Fractional Moments of Dirichlet L-Functions

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Abstract

Let k be a positive real number, and let $M_k(q)$ be the sum of $|L(\frac{1}{2},\chi)|^{2k}$ over all non-principal characters to a given modulus q. We prove that $M_k(q) \ll_k \phi(q)(\log q)^{k^2}$ whenever k is the reciprocal n^{-1} of a positive integer n. If one assumes the Generalized Riemann Hypothesis then the estimate holds for all positive real $k < 2$.

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1 Introduction

Mean-values of the type

$$I_k(T) := \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}dt,$$

with positive non-integral values of k, have been investigated by a number of authors, including Ramachandra [5], [6], Conrey and Ghosh [1] and Heath-Brown [3]. In particular the above papers by Ramachandra show, under the Riemann Hypothesis, that

$$I_k(T) \gg_k T(\log T)^{k^2} \quad (T \geq 2)$$

for all real $k \geq 0$, and that

$$I_k(T) \ll_k T(\log T)^{k^2} \quad (T \geq 2)$$

for all real $k \in [0,2]$.

It is natural to ask about the corresponding problem for Dirichlet L-functions in q-aspect, that is to say to investigate

$$M_k(q) := \sum_{\substack{\chi (\bmod\ q) \\ \chi \neq \chi_0}} |L(\tfrac{1}{2}, \chi)|^{2k}$$

for positive real k. However rather little is known about this in general. The method of Rudnick and Soundararajan [7], enables one to show unconditionally that

$$M_k(q) \gg_k \phi(q)(\log q)^{k^2}$$

for rational $k \geq 1$, at least when q is prime. The method does not immediately apply to the case $0 < k < 1$ and it would be interesting to establish lower bounds in this range.

In the reverse direction, Soundararajan [9, Section 4] shows under the Generalized Riemann Hypothesis that

$$M_k(q) \ll_{k\varepsilon} \phi(q)(\log q)^{k^2+\varepsilon},$$

for any real $k \geq 0$ and any fixed $\varepsilon > 0$. One would conjecture that the true order of magnitude for $M_k(q)$ should be $\phi(q)(\log q)^{k^2}$. The present paper will prove upper bound results of exactly this order, motivated by the author's work [3]. We establish the following theorems.

Theorem 1 *Assuming the Generalized Riemann Hypothesis we have*

$$M_k(q) \ll_k \phi(q)(\log q)^{k^2}$$

for all $k \in (0, 2)$.

Theorem 2 *Unconditionally we have*

$$M_k(q) \ll_k \phi(q)(\log q)^{k^2}$$

for any k *of the form* $k = 1/v$, *with* $v \in \mathbb{N}$.

Thus taking $v = 2$ we have

$$\sum_{\chi (\bmod\ q)} |L(\tfrac{1}{2}, \chi)| \ll \phi(q)(\log q)^{1/4}$$

in particular.

The approach in [3] is based on a convexity theorem for mean-value integrals, which appears to have no analogue for character sums. We therefore work with integrals, and extract the sum $M_k(q)$ at the end. While we can give lower bounds for the integrals that occur, as well as upper bounds, it is not clear how to give a lower bound for $M_k(q)$ in terms of an integral.

It seems plausible that our approach might apply to other families of L-functions. One interesting case would be the fractional moments of L-functions with quadratic characters, in the form

$$\sum_{q\le Q}\mu(2q)^2|L(\tfrac{1}{2},\left(\frac{*}{q}\right))|^k \ll_k Q(\log Q)^{k(k+1)/2},$$

for example. However the estimation of the sum corresponding to $K(\sigma)$ will be more difficult than in the present paper, although the techniques used by Soundararajan [8, Section 5] seem likely to suffice. In addition, with the argument in its current form, a crude bound for the analogue of $J^*(\sigma)$ will need to be found.

