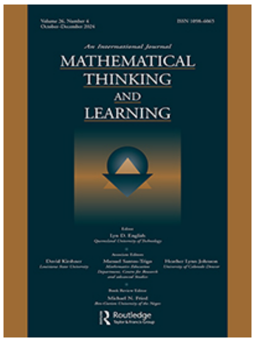

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RESEARCH ARTICLE

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Promoting elements of mathematical knowledge for teaching related to the notion of assumptions

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ABSTRACT

Although the notion of *assumptions* is important in mathematical activity as early as the elementary school, there is limited research on how to help elementary teachers develop *mathematical knowledge for teaching* related to assumptions. In this paper, we discuss the theoretical foundation and implementation of an intervention that aimed to promote three key elements of this knowledge among prospective elementary teachers. We developed the intervention in a 4-year design experiment that we conducted in an undergraduate mathematics course for prospective elementary teachers. The intervention's design utilized the notion of *productive ambiguity* in the context of a deliberately ambiguous task where the role of assumptions surfaced and was reflected upon in purposefully organized ways. We focus on the implementation of the intervention in the last of 5 research cycles of our design experiment to exemplify our theoretical framework and to discuss the promise of the intervention to promote the three targeted elements of knowledge. The approach to promoting mathematical knowledge for teaching that we discuss in the paper offers a paradigmatic case of how teacher educators can use productive ambiguity to design learning opportunities for prospective teachers to intertwine mathematical learning with pedagogical awareness thus developing pedagogically functional mathematical knowledge.

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Assumptions; design experiment; elementary school; mathematical knowledge for teaching; modeling; productive ambiguity; proof; task design; teacher education

Introduction

In mathematics, *assumptions* denote the statements doers of mathematics use or accept (often implicitly) and on which they base their mathematical claims (Fawcett, 1938). Accordingly, assumptions are fundamental to mathematical activity at all levels of education. An area where the notion of assumptions is particularly relevant in school mathematics, including in the elementary school, relates to when students engage in solving modeling tasks (Brown & Stillman, 2017; English, 2006; Suh et al., 2021). Modeling tasks tend to be open-ended problems that are not well defined, a characteristic which forces students to make assumptions in solving these tasks (Cai et al., 2022). As Brown and Stillman (2017) pointed out, "[i]n successfully passing from a real-world problem statement to a mathematical model [. . .] the modeler engages in *making assumptions*, formulating, and mathematizing" (p. 355; emphasis added).

Another area where assumptions are particularly relevant in school mathematics is when students engage in proving tasks with assumptions being the building blocks of arguments. In the context of proving, assumptions typically become the explicit focus of attention in secondary school, especially in the context of Euclidean geometry, and less so in the elementary school. According to A. J. Stylianides (2007), introducing elementary students to the notion of assumptions in proving can help promote

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mathematical sense-making and allow students to see how conclusions follow from accepted statements; can address a discontinuity in students' mathematical education when assumptions get explicitly addressed only in the secondary school; and can help create an "authentic" (Lampert, 1992) school mathematics curriculum that reflects appropriately a key notion of mathematical practice. Few studies explicitly examined the role of assumptions in proving at the elementary (A. J. Stylianides, 2007, 2016; Ball, 1993), secondary (Fawcett, 1938; Jahnke & Wambach, 2013; Komatsu et al., 2019; Komatsu, 2017), or university (Larsen & Zandieh, 2008) levels.

As far as teacher education is concerned, there is a scarcity of research on the design of interventions to effectively help teachers develop *Mathematical Knowledge for Teaching* (*MKT*; Ball et al., 2008) related to assumptions. This is concerning because, without appropriate knowledge, teachers are unlikely to offer their students productive learning opportunities about assumptions or exploit the full potential of proving or modeling tasks. The problem may be more intense for prospective elementary teachers who are non-specialists in mathematics. Based on the findings from a survey administered to 50 elementary teachers on their confidence in orchestrating mathematical modeling activities in their elementary classrooms, Suh et al. (2021) found "that making assumptions and defining variables to narrow and focus on the mathematical problem was the most challenging part of problem formulation" (p. 122). Findings from an analysis of 16 textbooks written for mathematics courses for prospective elementary teachers in the United States (McCrory & Stylianides, 2014) exacerbate the concern about the limited attention paid to the notion of assumptions in elementary mathematics teacher education: only two textbooks had "assumption" in their index and none of them had an explicit commentary on the role of assumptions in mathematical activity.

In this paper, we take a step toward responding to this need for research at the elementary teacher education level by addressing the following research question: *What is a promising intervention that can be used in teacher education to help prospective elementary teachers develop important elements of MKT related to assumptions?* In posing this question, we were guided by both a pragmatic concern to design an intervention that would promote the intended knowledge and a theoretical concern to explain the ways in which the intervention achieved its aims. This dual concern is characteristic of design-based research (Cobb et al., 2003). Indeed, our work to develop and theorize the intervention was part of a 4-year design experiment we conducted in an undergraduate mathematics course for prospective elementary teachers. In the Background and Theoretical Framework sections, we delineate three important elements of MKT related to assumptions that were the focus of the intervention and we identify theory-based design aspects of such an intervention. In the Methodology section, we revisit the research question to explain how we built on these theoretical aspects to design our intervention and examine its promise empirically.

Background

As a background for the paper, we first use an example from upper-level mathematics to introduce two of the three elements of knowledge about assumptions we targeted in the intervention. These elements are important for all users of mathematics and, as we illustrate later using an elementary classroom episode, are relevant also to elementary mathematics teaching. Using the same episode, we argue that tasks with purposefully ambiguous conditions offer a productive context for students' engagement with the notion of assumptions in mathematical activity, and we link this idea to the third element of knowledge we targeted in the intervention.

Upper-level mathematics

In his teaching experiment with advanced secondary students, Fawcett (1938) asked his students to analyze a proof of the theorem "the sum of the measures of the interior angles of a triangle is 180°" to help them see how the theorem depends on specific assumptions. Subsequently, he directed the students' attention to the parallel postulate of Euclidean geometry and, through perusal of the implications of modifying this postulate, he introduced the students to the elliptic and hyperbolic geometries where the sum of the measures of the interior angles of a triangle is more or less than 180°, respectively. Fawcett's students realized that the choice of assumptions in each of these geometries offered different starting points for their proofs and respective conclusions about the sum of the interior angles of a triangle. Although these conclusions (answers to the proving task) appeared to be contradictory with one another, they were all correct and meaningful within their underpinning assumptions and respective geometrical spaces (planar, spherical, etc.).

By engaging his students with the notion of assumptions in the context of axiomatic structures, Fawcett (1938) aimed for the students to understand that "conclusions are 'true' only within the limits of the assumptions on which they depend" (p. 71). This aim applies to different types of assumptions not limited to mathematical axioms; it can apply, for example, to assumptions not necessarily mathematical, made within a modeling or an open-ended task. From this general aim, we derive two important elements of knowledge about assumptions:

> Element 1: Understanding that a conclusion about a situation (e.g., an answer to a task) depends on the assumptions on which the argument that led to it was based.

> Element 2: Understanding that different conclusions (which sometimes may appear to be contradictory) about a situation may all be correct with respect to their underpinning assumptions.

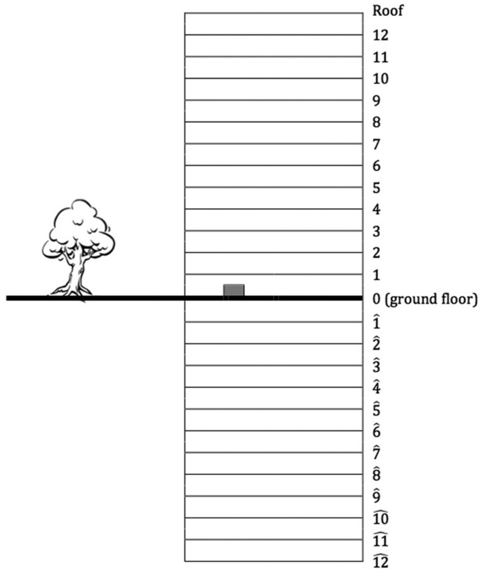
These elements of knowledge are not specific to the work of mathematicians or advanced secondary students but also to elementary teachers as we illustrate next.

Elementary school mathematics

The only elementary classroom episode we identified in the literature where assumptions was an explicit instructional focus was first reported by Ball (1993) as the teacher-researcher and was further analyzed by A. J. Stylianides (2007, 2016) from the perspective of task ambiguity and its interplay with the notion of assumptions. In this episode Ball used an ambiguous task, the Floors Problem (Figure 1), to introduce her third graders (7–8-year-olds) to the notion of assumptions.

Ball expected that some students would assume the person in the problem should follow a direct route to floor 2, write two-addend number sentences to represent the trips (e.g., $\hat{4} + 6 = 2$), and answer that there are 25 ways to get to floor 2. She expected further that some other students would assume the person could make multiple stops en route to floor 2, write multi-addend number sentences (e.g., $\hat{6} + 10 + 3 - 2 + 1 - 4 = 2$), and answer that there are infinitely many ways. According to Ball's journal entry, "*With each of the assumptions kids have made, the answer would be different* (25 vs. infinite) and I wanted the kids to see that, and to begin to develop a sense for the role of *assumptions*" (reported in A. J. Stylianides, 2007, p. 370; emphasis in original). Indeed, in the episode the third graders pursued different pathways based on different legitimate assumptions about the task's conditions and reached different conclusions. Ball resolved the contradictory situation that emerged by introducing the class to the notion of assumptions and helping students identify the answers to the problem to which different legitimate assumptions led.

