

RECONSTRUCTING COMPACT METRIZABLE SPACES

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ABSTRACT. The deck, $\mathcal{D}(X)$, of a topological space X is the set $\mathcal{D}(X) = \{[X \setminus \{x\}]: x \in X\}$, where $[Y]$ denotes the homeomorphism class of Y . A space X is (topologically) *reconstructible* if whenever $\mathcal{D}(Z) = \mathcal{D}(X)$, then Z is homeomorphic to X . It is known that every (metrizable) continuum is reconstructible, whereas the Cantor set is non-reconstructible.

The main result of this paper characterises the non-reconstructible compact metrizable spaces as precisely those where for each point x there is a sequence $\langle B_n^x: n \in \mathbb{N} \rangle$ of pairwise disjoint clopen subsets converging to x such that B_n^x and B_n^y are homeomorphic for each n , and all x and y .

In a non-reconstructible compact metrizable space the set of 1-point components forms a dense G_δ . For h -homogeneous spaces, this condition is sufficient for non-reconstruction. A wide variety of spaces with a dense G_δ set of 1-point components are presented, some reconstructible and others not reconstructible.

1. INTRODUCTION

The *deck* of a graph G is the set $\mathcal{D}(G) = \{[G - x]: x \in G\}$, where $[G - x]$ is the isomorphism class of the graph obtained from G by deleting the vertex x , and all incident edges. Then a graph G is *reconstructible* if whenever $\mathcal{D}(G) = \mathcal{D}(H)$, then G is isomorphic to H . Kelly and Ulam's well-known *Graph Reconstruction Conjecture* from 1941 proposes that every finite graph with at least three vertices is reconstructible. For more information, see for example [1].

Similarly, the *deck* of a topological space X is the set $\mathcal{D}(X) = \{[X \setminus \{x\}]: x \in X\}$, where $[X \setminus \{x\}]$ denotes the homeomorphism class of $X \setminus \{x\}$. Any space Y homeomorphic to some $X \setminus \{x\}$ is called a *card* of X . A space Z is a *reconstruction* of X if Z has the same cards as X , i.e. $\mathcal{D}(Z) = \mathcal{D}(X)$. Further, X is *topologically reconstructible* if whenever $\mathcal{D}(Z) = \mathcal{D}(X)$, then Z and X are homeomorphic.

Many familiar spaces such as $I = [0, 1]$, Euclidean n -space, n -spheres, the rationals and the irrationals are reconstructible, as are compact spaces containing an isolated point, [9]. However the Cantor set C and $C \setminus \{0\}$ have the same deck, $\{C \setminus \{0\}\}$. Hence the Cantor set is an example of a non-reconstructible compact, metrizable space. Note that $C \setminus \{0\}$ is non-compact. Indeed, [9, 5.2] states that if all reconstructions of a compact space are compact, then it is reconstructible.

In the other direction, the authors proved that every continuum, i.e. every connected compact metrizable space, is reconstructible, [3, 4.12]. The proof proceeds in two steps. First, every continuum has cards with a maximal finite compactification. Second, any compact space that has a card with a maximal finite compactification

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has only compact reconstructions—and so it is reconstructible by the result mentioned above, [3, 3.11].

We start this paper by showing that a compact metrizable space X without isolated points has a card with a maximal finite compactification if and only if $C_1(X) = \{x \in X : \{x\} \text{ is a 1-point component of } X\}$ is *not* a dense G_δ subset of X (Theorem 2.7). Hence, every non-reconstructible compact metrizable space contains a dense G_δ of 1-point components (Theorem 2.8). However, we will also give examples of compact metrizable spaces which are reconstructible despite having a dense set of 1-point components (Lemma 6.6). Hence, our approach for continua—looking for cards with maximal finite compactifications—does not work in general.

Instead, we characterise when a compact metrizable space is reconstructible via ‘universal sequences’. A space X is said to have a *universal sequence* if for each point x there is a sequence $\langle B_n^x : n \in \mathbb{N} \rangle$ of pairwise disjoint clopen subsets converging to x such that B_n^x and B_n^y are homeomorphic for every $x, y \in X$ and for each $n \in \mathbb{N}$.

Theorem 1.1 (Reconstruction Characterisation). *A compact metrizable space is reconstructible if and only if it does not have a universal sequence.*

The reverse implication is the content of Proposition 3.1, while the forward implication is Proposition 4.1.

Theorems 5.1 and 5.4 connect our characterisation of non-reconstructible compact metrizable spaces with the existence of certain types of dense subsets of 1-point components. If X is h -homogeneous, i.e. all non-empty clopen subsets of X are homeomorphic to X , then all that is required for X to be non-reconstructible is that $C_1(X)$ is dense. And if $C_1(X)$ has non-empty interior, then X is non-reconstructible if and only if the interior of $C_1(X)$ has no isolated points and is dense in X , which holds if and only if X is a compactification of $C \setminus \{0\}$. It follows, see Section 6.1 for details and more related results, that there are many compact, metrizable non-reconstructible spaces of this type.

For compact metrizable spaces where $C_1(X)$ is both dense and co-dense, the situation is considerably more complex. We provide two methods of constructing such spaces which are h -homogeneous and non-reconstructible (Corollary 6.5 and Theorem 6.8). Whereas non-reconstructible h -homogeneous spaces have a universal sequence with all terms homeomorphic to each other (and to X), we also construct examples without such a ‘constant’ universal sequence (Theorems 6.10 and 6.11).

In light of these results there remain a natural question and an open problem. Every example of a non-reconstructible compact metrizable space that we present appears to have a *unique* non-homeomorphic reconstruction.

Question 1.2. *Is there an example of a compact space with more than one non-homeomorphic reconstruction? What is the maximal number of non-homeomorphic reconstructions of a compact metrizable space?*

And second, our techniques used to establish the Reconstruction Characterisation Theorem rely on the spaces being metrizable.

Problem 1.3. *Find a characterisation of the reconstructible (first countable) compact spaces.*

2. CARDS WITH MAXIMAL FINITE COMPACTIFICATIONS

We prove that a compact metrizable space has a card with a maximal finite compactification if and only if the space does not contain a dense G_δ (a countable

intersection of open sets) of 1-point components. It follows that non-reconstructible compact metrizable spaces are highly disconnected.

We need the following two basic results from continuum theory. The component of a point $x \in X$ is denoted by $C_X(x)$, or $C(x)$ when X is clear from context.

