and R are given by

$$\begin{aligned} U &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1+2a\zeta\eta^2-4a^2\gamma\zeta\eta^4}{a(1+2a(\gamma+\zeta)\eta^2)} & 0 \\ 0 & 0 & \frac{2\gamma\zeta\sigma^2}{\gamma+\zeta} \end{pmatrix}, \\ V &= \begin{pmatrix} 0 & 0 & 0 \\ -2a_3\xi + \frac{\gamma\eta^2(1+2a\zeta\eta^2)}{1+2a(\gamma+\zeta)\eta^2} & 0 & 2a_3\xi + \frac{\gamma\eta^2(1+2a\zeta\eta^2)}{1+2a(\gamma+\zeta)\eta^2} \\ 0 & \frac{\gamma\zeta\sigma^2}{\gamma+\zeta} & 0 \end{pmatrix}, \end{aligned}$$

$$R = \begin{pmatrix} 2a_3 + \frac{\gamma\eta^2(1+2a\xi\eta^2)}{2+4a(\gamma+\xi)\eta^2} & 0 & -2a_3 + \frac{\gamma\eta^2(1+2a\xi\eta^2)}{2+4a(\gamma+\xi)\eta^2} \\ 0 & \frac{\gamma\xi\sigma^2}{2(\gamma+\xi)} & 0 \\ -2a_3 + \frac{\gamma\eta^2(1+2a\xi\eta^2)}{2+4a(\gamma+\xi)\eta^2} & 0 & 2a_3 + \frac{\gamma\eta^2(1+2a\xi\eta^2)}{2+4a(\gamma+\xi)\eta^2} \end{pmatrix},$$

and for the second equation, $C(t)$ and $E(t)$ are as follows:

$$C(t) = \begin{pmatrix} 0 & \frac{-a\gamma\eta^2(1+2a\xi\eta^2)+2a_3a\xi(1+2a(\gamma+\xi)\eta^2)+(1+2a\xi\eta^2-4a^2\gamma\xi\eta^2)g_{23}(t)}{a(1+2a(\gamma+\xi)\eta^2)} & -\frac{2\gamma\xi\sigma^2g_{24}(t)}{\gamma+\xi} \\ 0 & \frac{(1+2a\xi\eta^2-4a^2\gamma\xi\eta^2)g_{33}(t)}{a(1+2a(\gamma+\xi)\eta^2)} & -\frac{\gamma\xi\sigma^2(1+2g_{24}(t))}{\gamma+\xi} \\ 0 & \frac{-a(\gamma\eta^2(1+2a\xi\eta^2)+2a_3a\xi(1+2a(\gamma+\xi)\eta^2)+(1+2a\xi\eta^2-4a^2\gamma\xi\eta^2)g_{34}(t))}{a(1+2a(\gamma+\xi)\eta^2)} & -\frac{2\gamma\xi\sigma^2g_{44}(t)}{\gamma+\xi} \end{pmatrix},$$

$$E(t) = \begin{pmatrix} a_2\xi^2(1-4g_{23}(t)) & a_2(\nu + \xi^2g_{33}(t)) & 0 \end{pmatrix}^\top.$$

The equation for G_2 is a matrix Riccati equation. Let us introduce the matrices $C = I_3$ and $D = 0$, where I_3 denotes the identity matrix in $\mathbb{R}^{3 \times 3}$ and 0 denotes the zero matrix in the same space. The terminal condition of the equation is $G_2(T) = 0$, and it is therefore clear that $C + DG_2(T) + G_2(T)D = C > 0$. We now introduce the matrix Θ given by the following:

$$\Theta = \begin{pmatrix} -CV & -CU - V^\top D + DV^\top \\ 0 & -U^\top D \end{pmatrix}.$$

After some calculations, we see that $\Theta + \Theta^\top$ is equal to the following:

$$\begin{pmatrix} 0 & 2a_3\xi - \frac{\eta^2\gamma(2a\eta^2\xi+1)}{2a\eta^2(\gamma+\xi)+1} & 0 & 0 & 0 & 0 \\ 2a_3\xi - \frac{\eta^2\gamma(2a\eta^2\xi+1)}{2a\eta^2(\gamma+\xi)+1} & 0 & -2a_3\xi - \frac{\eta^2\gamma(2a\eta^2\xi+1)}{2a\eta^2(\gamma+\xi)+1} - \frac{\gamma\xi\sigma^2}{\gamma+\xi} & 0 & -\frac{4a^2\eta^2\gamma\xi-2a\eta^2\xi-1}{a(2a\eta^2(\gamma+\xi)+1)} & 0 \\ 0 & -2a_3\xi - \frac{\eta^2\gamma(2a\eta^2\xi+1)}{2a\eta^2(\gamma+\xi)+1} - \frac{\gamma\xi\sigma^2}{\gamma+\xi} & 0 & 0 & 0 & -\frac{2\gamma\xi\sigma^2}{\gamma+\xi} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{4a^2\eta^2\gamma\xi-2a\eta^2\xi-1}{a(2a\eta^2(\gamma+\xi)+1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2\gamma\xi\sigma^2}{\gamma+\xi} & 0 & 0 & 0 \end{pmatrix}$$

It is then clear that all the leading principal minors of the above matrix are nonpositive; in particular, they are all equal to zero, except for the determinant of

$$\begin{pmatrix} 0 & 2a_3\xi - \frac{\eta^2\gamma(2a\eta^2\xi+1)}{2a\eta^2(\gamma+\xi)+1} \\ 2a_3\xi - \frac{\eta^2\gamma(2a\eta^2\xi+1)}{2a\eta^2(\gamma+\xi)+1} & 0 \end{pmatrix}$$

which is given by the following:

$$-\left(2a_3\xi - \frac{\eta^2\gamma(2a\eta^2\xi+1)}{2a\eta^2(\gamma+\xi)+1}\right)^2 \leq 0.$$

Therefore, the matrix $\Theta + \Theta^\top$ is negative semi-definite, and Theorem 3.6.6 in Abou-Kandil et al. (2012) implies that there exists a well-defined solution of the Riccati equation on $(-\infty, T]$.

5 | Numerical Results

For the experiments below, we discretize $[0, T]$ in 10,000 timesteps with $T = 1$ (one day). We employ market data from Binance and Uniswap V2 in the pair ETH-USDC between January 1, 2022 and April 30, 2022 to calibrate model parameters. The initial exchange rate is $S_0 = Z_0 = 2820$ in the ETH-USDC pair and

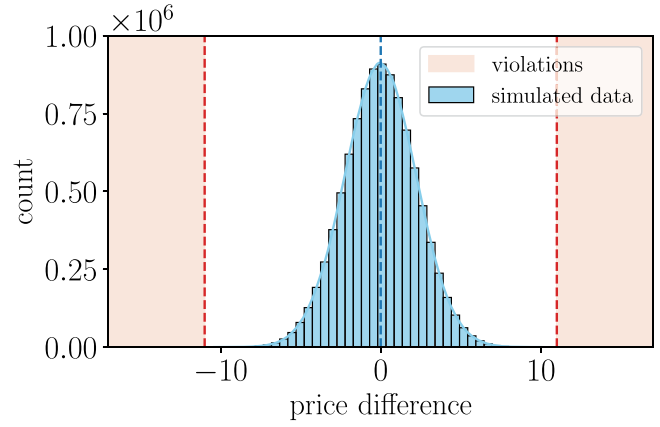


FIGURE 2 | Histogram of simulated price differences between Binance and Uniswap V2 for ETH-USDC using 1000 simulations and the model parameters at the start of the section. The red shaded area represents the region in which $a_1 \pm a_3(S_t - Z_t)$ becomes negative. We take $a_2 = 0$ for simplicity so that the violation boundary is fixed and does not depend on the number of ETH units in the pool. [Color figure can be viewed at wileyonlinelibrary.com]

the daily volatility in Binance is $\sigma = 0.0569 \times S_0$.¹¹ The average traded quantity in a 10-min window is $\xi = 300$, and the fee collected by the venue per transaction is approximated by the constant fee $\nu = 0.01 \times \xi \times Z_0$. Here, each jump in N^\pm captures the aggregate trading that happened in a 10-min window. The initial position in the pool is $Y_0 = 50,000$ ETH,¹² and the depth of the pool is $c_0 = Y_0 \times (Y_0 \times Z_0)$. We use $\eta = 10^{-10}$ ETH as the volatility of the Brownian motion used in the provision of liquidity. For $t \in \mathfrak{T}$, we estimate λ_t^\pm using data from the previous 10 min. We then fit a linear regression to calibrate a_1, a_3 in Equation (2.1); for now, we set $a_2 = 0$. We obtain the estimates $a_1 = 142.7$ and $a_3 = 13.6$. These values were scaled so that a jump corresponds to the activity in a 10-min window in line with ξ . Below, we study the strategies for the risk-averse case. The LP's risk aversion parameter is $\gamma = 10^{-18}$ and the venue's risk aversion is $\zeta = 10^{-6}$. Lastly, for the figures below, we let the temporary price impact parameter in Binance be $\alpha = 10^{-14}$ (negligible walking the book costs in the external venue). Below, we show that as we increase the value of this parameter, the strategy collapses around zero and the representative LP does not add liquidity to the pool.

Figure 2 shows the violations to the approximation assumption within the simulation. That is, the violations to the assumption that both $\lambda^-, \lambda^+ > 0$. We observe that there are no violations to the non-negative assumption in the simulations.

Figure 3 shows a sample path for the inventory Y_t , together with the prices S_t (external) and Z_t (pool).

As expected, arbitrageurs keep the prices in the pool aligned to those outside of the pool. For these parameters, with $a_2 = 0$, the changes in Y are relatively small and seem to be driven mostly by the liquidity taking activity; we confirm this below. Figure 4 shows the speed ν_t at which the LP adds or subtracts liquidity from the pool, and the cumulative liquidity provided (given by $\int_0^t \nu_s ds$).

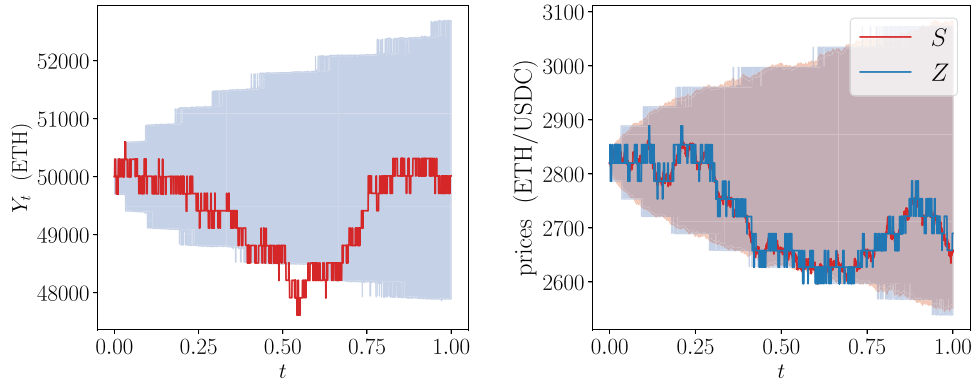


FIGURE 3 | Sample path (with 90% bands across time) for the inventory of ETH in the pool (left panel), and the instantaneous exchange rate in the pool and outside (right panel). [Color figure can be viewed at wileyonlinelibrary.com]

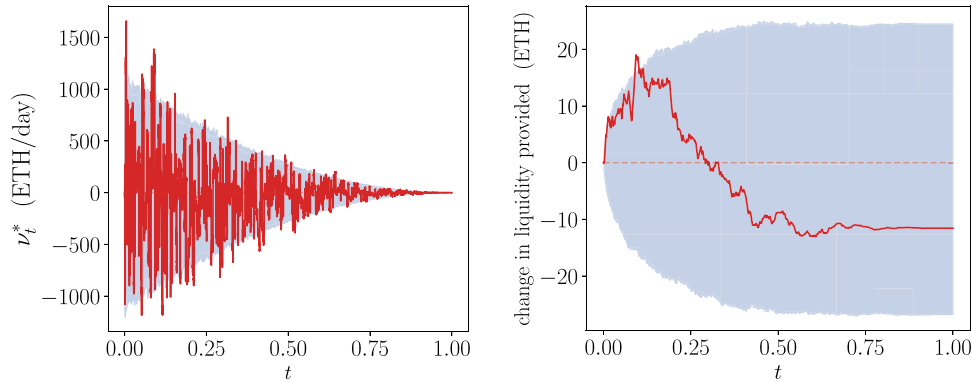


FIGURE 4 | Sample path (with 90% bands across time) for the speed at which the LP adds/removes liquidity from the pool (left panel), and the cumulative change in liquidity provided (right panels), given by $\int_0^t v_s ds$. [Color figure can be viewed at wileyonlinelibrary.com]

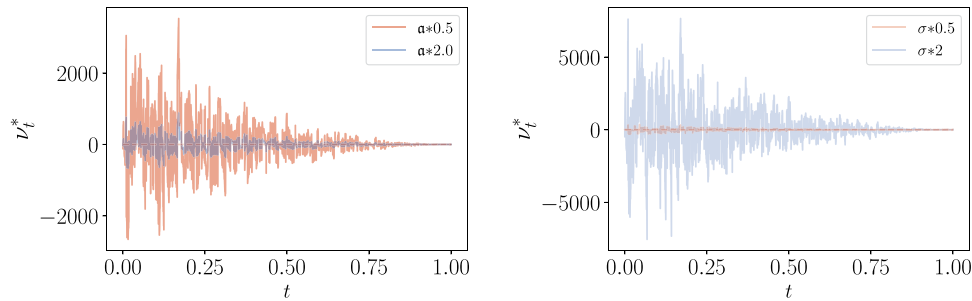


FIGURE 5 | Sample path of the optimal strategy of the LP as model parameters α and σ change. [Color figure can be viewed at wileyonlinelibrary.com]

Next, we study the optimal strategy v_t^* of the LP in more detail. Figure 5 shows a sample path of the optimal strategy as we stress the values of model parameters (the randomness is the same for all simulations).

As expected, the higher the volatility, the higher the variability of the strategy of the representative LP. The smaller the temporary price impact in the external venue α , the more trading activity the representative LP can afford. When we increase the transaction cost parameter for trading in the external venue to $\alpha \in \{10^{-13}, 10^{-12}\}$, we obtain that the strategy of the representative LP collapses around zero, as shown in Figure 6.

5.1 | Distributing the Fees Collected

The starting point of the contract representation, that is, P_0 , shifts the profitability of the LP to the right (the higher P_0 the more profitable), and it shifts the profitability of the venue to the left (the higher P_0 the less profitable). Here, we take P_0 such that both the venue and the representative LP have the same average wealth.

Figure 7 shows 10,000 Monte-Carlo simulations of the equilibrium of the model. We report the distribution of the final reward \mathfrak{R} and the venue's PnL. The red line is the mean of the distribution. On both sides, the mean is positive. The higher the

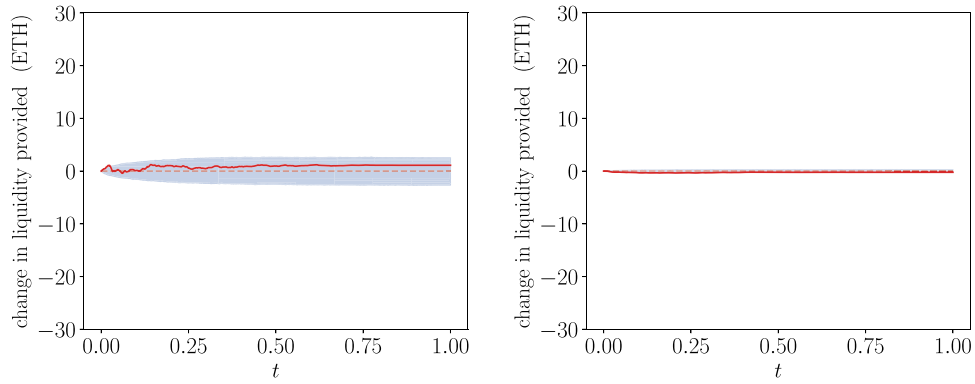


FIGURE 6 | Sample path (with 90% bands across time) for the cumulative change in liquidity provided given by $\int_0^t \nu_s ds$. The first plot is for $\alpha = 10^{-13}$ and the second plots is for $\alpha = 10^{-12}$. [Color figure can be viewed at wileyonlinelibrary.com]

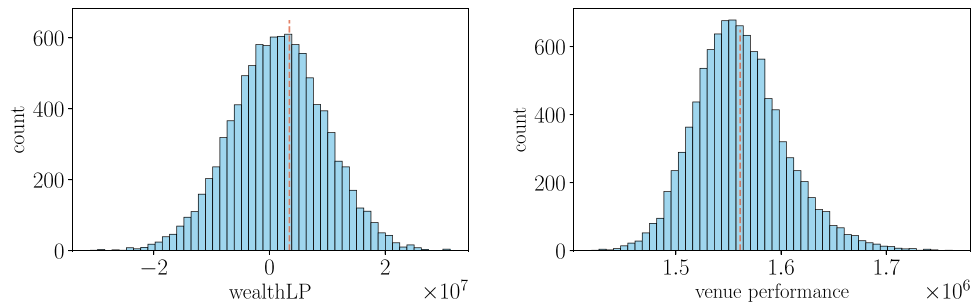


FIGURE 7 | Histograms for the reward \mathfrak{R} paid to the representative LP (left panel) and the performance of the venue $\mathfrak{r}(N_T^- + N_T^+) - \mathfrak{R}$ (right panel). The mean values are in vertical dotted lines. [Color figure can be viewed at wileyonlinelibrary.com]