Our account of this episode illustrates that Elements 1 and 2 are important also for an elementary teacher to organize and manage students' engagement with the notion of assumptions in the context of an ambiguous task. In fact, Ball not only understood that her students' answers to the Floors Problem depended on their assumptions about the nature of the person's trips (Element 1) and that their apparently contradictory answers could all be accepted as correct with respect to their underpinning assumptions (Element 2), but she also tried to make these understandings explicit to her students. The way Ball designed and implemented the Floors Problem demonstrated her understanding of important pedagogical implications of the interdependence between task ambiguity and multiple legitimate assumptions/conclusions; this was Element 3 of teacher knowledge that we targeted in the intervention, which we unpack later.



Problem statement as given to prospective teachers together with the building model on the left:

> How many ways are there for a person to get to the second floor? Prove your answer.

A note on notation:

The circumflex (ˆ) above the numerals was used in the place of the minus sign of negative integers (e.g., $\hat{4}$ meant "minus 4"). Ball (1993) considered that this notation could help her third graders focus "on the idea of a negative number as a number, not as an *operation* (i.e., subtraction) on a positive number" (p. 380). We followed the same notation in our work with the prospective teachers.

Figure 1. The floors problem (derived from Ball, 1993).

Theoretical framework

The theoretical framework underpinning the design of interventions in the undergraduate course that we used as the setting for our research had several interrelated features three of which are particularly relevant to the focal intervention. The first concerns our conceptualization of teacher knowledge and the delineation of the three elements of MKT we targeted in the intervention. The second concerns our instructional approach in the course to promote MKT through intertwining mathematical learning with pedagogical awareness. The third concerns our conceptualization of the interplay between the design and implementation of the interventions in the course using the notions of "hypothetical" and "actual learning trajectories;" the key role of the instructor during the implementation of the interventions is also considered under this feature.

Conceptualizing teacher knowledge related to assumptions: three elements of MKT

The three elements of MKT about assumptions that we identified earlier are summarized in Figure 2 (we introduce sub-Elements 3a and 3b later in the paper). First we discuss these elements using the MKT framework, and then we comment on their scope and relationship.

Mathematical Knowledge for Teaching and the three elements

Following Shulman's (1986) influential work on the nature of teachers' knowledge, notably, the notion of *pedagogical content knowledge*, several frameworks have been developed to describe components of knowledge that teachers of mathematics draw on, or need to have, for their professional practice (e.g., Ball et al., 2008; Rowland et al., 2009). These frameworks have spurred several other strands of research pertaining to mathematics teachers' knowledge, including research on elements of knowledge for teaching particular disciplinary concepts such as proof (A. J. Stylianides & Ball, 2008) and research on how teachers' mathematical knowledge is promoted in teacher education (A. J. Stylianides & Stylianides, 2014b; Adler & Davis, 2006; Silverman & Thompson, 2008).

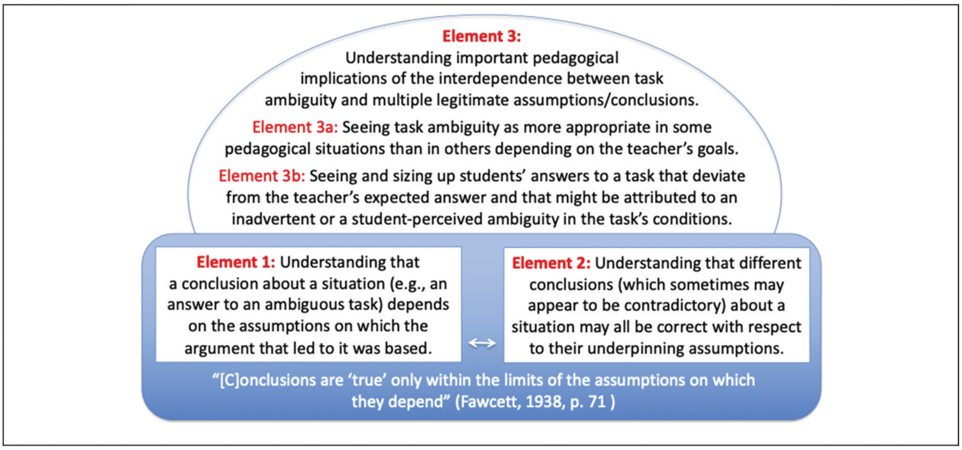


Figure 2. The three elements of knowledge about the role of assumptions in mathematical activity promoted by the focal intervention.

The notion of MKT (Ball et al., 2008) is associated with one of the mostly used frameworks that emerged from Shulman's work and denotes the kind of "pedagogically functional mathematical knowledge" (Ball & Bass, 2000, p. 95) that is necessary for teachers in managing the mathematical demands of their professional practice. According to Ball et al. (2008), MKT includes two of Shulman's (1986) initial categories: *subject matter knowledge* and *pedagogical content knowledge*. Within subject matter knowledge they identified two important subdomains: *specialized content knowledge*, which is content knowledge unique to teaching, and *common content knowledge*, which is content knowledge needed by teachers and nonteachers alike. They also identified two important subdomains within pedagogical content knowledge: *knowledge of content and students* and *knowledge of content and teaching*.[1] Ball et al. acknowledged that it is not easy to discern the boundaries between the categories, noting that "this affects the precision (or lack thereof) of [their] definitions" (p. 403). Given this caveat and our research purposes whereby a precise classification of the elements of MKT targeted by the intervention is not necessary, we situate Elements 1 and 2 in the broader subject matter knowledge part of the MKT framework, that is, in the combined domain of common content and specialized content knowledge.

Notwithstanding the importance of subject matter knowledge, we argue that mathematics courses for prospective teachers like the course in our study should not merely promote subject matter knowledge and expect subsequent mathematics pedagogy (methods) courses to promote pedagogical content knowledge that prospective teachers will then be able to integrate in teaching. This "integrative model" of teacher knowledge (Gess-Newsome, 1999) appears to be problematic, partly because many prospective teachers finish teacher training unable to effectively integrate their mathematical and pedagogical understandings (Silverman & Thompson, 2008). On the other hand, the "transformative model" of teacher knowledge (Gess-Newsome, 1999), whereby prospective teachers are offered purposefully designed opportunities to create a "new" knowledge by extending and connecting their mathematical and pedagogical understandings, has been viewed as "a solid starting point for thinking about the development of MKT" (Silverman & Thompson, 2008, p. 502). It is also consistent with other views in the literature such as that mathematics courses for prospective teachers need to maintain sight of the domain of application of the targeted knowledge (A. J. Stylianides & Stylianides, 2014b) and promote understandings that are specific "to the mathematics that teachers need to know and know how to use" (Adler & Davis, 2006, p. 271). Element 3 fits in the broader pedagogical content knowledge part of the MKT framework, that is, the combined domain of knowledge of content and students and the domain of knowledge of content and teaching. Also, the development of Element 3 can support the transformation of the mathematical understandings reflected in Elements 1 and 2 to ones that are also "pedagogically powerful" (see Silverman & Thompson, 2008). This, in turn, can empower prospective teachers to see how these understandings might be used in relevant pedagogical situations.

In more detail, Element 3 is motivated by our earlier discussion of task ambiguity being one key way in which elementary teachers can productively engage their students with the notion of assumptions and draws on Grosholz's (2007) notion of *productive ambiguity* as elaborated on by Foster (2011). According to Foster, an ambiguity is not the same as an error but "derives from a significant degree of uncertainty, caused by a lack of specification regarding a particular feature or an unstated assumption, paradigm or frame of reference" (p. 3); this allows one to see the same situation in multiple or new ways that can be useful (productive). Foster (2011, p. 4) discussed the potential pedagogical usefulness of productive ambiguity in the mathematics classroom as follows:

> If teachers specify mathematical tasks and definitions too tightly, they leave students little room for maneuver, restricting their freedom to explore interesting tensions and possibilities. Sometimes an ambiguity can be quickly resolved by providing additional information but, where the ambiguity is potentially productive, the dilemma, the tension and the contrast is lost and the energy is dissipated.

Several researchers have utilized productive ambiguity to create learning opportunities for students and prospective teachers related to particular mathematical ideas such as fractions (e.g., Marmur & Zazkis, 2022). Also productive ambiguity is a common feature of modeling tasks, which "are often characterized as being open-ended problems that are not well defined, allow multiple solutions, and lack necessary information (which in turn forces students to make assumptions or gather data on their own)" (Cai et al., 2022, p. 433).

The two sub-elements of Element 3 (Figure 2) delineate the precise focus of the intervention with respect to this element. We see no hierarchical relationship between Elements 3a and 3b as they correspond to different components of MKT: Element 3a refers to a teacher's pedagogical judgment about when it might be appropriate to use an ambiguous task to promote certain goals (see *knowledge of content and teaching*), whereas Element 3b refers to a teacher's ability to "see and size up" (Ball et al., 2005) unexpected student answers to a task and their possible link to task ambiguity (see *knowledge of content and students*). Regarding Element 3a, a teacher might deliberately design an ambiguous task to raise students' awareness of mathematical language and introduce them to the role of assumptions in mathematical activity (Ball, 1993; Komatsu et al., 2019). This exemplifies a pedagogical situation when productive ambiguity is "purposefully exploited for the learning of mathematics" (Foster, 2011, p. 7). Yet ambiguity may not always be beneficial to learners (Foster, 2011) such as in a written assessment of a mathematical concept different from assumptions. However, task ambiguity may still arise unexpectedly (Foster, 2011), and it can be actual or student perceived. In this kind of pedagogical situation, which relates to Element 3b, teachers should be able to see and size up students' unexpected answers to determine, for example, whether two students who reached contradictory conclusions have both applied sound reasoning following different legitimate interpretations of the task's conditions (A. J. Stylianides, 2007). Research in the area of modeling tasks shows that teachers can feel overwhelmed by task ambiguity or when responding to students' contributions (Cai et al., 2022; Suh et al., 2021).