Lemma 2.1 (Second Šura-Bura Lemma, [6, A.10.1]). *A component C of a compact Hausdorff space X has a clopen neighbourhood base in X (for every open set $U \supseteq C$ there is a clopen set V such that $C \subseteq V \subseteq U$).*

Lemma 2.2 (Boundary Bumping Lemma [2, 6.1.25]). *The closure of every component of a non-empty proper open subset U of a Hausdorff continuum intersects the boundary of U , i.e. $\overline{C_U(x)} \setminus U \neq \emptyset$ for all $x \in U$.*

A Hausdorff compactification γX of a space X is called an N -point compactification (for $N \in \mathbb{N}$) if its remainder $\gamma X \setminus X$ has cardinality N . A finite compactification of X is an N -point compactification for some $N \in \mathbb{N}$. We say νX is a maximal N -point compactification if no other finite compactification γX has a strictly larger remainder, i.e. whenever $|\gamma X \setminus X| = M$, then $M \leq N$.

For $N \geq 1$, a point $x \in X$ is N -splitting in X if $X \setminus \{x\} = X_1 \oplus \dots \oplus X_N$ such that $x \in \overline{X_i}$ for all $i \leq N$. Further, we say that x is locally N -splitting (in X) if there exists a neighbourhood U of x , i.e. a set U with $x \in \text{int}(U)$, such that x is N -splitting in U . Moreover, a point x is N -separating in X if $X \setminus \{x\}$ has a disconnection into N (clopen) sets $A_1 \oplus \dots \oplus A_N$ such that all A_i intersect $C_X(x)$. Similarly, we say x is locally N -separating in X if there is a neighbourhood U of x such that x is N -separating in U .

We will also need the following three results from [3].

Lemma 2.3 ([3, 3.4]). *A card $X \setminus \{x\}$ of a compact Hausdorff space X has an N -point compactification if and only if x is locally N -splitting in X .*

Lemma 2.4 ([3, 4.8]). *A card $X \setminus \{x\}$ of a Hausdorff continuum X has an N -point compactification if and only if x is locally N -separating in X .*

Theorem 2.5 ([3, 4.11]). *The number of locally 3-separating points in a T_1 space X does not exceed the weight of X .*

Now we generalize the second of the above results beyond continua.

Lemma 2.6. *Let X be a compact metric space and suppose there exists $\delta > 0$ such that the diameter of every component is at least δ . Then $X \setminus \{x\}$ has an N -point compactification if and only if x is locally N -separating in X .*

Proof. The backwards direction is immediate from Lemma 2.3, so we focus on the direct implication. Let us assume that $X \setminus \{x\}$ has an N -point compactification. By Lemma 2.3, there is a compact neighbourhood $U = \overline{B_\epsilon(x)}$ of x such that $U \setminus \{x\} = U_1 \oplus \dots \oplus U_N$ and $x \in \overline{U_i}^X$ for all i . Assume without loss of generality that $\epsilon < \delta/2$. We have to show that $C_U(x)$, the component of x in U , intersects every U_i . Suppose for a contradiction this is not the case, i.e. that say $C(x) \cap U_1 = \emptyset$. Since x lies in the closure of U_1 , we can find a sequence $\{x_n : n \in \omega\} \subset U_1$ converging to x .

Note that if $x \in \overline{C_{U_1}(x_n)}$, then $C_{U_1}(x_n) \subset C_U(x)$, a contradiction. Next, since $\text{diam}(C_X(x_n)) > 2\epsilon$, we have $C_X(x_n) \not\subseteq U$. Hence, applying Lemma 2.2 to $C_X(x_n) \cap U_1$ yields that $C_{U_1}(x_n)$ limits onto the boundary of U . Hence, whenever $d(x, x_n) < \epsilon/2$, then $\text{diam}(C_{U_1}(x_n)) \geq \epsilon/2$.

To conclude the proof, note that $C_U(x) \cap U_1 = \emptyset$ implies that x is a one-point component of $\overline{U_1}$. By Lemma 2.1 there is a clopen neighbourhood V of x in $\overline{U_1}$ such that the diameter of V is at most $\epsilon/4$. However, we have $x_n \in V$ for n large enough, implying that $\text{diam}(C_{U_1}(x_n)) \leq \text{diam}(V) \leq \epsilon/4$, a contradiction. \square

Our characterisation of compact metrizable spaces having a card with a maximal finite compactification follows.

Theorem 2.7. *For a compact metrizable space X without isolated points the following are equivalent:*

- (1) X has a card with a maximal 1- or 2-point compactification,
- (2) X has a card with a maximal finite compactification,
- (3) $C_1(X)$, the set of 1-point components of X , is not dense in X , and
- (4) $C_1(X)$ does not form a dense G_δ of size \mathfrak{c} .

Proof. The implications (1) \Rightarrow (2) and (3) \Rightarrow (4) are trivial.

For (2) \Rightarrow (3), assume $C_1(X)$ is dense in X . Let x be an arbitrary point of X . We need to show that the card $X \setminus \{x\}$ has arbitrarily large finite compactifications.

Fix a nested neighbourhood base $\{V_n : n \in \mathbb{N}\}$ of x such that $\overline{V_{n+1}} \subsetneq V_n$. Since the 1-point components are dense, it follows from Lemma 2.1 that for all n there is a non-empty clopen set $F_n \subsetneq V_n \setminus \overline{V_{n+1}}$. It follows that whenever A_1, \dots, A_{N-1} partitions \mathbb{N} into infinite subsets, the sets

$$G_i = \bigcup_{n \in A_i} F_n \quad \text{and} \quad G_0 = (X \setminus \{x\}) \setminus \bigcup_{n \in \mathbb{N}} F_n$$

form a partition of $X \setminus \{x\}$ into N non-empty non-compact clopen subsets. By Lemma 2.3, the card $X \setminus \{x\}$ has an N -point compactification. Since N was arbitrary, the card $X \setminus \{x\}$ does not have a maximal finite compactification.

For (4) \Rightarrow (1), we will prove the contrapositive. Assume that all cards of X have a three-point compactification. Set, for $n \in \mathbb{N}$, $X_n = \{x \in X : \text{diam}(C(x)) \geq 1/n\}$.

Every X_n is a closed subset of X . To see this, suppose for a contradiction that x lies in the closure of X_n and $\text{diam}(C(x)) < 1/n$. Find $\epsilon > 0$ such that $\text{diam}(C(x)) + 2\epsilon < 1/n$. By the Lemma 2.1, there is a clopen set F of X such that $C(x) \subseteq F \subseteq B_\epsilon(C(x))$. Since F is a neighbourhood of x , there is a point $y \in F \cap X_n$ witnessing that x lies in the closure of X_n . It follows $y \in C(y) \subseteq F$, and hence $\text{diam}(C(y)) < 1/n$, contradicting $y \in X_n$. This shows that X_n is closed.

