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Compactly supported radial basis functions: how and why?

by

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COMPACTLY SUPPORTED RADIAL BASIS FUNCTIONS: HOW AND WHY? *

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Abstract. The use of radial basis functions have attracted increasing attention in recent years as an elegant scheme for high-dimensional scattered data approximation, an accepted method for machine learning, one of the foundations of mesh-free methods, an alternative way to construct higher order methods for solving partial differential equations (PDEs), an emerging method for solving PDEs on surfaces, a novel method for mesh repair and so on. All these applications share one mathematical foundation: high dimensional approximation/interpolation. This paper explains why radial basis functions are preferred to multi-variate polynomials for scattered data approximation in high-dimensional space; and gives a brief description on how to construct the most commonly used compactly supported radial basis functions. Without sophisticated mathematics, one can construct a compactly supported (radial) basis function with required smoothness according to procedures described here. Short programs and tables for compactly supported radial basis functions are supplied.

Key words. Compact support, radial basis functions, high dimensional approximation, scattered data approximation, Wendland functions, missing Wendland functions.

AMS subject classifications. 00A02, 26A33, 33C90, 41A05, 41A30, 41A63, 65D05, 97N50

1. Introduction.

Radial basis functions (RBFs) have emerged as one powerful tool for scattered data approximation in high dimensional space. They have been successfully applied in various applications including

- · geography and digital terrain modelling [23][24][25];
- · data assimilation in geodesy and metrology [39][17];
- · engineering design and mesh generation [28][29] [35];
- · neural networks and artificial intelligence [15][38][43];
- · expensive function optimization and finding resource [11][21];
- · kinds of mesh-free methods [12][13][14][30][31][56][59][61][48];
- · solving partial differential equations(PDEs) on surfaces [42][16];
- · post-processing of simulation and 3D surface reconstruction [6][40];
- · sampling, signal processing and machine learning [44][1][41][18][26][45];
- · ......

Although these applications arise from different backgrounds, they share the same mathematical foundation: *multivariate* approximation/interpolation—finding a function $s(\mathbf{x})$ which can approximate/interpolate observations $f_1, f_2, \ldots, f_n$ on the corresponding data points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^d$, $d > 1$ i.e. $s(\mathbf{x}_i) \stackrel{\approx}{=} f_i$, for $i = 1, 2, \ldots, n$. It is noted that this problem in high-dimensional space is clearly *non-trivial.*

In the basis function framework, $s(\mathbf{x})$ consists of a linear combination of simple basis functions, say, $s(\mathbf{x}) = \sum_{j=1}^{n} \alpha_j \phi_j(\mathbf{x})$. For a given set of basis functions, the weights α_j for each basis function are determined by solving the following linear system:

$$\begin{pmatrix} \phi_1(\mathbf{x}_1) & \phi_2(\mathbf{x}_1) & \cdots & \phi_n(\mathbf{x}_1) \\ \phi_1(\mathbf{x}_2) & \phi_2(\mathbf{x}_2) & \cdots & \phi_n(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(\mathbf{x}_n) & \phi_2(\mathbf{x}_n) & \cdots & \phi_n(\mathbf{x}_n) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \stackrel{\approx}{=} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}. \tag{1.1}$$

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One may ask the following questions: what kind of basis functions shall we choose from? does the linear system (1.1) have a unique solution? is the linear systems easy to solve? We shall answer these questions step by step.

2. Why radial basis functions in $\mathbb{R}^d$?. In one dimensional space, commonly-used basis functions come from the polynomial space of degree at most $n-1$. We can, for example, chose $\phi_j(x) = x^{j-1}$, $x \in \mathbb{R}, j = 1, \ldots, n$. If the n interpolation points are distinct, then the linear system (1.1) has an unique solution, since it is a non-singular Vandermonde linear system. While, as illustrated in the Mairhuber-Curitis theorem [60, p.19][33][34], uniqueness of solution to the linear system (1.1) with multi-variate polynomial basis can not always be guaranteed. Such uncertainty was possibly first noted and proved by Haar [22][33, p.610]. He pointed out that the linear system (1.1) can be singular even for distinct points in $\mathbb{R}^d$, $d > 2$.

His arguments are based on the following basic facts of linear algebra: (a) uniqueness of solution to (1.1) is equivalent to the determinant of the interpolation matrix in (1.1) being non-zero; (b) the determinant of a matrix is a continuous function of its elements; and (c) switching two rows of a matrix will change the sign of its determinant. Based on these facts, one can find two points, say, $\mathbf{x}_1$ and $\mathbf{x}_2$ and construct two distinct curves $\xi_1(t)$ and $\xi_2(t)$ connecting these two points such that $\xi_1(0) = \mathbf{x}_1, \xi_1(1) = \mathbf{x}_2$, $\xi_2(0) = \mathbf{x}_2, \xi_2(1) = \mathbf{x}_1$, where the two curves have no other common points and do not intersect with the remaining $n-2$ interpolation points. When $\mathbf{x}_1$ goes along $\xi_1(t)$ to $\mathbf{x}_2$ and $\mathbf{x}_2$ goes along $\xi_2(t)$ to $\mathbf{x}_1$, the first two rows in (1.1) change continuously, and finally when $t = 1$, $\mathbf{x}_1$ and $\mathbf{x}_2$ get switched. Therefore, the determinant of the matrix continuously changes and finally changes sign, and thus there must be some $t \in [0, 1]$ which makes the determinant zero.