Scope of and relationship between Elements 1–3

Elements 1–3 do not form a comprehensive package of MKT related to assumptions, but they are nevertheless essential elements of teacher knowledge when managing or planning instruction related to assumptions in the context of ambiguous tasks. Although task ambiguity relates both to the notion of assumptions and to other areas of mathematics (Byers, 2007) such as fractions (Marmur & Zazkis, 2022), it does constitute one key way to engage elementary students with the notion of assumptions (A. J. Stylianides, 2007, 2016; Ball, 1993). Another way would be, for example, Fawcett's (1938) approach to proof analysis discussed earlier or to have students apply a procedure in a situation where some of its underlying assumptions were violated and ask students to interpret a nonsensical result (e.g., divide by zero and reach the conclusion $1 = 2$) (Powers, 2013).

Regarding the relationship between Elements 1–3, Elements 1 and 2 each capture complementary aspects of the fundamental idea that "conclusions are 'true' only within the limits of the assumptions on which they depend" (Fawcett, 1938, p. 71), and so we addressed them together in the intervention. We addressed Element 3 last as it presupposes mathematical understandings about the interdependence between task ambiguity and multiple legitimate assumptions/conclusions, that is, the understandings captured in Elements 1 and 2.

Promoting Mathematical Knowledge for Teaching through the intertwining of mathematical learning and pedagogical awareness

Our approaches in the course to promote MKT embodied the view of MKT as "pedagogically functional mathematical knowledge" (Ball & Bass, 2000, p. 95). They aimed to intertwine prospective teachers' mathematical learning with pedagogical awareness to facilitate the development of mathematical knowledge that is pedagogically powerful (Silverman & Thompson, 2008), and useful in and for teaching (A. J. Stylianides & Stylianides, 2014b; Ball & Bass, 2000).

One of the approaches, discussed in A. J. Stylianides and Stylianides (2014b), was the design and implementation of a special kind of mathematics tasks contextualized in a pedagogical context, called "pedagogy-related mathematics tasks." The task's pedagogical context gives information about a classroom situation or points to didactical considerations related to such a situation that are essential for prospective teachers' response to the task's mathematical question. The general idea underpinning the design of pedagogy-related mathematics tasks is the use of a pedagogical context to situate or motivate the study of mathematical ideas, which has been used also in other approaches to promoting MKT reported in the literature (e.g., Heid & Wilson, 2015; Wasserman et al., 2019).

The approach to promoting MKT that is illustrated in the focal intervention is similar to the previous approach in that it aims to intertwine prospective teachers' mathematical learning with pedagogical awareness. However, it does so not in the domain of a self-contained task, as in the case of pedagogy-related mathematics tasks, but across two parts:

- Part 1 is *a mathematics task devoid of a pedagogical context* that can support mathematical learning and create a potential space for pedagogical reflection on that learning.
- Part 2 are *follow-up prompts* that aim to help actualize the potential for pedagogical reflection and intertwine it with mathematical understandings emerging from Part 1.

The extent to which the task in Part 1 and prompts in Part 2 fulfill their intended purposes is an empirical question that design-based research is well positioned to examine. Also, the iterative aspect of design-based research can support improvements to both parts so as to better fulfill their purposes. In the focal intervention Part 1 was the Floors Problem (Figure 1) that lacks a pedagogical context (and thus could not be classified as a pedagogy-related mathematics task) but whose potential to support mathematical learning related to assumptions even at the elementary school level was nevertheless established empirically (A. J. Stylianides, 2007, 2016; Ball, 1993). Part 2 took the form of "conceptual awareness pillars" (G. J. Stylianides & Stylianides, 2009).

The notion of *conceptual awareness pillars*, or simply *pillars*, emerged from our design-based research and denotes "instructional activities that aim to direct students' [including prospective teachers'] attention to their conceptions [understandings, beliefs, pedagogical dispositions, etc.] about a particular mathematical topic" (G. J. Stylianides & Stylianides, 2009, p. 322). In the focal intervention, the pillars were reflection/discussion questions (summarized in Table 2, to be discussed later) that aimed to support prospective teachers' "pedagogical conceptualizations of the mathematics" (Silverman & Thompson, 2008, p. 507) by directing their attention to a mathematico-pedagogical space pertaining to assumptions in the context of an ambiguous task. In doing so, the pillars created opportunities for expression and development of prospective teachers' awareness of the pedagogical implications of their mathematical understandings from Part 1.

The notion of pillars revolves around that of "awareness." According to Mason (1998), "[b]eing aware is a state in which attention is directed to whatever it is that one is aware of" (p. 254), with this phrase referring "both to being explicitly aware and to being potentially aware" (p. 254). Mason views the turning of a potential awareness to an explicit awareness by means of directed attention as a key aspect of both learning mathematics and learning to teach mathematics; in our study this aspect of learning was served primarily by the implementation of the pillars. We note, however, that, while the pillars designed objects of attention around which prospective teachers' reflection/discussion centered purposefully rather than serendipitously (Silverman & Thompson, 2008), the pillars tended to be "stealthy" and not "funnel" (Herbel-Eisenmann & Breyfogle, 2005) prospective teachers' responses toward what the instructor might have expected them to say (A. J. Stylianides & Stylianides, 2014a, 2022; G. J. Stylianides & Stylianides, 2009, 2014).

Conceptualizing the interplay between the design and implementation of the interventions using the notion of learning trajectory, and the key role of the instructor

First we clarify our perspective on the notions of hypothetical and actual learning trajectories, and then we explain their relationship to the design and implementation of the focal and other interventions in the course. We finish with a discussion of the key role of the instructor.

The notions of hypothetical and actual learning trajectories

Since its introduction by Simon (1995), the notion of *hypothetical learning trajectory* has attracted considerable research attention, has been used variably in the literature, and has given rise to other related notions including that of *actual learning trajectory* (Leikin & Dinur, 2003). By "hypothetical learning trajectory" we refer to the learning route a classroom community is anticipated to follow or progress through, including the learning milestones the community is anticipated to achieve, during the implementation of an intervention that aims to promote a particular learning goal (A. J. Stylianides & Stylianides, 2022; G. J. Stylianides & Stylianides, 2009, 2014). The learning goal of the focal intervention was for prospective teachers to develop their knowledge of the elements in Figure 2. In broad terms (to be expanded in the Methodology section), the first milestone in this learning trajectory was for prospective teachers to develop knowledge of Elements 1 and 2; achieving this milestone was expected to create the foundation for the subsequent development of knowledge of Element 3, the final milestone in the learning trajectory. By "actual learning trajectory" we refer to the learning route a classroom community seemed to have followed, including the learning milestones the community seemed to have achieved (or not), during the implementation of the intervention (A. J. Stylianides & Stylianides, 2022; G. J. Stylianides & Stylianides, 2009, 2014). While a hypothetical learning trajectory is formulated by reasoned predictions about a community's expected learning route in an intervention (based on theory, research, or experience from previous implementations), an actual learning trajectory can only be specified empirically, using implementation data.[2]

As suggested by our definitions, we view a learning trajectory as encompassing a learning goal and a progression through learning milestones, two aspects that are included in most other perspectives on the construct of a learning trajectory (e.g., Clements et al., 2020; Marmur & Zazkis, 2021; Simon, 1995; Wilson et al., 2013). These two parts refer to the learner-content side of 2003) "instructional triangle" metaphor. According to this metaphor, learning is the outcome of interactions among learners and the instructor around content, in a classroom environment. Thus, the learning goal and associated progression through learning milestones in a learning trajectory cannot be considered in isolation from the intervention designed to promote them as implemented by the instructor. Yet there is no universal agreement among researchers as to whether the intervention should be considered a part of the learning trajectory; we view these different perspectives as being about semantics rather than substance. In our research we opted to label separately the intervention from the learning trajectory. Under "intervention" we included mainly the tasks (the Floors Problem), any follow-up prompts (the pillars), and the instructor's actions while implementing these tasks/prompts to enable the desirable learner-content relationship.

Design and implementation of interventions and close matching between hypothetical and actual learning trajectories

The hypothetical learning trajectory reflected our predictions about what would happen in the classroom in terms of participants' learning route during the implementation of an intervention, while the actual learning trajectory reflected the empirical examination of the extent to which our predictions were confirmed in practice. An indication of an intervention's promise (or success), then, was whether there was a close matching between the two trajectories (G. J. Stylianides & Stylianides, 2014). Such a close matching meant that the actual learning trajectory achieved the targeted elements of MKT. Also, as the hypothetical learning trajectory encompassed over the cycles of the design experiment our evolving theoretical understanding of how the intervention promoted these elements of MKT, a close matching between the two trajectories offered further supportive evidence for our theoretical framework.

Achieving a close matching between the two trajectories was typically a lengthy process requiring several research cycles to develop, empirically fine-tune, and theorize the respective interventions (see, e.g., A. J. Stylianides & Stylianides, 2014a, 2022; G. J. Stylianides & Stylianides, 2009). Our investment in designing and iteratively refining tasks and conceptual awareness pillars played a key role in facilitating this matching while also reducing the demands on the instructor for improvisation and in-the-moment decision making (G. J. Stylianides & Stylianides, 2014). We elaborate on the instructor's role next.

