The geometry of some moduli schemes arising in enumerative geometry

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to my families

Abstract

In this thesis, we work on the explicit geometric description of certain moduli schemes. Pandharipande-Thomas (PT) stable pairs provide an example of moduli spaces of objects in the derived category, whose scheme-theoretic geometry can be explored to a reasonable level of details. We consider PT-pairs supported at the double of the zero-section inside a particular bundle over the projective line, called the resolved conifold.

Their geometry can be probed in two ways. The first is sheaf-theoretic, and consists in looking at the non-reduced structures with which we can endow a reduced curve: this relies on a procedure by Ferrand, reducing the study to line bundles. Degenerations of those line bundles are relevant, as they carry information about the corresponding PT-pairs.

The second way involves representation theory. Thanks to a result by Nagao-Nakajima, PT-pairs on the resolved conifold correspond to stable representations of a particular quiver with potential, for a suitable stability condition. By finding an appropriate basis, we can read off the equations for the moduli scheme and recognise the same geometry observed through the first approach.

There is a stratification of the moduli scheme, which admits a simple sheaf-theoretic interpretation, but the actual equations are given via the quiver representation approach. The main technical obstacle is that, while stability for the Hilbert scheme (leading to Donaldson-Thomas invariants) translates to a simple algebraic condition, stability of PT-pairs involves a case-by-case analysis to obtain suitable algebraic conditions on the corresponding quiver representations.

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Contents

List of Figures

Chapter 0

Introduction

It's a sad bad and mad glad scheme
It's the best scheme I have
And that's bad enough for me
And I won't get sad if you don't
See just how mad glad my schemes can be
Bill Wurtz

This thesis concerns the explicit description of certain moduli schemes, of interest in enumerative geometry and generally in the context of moduli spaces of coherent sheaves over Calabi-Yau three-folds. Among those moduli schemes, the Hilbert scheme of points has been studied extensively, and is the subject of active research. It is worth recalling classical and recent results about what is known of their geometry. Consider the Hilbert scheme of points $\operatorname{Hilb}^n X$, parametrizing zero-dimensional subschemes $Z \subset X$ of length n, and call $\mathbf{hc}\colon \operatorname{Hilb}^n(X) \to \operatorname{Sym}^n(X)$ the *Hilbert-Chow* morphism sending Z to its underlying 0-cycle. When X is a smooth curve, it is a standard result that $\mathbf{hc}$ is an isomorphism (see for example [Kol96, §1.4.1]); for X an irreducible nonsingular quasiprojective surface, Fogarty [Fog68, Thm. 2.9] proves that the Hilbert scheme is smooth and irreducible for all n, and in fact $\mathbf{hc}$ is a resolution of singularities. Moreover, there is a plethora of results studying the structure of Hilbert schemes of points on surfaces via Nakajima quiver varieties [Nak99].

When the ambient variety X has dimension 3, already simple geometric questions concerning reducedness or irreducibility have only partial results, several of which provide bounds through explicit examples, often relying on computer algebra as in [HMS21] or [DJNT17]. As the question is local, fix for now $X = \mathbb{A}^3_{\mathbb{C}}$. We have that $\operatorname{Hilb}^n(\mathbb{A}^3_{\mathbb{C}})$ is smooth for $n \leq 3$, and reduced and irreducible but singular for $n = 4$ (originally proven by Fogarty [Fog68]; for an exposition see for example [Kat92]); hence, it is singular for $n \geq 4$ by considering subschemes containing a *fat point* with multiplicity 4. It is reducible for $n \geq 78$ by work of Iarrobino [Iar72], and irreducible for $n \leq 11$, with this lower bound attained in [DJNT17] - it is unknown which n is the threshold of reducibility. This threshold is however known for $d \geq 4$ where it has been shown [CEVV09] that $\mathrm{Hilb}^n(\mathbb{A}^d)$ is reducible for $n = 8$, hence for $n \geq 8$, and irreducible for $n < 8$. We provided here only a brief account of results, and we refer, for example, to [DJNT17, §1] for a more comprehensive summary.

In terms of reducedness, Fogarty [Fog68] conjectured $\mathrm{Hilb}^n(\mathbb{A}^d_k)$ to always be reduced for all n, d and all fields k of characteristic 0. A negative answer was recently provided by Jelisiejew [Jel20] and later further refined by Szachniewicz [Sza21], who construct counterexamples for $d = 16$ and $d = 6$ respectively. Jelisiejew conjectures that the same technique could be refined to work for $d = 4$, and notes that the borderline case $d = 3$ admits a further approach via representations of quiver with superpotential, as done for $\mathrm{Hilb}^4(\mathbb{A}^3_{\mathbb{C}})$ by Dimca-Szendrői [DS09]; it is currently an open question whether $\mathrm{Hilb}^5(\mathbb{A}^3_{\mathbb{C}})$ is reduced.

If we consider the Hilbert scheme of *curves*, that is 1-dimensional subschemes, even less is known about the explicit geometry. The question of irreducibility goes back to Severi [Sev21], and we know that $\mathrm{Hilb}^{3t+1}(\mathbb{P}^3_{\mathbb{C}})$, the Hilbert scheme of cubic curves, having Hilbert polynomia $3t + 1$, is reducible [PS85]. Moreover, Harris and Eisenbud [Har82] prove that, given any dimension d and genus g, if $\mathrm{Hilb}^{nt+1-g}(\mathbb{P}^d_{\mathbb{C}})$ is non-empty then it is reducible for sufficiently large degree n. On threefolds $d = 3$, their lower bound on n is rather coarse, compared to other results [Ili06].

Considering the issue of reducedness, Mumford [Mum62, §2] proves the following. Let $\mathrm{Hilb}^{14t-23}(\mathbb{P}^3_{\mathbb{C}})$ be the Hilbert scheme parametrising curves of degree 14 and arithmetic genus 24. This scheme has an irreducible component, generically parametrising smooth irreducible curves, which is everywhere nonreduced. This is arguably the most famous specimen, among the bestiary of known example of Hilbert schemes of curves. Another example with lower genus and degree is provided in [GP82, p. 417], where the authors prove the existence of an irreducible and nonreduced component of $\mathrm{Hilb}^{13t-17}(\mathbb{P}^3_{\mathbb{C}})$, parametrising smooth curves of degree 13 and genus 18 satisfying certain incidence conditions. To the best of our knowledge, these are currently the lowest values of degree and genus for which such a component parametrising smooth curves is known to exist.

There are more results aiming at exhibiting non-reduced components of Hilbert schemes of curves, for example [Ric20]. More generally, Martin–Deschamps and Perrin [MP96] show that, for "almost all" arithmetic genus and degree, the Hilbert scheme contains a non-reduced component, but those components parametrise possibly reducible or non-reduced curves. We refer to the summary by Hartshorne [Har10, end of §2.13] for a more in-depth overview. In general, to prove nonreducedness of a Hilbert scheme, one could either show that an entire component is nonreduced (as done in [Mum62; GP82]), or show that some component is parametrised by a scheme which can be explicitly described: this is for example the approach in [AV92]. A famous result of Hartshorne [Har66] shows that, for any Hilbert polynomial P, the Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^d_{\mathbb{C}})$ is connected; moreover, a recent work [SS20] characterises those polynomials P for which this Hilbert scheme is smooth.

Most of the results above yield coarse bounds on the discrete parameters (like degree, genus, or dimension of the ambient space), so that certain geometric properties are satisfied. In order to improve them, it is necessary to study in detail special cases. Given the scarcity of full examples, there is a legitimate interest in the explicit geometry of moduli spaces of curves, in particular around questions about their reducedness or reducibility. Instead of considering the Hilbert scheme of curves, we focus on moduli spaces of Pandharipande-Thomas (PT) stable pairs, introduced in [PT09a]. PT-moduli spaces provide a different compactification of the moduli space of smooth curves, and are better behaved than the Hilbert scheme of curves: for example, 1-dimensional schemes with isolated points are excluded. Therefore they are good examples of moduli spaces (of objects in the derived category of a smooth complex algebraic variety), whose scheme-theoretic geometry can be described to a reasonable level of detail. In particular, we look at moduli schemes $\mathcal{N}_{l,n}$ of PT-pairs on the resolved conifold $X = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)^{\oplus 2})$, parametrising pairs $(\mathcal{F}, s)$ of sheaves with given sections (satisfying certain conditions), such that $\mathcal{F}$ has Euler characteristic $\chi(\mathcal{F}) = n$, and it is schematically supported on a thickening of degree l of the zero section (cf. Definition 2.1.6).

The cases of stable pairs with reduced support $\mathcal{N}_{1,n}$, along with the first two cases with degree-two support $\mathcal{N}_{2,3}$ and $\mathcal{N}_{2,4}$, were described by Pandharipande-Thomas in their original paper [ibid.]: they are irreducible and non-singular, except for the last one which is non-reduced. Szendrői [Sze12], further studied their critical locus structure, as detailed below. The geometry of such moduli schemes can be probed in two ways.

The first is sheaf-theoretic and consists in looking at the non-reduced structures with which we can endow a reduced curve: this relies on a procedure by Ferrand, reducing the study to line bundles. Degenerations of those line bundles are relevant, as they carry information about the corresponding PT-pairs.

The second way involves representation theory. Thanks to a result by Nagao-Nakajima [NN11], PT-pairs on the resolved conifold correspond to stable representations of a particular quiver with potential, for a suitable stability condition. By finding an appropriate basis, we can read off the equations for the moduli scheme and recognise the same geometry observed through the first approach.

In the first interesting case $\mathcal{N}_{2,4}$, we prove a result confirming a conjecture made in [Sze12] about the local form of the potential, that is a regular function $f\colon U \to \mathbb{C}$ defined on a smooth embedding space U, whose critical locus $\mathrm{Crit}(f)$ is isomorphic to the moduli scheme being considered.

Theorem 4.3.6. *The moduli scheme $\mathcal{N}_{2,4}$ is isomorphic to* $\mathrm{Crit}(q) \subset \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^3}(-1))$, *where*

$$q = t^2 \begin{vmatrix} x & z \\ y & w \end{vmatrix} .$$

We examine then the first hitherto unstudied example of the moduli scheme of degree-two PT-pairs on the resolved conifold, $\mathcal{N}_{2,5}$, and we give a description of its schematic structure, which is the main result of this thesis. We proceed using both approaches described above (sheaf-theoretic and via quiver representations). The former offers a useful geometric insight, showing that the underlying reduced scheme is reducible, and has two components glued along the locus of curves with double marked points.

Theorem 3.4.1. *The moduli scheme $\mathcal{N}_{2,5}$ has the following description. The underlying reduced scheme $(\mathcal{N}_{2,5})_{\mathrm{red}}$ is reducible:*

$$(\mathcal{N}_{2,5})_{\mathrm{red}} \simeq \mathbb{P}^5 \cup_{\mathbf{F}} \tilde{\mathbf{F}} .$$

On the first component $\mathbb{P}^5$, there is a stratification

$$\mathbb{P}^5 \supset \mathbf{H} \supset \mathbf{G} \supset \mathbf{F} ,$$

where $\mathbf{H} \subset \mathbb{P}^5$ is a singular quartic divisor, $\mathbf{G} = \mathrm{Sing}(\mathbf{H})_{\mathrm{red}} \simeq \mathbb{P}^1 \times \mathbb{P}^2$ and $\mathbf{F} \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

The second component $\tilde{\mathbf{F}}$ is described in Proposition 3.4.15. It is a certain flag-bundle with incidence conditions, and it is an $\mathbb{F}_3$-fibration over $\mathbb{P}^1$, where $\mathbb{F}_3$ is the third Hirzebruch surface*; the section $\mathbf{F} \hookrightarrow \tilde{\mathbf{F}}$ arises by considering the choice of the second summand, using the isomorphism $\mathbb{F}_3 \simeq \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)\right)$.*

The scheme structure of $\mathcal{N}_{2,5}$ has embedded components along the divisor $\mathbf{H}$, where the Zariski tangent space has generic dimension 6, and along the threefold $\mathbf{G}$, where the Zariski tangent space is 7-dimensional. The Zariski tangent space at the generic point of $\tilde{\mathbf{F}} \setminus \mathbf{F}$, has dimension 3.

The latter approach, using representation theory, while agreeing with the results of the sheaf-theoretic description, provides in addition an explicit potential on a large (in a suitable sense) open subscheme, which is a new piece of information.

Theorem 4.4.18. *The scheme* $\mathcal{N}_{2,5}$ *admits an open subscheme* $\mathcal{N}_{2,5}^{\mathcal{U}}$*, containing the* $\mathbb{P}^5$ *component and all embedded components, and which sits inside* $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^5}(-1)^{\oplus 2})$ *as the critical locus of*

$$s^2 \begin{vmatrix} u & x \\ v & y \end{vmatrix} + 2st \begin{vmatrix} u & x \\ w & z \end{vmatrix} + t^2 \begin{vmatrix} v & y \\ w & z \end{vmatrix} ,$$

where $[x : y : z : u : v : w]$ *and* (s,t) *are coordinates on* $\mathbb{P}^5$ *and the line bundles, respectively.*

It is useful to notice that the open subscheme $\mathcal{N}_{2,5}^{\mathcal{U}}$ contains all singularities of $\mathcal{N}_{2,5}$: hence, little geometric information is lost about the whole moduli scheme.

To any scheme M written as $M \simeq \mathrm{Crit}(f)$, for f a potential as above, we can associate a *DT sheaf*, which is the perverse sheaf of vanishing cycles Φ_f; this perverse sheaf categorifies numerical invariants in a precise sense (cf. Section 1.4). The cohomology of the DT sheaf Φ_f is known for all moduli schemes $\mathcal{N}_{l,n}$. It is studied in [DMSS15] using purity results, and the motive is computed in [MMNS12]. Szendrői [Sze12] studied the case $\mathcal{N}_{2,4}$, using his conjectural f and computing the actual sheaf Φ_f in topological terms, whose topology agrees with the result computable via a motivic calculation and purity results. The new information available on $\mathcal{N}_{2,5}$ makes it possible to follow the steps of the same computation, describing the DT sheaf. We leave this for future work.

Finally, there is a class of examples of moduli spaces of sheaves, which could be studied using techniques similar to this thesis. Fixing $\Gamma < \mathrm{SL}(3,\mathbb{C})$ a finite subgroup and a natural number n, the *n-equivariant Hilbert scheme* $\mathrm{n}\Gamma\text{-Hilb}(\mathbb{C}^d)$ parametrises Γ-invariant ideals I of $R := \mathbb{C}[x_1,\dots,x_d]$ such that $R/I \simeq_\Gamma n \cdot \mathrm{reg}$, that is R/I is isomorphic, as a representation, to n copies of the regular representation. This scheme is related to the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^d/\Gamma)$ of points on a quotient singularity, and is isomorphic to the moduli scheme of stable representations of the framed McKay quiver, with a certain potential, associated to the action $\Gamma \circlearrowright \mathbb{C}^3$, for a suitable stability condition and where these representations have dimension depending on n. While there is a rich literature on equivariant Hilbert schemes on surfaces, exploiting the structure of Nakajima quiver varieties (see for instance the notes [Cra21]), already the case $d = 3, n \geq 2$ has not been explored in details. There are basic geometric questions (singularity, reducedness, reducibility, d-critical structure) which could be answered by providing an explicit schematic description, obtained by considering quiver representations and applying reduction techniques as in the cases of $\mathcal{N}_{2,4}$ and $\mathcal{N}_{2,5}$.

Organization of contents

Chapter 1 covers standard material, collecting technical tools. One of these is Ferrand's construction, where a *double structure* is assigned on a reduced curve $Z \subset X$ by assigning a line bundle $\mathcal{L} \subset \mathcal{N}_{Z/X}$. We also provide a broad overview of enumerative theories like Donaldson-Thomas and Pandharipande-Thomas invariants, and how they are related. Another essential tool is the equivalence existing between PT-pairs with fixed Gorenstein support curve C, and the datum of a zero-dimensional subscheme of C. We will make extensive use of this approach for our geometric description of $\mathcal{N}_{2,n}$.

Chapter 2 focuses on the resolved conifold X, which has a simple but interesting geometry, as it is a toric variety. This allows us to specialise Ferrand's construction to thickenings of the zero-section of X, reaching a particularly simple parametrisation of double structures. The toric geometry of X also lends itself to implement the 3-fold vertex formalism [PT09b], which offers in turn a convenient means to explore the deformation theory of the moduli schemes $\mathcal{N}_{l,n}$, by considering torus-fixed stable pairs and using localisation techniques. Finally, the derived category of X is itself relatively simple, as it can be described via a Morita equivalence to a category of modules over a certain quiver, called appropriately the *conifold quiver*. This description is the starting point of the result by Nagao-Nakajima.

Chapters 3 and 4 are the technical core of the present work, where we explore the geometry of several examples of moduli spaces $\mathcal{N}_{l,n}$ using the two approaches, sheaf-theoretic and representation-theoretic, outlined above. The case $l = 1$ does not offer a rich geometry, but already for $l = 2$ interesting non-reduced structures appear. The cases $n = 3, 4$ were already covered in the original paper [PT09a], albeit rather concisely, and in [Sze12]. We expand those descriptions, and proceed to the new case $\mathcal{N}_{2,5}$.

We conclude in Chapter 5 with a general discussion, observations, and conjectures about $\mathcal{N}_{2,n}$ for $n > 5$.

Statement of Originality

I declare that the work contained in this thesis is, to the best of my knowledge, original and my own work, unless indicated otherwise. I declare that the work contained in this thesis has not been submitted towards any other degree or award at this institution or at any other institution. Section 4.4 is based on joint work with Professor Balázs Szendrői.

Aurelio Carlucci
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Chapter 1

Background

1.1 Non-Reduced Structures

1.1.1 Sheafy algebra

We write here for convenience some definitions, following [HL10]. We work over $\mathbb{C}$ throughout. Let X be a noetherian scheme.

Definition 1.1.1 (Algebraic Support)**.** If A is a ring and M an A-module, the *support of* M is $\operatorname{Supp}(M) := \{\mathfrak{p} \in \operatorname{Spec} A \mid M_{\mathfrak{p}} \neq 0\}$. Sheafifying: given $E \in \mathsf{Coh}(X)$, the *support* is the *set* $\operatorname{Supp}(E) = \{x \in X | E_x \neq 0\}$ ♢

The set-theoretic support of a coherent sheaf can be given the structure of a subscheme of X: indeed, there is a sheaf of ideals, the *annihilator ideal* of E, which is the kernel of $\mathcal{O}_X \to \mathcal{E}nd(E)$, $f \mapsto (s \mapsto fs)$. The precise statement is Lemma 1.1.7 below.

Definition 1.1.2 (Purity)**.** The *dimension* of $E \in \mathsf{Coh}(X)$ is the dimension of $\operatorname{Supp}(E)$ and denoted $\dim(E)$; we say E is *pure of dimension* d if any non-trivial subsheaf $F \subset E$ satisfies $\dim(F) = d$. ♢

Definition 1.1.3 (Associated primes)**.** Let A be a noetherian ring and M an A-module. A prime ideal $\mathfrak{p} \subseteq A$ is an *associated prime of* M if one of the following equivalent conditions hold:

1. there is an element $x \in M$ such that $\mathfrak{p} = \operatorname{Ann}(x)$,
2. M contains a submodule isomorphic to $A/\mathfrak{p}$.

♢

Proposition 1.1.4 ([Mat80])**.**

1. *If $\mathfrak{p}$ is maximal in $\{\mathrm{Ann}(x) | x \in M, x \neq 0\}$, which is a set of* ideals*, then $\mathfrak{p}$ is prime. Hence it is clearly an associated prime of M.*

2. *If I is an ideal, the minimal associated primes of A/I are the minimal prime ideals containing I.*

3. *The minimal associated primes of M are the minimal elements of* $\operatorname{Supp} M$*. Non-minimal associated primes are called* embedded primes*.*

Example 1.1.5. If $Y \subset X$ is a subscheme, its structure sheaf $\mathcal{O}_Y$ has dimension $\dim(Y)$, and it is pure if Y has no components of dimension strictly less than $\dim(Y)$. ♣

Example 1.1.6. In dimension 1, a module is Cohen-Macaulay if and only if it is not mixed, that is all the associated primes have the same dimension. Consider $R = \mathbb{C}[x, y, z]$, I an ideal defining a subscheme of (pure) dimension 1, *e.g.* $I = (x, y)$ and $J \not\subseteq I$ another ideal of (pure) dimension 0, *e.g* $J = (x, y^2, z)$. Then the module $M = R/(I \cap J)$, in our case $\mathbb{C}[x, y, z]/(x, y^2, yz)$ is not Cohen-Macaulay. What we did was basically patching together geometric objects of dimension 0 and 1 to get a subscheme of mixed dimension. ♣

Lemma 1.1.7 ([Stacks, Tag 05JU])**.** *Let $\mathcal{F}$ be a finite-type quasi-coherent sheaf of $\mathcal{O}_X$-modules on a scheme X. There exists a closed subscheme $i : Z \hookrightarrow X$, minimal with respect to inclusion, and a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$, quasi-coherent and unique up to unique isomorphism, such that $i_*\mathcal{G} \simeq \mathcal{F}$.*

On an affine open subset $\operatorname{Spec} A \subset X$*, if $\mathcal{F}_{|\operatorname{Spec} A} \simeq \widetilde{M}$ for an A-module M, then*

$$Z \cap \operatorname{Spec} A \simeq \operatorname{Spec}\left({}^{A}\!\big/\!{}_{\mathrm{Ann}_A(M)} \right).$$

Definition 1.1.8. We call Z the *scheme-theoretic support* of $\mathcal{F}$. ♢

1.1.2 Ribbons

Let X be any smooth scheme over $\mathbb{C}$, and $Z \subset X$ a reduced, irreducible smooth curve. The aim is to construct a non-reduced scheme, supported at Z, in a controlled way.

Definition 1.1.9. Let Z be as above. A *multiple scheme structure* along Z, also called a *thickening* of Z, is a subscheme C of X which is set-theoretically supported on Z, scheme-theoretically satisfying $Z \subset C \subset Z^{(n)} \subset X$, for some n, where $Z^{(n)} \subset X$ has ideal sheaf $\mathcal{I}_Z^n$, given $\mathcal{I}_Z$ the ideal sheaf of Z. ◊

There are inclusions

$$\begin{tikzcd} Z \arrow[r, hook, "k"] \arrow[rr, hook, bend left, "i"] & C \arrow[r, "j"] & X \end{tikzcd},$$

which we will use when needed to avoid confusion. Moreover, we want the scheme structure on C to be *double* with respect to Z, in a way that we now make precise.

Definition 1.1.10 ([Stacks, Tag 0DR4])**.** Let C be a one-dimensional subscheme of X such that $C_{\text{red}} = Z$. The *multiplicity* of C is the multiplicity of the Artin ring $\mathcal{O}_{C,\xi}$, which is to say its length as a module over $\mathcal{O}_{Z,\xi}$, where ξ is the generic point of C. We say that C is *double* if it has multiplicity 2, and in such case we call it a *ribbon.* ◊

A convenient way to produce double curves is known as *Ferrand's construction* [Man03, §2.2][BF86, §1], outlined below.

Proposition 1.1.11. *Let $Z \subset X$ be as above. A Cohen-Macaulay double thickening C of Z is equivalent to the choice of a line sub-bundle $\mathcal{L} \subset \mathcal{N}_{Z/X}$ of the normal bundle of Z in X.*

Proof. Consider a double scheme C supported at Z, and let $\mathcal{I}_Z$ and $\mathcal{I}_C$ be the ideal sheaves in X of Z and C respectively: they must satisfy $\mathcal{I}_Z \supset \mathcal{I}_C \supset \mathcal{I}_Z^2$, corresponding to scheme-theoretic inclusions $Z \subset C \subset Z^{(2)} \subset X$, where $Z^{(2)}$ is the infinitesimal first-order thickening of Z given by $\mathcal{I}_Z^2$. Denoting $\mathcal{K}$ the cokernel of $\mathcal{I}_C \to \mathcal{I}_Z$, by universal properties we get a map γ fitting in the commutative diagram below

$$\begin{tikzcd} \mathcal{I}_Z^2 \arrow[dd, hook] \arrow[dr, hook] & & \mathcal{K} \\ & \mathcal{I}_Z \arrow[ur, two heads] \arrow[dr, two heads] & \\ \mathcal{I}_C \arrow[ur, hook] & & \mathcal{I}_Z/\mathcal{I}_Z^2 \arrow[uu, dashed, two heads, "\exists!\,\gamma"'] \end{tikzcd} \quad . \tag{1.1}$$

Moreover, we have an exact sequence

$$0 \longrightarrow \mathcal{I}_C/\mathcal{I}_Z^2 \longrightarrow \mathcal{I}_Z/\mathcal{I}_Z^2 \longrightarrow \mathcal{K} \longrightarrow 0\,. \tag{1.2}$$

Since $\mathcal{K}$ is a quotient of sheaves supported on Z, it is as well, and $\mathcal{K}$ is the pushforward of a sheaf on Z. Now, $\mathcal{O}_C$ has multiplicity 2 at the generic point, hence the sheaf induced on Z by $\mathcal{K}$ must locally have one generator, and being torsion-free over a PID it is in fact locally free by the Structure Theorem. Considering now the normal bundle $\mathcal{N}_{Z/X} = \left({}^{\mathcal{I}_Z}\!/\!_{\mathcal{I}_Z^2} \right)^\vee$, the desired line bundle $\mathcal{L}$ is the image of the inclusion $\mathcal{K}^\vee \hookrightarrow \mathcal{N}_{Z/X}$. Conversely, let us take a line bundle $\mathcal{L} \hookrightarrow \mathcal{N}_{Z/X}$. Its dual $\mathcal{K}$ fits in a surjection

$$p\colon {}^{\mathcal{I}_Z}\!/\!_{\mathcal{I}_Z^2} \twoheadrightarrow \mathcal{K}\,,$$

as in (1.2), and $\ker p \simeq {}^{\mathcal{I}_C}\!/\!_{\mathcal{I}_Z^2}$, where $\mathcal{I}_C$ is the ideal sheaf of a scheme C in which Z is a closed subscheme, and which satisfies $\mathcal{I}_Z^2 \subset \mathcal{I}_C$: this defines the required double scheme C. $\square$

Definition 1.1.12. With the same notation as Proposition 1.1.11, we call C the *double thickening* or *Ferrand doubling* of Z along $\mathcal{L}$ (or $\mathcal{K}$, where no ambiguity arises). $\lozenge$

Remark 1.1.13. From this point onwards, a thickening will always mean a *double* thickening, unless otherwise specified. ♠

Remark 1.1.14. A priori, a non-reduced scheme C such that $C_{\mathrm{red}} = Z$ could have embedded points. This kind of scheme will not arise when thickening Z along a line bundle $\mathcal{L} \subset \mathcal{N}_{Z/X}$. By only considering C Cohen-Macaulay, that is with $\mathcal{O}_C$ pure, we obtain the equivalence of Proposition 1.1.11. ♠

We can fit all information in exact sequences of sheaves *over* X:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & i_*\mathcal{K} \arrow[r] \arrow[u] & j_*\mathcal{O}_C \arrow[r] \arrow[u] & i_*\mathcal{O}_Z \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{I}_Z \arrow[r] \arrow[u] & \mathcal{O}_X \arrow[r] \arrow[u] & i_*\mathcal{O}_Z \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & \mathcal{I}_C \arrow[r, equal] \arrow[u] & \mathcal{I}_C \arrow[r] \arrow[u] & 0 \arrow[u] & \\
 & 0 \arrow[u] & 0\,, \arrow[u] & &
\end{tikzcd} \tag{1.3}$$

where the top row is exact by the Nine Lemma.

We notice that the extension in the top row of diagram (1.3) defines an element $[\mathcal{O}_C] \neq 0$ of $\operatorname{Ext}^1_X(i_*\mathcal{O}_Z, i_*\mathcal{K})$. Indeed, this extension is non split, for otherwise C would be isomorphic to the union of Z with an embedded point, which is not Cohen-Macaulay.

The double curves arising in this way retain substantial regularity, as the simple result below shows.

Proposition 1.1.15 ([BF86, §1.1])**.** *With notation as above, if Z is a local complete intersection, so is the Ferrand doubling C.*

Given a line bundle $\mathcal{L}$ on Z, the idea behind the construction of the thickening C is to impose $\mathcal{L}$ as a new direction in which C can have tangent vectors; alternatively, sections of $\mathcal{L}^\vee$ are new functions allowed on C. This is formalised in the statements below.

Proposition 1.1.16. *Let C be the double thickening of Z along $\mathcal{L}$. Then the normal sheaf $\mathcal{N}_{Z/C}$ of Z in C is isomorphic to $\mathcal{L}$.*

We recall first a technical result.

Lemma 1.1.17 ([Stacks, Tag 04CJ])**.** *Let $f\colon V \to W$ be a closed immersion of schemes. Then*

1. *The adjunction counit $f^*f_* \to \mathrm{Id}$ is a natural isomorphism of endofunctors on $\mathsf{Coh}(V)$;*
2. *the functor $f_*\colon \mathsf{Coh}(V) \to \mathsf{Coh}(W)$ is an* embedding*, that is exact and fully faithful.*

Proof of proposition. Let $\mathcal{I}_{Z/C}$ be the ideal sheaf of Z in C; by Lemma 1.1.17 above, the map $j\colon C \hookrightarrow X$ gives an embedding $j_*\colon \mathsf{D}^\flat\mathsf{Coh}(C) \to \mathsf{D}^\flat\mathsf{Coh}(X)$, and we get an exact sequence

$$0 \longrightarrow j_*\mathcal{I}_{Z/C} \longrightarrow j_*\mathcal{O}_C \longrightarrow i_*\mathcal{O}_Z \longrightarrow 0\,. \tag{1.4}$$

Comparing (1.4) with the top row of (1.3), we obtain that $j_*\mathcal{I}_{Z/C} \simeq i_*\mathcal{K}$. Since $i_* = j_* \circ k_*$ and j_* is an embedding, we deduce that

$$\mathcal{I}_{Z/C} \simeq k_*\mathcal{K}\,.$$

The conormal sheaf of Z in C is, by definition, $\mathcal{C}_{Z/C} \simeq k^*\left(\mathcal{I}_{Z/C}/\mathcal{I}^2_{Z/C}\right)$; however, given that $\mathcal{I}_Z^2 \subset \mathcal{I}_C \subset \mathcal{I}_Z$, we have

$$j_*\mathcal{I}^2_{Z/C} \simeq \left(\mathcal{I}_Z/\mathcal{I}_C\right)^2 \subset \mathcal{I}_Z^2/\mathcal{I}_C^2 \subset \mathcal{I}_C/\mathcal{I}_C^2\,,$$

therefore $\mathcal{I}^2_{Z/C} \simeq 0$, since it is a module over $\mathcal{O}_C \simeq j^*\left(\mathcal{O}_X/\mathcal{I}_C\right)$. In conclusion, we can write

$$\mathcal{C}_{Z/C} := k^*\left(\mathcal{I}_{Z/C}/\mathcal{I}^2_{Z/C}\right) \simeq k^*\mathcal{I}_{Z/C} \simeq k^*k_*\mathcal{K} \simeq \mathcal{K}\,,$$

where in the last isomorphism we used again Lemma 1.1.17. We have then

$$\mathcal{N}_{Z/C} := \mathcal{H}om_Z(\mathcal{C}_{Z/C}, \mathcal{O}_Z) \simeq \mathcal{K}^\vee =: \mathcal{L}\,.$$

□

Corollary 1.1.18. *At each reduced point $p \in Z \subset C$, the normal sheaf satisfies*

$$\mathcal{N}_{p/C} \simeq \mathrm{T}_{Z,p} \oplus \mathcal{L}_p\,.$$

Proof. Call $\iota\colon p \hookrightarrow Z$ the inclusion, and consider the conormal exact sequence [Stacks, Tag 06BB]

$$\iota^*\mathcal{C}_{Z/C} \longrightarrow \mathcal{C}_{p/C} \longrightarrow \mathcal{C}_{p/Z} \longrightarrow 0\,.$$

Since ι is a regular immersion, the sequence above is short exact. Using Proposition 1.1.16 we get then

$$\mathcal{C}_{p/C} \simeq \Omega_{Z,p} \oplus \mathcal{K}_p\,,$$

which yields the result. □

Example 1.1.19. Consider $X = \mathbb{A}^3 = \operatorname{Spec}\mathbb{C}\,[x,y,z]$, the curve $Z \hookrightarrow X$ defined by the ideal $I := \langle y, z\rangle$, and the rank-one module $L := \{z \mapsto 0\} \subset \operatorname{Hom}(\mathbb{C}\,y \oplus \mathbb{C}\,z, \mathbb{C}) = \mathcal{N}_{Z/X}$. To define a thickening of Z via $K = L^* \simeq \mathbb{C}\,y$, consider the affine realisation of the leftmost column of (1.3), which reads

$$0 \longrightarrow \langle y^2, z\rangle \longrightarrow \langle y, z\rangle \longrightarrow \mathbb{C}\,y \longrightarrow 0\,.$$

We obtain therefore the thickened curve $\operatorname{Spec} {\mathbb{C}\,[x,y,z]}/{\langle y^2, z\rangle}$. ♣

Example 1.1.20. Generalising the example above, given a straight line $Z \subset \mathbb{A}^3_{\mathbb{C}}$, we know a simple description of double curves C such that $C_{\mathrm{red}} = Z$, as shown for example in [EH00, § II.3.5]: if we assume x, y, z to be the affine coordinates and Z to be the z-axis, then the ideal of C is given by

$$I_C = I_{a,b} = \left(x^2, xy, y^2, b(z)x - a(z)y\right)\,,$$

where a and b are polynomials. If a, b have no common factor, then we are clearly constructing the thickening of Z along the normal direction given by $(a(z), b(z)) \in \mathbb{A}^2_{x,y}$ at each point $(0,0,z)$ of Z. Moreover, under these assumption of coprimality, one can show [ibid., Ex. II.34] that C has no embedded points and it is Cohen-Macaulay. Even more is true: there exists an automorphism[1] of $\mathbb{A}^3$ sending C to the planar double line with ideal (x, y^2), and any double Cohen-Macaulay curve has ideal of the form $I_{a,b}$ as above. ♣

[1]This is not true in the projective case.