We now argue that every X_n has empty interior in X . Otherwise, $\text{int}(X_n)$ would have to be uncountable, being locally compact without isolated points. As $X \setminus \{x\}$ has a 3-point compactification for all x , Lemma 2.3 implies that $X_n \setminus \{x\}$ has a 3-point compactification for all $x \in \text{int}(X_n)$. However, since all components of X_n have diameter at least $1/n$, Lemma 2.6 implies that all $x \in \text{int}(X_n)$ are locally 3-separating in X_n . But as compact metrizable spaces have countable weight, no such space can contain uncountably many locally 3-separating points by Lemma 2.5, a contradiction. Thus, every X_n has empty interior in X .

Finally, $C_1(X) = \bigcap_{n \in \mathbb{N}} X \setminus X_n$. Since every $X \setminus X_n$ is open dense, the Baire Category Theorem implies that $C_1(X)$ is a dense G_δ in X . In particular, $C_1(X)$ is completely metrizable without isolated points, and thus has cardinality \mathfrak{c} . \square

Since every compact Hausdorff space with a card with a maximal finite compactification is reconstructible, [3, 3.11], we deduce:

Theorem 2.8. *Every compact metrizable space in which the union of all 1-point components of X does not form a dense G_δ of cardinality \mathfrak{c} is reconstructible.* \square

Corollary 2.9. *The Cantor set is characterised topologically as the unique compact metrizable homogeneous non-reconstructible space.* \square

3. UNIVERSAL SEQUENCES OF CLOPEN SETS

In this section, we formally introduce universal sequences, show that every non-reconstructible compact metrizable space has a universal sequence, and provide a sufficient condition for the converse.

Recall that a sequence $\langle B_n : n \in \mathbb{N} \rangle$ of non-empty subsets of a space X is said to converge to a point x ($B_n \rightarrow x$) if for every neighbourhood U of x there exists $N \in \mathbb{N}$ such that $B_n \subseteq U$ for all $n \geq N$. Suppose $\langle T \rangle$ and $\langle T_n : n \in \mathbb{N} \rangle$ are sequences of topological spaces. A sequence $\langle B_n : n \in \mathbb{N} \rangle$ is of *type* $\langle T_n \rangle$ if $B_n \cong T_n$ for all $n \in \mathbb{N}$, and is of *constant type* $\langle T \rangle$ if $B_n \cong T$ for all $n \in \mathbb{N}$.

We say that a topological space X has a *universal sequence of type* $\langle T_n \rangle$ if every point of X is the limit of a sequence of disjoint clopen sets of type $\langle T_n \rangle$. The abbreviation ‘ X has a universal sequence’ means that for some type $\langle T_n : n \in \mathbb{N} \rangle$, X has a universal sequence of that type $\langle T_n \rangle$. Lastly, X has a *constant universal sequence* if there is $\langle T \rangle$ such that every point of X is the limit of a sequence of disjoint clopen sets of constant type $\langle T \rangle$.

Proposition 3.1 (Universal Sequence Existence). *Every non-reconstructible compact metrizable space has a universal sequence.*

Proof. Let X be a compact metrizable space with a non-homeomorphic reconstruction Z . Then Z is non-compact [9, 5.2]. It follows from results of [9], but in this case can also be proved directly, that Z is locally compact, separable and metrizable. So the 1-point compactification, αZ , of Z , is metrizable. Moreover, since X contains 1-point components by Theorem 2.8, every component of Z is compact. In particular, the point $\infty \in \alpha Z$ is a 1-point component. By Lemma 2.1, the point ∞ has a (countable) neighbourhood base of clopen sets in αZ , and hence Z can be written as $\bigoplus_{n \in \mathbb{N}} T_n$ for disjoint clopen compact subsets $T_n \subset Z$. We will show that a tail of $\langle T_n : n \in \mathbb{N} \rangle$ is the type of the desired universal sequence.

Claim. *For every $x \in X$ there is $N_x \in \mathbb{N}$ such that x is the limit of a sequence of disjoint clopen sets of type $\langle T_n : n > N_x \rangle$.*

To see the claim, note that $X \setminus \{x\} \cong Z \setminus \{z\}$ for some suitable $z \in Z$. But if $z \in T_{N_x}$, then $\langle T_n : n > N_x \rangle$ is an infinite sequence of disjoint compact clopen sets in $X \setminus \{x\}$ such that their union is a closed set. Compactness of X now implies that $\langle T_n : n > N_x \rangle$ converges to x .

Claim. *There is $N \in \mathbb{N}$ such that every $x \in X$ is the limit of a sequence of disjoint clopen sets of type $\langle T_n : n > N \rangle$.*

Choose non-empty open sets U and V with disjoint closures and $N \in \mathbb{N}$ such that both U and V contain a copy of $\bigoplus_{n > N} T_n$. If $x \in X \setminus \bar{U}$, apply the previous claim and use the copies of T_n for $N < n \leq N_x$ in U to obtain a sequence as claimed. By symmetry, the same holds for $x \in X \setminus \bar{V}$, completing the proof. \square

If a space X has a universal sequence of type $\langle T_n \rangle$, there is a witnessing indexed family $\{\mathcal{U}_x : x \in X\}$ where every $\mathcal{U}_x = \langle U_n^x : n \in \mathbb{N} \rangle$ is a sequence of disjoint clopen

subsets of X of type $\langle T_n \rangle$ converging to x . Let us call such an indexed family a *universal sequence system (of type $\langle T_n \rangle$)*. We distinguish the following additional properties. A universal sequence system $\{\mathcal{U}_x : x \in X\}$

- is *complement-equivalent* if for all $x, y \in X$, and $N \in \mathbb{N}$ with $\bigcup_{n>N} U_n^x \cap \bigcup_{n>N} U_n^y = \emptyset$, we have $X \setminus \bigcup_{n>N} U_n^x \cong X \setminus \bigcup_{n>N} U_n^y$,
- has *augmentation* if detaching a sequence \mathcal{U}_x from its unique limit point x and making it converge onto any $y \in X$ produces a space homeomorphic to X , i.e. for all $x, y \in X$, we have

$$\left((X \setminus \bigcup \mathcal{U}_x) \oplus \alpha \left(\bigcup \mathcal{U}_x \right) \right) / \{y, \infty\} \cong X,$$

- is *point-fixing* if for all $x, y \in X$ and $N \in \mathbb{N}$ with $\bigcup_{n>N} U_n^x \cap \bigcup_{n>N} U_n^y = \emptyset$, there is a homeomorphism $f_{xy} : X \setminus \bigcup_{n>N} U_n^x \rightarrow X \setminus \bigcup_{n>N} U_n^y$ fixing the points x and y , and
- is *thin* if for all $x \in X$, the set $\{x\} \cup \bigcup \mathcal{U}_x$ is not a neighbourhood of x .

