Using variation to develop deep conceptual understanding of trigonometry

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Abstract

The aims of this research were to investigate

1. How students learn trigonometry at GCSE and A level and in particular to follow the development of students' constructions of trigonometry throughout the A-level course.
2. The use of variation in helping students to gain a conceptual understanding of trigonometry.

In the first cycle of research, the progress of two students were followed through their A-level trigonometry course. Their understanding of trigonometry evolved from a simple ratio concept to a more developed functional understanding. Students completed a unit circle task that utilised the principles of variation theory to help them bridge the conceptual gap between ratio and function views of trigonometry.

The use of variation was developed further during the second cycle of research with year-9 trigonometry. Beginning with the concept of straight line gradient, the ideas of ratio trigonometry were gradually and carefully developed using similar triangles and trigonometry tables. The gradual variation was very effective in helping students understand the relationship between similar triangles and angles. Some students demonstrated uncertainties regarding the notation of trigonometry and further cycles of this research would be required to consider alternative approaches to introducing trigonometric notation.

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Chapter 1

Introduction

Trigonometry is an important topic in the secondary school mathematics curriculum. Typically it is introduced to students in year 9 in the context of right-angled triangles. In years 10 and 11, its use is extended to non-right-angled triangles through the sine and cosine rules. At A-level, trigonometry knowledge is extended to functional representations and identities, and in Further Maths its use is extended further still to include series and complex representations. This research will study the use of variation in the design and implementation of tasks and explore the efficacy of variation in enabling students to develop a conceptual understanding of functional trigonometry at A-level and an understanding of the use of trigonometric functions as ratios at the year-9 level.

The research for this project took place in a girls' secondary school in England, which for ethical considerations will be known as St Frances's School for Girls (or St Frances's) in this report. St Frances's is a selective entry independent school with a record of high attainment at both GCSE and A level. It is therefore important to note that any results, analysis or conclusions in this report should be read in the context of St Frances's and are unlikely to generalise readily to schools that are more representative of the national average. For maths lessons, students in years 7 to 11 are separated into between four and six sets based on prior attainment. In sixth form, students can choose to study either single or double maths. Single maths involves the students studying A-level maths over two years, and double maths involves students studying all of A-level maths in year 12 and then typically, although not universally, studying further maths in year 13. This year,

sixteen year-12 girls have chosen to study double maths, and they have been grouped into two teaching sets.

I am a maths teacher with five years' teaching experience, most of them at St Frances's. This year, among others, I taught the top set of year 9 and one of the year-12 double maths sets. The scheme of work for both of those sets involved trigonometry: in year 9, students were introduced to the concept of trigonometry for the first time, and in year 12 I taught the whole of A-level trigonometry.

Trigonometry is often seen as a difficult topic at all levels (see section 2.3.1). It therefore tends to be one that, despite a teacher's best intentions, students are often only able to grasp superficially and end up treating as a set of rules and procedures. This difficulty has been reported to have only caught out students when they reached sixth form — these students have been able to get by at GCSE by remembering trigonometry as a superficial collection of rules but adopting the same approach to A-level trigonometry has been overwhelming for them (eg Challenger 2009).

Given the reported difficulties of A-level trigonometry in particular, and the fact that I was teaching trigonometry at both introductory and A level, and also because trigonometry is a topic that I particularly enjoy, I chose to investigate students' learning and understanding of trigonometry for this report.

The maths department at St Frances's contains a knowledgeable group of teachers, discussions with whom provided a range of views on the teaching of trigonometry. Preliminary discussions with other teachers have revealed that there exists at least four different preferences for introducing students to trigonometric functions with real inputs and the consequent approach to solving trigonometric equations. One of the other teachers taught the parallel double maths set to mine, and in addition to discussions on approaches to teaching the topic, I worked with one of her students on developing conceptual understanding of trigonometry using a unit circle approach and variation. These discussions and interventions are intended to satisfy the collaboration requirement of this research.

Variation as a theory of structuring tasks is an established theory that has gained popularity in recent times, although the principles were applied long before the theory was articulated in western research literature (see section 2.4). Variation is a way of directing students attention to the important aspects of an idea or concept by changing either the essential or non-essential features. I have been making use of some of the elements of variation theory in my teaching for many years, hence my interest in applying it to the research for this report.

My research questions are as follows:

RQ1 How do students learn and understand trigonometry, both at year 9 and A level, particularly within a congnitive constructivist framework?

RQ2 Is there any evidence of students understanding trigonometric functions as actions, processes or objects?

RQ3 Can variation be used effectively in the design of mathematical tasks to help students develop a conceptual understanding of trigonometry at year 9 and A level?

RQ1 will be addressed in the literature review (Chapter 2). RQ2 will involve analysing the data in the context of the literature review to see if some of the theory is applicable to my students. Finally RQ3 will be answered by reflecting on my task design in the light of variation theory and using the data to attempt to determine the effectiveness of the tasks.

The research consists of two cycles. Chapter 4 describes the data collection and analysis of cycle 1, which is on the teaching and learning of the year-12 students. Chapter 5 contains the details of cycle 2, which relates to my work with year-9 trigonometry teaching. Finally Chapter 6 summarises the results and analysis of both cycles.

Chapter 2

Literature review

2.1 Styles of learning

There are a number of different theories that attempt to explain the way people learn. Theories of learning have been studied since at least the time of Plato c 400 BC (Sfard 1998), but predominantly since the early 20th century (Baird et al 2014) and currently the three most prominent learning theories are behaviourism, cognitive constructivism and sociocultural (eg Baird et al 2014; James 2008).

Behaviourism has its origins in the experiments of Hull (1943) and Skinner (1938), and sees learning as a response to external stimuli. Behaviourists break complex tasks down into discrete elements that children practise individually and learning is seen as a gradual accumulation of these discrete skills (eg James 2006). It was heavily criticised during the second half of the 20th century for many reasons (eg Schnaitter 1999) but the most relevant shortcoming of behaviourism for the purposes of this discussion is its failure to consider how learning manifests in the brain of the learner. The early behaviourists were biological scientists and were only concerned with things that they could physically measure, which did not include mental processes (James 2008).

Exploration of the mental processes became the concern of a new theory of learning known as cognitive constructivism, which is usually credited to the Swiss psychologist Jean Piaget (Baird et al 2014; Piaget 1952). Piaget attempted to understand and explain how the mind organised and made sense of new information and assimilated it with

pre-existing understanding (Mason and Johnston-Wilder 2004). Piaget and Duckworth (1970) saw mathematics as a set of abstractions based on *actions* on *objects* (Mason and Johnston-Wilder 2004, p 165). The concept of mathematics as a set of actions and objects will be explored further in section 2.2.2. The abstracted actions can then become 'items of thought' themselves, which can then be subject to further actions and ultimately establish a 'hierarchical cognitive structure' that becomes a person's understanding of mathematics (Challenger 2009). Work has continued since Piaget to understand the hierarchical cognitive structure and explore its creation. This will be discussed in section 2.2. Although many writers have sought to elaborate the way knowledge is constructed, the concepts are not universally accepted (Davis and Sumara 2002). The translation of Piaget's French text is better done as 'construe', which has connotations for something more fluid and less predictable than architectural term 'construct' (Davis and Sumara 2002).

Some writers felt that the ideas of constructivism focused too heavily on individual thinking and disregarded the social environment in which learning typically takes place (Mason and Johnston-Wilder 2004, p 95). Lewin (1947) emphasised the importance the social environment plays in developing understanding. This leads to the third of the main categories of learning theory — sociocultural — which has gained recent popularity due to the work of Vygotsky (1980).

Learning is seen as our way of adapting to changes in our social environment. Social environment can include not only other students, but also teachers and less animate objects such as books, images, symbols etc (Mason and Johnston-Wilder 2004, p 75). Informal learning such as learning to walk, talk, run, etc can be explained from a sociocultural perspective. Children learn these skills by interacting with their environment, imitating others and participating in a culture, rather than following a set of instructions; very often without anyone being aware that any learning is taking place (Sierpinska 1994, p 162). The theory can be seen as a shift from the manner in which an individual constructs the world to the manner in which the world constructs the individual (Davis and Sumara 2002).

No individual learning model is self-sufficient and comprehensive (Davis and Sumara 2010) and all of them can be seen to contribute to our understanding of learning. In addition to the constructivisms of Piaget and Vygotsky, there are many variations often stemming from socialogical, epistemological or philosophical perspectives, rather than school education (Davis and Sumara 2002; Phillips 1995). The origins of the constructivist theories can create problems when trying to apply them to modern education (Davis and Sumara 2003), but ultimately they can be classified by whether *individuals construct their own understandings* or whether *all knowledge is socially constructed* (Davis and Sumara 2002), and it is beyond the scope of this research to examine the variations of the constructivist literature. Cognitive constructivism as the theory that focuses on the way individuals make sense of new information is the most commonly used in the research on learning and trigonometry that is discussed in the following sections. Any of the other models or theories could be used as the context for the analysis of the data collected in this research but for consistency with prior research and due to the nature of prior learning by the students at St Frances's, it is most appropriate to consider the data and their learning in the context of cognitive constructivism. Most of the remainder of this chapter will seek to explore the constructivist understanding in more detail and explain the way a learner's mind may absorb (construct/construe) and grasp new mathematics.

2.2 Students' development of mathematical understanding

2.2.1 Instrumental v relational understanding

Skemp (1976) is one of a number of writers who has attempted to grasp what is meant by the word *understanding*. He observes that the same word is used to mean more than one thing. Many students (and some teachers) commonly use it to mean that they are able to obtain the correct answer to a given problem. 'Rules without reasons', as Skemp (1976, p 20) puts it. This type of understanding he terms *instrumental understanding*.

The other type of understanding that Skemp defines is *relational understanding*, which he describes as knowing both what to do and why. Skemp explicitly accepts that relational understanding is better than instrumental understanding. He sees relational understanding as being more adaptable to new tasks, placing less burden on students' memories and being more compatible with the concept of cognitive constructivism, in particular the formulation of a *relational schema*. A schema is a way of explaining the way one's mind organises knowledge (see section 2.2.2).

Having observed that many teachers teach mathematics for an instrumental understanding, even if they themselves have a relational understanding, Skemp sought the advantages of it. He observed that instrumental teaching and learning satisfied the short-term goals of right answers quickly and with minimal effort. Skemp accepts that a relational understanding takes longer to obtain and so in the time-pressured world of classroom mathematics, teachers may choose the path of least resistance. Students often see their goal in mathematics as being to obtain a page of correct answers and judge their success or failure on this basis (Boaler 2015), so giving students an algorithm for obtaining those correct answers can be a way of giving students some self-confidence in their mathematics (Skemp 1976, p 23).

From a researcher's (and hence teacher's) perspective, it can be impossible to determine from looking at a student's writing, whether the marks on a paper are made from an instrumental or relational understanding. 'Talking with the pupil is almost certainly the best way to find out' (Skemp 1976, p 24; Star and Stylianides 2013, p 177).

More recently, the language of procedural knowledge and conceptual knowledge (instead of instrumental and relational) has been used (Star and Stylianides 2013, p 169). The definitions of procedural and conceptual are given in Hiebert and Lefevre (1986) and are analogous to Skemp's definitions. Star and Stylianides (2013) are cautious about the distinction between the two types of understanding. Many commentators use the terms to imply deep understanding and superficial understanding, but it is possible, they argue, for students to have superficial relational understanding or a deep instrumental

understanding. The quality of a student's knowledge is more important than the type of knowledge and in fact some maths educators question whether these the distinction is worth making (Star and Stylianides 2013, p 179). Gray and Tall (1994) and Sfard (1991) cite process and concept as parts of the same knowledge structure, merging them as *procepts* (see section 2.2.3).

2.2.2 Actions, Processes, Objects and Schemas

Often known as APOS theory, this is an attempt to explain Piaget's notion of abstraction in the formation of mathematical understanding. Abstraction draws knowledge from objects by performing (or imagining) actions on them (Dubinsky 1991, p 98). These actions then form new objects. An object could be something physical, such as five sheep in a field, or more abstract, like the number five, or a collection of objects that itself becomes an object, such as a function or group.

Consider the idea of the fraction $\frac{3}{8}$. A fraction is often initially seen by children as the process of division — a *dynamic* process (Dubinsky 1991, p 101). Learners may instinctively want to write something like $\frac{3}{8} = 0.375$, but after performing a series of actions on numbers like $\frac{3}{8}$, learners *encapsulate* the dynamic process into a static object (Cottrill et al 1996; Dubinsky 1991; Piaget 1985). The fraction becomes a number, or object, in its own right. It can be manipulated in the same way as other numbers and it becomes incorporated into an overall framework, or *schema*, of numbers.

An *action* is 'any physical or mental transformation of objects to obtain other objects' (Cottrill et al 1996, p 171). An action seems at first glance to be much the same as a process. Cottrill et al (1996) give them the same initial definition: a transformation of an object; the difference, they say, is that the individual is 'in control' of a process whereas an action is more of a physical reflex. Dubinsky (1991, p 107) describes a process as an interiorised action. Interiorisation permits one to be conscious of the action, aware of all the steps without actually needing to perform them, able to estimate its outcome, and able to reverse it or combine it with other processes (Mason and Johnston-Wilder 2004).

This theory of the transformation of knowledge structure is an incarnation of Piaget's notions of empirical abstraction (Dubinsky 1991; Gray et al 1999) and accordingly should be read in the context of cognitive constructivist theory (see section 2.1).

A schema is a 'coherent collection of objects and processes' (Dubinsky 1991). The ability of a person to invoke a schema to make sense of a problem is the person's knowledge of a concept in maths. The number and connectedness of individual schemas is important. Initially students may have a number of disjointed, small schemas, but as they progress they may start to link smaller schemas together. Exactly where a process becomes a schema is difficult to identify, and of course one of the problems with constructivism in general is the difficulty in directly observing someone's schema, objects or processes (Dubinsky 1991). 'We can only infer them from our observations of individuals' in situations where they are trying to solve a problem or understand a phenomenon. The schema may in fact be the observer's way of explaining the subject's knowledge, rather than the reverse (Dubinsky 1991, p 103).

Students should be ready to de-encapsulate objects back to processes (Challenger 2009). Within a problem, it may be necessary to think of something as both an object and a process simultaneously (or at least in rapid succession) (Dubinsky 1991, p 119). For example $\int_0^\pi \sin^2 x \, dx$ requires thinking about $\sin^2 x$ as an object to apply the process of integration to, and then as a process when evaluating at its limits (Challenger 2009).

There is general agreement in the literature that process or operational conceptions 'must precede the development of structural or object notions' (Cottrill et al 1996; Sfard 1988, 1991). The movement from process to object (encapsulation) has been reported as difficult for some students to grasp in certain circumstances (eg Blackett 1990; Cottrill et al 1996). Techniques that have been suggested include the use of computer programmes, where the coding of an action helps it to be seen as static by the student (eg Blackett 1990; Cottrill et al 1996). However it is done, it is ultimately the students who must construct the mental objects and processes themselves — they cannot be transmitted from a teacher verbally or even with pictures (Dubinsky 1991, p 121). Mason and

The figure originally presented here cannot be made freely available via ORA because of copyright. The figure was sourced at Sfard, A (1991) On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. Educational Studies in Mathematics 22 (1), 1–36.

Figure 2.1: *Different organisations of a hierarchical schema.*

Johnston-Wilder (2004, p 173) observe that as soon as you try to make the students construct the objects and processes 'it becomes instruction rather than education; procedural content to be memorised rather than reconstructed'.

One of the advantages to having well structured schema is that it reduces the demand in recalling ideas when trying to solve a problem (Sfard 1991; Skemp 1976). Figure 2.1 shows two examples of possible schema hierarchies. Schema A is wide and shallow and represents a person with instrumental knowledge of a number of concepts. Schema B contains the same information but with a new organisation that allows faster retrieval and storage (Sfard 1991, p 27).

2.2.3 Procepts

Gray and Tall (2001) used similar language to Dubinsky (1991) and Cottrill et al (1996) to describe a similar phenomenon. An individual begins with a *procedure* — a 'step-by-step solution for a routine problem'. As the procedure becomes interiorised and can be seen as a whole without needing to refer to the individual steps then it has evolved into a *process*. A process can be thought of in a more flexible manner than a procedure.

When the learner develops the flexibility to switch freely between the concept of the process and carrying out the process itself, it becomes a *procept* (a portmanteau of 'process' and 'concept'). Proceptual thinking is characterised by the ability to compress stages in symbol manipulation to the point where symbols are viewed as objects that can be decomposed and recomposed in flexible ways (Gray and Tall 1994). The ability to see an

entity as both an object and a process 'is indispensible' for a deep (relational) understanding of mathematics (Sfard 1991; Skemp 1976).

Examples of procepts include $3x - 2$, where x could represent a specific unknown or be a variable; and $\frac{2}{3}$, which is both a division operation and the answer which has a place on the number line (Mason and Johnston-Wilder 2004, p 166). Process and object have also been described as *operational* and *structural* representations of mathematical notions (Sfard 1991, p 5). The dual use of mathematical symbols to represent both processes and objects and the necessity to switch between the different meanings is something that some learners find more challenging than others, leading to a *proceptual divide* Gray and Tall (1994).

The notion of *procept* can, in some senses, be thought of as analogous to an *encapsulated object* (Tall et al 1999). They are both the end result of cognitive construction that results in a concept that can be manipulated in its own right. A *reified object* (Sfard 1991) (see section 2.2.4) is another example of essentially the same thing (Tall et al 1999). Unlike the encapsulated or reified object, a procept is narrower in scope because a procept requires a *symbol* that can evoke a process or concept (Tall et al 1999).

2.2.4 Reification

Analogies can be drawn between the evolution of number theory and mathematical learning in general. Counting began as a process, which led to the integers becoming objects. This can be observed in young children who will repeat the counting process when asked how many objects there are (Sfard 1991, p 11). The process of subtraction produced negative numbers, which later become objects. The process of division yielded rational numbers. The process of determining diagonals of squares produced irrational numbers. And so on; this process-object cycle repeating (Sfard 1991).

When a process has become sufficiently familiar that it can be analysed without actually needing to be performed, it is said to be *interiorised* (the same term as Cottrill et al (1996)). When a lengthy sequence of operations can be seen as a whole process (eg finding the

equation of a tangent), the process has been *condensed*, and a new concept has officially been born (Sfard 1991, p 19). 'The condensation phase lasts as long as a new entity remains tightly connected to a certain process. Only when a person becomes capable of conceiving the notion as a full-fledged object, do we say that the concept has been *reified*' (Sfard 1991, p 19; Mason and Johnston-Wilder 2004, pp 167–169). The newly reified object can then undergo the next round of interiorisation-condensation-reification.

2.2.5 Summary

Within the framework of cognitive constructivism, there are a number of theories that describe some of the details of how knowledge/understanding is construed in the mind of the learner. Dynamic actions operate on static objects. When sufficient reflection is given to these actions, they become interiorised (Cottrill et al 1996; Sfard 1991) as processes. A process is something the learner has control over, can think about its results without actually needing to do it, and can reverse it. Over time, processes can become encapsulated (Cottrill et al 1996; Dubinsky 1991) or reified (Sfard 1991) to become new objects or encapsulated into procepts (eg Gray et al 1999).

The collection of objects, actions and processes comprises a learner's understanding of a concept and has been collectively called a schema (eg Cottrill et al 1996) Analysis of trigonometric understanding could be undertaken through any of the frameworks discussed above, but APOS theory has been chosen for this report primarily because a number of other related studies have also done so (eg Blackett 1990; Challenger 2009; Martínez-Planell and Delgado 2016; Pournara 2001); however consideration of the other theories is also given in section 2.3 in order to adequately represent the literature.

2.3 Students' understanding of trigonometry

There has been some research into students' learning and understanding of trigonometry at the GCSE level (eg Blackett 1990; Chin and Tall 2012; Kendal and Stacey 1998;

Pournara 2001; Watson 2008; Weber 2005), but less that specifically examines the development of understanding as students move from a ratio conception of trigonometry to a circular or functional understanding, which typically happens at A-level in the UK. (eg Altman and Kidron 2016; Challenger 2009; Kamber and Takaci 2017).

Trigonometry is seen as an important topic because a large body of future maths relies on understanding the basic principles — this includes not only A-level and university maths, but also areas of science such as harmonic and circular motion, engineering and architecture (Weber 2005).

Trigonometry is a school maths topic that is first introduced typically at year 9 with right angled triangles, then broadened in years 10 and 11 to include the sine and cosine rules for scalene triangles (Blackett 1990). At A-level, trigonometry is generalised further to include its domain of the real numbers, often utilising the unit circle or graph representation (Challenger 2009). In further maths, the functional nature of trigonometry is explored through the use of Taylor series representations (Chin and Tall 2012; Kamber and Takaci 2017).

2.3.1 Why is trigonometry hard?

Trigonometry is often seen as particularly challenging for students, both at GCSE (eg Watson 2008) and at A-level (eg Challenger 2009). Some of the reasons cited for this include the range of prerequisite concepts, many of which are challenging in their own right: angles, similarity, ratio and proportion, and functions (Watson 2008). The trigonometric functions are probably the first functions that students encounter that are symbolised by the name of the function followed by its argument. They are also among the first non-linear functions that students spend any time exploring (Watson 2008).

As students move into A-level, they can find the change from ratio trigonometry to functional, circular trigonometry confusing. Altman and Kidron (2016) and Challenger (2009, p 179) reported comments from students such as 'I didn't find it difficult at GCSE but at A-level it takes more understanding' and 'what is sine exactly? I thought I knew but

now it is so confusing'. Many students report the increased demand on memory required by A-level trigonometry (Challenger 2009; Chin and Tall 2012), which is consistent with students being taught to understand the concepts instrumentally (eg Blackett 1990; Kendal and Stacey 1998; Skemp 1976; Weber 2005). Instrumental understanding is adequate for many students at GCSE, especially because, until recently formulas like $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ were given on the formula sheet. Poor understanding of radian measure for angles has also been cited (eg Akkoc 2008; Challenger 2009; Kamber and Takaci 2017; Moore 2012).

Trigonometry is one of the few GCSE topics that is typically introduced with the definition first (Challenger 2009; Gray et al 1999). Students then explore the concept through the teacher's choice of examples and problems. This 'didactic reversal' can be problematic for students (eg Blackett 1990; Challenger 2009; Gray et al 1999; Weber 2005).

The range of different ways to represent and consider trigonometric functions can lead to uncertainties. They can be seen as ratios or functions (eg Challenger 2009; Pournara 2001; Weber 2008), they can be viewed as processes or objects (or procepts) (Challenger 2009; Pournara 2001; Weber 2005), they are related to triangles and circles, but their graphs are neither circular nor triangular (Kendal and Stacey 1998) and students need to be adept at switching between algebraic and pictorial or graphical representations of each of the functions (Challenger 2009). Unfortunately students can be reluctant to use visual representations when algebraic or other symbolic representations are available (Presmeg 2005).

Despite it being seen as central to a full understanding of trigonometry, some studies have reported that students do not view trigonometric functions as part of the general class of functions, even after two years of A-level study with repeated reference in lessons to trigonometric *functions* (Challenger 2009; Weber 2005). Other studies have observed the opposite — despite not being explicitly taught about functions, students had spontaneously developed a concept of the trigonometric functions as being an operator

that converts an input angle into an output ratio (Pournara 2001). The students' use of calculators was seen as the reason for this.