While the curve C is singular and we cannot define a tangent/normal exact sequence, there is the result below, originally due to Bănică and Forster.

Proposition 1.1.21 ([BF86, §1.2])**.** *Let $Z \subset X$ be a smooth reduced curve and $C \subset$ the Ferrand doubling along the surjection $\mathcal{C}_{Z/X} \twoheadrightarrow \mathcal{K}$, with K a line bundle. Denoting $k\colon Z \to C$ the immersion, there is an exact sequence of sheaves on Z as below:*

$$0 \longrightarrow \mathcal{K}^{\otimes 2} \longrightarrow k^*\mathcal{C}_{C/X} \longrightarrow \mathcal{C}_{Z/X} \longrightarrow \mathcal{K} \longrightarrow 0\,. \tag{1.5}$$

Proof. We write first of all

$$k^*\mathcal{C}_{C/X} = \mathcal{C}_{C/X} \otimes \mathcal{O}_Z = {}^{\mathcal{I}_C}\!/\!_{\mathcal{I}_C^2} \otimes {}^{\mathcal{O}_X}\!/\!_{\mathcal{I}_Z} \simeq {}^{\mathcal{I}_C}\!/\!_{\mathcal{I}_Z\mathcal{I}_C}\,.$$

Then, recalling from (1.1) that $\mathcal{K} \simeq {}^{\mathcal{I}_Z}\!/\!_{\mathcal{I}_C}$, we compute (see for example [Liu06, Ex. 1.3])

$$\mathcal{K}^{\otimes 2} \simeq {}^{\mathcal{I}_Z^2}\!/\!_{\mathcal{I}_Z\mathcal{I}_C}\,.$$

Finally, consider the short exact sequence

$$0 \longrightarrow {}^{\mathcal{I}_Z^2}\!/\!_{\mathcal{I}_Z\mathcal{I}_C} \simeq \mathcal{K}^{\otimes 2} \longrightarrow {}^{\mathcal{I}_C}\!/\!_{\mathcal{I}_Z\mathcal{I}_C} \simeq k^*\mathcal{C}_{C/X} \longrightarrow {}^{\mathcal{I}_C}\!/\!_{\mathcal{I}_Z^2} \longrightarrow 0\,,$$

which can be spliced to the previous sequence (1.2), obtaining the desired 4-term exact sequence. □

The machinery developed above can be applied as well to the problem of thickening reduced points $p \in X$, by repeating the proofs above with Z replaced by p. In particular, we will be interested in Section 3.4 in the thickening of a reduced point inside a non-reduced curve. In this case, the construction of a double structure on a reduced point $p \in X$ only depends on a line sub-bundle of $\mathcal{N}_{p/X} = \mathrm{T}_{X,p}$. The proof of the following two statements is immediate.

Proposition 1.1.22. *Let $p \in X$ a reduced point. The datum of a* double point at p*, that is a zero-dimensional subscheme $\mathbf{p} \subset X$ such that $\mathbf{p}_{\mathrm{red}} = p$ and $\mathcal{O}_{\mathbf{p}}$ has length 2 as $\mathcal{O}_{X,p}$-module, is equivalent to the choice of a line $\mathcal{R} \subset \mathrm{T}_{X,p}$.*

Corollary 1.1.23. *Let $Z \subset X$ a reduced curve, C a thickening of Z along a line bundle $\mathcal{L}$, and p a smooth point of Z. A double point at p corresponds to the choice of a line $\mathcal{R}$ inside a vector space $V = (\Omega_{C,p})^\vee$ fitting in an exact sequence*

$$0 \longrightarrow \mathrm{T}_{Z,p} \longrightarrow V \longrightarrow \mathcal{L}_p\,, \longrightarrow 0$$

which is split, albeit non canonically, giving $V \simeq \mathrm{T}_{Z,p} \oplus \mathcal{L}_p$.

Remark 1.1.24. Notice that in the construction above Z being a curve had little relevance. There is a general approach to producing *multiple structures*, that is non-reduced nilpotent schemes Y such that $Y_{\text{red}} = Z$ and Y is contained in some infinitesimal thickening $Z^{(l)}$ of Z. The construction is developed by Vatne in his thesis, the paper [Vat09] and the more extensive preprint [Vat02]. The starting point is a locally free sheaf $\mathcal{K}$ on X, which is a sub-sheaf of the conormal bundle for some embedding $Z \hookrightarrow X$. In this section we only focused on the two cases of interest in this work, which are curves and points. ♠

1.1.3 Degenerations of Ribbons

We have seen that, under mild conditions, the Ferrand doubling C of a reduced curve Z is Cohen-Macaulay. We are interested in degenerations of ribbons as described above, as they will be relevant in the study of stable pairs in Chapter 3. This section is mostly expository.

If we think of double curves as ribbons, one might ask what happens when this ribbon twists until it develops a *pinch*. The phenomenon we are considering is local, hence we assume to be working in the affine case: we see below what happens when, in the construction of Example 1.1.20, the polynomials a and b share a factor.

Example 1.1.25. Let $a(z) = b(z) = z$, and consider $I = (x^2, xy, y^2, zx - zy)$. The corresponding curve C has an embedded point $\mathbf{q}$ supported at the origin: this point has ideal (x^2, xy, y^2, z), with a 3-dimensional Zariski tangent space at $\mathbf{q}$. ♣

To make Example 1.1.25 less surprising, let us see a few examples of flat limits of reduced schemes.

Example 1.1.26 ([PT14, § $3\frac{1}{2}$]). Consider the curve

$$C_t = \{x = 0 = z\} \cup \{y = 0 = z - t\} .$$

The flat limit of the ideal $I_t = (x, y) \cap (y, z - t)$ is $I_0 = (xy, xz, yz, z^2)$, and has an embedded point at the origin, "remembering" the direction of the collision. ♣

Example 1.1.27. Take now the union of two lines with a crossing point

$$I_t = (x, y) \cap (x, y - tz) = (x, yzt - y^2) ,$$

where in the limit the two lines coincide. The flat limit of the ideal above is simply

$$I_0 = (x, y^2) ,$$

which defines a Cohen-Macaulay double curve, and at each point the thickening direction "remembers" the direction along which the two lines collided. ♣

Example 1.1.28. If we start with a limit as in Example 1.1.26, we construct a family where for each $t \neq 0$ there are two lines crossing at an embedded point supported at the origin:

$$I_t = \left(xz, y^2 - tyz, xy, x^2\right) .$$

As t approaches zero, one line rotates towards the other: the flat limit of this ideal is $I_0 = (x^2, xy, y^2, xz)$, which is precisely $I_{a,b}$ as above, with $a = 0$ and $b = z$, that is, a singular ribbon with a pinch at the intersection. The embedded point at the origin has non-reduced directions coming from both the collision of two lines as in Example 1.1.26 and from the rotation. ♣

The previous sequence of examples should give an idea of how pinched ribbon might arise. Of course, we can obtain Cohen-Macaulay ribbons as well, as seen below.

Example 1.1.29. Take now two reduced curves, with one twisting around the other without intersecting. Consider two coprime polynomials $a(z)$ and $b(z)$, and take the ideal

$$I_t = (x, y) \cap (x - ta(z), y - tb(z)) ,$$

whose flat limit is precisely $I_{a,b}$. This example showcases ribbons as limits of two reduced non-intersecting curves, with one twisting around the other. ♣

Finally, we can see an example of a flat family of Cohen-Macaulay ribbons degenerating to a pinch.

Example 1.1.30. Take a family of Cohen-Macaulay ribbons, with ideal

$$I_t = \left(x^2, xy, y^2, zx - ty\right) ,$$

whose flat limit is (x^2, xy, y^2, zx), that is, the ideal of a pinched ribbon. This is a special case of Example 1.1.20, setting the polynomials $a(z) = t$ and $b(z) = z$, which have indeed a common root for $t = 0$. ♣

Remark 1.1.31. All the flat limits in this section can be conveniently determined via computer algebra, if desired. For example, one can use `Macaulay2` as explained in [EGSS02, p. 62]. ♠

1.2 Donaldson-Thomas theory

1.2.1 Overview

Our main references for this section are [PT14; Sze15; Mei16]. We are interested in moduli spaces of sheaves over a projective Calabi-Yau three-fold X. Consider the functor $\mathfrak{M} = \mathfrak{M}(X)\colon (\mathsf{Sch}/\mathbb{C})^{\mathrm{op}} \to \mathsf{Grpd}$, taking a $\mathbb{C}$-scheme S to the grupoid of S-flat $\mathcal{O}_{X_S}$-modules on $X_S := X \times_{\operatorname{Spec}\mathbb{C}} S$.

More precisely, we focus on the following subfunctors. Having fixed an even cohomology class $\beta \in \mathrm{H}^{2*}(X, \mathbb{Q})$, we consider the functor $\mathfrak{M}(X, \beta)$ giving S-flat coherent sheaves on X_S with Chern character β, up to isomorphism. There is then a functorial decomposition $\mathfrak{M}(X) = \coprod_\beta \mathfrak{M}(X, \beta)$.

As yet, each $\mathfrak{M}(X, \beta)$ is an Artin stack; if we impose a suitable *stability condition* σ, there is an open substack $\mathfrak{M}^\sigma(X, \beta) \subset \mathfrak{M}(X, \beta)$ parametrising σ-stable sheaves. In the present section we shall not deal with the general definition of stability conditions, although this is a broad and beautiful topic: for example, Bridgeland stability conditions [Bri07] are the subject of active research. In this discussion, we will assume that appropriate X and σ have been chosen. The stabiliser of a σ-stable sheaf is $\mathbb{C}^*$: once we rigidify $\mathfrak{M}^\sigma(X, \beta)$ to remove the $\mathbb{C}^*$-stabiliser, this stack is represented by a finite-type scheme $\mathcal{M}^\sigma(X, \beta)$ over $\mathbb{C}$. As said below, this will be the case in our specific situation.

The space $\mathcal{M}^\sigma(X, \beta)$ can have serious singularities: a good compendium is for example [Vak06]. The infinitesimal deformations of a coherent sheaf $\mathcal{F}$ on X are encoded in the tangent space $\mathrm{T}_{[\mathcal{F}]}\mathcal{M}^\sigma(X, \beta)$ at the isomorphism class $[\mathcal{F}]$, which is isomorphic to $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$; an obstruction space $\mathrm{Ob}_{[\mathcal{F}]}\mathcal{M}^\sigma(X, \beta)$ can be taken to be $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$: this is a general fact about moduli spaces of sheaves [HL10]. There is a highly non-trivial and non-canonical map $\mathrm{T}_{[\mathcal{F}]}\mathcal{M}^\sigma(X, \beta) \to \mathrm{Ob}_{[\mathcal{F}]}\mathcal{M}^\sigma(X, \beta)$, the Kuranishi map, whose zeroes locally describe the moduli space near $[\mathcal{F}]$.

The reason why Calabi-Yau three-folds are interesting in this context is that, using Serre duality with a trivial canonical bundle $\mathrm{K}_X \simeq \mathcal{O}_X$, we get

$$\mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) \simeq \mathrm{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \mathrm{K}_X)^\vee \simeq \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})^\vee \,.$$

With extensive work, one can then prove that $\mathcal{M}^\sigma(X, \beta)$ carries a perfect obstruction theory, which being *symmetric* yields in particular virtual dimension zero, and hence it is possible to obtain a virtual fundamental class

$$[\mathcal{M}^\sigma(X, \beta)]^{\mathrm{vir}} \in \mathrm{H}_0(\mathcal{M}^\sigma(X, \beta), \mathbb{Z}) \,.$$

If there are no strictly semistable sheaves, then the space $\mathcal{M}^\sigma(X, \beta)$ has the key feature of being compact: hence we can take the degree of the virtual fundamental class to define the invariant.

Definition 1.2.1 ([Tho00])**.** The *Donaldson-Thomas* invariant on X, with class β and stability σ, is the integer

$$\mathrm{DT}^\sigma(\beta) := \deg\,[\mathcal{M}^\sigma(X,\beta)]^{\mathrm{vir}} = \int_{[\mathcal{M}^\sigma(X,\beta)]^{\mathrm{vir}}} 1\,.$$

$\Diamond$

We summarise below some properties of the DT-invariants of the moduli scheme $\mathcal{M} = \mathcal{M}^\sigma(X,\beta)$:

- $\mathrm{DT}^\sigma(\beta)$ is invariant under deformations of the complex structure of X;
- if $\mathcal{M}$ is smooth, then $\mathrm{DT}^\sigma(\beta) = (-1)^{\dim \mathcal{M}}\chi_{\mathrm{top}}(\mathcal{M})$;
- if $\mathcal{M}$ consists of a finite set of reduced points, then $\mathrm{DT}^\sigma(\beta)$ is precisely the number of points;
- if $\mathcal{M}$ consists of a finite disjoint union of non-reduced point, each of which is a global critical locus, then $\mathrm{DT}^\sigma(\beta)$ equals the sum of the lengths of the non-reduced points (see Remark 1.2.4 and Section 1.4.1).

1.2.2 Counting Curves

One first, more concrete application of DT-invariants is to the problem of counting curves inside an assigned variety. We summarise this approach following the survey by Pandharipande-Thomas [PT14, §$3\frac{1}{2}$]. Consider a scheme X of dimension 3, and let $Z \subset X$ be a *sub-curve* if it is a sub-scheme of dimension 1; denote by $[Z] \in \mathrm{H}_2(X)$ its homology class. Our goal is to parametrise, for any $n \in \mathbb{Z}$ and $\beta \in \mathrm{H}_2(X)$, sub-curves Z such that $[Z] = \beta$ and $\chi(\mathcal{O}_Z) = n$, where χ is the holomorphic Euler characteristic. For any such Z, its ideal sheaf $\mathcal{I}_Z$ is stable as it has rank 1 [HL10, Example 1.2.10], has trivial determinant, and Chern character $\mathrm{ch}(\mathcal{I}_Z) = (1, 0, -\beta, -n) \in \mathrm{H}^{2*}$; moreover, any such sheaf is the ideal sheaf of a sub-curve. Therefore, our problem is configured as a DT-like counting over a suitable moduli space of isomorphism classes of sheaves. Let $I_n(X,\beta)$ denote the Hilbert scheme, which compactifies the space of embedded curves Z, though in fact it allows degenerations to arbitrary sub-curves: it hence parametrises subschemes which are unions of possibly nonreduced points and curves in X. For example, for a reduced curve of genus g with k free embedded points, we have $n = 1 - g + k$: hence we can increase g provided we add points accordingly.

When X is a Calabi-Yau three-fold, the Hilbert scheme admits a virtual fundamental class whose degree provides the desired numerical *DT-invariant*

$$I_{n,\beta} := \int_{[I_n(X,\beta)]^{\mathrm{vir}}} 1\,,$$

which is an integer-valued deformation invariant: the numerical DT-invariants can be organized in the *DT partition function*

$$Z_\beta^{\mathrm{DT}}(q) := \sum_{n\in\mathbb{Z}} I_{n,\beta}\, q^n\,.$$

It can be shown that the moduli space satisfies $I_n(X,\beta) = \emptyset$, so $I_{n,\beta} = 0$, for n sufficiently negative: therefore $Z_\beta^{\mathrm{DT}}(q)$ is a Laurent series.

Remark 1.2.2. There is an analogous notion of generating series for invariants $\mathrm{DT}^\sigma(\beta)$, where the indexing set $\mathbb{Z}$ is replaced by a given lattice in $\mathrm{H}^{2*}(X,\mathbb{Q})$. Even though this leads to a very rich theory, originating in the Physics literature (see for example [Nek04]), we will not deal with it in the present discussion, as we are mostly focused on curve counting invariants. More on this subject is covered [JS12], and more references are in [Sze15]. ♠

1.2.3 The Behrend Function

The definition of DT-invariants given is global, and requires the moduli space $\mathcal{M}^\sigma(Y,\beta)$ to be proper in order to define the degree of its virtual fundamental class. A local approach is possible, thanks to a result by Behrend [Beh09]. Let M be any (moduli) scheme with a perfect symmetric obstruction theory; if it is smooth, then the obstruction sheaf $\mathrm{Ob}_M = \Omega_M$ is a bundle and the virtual fundamental class is the top Chern class, that is the Euler class $e(\mathrm{Ob}_M) = e(\Omega_M)$: therefore, the virtual degree is just the signed Euler characteristic of M

$$\deg\,[M]^{\mathrm{vir}} = \int_{[M]} e(\Omega_M) = (-1)^{\dim M}\chi_{\mathrm{top}}(M)\,.$$

Behrend generalises this construction proving that to any scheme M, locally of finite type but non necessarily smooth, is associated a canonical constructible function $\nu = \nu_M \colon M \to \mathbb{Z}$, such that its value at any nonsingular point is $(-1)^{\dim M}$. This *Behrend function* is obtained in terms of local data, and is used to define the *weighted Euler characteristic* as the integer

$$\chi_\nu(M) = \chi(M,\nu) := \int_M \nu_M\,\mathrm{d}\chi = \sum_{n\in\mathbb{Z}} n\cdot\chi_{\mathrm{top}}(\nu_M^{-1}(n))\,. \tag{1.6}$$

Theorem 1.2.3 ([Beh09]). *Let M be a scheme, proper and with a symmetric perfect obstruction theory (which implies that it admits a zero-dimensional virtual fundamental class). We have the equality*

$$\deg\,[M]^{\mathrm{vir}} = \chi_\nu(M)\,.$$

This holds true independently of the particular symmetric obstruction theory used to define $[M]^{\mathrm{vir}}$.

An immediate consequence of this is that DT invariants only depend on the underlying scheme structure of the moduli space $\mathcal{M}^\sigma(X,\beta)$, and not on the symmetric obstruction theory used to define them. Moreover, even when M is of finite type but not proper and the virtual fundamental class does not make sense, we can consider the weighted Euler characteristic χ_ν instead of the virtual count $\deg\,[-]^{\mathrm{vir}}$: in this context, the value $\nu_M(P)$ at a point $P \in M$ corresponds to the contribution of P to the virtual count.

Remark 1.2.4. If M is a fat point, that is a non-reduced zero-dimensional scheme, it is not true that $\nu(M) = \operatorname{length} \mathcal{O}_M$. For example, if $\mathfrak{m}$ is the maximal ideal of the origin in $\mathbb{A}^2_{\mathbb{C}}$, then the Behrend function satisfies $\nu(\operatorname{Spec} \mathbb{C}\,[x,y]\,/\mathfrak{m}^d) = d$. Already for $d = 2$, this value is different from the length of the structure sheaf, which is 3. See the preprint [GR22] for a detailed study of this phenomenon. However, in the case where M is a fat point *admitting a symmetric obstruction theory*, then indeed $\nu(M) = \operatorname{length} \mathcal{O}_M$.

♠

1.3 Pandharipande-Thomas invariants

1.3.1 Stable Pairs

A drawback of the Hilbert scheme of curves is that it allows one-dimensional subschemes with possibly free points. This leads to having in the same component of the moduli space curves of different genus, and there are simple examples when a jump in genus is the limit of a family, as shown for example in the survey [PT14]; therefore the Hilbert scheme has large components corresponding to curves with free points and is much larger than the moduli space of smooth curves it tries to compactify: in this sense, it is a rather inefficient compactification of the moduli space of embedded curves. Stable pairs, introduced by Pandharipande and Thomas [PT09a], provide a more manageable approach.

Definition 1.3.1. Let X be a non-singular projective three-fold. A *stable pair* is the datum of a coherent sheaf $\mathcal{F}$ on X, together with a section $s\colon \mathcal{O}_X \to \mathcal{F}$ satisfying the following properties.

1. The sheaf $\mathcal{F}$ is purely 1-dimensional, that is it has no proper sub-sheaf with 0-dimensional support.

2. The section s has 0-dimensional cokernel.

The scheme-theoretic support of $\mathcal{F}$, simply called *support of the pair*, is denoted $C_{\mathcal{F}}$ or just C when unambiguous; the *zero-locus of* s is the reduced support $\operatorname{Supp}_{\mathrm{red}}(\operatorname{coker} s)$. As explained in Section 1.3.2 below, the scheme $C_{\mathcal{F}}$ is a curve with no embedded points.

A morphism of stable pairs $(\mathcal{F}, s) \to (\mathcal{F}', s')$ is given by a commuting triangle

$$\begin{tikzcd} & \mathcal{F} \arrow[dd] \\ \mathcal{O}_X \arrow[ur, "s"] \arrow[dr, "s'"'] & \\ & \mathcal{F}' \end{tikzcd}$$

and we say that two pairs are *isomorphic* if the vertical arrow is. $\Diamond$

In general, a pair $(\mathcal{F}, s)$ defines a curve $C = C_{\mathcal{F}}$, and a finite length $\mathcal{O}_X$-module $\mathcal{Q} := \operatorname{coker} s$; moreover, the support $Q = \operatorname{Supp}_{\mathrm{red}}(\mathcal{Q})$ needs to lie on the curve $C_{\mathcal{F}}$. We can record all these sheaves in the sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_X \xrightarrow{s} \mathcal{F} \to \mathcal{Q} \to 0\,. \tag{1.7}$$

Example 1.3.2. Consider the following in $\mathbb{C}^3$, which is not a stable pair since $\mathcal{F}$ fails to be pure.

$$\begin{tikzcd} \mathcal{I}_C \arrow[r, hook] & \mathcal{O}_X \arrow[r, two heads] \arrow[rr, bend left, "s"] & \mathcal{O}_C \arrow[r, hook] & \mathcal{F} \arrow[r, two heads] & \mathcal{Q} \\ (y,z) & \mathbb{C}\left[x,y,z\right] & \frac{\mathbb{C}[x,y,z]}{(y,z)} & \frac{\mathbb{C}[x,y,z]}{(xy,y^2,z)} & \frac{\mathbb{C}[x,y]}{(x,y^2,z)} \end{tikzcd}$$

♣

Example 1.3.3. We can see that the curve described in Example 1.1.19 is indeed a stable pair, albeit with empty zero-locus. We shall see more elaborated examples when constructing stable pairs from the data of a support curve and a prescribed zero locus, as done in the next sections.

♣

1.3.2 Cohen-Macaulay Support

The requirement of the sheaf $\mathcal{F}$ being *pure* implies that the scheme-theoretic support $C_{\mathcal{F}}$ of $\mathcal{F}$ is a Cohen-Macaulay curve. Moreover, imposing $\mathrm{coker}(s)$ to be zero-dimensional implies that $\mathcal{F}$ is isomorphic to the structure sheaf $\mathcal{O}_{C_{\mathcal{F}}}$, away from finitely many points on $C_{\mathcal{F}}$: the cokernel $\mathcal{Q}$ in the sequence (1.7) measures this lack of isomorphism. The sheaf $\mathcal{F}$ has therefore rank 1 over $C_{\mathcal{F}}$, and the support of $\mathrm{Im}(s)$ is precisely $C_{\mathcal{F}}$. We can fit all this information in the more complete diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}_{C_{\mathcal{F}}} \arrow[r] & \mathcal{O}_X \arrow[rr, "s"] \arrow[dr] & & \mathcal{F} \arrow[r] & \mathcal{Q} \arrow[r] & 0 \\ & & & i_*\mathcal{O}_{C_{\mathcal{F}}} \arrow[ur, hook, "s'"] & & & \end{tikzcd} \tag{1.8}$$

where $C_{\mathcal{F}} \overset{i}{\hookrightarrow} X$ is the inclusion. In the future, when no confusion can arise, we will omit the push-forward.

Conversely, once we fix a Cohen-Macaulay curve C, there is a convenient way to determine stable pairs supported on that curve, as proved in [PT09a, Prop. 1.8].

Proposition 1.3.4. *Let $\mathfrak{m} \subset \mathcal{O}_C$ be the ideal of a finite union of disjoint points. There is a bijection between stable pairs $(\mathcal{F}, s)$, with cokernel $\mathcal{Q}$, such that*

$$\mathrm{Supp}(\mathcal{F}) = C \quad \textit{and} \quad \mathrm{Supp}_{\mathrm{red}}(\mathcal{Q}) \subset \mathrm{Supp}(\mathcal{O}_C/\mathfrak{m})\,,$$

and subsheaves $\mathcal{Q}$ of $\mathcal{H}om(\mathfrak{m}^r, \mathcal{O}_C)/\mathcal{O}_C$, with r sufficiently large for a given $\mathcal{Q}$.

We briefly present this correspondence. Let $(\mathcal{F}, s)$ be a stable pair as in Proposition 1.3.4. There is a zero-dimensional sub-scheme $\mathfrak{p} \overset{j}{\hookrightarrow} C$, such that

$$\mathcal{I}_{\mathfrak{p}} = \mathcal{H}om(\mathcal{F}, \mathcal{O}_C) \hookrightarrow \mathcal{O}_C \quad \text{and} \quad \mathfrak{p}_{\mathrm{red}} \subset \mathrm{Supp}_{\mathrm{red}}(Q)\,,$$

and the inclusion $\mathcal{I}_{\mathfrak{p}} \hookrightarrow \mathcal{O}_C$ yields a canonical section of $\mathcal{H}om(\mathcal{I}_{\mathfrak{p}}, \mathcal{O}_C)$ which factors through s:

$$\begin{tikzcd} \mathcal{O}_C \arrow[r, hook, "\mathrm{can}"] \arrow[dr, hook, "s"'] & \mathcal{H}om(\mathcal{I}_{\mathfrak{p}}, \mathcal{O}_C) \\ & \mathcal{F}\,. \arrow[u, hook] \end{tikzcd} \tag{1.9}$$

For sufficiently large r, we have that $\mathfrak{m}^r \hookrightarrow \mathcal{I}_{\mathfrak{p}}$ with zero-dimensional cokernel, and hence an inclusion $\mathcal{H}om(\mathcal{I}_{\mathfrak{p}}, \mathcal{O}_C) \hookrightarrow \mathcal{H}om(\mathfrak{m}^r, \mathcal{O}_C)$. We have the diagram

$$\begin{tikzcd} \mathcal{O}_C \arrow[r, hook, "\mathrm{can}"] \arrow[dr, hook, "s"'] & \mathcal{H}om(\mathfrak{m}^r, \mathcal{O}_C) \arrow[r, two heads] & \mathcal{H}om(\mathfrak{m}^r, \mathcal{O}_C)/\mathcal{O}_C \\ & \mathcal{F} \arrow[u, hook] \arrow[r, two heads] & \mathcal{Q} \arrow[u, hook] \end{tikzcd} \tag{1.10}$$

where the left triangle factors through (1.9): we can then express $\mathcal{F}$ as a subsheaf of $\mathcal{H}om(\mathfrak{m}^r, \mathcal{O}_C)$ containing the canonical section induced by $\mathfrak{m}^r \hookrightarrow \mathcal{O}_C$, and $\mathcal{Q}$ as a subsheaf of $\mathcal{H}om(\mathfrak{m}^r, \mathcal{O}_C)/\mathcal{O}_C$.

On the other hand, given a subsheaf $\mathcal{Q} \subset \mathcal{H}om(\mathfrak{m}^r, \mathcal{O}_C)/\mathcal{O}_C$, we construct a stable pair with cokernel $\mathcal{Q}$ by taking $\mathcal{F}$ as the pull-back in the bottom-left corner of the square in (1.10), and section s induced by the canonical section.

1.3.3 Gorenstein Support

In the special case where the support C is Gorenstein, Pandharipande and Thomas [PT10] show that a stable pair is determined by a zero-dimensional subscheme. This characterisation is essential in Chapter 3, since we will construct stable pairs by thickening a reduced curve and then dualising the ideal sheaf of a zero-dimensional subscheme of the corresponding double curve.

Proposition 1.3.5 ([ibid., Prop. B.5]). *Let X be a nonsingular, projective, Calabi-Yau 3-fold, and $C \subset X$ a Gorenstein curve of genus g. A stable pair $(\mathcal{F}, s)$ supported on C, such that $\mathcal{F}$ has holomorphic Euler characteristic $1-g+n$, is equivalent to a length-n 0-dimensional subscheme of C.*

We outline the construction, following [ibid., Appendix B]. A key technical result is the following.

Lemma 1.3.6 ([ibid., Prop. B.2]). *With the same setting as Proposition 1.3.5, a generically locally free sheaf $\mathcal{F}$ on C is pure if and only if $\mathcal{E}xt^i_C(\mathcal{F}, \mathcal{O}_C) = 0$ for all $i > 0$.*

Consider a length-n, 0-dimensional subscheme $Q \subset C$, that is a point $[Q] \in \mathrm{Hilb}^n(C)$, giving an exact sequence $0 \to \mathcal{I}_Q \to \mathcal{O}_C \to \mathcal{O}_Q \to 0$. We dualise this sequence *over* C, that is apply the functor $\mathcal{H}om_C(-, \mathcal{O}_C)$. Since C is Gorenstein, so in particular Cohen-Macaulay, by purity of $\mathcal{O}_C$ we have that $\mathcal{H}om_C(\mathcal{O}_Q, \mathcal{O}_C) = 0$, and applying Lemma 1.3.6 we see that $\mathcal{E}xt^i_C(\mathcal{I}_Q, \mathcal{O}_C) = 0$ for all $i > 0$, as well as $\mathcal{E}xt^i_C(\mathcal{O}_C, \mathcal{O}_C) = 0$ for all $i > 0$. Therefore we have the sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0\,,$$

where $\mathcal{F} := \mathcal{H}om_C(\mathcal{I}_Q, \mathcal{O}_C)$ and $\mathcal{Q} := \mathcal{E}xt^1(\mathcal{O}_Q, \mathcal{O}_C)$. The sheaf $\mathcal{Q}$ has zero-dimensional support, and since $\mathcal{E}xt^i_C(\mathcal{O}_Q, \mathcal{O}_C) = 0$ for all $i > 1$, we have that $\mathcal{Q}$ has length n. It remains to show that $\mathcal{F}$ is pure. Notice that, by the vanishing of all $\mathcal{E}xt$'s, $\mathcal{F} \simeq \mathbb{R}\mathcal{H}om_C(\mathcal{I}_Q, \mathcal{O}_C)$; therefore $\mathbb{R}\mathcal{H}om_C(F, \mathcal{O}_C) \simeq \mathcal{I}_Q$, which implies

$\mathcal{E}xt^1_C(\mathcal{F}, \mathcal{O}_C)$ $\forall i \geq 1$: that is, $\mathcal{F}$ is pure by Lemma 1.3.6 again. Since $\chi(\mathcal{O}_C) = 1 - g$, we have $\chi(\mathcal{F}) = 1 - g + n$ as required.