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2 Mean-Value Integrals

Throughout our argument we will write $v = 1$ for the proof of Theorem 1, and $v = k^{-1}$ in handling Theorem 2. In both cases the primary mean-value integral we will work with is

$$J(\sigma,\chi) := \int_{-\infty}^{\infty} |L(\sigma+it,\chi)|^{2k}|W(\sigma+it)|^6 dt,$$

where the weight function $W(s)$ is defined by

$$W(s) := \frac{q^{\delta(s-1/2)}-1}{(s-1/2)\log q},$$

with $\delta > 0$ to be specified later, see (6) and (7). We emphasize that, for the rest of this paper, all constants implied by the Vinogradov $\ll$ symbol will be uniform in σ for the ranges specified. However they will be allowed to depend on the values of k and δ, so that the symbol $\ll$ should be read as $\ll_{k,\delta}$ throughout.

In addition to the integral $J(\sigma,\chi)$ we will use

$$K(\sigma,\chi) := \int_{-\infty}^{\infty} |S(\sigma+it,\chi)|^{2}|W(\sigma+it)|^6 dt,$$

where

$$S(s) := \sum_{n \le q} d_k(n)\chi(n)n^{-s}$$

Notice here that a little care is needed in defining $d_k(n)$ when k is not an integer, see [3, §2].

When χ is a non-principal character the function $L(s, \chi)$ is entire. Moreover, if we assume the Generalized Riemann Hypothesis then there are no zeros for $\sigma > \frac{1}{2}$, so that one can define a holomorphic extension of

$$L(s, \chi)^k = \sum_{m=1}^{\infty} d_k(m)\chi(m)m^{-s} \quad (\sigma > 1)$$

in the half-plane $\sigma > \frac{1}{2}$. Having defined $L(s, \chi)^k$ in this way we now set

$$G(\sigma, \chi) := \int_{-\infty}^{\infty} |L(\sigma + it, \chi)^k - S(\sigma + it, \chi)|^2 |W(\sigma + it)|^6 dt, \quad (\sigma > \tfrac{1}{2}).$$

This integral will be used in the proof of Theorem 1, while for the unconditional Theorem 2 we will employ

$$H(\sigma, \chi) := \int_{-\infty}^{\infty} |L(\sigma + it, \chi) - S(\sigma + it, \chi)^v|^{2/v} |W(\sigma + it)|^6 dt.$$

In addition to $J(\sigma, \chi), K(\sigma, \chi), G(\sigma, \chi)$ and $H(\sigma, \chi)$ we will consider their averages over non-principal characters,

$$J(\sigma) := \sum_{\substack{\chi (\mathrm{mod}\ q) \\ \chi \ne \chi_0}} J(\sigma, \chi), \qquad K(\sigma) := \sum_{\substack{\chi (\mathrm{mod}\ q) \\ \chi \ne \chi_0}} K(\sigma, \chi)$$

$$G(\sigma) := \sum_{\substack{\chi (\mathrm{mod}\ q) \\ \chi \ne \chi_0}} G(\sigma, \chi), \qquad \text{and} \qquad H(\sigma) := \sum_{\substack{\chi (\mathrm{mod}\ q) \\ \chi \ne \chi_0}} H(\sigma, \chi).$$

To derive estimates relating values of these integrals we begin with the following convexity estimate of Gabriel [2, Theorem 2].

Lemma 1 *Let F be a complex-valued function which is regular in the strip $\alpha < \Re(z) < \beta$, and continuous for $\alpha \le \Re(z) \le \beta$. Suppose that $|F(z)|$ tends to zero as $|\Im(z)| \to \infty$, uniformly for $\alpha \le \Re(z) \le \beta$. Then for any $\gamma \in [\alpha, \beta]$ and any $a > 0$ we have*

$$I(\gamma) \le I(\alpha)^{(\beta-\gamma)/(\beta-\alpha)} I(\beta)^{(\gamma-\alpha)/(\beta-\alpha)}$$

where

$$I(\eta) := \int_{-\infty}^{\infty} |F(\eta + it)|^a dt.$$

The inequality should be interpreted appropriately if any of the integrals diverge. From Lemma 1 we will deduce the following variant.