2.3.2 Sine as a procept

All of the trigonometric functions are procepts (Gray and Tall 1992). The formula

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

describes a process of dividing the lengths of the opposite and hypotenuse sides of a triangle with angle θ. The result of the division is the value of $\sin\theta$ and is an object. This object is then required for more advanced calculations such as

$$\text{opposite} = \text{hypotenuse} \times \sin\theta$$

These representations of $\sin\theta$ are both ratio orientations (Pournara 2001).

Similar process and object orientations can exist in learners' minds when $\sin\theta$ is viewed as a function. The process orientation sees the sine function *operating* on an input angle θ. The object orientation is the result of sine operating on angle θ, for example the coordinates of a point on the unit circle are $(\cos\theta, \sin\theta)$ (Pournara 2001). When taking a functional view of $\sin\theta$, students can find difficulties in the units (Pournara 2001). The input variable is in degrees but the output is a dimensionless quantity. This can be seen as an example of cognitive discontinuity — where a new concept is inconsistent with a learner's existing schema (Pournara 2001, p 129; Tall et al 2001).

In order make sense of the inverse sine function, where the input is dimensionless and the output is an angle, it is necessary to have a function orientation of sine (Challenger 2009; Pournara 2001). Students who lack the functional concepts refer to things like 'shift sine' or '2nd sine', which are the required keystrokes on a calculator (Challenger 2009; Pournara 2001), or the classic *dividing by sine* (Pritchard and Simpson 1999). There are also accounts of students using 'inverse' to mean 'rearrange', such as when solving $\sin 30° = \frac{12}{x}$, which is

consistent with the procedure for finding the inverse of linear functions (Challenger 2009). All of these misconceptions for inverse trigonometric functions suggest students holding an action view of the functions (Challenger 2009; Pritchard and Simpson 1999), and further work is required for the action to be interiorised as a process (eg Cottrill et al 1996).

Challenger (2009) and Weber (2005) found that students were more likely to hold a proceptual view of trigonometric functions when given the opportunity to view them dynamically as geometric processes and reflect on the processes, resulting in greater opportunity to interiorise and condense the processes (consistent with the process/object dialectic of Sfard 1991). Supplying geometric or symbolic images that had no intrinsic meaning to students did not yield the same proceptual view in the students (Challenger 2009, p 53).

One of the signs of proceptual understanding is the capacity to estimate the result of a process, rather than having to actually carry out the process (Cottrill et al 1996; Sfard 1991; Weber 2005). This occurs because the process has been interiorised (Sfard 1991). Weber (2005) tested this idea by asking students to estimate $\sin 340°$. Only 16% of students were able to give satisfactory answers to this, which Weber implied was the result of students having failed to interiorise the process of $\sin\theta$. Blackett (1990) took the reverse view that estimating values of trigonometric functions led to a proceptual understanding of them (see also Chin and Tall 2012).

2.3.3 Trigonometric schemas and concept maps

As discussed in section 2.2.2, a person's schema cannot be directly observed. Tall et al (2001) suggest that one way of gathering information about someone's schema is to ask them to draw a concept map, which is a diagram that looks essentially like the networks described in Nickerson (1985) — nodes containing objects/processes/etc connected by lines where students consider the concepts to be connected. The concept maps shown in Tall et al (2001, p 12) are quite detailed and show a great deal of interconnectedness for the topic of functions — resembling schema B in figure 2.1.

Challenger (2009) asked students to draw concept maps of their trigonometric understanding at various stages through the A-level course. As expected, prior to the A-level course students' views of trigonometry were primarily triangle based, with some awareness of the basic graphs. The details in the concept maps changed over the course of the course, but they all tended to resemble schema A in figure 2.1. When questioned, students did not appear to have followed the instructions for drawing the concept maps, intentionally omitting concepts that they were quite familiar with. Consequently the data was found to be of minimal use in drawing meaningful conclusions about students' trigonometric schemas.

2.3.4 Developing trigonometric understanding

The basic ideas for trigonometry are all 'hard to teach and hard to learn' (Watson 2008) and often result in teachers taking the path of least resistance by adopting algorithmic approaches (eg Blackett 1990; Challenger 2009; Pournara 2001; Watson 2008). Many researchers have consequently found a general lack of relational understanding of trigonometry amongst high school and university students (eg Blackett 1990; Kamber and Takaci 2017; Watson 2008; Weber 2008).

A survey by Watson (2008) found that a similarity approach was the most popular — ie similar triangles have the same angles and by measuring, students find the sides are also in fixed ratios. The link between $\tan\theta$ and gradient is an equivalent starting point. An almost equally popular starting point is a functional approach whereby students fix the hypotenuse length and measure the length ratios for angles of 10°, 20°, etc. A few teachers use a multiplier approach, which is a variation on the similarity approach — sine and cosine are established as lengths of the legs of a triangle with unit hypotenuse and then scaled up or down as required. (eg Blackett 1990; Challenger 2009; Pournara 2001; Watson 2008).

A study by Kendal and Stacey (1998) compared the merits of introducing the trigonometric functions as ratios of lengths in triangles versus lengths in the unit circle.

The post-test showed that students who had been taught instrumentally performed better on questions that lended themselves to an algorithmic approach. The authors imply that questions were only asked about right angled triangles, which begs the questions as to how they were testing the merits of the unit circle approach (Watson, Jones and Pratt 2013). Perversely, Kendal and Stacey (1998) conclude by recommending the unit circle approach, which is supported by other researchers (eg Lakoff and Núñez 2000; Watson, Jones and Pratt 2013). Challenger (2009) found that it did not matter when technique students used for solving trigonometric equations — unit circle, graph, or otherwise — as long as they had a procedure that was reliable and which could be interiorised as a process. Once a reliable procedure has been interiorised, the individual steps are unimportant as the process is able to be reflected up as a whole and then encapsulated as an object.

Challenger (2009), who studied a group of A-level students, found that the more successful students were adept at mentally switching between algebraic and geometric images of trigonometry. These findings are supported by results in Pritchard and Simpson (1999) and Blackett (1990).

2.3.5 Summary

Trigonometry is a challenging topic at GCSE, and an important and even more challenging topic at A-level. Success at both levels, but at A level in particular requires students to view trigonometric functions as both ratios and functions, and as both processes and objects. A-level students must expand their schemas to see trigonometric functions as circular functions that can accept any real input, including inputs in radians, and operate with and on trigonometric functions in new ways. This requires students to have a good grasp of the underlying concepts of ratio and proportion, and functions; and the ability to readily switch between various representations of trigonometric functions.

2.4 Task design

Having identified many of the considerations in students' learning and understanding of trigonometry, a question that naturally arises is how can a teacher guide students to develop the necessary schema; to proceed from action to object; to develop a relational understanding; to interiorise, condense and reify the ideas. To explore all aspects of this question would be too extensive for this report, and so, without intending to diminish the importance of all other aspects of the learning process, task design, and specifically variation within task design was chosen as a focus.

Because of the wide range of mathematical concepts that contribute to trigonometry, it is important that students' attention is focused on those aspects that are most relevant (Mason and Johnston-Wilder 2004, p 60). There is discussion in the cognitive constructivist literature that a teacher's role is to *educate awareness* (Gattegno 1987; Mason 1998), although this is just a part of the rich tapestry of a teacher's job (Davis and Sumara 2002). One of the difficulties in a teacher's job from a cognitive constructivist aspect stems from the theory being one of individual learning — it offers an explanation for how learners learn, but not how to cause them to learn something in particular (Davis and Sumara 2002, p 413). Davis and Sumara (2002) continue, 'teaching is cast as a matter of back-and-forth efforts at *perturbation* (trying to prompt new ways of sense-making by interrupting established habits of interpretation)'.

Task design is one forum where such perturbation can occur that the teacher has some control over. Of course the learning activity that takes place from engaging in the task may differ from the teacher's proposed activity (Christiansen and Walther 1986), which reflects the *back-and-forth efforts* of Davis and Sumara (2002). The idea of perturbation has been developed into *variation theory*, the ideas for which have been around for over a century (Mason 2017; Watson 2017), especially in East Asian education (Gu, Huang and Marton 2004), but which has only been discussed in western literature since relatively recently (first published by Marton et al 2004).

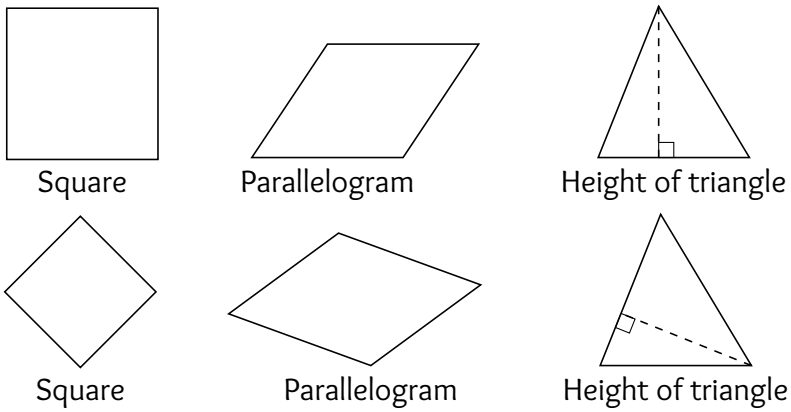


Figure 2.2: *Standard and non-standard figures. Using variation to highlight the essential features of some shapes.*

2.4.1 Variation theory

Variation is a necessary component in teaching in order for students to notice what is being learned (Kullberg, Runesson Kempe and Marton 2017, p 559). Put another way, Kullberg, Runesson Kempe and Marton (2017) and others claim that students will be unable to learn something until it is varied in some way. For example we can only recognise colours by seeing how they vary from other colours; we can only distinguish the number 3 by comparing it with numbers that are not 3 (Gu, Huang and Marton 2004). The use of standard figures can be helpful for illustrating concepts, eg figure 2.2. But unless the unessential features of the shapes are varied, students will be unable to distinguish the 'essential features of the concepts' (Gu, Huang and Marton 2004, p 317). This connects to the analysis of the mental conceptions of objects (Tall et al 1999, p 229) where a range of visual images contribute to the formation and understanding of mathematical objects (see section 2.2.3). The features of mathematical examples that can be varied depend on what a teacher wants to draw students' attention to. Figures 2.3 and 2.4 show examples from Watson and Mason (2006a, p 3) that demonstrate other contexts for variation.

The figure originally presented here cannot be made freely available via ORA because of copyright. The figure was sourced at Watson, A and Mason, J (2006a) Variation and mathematical structure. Mathematics Teaching 194, 3–5

Figure 2.3: *Examples of variation in linear graphs. Different features are emphasised depending on the amount of variation utilised.*

$$(x-2)(x+1) \equiv x^2 - x - 2$$
$$(x-3)(x+1) \equiv x^2 - 2x - 3$$
$$(x-4)(x+1) \equiv x^2 - 3x - 4$$

Figure 2.4: *An example of systematic variation in algebra.*

If the graphs in figure 2.3 are being used to show gradient, then the first graph shows too little variation, while the third graph show too much — the focus becomes the invariant point, which might be a useful image if students are studying point-slope form, eg $y - 2 = m(x - 1)$, but less useful for the concept of gradient.

The structured variation in figure 2.4 might be used to draw students' attention to the relationship between expanded and factorised forms. The vertical pattern can allow students to generate many more examples quickly, which can direct their attention to the structure and also encourages generalisation (see also Watson et al 2016). There is a 'danger' that examples like these are reduced in the learners' minds to number patterns and care must be taken to ensure that attention is paid to the structure (Watson et al 2016, p 15), but in this case the examples offer 'a way to deduce relationships and methods through observing and extending controlled patterns' (Watson and Mason 2006a, p 4).

In general, it is claimed that tasks with 'carefully constrained variation' are more likely to result in student 'progress' than unstructured sets of tasks (eg Kullberg, Runesson Kempe and Marton 2017; Watson and Mason 2006b). Using the language of section 2.2.2, a set of mathematical questions can be seen as an object in its own right, and variations within

it as properties (Watson and Mason 2006b). 'Exercises that offer controlled variation have the potential to lay the groundwork for rapid engagement with structure, relationships and properties' (Watson and Mason 2006b). By observing 'what changes and what stays the same', learners are more easily able to engage with a concept and can more readily generalise from the exercise compared with a set of randomly chosen problems.

2.4.2 Types of variation

Gu, Huang and Marton (2004) divide variation into two categories: *conceptual variation* and *procedural variation*.

The diagrams in figure 2.2 are an example of conceptual variation. Conceptual variation is concerned with illustrating a concept with multiple representations through varying all the features, both essential and non-essential features. The figure 2.2 examples show variation of some non-essential features. Varying the essential features will create counter-examples, which is an equally important element in conceptual variation (Gu, Huang and Marton 2004). A square can only be fully understood, Gu, Huang and Marton (2004) would suggest, by also considering shapes that are not squares. The idea of conceptual variation — multiple perspectives and experiences of mathematical concepts — is considered a useful approach to teaching trigonometry — students who are adept at mentally switching between representations of trigonometric functions are seen as having a better understanding of the concept (see section 2.3.4).

Procedural variation is associated with 'progressively unfolding mathematical activities' (Gu, Huang and Marton 2004), a stage-by-stage formation of a concept. Gu, Huang and Marton (2004) also use the idea of scaffolding to explain their categorisation. Procedural variation is concerned with the process of forming knowledge logically and sequentially so that the learner's understanding is well structured (Park and Leung 2006). The quadratic pattern in figure 2.4 could be the beginning of a set of procedural variation problems. By setting the expressions out in sequence, students are scaffolded to uncovering the general case of expanding $(x - \alpha)(x - \beta)$.

The literature is broadly supportive of the ideas of variation theory, but somewhat sceptical of the division of procedural and conceptual. The terms invoke thoughts of conceptual and procedural understanding/knowledge (eg Hiebert and Lefevre 1986; Star and Stylianides 2013), which is analogous the relational/instrumental dyad of Skemp (1976). The terms do not reflect well the meaning Gu, Huang and Marton (2004) attached to them (Park and Leung 2006; Watson et al 2016). Alternative terms such as *multi-dimensional variation* and *developmental variation* (Park and Leung 2006, p 83), and many others (Sun 2011) have been proposed.

2.4.3 Variation in trigonometry

Variation can be applied to any area of maths, including trigonometry. The importance of multiple representations for understanding trigonometric functions has already been discussed, and it is important to emphasise that the way variation is used is highly dependent on the intended focus of attention (Mason 2017). The following example is from Watson and Mason (2006b):

> If the intention is that learners should understand, say, the invariance of sine for a fixed angle, it makes sense to keep the angle and hence the ratio constant while changing other features so many times that the invariance of the ratio becomes more and more insistent and obvious in students' perceptions. The generalisation that 'opposite is related to hypotenuse in some way' (or equivalent expression in algebraic or diagrammatic form) then becomes a property of right-angled triangles to explore. The angle can now be varied, while other features are kept constant, so that the nature of the co-variation of 'opposite' and hypotenuse becomes more and more obvious — a property of angles that can eventually be named and discussed and compared to [sic] other properties.

Chapter 3

Research methodology

3.1 Selection of students

As discussed in chapter 1, this research investigates the learning and understanding of trigonometry by students in years 9 and 12 at St Frances's School for Girls. In order to address this question, two sets of students were chosen to be subjects in this research: one year 12 set, known as L6PMa1, and one year 9 set, known as U4Ma1; both are top sets in their years and were chosen because I was timetabled to be teaching them this year. For the purpose of making comparisons or conclusions about students more generally, it may have been preferable to use a wider range of students, but this was not possible.

L6PMa1 is a set of 10 girls who all obtained A* grades in GCSE maths (or equivalent). Three of the ten were students at St Frances's in year 11, the remainder were all new to the school in year 12. Of the new students, six are from East Asian countries, including China, Hong Kong, South Korea and Singapore. All students in the set had last year studied Additional Maths, a free-standing qualification popular amongst schools like St Frances's. Additional Maths contains many of the topics from A-level maths, including the introduction of trigonometric functions with unrestricted domains, trigonometric identities and solving simple trigonometric equations.

U4Ma1 is a set of 23 girls, ten of whom are new to the school this year. The students are a mix of overseas and local students and were placed in the set based either on their prior attainment at the school, or the results of their entrance exam. I knew from the occasional

comment overheard in lessons that a few students had encountered trigonometry before, but this experience was not widespread in the set, and with one exception, students had no experience solving problems or doing calculations using trigonometric concepts.

3.2 Collaborative work

One student outside these sets has been involved in the research. Student Y is a student in the parallel year 12 double maths set who was having difficulty with trigonometry. In collaboration with her class teacher, she became involved in the research because her teacher and I agreed that it would be worth trying the variation-based task with her in the hope that it would help her develop greater conceptual understanding of the functional nature of trigonometry (see also section 3.3).

In addition, discussions on various approaches to teaching trigonometry were held with other teachers in the maths department at St Frances's. The majority of discussions concerned the strongly held beliefs by other teachers on the superiority of the use of graphs or CAST diagrams for solving trigonometric equations. The strength of conviction of some teachers on this matter verges on a kind of trigonometric zealotry and the discussions were entertaining, if not entirely academically open minded.

3.3 Action research

Action research is a particular style of research that is common in projects of this type. Ary et al (2018, p 482) define action research as 'a form of research by practitioners to study a particular context and use findings to change practice'. It tends to be cyclical in nature as the research data is used to make changes to practice, which are then implemented in the next phase (see also Mertler 2009; Pelton 2010). This project consists of two cycles of data collection and analysis for which an overview is given here, but the details are contained in their respective chapters.

The project began with an intention to investigate and understand the way students' understanding of trigonometry evolved throughout the A-level course. The students of L6PMa1 attended their normal trigonometry lessons and were introduced to the relevant concepts in ways I have used with other students in the past. After a pre-test, normal classwork and a verbal task, it became clear that one student (who will be known as student Z) had weak grasp of the trigonometric functions as functions. At around the same time, student Y was also experiencing difficulties with trigonometry and was referred to me by her teacher. It was evident that the teaching methods and tasks I had used up to this point had failed to achieve their goals for student Z. And after discussing with her, it appeared that student Y's teacher had used a similar approach to me and thus any intervention designed for student Z would also be worth trialling with student Y. The theory of variation (see Section 2.4) was used to design the intervention task, known as the *unit circle task* in this report. The construction and implementation of the unit circle task is discussed in detail in section 4.3.

After implementing the unit circle task, I was encouraged by the feedback I received from the students and decided to make further use of the theory of variation in the next cycle of research with U4Ma1. The year-9 trigonometry task was written based on theory, and my experiences and feedback from the unit circle task. If time were available further cycles would be conducted. The details of the year 9 task, its construction, implementation and feedback are all contained in Chapter 5.

3.4 Ethical considerations and research limitations

This research explores the learning of a group of year-12 double maths students and a group of year-9 maths students at a girls' secondary school in England. The name of the school and the identity of the parties involved in the research have been anonymised for the purposes of this report. The purposes of anonymisation are to encourage participants to be involved in the research and to ensure their open and frank participation without fear of any reprisal or judgement by any third parties. Prior to their inclusion in the report,

students were advised of their right to anonymity and of their entitlement to withdraw from inclusion in the report prior to publication.

Despite the nature of the research and guarantees of anonymity, students were also informed that any data included in this report that would have been gathered in the normal course of their education would be shared amongst the school community if such sharing would ordinary practice and for their benefit as students. For example, information about the trigonometry knowledge and understanding of the one student who was not taught by me was shared with her normal class teacher because, beyond gathering data, as educators we have a duty to ensure the academic progress of our students and in this case it was considered important to discuss the girls' progress in trigonometry and also to discuss relevant ideas regarding teaching of trigonometry in general.

The majority of the data collected for this research would have been collected anyway in the normal course of teaching. It is generally considered good teaching practice to assess students or otherwise determine their level of knowledge so from the students' perspective, their trigonometry lessons were the same as any other lessons. Consequently they were not given an option to not sit the tests. The one piece of data that was unique to this research was the verbal task. In this case all students chose to participate after being informed that such participation was voluntary.

After data was collected, it was either scanned and returned to students, or kept in paper form in a locked filing cabinet at school. All electronic documents, which include scanned students' work and this report were stored on the school network, in a directory that is password protected or on my own computer, which is accessible only by me.

Approval for this research was considered and granted by the University of Oxford Central University Research Ethics Committee, and by St Frances's headmistress on behalf of the school.

The outcomes and conclusions of this research should be considered in the context of the school and the participants. St Frances's is an academically selective school — each year over 90% of students obtain A* grades at GCSE and over 40% A* at A level. All students in

this study are in the top sets in their year group, and accordingly are extremely competent mathematicians by national standards. All of the sixth form girls in the study had learnt A-level trigonometry in the previous year, and so their understanding of it and approach to the topic was coloured by their past experiences. This is not necessarily a negative aspect of the study, but as the literature notes, the formation of our knowledge structures is a function of all that has come before; accordingly the students learnt trigonometry in a way that is different from if they had been learning it for the first time.

The progress made in understanding triangle trigonometry by the girls in U4Ma1 is not generalisable. The students had previously demonstrated the capacity to learn many new topics to a very high level of competency and the likelihood is that no matter how well the material was introduced to them, many of them would have been able to form the requisite conceptual knowledge from their already strong grasp of the subject and their pre-existing capacity for logical mathematical thought.

Nevertheless, these were the students I was assigned to teach this year, and through their involvement in the research I have been able to reflect on various aspects of my teaching that will help me develop as a maths teacher for future years. The variation tasks that were developed during this study will need to be adapted if they are to be of use for very weak students, but the principles involved in the task design could be expected to apply to any students, regardless of their prior attainment.

Chapter 4

Cycle 1 — Year 12

4.1 Pre-test

4.1.1 Pre-test method

Before any teaching of trigonometry had taken place, a pre-test was undertaken by the girls in L6PMa1 to determine their knowledge of trigonometry from previous years. Since all of them had studied Additional Maths and most of them had studied at other schools in the previous year, it was deemed to be a good idea to determine their level of knowledge of trigonometry and to observe what methods and techniques they had at their disposal. The full text of the pre-test is contained in Appendix A.

The questions were designed to assess the students' ability to solve trigonometry problems ranging from basic GCSE-level right-angled triangle and non-right-angled triangle problems up to simple A-level-style problems that utilise the knowledge they should have gained in the previous year. The final question would usually require the use of the compound angle formula, and was included to determine whether any students had encountered the concept before. The problem could also be solved using derived knowledge from question 13, although it was not expected that students would make this connection. Following Pournara's (2001) classification of trigonometry knowledge, I wrote questions to attempt to determine whether students held a ratio or functional view of trigonometry and in either case whether they held process or object views. This was done

by asking similar questions in different ways, such as giving the values of $\sin$ as fractions or decimals, asking students to determine the value of $\cos\theta$ or the maximum value of $p\sin\theta + q\cos\theta$. The view of trigonometry as functions was primarily assessed in questions 13 – 22. Awareness of the infinite nature of the domain of cosine was tested in question 14 and the relationship between trigonometric functions was the focus of many of the last ten questions.

4.1.2 Pre-test results

With ten students each doing twenty-two questions, there is a lot of analysis that could be done of the pre-test. The number of students answering each question correctly is presented in Figure 4.1, and below I discuss some of the common themes of the students' work.