The key role of the instructor during the implementation of interventions

The first main aspect of the instructor's role was to be the representative of the wider mathematical (e.g., G. J. Stylianides & Stylianides, 2009; Lampert, 1992; Yackel & Cobb, 1996) and teacher education communities in the classroom. The latter meant, for example, offering to prospective teachers a glimpse of the relevant "mathematical horizon" (Ball, 1993) so as to help them appreciate the mathematical foundation teachers can set for what their students will encounter later in school (Ball et al., 2008).

The second main aspect of the instructor's role was to initiate *situations for institutionalization* (Brousseau, 1981) in order to give official status to pieces of knowledge that were important for the classroom's future work (Balacheff, 1990). This was especially relevant in the transition from Part 1 to Part 2 in the approach to promoting MKT we presented earlier: some pieces of knowledge from Part 1 would have to be institutionalized and "taken-as-shared" (Simon & Blume, 1996; Yackel & Cobb, 1996) within the community's work in Part 2.

The third main aspect of the instructor's role was to foster a dynamic pattern of classroom interaction as in "ambitious instruction" (e.g., G. J. Stylianides & Stylianides, 2014). Specifically, the instructor was responsive to prospective teachers' ideas, including when orchestrating whole class discussions (G. J. Stylianides & Stylianides, 2014; Stein et al., 2008), while encouraging the prospective teachers to engage in classroom discourse and build on each other's ideas through dialogue and small group work. In the context of ambitious instruction, it is inconsistent for the instructor to "funnel" (Herbel-Eisenmann & Breyfogle, 2005) learners' contributions toward particular ideas. Rather, the instructor needs to be "flexible" (Leikin & Dinur, 2003) and prepared to modify the instructional plan when learners' contributions deviate from the hypothetical learning trajectory.

The final main aspect of the instructor's role concerned the use of *four practices while orchestrating whole class discussions* (Table 1). In G. J. Stylianides and Stylianides (2014) we discussed these practices and connected them to relevant literature. The use of Practices 1–3 allowed for a multiplicity of voices to be expressed while relieving the instructor from the expectation to immediately respond to individual contributions. For example, even if the instructor found an individual contribution opaque (Practice 1), the situation would likely become clearer for him/her once other prospective teachers responded to it (Practice 2) or expressed their own ideas (Practice 3). Also, the use of a well-designed set of prompts (pillars) in whole class discussions facilitated certain patterns to emerge in prospective teachers' contributions. The clustering of contributions around particular ideas (ideally the intended learning milestones) was often facilitated by the use of Practice 4, which the instructor used to highlight relevant contributions.

Methodology

Revisiting the research question

Given our previous discussion, we revisit and further specify parts of the research question we address in this paper. The "important elements of MKT related to assumptions" referred to in the question were Elements 1–3 presented in Figure 2 and discussed under Theoretical Framework. The operational way to judge the intervention's "promise" was based on the degree of matching between the teacher education classroom community's hypothetical and actual learning trajectories: as per our Theoretical Framework, the closer the matching between the two trajectories, the more successful the intervention would be judged to be in promoting the targeted elements of MKT.

The design experiment

Our design experiment in a mathematics course for prospective elementary teachers in the United States comprised 5 research cycles of implementation, analysis, and refinement of interventions that targeted elements of prospective teachers' MKT and mathematics-related beliefs. We aimed for interventions of relatively short duration, each focusing on a well-defined but narrow set of learning goals (A. J. Stylianides & Stylianides, 2014a, 2022; G. J. Stylianides & Stylianides, 2009). As is typical of design experiments (Cobb et al., 2003), the cycles of analysis and refinement of the interventions were paralleled by refinements of our theoretical framework. The three features we presented earlier comprised the final version of the theoretical framework pertaining to the focal intervention.

The data for this paper derive from the last research cycle, which included two parallel sections of the course taught by the first author. The course was a prerequisite for admission to a masters-level elementary teacher education program, and it covered many topics across a broad range of mathematical domains (arithmetic, algebra, geometry, etc.), paying attention to the relevance of the targeted learning goals to mathematics teaching. The students were primarily third year undergraduates majoring in different fields of study; for many of them this was their first mathematics course since secondary school.

The focal intervention

Outline of the intervention

The intervention lasted about 80 minutes, was implemented toward the end of the semester, and was the only intervention in the course that explicitly targeted prospective teachers' MKT related to assumptions. We introduced the intervention in research cycle 4. In the previous cycles we addressed the notion of assumptions on an ad hoc basis, which was deemed unsatisfactory thus motivating the design of a dedicated intervention to promote Elements 1–3 of MKT. Analysis of how the intervention played out in cycle 4 led to its final form in cycle 5, as we describe next. The design of the intervention benefitted from knowledge we developed in earlier research cycles as part of the design of other interventions, such as knowledge about the importance of conceptual awareness pillars; this knowledge enabled us to develop the focal intervention in only two cycles.

Figure 3 outlines the intervention in cycle 5 and its respective hypothetical learning trajectory. As per the two-part instructional approach to promoting MKT that we described under Theoretical Framework, Part 1 of the intervention comprised the Floors Problem followed in Part 2 by prompts in

Table 1. Four practices used by the instructor while orchestrating whole class discussions.

Practice	Description
1	Asking prospective teachers to explain or clarify the thinking underlying their contributions
2	Encouraging prospective teachers to listen and respond to each other's contributions (e.g., by expressing and justifying their agreement or disagreement)
3	Soliciting contributions from several prospective teachers thus offering space for expression of a variety of contributions (including possible conflicting contributions)
4	Revoicing selected prospective teacher contributions in order, for example, to help direct the attention of the class to these contributions or to connect the contributions with the corresponding learning milestone

Note: The practices are slightly adapted versions of those in G. J. Stylianides and Stylianides (2014, p. 386).

the form of two groups of conceptual awareness pillars (Table 2). The design of the pillars followed a "progressive focusing" approach (Stake, 2010) as we explain below.

Figure 3 outlines also the three milestones in the hypothetical learning trajectory. As we explained earlier, Elements 1 and 2 were addressed together in milestone 1, while Element 3 capitalized on the other elements and was addressed in milestone 3. Milestone 2 related to prospective teachers recognizing the importance for elementary students to develop a sense of the role of assumptions in mathematical activity, and it was a precursor to milestone 3. The interposition of a *situation for institutionalization* (Brousseau, 1981) between Parts 1 and 2 of the intervention was intended to ensure that knowledge pertaining to Elements 1 and 2 would get an official status so that this knowledge could be used in the classroom's work in milestones 2 and 3.

Part 1: the Floors Problem

We hypothesized that an appropriately adapted version of Ball's (1993) implementation plan of the Floors Problem could help promote Elements 1 and 2 of MKT (milestone 1). Similar to how the task played out in Ball's class, we expected that the ambiguity in the problem's conditions would prompt prospective teachers to pursue different pathways based on different assumptions about these conditions, thus reaching different answers. This work would then create a productive space for prospective teachers to understand both that the answers depended on certain assumptions (Element 1) and that, although the answers might appear to be contradictory, they could all be correct given their underpinning assumptions (Element 2). However, our work in research cycle 4 showed that, unlike Ball's third graders who pursued their individual solution paths based on their specific interpretations of the problem's conditions, many prospective teachers identified from the outset the ambiguity in the problem's conditions and asked the instructor to clarify them. In cycle 4 the instructor was quick to acknowledge the ambiguity, and this compromised the natural emergence of the notion of assumptions in the discussion. To combat this issue, we planned in cycle 5 for the instructor to say upfront that he would not respond to any clarifying questions about the problem. We expected that this change to our plan would enable the word "assumption" to come up naturally in the discussion and that it would even be mentioned by the prospective teachers first. We describe more changes to our plan, from cycle 4 to cycle 5 later, under Part 2 of the intervention.

The situation for institutionalization

The situation for institutionalization that we interposed between the Floors Problem and the pillars would comprise a brief commentary by the instructor using the PowerPoint slides in Figure 4. Specifically, the instructor would act similarly to Ball's (1993) to introduce the class to the notion of assumptions in mathematical activity. Also, he would paraphrase Fawcett's (1938) idea that "conclusions are 'true' only within the limits of the assumptions on which they depend" (p. 71). As we explained earlier (Figure 2), this idea captured the essence of Elements 1 and 2, which were the pieces of knowledge we expected would gain official status at this stage in the intervention.

In enacting the situation for institutionalization, the instructor would also make some brief remarks about the role of assumptions in the discipline of mathematics (slide 2 in Figure 4). These remarks aimed to give to prospective teachers a glimpse of the relevant "mathematical horizon" (Ball, 1993) and help them appreciate the mathematical foundation in relation to the notion of assumptions they can set for their students (Ball et al., 2008). Also, the remarks are consistent with our view of the instructor as the representative of the broader mathematical and teacher education communities in the classroom as we discussed under Theoretical Framework.

Part 2: follow-up prompts in the form of conceptual awareness pillars

We expected that the implementation of the two groups of pillars in Part 2 of the intervention (Table 2) would help direct prospective teachers' attention to the mathematico-pedagogical space of interest. In research cycle 4, we used a small number of pillars that only resembled the group A pillars in research cycle 5. Our analysis of the implementation of the pillars in cycle 4 showed that these helped direct prospective teachers' attention to the *general* mathematico-pedagogical space of interest in the context of the Floors Problem. While this was desirable, it was insufficient in supporting prospective teachers' progression toward milestones 2 and 3. To combat this issue, in cycle 5 we used a "progressive focusing" approach (Stake, 2010) and added the group B pillars that aimed to direct prospective teachers' attention to a *better delineated* space so as to further develop their reflective insights from the group A pillars.

