Taylor & Francis
Taylor & Francis Group

British Journal for the History of Mathematics

ISSN: 2637-5451 (Print) 2637-5494 (Online) Journal homepage: www.tandfonline.com/journals/tbsh21

Proving the Binomial Theorem in Britain, 1750–1830

Christopher D. Hollings & Benjamin Wardhaugh

To cite this article: Christopher D. Hollings & Benjamin Wardhaugh (04 Jun 2026): Proving the Binomial Theorem in Britain, 1750–1830, British Journal for the History of Mathematics, DOI: 10.1080/26375451.2026.2648415

To link to this article: https://doi.org/10.1080/26375451.2026.2648415

Published online: 04 Jun 2026.

British Journal for the History of Mathematics, 2026
https://doi.org/10.1080/26375451.2026.2648415

Proving the Binomial Theorem in Britain, 1750–1830

CHRISTOPHER D. HOLLINGS
Mathematical Institute, University of Oxford, Oxford, United Kingdom

BENJAMIN WARDHAUGH *
Faculty of History, University of Oxford, Oxford, United Kingdom

This article examines the contested status and evolving proofs of the Binomial Theorem in Britain during the period 1750–1830. Although universally acknowledged as true and widely used in calculus, algebra, and the theory of infinite series, the theorem's general proof remained a source of prolonged mathematical and philosophical debate. The authors investigate why over forty British publications from this era sought to re-prove or reinterpret the theorem, linking this phenomenon to broader shifts in mathematical rigour and the eventual decline of the Newtonian fluxional calculus. The paper analyzes challenges surrounding the multiplicity of binomial forms and exponents, the lack of accepted general principles governing infinite series, and deep unease over Newton's own inductive, non-proof-based approach. Despite its central role in British mathematical education and its celebrated association with Newton, the Binomial Theorem's exact scope and justification remained elusive for decades. The authors argue that the persistence of divergent proofs and unresolved doubts reflects a transitional era in British mathematics—one marked by growing awareness of foundational uncertainty and the influence of more rigorous continental methods. This study thus offers insight into how mathematical authority, legacy, and proof were contested concepts in Enlightenment and post-Enlightenment Britain.

1. Introduction: the Binomial Theorem as fashion and problem in eighteenth-century Britain

This paper offers the first systematic study of why proving the Binomial Theorem became a sustained 'fashion' in Britain between 1750 and 1830, using around forty British publications to reinterpret the theorem's status as both celebrated and yet persistently regarded as not quite proved. It is original in showing how a cluster of technical issues – multiple binomial forms, exponent cases, the absence of generally accepted series methods, and the limited uptake of Bernoulli's combinatorial approach – combined with Newton's own justificatory practices to give the theorem an anomalous and unusually long-lived problematic status in British mathematics.[1]

*Benjamin Wardhaugh benjamin.wardhaugh@history.ox.ac.uk Faculty of History, University of Oxford, Oxford, United Kingdom

[1]For earlier discussions of the Binomial Theorem in this period, see in particular Coolidge (1949), Whiteside (1961), Noble (2022) and Hollings (2025).

Following its explicit introduction by Isaac Newton in the late 17th century, the General Binomial Theorem quickly became a topic of great concern and utility within the development of the Newtonian calculus. Indeed, Guicciardini (1989, 104) has remarked that 'algebraical proofs of the binomial theorem became a fashionable exercise in the late eighteenth century'. The present authors have identified around forty British publications from the period 1750–1830 that appear to intend to say something new about the proof of the Binomial Theorem. The purpose of this paper is to discover why this fashion should have existed at this particular place and time, and what it reveals about the status of the Binomial Theorem in 18th-century Britain, as well as about the collective state of mind of British mathematicians more generally during that interesting period.

The Binomial Theorem was highly visible; by the second quarter of the 18th century – if not before – it had become a staple of algebra textbooks and of expositions of the fluxional calculus, appearing for example in Nicholas Saunderson's *Elements of Algebra* (Saunderson 1740) and John Stewart's edition of Newton's work on quadratures (Stewart 1745). More than that, it was important for the development of several branches of mathematics. Treatments of the fluxional calculus usually assumed it, either explicitly or implicitly, in order to arrive at a general statement of the fluxion of x^n. Discussions of logarithms and their calculation required it in order to achieve series expansions for use in this context. And in any more general discussion of infinite series, it was likely to be one of the first examples discussed, and to be relied upon for the proof of more advanced results (see Noble 2022).

For this combination of reasons, the Binomial Theorem from an early stage – indeed already during the lifetime of its discoverer Newton – became 'celebrated', the word perhaps most frequently used to describe the theorem later in the century (Stewart 1745; Simpson 1752; Maseres 1778; Hutton 1812; Evans 1824). Stewart (1745, 380) claimed, for instance, that it was 'one of the most general and useful Theorems, that ever were discovered'; Simpson (1752, 20) that it was 'ranked, by many, among the principal discoveries of its illustrious author'. It was one of the specimen pieces of mathematics carved on Newton's memorial in Westminster Abbey, 'as one of his greatest discoveries', as Hutton (1795, vol. 1, 231) remarked.

The remainder of this paper will set out some of the ways in which the Binomial Theorem was problematic, but it is worth pointing out at this stage that the theorem was also universally regarded as true: none of those who raised questions about its form, status or proof ever suggested that it might be false in certain cases. This paper is concerned specifically with the history of the Binomial Theorem in the English-speaking world: its rather different – though equally complex – trajectory on the European continent has been surveyed, particularly for the first half of the 18th century, by Noble (2022).

We begin the paper proper in §2 with some comments on the status of the Binomial Theorem in 18th- and 19th-century Britain, and the forms in which it appeared. §3 treats, in general terms, the different strategies that were employed in attempts to prove the theorem, while §4 picks up some of the conceptual issues that arise in connection with these. Stepping away from the Binomial Theorem purely as a technical mathematical result, §5 further sets the theorem into the context of late-18th- and early-19th-century British mathematics. §6 offers some conclusions, as well as some suggestions as to future lines of research. Outlines of a small selection of the proofs of the Binomial Theorem that are mentioned in the paper are provided in Appendix A.

2. A celebrated yet perpetually unproven theorem

Despite and to some degree because of its visibility, importance and celebrity, the Binomial Theorem also acquired a problematic status during the 18th century, and came to occupy an anomalous position, at least in British mathematics. Although it was universally regarded as true, the question of whether or not it had been proved remained open in the English-speaking world until at least the 1820s, in other words until after the Newtonian school of mathematics and its showpiece the fluxional calculus had been effectively replaced by the self-conscious importation of continental methods (Dubbey 1963; Stenhouse 2020). Giovanni Salvemini (1709–1791) stated that no-one was unaware of the Binomial Theorem but that no-one he knew of had proved it (Salvemini 1742); Charles Hutton (1737–1823) remarked that 'many eminent mathematicians to this day account the demonstration not fully accomplished, and still a thing greatly to be desired' (Hutton 1786, 67),[2]and Francis Maseres (1731–1824) that 'as this famous Theorem has now been published during more than an hundred years, it seems to be high time that it should be demonstrated in all it's cases' (Maseres 1791–1807, vol. V, 86). Robert Woodhouse (1773–1827) still believed that all extant proofs were 'imperfect' (Woodhouse 1803, 34), enumerating several methods that he deemed 'inaccurate', and in 1812 Hutton reprinted his own assertion unchanged.[3]As late as 1827, Thomas Swinburne and Thomas Tylecote (1798–1882) would survey the proofs known to them and conclude that

> Amid this variety of demonstrations, one part of the mathematical world has acquiesced in this proof, another in that, while another part has considered every known demonstration as defective (Swinburne and Tylecote 1827, 3).

What can account for this multiplicity of proofs and this long – perhaps exceptionally long – period when the status of the Binomial Theorem as proven or unproven remained disputed?

2.1. *What binomial?*

A first difficulty experienced by those engaging with the Binomial Theorem in this period was the multiplicity of forms in which it had been stated or could be stated. Newton's first statement (as divulged to Henry Oldenburg in the so-called 'epistola prior' of June 1676 and subsequently published in the *Commercium epistolicum*: Collins 1712) provided a series expansion[4]for $\overline{P+PQ}^{\frac{m}{n}}$. This became perhaps the closest to a canonical form of the Binomial Theorem for British writers, and as late as the 1820s, Lewis Evans (1755–1827) took pains to prove specifically this rather than any alternative or equivalent (Evans 1824).

But Newton also gave series expansions for (certain powers of) other expressions such as (c^2+x^2) and $(d+e)$, and $(1-x^2)$, the latter in a subsequent letter to

[2]The comment appears within Hutton's Tract 6: 'Of the binomial theorem. With a demonstration of the truth of it in the General Case of Fractional Exponents'.
[3]Tract 12 in Hutton (1812) = Tract 6 in Hutton (1786).
[4]This period saw, in a British context, the transition from the 'overbar' to the use of parentheses to aggregate algebraic terms: for instance, from $\overline{x+y}$ to $(x+y)$ with identical meaning. For the remainder of this paper the second form will be used without further comment.

Oldenburg, the 'epistola posterior'. All of these are 'binomial theorems' properly so called. Later writers for various reasons undertook to derive expansions for binomials such as $(a + b)$, $(a - b)$, $(1 + x)$ or $(1 - x)$.

Eighteenth-century Britain possessed few if any general theorems about the manipulation of infinite series whose validity was widely accepted, and one consequence was that the transformation of one of these forms into another was a doubtful matter. Multiple writers, for instance, would provide separate proofs for $(1 - x)^n$ and $(1 + x)^n$ (Maseres 1791–1807, vol. I). Some would handle $(x + 1)^n$ and $(x + y)^n$ as separate cases. Motivations for this caution varied. Francis Maseres was one of the small group of late 18th-century mathematicians who held out for caution in the handling of negative quantities in geometrical or geometrically-derived contexts, and altogether against the use of imaginaries,[5]and this was one reason why he treated his two cases separately, even dignifying them with separate names as the binomial and the residual theorems (Maseres 1791–1807, vol. III, *130; for details of Maseres's proof, see Appendix A.6). Yet even the less extreme and more widespread position in which explicit reference to the square roots of negative quantities was rigorously avoided (a rejection, that is, of the reference of older writers to 'impossible' and of continental writers to 'imaginary' quantities) led in a similar direction: for exponent 1/2, for instance, $(1 - x)$ and $(1 + x)$ required different limitations on the values of x.

The total number of 'binomial theorems' proved in the 18th century is therefore large, certainly reaching a dozen or more, and their mutual equivalence was uncertain for most of those who attempted to address it, as was the ease with which one might be proved from another. These factors contributed to uncertainty on the part of many writers as to what exactly had or had not been (validly) proved.

2.2. *And what exponent?*

To a modern eye, a perhaps more obvious source of variation as to what the Binomial Theorem actually was is provided by the question of how the exponent is restricted (or not). Cases that some authors chose to treat separately included the natural numbers, negative whole numbers, reciprocals of natural numbers, reciprocals of negative whole numbers, positive fractions, or negative fractions. A minority of authors presented proofs which they claimed (sometimes with conspicuous lack of caution) were valid for several of these cases (Sewell 1796), or even for surd exponents (Hutton 1786).[6]The majority, though, considered at least three cases – positive integer, negative integer, fraction – and presented at least three separate proofs (cf. the remarks in §4.4). Some considered the case where the exponent was the reciprocal of an integer separately from that of fractions in general, and Maseres (1791–1807, vols. II, V) insisted that the case of the negative reciprocal of an integer required its own separate proof; Swinburne and Tylecote (1827) would cover separately the four cases m, $-m$, m/n, $-m/n$. Once again, it was seldom clear which cases could be deduced from others without either making unwarranted assumptions or doing a great deal of additional work. Questions about the validity of any particular approach to the Binomial Theorem therefore tended to fragment, with most proofs for natural-

[5]See, for example, Pycior (1982).