The questions in the pre-test can be coded by the type of view of trigonometry that is required in order for a student to make sense of them (Pournara 2001). The bars representing the number of correct answers to each question have been coloured in

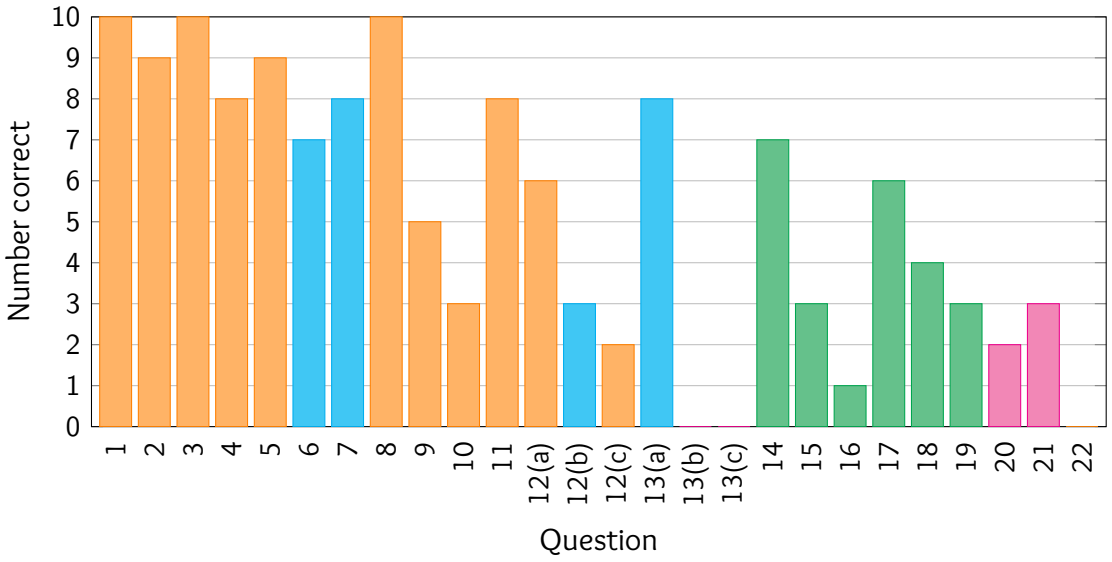


Figure 4.1: *Number of students (out of 10) who answered each pre-test question correctly.*
Orange: ratio orientation, process view.
Cyan: ratio orientation, object view.
Green: functional orientation, process view.
Magenta: functional orientation, object view.

Figure 4.1 to represent each question type, although in many cases it is difficult to single out one particular orientation/view for each question. Discussion regarding the distinction between ratio and function orientations and action, process and object views of trigonometry is contained in section 2.2.3. Questions 5 and 6 are good examples of the difficulty in assigning labels to the questions. Question 5 is 'write down the angle whose cosine is 0.1805'. The quantity 0.1805 is the value of $\cos\theta$, and would be considered an object, but the procedure for calculating the angle is a standard process for finding angles in triangles. It was felt that students would likely be able to find the angle without necessarily seeing $\cos\theta$ as an object, and hence the assignment of 'process' to this question. Contrastingly with question 6: 'draw a triangle for which $\cos\theta = \frac{3}{5}$' (and also question 7). Students were given a value of $\cos\theta$, which again is an object, and asked to do something with it. This question involves a combination of object and process views and probably could have been designated either way.

Question 12(b) required students to prove an equation in the form $\tan\theta = \frac{1}{2}(\tan\phi + \tan\psi)$. An equation comprised of trigonometric functions definitely requires an object view of trigonometry, so even though the proof required only two lines of working from the result in (a), very few students were able to complete it satisfactorily. The results suggest that, within the ratio orientation, students in L6PMa1 tend to hold more of a process view of ratio trigonometry than an object view.

Within the process questions, it was evident that some students had not interiorised the action of computing with sine, cosine and tangent. Question 4 gave a picture of a 5/12/13 triangle and asked students to write down the value of $\sin\theta$. Two students calculated the value of θ, instead, which is an indication of the lack of control over the process (Cottrill et al 1996). Alternatively, it may be that the students do not have an object view of $\sin\theta$ and instead see it as a value to be used in a calculation, in this case for finding an unknown angle. The students in question also made mistakes in questions 6 and 7, which lends weight to the latter theory. An equivalent misconception was observed in the Year-9 post-test (see Section 5.1).

From question 13(b), the questions switch from ratio to function orientation. Function orientation implies that the values of sine etc may be variable, that sine should be seen as something that converts input angle to output ratio/length, that sine accepts angles outside 0° – 90°, or that an output value of sine may have multiple input values. The variable nature of sine and cosine is raised in 13(b) and the results show clearly that students definitely lacked a functional object view of sine and cosine. Throughout GCSE, trigonometric functions tend to be seen as static quantities, so when presented with one that was varying, students did not have the cognitive structures in place to handle that thought process. Even though they all had a rectangle of paper on their desk that they could have used for an experiment, which would have given them the result immediately, none of them thought to do that. Other opportunities for holding an object view within a function orientation in questions 20–22 also showed that most students had not developed the required understanding. This result is not surprising as one of the points of the A-level trigonometry course is to develop this broader understanding of trigonometry, and at this stage none of the students had learnt any trigonometry at A-level.

Questions 14–19 involved computations or processes with a functional orientation and some students showed that they could recall some of their knowledge from Additional Maths to solve, eg $\cos\theta = -\frac{1}{2}$ or sketch $y = \sin x$. Only one student showed an awareness that the domain for $\cos\theta$ is unrestricted (question 14) and only one student had a concept of $\tan\theta$ that allowed her to solve $\frac{1}{\tan\theta} = 0$ (question 16). Most students struggled with question 16 at least partly because their understanding of fractions was not developed enough. Three students attempted to solve it such that $\frac{1}{0} = 0$.

Preliminary conclusions that can be drawn from the pre-test are that students generally had a process view of trigonometry and were more familiar with considering trigonometric functions as ratios than functions. This is not at all remarkable considering their stage of education. There is much more analysis that could be done of the results from the pre-test; however, for most of the remainder of this report, I will consider the trigonometric understanding of two students in particular: Y and Z (see section 3.3). Student Z is a year 12 student from L6PMa1 (taught by me) and student Y is in the same year from the

parallel set, taught by another teacher. Y did not undertake the pre-test, but Z did and showed similar trends to the others in the set, albeit a bit more extreme. Z did not attempt questions 5, 6, 12, 13, 21 or 22. She was aware of the graphs of sine and cosine, and the cast diagram, which is a tool used by some students for solving trigonometric equations, but was generally unable to use them to solve the problems.

Some interesting aspects of the students' trigonometric knowledge was revealed by the pre-test but ultimately there is a limit to the insight into students' thinking that can be gained through a written test (see section 2.2.1), so the next stage of data collection was to ask students to complete a set of verbal tasks.

4.2 Verbal task

4.2.1 Verbal task method

The verbal task was undertaken as a set of one-to-one meetings with the students who chose to participate. The purpose of the questions was to delve more deeply into the students' understanding of trigonometry than was possible in the pre-test by asking them to explain their reasoning or give greater detail to their answers, or in some cases to follow up their answers with further questions. The exact questions and format were unique to each student, but the general structure was as follows:

1. (a) What does $\sin\theta$ mean to you?

   (b) Can you draw a picture of $\sin\theta$?

   (c) Can you draw/write any other representations of $\sin\theta$?

2. Without a calculator, estimate $\sin 160°$.

3. Without a calculator, what is the exact value of $\cos 30°$?

4. Solve $\cos\theta = \frac{1}{2}, 0 \leqslant \theta \leqslant 360°$.

   Explain how you solved it.

   Illustrate your solutions on the unit circle.

5. Which of the following are equivalent? [Cut-out cards containing the following expressions were given to the students to arrange on the table with the instruction to group any that are equivalent, and with the caveat that there may be groups of more than two expressions and there may be expressions that have no equivalent on the cards.]

   $\sin x$ $\cos x$ $\tan x$ $\sin(x - 90°)$ $\sin(90° - x)$ $2\sin x$

   $\cos(x - 90°)$ $\sin 2x$ $\sin(x + 1)$ $\sin(x + 90°)$ $\sin x + 1$

   $1 + \sin x$ $\dfrac{\sin x}{\cos x}$

Question 2 is based on the work in a number of the theory papers that suggest that an ability to estimate the result of a process is evidence that the knowledge is stored as a process (Dubinsky 1991; Mason and Johnston-Wilder 2004) or procept (Cottrill et al 1996; Sfard 1991; Weber 2005) (see Section 2.3.2). Weber (2005, p 107) asked a similar question in his study of students' understanding of trigonometric functions, in order to assess the proceptual knowledge of trigonometry by groups of students. He found that students in the control group were less capable of accurate estimates than students in the experimental group, although students in the experimental group had previously completed estimation tasks in their lessons. The students in my study had not been asked

to estimate the value of any trigonometric functions prior to the verbal task. The explanation of how each student formed their estimate would be likely to provide an insight into their dominant representation of $\sin\theta$.

Question 5 comes from Challenger (2009, p 149), who asked it of some year 12 students as a way of testing their knowledge of trigonometric graphs, but perhaps in a way that might be unfamiliar to them.

Throughout the verbal task, students were encouraged to write or draw anything that might be helpful to them in answering the questions. If students were stuck on any questions, I provided minimal hints in order to help jog their memory if I thought it appropriate. The verbal tasks took place approximately six lessons into the trigonometric topic. Since beginning the topic, students had had the opportunity to revise any trigonometric knowledge they had forgotten since the previous year, and had had the opportunity to consider a range of different representations of trigonometric functions through classroom discussion and worksheet exercises. The verbal task therefore represented an opportunity for me to assess how the use of multiple representations in lessons had translated into students being adept at using multiple representations in their thinking — an important factor in students' conceptual understanding of trigonometry (Blackett 1990; Challenger 2009; Pritchard and Simpson 1999).

I made written notes during the verbal task of comments made by me and the students. The students also wrote down their ideas for each question, so the combination of both sets of notes provide a reasonable record of the task. A video or audio recording of the task would have provided a more comprehensive record but my experience has shown that this also results in constraints on thinking and discussion by both me and the students. The students were already more nervous than usual because of the artificial nature of a one-to-one verbal task, so it was felt that adding an extra level of formality by recording it would be counterproductive to gauging the students' thoughts and knowledge in as detailed and relaxed a manner as possible.

There are different styles of clinical interview, each of which is suitable in certain circumstances, depending on what the interviewer wishes to achieve. Writing from a Piagetian perspective, Ginsburg (1981) identified three separate purposes for interviews and observed that different objectives require different techniques from the interviewer. If the interviewer wishes to **discover** the 'cognitive processes' of the student then Ginsburg (1981) recommends setting an initial open-ended task followed by questions that are contingent on the student's comments. Standardised tests are of 'limited value' for discovery interviews involving individuals, although may have some benefits for discovering trends within groups. **Identifying** cognitive structures is the second possible aim for interviewers. The difficulty with this goal is that it requires the interviewer to reflect on the student's answers and make quick in situ judgements in order to ask useful contingent questions — something which I suggest may not be possible in the heat of the moment. The third aim is to assess **competence**, which Ginsburg (1981) defines as a combination of motivation and understanding. Having identified three categories of clinincal interview objective, Ginsburg (1981) then concedes that the boundaries of the objectives are not distinct and that any interview may overlap all three. It is therefore of questionable value to categorise clinical interviews in this way, but Ginsburg (1981) does identify a common thread amongst all three objectives, which is contingent questioning, and that is what distinguishes a clinical interview from a standardised test.

In view of my objective of identifying cognitive structures and processes, I determined that a clinical interview, rather than a standard test, would be most suitable. A standard test had already been undertaken, so the verbal task should be complementary to that. I used a combination of open and closed questions that were predetermined, but then follow-up questions that were contingent on the students' initial responses were asked that resulted in each interview being unique, but still following a similar structure.

All students in the study completed the verbal task, but only the results of students Z and Y will be reported, as these led to the follow-up unit circle variation task.

4.2.2 Verbal task results

Student Z

In response to the first question, 'what does $\sin\theta$ mean to you?', Z replied 'to find $\sin\theta$, you divide the opposite by the hypotenuse'. This response emphasises her view that the trigonometric functions are processes — means of figuring out lengths or angles. When pressed for other things $\sin\theta$ might mean, she also referred to the cast diagram and graph. Figure 4.2 shows an extract from Z's answer to the meaning of $\sin\theta$. It is clear from Z's pictures of the cast diagram and graph that both representations are used for doing calculations, especially for finding unknown angles. She does not necessarily see $\sin\theta$ as an object in its own right.

When asked how the ratio representation was related to the graph, Z attempted an explanation relating to the cast diagram, but it quickly became clear that she did not know. When I asked her 'is sine a function', she answered

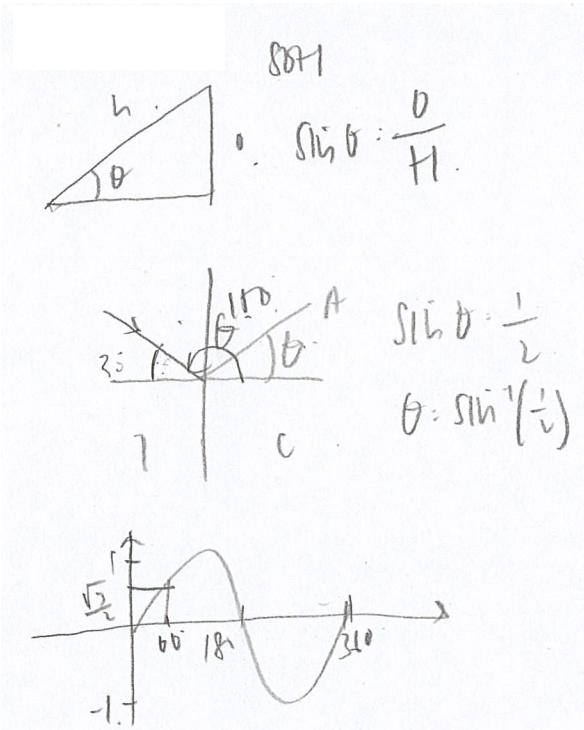


Figure 4.2: *Z's drawings in answer to the question 'what does $\sin\theta$ mean to you?'*

> Yes, because you can represent it as a graph; and you can transform the graph, like other functions.

Z's estimate of $\sin 160°$ was 0.25. Her reasoning was that $\sin 135° = 0.5$ and $\sin 180° = 0$, and 160° is approximately halfway between 135° and 180°. This misconception of sine as a linear function was surprisingly (to me) common amongst the girls in the set. In Z's case she held this misconception despite having drawn two (clearly non-linear) sine curves on the page in front of her. This misconception has not been reported in any of the literature, other than Gooya and Rabanifard (2008), although as noted in Watson (2008), trigonometric functions are amongst the first non-linear functions that students spend any time exploring (see section 2.3.1), so it may be the case that Z initially formed that misconception and is still struggling to overcome it.

In solving $\cos\theta = \frac{1}{2}$ for $0 \leqslant \theta \leqslant 360°$, Z correctly obtained the solutions using the 30/60/90 triangle for the principal solution and the cast diagram for the other solution. When asked to illustrate her solution on a unit circle, Z said she did not know how to use the unit circle.

Even though at a simplified level the unit circle is just a cast diagram with a circle drawn over it, it represents the connection between the ratio and functional orientations for the trigonometric functions and in my view having an understanding of it is fundamental to a conceptual understanding of trigonometry. This view is not necessarily shared by all members of the maths education community. Kendal and Stacey (1998) found that students performed better when taught a ratio approach, rather than unit circle, for solving basic triangle trigonometry problems, which is reasonable as the unit circle adds a layer of complexity to the simpler triangle approach. However in order to move to a functional understanding of trigonometry, that extra layer of complexity is necessary — the trigonometric functions are defined in terms of the unit circle, and knowing this helps students understand and predict their behaviour (eg Bressoud 2010; Kendal and Stacey 1998). Other teachers in the maths department at St Frances's are satisfied with students

knowing the triangle definitions and the graphs of the functions, but not necessarily the connection between the two.

In order to see if a stronger grasp of the concept of the unit circle would be useful for Z, I wrote a set of problems that were designed to help her see the features and symmetry of the unit circle and perhaps incorporate it into her trigonometry schema. The problems utilised variation theory in order to help focus Z's attention on the important aspects of the unit circle (see section 4.3).

Student Y

Student Y gave an equivalent answer to Z to explain what $\sin\theta$ meant to her. Her first answer was $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$. She also drew a picture of a cast diagram and fairly accurate graphs of sine and cosine (see figure 4.3). She did not mention the unit circle at this point, although she later drew a unit circle to explain question 4.

Her estimate of $\sin 160°$ was fairly accurate — reasoning that $\sin 150° = \sin 30°$, which she eventually worked out was $\frac{1}{2}$. And from looking at the graph, Y could see that

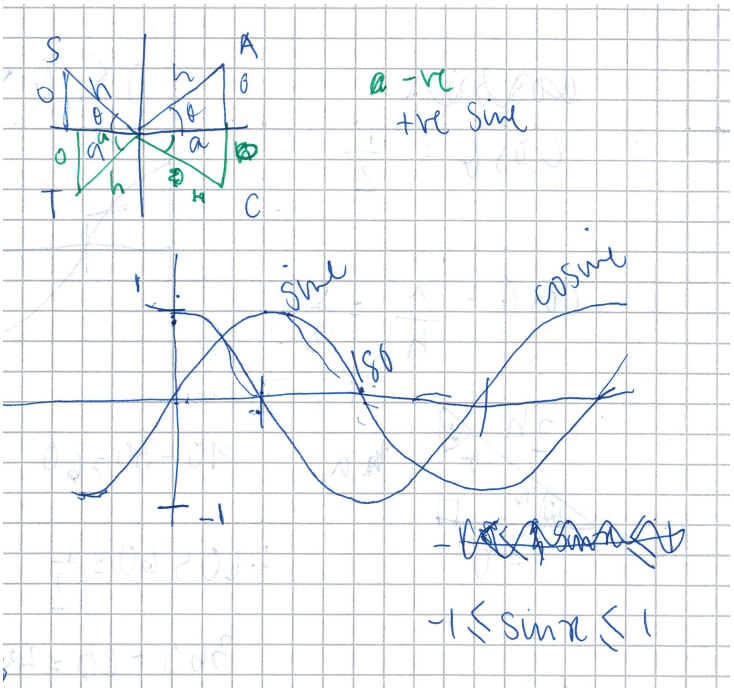


Figure 4.3: *Y's drawings in answer to the question 'what does $\sin\theta$ mean to you?'*

$\sin 160° < \sin 150°$, and so she estimated 0.3, which is correct to the nearest tenth. The ability to accurately estimate the outcome of a procedure is an indication of an interiorised action, namely a process (Dubinsky 1991).

When asked to solve $\cos\theta = \frac{1}{2}$, Y was noticeably less confident. She initially appeared very confused about the idea of inversing the cosine function, and although she eventually obtained the correct answer and was able to draw her solutions on a graph of cosine and on the unit circle, she was unable to explain the connection between these representations and the right-angled triangle representation. The ability to reverse an action is another indication of the formation of a process (Mason and Johnston-Wilder 2004), and Y's comments were suggestive of not having fully interiorised the trigonometric actions.

With questions 2 and 4 giving conflicting results, I asked her to find where $y = 2\sin(2x + 150°)$ crosses the y-axis. On her first attempt she found two x-intercepts (by setting $y = 0$), not realising that they were not y-intercepts. I pointed out to her what she had done, and she corrected her mistake. Her solutions for the x-intercepts were a correct procedure, but I wanted to explore further. I asked her to find all the values of x between 0° and 360° for which $y = \sqrt{2}$. This time she did not draw a graph, but began with an algebraic solution: see figure 4.4.

In her solution, you can see correct algebra up to the point where understanding of the functional nature of trigonometry is required. Y finds a principal solution of 45° using an isosceles right-angled triangle, but then incorrectly assumes that solutions are equally spaced around the circle, 90° apart. Initially she only found two solutions, 37.5° and 127.5°, but without prompting realised there must be two more, 180° more than her first two. She explained that the existence of four solutions was due to the amplitude being 2. Thinking I may have misunderstood her explanation, I asked how many solutions we would find for $y = 3\sin(2x + 150°)$. She said six. After a little prompting, she attempted to correct herself and said that they would be the *same* four solutions as for $y = 2\sin(2x + 150°)$. She is likely mistaking this problem with one of finding roots, which would not change as the amplitude changes.

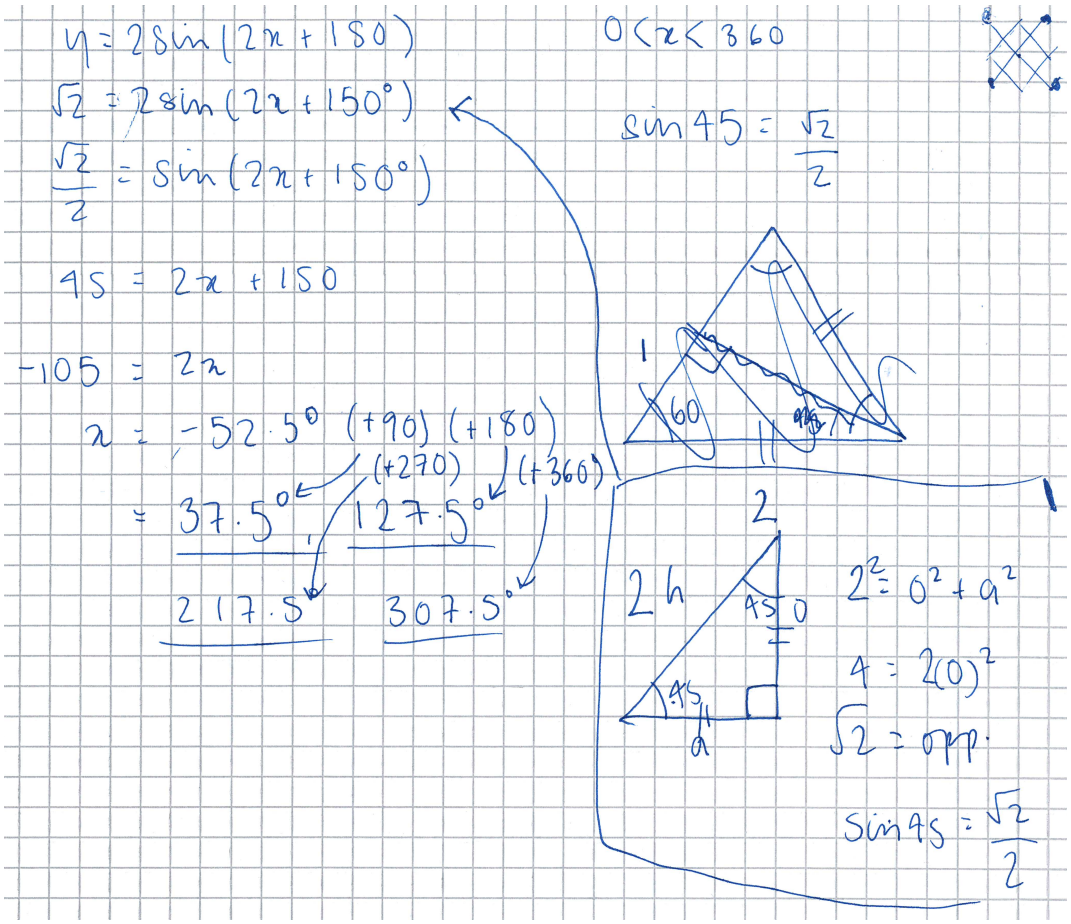


Figure 4.4: *Y's solution to 'for $y = 2\sin(2x + 150°)$, find the values of x ($0 \leqslant x \leqslant 360°$) for which $y = \sqrt{2}$'.*

In response to the question 'is $\sin\theta$ a function?', she said yes and explained that behind the word *sine* there is an algorithm/equation such that we put in angles and get out values between –1 and 1. She added that every x value has one y value.