Lemma 3.2. *Every point-fixing universal sequence system is complement-equivalent and has augmentation.*

Proof. Complement-equivalent is clear. Let $x, y \in X$. As $(X \setminus \bigcup \mathcal{U}_x) \oplus \bigcup \mathcal{U}_x = (X \setminus \bigcup_{n>N} U_n^x) \oplus \bigcup_{n>N} U_n^x$ for all $N \in \mathbb{N}$, we may assume $y \notin \bigcup \mathcal{U}_x$. Then

$$\begin{aligned} X &\cong \left((X \setminus \bigcup \mathcal{U}_y) \oplus \alpha \left(\bigcup \mathcal{U}_y \right) \right) / \{y, \infty\} \\ &\cong \left((X \setminus \bigcup \mathcal{U}_y) \oplus \alpha \left(\bigcup \mathcal{U}_y \right) \right) / \{f_{xy}(y), \infty\} \quad (\text{as } f_{xy} \text{ fixes } y) \\ &\cong \left((X \setminus \bigcup \mathcal{U}_x) \oplus \alpha \left(\bigcup \mathcal{U}_y \right) \right) / \{y, \infty\} \quad (\text{as } f_{xy} \text{ is a homeomorphism}) \\ &\cong \left((X \setminus \bigcup \mathcal{U}_x) \oplus \alpha \left(\bigcup \mathcal{U}_x \right) \right) / \{y, \infty\} \quad (\text{as } \bigcup \mathcal{U}_y \cong \bigcup \mathcal{U}_x). \quad \square \end{aligned}$$

Recall that a space is *pseudocompact* if every discrete family of open sets is finite. For Tychonoff spaces this coincides with the usual definition that every continuous real-valued function is bounded [2, §3.10]. Evidently, compact spaces are pseudocompact.

Lemma 3.3. *Every pseudocompact Hausdorff space with a complement-equivalent universal sequence system with augmentation (in particular: with a point-fixing universal sequence system) is non-reconstructible.*

Proof. Let X be a pseudocompact Hausdorff space with a universal sequence system $\{\mathcal{U}_x : x \in X\}$ which is complement-equivalent with augmentation. We verify that $Z = (X \setminus \bigcup \mathcal{U}_x) \oplus \bigcup \mathcal{U}_x$, or equivalently $(X \setminus \bigcup_{n>N} U_n^x) \oplus \bigcup_{n>N} U_n^x$, is a non-pseudocompact—and hence non-homeomorphic—reconstruction of X . Indeed, it is non-pseudocompact, as \mathcal{U}_x is an infinite discrete family of open sets in Z .

The inclusion “ $\mathcal{D}(X) \subseteq \mathcal{D}(Z)$ ” follows from complement-equivalence. Consider a card $X \setminus \{y\}$. We may assume $\bigcup \mathcal{U}_x \cap \bigcup \mathcal{U}_y = \emptyset$. Then for some suitable $z \in X$,

$$X \setminus \{y\} = X \setminus \left(\{y\} \cup \bigcup \mathcal{U}_y \right) \oplus \bigcup \mathcal{U}_y \cong X \setminus \left(\{z\} \cup \bigcup \mathcal{U}_x \right) \oplus \bigcup \mathcal{U}_x.$$

The inclusion “ $\mathcal{D}(X) \supseteq \mathcal{D}(Z)$ ” follows from augmentation. Consider a card $Z \setminus \{y\}$. Since $(\alpha Z) / \{\infty, y\} \cong X$, we have for some suitable $z \in X$ that

$$Z \setminus \{y\} = ((\alpha Z) / \{\infty, y\}) \setminus \{\{\infty, y\}\} \cong X \setminus \{z\}. \quad \square$$

4. UNIVERSAL SEQUENCES IMPLY NON-RECONSTRUCTIBILITY

In this section we prove the forward implication of the Reconstruction Characterisation Theorem 1.1.

Proposition 4.1. *A compact metrizable space with a universal sequence is non-reconstructible.*

In order to apply Lemma 3.3, we will show that every compact metrizable space with a universal sequence has a universal sequence *refinement* with a corresponding point-fixing universal sequence system. Indeed we show in Lemma 4.3 that every compact metrizable space with a universal sequence has a thin universal sequence system. Lemmas 4.2, 4.4 and 4.5, imply that this thin system can be refined to a (thin) point-fixing universal sequence system, as required.

Here, a sequence of topological spaces $\langle T'_n : n \in \mathbb{N} \rangle$ is a *refinement* of $\langle T_n : n \in \mathbb{N} \rangle$ if there is an injective map $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that $T'_n \hookrightarrow T_{\phi(n)}$ embeds as a clopen subset for all $n \in \mathbb{N}$. Note that if $\{\mathcal{U}_x : x \in X\}$ is a universal sequence system of type $\langle T_n \rangle$, then any refinement $\langle T'_n \rangle$ of $\langle T_n \rangle$ naturally induces a refinement $\{\mathcal{U}'_x : x \in X\}$ of $\{\mathcal{U}_x : x \in X\}$.

Lemma 4.2. *Any refinement of a (thin) universal sequence system is again a (thin) universal sequence system.* \square

Lemma 4.3. *Every universal sequence system has a thin universal sequence system refinement.*

Proof. For a universal sequence system of type $\langle T_n \rangle$, the refinement $\langle T_{2n} : n \in \mathbb{N} \rangle$ is easily seen to induce a thin universal sequence system. \square

In the following two lemmas, we use the shorthand “ x has a neighbourhood basis homeomorphic to $\bigoplus V_n \rightarrow x$ ” to mean: for every neighbourhood U of x there is $N \in \mathbb{N}$ and a clopen V with $x \in V \subseteq U$ such that $V \setminus \{x\} \cong \bigoplus_{n>N} V_n$.

Lemma 4.4. *Suppose X is compact metrizable with a universal sequence of type $\langle T_n \rangle$. Then X has a universal sequence refinement $\langle T'_n \rangle$ such that a dense collection of points x have neighbourhood basis homeomorphic to $\bigoplus T'_n \rightarrow x$.*

Proof. Every non-empty open set contains a clopen set from the tail of the universal sequence. Thus, by recursion, we can find a nested collection of non-empty clopen sets F_k of vanishing diameter such that $F_k \cong T_{n_k}$ for a subsequence $\langle n_k : k \in \mathbb{N} \rangle$.

Compactness and $\text{diam}(F_k) \rightarrow 0$ imply that $\bigcap F_k = \{x\} = C(x)$ for some point $x \in X$. By the Šura-Bura Lemma, x has a neighbourhood basis homeomorphic to $\bigoplus F_k \setminus F_{k+1} \rightarrow x$. Moreover, since every copy of T_{n_k} contains a point with this property, there is a dense set of points y with a neighbourhood looking like $\bigoplus F_k \setminus F_{k+1} \rightarrow y$.