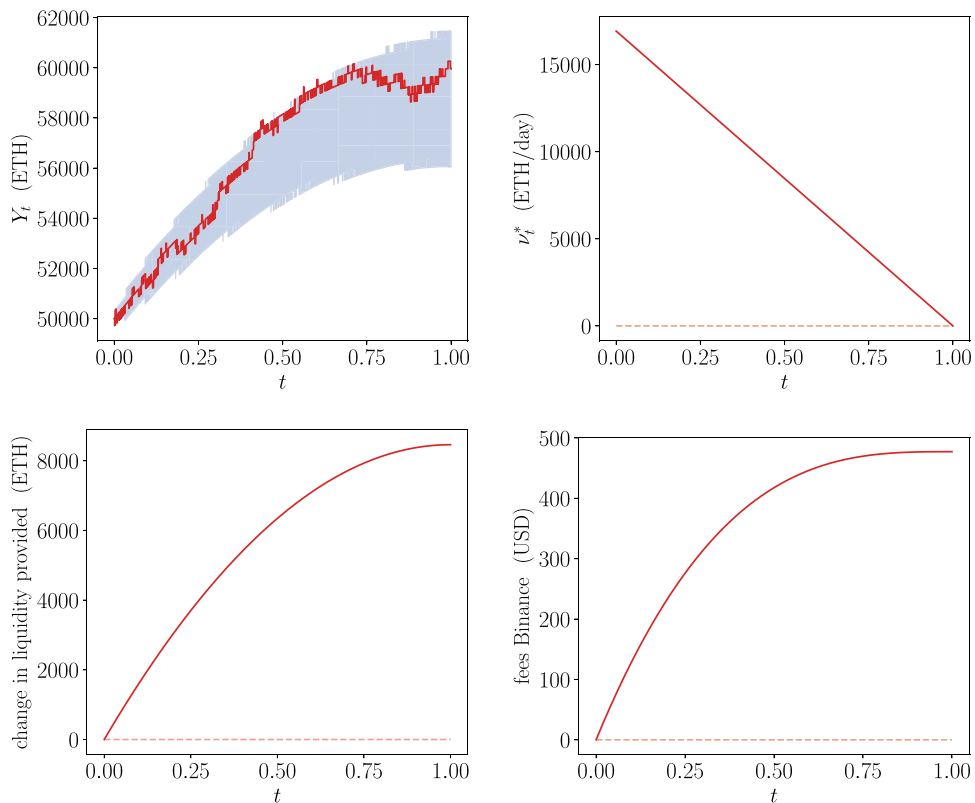


FIGURE 8 | Sample path (together with 90% confidence bands) of the (i) quantity of ETH in the pool (top left panel), (ii) the optimal speed at which the representative LP adds liquidity to the pool (top right panel), (iii) the cumulative liquidity provided (bottom left panel), and (iv) the cumulative fees paid to the external venue, given by $\int_0^t \alpha (\nu_u^*)^2 du$ (bottom right panel). [Color figure can be viewed at wileyonlinelibrary.com]

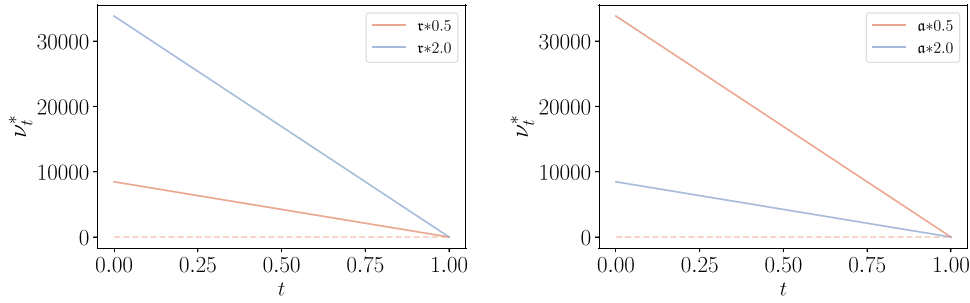


FIGURE 9 | Sample path of the optimal strategy of the LP as model parameters τ and α change. [Color figure can be viewed at wileyonlinelibrary.com]

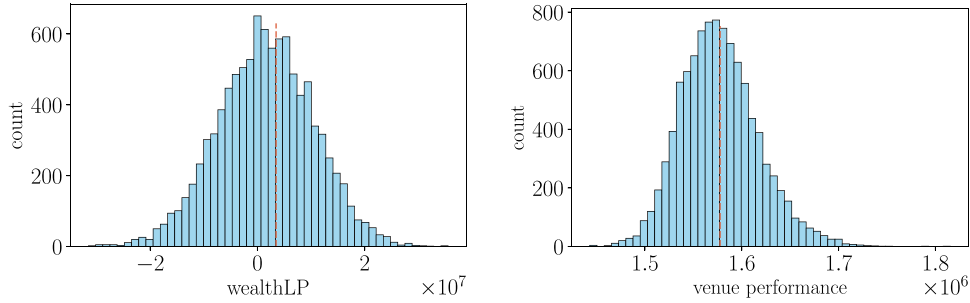


FIGURE 10 | Histograms for the reward \mathfrak{R} paid to the representative LP (left panel) and the performance of the venue $\tau(N_T^- + N_T^+) - \mathfrak{R}$ (right panel). The mean values are in vertical dotted lines. [Color figure can be viewed at wileyonlinelibrary.com]

right aversion of the venue, the more concentrated these profits would be.

The optimal contract benefits both the LPs and the venue. This can be seen in the positive values of the means of both histograms.

5.2 | Attracting Noise Trading

In the analysis, we carried out so far, the intensity of order arrivals does not react to the quantity of asset Y in the pool, that is, $a_2 = 0$. As shown above, this implies that the strategy of the representative LP collapses around zero for meaningful values of the transaction cost parameter for trading in the external venue. Here, we show that this is no longer the case if $a_2 > 0$. In what follows, we use $a_2 = 10^{-5}$ and $\alpha = 5 \times 10^{-6}$. Note that this value of a_2 only represents around 0.5 ETH/10-min of added intensity (given that the amount of ETH in the pool is around 50,000).

Figure 8 shows a trajectory with their confidence bands of the quantity of ETH in the pool, the optimal speed at which the representative LP adds liquidity to the pool, the cumulative liquidity provided, and the cumulative fees paid to the external venue.

With these model parameters, the representative LP adds roughly 8000 ETH to the pool for which she pays 500 USDT in fees; for comparison, the spread in Binance for ETH-USDT is typically 0.01, which implies that the trading costs in Binance are in the correct order of magnitude. Unlike the case in the previous section, to the naked eye, there is no variability in the trading speed. The confidence bands in the amount of ETH in the pool is due to the trading activity of LTs. Next, we study the effect of τ

and α in the optimal trading strategy of the representative LP in Figure 9.

As expected, all else being equal, the higher the fee charged to LTs, the more the venue collects and the better the contract that the venue offers the representative LP, which translates in more liquidity added to the pool. The effect of the transaction cost parameter is similar to that in the previous section. Higher transaction costs in the external venue diminish the activity in the pool.

Finally, we repeat the analysis in Figure 7 for this new set of model parameters in Figure 10.

We see that the performance of both players is still positive in line with the results in Figure 7. All else being equal, when $a_2 > 0$, both buy order arrivals and sell order arrivals increase with the liquidity in the pool, and this additional order flow is not directional (noise trading). The representative LP exploits this by adding liquidity to the pool and attracting more of these type of trades, which in turn allows the venue to offer a higher reward to the representative LP. We remark that for the above histogram, the transaction cost parameter for trading in the external venue is much higher than the one used in Figure 7.

6 | Extension to the Multi-Asset Setup

In classical LOB settings, multi-asset market making have been extensively studied; see, for example, Guéant (2016), Bergault et al. (2021), and Rosenbaum and Zhang (2022). In this section, we study how to extend our AMM framework to a market with

d assets with X^1 being the reference currency (e.g., USDC). As in the one-dimensional model, in addition to the CFM, let there be an external LOB venue where trading in X^1, \dots, X^d occurs. We let $S^i = (S_t^i)_{t \in \mathfrak{T}}$ denote the external mid-price of asset X^i in terms of asset X^1 , with $S_t^1 = 1$ for all $t \in \mathfrak{T}$. Within the CFM, we let X_t^1, \dots, X_t^d be the quantities of assets X^1, \dots, X^d , respectively, in the pool at time $t \in \mathfrak{T}$.

The simplest and most widely adopted multi-asset formulation is the geometric mean market maker (G3M), whose trading function is defined by

$$f(x^1, \dots, x^d) = \prod_{i=1}^d (x^i)^{w_i},$$

for weights $w^1, \dots, w^d > 0$ such that $\sum_{i=1}^d w_i = 1$.

For a given vector $x = (x^1, \dots, x^d) \in \mathbb{R}^d$, we denote by x^{-i} the vector in \mathbb{R}^{d-1} obtained by removing the i th coordinate of x :

$$x^{-i} = (x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^d).$$

We let $\varphi_{c_t}^i(x^{-i})$ be the level function of f such that $\varphi_{c_t}^i(X_t^{-i}) = X_t^i$ with

$$c_t = f(X_{t-}^1, \dots, X_{t-}^d) = \prod_{i=1}^d (X_{t-}^i)^{w_i}.$$

It follows that

$$\varphi_{c_t}^i(x^{-i}) = \frac{c^{1/w_i}}{\prod_{j \neq i} (x^j)^{w_j/w_i}}.$$

At any time $t \in \mathfrak{T}$, the marginal price $Z_t^{i,j}$ of asset X^i in terms of asset X^j is given by the following:

$$Z_t^{i,j} = -\partial_{x^i} \varphi_{c_t}^j(X_t^{-j}).$$

The external mid-prices follow

$$dS_t^i = \sigma_i dW_t^i, \quad S_0^i \in \mathbb{R}^+,$$

where $\sigma_i > 0$ is the volatility parameter of asset i (with $\sigma_1 = 0$), and (W^1, \dots, W^d) is a d -dimensional Brownian motion satisfying $d\langle W^i, W^j \rangle_t = \rho_{i,j} dt$. We denote by Σ the covariance matrix of $S = (S^1, \dots, S^d)$, that is,

$$\Sigma_{i,j} = \rho_{i,j} \sigma_i \sigma_j.$$

In the CFM, LTs arrive according to counting processes $N^{i,j}$ (an LT sells X^i and buys X^j) that model the number of trades over time. In this multi-asset framework, it is convenient to assume that trades are of constant size for the currency being bought, that is, when an LT buys currency X^i , they buy a constant amount $\xi^i > 0$. Alternative conventions such as random sizes or constant sizes in a reference currency can also be handled at the expense of higher complexity. For the sake of simplicity, for $x^{-i} = (x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^d)$, we denote $x^{-i} \ominus^j \xi^j = (x^1, \dots, x^j - \xi^j, \dots, x^{i-1}, x^{i+1}, \dots, x^d)$ if $j < i$ or $x^{-i} \ominus^j \xi^j =$

$(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^j - \xi^j, \dots, x^d)$ if $j > i$. The operator \ominus^i is defined similarly. With these conventions, the resulting dynamics can be written as follows:

$$\begin{aligned} dX_t^i &= \underbrace{Z_t^{i,i} \eta dB_t}_\text{activity from LPs} + \underbrace{\sum_{j \neq i} \left([\varphi_{c_t}^i(X_{t-}^{-i} \ominus^j \xi^j) - \varphi_{c_t}^i(X_{t-}^{-i})] dN_t^{i,j} - \xi^i dN_t^{j,i} \right)}_\text{activity from LTs}, \\ dZ_t^{i,j} &= \left[-\partial_{x^i} \varphi_{c_t}^j \left(X_t^{-j} \ominus^i \left[\varphi_{c_t}^i(X_{t-}^{-i} \ominus^j \xi^j) - \varphi_{c_t}^i(X_{t-}^{-i}) \right] \right) + \partial_{x^i} \varphi_{c_t}^j \left(X_t^{-j} \right) \right] dN_t^{i,j} \\ &\quad + \left[-\partial_{x^i} \varphi_{c_t}^j \left(X_t^{-j} \ominus^i \xi^i \right) + \partial_{x^i} \varphi_{c_t}^j \left(X_t^{-j} \right) \right] dN_t^{j,i}, \\ dN_t^{i,j} &= \mathbb{1}_{x_{t-}^i > 2\xi^i} \mathbb{1}_{x_{t-}^j > 2\xi^j} d\tilde{N}_t^{i,j}, \end{aligned}$$

with $\eta > 0$ representing the volatility in liquidity provision (expressed in the reference currency), $N_0^{i,j} = 0$, and $X_0 \in (\mathbb{R}^+)^d$.

At this stage, one is able to write the performance criterion to be optimized. As expected, these dynamics lead to a highly coupled and high-dimensional system that cannot be solved (even approximately) by the analytical methods employed in the present work. Fortunately, there are a number of methods tailored to solve high-dimensional PDEs such as those proposed by Sirignano and Spiliopoulos (2018), Han et al. (2018), Lu et al. (2025), Cohen et al. (2025), and the references therein.

7 | Conclusion

We characterized the Stackelberg equilibrium of a venue and the representative LP. Trading took place in an AMM and while the representative LP aims to maximize profitability, the venue wishes to collect as much fees and order flow as possible. We find that even when the venue employs the optimal contract, if higher liquidity does not attract noise trading, the representative LP does not have incentives to add liquidity to the pool. On the other hand, if higher liquidity in the AMM attracts noise trading and the venue employs the optimal contract, then the representative LP adds as much liquidity as the transaction costs in the external venue allow. Using data from January 1, 2022 to April 30, 2022, we find a positive correlation of 11% between the daily average depth in Y for the ETH-USDC pool and the daily order flow, furthermore, the slope of the linear regression is positive but there is no evidence to reject the hypothesis that the slope is equal to zero (the p value is 0.232). Our work provides insights into the optimal design of these venues going forward and the testable conditions for the well functioning of these venues. The extension of our results to the case of competing LPs is a promising direction for future research. Informed by the results in Baldacci et al. (2021), one might expect that there will be new phenomena appearing concerning the optimal number of market makers or changes to the conditions under which LPs decide to add liquidity to the pool.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

The data that support the findings of this study are available from the authors upon reasonable request.

Endnotes

- ¹Models with more complex price dynamics have been considered in Bergault et al. (2024b); recently, Bergault et al. (2024a) study the dynamics of stablecoins.
- ²Cartea et al. (2023) is an earlier version of this paper where arithmetic liquidity pools were introduced.
- ³Similar to Fukasawa et al. (2025), we study an aggregate of LPs as a representative provider.
- ⁴In their paper, the regulator is the leader and the LPs are the followers.
- ⁵The independence assumption between W and B can be relaxed.
- ⁶More general multi-asset pool specifications could be considered, at the cost of significant additional mathematical complexity. Focusing on two-asset CFMs remains both realistic and practically relevant: The largest AMM by traded volume and total value locked (TVL), Uniswap, supports only two-asset pools (with a TVL of about \$5 billion as of November 2025, according to DefiLlama). By contrast, multi-asset AMMs such as Balancer, allowing up to eight tokens per pool, remain much smaller in scale (with a TVL of roughly \$315 million in November 2025).
- ⁷The rationale for this is that more depth implies less slippage for any liquidity taking trade executed in the pool. Thus, all else being the same, it is more desirable to trade in a pool with more depth. For mathematical tractability, we employ the quantity $a_2 Y_t$ as a proxy for the depth. This is a reasonable approximation over a short time horizon, during which large moves in Y_t are mainly driven by LPs activity.
- ⁸The calculations below can be carried out if one also includes a running penalty of the form $-\int_s^t p(u, Z_u, Y_u, S_u)du$. For simplicity of formulae, we do not include such running penalty.
- ⁹Using a 5- or 15-min window yields similar results.
- ¹⁰The coefficients \hat{a}_1, \hat{a}_3 are both statistically significant with a p value $p \ll 0.01$.
- ¹¹We obtained the volatility estimate from [market-milk daily volatility](#).
- ¹²This value is motivated by those in [Uniswap's ETH-USDC pool](#) —accessed February 20, 2025.
- ¹³Note that even though the jump processes included in $Q_{0,t}^v$ are N^- and not \tilde{N}^- , we have $Q_{0,t}^v = Q_{0,t^-}^v$ and $\Delta_t^- = 0$ on the event $\{N_t^- = N_{t^-}^- + 1, Y_{t^-} \leq \xi\}$.
- ¹⁴We assume a_0 to be sufficiently small so that $a_1 - a_0 \geq 0$.

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Appendix A: Proof of Theorems 1 and 2

The following proofs for Theorems 1 and 2 follow the arguments of El Euch et al. (2021); however, we have to introduce Lemma 2 to prove the strict negativity of the LP's value function (V_t), which is necessary to derive the contract representation.