Such a essential difference between multi-variate and uni-variate polynomial interpolation on uniqueness of solution to linear system (1.1) can be another myth of polynomial interpolation [54], which motivates us to find non-polynomial basis functions.

If we choose $\phi_j(\mathbf{x}) = \phi(\mathbf{x} - \mathbf{x}_j)$, where $\phi : \mathbb{R}^d \mapsto \mathbb{R}$, then when two rows in the interpolation matrix are switched, two columns (two basis functions) will also be switched. Therefore, the sign of the determinant is thus unchanged. Such basis functions have the potential to avoid a singularity of the linear system (1.1) and thus are good candidates for high-dimensional approximation . One of the simplest such basis functions is $\phi(\mathbf{x}) = \|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$ which has *radial symmetry.* In this case, $\phi_j(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_j\|_2$, and the interpolation matrix is a distance matrix in $\mathbb{R}^d$, which is always invertible provided the n points are distinct.[1] Precisely, if the n points are distinct, a distance matrix has one positive eigenvalue and $n-1$ negative eigenvalues, [51, p.792]. Therefore a distance matrix is *almost negative definite* (only one positive eigenvalue). It seems that Schoenberg's results did not receive much attention until Micchelli [37] proved that a class of *radial* basis functions can always guarantee invertible interpolation matrices. This provides a sound foundation for scattered data approximation with RBFs [2]. (Otherwise on the regular tensor like mesh, one may choose, for example, a Fourier basis.)

Our next problem is whether the linear system (1.1) is easy to solve. In higher-dimensional space $\mathbb{R}^d$, $d \geq 2$, the linear system (1.1) often involves many unknowns, for example, when reconstructing a 3D surface from point clouds. Therefore, the sparsity of interpolation matrices is important and thus compactly supported radial basis functions (CSRBFs) are attractive for large number of data. Moreover, the linear system is expected

[1]This results is proved by Schoenberg who was motivated by proving what given n numbers in $\mathbb{R}^d$ can serve as the length of edges of a simplex(in $\mathbb{R}^2$ a simplex is a triangle) [50].

[2]Micchelli's work is motivated by a conjecture. The conjecture can be interpreted as the interpolation matrix in (1.1) with $\phi_j(\mathbf{x}) = \sqrt{1 + \|\mathbf{x} - \mathbf{x}_j\|_2}$ as basis functions is invertible. His proof is based on some results of distance geometry, conditionally positive definite functions and special functions that go beyond our discussion. But his results are encouraging: provided that the n points are distinct, interpolation matrices with some radial basis functions are invertible regardless the distribution of the interpolation points.

to be (symmetric) positive definite. A basis function is said to be *positive definite* if it can guarantee a positive linear system in (1.1) (Positive definiteness can be guaranteed provided the Fourier transform of the radial basis function is non-negative, see Appendix A). Positive definite linear system are relatively easy to solve, while indefinite matrices more likely lead to the failure of linear solvers. Therefore, those positive definite RBFs are the ones of our most interest. Those positive definite RBFs are thus the ones of our most interest.

3. Construction of compactly supported radial basis functions. There are several noticeable CSRBFs [4][5][20][62]. Due to the limited length of this paper, we only focus on those CSRBFs which make this paper more consistent. It is not difficult to construct compactly supported functions if constraints such as smoothness and positive definiteness are not required. For example, the truncated power function, which is also called Askey's power function [2], given by

$$\phi_\ell(r) = (1-r)_+^\ell, \text{ where } r = \|\mathbf{x}\|_2, a_+ = \max\{a, 0\}, \tag{3.1}$$

have a compact support in the ball $\|\mathbf{x}\|_2 \le 1$. Furthermore, it can be shown that [7][32][60, p.80]:

PROPOSITION 3.1. *Askey's truncated power function is positive definite on $\mathbb{R}^d$ if ℓ is an integer and $\ell \ge \lfloor d/2 \rfloor + 1$, where $\lfloor\cdot\rfloor$ is the floor function.*

However, Askey's power functions do not have continuous derivatives at $\|\mathbf{x}\|_2 = 0$ and $\|\mathbf{x}\|_2 = 1$, even when ℓ is large, i.e. $\phi_\ell \in C^0$. (See FIG. 3.2(a)).