Thus, the refinement $\langle T'_k = T_{n_k} \setminus T_{n_{k+1}} : k \in \mathbb{N} \rangle$ of $\langle T_n \rangle$ is as required. \square

Comparing the above lemma with Theorem 2.8, we note that the points we construct are indeed one-point components and hence having a universal sequence implies that the set of one-point components is dense in X .

Lemma 4.5. *Every thin universal sequence system $\{\mathcal{U}_x : x \in X\}$ of type $\langle T_n \rangle$ in a compact metrizable space X , with the property that a dense collection of points z has neighbourhoods looking like $\bigoplus T_n \rightarrow z$, is point-fixing.*

Proof. Let $x, y \in X$ arbitrary. Without loss of generality we may assume $\bigcup \mathcal{U}_x \cap \bigcup \mathcal{U}_y = \emptyset$. To see that $\{\mathcal{U}_x : x \in X\}$ is point-fixing, we need to construct a homeomorphism $f : X \setminus \bigcup \mathcal{U}_x \rightarrow X \setminus \bigcup \mathcal{U}_y$ with $f(x) = x$ and $f(y) = y$.

By thinness, the space $Z_0 = X \setminus (\bigcup \mathcal{U}_x \cup \bigcup \mathcal{U}_y \cup \{x, y\})$ is a non-empty open subset of X , so by assumption there is z_0 and a clopen $V_0 = \{z_0\} \cup \bigoplus_{n > n_0} V_n^{z_0}$ with $V_n^{z_0} \cong T_n$ for all $n > n_0$ such that $z_0 \in V_0 \subset Z_0$. Put $I_0 = [0, n_0] \cap \mathbb{N}$ and consider a homeomorphism

$$g_0 : \bigcup_{n \in I_0} U_n^y \mapsto \bigcup_{n \in I_0} U_n^x.$$

Next, consider the open subspace $Z_1 = X \setminus (\bigcup \mathcal{U}_x \cup \bigcup \mathcal{U}_y \cup V_0 \cup \{x, y\})$. Again by thinness, there are z_{-1} and z_1 with disjoint clopen neighbourhoods $V_{-1} = \{z_{-1}\} \cup \bigoplus_{n > n_1} V_n^{z_{-1}}$ and $V_1 = \{z_1\} \cup \bigoplus_{n > n_1} V_n^{z_1}$ with $n_1 > n_0$ and $V_n^{z_{-1}} \cong T_n \cong V_n^{z_1}$ for $n > n_1$ such that $V_{-1} \subset B_{\frac{1}{2}}(y) \cap Z_1$ and $V_1 \subset B_{\frac{1}{2}}(x) \cap Z_1$. Put $I_1 = (n_0, n_1] \cap \mathbb{N}$ and define

$$g_1 : \bigcup_{n \in I_1} U_n^y \mapsto \bigcup_{n \in I_1} V_n^{z_0} \mapsto \bigcup_{n \in I_1} U_n^x.$$

Continuing recursively, we get a double sequence $\{z_k : k \in \mathbb{Z}\}$ with disjoint clopen neighbourhoods $\{V_k : k \in \mathbb{Z}\}$ such that $V_k \xrightarrow[k \rightarrow -\infty]{} y$ and $V_k \xrightarrow[k \rightarrow \infty]{} x$, and maps g_k for intervals $I_k = (n_{k-1}, n_k] \cap \mathbb{N}$ sending

$$g_k : \bigcup_{n \in I_k} U_n^y \mapsto \bigcup_{n \in I_k} V_n^{z_{1-k}} \mapsto \bigcup_{n \in I_k} V_n^{z_{2-k}} \mapsto \dots \mapsto \bigcup_{n \in I_k} V_n^{z_{k-1}} \mapsto \bigcup_{n \in I_k} U_n^x.$$

Once the recursion is completed, consider the closed sets

$$A = \bigcup \mathcal{U}_y \cup \bigcup_{n \in \mathbb{Z}} V_n \cup \{x, y\} \quad \text{and} \quad B = X \setminus \left(\bigcup \mathcal{U}_x \cup \bigcup \mathcal{U}_y \cup \bigcup_{n \in \mathbb{Z}} V_n \right).$$

The resulting map $g = \bigcup g_k$ is continuous on A , and sends $g(y) = y$, $g(z_k) = z_{k+1}$ and $g(x) = x$. In particular, the partial maps $g|_A$ and $\text{id}|_B$ agree on $A \cap B = \{x, y\}$. Thus the map $f = g|_A \cup \text{id}|_B : X \setminus \bigcup \mathcal{U}_x \rightarrow X \setminus \bigcup \mathcal{U}_y$ is a continuous bijection, and hence, by compactness, a homeomorphism. \square

5. FURTHER RECONSTRUCTION RESULTS

In this section we apply the Reconstruction Characterisation Theorem 1.1 to better understand the role of the set $C_1(X)$ of 1-point components in the reconstruction problem. Theorem 2.8 showed that if X is compact, metrizable and non-reconstructible, then $C_1(X)$ is a dense G_δ subset of X . Now we investigate conditions on X or on $C_1(X)$ ensuring that the converse holds.

The characterisation of non-reconstructibility for compact metrizable spaces seems to depend crucially on metrizability. The results of this section also yield interesting sufficient conditions for non-reconstructibility in general compact Hausdorff spaces.

Recall that a space X is *h-homogeneous*, or *strongly homogeneous*, if every non-empty clopen subset of X is homeomorphic to X .

Theorem 5.1. *Every h-homogeneous, first-countable, compact space in which the 1-point components are dense is non-reconstructible.*

A collection \mathcal{B} of open sets is called a π -base for X if for every open set $U \subset X$ there is $B \in \mathcal{B}$ such that $B \subset U$. We call a space π -homogeneous if it has a clopen π -base of pairwise homeomorphic elements. If a space X is h -homogeneous and has a dense set of 1-point components, then the Second Šura-Bura Lemma 2.1 implies that X is π -homogeneous. Also note that every non-trivial h -homogeneous space has no isolated points. The following observation relating π -homogeneity and universal sequences in the realm of first-countable spaces is straightforward.

Lemma 5.2. *Let X be a first-countable Hausdorff space without isolated points. Then X is π -homogeneous if and only if X has a constant universal sequence. \square*

To prove Theorem 5.1 it remains to verify the following result which is of interest in its own right.