A.1 | Lemmas

For any stopping time τ valued in $[0, T]$, we define the following:

$$J(\tau, \nu) := \mathbb{E}_\tau^\nu \left[-\exp \left\{ -\gamma \left(\mathfrak{R} + Q_{\tau, T}^\nu \right) \right\} \right],$$

where ν is in $\mathcal{A}_{\tau, T}$ (the restriction of \mathcal{A} on time horizon $[\tau, T]$) and Q is the running wealth

$$\begin{aligned} Q_{s, t}^\nu &= - \int_s^t \{ \mathfrak{a} \nu_{u-}^2 \} du + \eta \int_s^t (S_{u-} + Z_{u-}) dB_u^\nu \\ &+ \sigma \int_s^t Y_{u-} dW_u + \sum_{i \in \{+, -\}} \int_s^t \left\{ \delta^i \xi \left(S_u - \frac{Z_{u-} Y_{u-}}{Y_{u-} + \delta^i \xi} \right) \right\} dN_u^i. \end{aligned}$$

Recall that the value function V is defined as follows:

$$V_\tau = \text{ess sup}_{\nu \in \mathcal{A}_{\tau, T}} J(\tau, \nu).$$

Lemma 1. *Let τ be a stopping time valued in $[t, T]$. Then,*

$$V_t = \text{ess sup}_{\nu \in \mathcal{A}_{t, \tau}} \mathbb{E}_t^\nu \left[\exp \{ -\gamma Q_{t, \tau}^\nu \} V_\tau \right],$$

Proof. By tower property and the definition of the value function V_t , we have that

$$\begin{aligned} V_t &= \text{ess sup}_{\nu \in \mathcal{A}_{t, \tau}} \mathbb{E}_t^\nu \left[\exp \{ -\gamma Q_{t, \tau}^\nu \} \mathbb{E}_\tau^\nu \left[-\exp \left\{ -\gamma \left(\mathfrak{R} + Q_{\tau, T}^\nu \right) \right\} \right] \right] \\ &\leq \text{ess sup}_{\nu \in \mathcal{A}_{t, \tau}} \mathbb{E}_t^\nu \left[\exp \{ -\gamma Q_{t, \tau}^\nu \} V_\tau \right]. \end{aligned}$$

For the other direction, by Proposition VI-1-1 in Neveu (1972), there exists $(\tilde{\nu}^k)_{k \in \mathbb{N}} \subset \mathcal{A}_{\tau, T}$ such that $J(\tau, \tilde{\nu}^k) \uparrow V_\tau$. By defining $\nu_s^k = \nu_s \mathbb{1}_{t \leq s < \tau} + \tilde{\nu}^k \mathbb{1}_{\tau \leq s \leq T}$, we obtain the following:

$$\frac{K_T^{\nu^k}}{K_\tau^{\nu^k}} = \frac{K_T^{\tilde{\nu}^k}}{K_\tau^{\tilde{\nu}^k}} \quad \text{and} \quad Q_{\tau, T}^{\nu^k} = Q_{\tau, T}^{\tilde{\nu}^k}.$$

Thus, it follows that

$$\begin{aligned} \mathbb{E}_\tau^{\nu^k} \left[-\exp \left\{ -\gamma \left(\mathfrak{R} + Q_{\tau, T}^{\nu^k} \right) \right\} \right] &= \mathbb{E}_\tau \left[-\frac{K_T^{\nu^k}}{K_\tau^{\nu^k}} \frac{L_T}{L_\tau} \exp \left\{ -\gamma \left(\mathfrak{R} + Q_{\tau, T}^{\nu^k} \right) \right\} \right] \\ &= \mathbb{E}_\tau \left[-\frac{K_T^{\tilde{\nu}^k}}{K_\tau^{\tilde{\nu}^k}} \frac{L_T}{L_\tau} \exp \left\{ -\gamma \left(\mathfrak{R} + Q_{\tau, T}^{\tilde{\nu}^k} \right) \right\} \right] \\ &= J(\tau, \tilde{\nu}^k). \end{aligned}$$

We then obtain the following:

$$\begin{aligned}
J(t, \nu^k) &= \mathbb{E}_t \left[-\frac{K_T^{\nu^k} L_T}{K_\tau^{\nu^k} L_\tau} \frac{K_\tau^{\nu^k} L_\tau}{K_t^{\nu^k} L_t} \exp \left\{ -\gamma Q_{t,\tau}^{\nu^k} \right\} \exp \left\{ -\gamma \left(\mathfrak{R} + Q_{\tau,T}^{\nu^k} \right) \right\} \right] \\
&= \mathbb{E}_t \left[\frac{K_\tau^{\nu^k} L_\tau}{K_t^{\nu^k} L_t} \exp \left\{ -\gamma Q_{t,\tau}^{\nu^k} \right\} \mathbb{E}_\tau \left[-\frac{K_T^{\nu^k} L_T}{K_\tau^{\nu^k} L_\tau} \exp \left\{ -\gamma \left(\mathfrak{R} + Q_{\tau,T}^{\nu^k} \right) \right\} \right] \right] \\
&= \mathbb{E}_t \left[\frac{K_\tau^{\nu^k} L_\tau}{K_t^{\nu^k} L_t} \exp \left\{ -\gamma Q_{t,\tau}^{\nu^k} \right\} \mathbb{E}_\tau^{\nu^k} \left[-\exp \left\{ -\gamma \left(\mathfrak{R} + Q_{\tau,T}^{\nu^k} \right) \right\} \right] \right] \\
&= \mathbb{E}_t \left[\frac{K_\tau^{\nu^k} L_\tau}{K_t^{\nu^k} L_t} \exp \left\{ -\gamma Q_{t,\tau}^{\nu^k} \right\} J(\tau, \bar{\nu}^k) \right].
\end{aligned}$$

Note that $K_T^{\nu^k} L_T / K_\tau^{\nu^k} L_\tau$ has mean of unity and is independent of \mathcal{F}_τ . Thus,

$$\begin{aligned}
\mathbb{E}_t \left[\frac{K_\tau^{\nu^k} L_\tau}{K_t^{\nu^k} L_t} \exp \left\{ -\gamma Q_{t,\tau}^{\nu^k} \right\} J(\tau, \bar{\nu}^k) \right] &= \mathbb{E}_t \left[\frac{K_\tau^{\nu^k} L_\tau}{K_t^{\nu^k} L_t} \exp \left\{ -\gamma Q_{t,\tau}^{\nu^k} \right\} J(\tau, \bar{\nu}^k) \right] \mathbb{E}_t \left[\frac{K_T^{\nu^k} L_T}{K_\tau^{\nu^k} L_\tau} \right] \\
&= \mathbb{E}_t \left[\frac{K_T^{\nu^k} L_T}{K_t^{\nu^k} L_t} \exp \left\{ -\gamma Q_{t,\tau}^{\nu^k} \right\} J(\tau, \bar{\nu}^k) \right] \\
&= \mathbb{E}_t^{\nu^k} \left[\exp \left\{ -\gamma Q_{t,\tau}^{\nu^k} \right\} J(\tau, \bar{\nu}^k) \right]
\end{aligned}$$

Due to Condition 1, the expression $\mathbb{E}_t^{\nu^k} \left[\exp \left\{ -\gamma Q_{t,\tau}^{\nu^k} \right\} J(\tau, \bar{\nu}^k) \right]$ is finite and we can use monotone convergence theorem to obtain the following:

$$\begin{aligned}
\lim_{k \rightarrow \infty} J(t, \nu^k) &= \mathbb{E}_t^{\nu^k} \left[\exp \left\{ -\gamma Q_{t,\tau}^{\nu^k} \right\} \lim_{k \rightarrow \infty} J(\tau, \bar{\nu}^k) \right] \\
&= \mathbb{E}_t^{\nu^k} \left[\exp \left\{ -\gamma Q_{t,\tau}^{\nu^k} \right\} V_\tau \right].
\end{aligned}$$

Thus, we obtain the following:

$$\begin{aligned}
V_t &\geq \lim_{k \rightarrow \infty} J(t, \nu^k) \\
&= \mathbb{E}_t^{\nu^k} \left[\exp \left\{ -\gamma Q_{t,\tau}^{\nu^k} \right\} V_\tau \right],
\end{aligned}$$

which concludes the proof. □

Lemma 2. Let $\Gamma \in \mathcal{F}$ such that $\mathbb{P}(\Gamma) > 0$. Then there exists $\varepsilon > 0$ such that

$$\mathbb{P}^{\nu}(\Gamma) > \varepsilon \quad \forall \nu \in \mathcal{A}.$$

Proof. We will prove this by contradiction. Assume there exists $(\nu^k)_{k \in \mathbb{N}}$ a sequence in \mathcal{A} such that $\lim_{n \rightarrow \infty} \mathbb{P}^{\nu^n}(\Gamma) = 0$. Thus,

$$\begin{aligned}
0 &= \lim_{n \rightarrow \infty} \mathbb{E}^{\nu^n} [\mathbb{1}_\Gamma] \\
&= \lim_{n \rightarrow \infty} \mathbb{E} \left[\left[K_T^{\nu^n} L_T \mathbb{1}_\Gamma \right] \right].
\end{aligned}$$

Then we can take a subsequence $(n_k)_{k \in \mathbb{N}} \subseteq (n)_{n \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} K_T^{\nu^{n_k}} L_T \mathbb{1}_\Gamma = 0$ \mathbb{P} -almost surely. Because Γ has \mathbb{P} -positive probability and $L_T > 0$, then $\mathbb{P}(\lim_{k \rightarrow \infty} K_T^{\nu^{n_k}} = 0) > 0$, or equivalently,

$$\begin{aligned}
&\mathbb{P} \left(\lim_{k \rightarrow \infty} \int_0^T \frac{\nu_t^{n_k}}{\eta} dB_t - \frac{1}{2} \int_0^T \frac{(\nu_t^{n_k})^2}{\eta^2} dt = -\infty \right) > 0, \\
&\Leftrightarrow \mathbb{P} \left(\lim_{k \rightarrow \infty} \int_0^T \frac{\nu_t^{n_k}}{\eta} dB_t = -\infty \right) > 0.
\end{aligned} \tag{A.1}$$

However, by Ito isometry,

$$\mathbb{E} \left[\left(\int_0^T \frac{\nu_u^{n_k}}{\eta} dB_u \right)^2 \right] = \mathbb{E} \left[\int_0^T \frac{(\nu_u^{n_k})^2}{\eta^2} du \right]$$

$$\leq \frac{T(\nu_\infty)^2}{\eta^2},$$

because ν^{n_k} is bounded by $\nu_\infty \, d\mathbb{P} \otimes dt$ a.e. given that $\nu_t^{n_k} \in \mathcal{A}$. Thus, the collection of random variables $\left\{ \int_0^T \frac{\nu_u^{n_k}}{\eta} dB_u \right\}_{k \in \mathbb{N}}$ is bounded in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. From Equation (A.1), we have the following:

$$\begin{aligned} & \mathbb{P} \left(\liminf_{k \rightarrow \infty} \left| \int_0^T \frac{\nu_t^{n_k}}{\eta} dB_t \right|^2 = \infty \right) > 0 \\ & \Rightarrow \mathbb{E} \left[\liminf_{k \rightarrow \infty} \left| \int_0^T \frac{\nu_t^{n_k}}{\eta} dB_t \right|^2 \right] = \infty. \end{aligned}$$

However, by Fatou's lemma,

$$\begin{aligned} \infty &= \mathbb{E} \left[\liminf_{k \rightarrow \infty} \left| \int_0^T \frac{\nu_t^{n_k}}{\eta} dB_t \right|^2 \right] \\ &\leq \liminf_{k \rightarrow \infty} \mathbb{E} \left[\left| \int_0^T \frac{\nu_t^{n_k}}{\eta} dB_t \right|^2 \right] \\ &\leq \frac{T(\nu_\infty)^2}{\eta^2}, \end{aligned}$$

which is a contradiction. □

Lemma 3. For all $t \in \mathfrak{Z}$, we have $V_t < 0$ \mathbb{P} -almost surely.

Proof. Let $\Gamma = \{V_t = 0\}$. By Neveu (1972), there exists $(\nu^k)_{k \in \mathbb{N}}$ a sequence in $\mathcal{A}_{t,T}$ such that $J(t, \nu^k) \uparrow 0$ on Γ . From monotone convergence theorem, $\mathbb{E} [\mathbb{1}_\Gamma J(t, \nu^k)] \rightarrow 0$. Then, there is a subsequence k_1, k_2, \dots such that

$$K_T^{\nu^{k_m}} L_T \exp \left\{ -\gamma \left(\mathfrak{R} + Q_{0,T}^{\nu^{k_m}} \right) \right\} \rightarrow 0 \quad \mathbb{P} - \text{a.s. on } \Gamma.$$

As $L_T > 0$ almost surely and does not depend on ν^{k_m} , we have the following:

$$K_T^{\nu^{k_m}} \exp \left\{ -\gamma \left(\mathfrak{R} + Q_{0,T}^{\nu^{k_m}} \right) \right\} \rightarrow 0 \quad \mathbb{P} - \text{a.s. on } \Gamma.$$

This implies the following:

$$\begin{aligned} & \int_0^T \frac{\nu_u^{k_m}}{\eta} dB_u + \int_0^T -\gamma \eta (S_{u-} + Z_{u-}) dB_u + \int_0^T \frac{\gamma \eta \nu_u^{k_m} (S_{u-} + Z_{u-})}{\eta} ds \\ & - \int_0^T \gamma \sigma Y_{u-} dW_u - \sum_{i \in \{+, -\}} \int_0^T \gamma \left\{ \delta^i \xi \left(S_u - \frac{Z_{u-} Y_{u-}}{Y_{u-} + \delta^i \xi} \right) \right\} dN_u^i \rightarrow -\infty \end{aligned}$$

\mathbb{P} -almost surely on Γ . However,

$$\sup_{t \in \mathfrak{Z}} Y_t \leq Y_0 + \xi (\hat{N}_T^- + \hat{N}_T^+) + \sup_{t \in \mathfrak{Z}} B_t, \quad \text{and} \quad \sup_{t \in \mathfrak{Z}} Z_t \leq Z_0 4^{\hat{N}_T^- + \hat{N}_T^+}.$$

Moreover, given that ν^{k_m} is bounded and using the above inequalities, we conclude that the last four terms in the above expression are finite a.s., therefore, it has to be the case that

$$\int_0^T \frac{\nu_u^{k_m}}{\eta} dB_u \rightarrow -\infty \quad \mathbb{P} - \text{a.s. on } \Gamma.$$

From Lemma 2, we should have $\mathbb{P}(\Gamma) = 0$. Thus, we have $V_t < 0$ \mathbb{P} -almost surely. □

A.2 | Contract Representation

Proof of Theorem 1. By the DPP in Lemma 1, one can show that for any $t \geq s$

$$V_s = \sup_{\nu \in \mathcal{A}} \mathbb{E}_s^\nu \left[\exp \left\{ -\gamma Q_{s,t}^\nu \right\} V_t \right].$$

Then, U^ν defined as

$$U_t^\nu = \exp \left\{ -\gamma Q_{0,t}^\nu \right\} V_t$$

is a \mathbb{P}^ν -supermartingale. By Doob–Meyer decomposition, we can write U^ν as

$$dU_t^\nu = dM_t^\nu - dA_t^{\nu,c} - dA_t^{\nu,d},$$

where M^ν is a $(\mathbb{F}, \mathbb{P}^\nu)$ -martingale, $A^{\nu,c}$ is a continuous nondecreasing process, $A^{\nu,d}$ is a pure-jump nondecreasing process, and $A_0^{\nu,c} = A_0^{\nu,d} = 0$. By Martingale Representation Theorem, there exists a \mathbb{F} -predictable process $(\tilde{A}^{\nu,W}, \tilde{A}^{\nu,B}, \tilde{A}^{\nu,-}, \tilde{A}^{\nu,+})$ such that

$$dM_t^\nu = \tilde{A}_t^{\nu,W} dW_t + \tilde{A}_t^{\nu,B} dB_t^\nu + \sum_{i \in \{+, -\}} \tilde{A}_t^{\nu,i} d\tilde{N}_t^i,$$

where

$$d\tilde{N}_t^\pm = d\hat{N}_t^\pm - \lambda_t^\pm dt.$$

By Ito's formula, we have that

$$\begin{aligned} dV_t &= U_{t-}^\nu d\left(\exp \left\{ \gamma Q_{0,t}^\nu \right\}\right) + \exp \left\{ \gamma Q_{0,t-}^\nu \right\} dU_t^\nu \\ &\quad + \sum_{i \in \{+, -\}} (U_t^\nu - U_{t-}^\nu) \left(\exp \left\{ \gamma Q_{0,t}^\nu \right\} - \exp \left\{ \gamma Q_{0,t-}^\nu \right\} \right) d\tilde{N}_t^i \\ &\quad + d\langle U^{\nu,c}, \exp \left\{ \gamma Q_{0,\cdot}^\nu \right\} \rangle_t \end{aligned}$$

where $U^{\nu,c}$ and $Q_{0,\cdot}^{\nu,c}$ are the continuous parts of U^ν and $Q_{0,\cdot}^\nu$, respectively. As $\langle \tilde{N}^i, A^{\nu,d} \rangle = 0$, we have the following:

$$(U_t^\nu - U_{t-}^\nu) \left(\exp \left\{ \gamma Q_{0,t}^\nu \right\} - \exp \left\{ \gamma Q_{0,t-}^\nu \right\} \right) d\tilde{N}_t^i = \tilde{A}_t^{\nu,i} \left(\exp \left\{ \gamma Q_{0,t}^\nu \right\} - \exp \left\{ \gamma Q_{0,t-}^\nu \right\} \right) d\tilde{N}_t^i.$$

We also calculate the following:

$$\begin{aligned} d\left(\exp \left\{ \gamma Q_{0,t}^\nu \right\}\right) &= \gamma \left(\exp \left\{ \gamma Q_{0,t-}^\nu \right\} \right) dQ_{0,t}^{\nu,c} + \frac{\gamma^2}{2} \left(\exp \left\{ \gamma Q_{0,t-}^\nu \right\} \right) d\langle Q_{0,\cdot}^{\nu,c} \rangle_t \\ &\quad + \sum_{i \in \{+, -\}} \left[\exp \left\{ \gamma Q_{0,t}^\nu \right\} - \exp \left\{ \gamma Q_{0,t-}^\nu \right\} \right] d\tilde{N}_t^i, \end{aligned} \quad 13$$

and finally,

$$\begin{aligned} d\langle U^{\nu,c}, \exp \left\{ \gamma Q_{0,\cdot}^\nu \right\} \rangle_t &= \gamma \exp \left\{ \gamma Q_{0,t-}^\nu \right\} \left(\sigma Y_{t-} \tilde{A}_t^{\nu,W} + \eta (S_{t-} + Z_{t-}) \tilde{A}_t^{\nu,B} \right) dt \\ &= \frac{\gamma V_{t-}^\nu}{U_{t-}^\nu} \left(\sigma Y_{t-} \tilde{A}_t^{\nu,W} + \eta (S_{t-} + Z_{t-}) \tilde{A}_t^{\nu,B} \right) dt. \end{aligned}$$

Thus, we are able to rewrite dV_t as follows:

$$\begin{aligned} dV_t &= \gamma U_{t-}^\nu \left(\exp \left\{ \gamma Q_{0,t-}^\nu \right\} \right) dQ_{0,t}^{\nu,c} + \frac{\gamma^2 U_{t-}^\nu}{2} \left(\exp \left\{ \gamma Q_{0,t-}^\nu \right\} \right) d\langle Q_{0,\cdot}^{\nu,c} \rangle_t \\ &\quad + \sum_{i \in \{+, -\}} U_{t-}^\nu \left[\exp \left\{ \gamma Q_{0,t}^\nu \right\} - \exp \left\{ \gamma Q_{0,t-}^\nu \right\} \right] d\tilde{N}_t^i \\ &\quad + \exp \left\{ \gamma Q_{0,t-}^\nu \right\} \tilde{A}_t^{\nu,W} dW_t + \exp \left\{ \gamma Q_{0,t-}^\nu \right\} \tilde{A}_t^{\nu,B} dB_t^\nu + \sum_{i \in \{+, -\}} \exp \left\{ \gamma Q_{0,t-}^\nu \right\} \tilde{A}_t^{\nu,i} (d\tilde{N}_t^i - \lambda_t^i dt) \end{aligned}$$

$$\begin{aligned}
& - \exp \left\{ \gamma Q_{0,t}^{\nu,c} \right\} dA_t^{\nu,c} - \exp \left\{ \gamma Q_{0,t}^{\nu,d} \right\} dA_t^{\nu,d} + \sum_{i \in \{+, -\}} \bar{A}_t^{\nu,i} \left(\exp \left\{ \gamma Q_{0,t}^{\nu,i} \right\} - \exp \left\{ \gamma Q_{0,t}^{\nu,-} \right\} \right) d\hat{N}_t^i \\
& + \frac{\gamma V_{t-}}{U_{t-}^{\nu}} \left(\sigma Y_{t-} \bar{A}_t^{\nu,W} + \eta (S_{t-} + Z_{t-}) \bar{A}_t^{\nu,B} \right) dt, \\
& = \gamma V_{t-} dQ_{0,t}^{\nu,c} + \frac{\gamma^2 V_{t-}}{2} d\langle Q_{0,\cdot}^{\nu,c} \rangle_t \\
& + \sum_{i \in \{+, -\}} V_{t-} \left(\exp \left\{ \gamma \Delta_t^i \right\} - 1 \right) d\hat{N}_t^i \\
& + \frac{V_{t-} \bar{A}_t^{\nu,W}}{U_{t-}^{\nu}} dW_t + \frac{V_{t-} \bar{A}_t^{\nu,B}}{U_{t-}^{\nu}} dB_t^{\nu} + \sum_{i \in \{+, -\}} \frac{V_{t-} \bar{A}_t^{\nu,i}}{U_{t-}^{\nu}} \left(d\hat{N}_t^i - \lambda_t^i dt \right) \\
& - \frac{V_{t-}}{U_{t-}^{\nu}} dA_t^{\nu,c} - \frac{V_{t-}}{U_{t-}^{\nu}} dA_t^{\nu,d} + \sum_{i \in \{+, -\}} \frac{V_{t-} \bar{A}_t^{\nu,i}}{U_{t-}^{\nu}} \left(\exp \left\{ \gamma \Delta_t^i \right\} - 1 \right) d\hat{N}_t^i \\
& + \frac{\gamma V_{t-}}{U_{t-}^{\nu}} \left(\sigma Y_{t-} \bar{A}_t^{\nu,W} + \eta (S_{t-} + Z_{t-}) \bar{A}_t^{\nu,B} \right) dt.
\end{aligned}$$