It is difficult to analyse Y's trigonometric knowledge in the context of APOS theory. She has a clear knowledge of the definition of function and trigonometry's place within that definition, but her view of trigonometric processes/objects has flaws in it. She does not fully understand the way $\sin\theta$ as an object behaves as θ changes, which leads to mistakes when executing the process of solving trigonometric equations. I asked Y to complete the written unit circle variation task, as I thought that working with the unit circle tasks would give her cognition an opportunity to repair some of the misconceptions it held regarding the trigonometric functions.

4.3 Unit circle task

4.3.1 Unit circle task method

Following lessons on the unit circle definitions of sine, cosine and tangent and the verbal task (section 4.2), two students were identified as not having a clear understanding of the relevant concepts. In order to address this shortfall in understanding, a written task was designed to lead the students through a set of structured exercises that it was intended would help them understand the nature of the trigonometric functions as periodic functions and mathematical objects. The worksheet is contained in Appendix B and is composed of two sections — a graphical solving section and a calculator solving section. Prior to doing the sheet, students were aware from lessons that any point on the unit circle had coordinates $(\cos\theta, \sin\theta)$, and that the gradient of the radius was equal to $\tan\theta$.

Questions 1 – 3 required students to find points on the unit circle using their protractor to measured an angle and write down the coordinates of the point and the gradient of the radius to that point. They were then asked to write down the values of sine, cosine and tangent for the given angle. The structure of questions 1 – 3 was the same, with only the angle changing. By keeping everything but the angle constant, I intended to emphasise that the trigonometric function definitions do not vary as the angle varies. The angles chosen were 130°, 230° and 305° so that students experienced angles in all quadrants. Question 4 reversed the order of previous questions: students were required to mark a point on the unit circle given its sine and cosine. The reversal of order was intended to reinforce the unit circle definitions of the functions. Questions 5, 6 and 7 were similar, but only one trigonometric value was given, thus introducing students to the idea of one-to-many functions and multiple solutions. Questions such as 'what do you notice' were included to encourage students to reflect on their answers —- an important aspect of learning through variation tasks (Watson 2017, p 90).

Having completed the graphical section, students were then asked to solve equivalent problems using their calculator to find the principal angle. In each case, students were still required to identify the locations of the solutions on a little unit circle picture, but the numerical values would come from the calculator instead of reading off a scale. This small step from reading off a scale to reading off a calculator screen, while still connecting the solutions to locations on the unit circle was intended to reinforce the role of the unit circle in the values of trigonometric functions.

Finally a few pictures of unit circle solutions were drawn and students were asked to suggest appropriate equations that the pictures might be being used to solve. This incorporates the idea of doing and undoing, which is an important part of developing mathematical thinking (Mason and Johnston-Wilder 2004, p 193).

Prior to undertaking the task, students had participated in class discussions where the details of the unit circle were discussed, and dynamic representations showing the behaviour of sine, cosine and tangent for variable θ were demonstrated using Geogebra. The Geogebra file was available to student Y when she completed her unit circle task.

4.3.2 Unit circle task results

The full working for both students' unit circle tasks are in Appendix F.

Student Y

Y showed a good awareness of the connection between sine, cosine and tangent, and the unit circle until the very last bit of part A. She correctly marked the points where $\tan\theta = -0.5$, and correctly wrote down the corresponding values of $\sin\theta$, but (c) asked her to 'work out $\cos\theta$', which should just involve reading off the x-coordinates at her two points, but instead she has written down $\frac{\sin}{\cos} = \tan$ and incorrectly rearranged it, obtaining $\cos\theta = \pm 0.22$ instead of ± 0.9. It is always difficult to know if any given mistake is due to some misconception or carelessness, but considering she has all the other answers

correct, it is probably just carelessness. The use of the phrase 'work out', instead of 'write down', may have made a difference, too.

In part B, she has correctly marked the locations of all the solutions on the unit circle, which was my main objective for the exercise. She did not know how to handle adjusted domains (questions 7 and 8), and so we discussed them and did some other examples after she submitted her work.

Student Z

Z obtained a nearly full set of correct answers. In part A, she explicitly noted the symmetry of her solutions for any constant value of sine, cosine or tangent and the connection between the solution angles in each case. In part B, there are three observations:

- She has an incorrect answer to question 5. She has correctly marked all three solutions on the picture, but despite her picture clearly showing $|\theta_3| = \theta_1$ and $\theta_3 > 90°$, she has written $\theta_3 = -(180° - \theta_1) = -46°$. If she thought about this for a moment she would realise her mistake.
- Question 7 requires solving $\sin 2\theta = \frac{1}{5}$. In all her other questions she has marked the solutions roughly to scale, but in 7 her solutions are marked for $\sin\theta > 0.5$. Contrast with question 2, which is $\sin\theta = -0.2$.
- Question 8 has no solutions within the domain, which Z has worked out, but she drew a graph of $y = \cos\frac{1}{2}\theta$ to satisfy herself of this. This suggests two things: she is readily able to switch between representations, a vital skill in understanding trigonometry (Challenger 2009), and she does not fully trust the unit circle representation even though she has just used it to solve seven other problems correctly.

4.3.3 Summary of unit circle task

The purpose of the unit circle task was to make explicit the important features of the unit circle and their connection to solving trigonometric equations. The use of variation within the task was to enable the students to focus their attention on the important aspects of the concepts without a teacher needing to be present. Both students demonstrated that they recognised the connection between trigonometric equation and location of solutions, even if they did not always find the numerical value of the solution.

Both students reported afterwards that they felt more confident in the meaning and purpose of the unit circle approach to solving equations, so to some extent the task achieved its aim. Further work is required to determine whether the activity has made any significant change to either students' conceptual understanding of trigonometry. Across questions 1 – 3, the only varying quantity was the angle. Within each question, however, many items were changing and all three base trigonometric functions were used. In addition, each angle was in a different quadrant, which may have led students to miss some details about how each trigonometric function was changing. Perhaps it would have been more beneficial to focus on one trigonometric function at a time to reduce the amount of variation to which students needed to attend. For example, if students drew angles and worked out coordinates at 10° increments, then it may create stronger trust in definitions and also emphasise the non-linearity of trigonometric functions — something which is evidently required for student Z (see section 2.3.2).

My reflections above contributed to the design of the year 9 trigonometry task (see section 5.1), which was designed to introduce trigonometry to students ab initio. The unit circle task had a different purpose as it was designed for students who had previously experienced the concepts in lessons and textbook exercises. It may have been expected that A-level double maths students would be able to handle a greater degree of variation within one activity.

4.4 Post-test

4.4.1 Post-test method

At the completion of the sequence of lessons on trigonometry, all students were given a standard test to determine their knowledge of the concepts and ability to solve mostly non-standard problems. The test tried to cover all elements of the course in some way, which included compound and double angle formulas and inverse trigonometric functions. The test questions are shown in Appendix C.

4.4.2 Post-test results

Due to timetabling constraints, student Y was unable to take the post-test. The questions were designed to assess students' understanding of the main trigonometric concepts in the A-level course, rather than be standard exam questions, and hence were challenging for most students in L6PMa1. Student Z's post-test answers can be seen in Appendix G. She showed a big improvement in her ability to hold a functional view of trigonometry. Question 3 was intended to identify whether students held a process or object view of solving trigonometric equations. Student Z appears to be somewhere in between the two. She began by trying to actually find all of the solutions, which is one way of answering the question, but was not required. After correctly finding all of the solutions for $0 < \theta \leqslant 2\pi$, she reasoned from the symmetry of the sine function that there must be the same number of solutions in $-2\pi < \theta \leqslant 0$. She drew a small picture of the unit circle in the margin next to her working, although it is unclear whether she used this to help her answer the question. Someone who holds an object view of $\sin\theta$ would be more likely to be able to write down the number of solutions without doing any working.

Question 4 was answered well — it is a very procedural question, but with a twist that the auxiliary angle is obtuse. Z showed an awareness of the overlap of positive sine and negative cosine to identify the correct angle. She used a cast diagram in part to answer

question 5, which she answered correctly although it appears she did not consider the changed domain of solutions after she changed variables. In question 6, she fell into my trap of cancelling $\cos^{-1}$ and $\cos$. Despite writing the range of $\cos^{-1} x$ above it, she then gave a value outside the range. The functional aspect of inverse trigonometric functions are new to the students at A-level, and it appears that Z has not interiorised that yet. It is likely that she ignored her work in (a) and applied her existing knowledge of functions: $f^{-1}f(x) = x$.

Question 7 was intended to test the students' knowledge of linear transformations of trigonometric functions. More typically students draw graphs from given functions, so asking the question in reverse was intended to assess whether they understand transformations as process or action (section 2.2.2). Z showed some good reasoning and thoughts on this question, identifying the amplitude and phase shift, but possibly due to the fractions involved she was mistaken about the angular frequency.

Overall the post-test shows that Z has developed her view of trigonometry from the original pre-test. She has interiorised many of the actions and now has a greater tendency to view trigonometry functionally. This development is not necessarily due to any particular teaching techniques, and is most likely just the consequence of attending A-level trigonometry lessons for a term. Earlier studies have reported that development of object and functional views of trigonometry are a necessary part of understanding at A-level (eg Challenger 2009) and although Z still has some scope for further development, she has made progress.

Chapter 5

Cycle 2 — Year 9

5.1 Year-9 trigonometry task

Following the relative success of the unit circle task (Section 4.3.3), I took the concept of variation to the learning of trigonometry by U4Ma1, the year-9 set. After reflecting on the reports that trigonometry is typically introduced with definition first, followed by exploration, and how this didactic reversal can be problematic for students (see section 2.3.1), and taking inspiration from Morgan (2014), I designed a series of exercises that began with similar triangles and led through to the concept of trigonometry. Trigonometric tables were used instead of calculators so that students did not have to worry about the notions of sine, cosine and tangent. Once students had become familiar with using the tables to look up required ratios or reverse lookup for angles, which took approximately two weeks of teaching, then the names sine, cosine and tangent were introduced and calculators became available for 'looking up' ratios. The worksheet is contained in Appendix D.

The variation from task to task is much more gradual than the unit circle task and initially only tangent was introduced and explored. The year-9 task began with a review of the concept of gradient of straight lines. Students had previously studied straight line graphs and were familiar with the idea that gradient represents the rise of a straight line for a given run. I used this idea to develop the concept of similar triangles (which St Frances's scheme of work provides for them to learn next year). The idea discussed was that the

gradient of a straight line is the same everywhere on the straight line. Most students were now able to complete Exercise 1. A few students found it difficult to deal with the 3/4/5 triangles in question 2, particularly when the scale factor was non-integer. Although I wanted students to find their own way to work out the side lengths, I helped a couple of them by reminding them that similar triangles had the same gradient, so they can write something like $m = \frac{4}{3} = \frac{5}{k}$.

After students had experienced the idea of similar triangles, I introduced the words *opposite* and *adjacent* to replace rise and run so that triangles could be drawn on any orientation. Exercise 2 is then equivalent to Exercise 1, but allows students the opportunity to familiarise themselves with identifying opposite and adjacent sides. In all questions, triangles were drawn to scale to help students identify which sides related to which in the similar triangles. Question 1 gives an example triangle in blue — identical to Exercise 1 — and students calculate the unknown adjacent or opposite, exactly as they do in Exercise 1. In question 2, the example triangle is replaced with $\frac{\text{opp}}{\text{adj}} = \frac{3}{2}$. Students were expected to notice that this is exactly analogous to the previous question and then proceed as before with their calculations. Questions 3 and 4 are equivalent to 1 and 2, but the ratio of opposite and adjacent is given as a decimal to help them move towards developing an object view of tangent within a ratio orientation (Pournara 2001, p 32). Pournara (2001) cited potential difficulties for students in giving the ratio as a decimal as it does not relate to two sides of a triangle as obviously as a fraction, but the students of U4Ma1 were very competent mathematicians for their age, with a good grasp of the concept of number, and so were not troubled by the ratio in decimal form. For the purpose of the exercise, it was important to move from fraction to decimal because ultimately they will be using their calculators to compute tangent ratios and so this exercise was intended to help them understand the meaning of the number their calculator gives them. The end of Question 4 asks students to reflect upon their work and to think further about tangent as an object (Watson 2017, p 90). With no guidance from me, all students were able to identify that multiplying by the ratio would calculate the opposite side, and dividing would calculate the adjacent side.

Students were now introduced to the table of tangents, and after a short explanation of how to read the table, students readily completed the tasks in Exercise 3. A couple of wordy questions and multi-stage questions were included to give students an opportunity to apply their new skill to more interesting problems. Some students had no trouble with the difficult problems, others needed more guidance/scaffolding.

All calculations had so far been done with the legs of a right-angled triangle. Now the hypotenuse was introduced and students had to choose which table to look up. This did not prove any impediment and students readily completed Exercise 4.

I gave no guidance at all as students began Exercise 5, and as expected, students required no guidance. They thought briefly and realised that they could use the table in reverse to find out unknown angles given two sides. The questions had been carefully made such that the exact ratios appeared in the table.

It was only at this point that I introduced the names of the ratios. Sine, cosine and tangent were explained merely as the names of the tables, and now students could use their calculators to look up values in the table by giving the name of the table, and the angle they wish to look up the ratio for. After an example of the kind of working one might be expected to show, students complete Exercise 6, although I asked them to answer Question 2 at the end as its position disrupted the flow from use of tables to use of calculators.

Finally the idea of inverse functions was introduced. Students were already familiar with the idea of a reverse lookup in the tables, and so this was again not a big step for them to take. Apologies were made for the awful notation (eg $\sin^{-1} x$). Again students had no trouble utilising their calculators to do the reverse lookup, although later in the term, when some students began referring to $\sin^{-1}$ as 'sine minus one', it became clear that not all students had fully understood the concept.

Exercise 8 was included to give students an opportunity to explore some of the connections between sine, cosine and tangent, especially the complementarity of them. In hindsight, I needed to expand on this exercise, as it jumps too quickly into the object view

of the trigonometric functions, and most students were not ready for it. It was undertaken a week after the previous exercise, so there was some degree of forgetting, and while many students were able to complete the exercise, most were using a computational approach and not appreciating or understanding the nature of the trigonometric functions. At Question 3, five separate students asked if they could 'divide by tan', which highlights the lack of understanding of the trigonometric functions as functions. This is perhaps fewer than might be expected from an average year-9 class considering that they have not explicitly learnt about functions and their inverses yet, but did highlight the conceptual jump from the previous exercise and was helpful in establishing the boundaries of their current understanding. Although I did not fully appreciate it at the time I wrote the question, the idea of inverse tan as the reverse table lookup for solving triangles looks conceptually different to students from its use in solving the equation $\tan\theta = 3\tan 20°$.

Overall the use of variation throughout the problem sheet allowed students to progress their understanding from gradient to right-angle triangle trigonometry through carefully controlled progressions. The choice of progression could be categorised as procedural variation — a stage-by-stage formation of a concept (Gu, Huang and Marton 2004), but there was much conceptual variation, too, with the use of triangles in varying orientations and different types of triangles, which allowed students to identify where trigonometry is an appropriate tool for solving problems.

I found that the structure of the worksheet allowed the students to move on their own through the exercises with minimal need for class discussion or exposition by the teacher. The majority of the teacher input involved introducing the trigonometric tables and notation. As discussed below in Section 5.2.2 there is evidence to suggest that all students were able to use trigonometry to solve triangle problems and many were able to see it as an extension of the concept of similar triangles, but others experienced difficulty seeing the trigonometric functions as objects within a ratio orientation.

5.2 Year-9 post-test

5.2.1 Year-9 post-test method

At the end of the trigonometry topic, students were given a worksheet, that I had initially intended to be a test, but due to various issues was not able to be done as a test, but which the students did during lesson time as a normal class exercise. For ease of reference I will still refer to it as a test. The questions are contained in Appendix E. The questions were a mixture of standard triangle-solving questions and some more unfamiliar questions designed to test students' conceptual understanding. Question 3(b) was designed to test their awareness of the connection between sine and cosine (which had been discussed in lessons). Question 4 could be done using similar triangles, rather than trigonometry, and was included to give an indication of those who still saw a connection between those two concepts, and who had displaced the idea of similar triangles with a more procedural trigonometry approach.

5.2.2 Year-9 post-test results

The percentage of correct answers to each question of the year-9 post-test are shown in Figure 5.1. Twenty-two of the twenty-three students took the test with the only missing student being the top student in the set who was already familiar with trigonometry before the beginning of the topic, and thus her absence has not affected results. The overall results indicate that all students were generally familiar with the basics of right-angled triangle trigonometry with nearly everyone able to apply the ratio definitions to solve triangle-based problems.

Question 2 of the non-calculator section was the most troublesome for students with only 59% answering it correctly. The question showed a 3/4/5 triangle with the lengths of the two legs given and one angle marked α. Students needed to identify the length of the hypotenuse and then write down $\sin\alpha = \frac{3}{5}$ (or 0.6). There were two different types of

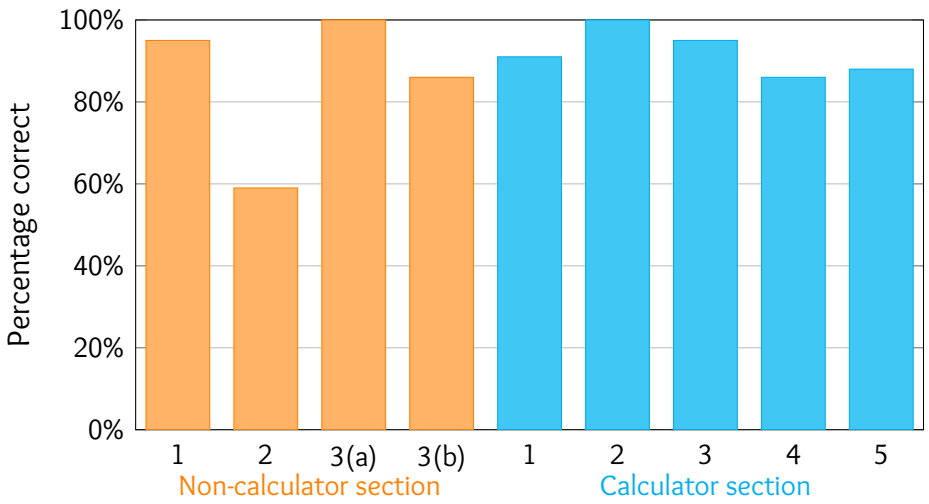


Figure 5.1: *Percentage of year-9 students who answered each post-test question correctly.*

mistakes. Two students answered $\frac{3}{4}$, possibly not realising that the hypotenuse was actually the unlabelled side. Both students had correctly used the sine ratio in the previous question, so this can probably be classified as a careless mistake. Eight students made the mistake of confusing $\sin\alpha$ with α and wrote $\alpha = \frac{3}{5}$ or $\alpha = 0.6$ or $\alpha = 0.6°$ or $\sin\alpha = 60°$ (possibly confusing 60° with 60% or realising that an angle of 0.6° was not sensible). All eight students had correctly applied the sine ratio to solve Question 1, and in seven of the eight cases they identified the correct two sides and ratioed them correctly — the fraction $\frac{3}{5}$ appearing somewhere in their working. There is evidently some uncertainty about the meaning of $\sin\alpha$ and its distinction from the angle α. A equivalent misconception was observed in the pre-test for L6PMa1 (Section 4.1.2). There had been minimal discussion during lessons about the units of, eg $\sin\alpha$, so several students knew that it was related to the angle and possibly assumed that it took the same units. The idea of a function taking an argument in degrees and returning a dimensionless number is one of the many new concepts in trigonometry and has been observed as one source of confusion for students (see Section 2.3.2). Pournara (2001, p 129) suggests that this indicates students are attempting to take a functional view of sine but the dimensionless output of the function does not fit with their existing schema.

When questioned, none of the students were able to explain their thoughts behind making the mistake and all were adamant that $\sin\alpha$ was a different quantity from α and that 0.6

did not represent the angle in degrees. Some of their responses did indicate a degree of uncertainty about what $\sin\alpha$ actually represented, and many required substantial guidance before being able to explain what the units of $\sin\alpha$ actually are. One student in particular was very clear that the statement $\sin\alpha = \frac{3}{5}$ represented an instruction to compute the angle α (even though the question said 'work out the value of $\sin\alpha$').

As I consider the trigonometry lessons and worksheet, there was insufficient time spent investigating what $\sin\alpha$ actually is. It was introduced as the name of the table that contains the ratios of opposite to adjacent sides, and was used almost exclusively as a means of solving triangle problems — either finding an unknown side or angle. There was only one question: Exercise 6, Q2(c), where students were asked to write the sine of an angle as a quantity in its own right. In all other cases it was just a step in a process that ended with an answer in length units or an angle in degrees. The idea of sine as a function that takes angle inputs and returns ratio outputs was discussed during lessons but to make sense of a question like 'write down $\sin\alpha$', students need to have some conception of sine as an object. Through the problems that students solved it is evident that a number of students had not had the opportunity to interiorise/condense the process of sine and see it as a (dimensionless) quantity in its own right (see section 2.3.2).

Five students showed an interesting thought process or made an important realisation in Question 4. The problem involves finding an unknown length, which one might expect to do by finding an unknown angle and then the unknown side using trigonometry. But because $\angle CAB$ is 90°, there are actually three similar triangles, so no trigonometry is required. One student spotted the similar triangles straight away and wrote down $\frac{13}{5} = \frac{CB}{13}$. A few more found they needed to calculate $\cos(\cos^{-1}\frac{5}{13})$, and hence didn't need to bother with trigonometry. And one more used a different approach that also involved similar triangles. The awareness to look for or recognise similar triangles is possibly related to the way I introduced them to trigonometry. All students had been doing standard trigonometry calculations for a couple of weeks and it would have been easy to forget about what trigonometry actually is — similar triangles with clumsy notation. Most students did just focus on the trigonometry in Question 4, which is understandable as the

question falls in the middle of a trigonometry test, but was pleasing to see some evidence that my chosen approach to trigonometry had resonated with at least a few students in the class.

5.3 Cycle 2 summary

Cycle 2 applied the ideas of variation introduced in cycle 1 to year-9 trigonometry. As the year-9 students were learning trigonometry for the first time, the style of the variation task had to be adapted from the unit circle task. This was done most significantly by decreasing the between-question variation. I tried as much as possible to vary only one thing from the previous question. In some instances larger jumps were necessary, but in most questions students were able to identify the varied concept and account for it in their working. As a result of the task, most students were able to see the connection between similar triangles and trigonometry, which was my overall intention.

There were some occasions when the variations were too big — especially with regard to the introduction of the notation of the trigonometric functions and their inverses. A future edition of the sheet will require greater thought to improve students' acceptance and understanding of trigonometric notation. The task contained little that allowed students to develop an object view of trigonometry, which is not unreasonable at the year-9 level, but the task could be adapted and extended to create questions that direct students' minds to an object view.

The theory of variation allows me to design tasks that ensure students' attention is directed to the important parts of a concept either by varying one aspect, or keeping one aspect invariant. In order to write a useful task, one must first be aware of exactly what one wants the students' attention directed to, which requires careful planning and thought. It is quite a time consuming task to write a sheet on the scale of the year-9 trigonometry task, but the benefit is that students can work at their own rate in lessons

and build their understanding individually through considering each question. Overall, the ideas developed during this cycle are promising and will lead to valuable future work.