[6]Glenie in (Maseres 1791–1807, vol. V) eventually proposed a proof for all real exponents, as did Spence (1809) and Knight (1816, 1817): see Craik (2013).

number exponents going unchallenged while a high proportion of proofs for other cases met with difficulties.

3. Eighteenth-century strategies for proving the Binomial Theorem

In the British tradition between 1750 and 1830, proofs of the Binomial Theorem formed a crowded and conceptually diverse field, and were central to the theorem's anomalous status as both indispensable and somehow not yet securely demonstrated. This section surveys the main proof strategies adopted in that period – from Newton's own justificatory practices, through combinatorial and fluxional arguments, to multinomial, equivalence-based, and long-division methods – and shows how each both responded to and reinforced wider doubts about algebraic rigour, infinitesimals, induction, and the manipulation of infinite series.

3.1. *Newton*

The difficulties began with Newton, and it is perhaps easiest to understand why by beginning with certain texts not directly visible to 18th-century commentators. The 1650s and 1660s saw a short-lived vogue for publishing algebraic arguments in which results were established not by presenting a general proof but by presenting a set of specific examples together with the assertion that the numerical pattern visible in those examples held generally. Important examples appear in the *Arithmetica infinitorum* of John Wallis and the *Logarithmo-technia* of Nicolaus Mercator (Wallis 1656; Mercator 1668). The phenomenon was arguably an offshoot of the longer-lived 17th-century tradition of relatively loose arguments about infinitesimal quantities, together perhaps with an impulse to expand the range of algebraic proof in much the same way that geometrical proof was contemporaneously expanding (Bos 2013).

For the Binomial Theorem, the important point is that Newton in his notebook of 1664–65 participated in this trend, investigating series for logarithms and for binomial expansions by methods explicitly inspired by those of Wallis: Newton observed the relationship of the binomial coefficients for natural-numbered exponents to the numbers in the arithmetical triangle, and proceeded to arrive at coefficients for other exponents by interpolating and extrapolating the triangle (Collins 1712; Trinity College, Cambridge, MS Add. 3958, ff. 70–72; Whiteside 1961; for more detailed comments, see Appendix A.1). Some of these results could be checked *post hoc*.

By using the theorem in his subsequent mathematics, Newton made it clear that he believed it, and made it clear also that he thought that belief was justified. He never published, nor as far as we know attempted to construct, a justification in the form of a conventional algebraic proof, and we must conclude that he thought his belief in the theorem was justified by the argument he had constructed in 1664–65. It is beyond the scope of this paper to probe this piece of Newtonian behavior further; suffice it to say that it proved crucial for giving the Binomial Theorem its anomalous status in the 18th century. When Newton came under pressure to divulge his series results in the 1670s, he communicated to Oldenburg and John Collins – and thus to Gottfried Wilhelm Leibniz and ultimately to print in the *Commercium epistolicum* – the statements he had arrived at in 1664–65, and the arguments with which he had there justified those statements (Collins 1712). It therefore became a matter of comment in a high proportion of later discussions of the Binomial Theorem that Newton possessed no proof of the theorem. Typical is the comment of Hutton (1786, 67), stating that the theorem 'was not demonstrated by Newton'; Woodhouse (1803, 32) would concur that 'Newton gave no demonstration; probably he possessed none'. What exactly Newton had provided by way of justification had to be framed rather carefully: Stewart (1745, 380) stated that 'this celebrated Theorem was discovered at first by the Method of Trial and Induction' (cf. Hutton 1795, vol. 1, 230). By 1808, Abram Robertson (1751–1826) could claim that 'it is well known to mathematicians, that what Sir Isaac said on the subject was never meant by him to be considered as a demonstration' (Robertson 1808, 7): this was to re-frame quite radically the material visible in the *Commercium*, which Newton to all appearances did think constituted a justification of the binomial expansions.

Newton's failure in this instance was a source of evident embarrassment to later commentators, and provided some of the impetus to supply a real proof of the Binomial Theorem in his form or at least in some form; yet the widely disseminated knowledge that Newton had possessed arguments that fell short of proof had the twofold effect of creating the impression that finding a proof had defeated Newton and was therefore difficult (and a prize worth attaining) and of placing the theorem itself in an ambiguous position: not quite proved, but not quite not proved either.

Analogously, another part of Newton's writings caused similar difficulty for the mathematical writers of the 18th century: Book I, Section I, Lemma I of the *Principia*, otherwise known as the 'method of first and last ratios'. Essentially a limiting process, Newton's method asserted that two varying quantities that come as close to each other as any given difference may be considered to be 'ultimately equal' (Guicciardini 1999, §3.3). Although the idea was probably conceived geometrically, Newton's readers spent much time and effort in trying to find the correct algebraic formulation of this method. As with the Binomial Theorem, Newton had left little to go on: a short contradiction proof, and a series of examples of the application of the method that suggested a degree of intuition that had not made it successfully onto the page. A popular approach by later readers, particularly those in continental Europe, was simply to rephrase things in terms of infinitesimal quantities, which of course brought with it different questions of rigour that were not satisfactorily resolved until the mid-19th century. In the meantime, British authors would often fall back on geometrical intuition. Unlike in the case of the Binomial Theorem, however, concerns were raised as to just how secure Newton's method was, and the approaches to calculus that were advocated by the likes of Landen and Woodhouse, as cited elsewhere in the present paper, attempted, among other things, to circumvent the need for first and last ratios.[7]

3.2. *The Binomial Theorem and combinatorics*

Among the many 18th-century proofs of various forms and cases of the Binomial Theorem, an important strand may justly be called the combinatoric approach. In 1713, the *Ars conjectandi* of Jacob Bernoulli (1655–1705) was published at Basel, concerned principally with questions about chance but also containing much

[7]See Hollings (2026) for an early 19th-century attempt to justify Newton's method to Cambridge undergraduates.

combinatoric material. This included a formula, with proof, for the numbers in the arithmetic triangle which could easily be applied to their appearance as the coefficients in the binomial expansion when the exponent is a natural number, a connection which Bernoulli explored at length (Bernoulli 1713, 72ff.). For continental mathematics, this inaugurated a long period of contact between the Binomial Theorem and the theory of combinatorics, which did much to determine the trajectory of the Binomial Theorem among French- and German-speaking mathematicians (Noble 2022). In the English-speaking world, however – and rather surprisingly in view of its later status as a founding text of both combinatorics and probability theory – this aspect of the *Ars conjectandi* went almost wholly unnoticed. Hutton (1786) believed that Stewart (1745) had used Bernoulli's text; Robertson (1795) similarly claimed that the 1742 proof by Salvemini used combinatorial material: but both claims seem uncertain, and the first explicit reference in English to Bernoulli's proof of the Binomial Theorem appears to be that of Hutton himself in 1786. Maseres (1791–1807, vol. II, 157–8) would state that the *Ars conjectandi* 'seems to have been but little known and read in England', and in 1796 he republished the relevant parts of the text with translation and commentary, making some fanfare of his rediscovery of an important 'lost' work (Maseres 1791–1807, vol. III).

Absent from British approaches to the Binomial Theorem for much of the 18th century, therefore, was any explicit invocation of results from combinatorics. All of the many mathematicians who offered proofs of the Binomial Theorem for natural-number exponents therefore carried out the – effectively – combinatorial work afresh, laboriously showing how the rearrangement of terms resulted in a predictable number of terms for each power of the variable. This resulted in a number of decidedly prolix proofs of this, the simplest case of the Binomial Theorem, and we believe contributed something to the reputation of the theorem for difficulty.

3.3. *Calculus*

A second strand in proofs of the Binomial Theorem, important more for its wider consequences than for its actual contribution to proofs with any longevity, was the use of calculus, that is in a British context of fluxions. A relatively easy and therefore attractive approach to the proof of the Binomial Theorem is to assume the existence of a series, that is to assume (Raphson 1715, 10, attributing the method to Heynes):

$$(a+x)^m = b + cx + dx^2 + ex^3 \text{ \&c.}$$

and then to take fluxions of both sides of this equation. By dividing by $\dot{x}$ (or in other versions of the proof by setting it to zero), a small amount of algebra then yields an alternative expression of the series. Equating coefficients then yields the law of coefficients for the series easily. (For more details of Raphson's method, see Appendix A.2.)

This proof is attractive for its brevity and for the fact that few conditions on the exponent seem to be required. Proofs of this kind were given not only by Joseph Raphson (1648–1715) but by Hodgson (1736),[8] Colson (1736), Maclaurin (1742) (repeating Colson's proof), Simpson (1752) (repeating Hodgson's proof), Horsley

[8] Further details of Hodgson's proof can be found in Appendix A.3.

(1779) (giving the Raphson/Heynes proof in an annotation to excerpts from the *Commercium epistolicum*), Vince (1815), and Evans (1824).

But it was dogged throughout the 18th century (and beyond) by the question of whether circular argument was being committed. In the *De analysi*, *De methodis*, and *De quadratura*, Newton had used the Binomial Theorem to arrive at the power rule, the general expression for the fluxion of x^n. Other presentations of the calculus tended to do the same, assuming the Binomial Theorem at an early stage. It became customary therefore to assert that a proof of the Binomial Theorem independent of fluxions was required, and to worry that if one could not be found, the fluxional calculus itself was ill-founded. John Colson (1680–1760), for instance, wrote that

> it can hardly be said that this, or any other that is developed from the Method of Fluxions, is a strict investigation of this Theorem. Because the Method itself is originally derived from the method of raising Powers, at least integral Powers (Colson 1736, 310; see Coolidge 1949, 155).

Some of the authors of such proofs, notably Thomas Simpson (1710–1761), were certainly guilty of logical fallacy in deriving the power law from the Binomial Theorem and the Binomial Theorem from the power law. On the continent, Leonhard Euler's (1707–1783) first attempt at a proof suffered from the same difficulty (Noble 2022).

The situation was not in fact as dire as most British mathematicians seem to have believed. In the *Principia*, Newton provided a purely algebraic derivation of the power rule, with no reliance on infinite series or the Binomial Theorem (book 2, lemma 2). Stewart (1745) was one of the minority to follow this line, explicitly making the Binomial Theorem dependent on the power rule rather than vice versa. James Hodgson (1672–1755) in (1736) also seems to have avoided the vicious circle.

Most, however, were unaware that the problem could be overcome: or if they were so aware, continued to feel strongly that a purely algebraic proposition required a purely algebraic – not a fluxional – demonstration, and therefore to insist that even if the Binomial Theorem had been demonstrated it had not been demonstrated 'satisfactorily' or 'legally' if fluxions had been involved. John Landen (1719–1790), for instance, wrote that the method of fluxions was 'not the most natural method of resolving many problems to which it is usually applied'; that

> peculiar principles borrowed from the doctrine of motion […] seem not so properly applicable to algebra as those on which that art was, before, very naturally founded (Landen 1758, 4).

Similar objections, similarly worded, were articulated by Hutton (1786), Robertson (1795), Knight (1816), and Cresswell (1817). Woodhouse (1803) adopted a particularly combative agenda to excise the 'principle of motion' from algebra:

> The project of extracting the square and cube roots of algebraical quantities by a principle of motion, is surely revolting from common sense (Woodhouse 1803, iii).