Lemma 2 *Let f and g be complex-valued functions which are regular in the strip $\alpha < \Re(z) < \beta$, and continuous for $\alpha \le \Re(z) \le \beta$. Let b and c be positive real numbers. Suppose that $|f(z)|^b|g(z)|^c$ and $|g(z)|$ tend to zero as $|\Im(z)| \to \infty$, uniformly for $\alpha \le \Re(z) \le \beta$. Set*

$$I(\eta) := \int_{-\infty}^{\infty} |f(\eta+it)|^b |g(\eta+it)|^c dt.$$

Then for any $\gamma \in [\alpha, \beta]$ we have

$$I(\gamma) \le I(\alpha)^{(\beta-\gamma)/(\beta-\alpha)} I(\beta)^{(\gamma-\alpha)/(\beta-\alpha)}. \tag{1}$$

To deduce Lemma 2 from Lemma 1 we choose a rational number $p/q > c/b$, and apply Lemma 1 with $F = f^q g^p$ and $a = b/q$. Since

$$|F| = (|f|^b|g|^c)^{q/b}|g|^{p-cq/b}$$

with $p - cq/b > 0$, we deduce that $|F|$ tends to zero as $|\Im(z)| \to \infty$, uniformly for $\alpha \le \Re(z) \le \beta$. We then obtain an inequality of the same shape as (1), but with the exponent c replaced by bp/q. Lemma 2 then follows on choosing a sequence of rationals p_n/q_n tending downwards to c/b.

We now apply Lemma 2 to $J(\sigma, \chi)$. When $\sigma = 3/2$ we have

$$W(s) \ll q^{\delta}/(1+|t|)$$

whence we trivially obtain

$$J(\tfrac{3}{2}, \chi) \ll q^{6\delta}.$$

An immediate application of Lemma 2 therefore yields

$$J(\sigma, \chi) \ll J(\tfrac{1}{2}, \chi)^{3/2-\sigma} q^{6\delta(\sigma-1/2)}$$

for $\frac{1}{2} \le \sigma \le \frac{3}{2}$, whence we trivially deduce that

$$J(\sigma) \ll J(\tfrac{1}{2})^{3/2-\sigma} q^{6\delta(\sigma-1/2)},$$

by Hölder's inequality. Since

$$J^f \le \left(\frac{\log q}{q}\right)^{1-f} \left(\frac{q}{\log q} + J\right) \ll q^{-(1-\delta)(1-f)} \left(\frac{q}{\log q} + J\right) \tag{2}$$

for any $J \ge 0$ and any $f \in [0,1]$, we conclude as follows.

Lemma 3 *We have*

$$J(\sigma) \ll q^{-(1-7\delta)(\sigma-1/2)} \left(\frac{q}{\log q} + J(\tfrac{1}{2}) \right)$$

for $\frac{1}{2} \le \sigma \le \frac{3}{2}$.

To obtain a second estimate involving $J(\sigma, \chi)$ we use Lemma 2 to show that if $\frac{1}{2} \le \sigma \le \frac{3}{4}$ and $1 - \sigma \le \gamma \le \sigma$ then

$$J(\gamma, \chi) \le J(\sigma, \chi)^{(\gamma-1+\sigma)/(2\sigma-1)} J(1-\sigma, \chi)^{(\sigma-\gamma)/(2\sigma-1)}.$$

An application of Hölder's inequality then shows that

$$J(\gamma) \le J(\sigma)^{(\gamma-1+\sigma)/(2\sigma-1)} J(1-\sigma)^{(\sigma-\gamma)/(2\sigma-1)}.$$

To handle $J(1-\sigma, \chi)$ we will use the functional equation for $L(s, \chi)$. If ψ is primitive, with conductor q_1, this yields

$$L(1-\sigma+it, \psi) \ll (1+|t|)^{\sigma-1/2} q_1^{\sigma-1/2} |L(\sigma+it, \psi)|$$

for $\frac{1}{2} \le \sigma \le \frac{3}{4}$ say. Thus if ψ induces a character χ modulo q we will have

$$L(1-\sigma+it, \chi) \ll (1+|t|)^{\sigma-1/2} q_1^{\sigma-1/2} \rho |L(\sigma+it, \chi)|$$

with

$$\rho = \prod_{p|q_2} \left(\frac{|1-\chi(p)p^{-\sigma-it}|}{|1-\chi(p)p^{\sigma-1-it}|} \right),$$

where $q_2 = q/q_1$. Thus

$$\log \rho \le (2\sigma - 1) \sum_{p|q_2} \frac{\log p}{p^{1-\sigma} - 1}.$$

However

$$\sum_{p|m} \frac{\log p}{p^{1/4} - 1} \le \tfrac{1}{2} \log m$$

for all sufficiently large m, whence $\rho \ll q_2^{\sigma-1/2}$. We therefore conclude that

$$L(1-\sigma+it, \chi) \ll (1+|t|)^{\sigma-1/2} q^{\sigma-1/2} |L(\sigma+it, \chi)|$$

when $\frac{1}{2} \le \sigma \le \frac{3}{4}$, for any character χ modulo q, whether primitive or not.

