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Distance Rationalization of Voting Rules

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Abstract The concept of *distance rationalizability* allows one to define new voting rules or rationalize existing ones via a consensus, i.e., a class of elections that have a unique, indisputable winner, and a distance over elections: A candidate is declared an election winner if she is the consensus candidate in one of the nearest consensus elections. Many classic voting rules are defined or can be represented in this way. In this paper, we focus on the power and the limitations of the distance rationalizability approach. Lerer and Nitzan (1985) and Campbell and Nitzan (1986) show that if we do not place any restrictions on the notions of distance and consensus then essentially all voting rules can be distance-rationalized. We identify a natural class of distances on elections—votewise distances—which depend on the submitted votes in a simple and transparent manner, and investigate which voting rules can be rationalized via distances of this type. We also study axiomatic properties of rules that can be defined via votewise distances.

1 Introduction

Voting is an important tool for making collective decisions that accommodate the preferences of all participating agents. Agents submit ballots, which usually are rankings of the alternatives (i.e., total orders over the set of all alternatives), and a *voting rule* is used to select the alternative that reflects the individual preferences of the agents in the best possible way. However, there is no universal agreement as to what 'the best possible way' means for a

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given society and, thus, there exist a multitude of remarkably diverse voting rules (see, e.g., Brams & Fishburn, 2002).

There are several explanations for this diversity. One reason is that each voting rule represents an agreement within the society as to what should be considered a just outcome, and this agreement may differ from one society to another. Another reason is the strikingly long list of impossibility results—starting with the famous Arrow's impossibility theorem (Arrow, 1951; revised edition, 1963)—which typically state that it is impossible to aggregate preferences so as to simultaneously satisfy several natural desiderata. Thus, in each real-life scenario the election designer has to decide which of the desired conditions should be sacrificed.

An earlier view, advocated by Marquis de Condorcet (Condorcet, 1785), is that the aim of voting is to determine the 'best' decision for the society when voters are prone to making mistakes in their judgements. This approach assumes that there is an objectively best choice, but voters have different opinions due to random errors of judgement; absent these errors, they would all agree on the most suitable alternative. Thus, one should aim to design a voting rule so as to maximize the probability of identifying the best choice (see Young, 1977; Young & Levenglick, 1978; Conitzer & Sandholm, 2005; Conitzer, Rognlie, & Xia, 2009; Xia, Conitzer, & Lang, 2010; Xia & Conitzer, 2011; Pivato, 2013; Caragiannis, Procaccia, & Shah, 2013; Elkind & Shah, 2014).

In this paper, we focus on yet another approach, which has been implicitly or explicitly suggested in a number of papers (Farkas & Nitzan, 1979; Nitzan, 1981; Lerer & Nitzan, 1985; Campbell & Nitzan, 1986; Baigent, 1987a; Meskanen & Nurmi, 2008; Elkind, Faliszewski, & Slinko, 2012); see also the survey of Eckert and Klamler (2011). This approach can be called consensus-based. It assumes that the society agrees on a notion of a consensus (for example, we could say that there is a consensus if all voters agree which candidate is the best, or if there exists a Condorcet winner), and the result of each election is viewed as an imperfect approximation to a consensus. If an election E is a consensus, then we pick the unique consensus winner, and otherwise we output the winner of a consensus election E' that is as close to E as possible. Alternatively, we may say that the society looks for a minimal change to the given election that turns it into an election with an indisputable winner. In the heart of this approach is the agreement as to (1) which situations should be viewed as electoral consensuses and (2) what is the appropriate notion of closeness among elections. The differences among voting rules can then be explained by different choices of these parameters. Explaining voting rules in terms of consensuses and distances is known as *distance rationalization of voting rules*. A survey on this topic is to appear soon (Elkind & Slinko, 2015). We remark that such a distance-based approach is also used in several other settings, including, e.g., weak-order aggregation (Baigent & Klamler, 2004), bargaining (Pfingsten & Wagener, 2003), and judgment aggregation (Miller & Osherson, 2009).

1.1 Goals and Motivation

Our aim is to study the properties of the distance rationalization framework. We focus on general mechanisms of distance rationalizability rather than on rationalizations of specific voting rules. This line of work was initiated in the mid-80s by Lerer and Nitzan (1985) and by Campbell and Nitzan (1986). These early papers provided two important results. First, it was proved that essentially every voting rule is distance rationalizable with respect to *some* (though, perhaps, very unnatural) distance.[1] Second, it was shown that, if we restrict our attention to a certain natural class of distances and to the unanimity consensus (where an election is said to be consensual if all voters agree on the top candidate), then the rules that can be distance rationalized are exactly the strong scoring rules, that is, scoring rules whose score vectors are strictly decreasing.[2] Together, these two results mean that, while it is essentially impossible to prove any nontrivial facts about all distance rationalizable voting rules—such results would have to, in effect, speak of almost all voting rules—it may be useful to seek natural restrictions to the distance rationalizability framework. Building upon and extending the approach of Lerer and Nitzan, we suggest such a restricted framework and study its strengths and limitations.

First, we limit our attention to the following consensus notions:

- the class $\mathcal{S}$ of strongly unanimous elections (all voters agree on the ranking of all candidates),
- the class $\mathcal{U}$ of unanimous elections (all voters agree which candidate is the best),
- the class $\mathcal{M}$ of elections with an absolute majority winner (a candidate ranked first by more than half of the voters),
- the class $\mathcal{C}$ of elections with a Condorcet winner (a candidate who is preferred to every other candidate by more than half of the voters).

Second, we restrict the class of distances, by requiring that a voting rule must depend on the submitted votes in a smooth, simple and transparent manner. To implement this principle, we limit ourselves to distances over elections that are obtained by extending distance over votes (preference orders).

For example, given a distance d over preference orders, and two elections, E and E', with the same candidate set and with lists $U = (u_1, \dots, u_n)$ and $V = (v_1, \dots, v_n)$ of submitted votes, we can define a distance $\widehat{d}$ between E and E' as the sum of the distances between the respective pairs of votes, i.e.,

[1] We refer to this result, proved by Lerer and Nitzan (1985) and generalized by Campbell and Nitzan (1986), as the 'universal distance rationalizability theorem'. Unaware of these papers, we have rediscovered the universal distance rationalizability theorem in a conference paper (Elkind, Faliszewski, & Slinko, 2010). We would like to thank the reviewer who brought the papers of Lerer and Nitzan and of Campbell and Nitzan to our attention.

[2] We have rediscovered a slightly stronger version of this result (Elkind et al., 2010; Elkind, Faliszewski, & Slinko, 2011).

we can set

$$\widehat{d}(E, E') = \sum_{i=1}^{n} d(u_i, v_i).$$

In other words, we can define a distance between E and E' as the ℓ_1-*norm* of the vector $(d(u_1, v_1), \dots, d(u_n, v_n))$. More generally, we can use any other norm in place of ℓ_1; the resulting function on pairs of elections is guaranteed to satisfy all distance axioms as long as d does (formally, a distance of this form is known as a *product metric*). We refer to distances obtained in this manner as *votewise* distances. Votewise distances generalize the *symmetric additively decomposable metrics* used by Lerer and Nitzan (1985).

The goal of this paper is to show that restricting the distance rationalizability framework to votewise distances and to the four consensus classes listed above leads to an interesting and useful concept. We demonstrate that, on the one hand, our framework is expressive enough to rationalize many interesting voting rules and, on the other hand, it enables us to derive some non-trivial general properties of the respective distance rationalizable voting rules.

1.2 Our Results

The paper is structured as follows. Sections 2 and 3 provide preliminaries and a formal definition of the distance rationalizability framework. We present our main results in Sections 4–6. Section 7 concludes.

In more detail, in Section 4.1 we present a catalog of votewise distance rationalizations, which contains many common voting rules. In particular, we review classic rationalizations of Plurality, the Borda rule, and the Kemeny rule; then, by tweaking parameters of these rationalizations, we obtain distance rationalizations of some less known rules such as the Threshold rule (Aleskerov, Chistyakov, & Kalyagin, 2010) and a variant of the Bucklin rule. These two rules are particularly interesting because their rationalizations make use of norms other than ℓ_1, whereas all votewise rationalizations that have appeared in the literature so far have relied on the ℓ_1 norm.

Modifying distance rationalizations of known voting rules may also result in new voting rules with appealing properties. In this paper, we consider two such examples. Our first example is the Dodgson$^\infty$ rule, which is obtained by replacing the ℓ_1 norm in the standard rationalization of the Dodgson rule with the ℓ_∞ norm. We show that Dodgson$^\infty$ is an approximation of the original Dodgson rule, which possesses many nice properties that are not shared by the Dodgson rule itself. Our second example is the class of $\mathcal{M}$-scoring rules. For every scoring rule we define its $\mathcal{M}$-analog, which is obtained by replacing the consensus class $\mathcal{U}$ in the canonical distance rationalization of this rule with the class $\mathcal{M}$. For example, under $\mathcal{M}$-Borda the score of each alternative is calculated not as the sum of all Borda scores given to this alternative by individual voters but as the sum of its best $\lfloor n/2 \rfloor + 1$ Borda scores, where n

is the number of voters. Rules of this type sometimes occur in real-life aggregation procedures (see, e.g., the discussion in a recent paper by Goldsmith, Lang, Mattei, & Perny, 2014).

In Section 4.2, we prove that some rules whose known distance rationalizations are very complicated (e.g., STV and Plurality with Runoff) are not votewise distance rationalizable with respect to the standard consensus classes. This shows that the dichotomy between the rules that are votewise and those that are not is indeed meaningful.

In Section 5 we show that for votewise rules it is possible to derive a number of standard properties—such as anonymity, neutrality, homogeneity and monotonicity—from fairly mild assumptions on the parameters of their votewise distance rationalizations.

Section 6 provides a characterization of scoring rules and $\mathcal{M}$-scoring rules in terms of votewise distance rationalizability. This result (together with a closely related result by Lerer & Nitzan, 1985) gives an alternative characterization of scoring rules, which complements the classic work of Young (1975).

2 Preliminaries

An *election* is a pair $E = (C, V)$, where $C = \{c_1, \dots, c_m\}$ is the set of *candidates* and $V = (v_1, \dots, v_n)$ is an ordered list of *voters*. The number of alternatives is denoted by $|C|$, and the number of voters is denoted by $|V|$. Each voter is identified with her *vote*, i.e., a strict linear order over C (also called a *preference order*); e.g., $c_i \succ c_j \succ \dots \succ c_k$ is a vote where c_i is ranked first, c_j is ranked second and c_k is ranked last. We denote by $P(C)$ the set of all possible preference orders over C; $P(C, c)$ denotes the set of all preference orders over C in which c is ranked first. We refer to the list V as a *preference profile*. We write $\mathrm{pos}(v_i, c)$ to denote the position of candidate c in vote v_i (the top candidate has position 1 and the last candidate has position m). We denote by $v_i(j)$ the candidate that is ranked j-th in v_i. A *Condorcet winner* of an election E is a candidate who is preferred to every other candidate by a strict majority of voters; note that not every election has a Condorcet winner.

We will often discuss various transformations of preference profiles. To this end, we will use the following notation. Let $E = (C, V)$ be an election with $C = \{c_1, \dots, c_m\}$, $V = (v_1, \dots, v_n)$, and let σ and π be permutations of $\{1, \dots, n\}$ and C, respectively. By $\sigma(V)$ we denote the preference profile $(v_{\sigma(1)}, \dots, v_{\sigma(n)})$ (abusing notation, we will sometimes refer to such σ as a *permutation of* V). We write $\tilde{\pi}(v)$ to denote the vote obtained from v by replacing c with $\pi(c)$ for all $c \in C$. We extend this notation to preference profiles and write $\tilde{\pi}(V)$ to denote the profile $(\tilde{\pi}(v_1), \dots, \tilde{\pi}(v_n))$. For each $C' \subseteq C$, we let $\pi(C') = \{\pi(c) \mid c \in C'\}$.

Let $E_1 = (C, V_1)$ and $E_2 = (C, V_2)$ be two elections over the same set of candidates C. We denote by $V_1 + V_2$ the preference profile V obtained by concatenating the lists V_1 and V_2, and write $E_1 + E_2$ to denote the election $(C, V_1 + V_2)$. Given an election $E = (C, V)$ and a positive integer k, we denote by kV the preference profile obtained by concatenating k copies of V, and write kE to denote (C, kV). When V consists of a single preference order v, we will write $k \times v$ instead of $k(v)$ to improve readability.

A *voting rule* $\mathcal{R}$ is a function that given an election $E = (C, V)$ returns a non-empty set of *election winners* $\mathcal{R}(E) \subseteq C$. When the set of candidates is clear from the context, we will sometimes identify an election $E = (C, V)$ with its preference profile V: e.g., we may write $\mathcal{R}(V)$ instead of $\mathcal{R}(E)$. We will use the following well-known voting rules as examples in our discussion (we omit discussion of tie-breaking mechanisms; our results are independent of them).

Scoring rules. For each m-element vector $\alpha = (\alpha_1, \dots, \alpha_m)$ of nonnegative reals with $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_m$, a *scoring rule* $\mathcal{R}_\alpha$ is defined as follows. Fix an election $E = (C, V)$, where $C = \{c_1, \dots, c_m\}$ and $V = (v_1, \dots, v_n)$. The α-*score* of a candidate $c \in C$ is given by $\sum_{i=1}^{n} \alpha_{\mathrm{pos}(v_i, c)}$. That is, candidate c receives α_j points from each voter that ranks her in the j-th position, and her score is the total number of points she receives. The winners of E under $\mathcal{R}_\alpha$ are the candidates with the maximum α-score. Note that $\mathcal{R}_\alpha$ is defined for a fixed number of candidates. However, many well-known voting rules correspond to families of scoring rules. For example, *Plurality* is defined by the family of vectors $(1, 0, \dots, 0)$, and the *Borda rule* is defined by the family of vectors $(m-1, m-2, \dots, 0)$.

Bucklin and Simplified Bucklin. Let $E = (C, V)$ be an election. Given a positive integer $k \leq |C|$, we say that a candidate c is a k-*majority winner* if more than $|V|/2$ voters rank c among the top k candidates. Let k' be the smallest positive integer such that E has at least one k'-majority winner. The *Bucklin score* of a candidate c is the number of voters that rank c in the top k' positions. The *Simplified Bucklin winners* are all k'-majority winners, whereas the *Bucklin winners* are the candidates with the highest Bucklin score (clearly, all of them are k'-majority winners).