Specifically, Pillar A1 was an open-ended prompt asking prospective teachers to describe anything that stood out to them from their work on the Floors Problem. We expected that this pillar would direct prospective teachers' attention to broad issues related to Elements 1 and 2. Pillar A2, another open-ended prompt, aimed to direct prospective teachers' attention to broad issues pertaining to teachers' use of ambiguous tasks. These issues related to milestone 2 as we viewed task ambiguity to be a key way of introducing elementary students to the role of assumptions, but also to milestone 3 as task ambiguity was at the core of Element 3. The group B pillars directed prospective teachers' attention to three position statements that were neutrally phrased so as not to "funnel" prospective teachers' responses toward certain ideas but, topic-wise, the statements related clearly to milestones 2 and 3. The first question in Pillar B1 aimed to re-focus prospective teachers' attention on the key idea that would emerge from Part 1 of the intervention; the second question in the same pillar asked them whether they considered important for elementary students to develop a sense of the role of assumptions. The discussion of the latter aimed to help establish a shared understanding in the class related to milestone 2, and to lay the ground for a more focused reflection in Pillars B2 and B3 that related to milestone 3.

The prospective teachers would respond to the group A pillars individually and in writing, while they would discuss the group B pillars in small groups and keep notes of their discussion. Whole class discussions would follow, orchestrated by the instructor using Practices 1–4 (Table 1).

Data

The data came from research cycle 5 and concerned the implementation of the intervention in one of the two parallel sections of the course with 16 prospective teachers present. In the two sections the

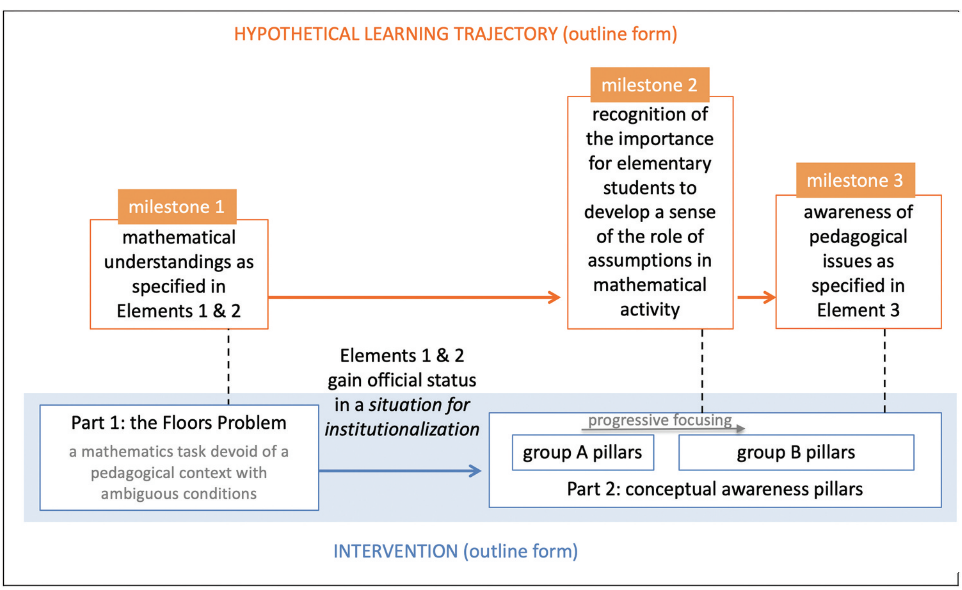


Figure 3. Outline of the intervention for promoting Elements 1–3 of MKT (lower part of the figure) and the corresponding hypothetical learning trajectory (upper part of the figure).

Table 2. The conceptual awareness pillars used in Part 2 of the intervention.

Pillars	Statements
Group A	A1. Is there anything that particularly stood out to you from our work on the Floors Problem?
	A2. The Floors Problem was purposefully designed to be ambiguous. Why might a teacher use a problem like that in his/her classroom?
Group B	B1. "Conclusions are 'true' only within the limits of the assumptions on which they are based." How do you understand this statement? Do you think it is important for elementary school students to develop a sense of the role of assumptions in mathematics? Explain.
	B2. "Teachers should always make sure that the mathematical tasks they give to their students have unambiguous conditions." What do you think about this statement? Explain.
	B3. There may be situations where teachers do not realize that a mathematical task they give to their students has ambiguous conditions. What might happen in these situations and how might teachers handle the situations?

Note: There was also a third pillar in group A that we do not present due to it not being directly relevant to our focus in this paper.

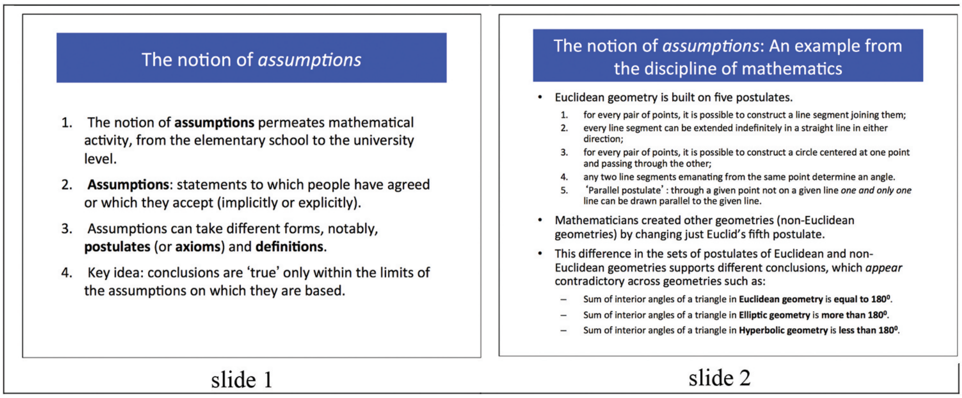


Figure 4. The slides used by the instructor for the *situation for institutionalization* in between Parts 1 and 2 of the intervention.

instructor was the same, the number and profile of students was similar (students were free to enroll in the section that better suited their timetable), the curriculum was the same as was the teaching time (3 hours per week for a semester), and the instructor established the same norms and used the same plans. Given the aforementioned, it is unsurprising that similar issues emerged in both implementations. The reason we chose the particular section for our analysis is for consistency with previous publications from this study where we reported analyses from the same section (e.g., A. J. Stylianides & Stylianides, 2014a, 2022; G. J. Stylianides & Stylianides, 2009).[3] The data included: transcribed video/audio records of the implementation including whole class discussions[4]; prospective teachers’ written work during the intervention, including their individual responses to the group A pillars and the notes from their small group discussion of the group B pillars; and field notes from a research assistant on the work of one small group. The class was organized in small groups of around four prospective teachers each. The groups were not formed based on a particular method but stayed the same for the entire semester; the research assistant picked one group at the beginning of the semester (following no specific selection criterion) and stayed with this group for the whole semester for continuity.

Analytic method

To address the research question, we compared the classroom community’s *hypothetical* and *actual* learning trajectories. Specifying a classroom community’s hypothetical learning trajectory is relatively unproblematic as it reflects the designers’ expectations of the learning route the community would follow during the intervention. The issue is raised, however, as to the relationship between the learning trajectory of a community as a whole and that of individual members; this applies equally to both hypothetical and actual learning trajectories.

Like Lampert (1992), we recognize that the learning of a classroom community does not necessarily reflect the learning of each individual member and that every act of “knowing” in the classroom occurs somewhere on a continuum between individual understandings and knowledge that can be considered as shared or accepted within the community. Accordingly, when we talk about the hypothetical learning trajectory of the classroom community, we do not imply that each learner is expected to follow the same learning trajectory. Likewise, when we talk about the classroom community’s actual learning trajectory, we do not imply that each member followed the same learning trajectory; rather, we refer to a best-fit description of knowledge advancements that can reasonably be assumed as shared or accepted within the community based on triangulation of available data from individual constructions and from whole class and small group discussions.

Using qualitative content analysis, we explored themes in prospective teachers’ individual constructions and examined the extent to which these related to milestones 1–3 as applicable. In our analysis of the small group and whole class discussions, we identified any ideas expressed by individuals or groups of prospective teachers that the class seemed to share or accept, and we examined again the extent to which these related to milestones 1–3. The generation of themes was mostly top-down as it was driven by the milestones; we started with initial codes that directly derived from the milestones and then we organized codes into broader themes. For example, our codes for prospective teachers' responses to Pillar A1 about things that stood out to them from their work on the Floors Problem all related to milestone 1 and were: reference to "assumptions," "one or more correct answers," and "one or more legitimate interpretations;" the themes that emerged from these codes are presented in Table 3 (Pillar A1, themes 1–4). However, the generation of themes was to some extent also bottom-up in the sense that not all codes could be predetermined in relation to the milestones. For example, theme 5 of Pillar A2 (Table 3) about teachers' purposeful use of ambiguous tasks to encourage discussion or reasoning emerged after our review of responses to the pillar. Both of us reviewed the data and discussed our interpretations of individual constructions and whole group discussions. In this approach, each of us was engaged in the textual meanings and interpretations of the responses (Krippendorf, 2004); we discussed our codings, reflecting the interpretive nature of qualitative research (O'connor & Joffe, 2020), until we reached agreement.