By proceeding in this way, the fluxionists, to Woodhouse's mind, pursued 'a method totally the reverse' of what he advocated (Woodhouse 1803, ii), namely the establishment of calculus on an algebraic basis, wherein root extraction would first be used to establish the Binomial Theorem (see §3.6 below), which would only then be applied to questions of motion. Some years later, when Woodhouse's successor in Cambridge mathematical reform, the Analytical Society, published its English translation of Sylvestre Lacroix's elementary text on differential and integral calculus (Lacroix 1816), the problem seemed at last to be solved.[9]Following Lacroix, a limit-based definition of a derivative paved the way for a calculus-based proof of the Binomial Theorem, which could then be used to deduce Taylor's Theorem (Lacroix 1816, § 20). But because the Analytical Society was otherwise committed to the Lagrangian approach to analysis, founded upon the existence of series expansions, an alternative point of view was provided in an appendix penned by George Peacock (1791–1858), wherein Taylor's Theorem ('unquestionably the most important in the whole range of Analysis': Lacroix 1816, 605) was derived first, and then used to prove the Binomial Theorem (Lacroix 1816, Note B).

3.4. *A preliminary: the Multinomial Theorem*

Turning to the strategies of proof used by those who rejected the use of fluxional methods to establish the Binomial Theorem, an important preliminary was the proof by Abraham De Moivre (1667–1754) of the Multinomial Theorem, that is a law for the expansion of a given algebraic expression – even an infinite series – to any positive integer power (Moivre 1697). De Moivre stated that his rule could also be applied to non-integer powers, but never proved any case other than exponent -1 (Noble 2022; Moivre 1730). The Multinomial Theorem was one of the few general results for the manipulation of infinite series widely accepted as having been validly proved, and was therefore a fairly common presence in proofs of the Binomial Theorem for non-integral powers. Hutton (1786), for instance, proposed a proof of the Binomial Theorem for exponent $1/n$ simply by assuming the existence of a series of the required form, using the Multinomial Theorem to raise it to the nth power and equating coefficients (see Appendix A.5).

3.5. *Equivalence*

Among the attempts at a purely algebraic – i.e. non-fluxional – proof of the Binomial Theorem, a large class of proofs have as their main idea the finding of two equivalent expressions for the same binomial expansion, in which the equating of coefficients will then lead with reasonable ease to a demonstration of the general coefficient of (as the case may be) x^k. Thomas Knight (1775–1853), for instance, assumed series of the required form for $(a+x)^m$ and $(a+y)^m$, multiplied them together and equated the two different expressions for the product which resulted, arriving at the Binomial Theorem as required by equating coefficients (Knight 1816) (see Appendix A.7).[10]Sewell (1796), in a broadly similar strategy, assumed series for $(1+x)^{m/n}$ and $(1+y)^{m/n}$ and considered their difference to arrive at a relatively succinct proof.

[9]On this process of reform, see Enros (1983) and Phillips (2006).

[10]This was essentially the approach taken to Newton's method of first and last ratios by the author, John Martin Frederick Wright (1793–1842), alluded to above in footnote 7 (Wright 1828, 4). Wright termed it the 'method of indeterminate coefficients' and referred the reader to the standard algebra textbooks of the day: Bonnycastle (1828) and Wood (1825).

Woodhouse (1803) assumed series for $(a+x)^m$ and for $(a+x+z)^m$ and multiplied out the latter to arrive at two equivalent expressions; William Spence (1777–1815) in (1809) equated series expressions for $\phi(1+x).\phi(1+y)$ and $\phi(1+x+y+xy)$, equating coefficients and eventually arriving at the Binomial Theorem as a special case.

In 18th-century terms (that is, setting aside the questions of whether a series of the form assumed must exist and whether it converges, on both of which issues see below), these tend to be among the more plausible of proofs of the Binomial Theorem. A pitfall is illustrated by Sewell (1796), who seems clearly inconsistent as to whether x and y are required to be distinct, dividing by $x-y$ at one stage but later assuming $x = y$. In this he was closely following a proof given by Landen (1758). More generally, proofs of this kind would be criticized by successors as 'shifting the hypothesis' (Cresswell 1817, 175), that is, of making assumptions inconsistent between one part of the proof and another. A further objection (Knight 1816, 331) amounted to a concern that allowing a quantity or a ratio to vanish was equivalent to reintroducing the principles of the fluxional calculus:

> it seems to make no difference whether, in $(a+x)^m$, we substitute $x+u$ for x, and take the coefficient of u, or substitute $x+\dot{x}$, and take the coefficient of $\dot{x}$.

More generally, a concern lingered as to what algebraic operations were really legitimate when applied to infinite series, and in what conditions the equality sign linking two operations would remain valid. Daniel Cresswell (1776–1844), refuting Woodhouse, objected to the assumption that

> if $(a+x+z)^{\pm m/n}$ be expanded as a binomial, first by considering $a+x$ as one quantity, then by taking $x+z$ as one quantity, the resulting series shall be identical, whatever the index of the expansion be (Cresswell 1817, 176).

This was in one sense the beginning of a concern about the conditions for convergence, but it was in another an articulation of a more basic uncertainty about what assumptions could validly be made in deriving this most fundamental of results about infinite series.

Perhaps the most celebrated proof of this general kind was given by Landen (1764) as part of his work on what he called 'residual calculus' (1758, 1764). This took its rise from a lemma providing an equivalent expression for $\dfrac{v^{m/r} - w^{m/r}}{v - w}$ and cannot be called lucid;[11] Maseres (1791–1807, vol. I, x) would state that it was

> extremely intricate and perplexed in the algebraick operations that are necessary to be performed in it, and is, upon the whole, exceedingly difficult to understand.

Although it was reproduced by William Hales (1747–1831) in (Hales 1784) and explained at length by Maseres, it does not seem to have done much to shift the consensus of British mathematicians that a valid proof was still awaited.

[11] On Landen's method, see Appendix A.4 and Phili (1994).

3.6. *Long division*

A strategy of proof that became important late in the period was an attempt at a purely algebraic approach that would avoid the explicit manipulation of series and the problems associated with substitutions and vanishing quantities that might or might not be (a) valid, or (b) equivalent to the fluxional calculus. It adapted the algorithms for long division and/or root extraction that were part of the curriculum of many schools in elementary arithmetic. The two algorithms as applied to numbers were visually and conceptually similar, involving extracting the digits of the solution one at a time and keeping continuous track of a 'remainder' on which computational work was still to be done. Some accounts of algebra presented versions of them adapted to the extraction of the quotients or roots of algebraic expressions.

Writers including Maseres (1791–1807, vol. III) and Thomas Manning (1772–1840) (Manning 1796–98) presented this as an appropriate approach to the derivation of binomial series expansions. Maseres performed an algebraic long division to arrive at an expansion for negative whole number exponents (see Appendix A.6); Manning instead used algebraic root extraction to obtain the coefficients for exponent $1/n$ (which he then raised to the rth power to obtain a general series for power r/n).

There were several difficulties with this approach. In one sense, it was similar to Newton's original method of discovery, in that it found (the beginnings of) certain specimen series and merely asserted that their visible patterns continued indefinitely. A writer like Maseres, however, wished to present, or to appear to present, not a method of discovery but a synthetic proof, and to this extent he indulged in some pages of argument to the effect that the regularities of the long-division algorithm guaranteed the indefinite continuation of the patterns, both from one term to the next and from one value of the exponent to the next. But this, at best, merely pushed the problem into another location, since although the algorithm for long division (and that for root extraction) was widely taught and universally recognized as giving correct answers, an explicit proof of its validity was not normally presented in a textbook context, and it is not clear that a proof of its validity as applied to algebraic expressions could have been provided without assuming the Binomial Theorem or something equivalent to it. Thus, although some presented proofs using this strategy, it does not seem to have become a highly regarded approach to the problem of the Binomial Theorem and its missing proof. In the words of Swinburne and Tylecote (1827, 6),

> the rule exhibits no pretensions whatever to a larger compass […] the unceremonious application of it, in order to find the nth root of $(a + x)$, or of $(a + x)^p$, &c. is a most arbitrary, violent, and unsafe extension.

4. Induction, existence, and convergence in binomial series

The above discussion exhausts the main strategies of proof applied by British mathematicians to the Binomial Theorem in this period, although there was a good deal of variation in the details of the proofs published, many of which were of a length and complexity to deter close scrutiny either from contemporaries or from historians (a small selection of proofs may be found in outline in Appendix A). A few further conceptual issues arose in the course of these proofs which are worth brief acknowledgement here, not least because they raise issues which rarely seem to have been discussed elsewhere in the British mathematics of the period.

4.1. *The inductive question*

One such issue is the validity or the acceptability of inductive proofs. While many authors attempted to prove the Binomial Theorem directly, a significant minority chose instead to present a proof that was inductive either over the values of the exponent (where those were positive integers) or over the terms of the series; or in some cases both. An example is the exchange between Hutton and Maseres; the latter had proposed that:

> if Q be the coefficient of one of the terms of the series which is equal to $(1+x)^{1/n}$, and P the coefficient of the next preceding term, and R the coefficient of the next following term; then, if Q be $= \frac{a}{b} \times P$, to prove that R will be $= \frac{a-n}{b+n} \times Q$ (Maseres quoted in Hutton 1786, 74).

Maseres had apparently observed that this 'would be quite perfect and satisfactory, as it would include all the terms of the series, as well those that are omitted, as those that are actually set down' (Hutton 1786, 74–75). Later Cresswell, attempting to justify a strategy of proof using the long division algorithm, asserted explicitly the propriety of 'the inference, drawn from a partial division of unity by $1-x$, that the mth term of the quotient must be x^{m-1}' by arguing that

> the connexion between the subject and the prædicate, the form, and the law of continuation, of the series, is intuitively known; and the mind is as fully satisfied of the truth of the assertion, as it can be of that of any other proposition in Euclid's Elements (Cresswell 1817, 181).

Thus, if 'the *law* of the formation of the terms be found for *any* two successive terms whatever, the pth and the $(p+1)$th, it may be fairly concluded to obtain in them all' (*ibid.*).

Explicit reflection on this strategy of proof was typically limited to brief remarks such as these, and it is of note that the kinds of conceptual difficulties that would later be raised about induction as a method were not in view here. Nevertheless, it does appear that writers of the more cautious stamp preferred to avoid induction.

4.2. *Existence*

To a modern reader aware of Niels Henrik Abel's eventual strategy of proof and of the apparatus of Cauchy sequences that facilitated it (Abel 1826), obvious issues with nearly every 18th-century proof of the Binomial Theorem are the existence of a series of the form required and its convergence, or more precisely the conditions on the various variables (depending on the form in which the theorem was proved) under which it would converge. Most authors neither addressed nor mentioned these issues, though there are good reasons to suppose most were aware of the question about convergence at least. Concerning the question about existence, it is striking that a high proportion of proofs begin with the explicit, unjustified assumption that a series in ascending powers of the variable exists, the work to be done consisting merely of finding the coefficients, or – as many put it – the 'law of continuation' of those coefficients: the rule that would generate as many of them as required.[12] We have found just one explicit attempt to justify that starting assumption, by Hutton (1786). Hutton wrote that

> the form of the series, as to the powers of x, [has] never been disputed, but taken for granted, either as incapable of receiving a demonstration, or as too evident to need one (Hutton 1786, 79)

but noted that 'I have been required, by a learned friend, to vindicate the propriety of that assumption' (Hutton 1786, 79–80).