Chapter 6

Conclusion

This project was a piece of action research with a small number of students at St Frances's School for Girls. Due to the nature of the school and the small numbers of students involved, any results reported are applicable only to the students and teachers involved in the research and cannot readily be generalised to UK secondary school students.

This research has explored the way students learn and understand trigonometry, both at A level and year 9. Trigonometry has been seen as one of the more difficult maths topics for students to learn as it introduces some brand new concepts at the year-9 level, and requires understanding in a variety of different ways at A level. A number of different models have been used to understand and categorise the way students learn trigonometry, and although the models use different language, their ideas are broadly similar. One of the challenges in trigonometry is that trigonometric functions are both processes and objects, or procepts, and are both ratios of side lengths and functions with angle inputs and dimensionless outputs. They can be represented with triangles, graphs and circles and although all the definitions and representations are mathematically consistent, they do not look very similar to students learning them for the first time. As students gradually incorporate the different representations and concepts in their understanding they develop a trigonometry schema. A schema describes a collection of processes and objects that exist in someone's mind. Using the action-process-object-schema model (APOS), the students' understanding of trigonometry was assessed in this project.

The concept of variation was used to design tasks that helped direct students' attention to particular elements of trigonometry. The unit circle task was given to two students in year 12 to help them understand some aspects of the functional nature of the trigonometric functions. The use of variation was refined further to produce the year-9 trigonometry task.

6.1 Cycle 1 — Year 12

Two A-level students were reported in detail. Both students initially found A-level trigonometry difficult, at least partly due to the numerous representations and increased need to hold an object-oriented view of the functions. During interviews, both students had difficulty explaining the connection between the right-angled triangle definitions of the trigonometric functions and their graphs. Being able to connect the two concepts was expected to enable them to make more sense of the topic and to deal with solving unusual or unfamiliar problems. To help them bridge this gap and hopefully better understand the trigonometric functions as objects and functions, the students undertook a unit circle task that was designed using the principles of variation. After completing the task, both students reported that they understood better the connections between the trigonometric functions and their representations. The students did not necessarily choose the unit circle representation as their first choice to solve problems, but experiencing the behaviour of the trigonometric functions through their base definitions in the unit circle appeared to at least help their confidence in dealing with trigonometric functions.

Post-tests with the students indicated some improvement in their understanding of trigonometry from earlier data, although given the limited data it is not possible to credit the unit circle task with this outcome — many other factors were involved. Some possible areas for improvement in the use of variation in the unit circle task were identified, in particular more carefully controlling the degree of variation from question to question. Although the unit circle task may have been suitable for A-level students seeking to refine their understanding of a particular aspect of trigonometry, the degree of variation would need to be reduced for introducing trigonometry to the year-9 students.

6.2 Cycle 2 — Year 9

More careful consideration was given to the variation within the trigonometry task used with the year-9 set. The learning began with the idea of gradient, which the students were familiar with, and gradually moved through to trigonometric functions and their inverses. The careful variation allowed students to explore each idea before moving on to the next concept, and gave them time to reach their own understanding. Trigonometry was introduced as a way of solving similar triangles using angles rather than just side lengths and there was evidence that at least some of the students saw and recognised this connection at the end of the teaching period. The examples chosen generally required the application of a right-angled triangle solution technique, with the similar triangles/trigonometry ideas used as a means of solving problems, rather than objects in their own right. Thus, in hindsight, it was not unexpected that students would come to see trigonometric functions as part of a geometry-problem-solving process, but at the year-9 level, this is sufficient.

Feedback at the end of the task indicated that there was room for improvement in the introduction of the names of trigonometric functions, the notation, and the concept of inverse functions. Students were generally adept at carrying out the procedures, but comments such as 'divide by $\tan$' and 'sine minus one' indicate some misconceptions amongst at least some of the set. A number of students referred to a value of $\sin\alpha$ as a quantity in degrees. Even though they knew that $\sin\alpha$ is a ratio of two lengths, the fact that α, an angle measured in degrees, is in the expression seemed to suggest that the output of $\sin\alpha$ must therefore also be in degrees.

6.3 Research questions

RQ1 was addressed in the literature review (Chapter 2). APOS was the model chosen to analyse students' understanding of trigonometry in this report, although there are other

models they may have been equally effective. Trigonometry is a particularly challenging topic to analyse because of the wide range of understanding that is required. Students require particularly detailed schemas to hold all the connections between the various representations and views.

There is some evidence of students understanding trigonometric functions as either processes or objects (RQ2), but this distinction has been problematic because the two categories are not necessarily mutually exclusive or binary in any given situation. One may claim that a student could not answer a question because it required an object orientation of sine, but then they may have answered the next question correctly which also required an object orientation. The APOS model is one of many tools that may be helpful in analysing student understanding but determining which type of understanding or knowledge is applicable to any particular situation is not an exact science and perhaps categorising knowledge into process or object pigeonholes is unnecessarily restrictive.

Variation can be effective in designing tasks that help students develop a conceptual understanding of trigonometry (RQ3). It cannot be claimed that the particular tasks employed in this research were effective, as there is insufficient evidence to support this claim; however there is evidence to support the claim to some extent. It may be the case that using other methods would have been equally effective or more effective, but this can be the subject of further research. The principles of variation were used to design tasks that allowed the students to work through with minimal input from their teacher, which suggested students had the opportunity to individually construct their own knowledge of the new concepts. Upon analysis, the year-9 task was very focussed on developing a process-level understanding of trigonometric functions, and the tasks would need to be altered or adapted to give students an opportunity to condense their knowledge to see trigonometric functions as objects.

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Appendices

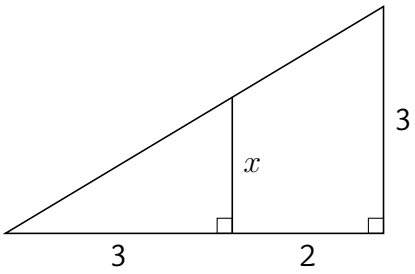
A Trigonometry pre-test

L6PMa1 Maths

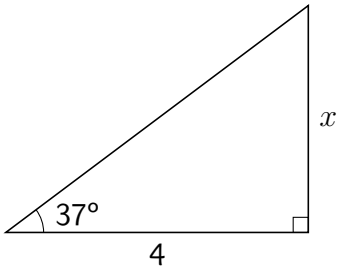
Trigonometry

Show all working. No calculator allowed for questions numbered in red.

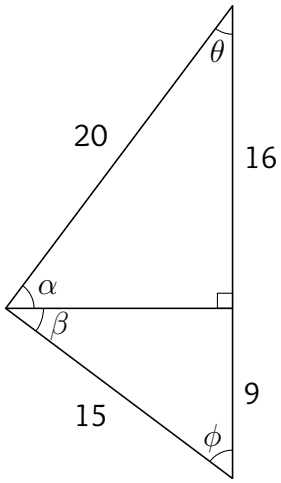
1. Find x.



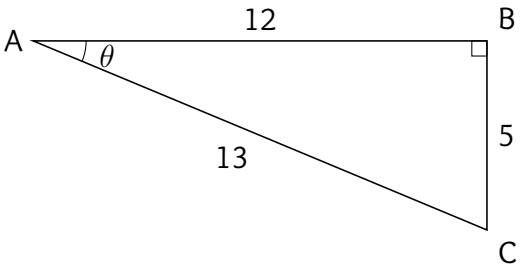
2. Find x



3. Find α, β, θ and ϕ.

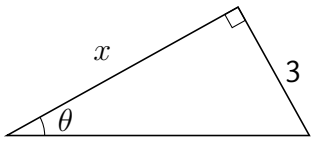


4. For $\triangle$ABC, write down the value of $\sin\theta$.



5. Write down the angle whose cosine is 0.1805.
6. Draw a triangle for which $\cos\theta = \frac{3}{5}$

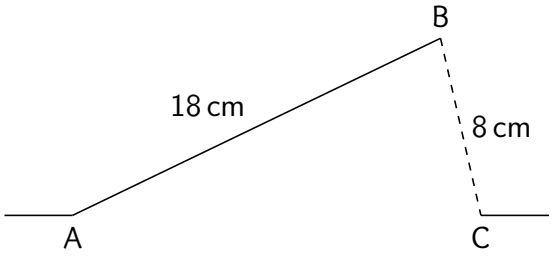
7. Find x given $\tan\theta = \frac{9}{5}$



8. If $\sin\theta = 0.6$ and θ is acute, what is $\tan\theta$?

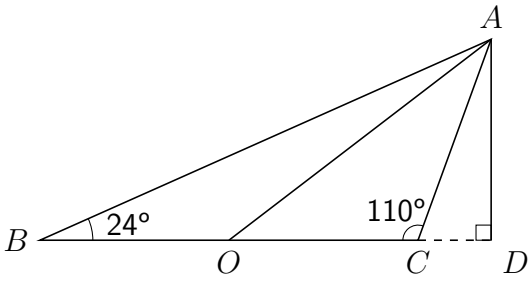
9. $\sin\theta = \dfrac{m^2-1}{m^2+1}$. Give $\cos\theta$ and $\tan\theta$ in terms of m.

10. A skylight AB is 18 cm long and is being kept open by a stick (BC) 8 cm long. Through what angle has AB been opened?



11. Izzy is standing at the window of a train carriage on her way back from her family's seaside holiday home in Devon after exeat. There are autumn leaves on the line so the train is travelling at a leisurely 48 km h^{-1} along a straight track. Izzy notices the tower of Exeter cathedral at an angle of 50° to her left. 5 min later it is at an angle of 40° to her right. How far away is the cathedral on the second occasion?

12. Triangle ABC has $\angle ABC = 24°$ and $\angle ACB = 110°$. O is the midpoint of BC, and AD meets the extension of BC at right angles.

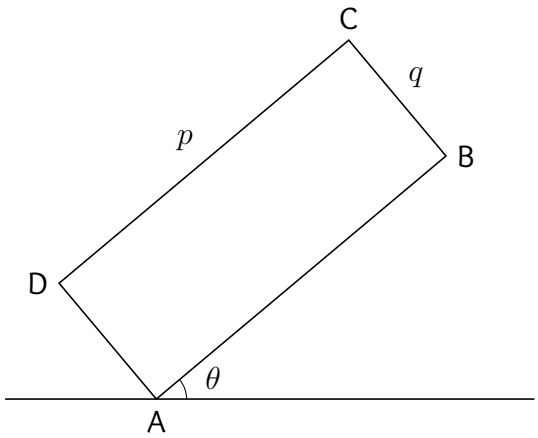


(a) Show that $DO = \frac{1}{2}(DB + DC)$

(b) Hence prove that $\tan(\angle OAD) = \frac{1}{2}\left(\tan(\angle BAD) + \tan(\angle CAD)\right)$

(c) Calculate $\angle AOC$

13. $ABCD$ is a rectangular box (dimensions: $p \times q$) that is rotated in the vertical plane such that point A is in contact with the horizontal floor.



(a) Show that the height of C above the floor is $p\sin\theta + q\cos\theta$.

(b) Hence or otherwise determine the maximum value of $p\sin\theta + q\cos\theta$.

(c) For $p = 5$, $q = 2$, calculate θ for the maximum height of C.

14. Solve $\cos\theta = -\frac{1}{2}$

15. Solve $\sin 2\theta = 1$ for $0 \leqslant \theta < 360°$

16. Solve $\dfrac{1}{\tan\theta} = 0$ for $-\pi \leqslant \theta < \pi$

17. Accurately sketch the graphs of $y = \sin x$ and $y = \cos x$ on the same axes. Label the coordinates of any axis intercepts or stationary points.

18. What transformation maps $y = \cos x$ on to $y = \sin x$? Hence write $\cos\theta$ in terms of $\sin$.

19. Evaluate $\cos^2 63° + \cos^2 27°$

20. Prove* $\sin(-\theta) \equiv -\sin\theta$

21. (a) Prove* $\sin(90° + \theta) \equiv \cos\theta$

    (b) Hence solve $\sin\theta = \cos 70°$ for $0 \leqslant \theta < 180°$

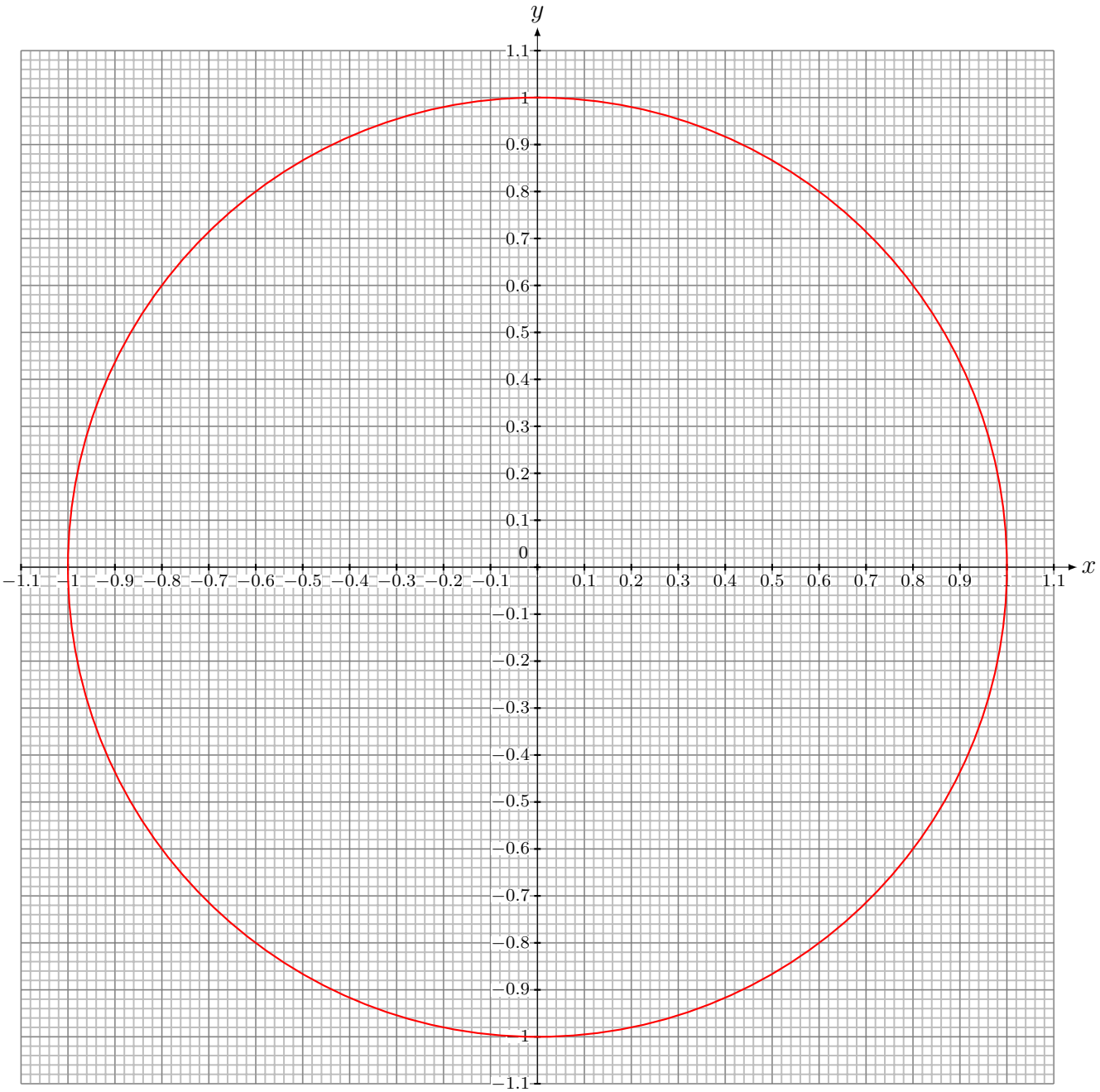
22. Find $\sin(x + y)$ given $\sin x = 0.4$ and $\sin y = 0.8$.

*Does not need to be an algebraic proof.

B Unit circle task

L6PMa1 Maths

Unit circle trigonometry

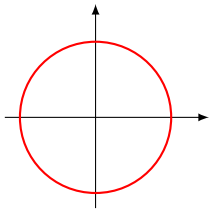


Do the following questions using the picture on page 1. Where necessary, assume $0 \leqslant \theta < 360°$.

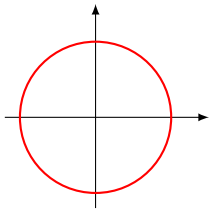
1. (a) Use your protractor to mark the point P on the circumference where $\theta = 130°$.
   (b) Write down the coordinates of P.
   (c) Hence work out the gradient of the line OP.
   (d) Write down the value of
   (i) $\sin 130°$ (ii) $\cos 130°$ (iii) $\tan 130°$
2. (a) Use your protractor to mark the point Q on the circumference where $\theta = 230°$.
   (b) Write down the coordinates of Q.
   (c) Hence work out the gradient of the line OQ.
   (d) Write down the value of
   (i) $\sin 230°$ (ii) $\cos 230°$ (iii) $\tan 230°$
3. (a) Use your protractor to mark the point R on the circumference where $\theta = 305°$.
   (b) Write down the coordinates of R.
   (c) Hence work out the gradient of the line OR.
   (d) Write down the value of
   (i) $\sin 305°$ (ii) $\cos 305°$ (iii) $\tan 305°$
4. Point S lies at the point where $\sin\theta = -0.6$ and $\cos\theta = 0.8$.
   (a) Write down the coordinates of S.
   (b) Mark point S on the diagram.
   (c) What is the value of θ?
   (d) What is the value of $\tan\theta$?
5. (a) Mark all the points on the circumference where $\sin\theta = 0.4$.
   (b) Write down the value of $\cos\theta$ for all of your points. What do you notice?
   (c) Work out $\tan\theta$ at your points. What do you notice?
   (d) Measure the values of θ at your points.
6. (a) Mark all the points on the circumference where $\cos\theta = -0.5$.
   (b) Write down the value of $\sin\theta$ for all of your points. What do you notice?
   (c) Work out $\tan\theta$ at your points. What do you notice?
   (d) Measure the values of θ at your points.
7. (a) Mark all the points on the circumference where $\tan\theta = -0.5$.
   (b) Write down the value of $\sin\theta$ for all of your points. What do you notice?
   (c) Work out $\cos\theta$ at your points. What do you notice?
   (d) Measure the values of θ at your points.

For the following questions, mark the relevant points on the little unit circle, but use your calculator to solve them. Domain is $0 \leqslant \theta < 360°$ unless specified.

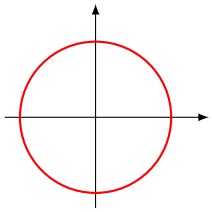
1. Solve $\cos\theta = 0.3$



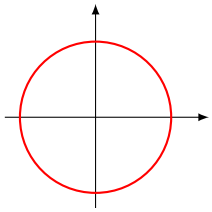
2. Solve $\sin\theta = -0.2$



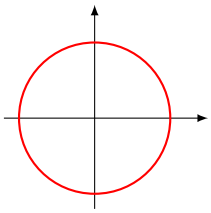
3. Solve $\tan\theta = 0.4$



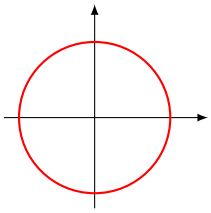
4. Solve $\sin\theta = 0.5, \quad -180° \leqslant \theta < 180°$



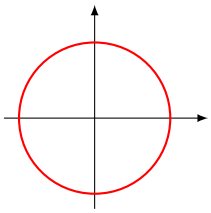
5. Solve $\cos\theta = -0.7, \quad -180° \leqslant \theta < 360°$



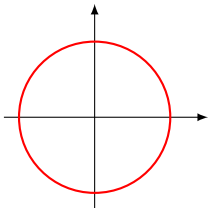
6. Solve $\tan\theta = -1.5, \quad -360° \leqslant \theta < 0°$



7. Solve $5\sin 2\theta = 1$

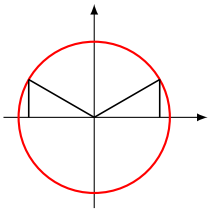


8. Solve $\cos\frac{1}{2}\theta = -0.8, \quad -180° \leqslant \theta < 180°$

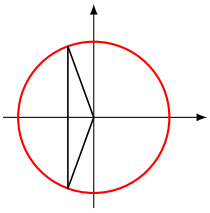


Write down an equation you might use the following pictures to solve.

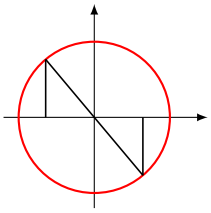
(a)



(b)



(c)



C Trigonometry post-test

L6PMa1 Maths

Trigonometry test

Show all working.

1. (a) What is a radian?

   (b) How many degrees in 1 radian?

2. What can be said about θ if $\cos\theta > 0$ and $\tan\theta < 0$?

3. State the number of solutions to $\sin\left(5\theta - \frac{2\pi}{3}\right) = \frac{1}{2}$ in $-2\pi < \theta \leqslant 2\pi$.

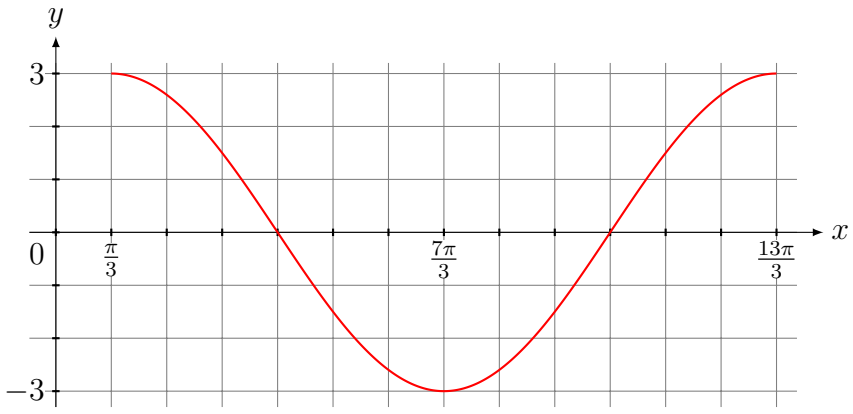
4. Write $\sin x - \cos x$ in the form $R\cos(x - \alpha)$, giving the exact value of α in radians.

5. Solve $4\sin\dfrac{\theta}{2} = \sec\dfrac{\theta}{2}$ for $-\pi \leqslant \theta < \pi$.

6. (a) Sketch $y = \cos^{-1} x$. Label axis intercepts. State or label the domain and range.

   (b) Find the exact value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$.

7. One period of a function of the form $y = a\cos[\omega(x - \phi)]$ is shown in the figure. Determine the values of the constants.



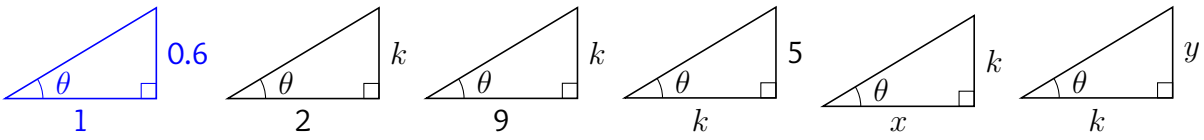
D Year-9 trigonometry worksheet

U4Ma1 Maths

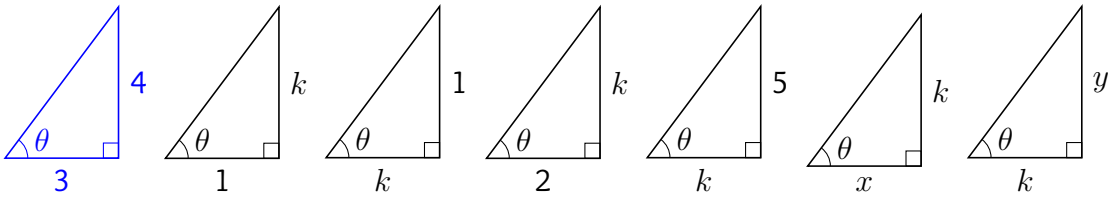
Similar triangles and trigonometry

Exercise 1.