Let $P_t = -\frac{1}{\gamma} \log(-V_t)$. Then, by Ito's formula,

$$\begin{aligned}
dP_t & = -\frac{1}{\gamma V_{t-}} dV_t^c + \frac{1}{2\gamma V_{t-}^2} d\langle V^c \rangle_t - \frac{1}{\gamma} \sum_{i \in \{+, -\}} \log \left(\frac{V_t}{V_{t-}} \right) d\hat{N}_t^i - \frac{1}{\gamma \Delta A_t^{\nu,d}} \log \left(\frac{V_t}{V_{t-}} \right) dA_t^{\nu,d} \\
& = -\frac{1}{\gamma V_{t-}} dV_t^c + \frac{1}{2\gamma V_{t-}^2} d\langle V^c \rangle_t - \frac{1}{\gamma} \sum_{i \in \{+, -\}} \log \left(1 + \frac{\Delta^i V_t}{V_{t-}} \right) d\hat{N}_t^i - \frac{1}{\gamma \Delta A_t^{\nu,d}} \log \left(1 + \frac{\Delta^d V_t}{V_{t-}} \right) dA_t^{\nu,d},
\end{aligned}$$

with V^c is the continuous part of V and $Y_T = \mathfrak{R}$. We calculate the following:

$$\begin{aligned}
\frac{d\langle V^c \rangle_t}{dt} & = \left(\frac{V_{t-} \bar{A}_t^{\nu,W}}{U_{t-}^{\nu}} + \gamma V_{t-} \sigma Y_{t-} \right)^2 + \left(\frac{V_{t-} \bar{A}_t^{\nu,B}}{U_{t-}^{\nu}} + \gamma V_{t-} \eta (S_t + Z_{t-}) \right)^2, \\
\Delta^i V_t & = \left(1 + \frac{\bar{A}_t^{\nu,i}}{U_{t-}^{\nu}} \right) V_{t-} \exp \left\{ \gamma \Delta_t^i \right\} - V_{t-}, \quad \Delta^d V_t = -\frac{V_{t-} \Delta A_t^{\nu,d}}{U_{t-}^{\nu}},
\end{aligned}$$

$$\begin{aligned}
dV_t^c & = \gamma V_{t-} dQ_{0,t}^{\nu,c} + \frac{\gamma^2 V_{t-}}{2} d\langle Q_{0,\cdot}^{\nu,c} \rangle_t + \frac{\gamma V_{t-}}{U_{t-}^{\nu}} \left(\sigma Y_{t-} \bar{A}_t^{\nu,W} + \eta (S_{t-} + Z_{t-}) \bar{A}_t^{\nu,B} \right) dt \\
& + \frac{V_{t-} \bar{A}_t^{\nu,W}}{U_{t-}^{\nu}} dW_t + \frac{V_{t-} \bar{A}_t^{\nu,B}}{U_{t-}^{\nu}} dB_t^{\nu} - \sum_{i \in \{+, -\}} \frac{\lambda_t^i V_{t-} \bar{A}_t^{\nu,i}}{U_{t-}^{\nu}} dt - \frac{V_{t-}}{U_{t-}^{\nu}} dA_t^{\nu,c}.
\end{aligned}$$

We may rewrite dP_t as follows:

$$\begin{aligned}
dP_t & = -dQ_{0,t}^{\nu,c} - \frac{\gamma}{2} d\langle Q_{0,\cdot}^{\nu,c} \rangle_t - \frac{\bar{A}_t^{\nu,W}}{\gamma U_{t-}^{\nu}} dW_t - \frac{\bar{A}_t^{\nu,B}}{\gamma U_{t-}^{\nu}} dB_t^{\nu} + \sum_{i \in \{+, -\}} \frac{\lambda_t^i \bar{A}_t^{\nu,i}}{\gamma U_{t-}^{\nu}} dt + \frac{1}{\gamma U_{t-}^{\nu}} dA_t^{\nu,c} \\
& + \frac{1}{2\gamma} \left[\left(\frac{\bar{A}_t^{\nu,W}}{U_{t-}^{\nu}} + \gamma \sigma Y_{t-} \right)^2 + \left(\frac{\bar{A}_t^{\nu,B}}{U_{t-}^{\nu}} + \gamma \eta (S_t + Z_{t-}) \right)^2 \right] dt \\
& - \frac{1}{U_{t-}^{\nu}} \left(\sigma Y_{t-} \bar{A}_t^{\nu,W} + \eta (S_{t-} + Z_{t-}) \bar{A}_t^{\nu,B} \right) dt \\
& - \frac{1}{\gamma} \sum_{i \in \{+, -\}} \left\{ \log \left(1 + \frac{\bar{A}_t^{\nu,i}}{U_{t-}^{\nu}} \right) + \gamma \Delta_t^i \right\} d\hat{N}_t^i - \frac{1}{\gamma \Delta A_t^{\nu,d}} \log \left(1 - \frac{\Delta A_t^{\nu,d}}{U_{t-}^{\nu}} \right) dA_t^{\nu,d} \\
& = \left\{ \alpha \nu_{t-}^2 + \frac{1}{2\gamma} \left[\left(\frac{\bar{A}_t^{\nu,W}}{U_{t-}^{\nu}} + \gamma \sigma Y_{t-} \right)^2 + \left(\frac{\bar{A}_t^{\nu,B}}{U_{t-}^{\nu}} + \gamma \eta (S_t + Z_{t-}) \right)^2 \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& -\frac{\gamma}{2} [(\sigma Y_{t-})^2 + \eta^2 (S_t + Z_{t-})^2] + \sum_{i \in \{+, -\}} \frac{\lambda_t^i \bar{A}_t^{v,i}}{\gamma U_{t-}^v} + \frac{1}{\gamma U_{t-}^v} \frac{dA_t^{v,c}}{dt} \\
& - \frac{1}{U_{t-}^v} \left(\sigma Y_{t-} \bar{A}_t^{v,W} + \eta (S_{t-} + Z_{t-}) \bar{A}_t^{v,B} \right) \Big\} dt - \frac{1}{\gamma} \sum_{i \in \{+, -\}} \left\{ \log \left(1 + \frac{\bar{A}_t^{v,i}}{U_{t-}^v} \right) + \gamma \Delta_t^i \right\} d\hat{N}_t^i \\
& - \frac{1}{\gamma \Delta A_t^{v,d}} \log \left(1 - \frac{\Delta A_t^{v,d}}{U_{t-}^v} \right) dA_t^{v,d} \\
& - \left\{ \sigma Y_{t-} + \frac{\bar{A}_t^{v,W}}{\gamma U_{t-}^v} \right\} dW_t - \left\{ \eta (S_{t-} + Z_{t-}) + \frac{\bar{A}_t^{v,B}}{\gamma U_{t-}^v} \right\} dB_t^v \\
& = \left\{ \alpha v_{t-}^2 + \frac{1}{2\gamma} \left[\left(\frac{\bar{A}_t^{v,W}}{U_{t-}^v} + \gamma \sigma Y_{t-} \right)^2 + \left(\frac{\bar{A}_t^{v,B}}{U_{t-}^v} + \gamma \eta (S_t + Z_{t-}) \right)^2 \right] \right. \\
& - \frac{\gamma}{2} [(\sigma Y_{t-})^2 + \eta^2 (S_t + Z_{t-})^2] + \sum_{i \in \{+, -\}} \frac{\lambda_t^i \bar{A}_t^{v,i}}{\gamma U_{t-}^v} + \frac{1}{\gamma U_{t-}^v} \frac{dA_t^{v,c}}{dt} \\
& - \frac{1}{U_{t-}^v} \left(\sigma Y_{t-} \bar{A}_t^{v,W} + \eta (S_{t-} + Z_{t-}) \bar{A}_t^{v,B} \right) - v_{t-} (S_{t-} + Z_{t-}) - \frac{\bar{A}_t^{v,B} v_{t-}}{\gamma \eta U_{t-}^v} \Big\} dt \\
& - \frac{1}{\gamma} \sum_{i \in \{+, -\}} \left\{ \log \left(1 + \frac{\bar{A}_t^{v,i}}{U_{t-}^v} \right) + \gamma \Delta_t^i \right\} d\hat{N}_t^i \\
& - \frac{1}{\gamma \Delta A_t^{v,d}} \log \left(1 - \frac{\Delta A_t^{v,d}}{U_{t-}^v} \right) dA_t^{v,d} \\
& - \left\{ \sigma Y_{t-} + \frac{\bar{A}_t^{v,W}}{\gamma U_{t-}^v} \right\} dW_t - \left\{ \eta (S_{t-} + Z_{t-}) + \frac{\bar{A}_t^{v,B}}{\gamma U_{t-}^v} \right\} dB_t.
\end{aligned}$$

Simplifying the above equations, we get the following:

$$\begin{aligned}
dP_t & = \left\{ \alpha v_{t-}^2 + \frac{1}{2\gamma} \left[\left(\frac{\bar{A}_t^{v,W}}{U_{t-}^v} \right)^2 + \left(\frac{\bar{A}_t^{v,B}}{U_{t-}^v} \right)^2 \right] \right. \\
& + \sum_{i \in \{+, -\}} \frac{\lambda_t^i \bar{A}_t^{v,i}}{\gamma U_{t-}^v} + \frac{1}{\gamma U_{t-}^v} \frac{dA_t^{v,c}}{dt} + v_{t-} (S_{t-} + Z_{t-}) + \frac{\bar{A}_t^{v,B} v_{t-}}{\gamma \eta U_{t-}^v} \Big\} dt \\
& - \frac{1}{\gamma} \sum_{i \in \{+, -\}} \left\{ \log \left(1 + \frac{\bar{A}_t^{v,i}}{U_{t-}^v} \right) + \gamma \Delta_t^i \right\} d\hat{N}_t^i \\
& - \frac{1}{\gamma \Delta A_t^{v,d}} \log \left(1 - \frac{\Delta A_t^{v,d}}{U_{t-}^v} \right) dA_t^{v,d} \\
& - \left\{ \sigma Y_{t-} + \frac{\bar{A}_t^{v,W}}{\gamma U_{t-}^v} \right\} dW_t - \left\{ \eta (S_{t-} + Z_{t-}) + \frac{\bar{A}_t^{v,B}}{\gamma U_{t-}^v} \right\} dB_t.
\end{aligned}$$

If we rewrite

$$\begin{aligned}
A_t^i & = -\frac{1}{\gamma} \left(\log \left(1 + \frac{\bar{A}_t^{v,i}}{U_{t-}^v} \right) + \gamma \Delta_t^i \right) \\
A_t^W & = -\left(\sigma Y_{t-} + \frac{\bar{A}_t^{v,W}}{\gamma U_{t-}^v} \right) \\
A_t^B & = -\left(\eta (S_{t-} + Z_{t-}) + \frac{\bar{A}_t^{v,B}}{\gamma U_{t-}^v} \right)
\end{aligned}$$

where $i \in \{+, -\}$, we have the following:

$$\begin{aligned} dP_t = & \left\{ \alpha \nu_{t-}^2 + \frac{\gamma}{2} \left[(A_t^W + \sigma Y_{t-})^2 + (A_t^B + \eta(S_{t-} + Z_{t-}))^2 \right] \right. \\ & - \sum_{i \in \{+, -\}} \frac{\lambda_t^i (1 - e^{-\gamma(A_t^i + \Delta_t^i)})}{\gamma} + \frac{1}{\gamma U_{t-}^\nu} \frac{dA_t^{\nu,c}}{dt} - \frac{A_t^B \nu_{t-}}{\eta} \left. \right\} dt \\ & + \sum_{i \in \{+, -\}} A_t^i d\tilde{N}_t^i - \frac{1}{\gamma \Delta A_t^{\nu,d}} \log \left(1 - \frac{\Delta A_t^{\nu,d}}{U_{t-}^\nu} \right) dA_t^{\nu,d} \\ & + A_t^W dW_t + A_t^B dB_t. \end{aligned}$$

Note that

$$\begin{aligned} P_u = \mathfrak{R} - \int_u^T & \left\{ \alpha \nu_{t-}^2 + \frac{\gamma}{2} \left[(A_t^W + \sigma Y_{t-})^2 + (A_t^B + \eta(S_{t-} + Z_{t-}))^2 \right] \right. \\ & - \sum_{i \in \{+, -\}} \frac{\lambda_t^i (1 - e^{-\gamma(A_t^i + \Delta_t^i)})}{\gamma} + \frac{1}{\gamma U_{t-}^\nu} \frac{dA_t^{\nu,c}}{dt} - \frac{A_t^B \nu_{t-}}{\eta} \left. \right\} dt \\ & - \sum_{i \in \{+, -\}} \int_u^T A_t^i d\tilde{N}_t^i - \int_u^T A_t^W dW_t - \int_u^T A_t^B dB_t + \int_u^T dA_t^d, \end{aligned}$$

with $dA_t^d = \frac{1}{\gamma \Delta A_t^{\nu,d}} \log \left(1 - \frac{\Delta A_t^{\nu,d}}{U_{t-}^\nu} \right) dA_t^{\nu,d}$, that is,

$$A_t^d = \frac{1}{\gamma} \sum_{s \leq t} \log \left(1 - \frac{\Delta A_s^{\nu,d}}{U_{s-}^\nu} \right).$$

This shows in particular that $\Delta a_t = \frac{-\Delta A_t^{\nu,d}}{U_{t-}^\nu} \geq 0$ is independent from $\nu \in \mathcal{A}$.

Therefore, we have $P_T = \mathfrak{R}$ and

$$dP_t = - \left\{ \bar{h}(\nu_t, A_t, Z_t, Y_t, S_t) - \frac{1}{\gamma U_{t-}^\nu} \frac{dA_t^{\nu,c}}{dt} \right\} dt + \sum_{i \in \{+, -\}} A_t^i d\tilde{N}_t^i + A_t^W dW_t + A_t^B dB_t - dA_t^d$$

with

$$\bar{h}(\nu, A, Z, Y, S) = h(\nu, A) + \sum_{i \in \{+, -\}} \frac{\bar{\lambda}^i(Z, Y, S)(1 - e^{-\gamma(A^i + \Delta^i)})}{\gamma} - \frac{\gamma}{2} \left[(A^W + \sigma Y)^2 + (A^B + \eta(S + Z))^2 \right].$$

We introduce the process I given by the following:

$$I_t = \int_0^t \left\{ \bar{h}(\nu_s, A_s, Z_s, Y_s, S_s) ds - \frac{1}{\gamma U_{s-}^\nu} dA_s^{\nu,c} \right\}.$$

By the DPP, we have the following:

$$\begin{aligned} 0 = \sup_{\nu \in \mathcal{A}} \mathbb{E}^\nu [U_T^\nu] - V_0 &= \sup_{\nu \in \mathcal{A}} \mathbb{E}^\nu [U_T^\nu - M_T^\nu] = \sup_{\nu \in \mathcal{A}} \mathbb{E}^\nu \left[\int_0^T (-dA_t^{\nu,c} - dA_t^{\nu,d}) \right] \\ &= \gamma \sup_{\nu \in \mathcal{A}} \mathbb{E}^\nu \left[\int_0^T U_{t-}^\nu \left(dI_t - \bar{h}(\nu_t, A_t, Z_t, Y_t, S_t) dt + \frac{da_t}{\gamma} \right) \right] \end{aligned}$$

where $da_t = \frac{-dA_t^{\nu,d}}{U_{t-}^\nu} \geq 0$.