Single Transferable Vote (STV). Under *STV*, also known as the *Alternative Vote*, the winner is chosen by repeated elimination of 'inferior' candidates. In each round a candidate ranked first the least number of times is eliminated and removed from all preference orders (this may require tie-breaking). This step is repeated until a single candidate remains; this candidate is declared to be the winner.

Plurality with Runoff. Under *Plurality with Runoff* we first pick two candidates with the highest Plurality scores and then output as winner the one who is preferred to the other by a majority of voters; if the voters are split evenly between these two candidates, both are declared to be the winners.

Dodgson. The *Dodgson score* of a candidate c is the smallest number of swaps of adjacent candidates that have to be performed on the votes to make c the Condorcet winner. The *Dodgson winners* are the candidates with the lowest Dodgson score.

Kemeny. A *disagreement* between two preference orders u and v over C is a pair $(c, c') \in C \times C$ such that u ranks c above c', but v ranks c' above c; we denote the number of disagreements between u and v by $t(u, v)$. A candidate c is a *Kemeny winner* of the election $E = (C, V)$ if she is ranked first in some vote $v' \in \arg\min_{v \in P(C)} \sum_{i=1}^{n} t(v, v_i)$.

Litvak. The *Litvak rule* (Litvak, 1982; Meskanen & Nurmi, 2008) is similar to the Kemeny rule. Given two preference orders u and v over C, we set $s(u, v) = \sum_{c \in C} |\mathrm{pos}(u, c) - \mathrm{pos}(v, c)|$; a candidate c is a *Litvak winner* (sometimes called a *Litvak median*) of an election $E = (C, V)$ if she is ranked first in some vote $v' \in \arg\min_{v \in P(C)} \sum_{i=1}^{n} s(v, v_i)$.

Threshold. Under the *Threshold rule*, the score of a candidate c in an election $E = (C, V)$ is $\min_{i=1,\dots,n}\{m - \mathrm{pos}(v_i, c)\}$, and the winners are the candidates with the highest score. The threshold rule was introduced by Aleskerov et al. (2010) (see also the references therein).

One of the standard ways of evaluating a voting rule is to check whether it satisfies common normative properties. We consider the following ones:

Anonymity. A voting rule $\mathcal{R}$ is *anonymous* if for every election $E = (C, V)$ and every permutation σ of V the election $E' = (C, \sigma(V))$ satisfies $\mathcal{R}(E) = \mathcal{R}(E')$.

Neutrality. A voting rule $\mathcal{R}$ is *neutral* if for every election $E = (C, V)$ and every permutation π of C the election $E' = (C, \tilde{\pi}(V))$ satisfies $\pi(\mathcal{R}(E)) = \mathcal{R}(E')$.

Continuity. A voting rule $\mathcal{R}$ is *continuous* if for every pair of elections $E_1 = (C, V_1)$, $E_2 = (C, V_2)$ such that $\mathcal{R}(E_1) = \{a\}$ the election $E = (C, kV_1 + V_2)$ satisfies $\mathcal{R}(E) = \{a\}$ for all sufficiently large values of k. This property was introduced by Young (1975), and is an analogue of the Archimedean property from analysis (see, e.g., Schechter, 1997).

Consistency. A voting rule $\mathcal{R}$ is *consistent* if for every pair of elections $E_1 = (C, V_1)$, $E_2 = (C, V_2)$ with $\mathcal{R}(E_1) \cap \mathcal{R}(E_2) \neq \emptyset$, the election $E = (C, V_1 + V_2)$ satisfies $\mathcal{R}(E) = \mathcal{R}(E_1) \cap \mathcal{R}(E_2)$. Consistency was introduced by Young (1975); it is also known as *reinforcement* (Chebotarev & Shamis, 1998).

Homogeneity. A voting rule $\mathcal{R}$ is *homogeneous* if for every election $E = (C, V)$ and every positive integer k it holds that $\mathcal{R}(E) = \mathcal{R}(kE)$. Note that consistency implies homogeneity.

Monotonicity. A voting rule $\mathcal{R}$ is *monotone* if for every election $E = (C, V)$, every $c \in \mathcal{R}(E)$ and every voter $v \in V$ with $\mathrm{pos}(v, c) \neq 1$ we have $c \in \mathcal{R}(E')$, where E' is the election obtained from E by swapping c with the candidate ranked right above her in v. Moulin (1991) calls this property *ceteris paribus monotonicity* to distinguish it from Maskin monotonicity.

3 Unrestricted Distance Rationalizability

Intuitively, the idea behind the notion of distance rationalizability is that election winners should be the candidates that are, in some sense, the closest to representing a consensus among the voters. Naturally, the opinions about what constitutes a consensus and how closeness should be measured may differ. As we will see, by combining different notions of consensus and distance we can synthesize a very diverse set of voting rules. Conversely, we can analyze a voting rule to uncover a consensus/distance pair that defines it (though such a pair need not be unique!) in order to learn more about the properties of the rule and the rationale behind it. Our focus is on the properties of the distance rationalization framework itself, and so we need a careful formal definition.

3.1 Distances

Let X be a set. A function $d\colon X \times X \to \mathbb{R} \cup \{+\infty\}$ is a *distance* (or, a *metric*) if for every $x, y, z \in X$ it satisfies the following four conditions:

(1) $d(x, y) \geq 0$ (nonnegativity),
(2) $d(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles),
(3) $d(x, y) = d(y, x)$ (symmetry),
(4) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

A *pseudodistance* is required to satisfy (1), (2), and (4), whereas a *quasidistance* needs to satisfy (1), (2), and (3).[3]

We are interested in distances on elections. Two classic examples of such distances are the *Hamming distance* and the *swap distance*. Both can be defined by taking a suitable distance d over $P(C)$ and extending it to a distance $\widehat{d}$ over elections as follows. Given two elections $E = (C, U)$ and $E' = (C, V)$, where $U = (u_1, \dots, u_n)$, $V = (v_1, \dots, v_n)$, we set $\widehat{d}(E, E') = d(u_1, v_1) + \dots + d(u_n, v_n)$.[4] For the Hamming distance, the underlying distance d on $P(C)$ is simply the *discrete distance* given by $d_{\mathrm{discr}}(u, v) = 0$ if $u = v$ and $d_{\mathrm{discr}}(u, v) = 1$ if $u \neq v$. For the swap distance, the underlying distance on $P(C)$ is the swap distance between individual votes: $d_{\mathrm{swap}}(u, v)$ is the number of pairwise disagreements between u and v. Thus, given two n-voter elections $E = (C, U)$ and $E' = (C, V)$, the Hamming distance $\widehat{d}_{\mathrm{discr}}(E, E')$ counts the number of positions in which U and V differ, and the swap distance $\widehat{d}_{\mathrm{swap}}(E, E')$ counts the overall number of inversions between the respective votes of U and V.

We will often discuss how distances change under various transformations of preference orders and preference profiles. In particular, we will be interested in variants of neutrality and anonymity for distances.

Definition 1 Let C be a set of candidates and let d be a distance on preference orders over C. We say that d is *neutral* if for every permutation π of C and every pair of preference orders $(u, v) \in P(C) \times P(C)$ it holds that $d(u, v) = d(\tilde{\pi}(u), \tilde{\pi}(v))$.

[3] We follow the terminology of Lerer and Nitzan (1985) here.

[4] We remark that Lerer and Nitzan (1985) refer to distances that are defined in this way as *symmetric additively decomposable metrics*; generalizing this class of distances is at the heart of this paper.

Definition 2 Let D be a distance over elections. We say that D is *anonymous* if for every pair of elections $E = (C, U), E' = (C, V)$ such that $|U| = |V| = n$ and every permutation σ of $\{1, \ldots, n\}$ it holds that $D(E, E') = D((C, \sigma(U)), (C, \sigma(V)))$. Further, we say that D is *neutral* if for every pair of elections $E = (C, U), E' = (C, V)$ and every permutation π of C it holds that $D(E, E') = D((C, \tilde{\pi}(U)), (C, \tilde{\pi}(V)))$.

It is easy to see that the distances d_{discr} and d_{swap} are neutral, and that the distances $\widehat{d}_{\text{discr}}$ and $\widehat{d}_{\text{swap}}$ are anonymous and neutral.

3.2 Consensus classes

Informally, we say that an election $E = (C, V)$ is a consensus if it has an undisputed winner reflecting a certain concept of fairness in the society. Formally, a *consensus class* is a pair $(\mathcal{E}, \mathcal{W})$ where $\mathcal{E}$ is a set of elections and $\mathcal{W} \colon \mathcal{E} \to C$ is a mapping that assigns a unique alternative to each election in $\mathcal{E}$; this alternative is called the *consensus alternative (winner)*.[5] The following four classes of elections have been historically viewed as situations of consensus:

Strong unanimity. This class, denoted $\mathcal{S}$, consists of elections where all voters report the same preference order. The consensus alternative is the candidate ranked first by all voters.

Unanimity. This class, denoted $\mathcal{U}$, consists of elections where all voters rank some candidate c first (but may disagree on the ranking of the remaining candidates). The consensus alternative is this candidate c.

Majority. This class, denoted $\mathcal{M}$, consists of elections where more than half of the voters rank some candidate c first. The consensus alternative is this candidate c.

Condorcet. This class, denoted $\mathcal{C}$, consists of elections with a Condorcet winner. The consensus alternative is the Condorcet winner.

There are other consensus classes one could consider: for example, one could study a 2/3-variant of the majority consensus $\mathcal{M}$, etc. Consensus classes can be viewed as partial voting rules, i.e., rules that are defined on consensus elections only. Thus we can extend the definitions of the standard normative properties of voting rules to consensus classes. In particular, we say that a consensus class $\mathcal{K} = (\mathcal{E}, \mathcal{W})$ is *anonymous* if for every election $E \in \mathcal{E}$ it holds that (a) every election E' that can be obtained by permuting the preference orders in E is also in $\mathcal{E}$ and, (b) for every such E' it holds that $\mathcal{W}(E) = \mathcal{W}(E')$.

Since we are interested in relations between voting rules and consensus notions, we need the following basic notion of compatibility between them.

[5] One can also consider situations in which the voters reach a consensus that several candidates are equally well qualified to be elected, but we do not discuss this possibility.

Definition 3 We say that a voting rule $\mathcal{R}$ is *compatible* with a consensus class $\mathcal{K} = (\mathcal{E}, \mathcal{W})$, or $\mathcal{K}$-*compatible*, if for every election E in $\mathcal{E}$ it holds that $\mathcal{R}(E) = \{\mathcal{W}(E)\}$.[6]

Almost all voting rules are $\mathcal{S}$-compatible (with the notable exceptions of Veto (Antiplurality), k-Approval, and similar rules). On the other hand, many voting rules, including all scoring rules, fail $\mathcal{C}$-compatibility.

3.3 Unrestricted Distance Rationalizability of Voting Rules Is Vacuous

We are now ready to define the distance rationalizability framework and explain the controversies surrounding it.

Definition 4 Let d be a distance over elections and let $\mathcal{K} = (\mathcal{E}, \mathcal{W})$ be a consensus class. We define the $(\mathcal{K}, d)$-*score* of a candidate c in an election E to be the distance (according to d) between E and a closest consensus election $E' \in \mathcal{E}$ such that c is the consensus winner of E'. The set of $(\mathcal{K}, d)$-*winners* of $E = (C, V)$ consists of all candidates in C whose $(\mathcal{K}, d)$-score is the smallest.

Definition 5 A voting rule $\mathcal{R}$ is *distance rationalizable* via a consensus class $\mathcal{K}$ and a distance d over elections (or, $(\mathcal{K}, d)$-*rationalizable*) if, for every election E, a candidate c is an $\mathcal{R}$-winner of E if and only if she is a $(\mathcal{K}, d)$-winner of E.

Many common voting rules are known to be distance rationalizable (see Table 1). In fact, Campbell and Nitzan (1986) demonstrate that essentially every voting rule is distance rationalizable. We will now state their 'universal distance rationalizability' result, tailored to our setting. A voting rule $\mathcal{R}$ (respectively, a consensus class) over a set of candidates C is said to satisfy *nonimposition* if for every $c \in C$ there exists an election (respectively, a consensus election) with candidate set C in which c is the unique winner under $\mathcal{R}$ (respectively, in which c is the consensus winner).

Theorem 1 (Campbell and Nitzan (1986)) *Let C be a set of candidates, let $\mathcal{R}$ be a voting rule over C that satisfies nonimposition, and let $\mathcal{K} = (\mathcal{E}, \mathcal{W})$ be a consensus class over C that satisfies nonimposition. Then there is a distance d such that $\mathcal{R}$ is $(\mathcal{K}, d)$-rationalizable if and only if $\mathcal{R}$ is $\mathcal{K}$-compatible.*

Consequently, distance rationalizability with respect to each of our consensus classes $\mathcal{S}$, $\mathcal{U}$, $\mathcal{M}$, and $\mathcal{C}$ is equivalent to compatibility with respect to this class.

Corollary 1 (Campbell and Nitzan (1986)) *For every consensus class $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}, \mathcal{M}, \mathcal{C}\}$, a voting rule $\mathcal{R}$ that satisfies non-imposition is $(\mathcal{K}, d)$-rationalizable for some distance d if and only if $\mathcal{R}$ is $\mathcal{K}$-compatible.*

[6] One might think that the term "$\mathcal{K}$-consistent" would be more appropriate than "$\mathcal{K}$-compatible." Indeed, a voting rule that elects the Condorcet winner whenever one exists is usually referred to as Condorcet-consistent. Nonetheless, we decided to use the term "$\mathcal{K}$-compatible" to avoid confusion with the normative axiom of consistency.

Further, for every voting rule that satisfies nonimposition, there is a consensus class with respect to which it is distance rationalizable.

Corollary 2 *For every voting rule $\mathcal{R}$ over a set of candidates C that satisfies nonimposition there exist a consensus class $\mathcal{K} = (\mathcal{E}, \mathcal{W})$ and a distance d such that $\mathcal{R}$ is $(\mathcal{K}, d)$-rationalizable.*

Proof Let $\mathcal{E} = \{E \mid |\mathcal{R}(E)| = 1\}$, and for each election $E \in \mathcal{E}$ let $\mathcal{W}(E)$ be the unique $\mathcal{R}$-winner of E. The resulting consensus class $\mathcal{K} = (\mathcal{E}, \mathcal{W})$ satisfies nonimposition, so by Theorem 1 $\mathcal{R}$ is $(\mathcal{K}, d)$-rationalizable for some distance d. □

In contrast with distance rationalizations listed in Table 1, the ones provided by Theorem 1 and its corollaries are very unnatural: they use artificial distances, artificial consensus classes, or both. Thus, merely knowing that a rule is distance rationalizable provides no insight into the properties of the rule. In other words, to be able to derive conclusions about a voting rule from its distance rationalization, we should restrict both the set of consensus classes (as we already did in Section 3.2) and the set of distances that we consider. In the next two sections we define and analyze a class of distances that, on the one hand, appear to be expressive enough to capture many of the interesting rules, and, on the other hand, allow us to draw nontrivial conclusions about rules rationalizable via these distances.