1. The following triangles are all similar. Find the length of the side with the k in each case.

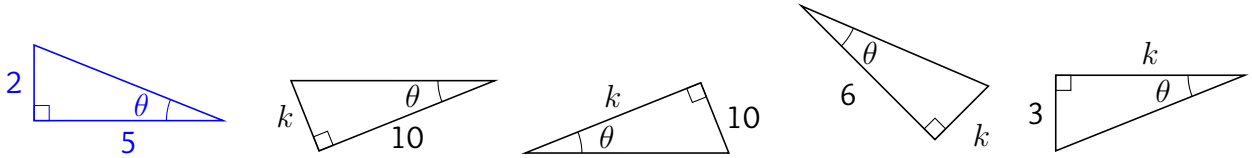


2. The following triangles are all similar. Find the length of the side with the k in each case.

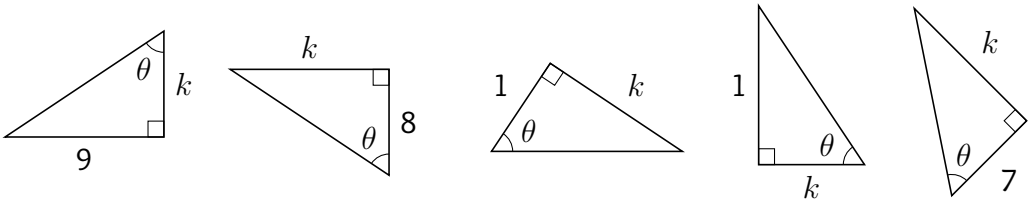


Exercise 2.

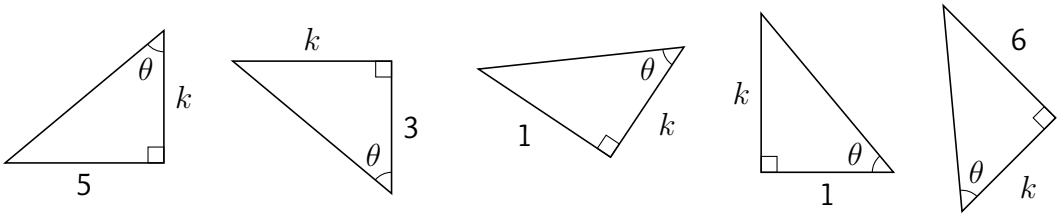
1. The following triangles are all similar. Find the length of the side with the k in each case.



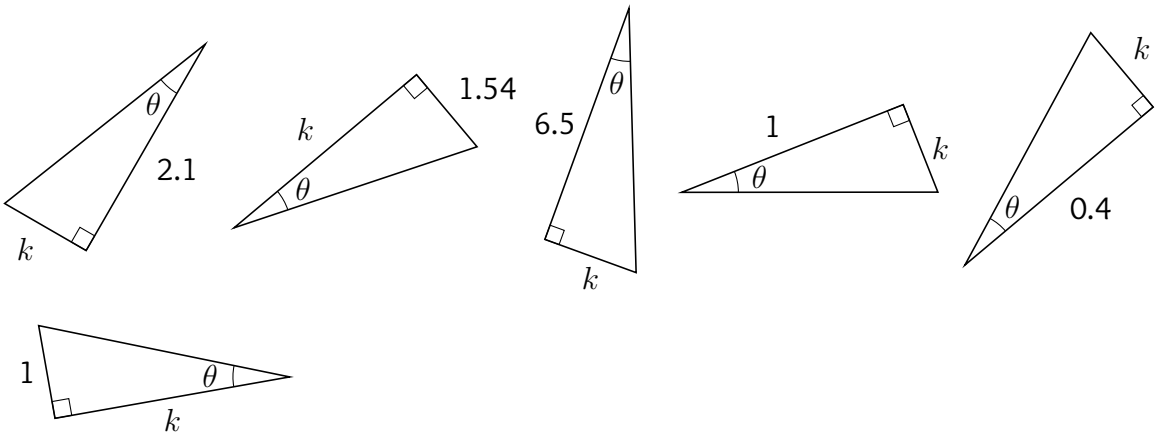
2. In all these triangles, $\frac{\text{opp}}{\text{adj}} = \frac{3}{2}$. Find the length of the side with the k in each case.



3. In all these triangles, $\frac{\text{opp}}{\text{adj}} = 1.2$. Find the length of the side with the k in each case.



4. In all these triangles, $\frac{\text{opp}}{\text{adj}} = 0.395$. Find the length of the side with the k in each case.



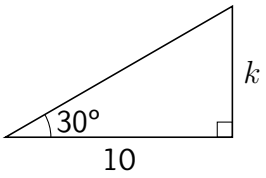
Notice how all your answers are obtained by either multiplying or dividing the known side by 0.395?

Explain in your own words when you should multiply by the 0.395 and when you should divide by it.

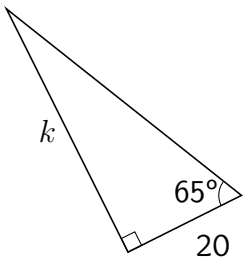
Exercise 3.

Find the length of the side marked k. Use the trigonometry table to find the relevant ratio.

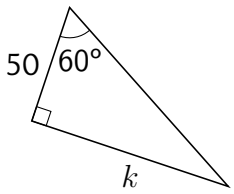
(a)



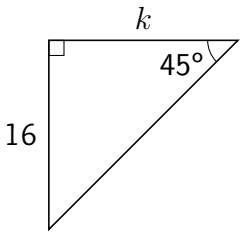
(b)



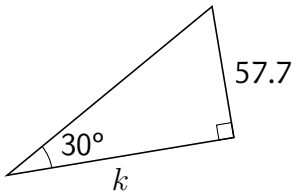
(c)



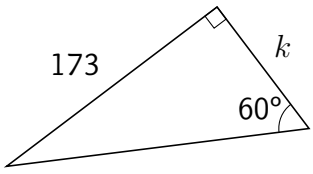
(d)



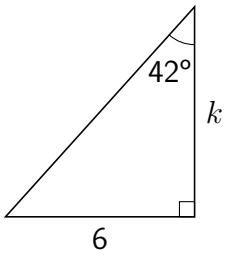
(e)



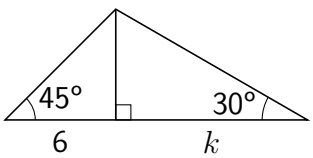
(f)



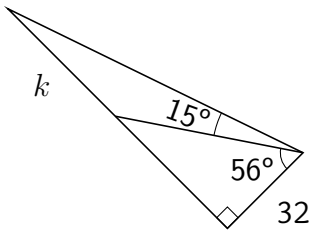
(g)



(h)



(i)



(j) The angle of elevation of the top of a cliff from a boat 125 m away from the bottom of the cliff is 35°. Find the height of the cliff.

(k) The angle of depression from a window 10 m above the ground to a coin on the ground is 15°. Find the distance of the coin from the base of the building.

(l) **(Extension)** Triangle ABC is such that

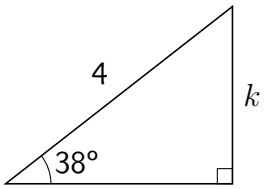
$$BC = 8\,\text{cm}, \quad \angle ABC = 46°, \quad \angle ACB = 62°$$

A circle is inscribed in the triangle, ie touches the three sides. Calculate its radius.

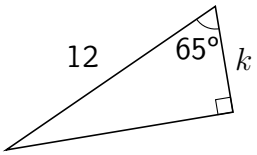
Exercise 4.

1. Find the length of the side marked k. Use the trigonometry tables to find the relevant ratio.

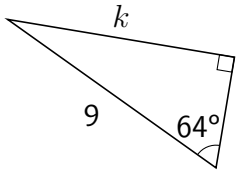
(a)



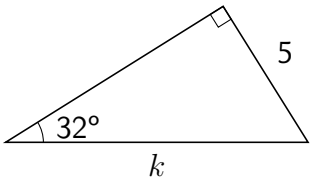
(b)



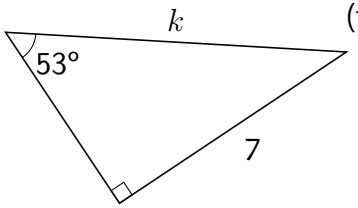
(c)



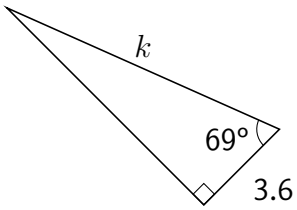
(d)



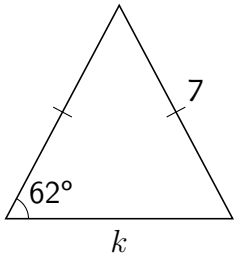
(e)



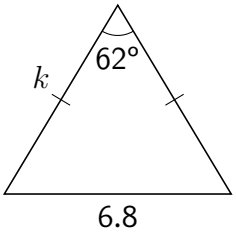
(f)



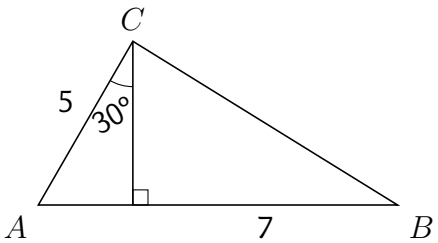
(g)



(h)



2. Work out the perimeter of triangle ABC.



3. The string of a kite is 45 m long and makes an angle of 62° with the horizontal. How high is the kite above the ground?

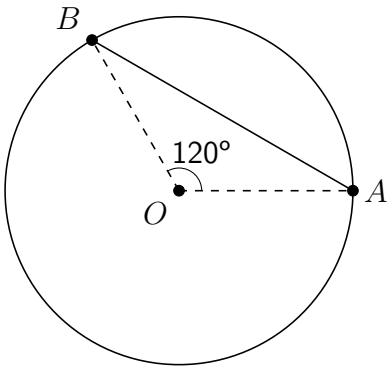
4. A ladder 6 m long leans against the side of a house.

   (a) The angle between the ladder and the ground is 60°. How far is the foot of the ladder from the house?

   (b) How far must the foot of the ladder be pushed to increase the angle of lean to 65°?

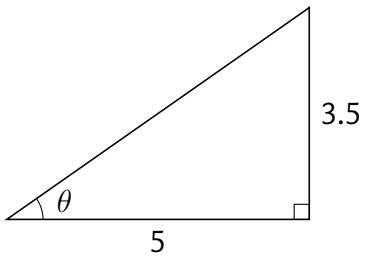
5. The diagonals of a rectangle are 12 cm long and contain an angle of 18°. Find the length of the base and height of the rectangle.

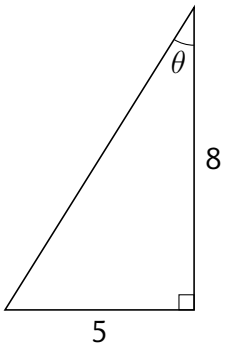
6. Find the length of the chord AB of the circle with radius 5 cm.

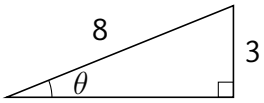


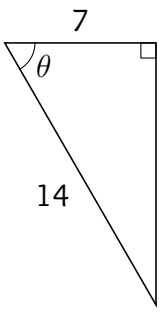
7. **(Extension)** A regular pentagon is inscribed in a circle of radius 5 cm.
   (a) What is the side length of the pentagon?
   (b) What is the area of the pentagon?

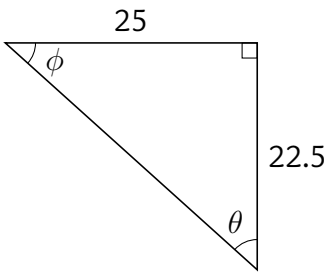
Exercise 5.
Use the tables to find the marked angles.

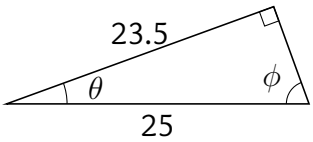
(a)


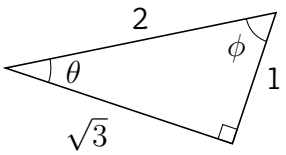
(b)


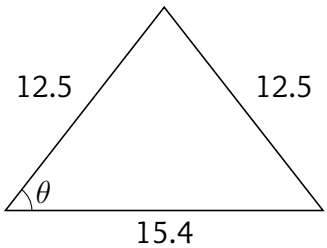
(c)


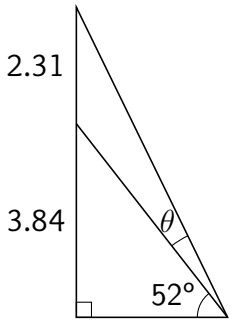
(d)


(e)


(f)


(g)


(h)


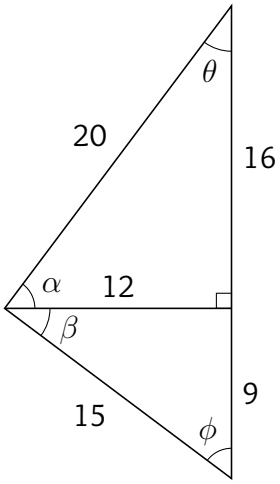
(i)


Exercise 6.

1. Use your calculator to work out *sine*, *cosine* and *tangent* of the following

   (a) 23.36° (b) 62.4° (c) 89.48°

2. Consider these triangles.



   (a) Show that the figure contains three similar triangles.

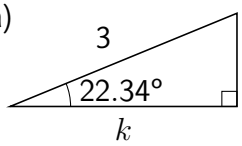
   (b) Hence write down the value of $\alpha + \beta$.

   (c) Write down *sine*, *cosine* and *tangent* of α, β, θ and ϕ.

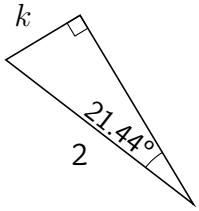
   (d) Comment on your answers.

3. Find the side labelled k. Give your answers to 3 decimal places.

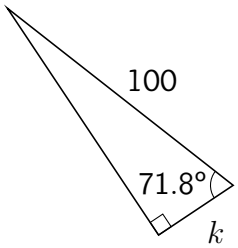
(a)



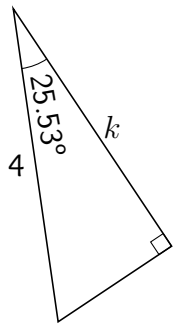
(b)



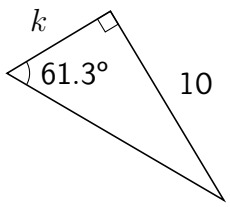
(c)



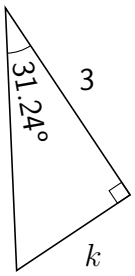
(d)



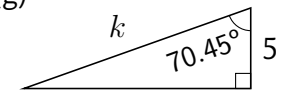
(e)



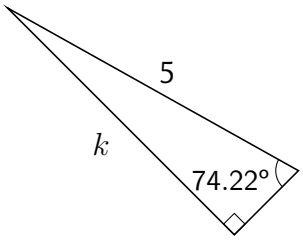
(f)



(g)



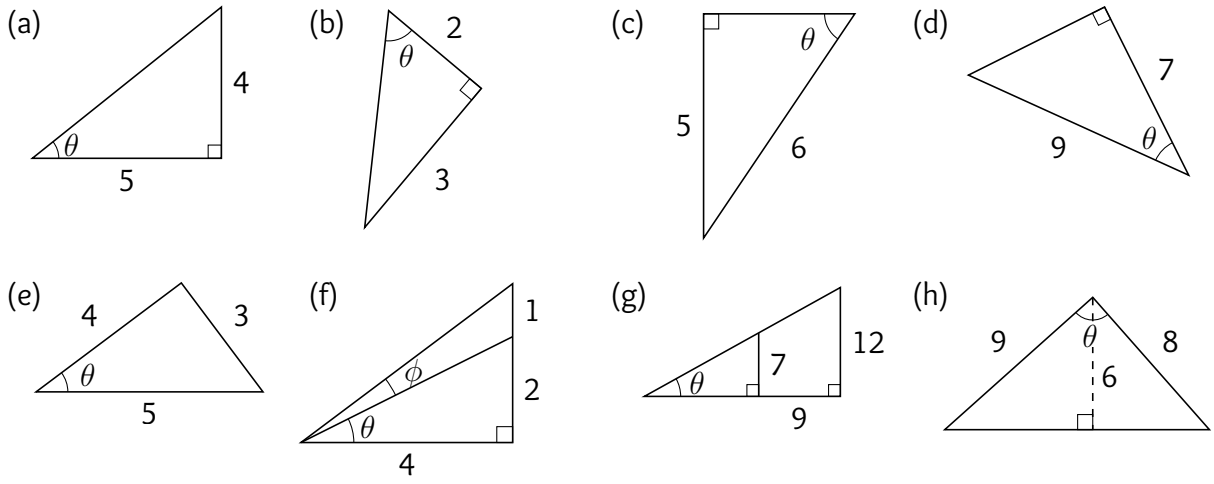
(h)



4. The vertical angle of a cone is 23° and the length of a slant edge is 2.5 cm. What is the diameter of the base?

Exercise 7.

1. Find the marked angles. Give your answers to 2 decimal places.



   (i) A chord AB of a circle of radius 6 cm is 5 cm long. Calculate $\angle AOB$ (where O is the centre of the circle).

   (j) A pencil 15 cm long casts a shadow 12 cm long when the sun is vertically overhead. What is the angle of inclination of the pencil to the vertical?

   (k) The sides of a parallelogram are 4 and 5 cm and its area is 12 cm^2. Calculate its angles.

Exercise 8.

1. What are the values of $\cos 32°$ and $\sin 58°$? Why are they equal?
2. Find a value of θ such that
   (a) $\cos\theta = \sin 48°$.
   (b) $\sin\theta = \cos 67.5°$.
   (c) $\cos\theta = \sin 21.78°$.
   (d) $\sin\theta = \cos 15.21°$.
3. Find a value of θ if $\tan\theta = 3\tan 20°$.
4. Simplify
   (a) $\tan 20° \times \tan 70°$
   (b) $\tan 34° - \dfrac{1}{\tan 56°}$

   Can you explain your answers?
5. What can you say about triangle ABC in which $\cos A = \sin B$ and B is acute?
6. One solution to the equation $\sin\theta + \cos\theta = 1.292$ is $\theta = 21°$. Find another solution.
7. Write down an equation that connects α and β if
$$\cos\alpha = \sin\beta$$
   Hence find a value of θ such that $\cos\theta = \sin(2\theta)$.
8. Find a value of θ such that
   (a) $\cos\theta = \sin(\theta + 20°)$
   (b) $\sin\theta = \cos(\theta - 16°)$

Trigonometry tables

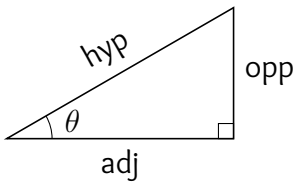


Table 1: $\frac{\text{opp}}{\text{adj}}$

θ	+0°	+1°	+2°	+3°	+4°	+5°	+6°	+7°	+8°	+9°
0°	0	0.017	0.035	0.052	0.070	0.087	0.105	0.123	0.141	0.158
10°	0.176	0.194	0.213	0.231	0.249	0.268	0.287	0.306	0.325	0.344
20°	0.364	0.384	0.404	0.424	0.445	0.466	0.488	0.510	0.532	0.554
30°	0.577	0.601	0.625	0.649	0.675	0.700	0.727	0.754	0.781	0.810
40°	0.839	0.869	0.900	0.933	0.966	1	1.036	1.072	1.111	1.150
50°	1.192	1.235	1.280	1.327	1.376	1.428	1.483	1.540	1.600	1.664
60°	1.732	1.804	1.881	1.963	2.050	2.145	2.246	2.356	2.475	2.605
70°	2.747	2.904	3.078	3.271	3.487	3.732	4.011	4.331	4.705	5.145
80°	5.671	6.314	7.115	8.144	9.514	11.430	14.301	19.081	28.636	57.290

Table 2: $\frac{\text{opp}}{\text{hyp}}$

θ	+0°	+1°	+2°	+3°	+4°	+5°	+6°	+7°	+8°	+9°
0°	0	0.017	0.035	0.052	0.070	0.087	0.105	0.122	0.139	0.156
10°	0.174	0.191	0.208	0.225	0.242	0.259	0.276	0.292	0.309	0.326
20°	0.342	0.358	0.375	0.391	0.407	0.423	0.438	0.454	0.469	0.485
30°	0.5	0.515	0.530	0.545	0.559	0.574	0.588	0.602	0.616	0.629
40°	0.643	0.656	0.669	0.682	0.695	0.707	0.719	0.731	0.743	0.755
50°	0.766	0.777	0.788	0.799	0.809	0.819	0.829	0.839	0.848	0.857
60°	0.866	0.875	0.883	0.891	0.899	0.906	0.914	0.921	0.927	0.934
70°	0.940	0.946	0.951	0.956	0.961	0.966	0.970	0.974	0.978	0.982
80°	0.985	0.988	0.990	0.993	0.995	0.996	0.998	0.999	0.999	1.000

Table 3: $\frac{\text{adj}}{\text{hyp}}$

θ	+0°	+1°	+2°	+3°	+4°	+5°	+6°	+7°	+8°	+9°
0°	1	1.000	0.999	0.999	0.998	0.996	0.995	0.993	0.990	0.988
10°	0.985	0.982	0.978	0.974	0.970	0.966	0.961	0.956	0.951	0.946
20°	0.940	0.934	0.927	0.921	0.914	0.906	0.899	0.891	0.883	0.875
30°	0.866	0.857	0.848	0.839	0.829	0.819	0.809	0.799	0.788	0.777
40°	0.766	0.755	0.743	0.731	0.719	0.707	0.695	0.682	0.669	0.656
50°	0.643	0.629	0.616	0.602	0.588	0.574	0.559	0.545	0.530	0.515
60°	0.5	0.485	0.469	0.454	0.438	0.423	0.407	0.391	0.375	0.358
70°	0.342	0.326	0.309	0.292	0.276	0.259	0.242	0.225	0.208	0.191
80°	0.174	0.156	0.139	0.122	0.105	0.087	0.070	0.052	0.035	0.017

Answers

Exercise 1.

1. 1.2, 5.4, 8.33, 0.6x, $\frac{y}{0.6}$
2. $\frac{4}{3}$, $\frac{3}{4}$, $\frac{8}{3}$, $\frac{15}{4}$, $\frac{4}{3}x$, $\frac{3}{4}y$

Exercise 2.

1. 4, 25, $\frac{12}{5}$, $\frac{15}{2}$
2. 6, 12, $\frac{3}{2}$, $\frac{2}{3}$, $\frac{21}{2}$
3. 4.167, 3.6, 0.833, 1.2, 5
4. 0.8295, 3.899, 2.5675, 0.395, 0.158, 2.532

Exercise 3.