Theorem 5.3. *Every pseudocompact Hausdorff space with a constant universal sequence is non-reconstructible.*

Proof. If X has a universal sequence system $\{\mathcal{U}_x : x \in X\}$ of constant type, then $\mathcal{U}'_x = \langle U_{2n}^x : n \in \mathbb{N} \rangle$ gives a point-fixing universal sequence system. Indeed, the map $X \setminus (\bigcup \mathcal{U}'_x) \rightarrow X \setminus (\bigcup \mathcal{U}'_y)$ sending U_{2n+1}^x homeomorphically to U_n^x and U_n^y homeomorphically to U_{2n+1}^y , and being the identity elsewhere (in particular: $x \mapsto x$, $y \mapsto y$) is a homeomorphism. So the claim follows from Lemma 3.3. \square

An example of a pseudocompact non-compact space with a constant universal sequence is mentioned in [4, p.19]. Start with a (pseudocompact) Ψ -space on \mathbb{N} [2, 3.6.I], and replace every isolated point by a clopen copy of the Cantor set. Every Ψ -space is first-countable, so this gives a pseudocompact space with a constant universal sequence of type $\langle C \rangle$.

Now we consider the case when $C_1(X)$ is not just a G_δ set, but is open or contains a non-empty open subset (in other words, is dense but not co-dense).

Theorem 5.4. *For a compact metrizable space X in which $C_1(X)$ forms a dense G_δ with non-empty interior the following are equivalent:*

- (1) X is not reconstructible,
- (2) X has a constant universal sequence (of type $\langle C \rangle$),
- (3) the interior of $C_1(X)$ contains no isolated points and is dense in X , and
- (4) X is a compactification of $C \setminus \{0\}$.

Proof. First we show the equivalence of (1), (2) and (3). If the interior of $C_1(X)$ is non-empty and not dense in X , there cannot exist a universal sequence: any sequence of disjoint clopen sets converging to a point in the interior of $C_1(X)$ is eventually of type $\langle C \rangle$, which is not the case for a point outside the closure of $\text{int}(C_1(X))$. Thus, X is reconstructible by the Reconstruction Characterisation Theorem, so (1) fails. Equivalently, (1) implies (3).

If (3) holds, the clopen sets of X homeomorphic to the Cantor set form a π -base for X . Thus, X has a constant universal sequence of type $\langle C \rangle$ by Lemma 5.2 and (2) holds. And (2) implies (1) by Theorem 5.3.

Now we verify (3) implies (4). Let U be an open dense, proper subset of X contained in $C_1(X)$. Then U is a zero-dimensional, locally compact, non-compact metrizable space without isolated points, and hence homeomorphic to $C \setminus \{0\}$ [2, 6.2.A(c)], and X is a compactification of this latter space.

It remains to show (4) implies (3). But if X contains a dense subset A homeomorphic to $C \setminus \{0\}$, then A is open by local compactness, dense, without isolated points, and contained in $C_1(X)$. \square

6. BUILDING NON-RECONSTRUCTIBLE COMPACT SPACES

This final section contains examples of compact metrizable spaces illustrating the frontier between reconstructibility and non-reconstructibility. One objective is to present a broad variety of non-reconstructible compact metrizable spaces. A second objective is to show that some natural strengthenings of the Reconstruction Characterisation do not hold.

Consider a compact metrizable space X without isolated points. We know that if X is not reconstructible, then $C_1(X)$, the set of 1-point components in X , is a dense G_δ . If X is h -homogeneous, then density of $C_1(X)$ suffices for non-reconstructibility (via a constant universal sequence). And non-reconstructibility (again via constant universal sequences) also follows if $C_1(X)$ is a dense *open* G_δ . Certain questions now arise:

- (1) Is X non-reconstructible if (and only if) $C_1(X)$ is a dense G_δ ?
- (2) If X is non-reconstructible, then is $C_1(X)$ a dense open (or dense, not co-dense) set? Equivalently, is X non-reconstructible if and only if it is the compactification of $C \setminus \{0\}$?
- (3) If X is non-reconstructible, must it have a *constant* universal sequence?

We give negative answers to all these below, providing h -homogeneous examples where possible.

6.1. Non-reconstructible spaces with open dense $C_1(X)$: Compactifications of $C \setminus \{0\}$. There are many non-reconstructible spaces X where $C_1(X)$ is dense but not co-dense.

Lemma 6.1. *For every compact metrizable space K there is a compact metrizable space $X = X_K$ with an open dense set U homeomorphic to $C \setminus \{0\}$ such that $X \setminus U$ is homeomorphic to K .*

Since all these X_K are non-reconstructible, the variety of non-reconstructible compact metrizable spaces is the same as that of all compact metrizable spaces. Note that if K is not zero-dimensional, then X_K is not h -homogeneous.

Proof. Note that $C \setminus \{0\}$ is homeomorphic to $\omega \times C$ and $(\omega + 1) \times C \cong C$, and fix a continuous surjection $f: \{\omega\} \times C \rightarrow K$. The adjunction space $C \cup_f K$ —the quotient $X_K = C/\mathcal{P}$ for $\mathcal{P} = \{f^{-1}(x): x \in K\} \cup \{\{x\}: x \in \omega \times C\}$ —is a metrizable compactification of $\omega \times C$ with remainder homeomorphic to K [6, A.11.4]. \square

With an eye on general compact Hausdorff spaces we state and prove a more general construction. Recall that a subset D of X is *sequentially dense* if every point of X is the limit of a converging sequence of points in D . A space is *sequentially separable* if it has a countable, sequentially dense subset. In general, the *sequential density* of a space is the least cardinal κ such that there exists a subset of X of size κ which is sequentially dense in X .

In [10], V. Tkachuk proved that every compact Hausdorff space K is a remainder of a compactification of a discrete space of cardinality at most $|K|$. The following result requires only minor modifications in Tkachuk's original proof.

Theorem 6.2 (cf. Tkachuk [10]). *For every compact Hausdorff space K of sequential density κ there is a compactification X_K of the discrete space of size κ such that*

- (1) *the remainder $X_K \setminus \kappa$ is homeomorphic to K , and*
- (2) *the discrete space κ is sequentially dense in X_K .*

Corollary 6.3. *For every pair of compact Hausdorff spaces K of sequential density κ , and Z with a constant universal sequence of type $\langle T \rangle$, there is a compactification $X_{K,Z}$ of $Z \times \kappa$ such that*

- (1) *the remainder of that compactification is homeomorphic to K , and*
- (2) *the compactification has a constant universal sequence of type $\langle T \rangle$.*

In particular, this compactification is non-reconstructible.