3.1. Increasing smoothness. Smoother CSRBFs can be constructed by convolution. The well-know cardinal B-splines [52] and the *Euclidean's hat* functions [8, p.81][19][57] in $\mathbb{R}^d$ are constructed in this way (see FIG. 3.1). The Euclidean's hat function is the self-convolution of an Enclidean ball with diameter 1 in $\mathbb{R}^d$. Smoother CSRBFs than the Euclidean's hat can be constructed recursively like the cardinal B-splines. However it turns out that computing convolution in $\mathbb{R}^d$ is not so easy. FIG 3.1(f) shows a compactly supported function obtained by convolution, Wolfram Mathematica® shows that it is a 17-piece-wise polynomial in $\mathbb{R}^2$. Simpler methods than recursive convolution is needed.

Another way to obtain smoother functions is integration. Suppose $\varphi(t) \in C^0$, continuous function without continuous derivatives on $\mathbb{R}$, then $\int_a^x \varphi(t)\, dt \in C^1$ and $\int_a^x \int_a^s \varphi(t)\, dtds \in C^2$. It can be shown that [9, p.6]

$$\int_a^x \int_a^s \varphi(t)\, dtds = \int_a^x x\varphi(t)\, dt - \int_a^x t\varphi(t)\, dt. \tag{3.2}$$

Both $\int_a^x x\varphi(t)\, dt$ and $\int_a^x t\varphi(t)\, dt$ are smoother than $\varphi(t)$. It turns out that a similar integral operator to $\int_a^x t\varphi(t)\, dt$ simplifies computations of constructing CSRBFs in higher-dimensional space.

3.2. Dimension walk and Wu's construction. The following integral operators were first introduced by Wu in the context of constructing CSRBFs [62][49]:

$$(\mathcal{I}\phi)(r) := \int_r^\infty t\phi(t)dt, \text{ for } r \ge 0 \text{ and } \phi(t)t \in L_1[0,\infty); \tag{3.3}$$

$$(\mathcal{D}\phi)(r) := -\frac{1}{r}\phi'(r), \text{ for } r \ge 0 \text{ and } \phi \in C^2(\mathbb{R}). \tag{3.4}$$

Similar to the integral transform $\int_a^x \varphi(t)\, dt$, $\mathcal{I}\phi$ is a smoother function than ϕ. While ϕ is smoother than $\mathcal{D}\phi$. Furthermore, if ϕ is compactly supported on $[0,1]$, so are $\mathcal{I}\phi$ and $\mathcal{D}\phi$; $\mathcal{D}\mathcal{I}\phi = \phi$ for $t\phi \in L_1[0,\infty$ and $\mathcal{I}\mathcal{D}\phi = \phi$ for $\phi \in C^2(\mathbb{R})$. The most attractive property of the two operators is the *dimension walk* property[60, p.121][49], which can reduce computations (Fourier transform of a radial function) in $\mathbb{R}^d$ to computations in the one dimensional space $\mathbb{R}$.

PROPOSITION 3.2. *Let ϕ be continuous function satisfying* (3.3) *and* (3.4) *respectively, then the radial function $\phi(r)$ with $r = \|\mathbf{x}\|_2$ is positive definite on $\mathbb{R}^d$ if and only if*

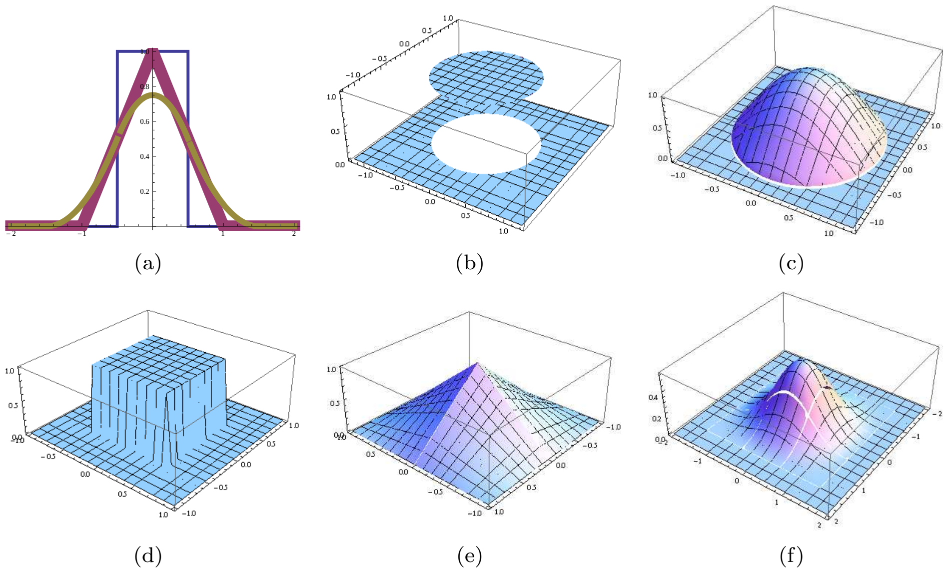


FIG. 3.1. *Constructing smoother function by self-convolution. (a) shows the first 3 cardinal B-spline B_k, $k = 0, 1, 2$. B_0 is the indicator function of $[-\frac{1}{2}, \frac{1}{2}]$, which is discontinuous. B_k are defined recursively as the convolution product $B_k := B_0 \star B_{k-1}$, $k = 1, 2, \ldots$. B_k has a compact support on $[-\frac{k+1}{2}, \frac{k+1}{2}]$ and $B_k \in C^k$. (b) and (d) are indicator functions. (c) and (e) are the self-convolutions of the indicator functions in (b) and (d) respectively. (f) is the convolution of the functions in (d) and (e).*