(a) 5.77 (b) 42.9 (c) 86.6 (d) 16 (e) 100 (f) 99.885 (g) 6.667 (h) 10.399 (i) 45.472 (j) 87.5 m (k) 37.313 m (l) 1.989 cm

Exercise 4.

1. (a) 2.464 (b) 5.076 (c) 8.091 (d) 9.434 (e) 8.761 (f) 10.056 (g) 6.566 (h) 6.602
2. 22.731
3. 39.735 m
4. (a) 3 m (b) 0.462 m
5. 1.872 cm $\times$ 11.856 cm
6. 8.66 cm
7. (a) 5.88 cm (b) 59.462 cm^2

Exercise 5.

(a) 35° (b) 32° (c) 22° (d) 60° (e) $\theta = 48°, \phi = 42°$ (f) $\theta = 20°, \phi = 70°$ (g) $\theta = 30°, \phi = 60°$ (h) 52° (i) 12°

Exercise 6.

1. (a) 0.397, 0.918, 0.432 (b) 0.886, 0.463, 1.913 (c) 0.999 96, 0.009 08, 110.181
3. (a) 2.775 (b) 0.731 (c) 31.233 (d) 3.609 (e) 5.475 (f) 1.820 (g) 14.942 (h) 4.812
4. 0.9968 cm

Exercise 7.

(a) 38.66° (b) 56.31° (c) 56.44° (d) 38.94° (e) 36.87° (f) $\theta = 26.57°, \phi = 10.30°$ (g) 29.05° (h) 89.60° (i) 49.25° (j) 53.13° (k) 36.87°, 143.13°

Exercise 8.

1. 0.848
2. (a) 42° (b) 22.5° (c) 68.22° (d) 74.79°
3. 47.52°
4. (a) 1 (b) 0
5. Isosceles, right angled.
6. 69°
7. $\theta = 30°$
8. (a) 35° (b) 53°

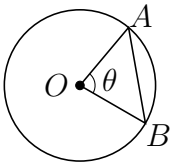
Trigonometry extension problems

1. (a) What angle does a line of slope 2 make with the x-axis?

   (b) Investigate the angle with the x-axis for lines of other slopes. You might want to plot a graph of slope v angle. How will you deal with negative slopes?

   (c) Find where $y = 2x - 1$ and $y = \frac{1}{2}x + 4$ intersect and show using trigonometry that they intersect at 90°. Hint: a good drawing will help you figure out which angles are where.

2. The diagram shows a chord AB and its central angle $\angle AOB = \theta$.

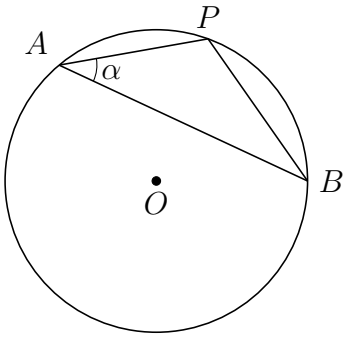


   (a) Given the diameter of the circle is 1, how is the length of AB related to θ?

   (b) Using the same diagram, suppose now that the *radius* of the circle is 1. How is the length of AB related to θ now?

3. This question generalises the previous question. Prove that if α is the angle subtended by a chord PB at a point on a circle of radius r (as shown in the diagram) then

$$\sin\alpha = \frac{PB}{2r}$$



4. Using a right angled triangle with hypotenuse 1 and acute angle θ,

   (a) Show that $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$

   (b) By convention, if we want to square a trigonometric function we write the squared symbol in the middle. Eg $(\sin\theta)^2 \equiv \sin^2\theta$. Show that

$$\sin^2\theta + \cos^2\theta \equiv 1$$

5. (a) If $p^2 + q^2 = 1$, show that $(p^2 - q^2)^2 + (2pq)^2 = 1$.

   (b) Suppose that t is a number between 0 and 1. If

$$a = \frac{1 - t^2}{1 + t^2} \qquad \text{and} \qquad b = \frac{2t}{1 + t^2}$$

   show that there is an angle θ such that

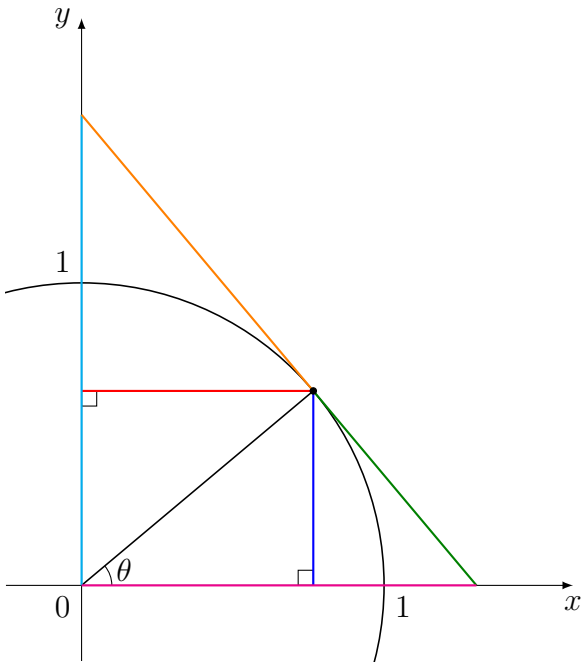
$$a = \cos\theta \qquad \text{and} \qquad b = \sin\theta$$

6. Which inequality is correct?

   (a) $\sin\alpha + \cos\alpha < 2$

   (b) $\sin\alpha + \cos\alpha \leqslant 2$

7. The diagram shows part of a circle of radius 1. The orangegreen line is tangent to the circle.



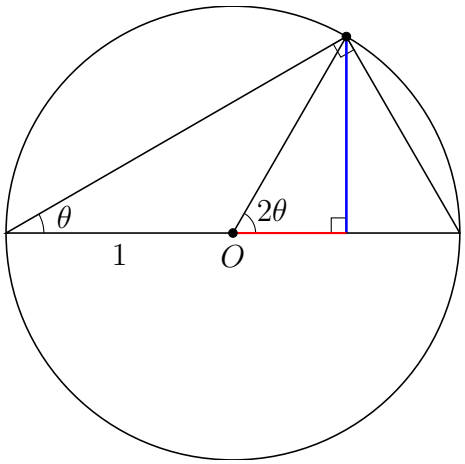
   (a) Find the lengths of the red and blue lines in terms of θ.

   (b) Show that the length of the green line is $\tan\theta$.

   (c) Find trigonometric expressions for the lengths of the orange, cyan and magenta lines.

8. Compute the lengths of the red and blue lines in two or three different ways and hence show that

$$\sin 2\theta \equiv 2\sin\theta\cos\theta$$
$$\cos 2\theta \equiv 2\cos^2\theta - 1 \equiv 1 - 2\sin^2\theta$$



   Use the identities above to show further that $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$.

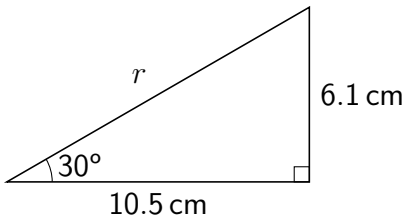
E Year-9 trigonometry post-test

U4Ma1 Maths

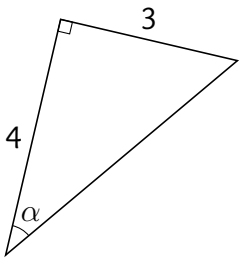
Trigonometry not test

Non-calculator section

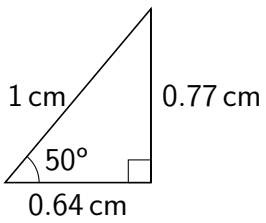
1. Work out the length r, given that $\sin 30° = 0.5$.

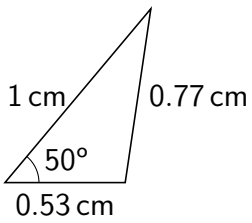


2. Work out the value of $\sin \alpha$.



3. (a) Here are sketches of two triangles, both with a longest side of 1 cm.

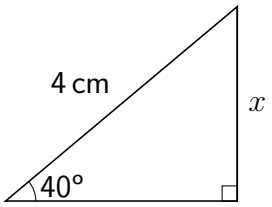




Circle the value of $\cos 50°$.

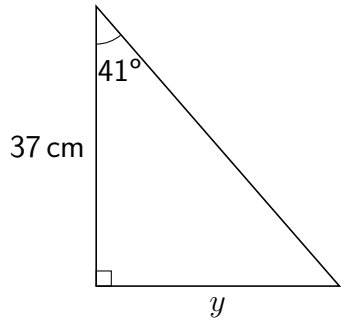
A. 0.77 B. 0.53 C. 0.64 D. 1

(b) Find the length x.

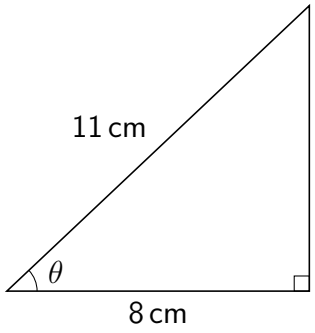


Calculator section

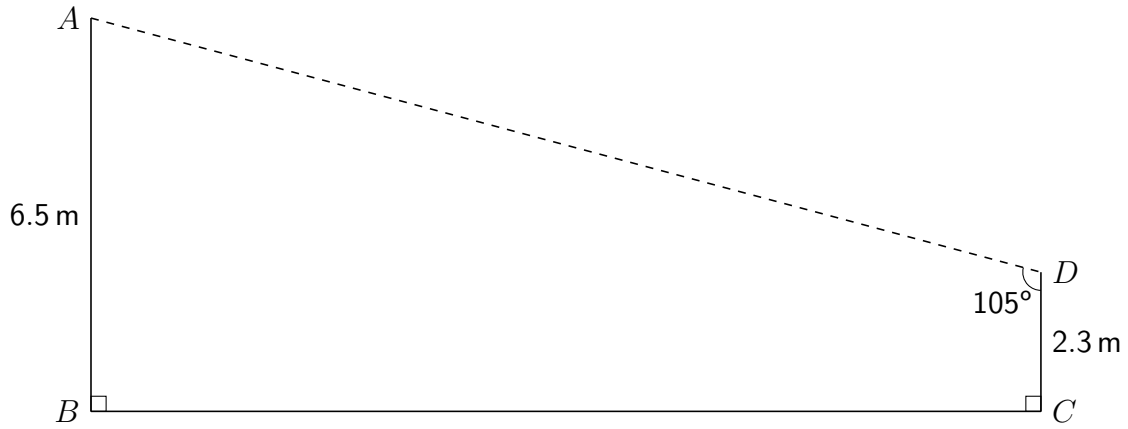
1. Work out the length y.



2. Work out the size of angle θ.



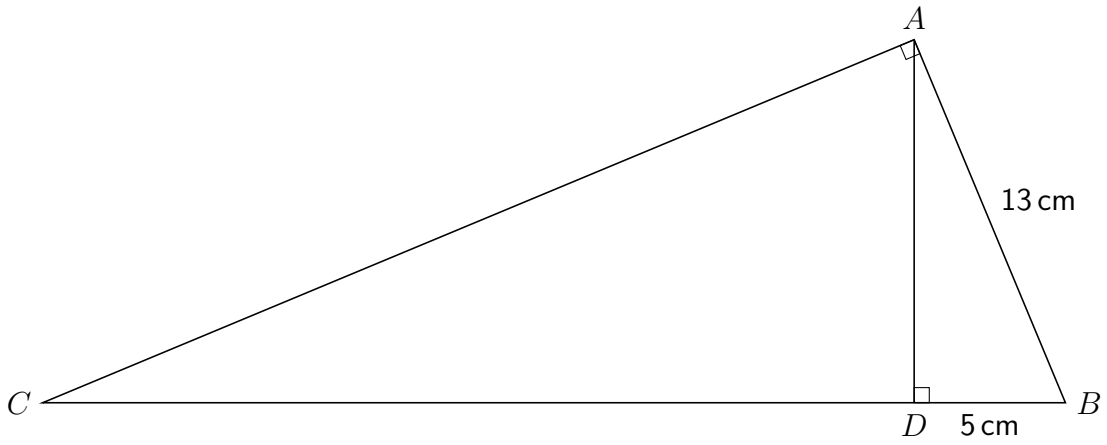
3. The diagram shows the design of a zipwire.



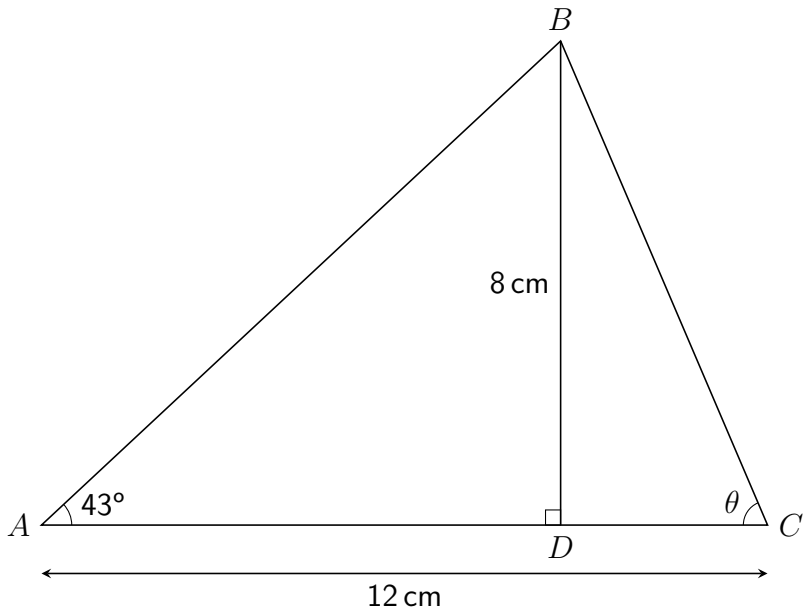
The zipwire will run between the top of two vertical posts, AB and CD.

Work out the distance AD.

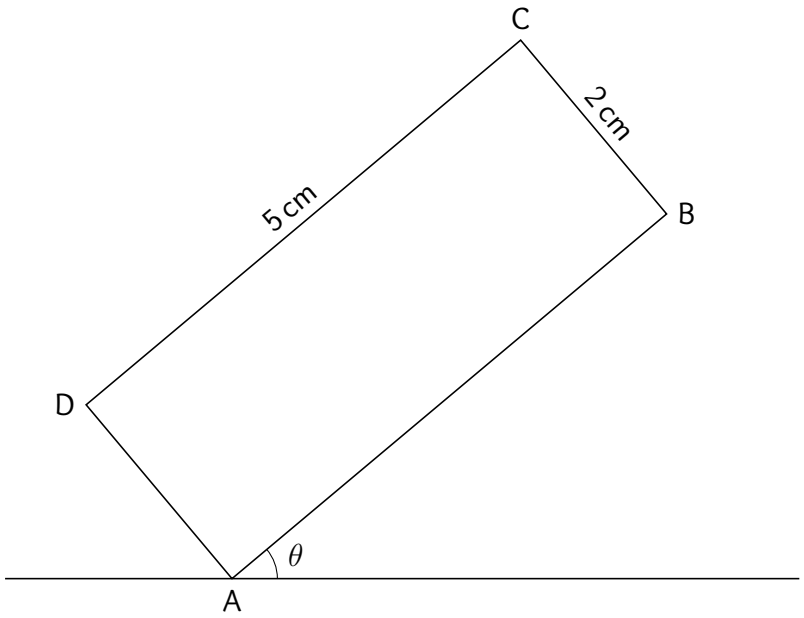
4. Work out the length CB.



5. Calculate angle θ.



6. (**Challenge**) $ABCD$ is a rectangular box (dimensions: 5 cm $\times$ 2 cm) that is rotated in the vertical plane such that point A is in contact with the horizontal floor.



(a) Explain clearly why the height of C above the floor is $5\sin\theta + 2\cos\theta$.

(b) Work out the angle that AC makes with the horizontal (in terms of θ).

(c) By thinking about the location of C as the box rotates, determine the maximum height of C.

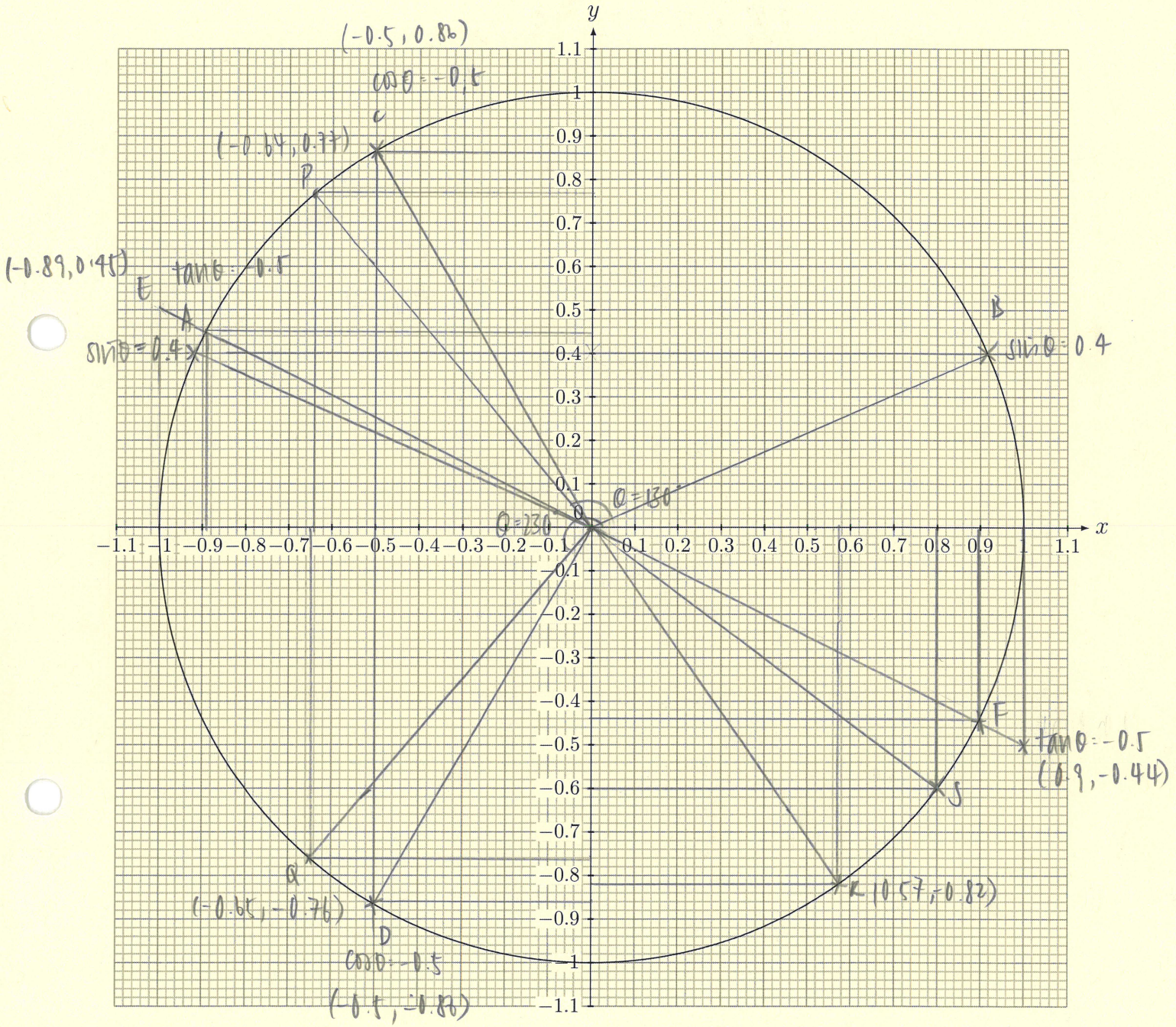
F Unit circle task results

Student Z, followed by Student Y.

29th January 2018

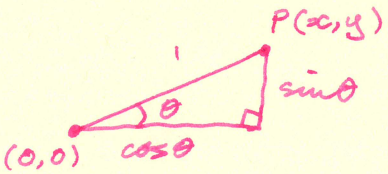
L6PMa1 Maths

Unit circle trigonometry



$P(x, y) = P(\cos\theta, \sin\theta)$

gradient of $OP = \tan\theta$



Do the following questions using the picture on page 1. Where necessary, assume $0 \leqslant \theta < 360°$.

1. (a) Use your protractor to mark the point P on the circumference where $\theta = 130°$.
   (b) Write down the coordinates of P. $(-0.64, 0.77)$
   (c) Hence work out the gradient of the line OP. gradient: $\frac{0.77}{-0.64} = -1.203125 = -1.20$ (3sf)
   (d) Write down the value of
   (i) $\sin 130°$ 0.77 (ii) $\cos 130°$ -0.64 (iii) $\tan 130°$ -1.20

2. (a) Use your protractor to mark the point Q on the circumference where $\theta = 230°$.
   (b) Write down the coordinates of Q. $(-0.65, -0.76)$
   (c) Hence work out the gradient of the line OQ. gradient: $\frac{-0.76}{-0.65} = 1.16923.. = 1.17$ (3sf)
   (d) Write down the value of
   (i) $\sin 230°$ -0.76 (ii) $\cos 230°$ -0.65 (iii) $\tan 230°$ 1.17

3. (a) Use your protractor to mark the point R on the circumference where $\theta = 305°$.
   (b) Write down the coordinates of R. $(0.57, -0.82)$
   (c) Hence work out the gradient of the line OR. gradient: $\frac{-0.82}{0.57} = -1.4387.. = -1.44$ (3sf)
   (d) Write down the value of
   (i) $\sin 305°$ -0.82 (ii) $\cos 305°$ 0.57 (iii) $\tan 305°$ -1.44

4. Point S lies at the point where $\sin\theta = -0.6$ and $\cos\theta = 0.8$.
   (a) Write down the coordinates of S. $(\cos\theta, \sin\theta)$ $\therefore S\ (0.8, -0.6)$
   (b) Mark point S on the diagram.
   (c) What is the value of θ? $360 - 37 = 323°$ $\theta = 323°$
   (d) What is the value of $\tan\theta$? $\frac{\sin\theta}{\cos\theta} = \frac{-0.6}{0.8} = -0.75$

5. (a) Mark all the points on the circumference where $\sin\theta = 0.4$. $\sin\theta$ = y coordinates
   (b) Write down the value of $\cos\theta$ for all of your points. What do you notice? A $(-0.92, 0.4)$ they are the opposite, B $(0.92, 0.4)$ negative values of each other
   (c) Work out $\tan\theta$ at your points. What do you notice? A $\tan\theta = -0.435$ B $\tan\theta = 0.435$ opposite neg values of each other.
   (d) Measure the values of θ at your points. $\theta_A = 156°$ $\theta_B = 24°$ - Add up to $180°$.

6. (a) Mark all the points on the circumference where $\cos\theta = -0.5$. $\cos\theta$ = x coordinates
   (b) Write down the value of $\sin\theta$ for all of your points. What do you notice? C $(-0.5, 0.87)$ the opposite negative values of each other, D $(-0.5, -0.87)$
   (c) Work out $\tan\theta$ at your points. What do you notice? C $\tan\theta = -1.72$ D $\tan\theta = 1.72$ opposite neg values of each other
   (d) Measure the values of θ at your points. $\theta_C = 120°$ $\theta_D = 240°$ - add to 360.