Conversely, let $0 \to \mathcal{O}_C \xrightarrow{s} \mathcal{F} \to \mathcal{Q} \to 0$ be a stable pair supported on C, with $\chi(\mathcal{F}) = 1 - g + n$, and dualise the sequence over C, applying $\mathcal{H}om_C(-, \mathcal{O}_C)$. We have that $\mathcal{H}om_C(\mathcal{Q}, \mathcal{O}_C) = 0$, and by the purity of $\mathcal{F}$ Lemma 1.3.6 implies $\mathcal{E}xt^i_C(\mathcal{F}, \mathcal{O}_C) = 0$, as well as $\mathcal{E}xt^i_C(\mathcal{O}_C, \mathcal{O}_C) = 0$, for all $i \geq 1$. The immediate consequence is that $\mathcal{E}xt^i_C(\mathcal{Q}, \mathcal{O}_C) = 0$ for all $i \geq 2$. Therefore, the only surviving terms of the dualised sequence are

$$0 \longrightarrow \mathcal{H}om_C(\mathcal{F}, \mathcal{O}_C) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E}xt^1(\mathcal{Q}, \mathcal{O}_C) \longrightarrow 0\,,$$

which define a subscheme $Q \subset C$ via $\mathcal{I}_Q := \mathcal{H}om_C(\mathcal{F}, \mathcal{O}_C)$ and $\mathcal{O}_Q := \mathcal{E}xt^1_C(\mathcal{Q}, \mathcal{O}_C)$. Since $\mathcal{F}$ is pure 1-dimensional, and isomorphic to $\mathcal{O}_C$ away from the support of $\mathcal{Q}$, then $\mathcal{I}_Q$ is 1-dimensional as well, and the cokernel $\mathcal{O}_Q$ must be 0-dimensional. Finally, by the vanishing of $\mathcal{E}xt^{\geq 2}(\mathcal{Q}, \mathcal{O}_C)$ we have that $\mathcal{O}_Q = \mathbb{R}\mathcal{H}om_C(\mathcal{Q}, \mathcal{O}_C)$, from which we deduce that $\mathcal{O}_Q$ and $\mathcal{Q}$ have both length n.

A good source of examples of Gorenstein schemes is provided by the algebraic result below.

Proposition 1.3.7 ([Eis95, Cor. 21.19])**.** *If R is a regular local ring and I is an ideal generated by a regular sequence, then $A = R/I$ is a Gorenstein ring.*

Corollary 1.3.8. *In particular, any local complete intersection or hypersurface in a smooth variety is a Gorenstein scheme.*

Corollary 1.3.9. *Let $Z \subset X$ be a reduced smooth curve, and C a "double" thickening of Z along a line bundle $\mathcal{L} \subset \mathcal{N}_{Z/X}$, as detailed in Section 1.1.2. Then C is Gorenstein.*

Proof. Any regular local ring is a complete intersection ring, hence Z is a local complete intersection and we can apply Proposition 1.1.15. □

1.3.4 PT Moduli Spaces

There is, thanks to the work of Le Potier, [Le 93], a projective moduli space of stable pairs.

Definition 1.3.10. For fixed n and $\beta \in \mathrm{H}_2(X, \mathbb{Z})$, denote by $P_n(X, \beta)$ the projective moduli scheme which parametrises equivalence classes of stable pairs such that $\chi(\mathcal{F}) = n$ and the support satisfies $[C_{\mathcal{F}}] = \beta$. ♢

We can associate to a stable pair $(\mathcal{F}, s)$ a complex $I^\bullet := \{\mathcal{O}_X \xrightarrow{s} \mathcal{F}\}$ concentrated in degrees 0, 1. A key result of the original paper [PT09a, Prop. 1.21] is the fact that stable pairs are isomorphic if and only if their associated complexes are quasi-isomorphic. Therefore $P_n(X, \beta)$ can be thought of as a moduli space of objects in the derived category of coherent sheaves.

Remark 1.3.11. We could have considered X of any dimension. In general, isomorphic stable pairs will clearly have quasi-isomorphic associated complexes, with the converse result being not true if $\dim X \leq 2$ and true if $\dim X \geq 3$. ♠

There are exact triangles in $\mathsf{D}^\flat\mathsf{Coh}(X)$

$$I^\bullet \longrightarrow \mathcal{O}_X \xrightarrow{s} \mathcal{F} \xrightarrow{+1} \tag{1.11}$$

and

$$\mathcal{I}_C \longrightarrow I^\bullet \longrightarrow \mathcal{Q}\,[-1] \xrightarrow{+1} . \tag{1.12}$$

Remark 1.3.12. In any abelian category, a complex $C^\bullet = \{A \xrightarrow{f} B\}$, concentrated in degrees 0 and 1, sits in a distinguished triangle

$$\ker f \longrightarrow C^\bullet \longrightarrow (\operatorname{coker} f)\,[-1] \xrightarrow{+1},$$

with $\{A \to \operatorname{Im} f\} \overset{\text{q.i.}}{\simeq} \ker f$ and $\{\operatorname{Im} f \to B\} \overset{\text{q.i.}}{\simeq} (\operatorname{coker} f)\,[-1]$. In particular, this produces the exact triangle (1.12). ♠

1.3.5 Deformation of stable pairs

Definition 1.3.13 ([ibid., § 2.2])**.** A *family* of stable pairs over a quasi-projective base scheme B is given by a section

$$\mathcal{O}_{X\times B} \xrightarrow{s} \mathcal{F}$$

where $\mathcal{F}$ is flat over B, and for all closed points b in B the restriction $(\mathcal{F}_b, s_b)$ is a stable pair on $X \times \{b\}$. ♢

Following the original paper [ibid.] and the previous work [Le 93], we know that the deformations of a stable pair $(\mathcal{F}, s)$ are governed by $\operatorname{Hom}_{\mathsf{D}^\flat(X)}(I^\bullet, \mathcal{F})$ and the obstructions by $\operatorname{Hom}_{\mathsf{D}^\flat(X)}(I^\bullet, \mathcal{F}\,[1]) = \operatorname{Ext}^1(I^\bullet, \mathcal{F})$. However, $\mathbb{R}\operatorname{Hom}(I^\bullet, \mathcal{F})$ defines an obstruction theory which is not symmetric. For a complex of sheaves $\mathcal{E} \in \mathsf{D}^\flat\mathsf{Coh}(X)$, the *traceless k-th extension* $\operatorname{Ext}^k(\mathcal{E}, \mathcal{E})_0$ is the kernel of the trace map

$$\operatorname{Ext}^k(\mathcal{E}, \mathcal{E}) \xrightarrow{\operatorname{tr}} \mathrm{H}^k(X, \mathcal{O}_X)\,,$$

obtained by taking the cohomology of the corresponding trace map from $\mathbb{R}\mathcal{H}om(\mathcal{E},\mathcal{E})$ to $\mathcal{O}_X$. Using this, we know that the deformations and obstructions, with fixed determinant, of the complex $I^\bullet$ inside $\mathsf{D}^\flat\mathsf{Coh}(X)$, correspond respectively to $\mathrm{Ext}^1(I^\bullet, I^\bullet)_0$ and $\mathrm{Ext}^2(I^\bullet, I^\bullet)_0$. A deformation of a pair $(\mathcal{F}, s)$ induces a deformation of $I^\bullet$ with fixed determinant, and vice versa: this is the content of [PT09a, Thm. 2.7]. We obtain therefore an isomorphism

$$\mathrm{Hom}(I^\bullet, \mathcal{F}) \xrightarrow{\sim} \mathrm{Ext}^1(I^\bullet, I^\bullet)_0 \,. \tag{1.13}$$

The fixed-determinant deformations of $I^\bullet$ provide an obstruction theory for stable pairs, and for Calabi-Yau three-folds this obstruction theory is symmetric and in particular zero-dimensional. We can define the numerical *PT-invariant*

$$P_{\beta,n} := \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1 \,,$$

since from such an obstruction theory we can construct a virtual fundamental class; the *PT partition function* is

$$Z_\beta^{\mathrm{PT}}(q) := \sum_n P_{\beta,n}\, q^n \,.$$

1.3.6 Fixed support

We will mostly focus on stable pairs supported at a given curve C, necessarily Cohen-Macaulay, but possibly non-reduced.

Definition 1.3.14. For a fixed curve $C \subset X$ such that $[C] = \beta$, denote by $P_n(C) \subset P_n(X, \beta)$ the subscheme parametrising equivalence classes of stable pairs supported on C. ♢

Therefore, the numerical invariant in which we are interested is the weighted Euler characteristic

$$\chi(P_{1-g+n}(C)\,, \nu_{|P_{1-g+n}(C)})\,.$$

In the case where C is Gorenstein, of genus g, Proposition 1.3.5 gives a set-theoretic bijection between $P_{1-g+n}(C)$ and $\mathrm{Hilb}^n(C)$. This can be promoted to an isomorphism of moduli schemes, following [PT10, Prop. B.8].

Theorem 1.3.15. *Let C as above. There is an isomorphism of moduli spaces $P_{1-g+n}(C) \simeq \mathrm{Hilb}^n(C)$.*

1.3.7 Stable pairs supported at reduced curves

In their survey [PT14], Pandharipande and Thomas showcase how the moduli space of stable pairs is less pathological than the Hilbert scheme. Consider, for instance, the limit of the family in Example 1.1.26: in the Hilbert scheme (that is, taking the flat limit of the ideal sheaves) it converges to a curve with an embedded point. For each $t \neq 0$, we consider the structure sheaf of C_t which is a Cohen-Macaulay curve, with its canonical section, and get the family of stable stable pairs below

$$\mathcal{O}_X \longrightarrow \mathcal{O}_{C_t} .$$

This family of skew lines converges to a stable pair whose support has ideal (xy, z), and with a marked point (the origin) in the zero-locus. The apparition of a point in the zero-locus replaces the embedded point arising in the Hilbert scheme limit. We aim to show this phenomenon in details, working in affine coordinates, and from now on we will freely exchange sheaves and modules when no confusion arises.

Consider the rings $T = \mathbb{C}[x, y, z]$, $R = \mathbb{C}[t]$, and $S = R[x, y, z]$; we name the corresponding affine spaces $X := \mathbb{A}^3_{x,y,z} = \operatorname{Spec} T$, $B := \mathbb{A}^1_t = \operatorname{Spec} R$ and $X \times B = \operatorname{Spec} S$. We construct a family of stable pairs over the base B by defining the S-module F via the presentation below, where F arises as the cokernel of the map given by the matrix M_t below.

$$S^4 \xrightarrow[M_t]{\begin{pmatrix} y & z & 0 & 0 \\ 0 & 0 & x & z-t \end{pmatrix}} S^2 . \tag{1.14}$$

For $t \neq 0$, one considers the fibre F_t and computes $\operatorname{Ann}(F_t) = (z^2 - tz, yz - ty, xy, xz)$, which is the same ideal as the intersection $(y, z) \cap (x, z - t)$. In other words, F_t is supported at the union of the two skew lines C_t appearing in Example 1.1.26: as the two curves are reduced and Cohen-Macaulay, F_t is pure 1-dimensional.

When $t = 0$, the fibre F_0 is presented by the matrix $\left(\begin{smallmatrix} y & z & 0 & 0 \\ 0 & 0 & x & z \end{smallmatrix}\right)$, and $\operatorname{Ann}(F_0) = (z, xy)$. Hence, F_0 is supported at a curve without embedded points, which is *not* the flat limit of the curves C_t in Example 1.1.26.

There is a section s given as the composition

$$\begin{tikzcd} S = S^1 \arrow[r, "\binom{1}{1}"] \arrow[rr, bend right, "s"'] & S^2 \arrow[r] & F \end{tikzcd}$$

where the second map is simply the projection onto the cokernel of the presentation map (1.14). One can compute that the annihilator $\operatorname{Ann}_S(\operatorname{coker} s)$ is the ideal

(t, x, y, z), meaning that the map s fails to be surjective only for $t = 0$, with cokernel supported at the origin. Similarly, we have that $\mathrm{Ann}_S(F) = \ker s$: since the section s factors through the quotient $T = \mathcal{O}_X \to \mathcal{O}_{C_t} \to 0$ and $\mathrm{Im}(s) \simeq \mathcal{O}_{C_t}$ by [PT09a, Lemma 1.6], for $t \neq 0$ there is an isomorphism $\mathcal{O}_{C_t} \simeq F$.

So far, we have shown that for every $b \in B$, the fibre of $\mathcal{O}_X \xrightarrow{s} F$ is a stable pair, which for $b \neq 0 = (t)$ is simply isomorphic to the structure sheaf $\mathcal{O}_{C_t}$ of the union of two disjoint reduced lines, with the canonical section. We are left to verify flatness over B, that is F is R-flat: this follows from the fact that the matrix M_t is block-diagonal, so F splits as the direct sum of two submodules, each isomorphic to the structure sheaf of a reduced line.

In conclusion, we have shown an example of a 1-parameter family of stable pairs, where for $t \neq 0$ we simply obtain the union of two disjoint reduced curves, and in the limit $t = 0$ a marked point arises where the two lines collide. The marked point replaces the embedded point seen in the case of the Hilbert scheme.

1.3.8 Stable pairs supported at ribbons

We wish to now show that, similarly to the case of two colliding skew lines, a regularising phenomenon arises when considering families of ribbons which degenerate to a *pinch*, as described in Section 1.1.3. As the issue is local, we work in affine space, retaining the same notation as Section 1.3.7. Consider again $S = \mathbb{C}[t, x, y, z]$ and the family of curves C_t in Example 1.1.30 where

$$C_t := \mathrm{Spec}\, {}^{S}\!/_{I_t} \quad \text{with} \quad I_t = \left(x^2, xy, y^2, zx - ty\right),$$

recalling that, in the Hilbert scheme, the flat limit is the curve C_0 with ideal $I_0 = (x^2, xy, y^2, zx)$.

For each $t \neq 0$, C_t is a Cohen-Macaulay thickening of the reduced curve $Z = \{x = 0 = y\}$, and gives rise to a stable pair $\mathcal{O}_X \to \mathcal{O}_{C_t}$ with the canonical section. In order to understand the limit, we aim to construct a stable pair, supported at a double Cohen-Macaulay curve, which sits as the fibre over $t = 0$ of a flat family over $B = \mathrm{Spec}\, R$ with $R = \mathbb{C}[t]$, whose fibres for $t \neq 0$ are isomorphic to $\mathcal{O}_X \to \mathcal{O}_{C_t}$ as above.

Consider in $X = \mathbb{A}^3$ the curve C with ideal $(x, y^2) \subset T = \mathbb{C}[x, y, z]$, which differs from the flat limit for $t = 0$ of the ideals I_t. We wish to construct a stable pair supported at C, with zero-locus consisting of a single reduced point at the origin, having ideal $\mathfrak{m} = (x, y, z)$. Following the construction in the proof of Proposition 1.3.4, we look at subsheaves $\mathcal{F} \subset \mathcal{H}om(\mathfrak{m}^{\mathfrak{r}}, \mathcal{O}_C)$ as in diagram (1.10). Since in our case

$r = 1$ is *large enough*, it suffices to understand the module $F_0 := \mathrm{Hom}_T(\mathfrak{m}, \mathrm{H}^0(\mathcal{O}_C))$, which admits two generators

$$\sigma_1 \colon \begin{pmatrix} x & \mapsto & 0 \\ y & \mapsto & y \\ z & \mapsto & z \end{pmatrix} \quad \text{and} \quad \sigma_2 \colon \begin{pmatrix} x & \mapsto & 0 \\ y & \mapsto & 0 \\ z & \mapsto & y \end{pmatrix}.$$

We can conveniently think in terms of formal left multiplication $\sigma_1 = 1 \cdot (-)$, $\sigma_2 = yz^{-1} \cdot (-)$, and represent them pictorially as in Figure 1.1, together with their monomial span.

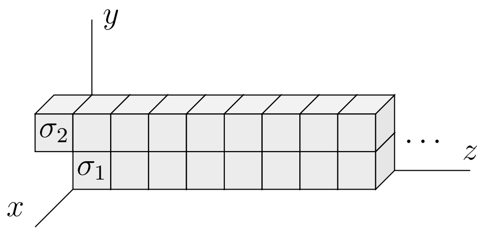


Figure 1.1: The module F_0, with generators $\sigma_1 = 1$ and $\sigma_2 = yz^{-1}$. Each box marks the degrees of a corresponding $\mathbb{Z}^3$-graded monomial in x, y, z. Notice that the box configuration is unbounded to the right.

The two generators satisfy relations $x\sigma_1 = x\sigma_2 = y^2\sigma_1 = y\sigma_2 = 0$ and $y\sigma_1 = z\sigma_2$, hence F_0 admits the presentation

$$T^5 \xrightarrow{\left(\begin{smallmatrix} x & y^2 & 0 & 0 & y \\ 0 & 0 & x & y & -z \end{smallmatrix}\right)} T^2 \longrightarrow F_0 \longrightarrow 0\,. \tag{1.15}$$

One verifies that $\mathrm{Ann}_T(F_0) = (x, y^2)$ and that, given the section $s_0 \colon T \to F_0$ mapping $1 \mapsto \sigma_1$, we have $\mathrm{Ann}_T(\operatorname{coker} s_0) = (x, y, z)$. In other words, $s_0 \colon \mathcal{O}_X = T \to F_0$ is a stable pair supported at the prescribed curve C, with zero-locus consisting of the reduced origin.

Consider now $S = \mathbb{C}\,[t, x, y, z]$ and the S-module F presented as the cokernel below, with the given section s.

$$\begin{tikzcd} S^5 \arrow[r, "\left(\begin{smallmatrix} x & y^2 & 0 & 0 & y \\ -t & 0 & x & y & -z \end{smallmatrix}\right)", "M_t"'] & S^2 \arrow[r] & F \arrow[r] & 0 \\ & S\,. \arrow[u, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] \arrow[ur, "s"'] & & \end{tikzcd} \tag{1.16}$$

Remark 1.3.16. The presentation of F_0 we provide is not minimal, but convenient when comparing it to the construction of F, as the generators of the former module are clearly understood. ♠

Our goal is to show that $s\colon \mathcal{O}_{B\times X} \to F$ is a family of stable pairs, with the argument proceeding as in Section 1.3.7. First of all, $\mathrm{Ann}_S(\operatorname{coker} s) = (t,x,y,z)$, hence s is surjective for $t \neq 0$ and has zero-dimensional cokernel otherwise. Over $t = 0$, we simply obtain the stable pair $s_0\colon T \to F_0$ described above. When $t \neq 0$, the annihilator of F_t is given by $(zx - ty, x^2, xy, y^2) = I_t$; moreover s_t is surjective, and it factors through the map $\mathcal{O}_{C_t} \to F_t$ which is then necessarily an isomorphism. The module F_t is exhibited as the structure sheaf of a Cohen-Macaulay curve, hence it is pure. In order to verify flatness of F over B, we check the condition locally at every closed point. All fibres for $t \neq 0$ are isomorphic via a change of variables rescaling y, which entails flatness at every closed point $\mathfrak{m} \neq (t)$ of B.

It remains to check flatness at $t = 0$. We rely on a key technical result, stated below for convenience.

Lemma 1.3.17 ([Stacks, Tag 00MI])**.** *Let R and S be local Noetherian rings, with $R \to S$ a local ring homomorphism making S into an R-module. Consider a finite complex of finite S-modules*

$$0 \longrightarrow E_k \longrightarrow E_{k-1} \longrightarrow \cdots \longrightarrow E_0\,.$$

Assume that E_i is a flat R-module for $i = k, \dots, 0$; calling $\mathfrak{m}$ the maximal ideal of R, suppose that the induced complex of R-modules

$$0 \longrightarrow E_k/\mathfrak{m} \longrightarrow E_{k-1}/\mathfrak{m} \longrightarrow \cdots \longrightarrow E_0/\mathfrak{m}$$

is exact. Then $\operatorname{coker}(E_1 \to E_0)$ *is a flat R-module.*

Consider the natural inclusion $R = \mathbb{C}[t] \to S = \mathbb{C}[t,x,y,z]$, and the following free resolution of F

$$0 \longrightarrow S^2 \xrightarrow{\begin{pmatrix} -y & 0 \\ x & 0 \\ z & -y \\ -t & x \end{pmatrix}} S^4 \xrightarrow{\begin{pmatrix} x & y & 0 & 0 \\ -t & -z & x & y \end{pmatrix}} S^2\,.$$

When we localise at $\mathfrak{m} = (t)$, we obtain a free resolution of $F_{\mathfrak{m}}$ by free $S_{\mathfrak{m}}$-modules $0 \to E_2 \to E_1 \to E_0$. Since all elements s of S are mapped to $s/1$, if $\mathbf{e}$ is a basis element of S^j, then $\mathbf{e}/1$ is a generator of $S^j_{\mathfrak{m}}$ (in fact, we get a basis). Given the S-linear morphism $\eta\colon S^j \to S^k$, suppose $\mathbf{e}$ maps to $\mathbf{f}$. By the universal property of localisation, given the composite map $S^j \to S^k \to S^k_{\mathfrak{m}}$, there is a unique natural morphism $S^j_{\mathfrak{m}} \to S^k_{\mathfrak{m}}$, which necessarily maps $\mathbf{e}/1$ to $\mathbf{f}/1$. This last map is precisely the one induced by the same matrice as η , with a slight abuse of notation consisting in omitting to rewrite an entry $a \in S$ as $a/1 \in S_{\mathfrak{m}}$.

Now, S is free (albeit not finite) over R, and so for each i we have that E_i is free over $R_{\mathfrak{m}}$, hence flat. Finally, taking the quotients $E_i/\mathfrak{m}$ with the induced complex simply entails replacing t with 0 in the matrices (up to abuse of notation, again), and we obtain a complex of $R_{\mathfrak{m}}$-modules which is readily verified to be exact: in fact, it is a resolution of the fibre F_0. This concludes our proof.

Remark 1.3.18. The discussion above offers a practical approach to verify flatness of finitely presented modules E over an affine base $\mathbb{A}^1_t$. One considers a *free* resolution of E as an S-module: if replacing t with $a \in \mathbb{C}$ yields a resolution of E_a, then E is flat over $\mathfrak{m} = (t-a)$. For example one, can observe that the two cases below *do not* satisfy the outlined criterion, where taking $S = \mathbb{C}[t, x]$ and considering the ideals

- (x, t), sudden appearance of a point,
- $(tx - t)$, sudden appearance of a line.

♠

This example shows how, given a family of Cohen-Macaulay ribbons, in the stable pair limit the marked point in the zero-locus replaces the embedded point arising at the pinch.

Remark 1.3.19. The same construction can be applied to any family of affine Cohen-Macaulay ribbons degenerating to a pinch. Locally, the thickening direction is given by two polynomials f, g in z, where z is the local coordinate of the underlying reduced smooth curve. In the limit, these two polynomials have a single common root α. One obtains a stable pair whose support is the Cohen-Macaulay curve given by the thickening by $f/(z-\alpha)$ and $g/(z-\alpha)$, and whose zero-locus tracks the location of the pinch point, that is the common root $z = \alpha$. ♠

1.4 Cohomological invariants

One extension of PT- and DT- theory is to replace numerical invariants with more complicated objects, which carry more information, both in Mathematics and Physics. One such strategy is to express the DT-invariant of a moduli space $\mathcal{M}$ as a weighted Euler characteristic: the first step is finding a finite-dimensional graded vector space $\mathcal{H}^*(\mathcal{M})$, a *cohomology theory* on $\mathcal{M}$, such that

$$\chi_\nu(\mathcal{M}) = \sum_i (-1)^i \dim \mathcal{H}^i(\mathcal{M})\,.$$

Example 1.4.1. As a reality check, consider that, if $\mathcal{M}$ is non singular, the Behrend function is identically $(-1)^{\dim \mathcal{M}}$, and we can easily compute

$$\chi_\nu(\mathcal{M}) = \sum_{n \in \mathbb{Z}} n \cdot \chi(\nu_{\mathcal{M}}^{-1}(n)) = (-1)^{\dim \mathcal{M}} \cdot \chi(\mathcal{M})\,;$$

hence we can choose as cohomology theory a shift of the classical cohomology $\mathrm{H}^*(\mathcal{M}, \underline{\mathbb{Q}}_{\mathcal{M}})\,[\dim \mathcal{M}] = \mathrm{H}^*(\mathcal{M}, \underline{\mathbb{Q}}_{\mathcal{M}}\,[\dim \mathcal{M}])$. ♣

1.4.1 Critical Charts

If $\mathcal{M}$ is the moduli space of coherent sheaves over a fixed Calabi-Yau three-fold, then locally $\mathcal{M}$ is a critical locus: more precisely, each point $p \in \mathcal{M}$ has a neighborhood which embeds in some ambient space U, where it is realised as the critical locus of a function $f \colon U \to \mathbb{C}$. This is a strong result, firstly proved in the non-algebraic case [Wit95; DT98; JS12]. A more recent, algebraic equivalent is provided in [BBJ13].

Theorem 1.4.2. *If $\mathcal{M}$ a moduli space of simple sheaves $\mathcal{F}$ over a Calabi-Yau three-fold X, then for each class $[\mathcal{F}]$ there exist:*

- *a smooth scheme U such that* $\dim U = \dim \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$,
- *a regular function* $f \colon U \to \mathbb{C}$,
- *a Zariski open neighborhood V of $[\mathcal{F}]$ in $\mathcal{M}$,*
- *an isomorphism* $\mathrm{Crit}(f) \xrightarrow{\sim} V$.

The data prescribed above are called a *critical chart*, and they describe locally the variety $\mathcal{M}$ as a critical locus. Thanks to the work of Behrend [Beh09] we know that the value $\nu_{\mathcal{M}}(p)$ is, up to sign, the Euler characteristic, with support at p, of the sheaf of vanishing cycles of f, which is the perverse sheaf of vanishing cycles $\Phi = \bigoplus_c \phi_{f-c}\,\underline{\mathbb{Q}}_U\,[\dim U]$, described for example in the book [Dim13]. The key point that Behrend proves is that $\nu_{\mathcal{M}}(p)$ is, up to sign, the topological Euler characteristic of the reduced Milnor fiber of f at $p \in \mathcal{M}$. This is a local construction, but if $\mathcal{M}$ is globally a critical locus, then clearly a good candidate cohomology theory to generalise DT-invariants is the cohomology of this perverse sheaf.

We only expect to be able to describe $\mathcal{M}$ *locally* in this way, and so far we have no notion of compatibility between critical charts: this sort of structure is desirable if we want to employ the machinery of perverse sheaves which allows gluing local data.

An effective approach to encoding this "gluing" data is provided by Joyce [Joy15]: to any scheme M is attached a sheaf $\mathcal{S}_M$, whose sections parametrise, in a precise sense, the various ways to describe open subsets of M as critical loci $\mathrm{Crit}(f) \subset U$, for $f \colon U \to \mathbb{C}$. The definition and proof of existence of this sheaf is quite technical, so we refer to the original work [ibid., Theorem 2.1]; for our treatment below, it suffices to say that, if locally $X = \mathrm{Crit}(f)$, then there is a section s of $\mathcal{S}_M$ satisfying $s = f + I_{\mathrm{d}f}^2$.

Example 1.4.3 ([ibid., Example 2.4],[Bus14, Remark 2.1.1])*.* While the construction of the sheaf $\mathcal{S}_M$ above does not require M to be a global critical locus, we will focus on this particular case in this work. Assume that f is regular and $M = \mathrm{Crit}(f) \subseteq U$ is a closed embedding into a a smooth $\mathbb{C}$-scheme, we obtain a natural global section of $\mathcal{S}_M$; calling $I_{\mathrm{d}f} \subset \mathcal{O}_U$ the ideal generated by $\mathrm{d}f$, we can write $s = f + I_{\mathrm{d}f}^2$, and $s_{|M} = f + I_{\mathrm{d}f}$. The function f is not necessarily locally constant on M, but only on the reduced analytic space M_{red}: now, locally constant functions on M_{red} extend uniquely to locally constant functions on U near M_{red}. ♣

In conclusion, the idea is that $\mathcal{S}_M$ only encodes f up to the information that makes sense on M intrinsically, independently on the ambient space U. In general, for any scheme M we have a decomposition $\mathcal{S}_M = \underline{\mathbb{C}}_M \oplus \mathcal{S}_M^0$, where $\mathcal{S}_M^0$ is the kernel of the composition $\mathcal{S}_M \to \mathcal{O}_M \to \mathcal{O}_{M^{\mathrm{red}}}$.

Definition 1.4.4 ([Joy15, Definition 2.5])**.** An *algebraic critical d-locus* is a pair (M, s) of a scheme M together with a section $s \in \mathrm{H}^0(M, \mathcal{S}_M^0)$, such that

- M admits a cover by Zariski open sets R_λ,
- there are closed embeddings $R_\lambda \xhookrightarrow{i_\lambda} U_\lambda$ into smooth schemes,
- there are regular functions $f_\lambda \colon U_\lambda \to \mathbb{C}$ with $\mathrm{Im}(i_\lambda) = \mathrm{Crit}(f_\lambda)$,
- the condition $s_{|R_\lambda} = f_\lambda + I_{\mathrm{d}f}^2$ holds, saying that $s_{|R_\lambda}$ is similar to Example 1.4.3.

The datum of $(R_\lambda, U_\lambda, f_\lambda, i_\lambda)$ is called a *critical chart* of the d-critical locus. ♢

Remark 1.4.5. Looking at Example 1.4.3, we see that by definition $M = \mathrm{Crit}(f) \xhookrightarrow{i} U$ admits a d-critical structure $(M, f + l_{\mathrm{d}f}^2)$ with a *global* critical chart (M, U, f, i). We will abuse notation and denote this global structure by $\mathrm{Crit}(f, U)$.

Similarly, if M is a smooth $\mathbb{C}$-scheme it has a trivial d-critical structure $(M, 0)$ with global chart $(M, M, 0, \mathrm{Id}_M)$, and this structure $\mathrm{Crit}(0, M)$ is unique since $\mathcal{S}_M = \underline{\mathbb{C}}_M$ and $\mathcal{S}_M^0 = 0$ [ibid., Example 2.15]. ♠

Let U be a scheme, and $f\colon U \to \mathbb{C}$ a regular function. Assume that U' is a subscheme containing $(\mathrm{Crit}\, f)_{\mathrm{red}}$. Unless U' is an open subscheme, it is not necessarily true that $\mathrm{Crit}(f_{|U'}) \simeq \mathrm{Crit}(f)$ as subschemes.

Example 1.4.6. Take $U = \mathbb{A}^2$, $f = y^3$, and $U' = \operatorname{Spec} \mathbb{C}[x, y]/(y)$. ♣

A priori, we could endow the same scheme with more than one d-critical locus structure, and we need the language to compare them. There is a notion of morphism of d-critical loci [Joy15, Definition 2.5], relying on the existence of a map $\phi^\star\colon \phi^{-1}(\mathcal{S}_Y) \to \mathcal{S}_X$ [ibid., Proposition 2.3] for any morphism $\phi\colon X \to Y$, but for our purposes we can specialise it to the case of a single global critical chart.

Definition 1.4.7. A *morphism* of global critical charts, that is between critical loci $\mathrm{Crit}(f, U)$ and $\mathrm{Crit}(g, V)$, is a morphism of $\mathbb{C}$-schemes $\phi\colon U \to V$ such that $g \circ \phi - f \in I^2_{\mathrm{d}f}$. An *isomorphism* of global critical charts is a pair of morphisms $\phi\colon U \to V$ and $\psi\colon V \to U$ as above, which restrict to an isomorphism between the scheme-theoretic critical loci $\mathrm{Crit}(f)$ and $\mathrm{Crit}(g)$. ♢

There are cases when a critical chart can be simplified up to isomorphism. The results below, adapted from [DS09, Lemma 1.3.1], will be useful in Section 3.3 and Section 3.4.

Lemma 1.4.8. *Let U be a smooth scheme, $f\colon U \to \mathbb{C}$ a regular function; set $V := U \times \mathbb{A}^2_{x,y}$, with affine coordinates (x, y); call $\pi\colon V \to U$ the projection and consider $g\colon V \to \mathbb{C}$ given by $g := f\pi + xy$. Then $\mathrm{Crit}(f, U)$ and $\mathrm{Crit}(g, V)$ are isomorphic as global critical charts.*

Proof. It is immediate that the projection $\pi\colon V \to U$ restricts to an isomorphism $\mathrm{Crit}(f) \simeq \mathrm{Crit}(g)$ of $\mathbb{C}$-schemes, with inverse the section $\sigma\colon U \xrightarrow{\sim} U \times \{(0, 0)\} \hookrightarrow V$. We can write[2] in one direction $g\sigma = f$; in the other, $I_{\mathrm{d}g} = (\mathrm{d}f, x, y)$, and we verify that $f\pi - g = xy \in I^2_{\mathrm{d}g}$. Hence π and σ are morphisms of critical charts. □

Lemma 1.4.9. *Let U be a smooth scheme, $f\colon W := U \times \mathbb{A}^1_x \to \mathbb{C}$ a regular function, with x the affine coordinate; set $V := U \times \mathbb{A}^1_x \times \mathbb{A}^2_{y,z}$, with new affine coordinates (y, z). Set $\pi\colon V \to U \times \mathbb{A}^1_x$ the projection, and define $g\colon V \to \mathbb{C}$ by $g := f\pi + (x - y)z$. Then $\mathrm{Crit}(f, W)$ and $\mathrm{Crit}(g, V)$ are isomorphic as global critical charts.*

Proof. First of all, for $u \in U$ and $x \in \mathbb{A}^1_x$, we have the differential

$$\mathrm{d}f\colon \mathrm{T}_{(u,x)}W \simeq \mathrm{T}_u U \oplus \mathrm{T}_x \mathbb{A}^1_x \longrightarrow \mathrm{T}_{f(u,x)}\mathbb{C} \simeq \mathbb{C}\,,$$

[2]With a slight abuse of notation.

and we denote by $\mathrm{d}_U f$ the first component of the linear map above. Then, we write local generators of the sheaf of ideals

$$I_{\mathrm{d}g} = (\mathrm{d}_U f, \partial_x f + z, z, x - y) = (\mathrm{d}_U f, \partial_x f, z, x - y) = (\mathrm{d}f, z, x - y)\,,$$

where in the last equality we used that $(\mathrm{d}_U f, \partial_x f) = (\mathrm{d}f)$. Hence, the projection $\pi\colon V = U \times \mathbb{A}^3_{x,y,z} \to U \times \mathbb{A}^1_x$ restricts to a morphism of subschemes $\tilde{\pi}\colon \operatorname{Crit}(g) \to \operatorname{Crit}(f)$.