4 Votewise Distance Rationalizability

In a democratic society, the outcome of an election must depend on the votes cast in a simple and transparent way. Ideally, we would also like to see an analogue of continuity called ‘proximity preservation’ (Baigent, 1987b) that requires that a small change in opinions should not result in a large change in the outcome. This is not always achievable, and one should expect that when two or more candidates in an election are tied or almost tied, a small change in the votes may result in an abrupt change of the winner. Indeed, Baigent (1987b) showed that ‘proximity preservation’ cannot coexist with anonymity and unanimity. However, proximity preservation may be achievable for profiles that are far from being tied. Nitzan (1989) showed that proximity preservation near unanimous profiles is compatible with anonymity and unanimity.

In the context of our framework it is therefore natural to require distances between elections to be simple functions of preference orders in these elections. One way to construct a distance that satisfies this requirement is to first define a distance on preference orders and then extend it to elections using one of the classical norms. The reader may observe that the distances $\widehat{d}_{\mathrm{discr}}$ and $\widehat{d}_{\mathrm{swap}}$ defined in Section 3.1 are obtained in this way.

Definition 6 A *norm* on $\mathbb{R}^n$ is a mapping $N\colon \mathbb{R}^n \to \mathbb{R}$ that possesses the following properties:

(a) positive scalability: $N(\alpha u) = |\alpha| N(u)$ for all $u \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$;
(b) positive semidefiniteness: $N(u) \geq 0$ for all $u \in \mathbb{R}^n$, and $N(u) = 0$ if and only if $u = (0, 0, \dots, 0)$;
(c) triangle inequality: $N(u + v) \leq N(u) + N(v)$ for all $u, v \in \mathbb{R}^n$.

A well-known class of norms on $\mathbb{R}^n$ is that of *p-norms* ℓ_p, $p \in \mathbb{Z}^+ \cup \{\infty\}$:

$$\ell_p(x_1, \dots, x_n) = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \text{ for } p \in \mathbb{Z}^+, \quad \ell_\infty(x_1, \dots, x_n) = \max_{i=1,\dots,n} |x_i|.$$

In particular, $\ell_1(x_1, \dots, x_n) = |x_1| + \dots + |x_n|$. A norm N on $\mathbb{R}^n$ is said to be *symmetric* if it satisfies $N(x_1, \dots, x_n) = N(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for every permutation σ of $\{1, \dots, n\}$. Clearly, all p-norms are symmetric.

Definition 7 Consider a set of candidates C, a distance d on $P(C)$, an $n > 0$, and a norm N on $\mathbb{R}^n$. We say that a (pseudo)distance D on the space of all n-voter elections over C is *N-votewise* if for every pair of elections $E = (C, V)$ and $E' = (C, U)$ with $V = (v_1, \dots, v_n)$ and $U = (u_1, \dots, u_n)$ we have

$$D(E, E') = N(d(u_1, v_1), \dots, d(u_n, v_n)). \tag{1}$$

It is easy to check that for every (pseudo)distance d on $P(C)$ and every norm N on $\mathbb{R}^n$ the function defined by (1) is a (pseudo)distance. We will denote this (pseudo)distance by $\widehat{d^N}$. If $N = \ell_p$ for some $p \in \mathbb{Z}^+ \cup \{\infty\}$, we will write $\widehat{d^p}$ instead of $\widehat{d^{\ell_p}}$. Further, since many distance rationalizations use ℓ_1 as the underlying norm, we will write $\widehat{d}$ instead of $\widehat{d^1}$ (note that this notation is consistent with the one used in Section 3.1 for $\widehat{d}_{\text{swap}}$ and $\widehat{d}_{\text{discr}}$). Given a norm[7] N, we will say that a rule is *N-votewise* if it can be distance rationalized via an N-votewise distance; we say that a rule is *votewise* if it is N-votewise for some norm N. When the set of candidates C is clear from the context, we will omit it from the notation and write $\widehat{d^N}(U, V)$ instead of $\widehat{d^N}((C, U), (C, V))$, i.e., we will view $\widehat{d^N}$ also as a distance over preference profiles.

Usually, $\widehat{d^N}$ inherits some of the desirable properties of d and N.

Proposition 1 *Let N be a norm on $\mathbb{R}^n$ and let d be a (pseudo)distance on $P(C)$. If d is neutral then the (pseudo)distance $\widehat{d^N}$ is also neutral. If N is symmetric then $\widehat{d^N}$ is anonymous.*

Lerer and Nitzan (1985) studied symmetric additively decomposable rationalizations, which in our terminology are ℓ_1-votewise rationalizations. Throughout this paper, we will mostly focus on ℓ_p-votewise rules. Interestingly, all votewise rationalizations of common voting rules that have been discovered so far use either ℓ_1 or ℓ_∞ as the underlying norm; we see no obvious explanation to that. We note, however, that ℓ_2-votewise distances have been used in the preference aggregation literature (see, e.g., Bogard, 1973, 1975; Cook & Seiford, 1978, 1982; Litvak, 1983).

[7] Technically, a norm is defined for a fixed value of n, whereas voting rules are usually defined for any number of voters, i.e., we require a family of norms, one for each value of n. Note that each p-norm, $p \in \mathbb{Z}^+ \cup \{\infty\}$, is indeed defined for all values of n.

4.1 A Catalog of Votewise Distance Rationalizations

Our goal now is to see which voting rules are votewise distance rationalizable, and to collect these results into a convenient catalog. To do so, we have to introduce a few more distances on preference orders.

4.1.1 Distances on Preference Orders

Let C be a set of candidates and let u and v be two preference orders from $P(C)$. We consider the following distances over $P(C)$ (for completeness, we include the already-defined discrete distance and swap distance):

Discrete distance. The *discrete distance* $d_{\text{discr}}(u, v)$ is defined to be 0 if $u = v$ and to be 1 otherwise.

Swap distance. The *swap distance* $d_{\text{swap}}(u, v)$ is the number of pairs $(c, c') \in C \times C$ such that u ranks c above c', but v ranks c' above c.

Sertel distance. *Sertel's distance* $d_{\text{sert}}(u, v)$ is defined as $d_{\text{sert}}(u, v) = \max\{i \mid u(i) \neq v(i)\}$, with the convention that $d_{\text{sert}}(u, v) = 0$ if $u = v$.

Spearman's distance. *Spearman's distance*[8] $d_{\text{spear}}(u, v)$ is defined as

$$d_{\text{spear}}(u, v) = \textstyle\sum_{i=1}^{m} |\text{pos}(u, c_i) - \text{pos}(v, c_i)|. \quad (2)$$

We also define $d_{\infty\text{-spear}}(u, v) = \max_{i=1,\dots,m} |\text{pos}(u, c_i) - \text{pos}(v, c_i)|$.

Weighted distances. Many of the above distances have their weighted variants. Let $\alpha = (\alpha_1, \dots, \alpha_m)$ be a vector of m nonnegative rationals (weights). We define weighted Spearman's (pseudo)distance $d_\alpha(u, v)$ as

$$d_\alpha(u, v) = \textstyle\sum_{i=1}^{m} |\alpha_{\text{pos}(u,c_i)} - \alpha_{\text{pos}(v,c_i)}|. \quad (3)$$

Clearly, for $\alpha = (m, \dots, 1)$, i.e., $\alpha_i = i$ for all $i = 1, \dots, m$, we recover the original Spearman's distance: $d_{(m,\dots,1)}(u, v) = d_{\text{spear}}(u, v)$.
Similarly, following Lerer and Nitzan (1985), we define a weighted variant of the swap distance, $d_{\alpha\text{-swap}}$, as follows. The cost of transforming a preference order by swapping the candidate currently in position j, $2 \leq j \leq m$, with the one currently in position $j - 1$ is $|\alpha_{j-1} - \alpha_j|$. The distance between preference orders u and v is the total cost of the cheapest sequence of swaps that transforms u into v. Again, for $\alpha = (m, \dots, 1)$ we recover the swap distance, i.e., $d_{\alpha\text{-swap}}(u, v) = d_{\text{swap}}(u, v)$. Note that d_α and $d_{\alpha\text{-swap}}$ are different; for instance, for $\alpha = (3, 2, 1)$ and $u = a \succ b \succ c$, $v = c \succ b \succ a$ we obtain $d_\alpha(u, v) = 4$ and $d_{\alpha\text{-swap}}(u, v) = 3$.

For arbitrary weights, d_α and $d_{\alpha\text{-swap}}$ are pseudodistances; they are distances if all weights are distinct.

There are many other natural distances on $P(C)$, which are used in statistics, computer science and biology; we point the reader to the Encyclopedia of Distances (Deza & Deza, 2009) for a comprehensive overview.

[8] Known in statistics as Spearman's footrule (Kendall & Gibbons, 1990)

Voting rule	Consensus class	Distance over votes	Norm	Reference
Plurality	$\mathcal{U}$	$d_{\rm discr}$	ℓ_1	(Nitzan, 1981)
Plurality	$\mathcal{M}$	$d_{\rm discr}$	ℓ_1	Implicit in (Nitzan, 1981)
Plurality	$\mathcal{S}$	no name	ℓ_1	Theorem 3
Voter replacement rule	$\mathcal{C}$	$d_{\rm discr}$	ℓ_1	(Elkind et al., 2012)
Kemeny	$\mathcal{S}$	$d_{\rm swap}$	ℓ_1	(Meskanen & Nurmi, 2008)
Borda	$\mathcal{U}$	$d_{\rm swap}$	ℓ_1	(Farkas & Nitzan, 1979)
Threshold	$\mathcal{U}$	$d_{\rm swap}$	ℓ_∞	Proposition 2
$\mathcal{M}$-Borda	$\mathcal{M}$	$d_{\rm swap}$	ℓ_1	Definition 9
Dodgson	$\mathcal{C}$	$d_{\rm swap}$	ℓ_1	(Meskanen & Nurmi, 2008)
Dodgson$^\infty$	$\mathcal{C}$	$d_{\rm swap}$	ℓ_∞	Definition 8
Borda	$\mathcal{U}$	$d_{\rm spear}$	ℓ_1	Section 6
Borda	$\mathcal{U}$	$d_{\rm sert}$	ℓ_1	Section 6
scoring rule $\mathcal{R}_\alpha$	$\mathcal{U}$	$d_{\alpha\text{-swap}}$	ℓ_1	(Lerer & Nitzan, 1985)
scoring rule $\mathcal{R}_\alpha$	$\mathcal{U}$	d_α	ℓ_1	Section 6
$\mathcal{M}$-scoring rule $\mathcal{M}$-$\mathcal{R}_\alpha$	$\mathcal{M}$	d_α	ℓ_1	Section 6
Simplified Bucklin	$\mathcal{M}$	$d_{\infty\text{-spear}}$	ℓ_∞	Theorem 2
Simplified Bucklin	$\mathcal{M}$	$d_{\rm sert}$	ℓ_∞	Theorem 2
Litvak	$\mathcal{S}$	$d_{\rm spear}$	ℓ_1	(Meskanen & Nurmi, 2008)

Table 1 A list of known votewise distance rationalizations. Note that d_α for certain scoring vectors α is a pseudodistance, but not a distance.

4.1.2 Some Votewise Rationalizations

Table 1 lists a number of votewise distance rationalizations, including those first described in this paper. Let us now go briefly through this table. The (easy) rationalization of Plurality via $\mathcal{U}$ and $\widehat{d}_{\rm discr}$ is due to Nitzan (1981). The $(\mathcal{S}, \widehat{d}_{\rm swap})$-rationalization of the Kemeny rule, the $(\mathcal{U}, \widehat{d}_{\rm swap})$-rationalization of the Borda rule, and the $(\mathcal{C}, \widehat{d}_{\rm swap})$-rationalization of the Dodgson rule are essentially direct invocations of the definitions of these rules. However, replacing the ℓ_1 norm with the ℓ_∞ norm in these rationalizations leads to interesting observations. For example, applying this tweak to the Borda rule turns it into the Threshold rule (the proof follows directly from the definition of the rule).

Proposition 2 *The Threshold rule is* $(\mathcal{U}, \widehat{d}^{\infty}_{\rm swap})$*-rationalizable.*

On the other hand, if we use ℓ_∞ instead of ℓ_1 in the rationalization of the Dodgson rule, we obtain a new voting rule, which we will call Dodgson$^\infty$.

Definition 8 *Dodgson*$^\infty$ is the $(\mathcal{C}, \widehat{d}^{\infty}_{\rm swap})$-rationalizable rule.

This version of the Dodgson rule is remarkable in several ways. In particular, we will show that it is homogeneous (Theorem 10), and it can be shown to admit a polynomial-time winner determination algorithm, whereas the original Dodgson rule has neither of these properties—see, e.g., the survey of Brandt (2009) for a list of deficiencies of the Dodgson rule, and the work of Bartholdi, Tovey, and Trick (1989) and Hemaspaandra, Hemaspaandra, and Rothe (1997) for hardness of the winner determination problem.

Instead of changing the norm, we can vary other parameters of votewise rationalizations: e.g., replacing $\mathcal{U}$ with $\mathcal{M}$ in the $(\mathcal{U}, \widehat{d}_{\rm swap})$-rationalization of the Borda rule results in a new rule, which we will call $\mathcal{M}$-Borda.

Definition 9 $\mathcal{M}$*-Borda* is the $(\mathcal{M}, \widehat{d}_{\rm swap})$-rationalizable rule.

Intuitively, $\mathcal{M}$-Borda is a variant of the Borda rule that for each candidate c discards the bottom half of c's scores and calculates the sum of the remaining ones. This way we can define $\mathcal{M}$-variants of all scoring rules (see Section 6).

The consensus class $\mathcal{M}$ also allows us to distance rationalize the Simplified Bucklin rule whose distance rationalization was previously unknown.