His strategy for fixing this difficulty began by assuming instead a more general form of series for $(1+x)^{1/n}$, with arbitrary powers of x. This he raised to the nth power using the Multinomial Theorem, which by equating corresponding terms enabled him to state that the terms of the series must be integer powers of x only. Unsatisfactory though this may be, it appears to be the only British take on this particular question in the period, and was reprinted in (Hutton 1812) as well as taken up by Maseres (1791–1807, vol. V) and Cresswell (1817), with explicit acknowledgement to Hutton. It must be supposed that other authors thought either that there was no problem to solve or that Hutton had adequately solved it.

4.3. *Convergence*

The question of convergence was certainly easier for 18th-century writers to formulate, and it is one of which they can hardly be supposed unaware even in those cases where absolutely no mention was made of the issue. In 18th-century conditions, the use of infinite series for practical calculation meant that rapidly converging series were at a premium, and a small strand of publication addressed ways to transform series so as to achieve faster convergence (Hutton 1776, 1780). Series that did not converge at all, or whose convergence was slow enough to require the calculation of more than perhaps half a dozen terms, were an obvious problem in these conditions.

Yet there were, of course, no general results about the convergence of series in this period of British mathematics, and the conceptual apparatus for discussing convergence precisely did not exist.[13] As mentioned above, it was simply unclear what algebraic operations might be performed on a series, or what substitutions applied to its variables, without affecting its eventual convergence. Writers on the Binomial Theorem seem overwhelmingly to have been content to ignore the issue, effectively assuming that if the binomial expression in question had a well-defined value its series would converge to that value.

There is one intriguing exception. In 1796, Maseres presented what we believe is the first and only discussion of the convergence of binomial series from the 18th-century British tradition. He wished to address 'a difficulty which may, perhaps,

[12] Authors who took this approach could at least cite a respectable pedigree in making such an assumption: it had been the starting point for Lagrange's attempt at a new foundation for the calculus (Lagrange 1797), and earlier still had been made by Maclaurin in connection with the series that now bear his name (Maclaurin 1742).

[13] It was only in the late-19th century that divergent series entered the mathematical mainstream (Hardy 1949), although there had been earlier attempts to take them seriously in the British context, notably by Augustus De Morgan (1844).

occur to the reader's mind', which amounted to the question of whether the series for $(1-x)^{-n}$ converges if x is 'but little less than 1' (Maseres 1791–1807, vol. III, *131). He showed that

> we may, by continuing the series of these generating fractions to a great number of terms, come to one in which the ratio of the numerator to the denominator shall be less than any proposed ratio of majority.

Therefore, 'however nearly the quantity x may approach to an equality with 1', there will always be a term beyond which the terms decrease

> in a greater and greater proportion continually, as the series advances. And consequently the said series will in all cases be of a finite magnitude, however nearly the quantity x may approach to an equality with 1.

It is striking that he introduced the concept of finding an n large enough to bring the terms below a given limit, and it is very tempting indeed to suppose that a continental source was involved. But no such citation appeared in the text, making this passage something of a mystery. We have not discovered that it was taken up by later writers in the purely British tradition, although of course by this date that tradition had only a few years more to run before the wholesale adoption of continental methods of analysis.[14]

A related question canvassed during that brief period was what precisely the sign of equality meant when an infinite series was involved. Woodhouse (1803) addressed this at length, and Cresswell (1817) reverted to the same question, both in the context of the Binomial Theorem and its proof. Cresswell explicitly attempted to specify the residue remaining if the series were terminated after p terms, and Swinburne and Tylecote, as part of their survey of methods of proof of the Binomial Theorem, explicitly objected to methods involving substitutions because 'numerical equality does not obtain between either $((a+x)+z)^{\pm m/n}$ and its expansion; or between $(a+(x+z))^{\pm m/n}$ and its expansion' and thus 'it surely cannot be fairly predicated, that the two expansions are identical' (Swinburne and Tylecote 1827, 29).

4.4. *The permanence principle*

The final mathematical issue that we raise here is the (mostly implicit) presence in several approaches to the Binomial Theorem of some version of the 'principle of the permanence of equivalent forms' (or 'permanence principle' for short). Formalized under the latter name by George Peacock in his *Treatise on Algebra* and taken as the basis for his notion of 'symbolical algebra', this was the principle whereby

> *Whatever form is Algebraically equivalent to another, when expressed in general symbols, must be true, whatever those symbols denote.* (Peacock 1830, 104)

Thus, in practical terms, if a general formula has been established for, say, integer values, then we may legitimately extend the application of the formula to a wider

[14] Questions of convergence are considered throughout (Lacroix 1816), for example.

class of values, such as fractional ones.[15]The utility of this principle in application to the Binomial Theorem is clear, and indeed it was present in several proofs, long before Peacock gave it a label. It was a key part, for instance, of a proof by Euler (1775), and was also employed by Robertson (1806, 318) in extending his multiplicative proof of the theorem from integer to fractional powers.[16]Moreover, a version of the permanence principle was given by Woodhouse (1803, 8–9) and invoked by him (not with the greatest clarity) in his approach to the Binomial Theorem. Unsurprisingly, Peacock (1830, 516) also used the permanence principle to establish the Binomial Theorem, and went still further in asserting that this was the *only* way to attain a general proof of the result.

5. Isolation, publication practices, and the circulation of binomial proofs

This discussion has so far treated the Binomial Theorem as a technical or conceptual problem. Something should also be said about its status as what might be called a problem in mathematics communication or even the sociology of mathematics. Its anomalous status was as much as anything a failure of the admittedly limited mechanisms that would normally have produced a consensus among mathematicians as to whether or not a valid proof had been published.

5.1. *Isolation*

It has been mentioned above, and should be emphasized, that mathematics was a divided field for most of the 18th century, with British and continental schools strikingly reluctant to read each other's publications or take seriously even such of their results as they became aware of. This is why it is feasible and worthwhile to discuss British proofs of the Binomial Theorem in isolation from the contemporaneous continental work. In John Playfair's (1748–1819) much-quoted comment,

> a man may be perfectly acquainted with every thing on mathematical learning that has been written in this country [sc. the United Kingdom], and may yet find himself stopped at the first page of the works of Euler or D'Alembert (Playfair 1808, 281).

Hutton, writing in 1786 and with a fair claim to be the Englishman best acquainted with continental mathematics at that time, stated that the proofs by Bernoulli, De Moivre and Landen were 'the principal demonstrations and investigations' of the Binomial Theorem; he made no mention of Euler or of any other continental writer on the subject (Hutton 1786, 74). The case of Bernoulli's *Ars conjectandi* has been mentioned above; it is also the case that Euler's various proofs of the Binomial Theorem were mentioned by no British writer on the Binomial Theorem before Woodhouse (1803).[17]

[15]On Peacock and the permanence principle, see Pycior (1981).

[16]Indeed, this common feature was the basis for an accusation of plagiarism that was leveled against Robertson: see footnote 17 below.

[17]Woodhouse (1807) also accused (Robertson 1806) of plagiarizing Euler's proof, to which the defence was of course that 'I did not see Euler's demonstration till my own was printed' (Robertson 1808, 15). On this episode, see Hollings (2025).

Part of the conditions of work for British algebraists in the 18th century, then, was a determination to provide results and proofs that relied only on earlier work in the British tradition: which accounts to some degree for the particular set of methods used and issues raised around the Binomial Theorem; it accounts in particular for the absence until late in the period of real engagement with the combinatorial theory; it also accounts for the persistence of the belief that the circularity of proofs involving calculus could not be resolved. More profoundly, it accounts for the long-lived conviction that that problem *should not* be resolved: that algebraic results should be founded on pure algebra, and that infinitesimal analysis should not be allowed to become the foundation of other branches of mathematics. This was a striking contrast, of course, with what was taking place on the continent at the hands of the Bernoullis and Euler and their successors.

5.2. *Venues of publication*

Internally to British mathematics and its own conditions of work, the publications on the Binomial Theorem surveyed here reveal something about the limitations of the available modes of publication as far as the dissemination of new work and the creation of consensus was concerned. Commenting on Spence's proof in 1818 – late enough that the British visibility of continental work could be assumed – an anonymous reviewer (probably Woodhouse[18]) could nevertheless still remark on

> the great neglect that English mathematicians manifest towards each other's performances. We have little doubt that, if Mr. Spence's work had been written in French instead of English, it would have found its way into nearly every mathematical library of the kingdom (Anon 1818, 259).

One of the obvious conclusions to be drawn from the multiplicity of British proofs of the Binomial Theorem is that their authors were seldom more than scantily aware of the work even of their British predecessors.

Robertson and William Sewell published their proofs in the *Philosophical Transactions*, but it was and is notorious that that venue and the Royal Society became hostile towards (some) mathematics and mathematicians in this period, particularly under the long presidency of Joseph Banks. Matters came to a head when Banks forced the resignation of Charles Hutton as Foreign Secretary of the Society, precipitating an embarrassing, widely reported row which held up the scientific business of the Society for several weeks early in 1784 (see Wardhaugh 2017a). One consequence was that some mathematicians, including Hutton himself, became unwilling to attempt publication in the *Transactions*. It is also the case that the Society's convention of having papers read at meetings by a secretary was profoundly ill-suited to the presentation of technical material in general, and in particular of papers involving significant quantities of algebra (or for that matter requiring geometrical or other diagrams for their comprehension). When the *Transactions* states that, say, Robertson's 1795 paper titled 'The binomial theorem demonstrated by the principles of multiplication' was 'read before' the Society on 21 May, there is some unclarity as to what that

[18] It was an open secret that Woodhouse was the supposedly anonymous mathematical reviewer of the journal in question (Wardhaugh 2017b, 128 and letter 87).

meant or could have meant, and some evidence that the reading was limited to the covering letter or to a presumptively non-technical summary. All of which contributed to the declining fitness of the Royal Society and its *Transactions* as venues for disseminating new mathematical work.

Some mathematicians resorted to pamphlet publication for their work (like Robertson 1808), or to the production of their own volumes of mathematical tracts or miscellanea (such as Hutton 1786). We know virtually nothing about print runs and sales of these items, although subscription lists have the potential to be revealing and suggest that in at least some cases an author was able both to support publication and to ensure dissemination to a relevant audience by assembling a list of up to (in extreme cases) three or four hundred subscribers, more commonly several dozen. Wardhaugh (2019) has suggested that one route for authors to assemble such lists of subscribers was the philomath journals such as *The Ladies' Diary* and *The Gentleman's Diary* and the network of communication they embodied; despite their unpromising titles, these annual publications contained by this date long series of difficult mathematical problems and their reader-contributed solutions. That said, this was always a laborious and doubtful way of reaching an audience (Despeaux 2014).

The philomath journals themselves were venues for the publication of a good deal of original mathematics, but their concept was to remain within the reach of the amateur, and it appears that new proofs of the Binomial Theorem fell above the level of difficulty, or the sheer length, that authors would submit or editors agree to publish in those venues; we have not in fact found significant discussion of the Binomial Theorem in those locations.