Moreover, by Lemma 3, we have the following:

$$U_t^\nu = e^{-\gamma Q_{0,t}^\nu} V_t \leq -\beta_t V_t < 0,$$

where we write

$$\beta_t = -e^{-\gamma \left(+\eta \int_0^t (S_u + Z_{u-}) dB_u + \int_0^t |S_u + Z_{u-}| \nu_\infty du + \sigma \int_0^t Y_{t-} dW_u + \sum_{i=a,b} \int_0^t \left\{ \delta^i \xi \left(S_u - \frac{Z_u - Y_{u-}}{Y_{u-} + \delta^i \xi} \right) \right\} dN_u^i \right)}.$$

Therefore, we have the following:

$$\begin{aligned} 0 &\leq \sup_{\nu \in \mathcal{A}} \mathbb{E}^\nu \left[\int_0^T -\beta_t V_t \left(dI_t - \bar{h}(\nu_t, A_t, Z_t, Y_t, S_t) dt + \frac{da_t}{\gamma} \right) \right] \\ &\leq \sup_{\nu \in \mathcal{A}} \mathbb{E}^\nu \left[\int_0^T -\beta_t V_t \left(dI_t - \bar{H}(A_t, Z_t, Y_t, S_t) dt + \frac{da_t}{\gamma} \right) \right]. \end{aligned}$$

Let us denote by Θ the random variable

$$\Theta = \int_0^T -\beta_t V_t \left(dI_t - \bar{H}(A_t, Z_t, Y_t, S_t) dt + \frac{da_t}{\gamma} \right) \leq 0.$$

We then have the following:

$$0 \leq \sup_{\nu \in \mathcal{A}} \mathbb{E}^\nu[\Theta].$$

Let $\varepsilon > 0$. For all $\nu \in \mathcal{A}$, we have the following:

$$\mathbb{E}^\nu[\Theta] = \mathbb{E}^\nu[\Theta \mathbf{1}_{\Theta \leq -\varepsilon} + \Theta \mathbf{1}_{\Theta > -\varepsilon}] \leq -\varepsilon \mathbb{P}^\nu(\Theta \leq -\varepsilon).$$

Therefore,

$$0 \leq \sup_{\nu \in \mathcal{A}} -\varepsilon \mathbb{P}^\nu(\Theta \leq -\varepsilon),$$

which implies that

$$\inf_{\nu \in \mathcal{A}} \mathbb{P}^\nu(\Theta \leq -\varepsilon) = 0.$$

By Lemma 2, this implies that

$$\mathbb{P}(\Theta \leq -\varepsilon) = 0.$$

This is true for all $\varepsilon > 0$, therefore we have that

$$\Theta = \int_0^T -\beta_t V_t \left(dI_t - \bar{H}(A_t, Z_t, Y_t, S_t) dt + \frac{da_t}{\gamma} \right) = 0 \quad \text{a.s.}$$

Since $\beta V > 0$, $dI_t - \bar{H}(A_t, Z_t, Y_t, S_t) dt \geq 0$, and $da_t \geq 0$, this finally implies that

$$dI_t = \bar{H}(A_t, Z_t, Y_t, S_t) dt \quad \text{and} \quad da_t = dA_t^d = 0.$$

It remains to prove that $A \in \Lambda$, that is,

$$\sup_{\nu \in \mathcal{A}} \sup_{t \in \mathfrak{T}} \mathbb{E}^\nu \left[\exp \left\{ -\gamma' P_t \right\} \right] < \infty \quad \text{for some } \gamma' > \gamma.$$

Condition 1 together with Hölder inequality guarantees that there exists $r > 0$ such that

$$\sup_{\nu \in \mathcal{A}} \mathbb{E}^\nu \left[|U_T^\nu|^{r+1} \right] < \infty.$$

Therefore, as U^ν is a negative \mathbb{P}^ν -supermartingale, we get

$$\sup_{\nu \in \mathcal{A}} \sup_{t \in \mathfrak{T}} \mathbb{E}^\nu \left[|U_t^\nu|^{r+1} \right] = \sup_{\nu \in \mathcal{A}} \mathbb{E}^\nu \left[|U_T^\nu|^{r+1} \right] < \infty,$$

which leads to the result by using again Hölder inequality and the fact that $e^{-\gamma P_t} = U_t^\nu e^{\gamma Q_{0,t}^\nu}$. □

Proof of Theorem 2. Let $\mathfrak{R} = P_T^{P_0, A}$, $\tilde{P}_t = P_t^{P_0, A} + Q_{0,t}^\nu$. Then,

$$de^{-\gamma \tilde{P}_t} = \gamma e^{-\gamma \tilde{P}_{t-}} \left\{ (H(A_t) - h(\nu_t, A_t)) dt - A_t^W dW_t - A_t^B dB_t^\nu + \sum_{i \in \{+, -\}} \frac{e^{-\gamma(\Delta_t^i + A_t^i)} - 1}{\gamma} d\tilde{N}_t^i \right\}.$$

We observe that $H(A) \geq h(\nu, A)$. As $A \in \Lambda$, the process $(-e^{-\gamma \tilde{P}_t})_{t \in \mathfrak{T}}$ is a \mathbb{P}^ν -supermartingale. Thus,

$$\mathbb{E}^\nu[-e^{-\gamma \tilde{P}_T}] \leq -e^{-\gamma P_0}.$$

Equality applies if and only if $H(A_t) = h(\nu_t, A_t)$, $dt \otimes d\mathbb{P}$ a.s., that is, $\nu_t = \bar{\nu}(A_t)$. □

Appendix B: Viscosity Properties of the Value Function

In this section, we study the viscosity properties of the value function. We start by showing the local boundedness properties.

B.1 | The Boundedness of the Venue's Value Function

Lemma 4. *There exist constants $\tilde{C}_0, \tilde{C}_1, \tilde{C}_2$ such that*

$$\sup_{\nu \in \mathcal{A}} \mathbb{E}_{t_1, z, y, s}^\nu \left[\int_{t_1}^{t_2} Z_{u-}^2 + Y_{u-}^2 du \right] \leq (t_2 - t_1) e^{\tilde{C}_2(t_2 - t_1)} (y^2 + z^2 + \tilde{C}_0 + \tilde{C}_1 (|s|^2 + |s|^3 + |s|^4)).$$

Proof. Given that Z is a pure jump process, we have that

$$d(Z_t)^2 = Z_{t-}^2 \left\{ \left[\left(\frac{Y_{t-}}{Y_{t-} + \xi} \right)^4 - 1 \right] dN_t^+ + \left[\left(\frac{Y_{t-}}{Y_{t-} - \xi} \right)^4 - 1 \right] dN_t^- \right\}.$$

Thus, for $t_1, t_2 \in [0, T]$, $t_1 < t_2$, we have the following:

$$\begin{aligned} & \mathbb{E}_{t_1, z, y, s}^\nu \left[Z_{t_2}^2 \right] \\ &= z^2 + \mathbb{E}_{t_1, z, y, s}^\nu \left[\int_{t_1}^{t_2} Z_{u-}^2 \left\{ \left(\left(\frac{Y_{u-}}{Y_{u-} + \xi} \right)^4 - 1 \right) \bar{\lambda}^+(Z_{u-}, Y_{u-}, S_u) \right. \right. \\ & \quad \left. \left. + \left(\left(\frac{Y_{u-}}{Y_{u-} - \xi} \right)^4 - 1 \right) \bar{\lambda}^-(Z_{u-}, Y_{u-}, S_u) \mathbb{1}_{Y_{u-} > 2\xi} \right\} du \right] \\ &= z^2 + \mathbb{E}_{t_1, z, y, s}^\nu \left[\int_{t_1}^{t_2} Z_{u-}^2 \left\{ \left(\left(\frac{Y_{u-}}{Y_{u-} + \xi} \right)^4 - 1 \right) \bar{\lambda}^+(Z_{u-}, Y_{u-}, S_u) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{|Z_{u-} - S_u| \leq \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right. \right. \\ & \quad + \left(\left(\frac{Y_{u-}}{Y_{u-} + \xi} \right)^4 - 1 \right) \bar{\lambda}^+(Z_{u-}, Y_{u-}, S_u) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{Z_{u-} - S_u > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \\ & \quad + \left(\left(\frac{Y_{u-}}{Y_{u-} + \xi} \right)^4 - 1 \right) \bar{\lambda}^+(Z_{u-}, Y_{u-}, S_u) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{S_u - Z_{u-} > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \\ & \quad + \left(\left(\frac{Y_{u-}}{Y_{u-} - \xi} \right)^4 - 1 \right) \bar{\lambda}^-(Z_{u-}, Y_{u-}, S_u) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{|Z_{u-} - S_u| \leq \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \\ & \quad + \left(\left(\frac{Y_{u-}}{Y_{u-} - \xi} \right)^4 - 1 \right) \bar{\lambda}^-(Z_{u-}, Y_{u-}, S_u) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{Z_{u-} - S_u > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \\ & \quad \left. \left. + \left(\left(\frac{Y_{u-}}{Y_{u-} - \xi} \right)^4 - 1 \right) \bar{\lambda}^-(Z_{u-}, Y_{u-}, S_u) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{S_u - Z_{u-} > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right\} du \right]. \end{aligned}$$

Observe that on the event $\{Y_{u-} > 2\xi\}$, the following expressions,

$$Y_{u-} \left(\left(\frac{Y_{u-}}{Y_{u-} \pm \xi} \right)^4 - 1 \right) = \frac{-Y_{u-} (\pm 4 Y_{u-}^3 \xi + 6 Y_{u-}^2 \xi^2 \pm 4 Y_{u-} \xi^3 + \xi^4)}{(Y_{u-} \pm \xi)^4},$$

$$\left(\frac{Y_{u-}}{Y_{u-} \pm \xi} \right)^4 - 1, \quad \text{and} \quad \frac{a_1}{Y_{u-}} + a_2,$$

are all bounded by a deterministic constant C_1 .

Observe that if $|Z_{u-} - S_u| \leq \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}$,¹⁴ then $\bar{\lambda}^\pm((Z_{u-}, Y_{u-}, S_u)) = a_1 + a_2 Y_{u-} \pm a_3 (Z_{u-} - S_{u-})$. Thus,

$$\begin{aligned} & \int_{t_1}^{t_2} Z_{u-}^2 \left\{ \left(\left(\frac{Y_{u-}}{Y_{u-} + \xi} \right)^4 - 1 \right) \bar{\lambda}^+(Z_{u-}, Y_{u-}, S_u) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{|Z_{u-} - S_u| \leq \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right. \\ & \quad \left. + \left(\left(\frac{Y_{u-}}{Y_{u-} - \xi} \right)^4 - 1 \right) \bar{\lambda}^-(Z_{u-}, Y_{u-}, S_u) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{|Z_{u-} - S_u| \leq \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right\} du \\ & \leq \int_{t_1}^{t_2} Z_{u-}^2 \left\{ C_1 \left(\frac{a_0}{Y_{u-}} + a_2 + \frac{a_3 (Z_{u-} - S_u)}{Y_{u-}} \right) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{|Z_{u-} - S_u| \leq \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right. \\ & \quad \left. + C_1 \left(\frac{a_0}{Y_{u-}} + a_2 - \frac{a_3 (Z_{u-} - S_u)}{Y_{u-}} \right) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{|Z_{u-} - S_u| \leq \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right\} du \\ & = \int_{t_1}^{t_2} 2 Z_{u-}^2 \left\{ C_1 \left(\frac{a_0}{Y_{u-}} + a_2 \right) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{|Z_{u-} - S_u| \leq \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right\} du \\ & \leq 2 C_1^2 \int_{t_1}^{t_2} Z_{u-}^2 du. \end{aligned}$$

If $Z_{u-} - S_u > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}$, then

$$\begin{aligned} \bar{\lambda}^+(Z_{u-}, Y_{u-}, S_u) &= a_1 + a_2 Y_{u-} + a_3 (Z_{u-} - S_{u-}), \\ \bar{\lambda}^-(Z_{u-}, Y_{u-}, S_u) &= a_0, \end{aligned}$$

thus,

$$\begin{aligned} & \int_{t_1}^{t_2} Z_{u-}^2 \left\{ \left(\left(\frac{Y_{u-}}{Y_{u-} - \xi} \right)^4 - 1 \right) \bar{\lambda}^-(Z_{u-}, Y_{u-}, S_u) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{Z_{u-} - S_u > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right\} du \\ & \leq a_0 C_1 \int_{t_1}^{t_2} Z_{u-}^2 du. \end{aligned}$$

Otherwise, if $S_u - Z_{u-} > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}$, then

$$\begin{aligned} \bar{\lambda}^+(Z_{u-}, Y_{u-}, S_u) &= a_0, \\ \bar{\lambda}^-(Z_{u-}, Y_{u-}, S_u) &= a_1 + a_2 Y_{u-} - a_3 (Z_{u-} - S_{u-}), \end{aligned}$$

thus,

$$\begin{aligned} & \int_{t_1}^{t_2} Z_{u-}^2 \left\{ \left(\left(\frac{Y_{u-}}{Y_{u-} + \xi} \right)^4 - 1 \right) \bar{\lambda}^+(Z_{u-}, Y_{u-}, S_u) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{S_u - Z_{u-} > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right\} du \\ & \leq a_0 C_1 \int_{t_1}^{t_2} Z_{u-}^2 du. \end{aligned}$$

Moreover, we have the following:

$$\int_{t_1}^{t_2} Z_{u-}^2 \left\{ \left(\left(\frac{Y_{u-}}{Y_{u-} + \xi} \right)^4 - 1 \right) \bar{\lambda}^+(Z_{u-}, Y_{u-}, S_u) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{Z_{u-} - S_u > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right\} du$$

$$\begin{aligned}
& + \left(\left(\frac{Y_{u-}}{Y_{u-} - \xi} \right)^4 - 1 \right) \tilde{\lambda}^-(Z_{u-}, Y_{u-}, S_u) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{S_u - Z_{u-} > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \Big\} du \\
= & \int_{t_1}^{t_2} Z_{u-}^2 \left\{ \left(\left(\frac{Y_{u-}}{Y_{u-} + \xi} \right)^4 - 1 \right) (a_1 + a_2 Y_{u-} + a_3 (Z_{u-} - S_u)) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{Z_{u-} - S_u > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right. \\
& \left. + \left(\left(\frac{Y_{u-}}{Y_{u-} - \xi} \right)^4 - 1 \right) (a_1 + a_2 Y_{u-} + a_3 (S_u - Z_{u-})) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{S_u - Z_{u-} > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right\} du \\
\leq & 4 C_1 \int_{t_1}^{t_2} Z_{u-}^2 du + a_3 \int_{t_1}^{t_2} Z_{u-}^2 \left\{ \left(\left(\frac{Y_{u-}}{Y_{u-} + \xi} \right)^4 - 1 \right) (Z_{u-} - S_u) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{Z_{u-} - S_u > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right. \\
& \left. + \left(\left(\frac{Y_{u-}}{Y_{u-} - \xi} \right)^4 - 1 \right) (S_u - Z_{u-}) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{S_u - Z_{u-} > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right\} du.
\end{aligned}$$

Next, we have that $\left(\frac{Y_{u-}}{Y_{u-} + \xi}\right)^4 - 1$ is negative. Moreover, on the event that $S_u - Z_{u-} > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}$, we have $Z_{u-} < S_u - \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}$. On this event, we cannot have $Z_{u-} \leq 0$ as we are considering the case $Y_{u-} > 2\xi$, so $Z_{u-}^2 < \left(S_u - \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}\right)^2$. Thus,

$$\begin{aligned}
& \int_{t_1}^{t_2} Z_{u-}^2 \left\{ \left(\left(\frac{Y_{u-}}{Y_{u-} + \xi} \right)^4 - 1 \right) (Z_{u-} - S_u) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{Z_{u-} - S_u > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right. \\
& \quad \left. + \left(\left(\frac{Y_{u-}}{Y_{u-} - \xi} \right)^4 - 1 \right) (S_u - Z_{u-}) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{S_u - Z_{u-} > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right\} du \\
\leq & \int_{t_1}^{t_2} Z_{u-}^2 \left\{ \left(\left(\frac{Y_{u-}}{Y_{u-} + \xi} \right)^4 - 1 \right) \left(\frac{a_1 - a_0 + a_2 Y_{u-}}{a_3} \right) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{Z_{u-} - S_u > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right. \\
& \quad \left. + \left(\left(\frac{Y_{u-}}{Y_{u-} - \xi} \right)^4 - 1 \right) (S_u - Z_{u-}) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{S_u - Z_{u-} > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right\} du \\
\leq & \left(\frac{a_1 - a_0}{a_3} \right) C_1 \int_{t_1}^{t_2} Z_{u-}^2 du + \left(\frac{a_2}{a_3} \right) C_1 \int_{t_1}^{t_2} Z_{u-}^2 du \\
& + \int_{t_1}^{t_2} \left(S_u - \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3} \right)^2 \left\{ \left(\left(\frac{Y_{u-}}{Y_{u-} - \xi} \right)^4 - 1 \right) (S_u - Z_{u-}) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{S_u - Z_{u-} > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right\} du \\
\leq & \left(\frac{a_1 - a_0}{a_3} \right) C_1 \int_{t_1}^{t_2} Z_{u-}^2 du + \left(\frac{a_2}{a_3} \right) C_1 \int_{t_1}^{t_2} Z_{u-}^2 du \\
& + C_1 \int_{t_1}^{t_2} |S_u|^3 + S_u^2 |Z_{u-}| du + \frac{C_1 a_2^2}{a_3^2} \int_{t_1}^{t_2} |Y_{u-}| (|S_u| + |Z_{u-}|) du \\
& + \frac{C_1 (a_1 - a_0)^2}{a_3^2} \int_{t_1}^{t_2} |S_u| + |Z_{u-}| du + \frac{2C_1 (a_1 - a_0)}{a_3} \int_{t_1}^{t_2} S_u^2 + |S_u Z_{u-}| du \\
& + \frac{2C_1 a_2}{a_3} \int_{t_1}^{t_2} S_u^2 + |S_u Z_{u-}| du + \frac{2C_1 a_2 (a_1 - a_0)}{a_3} \int_{t_1}^{t_2} |S_u| + |Z_{u-}| du.
\end{aligned}$$

Using that S is a scaled Brownian motion for any admissible measure $\nu \in \mathcal{A}$, we have that

$$\sup_{\nu \in \mathcal{A}} \mathbb{E}_{t_1, z, y, s}^\nu \left[\sup_{u \in [t_1, t_2]} |S_u|^p \right] \leq M_p (|s|^p + 1)$$

for a deterministic constant M_p . By applying the inequality of arithmetic and geometric means, there exist constants C_2, C_3, C_4, C_5 (which do not depend on the measure \mathbb{P}^ν) such that

$$\begin{aligned}
& \mathbb{E}_{t_1, z, y, s}^\nu \left[\int_{t_1}^{t_2} Z_{u-}^2 \left\{ \left(\left(\frac{Y_{u-}}{Y_{u-} + \xi} \right)^4 - 1 \right) (Z_{u-} - S_u) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{Z_{u-} - S_u > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right. \right. \\
& \quad \left. \left. + \left(\left(\frac{Y_{u-}}{Y_{u-} - \xi} \right)^4 - 1 \right) (S_u - Z_{u-}) \mathbb{1}_{Y_{u-} > 2\xi} \mathbb{1}_{S_u - Z_{u-} > \frac{a_1 - a_0 + a_2 Y_{u-}}{a_3}} \right\} du \right]
\end{aligned}$$

$$\leq C_2 + C_3 (|s|^2 + |s|^3 + |s|^4) + C_4 \mathbb{E}_{s,z,y,s}^\nu \left[\int_{t_1}^{t_2} Z_{u-}^2 du \right] + C_5 \mathbb{E}_{s,z,y,s}^\nu \left[\int_{t_1}^{t_2} Y_{u-}^2 du \right].$$