Theorem 2 *The Simplified Bucklin rule is* $(\mathcal{M}, \widehat{d}^{\infty}_{\rm sert})$*-rationalizable as well as* $(\mathcal{M}, \widehat{d}^{\infty}_{\infty\text{-spear}})$*-rationalizable.*

Proof Consider an election $E = (C, V)$ with $V = (v_1, \dots, v_n)$. Let k be the smallest integer such that V has a k-majority winner. By definition, the Simplified Bucklin rule outputs all k-majority winners of E. Let c be one of the k-majority winners of E. We will argue that for any $\mathcal{M}$-consensus election E' we have $\widehat{d}^{\infty}_{\rm sert}(E, E') \geq k$ and for some $\mathcal{M}$-consensus election E_c with winner c we have $\widehat{d}^{\infty}_{\rm sert}(E, E_c) = k$.

For the sake of contradiction, let $E' = (C, U)$, $U = (u_1, \dots, u_n)$, be an $\mathcal{M}$-consensus election with consensus winner b such that $\widehat{d}^{\infty}_{\rm sert}(E, E') < k$. We have $d_{\rm sert}(v_i, u_i) < k$ for $i = 1, \dots, n$, so whenever u_i ranks b first, v_i ranks b in top $k-1$ positions. Since b is an $\mathcal{M}$-consensus winner of E', he is ranked first in more than half of the votes in U. Hence, b must be a $(k-1)$-majority winner of E, a contradiction.

On the other hand, there is an $\mathcal{M}$-consensus election $E_c = (C, W)$ with $W = (w_1, \dots, w_n)$ that has c as its consensus winner and satisfies $\widehat{d}^{\infty}_{\rm sert}(E, E_c) \leq k$. Indeed, we can construct W from V by shifting c to the top in each vote that ranks c among the top k candidates (without changing anything else in those votes). We have $\widehat{d}^{\infty}_{\rm sert}(E, E_c) \leq k$. Further, since c is a k-majority winner, these shifts will be made in a majority of votes, which guarantees that E_c is in $\mathcal{M}$. Thus the set of $(\mathcal{M}, \widehat{d}^{\infty}_{\rm sert})$-winners coincides with the set of the Simplified Bucklin winners.

For the $(\mathcal{M}, \widehat{d}^{\infty}_{\infty\text{-spear}})$-rationalization the proof is similar. □

It is quite remarkable how robust Plurality is with respect to votewise distance rationalizability. Even though $\mathcal{M}$-Borda is quite different from the Borda rule (e.g., the Borda rule is homogeneous and $\mathcal{M}$-Borda is not; see Appendix A), one can verify that $\mathcal{M}$-Plurality is exactly Plurality, and, in particular, the Plurality rule is $(\mathcal{M}, \widehat{d}_{\rm discr})$-rationalizable. Further, Plurality can also be rationalized with respect to $\mathcal{S}$ (see Appendix B for the proof).

Theorem 3 *There exists an* ℓ_1*-votewise distance* $\widehat{d}$ *such that Plurality is* $(\mathcal{S}, \widehat{d})$*-rationalizable.*

4.2 Rules That Are Not Votewise

So far, we have focused on positive results, i.e., on proving that some voting rules are votewise. Now, let us focus on impossibility of votewise rationalizations. A very basic observation, which follows from Corollary 1 and the fact that no scoring rule is $\mathcal{C}$-compatible (Moulin, 1991), is that no scoring rule is (votewise) distance rationalizable with respect to $\mathcal{C}$. It is also not too difficult to prove that there is no neutral distance d over preference orders such that the Borda rule is $(\mathcal{S}, \widehat{d})$-rationalizable.[9]

In contrast, for STV and Plurality with Runoff we will show that there are no votewise rationalizations with respect to any of our four standard consensus classes and any 'reasonable' norm. All of our impossibility results are limited to rationalizations by neutral distances; however, since these rules themselves are neutral, their rationalizations by nonneutral distances are unlikely to exist and, even if they do exist, they are bound to be unnatural.

Each of the following impossibility-of-rationalization proofs considers elections over the candidate set $C = \{a, b, c\}$. To simplify notation, we write xyz instead of $x \succ y \succ z$ for $\{x, y, z\} = \{a, b, c\}$.

Consider a neutral distance d over $P(C)$. By neutrality, it is completely described by its values on the pairs (abc, abc), (abc, acb), (abc, bac), (abc, bca), (abc, cab), and (abc, cba). Further, we have $d(abc, abc) = 0$, and by neutrality and the symmetry axiom we have $d(abc, bca) = d(abc, cab)$ (to see this, note that the permutation π given by $\pi(a) = c$, $\pi(b) = a$, $\pi(c) = b$ transforms abc into cab and bca into abc). This argument shows that d can be completely described by four positive numbers, namely, $T = d(abc, acb)$, $B = d(abc, bac)$, $M = d(abc, cba)$, and $S = d(abc, bca)$ (see Table 2).

	abc	*acb*	*bac*	*bca*	*cab*	*cba*
abc	0	T	B	S	S	M
acb	T	0	S	M	B	S
bac	B	S	0	T	M	S
bca	S	M	T	0	S	B
cab	S	B	M	S	0	T
cba	M	S	S	B	T	0

Table 2 Values of distance d over $P(\{a, b, c\})$. For example, $d(bac, cab) = M$. While not all choices of values for T, M, B, S would result in a distance, for every neutral distance over $P(\{a, b, c\})$ there are values of T, M, B, S that define it.

We will show now that neither STV nor Plurality with Runoff are votewise rationalizable with respect to $\mathcal{S}$, as long as we restrict ourselves to neutral distances and norms that are symmetric and *monotonic in the positive orthant.*

Definition 10 (Bauer, Stoer, and Witzgall (1961)) A norm N on $\mathbb{R}^n$ is *monotonic in the positive orthant*, or $\mathbb{R}^n_+$*-monotonic*, if for every pair of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$ such that $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$ and $x_i \leq y_i$ for all $i = 1, \ldots, n$ we have $N(\mathbf{x}) \leq N(\mathbf{y})$.

[9] The proof is available on request.

This is a very natural notion of monotonicity. Note that all p-norms satisfy it.

Theorem 4 *There is no pair (d, N), where d is a neutral distance over preference orders and N is a symmetric norm that is monotonic in the positive orthant, such that either STV or Plurality with Runoff (together with any intermediate tie-breaking rule) is $(\mathcal{S}, \widehat{d}^N)$-rationalizable.*

Proof Let the candidate set C be $\{a, b, c\}$. For three candidates, the set of STV winners coincides with the set of Plurality with Runoff winners, and thus it suffices to focus on one of these rules, say, STV.

For the sake of contradiction, suppose that there exist a neutral distance d over $P(C)$ and a norm N that is monotonic in the positive orthant such that STV is $(\mathcal{S}, \widehat{d}^N)$-rationalizable. Consider the profile $V = k \times abc + k \times bca + cab$. We have argued that there exist nonnegative numbers B, M, S, T such that $d(abc, bca) = d(abc, cab) = S$, $d(abc, acb) = T$, $d(abc, bac) = B$, and $d(abc, cba) = M$. Hence we have

$$d_1 = \widehat{d}^N(V, (2k+1) \times abc) = N(0, \dots, 0, S, \dots, S, S),$$
$$d_2 = \widehat{d}^N(V, (2k+1) \times acb) = N(T, \dots, T, M, \dots, M, B),$$
$$d_3 = \widehat{d}^N(V, (2k+1) \times bca) = N(S, \dots, S, 0, \dots, 0, S),$$
$$d_4 = \widehat{d}^N(V, (2k+1) \times bac) = N(B, \dots, B, T, \dots, T, M).$$

Clearly, under STV candidate a is the unique winner of (C, V). Thus it must be the case that $\min\{d_1, d_2\} < \min\{d_3, d_4\}$. By symmetry we have $d_1 = d_3$, and hence $d_2 < d_4$. Also, by symmetry we get $d_4 = N(T, \dots, T, M, B, \dots, B)$. Hence, by monotonicity $M < B$.

Now, consider the profile W obtained by replacing the last voter in V by a voter whose preference order is cba. We have

$$d'_1 = \widehat{d}^N(W, (2k+1) \times abc) = N(0, \dots, 0, S, \dots, S, M),$$
$$d'_2 = \widehat{d}^N(W, (2k+1) \times acb) = N(T, \dots, T, M, \dots, M, S),$$
$$d'_3 = \widehat{d}^N(W, (2k+1) \times bca) = N(S, \dots, S, 0, \dots, 0, B),$$
$$d'_4 = \widehat{d}^N(W, (2k+1) \times bac) = N(B, \dots, B, T, \dots, T, S).$$

The STV winner of (C, W) is b, so we have $\min\{d'_1, d'_2\} > \min\{d'_3, d'_4\}$. Furthermore, by symmetry, we have $d'_3 = N(0, \dots, 0, S, \dots, S, B)$. As $M < B$, by monotonicity we conclude that $d'_1 \le d'_3$. This implies that $d'_2 > d'_4$. However, by symmetry we have $d'_4 = N(T, \dots, T, B, \dots, B, S)$, so by monotonicity $d'_2 \le d'_4$, a contradiction. □

We can use similar ideas to show that STV and Plurality with Runoff are not distance rationalizable with respect to either $\mathcal{U}$ or $\mathcal{M}$. The proof of the following theorem is provided in Appendix C.

Theorem 5 *There is no pair (d, N) where d is a neutral distance d over preference orders and N is a symmetric norm that is monotonic in the positive orthant, such that STV or Plurality with Runoff (together with any intermediate tie-breaking rule) are either $(\mathcal{U}, \widehat{d}^N)$-rationalizable or $(\mathcal{M}, \widehat{d}^N)$-rationalizable.*

Note that for each $p \in \mathbb{Z}^+ \cup \{\infty\}$, all ℓ_p-votewise distances satisfy the requirements of Theorems 4 and 5. Thus neither STV nor Plurality with Runoff can be rationalized with respect to strong unanimity, weak unanimity, or majority via any neutral ℓ_p-votewise distance. Further, neither STV nor Plurality with Runoff is distance rationalizable with respect to $\mathcal{C}$, because neither of these rules is $\mathcal{C}$-compatible.

Thus, even though Meskanen and Nurmi (2008) show that STV can be distance rationalized with respect to $\mathcal{U}$, of all natural rules we have considered so far, STV is distance rationalizable in the weakest possible sense.

5 Properties of Votewise Rules

In this section, we present our second argument in favor of votewise distances. Namely, we show that many standard properties of voting rules (such as, for example, anonymity, neutrality, monotonicity, or homogeneity) can be linked to those of the components of their votewise rationalizations (i.e., the consensus class, the norm, or the underlying distance over preference orders).

5.1 Anonymity and Neutrality

We start by showing that for votewise rules, a symmetric norm and an anonymous consensus class produce an anonymous rule.

Proposition 3 *Suppose that a voting rule $\mathcal{R}$ is $(\mathcal{K}, \widehat{d^N})$-rationalizable for some pseudodistance d over preference orders, an anonymous consensus class $\mathcal{K}$, and a symmetric norm N. Then $\mathcal{R}$ is anonymous.*

Proof Let V be a profile over a candidate set C, and let σ be a permutation of V. Fix a candidate $c \in \mathcal{R}(V)$, and let (C, U) be a $\mathcal{K}$-consensus profile closest to (C, V) among those whose winner is c. As $\mathcal{K}$ is anonymous, the election $(C, \sigma(U))$ is also a $\mathcal{K}$-consensus with consensus winner c, and by Proposition 1 we obtain $\widehat{d^N}(V, U) = \widehat{d^N}(\sigma(V), \sigma(U))$. Now, suppose that there exists a $\mathcal{K}$-consensus election (C, W) such that $\widehat{d^N}(\sigma(V), W) < \widehat{d^N}(\sigma(V), \sigma(U))$. Then $(C, \sigma^{-1}(W))$ is also a consensus, and

$$\widehat{d^N}(V, \sigma^{-1}(W)) = \widehat{d^N}(\sigma(V), W) < \widehat{d^N}(\sigma(V), \sigma(U)) = \widehat{d^N}(V, U),$$

a contradiction with our choice of U. Since (C, U) and $(C, \sigma(U))$ have the same winner, we have $c \in \mathcal{R}(\sigma(V))$, and hence $\mathcal{R}(V) \subseteq \mathcal{R}(\sigma(V))$. Using the permutation σ^{-1}, we obtain $\mathcal{R}(\sigma(V)) \subseteq \mathcal{R}(V)$. Thus $\mathcal{R}(\sigma(V)) = \mathcal{R}(V)$. □

Proposition 3 shows that anonymity of a votewise rule is essentially a property of the underlying norm. In contrast, neutrality is inherited from the underlying distance over preference orders.

Proposition 4 *Suppose that a voting rule $\mathcal{R}$ is $(\mathcal{K}, \widehat{d}^N)$-rationalizable for some norm N, a neutral consensus class $\mathcal{K}$ and a neutral pseudodistance d over preference orders. Then $\mathcal{R}$ is neutral.*

The proof is similar to the proof of Proposition 3 and thus we omit it.

One may wonder if the converse of Propositions 3 and 4 is also true, that is, if every neutral (anonymous) votewise rule can be rationalized via a neutral (anonymous) distance. Conitzer et al. (2009) provide a positive answer to a similar question in the context of representing voting rules as maximum likelihood estimators. Yet, it is not clear if such a result holds in our setting.

5.2 Continuity, Consistency, and Homogeneity

Let us now consider three normative properties of voting rules that deal with merging electorates. While in this case we cannot hope for such sweeping results as for anonymity and neutrality—for example, the Kemeny rule is not consistent and the Dodgson rule is not even homogeneous—we can still obtain quite general characterization theorems. For example, it turns out that for consensus classes $\mathcal{S}$ and $\mathcal{U}$ all ℓ_p-votewise rules, $p \in \mathbb{Z}^+$, are continuous.