A possibly more reliable way to sell mathematics was to write a textbook, and certain authors used such venues to disseminate their proofs of the Binomial Theorem. Woodhouse, for instance, published a new proof of the Binomial Theorem within a general exposition of algebra of the broadly textbook kind (Woodhouse 1803). The question with this mode of presentation must be whether it reached the minority of mathematicians competent to judge its value as novelty, a question which we are not yet in a position to answer: but there must remain some suspicion that a mathematical culture that resorted to burying new proofs or results in the pages of textbooks was not taking the most effective way to communicate internally. A similar example is furnished by the 1830–31 *Private Tutor* of J. M. F. Wright, a short-lived journal whose goal was to provide hints, explanations, examples, and practice problems for mathematical learners in Cambridge.[19]In his first issue (21–27), Wright presented a method of proof of the Binomial Theorem involving manipulations of power series and comparison of coefficients that were somewhat novelly framed, and which he introduced with typical hubris:

> Numerous as have been the unsuccessful attempts to prove the Binomial Theorem of Newton for a general index, we are inclined to imagine the following Demonstration more worthy of that name than most, if not all, of its predecessors.

[19]The full title of the journal, part of Wright's bid to make a living as a mathematical writer and private tutor, was *The Private Tutor, and Cambridge Mathematical Repository; comprising illustrations and examples in every branch of the mathematics; and other contributions.* It ran as a weekly publication during 1830–31, with the issues then being collected together into a two-volume publication (Wright 1830–31), and some parts being collected into stand-alone treatises (such as Wright 1831a). On Wright, see Wright (2023).

As with Wright's other mathematical writings, however, it is unclear whether this had any impact.[20] We will offer further comments on appearances of proofs of the Binomial Theorem in educational contexts in §5.3.

Finally, after the row of 1784, a number of multi-author series existed whose purpose was to provide an alternative venue for mathematical publication for those who considered the *Philosophical Transactions* closed to them or their subject. Hutton's 1786 *Tracts* were for his own work only, but Maseres' *Scriptores Logarithmici* contained pieces from several living mathematical authors as well as items of historical interest and a good deal of his own work (Maseres 1791–1807). Hutton's three volumes of tracts in 1812 contained work by a few contemporaries (Hutton 1812), and the *Mathematical Repository* (1806–33) edited by Thomas Leybourne (1770–1840) both reprinted and contained new mathematics from a handful of the most active British mathematicians of its period. Proofs of the Binomial Theorem appeared in all of these series, and it may be argued that they represent the most significant venue for mathematical publication Britain possessed in the decades either side of 1800. How large an audience they in fact reached is an unexplored question.

5.3. *Pedagogy*

Having outlined the routes of publication that were open to the various authors mentioned here, it is worth focusing on those whose motivations were educational. The Binomial Theorem in its original Newtonian form was certainly not aimed at a student readership, nor were many of the proofs that we have discussed, and yet a number – including, unsurprisingly, those published within a textbook format – were self-consciously pedagogical in their goals. A good example of this appears in the papers of Robertson (1795, 1806), whose critical views of prior proofs of the Binomial Theorem we have already noted. To Robertson, the Binomial Theorem was a multiplicative result, and so a purely multiplicative proof of it was not only 'much to be desired', but would also furnish a proof 'as simple and perspicuous as the subject will permit' (Robertson 1795, 298). Other perhaps than describing himself as 'a strenuous advocate for smoothing the way to the acquisition of useful knowledge' (Robertson 1806, 305), Robertson made no direct reference to teaching the Binomial Theorem, and yet we can see that this was indeed his motivation by examining the notes of the lectures that he delivered in Oxford as Savilian Professor of Geometry:[21] they contain precisely the proof, by 'naive' manipulation of infinite series, that appears in his *Philosophical Transactions* papers.

[20] At the heart of Wright's approach was his method of indeterminate coefficients (see footnote 10 above). Elsewhere, he had already combined this with some basic calculus to provide a shorter proof of the Binomial Theorem (Wright 1828, vol. I, 6). That it was appropriate for Wright to try to present Cambridge undergraduates with an easier proof of the Binomial Theorem is borne out by the fact that the problem 'Prove the Binomial Theorem for all values of the index, and write down the first four terms of the expansion of $\frac{1}{\sqrt{(x^2-1)}}$' appeared on an examination paper for St John's College in May 1831 (as preserved in a collection of papers collated by Wright 1833, 260). Curiously, Trinity College had set a very similar problem two years earlier: 'Prove the binomial theorem when the index is a positive integer, and write down the nth term of the expansion of $\frac{1}{\sqrt{(1-x^2)}}$' (Wright 1831b, 202). In neither case is it clear which proof the examiners were expecting.

[21] Bodleian Library: MS Rigaud 16.

We may also return here to instances of the Binomial Theorem in textbook publications, where it appeared in books on algebra and those on calculus. Not unreasonably, the degree to which textbook authors discussed the difficulties of the Binomial Theorem depended on the level at which their work was pitched. For instance, one of the main routes to an elementary education in algebra during the late 18th and early 19th centuries was through the *Introduction to Algebra* by John Bonnycastle (1751–1821), written originally for use at the Royal Military Academy at Woolwich, but soon in much wider circulation. Aimed at the complete beginner in the subject, Bonnycastle's *Introduction* simply states the Binomial Theorem as a rule ('Sir Isaac Newton's rule', using the form $(P + PQ)^{\frac{m}{n}}$), and illustrates it by several examples, but is silent on the matter of proof (Bonnycastle 1782, 33).[22]The determined reader might eventually work up to Bonnycastle's more advanced *Treatise on Algebra*, which proved the Binomial Theorem by writing down two different expressions for an expansion and equating coefficients (Bonnycastle 1813, vol. II, 169–181).

Another book that was pitched at a similar level to Bonnycastle's *Introduction*, this time for students at the East India Company's College, was the *Algebra* of Bewick Bridge (1767–1833), another widely read standard of the subject (see Hollings 2024). In contrast to Bonnycastle's exclusive use of specific examples, Bridge's approach to the Binomial Theorem was slightly more general: by way of motivation, he examined the expansions of $(a + b)^n$ for small positive integer values of n, leading to general forms for the first few terms. From here, seeing the evident pattern, it was only a short intuitive leap to the general Binomial Theorem (Bridge 1810, 54–55), but nothing approaching a proof was offered, at least in the early editions of the book. In later editions he went a very little further by including a combinatorial proof of the Binomial Theorem for positive integer powers; to find expansions for negative or fractional powers, Bridge simply substituted these into the formula established for positive integers, noting that

> To prove the truth of [the Binomial Theorem] when n is *negative* or *fractional*, requires a species of analytical investigation not well adapted to an Elementary Treatise (Bridge 1818, 184).

Bridge instead referred the interested reader to Bonnycastle's *Treatise on Algebra*.

Turning next to books aimed at undergraduate students, we observe that two of the main texts that were in use in Cambridge in the second half of the 18th century were those of Colin Maclaurin (1698–1746): his *Treatise of Fluxions* (Maclaurin 1742) and his *Treatise of Algebra* (Maclaurin 1748). We have already noted that the former contains a fluxional proof of the Binomial Theorem; the latter features no formal proof of the result, instead taking an approach similar to that later adopted by Bridge (Maclaurin 1748, 38–41). Around the middle of our period of interest, Maclaurin's works were replaced as standard Cambridge undergraduate texts by James Wood's *Elements of Algebra* (Wood 1795) and Samuel Vince's (1749–1821)

[22]We are unsure how much to read into the rewording of a later edition: when introducing the rule for expanding $(P + PQ)^{\frac{m}{n}}$, Bonnycastle's note that it was 'first *proposed* by Newton' (our emphasis) may be a nod to the uncertain proof status of the Binomial Theorem (Bonnycastle 1824, 143).

Principles of Fluxions (Vince 1795).[23]Although some aspects of the relevant mathematical learning were updated thereby, students would have seen little change in how the Binomial Theorem was handled: Vince (1795, 45–47) gave a fluxional proof early in his book, and proceeded immediately to use the Binomial Theorem for computational purposes, while Wood (1795, 109–110) provided some justification via multiplication for the Binomial Theorem for positive integer powers, but then extended this to negative and fractional powers with little further comment. Wood's *Algebra* would eventually run to multiple editions, spanning much of the 19th century, and by the sixth edition, a little further justification of the Binomial Theorem for positive fractional powers had been added in the form of some elaborate manipulations of certain power series that were assumed to exist (Wood 1815, 119–120).[24]

We cannot survey all textbook approaches to the proof of the Binomial Theorem in the late 18th and early 19th centuries, but we do perhaps begin to see a pattern emerging from the few examples given here, namely that educational authors were often inclined to sidestep the difficulties of the Binomial Theorem altogether, simply by setting it up as a rule without proof. Where any justification was given, the inclusion of fluxional proofs in calculus texts may arguably have been a pragmatic choice in a context that was skewed heavily towards the applications of the subject. In algebraic settings, attempts at proof were much less extensive, often omitting the difficult cases, and relying as far as possible on pure algebraic manipulation: the 'simple and perspicuous' approach advocated by Robertson, among others. But this tendency quietly to omit the difficulties did not find favour with all readers. Augustus De Morgan, reviewing the ninth edition of Wood's *Algebra*, took issue with aspects of Wood's treatment of the Binomial Theorem (Wood 1830). For instance, Wood's assertion, without any further justification, that the theorem holds for negative powers, was deemed by De Morgan to be 'an omission of considerable magnitude', since it was unlikely that the pupil at this stage of their education would be able to 'supply the deficiency' (De Morgan 1832, 282).[25]De Morgan seems to have seen this as a rare deviation on Wood's part from the 'honest' pedagogy of which the former was a strong advocate (De Morgan 1831): that mathematical learners ought to be made aware of points of difficulty, even if they are not able to resolve them for the time being.[26]And De Morgan practiced what he preached: his own treatment of the Binomial Theorem published soon afterwards was the most thorough to date, with an attempt to combine what he deemed to be the best parts of previous proofs (De Morgan 1835, ch. XI). De Morgan acknowledged the problems associated with the Binomial Theorem:

[23]These belonged to a new series of books for undergraduates, titled *The Principles of Mathematics and Natural Philosophy*: see (Hollings 2024).

[24]After Wood's death in 1839, his *Algebra* went through several further editions under the direction of a former colleague, Thomas Lund (see Hollings 2024). The eleventh edition boasted a section on the Binomial Theorem that was 'entirely remodeled' (Wood 1841, Advertisement) and now included Euler's proof of the theorem.

[25]Wood's proof of the Binomial Theorem may indeed have caused difficulties for Cambridge undergraduates: one former student encountered by the mathematician Thomas Penyngton Kirkman (1806–1985) described Wood's proof as 'mystery inexplicable' (Crilly 2024, 101).

[26]Elsewhere in the review, De Morgan praised Wood's acknowledgement of inadequacies in his proof of the Fundamental Theorem of Algebra.

> Every proof which has ever been given of this theorem has been contested; that is, no one has ever disputed the truth of the theorem itself, but only the method of establishing it. And the general practice is, for each proposer of a new proof to be very much astonished at the want of logic of his predecessors (De Morgan 1835, 213).

The method presented by De Morgan combined ideas about limits with the well-known proof by Euler, which rested upon a version of the permanence principle. De Morgan acknowledged that there were objections to each of these methods, but asserted that each resolved those of the other: Euler's approach establishes the existence of a series that the limit-based method assumes, and limits supply an analytical version of the merely synthetical arguments made by Euler. To what extent De Morgan's student readers actually worried over these points remains an open question, however.

6. Concluding remarks: open questions raised by the British binomial tradition

This discussion has shown that British interest in new proofs of the Binomial Theorem in the late-18th and early-19th centuries arose from a complex interplay of factors rather than from any single dominant issue. The anomalous status of the theorem as universally assumed true but not actually proved by its discoverer created a powerful initial stimulus. British mathematical isolation from continental developments shaped local approaches, while technical worries about the foundations of the calculus, the legitimacy of infinitesimal methods in algebra, the acceptability of inductive proofs, and the common but rarely justified assumption of the existence and convergence of appropriate series all contributed to sustained unease. The absence of a general theory of infinite series, and the fact that many of the most interesting results about series presupposed the Binomial Theorem, ensured that doubts about the validity of extant proofs could persist for decades. The proliferation of distinct forms and exponent cases for the Binomial Theorem added further complexity, fostering both genuine confusion and opportunities for authors to present yet another 'new' proof of what was by then an old and much-discussed theorem.