The act of "accepting" or "sharing" of an idea within the classroom community could be explicit, as in the case when several prospective teachers agreed with it or expressed it in their own words without being challenged by others, or implicit, as in the case when an expressed idea went unchallenged even after the instructor had encouraged further comments on it (G. J. Stylianides & Stylianides, 2009). The instructor's use of Practices 1–3 (Table 1) while managing whole class discussions, in conjunction with the broader culture we aimed to cultivate from the start in the class where prospective teachers were expected to critique and debate ideas that did not make sense to them, helped ensure that multiple viewpoints would emerge and that any disagreements would be expressed freely. The latter was particularly true for the focal intervention as it was implemented toward the end of the semester when relevant social norms were well-established.[5] Furthermore, to mitigate the problem of only some prospective teachers contributing to whole class discussions, we often preceded those discussions by individual written responses to relevant prompts (as in the group A pillars) or by discussions in small groups with each group being expected to write down their agreed views (as in the group B pillars). Also, the research assistant's fieldnotes from one small group allowed us to gather more information about the development of ideas in the classroom.

The implementation of the focal intervention in research cycle 5

In this section, we present the implementation of the intervention in cycle 5, and we analyze the degree of matching between the community's hypothetical and actual learning trajectories. We organize the section around the two main parts of the intervention (Figure 3). We do not discuss the situation for institutionalization that was interposed between the two parts, because it involved no learner input and was implemented by the instructor according to the plan we discussed earlier.

Part 1: the Floors Problem

The instructor showed the problem and explained the notation for negative numbers as in Figure 1. After engaging the class in representing particular trips using number sentences, he asked the prospective teachers to work on the problem first individually and then in groups of four. Also, as we explained earlier, he said he would not answer any clarifying questions about the problem.

The following discussion in the small group where the research assistant was sitting is indicative of prospective teachers' original engagement with the problem. They were uncertain about the task's conditions, notably the nature of the person's trips.[6]

Table 3. Findings from the analysis of prospective teachers' individual (written) responses to the group A pillars.

Pillars[1]	Themes and frequencies (in parentheses)[2]
A1	1. Multiple legitimate interpretations (8) 2. Multiple assumptions (5) 3. Multiple correct answers (4) 4. Importance of assumptions (4) 5. Other (2)
A2	1. To encourage sensitivity to language in the formulation of a task (7) 2. To increase awareness of the role of assumptions (6) 3. To enhance appreciation of different possible interpretations of the same (ambiguous) text (6) 4. To enhance appreciation of the interdependency between different assumptions/interpretations and different answers and solution paths (5) 5. To encourage discussion or reasoning (4) 6. To open a window into students' thinking (4)

Notes. [1]The statements of Pillars A1 and A2 are presented in Table 2.
[2]A total of 16 prospective teachers responded individually and in writing to Pillars A1 and A2. Each response could receive multiple codes but the frequency of each code per prospective teacher could be either 0 or 1.

(1) Amanda: So … can you pass the second floor and then go back to it? Or do you have to stop, because you've technically gotten there? So … you just have to look at how to get *directly* there? If not, it's going to be infinity!
(2) Beth: That's what I did … the direct [way]. I just counted how many up and how many down …
(3) Monica: But that *doesn't say that* you have to come in the entrance … So you can start anywhere?

At this point Stylianides passed by the small group and Monica asked him whether the trips had to start from the ground floor. Stylianides kindly reminded her he would not respond to clarifying questions. The small group discussion continued:

(4) Victor: So … the solution of this problem depends on our *assumption*.
(5) Amanda: Well, fine then. I say 15 … because I'm *assuming* that you start on the ground floor and can't pass [floor] 2 and can't change direction more than once.
(6) Victor: I say 25 because … [He did not finish his sentence because the whole class discussion was about to begin.]

As illustrated by this exchange, our decision for the instructor not to respond to clarifying questions allowed for the notion of assumptions to emerge naturally in the discussion: Victor (line 4) realized that the answer was dependent on one's assumption about the nature of the person's trip (Element 1), while Amanda and Victor (lines 5–6) both began to offer different answers based on different legitimate assumptions about the problem's ambiguous conditions (Element 2).

The whole class discussion started with Stylianides inviting all groups to state their answers to the problem (15, 25, 51, and infinity were mentioned). Then he asked each group to explain their answer, beginning with Sherrill's group answer of 25.

(7) Sherrill: Um, I actually wrote it all out [all number sentences] … I said that if you started at, um, negative 12 and added 14 because you have to go up 14 times to get to the second floor, and I just wrote it all out for all of them … [Stylianides writes the relevant number sentence on the board; Sherrill confirms.]
(8) Stylianides: So you did that all the way until … ?
(9) Sherrill: Until the roof minus 11.
(10) Stylianides: Okay, so you also had 12 minus 10 equals 2, and then you had the roof minus 11 equals 2 [he writes the relevant number sentences on the board]. Did you also include the second floor [as a possible starting point]?
(11) Sherrill: No.
(12) Stylianides: Okay, and you had 25 [ways]. [To the class:] What do you think about this solution?
(13) Helen: So … the second floor wasn't included? You couldn't just go straight to the second floor?
(14) Sherrill: Well, if you're *on* the second floor, you're already there!
(15) Helen: Oh, *starting* from the second floor …
(16) Sherrill: Yeah, we're starting on the second floor, so it wouldn't make sense to say we're starting on the second floor and going nowhere …
(17) Stylianides: […] [To the class:] What do you think?

In the above excerpt Stylianides implemented Practices 1–3 (Table 1). Regarding Practice 1, he invited Sherrill to explain the thinking of her small group that led to the answer of 25, and he asked clarifying questions so that this thinking became clear to the class (lines 8, 10, 12). Regarding Practice 2, he encouraged the class to respond to Sherrill's contribution (line 12), and this led to Helen asking Sherrill clarifying questions (lines 13, 15). Sherrill responded directly to Helen (lines 14, 16), which illustrates that the classroom participants were used to listening and responding to each other's contributions. Finally, regarding Practice 3, Stylianides solicited contributions from several students so that all ways of thinking were considered (lines 12, 17). Sophie responded next to Stylianides' invitation for further contributions, and she started explaining infinity as an answer.

(18) Sophie: We picked infinity because [. . .] you can go up and down and up and down as many times as you wanted before you reach the second floor. I think that *the question isn't specific enough*. Like they [Sherrill's group] *assumed* you can only go up or down one time.
(19) Stylianides: What did this group [Sherrill's group] *assume*?
(20) Sophie: That you can only go up or down once.

Here the instructor picked up on Sophie's use of "assume" (line 18); given the importance of this term, he revoiced it in a question to the class (line 19) thus implementing Practice 4. Victor and Amanda had mentioned this term during their small group work (lines 4–5), but Sophie was the first to mention it in the whole class discussion.

(21) Stylianides: [. . .] So you say [referring to Sophie] if we *assume* that you're allowed to travel up and down as many times as you want, then the answer would be infinity? [Sophie nods in agreement. To the class:] What do you think about that? Is one of them [Sherrill or Sophie's group] wrong? Are both of them right? [. . .]
(22) Lindsey: I think they're all [right] . . . *They just made different rules, and like, thought processes*. [. . .] You *could* have a direct route and that's it, no going back and forth . . . like, you *could*, when you were talking about going up and down and up and down and that's not really a direct route . . . So really, you could do anything that you wanted to, but *it just depends* on what the person . . . what the problem is looking for, I guess.
(23) Sherrill: It's up to the *interpretation* of the reader. . . Like I read it and I just *assumed* that it had to be a direct route, but it clearly doesn't state that . . . so . . . it's open to [interpretation] . . .

The whole class discussion in lines 18–23 mirrored the previous small group discussion in lines 1–6. Sophie (line 18), Lindsey (line 22), and Sherrill's (line 23) comments reflected mathematical understandings related to Elements 1 and 2 of MKT: they all recognized that the task allowed for different legitimate interpretations of its conditions and that the answer depended on one's assumptions about these conditions (Element 1). Lindsey and Sherrill's comments suggested further a realization that different answers could be correct based on different legitimate interpretations of the task's conditions (Element 2).

The discussion continued similarly with consideration of the remaining answers in relation to their underpinning assumptions. Prospective teachers' contributions continued to reflect mathematical understandings related to Elements 1 and 2 of MKT, thus indicating achievement of milestone 1 of the hypothetical learning trajectory (Figure 3). This allowed the instructor to proceed with the preplanned *situation for institutionalization*, giving official status to the two elements of knowledge that were important for the classroom community's subsequent work.

Part 2: follow-up prompts in the form of conceptual awareness pillars

Part 2 of the intervention comprised sequential implementation of the pillars in Table 2. As we explained earlier, the group A pillars aimed to draw on prospective teachers' experience with the Floors Problem to direct their attention to the general mathematico-pedagogical space of interest, while the group B pillars aimed to direct their attention to a better delineated space to further develop their reflective insights from the group A pillars and progress toward milestones 2 and 3.

The group A pillars

In Table 3 we summarize the themes that emerged from our analysis of the prospective teachers' individual written responses to Pillars A1 and A2, and in Table 4 we offer illustrative responses. The frequencies are offered as an indication of the prominence of each theme among the responses, not for generalization.

Pillar A1 was an open-ended prompt that was not asking for responses relating solely or specifically to Elements 1 and 2 of MKT (milestone 1). Yet, except for two contributions classified under Other (Pillar 1, theme 5, Table 3), all aspects of the experience of working on the Floors Problem that were mentioned by the prospective teachers related to the essence of these elements (Pillar A1, themes 1–4). Overall, the findings indicated that the prospective teachers' attention resided in the general mathematical space of interest, as we had intended.