Consider now the closed inclusion $\sigma\colon U \times \mathbb{A}^1_x \to V = U \times \mathbb{A}^3_{x,y,z}$, defined by $\sigma\colon (u, x) \mapsto (u, x, x, 0)$, where $u \in U$. Notice that, if $\mathrm{d}f(u, x) = 0$ then $(u, x, x, 0) \in \operatorname{Crit}(g)$, hence σ restricts to a morphism of schemes $\tilde{\sigma}\colon \operatorname{Crit}(f) \to \operatorname{Crit}(g)$, which is the inverse of $\tilde{\pi}$.

Finally, we have that $g\sigma - f = 0 \in I^2_{\mathrm{d}f}$ and $g - f\pi = (x - y)z \in I^2_{\mathrm{d}g}$, proving that π and σ define a pair of morphisms of critical loci which restrict to an isomorphism of schemes $\operatorname{Crit}(f) \simeq \operatorname{Crit}(g)$. □

1.4.2 The DT-Sheaf

As described above, the moduli spaces $\mathcal{M}$ that we are interested in are locally critical loci, that is they admit critical charts: moreover, the results by Brav, Bussi and Joyce [BBJ13] ensure that $M = \mathcal{M}$ has indeed the structure of a d-critical locus as in Definition 1.4.4. Therefore, it would be desirable to use this structure to glue together the collection of perverse sheaves $\Phi_\lambda = \bigoplus_c \phi_{f_\lambda - c}\, \underline{\mathbb{Q}}_{U_\lambda}\, [\dim U_\lambda]$. Thanks to the work of Brav, Bussi, Dupont, Joyce and Szendrői [BBDJS15], there are gluing isomorphisms between the perverse sheaves of vanishing cycles defined on the overlap of critical charts, defined up to sign; an analogous result is true for stacks thanks to Ben-Bassat, Brav, Bussi and Joyce [BBBJ15]. To resolve the issue, one need assign a choice of sign to every gluing isomorphism, and this is done in a sheaf-theoretic way, by assigning a certain real line bundle. We won't cover this subject in detail, as it becomes quite technical; good surveys are contained in Davison's thesis [Dav10, Chapter 2] and in the paper by Joyce and Upmeier [JU21]. The notion of orientation data was introduced in general triangulated categories by Kontsevich and Soibelman [KS08]. In the case of moduli spaces of coherent sheaves over a Calabi-Yau three-fold, Nekrasov and Okounkov [NO16] prove existence of such structure in a special case, and more recently Cao-Gross-Joyce [CGJ20] proved the existence of orientation data in more generality, including Calabi-Yau 4-folds.

In conclusion, on a Calabi-Yau three-fold X, we can assign d-critical locus structure to a moduli space $\mathcal{M}$ of coherent sheaves on X: once orientation is defined on such a moduli space, we can glue together the perverse sheaves Φ_λ constructed locally on the critical charts.

Definition 1.4.10. The *DT-sheaf* is the perverse sheaf Φ obtained as above. The *cohomological DT-invariant* is the hypercohomology $\mathbb{H}^*(\mathcal{M}, \Phi)$. $\lozenge$

The cohomological DT-invariant Φ is a graded vector space, and its Euler characteristic satisfies

$$\sum_i (-1)^i \dim \mathbb{H}^i(\mathcal{M}, \Phi) = \int_{\mathcal{M}} \nu_{\mathcal{M}} \, \mathrm{d}\chi \,.$$

In other words, it agrees with the numerical DT-invariant obtained from the Behrend function.

Remark 1.4.11. The local description of M as critical locus, with a suitable notion of compatibility between the critical charts and the choice of an orientation data, is all that is required to define the DT-sheaf. Therefore, for such moduli spaces we can define *DT-like* invariants. One salient example of such case is the moduli space of stable representations of a quiver compatible with an assigned potential, under appropriate stability conditions, as explained in the survey [Sze15, §3] and in Section 2.3. ♠

Chapter 2

PT moduli spaces on the resolved conifold

2.1 Preliminaries

We consider the variety $X := \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \longrightarrow \mathbb{P}^1)$, called the *resolved conifold*, which is a non-compact Calabi-Yau 3-fold; it can be thought of as the locus

$$X = \left\{ \begin{pmatrix} X_1 & X_0 \\ Y_1 & Y_0 \end{pmatrix} \begin{pmatrix} \zeta_0 \\ -\zeta_1 \end{pmatrix} = 0 \right\} \subset \mathbb{P}^1_{[\zeta_0:\zeta_1]} \times \mathbb{C}^4_{(X_0,X_1,Y_0,Y_1)} .$$

On the standard affine cover of $\mathbb{P}^1$ via $U_i := \{\zeta_i \neq 0\}$, for $i = 0, 1$, we have coordinates (X_0, Y_0, Z_0) and (X_1, Y_1, Z_1) respectively, where $Z_0 = Z_1^{-1} = {}^{\zeta_1}\!/_{\zeta_0}$, and $X_1 = Z_0 X_0$, $Y_1 = Z_0 Y_0$ on $U_0 \cap U_1$. This is just another way to say that the transition functions for the bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ are given by

$$g_{10} = \begin{pmatrix} {}^{\zeta_1}\!/_{\zeta_0} & 0 \\ 0 & {}^{\zeta_1}\!/_{\zeta_0} \end{pmatrix} .$$

The zero section $Z \cong \mathbb{P}^1_{[\zeta_0:\zeta_1]}$ is clearly cut out by the equations $X_i = Y_i = 0$ on U_i; we call $i \colon Z \hookrightarrow X$ the inclusion and $\pi \colon X \to Z$ the projection.

Remark 2.1.1. Let $B := \mathrm{Bl}_0\, \mathbb{A}^2$ be the blow-up of the affine plane at the origin. Then the resolved conifold X can be seen as $X \simeq \mathrm{Tot}(\mathcal{O}_Z(-1)^{\oplus 2}) \simeq \mathrm{Tot}(\omega_B)$, where $Z \subset B$ is the exceptional curve. ♠

Remark 2.1.2. As stated before, all sheaves will be considered on X unless otherwise specified, and the pushforward i_* omitted. Moreover, on X there is a family of invertible sheaves given by $\pi^*\mathcal{O}_Z(k)$, for $k \in \mathbb{Z}$: all twistings of sheaves over X will be performed by tensoring with $\pi^*\mathcal{O}_Z(k)$, omitting the pullback π^*. ♠

We can conveniently describe sheaves on X. If we consider the sheaf of $\mathcal{O}_{\mathbb{P}^1}$-algebras $\mathcal{A} := \mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2})$, using the relative spectrum construction we write $X \simeq \underline{\mathrm{Spec}}_{\mathbb{P}^1}(\mathcal{A})$.

Remark 2.1.3. Our convention is different from what is used, for example, in the books [Har77, Ex II.5.17] or [Stacks, Tag 01LL]: there, given a quasicoherent sheaf $\mathcal{E}$ on X, the associated vector bundle is $\underline{\mathrm{Spec}}_X(\mathrm{Sym}\,\mathcal{E})$, instead of $\underline{\mathrm{Spec}}_X(\mathrm{Sym}\,\mathcal{E}^\vee)$ as in our case. ♠

Given a quasicoherent sheaf $\mathcal{F} \in \mathsf{QCoh}(X)$, consider $\mathcal{G} := \pi_*\mathcal{F} \in \mathsf{QCoh}(\mathbb{P}^1)$. Since $\mathcal{A} \simeq \pi_*\mathcal{O}_X$, the sheaf $\mathcal{G}$ is endowed with an $\mathcal{A}$-module structure, corresponding to a map $\mathcal{A} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{G} \to \mathcal{G}$; as the algebra $\mathcal{A}$ is generated in degree 1, such a module structure is determined by two maps $\mathcal{G}(1) \to \mathcal{G}$, commuting in the appropriate sense. Conversely, given a quasicoherent sheaf $\mathcal{G}$ on $\mathbb{P}^1$, together with an $\mathcal{A}$-module structure extending the $\mathcal{O}_{\mathbb{P}^1}$-module structure, there is a unique sheaf $\mathcal{F}$ on X satisfying $\pi_*\mathcal{F} = \mathcal{G}$. We condense the above discussion in the simple result below.

Proposition 2.1.4. *The datum of a quasicoherent sheaf on the resolved conifold X is equivalent to a quasicoherent sheaf $\mathcal{G}$ on $\mathbb{P}^1$, together with two maps*

$$\varphi_1\,, \varphi_2 \in \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{G}\,, \mathcal{G}(-1))$$

satisfying $\varphi_1(-1) \circ \varphi_2 = \varphi_2(-1) \circ \varphi_1$.

Remark 2.1.5. Notice that, even if $\mathcal{F}$ is coherent on X, $\mathcal{G}$ will not necessarily be. However, if we assume $\mathcal{F}$ coherent with proper support, then $\mathcal{G}$ is also coherent. ♠

2.1.1 Basic definitions

We shall consider moduli spaces of stable pairs on the resolved conifold X, defined above. Even though X is not projective, we can construct moduli spaces of stable pairs on it. Pandharipande and Thomas [PT09a] prove that there exist proper moduli spaces of stable pairs supported at the zero-section $Z \simeq \mathbb{P}^1$: this comes from the fact that the sheaf $\mathcal{F}$ of a stable pair must have proper support, and that Z is the unique proper reduced embedded curve: it is the only proper curve since it is rigid given that its normal bundle $\mathcal{N}_{Z/X} \simeq \mathcal{O}_Z(-1)^{\oplus 2}$ does not admit global sections; more generally, there is a contraction $X \to \bar{X}$ mapping Z to a point $\mathrm{pt} \in \bar{X}$, with $\bar{X}$ affine, so containing no proper curves.

In the following, to avoid confusion, we will use the notation $Z \simeq \mathbb{P}^1$ when referring to the projective line in the conifold stressing that it is viewed as the reduced zero-section of $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. We follow the notation in [Sze12].

Definition 2.1.6. Let $X = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})$ be the resolved conifold. We define, for integers $l, n \in \mathbb{N}$, the moduli schemes

$$\mathcal{N}_{l,n} := \left\{ (\mathcal{F}, s) \text{ a stable pair on } X \,\middle|\, [\mathrm{Supp}(\mathcal{F})] = l \left[\mathbb{P}^1\right], \chi(\mathcal{F}) = n \right\} \big/ \sim,$$

with $\sim$ denoting isomorphism of stable pairs. ◊

Remark 2.1.7. Following the notation in Definition 1.3.14, we have that $\mathcal{N}_{1,n} \simeq P_n(Z)$. ♠

Nagao and Nakajima [NN11] prove that the moduli schemes $\mathcal{N}_{l,n}$ are global critical loci; this property is deduced by showing an isomorphism with the moduli spaces of stable representations of a certain quiver with potential, as detailed in Section 2.3.3.

The DT invariants were computed for local Calabi-Yau geometries in [MNOP06]. Combining this with the proof by Bridgeland [Bri11], of the DT/PT correspondence in the Calabi-Yau case, the PT-invariants $P_{l,n} := P_{l[\mathbb{P}^1],n}$, for stable pairs supported on $l\,[\mathbb{P}^1]$, are determined by the generating series

$$\mathrm{Z}_{\mathrm{PT}}(q,T) := \sum_{l \in \mathbb{N}} Z^{\mathrm{PT}}_{l[\mathbb{P}^1]}(q) \cdot T^l = \sum_{\substack{l \in \mathbb{N} \\ n \in \mathbb{Z}}} P_{l,n}\, q^n T^l = \prod_{m \geq 1} (1 - (-q)^m T)^m \,. \tag{2.1}$$

We shall use extensively the knowledge of the Euler characteristic of the push-forward to X of line bundles on Z. If our line bundles were just on $\mathbb{P}^1$, we could simply compute $\chi(\mathcal{O}_{\mathbb{P}^1}(d)) = d + 1$. A simple argument shows the same holds true on X.

Proposition 2.1.8. *Let $f \colon V \hookrightarrow W$ a closed immersion of schemes. Then $\chi(f_*\mathcal{F}) = \chi(\mathcal{F})$ for all $\mathcal{F} \in \mathsf{Coh}(V)$.*

Proof. We have that, using that f_* is exact and $\mathcal{O}_W$ is free,

$$\begin{aligned}
\mathrm{H}^i(W, f_*\mathcal{F}) &= \mathrm{Ext}^i(\mathcal{O}_W, f_*\mathcal{F}) = \mathrm{Hom}_{\mathsf{D^bCoh}(W)}(\mathcal{O}_W, f_*\mathcal{F}\,[i]) \\
&= \mathrm{Hom}_{\mathsf{D^bCoh}(V)}(\mathbb{L}f^*\mathcal{O}_W, \mathcal{F}\,[i]) = \mathrm{Hom}_{\mathsf{D^bCoh}(V)}(\mathcal{O}_V, \mathcal{F}\,[i]) \\
&= \mathrm{H}^i(V, \mathcal{F})\,,
\end{aligned}$$

where $\mathbb{L}f^*\mathcal{O}_W = f^*\mathcal{O}_W \simeq \mathcal{O}_V$ since $\mathcal{O}_W$ is a locally free sheaf on W. The proof follows from taking the alternating sum over i. □

Remark 2.1.9. A more elegant argument is the following. Consider the morphisms to the point

$$\begin{tikzcd} V \arrow[r, hook, "f"] \arrow[rr, bend left, "p"] & W \arrow[r, "q"] & \bullet \end{tikzcd}$$

where $p = q \circ f$; we can write $\mathbb{R}p_* = \mathbb{R}q_* \circ \mathbb{R}f_* = \mathbb{R}q_* \circ f_*$, as f_* is exact, coming from a closed embedding. Now, the functor $\mathrm{H}^i(W, -)$ is just $\mathrm{R}^i\, q_*$, and similarly for V: therefore considering cohomologies of the total derived functors above yields the required equality. ♠

Corollary 2.1.10. *Let $i : Z \hookrightarrow X$ be the inclusion of the zero-section. Then* $\chi(i_*\mathcal{O}_Z(d)) = d + 1$.

2.1.2 Thickenings of the zero-section

We want to specialise the construction in Section 1.1.2 to the conifold case, as we want to define a double thickening C of the zero-section Z, and consider its structure sheaf as a stable pair. Consider that, since X is the total space of a vector bundle, we also have retractions π and r fitting in the diagram

$$\begin{tikzcd} Z \arrow[rr, hook, "i"] \arrow[dr, "k"] & & X \arrow[ll, bend right, "\pi"'] \\ & C \arrow[ul, bend left, "r"] \arrow[ur, hook, "j"'] & \end{tikzcd}$$

Notice that the retraction gives $\mathcal{O}_C$ the structure of a $\mathcal{O}_Z$-module: in this case, to say that C is a *double* thickening of Z following Definition 1.1.10 is equivalent to $r_*\mathcal{O}_{C,p}$ having length-2 as an $\mathcal{O}_{Z,p}$-module for all $p \in Z$.

The zero section $Z \simeq \mathbb{P}^1 \subset X$ has normal sheaf $\mathcal{N}_{Z/X} \simeq \mathcal{O}_Z(-1)^{\oplus 2}$, which does not admit global sections: this means that there are no infinitesimal deformations of Z inside of X. The group $\mathrm{Ext}^1_X(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z)$ measures [Huy06; HL10] all deformations, including those of $\mathcal{O}_Z$ on Z (for example, deformations to non-trivial line bundles). By considering

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z \longrightarrow 0$$

and applying $\mathrm{Hom}(-, i_*\mathcal{O}_Z)$, we obtain the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^0(i_*\mathcal{O}_Z) \longrightarrow \mathrm{H}^0(i_*\mathcal{N}_{Z/X}) \longrightarrow \mathrm{Ext}^1_X(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z) \longrightarrow \mathrm{H}^1(i_*\mathcal{O}_Z) \longrightarrow \cdots,$$

and using $\mathrm{H}^1(\mathcal{O}_Z) = 0$ we show that $\mathrm{Ext}^1_X(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z)$ is therefore trivial. Thanks to Serre duality for compactly supported sheaves (see for example [Tar95, § 5.1]) we have then

$$\mathrm{Hom}_X(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z) \simeq \mathrm{Hom}_X(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z\,[3])^\vee\,,$$

and similarly

$$\mathrm{Hom}_X(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z\,[2])^\vee \simeq \mathrm{Hom}_X(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z\,[1]) \simeq \mathrm{Ext}^1_Z(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z) = 0\,.$$

A line bundle on Z will be isomorphic to $\mathcal{O}_Z(-d)$, for some $d \in \mathbb{Z}$. To choose a thickening line bundle $\mathcal{L} \subset \mathcal{N}_{Z/X}$, we can start from a map $\alpha\colon \mathcal{O}_Z(-d) \hookrightarrow \mathcal{O}_Z(-1)^{\oplus 2}$: clearly we require $\alpha \neq 0$, for otherwise we would not get a thickened curve $C \supset Z$, as C would be reduced; however, the image of such a map might not be a line bundle. The map α is parametrised by $\mathrm{Hom}_Z\left(\mathcal{O}_Z(-d), \mathcal{O}_Z(-1)^{\oplus 2}\right) \simeq \mathrm{H}^0\left(Z, \mathcal{O}_Z(d-1)^{\oplus 2}\right)$, and will be locally given by two polynomials P, Q in the coordinates X_i on Z, of degree $d-1$. If these polynomials have a non-invertible greatest common divisor M, the image of α will have base-points corresponding to the roots of M: in such case, we divide P and Q by M and consider the associated map $\mathcal{O}_Z(-d') \hookrightarrow \mathcal{O}_Z(-1)^{\oplus 2}$, with $d' < d$, whose image is now an honest line bundle $\mathcal{L} \simeq \mathcal{O}_Z(-d')$ on Z. This construction will be carried out in the following sections, with a more detailed discussion on the general situation in Chapter 5.

There is a more general sheaf-theoretic approach to the construction outlined above. For any map of sheaves $f\colon \mathcal{A} \to \mathcal{B}$, we can consider the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "f"] \arrow[dr, dashed] & \mathcal{B} \arrow[r, two heads] \arrow[rr, bend left, two heads, "\mathrm{ck}_{\mathrm{TF}}"] & \mathrm{coker}(f) \arrow[r, two heads] & \mathrm{coker}(f)_{\mathrm{TF}} \\
& \mathcal{C}\,, \arrow[u, hook] & &
\end{tikzcd}$$

where $\mathrm{coker}(f)_{\mathrm{TF}}$ is the torsion-free part of the cokernel, that is the quotient by the submodule of torsion elements. We take the *saturation* of $A \to B$, which is by definition $\mathcal{C} := \ker(\mathrm{ck}_{\mathrm{TF}})$, and the dashed arrow exists by universal properties. What we did was considering $\mathcal{A} = \mathcal{O}_Z(-d)$, $\mathcal{B} = \mathcal{O}_Z(-1)^{\oplus 2}$, $f = \alpha$ and $\mathcal{C} = \mathcal{O}_Z(-d')$.

In conclusion, a thickening C of the zero-section Z will be given, up to isomorphism, by a non-split extension of the form

$$0 \longrightarrow i_*\mathcal{O}_Z(d') \longrightarrow j_*\mathcal{O}_C \longrightarrow i_*\mathcal{O}_Z \longrightarrow 0\,. \tag{2.2}$$

Remark 2.1.11. Let C be a thickening of the zero section Z as above. Since Z is smooth, then C is Gorenstein by Corollary 1.3.9. ♠

Proposition 2.1.12.

$$\mathbb{L}i^* i_* \mathcal{O}_Z \simeq \mathcal{O}_Z \oplus \mathcal{O}_Z(1)^{\oplus 2}\,[1] \oplus \mathcal{O}_Z(2)\,[2]\ .$$

Proof. We aim at writing Z as the zero-locus of a section of a vector bundle E on X, so as to resolve $i_*\mathcal{O}_Z$ via the Koszul complex. This can be accomplished by considering $E := \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, which is a rank-2 vector bundle on X, with projection $\pi' \colon E \to X$ given by $v \mapsto x$ for all $v \in \mathcal{O}_{\mathbb{P}^1,\pi(x)}(-1)^{\oplus 2} = E_x$. There is a tautological section $\sigma \colon X \to E$ sending $X \ni x \mapsto p(x)$, where $p(x) \in E_x$ is just x viewed in the fibre of the vector bundle over Z. It is immediate that $\sigma^{-1}(0) \simeq Z$.

As our curve Z is a complete intersection, the Koszul complex gives an exact sequence

$$0 \longrightarrow \bigwedge^2 E^\vee \longrightarrow E^\vee \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z \longrightarrow 0\,,$$

with morphisms given by contraction with σ. In $\mathsf{D}^\flat\mathsf{Coh}(X)$ we can write

$$i_*\mathcal{O}_Z \simeq \left(\bigwedge^2 E^\vee \to E^\vee \to \mathcal{O}_X\right)\,,$$

and since on the level of vector bundles $\mathbb{L}i^*$ acts by restriction, we have

$$\mathbb{L}i^* i_*\mathcal{O}_Z \simeq \left(\bigwedge^2 E^\vee_{|Z} \to E^\vee_{|Z} \to \mathcal{O}_Z\right)\,.$$

By definition, $i^*E \simeq i^*\pi^*\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, therefore

$$E^\vee_{|Z} \simeq \mathcal{O}_Z(1)^{\oplus 2} \quad \text{and} \quad \bigwedge^2 E^\vee_{|Z} \simeq \mathcal{O}_Z(2)\,.$$

Since $\sigma_{|Z} = 0$, we conclude the proof. □

We want to compute possible extensions between sheaves of the form $i_*\mathcal{O}_Z(d)$, for $d \in \mathbb{Z}$. We have on X a line bundle $\mathcal{O}_X(1) := \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ whose restriction satisfies $i^*\mathcal{O}_X(1) \simeq \mathcal{O}_Z(1)$, whence

$$\operatorname{Ext}^k_X\left(i_*\mathcal{O}_Z(d_1), i_*\mathcal{O}_Z(d_2)\right) \simeq \operatorname{Ext}^k_X\left(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z(d_2 - d_1)\right)\ .$$

Therefore we are only interested in computing extensions of this form. From Proposition 2.1.12 we deduce the following.

Corollary 2.1.13.

$$\mathrm{Hom}_X\left(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z(d)\right) \simeq \begin{cases} \mathbb{C}^{d+1} & \textit{if } d \geq 0 \\ 0 & \textit{if } d < 0 \end{cases}$$

$$\mathrm{Ext}^1_X\left(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z(d)\right) \simeq \begin{cases} \mathbb{C}^{2d} & \textit{if } d \geq 0 \\ \mathbb{C}^{-d-1} & \textit{if } d \leq -2 \\ 0 & \textit{if } d = -1 \end{cases}$$

$$\mathrm{Ext}^2_X\left(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z(d)\right) \simeq \begin{cases} \mathbb{C}^{d-1} & \textit{if } d \geq 1 \\ \mathbb{C}^{-2d} & \textit{if } d \leq -1 \\ 0 & \textit{if } d = 0 \end{cases}$$

$$\mathrm{Ext}^3_X\left(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z(d)\right) \simeq \begin{cases} \mathbb{C}^{1-d} & \textit{if } d < 1 \\ 0 & \textit{if } d \geq 1 \end{cases}$$

Proof. By adjunction $\mathbb{L}i^* \dashv i_*$, we can write

$$\begin{aligned}\mathrm{Ext}^k_X\left(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z(d)\right) &\simeq \mathrm{Ext}^k_Z\left(\mathbb{L}i^*i_*\mathcal{O}_Z, \mathcal{O}_Z(d)\right) \simeq \mathrm{Hom}_Z\left(\mathbb{L}i^*i_*\mathcal{O}_Z, \mathcal{O}_Z(d)\,[k]\right) \\ &\simeq \mathrm{Hom}_Z\left(\mathcal{O}_Z \oplus \mathcal{O}_Z(1)^{\oplus 2}\,[1] \oplus \mathcal{O}_Z(2)\,[2]\ ,\ \mathcal{O}_Z(d)\,[k]\right)\ .\end{aligned}$$

Specialising, we have

$$\begin{aligned}\mathrm{Hom}_X\left(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z(d)\right) &\simeq \mathrm{Hom}_Z\left(\mathcal{O}_Z, \mathcal{O}_Z(d)\right)\ , \\ \mathrm{Ext}^1_X\left(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z(d)\right) &\simeq \mathrm{H}^0\left(\mathcal{O}_Z(d-1)\right)^{\oplus 2} \oplus \mathrm{H}^1\left(O_Z(d)\right)\ , \\ \mathrm{Ext}^2_X\left(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z(d)\right) &\simeq \mathrm{H}^0\left(\mathcal{O}_Z(d-2)\right) \oplus \mathrm{H}^1\left(O_Z(d-1)^{\oplus 2}\right)\ , \\ \mathrm{Ext}^3_X\left(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z(d)\right) &\simeq \mathrm{H}^1\left(O_Z(d-2)\right)\ .\end{aligned}$$

□

Remark 2.1.14. From Corollary 2.1.13 we notice that the dimensions satisfy

$$\mathrm{ext}^k_X\left(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z(d)\right) = \mathrm{ext}^{3-k}_X\left(i_*\mathcal{O}_Z, i_*\mathcal{O}(-d)\right)\ ,$$

as expected since $\mathsf{D}^\flat\mathsf{Coh}(X)$ is a 3-Calabi-Yau category. ♠

We can consider in particular $\mathrm{Ext}^1_X\left(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z(d)\right) \simeq \mathbb{C}^{2d}$, which is the parameter space of all extensions of the form

$$0 \longrightarrow i_*\mathcal{O}_Z(d) \longrightarrow \mathcal{F} \longrightarrow i_*\mathcal{O}_Z \longrightarrow 0\,, \tag{2.3}$$

and generic such extensions correspond to thickenings C of Z along a line bundle $\mathcal{L} \simeq \mathcal{O}_Z(-d)$ via $\mathcal{F} \simeq j_*\mathcal{O}_C$, where $j\colon C \hookrightarrow X$. We are interested in considering such sheaves, with the canonical section $\mathcal{O}_X \to j_*\mathcal{O}_C$, inside the moduli space of stable pairs: they correspond to the stratum with the simplest geometry.

Proposition 2.1.15. *Let C be the thickening of Z along $\mathcal{L}$, and consider $(\mathcal{F}, s)$ the stable pair obtained as $\mathcal{F} := j_*\mathcal{O}_C$ with the canonical section $\mathcal{O}_X \to j_*\mathcal{O}_C$. If $\mathcal{L} \simeq \mathcal{O}_Z(-d)$, then $(\mathcal{F}, s)$ belongs to $\mathcal{N}_{2,n}$ for $n = d+2$.*

Proof. The statement follows from the long exact sequence obtained applying $\mathrm{Hom}_X(\mathcal{O}_X, -)$ to (2.2). We have the vanishing $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all $i \geq 1$, and using Corollary 2.1.10 we can write $\chi(X, \mathcal{F}) = \chi(Z, \mathcal{O}_Z(d)) + \chi(Z, \mathcal{O}_Z) = d + 2$.

□

Remark 2.1.16. Notice that, as a consequence of the proposition above, the moduli spaces $\mathcal{N}_{2,n}$ are empty for $n \leq 2$. Indeed, for any stable pair $(\mathcal{F}, s)$, we have by sequence (1.8) that $\chi(\mathcal{F}) \geq \chi(\mathcal{O}_{C_\mathcal{F}})$, and $\chi(\mathcal{O}_C) \geq 3$ for any double Cohen-Macaulay curve supported at the zero section Z. ♠

We conclude this section with an explicit affine construction of ribbons whose reduced support is the zero section $Z \subset X$.

Example 2.1.17 (Affine construction). With the same notation as Section 2.1, take two degree-d homogeneous polynomials $P(\zeta_0, \zeta_1)$ and $Q(\zeta_0, \zeta_1)$, calling $P_i(Z_i)$ and $Q_i(Z_i)$, respectively, their de-homogenisation in the affine patch $U_i = \{\zeta_i \neq 0\}$, for $i = 0, 1$. Starting with U_1, we proceed as in Example 1.1.20 and consider the curve with ideal

$$I_1 := I_{P_1,Q_1} = \left(X_1^2, X_1Y_1, Y_1^2, Q_1(Z_1)X_1 - P_1(Z_1)Y_1\right) .$$

On $U_0 \cap U_1$, we perform the change of coordinates obtaining the ideal

$$\left(Z_0^2X_0^2, Z_0^2X_0Y_0, Z_0^2Y_0^2, \frac{1}{Z_0^{d-1}}Q_0(Z_0)X_0 - \frac{1}{Z_0^{d-1}}P_0(Z_0)Y_0\right) ,$$

which coincides with $I_0 := I_{P_0,Q_0}$ as modules over $\mathbb{C}\left[X_0, Y_0, Z_0, Z_0^{-1}\right]$. Hence I_0 and I_1 glue to define a curve $C_{P,Q}$ on the resolved conifold, whose reduced support is precisely the zero-section Z since it satisfies the equations $X_i = Y_i = 0$ for $i = 0, 1$. ♣

Remark 2.1.18. Notice how, in the construction above, the two polynomials must have the same degree for the gluing to occur, unless one of them is the *zero polynomial.* ♠

2.2 The toric formalism

2.2.1 The resolved conifold as toric variety

The resolved conifold X, as defined at the beginning of Section 2.1, admits a convenient description as a toric variety, well known also in the physics literature.

There is an action of the torus $\mathbb{C}^*$ on $\mathbb{C}^4$

$$\mathbb{C}^* \ni \lambda \colon (z_1, z_2, z_3, z_4) \longmapsto (\lambda^{Q_1} z_1, \lambda^{Q_2} z_2, \lambda^{Q_3} z_3, \lambda^{Q_4} z_z)$$

with charges $(Q_1, Q_2, Q_3, Q_4) = (1, 1, -1, -1)$. Then it is known that $X \simeq \mathbb{C}^4 /\!\!/_\chi \mathbb{C}^*$ is a GIT quotient of $\mathbb{C}^4$ by $\mathbb{C}^*$ corresponding to a nontrivial character χ of $\mathbb{C}^*$.

Alternatively, we can choose a real parameter $t > 0$ and consider the space in $\mathbb{C}^4$ where

$$|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = t\,,$$

and quotient by the action of $\mathrm{U}(1)$ given by $\vartheta \colon z_i \mapsto \mathrm{e}^{\mathrm{i}Q_i\vartheta} z_i$, which leaves the action of the real torus $\mathbf{T} \simeq (\mathrm{U}(1))^4/\mathrm{U}(1)$. Using standard computations in toric geometry (see for example Fulton's book [Ful93, §4.3]), we have that the condition $\sum_i Q_i = 0$ ensures the triviality of the canonical bundle, hence X is Calabi-Yau as expected. In order to find the fan Σ of X, we must find four lattice points $v_1, \dots, v_4$ in $\mathbb{Z}^3$ satisfying

$$\sum_i Q_i v_i = v_1 + v_2 - v_3 - v_4 = 0\,,$$

and an immediate solution is

$$\begin{aligned} v_1 &= (1, 1, 1)\,, \quad & v_2 &= (-1, -1, 1) \\ v_3 &= (-1, 1, 1)\,, \quad & v_4 &= (1, -1, 1)\,. \end{aligned}$$

Indeed, the Calabi-Yau condition on the charges ensures that $v_1, \dots, v_4$ lie on the same hyperplane in $\mathbb{Z}^3$, therefore we can consider the *toric graph* $\tilde{\Sigma}$, the intersection of Σ with said hyperplane, in Figure 2.1a. In addition, we note that X must necessarily be non-compact, as the cones corresponding to its fan do not cover the vector space $\mathbb{R}^3$. We consider also the *toric diagram* Γ, the dual of the toric graph, in Figure 2.1b.