The theorem's celebrity made it a natural target for mathematicians seeking to establish a reputation, yet this ambition was constrained by the limited number of outlets for substantial mathematical writing and the small readership that such publications typically enjoyed. In practice, many proofs seem to have attracted little close scrutiny, and even conscientious authors often remained unaware of significant earlier contributions. Against this backdrop, several broader lines of inquiry suggest themselves. Early modern 'arithmetical' proofs of algebraic results might be examined to see whether they generated difficulties comparable to those surrounding the Binomial Theorem, and other prominent 17th- and 18th-century achievements – such as the method of first and last ratios, the Fundamental Theorem of Algebra, or attempts to establish the Parallel Postulate – could be studied as parallel cases of results whose status remained unsettled over long periods. More generally, it seems fruitful to investigate how often important theorems acquired similarly ambiguous 'proof status', and, in the specific context of 18th-century British mathematics, what standards and practices were employed in order to negotiate, stabilize, or leave unresolved such ambiguities.

Disclosure statement

No potential conflict of interest was reported by the author(s).

ORCID

Benjamin Wardhaugh http://orcid.org/0000-0002-3383-7574
Christopher Hollings http://orcid.org/0000-0003-3181-1968

References

Abel, Niels Henrik, 'Beweis eines Ausdruckes, von welchem die Binomial-Formel ein einzelner Fall ist', *Journal für die reine und angewandte Mathematik*, 1 (1826), 159–160.

Anon, 'Philosophical Transactions of the Royal Society of London, for the year 1817. part II', *The Monthly Review*, 85(Mar) (1818), 254–261.

Bernoulli, Jacob, *Ars conjectandi, opus posthumum. Accedit Tractatus de seriebus infinitis, et epistola Gallicè scripta de ludo pilæreticularis*, Basel: Impensis Thurnisiorum, Fratrum, 1713.

Bonnycastle, John, *An introduction to algebra, with notes and observations; designed for the use of schools and other places of public education*, London: Printed for J. Johnson, 1782.

Bonnycastle, John, *A treatise on algebra, in practice and theory, in two volumes, with notes and illustrations; containing a variety of particulars relating to the discoveries and improvements that have been made in this branch of analysis*, London: Printed for J. Johnson, 1813.

Bonnycastle, John, *An introduction to algebra, with notes and observations; designed for the use of schools and other places of public education*, London: Printed for J. Nunn, Longman, Rees, Orme, Brown, and Green, etc, 13th edition, 1824.

Bonnycastle, John, *An introduction to algebra, with notes and observations; designed for the use of schools and other places of public education*, London: Printed for J. Nunn, Longman, Rees, Orme, Brown, and Green, etc, 14th edition, 1828.

Bos, Henk J M, *Redefining geometrical exactness: Descartes' transformation of the early modern concept of construction*, New York: Springer, 2013.

Bridge, Bewick, *Lectures on the elements of algebra*, London: Cadell & Davies, 1810.

Bridge, Bewick, *An elementary treatise on algebra*, London: Cadell & Davies 1815, 4th edition [of *Lectures on the Elements of Algebra*].1818.

Collins, John, *Commercium epistolicum*, Londini: typis Pearsonianis, 1712.

Colson, John, *The method of fluxions and infinite series; with its application to the geometry of curve-lines*, London: Printed by Henry Woodfall, 1736 (transl.).

Coolidge, J L, 'The story of the binomial theorem', *The American Mathematical Monthly*, 56(3) (1949), 147–157.

Craik, Alex D D, 'Polylogarithms, functional equations and more: the elusive essays of William Spence (1777–1815)', *Historia Mathematica*, 40(4) (2013), 386–422.

Cresswell, Daniel, *An elementary treatise on the geometrical and algebraical investigation of maxima and minima, being the substance of a course of lectures delivered conformably to the will of Lady Sadler: to which is added, a selection of propositions deducible from Euclid's Elements*, Cambridge: Printed by J. Smith, printer to the University; for J. Deighton & Sons; and Law & Whittaker, Ave-Maria-Lane, London, 2nd edition, 1817,

Crilly, Tony, *The Rev. Thomas P. Kirkman: A mathematician from the North Country*, Lulu, 2024.

De Morgan, Augustus, *On the study and difficulties of mathematics*, London: Society for the Diffusion of Useful Knowledge, 1831.

De Morgan, Augustus, 'Wood's algebra [9th ed.]', *Quarterly Journal of Education*, 3 (1832), 276–285.

De Morgan, Augustus, *The elements of algebra preliminary to the differential calculus and for the higher classes of schools in which the principles of arithmetic are taught*, London: Printed for John Taylor, 1835.

De Morgan, Augustus, 'On divergent series and various points of analysis connected with them', *Transactions of the Cambridge Philosophical Society*, 8 (1844), 182–203.

Despeaux, Sloan Evans, ‘Mathematical questions: A convergence of mathematical practices in British journals of the eighteenth and nineteenth centuries’, *Revue d’histoire des mathématiques*, 20 (2014), 5–71.

Dubbey, J M, ‘The introduction of the differential notation to Great Britain’, *Annals of Science*, 19(1) (1963), 37–48.

Enros, Philip C, ‘The Analytical Society (1812–1813): precursor of the renewal of Cambridge mathematics’, *Historia Mathematica*, 10 (1) (1983), 24–47.

Euler, Leonhard, ‘Demonstratio theorematis Neutoniani de evolutione potestatum binomii pro casibus, quibus exponentes non sunt numeri integri’, *Novi commentarii Academiae Scientiarum Imperialis Petropolitanae*, 19 (1775), 103–111 (E465).

Evans, Lewis, ‘An improved demonstration of Sir Isaac Newton’s binomial theorem, on fluxional principles’, *Philosophical Magazine*, 64 (318) (1824), 270–272.(MS version in History of Science Museum, Oxford, MSS Evans 23).

Guicciardini, Niccolò, *The development of Newtonian calculus in Britain 1700–1800*, Cambridge University Press, 1989.

Guicciardini, Niccolò, *Reading the* Principia*: The debate on Newton’s mathematical methods for natural philosophy from 1687 to 1736*, Cambridge University Press, 1999.

Hales, William, *Analysis æquationum*, Dublinii: excudebat Josephus Hill, 1784.

Hardy, G H, *Divergent series*, Oxford: Clarendon Press, 1949.

Hodgson, James, *The doctrine of fluxions, founded on Sir Isaac Newton’s method, published by himself in his tract upon the quadrature of curves*, London: printed by T. Wood, for the author, 1736.

Hollings, Christopher D, ‘Two books on the elements of algebra’, in Philip Beeley and Ciarán Mac an Bhaird (eds), *Mathematical book histories. Printing, provenance, and practices of reading*. (Trends in the History of Science), Cham: Birkhäuser, 113–164. 2024.

Hollings, Christopher D, ‘Abraham Robertson, the binomial theorem, and the nature of plagiarism in mathematics’, *The American Mathematical Monthly*, 132(6) (2025), 501–511.

Hollings, Christopher D, ‘J. M. F. Wright and Newton’s method of first and last ratios’, in Maria Zack and David Waszek (eds), *Research in history and philosophy of mathematics: The CSHPM 2024 Volume* (Annals of the Canadian Society for History and Philosophy of Mathematics/Société canadienne d’histoire et de philosophie des mathématiques), Cham: Birkhäuser, 47–65. 2026.

Horsley, Samuel (ed.), *Isaaci Newtoni opera quæexstant omnia*, Londini: Excudebat Joannes Nichols, 1779.

Hutton, Charles, ‘A new and general method of finding simple and quickly-converging series; by which the proportion of the diameter of a circle to its circumference may easily be computed to a great number of places of figures’, *Philosophical Transactions of the Royal Society of London*, 66 (1776), 476–492.

Hutton, Charles, ‘Of cubic equations and infinite series’, *Philosophical Transactions of the Royal Society of London*, 70 (1780), 387–450.

Hutton, Charles, *Tracts, mathematical and philosophical*, vol 1, London: Printed for G. G. J. and J. Robinson, 1786.

Hutton, Charles, *A mathematical and philosophical dictionary: containing an explanation of the terms, and an account of the several subjects, comprized under the heads mathematics, astronomy, and philosophy both natural and experimental: with an historical account of the rise, progress, and present state of these sciences: also memoirs of the lives and writings of the most eminent authors, both ancient and modern, who by their discoveries or improvements have contributed to the advance of them*, 2 vols, London: Printed by J. Davis, for J. Johnson, in St. Paul’s Church-Yard; and G. G. and J. Robinson, in Paternoster-Row, 1795.

Hutton, Charles, *A course of mathematics, in two volumes: composed, and more especially designed, for the use of the gentlemen cadets in the Royal Military Academy at Woolwich*, vol 2, London: Printed for G. G. and J. Robinson, Paternoster-Row, 1798.

Hutton, Charles, *Tracts on mathematical and philosophical subjects: comprising, among numerous important articles, the theory of bridges, with several plans of recent improvement. Also the results of numerous experiments on the force of gunpowder, with applications to the modern practice of artillery*, 3 vols, London: Printed for F. C. and J. Rivington; G. Wilkie and J. Robinson; J. Walker; Lackington, Allen and Co.; Cadell and Davies; J. Cuthell; B. and R. Crosby and Co.; J. Richardson; J. M. Richardson; R. Baldwin; and G. Robinson, 1812.

Knight, Thomas, 'A new demonstration of the binomial theorem', *Philosophical Transactions of the Royal Society of London*, 106 (1816), 331–334.

Knight, Thomas, 'Note respecting the demonstration of the binomial theorem inserted in the last volume of the Philosophical Transactions', *Philosophical Transactions of the Royal Society of London*, 107 (1817), 245–251.

Lacroix, Sylvestre-François, *An elementary treatise on the differential and integral calculus*, Cambridge, 1816.

Lagrange, Joseph-Louis, *Théorie des fonctions analytiques*, Paris: De l'Imprimerie de la République, 1797.

Landen, John, *A discourse concerning the residual analysis: a new branch of the algebraic art, of very extensive use, both in pure mathematics and natural philosophy*, London: Printed for J. Nourse, 1758.

Landen, John, *The residual analysis: a new branch of the algebraic art, of very extensive use, both in pure mathematics, and natural philosophy. Book I*, London: Printed for the author; and sold by L. Hawes, W. Clarke, and R. Collins, at the Red Lion in Pater-noster Row, 1764.

Maclaurin, Colin, *A treatise of fluxions*, 2 vols, Edinburgh: Printed by T. W. and T. Ruddimans, 1742.

Maclaurin, Colin, *A treatise of algebra*, London: A. Millar and J. Nourse, 1748.

Manning, Thomas, *An introduction to arithmetic and algebra*, 2 vols, Cambridge: Printed by Benjamin Flower: and sold by Nicholson, Lunn, and Deighton, Cambridge; Cooke, Oxford; Stevenson, Norwich, 1796–98.

Maseres, Francis, 'A method of finding, by the help of Sir Isaac Newton's binomial theorem, a near value of the very slowly-converging infinite series $x + \frac{xx}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \&c.$ when x is very nearly equal to 1', *Philosophical Transactions of the Royal Society of London*, 68 (1778), 895–901.