By adding the parts, there exists constants C_6, C_7, C, C_9 (which do not depend on the measure \mathbb{P}^ν) such that

$$\mathbb{E}_{t_1,z,y,s}^\nu [Z_{t_2}^2] \leq C_6 + C_7 (|s|^2 + |s|^3 + |s|^4) + C_8 \mathbb{E}_{s,z,y,s}^\nu \left[\int_{t_1}^{t_2} Z_{u-}^2 du \right] + C_9 \mathbb{E}_{s,z,y,s}^\nu \left[\int_{t_1}^{t_2} Y_{u-}^2 du \right].$$

Next, we bound $\mathbb{E}_{t,z,y,s}^\nu [Y_t^2]$. By Ito's formula,

$$\begin{aligned} dY_t^2 &= 2Y_{t-} dY_t^c + d\langle Y^c \rangle_t + ((Y_{t-} + \xi)^2 - Y_{t-}^2) dN_t^+ + ((Y_{t-} - \xi)^2 - Y_{t-}^2) dN_t^- \\ &= 2Y_{t-} \nu_t dt + 2\eta Y_{t-} dB_t^\nu + \eta^2 dt + \xi(2Y_{t-} + \xi) dN_t^+ + \xi(-2Y_{t-} + \xi) dN_t^-, \end{aligned}$$

thus,

$$\begin{aligned} \mathbb{E}_{t_1,z,y,s}^\nu [Y_{t_2}^2] &= y^2 + \mathbb{E}_{t_1,z,y,s}^\nu \left[\int_{t_1}^{t_2} 2Y_{u-} \nu_u du \right] + \mathbb{E}_{t_1,z,y,s}^\nu \left[\int_{t_1}^{t_2} 2\eta Y_{u-} dB_u^\nu \right] + \eta^2 (t_2 - t_1) \\ &\quad + \mathbb{E}_{t_1,z,y,s}^\nu \left[\int_{t_1}^{t_2} \xi(2Y_{u-} + \xi) dN_u^+ \right] + \mathbb{E}_{t_1,z,y,s}^\nu \left[\int_{t_1}^{t_2} \xi(-2Y_{u-} + \xi) dN_u^- \right] \\ &\leq y^2 + \mathbb{E}_{t_1,z,y,s}^\nu \left[\int_{t_1}^{t_2} Y_{u-}^2 du \right] + \mathbb{E}_{t_1,z,y,s}^\nu \left[\int_{t_1}^{t_2} \nu_\infty^2 du \right] + \eta^2 (t_2 - t_1) \\ &\quad + 2\xi^2 \mathbb{E}_{t_1,z,y,s}^\nu \left[\int_{t_1}^{t_2} a_1 + a_2 |Y_{u-}| + a_3 |Z_{u-} - S_u| du \right] + 2\xi \mathbb{E}_{t_1,z,y,s}^\nu \left[\int_{t_1}^{t_2} Y_{u-}^2 du \right] \\ &\quad + 2\xi \mathbb{E}_{t_1,z,y,s}^\nu \left[\int_{t_1}^{t_2} (a_1 + a_2 |Y_{u-}| + a_3 |Z_{u-} - S_u|)^2 du \right] \\ &\leq y^2 + \mathbb{E}_{t_1,z,y,s}^\nu \left[\int_{t_1}^{t_2} Y_{u-}^2 du \right] + \nu_\infty^2 (t_2 - t_1) + \eta^2 (t_2 - t_1) \\ &\quad + 2\xi^2 \mathbb{E}_{t_1,z,y,s}^\nu \left[\int_{t_1}^{t_2} a_1 + a_2 |Y_{u-}| + a_3 |Z_{u-} - S_u| du \right] + 2\xi \mathbb{E}_{t_1,z,y,s}^\nu \left[\int_{t_1}^{t_2} Y_{u-}^2 du \right] \\ &\quad + 6\xi \mathbb{E}_{t_1,z,y,s}^\nu \left[\int_{t_1}^{t_2} a_1^2 + a_2^2 |Y_{u-}|^2 + a_3^2 |Z_{u-} - S_u|^2 du \right]. \end{aligned}$$

Then, there exist deterministic constants $\tilde{C}_0, \tilde{C}_1, \tilde{C}_2$ (independent of t_1, t_2 and measure \mathbb{P}^ν) such that

$$\mathbb{E}_{t_1,z,y,z}^\nu [Y_{t_2}^2 + Z_{t_2}^2] \leq y^2 + z^2 + \tilde{C}_0 + \tilde{C}_1 (|s|^2 + |s|^3 + |s|^4) + \tilde{C}_2 \mathbb{E}_{t_1,z,y,z}^\nu \left[\int_{t_1}^{t_2} Y_{u-}^2 + Z_{u-}^2 du \right].$$

By Grönwall's inequality, we have the following:

$$\mathbb{E}_{t_1,z,y,z}^\nu [Y_{t_2}^2 + Z_{t_2}^2] \leq (y^2 + z^2 + \tilde{C}_0 + \tilde{C}_1 (|s|^2 + |s|^3 + |s|^4)) e^{\tilde{C}_2 (t_2 - t_1)},$$

thus,

$$\sup_{v \in \mathcal{A}} \mathbb{E}_{t_1,z,y,s}^\nu \left[\int_{t_1}^{t_2} Z_{u-}^2 + Y_{u-}^2 du \right] \leq (t_2 - t_1) e^{\tilde{C}_2 (t_2 - t_1)} (y^2 + z^2 + \tilde{C}_0 + \tilde{C}_1 (|s|^2 + |s|^3 + |s|^4)), \quad (\text{B.1})$$

which concludes the proof. \square

Corollary 1. *The function $v : [0, T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ as defined in Equation (4.1) is locally bounded. Moreover, there exists deterministic constants \hat{C}_0, \hat{C}_1 such that*

$$|v(t, z, y, s)| \leq (T - t) e^{\hat{C}_1 (T - t)} \hat{C}_0 (1 + |y|^2 + |z|^2 + |s|^4). \quad (\text{B.2})$$

Proof. Observe that for any $A \in \Lambda$, we have the following:

$$\begin{aligned}
& \mathbb{E}_{t,z,y,s}^{v^*(P_T^{P_0,A})} \left[\int_t^T \sum_{i \in \{+, -\}} \frac{\lambda_s^i (1 - e^{-\gamma(A_s^i + \Delta_s^i)})}{\gamma} ds - \sum_{i \in \{+, -\}} \int_t^T A_s^i d\tilde{N}_s^i \right] \\
&= \mathbb{E}_{t,z,y,s}^{v^*(P_T^{P_0,A})} \left[\int_t^T \sum_{i \in \{+, -\}} \lambda_s^i \mathbb{1}_{Y_{s-} > 2\xi} \left(-A_s^i + \frac{1 - e^{-\gamma(A_s^i + \Delta_s^i)}}{\gamma} \right) ds \right] \\
&\quad + \mathbb{E}_{t,z,y,s}^{v^*(P_T^{P_0,A})} \left[\int_t^T \sum_{i \in \{+, -\}} \lambda_s^i \mathbb{1}_{Y_{s-} \leq 2\xi} \left(-A_s^i + \frac{1 - e^{-\gamma A_s^i}}{\gamma} \right) ds \right] \\
&\leq \mathbb{E}_{t,z,y,s}^{v^*(P_T^{P_0,A})} \left[\int_t^T \sum_{i \in \{+, -\}} \lambda_s^i \mathbb{1}_{Y_{s-} > 2\xi} \Delta_s^i ds \right],
\end{aligned}$$

where equality occurs if and only if $A_s^i = \Delta_s^i \mathbb{1}_{Y_{s-} > 2\xi} = \Delta_s^i$. Moreover, the choice of A^i does not affect the measure $\mathbb{P}^{v^*(P_T^{P_0,A})}$ as the measure depends only on A^B . Thus, to maximize the venue's performance criterion, they have to choose $A_s^i = \Delta_s^i \mathbb{1}_{Y_{s-} > 2\xi}$.

By substituting this choice for A^i , expanding the terms in Equation (4.1) and applying Lemma 4, we obtain that v is locally bounded. Moreover, the bound in Equation (B.2) is obtained from Equation (B.1). \square

B.2 | Viscosity Property of the Venue's Value Function

Next, we define v_* as the lower semicontinuous envelope of v (respectively v^* as the upper semi continuous envelope), that is,

$$v_*(t, z, y, s) := \liminf_{(t', z', y', s') \rightarrow (t, z, y, s)} v(t', z', y', s'), \quad (\text{B.3})$$

$$v^*(t, z, y, s) := \limsup_{(t', z', y', s') \rightarrow (t, z, y, s)} v(t', z', y', s'). \quad (\text{B.4})$$

Let us define $\mathcal{O} = \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ and $\tilde{\alpha} : \mathcal{O} \times \mathbb{R} \rightarrow \mathbb{R}$,

$$\tilde{\alpha}(z, y, s, p_2) := \frac{\frac{p_2}{a\eta} - 2(s+z)\gamma\eta}{2\gamma + \frac{1}{a\eta^2}}, \quad \forall (z, y, s) \in \mathcal{O}, p_2 \in \mathbb{R},$$

and $I : [0, T] \times \mathcal{O} \times C^2(\mathbb{R}^3) \times \mathbb{R}$,

$$I(z, y, s; \varphi; u) := \sum_{i \in \{+, -\}} \tilde{\lambda}^i(z, y, s) [\mathbb{1}_{y > 2\xi} (\varphi(z^i, y^i, s) - u) + \mathbb{1}_{y \leq 2\xi} (\varphi(z, y, s) - u)],$$

where $y^\pm = y \pm \xi$ and $z^i = z y^2 / (y^i)^2$, $i \in \{+, -\}$.

Let \mathbb{S}^N be the space of $N \times N$ symmetric matrices and $F : [0, T] \times \mathcal{O} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{S}^3 \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\begin{aligned}
& F(t, z, y, s, u, q, (p_i)_{i=1,2,3}, (A_{i,j})_{i,j=1,2,3}, I) \\
&:= -I + \sum_{i \in \{+, -\}} \tilde{\lambda}^i(z, y, s) \mathbb{1}_{y > 2\xi} (\mathbf{r} + \Delta^i) \\
&\quad - \left(-\frac{1}{2} \gamma \eta^2 s^2 - \gamma \eta^2 s z - \frac{1}{2} \gamma \eta^2 z^2 - \frac{1}{2} \gamma \sigma^2 y^2 + \frac{1}{2} \sigma^2 A_{3,3} + \frac{1}{2} \eta^2 A_{2,2} + q \right) - \frac{\gamma}{2} \sigma^2 y^2 \\
&\quad - \left(\frac{\gamma}{2} + \frac{1}{4a\eta^2} \right) (\tilde{\alpha}(z, y, s, p_2))^2 \mathbb{1}_{|\tilde{\alpha}(z, y, s, p_2)| \leq 2a\eta v_\infty} \\
&\quad - \left(\frac{\gamma}{2} (s+z)^2 \eta^2 - a v_\infty^2 + p_2 v_\infty \right) \mathbb{1}_{|\tilde{\alpha}(z, y, s, p_2)| > 2a\eta v_\infty} \mathbb{1}_{-(s+z)\eta > 2a\eta v_\infty} \\
&\quad - \left(-2\gamma a^2 \eta^2 v_\infty^2 - 2(s+z)\gamma a \eta^2 v_\infty - a v_\infty^2 + p_2 v_\infty \right) \mathbb{1}_{|\tilde{\alpha}(z, y, s, p_2)| > 2a\eta v_\infty} \mathbb{1}_{|(s+z)\eta| \leq 2a\eta v_\infty} \\
&\quad - \left(\frac{\gamma}{2} (s+z)^2 \eta^2 - a v_\infty^2 - p_2 v_\infty \right) \mathbb{1}_{|\tilde{\alpha}(z, y, s, p_2)| > 2a\eta v_\infty} \mathbb{1}_{-(s+z)\eta < -2a\eta v_\infty} \\
&\quad - \left(-2\gamma a^2 \eta^2 v_\infty^2 + 2(s+z)\gamma a \eta^2 v_\infty - a v_\infty^2 - p_2 v_\infty \right) \mathbb{1}_{|\tilde{\alpha}(z, y, s, p_2)| < -2a\eta v_\infty} \mathbb{1}_{|(s+z)\eta| \leq -2a\eta v_\infty}.
\end{aligned}$$

For $\varphi \in C^2(\mathfrak{X} \times \mathcal{O}; \mathbb{R})$, we define the operator,

$$F[\varphi](t, z, y, s) := F(t, z, y, s, \varphi(t, z, y, s), \partial_t \varphi(t, z, y, s), D\varphi(t, z, y, s), D^2\varphi(t, z, y, s), I(z, y, s; \varphi(t, \cdot); \varphi(t, z, y, s))).$$

We observe that

$$(t, z, y, s, u, q, (p_i)_{i=1,2,3}, (A_{i,j})_{i,j=1,2,3}, I) \mapsto F(t, z, y, s, u, q, (p_i)_{i=1,2,3}, (A_{i,j})_{i,j=1,2,3}, I)$$

is continuous on the following boundary cases:

- i. $-(s+z)\eta = 2\alpha\eta\nu_\infty$ and $\tilde{\alpha}(z, y, s, p_2) \geq 2\alpha\eta\nu_\infty$,
- ii. $-(s+z)\eta = -2\alpha\eta\nu_\infty$ and $\tilde{\alpha}(z, y, s, p_2) \leq -2\alpha\eta\nu_\infty$,
- iii. $\tilde{\alpha}(z, y, s, p_2) = 2\alpha\eta\nu_\infty$ and $-(s+z)\eta < 2\alpha\eta\nu_\infty$,
- iv. $\tilde{\alpha}(z, y, s, p_2) = -2\alpha\eta\nu_\infty$ and $-(s+z)\eta > -2\alpha\eta\nu_\infty$.

However, F is not continuous on the following boundary cases:

- i. $y = 2\xi$, and
- ii. $|\tilde{\alpha}(z, y, s, p_2)| = 2\alpha\eta\nu_\infty$ and $|(s+z)\eta| > 2\alpha\eta\nu_\infty$ (source of discontinuity: deciding $(A^B)^* = \tilde{\alpha}(z, y, s, \partial_y v)$ or $-(s+z)\eta$).

Thus, we are working with a possibly discontinuous viscosity solution. As such, we are going to use the definition of viscosity subsolution and supersolution from Barles (2013) (Section 6.1) as follows.

Definition 1. A locally bounded upper semicontinuous function u^* is a discontinuous viscosity subsolution of the HJB equation (4.2) if for any $\varphi \in C^{1,2}(\mathfrak{X} \times \mathcal{O}; \mathbb{R})$ and $0 = (u^* - \varphi)(\hat{t}, \hat{z}, \hat{y}, \hat{s}) = \max_{(t,z,y,s) \in \mathfrak{X} \times \mathcal{O}} (u^* - \varphi)(t, z, y, s)$, we have the following:

$$F_*[\varphi](\hat{t}, \hat{z}, \hat{y}, \hat{s}) := \liminf_{(t,z,y,s) \rightarrow (\hat{t}, \hat{z}, \hat{y}, \hat{s})} F[\varphi](t, z, y, s) \leq 0.$$

Similarly, a locally bounded lower semicontinuous function u_* is a discontinuous viscosity supersolution of the HJB equation (4.2) if for any $\varphi \in C^{1,2}(\mathfrak{X} \times \mathcal{O}; \mathbb{R})$ and $0 = (u_* - \varphi)(\hat{t}, \hat{z}, \hat{y}, \hat{s}) = \min_{(t,z,y,s) \in \mathfrak{X} \times \mathcal{O}} (u_* - \varphi)(t, z, y, s)$, we have the following:

$$F^*[\varphi](\hat{t}, \hat{z}, \hat{y}, \hat{s}) := \limsup_{(t,z,y,s) \rightarrow (\hat{t}, \hat{z}, \hat{y}, \hat{s})} F[\varphi](t, z, y, s) \geq 0.$$

We can rewrite our function F as follows:

$$\begin{aligned} & F(t, z, y, s, u, q, (p_i)_{i=1,2,3}, (A_{i,j})_{i,j=1,2,3}, I) \\ & := -I + \sum_{i \in \{+, -\}} \tilde{\lambda}^i(z, y, s) (\mathbf{r} + \Delta^i) \mathbb{1}_{y > 2\xi} \\ & \quad - \left(-\frac{1}{2} \gamma \eta^2 s^2 - \gamma \eta^2 s z - \frac{1}{2} \gamma \eta^2 z^2 - \frac{1}{2} \gamma \sigma^2 y^2 + \frac{1}{2} \sigma^2 A_{3,3} + \frac{1}{2} \eta^2 A_{2,2} + q \right) - \frac{\gamma}{2} \sigma^2 y^2 \\ & \quad - \left[-\frac{\gamma}{2} (\tilde{\alpha}^*(z, y, s, p_2))^2 - \frac{(\tilde{\alpha}^*(z, y, s, p_2))^2}{4\alpha\eta^2} \right. \\ & \quad \quad \left. - (s+z)\gamma\eta\tilde{\alpha}^*(z, y, s, p_2) + \frac{p_2\tilde{\alpha}^*(z, y, s, p_2)}{2\alpha\eta} \right] \mathbb{1}_{|s+z|\eta \leq 2\alpha\eta\nu_\infty} \\ & \quad - \left[-\frac{\gamma}{2} (\tilde{\alpha}(z, y, s, p_2))^2 - \frac{(\tilde{\alpha}(z, y, s, p_2))^2}{4\alpha\eta^2} \right. \\ & \quad \quad \left. - (s+z)\gamma\eta\tilde{\alpha}(z, y, s, p_2) + \frac{p_2\tilde{\alpha}(z, y, s, p_2)}{2\alpha\eta} \right] \mathbb{1}_{|s+z|\eta > 2\alpha\eta\nu_\infty} \mathbb{1}_{|\tilde{\alpha}(z, y, s, p_2)| \leq 2\alpha\eta\nu_\infty} \\ & \quad - \left[\frac{\gamma}{2} (s+z)^2 \eta^2 - \alpha\nu_\infty^2 + p_2\nu_\infty \right] \mathbb{1}_{-(s+z)\eta > 2\alpha\eta\nu_\infty} \mathbb{1}_{|\tilde{\alpha}(z, y, s, p_2)| > 2\alpha\eta\nu_\infty} \\ & \quad - \left[\frac{\gamma}{2} (s+z)^2 \eta^2 - \alpha\nu_\infty^2 - p_2\nu_\infty \right] \mathbb{1}_{-(s+z)\eta < -2\alpha\eta\nu_\infty} \mathbb{1}_{|\tilde{\alpha}(z, y, s, p_2)| > 2\alpha\eta\nu_\infty}. \end{aligned}$$

where

$$\tilde{\alpha}^*(z, y, s, p_2) := (\tilde{\alpha}(z, y, s, p_2) \vee 2\mathbf{a}\eta\nu_{-\infty}) \wedge 2\mathbf{a}\eta\nu_{\infty}.$$