We can parametrise the base of the fibration given by the moment map for $\mathbf{T}$ above,

(a) Intersection of the fan Σ with a hyperplane.

(b) Toric diagram Γ, dual of $\tilde{\Sigma}$.

Figure 2.1: Toric construction of the conifold

via coordinates $y_i := |z_i|^2$ satisfying $y_1 + y_2 - y_3 - y_4 - t = 0$ and $y_i \geq 0$. As y_4 is dependent on the other coordinates, the boundary of the base is then given by

$$y_1 = y_2 = y_3 = 0\,, \qquad \text{or} \qquad y_1 + y_2 - y_3 = t\,,$$

as when a coordinate y_i vanishes the corresponding $\mathrm{U}(1)$ degenerates. This gives four hyperplanes whose intersections form a three-dimensional graph as in Figure 2.2, which corresponds to Γ. The vertices of Γ correspond to $\mathbf{T}$-fixed points, and the

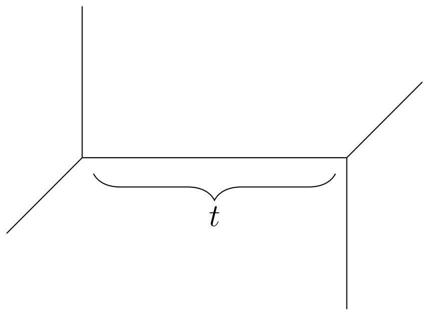


Figure 2.2: The conifold toric graph

edges to $\mathbf{T}$-fixed lines on X. In the case of Figure 2.2, there are two fixed points P_0 and P_1 corresponding to the trivalent vertices, and the fixed line joining them is the zero section $Z \simeq \mathbb{P}^1$ of the bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$.

Around each fixed point P_α, for $\alpha = 0, 1$, there is a canonical, $\mathbf{T}$-invariant affine open chart $U_\alpha \simeq \mathbb{C}^3$ centered at P_i; after choosing suitable coordinates t_i on $\mathbf{T}$ and x_i on U_α, such that the action $(t_1, t_2, t_3) \cdot x_i = t_i x_i$ is diagonal, the transition functions on $U_0 \cap U_1$ can be written as

$$(x_1, x_2, x_3) \longmapsto (x_1^{-1}, x_1^{-1} x_2, x_1^{-1} x_3)\,.$$

We recover the cordinates of X as the small resolution of the (singular) conifold. Finally, in these coordinates, the $\mathbf{T}$-invariant zero-section Z has equation $x_2 = x_3 = 0$; we also have that that the parameter t has a geometric meaning, as it is the volume of the zero section.

Remark 2.2.1. The description of the variety X as the total space of a vector bundle, as the resolution of the conifold singularity and as a toric variety are not substantially different. Indeed, the natural local coordinates we obtain in affine patches, up to renaming, are the same. ♠

2.2.2 The 3-fold vertex

Consider a complex non-compact Calabi-Yau threefold X, with the action of a torus $\mathbf{T} \simeq (\mathbb{C}^*)^3$. This action naturally extends to the moduli space of stable pairs, and its $\mathbf{T}$-fixed geometry is studied by Pandharipande and Thomas [PT09b]. We briefly present here the key results, mostly to establish the notation for Appendix B and Appendix A.1. Even though most results hold in much broader generality, we will specialise them to the case of the resolved conifold, since the combinatorics becomes much simpler. Firstly, we focus the support. If a pair $(\mathcal{F}, s)$ is $\mathbf{T}$-fixed, then its support $C = C_{\mathcal{F}}$ must be fixed as well: therefore, in order to understand torus-fixed stable pairs, one must know fixed curves. The edges of the toric diagram Γ correspond to the $\mathbf{T}$-fixed curves: in the case of the resolved conifold X, Γ only contains one edge, as per Figure 2.1b; the fixed line is precisely the zero-section Z.

In local coordinates, any torus-fixed ideal must be monomial, and a convenient way to represent a monomial ideal in two (or three) variables is via ordinary, two-dimensional (respectively, three-dimensional) partitions. The example below clarifies the situation for zero-dimensional subschemes of a three-fold.

Example 2.2.2. The partition in Figure 2.3 corresponds to the monomial ideal

$$\begin{aligned}(x_3^5, x_3^3x_1, x_3^2x_1^2, x_3^2x_1^3, x_1^4, x_3^4x_2, x_3^2x_2x_1, x_3^2x_2x_1^2,\\ x_3^3x_2x_1^3, x_2x_1^4, x_3^2x_2^2, x_3x_2^2x_1, x_2^2x_1^2, x_3x_2^3, x_2^3x_1, x_2^4)\,.\end{aligned}$$

Each box in the configuration is marked by the coordinates of its corner closest to the origin, giving a triple of natural numbers (i, j, k) which carries the information of a monomial $x_1^i x_2^j x_3^k$ which is *not* in the ideal.

This ideal defines a zero-dimensional subscheme of $\mathbb{A}^3$, supported at the origin. ♣

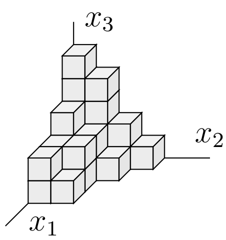


Figure 2.3: Partition of a monomial ideal

Following the same rationale, a $\mathbf{T}$-fixed curve will correspond, around a chart centered at one of the vertices of Γ, to an infinite partition extending indefinitely along at least one of the adjacent edges of Γ. Asymptotically, the infinite three-dimensional partition is the union of three cylinders along the axes: each of the three cylinders is determined by its base, which corresponds to a two-dimensional partition. Consider the monomial ideal I of such a torus-fixed curve C, in affine chart; if there is a monomial $m \notin I$ such that all $x_1 m, x_2 m, x_3 m \in I$, then C has an embedded point supported at the origin. Therefore, if the curve C is assumed Cohen-Macaulay, then its corresponding three-dimensional partition must be the union of cylinders. In other words, a $\mathbf{T}$-fixed curve is Cohen-Macaulay if and only if its ideal, around each vertex of Γ, is completely determined by the data of a plane partition along each of the edges leaving that vertex.

Example 2.2.3. The vertex in Figure 2.4, corresponding to the ideal

$$(x_1^2 x_2^2, x_1^2 x_3^2, x_2 x_3^3, x_1 x_2 x_3^2, x_1^2 x_2 x_3)\,,$$

has limit partitions $\mu_1 = \{2,1\}$, $\mu_2 = \{2,2,1\}$ and $\mu_3 = \{1,1\}$, with each expressed as a list of lengths corresponding to the rows of a Young diagram. Each μ_i describes the section of the limit cylinder along the axis x_i and following the natural orientation. The corresponding curve is Cohen-Macaulay. ♣

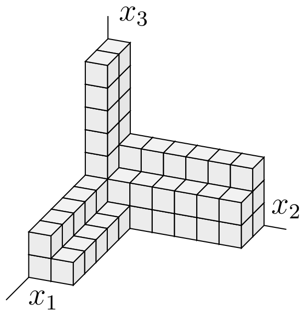
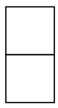
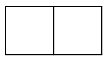


Figure 2.4: Three-dimensional vertex, with legs along the axis extending indefinitely

In particular, on the resolved conifold X, there is only one edge of Γ on which we can assign a plane partition. If we require the curve C to be a *double* thickening of Z, those partitions can only be

 or , (2.4)

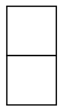
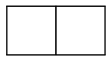
and the corresponding cylinders are shown in Figure 2.5.

Figure 2.5: Edge box configurations for a double curve C.

Secondly, we consider the zero-locus of a stable pair. For simplicity, from now on we will focus on the moduli spaces $\mathcal{N}_{2,n}$ of stable pairs on the resolved conifold X, supported on a double curve C. Clearly, if we want a pair to be $\mathbf{T}$-fixed, its zero-locus must be supported at a fixed point, that is the vertices of Γ: for the conifold X, this means only two possible points.

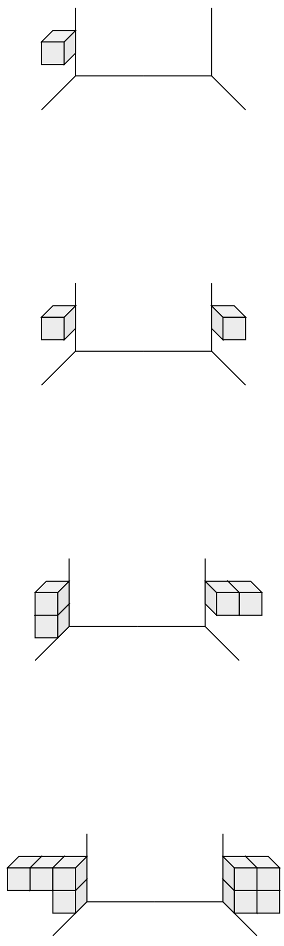
There is a specialisation of Proposition 1.3.4 to the $\mathbf{T}$-fixed geometry. With the same notation, a torus-fixed subsheaf $\mathcal{Q}$ of $\mathcal{H}om(\mathfrak{m}^r, \mathcal{O}_C)/\mathcal{O}_C$, for r sufficiently large, is determined by a *vertex configuration*, as we explain below. In any of the affine patches U_0, U_1 around one of the two vertices of Γ, a map in $\varinjlim \mathcal{H}om(\mathfrak{m}^r, \mathcal{O}_C)$ will correspond to a box in the infinite cylinder $\mathcal{C}$ obtained extending in the negative direction the configuration of C, as in Figure 2.5, which is to say the cylinder over one of the two plane partitions (2.4). Finally, considering $\varinjlim \mathcal{H}om(\mathfrak{m}^r, \mathcal{O}_C)/\mathcal{O}_C$, a sub-sheaf $\mathcal{Q}$ corresponds to a finite collection of boxes inside the negative half of $\mathcal{C}$, satisfying simple conditions as detailed below. Since this reasoning was local, we have two pieces of data, one for each vertex of Γ.

Proposition 2.2.4 ([PT09b])**.** *A $\mathbf{T}$-fixed stable pair in $\mathcal{N}_{2,n}$ is equivalent to the following data.*

1. *The choice of a length-2 plane partition μ, that is*  *or* .
2. *Two collections B_0 and B_1 of boxes, such that*
   (a) *for all $i = 0, 1$, the boxes in B_i belong to the negative part of the cylinder $\mathcal{C}$ over μ;*
   (b) *for all $i = 0, 1$, the boxes in B_i are justified against the $(-1, 1)$-corner;*
   (c) *the sum of the boxes in B_0 and B_1 is n.*

Proof. This is the content of [ibid., Prop. 2], specialised to the case of stable pairs of degree 2 on the conifold X. □

In order to denote vertex box configurations in a more convenient way, we will introduce the notion of *diagram* and *shape*: the former is a table representing the structure sheaf $\mathcal{O}_C$, where the squares corresponding to a vertex box have been marked; the latter is a bracketed list of the *heights* of the box configuration, relative to each vertex. The notation is self-explanatory and exemplified in Figure 2.6. Given a diagram or a shape, we can easily read off the *length* of the box configuration as either the number of marked squares in the diagram, or the sum of indices appearing in the shape. Both diagram and shape retain only the essential information on the vertex box configuration, relatively to the fixed double curve C.



$*$					...					
					...					

$[1 \,|\, 0]$

$*$					...					$*$
					...					

$[1 \,|\, 1]$

$*$					...				$*$	$*$
$*$					...					

$[2 \,|\, 1, 1]$

$*$	$*$	$*$			...				$*$	$*$
$*$					...				$*$	$*$

$[1, 1, 2 \,|\, 2, 2]$

Figure 2.6: Examples of vertex box configurations, with the respective diagram and shape.

Remark 2.2.5. For each of the vertex configurations of Figure 2.6, there is a mirror configuration corresponding to the choice of the other possible $\mathbf{T}$-fixed support C

in Figure 2.5. In other words, given a diagram or shape of length n, there are precisely *two* $\mathbf{T}$-fixed stable pairs in $\mathcal{N}_{2,n}$ with that diagram or shape. We will make extensive use of this fact to simplify notation and only list *half* of the torus-fixed stable pairs. ♠

There is a beautiful machinery which allows us to compute the dimension of the Zariski tangent space to a $\mathbf{T}$-fixed stable pair, starting from the data prescribed by Proposition 2.2.4. As it is quite technical, we will omit the details, which can be found in the original paper [PT09b, §4.4, §4.5]. A computer implementation of this machinery, relative to the moduli spaces $\mathcal{N}_{l,n}$, can be found in Appendix A.1; a collection of results for torus-fixed stable pairs in $\mathcal{N}_{2,n}$ for $n = 3, 4, 5, 6, 7$ is in Appendix B.

2.3 The quiver formalism

We wish to apply the results by Nagao and Nakajima [NN11], who establish an equivalence between moduli spaces of stable perverse coherent systems and moduli spaces of stable representations of quivers. In particular, they offer a quiver-theoretic description of moduli schemes of stable pairs on the resolved conifold.

2.3.1 Generalities

This section has the purpose of establishing notation and recalling general results, following the paper by King [Kin94, §3], books by Kirillov [Kir16] and Derksen-Weyman [DW17], and lecture notes by Hoskins [Hos18].

Definition 2.3.1. A *quiver* $Q = (Q_0, Q_1)$ consists of the data of a set of *vertices* Q_0 and *edges* Q_1, together with *head* and *tail* maps $h, t\colon Q_1 \to Q_0$. Given a field k, a *representation* of Q is the data of a k-vector space V_λ for every vertex $\lambda \in Q_0$ and a k-linear map $\phi_e\colon V_{t(e)} \to V_{h(e)}$ for each edge $e \in Q_1$. We denote a representation as $(V, \phi) = (V_\lambda\colon \lambda \in Q_0\,, \phi_e\colon e \in Q_1)$, omitting the edge maps when necessary.

The *dimension vector* of (V, ϕ) is $\mathbf{d} = \mathbf{d}^V = (d^V_\lambda)_{\lambda \in Q_0}$, where $d^V_\lambda = \dim V_\lambda$. We shall omit the superscript unless necessary.

A morphism between representations (V, ϕ) and (W, ψ) is a collection $(f_\lambda\colon \lambda \in Q_0)$ of k-linear maps $f_\lambda\colon V_\lambda \to W_\lambda$ such that for each edge $e \in Q_1$ it satisfies $f_{t(e)}\phi_e = \psi_e f_{h(e)}$, and we denote by $\operatorname{Hom}_Q(V, W)$ the abelian group of such morphisms. ♢

Definition 2.3.2. The *path algebra* kQ is the free vector space on paths on the quiver, which is a k-algebra with multiplication given by juxtaposition of composable paths. A representation as above is just a representation of kQ. We will denote by $\mathsf{Mod}\text{-}kQ$ the abelian category of representations of kQ: indeed, for a representation (V,ϕ) as above, and a path $p = e_1 \cdots e_j$ for edges $e_i \in Q_1$, we set

$$\phi \cdot p = \phi_{e_1} \circ \cdots \circ \phi_{e_j} \, ,$$

extended linearly on kQ. ♢

Remark 2.3.3. We will refer to representations of a quiver as modules over its algebra, meaning *right modules*. ♠

Consider a quiver Q and a fixed dimension vector $\mathbf{d}$. Given a representation (V,ϕ) with dimension vector $\mathbf{d}$, after the choice of a basis in each V_λ, we obtain a point in the affine *representation space*

$$\mathrm{Rep}_{\mathbf{d}}(Q) := \bigoplus_{e \in Q_1} \mathrm{Hom}_k(k^{d_{t(e)}}, k^{d_{h(e)}}) \, .$$

On the other hand, any point $M = (M_e)$ in the representation space is just a collection of matrices, which correspond to a representation (V,ϕ), where ϕ_e is given by M_e in the chosen basis. We have the group

$$\mathrm{GL}_{\mathbf{d}}(Q) := \prod_{\lambda \in Q_0} \mathrm{GL}(d_\lambda, k) \, ,$$

which acts on $\mathrm{Rep}_{\mathbf{d}}(Q)$ via conjugation: given $g = (g_\lambda)$ we define

$$g \cdot M := (g_{h(e)} M_e g_{t(e)}^{-1}) \, .$$

It is a standard result ([Hos18, §2.2],[Kin94, §3],[Kir16, §2.1]) that the orbits for the action of $\mathrm{GL}_{\mathbf{d}}(Q)$ on $\mathrm{Rep}_{\mathbf{d}}(Q)$ are in bijection with the isomorphism classes of representations of Q with dimension $\mathbf{d}$.

Theorem 2.3.4 ([ibid., Thm 2.1])**.** *Let $V = (V,\phi), W = (W,\psi) \in \mathsf{Mod}\text{-}kQ$ have dimension $\mathbf{d} = (v_\lambda)$; fix bases and call $M, N \in \mathrm{Rep}_{\mathbf{d}}(Q)$ the corresponding points. Then V and W are isomorphic if and only if M and N are in the same $\mathrm{GL}_{\mathbf{d}}(Q)$-orbit.*

We recall some basic facts concerning the homological algebra of quiver representations, following [DW17, §2.4] and [Hos18, §2.1].

Proposition 2.3.5. *Let Q be a quiver, and $V = (V,\phi), W = (W,\psi) \in \mathsf{Mod}\text{-}kQ$ two representations. Given a tuple $\upsilon = (\upsilon_e) \in \bigoplus_{e \in Q_1} \mathrm{Hom}(V_{t(e)}, W_{h(e)})$, the representation defined by*

$$\mathrm{E}(V,W,\upsilon) := \left((W_\lambda \oplus V_\lambda)_\lambda \ , \left(\begin{pmatrix} \psi_e & \upsilon_e \\ 0 & \phi_e \end{pmatrix} \right)_e \right)$$

sits in an extension

$$0 \longrightarrow W \longrightarrow \mathrm{E}(V,W,\upsilon) \longrightarrow V \longrightarrow 0\,,$$

where the maps are the natural ones. Moreover, there is an exact sequence

$$0 \to \mathrm{Hom}_Q(V,W) \xrightarrow{i} \bigoplus_{\lambda\in Q_0} \mathrm{Hom}(V_\lambda, W_\lambda) \xrightarrow{\alpha}$$
$$\xrightarrow{\alpha} \bigoplus_{e\in Q_1} \mathrm{Hom}(V_{t(e)}, W_{h(e)}) \xrightarrow{\beta} \mathrm{Ext}^1_Q(V,M) \to 0$$

where i is the inclusion, α is defined by $\alpha((f_\lambda)_e) := f_{h(e)} \circ \phi_e - \psi_e \circ f_{t(e)}$, and $\beta(\upsilon)$ is the equivalence class of $\mathrm{E}(V,W,\upsilon)$ as a Yoneda extension.

Corollary 2.3.6. *Denoting by* hom *and* ext *the dimensions of the respective spaces, we have*

$$\mathrm{hom}_Q(V,W) - \mathrm{ext}^1_Q(V,W) = \sum_{\lambda\in Q_0} d^V_\lambda d^W_\lambda - \sum_{e\in Q_1} d^V_{t(e)} d^W_{h(e)}\,.$$

This justifies the following.

Definition 2.3.7. Given a quiver Q, the *Euler form* or *Ringel form* on $\mathbb{Z}^{Q_0}$ is defined by

$$\langle \mathbf{d}, \mathbf{d}' \rangle_Q = \sum_{\lambda\in Q_0} d_\lambda d'_\lambda - \sum_{e\in Q_1} d_{t(e)} d'_{h(e)}\,,$$

and the associated *Tits quadratic form* is $q_Q(\mathbf{d}) := \langle \mathbf{d}, \mathbf{d} \rangle_Q$. ♢

Remark 2.3.8. The Euler form depends on the orientation of Q, but the Tits form only on the underlying graph. Moreover, the abelian category $\mathsf{Mod}\text{-}kQ$ has homological dimension 1, and given two representations V, W the Euler characteristic simply reads

$$\chi(V,W) = \mathrm{Hom}_Q(V,W) - \mathrm{Ext}^1_Q(V,W)\,.$$

Corollary 2.3.6 simply says that $\chi(V,W) = \left\langle \mathbf{d}^V, \mathbf{d}^W \right\rangle_Q$, hence the name. Finally, it is immediate that $q_Q(\mathbf{d}) = \dim \mathrm{GL}_{\mathbf{d}}(Q) - \dim \mathrm{Rep}_{\mathbf{d}}(Q)$. ♠

2.3.2 Stability

Throughout the rest of this chapter, we will work over $k = \mathbb{C}$.

Definition 2.3.9 ([Kin94, §3])**.** Let $\zeta = (\zeta_\lambda) \in \mathbb{Z}^{Q_0}$ be a *stability parameter.* For a $\mathbb{C}Q$-module $V \in \mathsf{Mod}\text{-}\mathbb{C}Q$ with dimension vector $(v_\lambda)_{\lambda\in Q_0}$, the *charge* with respect to ζ is

$$\theta_\zeta = \sum_{\lambda\in Q_0} \zeta_\lambda v_\lambda\,.$$

We say that V is *(semi)stable* with respect to θ_ζ if $\theta_\zeta(V) = 0$ and every non-zero proper sub-representation $U \subset V$ satisfies

$$\theta_\zeta(U) \underset{(=)}{<} 0\,.$$

$\Diamond$

Remark 2.3.10. There is a difference in sign conventions for stability parameters, between the original paper [Kin94] and the more recent work [NN11]. We will adhere to the notation in the latter, which is standard: that is a module is (semi)stable if any of its submodules has (weakly) smaller charge. ♠

There are open sets $\mathrm{Rep}_{\mathbf{d}}(Q)^{\zeta-\mathrm{st}} \subseteq \mathrm{Rep}_{\mathbf{d}}(Q)^{\zeta-\mathrm{ss}} \subseteq \mathrm{Rep}_{\mathbf{d}}(Q)$ of (semi)stable representations, and King [Kin94] proves that being (semi)stable[1] with respect to θ_ζ is equivalent to (semi)stability with respect to a certain character χ_ζ. Trerefore, we can construct the GIT quotient

$$\mathcal{R}_{\mathbf{d}}^{\zeta-\mathrm{ss}}(\mathbb{C}Q) := \mathrm{Rep}_{\mathbf{d}}(Q) /\!\!/_{\chi_\zeta} \mathrm{GL}_{\mathbf{d}}(Q)\,,$$

which contains an open subscheme

$$\mathcal{R}_{\mathbf{d}}^{\zeta}(\mathbb{C}Q) := \mathcal{R}_{\mathbf{d}}^{\zeta-\mathrm{st}}(\mathbb{C}Q) = \mathrm{Rep}_{\mathbf{d}}(Q)^{\zeta-\mathrm{st}}/\mathrm{GL}_{\mathbf{d}}(Q)\,,$$

of dimension $-q_Q(\mathbf{d})$, where q_Q is the Tits form of Definition 2.3.7. We will refer to this latter construction when talking about moduli spaces of stable representations.

2.3.3 The conifold quiver

Let $X := \mathrm{Tot}\,(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})$ be the resolved conifold, $\mathbb{P}^1 \simeq Z \overset{i}{\hookrightarrow} X$ the zero-section, and $C \overset{j}{\hookrightarrow} X$ be a double curve such that $C_{\mathrm{red}} = Z$, given by the thickening along a line bundle $\mathcal{L} \simeq \mathcal{O}_Z(-d)$ which is the image of a map $\mathcal{O}_Z(-d) \to \mathcal{N}_{Z/X} \simeq O_Z(-1)^{\oplus 2}$. Let $Q = (Q_0, Q_1)$ be the *conifold quiver*, having vertices $Q_0 = \{\mathbf{0}, \mathbf{1}\}$ and edges $Q_1 = (a_i, b_j)_{i,j=0,1}$, with $a_i \colon \mathbf{0} \to \mathbf{1}$ and $b_j \colon \mathbf{1} \to \mathbf{0}$. We consider the quiver algebra $\mathbb{C}Q$ and the potential $W = a_1b_1a_2b_2 - a_1b_2a_2b_1$, called Klebanov-Witten superpotential [KW99]. In this context, a *superpotential* is a formal linear combination of cycles. We denote by $A := \mathbb{C}Q/\partial W$ the Jacobi algebra, that is the quotient by the relations

[1] In fact, we must look at *geometrically stable* representations: however, this notion coincides with slope-stability, when the ground field is algebraically closed.

corresponding to the vanishing of all *formal* derivatives of W with respect to all edges. In our specific case, the relations are

$$\begin{aligned}
\partial_{a_1} W &:= b_1 a_2 b_2 - b_2 a_2 b_1 = 0 \\
\partial_{a_2} W &:= b_2 a_1 b_1 - b_1 a_1 b_2 = 0 \\
\partial_{b_1} W &:= a_2 b_2 a_1 - a_1 b_2 a_2 = 0 \\
\partial_{b_2} W &:= a_1 b_1 a_2 - a_2 b_1 a_1 = 0 \,.
\end{aligned} \tag{2.5}$$

Representations of A are characterised as those representations of $\mathbb{C}Q$ satisfying the relations (2.5) above. The notion of stability descends to representations of the Jacobi algebra, and we can use GIT to construct the moduli scheme $\mathcal{R}^{\zeta}_{\mathbf{d}}(A) := \mathcal{R}^{\zeta-\mathrm{st}}_{\mathbf{d}}(A)$. The following result is standard, see for example [Sze15, § 3.3].

Theorem 2.3.11. *Let Q be a quiver, with a superpotential W. Fix a stability parameter ζ, a dimension vector $\mathbf{d}$ and let $\mathcal{R}^{\zeta}_{\mathbf{d}}(\mathbb{C}Q)$ be the moduli space of stable representations of the quiver algebra $\mathbb{C}Q$ with fixed dimension.*

The map $\operatorname{tr} W$, acting by evaluation, is a regular function

$$\operatorname{tr} W \colon \mathcal{R}^{\zeta}_{\mathbf{d}}(\mathbb{C}Q) \longrightarrow \mathbb{C} \,.$$

If we denote by $A = \mathbb{C}Q/\partial W$ the Jacobi algebra, the moduli space of stable representations $\mathcal{R}^{\zeta}_{\mathbf{d}}(A)$ is isomorphic to the critical locus $\operatorname{Crit}(\operatorname{tr} W)$.

Remark 2.3.12. The potential W is a linear combination of closed paths along the edges, up to cyclic permutations of each loop. The scalar $\operatorname{tr}(\phi \cdot W)$, as in Definition 2.3.2, is independent of the choice of such cyclic permutation, for all representation ϕ. Hence there is a well defined map $\operatorname{tr} W$ as above. ♠

Corollary 2.3.13. $\mathcal{R}^{\zeta}_{\mathbf{d}}(A)$ *is endowed with the d-critical locus structure (cf. Definition 1.4.4) given by* $\operatorname{Crit}(\operatorname{tr} W, \mathcal{R}^{\zeta}_{\mathbf{d}}(\mathbb{C}Q)) = (\mathcal{R}^{\zeta}_{\mathbf{d}}(A), \mathcal{R}^{\zeta}_{\mathbf{d}}(\mathbb{C}Q), \operatorname{tr} W, i)$ *where* $i \colon \mathcal{R}^{\zeta}_{\mathbf{d}}(A) \to \mathcal{R}^{\zeta}_{\mathbf{d}}(\mathbb{C}Q)$ *is the inclusion.*

This result is crucial for two of our main results, that is Theorem 4.3.6 and Theorem 4.4.18. Once matrices of a representation are given, a routine to extract equations in the entries from the potential can be found in Appendix A.2.

We consider now an enrichment of quiver representations.

Definition 2.3.14. The *framed* conifold quiver $\tilde{Q}$, in Figure 2.7, is constructed by adding a vertex $\boldsymbol{\infty}$ and an arrow $\iota \colon \boldsymbol{\infty} \to \mathbf{0}$. A *framed representation* is simply a representation (V, ϕ) of Q with the extra datum of a vector space V_∞ and a linear map $\phi_\iota \colon V_\infty \to V_0$. Its dimension vector is the triple $\mathbf{d} = (d_0, d_1, d_\infty)$, with obvious notation.

We denote by $\tilde{A} := \mathbb{C}\tilde{Q}/\partial W$ the corresponding Jacobi algebra with respect to the same potential W. There is an immediate notion of *stability*, depending on a triple $\tilde{\zeta} = (\zeta_0, \zeta_1, \zeta_\infty)$. $\Diamond$

The result of Theorem 2.3.11 extend to framed representations as well. Therefore, in order to understand stable $\tilde{A}$-modules, it is helpful to describe stable $\mathbb{C}\tilde{Q}$-modules.

$$\begin{tikzcd}
\infty \arrow[r, "\iota"] & \mathbf{0} \arrow[r, bend left=40, "a_{1,2}"] \arrow[r, bend left=20] & \mathbf{1} \arrow[l, bend left=20] \arrow[l, bend left=40, "b_{1,2}"]
\end{tikzcd}$$

Figure 2.7: The framed conifold quiver $\tilde{Q}$.

Remark 2.3.15. The framing vertex is not affected by change of basis. In other words, an element $g = (g_\lambda) = (M_0, M_1)$ of the gauge group $\mathrm{GL}_{\mathbf{d}}(Q)$ acts on a framed representation $((\phi_e), \phi_\iota)$ by

$$g \cdot \phi := \left((g_{h(e)}\, \phi_e\, g_{t(e)}^{-1})_{e \in Q_1}\, ,\, g_{\mathbf{0}}\, \phi_\iota\right).$$

The action $g \cdot (\tilde{V}, \phi) = (\tilde{V}', \phi')$ is represented in the diagram below, where all squares are commutative.

$$\begin{tikzcd}
V_\infty \arrow[r, "\phi_\iota"] \arrow[d, equal] & V_0 \arrow[r, bend left=40, "\phi_{a_{1,2}}"] \arrow[r, bend left=20] \arrow[d, "M_0"'] & V_1 \arrow[l, bend left=20, "\phi_{b_{1,2}}"'] \arrow[l, bend left=40] \arrow[d, "M_1"] \\
V'_\infty \arrow[r, "\phi'_\iota"] & V'_0 \arrow[r, bend left=40] \arrow[r, bend left=20, "\phi'_{a_{1,2}}"'] & V'_1 \arrow[l, bend left=20] \arrow[l, bend left=40, "\phi'_{b_{1,2}}"]
\end{tikzcd} \tag{2.6}$$

The procedure of *framing* a quiver is more general than what we use in this case, and is key to the construction of certain moduli spaces of representations, called *Nakajima quiver varieties*. See for example [Kir16] for a comprehensive introductory text. ♠

Lemma 2.3.16 ([NN11, Lemma 4.4])**.** *Let $(V, \phi) = (V_i, \phi_{a_i}, \phi_{b_i} : i = 1, 2)$ be a stable A-module with respect to some stability parameter. Then at least one of the following holds:*

1. $\dim V_0 = \dim V_1$,

2. $\phi_{a_1} = \phi_{a_2} = 0$,

3. $\phi_{b_1} = \phi_{b_2} = 0$.

Corollary 2.3.17. *As part of the proof we have that, if case 1 above does not occur, then $\phi_{a_i}\phi_{b_j} = 0$ and $\phi_{b_i}\phi_{a_j} = 0$ for all $i, j = 0, 1$.*

We will consider representations $V = (V_0, V_1)$ of A, and framed representations $\tilde{V} = (V_0, V_1, V_\infty)$ of $\tilde{A}$. Thanks to [ibid., Rem 2.8], we can normalise a stability parameter $\tilde{\zeta} = (\zeta_0, \zeta_1, \zeta_\infty) \in \mathbb{R}^3$ for $\tilde{A}$-modules, choosing ζ_∞ in such a way that $\theta_{\tilde{\zeta}}(\tilde{V}) = 0$. The construction of $\mathcal{R}^{\zeta}_{\mathbf{d}}(\mathbb{C}Q)$ and $\mathcal{R}^{\zeta}_{\mathbf{d}}(A)$ as GIT quotients extends to framed representations, yielding moduli schemes which we will denote by $\mathcal{R}^{\tilde{\zeta}}_{\mathbf{d}}(\mathbb{C}\tilde{Q})$ and $\mathcal{R}^{\zeta}_{\mathbf{d}}(\tilde{A})$, respectively.

Remark 2.3.18. From a framed representation $\tilde{V}$, we can always obtain an unframed representation V by forgetting the framing data. However, this unframed representation might not be stable, with respect to any parameter. ♠

Remark 2.3.19. When no confusion might arise, we shall use ζ in place of $\tilde{\zeta}$, implicitly assuming the normalization above; similarly, we shall write θ_ζ for the charge of a representation of the *framed* quiver. Without loss of generality, we can assume that our space of stability conditions is just $\mathbb{R}^2$. ♠

Example 2.3.20. Clearly the choice of stability parameter determines different features of the moduli space of stable $\tilde{A}$-modules.