Maseres, Francis, *Scriptores logarithmici: or, a collection of several curious tracts on the nature and construction of logarithms, mentioned in Dr. Hutton's historical introduction to his new edition of Sherwin's mathematical tables: together with some tracts on the binomial theorem and other subjects connected with the doctrine of logarithms*, 6 vols, London: Printed by J. Davis, and sold by B. White and Son, Fleet-Street, 1791–1807.

Mercator, Nicolaus, *Logarithmo-technia, sive, Methodus construendi logarithmos nova, accurata, & facilis scripto antehác communicata, anno sc. 1667, nonis Augusti: cui nunc accedit, Vera quadratus hyperbolae, & Inventio summae logarithmorum*, Londini: Typis Guilielmi Godbid, & Impensis Mosis Pitt, 1668.

Moivre, Abraham De, 'A method of raising an infinite multinomial to any given power, or extracting any given root of the same', *Philosophical Transactions of the Royal Society of London*, 19(230) (1697), 619–625.

Moivre, Abraham De, *Miscellanea analytica de seriebus et quadraturis: Accessere* variæ considerationes *de methodis comparationum, combinationum & differentiarum, solutiones difficiliorum aliquot problematum ad sortem spectantium, itemque constructiones faciles orbium planetarum, una cum determinatione maximarum & minimarum quæ in motibus corporum cœlestium occurrunt*, Londini: Excudebant J. Tonson & J. Watts, 1730.

Noble, Eduardo, 'A history of the binomial and multinomial theorems', in *The rise and fall of the German combinatorial analysis*, Cham: Springer, 17–66. 2022.

Peacock, George, *A treatise on algebra*, Cambridge: Printed by J. Smith, Printer to the University, 1830.

Phili, Christine, 'John Landen: first attempt for the algebrization of infinitesimal calculus', in Kostas Gavroglu, Jean Christianidis, Efthymios Nicolaidis (eds), *Trends in the historiography of science*, Dordrecht: Springer, 279–293. 1994.

Phillips, Christopher, 'Robert Woodhouse and the evolution of Cambridge mathematics', *History of Science*, 44(1) (2006), 69–93.

Playfair, John, 'Traité de Méchanique Céleste. Par P. S. La Place', *Edinburgh Review*, 11(22) (1808), 249–284.

Pycior, Helena M, 'George Peacock and the British origins of symbolical algebra', *Historia Mathematica*, 8 (1) (1981), 23–45.

Pycior, Helena M, 'Early criticism of the symbolical approach to algebra', *Historia Mathematica*, 9 (4) (1982), 392–412.

Raphson, Joseph, *The history of fluxions, shewing in a compendious manner the first rise of, and various improvements made in that incomparable method*, London: Printed by William Pearson, and sold by Richard Mount, on Tower-Hill, 1715.

Robertson, Abram, 'The binomial theorem demonstrated by the principles of multiplication', *Philosophical Transactions of the Royal Society of London*, 85 (1795), 298–321.

Robertson, Abram, 'A new demonstration of the binomial theorem, when the exponent is a positive or negative fraction', *Philosophical Transactions of the Royal Society of London*, 96 (1806), 305–326.

Robertson, Abram, *A reply to a Critical and Monthly reviewer, in which is inserted Euler's demonstration of the binomial theorem*, Oxford: At the University Press, 1808.

Salvemini, Giovanni, 'J. Castillioneus Dno. De Montagny V. C. Philosophiæ Professori in academia Lauzannensi, RegiæSocietatis Londinensis Membro dignissimo, Sti Evangelii Ministro, &c. &c. S. P. D', *Philosophical Transactions of the Royal Society of London*, 42(463) (1742), 91–98.

Saunderson, Nicholas, *The elements of algebra, in ten books*, Cambridge: Printed at the University-Press: and sold by Mrs. Saunderson, at Cambridge, by John Whiston bookseller at Boyle's Head in Fleetstreet London, and Thomas Hammond bookseller in York, 1740.

Sewell, William, 'Newton's binomial theorem legally demonstrated by algebra', *Philosophical Transactions of the Royal Society of London*, 86 (1796), 382–384.

Simpson, Thomas, 'A general method for exhibiting the value of an algebraic expression involving several radical quantities in an infinite series: wherein Sir Isaac Newton's theorem for involving a binomial, with another of the same author, relating to the roots of equations', *Philosophical Transactions of the Royal Society of London*, 47 (1752), 20–27.

Spence, William, *An essay on the theory of the various orders of logarithmic transcendents: with an inquiry into their applications to the integral calculus and the summation of series*, London: Printed for John Murray and Archibald Constable and Co, 1809.

Stenhouse, Brigitte, 'Mary Somerville's early contributions to the circulation of differential calculus', *Historia Mathematica*, 51 (2020), 1–25.

Stewart, John, *Sir Isaac Newton's two treatises of the quadrature of curves and analysis by equations of an infinite number of terms*, London: Bettenham, 1745.

Swinburne, Thomas, and Tylecote, Thomas, *The true development of the binomial theorem, or more properly of its base, the monomial theorem, by a purely algebraical process; preceded by strictures exhibiting the defects of some of the most approved demonstrations*, London: C. and J. Rivington 1827.

Vince, Samuel, *The principles of fluxions, designed for the use of students in the university*, Cambridge: John Burges, 1795.

Vince, Samuel, 'Two proofs of the binomial theorem', *Transactions of the Royal Irish Academy*, 12 (1815), 31–33.

Wallis, John, *Arithmetica Infinitorum: sive de Curvilineorum quadratura, &c.*, Oxford: Lichfield and Robinson, 1656.

Wardhaugh, Benjamin, 'Charles Hutton and the 'Dissensions' of 1783–84: scientific networking and its failures', *Notes & Records of the Royal Society of London*, 71(1) (2017a), 41–59.

Wardhaugh, Benjamin, *The correspondence of Charles Hutton: mathematical networks in Georgian Britain*, Oxford: Oxford University Press, 2017b.

Wardhaugh, Benjamin, *Gunpowder and geometry: the life of Charles Hutton, pit boy, mathematician and scientific rebel*, London: William Collins, 2019.

Whiteside, D T, 'Newton's discovery of the general binomial theorem', *The Mathematical Gazette*, 45(353) (1961), 175–180.

Wood, James, *The elements of algebra, designed for the use of students in the university*, Cambridge: John Burges, 1795.

Wood, James, *The elements of algebra, designed for the use of students in the university*, Cambridge: J. Smith, 6th edition, 1815.

Wood, James, *The elements of algebra, designed for the use of students in the university*, Cambridge: Printed for J. Smith, Printer to the University; and sold by Deighton & Sons, 8th edition, 1825,

Wood, James, *The elements of algebra, designed for the use of students in the university*, Cambridge: Printed for J. Smith, Printer to the University; and sold by Deighton & Sons, 9th edition, 1830.

Wood, James, *The elements of algebra, designed for the use of students in the university*, Cambridge: Printed at the Pitt Press by John W. Parker, Printer to the University, 11th edition by Thomas Lund, 1841.

Woodhouse, Robert, *The principles of analytical calculation*, Cambridge: Printed at the University Press, 1803.

Woodhouse, Robert, 'Philosophical Transactions of the Royal Society of London, for the year 1806, part II', *The Monthly Review*, 52(Feb) (1807), 159–172.

Wright, J M F, *A commentary on Newton's* Principia. *With supplementary volume. Designed for the use of students at the universities*, vol 2, London: Black, Young and Young; Cambridge: J. & J. J. Deighton, 1828.

Wright, J M F, *The private tutor, and Cambridge mathematical repository: comprising illustrations and examples in every branch of the mathematics; with essays, problems, solutions, and other contributions*, vol. 2, Cambridge: W. P. Grant, 1830–31.

Wright, J M F, *An elementary treatise on the theory of numbers; from The Private Tutor*, Cambridge: W. P. Grant, 1831a.

Wright, J M F, *A collection of Cambridge mathematical examination papers; as given at the several colleges. Part II. Containing papers in the branches of mixed mathematics*, Cambridge: W. P. Grant, 1831b.

Wright, J M F, *Supplement to a collection of Cambridge mathematical examination papers; as given at the several colleges. Part II. Containing papers in mixed mathematics*, Cambridge: W. P. Grant, 1833.

Wright, J M F, *Student life in nineteenth-century Cambridge: John Wright's* Alma Mater, University of Exeter Press, 2023.

Appendix. Some proofs of the Binomial Theorem

We give here basic outlines of some of the proofs mentioned in the paper.

A.1. *Newton (1664–5)*

Although, as we have made clear in the foregoing paper, Newton did not *prove* the Binomial Theorem, we can nevertheless find some intuition for his discovery in the manuscript (CUL Add. MS 3958.3, f. 72) cited in §3.1. Newton was interested here in finding the quadrature of certain standard curves, and drew up tables to aid him in spotting a general pattern in the results. In one grid, for example, Newton considered the integral (to use an anachronistic term) of successive powers of the binomial $(1 + x)$ and tabulated the numerical coefficients of x, $\frac{x^2}{2}$, $\frac{x^3}{3}$, …, in the results. We give a fragment of this (unlabelled) table in Figure A1; the first row contains the coefficients of x, the second those of $\frac{x^2}{2}$, and so on. Ignoring the first column for the moment, the second column corresponds to the integral of $(1+x)^0$, the third to that of $(1+x)$, the fourth to that of $(1+x)^2$, and so on. Newton observed that within the table, we find a skewed version of Pascal's triangle, whose pattern we can extrapolate leftwards in order to find series for $(1+x)$ to negative integer powers. Once the top-left 1 has been filled in, either by other methods, or by asserting that the value here *ought* to be 1, given the appearance of the rest of the first row, the remainder of the first column can be filled in, giving the expression $x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ for the integral of $(1+x)^{-1}$.

Elsewhere on the same manuscript page, Newton similarly considered powers of the binomial $(1-x^2)$. A fragment of the corresponding table is given in Figure A2. This time, the successive rows contain the coefficients of $x, -\frac{x^3}{3}, \frac{x^5}{5}, -\frac{x^7}{7}, \ldots$. The third column corresponds to the integral of $(1-x^2)^0$, the fifth to that of $(1-x^2)$, the seventh to that of $(1-x^2)^2$, and so on. The skewed Pascal triangle is still present, provided we skip columns. Once again filling in some of the top values by other means, such as adapting the standard formula for binomial coefficients to fractional values, we may extend the pattern of the Pascal triangle to the interpolated columns, which correspond to half-integer powers of $(1-x^2)$. Thus, for example, the fourth column gives a series for the integral of $(1-x^2)^{\frac{1}{2}}$. The first and second columns give series for the integrals of $(1-x^2)^{-1}$ and $(1-x^2)^{-\frac{1}{2}}$.

By applying such methods to a variety of curves and carefully studying the resulting patterns, Newton amassed a wealth of numerical evidence that led him to, and convinced him of the truth of, the general Binomial Theorem.