For $\varphi \in C^2(\mathfrak{X} \times \mathcal{O}; \mathbb{R})$, we define the operator

$$\begin{aligned} F_A[\varphi](t, z, y, s) &:= - \left\{ \bar{\lambda}^-(z, y, s) [\mathbf{r} + \varphi(t, z^-, y - \xi, s) - \varphi(t, z, y, s)] \mathbb{1}_{y > 2\xi} \right. \\ &\quad + \bar{\lambda}^+(z, y, s) [\mathbf{r} + \varphi(t, z^+, y + \xi, s) - \varphi(t, z, y, s)] \mathbb{1}_{y > 2\xi} \\ &\quad + \frac{1}{4} [-2\gamma\eta^2 s^2 - 4\gamma\eta^2 s z - 2\gamma\eta^2 z^2 - 2\gamma\sigma^2 y^2 + 2\sigma^2 \partial_{SS}\varphi + 2\eta^2 \partial_{YY}\varphi + 4\partial_t\varphi] \\ &\quad + \left[-\frac{\gamma}{2} (A^W)^2 - y\gamma\sigma A^W \right] \\ &\quad + \left[-\frac{\gamma}{2} (A^B)^2 + h(\bar{v}(A), A) - \frac{A^B \bar{v}(A)}{\eta} - (s+z)\gamma\eta A^B + \partial_Y\varphi \bar{v}(A) \right] \\ &\quad + \bar{\lambda}^-(z, y, s) \left[-A^- + \frac{1 - e^{-\gamma(A^- + \Delta^-)}}{\gamma} \right] \mathbb{1}_{y > 2\xi} + \bar{\lambda}^- \left[-A^- + \frac{1 - e^{-\gamma A^-}}{\gamma} \right] \mathbb{1}_{y \leq 2\xi} \\ &\quad + \bar{\lambda}^+(z, y, s) \left[-A^+ + \frac{1 - e^{-\gamma(A^+ + \Delta^+)}}{\gamma} \right] \mathbb{1}_{y > 2\xi} + \bar{\lambda}^+ \left[-A^+ + \frac{1 - e^{-\gamma A^+}}{\gamma} \right] \mathbb{1}_{y \leq 2\xi} \left. \right\}. \end{aligned}$$

Observe that

$$-F[\varphi](t, z, y, s) = \sup_A -F_A[\varphi](t, z, y, s) = -\inf_A F_A[\varphi](t, z, y, s). \quad (\text{B.5})$$

We will prove that v is the solution to the PDE (4.2) in the (discontinuous) viscosity sense.

Proposition 1. *The function v^* as defined in Equation (B.4) is a subsolution to Equation (4.2) in $[0, T) \times \mathcal{O}$ with $v^*(T, \cdot) \leq 0$.*

Proof. First, we will show that $v^*(T, \cdot) \leq 0$. Let us take $((t_m, z_m, y_m, s_m))_{m=1}^{\infty} \subset [0, T) \times \mathcal{O}$ such that $(t_m, z_m, y_m, s_m) \rightarrow (T, z, y, s)$ and $v(t_m, z_m, y_m, s_m) \rightarrow v^*(T, z, y, s)$. Thus, from the bound of v in Equation (B.2)

$$\begin{aligned} v^*(T, z, y, s) &= \lim_{m \rightarrow \infty} v(t_m, z_m, y_m, s_m) \\ &\leq \lim_{m \rightarrow \infty} (T - t_m) e^{\hat{C}_1(T-t_m)} \hat{C}_0 (1 + |y_m|^2 + |z_m|^2 + |s_m|^4) \\ &= 0. \end{aligned}$$

Let $\varphi \in C^{1,2}(\mathfrak{X} \times \mathcal{O}; \mathbb{R})$ such that $0 = \max_{\mathfrak{X} \times \mathcal{O}} (v^* - \varphi) = (v^* - \varphi)(\hat{t}, \hat{z}, \hat{y}, \hat{s})$ with $(\hat{t}, \hat{z}, \hat{y}, \hat{s}) \in \mathfrak{X} \times \mathcal{O}$ is the unique maximizer of $v^* - \varphi$ and $\hat{t} < T$. We wish to prove that

$$F_*[\varphi](\hat{t}, \hat{z}, \hat{y}, \hat{s}) \leq 0.$$

We proceed by contradiction, that is, assume there exists $\rho > 0$ such that

$$F_*[\varphi](\hat{t}, \hat{z}, \hat{y}, \hat{s}) > 2\rho,$$

which implies

$$F(\hat{t}, \hat{z}, \hat{y}, \hat{s}, \varphi(\hat{t}, \hat{z}, \hat{y}, \hat{s}), \partial_t \varphi(\hat{t}, \hat{z}, \hat{y}, \hat{s}), D\varphi(\hat{t}, \hat{z}, \hat{y}, \hat{s}), D^2\varphi(\hat{t}, \hat{z}, \hat{y}, \hat{s}), I(\hat{z}, \hat{y}, \hat{s}; \varphi(\hat{t}, \cdot); \varphi(\hat{t}, \hat{z}, \hat{y}, \hat{s}))) > 2\rho.$$

By the lower semicontinuity of F_* , there exists an open neighborhood B_0 of $(\hat{t}, \hat{z}, \hat{y}, \hat{s})$ such that

$$F_*[\varphi](t, z, y, s) > \rho,$$

which implies

$$F(t, z, y, s, \varphi(t, z, y, s), \partial_t \varphi(t, z, y, s), D\varphi(t, z, y, s), D^2\varphi(t, z, y, s), I(z, y, s; \varphi(t, \cdot); \varphi(t, z, y, s))) > \rho,$$

for all $(t, z, y, s) \in B_0$. Let \bar{B}_0 be the closure of B_0 . Because $(\hat{t}, \hat{z}, \hat{y}, \hat{s})$ is an interior point of \bar{B}_0 and the unique maximizer of $v^* - \varphi$, there exists $\rho_1 > 0$ such that $v^* - \varphi < -\rho_1$ on $\partial\bar{B}_0$, that is, the boundary of \bar{B}_0 . Moreover, because of the uniqueness of the maximizer and the upper semicontinuity of $v^* - \varphi$, there exists $\rho_2 > 0$ such that $v^* - \varphi < -\rho_2$ in $\bar{B}_0 \setminus B_0$, where \bar{B}_0 is the translation of $B_0 \ni (t, z, y, s) \mapsto (t, z^\pm, y^\pm, s) \in \bar{B}_0$.

Next, we define $\bar{L} := \{(T, z, y, s) : \exists t \in \mathfrak{T} \text{ such that } (t, z, y, s) \in \bar{B}_0\}$. Because $\hat{t} < T$, there exists $\rho_3 > 0$ such that $v^* - \varphi < -\rho_3$ in \bar{L} . By the construction of v^* , there exists $((t_m, z_m, y_m, s_m))_{m=1}^\infty \subset B_0$ such that $(t_m, z_m, y_m, s_m) \rightarrow (\hat{t}, \hat{z}, \hat{y}, \hat{s})$ and $v(t_m, z_m, y_m, s_m) \rightarrow v^*(\hat{t}, \hat{z}, \hat{y}, \hat{s})$. We define

$$\tau_m := \inf\{t \geq t_m : (t, Z_t^{t_m, z_m, y_m, s_m}, Y_t^{t_m, z_m, y_m, s_m}, S_t^{t_m, z_m, y_m, s_m}) \notin \bar{B}_0\}.$$

Given that (Z, Y, S) is a right-continuous process and $(\bar{B}_0)^c$ is open, it follows that τ_m is a stopping time. Moreover, by the definition of τ_m and the right-continuity of (Z, Y, S) , we have the following:

$$(\tau_m, Z_{\tau_m}^{t_m, z_m, y_m, s_m}, Y_{\tau_m}^{t_m, z_m, y_m, s_m}, S_{\tau_m}^{t_m, z_m, y_m, s_m}) \in \partial\bar{B}_0 \cup (\bar{B}_0 \setminus B_0),$$

implying that $(v^* - \varphi)(t, Z_t^{t_m, z_m, y_m, s_m}, Y_t^{t_m, z_m, y_m, s_m}, S_t^{t_m, z_m, y_m, s_m}) < -\rho_4$ with $\rho_4 := \min\{\rho, \rho_1, \rho_2, \rho_3\}$. By Itô's formula and the fact that $F[\varphi]$ satisfies Equation (B.5), we have the following:

$$\begin{aligned} & \mathbb{E}_{t_m, z_m, y_m, s_m}^{v^*(P_T^{\hat{t}, \hat{z}, \hat{y}, \hat{s}})} \left[\varphi(T \wedge \tau_m, Z_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, Y_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, S_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}) \right] \\ & \leq \varphi(t_m, z_m, y_m, s_m) \\ & \quad - \mathbb{E}_{t_m, z_m, y_m, s_m}^{v^*(P_T^{\hat{t}, \hat{z}, \hat{y}, \hat{s}})} \left[\int_{t_m}^{T \wedge \tau_m} F[\varphi](u, Z_u^{t_m, z_m, y_m, s_m}, Y_u^{t_m, z_m, y_m, s_m}, S_u^{t_m, z_m, y_m, s_m}) du \right] \\ & \quad - \mathbb{E}_{t_m, z_m, y_m, s_m}^{v^*(P_T^{\hat{t}, \hat{z}, \hat{y}, \hat{s}})} \left[\int_{t_m}^{T \wedge \tau_m} \left\{ -\frac{\gamma}{2} \left[(A_u^W + \sigma Y_{u-})^2 + (A_u^B + \eta(S_{u-} + Z_{u-}))^2 \right. \right. \right. \\ & \quad \left. \left. \left. + \sum_{i \in \{+, -\}} \frac{\lambda_u^i (1 - e^{-\gamma(A_u^i + \Delta_u^i)})}{\gamma} + H(A_u) - \frac{A_u^B \nu_u^*}{\eta} \right\} ds \right. \\ & \quad \left. + \sum_{i \in \{+, -\}} \left(\int_{t_m}^{T \wedge \tau_m} (\mathbf{r} - A_u^i) dN_u^i - \int_{t_m}^{T \wedge \tau_m} A_u^i \mathbb{1}_{Y_{u-} \leq 2\xi} d\hat{N}_u^i \right) \right]. \end{aligned}$$

Thus, we have the following:

$$\begin{aligned} & \varphi(t_m, z_m, y_m, s_m) \\ & \geq \mathbb{E}_{t_m, z_m, y_m, s_m}^{v^*(P_T^{\hat{t}, \hat{z}, \hat{y}, \hat{s}})} \left[\int_{t_m}^{T \wedge \tau_m} \left\{ -\frac{\gamma}{2} \left[(A_u^W + \sigma Y_{u-})^2 + (A_u^B + \eta(S_{u-} + Z_{u-}))^2 \right. \right. \right. \\ & \quad \left. \left. \left. + \sum_{i \in \{+, -\}} \frac{\lambda_u^i (1 - e^{-\gamma(A_u^i + \Delta_u^i)})}{\gamma} + H(A_u) - \frac{A_u^B \nu_u^*}{\eta} \right\} ds \right. \\ & \quad \left. + \sum_{i \in \{+, -\}} \left(\int_{t_m}^{T \wedge \tau_m} (\mathbf{r} - A_u^i) dN_u^i - \int_{t_m}^{T \wedge \tau_m} A_u^i \mathbb{1}_{Y_{u-} \leq 2\xi} d\hat{N}_u^i \right) \right. \\ & \quad \left. + \varphi(T \wedge \tau_m, Z_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, Y_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, S_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}) \right] \\ & \quad + \rho \mathbb{E}_{t_m, z_m, y_m, s_m}^{v^*(P_T^{\hat{t}, \hat{z}, \hat{y}, \hat{s}})} [T \wedge \tau_m - t_m]. \end{aligned}$$

By continuity, $\varphi(t_m, z_m, y_m, s_m) \rightarrow \varphi(\hat{t}, \hat{z}, \hat{y}, \hat{s}) = v^*(\hat{t}, \hat{z}, \hat{y}, \hat{s})$. However, $v(t_m, z_m, y_m, s_m) \rightarrow \varphi(\hat{t}, \hat{z}, \hat{y}, \hat{s}) = v^*(\hat{t}, \hat{z}, \hat{y}, \hat{s})$ by construction. Thus, there exists $M \in \mathbb{N}$ such that $|\varphi(t_m, z_m, y_m, s_m) - v(t_m, z_m, y_m, s_m)| < \rho_4/2$ for all $m \geq M$. We obtain the following:

$$\begin{aligned} v(t_m, z_m, y_m, s_m) & \geq \mathbb{E}_{t_m, z_m, y_m, s_m}^{v^*(P_T^{\hat{t}, \hat{z}, \hat{y}, \hat{s}})} \left[\int_{t_m}^{T \wedge \tau_m} \left\{ -\frac{\gamma}{2} \left[(A_u^W + \sigma Y_{u-})^2 + (A_u^B + \eta(S_{u-} + Z_{u-}))^2 \right. \right. \right. \\ & \quad \left. \left. \left. + \sum_{i \in \{+, -\}} \frac{\lambda_u^i (1 - e^{-\gamma(A_u^i + \Delta_u^i)})}{\gamma} + H(A_u) - \frac{A_u^B \nu_u^*}{\eta} \right\} ds \right. \\ & \quad \left. + \sum_{i \in \{+, -\}} \left(\int_{t_m}^{T \wedge \tau_m} (\mathbf{r} - A_u^i) dN_u^i - \int_{t_m}^{T \wedge \tau_m} A_u^i \mathbb{1}_{Y_{u-} \leq 2\xi} d\hat{N}_u^i \right) \right] \end{aligned}$$

$$\begin{aligned}
& + \varphi(T \wedge \tau_m, Z_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, Y_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, S_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}) \quad \Big] \\
& - \frac{\rho_4}{2} + \rho \mathbb{E}_{t_m, z_m, y_m, s_m}^{v^*(P_T^{\hat{P}_0, A})} [T \wedge \tau_m - t_m].
\end{aligned}$$

Observe that

$$\begin{aligned}
& \varphi(T \wedge \tau_m, Z_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, Y_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, S_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}) \\
& = \underbrace{\varphi(T, Z_T^{t_m, z_m, y_m, s_m}, Y_T^{t_m, z_m, y_m, s_m}, S_T^{t_m, z_m, y_m, s_m}) \mathbb{1}_{T \leq \tau_m}}_{\geq v^*(T, Z_T^{t_m, z_m, y_m, s_m}, Y_T^{t_m, z_m, y_m, s_m}, S_T^{t_m, z_m, y_m, s_m}) + \rho_4} \\
& \quad + \underbrace{\varphi(\tau_m, Z_{\tau_m}^{t_m, z_m, y_m, s_m}, Y_{\tau_m}^{t_m, z_m, y_m, s_m}, S_{\tau_m}^{t_m, z_m, y_m, s_m}) \mathbb{1}_{T > \tau_m}}_{\geq v^*(\tau_m, Z_{\tau_m}^{t_m, z_m, y_m, s_m}, Y_{\tau_m}^{t_m, z_m, y_m, s_m}, S_{\tau_m}^{t_m, z_m, y_m, s_m}) + \rho_4} \\
& \geq v(T \wedge \tau_m, Z_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, Y_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, S_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}) + \rho_4,
\end{aligned}$$

thus,

$$\begin{aligned}
v(t_m, z_m, y_m, s_m) & \geq \mathbb{E}_{t_m, z_m, y_m, s_m}^{v^*(P_T^{\hat{P}_0, A})} \left[\int_{t_m}^{T \wedge \tau_m} \left\{ -\frac{\gamma}{2} \left[(A_u^W + \sigma Y_{u-})^2 + (A_u^B + \eta(S_{u-} + Z_{u-}))^2 \right] \right. \right. \\
& \quad \left. \left. + \sum_{i \in \{+, -\}} \frac{\lambda_u^i (1 - e^{-\gamma(A_u^i + \Delta_u^i)})}{\gamma} + H(A_u) - \frac{A_u^B \nu_u^*}{\eta} \right\} ds \right. \\
& \quad \left. + \sum_{i \in \{+, -\}} \left(\int_{t_m}^{T \wedge \tau_m} (\mathbf{x} - A_u^i) dN_u^i - \int_{t_m}^{T \wedge \tau_m} A_u^i \mathbb{1}_{Y_{u-} \leq 2\xi} dN_u^i \right) \right. \\
& \quad \left. + v(T \wedge \tau_m, Z_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, Y_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, S_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}) \quad \Big] \right. \\
& \quad \left. + \frac{\rho_4}{2} + \rho \mathbb{E}_{t_m, z_m, y_m, s_m}^{v^*(P_T^{\hat{P}_0, A})} [T \wedge \tau_m - t_m]. \right.
\end{aligned}$$

Taking the supremum over all $A \in \Lambda$, we have

$$\begin{aligned}
& v(t_m, z_m, y_m, s_m) \\
& \geq \frac{\rho_4}{2} + \sup_{A \in \Lambda} \mathbb{E}_{t_m, z_m, y_m, s_m}^{v^*(P_T^{\hat{P}_0, A})} \left[\int_{t_m}^{T \wedge \tau_m} \left\{ -\frac{\gamma}{2} \left[(A_u^W + \sigma Y_{u-})^2 + (A_u^B + \eta(S_{u-} + Z_{u-}))^2 \right] \right. \right. \\
& \quad \left. \left. + \sum_{i \in \{+, -\}} \frac{\lambda_u^i (1 - e^{-\gamma(A_u^i + \Delta_u^i)})}{\gamma} + H(A_u) - \frac{A_u^B \nu_u^*}{\eta} \right\} ds \right. \\
& \quad \left. + \sum_{i \in \{+, -\}} \left(\int_{t_m}^{T \wedge \tau_m} (\mathbf{x} - A_u^i) dN_u^i - \int_{t_m}^{T \wedge \tau_m} A_u^i \mathbb{1}_{Y_{u-} \leq 2\xi} dN_u^i \right) \right. \\
& \quad \left. + v(T \wedge \tau_m, Z_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, Y_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, S_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}) \quad \Big] \right. \\
& \quad \left. + \frac{\rho_4}{2} + \rho \mathbb{E}_{t_m, z_m, y_m, s_m}^{v^*(P_T^{\hat{P}_0, A})} [T \wedge \tau_m - t_m]. \right.
\end{aligned}$$

contradicting the Dynamic Programming Principle. □

Proposition 2. The function v_* as defined in Equation (B.3) is a supersolution to Equation (4.2) in $[0, T) \times \mathcal{O}$ with $v_*(T, \cdot) \geq 0$.