- Let $\zeta_0, \zeta_1 > 0$, and let $\tilde{V} = (V_0, V_1, V_\infty)$ be a stable $\tilde{A}$-module with dimension vector $\mathbf{d} = (v_0, v_1, 1)$. Assume that $\tilde{V}$ is non-zero, and consider the sub-module $\tilde{U} = (V_0, V_1, 0)$, with natural edge maps, whose charge is

  $$\theta_\sigma(\tilde{U}) = \zeta_0 v_0 + \zeta_1 v_1 > 0 \,.$$

  Therefore, for this choice of stability parameter ζ, there are no θ_ζ-stable $\tilde{A}$-modules unless $v_0 = v_1 = 0$.

- Let $\zeta_0, \zeta_1 < 0$, with $\tilde{V}$ as above. Let $\tilde{U} := V_\infty \tilde{A} \subseteq \tilde{V}$ be the sub-module generated by the framing vertex, with dimension vector $(u_0, u_1, 1)$. Its charge is

$$\theta_\zeta(U) = \zeta_0(u_0 - v_0) + \zeta_1(u_1 - v_1) \geq 0\,,$$

hence $\theta_\zeta(\tilde{U}) > 0$ if $\tilde{U} \subsetneq \tilde{V}$, leading to a contradiction; in this case we have that θ_ζ-stable $\tilde{A}$-modules, are precisely those generated by the framing.

♣

2.3.4 The PT chamber

From now on, we shall consider the case when $\dim V_\infty = 1$. In particular, notice that the dimension of $\mathcal{R}^\zeta_{\mathbf{d}}(\mathbb{C}\tilde{Q}) := \mathcal{R}^{\tilde{\zeta}}_{\mathbf{d}}(\mathbb{C}\tilde{Q})$ is $d_0 - q_Q(d_0, d_1) = d_0 + 4d_0d_1 - d_0^2 - d_1^2$.

Remark 2.3.21. In order to simplify the notation, we shall write ζ-(semi)stable in place of θ_ζ-(semi)stable, when no confusion arises. Moreover, when there is no risk of ambiguity, we will abuse notation by denoting with $a_1, a_2, b_1, b_2, \iota$ the maps of a *framed* representation. ♠

Theorem 2.3.22. *Let $\mathcal{R}^\zeta_{\mathbf{d}}(\tilde{A})$ be the moduli space of ζ-stable $\tilde{A}$-modules with dimension vector $\mathbf{d}$. For each $\mathbf{d}$, there is $\varepsilon > 0$ sufficiently small, such that for any stability parameter $\zeta = (\zeta_0, \zeta_1)$, if the following conditions are satisfied,*

1. *$\zeta_0 + \zeta_1 = \varepsilon$,*

2. *$\zeta_0 < 0$ (therefore $\zeta_1 > 0$),*

then $\mathcal{R}^\zeta_{\mathbf{d}}(\tilde{A})$ is isomorphic to $\mathcal{N}_{l,n}$ as moduli schemes, for some integers l, n.

Proof. This is just a specialisation of [NN11, Prop. 3.11]. □

Definition 2.3.23. Fixed the dimension vector, we will say that a stability parameter ζ is in the *PT-chamber* if it satisfies the two conditions of Theorem 2.3.22, for $\varepsilon > 0$ sufficiently small. ♢

We also have an explicit relation between $\mathbf{d}$ and (l, n), completing the previous theorem.

Proposition 2.3.24 ([ibid., §3.5])**.** *Let $\tilde{V}$ be a $\tilde{\zeta}$-stable $\tilde{A}$-module with dimension vector $\mathbf{d}$. Then $\tilde{V}$ is equivalent to a stable pair $(\mathcal{F}, s)$ in $\mathcal{N}_{l,n}$ if and only if*

$$\mathbf{d} = (n, n - l, 1)\,.$$

Remark 2.3.25. We use the stability condition that Nagao and Nakajima call in their paper $\zeta^+ = (-1+\varepsilon, 1)$, where the third component (corresponding to the framing vertex) is determined by normalization. Once the numerical class $\chi(\mathcal{F}) = n$ is fixed, the requirement that $\varepsilon \cdot \chi(\mathcal{F}) < 1$ ensures that stable $\tilde{A}$-modules correspond to stable pairs. For example, when considering $\mathcal{N}_{l,n}$ we can take our stability vector to be $\tilde{\zeta} = (\zeta_0, \zeta_1, \zeta_\infty) = (-n, n+1, n^2 - (n+1)(n-l))$. However, in the discussion below we shall not require knowledge of the explicit stability parameters. ♠

Remark 2.3.26. In fact, Nagao-Nakajima prove more. There are other chambers in a space of stability conditions, with each corresponding to different moduli spaces. Four chambers are shaded in Figure 2.8. Following the same reasoning as in Example 2.3.20, any stability parameter in the top-right quadrant will yield no non-zero stable modules; on the other hand, modules which are stable for stability parameters in the bottom-left are cyclic, generated by the framing vertex, and correspond to *non commutative Donaldson-Thomas* invariants introduced by Szendrői [Sze08]. The chamber for PT invariants, discussed above, is delimited by the bisector and the line of slope ζ^+, dependent on the dimension vector as detailed in Remark 2.3.25, and it is adjacent to the chamber corresponding to DT invariants.

While this theory is rich and beautiful, we will omit it here for the sake of brevity, and refer to the original paper [NN11] for all details. ♠

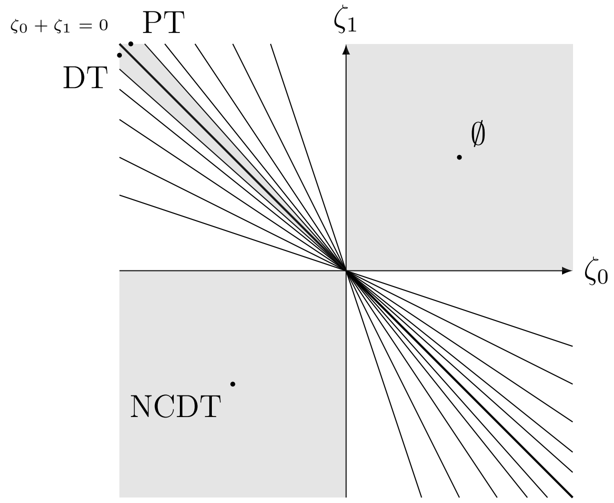


Figure 2.8: Wall and chamber structure for stability parameters on the resolved conifold.

We can determine an explicit condition about the dimension vector of an allowed subrepresentation in the PT-chamber. In the following, we will say that a subrepresentation of V is *allowed* if it is not destabilising.

Proposition 2.3.27. *Let $\zeta \in \mathbb{R}^2$ be a stability parameter in the PT-chamber, as in Theorem 2.3.22 , and let $\tilde{V}$ be a $\theta_{\tilde{\zeta}}$-stable $\tilde{A}$-module corresponding to a stable pair in $\mathcal{N}_{l,n}$, that is with dimension vector $(n, n-l, 1)$. If $\tilde{U} \subset \tilde{V}$ is a sub-module, then its dimension vector $\mathbf{d} = (d_0, d_1, d_\infty)$ must satisfy*

$$d_1 - d_0 + d_\infty l < 0\,.$$

Proof. Immediate, by expanding

$$0 > \theta_\zeta(\tilde{U}) = \zeta_0 d_0 + \zeta_1 d_1 + \zeta_\infty = \varepsilon(d_0 - n) + \zeta_1(d_1 - d_0 + d_\infty l)\,,$$

and recalling that $\zeta_1 > 0$. □

Remark 2.3.28. In the proof above, we did not use the potential W on the framed conifold quiver $\tilde{Q}$: therefore the analogous proposition holds for $\mathbb{C}\tilde{Q}$-modules. ♠

We can conveniently represent the criterion expressed by Proposition 2.3.27, as in Figure 2.9. Marked points in the light grey area correspond to allowed dimension vectors for un-framed subrepresentations, while the dark grey area marks allowed framed subrepresentations (that is, subrepresentations where the framing map factors through the inclusion morphism).

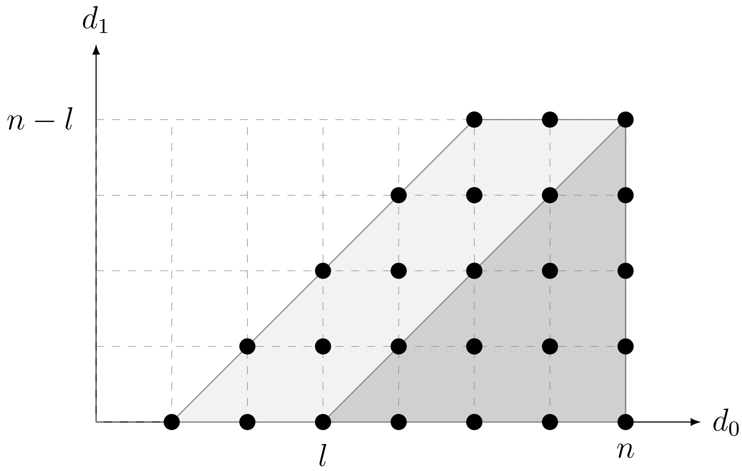


Figure 2.9: Allowed dimension vectors for submodules of a stable module in $\mathcal{N}_{l,n}$. In dark grey *framed* submodules; in light grey *unframed* submodules.

Chapter 3

The geometry of some moduli spaces

The next two chapters are devoted to describing the geometry of moduli schemes of stable pairs on the resolved conifold, supported on a doubling of the zero-section: that is, referring to Definition 2.1.6, we will deal with $\mathcal{N}_{2,n}$. The case $(l,n) = (2,4)$ was already discussed by Pandharipande and Thomas [PT09a], albeit omitting some technical details.

The case $(l,n) = (2,5)$, presented in Section 3.4, is a new result, stated in Theorem 3.4.1, and it is the main content of this chapter.

While for $l = 1$ the description is straightforward, the study of the moduli schemes $\mathcal{N}_{2,n}$ is accomplished in two complementary ways, one geometric and the other algebraic. In the geometric approach, performed in this chapter, we start from the zero section Z of the resolved conifold X and consider double thickenings C of Z obtained via Ferrand's construction, as described in Section 1.1.2 and Section 2.1.2. We then take a zero-dimensional subscheme of C, which allows us to construct a stable pair as explained in Section 1.3.3.

3.1 The space $\mathcal{N}_{1,n}$

We start by considering stable pairs with reduced support. In the case when the curve class has multiplicity $l = 1$, the moduli space $\mathcal{N}_{1,n}$ parametrises pairs $(\mathcal{F}, s)$ whose scheme-theoretic support $C = C_{\mathcal{F}}$ is just the zero-section $Z \simeq \mathbb{P}^1$ of X. Since this is reduced, such pairs are controlled (up to isomorphism), by the choice of their zero-locus.

Proposition 3.1.1 ([ibid., §4],[Tod18, Example 2.23])**.** *We have* $\mathcal{N}_{1,n} \simeq \mathbb{P}^{n-1}$.

Proof. Let $Q \xhookrightarrow{i} C$ be a divisor of degree $d = n-1$, and let $\mathfrak{m}_Q$ be its ideal sheaf in C:

$$0 \longrightarrow \mathfrak{m}_Q \simeq \mathcal{O}_C(-Q) \longrightarrow \mathcal{O}_C \longrightarrow i_*\mathcal{O}_Q \longrightarrow 0\,;$$

twisting, we get

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C(Q) \longrightarrow i_*\mathcal{O}_Q(Q) \longrightarrow 0\,.$$

We can then take the sheaf $\mathcal{F} := \mathcal{O}_C(Q)$ and $\mathcal{Q} := i_*\mathcal{O}_Q(Q)$; the section, giving a stable pair, is the canonical section of $\mathcal{F}$. Using Proposition 2.1.8, we have that $\chi(\mathcal{Q}) = d$, since it is a skyscraper sheaf supported at d points, and by Corollary 2.1.10 we know $\chi(\mathcal{O}_C) = 1$, since $C \simeq \mathbb{P}^1$: hence $\chi(\mathcal{F}) = 1 + d = n$. Therefore, a point in $\mathcal{N}_{1,n}$ corresponds to the choice of $d = n-1$ (not necessarily distinct) points on $C \simeq \mathbb{P}^1$, which are the zero locus of the pair, and we can write

$$\mathcal{N}_{1,n} \simeq \operatorname{Sym}^{n-1}\mathbb{P}^1 \simeq \mathbb{P}^{n-1}\,.$$

□

Remark 3.1.2. Notice that $\mathcal{N}_{1,n} \simeq \operatorname{Hilb}^{n-1}(C)$, that is, a stable pair with reduced support is completely determined by its zero locus. This is of course a particular instance of Theorem 1.3.15. ♠

There is another way to look at this. Once we fix an isomorphism $\mathcal{O}_C(Q) \simeq \mathcal{O}_C(d)$, which is defined up to scalar multiplication, we get a non-zero section s of the second sheaf:

$$\begin{tikzcd} \mathcal{O}_C \arrow[rr, hook, "s"] \arrow[dr, hook] & & \mathcal{O}_C(Q) \\ & \mathcal{O}_C(Q) \arrow[ur, "\simeq"] & \end{tikzcd}$$

so we can think of $\mathcal{N}_{1,n}$ as parametrising, up to scale, non-zero sections of $\mathcal{O}_C(n-1)$. We have

$$\mathcal{N}_{1,n} \simeq \mathbb{P}(\mathrm{H}^0(\mathcal{H}om(\mathcal{O}_C, \mathcal{O}_C(n-1)))) \simeq \mathbb{P}(\mathrm{H}^0(\mathcal{O}_C(n-1))) \simeq \mathbb{P}\mathbb{C}^{\binom{n}{1}} = \mathbb{P}^{n-1}\,.$$

Corollary 3.1.3. *We can compute the PT-invariant of $\mathcal{N}_{1,n}$ using the weighted Euler characteristic*

$$P_{1,n} = \int_{[P_n(X,[\mathbb{P}^1])]^{\mathrm{vir}}} 1 = (-1)^{\dim \mathbb{P}^{n-1}} \chi_{top}(\mathbb{P}^{n-1}) = (-1)^{n-1} n\,.$$

It is easy to check that this is consistent with the generating series (2.1). Indeed, in the case when the curve class has multiplicity $l = 1$, expanding the series (2.1) and considering terms with the monomial T^1, we get the partition function

$$Z^{\mathrm{PT}}_{[\mathbb{P}^1]}(q) = q - 2q^2 + 3q^3 - 4q^4 + 5q^5 - 6q^6 + 7q^7 + \ldots = \frac{q}{(1+q)^2}\,. \qquad (3.1)$$

3.2 The space $\mathcal{N}_{2,3}$

For stable pairs supported on a curve with class the double of the zero-section, expanding again (2.1) and selecting monomials with a T^2 factor, we get the partition function

$$Z^{\mathrm{PT}}_{2[\mathbb{P}^1]}(q) = -2q^3 + 4q^4 - 10q^5 + 16q^6 - 28q^7 + \dots \,. \tag{3.2}$$

In the first case, if we want $n = \chi(\mathcal{O}_C) = 3$, the thickening line bundle $\mathcal{L}$ must satisfy $\deg(\mathcal{L}^\vee) = 1$. Notice that $\mathcal{L} \neq 0$, for otherwise we would have $C \simeq Z \simeq \mathbb{P}^1$, that is reduced. We need $\mathcal{O}_C$ to be defined by a non-spit extension of $\mathcal{O}_Z$ via $\mathcal{L}^\vee$, as in (2.2).

Proposition 3.2.1 ([PT09a]). *The moduli scheme $\mathcal{N}_{2,3}$ is reduced, irreducible, and it satisfies $\mathcal{N}_{2,3} \simeq \mathbb{P}^1$.*

Proof. We start from a non-zero map

$$\lambda \in \mathbb{P}\mathrm{Hom}\left(\mathcal{O}_Z(-1), \mathcal{O}_Z(-1)^{\oplus 2}\right) \simeq \mathbb{P}^1\,,$$

whose image is a line bundle $\mathcal{L} \simeq \mathcal{O}_Z(-1) \hookrightarrow \mathcal{O}_Z(-1)^{\oplus 2}$. Note that λ is indeed defined up to scalar multiplication, since scaling gives isomorphic thickenings; moreover $\lambda \neq 0$, for otherwise we would not get an actual thickening, but a scheme isomorphic to Z. We define our stable pair $\mathcal{F}$ as the structure sheaf of C, the thickening of Z along $\mathcal{L}$, via the extension

$$0 \longrightarrow \mathcal{O}_Z(1) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_Z \longrightarrow 0\,,$$

together with the canonical section $\mathcal{O}_X \longrightarrow \mathcal{O}_C =: \mathcal{F}$. Any choice of the map λ will lead to a thickening line bundle isomorphic to $\mathcal{O}_Z(-1)$.

Finally, notice that if $(\mathcal{F}, s)$ is in $\mathcal{N}_{2,3}$, it is supported on a double Cohen-Macaulay C curve of minimal $\chi(\mathcal{O}_C)$ across such curves. Hence, the zero-locus of this stable pair must be empty, and $\mathcal{F} = \mathcal{O}_C$ is completely determined by the choice of the Ferrand doubling, as above.

□

Finally, we can compute the numerical invariant:

$$P_{3,2} = \int_{[P_3(X, 2[\mathbb{P}^1])]} 1 = (-1)^{\dim \mathbb{P}^1} \chi_{\mathrm{top}}(\mathbb{P}^1) = -2\,,$$

in accordance with the partition function (3.2).

Remark 3.2.2. Notice that all stable pairs $\mathcal{F}, s$ in $\mathcal{N}_{2,3}$ have empty zero-locus. Indeed, any thickening line bundle must be isomorphic to $\mathcal{O}_Z(-1)$ and every point in the zero locus would increase the Euler characteristic of $\mathcal{F}$. ♠

Remark 3.2.3. Consider the special case $\lambda = [1:0]$, that is $\mathcal{L}$ is just the first factor of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. In any of the two affine charts U_0 or U_1, where the coherent sheaves above are just modules, the construction above corresponds to Example 1.1.19. ♠

3.3 The space $\mathcal{N}_{2,4}$

This moduli space is the only non-trivial example available in the literature which has some interesting geometry. It first appeared in the original paper [PT09a], and a putative potential in local analytic coordinates was provided later in [Sze12].

Proposition 3.3.1 ([PT09a])**.** *The moduli scheme $\mathcal{N}_{2,4}$ can be described as follows. The underlying reduced scheme satisfies*

$$(\mathcal{N}_{2,4})_{\mathrm{red}} \simeq \mathbb{P}^3 \,.$$

The scheme structure of $\mathcal{N}_{2,4}$ has an embedded component isomorphic to a quadric $\mathbf{Q} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$, embedded via the Segre map, along which the Zariski tangent space is 4*-dimensional.*

We provide two proofs: the first, below, is an elaboration of the original proof in [ibid.]. The second proof, in Section 4.3, is quiver-theoretic and confirms a conjecture made in [Sze12] about the potential of $\mathcal{N}_{2,4}$.

3.3.1 Description via Thickenings

To obtain pairs in $\mathcal{N}_{2,4}$, we look at choices of sub-bundles $\mathcal{L} \simeq \mathcal{O}_Z(-2) \subset \mathcal{O}_Z(-1)^{\oplus 2}$, given by a map $\alpha \in \mathrm{Hom}\left(\mathcal{O}_Z(-2), \mathcal{O}_Z(-1)^{\oplus 2}\right)$ of the form

$$\mathcal{O}_Z(-2) \xhookrightarrow[\alpha]{\begin{pmatrix} x_0\zeta_0 + x_1\zeta_1 \\ y_0\zeta_0 + y_1\zeta_1 \end{pmatrix}} \mathcal{O}_Z(-1)^{\oplus 2} \,, \tag{3.3}$$

where ζ_i are local coordinates for the zero locus of the $\mathcal{O}_Z(-1)^{\oplus 2}$ bundle, following the notation in Section 2.1. The choices for α are parametrised by $\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))) \simeq \mathbb{P}^3$, since proportional maps have images producing the same thickening of Z.

When $\Delta := \det\left(\begin{smallmatrix} x_0 & x_1 \\ y_0 & y_1 \end{smallmatrix}\right) \neq 0$, the map α has image a line bundle without base-points, and we get indeed a sub-bundle $\mathcal{L} \simeq \mathcal{O}_Z(-2)$: as in the previous case, we can take $\mathcal{F} := \mathcal{O}_C$, fitting in the sequence

$$0 \longrightarrow \mathcal{O}_Z(2) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_Z \longrightarrow 0\,, \tag{3.4}$$

presenting C as the thickening of Z along $\mathcal{L}$: in this case, $\mathcal{F} := \mathcal{O}_C$ satisfies the requirement $\chi(\mathcal{F}) = 4$ and comes with a section $\mathcal{O}_X \to \mathcal{F}$. Thanks to the quiver description in Section 4.3.2, we know that the Zariski tangent space at stable pairs $(\mathcal{F}, s)$ as above has dimension 3.

So far, we are considering the open set inside $\mathbb{P}^3$ given by $\Delta \neq 0$. The locus where $\Delta = 0$ is a quadric $\mathbf{Q}$, the determinantal variety which is the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$.

Lemma 3.3.2. *A map $\alpha \in \mathbf{Q}$ factors as*

$$\begin{tikzcd} \mathcal{O}_Z(-2) \arrow[rr, hook, "\alpha"] \arrow[dr, hook] & & \mathcal{O}_Z(-1)^{\oplus 2} \\ & \mathcal{O}_Z(-1)\,. \arrow[ur, hook] & \end{tikzcd} \tag{3.5}$$

Proof. Consider a representative of α, since the construction below is well behaved with respect to scalar multiplication. After twisting by $\mathcal{O}_Z(2)$, we get a map $\mathcal{O}_Z \xhookrightarrow{\alpha} \mathcal{O}_Z(1)^{\oplus 2}$. Such a map is given by proportional polynomials $P = \lambda_1 l$ and $Q = \lambda_2 l$, with P, Q and l homogeneous of degree 1 and $\lambda_1, \lambda_2 \in \mathbb{C}$. Consider now the sections $l \in \mathrm{H}^0(Z, \mathcal{O}_Z(1))$ and $\lambda := \lambda_1 \oplus \lambda_2 \in \mathrm{H}^0(Z, \mathcal{O}_Z^{\oplus 2})$, where we omitted the obvious twist. Notice that those are also sections *on* X, as explained in Remark 2.1.9.

If we consider l and λ up to scalars, we can express them in terms of the Segre image, that is $[l] = [x_0 : x_1]$ and $[\lambda_1 : \lambda_2] = [x_0 : y_0]$. We immediately have

$$\begin{tikzcd} \mathcal{O}_Z \arrow[rr, hook, "\alpha"] \arrow[dr, hook, "l"'] & & \mathcal{O}_Z(1)^{\oplus 2} \\ & \mathcal{O}_Z(1)\,, \arrow[ur, hook, "\lambda"'] & \end{tikzcd} \tag{3.6}$$

which is, up to twist, the required factorisation. □

Remark 3.3.3. There is an immediate interpretation of the $\mathbb{P}^1 \times \mathbb{P}^1$ above, in terms of moduli count. Stable pairs parametrised by maps in $\mathbf{Q}$ are just the data of a thickenings along a sub-bundle $\mathcal{L} \simeq \mathcal{O}_Z(-1) \subset \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z(-1)$ and of a reduced point $p \in Z$. Both choices correspond to a $\mathbb{P}^1$-worth of parameters. ♠

Therefore, from a map $\alpha \in \mathbf{Q}$ we recover two pieces of information. Firstly, a line bundle $\mathcal{L} \simeq \mathcal{O}_Z(-1) \hookrightarrow \mathcal{O}_Z(-1)^{\oplus 2}$, along which we thicken Z via an extension

$$0 \longrightarrow \mathcal{O}_Z(1) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_Z \longrightarrow 0\,; \tag{3.7}$$

notice that $\chi(\mathcal{O}_C) = 3$, as was indeed in Section 3.2.

The second piece of information is the cokernel of the map l in Lemma 3.3.2, giving the structure sheaf of a reduced zero-dimensional length-1 subscheme p of Z

$$\mathcal{I}_{p/Z} \simeq \mathcal{O}_Z(-1) \overset{l}{\hookrightarrow} \mathcal{O}_Z \twoheadrightarrow \mathcal{O}_p\,.$$

In simple terms, p is the divisor on $\mathbb{P}^1$ given by the vanishing of l. Consider then the diagram of coherent sheaves on X

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_Z(1) \arrow[r] \arrow[d, equal] & \mathcal{I}_{p/C} \arrow[r] \arrow[d, hook] & \mathcal{I}_{p/Z} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{O}_Z(1) \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d, two heads] & \mathcal{O}_Z \arrow[r] \arrow[d, two heads] & 0 \\
& & \mathcal{O}_p \arrow[r, equal] & \mathcal{O}_p &
\end{tikzcd} \tag{3.8}$$

with $\mathcal{I}_{p/C}$ the ideal sheaf of p in C. Recalling that C is Gorenstein by Remark 2.1.11, we can now use the correspondence in Proposition 1.3.5 to construct a stable pair supported on C. Notice that all ideals $\mathcal{I}_{p/C}$ of length 1 (in particular *reduced*) zero-dimensional subschemes of C will arise as in diagram (3.8).

Remark 3.3.4. There is a different construction provided in [PT09a], for this particular case. Twisting diagram (3.8) by $\mathcal{O}_Z(1)$ we get

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_Z(2) \arrow[r] \arrow[d, equal] & \mathcal{I}_{p/C}(1) \arrow[r] \arrow[d, hook] & \mathcal{I}_{p/Z}(1) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{O}_Z(2) \arrow[r] & \mathcal{O}_C(1) \arrow[r] \arrow[d, two heads] & \mathcal{O}_Z(1) \arrow[r] \arrow[d, two heads] & 0 \\
& & \mathcal{O}_p \arrow[r, equal] & \mathcal{O}_p &
\end{tikzcd} \tag{3.9}$$

and therefore obtain a sheaf $\mathcal{F} := \mathcal{I}_{p/C}(1)$ such that $\chi(\mathcal{F}) = 4$. ♠

We shall see, in Section 3.3.2 below, that the bijection of sets $\mathbb{P}^3 \leftrightarrow \mathcal{N}_{2,4}$ extends to a morphism of schemes $\mathbb{P}^3 \simeq (\mathcal{N}_{2,4})_{\mathrm{red}}$. However, it is clear that $\mathcal{N}_{2,4}$ cannot be reduced, for otherwise we would be able to compute

$$P_{2,4} = (-1)^{\dim \mathbb{P}^3} \chi_{\mathrm{top}}(\mathbb{P}^3) = -4\,,$$

in contradiction with the expected result from the partition function (3.2), which is 4. The key is the non-reduced structure of the moduli space along $\mathbf{Q} \subset \mathcal{N}_{2,4}$.

Proposition 3.3.5 ([PT09a]). *Let $\mathcal{F} := \mathcal{I}_{p/C}(1)$ be a sheaf on X, where C is the thickening of the zero-section Z along a line bundle $\mathcal{L} \simeq \mathcal{O}_Z(-1)$, and $\mathcal{I}_{p/C}$ is the ideal of a reduced point $p \in C$. Then $\mathcal{F}$ admits a unique section s, up to isomorphism, such that $[(\mathcal{F}, s)] \in \mathcal{N}_{2,4}$; moreover, the Zariski tangent space to the moduli scheme $\mathcal{N}_{2,4}$, at the point $[(\mathcal{F}, s)]$, has dimension 4. The extra deformation direction is completely obstructed.*

The natural torus action on the conifold induces an action on the moduli space of stable pairs. We can use this information to determine the PT-invariants, thanks to the virtual localization formula of Graber and Pandharipande [GP99]. In the case of the moduli space $\mathcal{N}_{2,4}$ of stable pairs over the resolved conifold, we have four $\mathbf{T}$-fixed points $p_1, \ldots, p_4$, all lying on the quadric $\mathbf{Q}$, and all with four-dimensional Zariski tangent space. The box configurations of these points are listed in Appendix B (recall Remark 2.2.5 about each two-dimensional box configuration appearing twice). The virtual localization formula then reads

$$P_{4,2} = \sum_{i=1}^{4} (-1)^{\dim T_{p_i}\mathcal{N}_{2,4}} = 4\,,$$

as expected.

The proof of Proposition 3.3.5 above is provided in [PT09a]. The statement about the Zariski tangent space can be checked in two ways. The first consists in using the topological vertex formalism from Section 2.2.2, using the computations in Appendix B. The second approach exploits the description of $\mathcal{N}_{2,4}$ as a critical locus. We follow the latter in the next chapter.

3.3.2 Local affine construction of families

We have, up to this point, given a set-map $\mathbb{P}^3 \to (\mathcal{N}_{2,4})_{\mathrm{red}}$; for numerical reasons, a stable pair $\mathcal{N}_{2,4}$ is either the structure sheaf of a Cohen-Macaulay double curve such that $\chi(\mathcal{O}_C) = 4$, with empty zero locus, or its zero-locus contains precisely one reduced point, but in such case the support satisfies $\chi(\mathcal{O}_C) = 3$. So far, we showed that there is a set-theoretic correspondence between maps $\alpha\colon \mathcal{O}_Z(-2) \to \mathcal{O}_Z(-1)^{\oplus 2}$ and closed points of $\mathcal{N}_{2,4}$. We aim in this section to promote the correspondence to families, yielding an injection of schemes $\mathbb{P}^3 \hookrightarrow (\mathcal{N}_{2,4})_{\mathrm{red}}$.

The argument in Section 1.3.8 applies to this case as well: we can construct families of stable pairs on $\mathbb{P}^3 \setminus \mathbf{Q}$, defined as structure sheaves of Cohen-Macaulay ribbons, whose limit is on **Q**. This pair is constructed from the data of coprime polynomials (giving a Cohen-Macaulay thickening) and a marked point (the zero locus), and is precisely what we obtain taking the limit in the PT moduli space.

We begin by considering stable pairs on the affine space $\mathbb{A}^3_{x,y,z}$, supported at a doubling of the z-axis. The starting point is the presentation (1.16), corresponding to the family in Example 1.1.30: the aim is to construct a family of stable pairs whose Cohen-Macaulay support is the more generic ribbon described in Example 1.1.20. Let us define rings $T = \mathbb{C}[x, y, z]$, $R = \mathbb{C}[x_0, x_1, y_0, y_1]$, and $S = R[x, y, z]$, and take two polynomials $a(z) = x_0 z + x_1$ and $b(z) = y_0 z + y_1$, of degree $d = 1$. Now, consider the matrix

$$M_{a,b} := \begin{pmatrix} x^2 & xy & y^2 & b(z)x - a(z)y & x & y \\ 0 & 0 & 0 & 0 & a(z) & b(z)\,, \end{pmatrix} \tag{3.10}$$

and define the S-module $F := \operatorname{coker}(M_{a,b})$, which is equipped with a section s as per diagram below:

$$\begin{tikzcd} S^6 \arrow[r, "M_{a,b}"] & S^2 \arrow[r] & F \arrow[r] & 0 \\ & S\,. \arrow[u, "\binom{1}{0}"] \arrow[ur, "s"'] & & \end{tikzcd} \tag{3.11}$$

We verify that $S \xrightarrow{s} F$ is a family of stable pairs over the base $B = \operatorname{Spec} R$, and we will write $(a, b) \in B$ with slight abuse of notation. Over the generic point, assume $(a, b) \in B$ are coprime polynomials: one computes that indeed the support of the fibre $F_{(a,b)}$ has ideal given by

$$\operatorname{Ann} F_{(a,b)} = (x^2, xy, y^2, b(z)x - a(z)y)\,, \tag{3.12}$$

that is a curve $C_{(a,b)}$, a thickening of the z-axis, which is precisely the Cohen-Macaulay ribbon described in Example 1.1.20.

Remark 3.3.6. We know, as in Example 1.1.20, that all Cohen-Macaulay double curves $C = C_{(a,b)}$ and $\tilde{C} = C_{(\tilde{a},\tilde{b})}$ in $\mathbb{A}^3$, with fixed reduced support, are isomorphic via an automorphism of $\mathbb{A}^3$. However, when considering the stable pairs $\mathcal{O}_X \to \mathcal{O}_C$ and $\mathcal{O}_X \to \mathcal{O}_{\tilde{C}}$ with their respective canonical sections, these are *not* isomorphic as *stable pairs* unless (a, b) and $(\tilde{a}, \tilde{b})$ are proportional. ♠

Assuming now, without loss of generality, that $b = \lambda a$ for some $\lambda \in \mathbb{C}$, $\lambda \neq 0$, we have that

$$\operatorname{Ann} F_{(a,\lambda a)} = (x^2, xy, y^2, \lambda x - y) = (y^2, \lambda x - y)\,,$$

and given a linear form l, and scalars $(\lambda_1, \lambda_2) \neq (0, 0)$, we have

$$\operatorname{Ann} F_{(\lambda_1 l, \lambda_2 l)} = (x^2, xy, y^2, \lambda_2 x - \lambda_1 y)\,, \tag{3.13}$$

which is a *flat ribbon* whose reduced curve is the z-axis.