Pillar A2 was another open-ended prompt and was the first pillar to direct prospective teachers' attention to the general pedagogical space of interest pertaining to milestones 2 and 3, namely, teachers' use of ambiguous tasks. The breadth and depth of thinking reflected in prospective teachers' individual responses to Pillar A2, which were independent of any direct input from the instructor or other prospective teachers, indicated that the pillar offered a productive space for prospective teachers to begin to develop pedagogical awareness of issues pertaining to the classroom use of ambiguous tasks in preparation for the group B pillars. For example, six prospective teachers wrote that teachers' use of ambiguous tasks can help increase students' awareness of the role of assumptions in mathematical activity (Pillar 2, theme 2, Table 3); this suggested a possible recognition of the importance for introducing students to the role of assumptions (milestone 2), a matter that would be revisited in Pillar B1. Also, virtually all prospective teachers' responses to Pillar A2 could feed into their thinking about teachers' handling of task ambiguity in mathematics instruction (milestone 3), the focus of Pillars B2 and B3.

The prospective teachers' individual responses to Pillars A1 and A2 were followed by a brief whole class discussion where several prospective teachers shared their responses. Their contributions went along similar lines as in their individual contributions. The instructor implemented Practices 1–3 (Table 1), which helped bring to everyone's attention and reinforce ideas that not all prospective teachers might have thought independently.

The group B pillars

The prospective teachers were divided into four small groups to discuss the group B pillars. They were asked to prepare a brief, written group response to each pillar based on their agreed points and be prepared to share their ideas in a subsequent whole class discussion. Also, due to the limited time available, the groups were asked to consider the three pillars in different orders to ensure that all pillars received attention by at least some of the groups. For each pillar we ended up receiving written responses from three groups. We used the groups' written notes for triangulation of the conclusions we drew from our analysis of the whole class discussion of each pillar.

Pillar B1. The first question in Pillar B1 aimed to re-focus prospective teachers' attention on the key idea that emerged from Part 1 of the intervention that "conclusions are 'true' only within the limits of the assumptions on which they are based." As we explained, this idea encompassed Elements 1 and 2 of MKT already established within the classroom community. Unsurprisingly, the prospective teachers' responses to Pillar B1 reconfirmed the status of Elements 1 and 2 in the shared knowledge of the classroom. Due to space constraints, we omit details of those discussions.

Regarding the second question in Pillar B1, all three small groups that returned written notes for this pillar indicated their agreement for the importance of helping elementary students develop a sense of the role of assumptions, offering one or more of these justifications: (1) assumptions are "the building blocks" of mathematical reasoning; (2) knowledge of assumptions can help students understand that different legitimate interpretations of a problem's conditions can lead to different correct answers; and (3) knowledge of assumptions can highlight the need for students to explain their

Table 4. Illustrative prospective teacher responses to the group A pillars and respective themes.

Pillars[1]	Themes[2]	Responses
A1	1	*Amanda*: The lack of guidelines and detail in the question stood out to me. It was difficult to even start narrowing down ways to solve it because the problem was so open and left so much open to interpretation.
	1, 3	*Lorri*: What stood out to me was how all of our groups had different interpretations of how many ways there were to get to the second floor. The answers ranged from 15, 25, 51, and infinity. The problem was ambiguous. Some of the other groups' answers were readily understood by me. I could understand their rationales as to why their answer was correct. Our answers were not wrong under this ambiguous problem.
	2, 4	*Sherrill*: I was surprised at the multiple assumptions made. I had not even considered the fact that answers other than 25 existed because of the way people interpreted the problem. I now realize how greatly making assumptions can alter mathematics.
A2	1, 2, 3	*Helen*: To learn about assumptions and understand not only the importance of specifics and details, but also the freedom for one to think along the terms of their own assumptions.
	1, 3, 4	*Lorri*: A teacher could use this ambiguous problem to get their students to understand that sometimes their answers are not wrong and that there may be other answers to a problem. Students will be able to explore their different ways of interpretation and must be open to others' interpretations as well. They will understand that there will be different interpretations of answers that will depend on the information given in the problem, which may be ambiguous.
	3, 5	*Ellice*: Perhaps a teacher would use this to see how many ways a student looks at a problem. Also to see the skill of the students of providing an argument and convincing others.
	3, 6	*Victor*: Since all children operate on different levels, it is important to see how each child thinks. This problem would allow for the teacher to see how one may interpret things.

Notes. [1]The statements of Pillars A1 and A2 are presented in Table 2.
[2]The descriptions of Themes are presented in Table 3.

thinking. These justifications were underpinned by an understanding of Elements 1 and 2, particularly the first two justifications. The justifications were also consistent with themes that emerged from prospective teachers' earlier responses to the group A pillars, which suggests the merit of the "progressive focusing" approach we adopted in the design of the two groups of pillars. For example, the third justification relates to what several prospective teachers wrote in response to Pillar A2, namely, that teachers might use an ambiguous task to encourage discussion or reasoning and open a window into students' thinking (Pillar 2, themes 5 & 6, Table 3).

The whole class discussion around the second question in Pillar B1 confirmed prospective teachers' recognition of the importance of elementary students developing a sense of the role of assumptions and helped expand the range of relevant justifications for it. For example, another justification was that knowledge of the role of assumptions can help students appreciate the use of mathematical language and vocabulary. In orchestrating the discussion, the instructor primarily implemented Practices 1–3 (Table 1) so as to maximize the views expressed publicly. The prospective teachers were contributing similar ideas with no instances of disagreement. The discussion finished with the following contribution from Laura on behalf of her small group:

(24) Laura: We were also saying, too, that by teaching kids about the role of assumptions in mathematics, you're also teaching them about the process of how to solve math problems. You know, because I think traditionally in elementary school it's all about [. . .] number facts and that's pretty much it. When you start to go through those kinds of conversations, you're teaching kids how to look at problems and actually *think about them* and *use math language*, you know, to solve them. So in doing that you have to give them a context of where you're coming from [. . .] If you get them to do it through assumptions, mathematical assumptions, it's teaching them a process of how to *think*. You know, don't just blurt out the answer. Think about: "Okay, what do I know? What don't I know? How do I combine those two things together to come up with an answer I can explain?" [. . .] It's helping them develop a *math language* and a *better vocabulary* which is so much more . . . it's really important for like geometry and trig and calculus. [. . .] But I also think, too, that this is good for other subjects besides math. Um, we talked about kids in a debate being able to justify their argument; they have to explain, you know, their context of where their argument is coming from. So again it's a process of thinking . . .

Laura's contribution offered yet another justification for the importance of helping students develop a sense of the role of assumptions: understanding assumptions can be helpful also in other school subjects that involve development and understanding of the origins of arguments.

To conclude, the classroom community had developed at this point a shared recognition of the importance for elementary students to develop a sense of the role of assumptions in mathematical activity, thus achieving milestone 2 in the hypothetical learning trajectory. This was reflected in all prospective teacher contributions to the discussions, both in the small groups (written) and the whole class (verbal). Furthermore, the recognition was expressed with elaborate and thoughtful justifications rather than mere declarations of agreement; each new contribution in the whole class discussion either reiterated/reinforced earlier justifications or offered a new justification, with no disagreements expressed about what had been noted previously.

Pillar B2. Three small groups kept written notes of their discussion of Pillar B2. According to these notes, all groups disagreed with the position statement in the pillar and noted that the phrasing of a task should "depend" (a word used by all) on "the situation," "the end purpose of the exercise," or "what the teacher is trying to get out of the task [. . .], lesson or mathematical question." These responses capture the idea that there are certain circumstances under which ambiguity can be beneficial to learners (see Stylianides, 2007; Foster, 2011; Marmur & Zazkis, 2022). Also, the prospective teachers mentioned specific examples to illustrate their points as in the following excerpt from one small group: "If a teacher is giving a math test, or graded evaluation where they are looking for a specific answer then yes, the tasks should be unambiguous. However, if the teacher is trying to prove a point about assumptions or different approaches to the same problem, an ambiguous task may be beneficial." It is notable that, although the prompt in the pillar did not mention the term "assumptions," the prospective teachers made the connection themselves.

The whole class discussion included similar contributions. It started with the instructor's open call for contributions, and it finished once all prospective teachers who wished to contribute had done so (Practice 3). Kira spoke first.

(25) Kira: We kind of said that it depended on [. . .] what the expectations of the teacher are. Like, whether they . . . like, you, if you want one specific answer, you better give *unambiguous* conditions. If you're looking to see how the student can solve something, and looking for different ways to solve something and just the way they think or are looking at the problem, then you can give unambiguous conditions. [Kira meant to say *ambiguous conditions* and this was clarified by the instructor immediately afterward.] Because it kind of forces the kids to think on a different level. [. . .]

(26) Laura: It [the use of tasks with unambiguous or ambiguous conditions] depends on the end result that the teacher wants. For example, with the Floors Problem you gave us ambiguous conditions by the word "different" and that was interpreted . . . because you wanted a result of showing that there's different ways of interpreting information and different ways to come up with a conclusion. But if you want a very specific answer, like you wanted the answer of 15, then you would have had to give us all those assumptions and conditions and they would have had to have been very specific to guide us to the right answer.

(27) Stylianides: Other thoughts on [Pillar B2]? Amanda?

(28) Amanda: We agree: If it's a math test or some kind of graded evaluation, then it would be unfair to give ambiguous conditions, but then in other circumstances ambiguous conditions can be really beneficial.