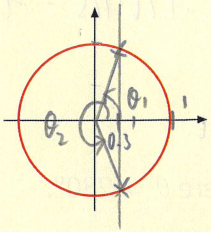
7. (a) Mark all the points on the circumference where $\tan\theta = -0.5$.
   (b) Write down the value of $\sin\theta$ for all of your points. What do you notice? E $(-0.89, 0.45)$ F $(0.9, -0.44)$ opposite and negative values of each other
   (c) Work out $\cos\theta$ at your points. What do you notice? E $\tan\theta = -0.506$ F $\tan\theta = -0.489$ ≈ -0.5
   (d) Measure the values of θ at your points. $\theta_E = 153°$ $\theta_F = 333.5°$ (333) - take away $180°$

$\frac{\sin\theta}{\cos\theta} = \tan\theta$ $\cos\theta = \frac{\sin\theta}{\tan\theta}$.

E $\cos\theta = -0.9$ F $\cos\theta = 0.88$ (opposite and negative values of each other)

For the following questions, mark the relevant points on the little unit circle, but use your calculator to solve them. Domain is $0 \leqslant \theta < 360°$ unless specified.

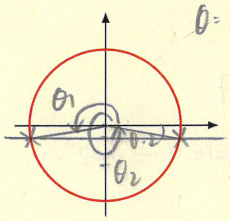
1. Solve $\cos\theta = 0.3$



$\theta_1 = 72.5$
$\theta_2 = 360 - 72.5$
$= 287.5$

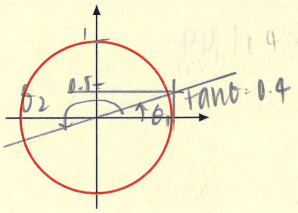
sin θ = y values

2. Solve $\sin\theta = -0.2$



$\theta = \sin^{-1}(-0.2) = -11.5$...
$\theta_2 = 360 + -11.5$...
$= 348.46... = 348°$
$\theta_1 = 180 - -11.5$...
$= 191.536... = 191.5$

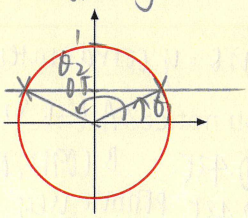
3. Solve $\tan\theta = 0.4$



$\theta_1 = 21.80...$
$= 21.8°$
$\theta_2 = 180 + 21.8°$
$= 201.8°$

4. Solve $\sin\theta = 0.5$, $-180° \leqslant \theta < 180°$

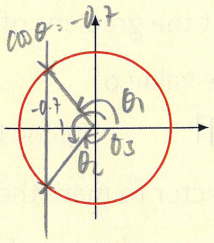
sin θ = y values



$\theta_1 = 30°$
$\theta_2 = 180 - 30° = 150°$

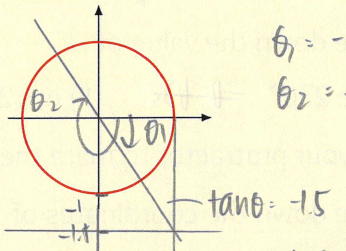
there are no values of $\sin\theta = 0.5$ between $-180° \leq \theta \leq 0$ because sine of the and thus can only take values of +0.5, not -0.5

5. Solve $\cos\theta = -0.7$, $-180° \leqslant \theta < 360°$



$\theta_1 = 134.427 = 134°$
$\theta_2 = (180 - 134.427...) + 180 = 225.57$
$= 226$
$\theta_3 = -(180° - \theta_1)$
$= -46°$

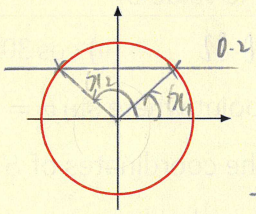
6. Solve $\tan\theta = -1.5$, $-360° \leqslant \theta < 0°$



$\theta_1 = -56.3099... = -56.3°$
$\theta_2 = -236.3099... = -236°$

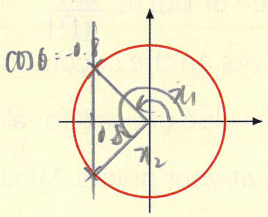
7. Solve $5\sin 2\theta = 1$

y value

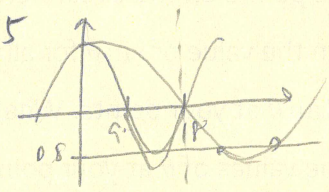


$0 \leq \theta \leq 360$
$0 \leq 2\theta \leq 720°$
$5\sin 2\theta = 1$ let $2\theta = x$
$\sin 2\theta = \frac{1}{5}$
$\sin x = \frac{1}{5} = 0.2$
$x = \sin^{-1}(\frac{1}{5}) = 11.5369...$
$x_1 = 11.5369 = 11.5°$
$x_2 = 180 - 11.5369... = 168.463... = 168°$ *

8. Solve $\cos\frac{1}{2}\theta = -0.8$, $-180° \leqslant \theta < 180°$



$-180° \leq \theta \leq 180$
$-90° \leq \frac{1}{2}\theta \leq 90°$
$\cos\frac{1}{2}\theta = -0.8$
$\frac{1}{2}\theta \cos^{-1}(-0.8) = 143.130...$
$\frac{1}{2}\theta = 143°, 217°; -143°, -217°$
$\theta = 286°, 434°, -286°, -434°$
no values exceed the range

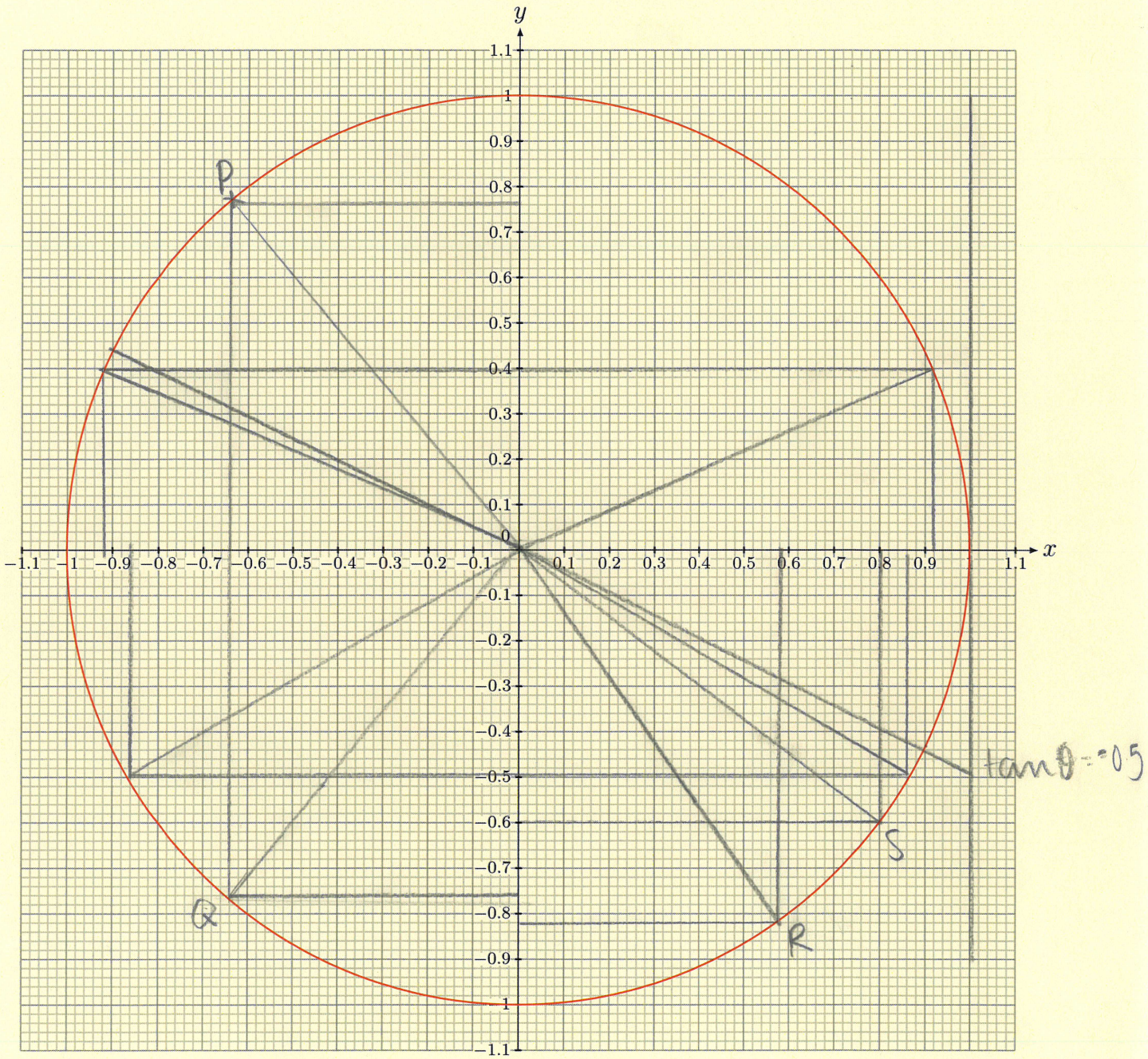


*7: $x_1 = 11.5$ $2\theta = x$
$x_2 = 168°$

$2\theta = 11.5°, 168°, 371.5°, 528°,$
$\theta = 5.75°, 84°, 185.75°, 268°$

L6PMa1 Maths

Unit circle trigonometry

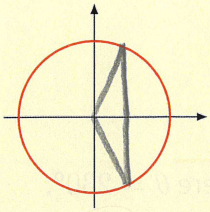


Do the following questions using the picture on page 1. Where necessary, assume $0 \leqslant \theta < 360°$.

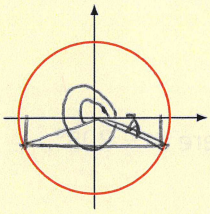
1. (a) Use your protractor to mark the point P on the circumference where $\theta = 130°$.
   (b) Write down the coordinates of P.
   (c) Hence work out the gradient of the line OP.
   (d) Write down the value of
   (i) $\sin 130°$ (ii) $\cos 130°$ (iii) $\tan 130°$
2. (a) Use your protractor to mark the point Q on the circumference where $\theta = 230°$.
   (b) Write down the coordinates of Q.
   (c) Hence work out the gradient of the line OQ.
   (d) Write down the value of
   (i) $\sin 230°$ (ii) $\cos 230°$ (iii) $\tan 230°$
3. (a) Use your protractor to mark the point R on the circumference where $\theta = 305°$.
   (b) Write down the coordinates of R.
   (c) Hence work out the gradient of the line OR.
   (d) Write down the value of
   (i) $\sin 305°$ (ii) $\cos 305°$ (iii) $\tan 305°$
4. Point S lies at the point where $\sin\theta = -0.6$ and $\cos\theta = 0.8$.
   (a) Write down the coordinates of S.
   (b) Mark point S on the diagram.
   (c) What is the value of θ?
   (d) What is the value of $\tan\theta$?
5. (a) Mark all the points on the circumference where $\sin\theta = 0.4$.
   (b) Write down the value of $\cos\theta$ for all of your points. What do you notice?
   (c) Work out $\tan\theta$ at your points. What do you notice?
   (d) Measure the values of θ at your points.
6. (a) Mark all the points on the circumference where $\cos\theta = -0.5$.
   (b) Write down the value of $\sin\theta$ for all of your points. What do you notice?
   (c) Work out $\tan\theta$ at your points. What do you notice?
   (d) Measure the values of θ at your points.
7. (a) Mark all the points on the circumference where $\tan\theta = -0.5$.
   (b) Write down the value of $\sin\theta$ for all of your points. What do you notice?
   (c) Work out $\cos\theta$ at your points. What do you notice?
   (d) Measure the values of θ at your points.

For the following questions, mark the relevant points on the little unit circle, but use your calculator to solve them. Domain is $0 \leqslant \theta < 360°$ unless specified.

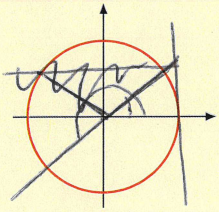
1. Solve $\cos\theta = 0.3$



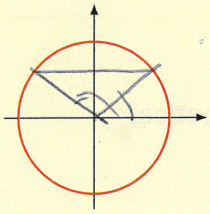
2. Solve $\sin\theta = -0.2$



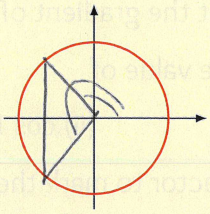
3. Solve $\tan\theta = 0.4$



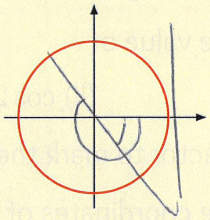
4. Solve $\sin\theta = 0.5, \quad -180° \leqslant \theta < 180°$



5. Solve $\cos\theta = -0.7, \quad -180° \leqslant \theta < 360°$

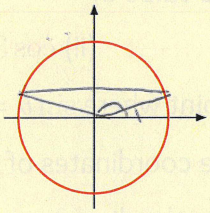


6. Solve $\tan\theta = -1.5, \quad -360° \leqslant \theta < 0°$

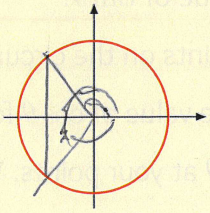


7. Solve $5\sin 2\theta = 1$

$\sin 2\theta = \frac{1}{5}$

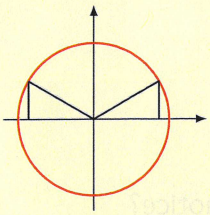


8. Solve $\cos\frac{1}{2}\theta = -0.8, \quad -180° \leqslant \theta < 180°$

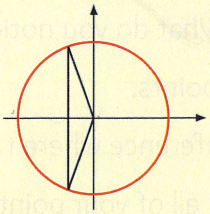


Write down an equation you might use the following pictures to solve.

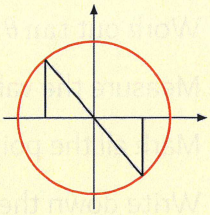
(a)



(b)



(c)



Unit Circle Trigonometry.

1.) b) $(-0.64, 0.76)$

c) $\frac{0.76}{-0.64}$

$= -1.1875$

d) $\sin 130 = 0.76$

$\cos 130 = -0.64$

$\tan 130 = -1.1875$

2.) b) $(-0.64, -0.76)$

c) $\frac{-0.76}{-0.64}$

$= 1.1875$

d) $\sin 230 = -0.76$

$\cos 230 = -0.64$

$\tan 230 = 1.1875$

3.) b) $(0.58, -0.82)$

c) $\frac{-0.82}{0.58}$

$= -1.4138$

d) $\sin 305 = -0.82$

$\cos 305 = 0.58$

$\tan 305 = -1.4138$

4.) a) $(0.8, -0.6)$

c) $323°$

d) $\frac{-0.6}{0.8}$

$= -0.75$

5.) b) 0.92 , -0.92

positive & negative.

c) $\frac{0.4}{0.92}$ $\frac{0.4}{0.92}$

$= 0.43$ $= -0.43$

+ve & -ve.

d) 24° , 156°

6.) b) 0.86 , -0.86

c) $\frac{0.86}{0.5}$ $\frac{-0.86}{0.5}$

$= 1.72$ $= -1.72$

d) 210° , 330°

7.) ~~a) 154°, 334°~~

b) 0.44 , -0.44

c) ~~$\frac{0.44}{154}$~~ ~~$\frac{-0.44}{334}$~~ $\frac{\sin}{\cos} = \tan$

0.44×0.5 -0.44×0.5

$= 0.22$ $= -0.22$

d) 154° 334°

1.) 72.5° , 287.5°

2.) 191.5° , 348.5°

3.) 21.8° , 201.8°

4.) 30° , -150°

5.) -134.4° , 134.4° , 225.6°

6.) -236.3° , -56.3°

7.) 5.77° , 174.23°

8.) 71.6° , -71.6°

a) $\sin\theta$ b) $\cos\theta$ c) $\tan\theta$

$\sin\theta = 0.5$ $\cos\theta = -0.5$ $\tan\theta = -1.5$

-0.3

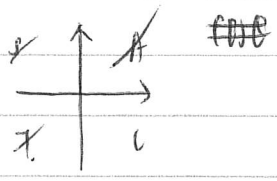
G Student Z post-test results

1. a) alternative form of a degree.

b) $2\pi = 360$

$1 = \frac{360}{2\pi} = \frac{180}{\pi}°$

2. $\cos\theta > 0$ $\tan\theta < 0$ θ in reflex.



3. $\sin\left(\theta - \frac{2\pi}{3}\right) = \frac{1}{2}$

$\sin\theta\cos\left(\frac{2\pi}{3}\right) - \cos\theta\sin\left(\frac{2\pi}{3}\right) = \frac{1}{2}$

$\sin\theta\left(-\frac{1}{2}\right) - \cos\theta\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{2}$

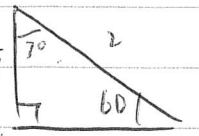
$\frac{\sin\theta}{\cos\theta}\left(-\frac{1}{2}\right) - \left(\frac{\sqrt{3}}{2}\right) = \frac{1}{\cos\theta}$

$\frac{1}{2}(\sin\theta - \cos\theta(\sqrt{3})) = \frac{1}{2}$

$\sin\theta - \cos\theta(\sqrt{3}) = 1$

$\cos\frac{2\pi}{3} = \cos\frac{\pi}{3}$.

$\cos\frac{\pi}{3} = \cos\frac{180}{3} = \cos 60°$



$\cos\theta = \frac{1}{2}$

$\sin\theta = \frac{\sqrt{3}}{2}$

$2\pi > \theta > -2\pi$

$\sin\left(\theta - \frac{2\pi}{3}\right) = \frac{1}{2}$

$-10\pi - \frac{2\pi}{3} < \theta - \frac{2\pi}{3} < 10\pi - \frac{2\pi}{3}$

$-\frac{32\pi}{3} < \theta - \frac{2\pi}{3} < \frac{28}{3}\pi$



$\theta - \frac{2\pi}{3} = \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}, (3\pi - \frac{\pi}{6}), (4\pi + \frac{\pi}{6}), (5\pi - \frac{\pi}{6}), (6\pi + \frac{\pi}{6}), (7\pi - \frac{\pi}{6}), (8\pi + \frac{\pi}{6}), (9\pi - \frac{\pi}{6}), (10\pi + \frac{\pi}{6})$

$= -\frac{7\pi}{6}, (-12\pi + \frac{\pi}{6}), (-3\pi - \frac{\pi}{6})$, ~~12~~

$= 20$.

20 solutions:

4. $\sin x - \cos x = R(\cos\alpha\sin x - \sin\alpha\cos x)$

$\sin x - \cos x = R(\cos x\cos\alpha + \sin x\sin\alpha)$

$\sin x - \cos x = R\cos x\cos\alpha + R\sin x\sin\alpha$

$\sin x - \cos x = (R\cos\alpha)\cos x + (R\sin\alpha)\sin x$

$R\sin\alpha = 1 \quad R^2\sin^2\alpha = 1$

$R\cos\alpha = -1 \quad R^2\cos^2\alpha = 1$

$R^2(1) = 2$

$R = \sqrt{2}$

$\sin\frac{\sqrt{2}}{2}$



$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\sin\alpha = \frac{\sqrt{2}}{2}$

$\alpha = 45° = \frac{\pi}{4}$

$\sqrt{2}\sin x = 1 \quad \sqrt{2}\cos x = -1$

$\sin x = 0.707 \quad \cos x = -0.707$.

$x = 0.785... \quad x = 2.356...$

$\pi - 0.785... \quad 2.356...$

$\alpha = \pi - \arcsin\left(\frac{1}{\sqrt{2}}\right) = \pi - \frac{\pi}{4}$

$\alpha = \frac{3\pi}{4}$

$\therefore \sin x - \cos x = \sqrt{2}\left(\cos x - \frac{3\pi}{4}\right)$

$\sec\theta = \frac{1}{\cos\theta}$ $\tan\frac{\theta}{2} = A$

5 $4\sin\frac{\theta}{2} = \sec\frac{\theta}{2}$ $-\pi \le \theta < \pi$

$4\sin A = \frac{1}{\cos A}$

$4\sin A\cos A = 1$

$2(2\sin A\cos A) = 1$

$2(\sin 2A) = 1$

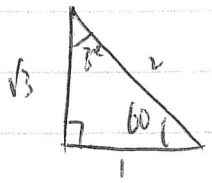
$\sin 2A = \frac{1}{2}$

$\sin 2A = \sin^{-1}\left(\frac{1}{2}\right)$

$2A = \frac{\pi}{6}$.

$2 \times \frac{\theta}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

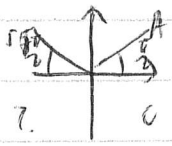


$\sin 30 = \frac{1}{2}$

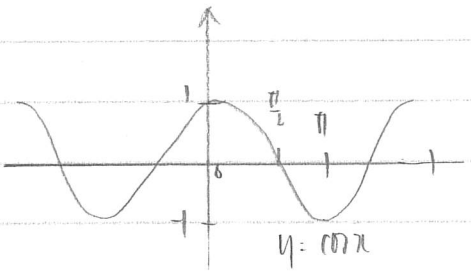
$30 = \frac{\pi}{6}$

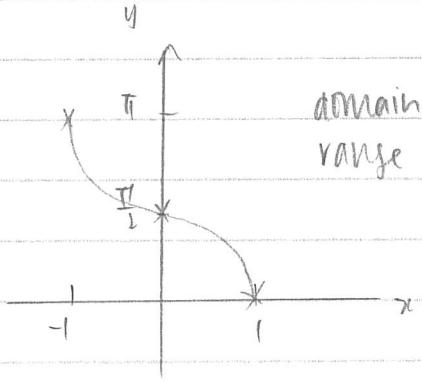
$\frac{\theta}{2} \times 2 = \sin^{-1}\left(\frac{1}{2}\right)$

$\theta = \frac{\pi}{6}$



6. a) $\boxed{y = \cos^{-1}(x) \quad -\pi \le x \le \pi}$





domain $(-1 \le x \le 1)$

range $(-\pi \le y \le \pi)$

b) $\cos^{-1}(\cos\frac{7\pi}{6})$.

$\cos\theta = x$

$\cos^{-1}(x) = \theta$

$\arccos x = \theta$

$\arccos(\cos\frac{7\pi}{6}) = \frac{5\pi}{6}$.

7. $y = A\cos[w(x-\phi)]$. $A\cos(R(x \pm \theta))$.

~~$A\cos(1) = 3$~~ ~~Period~~

~~$A(\cos(1)) = 3$~~ ~~Amplitude~~

$\boxed{A = 3}$

$\boxed{w = \frac{2\pi}{3}}$

$\boxed{\phi = \frac{\pi}{3}}$.

~~Made horizontal~~

~~Moved vertical~~

$y = 3\cos(w(x-\theta))$

$y = 3\cos\left(\frac{1}{3}(x-\phi)\right)$ $-\frac{1}{3}$ by 3

$y = 3\cos\left(\frac{1}{3}(x-\frac{\pi}{3})\right)$

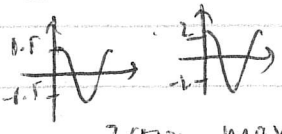
~~$\cos(\theta) = \theta$~~

$\cos(\pi) = -1$

$\cos\left(\frac{1}{3}\pi\right) = 0.5$

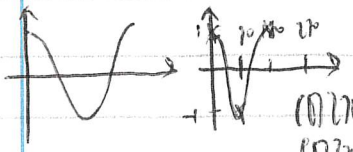
$\cos x$

$\frac{1}{2}\cos x$ $2\cos x$



$3\cos x$ - max at 3
min at -3.

$\cos 2x$ $\cos x$:



$\cos 2x$ - 1 scalar turn

$\cos 3x$ - scalar three times