1. *$\mathcal{I}\phi(r)$ is positive definite on $\mathbb{R}^{d-2}$ for $d > 3$;*
2. *$\mathcal{D}\phi(r)$ is positive definite on $\mathbb{R}^{d+2}$.*

Wu constructs CSRBFs by using the dimension walk property of the operator $\mathcal{D}$. He starts with a very smooth positive CSRBF $w_\ell(r) := \phi_\ell(r^2) \star \phi_\ell(r^2)$, where $\phi_\ell(\cdot)$ is Askey's power function defined in (3.1) and $\star$ denotes for the convolution operator. According to Proposition 3.2, $\mathcal{D}^k w_\ell(2r)$ is a positive definite CSRBFs on $\mathbb{R}^{2k+1}$. However, $\mathcal{D}^k w_\ell(2r)$ in $\mathbb{R}^{2k+1}$ is less smooth than w_ℓ in $\mathbb{R}$. To get a smooth CSRBFs in $\mathbb{R}^{2k+1}$, one has to start with much smoother CSRBFs in $\mathbb{R}$ which correspond higher degree of polynomials $w_\ell(r)$. Thus, at the end the paper of [62], Wu proposes the question: what is the lowest degree of a positive definite CSRBF with a given smoothness in $\mathbb{R}^d$?

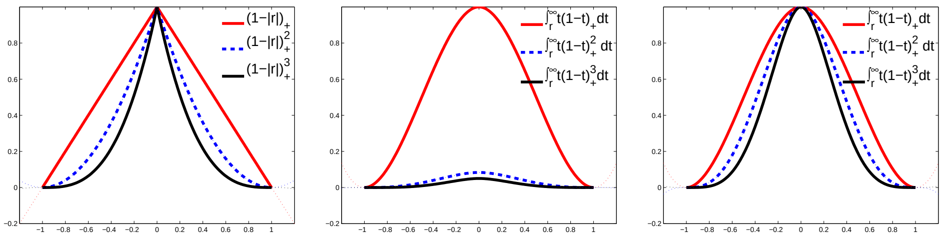
Wendland answers Wu's question by constructing his CSRBFs of minimal degree[58].

3.3. Construction of the Wendland's functions. Wendland functions are constructed via the integral operator $\mathcal{I}$ in (3.3). By repeatedly applying $\mathcal{I}$ to Askey's truncated power functions $\phi_\ell(r) = (1-r)_+^\ell$, Wendland obtains the following functions

$$\phi_{d,k}(r) = \mathcal{I}^k \phi_\ell, \text{ where } \ell = \lfloor d/2 \rfloor + k + 1, \text{ and } \phi_\ell = (1-r)_+^\ell. \tag{3.5}$$

Because $\lfloor d/2 \rfloor + k + 1 \ \geq \lfloor (d+2k)/2 \rfloor + 1$, according to the property of Askey's power function in (3.1), ϕ_ℓ with $\ell = \lfloor d/2 \rfloor + k + 1$ is positive definite on $\mathbb{R}^{d+2k}$ for non-negative integer k. According to Proposition 3.2, $\phi_{d,k}$ is positive definite on $\mathbb{R}^d$. Since ℓ defined in (3.5) is the smallest integer such that ϕ_ℓ is positive definite on $\mathbb{R}^{d+2k}$, and thus ℓ is also the smallest integer such that $\phi_{d,k}$ is positive definite on $\mathbb{R}^d$. In this sense Wendland's functions are also called CSRBFs of minimal degree.

$\phi_{d,k}(r)$ can be easily computed with the help of mathematical software. Table 4.1 is computed by a short Maple program provided in the appendix. As in Table 4.1, Wendland



(a) ϕ_ℓ, $\ell = 1, 2, 3$. (b) $\mathcal{I}(\phi_\ell), \ell = 1, 2, 3$. (c) scaled $\mathcal{I}(\phi_\ell), \ell = 1, 2, 3$.

FIG. 3.2. *Smoothing functions using the operator $\mathcal{I}$. Functions in (b) and (c) are even extensions of $\mathcal{I}(\phi_\ell)$.*

functions can be represented as follows:

$$\phi_{d,k}(r) = \mathcal{I}^k \phi_\ell = \phi_{\ell+k} p_{k,\ell}(r) = (1-r)_+^{(\lfloor d/2 \rfloor + k + 1) + k} p_{k,\ell}(r), \tag{3.6}$$

where $p_{k,\ell}(r)$ is a polynomial of degree k whose coefficients depend on ℓ, and $p_{0,\ell} := 1$.