Focusing now on the section s, we compute that

$$\mathrm{Ann}_S \operatorname{coker} s = (x, y, a(z), b(z), x_0 y_1 - x_1 y_0).$$

This means that, over the generic $(a, b) \in B$ with a, b coprime, the fibre $T \xrightarrow{s} F_{(a,b)}$ has empty support, since $x_0 y_1 - x_1 y_0$ is invertible. In other words, s is surjective and $F_{(a,b)} \simeq \mathcal{O}_{C_{(a,b)}}$, with $C_{(a,b)}$ the support curve, defined by (3.12).

Over $(\lambda_1 l, \lambda_2 l) \in B$, however, the section s has cokernel supported at the closed point $\mathfrak{m} = (x, y, l)$, which is precisely the point on the support curve marked by the vanishing of the common factor l.

Remark 3.3.7. The matrix in $M_{a,b}$ in (3.10) was not obtained immediately from M_t in (1.16), described in Section 1.3.8. However, we can still recognise the key features of the module F constructed above, by thinking of it as a submodule of either $\mathrm{Hom}(\mathcal{O}_C, \mathcal{O}_C)$ or $\mathrm{Hom}(\mathfrak{m}, \mathcal{O}_C)$, with C some ribbon and $\mathfrak{m} \subset \mathcal{O}_C$ maximal. There is a generator σ_1, whose annihilator defines the support C of F, and a generator σ_2 whose behaviour depends on the singularity of $(a, b) \in B$.

- For coprime (a, b), σ_2 acts as multiplication by the nilpotent direction of $C_{a,b}$; this is depicted in Figure 3.1.

- For $(a, b) = (\lambda_1 l, \lambda_2 l)$, σ_2 acts again as multiplication by the nilpotent direction of C_{λ_1,λ_2} *twisted* by the dual of the ideal $\mathfrak{m} = (x, y, l)$ of the marked point; Figure 3.2 provides a pictorial description.

In both cases, σ_2 is annihilated by the ideal of the *reduced* support (x, y). ♠

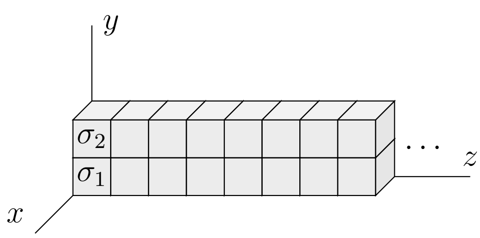


Figure 3.1: Intuition behind the module $F_{(a,b)}$, with generators $\sigma_1 = 1$ and $\sigma_2 = ax + by$. There is no correspondence between boxes and monomials

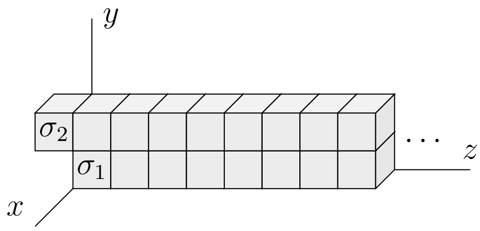


Figure 3.2: Intuition behind the module $F_{(\lambda_1 l, \lambda_2 l)}$, with generators $\sigma_1 = 1$ and $\sigma_2 = y/l = x/l$. There is no correspondence between boxes and monomials

The last step, to verify that $S \xrightarrow{s} F$ is indeed a family of stable pairs, consists in checking flatness, which is only necessary at singular points $(\lambda_1 l, \lambda_2 l) \in B$. In such cases, the proof is routine and performed as in Section 1.3.8, after an appropriate change of coordinates.

Remark 3.3.8. Notice that, for $\xi \in \mathbb{C}$, if $\xi \neq 0$ we can rescale a, b and consider the S-module $\tilde{F} := \operatorname{coker} M_{\xi a, \xi b}$. The latter is isomorphic to F via the map ψ defined by the commutative diagram below:

$$\begin{tikzcd} S^6 \arrow[r, "M_{\xi a,\xi b}"] & S^2 \arrow[r] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & 1/\xi \end{pmatrix}"'] & \tilde{F} \arrow[r] \arrow[d, "\psi"] & 0 \\ S^6 \arrow[r, "M_{a,b}"] & S^2 \arrow[r] & F \arrow[r] & 0\,. \end{tikzcd} \tag{3.14}$$

Notice that ψ is compatible with the respective sections $S \xrightarrow{s} F$ and $S \xrightarrow{\tilde{s}} \tilde{F}$, and is therefore an isomorphism of stable pairs. Hence, we are in fact constructing a family of stable pairs on $\mathbb{A}^3_{x,y,z}$ over the base $B = \mathbb{P}^3 = \operatorname{Proj} R$. ♠

It is now clear how we can proceed in the case of the resolved conifold X, using the notation at the beginning of Section 2.1. Let P and Q be the two homogeneous polynomials, of degree $d = 1$, defining α in (3.3).

Remark 3.3.9. While in our case the degree is just $d = 1$, we have chosen to leave it as a parameters, since the constructions in following sections will bear a formal similarity to what is done here: hence it helps to keep track of the degree. ♠

Consider the affine chart $U_1 = \{\zeta_1 \neq 0\}$, with coordinates $x = X_1$, $y = Y_1$, and $z = Z_1 = \zeta_0/\zeta_1$. We have de-homogenised polynomials $a = P_1$ and $b = Q_1$, and can take the matrix M_{P_1,Q_1} as in (3.10). The corresponding module $F_1 = \operatorname{coker} M_{P_1,Q_1}$, with s_1 its natural section (3.11), satisfies all the requirements and defines a family of stable pairs (F_1, s_1) on U_1, all supported at some Ferrand doubling of the zero section $\{X_1 = Y_1 = 0\}$, and which is parametrised by the base $B = \mathbb{P}^3$ by the last Remark 3.3.8.

We repeat the analogous construction on the affine chart U_0, obtaining the pair (F_0, s_0), noticing that the respective supports C_i of F_i glue nicely as in Example 2.1.17. On the intersection $U_0 \cap U_1$, the change of coordinates performed on

$$M_{P_1,Q_1} := \begin{pmatrix} X_1^2 & X_1Y_1 & Y_1^2 & Q_1X_1 - P_1Y_1 & X_1 & Y_1 \\ 0 & 0 & 0 & 0 & P_1 & Q_1 \end{pmatrix}$$

yields the matrix

$$\tilde{M}_{P_0,Q_0} := \begin{pmatrix} Z_0^2X_0^2 & Z_0^2X_0Y_0 & Z_0^2Y_0^2 & \frac{1}{Z_0^{d-1}}(Q_0X_0 - P_0Y_0) & Z_0^{d+1}X_0 & Z_0^{d+1}Y_0 \\ 0 & 0 & 0 & 0 & P_0 & Q_0 \end{pmatrix}$$

Remark 3.3.10. In the matrix above, we are at liberty to rescale individual columns (corresponding to relations among the two generators) by non-zero elements of $\mathbb{C}\left[Z_0, Z_0^{-1}\right]$, since such transformations does not affect the cokernel which is hence the *same*. ♠

We must compare the module $\tilde{F}_0 := \operatorname{coker} \tilde{M}_{P_0,Q_0}$ to $F_0 := \operatorname{coker} M_{P_0,Q_0}$, that is the stable pair constructed on U_0 as in (3.11). In fact, these two modules are isomorphic over $R\left[X_0, Y_0, Z_0, Z_0^{-1}\right]$ via the morphism ϕ, which is well defined by the commutative diagram below:

$$\begin{array}{ccccccc} S^6 & \xrightarrow{\tilde{M}_{P_0,Q_0}} & S^2 & \longrightarrow & \tilde{F}_0 & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle \begin{pmatrix} 1 & 0 \\ 0 & Z_0^{d+1} \end{pmatrix}} & & \Big\downarrow {\scriptstyle \phi} & & \\ S^6 & \xrightarrow{M_{P_0,Q_0}} & S^2 & \longrightarrow & F_0 & \longrightarrow & 0\,. \end{array} \tag{3.15}$$

Remark 3.3.11. We mentioned in Remark 3.3.7 that the generator σ_2 corresponds to multiplication by the nilpotent direction of the support, which is engineered to be a Ferrand doubling of the zero section $Z \subset X$, thickened along a line bundle $\mathcal{L} \simeq \mathcal{O}_Z(-2) = \mathcal{O}_Z(-d-1)$. Hence, it is not a surprise how the transition functions for σ_2 are those of $\mathcal{O}_Z(-d-1)$. ♠

The module $\tilde{F}_0$ admits a section $\tilde{s}_0$ obtained from (F_1, s_1): the morphism ϕ above is compatible with $\tilde{s}_0$ and s_0, and is hence an isomorphism of stable pairs. In conclusion, (F_i, s_i) for $i = 0, 1$ glue, yielding a family

$$\mathcal{O}_{X \times \mathbb{P}^3} \xrightarrow{s} \mathcal{F}$$

which in turns gives a map of schemes $\Psi\colon \mathbb{P}^3 \to \mathcal{N}_{2,4}$.

By the argument at the beginning of this section, a stable pair corresponding to a closed point of $\mathcal{N}_{2,4}$ is completely determined by a Ferrand doubling of the zero section $Z \subset X$ and the datum of the zero-locus of the pair. Now, the Ferrand doubling is provided by a line bundle $\mathcal{L}$ isomorphic to either $\mathcal{O}_Z(-2)$ or $\mathcal{O}_Z(-1)$, with the zero-locus being empty or consisting of a reduced point, respectively. All these can be realised by the family constructed above, showing that the induced morphism Ψ is a surjection on closed points: this implies that $(\mathcal{N}_{2,4})_{\mathrm{red}}$ is irreducible, and its normalisation is $\mathbb{P}^3$. Moreover, by Remark 3.3.6, Ψ is also injective on closed points. Thanks to the construction with quiver moduli spaces, to be seen in Section 4.3, one could then prove that Ψ is in fact a closed immersion.

3.4 The space $\mathcal{N}_{2,5}$

The following is one of the main results of this thesis.

Theorem 3.4.1. *The moduli scheme $\mathcal{N}_{2,5}$ has the following description. The underlying reduced scheme $(\mathcal{N}_{2,5})_{\mathrm{red}}$ is reducible:*

$$(\mathcal{N}_{2,5})_{\mathrm{red}} \simeq \mathbb{P}^5 \cup_{\mathbf{F}} \tilde{\mathbf{F}} .$$

On the first component $\mathbb{P}^5$, there is a stratification

$$\mathbb{P}^5 \supset \mathbf{H} \supset \mathbf{G} \supset \mathbf{F} ,$$

where $\mathbf{H} \subset \mathbb{P}^5$ is a singular quartic divisor, $\mathbf{G} = \mathrm{Sing}(\mathbf{H})_{\mathrm{red}} \simeq \mathbb{P}^1 \times \mathbb{P}^2$ and $\mathbf{F} \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

The second component $\tilde{\mathbf{F}}$ is described in Proposition 3.4.15. It is is an $\mathbb{F}_3$-fibration over $\mathbb{P}^1$, where $\mathbb{F}_3$ is the third Hirzebruch surface*; the section $\mathbf{F} \hookrightarrow \tilde{\mathbf{F}}$ is arises by considering the choice of the second summand, using the isomorphism $\mathbb{F}_3 \simeq \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)\right)$.*

The scheme structure of $\mathcal{N}_{2,5}$ has embedded components along the divisor $\mathbf{H}$, where the Zariski tangent space has generic dimension 6*, and along the threefold $\mathbf{G}$, where the Zariski tangent space is* 7*-dimensional. The Zariski tangent space at the generic point of $\tilde{\mathbf{F}} \setminus \mathbf{F}$, has dimension* 3*.*

Remark 3.4.2. In particular, along $\mathbf{F}$ the Zariski tangent space is 7-dimensional: there are 2 extra deformation directions, not coming from the ambient $\mathbb{P}^5$. One of these directions is unobstructed, and corresponds to a fibre of $\tilde{\mathbf{F}}$. ♠

The rest of the present section is devoted to a proof the statement about the scheme theoretic geometry of $\mathcal{N}_{2,5}$. The critical locus structure is described in Theorem 4.4.18, in Chapter 4. We start by considering choices of sub-bundles $\mathcal{L} \simeq \mathcal{O}_Z(-3) \subset \mathcal{O}_Z(-1)^{\oplus 2}$. A map α inside $\mathrm{Hom}_Z\left(\mathcal{O}_Z(-3), \mathcal{O}_Z(-1)^{\oplus 2}\right)$ will be of the form

$$\mathcal{O}_Z(-3) \xrightarrow[\alpha]{\begin{pmatrix} x_0\zeta_0^2+x_1\zeta_0\zeta_1+x_2\zeta_1^2 \\ y_0\zeta_0^2+y_1\zeta_0\zeta_1+y_2\zeta_1^2 \end{pmatrix}} \mathcal{O}_Z(-1)^{\oplus 2}\,,$$

with $[\zeta_0 \colon \zeta_1]$ coordinates on Z. For α generic, its image is a line sub-bundle $\mathcal{L} \simeq \mathcal{O}_Z(-3)$, whose dual $\mathcal{K} \simeq \mathcal{O}_Z(3)$ fits in an extension

$$0 \longrightarrow i_*\mathcal{K} \longrightarrow j_*\mathcal{O}_C \longrightarrow i_*\mathcal{O}_Z \longrightarrow 0\,, \tag{3.16}$$

and taking $\mathcal{F} := j_*\mathcal{O}_C$ with the canonical section we obtain indeed a stable pair in $\mathcal{N}_{2,5}$, since we compute $\chi(\mathcal{F}) = 4 + 1$. This happens precisely when the two polynomials defining α have *no common roots*.

Reasoning as in the previous sections, we shall prove that on an open subscheme of the moduli scheme, stable pairs are parametrised by

$$\mathbb{P}\left(\mathrm{Hom}_Z\left(\mathcal{O}_Z(-3), \mathcal{O}_Z(-1)^{\oplus 2}\right)\right) \simeq \mathbb{P}\left(\mathrm{H}^0\left(Z, \mathcal{O}(2)^{\oplus 2}\right)\right) \simeq \mathbb{P}^5\,,$$

with projective coordinates $[x_0 : x_1 : x_2 : y_0 : y_1 : y_2]$.

Remark 3.4.3. As will become clearer in the following sections, we aim to construct a family of stable pairs over the above $\mathbb{P}^5$ above, such that each fibre $(\mathcal{F}, s)$ satisfies the numerical conditions of $\mathcal{N}_{2,5}$. These pairs will be one of the following types, all with reduced support the zero-section Z:

- supported at Cohen-Macaulay thickenings of Z along a line bundle $\mathcal{L} \simeq \mathcal{O}_Z(-3)$, empty zero-locus;
- supported at Cohen-Macaulay thickenings of Z along a line bundle $\mathcal{L} \simeq \mathcal{O}_Z(-2)$, zero-locus consisting of a reduced points;
- supported at Cohen-Macaulay thickenings of Z along a line bundle $\mathcal{L} \simeq \mathcal{O}_Z(-1)$, zero-locus consisting of two points (possibly a double point) *inside* Z.

Which of the above occurs depends on the singularity of the two polynomials giving α, that is how many common roots they have. ♠

Proposition 3.4.4. *Assume the image of α above is a line sub-bundle $\mathcal{L} \simeq \mathcal{O}_Z(-3)$, and let $\mathcal{F} := j_*\mathcal{O}_C$, with the canonical section s, be the associated stable pair. Then the Zariski tangent space to $\mathcal{N}_{2,5}$ at the class $[(\mathcal{F}, s)]$ has dimension* 5.

Proof. This follows from the quiver description in Section 4.4, in particular Theorem 4.4.18. Once an explicit set of equations it obtained, one can compute the dimension; this can also be accomplished via computer algebra. □

We consider now the various pathologies of α, such that the corresponding line bundle $\mathcal{L}$ thickens the zero section Z to a curve C whose structure sheaf does *not* satisfy $\chi(\mathcal{O}_C) = 5$: this will lead to a stratification of $\mathbb{P}^5$. We define polynomials

$$\begin{aligned} P(\zeta_0, \zeta_1) &:= x_0\zeta_0^2 + x_1\zeta_0\zeta_1 + x_2\zeta_1^2\,, \\ Q(\zeta_0, \zeta_1) &:= y_0\zeta_0^2 + y_1\zeta_0\zeta_1 + y_2\zeta_1^2\,. \end{aligned} \tag{3.17}$$

Remark 3.4.5. As seen in Section 1.3.8, we know the behaviour of families of degenerating Cohen-Macaulay ribbons, whose limit is taken in the moduli space of stable pairs. Each time a pinch arises (that is, the polynomials defining the thickening direction have a common zero), we take the stable pair whose support is given by a thickening by coprime polynomials, with the removed factor marking points in the zero-locus. In the sections below, we will be transitioning from one construction to another, but it should be understood that the two are joined by flat families of stable pairs: indeed, the computation of the limit is local and can be checked explicitly. ♠

3.4.1 Case 1: P, Q with a common factor

The two polynomials have a common factor when their resultant $\mathrm{Res}(P, Q)$ satisfies

$$\mathrm{Res}(P, Q) := \det \begin{pmatrix} x_0 & x_1 & x_2 & 0 \\ 0 & x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 & 0 \\ 0 & y_0 & y_1 & y_2 \end{pmatrix} = 0\,, \tag{3.18}$$

an equation defining a hypersurface $\mathbf{H} \subset \mathbb{P}^5$. If we call α' the shared factor between P and Q, a twisted version of the proof of Lemma 3.3.2 implies that α factors as

$$\begin{tikzcd} \mathcal{O}_Z(-3) \arrow[rr, hook, "\alpha"] \arrow[dr, hook, "\alpha'"'] & & \mathcal{O}_Z(-1)^{\oplus 2} \\ & \mathcal{O}_Z(-2) \arrow[ur, hook, "\lambda"'] \arrow[dr, two heads] & \\ & & \mathcal{O}_p(-2)\,, \end{tikzcd} \tag{3.19}$$

where $p \in Z$ is just a reduced point given by the vanishing of α'. We thicken Z along the sub-bundle $\mathcal{L} \simeq \mathcal{O}_Z(-2)$ defined by λ, and obtain a double curve C such that $\chi(\mathcal{O}_C) = 4$. Consider then the diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_Z(2) \arrow[r] \arrow[d, equal] & \mathcal{I}_{p/C} \arrow[r] \arrow[d, hook] & \mathcal{I}_{p/Z} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{O}_Z(2) \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d, two heads] & \mathcal{O}_Z \arrow[r] \arrow[d, two heads] & 0 \\
 & & \mathcal{O}_p \arrow[r, equal] & \mathcal{O}_p \,, &
\end{tikzcd}
\tag{3.20}
$$

and the construct a stable pair $0 \to \mathcal{O}_C \to \mathcal{F}$ using the machinery of Proposition 1.3.5, where $\mathcal{F} \simeq \mathcal{H}om_C(\mathcal{I}_{p/C}, \mathcal{O}_C)$, satisfying indeed $\chi(\mathcal{F}) = 5$.

Proposition 3.4.6. *Let $(\mathcal{F}, s)$ be a stable pair on X, corresponding to a map $\alpha \in \mathbf{H}$ as above. The Zariski tangent space to $\mathcal{N}_{2,5}$, at the point $[(\mathcal{F}, s)]$, has dimension 6.*

Proof. This also is a consequence of Theorem 4.4.18. □

3.4.2 Case 2: P, Q proportional

This condition on α corresponds to asking that

$$
\operatorname{rk} \begin{pmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{pmatrix} \leq 1 \,,
$$

which is satisfied on a codimension-2 locus $\mathbf{G} \subset \mathbf{H} \subset \mathbb{P}^5$, and we can consider it as the image $\mathbf{G} \simeq \mathbb{P}^1_{[x_0:y_0]} \times \mathbb{P}^2_{[x_0:x_1:x_2]}$ of the Segre embedding. The map α factors as

$$
\begin{tikzcd}
\mathcal{O}_Z(-3) \arrow[rr, hook, "\alpha"] \arrow[dr, "\alpha'"'] & & \mathcal{O}_Z(-1)^{\oplus 2} \\
 & \mathcal{O}_Z(-1) \arrow[ur, hook, "l"'] \arrow[dr, two heads] & \\
 & & \mathcal{O}_{\mathbf{p}}(-1) \,,
\end{tikzcd}
\tag{3.21}
$$

with $[\alpha']$ given by $[x_0 : x_1 : x_2]$ and $[l]$ by $[x_0 : y_0]$ respectively. We obtain via l a line bundle $\mathcal{L} \simeq \mathcal{O}_Z(-1)$, and the vanishing locus of α' defines a length-2 zero-dimensional subscheme $\mathbf{p} \subset Z$: when P and Q are not squares, it is supported at *two distinct points* $\mathbf{p} = \{p_1, p_2\}$. As before, we thicken Z along $\mathcal{L}$ and consider the diagram below,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_Z(1) \arrow[r] \arrow[d, equal] & \mathcal{I}_{\mathbf{p}/C} \arrow[r] \arrow[d, hook] & \mathcal{I}_{\mathbf{p}/Z} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{O}_Z(1) \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d, two heads] & \mathcal{O}_Z \arrow[r] \arrow[d, two heads] & 0 \\
& & \mathcal{O}_{\mathbf{p}} \arrow[r, equal] & \mathcal{O}_{\mathbf{p}} &
\end{tikzcd} \tag{3.22}$$

construct again stable pair $0 \to \mathcal{O}_C \to \mathcal{F}$ using Proposition 1.3.5. Here $\mathcal{F} \simeq \mathcal{H}om_C(\mathcal{I}_{\mathbf{p}/C}, \mathcal{O}_C)$, and $\chi(\mathcal{F}) = 5$.

Proposition 3.4.7. *The Zariski tangent space to the moduli scheme $\mathcal{N}_{2,5}$, at any point of the stratum $\mathbf{G}$, has dimension 7.*

Proof. We exploit the machinery of the 3-fold vertex [PT09b].

Torus-fixed stable pairs in the stratum $\mathbf{G}$ correspond to box configurations with vertex datum of length 2 and height 1: there are six such configurations, with shapes

$$[1,1\,|\,0]\,, \quad [1\,|\,1]\,, \quad [0\,|\,1,1]\,,$$

where the first and last shapes correspond to points in $\mathbf{F}$. At all of these points the Zariski tangent space has dimension 7, as shown in Appendix B. □

Remark 3.4.8. Notice that this proof is independent of the quiver description, but it yields the same result as Theorem 4.4.18. ♠

3.4.3 Case 3: P, Q proportional and squares

This is the innermost stratum $\mathbf{F} \subset \mathbf{G} \subset \mathbf{H} \subset \mathbb{P}^5$, where an interesting behaviour is exhibited. We regard $\mathbf{F}$ as the image of the second Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^2$ given by $[u_0 : u_1] \mapsto [u_0^2 : 2u_0u_1 : u_1^2]$, that is

$$\mathbf{F} \simeq \mathbb{P}^1_{[x_0:y_0]} \times \mathbb{P}^1_{[u_0:u_1]} \subset \mathbb{P}^1_{[x_0:y_0]} \times \mathbb{P}^2_{[x_0:x_1:x_2]} \simeq \mathbf{G}\,.$$

Like before, the map α factors as

$$\begin{tikzcd}
\mathcal{O}_Z(-3) \arrow[rr, hook, "\alpha"] \arrow[dr, hook, "\alpha'"'] & & \mathcal{O}_Z(-1)^{\oplus 2} \\
& \mathcal{O}_Z(-1) \arrow[ur, hook, "l"'] \arrow[dr, two heads] & \\
& & \mathcal{O}_{\mathbf{q}}(-1)\,,
\end{tikzcd} \tag{3.23}$$

with a line bundle $\mathcal{L} \simeq \mathcal{O}_Z(-1)$ given by l, and a zero-dimensional length-2 subscheme of Z defined by the vanishing of α', which consists of a double point. Clearly the construction of Section 3.4.2 could be carried out, but there are other stable pairs that we are missing in that way.

Consider a length-2 zero-dimensional subscheme $\mathbf{q}$ *of* C, which is not a subscheme of Z. In that case, we get the diagram

$$\begin{tikzcd} & & \mathcal{J}_{\mathbf{q}} \arrow[d, hook] & & \\ 0 \arrow[r] & \mathcal{O}_Z(1) \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d, two heads] & \mathcal{O}_Z \arrow[r] & 0 \\ & & \mathcal{O}_{\mathbf{q}}\,. & & \end{tikzcd} \tag{3.24}$$

Consider the stable pair constructed following Proposition 1.3.5, with $\mathcal{F} := \mathcal{H}om_C(\mathcal{I}_{\mathbf{q}/C}, \mathcal{O}_C)$, which again satisfies $\chi(\mathcal{F}) = 5$ but does not come from any map α alone. Clearly, if we perform the construction above with $\mathbf{q} \subset Z$ it is possible to complete the top right square of (3.24), reverting back to Case 2 in Section 3.4.2.

There is a geometrical interpretation for this stratum of $\mathcal{N}_{2,5}$. In order to produce a stable pair as above, we need to choose

- one thickening line bundle $\mathcal{L} \simeq \mathcal{O}_Z(-1) \subset \mathcal{O}_Z(-1)^{\oplus 2}$,
- a reduced point $q \in Z$,
- a non reduced subscheme $\mathbf{q} \subset C$, of length 2, such that $\mathbf{q}_{\mathrm{red}} = q$.

All choices contribute a $\mathbb{P}^1$ worth of moduli; for the first two this is obvious. Recalling Section 1.1.2 and in particular Corollary 1.1.23 on the thickening of points, the choice of a non reduced structure on q amounts to finding a direction $\mathcal{R}$ inside $\mathrm{N}_{q/C}$, which following Corollary 1.1.18 is just the rank-2 vector space

$$\mathrm{N}_{q/C} \simeq \mathrm{T}_{Z,q} \oplus \mathcal{L}_q\,.$$

Therefore, the choice of $\mathbf{q}$ depends on an extra $\rho \in \mathbb{P}^1$ inside the moduli space.

Let us denote with $\rho = 0$ the choice of the first summand above. In such case, any stable pair constructed via a diagram (3.24) could deform to a stable pair in $\mathbf{G}$, that is with its zero-locus comprising two distinct points. However, if $\rho \neq 0$ certain deformations are obstructed, specifically those that would deform towards a length-2 zero-dimensional reduced subscheme *of* Z: in other words, those deformations where a double point in the zero-locus splits in two. A clearer picture is given by the diagram below, which shows the interplay between the thickening of Z along $\mathcal{L} \simeq \mathcal{O}_Z(-1)$ and the thickening of q inside C, and how $\mathcal{I}_{\mathbf{q}}$ is constructed.

$$
\begin{tikzcd}
 & & \mathcal{I}_{\mathbf{q}} \arrow[dd, hook] \arrow[dr, equal] & & & & \\
 & & & \mathcal{I}_{\mathbf{q}} \arrow[dd, hook] & & & \\
\mathcal{L}^\vee \arrow[rr, hook] \arrow[dr, equal] & & \mathcal{I}_{q/C} \arrow[rr, two heads] \arrow[dr, hook] \arrow[dd, two heads] & & \mathcal{I}_{q/Z} \arrow[dr, hook] & & \\
 & \mathcal{L}^\vee \arrow[rr, hook] & & \mathcal{O}_C \arrow[rr, two heads] \arrow[dr, two heads] \arrow[dd, two heads] & & \mathcal{O}_Z \arrow[dr, two heads] & \\
 & & \mathcal{R}^\vee \arrow[dr, hook] & & \mathcal{O}_q \arrow[rr, equal] \arrow[dd, equal] & & \mathcal{O}_q \\
 & & & \mathcal{O}_{\mathbf{q}} \arrow[dr, two heads] & & & \\
 & & & & \mathcal{O}_q & &
\end{tikzcd}
\tag{3.25}
$$

There is an extra component $\tilde{\mathbf{F}} \subset \mathcal{N}_{2,5}$, parametrising stable pairs of the form $\mathcal{F} = \mathcal{J}_{\mathbf{q}}(2)$ above, to be described in Section 3.4.5 and Section 3.4.6 below.

3.4.4 Local affine construction of families over $\mathbb{P}^5$

Before turning our attention to the extra component $\tilde{\mathbf{F}}$, let us construct explicitly a family of stable pairs over $\mathbb{P}^5$. We proceed as in section Section 3.3.2, from which we inherit notation and techniques. Consider, without loss of generality, the affine patch $U_0 = \{\zeta_0 \neq 0\} \subset X$, with affine coordinates X_0, Y_0, Z_0, and polynomials (3.17) dehomogenised to P_0 and Q_0. Setting $R = \mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2]$ and $S = R[X_0, Y_0, Z_0]$, we take the matrix

$$
M_{P_0,Q_0} := \begin{pmatrix} X_0^2 & X_0Y_0 & Y_0^2 & Q_0X_0 - P_0Y_0 & X_0 & Y_0 \\ 0 & 0 & 0 & 0 & P_0 & Q_0 \end{pmatrix} \tag{3.26}
$$

and define the S-module $F_0 = \operatorname{coker} M_{P_0,Q_0}$. This module has two generators σ_1 and σ_2, and it is endowed with a section s_0 defined as in (3.11), corresponding to the assignment $1 \to \sigma_1$. This section makes (F_0, s_0) a family of stable pairs over the base $\mathbb{P}^5 = \operatorname{Proj} R$, by Remark 3.3.8. On the affine chart $U_1 = \{\zeta_1 \neq 0\}$, we construct the corresponding stable pair (F_1, s_1), which glues to (F_0, s_0) yielding a family

$$
\mathcal{O}_{\mathbb{P}^5 \times X} \xrightarrow{s} \mathcal{F} .
$$

Remark 3.4.9. Checking that this is a well defined family, flat over $\mathbb{P}^5$, such that each fibre is a stable pair on X, is routine and performed as in Section 3.3.2. The outstanding check concerns a new phenomenon, that is limits in which two points in the zero-locus converge to one. We address that case at the end of this section. ♠

We show how the stable pair $(\mathcal{F}, s)$ agrees with the sheaf-theoretic constructions performed in the previous sections. For generic coprime (P, Q), the affine fibre satisfies

$$\operatorname{Ann}_S F_{(P_0,Q_0)} = (X_0^2, X_0Y_0, Y_0^2, Q_0(Z_0)X_0 - P_0(Z_0)Y_0)\,.$$

If P and Q share a non-trivial common factor L, then the ideal above equates

$$(X_0^2, X_0Y_0, Y_0^2, \frac{1}{L_0}(Q_0X_0 - P_0Y_0))\,,$$

where we slightly abused notation to mean the reduced polynomials. That is, at every fibre, the support of $\mathcal{F}_{(P,Q)}$ is the nilpotent thickening of the zero-section Z determined by the reduced polynomials.

Consider now the section s. In the affine chart U_0 we compute

$$\operatorname{Ann}_S \operatorname{coker} s = (X_0, Y_0, P_0, Q_0, \left|\begin{smallmatrix} x_0 & x_2 \\ y_0 & y_2 \end{smallmatrix}\right| Z_0 + \left|\begin{smallmatrix} x_1 & x_2 \\ y_1 & y_2 \end{smallmatrix}\right|, \left|\begin{smallmatrix} x_0 & x_1 \\ y_0 & y_1 \end{smallmatrix}\right| Z_0 + \left|\begin{smallmatrix} x_0 & x_2 \\ y_0 & y_2 \end{smallmatrix}\right|, \operatorname{Res}(P_0, Q_0))\ ,$$

where Res is the resultant (3.18). This shows that the zero-locus of a fibre $(\mathcal{F}_{(P,Q)}, s)$ is non-empty only when $\operatorname{Res}(P, Q) = 0$ is not invertible, that is P and Q share a non-trivial factor L. In such case, the zero-locus of the pair has ideal $(L) \subset \mathcal{O}_Z$.

It is worth investigating more in details what happens when P, Q have two common factors, that is they share all roots. We work in an affine chart, and not to overburden notation we will simply call the coordinates x, y, z. By enumerative arguments, in this case the Cohen-Macaulay support is a *flat ribbon*, that is locally given by equations $(x^2, xy, y^2, bx-ay)$ with $[a\colon b] \in \mathbb{P}^1$. Up to a change of coordinates, we assume that this support curve has ideal (x, y^2), and that the two points in the zero-locus of the stable pair are at $z = 0$ and $z = t$.