Theorem 6 *Suppose that a voting rule $\mathcal{R}$ is $(\mathcal{K}, \widehat{d}^p)$-rationalizable for some $p \in \mathbb{Z}^+$, some pseudodistance d over preference orders, and a consensus class $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}\}$. Then $\mathcal{R}$ is continuous.*

Proof Let $E_1 = (C, V_1)$ and $E_2 = (C, V_2)$ be two elections over the same set of candidates C, and suppose that $\mathcal{R}(E_1) = \{a\}$ for some $a \in C$. Then there exist some $D_1 \geq 0$, $\delta > 0$ and a $\mathcal{K}$-consensus election $E_1' = (C, U_1)$ with winner a such that $\widehat{d}^p(V_1, U_1) = D_1$ and $\widehat{d}^p(V_1, W) \geq D_1 + \delta$ for every $\mathcal{K}$-consensus election $E = (C, W)$ whose consensus winner differs from a. Pick some vote u in U_1, and let $U_2 = |V_2| \times u$. Then for every positive integer k the election $F_k = (C, kU_1 + U_2)$ is a $\mathcal{K}$-consensus with winner a. Set $D_2 = \widehat{d}^p(V_2, U_2)$. We have

$$\widehat{d}^p(kV_1 + V_2, kU_1 + U_2) = \sqrt[p]{kD_1^p + D_2^p}. \tag{4}$$

On the other hand, let (C, W_k) be some $\mathcal{K}$-consensus with $k|V_1| + |V_2|$ voters whose winner is $b \neq a$. Since $\mathcal{K} \in \{\mathcal{U}, \mathcal{S}\}$, we can decompose W_k as $W_k = W_k^1 + \ldots + W_k^k + W_k^{k+1}$, where for each $i = 1, \ldots, k$ the election (C, W_k^i) is a $\mathcal{K}$-consensus with winner b and $|W_k^i| = |V_1|$. Therefore,

$$\widehat{d}^p(kV_1 + V_2, W_k) \geq \sqrt[p]{\sum_{i=1}^{k} \left(\widehat{d}^p(V_1, W_k^i)\right)^p} \geq \sqrt[p]{k(D_1 + \delta)^p}. \tag{5}$$

For large enough values of k the right-hand side of (5) exceeds the right-hand side of (4). Therefore, for such values of k candidate a is the unique winner of $kE_1 + E_2$ under $\mathcal{R}$. □

The proof of Theorem 6 does not go through for $\mathcal{C}$ and $\mathcal{M}$. Specifically, this proof relies on the fact that for $\mathcal{U}$ and $\mathcal{S}$ any subset of votes of a consensus election still forms a consensus election, and this is not the case for $\mathcal{M}$ and $\mathcal{C}$.

Let us now move on to consistency and its weaker variant, homogeneity.

Theorem 7 *Suppose that a voting rule* $\mathcal{R}$ *is* $(\mathcal{U}, \widehat{d^p})$*-rationalizable for some* $p \in \mathbb{Z}^+$ *and a pseudodistance* d *over preference orders. Then* $\mathcal{R}$ *is consistent.*

Proof Let $E_1 = (C, V_1)$ and $E_2 = (C, V_2)$ be two elections over the same set of candidates C such that $\mathcal{R}(E_1) \cap \mathcal{R}(E_2) \neq \emptyset$, and let $E = (C, V_1 + V_2)$. First, we will show that $\mathcal{R}(E_1) \cap \mathcal{R}(E_2) \subseteq \mathcal{R}(E)$. Fix a candidate $c \in \mathcal{R}(E_1) \cap \mathcal{R}(E_2)$. By definition, there are two $\mathcal{U}$-consensus elections (C, U_1) and (C, U_2) such that for $i = 1, 2$ it holds that c is the unanimity winner of (C, U_i) and, moreover, (C, U_i) is a closest unanimous election to E_i.

Consider an arbitrary unanimous election $(C, W_1 + W_2)$ with $|W_1| = |V_1|$ and $|W_2| = |V_2|$. Note that (C, W_1) and (C, W_2) are unanimous elections as well, and by the choice of U_1 and U_2 we have $\widehat{d^p}(V_1, U_1) \leq \widehat{d^p}(V_1, W_1)$ and $\widehat{d^p}(V_2, U_2) \leq \widehat{d^p}(V_2, W_2)$. Consequently, we have

$$\sqrt[p]{\left(\widehat{d^p}(V_1, U_1)\right)^p + \left(\widehat{d^p}(V_2, U_2)\right)^p} \leq \sqrt[p]{\left(\widehat{d^p}(V_1, W_1)\right)^p + \left(\widehat{d^p}(V_2, W_2)\right)^p}, \quad (6)$$

or, equivalently, $\widehat{d^p}(V_1 + V_2, U_1 + U_2) \leq \widehat{d^p}(V_1 + V_2, W_1 + W_2)$. Thus, $(C, U_1 + U_2)$ is a closest unanimous election to $(C, V_1 + V_2)$, and therefore $c \in \mathcal{R}(E)$.

To show that $\mathcal{R}(E) \subseteq \mathcal{R}(E_1) \cap \mathcal{R}(E_2)$, consider a candidate $c' \in \mathcal{R}(E)$. Since c and c' are both in $\mathcal{R}(E)$, there exists a unanimous election $(C, X_1 + X_2)$ with winner c' such that $|X_1| = |V_1|$, $|X_2| = |V_2|$ and $\widehat{d^p}(V_1 + V_2, X_1 + X_2) = \widehat{d^p}(V_1 + V_2, U_1 + U_2)$. Thus, we have

$$\sqrt[p]{\left(\widehat{d^p}(V_1, X_1)\right)^p + \left(\widehat{d^p}(V_2, X_2)\right)^p} = \sqrt[p]{\left(\widehat{d^p}(V_1, U_1)\right)^p + \left(\widehat{d^p}(V_2, U_2)\right)^p}. \quad (7)$$

On the other hand, we have

$$\widehat{d^p}(V_1, U_1) \leq \widehat{d^p}(V_1, X_1), \qquad \widehat{d^p}(V_2, U_2) \leq \widehat{d^p}(V_2, X_2). \quad (8)$$

It follows from (7) that both of the inequalities in (8) are, in fact, equalities. Thus, by our choice of U_1, U_2, for $i = 1, 2$ it holds that (C, X_i) is a closest unanimous election to E_i. Since c' is the unanimity winner in X_1 and X_2, we obtain $c' \in \mathcal{R}(E_1) \cap \mathcal{R}(E_2)$. □

For consensus classes $\mathcal{S}$, $\mathcal{M}$, and $\mathcal{C}$ an analog of Theorem 7 is not true: The counterexamples are provided by the Kemeny rule, $\mathcal{M}$-Borda, and the Dodgson rule, respectively. Indeed, these rules are ℓ_1-votewise with respect to $\mathcal{S}$, $\mathcal{M}$, and $\mathcal{C}$, respectively, but none of them is consistent. For the case of the Kemeny rule and the Dodgson rule, both of which are $\mathcal{C}$-compatible, this is a consequence of Theorem 2 from the work of Young and Levenglick

(1978). As for $\mathcal{M}$-Borda, in Appendix A we provide a proof that it is not even homogeneous (recall that consistency implies homogeneity).

However, it is easy to modify the proof of Theorem 7 to show that all rules that can be rationalized with respect to $\mathcal{S}$ by an ℓ_p-votewise pseudodistance are homogeneous.

Theorem 8 *Suppose that a voting rule $\mathcal{R}$ is $(\mathcal{S}, \widehat{d^p})$-rationalizable for some $p \in \mathbb{Z}^+$, and a pseudodistance d over $P(C)$. Then $\mathcal{R}$ is homogeneous.*

We could use a similar argument for ℓ_∞-votewise rules, but for this class of rules a different approach yields a stronger result. Specifically, such rules turn out to be homogeneous under a very mild condition on the consensus class $\mathcal{K}$.

Definition 11 We say that a consensus class $\mathcal{K} = (\mathcal{E}, \mathcal{W})$ is *split-homogeneous* if for every positive integer k the following two conditions hold:

(a) If $E \in \mathcal{E}$ is a $\mathcal{K}$-consensus then so is kE, and $\mathcal{W}(kE) = \mathcal{W}(E)$.
(b) If $E_1, \dots, E_k$ are elections with the same number of voters such that the election $E = E_1 + \dots + E_k$ is a $\mathcal{K}$-consensus, then there exists an $i \in \{1, \dots, k\}$ such that E_i is a $\mathcal{K}$-consensus and $\mathcal{W}(E_i) = \mathcal{W}(E)$.

We will now show that a combination of a split-homogeneous consensus class with an ℓ_∞-votewise distance produces a homogeneous rule.

Theorem 9 *Suppose that a voting rule $\mathcal{R}$ is $(\mathcal{K}, \widehat{d^\infty})$-rationalizable for some pseudodistance d over preference orders and a split-homogeneous consensus class $\mathcal{K}$. Then $\mathcal{R}$ is homogeneous.*

Proof Fix a split-homogeneous consensus class $\mathcal{K}$, and consider an election $E = (C, V)$. Let c be a winner of E, and let (C, U) be a $\mathcal{K}$-consensus that witnesses this fact. Set $t = \widehat{d^\infty}(V, U)$. For every $\mathcal{K}$-consensus (C, U') we have

$$t = \widehat{d^\infty}(V, U) \le \widehat{d^\infty}(V, U'). \tag{9}$$

Fix a positive integer k. We will first argue that $\mathcal{R}(E) \subseteq \mathcal{R}(kE)$. By condition (a) of Definition 11, (C, kU) is a $\mathcal{K}$-consensus with winner c. If c is not a winner of kE, then there exists a $\mathcal{K}$-consensus $(C, X_1 + \dots + X_k)$ with winner $b \neq c$ and $|X_1| = \dots = |X_k| = |V|$ such that

$$\widehat{d^\infty}(kV, X_1 + \dots + X_k) < \widehat{d^\infty}(kV, kU) = \widehat{d^\infty}(V, U) = t,$$

which means that $\widehat{d^\infty}(V, X_i) < t$ for every $i = 1, \dots, k$. By condition (b) of Definition 11, at least one of the elections $(C, X_1), \dots, (C, X_k)$ is a $\mathcal{K}$-consensus, which contradicts (9).

It remains to show that $\mathcal{R}(kE) \subseteq \mathcal{R}(E)$. Let b be a winner of kE, and let (C, W) be a $\mathcal{K}$-consensus that witnesses this fact; we have $\widehat{d^\infty}(kV, W) \le \widehat{d^\infty}(kV, kU) = t$. We can write W as $W_1 + \dots + W_k$, where $|W_i| = |V|$ for each $i = 1, \dots, k$. By condition (b) of Definition 11, at least one of the elections $(C, W_1), \dots, (C, W_k)$ is a $\mathcal{K}$-consensus with winner b, and $\widehat{d^\infty}(V, W_i) \le t$ for each $i = 1, \dots, k$. Hence, $b \in \mathcal{R}(E)$. □

Since $\mathcal{S}$, $\mathcal{U}$, and $\mathcal{M}$ are split-homogeneous, we have the following corollary.

Corollary 3 *Suppose that a voting rule $\mathcal{R}$ is $(\mathcal{K}, \widehat{d^\infty})$-rationalizable for some pseudodistance d over preference orders and a consensus class $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}, \mathcal{M}\}$. Then $\mathcal{R}$ is homogeneous.*

Corollary 3 immediately implies that both the Simplified Bucklin rule and the Threshold rule are homogeneous.

On the other hand, the consensus class $\mathcal{C}$ is not split-homogeneous, and, indeed, there is a distance d over preference orders such that the $(\mathcal{C}, \widehat{d^\infty})$-rationalized rule is not homogeneous (see Appendix D). However, it turns out that the Dodgson$^\infty$ rule, which is $(\mathcal{C}, \widehat{d^\infty_{\mathrm{swap}}})$-rationalizable, *is* homogeneous, i.e., the conditions of Theorem 9 are sufficient, but not necessary.

Theorem 10 *Dodgson$^\infty$ is homogeneous.*

Proof Consider an election $E = (C, V)$ and a candidate $c \in C$. It is easy to verify that the following procedure computes the Dodgson$^\infty$-score of c in E: Until c becomes a Condorcet winner, execute the loop which in every iteration swaps c with its predecessor in every vote where c is not ranked first. The number of iterations of this loop is the Dodgson$^\infty$-score of c.

For every $c \in C$ and every positive integer k, this procedure terminates in s iterations on E if and only if it terminates in s iterations on kE, i.e., each candidate gets the same score in E and kE. This proves the theorem. □

5.3 Monotonicity

Characterizing monotone votewise rules appears to be more difficult than characterizing consistent or homogeneous ones. In what follows, we attempt to identify conditions on distances over preference orders that, coupled with suitable consensus classes, ensure monotonicity of the resulting voting rule.

In our discussion of monotonicity we will not consider the Condorcet consensus. The reason for this is that the Dodgson rule, which is $(\mathcal{C}, \widehat{d}_{\mathrm{swap}})$-rationalizable, is not monotone (see, e.g., the survey of Brandt, 2009). This does not leave much hope for finding a distance d that produces a monotone $(\mathcal{C}, \widehat{d})$-rationalizable rule, since the swap distance is perhaps the best-behaved distance one can think of.

Let d be a distance over preference orders. We are interested in identifying conditions on d that would ensure monotonicity of every $(\mathcal{K}, \widehat{d^N})$-rationalizable voting rule (for some consensus class $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}, \mathcal{M}\}$ and some norm N). Intuitively, we should require that if a candidate c is a winner and some voter moves him higher in her preference order, then the distance to votes that rank c first decreases at least as much as the distance to votes that rank other candidates first. The following definition expresses this idea formally.

Definition 12 Let C be a set of candidates and let d be a distance over $P(C)$. We say that d is *relatively monotone* if for every candidate $c \in C$ and every pair of preference orders x and z such that x ranks c first and z does not, the following holds: if two preference orders y and y' rank all candidates in $C \setminus \{c\}$ in the same order, but y' ranks c higher than y does, then

$$d(x, y) - d(x, y') \geq d(z, y) - d(z, y'). \tag{10}$$

As a quick sanity check, we note that the swap distance is relatively monotone.

Proposition 5 *The distance d_{swap} is relatively monotone.*

Proof Consider a set of candidates C and a candidate $c \in C$. Let $d = d_{\text{swap}}$, and let y, y', x, and z be as in Definition 12. Then $\text{pos}(y', c) = \text{pos}(y, c) - s$, where s is a positive integer. Let C' be the set of s candidates that are ranked above c in y, but below c in y'. Clearly, $d(y, y') = s$. Further, we have $d(x, y) - d(x, y') = s$. Indeed, since x ranks c first, every pair of the form (b, c), where $b \in C'$, contributes to $d(x, y)$, but does not contribute to $d(x, y')$, whereas every other pair of candidates either contributes to both $d(x, y)$ and $d(x, y')$ or does not contribute to either of them. On the other hand, by the triangle inequality we have $d(z, y) \leq d(z, y') + d(y', y) = d(z, y') + s$, hence $d(z, y) - d(z, y') \leq s$. This completes the proof. □

Relative monotonicity of a distance d naturally translates into monotonicity of a voting rule that is $(\mathcal{S}, \widehat{d})$-rationalizable or $(\mathcal{U}, \widehat{d})$-rationalizable.