We now deduce that

$J(1-\sigma, \chi)$

$$\ll q^{2k(\sigma-1/2)}\int_{-\infty}^{\infty}|L(\sigma+it,\chi)|^{2k}(1+|t|)^{2k(\sigma-1/2)}|W(1-\sigma+it)|^{6}dt.$$

The presence of the factor $(1+|t|)^{2k(\sigma-1/2)}$ is inconvenient. However, since $0<k<2$ we have

$$(1+|t|)^{2k(\sigma-1/2)}|W(1-\sigma+it)|^{6}\ll(\log q)^{-6}|t|^{-2},$$

for $|t|\geq 1$ and $\frac{1}{2}\leq\sigma\leq 1$. It follows that

$$J(1-\sigma,\chi)\ll q^{2k(\sigma-1/2)}\left(J(\sigma,\chi)+(\log q)^{-6}J^{*}(\sigma,\chi)\right),$$

where

$$J^{*}(\sigma,\chi):=\int_{-\infty}^{\infty}|L(\sigma+it,\chi)|^{2k}\frac{dt}{1+t^{2}}.$$

Thus

$$J(1-\sigma)\ll q^{2k(\sigma-1/2)}\left(J(\sigma)+(\log q)^{-6}J^{*}(\sigma)\right)$$

with

$$J^{*}(\sigma):=\sum_{\substack{\chi(\mathrm{mod}\ q)\\ \chi\neq\chi_0}}\int_{-\infty}^{\infty}|L(\sigma+it,\chi)|^{2k}\frac{dt}{1+t^{2}}.$$

Finally we observe that

$$J(\sigma)^{(\gamma-1+\sigma)/(2\sigma-1)}\left\{J(\sigma)+(\log q)^{-6}J^{*}(\sigma)\right\}^{(\sigma-\gamma)/(2\sigma-1)}$$
$$\leq J(\sigma)+(\log q)^{-6}J^{*}(\sigma).$$

On comparing our results we therefore conclude that

$$J(\gamma)\ll q^{k(\sigma-\gamma)}\left(J(\sigma)+(\log q)^{-6}J^{*}(\sigma)\right). \tag{3}$$

We have now to consider $J^{*}(\sigma)$. It was shown by Montgomery [4, Theorem 10.1] that

$$\sum_{\chi(\mathrm{mod}\ q)}{}^{*}\int_{-T}^{T}|L(\tfrac{1}{2}+it,\chi)|^{4}dt\ll\phi(q)T(\log qT)^{4}$$

for $T\geq 2$, where Σ^{*} indicates that only primitive characters are to be considered. (It should be noted that there is a misprint in the statement of [4, Theorem 10.1], in that $L(\frac{1}{2}+it,\chi)$ should be replaced by $L(\sigma+it,\chi)$. However we are only interested in the case $\sigma=\frac{1}{2}$. Moreover, in the proof of [4, Theorem 10.1], at the top of page 83, the reference to Theorem 6.3 should be to Theorem 6.5.)