It can be shown that Wendland's functions $\phi_{d,k}(r)$, $r = \|\mathbf{x}\|_2$ defined in (3.5) are polynomials of polynomials of degree $\lfloor d/2 \rfloor + 3k + 1$ on $\mathbb{R}^d$ with respect to r and positive definite on $\mathbb{R}^d$ with compact support in $\|\mathbf{x}\|_2 \leq 1r$. For each k, $\phi_{d,k}(r)$ it possesses $2k$ continuous derivatives around zeros and $k + \ell - 1 = 2k + \lfloor d/2 \rfloor$ continuous derivatives around $\|\mathbf{x}\|_2 = 1$. [58][60, p.128, Theorem 9.13, p.160, Theorem 10.35]. It also has the following so-called reproducing property.

PROPOSITION 3.3. *For $d \geq 3$, and k non-negative integer, $\phi_{d,k}$ is a reproducing kernel in Hilbert space, which is norm-equivalent to the Sobolev space $\mathcal{H}^{d/2+k+1/2}(\mathbb{R}^d)$.*

Proposition 3.3 suggests that there must be some CSRBFs missing in perhaps the interesting case $\mathbb{R}^2$.

3.4. Construction of the missing Wendland functions. CSRBFs which reproduce the Sobolev space $\mathcal{H}^{d/2+k+1/2}(\mathbb{R}^d)$ for even d and half-integer k, have only been found recently [47]. Such functions are called the missing Wendland functions. The missing Wendland functions are constructed by using a more general integral operator $\mathcal{I}_\alpha$, as mentioned above:

$$\mathcal{I}_\alpha(f)(t) := \int_t^\infty f(s) \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} ds, \tag{3.7}$$

where α can be a half-integer and $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$ is the Gamma function. The operator $\mathcal{I}_\alpha$ is a scaled integral operator which is closely related to fractional derivatives, and was used to simplify the multivariate Fourier transform for radial functions [49]. Applying this operator to a modified version of the truncated power function, $a_\mu(s) := (1 - \sqrt{2s})_+^\mu$, we get

$$\mathcal{I}_\alpha(a_\mu)(t) = \int_t^\infty (1 - \sqrt{2s})_+^\mu \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} ds. \tag{3.8}$$

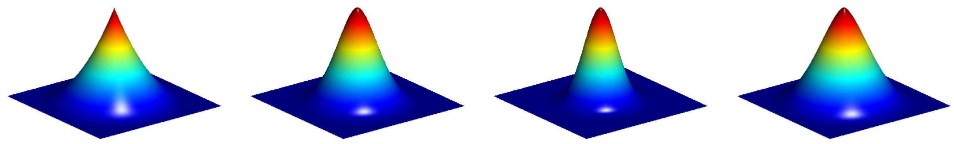
Defining $\Psi_{\mu,\alpha}(r) := \mathcal{I}_\alpha(a_\mu)(r^2/2)$, then we have

$$\Psi_{\mu,\alpha}(r) = \int_{r^2/2}^\infty (1 - \sqrt{2s})_+^\mu \frac{(s - r^2/2)^{\alpha-1}}{\Gamma(\alpha)} ds = \int_r^1 t(1-t)^\mu \frac{(t^2 - r^2)^{\alpha-1}}{\Gamma(\alpha) 2^{\alpha-1}} dt. \tag{3.9}$$

In particular, when $\alpha = 1$

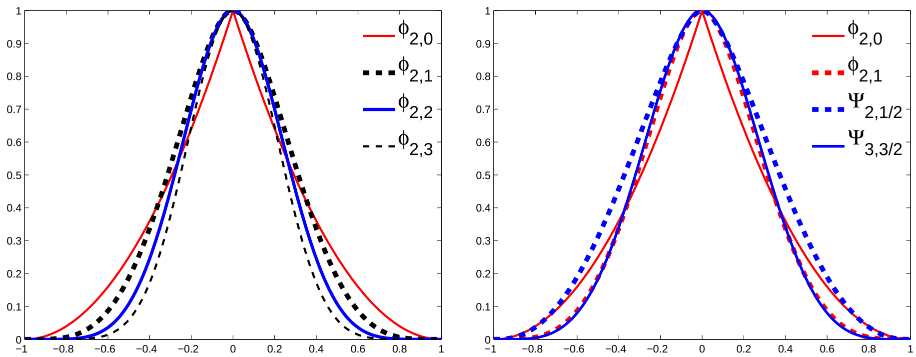
$$\Psi_{\mu,1} = \int_r^1 t(1-t)^\mu dt = \int_r^\infty t(1-t)_+^\mu dt = \mathcal{I}(\phi_\mu)(r). \tag{3.10}$$

It turns out that $\Psi_{\ell,1}(r)$ is simply the operator $\mathcal{I}$ defined in (3.3) acting on the truncated power functions $\phi_\ell(t)$. Furthermore, it can be shown that the following relationship between



(a) $\phi_{2,0}$ (b) $\phi_{2,1}$ (c) $\Psi_{2,1/2}$ (d) $\Psi_{2,3/2}$

FIG. 3.3. *Scaled Wendland functions and missing Wendland functions in* $\mathbb{R}^2$.