Kira's contribution (line 25) could be summarized in the excerpt, "[whether or not a teacher would use tasks with ambiguous conditions] depended on [. . .] what the expectations of the teacher are." Laura (line 26) and Amanda (line 28) reinforced and further elaborated on this position, which we take to be the collective view shared within the class in relation to Pillar B2. Indeed, all prospective teachers who contributed to the whole class discussion, talking on behalf of their groups (note the use of "we" at the beginning of Kira and Amanda's contributions), agreed that there are (1) situations when a teacher might want to use tasks with *unambiguous* conditions, such as when the teacher wants one specific answer to a task (Kira, line 25; Laura, line 26) like in a test (Amanda, line 28), but also (2) situations when a teacher might opt for tasks with *ambiguous* conditions such as when the teacher wants students to solve the problem in different ways or get an insight into students' thinking or interpretations of a problem (Kira, line 25; Laura, line 26).

To conclude, the thoughtfulness and consistency of prospective teachers' responses to Pillar B2 in the small group and whole class discussions show that the classroom community developed shared pedagogical awareness of situations when a teacher needs to make informed decisions as to whether to deliberately create or avoid task ambiguity to serve particular pedagogical purposes. Thus, the classroom community showed understanding of Element 3a in milestone 3.

Pillar B3. The last pillar directed prospective teachers' attention to teachers' decision-making in situations when teachers did not realize a mathematical task they gave to students had ambiguous conditions. We hypothesized that Pillar B3 would provide a context for the development of pedagogical awareness related to Element 3b, which was the remaining part of milestone 3.

Two of the three small groups that kept written notes of their discussion of Pillar B3 indicated, through their writings, awareness of the importance for teachers to be able to see, size up, and appreciate unexpected student answers to a task that might be due to different interpretations of the task's conditions. Here is an illustrative excerpt: "Children might come up with different interpretations of problems and therefore have a different answer. However, the teacher should not penalize the student for this. As long as they can explain their answers, they should be given full credit." The third group that returned notes only wrote one sentence (presumably due to lack of time) that was hard to interpret. However, Amanda, a member of this group, spoke first in the whole class discussion and expressed, on behalf of her group, similar ideas as the other two groups:

(29) Amanda: [. . .] [Teachers] need to be flexible about that, and if students can show their work and justify their answer, give them credit for the problem.

This general way of a teacher handling unexpected task ambiguity was elaborated on by the prospective teachers who contributed next in the whole class discussion. Following the instructor's call for further contributions (Practice 3), Helen described reasons for which students might have reached answers that deviated from the teacher's expected answer, and she highlighted the importance of the teacher tracing students' line of reasoning:

(30) Helen: We said that there are a couple of things that could happen . . . A student may arrive at a conclusion but not understand the process, or a student may understand the process but may have made just a small error and it might look like they don't understand it. So a teacher should start from where they went wrong and [. . .] understand there's a lot happening in class [. . .]

Joan built on Helen's contribution (Practice 2) by distinguishing between lines of reasoning that include an error and others that are the results of task ambiguity:

(31) Joan: You [referring to Helen] made a good point. That you have to think about presentation, and there are mistakes that can be more than a silly mistake or a minor little math error. Because, I mean, the premise here is that the teacher does not realize that something is ambiguous. [. . .] But, really, you need to ask the student their own reasoning, and then you can determine if it's just a step that they're missing, or if you have not given the information clearly.

The whole class discussion went on with similar contributions. On the basis of the consistency amongst prospective teachers' verbal contributions in the whole class discussion and the views expressed by small groups in their written responses to Pillar B3, we conclude that the classroom community had developed pedagogical awareness of a teacher's possible decisions when sizing up students' answers to a task that deviate from the teacher's expected answer and that might be attributed to an inadvertent or a student-perceived ambiguity in the task's conditions. Specifically, there was a shared appreciation of: (1) the need for teachers to recognize that some students' approaches to a task that appear faulty may in fact be mathematically sound and based on unforeseen (to the teacher) legitimate assumptions, and (2) the importance of teachers asking students to explain their thinking to see whether the students indeed operated on different assumptions. These aspects of pedagogical awareness correspond to Element 3b in milestone 3.

Conclusion

The notion of assumptions is central to mathematical activity in the elementary school and beyond, including in modeling and proving tasks (e.g., A. J. Stylianides, 2007; Brown & Stillman, 2017; Komatsu et al., 2019). Yet there is a scarcity of research on the design of interventions to help prospective teachers develop important elements of MKT related to assumptions. This is problematic, especially at the elementary teacher education level: prospective elementary teachers have difficulties dealing with assumptions in tasks whose conditions are not well defined (Cai et al., 2022; Suh et al., 2021) and the notion of assumptions received limited attention in textbooks for use in mathematics courses for prospective elementary teachers (McCrory & Stylianides, 2014).

In this paper, we took a step toward addressing this research gap by reporting the theoretical foundation and implementation of an intervention that aimed to promote three elements of MKT regarding assumptions in a mathematics course for prospective elementary teachers. The intervention's design utilized the idea of "productive ambiguity" (Grosholz, 2007; as elaborated by Foster, 2011) in the context of a deliberately ambiguous task where the role of assumptions surfaced and was explicitly reflected upon (A. J. Stylianides, 2007, 2016; Ball, 1993; Komatsu et al., 2019). Our findings, similarly to those of Marmur and Zazkis (2022), illustrate that an ambiguous task cannot only reveal teachers' knowledge, but it can also promote knowledge development. Also, our study provides a paradigmatic case in the context of teacher education of how productive ambiguity can be purposefully planned and exploited to promote students' learning of mathematics (Foster, 2011).

The close matching between the classroom community's hypothetical and actual learning trajectories, which we view as the outcome of the systematic approach we followed to both develop the intervention and the associated theoretical framework within the design-based research paradigm, suggests that the intervention fulfilled to a large extent its aim to promote the three elements of MKT regarding assumptions. Our documentation of the classroom community's actual learning trajectory was a best-fit description of knowledge advancements that could reasonably be assumed as shared or accepted within the community based on triangulation of data from multiple sources. This does not mean that every prospective teacher in the class followed the hypothetical learning trajectory. Indeed, our data do not allow any such claims about the learning path followed by each individual prospective teacher. This is a limitation of our study but also one that is hard to address in research conducted in actual classroom settings. Another limitation is that we investigated the development of prospective teachers' knowledge about assumptions in the context of only one task (the Floors Problem) and we did not examine whether the learning documented in this paper was sustained. We also acknowledge that, to solidify their knowledge, prospective teachers need multiple opportunities to learn about assumptions. This highlights both a direction for future research and the need for the notion of assumptions to receive more attention in textbooks and other resources for use with prospective teachers. The fact that the focal intervention included only one mathematical problem also had its advantages. Like the interventions we reported elsewhere (Stylianides & Stylianides, 2014a, 2022; Stylianides & Stylianides, 2009) from the same design experiment, this one also had a short duration and could stand alone, aspects that can facilitate its more flexible incorporation into existing instructional resources.

The paper's contribution is not limited to having shown that the intervention "works" or "can work" to support development of the three elements of MKT in the important area of assumptions. Consistent with another primary intent of design-based research (Cobb et al., 2003; Schoenfeld, 2006), we also offered an account of how the intervention promoted these elements, thus deepening theoretical understanding about ways to promote MKT. This is important as there is a scarcity of research on, and illustrative cases of, promoting MKT not only in the area of assumptions but also more broadly (Silverman & Thompson, 2008). The approach to promoting MKT that we discussed in this paper exemplifies how teacher educators can design learning opportunities for prospective teachers to intertwine mathematical learning with pedagogical awareness, thus developing "pedagogically functional mathematical knowledge" (Ball & Bass, 2000, p. 95). The two-part structure of our approach – beginning with a mathematics task devoid of a pedagogical context and following up on it with prompts in the form of "conceptual awareness pillars" (G. J. Stylianides & Stylianides, 2009) to direct prospective teachers' attention to the mathematico-pedagogical space of interest – complements other approaches to promoting MKT in the literature that utilized a pedagogical context to situate or motivate the study of mathematics in teacher education (A. J. Stylianides & Stylianides, 2014b; Heid & Wilson, 2015; Wasserman et al., 2019). The purposeful design and implementation of the pillars played a major role in the intervention in supporting prospective teachers' progression from learning mathematics for themselves to developing pedagogical conceptualizations of the mathematics they learnt, which Silverman and Thompson (2008) proposed is essential for developing MKT.

Notes

1. Ball et al. (2008) included two further knowledge categories – *horizon content knowledge* and *knowledge of content and curriculum* – within subject matter knowledge and pedagogical content knowledge, respectively.
2. We discuss our documentation of a community's actual learning trajectory in the Methodology section where we also discuss the relationship between the learning trajectory of a community and that of individual members.
3. The study received ethical approval from the University of Pittsburgh's Institutional Review Board (IRB Number: 0512096). All participants provided informed consent to participate in the study.
4. The transcription of the whole class discussions recorded all the words, including any false starts and errors, but cut out any stuttering or filler words.
5. The following excerpt from the course syllabus reflects our emphasis on prospective teachers' active participation in class discussions and related norms: "You are expected to participate in the class activities and discussions. Desirable participation is not limited to giving 'correct' responses; sharing undeveloped ideas and puzzlements with the group as well as responding to your colleagues' ideas, statements, and questions are considered equally important. Communicating about mathematics – both talking and listening – is a central part of mathematics teaching and, as such, is part of the goal of the course." Participation in classroom discussions and a portfolio entry accounted for 15% of the course grade.
6. *Italics* in classroom discussions are ours to draw attention to specific segments. All prospective teacher names are pseudonyms.

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