If χ is an imprimitive character modulo q, induced by a primitive character ψ with conductor q_1, then

$$|L(\tfrac{1}{2}+it,\chi)|^4 \le |L(\tfrac{1}{2}+it,\psi)|^4 \prod_{p|q,\, p\nmid q_1} (1+p^{-1/2})^4.$$

Thus if $\Sigma^{(1)}$ indicates summation over all characters χ modulo q for which the conductor has a given value q_1, we will have

$$\sum_{\chi}{}^{(1)} \int_{-T}^{T} |L(\tfrac{1}{2}+it,\chi)|^4 dt \ll \phi(q_1)T(\log q_1 T)^4 \prod_{p|q,\, p\nmid q_1} (1+p^{-1/2})^4.$$

If we now sum for $q_1|q$ we obtain

$$\sum_{\chi (\mathrm{mod}\ q)} \int_{-T}^{T} |L(\tfrac{1}{2}+it,\chi)|^4 dt \ll T(\log qT)^4 f(q),$$

where

$$f(q) = \sum_{q_1|q} \phi(q) \prod_{p|q,\, p\nmid q_1} (1+p^{-1/2})^4.$$

The function f is multiplicative, with

$$f(p^e) = (1+p^{-1/2})^4 + \phi(p) + \phi(p^2) + \ldots + \phi(p^e) = p^e \left(1 + O(p^{-3/2})\right).$$

Thus $f(q) \ll q$ and we conclude that

$$\sum_{\chi (\mathrm{mod}\ q)} \int_{-T}^{T} |L(\tfrac{1}{2}+it,\chi)|^4 dt \ll qT(\log qT)^4.$$

We may now deduce that if $f(s) = L(s,\chi)^2 s^{-1}$ then

$$\sum_{\chi (\mathrm{mod}\ q)} \int_{-\infty}^{\infty} |f(\tfrac{1}{2}+it)|^2 dt \ll q(\log q)^4.$$

Moreover the trivial bound $L(s,\chi) \ll 1$ for $\sigma = 3/2$ shows that

$$\sum_{\chi (\mathrm{mod}\ q)} \int_{-\infty}^{\infty} |f(\tfrac{3}{2}+it)|^2 dt \ll q.$$

We can therefore apply Lemma 1, together with Hölder's inequality, to deduce that

$$\sum_{\chi (\mathrm{mod}\ q)} \int_{-\infty}^{\infty} |f(\sigma+it)|^2 dt \ll q(\log q)^4$$

uniformly for $\frac{1}{2} \le \sigma \le \frac{3}{2}$. A final application of Hölder's inequality then implies that

$$J^*(\sigma) \ll q(\log q)^4.$$

We can now insert this into (3) and deduce as follows.

Lemma 4 *We have*

$$J(\gamma) \ll q^{k(\sigma-\gamma)}\left(\frac{q}{\log q} + J(\sigma)\right)$$

for $\frac{1}{2} \le \sigma \le 1$ *and* $1-\sigma \le \gamma \le \sigma$.

We now turn our attention to $G(\sigma,\chi)$ and $H(\sigma,\chi)$. By Lemma 2 we have

$$G(\sigma,\chi) \le G(\tfrac{1}{2},\chi)^{3/2-\sigma} G(\tfrac{3}{2},\chi)^{\sigma-1/2} \quad (\tfrac{1}{2} \le \sigma \le \tfrac{3}{2})$$

for non-principal characters χ modulo q. We then find via Hölder's inequality that

$$G(\sigma) \le G(\tfrac{1}{2})^{3/2-\sigma} G(\tfrac{3}{2})^{\sigma-1/2} \tag{4}$$

Since

$$W(\tfrac{3}{2}+it) \ll q^{\delta}(1+|t|)^{-1}$$

we see that

$$G(\tfrac{3}{2},\chi) \ll q^{6\delta} \int_{-\infty}^{\infty} |L(\tfrac{3}{2}+it,\chi)^k - S(\tfrac{3}{2}+it,\chi)|^2 \frac{dt}{1+|t|^2}.$$

However

$$L(\tfrac{3}{2}+it,\chi)^k - S(\tfrac{3}{2}+it,\chi) = \sum_{n>q} d_k(n)\chi(n)n^{-3/2-it}$$

whence

$$\int_{-\infty}^{\infty} |L(\tfrac{3}{2}+it,\chi)^k - S(\tfrac{3}{2}+it,\chi)|^2 \frac{dt}{1+|t|^2}$$
$$= \pi \sum_{m,n>q} d_k(m)d_k(n)\chi(m)\overline{\chi(n)} \min\left(m^{-1/2}n^{-5/2}\,,\, n^{-1/2}m^{-5/2}\right).$$