A.2. *Raphson (1715)*

To expand upon the comments made at the beginning of our §3.3, we recall that Raphson (1715, 10) started from the assumed expansion

$$(a+x)^m = b + cx + dx^2 + ex^3 \ \&\text{c.} \tag{A1}$$

Taking fluxions gave him

$$m(a+x)^{m-1}\dot{x} = c\dot{x} + 2dx\dot{x} + 3ex^2\dot{x} \ \&\text{c.},$$

which he divided by $\dot{x}$ to obtain

$$m(a+x)^{m-1} = c + 2dx + 3ex^2 \ \&\text{c.} \tag{A2}$$

Then

$$\frac{m(a+x)^{m-1}}{(a+x)^m} = m(a+x)^{-1} = \frac{m}{a+x} = \frac{c + 2dx + 3ex^2 + 4fx^4 \ \&\text{c.}}{b + cx + dx^2 + ex^3 \ \&\text{c.}}, \tag{A3}$$

using (A1) and (A2), and as a consequence

$$\begin{aligned} bm + cmx + dmx^2 + emx^3 \ \text{c.} = ac \ &+ \ 2adx \ + \ 3aex^2 \ + \ 4afx^3 \ \ \&\text{c.} \\ & \ \ cx \ \ + \ 2d^2x^2 \ + \ 3e^3x^3 \ \ \&\text{c.} \end{aligned} \tag{A4}$$

Setting $b = a^m$, 'as the nature of Powers requires', Raphson compared coefficients to derive that

$$c = ma^{m-1}, \quad d = \frac{m-1}{2} \times ma^{m-2}, \quad e = \frac{m-2}{3} \times \frac{m-1}{2} \times ma^{m-3}, \quad \text{and so on.}$$

1	1	1	1	1	1	⋯
−1	0	1	2	3	4	⋯
1	0	0	1	3	6	⋯
−1	0	0	0	1	4	⋯
1	0	0	0	0	1	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋱

Figure A1. Example of Newton's method of extrapolation.

A.3. *Hodgson (1736)*

Another calculus-based proof that was noted in passing in §3.3 was that given by Hodgson (1736, 24–26). This proceeds in a very similar manner to Raphson's method, but seeks to expand Newton's original binomial, namely $(p + px)$. Hodgson started from the assumption that

$$(1 + x)^n = 1 + Ax + Bxx + Cx^3 + Dx^4 + Ex^5 + Fx^6 \text{ \&c.},$$

took fluxions, divided by $\dot{x}$, and then performed the further division needed to obtain his own version of (A3). This led Hodgson in turn to his own version of (A4), and after comparing coefficients, the result that[27]

$$(1 + x)^n = 1 + nx + n \times \frac{n-1}{2}x^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3}x^3 \text{ \&c.} \qquad (A5)$$

Multiplying both sides of (A5) by p^n immediately gave Hodgson a series for $(p + px)^n$, whereupon he introduced the curious relabeling $p^n = A$, $np^n x = B$, $n \times \frac{n-1}{2} p^n x^2 = C$, and so on, which allowed him to write down a series that contained only explicit first powers of x.

A.4. *Landen (1764)*

Landen famously advocated for the reformulation of the calculus on a purely algebraic basis, although his efforts in this direction were not entirely successful. As we noted in §3.5, he based his approach to the Binomial Theorem (as also much of the rest of his text) on the foundational lemma that

$$\frac{x^{\frac{m}{n}} - v^{\frac{m}{n}}}{x - v} = x^{\frac{m}{n}-1} \times \frac{1 + q + q^2 + q^3 + \cdots}{1 + q^{\frac{m}{n}} + q^{\frac{2m}{n}} + q^{\frac{3m}{n}} + \cdots},$$

where m and n are integers, and $q = \frac{v}{x}$; the numerator of the fractional part of the right-hand side contains m terms in total, and the denominator n. Landen barely proved this result, but

1	1	1	1	1	1	1	⋯
−1	−1/2	0	1/2	1	3/2	2	⋯
1	3/8	0	−1/8	0	3/8	1	⋯
−1	−5/16	0	3/48	0	−1/16	0	⋯
1	35/128	0	−15/384	0	3/128	0	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Figure A2. Example of Newton's method of interpolation.

[27]Hodgson continued to list all terms up to and including that for x^6, but we curtail his series here in the interests of saving space.

asserted that it could be verified through simple algebraic manipulations (Landen 1764, 6). In what follows, we employ Hutton's slightly more transparent summary of Landen's approach to the Binomial Theorem (Hutton 1786, 73), which he included alongside his own treatment of the theorem (see Appendix A.5).

Like many authors before him, Landen began by assuming the existence of the series

$$(1+x)^{\frac{m}{n}} = 1 + ax + bx^2 + cx^3 \text{ \&c.} \tag{A6}$$

This naturally also gave him

$$(1+y)^{\frac{m}{n}} = 1 + ay + by^2 + cy^3 \text{ \&c.} \tag{A7}$$

Subtracting (A7) from (A6), dividing by $x-y$, and applying the foundational lemma gives:

$$\begin{aligned}\frac{(1+x)^{\frac{m}{n}} - (1+y)^{\frac{m}{n}}}{x-y} &= (1+x)^{\frac{m}{n}-1} \times \frac{1 + \frac{1+y}{1+x} + \left(\frac{1+y}{1+x}\right)^2 + \left(\frac{1+y}{1+x}\right)^3 + \cdots}{1 + \left(\frac{1+y}{1+x}\right)^{\frac{m}{n}} + \left(\frac{1+y}{1+x}\right)^{\frac{2m}{n}} + \left(\frac{1+y}{1+x}\right)^{\frac{3m}{n}} + \cdots} \\ &= a + b(x+y) + c(x^2+xy+y^2) + d(x^3+x^2y+xy^2+y^3) \text{ \&c.}\end{aligned}$$

Since this must hold for any value of y, we set $x = y$, which yields

$$\frac{m}{n} \times (1+x)^{\frac{m}{n}-1} = a + 2bx + 3cx^2 + 4cx^3 \text{ \&c.}$$

Multiplying both sides by $(1+x)$ and applying (A6) gives

$$\frac{m}{n} + \frac{m}{n}ax + \frac{m}{n}bx^2 + \frac{m}{n}cx^3 \text{ \&c.} = a + \left.\begin{matrix}2b\\a\end{matrix}\right\}x + \left.\begin{matrix}3c\\2b\end{matrix}\right\}x^2 + \left.\begin{matrix}4d\\3c\end{matrix}\right\}x^3 \text{ \&c.}$$

It remains to compare coefficients, and solve for $a, b, c, \ldots$.

A.5. *Hutton (1786)*

As noted in §3.4, Hutton's approach to the Binomial Theorem invoked the Multinomial Theorem (Hutton 1786, 78). He began by assuming the existence of the following series:

$$(1+x)^{\frac{1}{n}} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \text{\&c.}$$

for indeterminate $A, B, C, D, \ldots$. Raising both sides to the power n gives

$$1 + x = (1 + Ax + Bx^2 + Cx^3 + Dx^4 + \text{\&c.})^n.$$

The Multinomial Theorem for integer powers (elaborated by Hutton in the preceding pages) applies to the right-hand side of the above, giving[28]

$$1 + \frac{\overbrace{nA}^{a}}{1}x + \frac{\overbrace{2nB + (n-1)Aa}^{b}}{2}x^2 + \frac{\overbrace{3nC + (2n-1)Ba + (n-2)Ab}^{c}}{3}x^3 + \text{\&c.}$$

[28]Hutton also gave the term in x^4, but we omit that here in the interests of saving space.

Now, if this is to equal $1 + x$, we must have $nA = 1$ and $b = c = \cdots = 0$. Solving the resulting system of equations yields values for the original coefficients

$$B = \frac{1-n}{2n}Aa, \quad C = \frac{1-2n}{3n}Ba, \quad D = \frac{1-3n}{4n}Ca, \quad \text{and so on,}$$

and therefore a series for $(1+x)^{\frac{1}{n}}$:

$$1 + \frac{1}{n}x + \frac{1}{n}\cdot\frac{1-n}{2n}x^2 + \frac{1}{n}\cdot\frac{1-n}{2n}\cdot\frac{1-2n}{3n}x^3 + \&\text{c.}$$

A.6. *Maseres (1796)*

As we have already observed in §3.2, Maseres presented translations of parts of the *Ars conjectandi* by way of introducing his English readers to Bernoulli's ideas on combinatorics (Maseres 1791–1807, vol. III, 25–98). In an appendix to his translation, Maseres used Bernoulli's ideas to investigate the patterns found in expansions to positive integer powers of the binomial $(a + b)$ by writing out the expansions explicitly for several successive powers (Maseres 1791–1807, vol. III, *100–*133). A natural extension then occurred to Maseres: a similar expansion for negative integer powers may be obtained by dividing 1 repeatedly by $(a + b)$ (see §3.6). Simplifying matters slightly, he began by dividing 1 by $1 + x$ to obtain the series $1 - x + x^2 - x^3 + \cdots$ as an expansion for $\frac{1}{1+x}$. The further division of $1 - x + x^2 - x^3 + \cdots$ by $1 + x$ furnished a series for $\frac{1}{(1+x)^2}$, and so on. Maseres took the explicit calculations as far as $\frac{1}{(1+x)^6}$ and observed how the numerical coefficients that appear at each stage may be related to the tables of binomial coefficients (figurate numbers) that he had presented as part of his translation from the *Ars conjectandi*. A general expression for $\frac{1}{(1+x)^n}$ was the result, and a simple substitution yielded one for $\frac{1}{(a+b)^n}$. As we have noted (see §2.1), Maseres treated the expansions of $(1 - x)$ and $(1 + x)$ separately, but his approach to $\frac{1}{(1-x)^n}$ was much the same as before: repeated division by $(1 - x)$, again leading to a general pattern, and an expansion for negative integer powers.

A.7. *Knight (1816)*

The method presented by Knight in a paper of 1816 was one of those that involved the (to modern eyes) naive manipulation of power series, and hinged upon the fact that $(a + x)^m \times (a + y)^m = \{(a + x)(a + y)\}^m$. Working with any power m ('positive or negative, whole or fractional'), Knight wrote down the expression[29]

$$(a+x)^m = a^m + A'a^{m-1}x + A''a^{m-2}x^2 + A'''a^{m-3}x^3 + \cdots, \tag{A8}$$

where A', A'', A''', ... depend on m only. It is clear also that

$$(a+y)^m = a^m + A'a^{m-1}y + A''a^{m-2}y^2 + A'''a^{m-3}y^3 + \cdots. \tag{A9}$$

Noting that $\{(a + x)(a + y)\}^m = (a^2 + ax + ay + xy)^m = a^m(a + x + \pi y)^m$, where $\pi = 1 + \frac{x}{a}$, we also have

$$\begin{aligned}&\{(a+x)(a+y)\}^m \\ &\quad = a^m\{a^m + A'a^{m-1}(x+\pi y) + A''a^{m-2}(x+\pi y)^2 + A'''a^{m-3}(x+\pi y)^3 + \cdots\}.\end{aligned}$$

[29] Knight placed the dashes on the left-hand sides of his letters, but we shift them to the right for ease of typesetting.

Knight expanded further, neglecting all powers of y beyond the first, to rewrite this last expression as

$$\begin{aligned} &a^m\{a^m + A'a^{m-1}x + A''a^{m-2}x^2 + A'''a^{m-3}x^3 + \cdots\} \\ &\quad + a^m y\{A'a^{m-1}\pi + 2A''a^{m-2}\pi x + 3A'''a^{m-3}\pi x^2 + \cdots\} + \cdots \end{aligned} \tag{A10}$$

It now remained only to compare coefficients. Knight began by comparing the coefficient of y in the product of (A8) and (A9) with that in (A10). Replacing π by $1 + \frac{x}{a}$ in the resulting expression, Knight next compared coefficients of different powers of x in order to find A'', A''', ... in terms of A'. A brief case-by-case analysis of what happens when m is an integer, or a unit or general fraction, shows that in all instances $A' = m$.