Theorem 11 *Suppose that a voting rule $\mathcal{R}$ is $(\mathcal{K}, \widehat{d})$-rationalizable for some relatively monotone distance d over preference orders and a consensus class $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}\}$. Then $\mathcal{R}$ is monotone.*

Proof Consider an election $E = (C, V)$ with $V = (v_1, \ldots, v_n)$ and a candidate $c \in \mathcal{R}(E)$. Suppose v_1 does not rank c first. Let v'_1 be a preference order obtained from v_1 by shifting c upwards (without changing the relative order of the other candidates), let $V' = (v'_1, v_2, \ldots, v_n)$, and set $E' = (C, V')$. It suffices to show that $c \in \mathcal{R}(E')$.

Consider a $\mathcal{K}$-consensus (C, U) with $U = (u_1, \ldots, u_n)$ witnessing that $c \in \mathcal{R}(E)$, i.e., $\widehat{d}(V, U) \leq \widehat{d}(V, U')$ for every $\mathcal{K}$-consensus (C, U'), and c is the consensus winner of (C, U). Pick a candidate $b \neq c$, and let (C, W), $W = (w_1, \ldots, w_n)$, be some $\mathcal{K}$-consensus with winner b. Since $\mathcal{K} \in \{\mathcal{U}, \mathcal{S}\}$, we know that u_1 ranks c first, while w_1 ranks b first. As d is relatively monotone, this means that

$$d(u_1, v_1) - d(u_1, v'_1) \geq d(w_1, v_1) - d(w_1, v'_1). \tag{11}$$

Combining (11) with the fact that $\widehat{d}(V, U) \leq \widehat{d}(V, W)$, we obtain

$$\begin{aligned} \widehat{d}(V', U) &= \widehat{d}(V, U) + d(v'_1, u_1) - d(v_1, u_1) \\ &\leq \widehat{d}(V, W) + d(v'_1, w_1) - d(v_1, w_1) = \widehat{d}(V', W). \end{aligned}$$

As this holds for every $b \neq c$ and every $\mathcal{K}$-consensus (C, W) with winner b, it follows that c is an $\mathcal{R}$-winner of E'. □

Unfortunately, the power of Theorem 11 is limited by the fact that relative monotonicity is a remarkably strong condition. Our next example shows that it is failed by some very natural distances which, intuitively, should be monotone.

Example 1 We will show that the distance d_{spear} is not relatively monotone. Consider the set of candidates $C = \{c, b, a_1, a_2, a_3, a_4\}$ and the following profile over C:

$$x\ : c \succ b \succ a_1 \succ a_2 \succ a_3 \succ a_4, \qquad z\ : a_1 \succ c \succ a_2 \succ a_3 \succ a_4 \succ b,$$
$$y\ : a_1 \succ a_2 \succ b \succ c \succ a_3 \succ a_4, \qquad y' : a_1 \succ a_2 \succ c \succ b \succ a_3 \succ a_4.$$

Note that x ranks c first, while z does not, and y' is obtained from y by shifting c upwards, but $d_{\text{spear}}(x, y) - d_{\text{spear}}(x, y') = 0$, $d_{\text{spear}}(z, y) - d_{\text{spear}}(z, y') = 2$. Thus, d_{spear} is not relatively monotone.

If we focus on neutral distances and the consensus classes $\mathcal{U}$ and $\mathcal{M}$, it becomes much easier to provide a condition that ensures monotonicity. Recall that $P(C, c)$ denotes the set of preference orders over the candidate set C that rank candidate c first. We use the following result of Lerer and Nitzan (1985).

Proposition 6 (Lerer and Nitzan (1985)) *For every set of candidates C, $|C| = m$, and a neutral distance d over $P(C)$, there exists a vector $\{\beta_1, \dots, \beta_m\}$ of nonnegative reals such that for each $v \in P(C)$ and each $c \in C$ it holds that* $\text{pos}(v, c) = k$ *implies* $\min_{w \in P(C,c)} d(v, w) = \beta_k$.

In other words, for a neutral distance d, the distance between a preference order v and the closest preference order where a candidate c is ranked first depends on $\text{pos}(v, c)$ only. Below we illustrate this point for the case of weighted Spearman's distance, by showing that we can set $\beta_k = 2|\alpha_1 - \alpha_k|$.

Proposition 7 *For every set of candidates C, $|C| = m$, every vector $\alpha = (\alpha_1, \dots, \alpha_m)$ of nonnegative reals, each $v \in P(C)$ and each $c \in C$ it holds that* $\text{pos}(v, c) = k$ *implies* $\min_{w \in P(C,c)} d_\alpha(w, v) = 2|\alpha_1 - \alpha_k|$.

Proof Fix a set of candidates $C = \{c_1, \dots, c_m\}$ and a candidate $c \in C$. Consider a pair of votes v, w, where $v \in P(C)$, $w \in P(C, c)$, and $\text{pos}(v, c) = k$. Observe that

$$\sum_{j=1}^{m} \left(\alpha_{\text{pos}(w,c_j)} - \alpha_{\text{pos}(v,c_j)}\right) = \sum_{j=1}^{m} \alpha_{\text{pos}(w,c_j)} - \sum_{j=1}^{m} \alpha_{\text{pos}(v,c_j)} = \sum_{i=1}^{m} \alpha_i - \sum_{i=1}^{m} \alpha_i = 0.$$

For any m real numbers $x_1, \dots, x_m$ such that $\sum_{i=1}^{m} x_i = 0$ and any $j \in [m]$ we have $\sum_{i=1}^{m} |x_i| \ge |x_j| + |\sum_{i \ne j} x_i| = 2|x_j|$. As $\alpha_{\text{pos}(w,c)} - \alpha_{\text{pos}(v,c)} = \alpha_1 - \alpha_k$, we obtain

$$d_\alpha(v, w) = \sum_{j=1}^{m} \left|\alpha_{\text{pos}(w,c_j)} - \alpha_{\text{pos}(v,c_j)}\right| \ge 2 \left|\alpha_{\text{pos}(w,c)} - \alpha_{\text{pos}(v,c)}\right| = 2|\alpha_1 - \alpha_k|.$$

On the other hand, for the preference order w' that is obtained from v by swapping c with the top candidate in v, we have $d_\alpha(v, w') = 2|\alpha_1 - \alpha_k|$. Hence, the d_α-distance from v to the nearest vote in $P(C, c)$ is exactly $2|\alpha_1 - \alpha_k|$. □

Proposition 6 inspires the following notion of monotonicity, which we will call *rank monotonicity.*[10]

Definition 13 (Lerer and Nitzan (1985)) Given a set of candidates C, we say that a distance d over $P(C)$ is *rank-monotone* if for every pair of candidates a, b and every preference order v, if $\mathrm{pos}(v, a) < \mathrm{pos}(v, b)$ then

$$\min_{u \in P(C,a)} d(v, u) \le \min_{w \in P(C,b)} d(v, w).$$

Now, suppose that d is both neutral and rank-monotone. The definition of rank monotonicity then implies that the vector $(\beta_1, \dots, \beta_m)$ mentioned in Proposition 6 satisfies $\beta_1 \le \dots \le \beta_m$. This makes rank monotonicity and neutrality a very powerful combination: indeed, it turns out that if we use a distance that has both of these properties and our consensus class is either $\mathcal{U}$ or $\mathcal{M}$, we obtain a monotone rule.

Theorem 12 *Suppose that a voting rule $\mathcal{R}$ is $(\mathcal{K}, \widehat{d}^N)$-rationalizable for some rank-monotone neutral distance d over preference orders, a norm N that is monotonic in the positive orthant, and a consensus class $\mathcal{K} \in \{\mathcal{U}, \mathcal{M}\}$. Then $\mathcal{R}$ is monotone.*

Proof Again, consider an election $E = (C, V)$ with $V = (v_1, \dots, v_n)$ and a candidate $c \in \mathcal{R}(E)$. Suppose that $\mathrm{pos}(v_1, c) = j > 1$, and let v'_1 be a preference order obtained from v_1 by shifting c upwards (without changing the relative order of the other candidates); we have $\mathrm{pos}(v'_1, c) = \ell < j$. Let $V' = (v'_1, v_2, \dots, v_n)$, and set $E' = (C, V')$. It suffices to show that $c \in \mathcal{R}(E')$.

Let $(\beta_1, \dots, \beta_m)$ be the vector associated with d (see Proposition 6); as argued above, by neutrality of d we have $\beta_1 \le \dots \le \beta_m$. Moreover, Proposition 6 implies that $\min_{u \in P(C,c)} d(v_1, u) = \beta_j$, $\min_{u' \in P(C,c)} d(v'_1, u') = \beta_\ell$.

Consider a $\mathcal{K}$-consensus (C, U) with $U = (u_1, \dots, u_n)$ witnessing that $c \in \mathcal{R}(E)$. We construct an election (C, U'), where $U' = (u'_1, u_2, \dots, u_n)$, and u'_1 is selected as follows: if u_1 ranks c first, we pick u'_1 from $\arg\min_{u' \in P(C,c)} d(v'_1, u')$ and otherwise we set $u'_1 = v'_1$ (note that the latter case is possible only if $\mathcal{K} = \mathcal{M}$). Observe that (C, U') is a $\mathcal{K}$-consensus with winner c, too: the number of voters who rank c first is the same in (C, U) and (C, U'). Moreover, we claim that $\widehat{d}^N(V', U') \le \widehat{d}^N(V, U)$. Since N is monotonic in the positive orthant and U and U' only differ in the first vote, to prove this claim, it suffices to show that $d(v'_1, u'_1) \le d(v_1, u_1)$. But this is easy to see: if u_1 ranks c first, we have

$$d(v_1, u_1) \ge \min_{u \in P(C,c)} d(v_1, u) = \beta_j, \quad d(v'_1, u'_1) = \min_{u' \in P(C,c)} d(v'_1, u') = \beta_\ell \le \beta_j,$$

whereas if u_1 does not rank c first, we have $d(v'_1, u'_1) = 0 \le d(v_1, u_1)$.

On the other hand, consider a candidate $b \ne c$, and let (C, W'), $W' = (w'_1, \dots, w'_n)$, be an arbitrary $\mathcal{K}$-consensus with winner b. We construct an

[10] This notion is due to Lerer and Nitzan (1985), who did not use the term "rank-monotonicity." We introduce this term to distinguish this notion from relative monotonicity (Definition 12).

election (C, W), where $W = (w_1, w'_2, \dots, w'_n)$, and w_1 is selected as follows: if w'_1 ranks b first, we pick w_1 from $\arg\min_{w \in P(C,b)} d(v_1, w)$ and otherwise we set $w_1 = v_1$. An argument similar to the one given in the previous paragraph shows that (C, W) is also a $\mathcal{K}$-consensus with winner b and $\widehat{d}^N(V, W) \le \widehat{d}^N(V', W')$; the proof uses the fact that v_1 ranks b at least as highly as v'_1 does.

In effect, we get $\widehat{d}^N(V', U') \le \widehat{d}^N(V, U) \le \widehat{d}^N(V, W) \le \widehat{d}^N(V', W')$, where the second inequality holds by our choice of U and the fact that (C, W) is a $\mathcal{K}$-consensus. As this is true for every $b \in C \setminus \{c\}$ and every $\mathcal{K}$-consensus (C, W') with winner b, it follows that c is an $\mathcal{R}$-winner of E'. □

Interestingly, even though rank monotonicity seems to be a much weaker condition than relative monotonicity, there is no formal connection between these two notions: it is possible to construct a pseudodistance that is relatively monotone, but not rank-monotone.

6 Classical Scoring Rules and Their $\mathcal{M}$-Counterparts

We will now use the tools developed in Sections 4 and 5 to provide a complete characterization of rules that are ℓ_1-votewise with respect to $\mathcal{U}$ and $\mathcal{M}$. In essence, the case of $\mathcal{U}$ has been treated by Lerer and Nitzan (1985). Our approach is very similar to theirs, but there are some important differences as well. In particular, for scoring rules that are not strong (i.e., scoring rules with vectors $(\alpha_1, \dots, \alpha_m)$ where $\alpha_i = \alpha_j$ for some i, j) Lerer and Nitzan gave quasidistance rationalizations (which become distance rationalizations for strong scoring rules), whereas we give pseudodistance rationalizations. Moreover, we prove that such rules can be distance rationalized if and only if $\alpha_1 \ne \alpha_2$.

The classical scoring rules (i.e., rules $\mathcal{R}_\alpha$, where α is a vector of nonnegative reals) were defined in Section 2. Let us now define their $\mathcal{M}$-counterparts.

Definition 14 Let $\alpha = (\alpha_1, \dots, \alpha_m)$ be a vector of nonnegative reals satisfying $\alpha_1 \ge \dots \ge \alpha_m$. The $\mathcal{M}$-*scoring rule* $\mathcal{R}^{\mathcal{M}}_\alpha$ is defined as follows. Fix an election $E = (C, V)$ with $C = \{c_1, \dots, c_m\}$ and $V = (v_1, \dots, v_n)$. For each candidate $c \in C$, let $M(c)$ be the multiset of numbers $\{\alpha_{\mathrm{pos}(v_j,c)} \mid 1 \le j \le n\}$. The $\mathcal{R}^{\mathcal{M}}_\alpha$-*score* of a candidate c is the sum of the $\lfloor \frac{n}{2} \rfloor + 1$ largest numbers in $M(c)$. The $\mathcal{R}^{\mathcal{M}}_\alpha$-*winners* are the candidates with the highest $\mathcal{R}^{\mathcal{M}}_\alpha$-score.

Proposition 8 *For each vector $\alpha = (\alpha_1, \dots, \alpha_m)$ of nonnegative reals such that $\alpha_1 \ge \dots \ge \alpha_m$, the scoring rule $\mathcal{R}_\alpha$ is $(\mathcal{U}, \widehat{d_\alpha})$-rationalizable and the $\mathcal{M}$-scoring rule $\mathcal{R}^{\mathcal{M}}_\alpha$ is $(\mathcal{M}, \widehat{d_\alpha})$-rationalizable.*

Proof Consider an election $E = (C, V)$ with $|C| = m$, a candidate $c \in C$, and a voter $v \in V$ that ranks c in position k. By Proposition 7, the d_α-distance from v to the nearest preference order in $P(C, c)$ is $2|\alpha_1 - \alpha_k| = 2(\alpha_1 - \alpha_k)$. Hence, the $\widehat{d_\alpha}$-distance from E to the nearest $\mathcal{U}$-consensus with winner c is

$$\sum_{i=1}^{n} 2(\alpha_1 - \alpha_{\mathrm{pos}(v_i,c)}) = 2n\alpha_1 - 2\sum_{i=1}^{n} \alpha_{\mathrm{pos}(v_i,c)}.$$

On the other hand, the score that c receives under $\mathcal{R}_\alpha$ is equal to $\sum_{i=1}^{n} \alpha_{\mathrm{pos}(v_i,c)}$. Thus, every candidate in E with the highest score under $\mathcal{R}_\alpha$ is a $(\mathcal{U}, \widehat{d}_\alpha)$-winner of E and vice versa. The statement about $\mathcal{R}_\alpha^{\mathcal{M}}$ is proved similarly. □

If $\alpha_i = \alpha_{i+1}$ for some $i \in \{1, \ldots, m-1\}$, then d_α is a pseudodistance, but not a distance. Thus, for any such α the proof of Proposition 8 only shows that $\mathcal{R}_\alpha$ and $\mathcal{R}_\alpha^{\mathcal{M}}$ are pseudodistance rationalizable. Can these rules be distance rationalized? It turns out that the answer is "yes" as long as $\alpha_1 \neq \alpha_2$.