It follows that

$$\sum_{\chi \pmod q} \int_{-\infty}^{\infty} |L(\tfrac{3}{2}+it,\chi)^k - S(\tfrac{3}{2}+it,\chi)|^2 \frac{dt}{1+|t|^2}$$

$$= \pi\phi(q) \sum_{\substack{m,n>q \\ q|m-n,\,(mn,q)=1}} d_k(m)d_k(n) \min\left(m^{-1/2}n^{-5/2}\,,\, n^{-1/2}m^{-5/2}\right)$$

To estimate this double sum we use that fact that $d_k(n) \ll_\varepsilon n^\varepsilon$ for any fixed $\varepsilon > 0$. This leads to the bound

$$\sum_{\substack{m,n>q \\ q|m-n}} d_k(m)d_k(n) \min\left(m^{-1/2}n^{-5/2}\,,\, n^{-1/2}m^{-5/2}\right) \ll_\varepsilon q^{2\varepsilon-2}.$$

It therefore follows that

$$\sum_{\chi (\mathrm{mod}\ q)} \int_{-\infty}^{\infty} |L(\tfrac{3}{2}+it,\chi)^k - S(\tfrac{3}{2}+it,\chi)|^2 \frac{dt}{1+|t|^2} \ll_\varepsilon q^{2\varepsilon-1}.$$

Inserting this bound into (4) we obtain

$$G(\sigma) \ll_\varepsilon G(\tfrac{1}{2})^{3/2-\sigma} q^{(\sigma-1/2)(6\delta+2\varepsilon-1)}.$$

Using (2) again, we see that

$$G(\sigma) \ll_\varepsilon q^{1-2\sigma+(7\delta+2\varepsilon)(\sigma-1/2)} \left(\frac{q}{\log q} + G(\tfrac{1}{2})\right)$$

for $\sigma \in [\frac{1}{2}, \frac{3}{2}]$. The positive number ε is at our disposal, and we choose it to be $\varepsilon = \delta/2$, whence

$$G(\sigma) \ll q^{-(1-4\delta)(2\sigma-1)} \left(\frac{q}{\log q} + G(\tfrac{1}{2})\right).$$

The treatment of $H(\sigma,\chi)$ is similar. This time, since $k = 1/v$, we have

$$H(\tfrac{3}{2},\chi) \le \left\{\int_{-\infty}^{\infty} |W(\tfrac{3}{2}+it)|^6 dt\right\}^{1-k}$$
$$\times \left\{\int_{-\infty}^{\infty} |L(\tfrac{3}{2}+it,\chi) - S(\tfrac{3}{2}+it,\chi)^v|^2 |W(\tfrac{3}{2}+it)|^6 dt\right\}^k$$

by Hölder's inequality. The first integral on the right is trivially $O(q^{6\delta})$. Moreover

$$L(\tfrac{3}{2}+it,\chi) - S(\tfrac{3}{2}+it,\chi)^v = \sum_{n>q} a_k(n)\chi(n)n^{-3/2-it}$$

with certain coefficients $a_k(n) \ll_\varepsilon n^\varepsilon$. The argument then proceeds as before, noting that

$$\sum_{\substack{m,n>q \\ q|m-n}} a_k(m)a_k(n) \min\left(m^{-1/2}n^{-5/2}\,,\, n^{-1/2}m^{-5/2}\right) \ll_\varepsilon q^{2\varepsilon-2}.$$

It follows that

$$\sum_{\chi (\bmod\ q)} \int_{-\infty}^{\infty} |L(\tfrac{3}{2}+it,\chi) - S(\tfrac{3}{2}+it,\chi)^v|^2 |W(\tfrac{3}{2}+it)|^6 dt \ll q^{2\delta+\varepsilon-1}.$$

we then deduce, by the same line of argument as before, that

$$H(\sigma) \ll q^{-(k-4\delta)(2\sigma-1)} \left(\frac{q}{\log q} + H(\tfrac{1}{2})\right)$$

for $\sigma \in [\frac{1}{2}, \frac{3}{2}]$.

We record these results formally in the following lemma.