Proof. In Theorem 6.2, replace every point of κ by a clopen copy of Z . By sequential density, the resulting space has a constant universal sequence of type $\langle T \rangle$. \square

6.2. Non-reconstructible spaces with dense, co-dense C_1 : h -homogeneous spaces. We present a simple method to obtain non-reconstructible h -homogeneous spaces with dense and co-dense $C_1(X)$. This shows that our second question above has a negative answer. Indeed, note that in a h -homogeneous compact metrizable space different from the Cantor set, $C_1(X)$ must always be co-dense.

In Lemma 6.6, these spaces are modified to give an example of a compact metrizable space without isolated points which is reconstructible even though $C_1(X)$ is a dense and co-dense G_δ —thus answering our first question in the negative.

Finding h -homogeneous spaces is simplified by the next result.

Theorem 6.4 (Medini [5, Thm. 18]). *If a Hausdorff space X has a dense set of isolated points, then X^κ is h -homogeneous for every infinite cardinal κ .*

Corollary 6.5. *If X is a first-countable compact space with a dense set of isolated points, then $X^\mathbb{N}$ is h -homogeneous and non-reconstructible.*

Proof. Note that if X has a dense set of isolated points, then $X^\mathbb{N}$ has a dense set of 1-point components. By Theorem 6.4, the space $X^\mathbb{N}$ is h -homogeneous, and since $X^\mathbb{N}$ is first-countable, the result now follows from Theorem 5.1. \square

In other words, for every first-countable compactification $\gamma\mathbb{N}$ of the countable discrete space, the product $(\gamma\mathbb{N})^\mathbb{N}$ is non-reconstructible. We refer the reader back to Theorem 6.2, where such compactifications are constructed.

Lemma 6.6. *There is a compact metrizable space X such that $C_1(X)$ is a dense co-dense G_δ but X is reconstructible.*

Proof. Let Y and Z be non-homeomorphic compact metrizable, h -homogeneous non-reconstructible spaces, as given by Corollary 6.5, different from the Cantor set. Let $X = Y \oplus Z$. Then X is a compact metrizable space, and $C_1(X)$ is a dense co-dense G_δ . However, X does not have a universal sequence, because the sequence would need to be constant of type $\langle Y \rangle$ at points of Y , but constant of type $\langle Z \rangle$ at points of Z . Hence, X is reconstructible. \square

6.3. A geometric construction of spaces with dense and co-dense C_1 . We now present machinery for geometric constructions of compact metrizable non-reconstructible spaces. Our construction takes place in the unit cube I^3 and works by constructing a decreasing sequence of compact subsets of I^3 .

An informal description. The basic building block of our construction consists of a planar continuum and a countable sequence of cubes of exponentially decreasing diameter ‘approaching’ this continuum from above so that every point of the continuum is a limit of cubes. In other words, our basic building block is a compactification of $I^3 \times \mathbb{N}$ embedded in I^3 with remainder a planar continuum. If the planar continuum is E , we will call a space of this type a block with basis E .

Let X_1 be such a block with basis E_\emptyset . To obtain X_2 we replace the k th cube, denoted by $F_{\langle k \rangle}$, by an appropriately scaled block with basis $E_{\langle k \rangle}$, and we do this for each k . In X_2 we index the new cubes by elements of \mathbb{N}^2 . Clearly, $X_2 \subset X_1$. We repeat this procedure to recursively construct the X_n .

Relativizing to cubes. For a closed cube $C = \overline{B}_r^\infty(c) = \{x \in \mathbb{R}^3 : d_\infty(x, c) \leq r\}$ with center c and side-length $2r$ in \mathbb{R}^3 we let $a_C : I^3 \rightarrow C$ be the natural affine map given by $x \mapsto 2r(x - (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})) + c$. If A is a subset of I^3 we say that its image under a_C is the relativization of A to C and write $A|C$ for this image.

The basic building block. Given a planar continuum $E \subset [\frac{1}{4}, \frac{3}{4}]^2 \times \{\frac{1}{2}\} \subset I^3$ and a countable dense enumerated subset $D = \{d_k : k \in \mathbb{N}\}$ of E , we define

$$F_k^{E,D} = \overline{B}_{2^{-2k-1}}^\infty(d_k + (0, 0, 2^{-2k}))$$

and

$$C_{E,D} = E \cup \bigcup_{k \in \mathbb{N}} F_k^{E,D}.$$

We have that $C_{E,D} \subset I^3$, and for distinct $k, l \in \mathbb{N}$ that $F_k^{E,D} \cap F_l^{E,D} = \emptyset$.

The recursive construction. Our construction uses as input a countable list $\mathcal{E} = \{(E_f, D_f) : f \in \mathbb{N}^{<\mathbb{N}}\}$ (where $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}^{<\mathbb{N}} = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$) of non-trivial planar continua $E_f \subset [\frac{1}{4}, \frac{3}{4}]^2 \times \{\frac{1}{2}\} \subset I^3$, and countable dense subsets $D_f \subset E_f$ with a fixed enumeration $D_f = \{d_{f,k} : k \in \mathbb{N}\}$.

From this list \mathcal{E} , we will build a decreasing sequence of compact metrizable spaces $X_n \subset I^3$, each with a designated collection of closed cubes $\{F_f : f \in \mathbb{N}^n\}$.

We will start by defining

$$X_1 = C_{E_{\langle \emptyset \rangle}, D_{\langle \emptyset \rangle}}|I^3 \text{ and, for all } k \in \mathbb{N}, F_{\langle k \rangle} = F_k^{E_{\langle \emptyset \rangle}, D_{\langle \emptyset \rangle}}|I^3.$$

The relativization at this point is trivial and has only been included for the sake of clarity. Having defined X_n and $\{F_f : f \in \mathbb{N}^n\}$ for some $n \in \mathbb{N}$, we set

$$X_{n+1} = \left(X_n \setminus \bigcup_{f \in \mathbb{N}^n} F_f \right) \cup \bigcup_{f \in \mathbb{N}^n} C_{E_f, D_f}|F_f$$

and for $k \in \mathbb{N}$ and $f \in \mathbb{N}^n$

$$F_{f \frown k} = F_k^{E_f, D_f}|F_f.$$

Since the X_n form a decreasing sequence of non-empty compact sets, the space $X_{\mathcal{E}} = \bigcap_n X_n$ is a non-empty compact metric space.

Key properties of the construction. The next lemma gathers the necessary particulars about our construction, and can be easily verified.