(a) Wendland functions in $\mathbb{R}^2$ (b) Missing Wendland functions in $\mathbb{R}^2$

FIG. 3.4. *Comparison between Wendland functions and missing Wendland functions on* $\mathbb{R}^2$. *The missing Wendland function* $\Psi_{3,3/2}$ *is very similar to and overlaps the Wendland function* $\phi_{2,1}$.

Wendland's function $\phi_{d,k}$ and the function defined in (3.9) holds for non-negative integer k (see Appendix D for details):

$$\phi_{d,k} = \Psi_{\lfloor d/2 \rfloor + k + 1, k}. \tag{3.11}$$

The above relationship shows that $\Psi_{\mu,\alpha}$ are a larger class of CSRBFs which includes Wendland's functions. It can also be shown that $\Psi_{\mu,\alpha}$ is positive definite when μ and α satisfy some constraint [47][49].

PROPOSITION 3.4. *For all non-negative integers* $\mu \in \mathbb{N}$ *and all half-integer* $\alpha = n + 1/2, n \in \mathbb{N}$, *the generalized Wendland function defined in* (3.9) *is positive definite on* $\mathbb{R}^d$, *if* $\mu \geq \lfloor d/2 + \alpha \rfloor + 1$. Schaback also proves that the function $\Psi_{\mu,\alpha}$ has similar reproducing property as Wendland's function in Proposition 3.3 in even dimensional space $\mathbb{R}^d$ [47, p.75 Collollary 1]:

PROPOSITION 3.5. *For integers* $m \geq 1, n \geq 0, d = 2m$, $\Psi_{\mu,n+1/2}$ *reproduce a Hilbert space which is isomorphic to Sobolev space* $\mathcal{H}^{m+n+1}(\mathbb{R}^d) = \mathcal{H}^{d/2+\alpha+1/2}(\mathbb{R}^d)$, *where* $\alpha = n + 1/2$
For such functions $\Psi_{\mu,\alpha}$, where μ is an integer and $\alpha = n + 1/2$ is a half integer, they are the so-called missing Wendland functions.

The generalized Wendland functions can be computed by a 6-line Maple program in Appendix C. It turns out that the missing Wendland functions $\Psi_{\mu,\alpha}$ involve two non-polynomial terms, and can be written as

$$\Psi_{\mu,\alpha}(r) = \mathcal{P}_{\mu,\alpha} \log\left(\frac{r}{1+\sqrt{1-r^2}}\right) + \mathcal{Q}_{\mu,\alpha}\sqrt{1-r^2}, \tag{3.12}$$

where $\mathcal{P}_{\mu,\alpha}$ and $\mathcal{Q}_{\mu,\alpha}$ are polynomials in r^2. For a detailed derivation and property of $\mathcal{P}_{\mu,\alpha}$ and $\mathcal{Q}_{\mu,\alpha}$, the reader is directed to [47]. For more details on the Wendland and the missing Wendland functions, one can refer to a recent paper [27]. Several missing Wendland functions of interest are listed in Table 4.2.

3.5. Construction by convolution and others. Provided some CSRBFs have been found, we can construct a class of CSRBFs by convolution. This is based on two facts: a function is positive definite if its Fourier transform is positive definite (see Appendix A); and the Fourier transform of the convolution of two functions is the product of the Fourier transforms of the two functions, namely, if $h(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\, dy$, then the following equation holds $\hat{h}(\xi) = \hat{f}(\xi)\hat{h}(\xi)$. Therefore, any two positive definite radial basis functions give another positive definite basis functions (which is not necessary radial symmetric); if one of them is compacted supported, then the resulting function is compactly supported.

Furthermore, we can construct positive definite compactly supported basis functions on a square. For example, if $\phi_1(x)$ and $\phi_2(x)$ are positive definite with a compact support $[-1,1]$, then their tensor product $\phi(x,y) = \phi_1(x)\phi_2(y)$ is also positive definite with a compact support on $[-1,1]\times[-1,1]$, but ϕ not radial symmetric. For compactly supported basis functions on a general polygon, readers are referred to box-spline [10].

As seen, Wendland functions and missing Wendland functions are finite smooth. A natural and interesting question is the existence of infinite smooth *positive definite* CSRBFs in $\mathbb{R}^d$. Schaback points out this is a open problem [46]. If there is no positive definiteness constraint, the well-known mollifier given by

$$\phi(\mathbf{x}) = \begin{cases} e^{-\frac{1}{1-\|\mathbf{x}\|^2}} & \text{if } \|\mathbf{x}\| \le 1; \\ 0 & \text{if } \|\mathbf{x}\| \ge 1, \end{cases} \tag{3.13}$$

is infinitely differentiable with compact support. While the Mollifier may be positive definite on a lower dimensional space, it is likely not to be positive definite on some higher dimensional space, because it has been proven that a continuous CSRBF can not be positive definite on every $\mathbb{R}^d$ [60, p.120].