Lemma 5 *For $\sigma \in [\frac{1}{2}, \frac{3}{2}]$ we have*

$$G(\sigma) \ll q^{-(1-4\delta)(2\sigma-1)} \left(\frac{q}{\log q} + G(\tfrac{1}{2})\right)$$

and

$$H(\sigma) \ll q^{-(k-4\delta)(2\sigma-1)} \left(\frac{q}{\log q} + H(\tfrac{1}{2})\right).$$

We end this section by considering $K(\sigma)$. We have

$$K(\sigma) \le \sum_{\chi (\bmod\ q)} K(\sigma,\chi) = \sum_{m,n\le q} \frac{d_k(m)d_k(n)}{(mn)^\sigma} S(m,n)I(m,n),$$

where

$$S(m,n) = \sum_{\chi (\bmod\ q)} \chi(m)\overline{\chi(n)}$$

and

$$I(m,n) = \int_{-\infty}^{\infty} \left(\frac{n}{m}\right)^{it} |W(\sigma+it)|^6 dt.$$

Evaluating the sum $S(m,n)$ we find that

$$\begin{aligned}
&\sum_{m,n\le q} \frac{d_k(m)d_k(n)}{(mn)^\sigma} S(m,n) I(m,n) \\
&\qquad = \phi(q) \sum_{\substack{m,n\le q \\ q|m-n,\,(mn,q)=1}} \frac{d_k(m)d_k(n)}{(mn)^\sigma} I(m,n) \\
&\qquad = \phi(q) \sum_{\substack{n\le q \\ (n,q)=1}} \frac{d_k(n)^2}{n^{2\sigma}} \int_{-\infty}^{\infty} |W(\sigma+it)|^6 dt.
\end{aligned}$$

We then observe that

$$\sum_{\substack{n\le q \\ (n,q)=1}} \frac{d_k(n)^2}{n^{2\sigma}} \le \sum_{n\le q} \frac{d_k(n)^2}{n} \ll (\log q)^{k^2},$$

and that

$$\int_{-\infty}^{\infty} |W(\sigma+it)|^6 dt \ll q^{3\delta(2\sigma-1)} (\log q)^{-1}.$$

These bounds allow us to conclude as follows.

Lemma 6 *For $\frac{1}{2} \le \sigma \le \frac{3}{2}$ we have*

$$K(\sigma) \ll \phi(q) q^{3\delta(2\sigma-1)} (\log q)^{k^2-1}.$$

3 Proof of the Theorems

By definition of $G(\sigma,\chi)$ and $H(\sigma,\chi)$ we have

$$J(\sigma) \ll K(\sigma) + G(\sigma)$$

under the Generalized Riemann Hypothesis, and

$$J(\sigma) \ll K(\sigma) + H(\sigma)$$

unconditionally. In view of Lemma 5 these produce

$$J(\sigma) \ll K(\sigma) + q^{-(1-4\delta)(2\sigma-1)} \left(\frac{q}{\log q} + G(\tfrac{1}{2}) \right)$$

and

$$J(\sigma) \ll K(\sigma) + q^{-(k-4\delta)(2\sigma-1)} \left(\frac{q}{\log q} + H(\tfrac{1}{2}) \right)$$

respectively. However we also have

$$G(\tfrac{1}{2}) \ll K(\tfrac{1}{2}) + J(\tfrac{1}{2})$$

and

$$H(\tfrac{1}{2}) \ll K(\tfrac{1}{2}) + J(\tfrac{1}{2})$$

from the definitions again, so that

$$J(\sigma) \ll K(\sigma) + q^{-(1-4\delta)(2\sigma-1)} \left(\frac{q}{\log q} + K(\tfrac{1}{2}) + J(\tfrac{1}{2}) \right)$$

and

$$J(\sigma) \ll K(\sigma) + q^{-(k-4\delta)(2\sigma-1)} \left(\frac{q}{\log q} + K(\tfrac{1}{2}) + J(\tfrac{1}{2}) \right)$$

in the two cases respectively.