Lemma 6.7. *Let $\mathcal{E} = \{(E_f, D_f) : f \in \mathbb{N}^{<\mathbb{N}}\}$ be a list of non-trivial planar continua, and consider the compact metrizable space $X_{\mathcal{E}}$ as described above.*

- (1) *For all $g \in \mathbb{N}^{<\mathbb{N}}$ we have $F_g \cap X_{\mathcal{E}}$ is homeomorphic to $X_{\tilde{\mathcal{E}}}$ where $\tilde{\mathcal{E}} = \{(\tilde{E}_f, \tilde{D}_f) : f \in \mathbb{N}^{<\mathbb{N}}\}$ with $\tilde{E}_f = E_{g \smallfrown f}$ and $\tilde{D}_f = D_{g \smallfrown f}$.*
- (2) *The union over $\mathcal{F}_n = \{F_f \cap X_{\mathcal{E}} : f \in \mathbb{N}^n\}$ equals $X_{\mathcal{E}} \setminus X_{n-1}$ and is a dense open subspace of $X_{\mathcal{E}}$.*
- (3) *The collection $\mathcal{F} = \bigcup_n \mathcal{F}_n$ forms a clopen π -basis for $X_{\mathcal{E}}$.*
- (4) *We have $C_1(X_{\mathcal{E}}) = X_{\mathcal{E}} \setminus \bigcup_{f \in \mathbb{N}^{<\mathbb{N}}} E_f | F_f$ is a dense, co-dense G_{δ} in $X_{\mathcal{E}}$.*

6.4. More non-reconstructible h -homogeneous spaces. Applying the above construction in the special case that we choose all E_f to be identical, we get a non-reconstructible h -homogeneous space. Of course, taking all the E_f to be a single point, we obtain the Cantor set which is the inspiration for the above construction.

Theorem 6.8. *Suppose $\mathcal{E}^{(0)} = \{(E, D)\}$ for some planar continuum E and a fixed enumerated countable dense set $D \subset E$. Then the space $X_{\mathcal{E}^{(0)}}$ is a non-reconstructible compact metrizable h -homogeneous space.*

For the proof we need the following lemma.

Lemma 6.9 (Matveev [4]). *Assume that X has a π -base consisting of clopen sets that are homeomorphic to X . If there exists a sequence $\langle U_n : n \in \omega \rangle$ of non-empty open subsets of X converging to a point of X , then X is h -homogeneous.* \square

Proof of Theorem 6.8. The sets F_f for $f \in \mathbb{N}^{<\mathbb{N}}$ form a clopen π -basis, all elements of which are homeomorphic to $X_{\mathcal{E}^{(0)}}$ by Lemma 6.7(1). Moreover, $X_{\mathcal{E}^{(0)}}$ has a dense set of one-point components by Lemma 6.7(4). Thus, $X_{\mathcal{E}^{(0)}}$ is h -homogeneous by Lemma 6.9, and it follows from Theorem 5.1 that it is non-reconstructible. \square

6.5. Non-reconstructible spaces without a constant universal sequence.

All our examples of non-reconstructible spaces so far had a constant universal sequence. In this section, we answer our third question and show that this need not be the case: We use the construction presented in Section 6.3 to build two non-reconstructible compact metrizable spaces without a constant universal sequence.

Despite not having a constant universal sequence, our first example is sufficiently self-similar so that non-reconstructibility can be verified directly. The second construction is much more subtle since it has a lot of rigidity built in. Analysing these two spaces in detail led the authors to the main characterisation in Theorem 1.1.

A non-reconstructible space without constant universal sequence I. For the first space, let $\{E_n : n \in \omega\}$ be a list of pairwise non-homeomorphic planar continua, and consider the list $\mathcal{E}^{(1)} = \{(E_f, D_f) : f \in \mathbb{N}^{<\mathbb{N}}\}$ such that for $f : k \rightarrow \mathbb{N}$ we have $E_f = E_{f(k-1)}$ and $D_f = D_{f(k-1)}$. Consider the space $X_{\mathcal{E}}$ as described in Section 6.3. Our list ensures that at the n th step in the recursion we replace the cubes always by the same building block C_{E_n, D_n} . Thus the homeomorphism type of $F_f \cap X_{\mathcal{E}}$ only depends on the length of $f \in \mathbb{N}^{<\omega}$ and we write T_n for the homeomorphism type of $F_f \cap X_{\mathcal{E}}$ for any $f \in \mathbb{N}^n$.

Theorem 6.10. *The space $X_{\mathcal{E}^{(1)}}$ is an example of a non-reconstructible compact metrizable space without a constant universal sequence.*

Proof. Since T_{n+1} is a clopen subset of T_n , every point $x \in E_f|F_f$ is the limit of a sequence homeomorphic to some tail of $\langle T_n : n \in \mathbb{N} \rangle$, living in F_f . By density (Lemma 6.7(4)), every point of $X_{\mathcal{E}(1)}$ is the limit of a sequence homeomorphic to some tail of $\langle T_n : n \in \mathbb{N} \rangle$. As in Proposition 3.1, this implies that $X_{\mathcal{E}(1)}$ has a universal sequence of type $\langle T_n : n > N \rangle$. Hence, this space is non-reconstructible.

To see that $X_{\mathcal{E}(1)}$ does not contain a constant universal sequence, we argue it is not π -homogeneous (Lemma 5.2). Indeed, any clopen subset $B \subset X_{\mathcal{E}(1)}$ contains a non-trivial component, say E_n . But T_k for $k > n$ does not contain a copy of E_n , and hence subsets homeomorphic to B cannot form a π -base for $X_{\mathcal{E}(1)}$. \square

A non-reconstructible space without constant universal sequence II. For the second space we use a list $\mathcal{E}^{(2)} = (E_f, D_f)$ where every continuum appears only finitely often. So let E_n be countably many distinct planar continua embedded in $[\frac{1}{4}, \frac{3}{4}]^2 \times \{\frac{1}{2}\}$ and $D_n \subset E_n$ enumerated countable dense subsets of E_n . For each n , partition D_n into countably many finite consecutive subsequences $D_m^n = \{d_k : k = N_m, \dots, N_{m+1} - 1\}$ such that D_m^n is 2^{-m} -dense in E_n .

We now let $E_{\langle \emptyset \rangle} = E_0$, $E_{\langle n \rangle} = E_k$ if $n \in D_k^0$. We then inductively define $E_{f \cap n} = E_k$ if $n \in D_k^m$ where m is such that $E_f = E_m$.

Our choices above ensure that for any two F_f, F_g for which $E_f = E_g$, the space $X \cap F_f$ and $X \cap F_g$ are homeomorphic (via the relativization homeomorphisms). We will thus define $X_n = X \cap F_f$ where n is such that $E_f = E_n$.

Theorem 6.11. *The space $X_{\mathcal{E}(2)}$ is an example of a non-reconstructible compact metrizable space in which every non-trivial component appears only finitely often.*

Proof. By the partitioning of D_n , every point $x \in E_f|F_f$ is the limit of a sequence homeomorphic to some tail of $\langle X_n : n > N \rangle$. Now proceed as above. \square

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