4. Conclusion. In this paper we have considered high-dimensional approximation problems. These problems are challenging because on one hand, as seen, some well-accepted results in one-dimensional space may not be valid in higher-dimensional spaces; on the other hand, there are some challenging computational issues which go beyond our discussion. Radial basis functions are good candidates for high-dimensional scattered data approximation because they can avoid a singular interpolation matrix and there are simple and efficient ways to construct compactly supported radial basis functions with given smoothness. We want to emphasize that *"in almost every area of numerical analysis, sooner or later, the discussion comes down to approximation theory"* [53, p.605]; and radial basis function is one *"major newer topic "* in this fundamental area (compared with polynomial and rational minimax approximation) . Recent years have seen there are many advancement in this field, but further research is still needed to make these methods more effective and applicable to an even broader range of real-life applications.

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Appendix .

A. Suppose $\phi_j(\mathbf{x}) = \phi(\mathbf{x}-\mathbf{x}_j)$, where $\phi(\mathbf{x})$ is radially symmetric and has an integrable Fourier transform $\hat{\phi}$. According to the inverse Fourier transform, we have

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{\phi}(\omega) e^{i\mathbf{x}^T\omega} d\omega. \tag{4.1}$$

The positive definiteness of the linear systems (1.1) is equivalent to the positiveness of the following quadratic form:

$$\sum_{k,j=1}^{n} \alpha_k \bar{\alpha}_j \phi(\mathbf{x}_k - \mathbf{x}_j) = \frac{1}{(2\pi)^{d/2}} \sum_{j,k=1}^{N} \alpha_j \bar{\alpha}_k \int_{\mathbb{R}^d} \hat{\phi}(\omega) e^{i\omega^T(\mathbf{x}_j - \mathbf{x}_k)} d\omega \tag{4.2}$$

$$= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{\phi}(\omega) \left| \sum_{j=1}^{n} \alpha_j e^{i\mathbf{x}_j \omega} \right|^2 d\omega. \tag{4.3}$$

From (4.2) to (4.3), we need to write $e^{i\omega^T(\mathbf{x}_j - \mathbf{x}_k)}$ as $e^{i\omega^T \mathbf{x}_j} e^{-i\omega^T \mathbf{x}_k}$ and use the relationship $\sum_{k=1}^{n} \bar{\alpha}_k e^{-i\omega^T \mathbf{x}_k} = \overline{\sum_{j=1}^{n} \alpha_j e^{i\omega^T \mathbf{x}_j}}$. According to (4.3), a function ϕ whose Fourier transform $\hat{\phi}$ is positive guarantees a positive definite linear system (1.1), and thus is said to be *positive definite*. Using Fourier transforms to characterize a positive definite function dates back to Mathias [36], Bochner [3][60, p.67], and followed by von Neumann, Schoenberg [55] among others; and it provides simple way to verify whether the linear system (1.1) is positive definite or not. Generally speaking, finding a multi-variate Fourier transforms is not easy, but finding Fourier transform for radial functions can be carried out in univariate operations as shown in Schaback's and Wu's work [49].

B. Maple program for computing Wendland functions.

```
wd := proc (d, k, r)
local wd, kk;
wd := (1-r)^ (floor((1/2)*d)+k+1);
for kk form 1 by 1 to k do
wd := int(t*subs(r = t, wd), t = r .. 1)
end do;
return factor(wd)
end proc
```

C . Maple program for computing missing Wendland functions . The following program is a revised version of that in [47]

```
mswd := proc (mu, alpha, r)
local mswd;
mswd := t*(1-t)^ mu*(t^ 2-r^ 2)^ (alpha-1)/(GAMMA(alpha)*2^ (alpha-
    1));
mswd := int(mswd, t = r ..1);
return combine(simplify(mswd), ln)
end proc
```

It is noted that this program does not work when both μ and α are half-integer.

D. First, It can be shown that the operator defined in 3.8 have the following property [47]:

PROPOSITION 4.1. *$\mathcal{I}_\alpha \circ \mathcal{I}_\beta := \mathcal{I}_\alpha(\mathcal{I}_\beta(a_\mu)) = \mathcal{I}_{\alpha+\beta}$ and $\mathcal{I}_\alpha^k = \mathcal{I}_{k\alpha}$.* Using Proposition 4.1 and the construction process in the section 3.3, we can prove $\phi_{d,k} = \Psi_{\lfloor d/2 \rfloor + k+1,k}$. Define $\tilde{\phi}_{d,k,\alpha} := \mathcal{I}_\alpha^k(a_\mu)(r^2/2)$, where $\mu = \lfloor d/2 \rfloor + k\alpha + 1$, for $k = 1, 2, 3, 4, ...$, then by Proposition 4.1, we see that

$$\tilde{\phi}_{d,1,\alpha} = \mathcal{I}_\alpha(a_\mu)(r^2/2) = \Psi_{\mu,\alpha} = \Psi_{\lfloor d/2 \rfloor + \alpha+1,\alpha}(r), \tag{4.4}$$

$$\tilde{\phi}_{d,2,\alpha} = \mathcal{I}_\alpha^2(a_\mu)(r^2/2) = \mathcal{I}_{2\alpha}(a_\mu)(r^2/2) = \Psi_{\mu,2\alpha} = \Psi_{\lfloor d/2 \rfloor + 2\alpha+1,2\alpha}(r), \tag{4.5}$$

$$\tilde{\phi}_{d,k,\alpha} = \mathcal{I}_\alpha^k(a_\mu)(r^2/2) = \mathcal{I}_{k\alpha}(a_\mu)(r^2/2) = \Psi_{\mu,2\alpha} = \Psi_{\lfloor d/2 \rfloor + k\alpha+1,k\alpha}(r). \tag{4.6}$$