If we now call on Lemma 6 we then find that

$$\begin{aligned} J(\sigma) &\ll \phi(q) q^{3\delta(2\sigma-1)} (\log q)^{k^2-1} + q^{-(1-4\delta)(2\sigma-1)} \left(\frac{q}{\log q} + J(\tfrac{1}{2}) \right) \\ &\ll q^{4\delta(2\sigma-1)} \left(\phi(q)(\log q)^{k^2-1} + q^{1-2\sigma} J(\tfrac{1}{2}) \right) \end{aligned}$$

under the Generalized Riemann Hypothesis, since

$$\frac{q}{\log q} \ll \phi(q)(\log q)^{k^2-1} \tag{5}$$

for $0 < k < 2$. Similarly we have

$$J(\sigma) \ll q^{4\delta(2\sigma-1)} \left(\phi(q)(\log q)^{k^2-1} + q^{k(1-2\sigma)} J(\tfrac{1}{2}) \right)$$

unconditionally.

Finally we apply Lemma 4 with $\gamma = \frac{1}{2}$ and use (5) again, to deduce that

$$J(\sigma) \ll q^{4\delta(2\sigma-1)} \left(\phi(q)(\log q)^{k^2-1} + q^{-(2-k)(\sigma-1/2)} J(\sigma) \right)$$

under the Generalized Riemann Hypothesis. Similarly we may derive the unconditional bound

$$J(\sigma) \ll q^{4\delta(2\sigma-1)} \left(\phi(q)(\log q)^{k^2-1} + q^{-k(\sigma-1/2)} J(\sigma) \right).$$

We are now ready to choose our value of δ. For Theorem 1 we take

$$\delta = \frac{2-k}{10}, \tag{6}$$

and for Theorem 2 we choose

$$\delta = \frac{k}{10}. \tag{7}$$

Then in either case we will have

$$J(\sigma) \ll q^{4\delta(2\sigma-1)}\phi(q)(\log q)^{k^2-1} + q^{-\delta(2\sigma-1)}J(\sigma).$$

We write c_k for the implied constant in this last estimate, and note that c_k depends only on k. We then take

$$\sigma = \sigma_0 := \frac{1}{2} + \frac{\kappa}{\log q}$$

with

$$\kappa = (2\delta)^{-1}\max(1\,,\,\log 2c_k).$$

These choices ensure that

$$c_k q^{-\delta(2\sigma_0-1)} \le \frac{1}{2},$$

and hence imply that

$$J(\sigma_0) \ll q^{4\delta(2\sigma_0-1)}\phi(q)(\log q)^{k^2-1} \ll \phi(q)(\log q)^{k^2-1}.$$

Finally, we may apply Lemma 4 to deduce the following

Lemma 7 *With σ_0 as above we have*

$$J(\gamma) \ll \phi(q)(\log q)^{k^2-1}$$

uniformly for $1-\sigma_0 \le \gamma \le \sigma_0$.

All that remains is to bound $M_k(q)$ from above, using averages of $J(\gamma)$. Since $|L(s,\chi)|^{2k}$ is subharmonic we have

$$|L(\tfrac{1}{2},\chi)|^{2k} \le \frac{1}{2\pi}\int_0^{2\pi}|L(\tfrac{1}{2}+re^{i\theta},\chi)|^{2k}d\theta.$$

We now multiply by r and integrate for $0 \le r \le R$ to show that

$$|L(\tfrac{1}{2},\chi)|^{2k} \le \frac{1}{\mathrm{Meas}(D)}\int_D |L(\tfrac{1}{2}+z,\chi)|^{2k}dA,$$

where $D = D(0,R)$ is the disc of radius R about the origin, and dA is the measure of area. We take

$$R = \frac{\min(\kappa\,,\,\delta^{-1})}{\log q},$$

so that if $z \in D$ then $1-\sigma_0 \le \Re(\frac{1}{2}+z) \le \sigma_0$ and $|W(\frac{1}{2}+z)| \gg 1$. It follows that

$$\int_D |L(\tfrac{1}{2}+z,\chi)|^{2k} dA \ll \int_{1-\sigma_0}^{\sigma_0} J(\gamma,\chi)d\gamma$$

whence

$$M_k(q) \ll \frac{1}{\text{Meas}(D)} \int_{1-\sigma_0}^{\sigma_0} J(\gamma)d\gamma.$$

Since $\text{Meas}(D) \gg (\log q)^{-2}$ we now deduce from Lemma 7 that

$$M_k(q) \ll \phi(q)(\log q)^{k^2},$$

as required.

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