Proof. First, we will show that $v_*(T, \cdot) \geq 0$. Let us take $((t_m, z_m, y_m, s_m))_{m=1}^\infty \subset [0, T) \times \mathcal{O}$ such that $(t_m, z_m, y_m, s_m) \rightarrow (T, z, y, s)$ and $v(t_m, z_m, y_m, s_m) \rightarrow v_*(T, z, y, s)$. Thus, from the bound of v in Equation (B.2),

$$\begin{aligned}
v_*(T, z, y, s) & = \lim_{m \rightarrow \infty} v(t_m, z_m, y_m, s_m) \\
& \geq \lim_{m \rightarrow \infty} -(T - t_m) e^{\hat{C}_1 (T - t_m)} \hat{C}_0 (1 + |y_m|^2 + |z_m|^2 + |s_m|^4) \\
& = 0.
\end{aligned}$$

Let $\varphi \in C^{1,2}(\mathfrak{X} \times \mathcal{O}; \mathbb{R})$ such that $0 = \min_{\mathfrak{X} \times \mathcal{O}}(v_* - \varphi) = (v_* - \varphi)(\hat{t}, \hat{z}, \hat{y}, \hat{s})$ with $(\hat{t}, \hat{z}, \hat{y}, \hat{s}) \in \mathfrak{X} \times \mathcal{O}$ is the unique minimizer of $v_* - \varphi$ and $\hat{t} < T$. We wish to prove that

$$F^*[\varphi](\hat{t}, \hat{z}, \hat{y}, \hat{s}) \geq 0.$$

We prove this by contradiction. Suppose that there exists $\rho > 0$ such that

$$F^*[\varphi](\hat{t}, \hat{z}, \hat{y}, \hat{s}) < -2\rho,$$

which implies

$$F(\hat{t}, \hat{z}, \hat{y}, \hat{s}, \varphi(\hat{t}, \hat{z}, \hat{y}, \hat{s}), \partial_t \varphi(\hat{t}, \hat{z}, \hat{y}, \hat{s}), D\varphi(\hat{t}, \hat{z}, \hat{y}, \hat{s}), D^2\varphi(\hat{t}, \hat{z}, \hat{y}, \hat{s}), I(\hat{z}, \hat{y}, \hat{s}; \varphi(\hat{t}, \cdot); \varphi(\hat{t}, \hat{z}, \hat{y}, \hat{s}))) < -2\rho.$$

By the upper semicontinuity of F^* , there exists an open neighborhood B_0 of $(\hat{t}, \hat{z}, \hat{y}, \hat{s})$ such that

$$F^*[\varphi](t, z, y, s) < -\rho,$$

which implies

$$F(t, z, y, s, \varphi(t, z, y, s), \partial_t \varphi(t, z, y, s), D\varphi(t, z, y, s), D^2\varphi(t, z, y, s), I(z, y, s; \varphi(t, \cdot); \varphi(t, z, y, s))) < -\rho,$$

for all $(t, z, y, s) \in B_0$. Let \bar{B}_0 be the closure of B_0 . Given that $(\hat{t}, \hat{z}, \hat{y}, \hat{s})$ is an interior point of \bar{B}_0 and the unique minimizer of $v_* - \varphi$, there exists $\rho_1 > 0$ such that $v_* - \varphi > \rho_1$ on ∂B_0 , that is, the boundary of \bar{B}_0 . Moreover, by the uniqueness of the minimizer and the lower semicontinuity of $v_* - \varphi$, there exists $\rho_2 > 0$ such that $v_* - \varphi > \rho_2$ in $\bar{B}_0 \setminus B_0$, where \bar{B}_0 is the translation of $B_0 \ni (t, z, y, s) \mapsto (t, z^\pm, y^\pm, s) \in \bar{B}_0$. Next, we define $\bar{L} := \{(T, z, y, s) : \exists t \in \mathfrak{X} \text{ such that } (t, z, y, s) \in \bar{B}_0\}$. Because $\hat{t} < T$, there exists $\rho_3 > 0$ such that $v_* - \varphi > \rho_3$ in \bar{L} . By the construction of v_* , there exists $((t_m, z_m, y_m, s_m))_{m=1}^\infty \subset B_0$ such that $(t_m, z_m, y_m, s_m) \rightarrow (\hat{t}, \hat{z}, \hat{y}, \hat{s})$ and $v(t_m, z_m, y_m, s_m) \rightarrow v_*(\hat{t}, \hat{z}, \hat{y}, \hat{s})$. As above, we define the following:

$$\tau_m := \inf\{t \geq t_m : (t, Z_t^{t_m, z_m, y_m, s_m}, Y_t^{t_m, z_m, y_m, s_m}, S_t^{t_m, z_m, y_m, s_m}) \notin \bar{B}_0\}.$$

Since (Z, Y, S) is a right-continuous process and $(\bar{B}_0)^c$ is open, it follows that τ_m is a stopping time. Moreover, by the definition of τ_m and the right-continuity of (Z, Y, S) , we have

$$(\tau_m, Z_{\tau_m}^{t_m, z_m, y_m, s_m}, Y_{\tau_m}^{t_m, z_m, y_m, s_m}, S_{\tau_m}^{t_m, z_m, y_m, s_m}) \in \partial \bar{B}_0 \cup (\bar{B}_0 \setminus B_0),$$

implying that $(v_* - \varphi)(t, Z_t^{t_m, z_m, y_m, s_m}, Y_t^{t_m, z_m, y_m, s_m}, S_t^{t_m, z_m, y_m, s_m}) > \rho_4$ with $\rho_4 := \min\{\rho, \rho_1, \rho_2, \rho_3\}$.

Using Ito's formula, we have that

$$\begin{aligned} & \mathbb{E}_{t_m, z_m, y_m, s_m}^{v_* (P_T^{\beta_0, A})} \left[\varphi(T \wedge \tau_m, Z_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, Y_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, S_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}) \right] \\ &= \varphi(t_m, z_m, y_m, s_m) - \mathbb{E}_{t_m, z_m, y_m, s_m}^{v_* (P_T^{\beta_0, A})} \left[\int_{t_m}^{T \wedge \tau_m} F_{A_u}[\varphi](u, Z_u^{t_m, z_m, y_m, s_m}, Y_u^{t_m, z_m, y_m, s_m}, S_u^{t_m, z_m, y_m, s_m}) du \right] \\ & \quad - \mathbb{E}_{t_m, z_m, y_m, s_m}^{v_* (P_T^{\beta_0, A})} \left[\int_{t_m}^{T \wedge \tau_m} \left\{ -\frac{\gamma}{2} \left[(A_u^W + \sigma Y_{u-})^2 + (A_u^B + \eta(S_{u-} + Z_{u-}))^2 \right] \right. \right. \\ & \quad \quad \quad \left. \left. + \sum_{i \in \{+, -\}} \frac{\lambda_u^i (1 - e^{-\gamma(A_u^i + \Delta_u^i)})}{\gamma} + H(A_u) - \frac{A_u^B v_u^*}{\eta} \right\} ds \right. \\ & \quad \quad \left. + \sum_{i \in \{+, -\}} \left(\int_{t_m}^{T \wedge \tau_m} (\mathbf{r} - A_u^i) dN_u^i - \int_{t_m}^{T \wedge \tau_m} A_u^i \mathbb{1}_{Y_{u-} \leq 2\xi} d\tilde{N}_u^i \right) \right], \end{aligned}$$

thus,

$$\begin{aligned} & \varphi(t_m, z_m, y_m, s_m) \\ &= \mathbb{E}_{t_m, z_m, y_m, s_m}^{v_* (P_T^{\beta_0, A})} \left[\int_{t_m}^{T \wedge \tau_m} \left\{ -\frac{\gamma}{2} \left[(A_u^W + \sigma Y_{u-})^2 + (A_u^B + \eta(S_{u-} + Z_{u-}))^2 \right] \right. \right. \\ & \quad \quad \quad \left. \left. + \sum_{i \in \{+, -\}} \frac{\lambda_u^i (1 - e^{-\gamma(A_u^i + \Delta_u^i)})}{\gamma} + H(A_u) - \frac{A_u^B v_u^*}{\eta} \right\} ds \right. \\ & \quad \quad \left. + \sum_{i \in \{+, -\}} \left(\int_{t_m}^{T \wedge \tau_m} (\mathbf{r} - A_u^i) dN_u^i - \int_{t_m}^{T \wedge \tau_m} A_u^i \mathbb{1}_{Y_{u-} \leq 2\xi} d\tilde{N}_u^i \right) \right] \end{aligned}$$

$$\begin{aligned}
& + \varphi(T \wedge \tau_m, Z_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, Y_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, S_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}) \Big] \\
& + \mathbb{E}_{t_m, z_m, y_m, s_m}^{v^*(P_T^{\beta_0, A})} \left[\int_{t_m}^{T \wedge \tau_m} F_{A_u}[\varphi](u, Z_u^{t_m, z_m, y_m, s_m}, Y_u^{t_m, z_m, y_m, s_m}, S_u^{t_m, z_m, y_m, s_m}) du \right].
\end{aligned}$$

By continuity, $\varphi(t_m, z_m, y_m, s_m) \rightarrow \varphi(\hat{t}, \hat{z}, \hat{y}, \hat{s}) = v_*(\hat{t}, \hat{z}, \hat{y}, \hat{s})$. However, $v(t_m, z_m, y_m, s_m) \rightarrow \varphi(\hat{t}, \hat{z}, \hat{y}, \hat{s}) = v_*(\hat{t}, \hat{z}, \hat{y}, \hat{s})$ by construction. Thus, there exists $M \in \mathbb{N}$ such that $|\varphi(t_m, z_m, y_m, s_m) - v(t_m, z_m, y_m, s_m)| < \rho_4/2$ for all $m \geq M$. We obtain the following:

$$\begin{aligned}
v(t_m, z_m, y_m, s_m) & \leq \mathbb{E}_{t_m, z_m, y_m, s_m}^{v^*(P_T^{\beta_0, A})} \left[\int_{t_m}^{T \wedge \tau_m} \left\{ -\frac{\gamma}{2} \left[(A_u^W + \sigma Y_{u-})^2 + (A_u^B + \eta(S_{u-} + Z_{u-}))^2 \right] \right. \right. \\
& \quad \left. \left. + \sum_{i \in \{+, -\}} \frac{\lambda_u^i (1 - e^{-\gamma(A_u^i + \Delta_u^i)})}{\gamma} + H(A_u) - \frac{A_u^B v_u^*}{\eta} \right\} ds \right. \\
& \quad \left. + \sum_{i \in \{+, -\}} \left(\int_{t_m}^{T \wedge \tau_m} (\mathbf{r} - A_u^i) dN_u^i - \int_{t_m}^{T \wedge \tau_m} A_u^i \mathbb{1}_{Y_{u-} \leq 2\xi} d\hat{N}_u^i \right) \right. \\
& \quad \left. + \varphi(T \wedge \tau_m, Z_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, Y_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, S_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}) \right] \\
& \quad + \frac{\rho_4}{2} + \mathbb{E}_{t_m, z_m, y_m, s_m}^{v^*(P_T^{\beta_0, A})} \left[\int_{t_m}^{T \wedge \tau_m} F_{A_u}[\varphi](u, Z_u^{t_m, z_m, y_m, s_m}, Y_u^{t_m, z_m, y_m, s_m}, S_u^{t_m, z_m, y_m, s_m}) du \right].
\end{aligned}$$

Observe that

$$\begin{aligned}
& \varphi(T \wedge \tau_m, Z_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, Y_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, S_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}) \\
& = \underbrace{\varphi(T, Z_T^{t_m, z_m, y_m, s_m}, Y_T^{t_m, z_m, y_m, s_m}, S_T^{t_m, z_m, y_m, s_m})}_{\leq v_*(T, Z_T^{t_m, z_m, y_m, s_m}, Y_T^{t_m, z_m, y_m, s_m}, S_T^{t_m, z_m, y_m, s_m}) - \rho_4} \mathbb{1}_{T \leq \tau_m} \\
& \quad + \underbrace{\varphi(\tau_m, Z_{\tau_m}^{t_m, z_m, y_m, s_m}, Y_{\tau_m}^{t_m, z_m, y_m, s_m}, S_{\tau_m}^{t_m, z_m, y_m, s_m})}_{\leq v_*(\tau_m, Z_{\tau_m}^{t_m, z_m, y_m, s_m}, Y_{\tau_m}^{t_m, z_m, y_m, s_m}, S_{\tau_m}^{t_m, z_m, y_m, s_m}) - \rho_4} \mathbb{1}_{T > \tau_m} \\
& \leq v(T \wedge \tau_m, Z_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, Y_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, S_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}) - \rho_4,
\end{aligned}$$

thus,

$$\begin{aligned}
& v(t_m, z_m, y_m, s_m) \\
& \leq \mathbb{E}_{t_m, z_m, y_m, s_m}^{v^*(P_T^{\beta_0, A})} \left[\int_{t_m}^{T \wedge \tau_m} \left\{ -\frac{\gamma}{2} \left[(A_u^W + \sigma Y_{u-})^2 + (A_u^B + \eta(S_{u-} + Z_{u-}))^2 \right] \right. \right. \\
& \quad \left. \left. + \sum_{i \in \{+, -\}} \frac{\lambda_u^i (1 - e^{-\gamma(A_u^i + \Delta_u^i)})}{\gamma} + H(A_u) - \frac{A_u^B v_u^*}{\eta} \right\} ds \right. \\
& \quad \left. + \sum_{i \in \{+, -\}} \left(\int_{t_m}^{T \wedge \tau_m} (\mathbf{r} - A_u^i) dN_u^i - \int_{t_m}^{T \wedge \tau_m} A_u^i \mathbb{1}_{Y_{u-} \leq 2\xi} d\hat{N}_u^i \right) \right. \\
& \quad \left. + v(T \wedge \tau_m, Z_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, Y_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, S_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}) \right] \\
& \quad - \frac{\rho_4}{2} + \mathbb{E}_{t_m, z_m, y_m, s_m}^{v^*(P_T^{\beta_0, A})} \left[\int_{t_m}^{T \wedge \tau_m} F_{A_u}[\varphi](u, Z_u^{t_m, z_m, y_m, s_m}, Y_u^{t_m, z_m, y_m, s_m}, S_u^{t_m, z_m, y_m, s_m}) du \right] \\
& \leq \sup_{A \in \Lambda} \left\{ \mathbb{E}_{t_m, z_m, y_m, s_m}^{v^*(P_T^{\beta_0, A})} \left[\int_{t_m}^{T \wedge \tau_m} \left\{ -\frac{\gamma}{2} \left[(A_u^W + \sigma Y_{u-})^2 + (A_u^B + \eta(S_{u-} + Z_{u-}))^2 \right] \right. \right. \right. \\
& \quad \left. \left. + \sum_{i \in \{+, -\}} \frac{\lambda_u^i (1 - e^{-\gamma(A_u^i + \Delta_u^i)})}{\gamma} + H(A_u) - \frac{A_u^B v_u^*}{\eta} \right\} ds \right. \\
& \quad \left. \left. + \sum_{i \in \{+, -\}} \left(\int_{t_m}^{T \wedge \tau_m} (\mathbf{r} - A_u^i) dN_u^i - \int_{t_m}^{T \wedge \tau_m} A_u^i \mathbb{1}_{Y_{u-} \leq 2\xi} d\hat{N}_u^i \right) \right. \right.
\end{aligned}$$

$$\begin{aligned}
& +v(T \wedge \tau_m, Z_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, Y_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, S_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}) \quad] \quad \Big\} \\
& - \frac{\rho_4}{2} + \mathbb{E}_{t_m, z_m, y_m, s_m}^{y^*(P_T^{\beta_0, A})} \left[\int_{t_m}^{T \wedge \tau_m} F_{A_u}[\varphi](u, Z_u^{t_m, z_m, y_m, s_m}, Y_u^{t_m, z_m, y_m, s_m}, S_u^{t_m, z_m, y_m, s_m}) du \right].
\end{aligned}$$

Given that $\inf_A F_A[\varphi] = F[\varphi] < -\rho$, we can choose $A \in \Lambda$ such that

$$F_{A_u}[\varphi](u, Z_u^{t_m, z_m, y_m, s_m}, Y_u^{t_m, z_m, y_m, s_m}, S_u^{t_m, z_m, y_m, s_m}) du \leq 0$$

on $[t_m, T \wedge \tau_m]$. Thus, we have the following:

$$\begin{aligned}
& v(t_m, z_m, y_m, s_m) \\
& \leq -\frac{\rho_4}{2} + \sup_{A \in \Lambda} \left\{ \mathbb{E}_{t_m, z_m, y_m, s_m}^{y^*(P_T^{\beta_0, A})} \left[\int_{t_m}^{T \wedge \tau_m} \left\{ -\frac{\gamma}{2} \left[(A_u^W + \sigma Y_{u-})^2 + (A_u^B + \eta(S_{u-} + Z_{u-}))^2 \right] \right. \right. \right. \\
& \qquad \qquad \qquad \left. \left. \left. + \sum_{i \in \{+, -\}} \frac{\lambda_u^i (1 - e^{-\gamma(A_u^i + \Delta_u^i)})}{\gamma} + H(A_u) - \frac{A_u^B v_u^*}{\eta} \right\} ds \right. \right. \\
& \qquad \qquad \qquad \left. \left. + \sum_{i \in \{+, -\}} \left(\int_{t_m}^{t \wedge \tau_m} (v - A_u^i) dN_u^i - \int_{t_m}^{t \wedge \tau_m} A_u^i \mathbb{1}_{Y_{u-} \leq 2\xi} d\hat{N}_u^i \right) \right. \right. \\
& \qquad \qquad \qquad \left. \left. +v(T \wedge \tau_m, Z_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, Y_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}, S_{T \wedge \tau_m}^{t_m, z_m, y_m, s_m}) \quad] \quad \right\}.
\end{aligned}$$

contradicting the Dynamic Programming Principle. □