It is noted that $\Psi_{\mu,1}$ (3.10) is equal to $\phi_{d,0}$, which proves the result.

More generally, we can apply different operator $\mathcal{I}_\alpha$ in different steps, for example, $\mathcal{I}_\beta \mathcal{I}_\alpha(a_\mu)(r^2/2) = \Psi_{\mu,\alpha+\beta}$.

TABLE 4.1

Wendland's compactly supported radial basis functions of minimal degree

d	Wendland function $\phi_{d,k}(r), r = \|\mathbf{x}\|_2$	Smoothness
	$\phi_{1,0}(r) = (1-r)_+$	C^0
	$\phi_{1,1}(r) = (1-r)_+^3(1+3r)/12$	C^2
$d = 1$	$\phi_{1,2}(r) = (1-r)_+^5(3+15r+24r^2)/840$	C^4
	$\phi_{1,3}(r) = (1-r)_+^7(15+105r+285r^2+315r^3)/151200$	C^6
	$\phi_{1,4}(r) = (1-r)_+^9(105+945r+3555r^2+6795r^3+5760r^4)/51891840$	C^8
	$\phi_{3,0}(r) = (1-r)_+^2$	C^0
	$\phi_{3,1}(r) = (1-r)_+^4(1+4r)/20$	C^2
$d \leq 3$	$\phi_{3,2}(r) = (1-r)_+^6(3+18r+35r^2)/1680$	C^4
	$\phi_{3,3}(r) = (1-r)_+^8(15+120r+375r^2+480r^3)/332640$	C^6
	$\phi_{3,4}(r) = (1-r)_+^{10}(105+1050r+4410r^2+9450r^3+9009r^4)/121080960$	C^8
	$\phi_{5,0}(r) = (1-r)_+^3$	C^0
	$\phi_{5,1}(r) = (1-r)_+^5(1+5r)/30$	C^2
$d \leq 5$	$\phi_{5,2}(r) = (1-r)_+^7(3+21r+48r^2)/3024$	C^4
	$\phi_{5,3}(r) = (1-r)_+^9(15+135r+477r^2+693r^3)/665280$	C^6
	$\phi_{5,4}(r) = (1-r)_+^{11}(105+1155r+5355r^2+12705r^3+13440r^4)/259459200$	C^8
	$\phi_{7,0}(r) = (1-r)_+^4$	C^0
	$\phi_{7,1}(r) = (1-r)_+^6(1+6r)/42$	C^2
$d \leq 7$	$\phi_{7,2}(r) = (1-r)_+^8(3+24r+63r^2)/5040$	C^4
	$\phi_{7,3}(r) = (1-r)_+^{10}(15+150r+591r^2+960r^3+591r^2+960r^3)/1235520$	C^6
	$\phi_{7,4}(r) = (1-r)_+^{12}(105+1260r+6390r^2+16620r^3+19305r^4)/518918400$	C^8

TABLE 4.2

Missing Wendland functions

$\Psi_{\mu,\alpha}$	Function
$\Psi_{2,1/2}$	$\frac{\sqrt{2}}{3\Gamma(1/2)}\left(3r^2\mathcal{L}+(2r^2+1)\mathcal{S}\right)$
$\Psi_{3,3/2}$	$\frac{-\sqrt{2}}{480\Gamma(3/2)}\left((15r^6+90r^4)\mathcal{L}+(81r^4+28r^2-4)\mathcal{S}\right)$
$\Psi_{4,5/2}$	$\frac{\sqrt{2}}{40320\Gamma(5/2)}\left((945r^8+2520r^6)\mathcal{L}+(256r^8+2639r^6+690r^4-136r^2+16)\mathcal{S}\right)$
$\Psi_{5,7/2}$	$\frac{-\sqrt{2}}{5677056\Gamma(7/2)}\left(\mathcal{P}_{5,7/2}\mathcal{L}+\mathcal{Q}_{5,7/2}\mathcal{S}\right)$
	$\mathcal{P}_{5,7/2} = 3465r^{12}+83160r^{10}+13860r^8$
	$\mathcal{Q}_{5,7/2} = 37495r^{10}+160290r^8+33488r^6-724r^4+1344r^2-128$
	$\mathcal{L}(r) = \log(\frac{r}{1+\sqrt{1-r^2}}) \quad \mathcal{S}(r) := \sqrt{1-r^2}$

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