Proposition 9 *For each vector $\alpha = (\alpha_1, \ldots, \alpha_m)$ of nonnegative reals such that $m \geq 2$, $\alpha_1 \geq \ldots \geq \alpha_m \geq 0$, the rules $\mathcal{R}_\alpha$ and $\mathcal{R}_\alpha^{\mathcal{M}}$ are votewise distance rationalizable with respect to consensus classes $\mathcal{U}$ and $\mathcal{M}$, respectively, if and only if $\alpha_1 \neq \alpha_2$.*

Proof Suppose that $\alpha_1 = \alpha_2$. Then $\mathcal{R}_\alpha$ is not compatible with $\mathcal{U}$ and $\mathcal{R}_\alpha^{\mathcal{M}}$ is not compatible with $\mathcal{M}$. Indeed, consider an election over a candidate set $C = \{c_1, \ldots, c_m\}$ where all voters rank c_1 first and c_2 second. This election is both a $\mathcal{U}$-consensus and an $\mathcal{M}$-consensus (with consensus winner c_1), yet under both $\mathcal{R}_\alpha$ and $\mathcal{R}_\alpha^{\mathcal{M}}$ the set of winners is $\{c_1, c_2\}$. Hence, Corollary 1 implies that $\mathcal{R}_\alpha$ and $\mathcal{R}_\alpha^{\mathcal{M}}$ are not distance rationalizable with respect to $\mathcal{U}$ and $\mathcal{M}$, respectively.

On the other hand, suppose that $\alpha_1 \neq \alpha_2$. Set $\varepsilon = \min\{|\alpha_j - \alpha_k| \mid \alpha_j \neq \alpha_k\}$ and let $d'_\alpha(u, v) = 0$ if $u = v$ and $d'_\alpha(u, v) = \max\{d_\alpha(u, v), \varepsilon\}$ otherwise. We will now argue that d'_α is a distance. Symmetry and nonnegativity are immediate, and by construction we have $d'_\alpha(u, v) \neq 0$ for $u \neq v$. To see that the triangle inequality is satisfied, note that whenever $d_\alpha(u, w) > 0$, by our choice of ε we have $d'_\alpha(u, w) = d_\alpha(u, w)$, and hence $d'_\alpha(u, v) + d'_\alpha(v, w) \geq d_\alpha(u, v) + d_\alpha(v, w) \geq d_\alpha(u, w) = d'_\alpha(u, w)$. On the other hand, if $d_\alpha(u, w) = 0$, we have $d'_\alpha(u, w) = \varepsilon \leq d'_\alpha(u, v) + d'_\alpha(v, w)$ by construction.

Now, consider a vote v over a candidate set C, $|C| = m$, a candidate $c \in C$ with $\mathrm{pos}(v, c) = j$, and a vote $u \in \arg\min_{u \in P(C,c)} d_\alpha(v, u)$. If $j = 1$, we can pick u so that $v = u$ and therefore $d'_\alpha(v, u) = d_\alpha(v, u) = 0$. If $j > 1$, we have $d_\alpha(v, u) = 2|\alpha_1 - \alpha_j| \geq 2|\alpha_1 - \alpha_2| \geq \varepsilon$, so $d'_\alpha(u, v) = d_\alpha(u, v)$. Therefore $\mathcal{R}_\alpha$ is $(\mathcal{U}, \widehat{d'}_\alpha)$-rationalizable and $\mathcal{R}_\alpha^{\mathcal{M}}$ is $(\mathcal{M}, \widehat{d'}_\alpha)$-rationalizable. □

We will now give a characterization of scoring rules and $\mathcal{M}$-scoring rules in terms of distance rationalizability.

Theorem 13 *A voting rule is $(\mathcal{U}, \widehat{d})$-rationalizable (respectively, $(\mathcal{M}, \widehat{d})$-rationalizable) for some rank-monotone neutral pseudodistance d on preference orders if and only if it is a scoring rule $\mathcal{R}_\alpha$ (respectively, an $\mathcal{M}$-scoring rule $\mathcal{R}_\alpha^{\mathcal{M}}$) for some vector $\alpha = (\alpha_1, \ldots, \alpha_m)$ such that $\alpha_1 \geq \ldots \geq \alpha_m \geq 0$.*

Proof We will only give the proof for $\mathcal{U}$; for $\mathcal{M}$ the argument is similar. The "if" direction follows from Proposition 8 and the fact that d_α is rank-monotone when $\alpha_1 \geq \ldots \geq \alpha_m$ (see Proposition 7).

For the "only if" direction, let $C = \{c_1, \ldots, c_m\}$ be a set of candidates and let $\mathcal{R}$ be a $(\mathcal{U}, \widehat{d})$-rationalizable voting rule, where d is a rank-monotone neutral pseudodistance on $P(C)$. Given a $k \in \{1, \dots, m\}$, pick a preference order $v \in P(C)$, and let c be the candidate that is ranked in position k in v. Set $\beta_k = \min_{u \in P(C,c)} d(u, v)$. By Proposition 6, β_k does not depend on our choice of v. Clearly, $\beta_1 = 0$, and by rank monotonicity of d we have $\beta_1 \leq \dots \leq \beta_m$. Set $\alpha_i = \beta_m - \beta_i$ for $i = 1, \dots, m$, and consider the vector $\alpha = (\alpha_1, \dots, \alpha_m)$. Note that $\alpha_1 \geq \dots \geq \alpha_m \geq 0$.

Consider an election (C, V) with $V = (v_1, \dots, v_n)$, and for each $c \in C$ let (C, U_c) be a $\mathcal{U}$-consensus with winner c that is closest to (C, V). Then

$$\widehat{d}(V, U_c) = \sum_{i=1}^{n} \min_{u_i \in P(C,c)} d(v_i, u_i) = \sum_{i=1}^{n} \beta_{\mathrm{pos}(v_i,c)} = n\beta_m - \sum_{i=1}^{n} \alpha_{\mathrm{pos}(v_i,c)}.$$

This expression is minimized if c is an $\mathcal{R}_\alpha$-winner of E. □

7 Conclusions and Further Research

The novelty of this work is in subjecting the concept of distance rationalizability of voting rules, which existed in various forms and shapes for at least 30 years (Nitzan, 1981; Lerer & Nitzan, 1985; Campbell & Nitzan, 1986; Baigent, 1987a; Meskanen & Nurmi, 2008; Elkind et al., 2012; Hudry & Monjardet, 2010) to an extended, rigorous study.

It was known that the broad and intuitively appealing definition of this concept, when all distances are allowed, is vacuous: Lerer and Nitzan (1985) and Campbell and Nitzan (1986) have shown that without any restrictions on distances essentially every reasonable voting rule is distance rationalizable. Motivated by this observation, we have identified a natural class of distances, which we called votewise distances, and proved that if we restrict ourselves to distances from this class, the concept of distance rationalizability becomes meaningful. In particular, some voting rules, e.g., STV and Plurality with Runoff, are not distance rationalizable via votewise distances. Nonetheless, a number of interesting rules can be rationalized in this way, and we showed that many desirable properties of voting rules, such as neutrality, anonymity, consistency, homogeneity, and monotonicity, can be deduced from the properties of the underlying distances over votes, the norms used to aggregate them, and the consensus classes. We gave a characterization of scoring rules as the class of rules that are distance rationalizable with respect to the weak unanimity consensus via votewise distances that satisfy certain simple axioms.

An attractive feature of the votewise distance rationalizability framework is its modularity: we can combine distances over votes, norms and consensus classes in a variety of ways to construct new voting rules. Moreover, the results of Section 5 tell us which components to use in order to ensure that the resulting rule has desired normative properties. Further, if our goal is to derive a voting rule that is similar to an existing rule, we can start with the distance rationalization of the latter and replace one of its components. We used this method to derive the Dodgson$^\infty$ rule and the class of $\mathcal{M}$-scoring rules.

Our work suggests several interesting research questions. Perhaps the most obvious of them is to determine the status of classical voting rules not considered in this paper with respect to the votewise distance rationalizability framework. For instance, Meskanen and Nurmi (2008) and Elkind et al. (2012) provide quite natural distance rationalizations for the Copeland rule, the Slater rule, Maximin, and the Young rule, which nonetheless are obviously not votewise. It would be interesting to see if these rules can also be rationalized by votewise distances. Another direction that may be worth exploring is to extend our framework to settings where voters' preferences may have cycles or be incomplete. Yet another research direction, initiated by Boutilier and Procaccia (2012), is to seek further justification for the distance rationalizability framework by, for example, showing that one obtains distance rationalizable rules as optimal solutions in various voting scenarios.

References

Aleskerov, F., Chistyakov, V., & Kalyagin, V. (2010). The threshold aggregation. *Econ Lett*, *107*(2), 261–262.

Arrow, K. (1951; revised edition, 1963). *Social choice and individual values.* John Wiley and Sons.

Baigent, N. (1987a). Metric rationalisation of social choice functions according to principles of social choice. *Math Soc Sci*, *13*(1), 59–65.

Baigent, N. (1987b). Preference proximity and anonymous social choice. *Q J Econ*, *102*(1), 161–169.

Baigent, N., & Klamler, C. (2004). Transitive closure, proximity and intransitivities. *Econ Theory*, *23*(1), 175-181.

Bartholdi, J., III, Tovey, C., & Trick, M. (1989). Voting schemes for which it can be difficult to tell who won the election. *Soc Choice Welf*, *6*(2), 157–165.

Bauer, F., Stoer, J., & Witzgall, C. (1961). Absolute and monotonic norms. *Numerische Matematic*, *3*, 257–264.

Bogard, K. (1973). Preference structures I: Distances between transitive preference relations. *J. Math. Sociol*, *3*, 49–67.

Bogard, K. (1975). Preference structures II: Distances between transitive preference relations. *SIAM J Appl Math*, *29*, 254–262.

Boutilier, C., & Procaccia, A. (2012, July). A dynamic rationalization of distance rationalizability. In *Proceedings of the 26th AAAI Conference on Artificial Intelligence* (pp. 1278–1284). AAAI Press.

Brams, S., & Fishburn, P. (2002). Voting procedures. In K. Arrow, A. Sen, & K. Suzumura (Eds.), *Handbook of social choice and welfare, volume 1* (pp. 173–236). Elsevier.

Brandt, F. (2009). Some remarks on Dodgson's voting rule. *Math Logic Quart*, *55*(4), 460–463.

Campbell, D., & Nitzan, S. (1986). Social compromise and social metrics. *Soc Choice Welf*, *3*(1), 1-16.

Caragiannis, I., Procaccia, A., & Shah, N. (2013). When do noisy votes reveal the truth? In *Proc. 13th ACM Conf. on Electronic Commerce* (pp. 143–160).

Chebotarev, P. Y., & Shamis, E. (1998). Characterizations of scoring methods for preference aggregation. *Ann Oper Res*, *80*, 299–332.

Condorcet, J. (1785). *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix.* (Facsimile reprint of original published in Paris, 1972, by the Imprimerie Royale)

Conitzer, V., Rognlie, M., & Xia, L. (2009, July). Preference functions that score rankings and maximum likelihood estimation. In *Proc. of the 21st Int. Joint Conf. on Artificial Intelligence* (pp. 109–115). AAAI Press.

Conitzer, V., & Sandholm, T. (2005, July). Common voting rules as maximum likelihood estimators. In *Proceedings of the 21st Conference in Uncertainty in Artificial Intelligence* (pp. 145–152). AUAI Press.

Cook, W., & Seiford, L. (1978). Priority ranking and consensus information. *Management Science*, *24*, 1721–1732.

Cook, W., & Seiford, L. (1982). On the Borda–Kendall consensus method for priority ranking problems. *Management Science*, *28*, 621–637.

Deza, M. M., & Deza, E. (2009). *Encyclopedia of distances.* Springer.

Eckert, D., & Klamler, C. (2011). Distance-based aggregation theory. In E. Herrera-Viedma, J. L. Garca-Lapresta, J. Kacprzyk, M. Fedrizzi, H. Nurmi, & S. Zadrozny (Eds.), *Consensual processes* (pp. 3–22). Springer Verlag.

Elkind, E., Faliszewski, P., & Slinko, A. (2010, May). On the role of distances in defining voting rules. In *Proc. 9th Int. Conf. on Autonomous Agents and Multiagent Systems* (pp. 375–382).

Elkind, E., Faliszewski, P., & Slinko, A. (2011). Homogeneity and monotonicity of distance-rationalizable voting rules. In *Proc. 10th Int. Conf. on Autonomous Agents and Multiagent Systems* (pp. 821–828).

Elkind, E., Faliszewski, P., & Slinko, A. (2012). Rationalizations of Condorcet-consistent rules via distances of Hamming type. *Soc Choice Welf*, *4*(39), 891–905.

Elkind, E., & Shah, N. (2014, July). Electing the most probable without eliminating the irrational: Voting over intransitive domains. In *Proc. 30th Conf. on Uncertainty in Artificial Intelligence.*

Elkind, E., & Slinko, A. (2015). Rationalizations of voting rules. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, & A. D. Procaccia (Eds.), *Handbook of computational social choice* (p. Chapter 8). Cambridge University Press.

Farkas, D., & Nitzan, S. (1979). The borda rule and pareto stability : A comment. *Econometrica*, *47*, 1305–1306.

Goldsmith, J., Lang, J., Mattei, N., & Perny, P. (2014, July). Voting with rank dependent scoring rules. In *Proceedings of the 28th AAAI Conference on Artificial Intelligence* (pp. 698–704). AAAI Press.

Hemaspaandra, E., Hemaspaandra, L., & Rothe, J. (1997). Exact analysis of Dodgson elections: Lewis Carroll's 1876 voting system is complete for

parallel access to NP. *Journal of the ACM*, *44*(6), 806–825.
Hudry, O., & Monjardet, B. (2010). Consensus theories. An oriented survey. *Mathematics and Social Sciences*, *190*, 139–167.
Kendall, M., & Gibbons, J. (1990). *Rank correlation methods.* Oxford University Press.
Lerer, E., & Nitzan, S. (1985). Some general results on the metric rationalization for social decision rules. *J Econ Theory*, *37*(1), 191-201.
Litvak, B. (1982). *Information given by the experts. Methods of acquisition and analysis.* Radio and Communication, Moscow.
Litvak, B. (1983). Distances and consensus rankings. *Cybernetics and systems analysis*, *19*(1), 71–81. (Translated from Kibernetika, No. 1, pp. 57–63, January–February 1983.)
Meskanen, T., & Nurmi, H. (2008). Closeness counts in social choice. In M. Braham & F. Steffen (Eds.), *Power, freedom, and voting.* Springer-Verlag.
Miller, M., & Osherson, D. (2009). Methods for distance-based judgment aggregation. *Soc Choice Welf*, *4*(32), 575–601.
Moulin, H. (1991). *Axioms of cooperative decision making.* Cambridge University Press.
Nitzan, S. (1981). Some measures of closeness to unanimity and their implications. *Theory and Decision*, *13*(2), 129–138.
Nitzan, S. (1989). More on preservation of preference proximity and anonymous social choice. *Q J Econ*, *104*(1), 187–190.
Pfingsten, A., & Wagener, A. (2003). Bargaining solutions as social compromises. *Theory and Decision*, *55*(4), 359-389.
Pivato, M. (2013). Voting rules as statistical estimators. *Soc Choice Welf*, *40*(2), 581–630.
Schechter, E. (1997). *Handbook of analysis and its foundations.* Acad. Press.
Xia, L., & Conitzer, V. (2011, July). A maximum likelihood approach towards aggregating partial orders. In *Proc. 22nd Int. Joint Conf. on Artificial Intelligence* (pp. 446–451).
Xia, L., Conitzer, V., & Lang, J. (2010). Aggregating preferences in multi-issue domains by using maximum likelihood estimators. In *Proc. 9th Int. Conf. on Autonomous Agents and Multiagent Systems* (pp. 399–406).
Young, H. (1975). Social choice scoring functions. *SIAM J Appl Math*, *28*(4), 824–838.
Young, H. (1977). Extending Condorcet's rule. *J Econ Theory*, *16*(2), 335–353.
Young, H., & Levenglick, A. (1978). A consistent extension of Condorcet's election principle. *SIAM J Appl Math*, *35*(2), 285–300.

A $\mathcal{M}$-Borda is not Homogeneous

We show that Theorem 9 cannot be extended to ℓ_1-votewise rules by showing that $\mathcal{M}$-Borda is not homogeneous. Consider election E:

v_1	v_2	v_3	v_4	v_5	v_6
b	a	c	c	c	d
a	b	b	b	d	a
d	d	a	a	a	b
e	e	e	e	e	e
c	c	d	d	b	c

A simple calculation shows that, to become a majority winner, a needs 4 swaps, b needs 3 swaps, c needs 4 swaps, and d needs 5 swaps. Thus, b is the unique winner of this election under $\mathcal{M}$-Borda. However, in the election $2E$ candidate b needs 5 swaps to become a majority winner, while c requires only 4 swaps.

B Proof of Theorem 3

We will show that Plurality is ℓ_1-votewise rationalizable with respect to strong unanimity. Given a set of candidates $C = \{c_1, \dots, c_m\}$ and two preference orders $u, v \in P(C)$, we say that v can be obtained from u by a *cyclic shift* if there exists an $i \in [m]$ and a permutation $\pi\colon C \to C$ such that $v = \pi(c_1) \succ \dots \succ \pi(c_m)$, and $u = \pi(c_i) \succ \dots \succ \pi(c_m) \succ \pi(c_1) \succ \dots \succ \pi(c_{i-1})$.

We partition $P(C)$ into m groups $L_1, \dots, L_m$ by setting $L_i = \{v \in P(C) \mid \mathrm{pos}(v, c_i) = 1\}$. Set $s = (m-1)!$ and, for each $i \in [m]$, enumerate the preference orders in L_i as $v_i^1, \dots, v_i^s$ so that for every $i, j \in [m]$ and every $t = 1, \dots, s$ the vote v_j^t can be obtained from the vote v_i^t by a cyclic shift. To see that this is possible, note that for each v_i^t, $i \in [m]$, $t \in [s]$, and each $j \in [m]$, there is exactly one vote in L_j that can be obtained from v_i^t by a cyclic shift.

The distance d is defined as follows. We set $d(v_i^t, v_j^r) = 0$, if $i = j$ and $t = r$; $d(v_i^t, v_j^r) = 1$, if $i = j$ or $t = r$ but $(i, t) \neq (j, r)$, and $d(v_i^t, v_j^r) = 2$, if $i \neq j$ and $t \neq r$. In other words, $d(u, u) = 0$ for all $u \in P(C)$, $d(u, v) = 1$ if and only if $u \neq v$ and u and v rank the same candidate first or v is obtained from u by a cyclic shift, and $d(u, v) = 2$ in all other cases. Observe that since $d(u, v) \in \{1, 2\}$ for $u \neq v$, the mapping d satisfies the triangle inequality; it is also obviously symmetric.

Consider an election $E = (C, V)$. For every $j \in [m]$ and $r \in [s]$ let a_j^r be the number of voters in V with preference order v_j^r. Let us calculate the distance from E to the $\mathcal{S}$-consensus election (C, X_i^t) with $X_i^t = |V| \times v_i^t$:

$$\widehat{d}(V, X_i^t) = \sum_{r \in [s] \setminus \{t\}} a_i^r \;+ \sum_{j \in [m] \setminus \{i\}} a_j^t \;+\; 2 \sum_{j \neq i} \sum_{r \neq t} a_j^r = \sum_{j \neq i} \sum_{r \in [s]} a_j^r \;+ \sum_{j \in [m]} \sum_{r \neq t} a_j^r.$$

As the first summand does not depend on t, the distance from E to the nearest $\mathcal{S}$-consensus with winner c_i is $\min_{t \in [s]} \widehat{d}(V, X_i^t) = \sum_{j \neq i} \sum_{r \in [s]} a_j^r + \min_{t \in [s]} \sum_{j \in [m]} \sum_{r \neq t} a_j^r$. The second component of this expression does not depend on i, while its first component counts the number of voters who do not rank c_i first. Thus, the nearest strong unanimity consensus to E has c_i as its winner if and only if i minimizes the sum $\sum_{j \in [m] \setminus \{i\}} \sum_{r \in [s]} a_j^r$ over all $i \in [m]$, i.e., c_i has the largest number of first-place votes. Thus, Plurality is $(\mathcal{S}, \widehat{d})$-rationalizable. □

C Proof of Theorem 5

As in the proof of Theorem 4, we set $C = \{a, b, c\}$ and use the same notation as throughout Section 4.2. We focus on STV, and define V and W as in the proof of Theorem 4. Also, we set $m_{sm} = \min\{S, M\}$, $m_{sb} = \min\{S, B\}$.

We will consider the majority consensus first. For the sake of contradiction, suppose that STV is $(\mathcal{M}, \widehat{d^N})$-rationalizable for some neutral distance d over $P(C)$ and a symmetric norm N that is monotonic in the positive orthant. Let (C, M_a) be an $\mathcal{M}$-consensus with winner a that is closest to (C, V) and let (C, M_b) be an $\mathcal{M}$-consensus with winner b that is closest to

(C, V). We have $\widehat{d^N}(V, M_a) = N(0, \dots, 0, \min\{m_{sb}, m_{sm}\}) = \widehat{d^N}(V, M_b)$. However, this is a contradiction as a is the unique winner of (C, V). Thus, STV is not $(\mathcal{M}, \widehat{d^N})$-rationalizable.

Similarly, suppose that STV is $(\mathcal{U}, \widehat{d^N})$-rationalizable. Let (C, U_a) be a $\mathcal{U}$-consensus with winner a that is closest to (C, V) and let (C, U_b) be a $\mathcal{U}$-consensus with winner b that is closest to (C, V). We have

$$\widehat{d^N}(V, U_a) = N(0, \dots, 0, m_{sm}, \dots, m_{sm}, m_{sb}).$$
$$\widehat{d^N}(V, U_b) = N(m_{sb}, \dots, m_{sb}, 0, \dots, 0, m_{sm}).$$

As a is the unique winner of (C, V), we have $\widehat{d^N}(V, U_a) < \widehat{d^N}(V, U_b)$, and therefore by symmetry and monotonicity $m_{sm} < m_{sb}$.

Now, let (C, U'_a) be a $\mathcal{U}$-consensus with winner a that is closest to (C, W) and let (C, U'_b) be a $\mathcal{U}$-consensus with winner b that is closest to (C, W). We have

$$\widehat{d^N}(W, U'_a) = N(0, \dots, 0, m_{sm}, \dots, m_{sm}, m_{sm}),$$
$$\widehat{d^N}(W, U'_b) = N(m_{sb}, \dots, m_{sb}, 0, \dots, 0, m_{sb}).$$

As b is the unique winner of (C, W), we have $\widehat{d^N}(W, U'_a) > \widehat{d^N}(W, U'_b)$, and hence $m_{sm} > m_{sb}$, a contradiction. Thus, STV is not $(\mathcal{U}, \widehat{d^N})$-rationalizable.

D Condorcet Consensus and Homogeneity

The following example shows that the Condorcet consensus is not split-homogeneous. Consider the following election $E = (C, V)$ with $C = \{a, b, c, d, e\}$ and $V = (v_1, \dots, v_{12})$.

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
a	b	c	d	e	c	e	a	b	c	d	c
b	c	d	e	a	a	d	e	a	b	c	a
c	d	e	a	b	b	c	d	e	a	b	b
d	e	a	b	c	d	b	c	d	e	a	d
e	a	b	c	d	e	a	b	c	d	e	e

Set $V_1 = (v_1, \dots, v_5)$, $V_2 = (v_7, \dots, v_{11})$, $V'_1 = (v_1, \dots, v_6)$, $V_2 = (v_7, \dots, v_{12})$. The profile V_1 is a Condorcet cycle, and the profile V_2 is obtained from V_1 by reversing the preference order of each voter. Voters v_6 and v_{12} are identical and rank candidate c first. Since voters from the groups V_1 and V_2 'cancel' each other, c is the Condorcet winner in E. On the other hand, consider elections $E_1 = (C, V'_1)$ and $E_2 = (C, V'_2)$. In E_1, b is ranked above c in 4 votes, so c is not a Condorcet winner in E_1. Similarly, in E_2, d is ranked above c in 4 votes, so c is not a Condorcet winner in E_2 either.

While Dodgson$^\infty$ is homogeneous, there are ℓ_∞-votewise distances that, paired with the Condorcet consensus, yield rules that fail homogeneity.

Proposition 10 *There exists a set of candidates C and a distance d on $P(C)$ such that the voting rule rationalized by $(\mathcal{C}, \widehat{d^\infty})$ is not homogeneous.*

Proof We will use two types of operations on preference profiles. Operations of the first type are the usual swaps of adjacent candidates. Let us now define an operation of the second type: a *two-candidate shift*. Given a preference order v and two candidates c and d that are ranked consecutively in v, a two-candidate shift of (c, d) pushes both c and d two positions upwards maintaining their relative ranking; if this is not possible, then the two-candidate shift is not defined. For example, given a preference order $u = a \succ b \succ c \succ d \succ e$ over the candidate set $C = \{a, b, c, d, e\}$, a two-candidate shift of (c, d) transforms u into $v = c \succ d \succ a \succ b \succ e$. Note that v can be transformed back into u by a two-candidate shift of (a, b); generally, if a vote u' can be transformed into a vote v' by a two-candidate shift, then v' can be transformed into u' by a two-candidate shift as well.

We define the distance d between two preference orders u and v as the minimum number of operations of both types that transform u into v. It is easy to see that d is indeed a distance, because it counts the number of reversible operations that are required to transform one preference order into the other. Let $\mathcal{R}$ be the voting rule that outputs all $(\mathcal{C}, \widehat{d^\infty})$-winners.

We will now build an election $E = (C, V)$ such that $\mathcal{R}(E) \neq \mathcal{R}(2E)$. Set $C = \{a, b, c, d\} \cup S$, where $S = \cup_{j=1}^{18} S_j$ is disjoint from $\{a, b, c, d\}$ and $S_1, \dots, S_{18}$ are pairwise disjoint sets of candidates of size 4 each. Let $V = (v_1, v_2, \dots, v_6)$; the voters' preference orders are given in the following table, where the candidates in each set S_j, $j = 1, \dots, 18$, are listed in an arbitrary order and each symbol $\downarrow$ replaces all candidates that have not been listed explicitly in the vote (for instance, in the first vote $\downarrow$ replaces $S \setminus (S_1 \cup S_6 \cup S_{12})$).

v_1	v_2	v_3	v_4	v_5	v_6
c	a	d	d	d	c
S_1	b	S_2	S_3	a	a
b	c	c	b	S_4	S_5
S_6	d	S_7	S_8	b	b
a	S_9	a	a	S_{10}	S_{11}
S_{12}	S_{13}	S_{14}	S_{15}	c	d
d	S_{16}	b	c	S_{17}	S_{18}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$

Note that each candidate in $\{a, b, c, d\}$ is preferred to each candidate in S by at least five voters. Further, four voters prefer a to b and only two voters prefer b to a, while all other head-to-head contests among the candidates in $\{a, b, c, d\}$ are tied. Note also that one needs at least two operations (swaps or shifts) to change the relative order of two candidates separated by a set S_j, $j = 1, \dots, 18$.

It is easy to verify that $\mathcal{R}(E) = \{a\}$. Indeed, a's $(\mathcal{C}, \widehat{d^\infty})$-score equals 1 (to make a the Condorcet winner, it suffices to swap it with d in v_5 and with c in v_6), whereas every other candidate requires at least two operations per vote to become the Condorcet winner.

On the other hand, we have $d \in \mathcal{R}(2E)$. Indeed, it is easy to see that each candidate's $(\mathcal{C}, \widehat{d^\infty})$-score in $2E$ is at least 1. Further, the $(\mathcal{C}, \widehat{d^\infty})$-score of d in $2E$ is 1: to make d the Condorcet winner, it suffices to swap d and c in one copy of v_2, and perform a two-candidate shift of (c, d) in the other copy of v_2. □