Consider the matrix

$$M_t := \begin{pmatrix} x & y^2 & 0 & 0 & y \\ 0 & 0 & x & y & tz - z^2 \end{pmatrix} \tag{3.27}$$

and define $F_t := \operatorname{coker} M_t$, which is a module over $\mathbb{C}\,[t, x, y, z]$.

Remark 3.4.10. Of course, the matrix (3.27) is obtained from (3.26), setting $Q_0 = z(t-z)$ and $P_0 = 0 \cdot Q_0 = 0$, and rearranging the relations. Calling σ_1 and σ_2 the generators of the cokernel, we can replace the relations $x^2\sigma_1 = 0$ and $xy\sigma_1 = 0$, with $x\sigma_2 = y\sigma_2 = 0$. Hence $F_t = F_{(0,z(t-z))}$. ♠

This presentation showcases how σ_1 is supported at the prescribed Cohen-Macaulay curve, while σ_2 at the reduced underlying curve. Figure 3.3 depicts the situation.

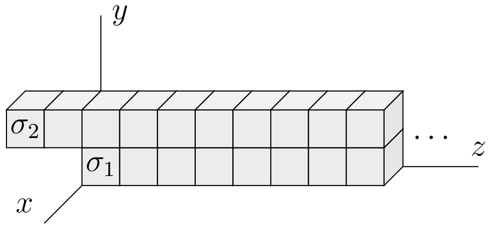


Figure 3.3: Intuition behind the module F_t, with generators $\sigma_1 = 1$ and $\sigma_2 = y/(z^2 - tz)$. There is no correspondence between boxes and monomials.

It is immediate to show that $\operatorname{Ann} F_t = (x, y^2)$, and that given the section $s\colon \mathbb{C}(t,x,y,z) \to F_t$ such that $s(1) = \sigma_1$, we have $\operatorname{Ann}\operatorname{coker} s = (x, y, z(t-z))$. This family is flat over $t = 0$, as can be easily checked computing a free resolution of F_t and applying Lemma 1.3.17.

3.4.5 Local affine construction of families over $\tilde{\mathrm{F}}$

Further to the previous section, we begin by understanding the affine presentation of stable pairs supported at a flat ribbon C, with zero-locus consisting of a *double point*. Again, up to a change of coordinates we can assume that the ribbon has equations (x, y^2), and that the reduced zero-locus has maximal ideal $\mathfrak{m}(x, y, z)$.

Take $\mathfrak{m}^2 = (yz, z^2) \subset \mathcal{O}_C$. Following the proof of Proposition 1.3.4, we need consider subsheaves $\mathcal{F} \subset \mathcal{H}om(\mathfrak{m}^2, \mathcal{O}_C)$: hence, we are interested in sub-modules of $G = \operatorname{Hom}(\mathfrak{m}^2, \mathrm{H}^0(\mathcal{O}_C))$. This module G has generators below:

$$\sigma_1\colon \begin{pmatrix} x & \mapsto & 0 \\ yz & \mapsto & yz \\ z^2 & \mapsto & z^2 \end{pmatrix} \qquad \sigma_2\colon \begin{pmatrix} x & \mapsto & 0 \\ yz & \mapsto & 0 \\ z^2 & \mapsto & y \end{pmatrix} \qquad \sigma_3\colon \begin{pmatrix} x & \mapsto & 0 \\ yz & \mapsto & y \\ z^2 & \mapsto & z \end{pmatrix}.$$

Remark 3.4.11. Notice that the set of generators above is not minimal, as, for example $\sigma_1 = z\sigma_3$. However, including the identity σ_1 among the generators is useful when defining a section for the stable pair. We can visualise the generators of G in Figure 3.4, from which it is easy to read off relations, obtaining an explicit presentation.. ♠

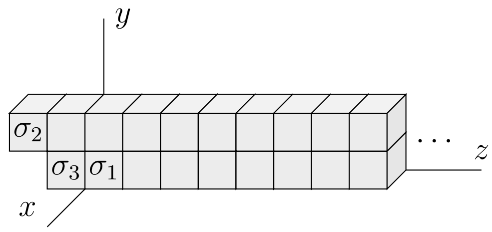


Figure 3.4: The module $\mathrm{Hom}(\mathfrak{m}^2, \mathcal{O}_C)$, with generators $\sigma_1 = 1$, $\sigma_2 = y/z^2$, and $\sigma_3 = 1/z$.

We construct $F_u := \langle \sigma_1, \sigma_2 + u\sigma_3 \rangle_T \subset G$, where $T = \mathbb{C}[x,y,z]$, and view F_u as a module over $\mathbb{C}[u,x,y,z]$. Starting from a presentation of G, one obtains that $F_u = \operatorname{coker} M_u$, with M_u the matrix below:

$$M_u := \begin{pmatrix} x & y^2 & 0 & u^2 & y+uz & uy & ux \\ 0 & 0 & x & y-uz & -z^2 & -yz & -xz \end{pmatrix}. \tag{3.28}$$

Given the section $s\colon 1 \mapsto \sigma_1$, it is routine to check that (F_u, s) is a flat family of stable pairs over the base $\mathbb{A}^1_u$. At each fibre, the support is $\operatorname{Ann} F_u = (x, y^2)$, while the zero locus has equation

$$\operatorname{Ann}\operatorname{coker} s = (x, y^2, yz, z^2, uz - y),$$

that is a double point, a thickening of the origin, sitting inside C.

Remark 3.4.12. In fact, it is straightforward to construct a flat family over the base $\mathbb{P}^1 = \operatorname{Proj}\mathbb{C}[u,v]$, by considering $F_{[u:\,v]} := \langle \sigma_1, v\sigma_2 + u\sigma_3 \rangle$ with the same section $s\colon 1 \mapsto \sigma_1$. In particular, for $v = 0$, we will say that the corresponding zero locus is *vertical*, while for $u = 0$ it is *horizontal*. ♠

Now, we clearly have that setting $t = 0$ in (3.27) or $u = 0$ in (3.28) yield the same presentation, that is that of $F_0 = \langle \sigma_1, \sigma_2 \rangle = \operatorname{coker} M_0$. We can harmonise the two by working over $\mathbb{C}[t,u,x,y,z]$ and considering the matrix

$$M_{t,u} := \begin{pmatrix} x & y^2 & 0 & u^2 & y+uz & uy & ux \\ 0 & 0 & x & y-uz & z(t-z) & -yz & -xz \end{pmatrix}. \tag{3.29}$$

The corresponding module $F_{t,u} := \operatorname{coker} M_{t,u}$ is supported at

$$\operatorname{Ann} F_{t,u} = (x, y^2, tu \cdot yz, tu^2 \cdot z),$$

hence we must require the incidence condition $tu = 0$. We have that $F_t = F_{t,0}$ and $F_u = F_{0,u}$, meaning that $F_{t,u}$ is a flat module when restricted to each component of the reduced scheme $B = \operatorname{Spec}\mathbb{C}[t,u]/(tu)$. This immediately entails flatness over all closed points, except for the case $t = u = 0$ which needs more attention. Here, we apply the valuative criterion for flatness (see for example [EGA-IV.3, Théorème 11.8.1]), showing that $F_{t,u}$ is flat over B.

Remark 3.4.13. We cannot promote $F_{t,u} = \operatorname{coker} M_{t,u}$ as in (3.29) to a family of stable pairs over $\mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[t,u]$. If $tu \neq 0$, then $F_{t,u}$ is supported at (x, y^2, z), and it fails to be 1-dimensional. ♠

In this section, we assumed our Cohen-Macaulay support to be fixed, as well as the reduced zero-locus. However, we repeat the construction above for any choice of flat ribbon $(x^2, xy, y^2, x - ay)$ (depending on $a \in \mathbb{A}^1$) and any choice of reduced support $\mathfrak{m} = (x, y, z - p)$ (depending on $p \in Z$) with a simple change of coordinates, recovering families over $\mathbb{A}^1_a \times Z \times \operatorname{Spec} \mathbb{C}[t,u]/(tu)$.

However, this is not enough to describe the geometry of the extra component $\tilde{\mathbf{F}}$, as we want to consider ribbons $(x^2, xy, y^2, bx - ay)$, depending on $[a\colon b] \in \mathbb{P}^1$.

3.4.6 The extra component $\tilde{\mathbf{F}}$, globally

We saw that, over $\mathbf{F}$, locally the construction of a stable pair on this component amounts to choosing an extra $\mathbb{P}^1$ worth of parameters, however the global geometry of $\tilde{\mathbf{F}}$ is not just $\mathbf{F} \times \mathbb{P}^1$. Recall that $\mathbf{F} \simeq \mathbb{P}^1 \times Z \ni ([l], q)$, with $[l]$ giving the thickening C and q the reduced support of the stable pair; if we call $\Lambda := \mathbb{P}^1 \ni [l]$ there is a map

$$\tilde{\mathbf{F}} \longrightarrow \Lambda \times Z \simeq \mathbf{F}$$

sending a stable pair to the thickening data $[l]$ and the location of its reduced support: the fibre of this map over a point $([l], q)$ is clearly $\mathbb{P}(\mathcal{N}_{q/C}) \simeq \mathbb{P}^1$. There is a section

$$\tilde{\sigma}\colon \Lambda \times Z \simeq \mathbf{F} \longrightarrow \tilde{\mathbf{F}}$$

which sends the data $([l], q)$ to the stable pair whose support C is thickened along $[l]$ and with zero locus given by a double point $\mathbf{q}$ inside Z, such that $(\mathbf{q})_{\text{red}} = q$.

In order to determine the global geometry, we begin by fixing $[l] \in \Lambda = \mathbb{P}^1$ and the corresponding double thickening C of Z. Recall that we are constructing a Ferrand doubling along the surjection

$$\lambda\colon \mathcal{C}_{Z/X} \simeq \mathcal{O}_Z(1)^{\oplus 2} \twoheadrightarrow \mathcal{K} \simeq \mathcal{O}_Z(1),$$

where λ is the dual of the map l appearing in (3.23).

Keeping the support of the stable pair C fixed and given by $[l] \in \Lambda = \mathbb{P}^1$, a stable pair is determined by the choice of the data $(q, \mathcal{R})$, with $\mathcal{R} \subset \mathcal{N}_{q/C}$ a line. From the sequence

$$0 \longrightarrow \mathcal{C}_{Z/C} \longrightarrow \Omega_{C|Z} \longrightarrow \Omega_Z \longrightarrow 0$$

as $\mathrm{Ext}^1(\mathcal{O}_Z(-2), \mathcal{O}_Z(1)) = \mathrm{H}^1(\mathcal{O}_Z(3)) = 0$, we have the splitting

$$\Omega_{C|Z} = \mathcal{C}_{Z/C} \oplus \Omega_Z \,.$$

Globally, over this fixed $[l]$, the fibre of $\tilde{\mathbf{F}}$ is then

$$\mathbb{P}\left(\mathcal{K} \oplus \Omega_Z\right) \simeq \mathbb{P}\left(\mathcal{O}_Z(1) \oplus \mathcal{O}_Z(-2)\right) \simeq \mathbb{P}\left(\mathcal{O}_Z(3) \oplus \mathcal{O}_Z\right) =: \mathbb{F}_3 \tag{3.30}$$

where $\mathbb{F}_3$ is the *third Hirzebruch surface*. The second isomorphism appearing in (3.30) follows from the fact that, given a vector bundle E and a line bundle L on X, then $\mathbb{P}(L \otimes E) \simeq \mathbb{P}(E)$. There are two distinguished sections

$$\begin{tikzcd} \mathbb{F}_3 \arrow[d] \\ Z \arrow[u, bend left=40, "\sigma_\infty"] \arrow[u, bend right=40, "\sigma_0"'] \end{tikzcd}$$

corresponding to choosing the first or second sub-bundle, respectively, in the decomposition $\mathcal{O}_Z(1) \oplus \mathcal{O}_Z(-2)$. Choosing the $\mathcal{O}_Z(-2)$ summand gives a length-2 subscheme $\mathbf{q}$ in Z, while choosing the other one gives a *vertical* subscheme of C, which is to say a length-2 subscheme $\mathbf{q} \subset C$ such that $\mathrm{T}_{\mathbf{q},q} \simeq \mathcal{N}_{Z/C,q}$.

We aim now to understand the fibre of $\tilde{\mathbf{F}}$ over $q \in Z$, which amounts to consider stable pairs with a double marked point supported at q. The construction is local, so we take an affine chart $\mathbb{A}^3$ around q and consider $V = \mathrm{T}_q\mathbb{A}^3$ a three-dimensional vector space with a fixed one-dimensional subspace U: q is at the origin, and the special line is the restriction of $\mathrm{T}Z$ to the affine chart. In this setup, a thickening of Z along a line bundle $\mathcal{L} \simeq \mathcal{O}_Z(-1)$ amounts to choose a plane $\Pi \subset V$ containing U; on the other hand, a double point supported at q comes from the datum of a line in $L \subset \Pi$.

Let $\mathbb{P}(V) \simeq \mathbb{P}^2$ be the space of lines in V, and calling $V^\vee$ the dual, $\mathbb{P}(V^\vee) \simeq \mathbb{P}^2$ is the space of planes in V. Now, we can consider the full two-step flag variety

$$\mathrm{Fl}(1,2) \subset \mathbb{P}(V) \times \mathbb{P}(V^\vee) \simeq \mathbb{P}^2 \times \mathbb{P}^2 \,,$$

which is a threefold. The plane Π contains U if and only if the linear functional defining Π vanishes on U; in other words, the space of planes $\Pi \subset V$ such that

$\Pi \supset U$ is given by $\mathbb{P}(\operatorname{Ann} U) \subset \mathbb{P}(V^\vee)$. Hence, the space of flags $L \subset \Pi$ subject to the incidence condition $U \subset \Pi$ is given by

$$\mathrm{Fl}_U(1,2) := \mathrm{Fl}(1,2) \cap (\mathbb{P}(V) \times \mathbb{P}(\operatorname{Ann} U)) \subset \mathbb{P}(V) \times \mathbb{P}(V^\vee)\,.$$

We have two projection maps, sending a flag to its components:

$$\tau_{\text{line}} \colon \mathrm{Fl}_U(1,2) \longrightarrow \mathbb{P}(V) \simeq \mathbb{P}^2 \quad \text{and} \quad \tau_{\text{plane}} \colon \mathrm{Fl}_U(1,2) \longrightarrow \mathbb{P}(\operatorname{Ann} U) \simeq \mathbb{P}^1\,. \tag{3.31}$$

Notice that the map τ_{plane} is a $\mathbb{P}^1$-bundle: the fibre over a point is the set of lines contained in a plane; moreover, there is a tautological section

$$\upsilon \colon \mathbb{P}^1 \longrightarrow \mathrm{Fl}_U(1,2)\,, \tag{3.32}$$

which corresponds to choosing a plane Π and taking the flag $\{U \subset \Pi\}$. On the other hand, the map τ_{line} is birational: if $L \in \mathbb{P}(V)$ is different from U, then L and U span a unique plane, that is the fibre of τ_{line} over L is just a point. However, the fibre of τ_{line} over the chosen $U \in \mathbb{P}(V)$ is the whole $\mathbb{P}(\operatorname{Ann} U) \simeq \mathbb{P}^1$, and this fibre is precisely the image of the section υ above.

Remark 3.4.14. The choice of a flag $L \subset \Pi$, subject to the incidence $U \subset \Pi$ around $q \in Z$ determines a stable pair on the entire resolved conifold X. By construction the reduced zero-locus is supported at q, and the plane Π determines a flat Cohen-Macaulay ribbon. ♠

We conclude that $\mathrm{Fl}_U(1,2) \simeq \mathbb{F}_1$, where $\mathbb{F}_1 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ is the first Hirzebruch surface. The map τ_{plane} is just the bundle map $\mathbb{F}_1 \to \mathbb{P}^1$, while the section υ is the choice of the $\mathcal{O}(-1)$-summand.

After this local construction, we can understand that the fibre of the extra component $\tilde{\mathbf{F}}$ over $q \in Z$ is precisely $\mathbb{F}_1$. To probe the geometry of this $\mathbb{F}_1$-fibration for varying q, notice that the normal bundle to a reduced point satisfies $\mathcal{N}_{q/X} \simeq \mathcal{N}_{Z/X,p} \oplus \mathrm{T}_{Z,p} \simeq \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z(2)$. Hence, we can repeat the procedure above defining a *flag bundle*, proving the following.

Proposition 3.4.15. *Consider the rank-*3 *bundle* $\mathcal{E} := \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z(2)$ *on* $Z \simeq \mathbb{P}^1$*. The extra component* $\tilde{\mathbf{F}}$ *is isomorphic to* $\underline{\mathrm{Fl}}_Z(1,2,\mathcal{E}) \to Z$*, which is the* flag bundle *whose fibres are flags varieties* $\mathrm{Fl}_U(1,2)$ *as above. The flags* $\{L \subset \Pi\} \in \mathrm{Fl}_U(1,2)$ *satisfy the incidence condition* $\Pi \supset U$*, where the special 1-dimensional subspace* U *corresponds to the* $\mathcal{O}(2)$*-summand in* $\mathcal{E}$*.*

The maps τ_{plane} appearing in (3.31) *glue to give a map $\underline{\mathrm{Fl}}_Z(1,2,\mathcal{E}) \to \Lambda \simeq \mathbb{P}^1$, and by universal property there is a projection*

$$\underline{\mathrm{Fl}}_Z(1,2,\mathcal{E}) \longrightarrow \Lambda \times Z\,, \tag{3.33}$$

which is an $\mathbb{F}_3$-fibration on Λ and an an $\mathbb{F}_1$-fibration on Z, where $\mathbb{F}_k$ is the k-th Hirzebruch surface. Finally, there is is a section

$$\tilde{\sigma}\colon \Lambda \times Z \longrightarrow \underline{\mathrm{Fl}}_Z(1,2,\mathcal{E}) \tag{3.34}$$

obtained by gluing the maps υ appearing in (3.32) *above.*

In addition to the discussion so far, we have the result below.

Proposition 3.4.16. *The Zariski tangent space to the moduli scheme $\mathcal{N}_{2,5}$, at the generic point of $\tilde{\mathbf{F}} \setminus \mathbf{F}$, has dimension* 3*.*

Proof. We proceed as in the proof of Proposition 3.4.7. We are looking for box configurations with vertex datum of length 2 and height 2: there are four such configurations, corresponding to shapes

$$[2\,|\,0]\,,\quad [0\,|\,2]\,,$$

and at these points the Zariski tangent space has dimension 3, as shown in Appendix B. ∎

3.4.7 Global picture

We summarise below the geometry of $\mathcal{N}_{2,5}$. There is a stratification

$$\begin{tikzcd}
 & & & \tilde{\mathbf{F}} \\
\mathbb{P}^5 \arrow[d, phantom, "\cong" rotate=-90] & \mathbf{H} \arrow[l, hook'] & \mathbf{G} \arrow[l, hook'] \arrow[d, phantom, "\cong" rotate=-90] & \mathbf{F} \arrow[l, hook'] \arrow[u, hook] \arrow[d, phantom, "\cong" rotate=-90] \\
\mathbb{P}\mathrm{H}^0\left(\mathcal{O}_Z(-1)^{\oplus 2}\right) & & \mathbb{P}^2 \times \mathbb{P}^1 & \mathbb{P}^1 \times \mathbb{P}^1
\end{tikzcd}$$

where in terms of the coordinates $[x_0 : x_1 : x_2 : y_0 : y_1 : y_2]$ on $\mathbb{P}^5$

- $\mathbf{H}$ is given by the quartic equation $\begin{vmatrix} x_0 & x_1 & x_2 & 0 \\ 0 & x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 & 0 \\ 0 & y_0 & y_1 & y_2 \end{vmatrix} = 0,$

- $\mathbf{G}$ is given by the quadric equations $\operatorname{rk}\begin{pmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{pmatrix} \leq 1$,
- $\mathbf{F}$ is cut out by the additional quadric equation $x_1^2 - 4x_0x_2 = 0$.

Concerning the maps,

- the inclusion $\mathbf{G} \simeq \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ is the second Veronese embedding
$$([\lambda_0 : \lambda_1], [\mu_0 : \mu_1 : \mu_2]) \mapsto [\lambda_0\mu_0 : \lambda_0\mu_1 : \lambda_0\mu_2 : \lambda_1\mu_0 : \lambda_1\mu_1 : \lambda_1\mu_2] ,$$
- the inclusion $\mathbf{F} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^2 \simeq \mathbf{G}$ is given by
$$([\lambda_0 : \lambda_1], [u_0 : u_1]) \longmapsto \left([\lambda_0 : \lambda_1], \left[u_0^2 : 2u_0u_1 : u_2^2\right]\right) ,$$
corresponding to squaring a linear form
$$(u_0X_0 + u_1X_1)^2 = u_0^2X_0^2 + 2u_0u_1X_0X_1 + u_1^2X_1^2$$
on the base $Z \simeq \mathbb{P}^1$;
- the inclusion $\tilde{\sigma} \colon \mathbf{F} \hookrightarrow \tilde{\mathbf{F}}$ is given by sending $([l], q)$ to the stable pair whose support C is thickened along $[l]$ and with zero locus given by a double point $\mathbf{q}$ inside Z, such that $(\mathbf{q})_{\mathrm{red}} = q$.

In order to complete the description, we have to show that there are no other stable pairs in $\mathcal{N}_{2,5}$ which we are excluding. But the extra tangent directions along $\mathbf{H}$ and $\mathbf{G}$ are obstructed: those directions correspond to moving the point(s) in the zero-locus in the nilpotent direction of the Cohen-Macaulay support. These deformations cannot be realised as families over a reduced positive-dimensional basis: see for reference Remark 3.4.13.

Along the component $\mathbf{F}$, where the zero-locus consists of a double point in Z, one of the extra tangent directions in the moduli space corresponds to rotating the zero-locus inside the Cohen-Macaulay support: these are precisely the stable pairs parametrised by $\tilde{\mathbf{F}}$. Finally, along $\tilde{\mathbf{F}}$ we have Zariski tangent space of dimension 3, so there are no more stable pairs there either.

Hence there are no more deformation directions that we have not accounted for, and there are no more components since there are no other ways to thicken the base curve Z. This concludes the proof of Theorem 3.4.1.

We can easily check that the geometric description here provided agrees with the expected numerical PT invariant, which should be $P_{2,5} = -10$. Considering the natural torus action, there are in total 10 fixed points $p_1, \dots, p_{10}$. Two fixed points p_1, p_2 correspond to stable pairs whose zero locus consists of two distinct reduced points, and so $p_1, p_2 \in \mathbf{G} \setminus \mathbf{H}$. Eight more fixed points correspond to stable pairs with non-reduced zero-locus. Of these, four points $p_3, \dots, p_6$ are in $\mathbf{F}$, and are such that the zero-locus is scheme-theoretically contained in the reduced zero-section $Z \simeq \mathbb{P}^1$ (in other words, they are *horizontal*). The remaining four points $p_7, \dots, p_{10} \in \tilde{\mathbf{F}}$ correspond to stable pairs with *vertical* zero-locus.

All these torus-fixed points have odd-dimensional Zariski tangent space, as shown in Appendix B: for $p_1, \dots, p_6$ it is 7-dimensional, and for $p_7, \dots, p_{10}$ it is 3-dimensional. We then compute, using the virtual localization formula:

$$P_{5,2} = \sum_{i=1}^{10} (-1)^{\dim T_{p_i} \mathcal{N}_{2,5}} = -10\,,$$

in accordance with the partition function (3.2).

3.4.8 Scheme-theoretic remarks

Similarly to the discussion at the end of Section 3.3.2, the constructions in Section 3.4.4 and Section 3.4.5 amount to providing two maps

$$\varphi_1 \colon \mathbb{P}^5 \to (\mathcal{N}_{2,5})_{\mathrm{red}} \quad \text{and} \quad \varphi_2 \colon \tilde{\mathbf{F}} \to (\mathcal{N}_{2,5})_{\mathrm{red}}\,.$$

Proposition 3.4.17. *We have that* $\mathrm{Im}(\varphi_1) \cup \mathrm{Im}(\varphi_2) = (\mathcal{N}_{2,5})_{\mathrm{red}}$

Proof. Consider a stable pair $(\mathcal{F}, s)$ with reduced support $Z \in X$, scheme theoretic support C a double Cohen-Macaulay curve, and $\chi(\mathcal{F}) = 5$. We know that C must be a thickening of Z along a line bundle, which is necessarily $\mathcal{O}_Z(-d)$ for $d = 1, 2, 3$. The numerical requirement $\chi(\mathcal{F}) = 5$ forces the zero-locus to contain two, one, or zero points respectively.

We can rely on Proposition 1.3.5 to prove our claim. We know that a stable pair is completely determined by its support and zero locus: if the zero-locus is reduced, we can use the construction φ_1; while in the case where $d = 1$ and where the zero-locus is non-reduced, we use φ_2. □

Moreover, a similar argument shows the statement below.

Proposition 3.4.18. *We have that* $\varphi_{1|\mathbf{F}}$ *and* $\varphi_{2|\mathbf{F}}$ *provide pointwise isomorphic pairs. Moreover, there is no pair* $(\mathcal{F}, s)$ *in* $\mathrm{Im}\, \varphi_{1|\mathbb{P}^5 \setminus \mathbf{F}}$ *which is isomorphic to a pair in* $\mathrm{Im}\, \varphi_{2|\tilde{\mathbf{F}} \setminus \mathbf{F}}$.

From the above, we deduce that $(\mathcal{N}_{2,5})_{\mathrm{red}}$ is reducible, with two components dominated by $\mathbb{P}^5$ and $\tilde{\mathbf{F}}$, respectively. A much stronger statement is given in Theorem 2.3.22, where the moduli space $\mathcal{N}_{2,5}$ is isomorphic *as a scheme* to a certain space of representations which we describe in Section 4.4.3.

3.4.9 Reduced geometry as critical locus

We want now to investigate in detail the affine geometry of the situation above, ignoring for now the extra component $\tilde{\mathbf{F}}$. The information we have so far allows us to interpret the stratification of $\mathcal{N}_{2,5}$ by looking at the *zero locus* of a certain regular function p: this is *not* the potential q obtained via quiver techniques in the next chapter, which exhibits our moduli scheme as a *critical locus*. Indeed, the description provided below fails to capture the non-reduced geometry of our scheme, but it is nonetheless useful as a comparison to Section 4.4.3, where a q is used to perform a similar geometric analysis.

We start by considering the singularities of $\mathbf{H}$. Choose affine coordinates around $x_0 \neq 0$

$$v := \frac{x_1}{x_0}\,, \quad b := \frac{x_2}{x_0}\,, \quad c := \frac{y_1}{x_0}\,, \quad d := \frac{y_2}{x_0}\,, \quad w := \frac{y_0}{x_0}\,,$$

and call H, G, F, the affine images of $\mathbf{H}$, $\mathbf{G}$, $\mathbf{F}$ respectively. By Gaussian elimination, the equation for H becomes

$$\begin{vmatrix} 1 & v & b & 0 \\ 0 & 1 & v & b \\ w & c & d & 0 \\ 0 & w & c & d \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a & b \\ 0 & c-vw & d-bw & 0 \\ 0 & 0 & c-vw & d-bw \end{vmatrix} = 0\,.$$

Applying the invertible change of coordinates $x := c - vw$, $\xi := d - bw$, the equation above becomes

$$\begin{vmatrix} 1 & v & b \\ x & \xi & 0 \\ 0 & x & \xi \end{vmatrix} = \xi^2 + bx^2 - vx\xi = \left(\xi - \frac{vx}{2}\right)^2 + \left(b - \frac{v^2}{4}\right)x^2 = 0\,.$$

Finally, in the new coordinates $z := \xi - \frac{vx}{2}$ and $y := b - \frac{v^2}{4}$, we obtain the locus

$$\mathrm{H} = \left\{x^2y + z^2 = 0\right\} \subset \mathbb{A}^5_{(x,y,z,v,w)}\,.$$

In these coordinates, we have that

$$\begin{aligned} \mathrm{H}^{\mathrm{sing}}_{\mathrm{red}} = \{z = x = 0\} = \{x = \xi = 0\} &= \{c = vw,\, d = bw\} \\ &= \{(w,c,d) = e(1,v,b)\} = \mathrm{G} \subset \mathbb{A}^5_{(x,y,z,v,w)}\,. \end{aligned}$$

Moreover, $\mathrm{H}^{\mathrm{sing}}$ has embedded components along

$$\{x = y = z = 0\} = \left\{c = vw,\ d = bw,\ b = \frac{v^2}{4}\right\} = \mathrm{F} \subset \mathbb{A}^5_{(x,y,z,v,w)}$$

An interesting way to see this is to consider the polynomial

$$p(x, y, z, v, w) := x^2 y + z^2 \,,$$

giving the stratification

$$\begin{tikzcd}
 & & \mathbb{A}^3_{(y,v,w)} \arrow[d, phantom, "\shortparallel"] & \mathbb{A}^2_{(v,w)} \arrow[d, phantom, "\shortparallel"] \\
\mathbb{A}^5_{(x,y,z,v,w)} & \mathrm{H} \arrow[l, hook'] \arrow[d, phantom, "\shortparallel"] & \mathrm{G} \arrow[l, hook'] \arrow[d, phantom, "\shortparallel"] & \mathrm{F} \arrow[l, hook'] \\
 & \{p = 0\} & \{\mathrm{d}p = 0\}_{\mathrm{red}} &
\end{tikzcd}$$

where $\mathrm{G} = \mathrm{Sing}(\mathrm{H})$ is the singular locus of H, and F is the embedded component of $\mathrm{Crit}(p)$ as a scheme.

Remark 3.4.19. Recall that this is only a description of the *affine* and *reduced* geometry. In order to obtain a proper potential, one must employ the quiver description, as done in the next chapter. ♠

Chapter 4

Description via Quiver Representations

In this chapter, we look at $\mathcal{N}_{l,n}$ from the algebraic viewpoint: we employ the results by Nagao and Nakajima [NN11] to explicitly describe our moduli schemes as parametrising representations of the framed conifold quiver, following Theorem 2.3.22. We then show that the resulting description matches the one obtained geometrically in Chapter 3. In addition, the quiver approach allows us to describe the schemes $\mathcal{N}_{2,n}$ as global critical loci, which in turn can be used to compute cohomological Donaldson-Thomas invariants.

Szendrői [Sze12] made a conjecture about the potential for $\mathcal{N}_{2,4}$; in Section 4.3 we give a proof of that conjecture, as Theorem 4.3.6. Finally, we consider $\mathcal{N}_{2,5}$ in Section 4.4, where the key technical step is to translate the appropriate stability condition into a collection of algebraic statements: this allows us to define an open set inside $\mathcal{N}_{2,5}$, containing the all the singular strata, which admits a critical locus description, as provided in Theorem 4.4.18.

4.1 Quiver description of $\mathcal{N}_{1,n}$

We recall our setup from Section 2.3.3 and 2.3.4. Consider the framed conifold quiver, introduced in Definition 2.3.14. It has vertices $(\mathbf{0}, \mathbf{1}, \boldsymbol{\infty})$, maps $a_i \colon \mathbf{0} \to \mathbf{1}$ and $b_j \colon \mathbf{1} \to \mathbf{0}$ for $i, j = 1, 2$, and framing $\iota \colon \boldsymbol{\infty} \to \mathbf{0}$ by a 1-dimensional vector space. By taking formal partial derivatives of the Klebanov-Witten potential $W = a_1b_1a_2b_2 - a_1b_2a_2b_1$, we construct the Jacobi algebra $\tilde{A} := \mathbb{C}\tilde{Q}/\partial W$.

Fix an appropriate choice of stability parameter ζ in the PT chamber (cf. Definition 2.3.23), and dimension vector $\mathbf{d} = (n, n-l, 1)$. Then by Theorem 2.3.22, $\mathcal{N}_{l,n}$ is isomorphic, as moduli scheme, to the GIT quotient $\mathcal{R}^{\zeta}_{\mathbf{d}}(\tilde{A})$, that is the moduli space of ζ-stable $\tilde{A}$-modules with dimension vector $\mathbf{d}$. Therefore, the key step to produce coordinates for $\mathcal{N}_{l,n}$ is to understand which constraints ζ-stability imposes on modules over $\mathbb{C}\tilde{Q}$ and $\tilde{A}$.

4.1.1 $n = 2$

We can analyse this first case separately, to illustrate the main ideas developed in later sections. First of all, a ζ-stable representation $\tilde{V}$ of the Jacobi algebra $\tilde{A}$ must have dimension vector $\mathbf{v} = (1, 2, 1)$: therefore $\zeta_\infty = -2\zeta_0 - \zeta_1 < 0$, following Theorem 2.3.22. An immediate consequence is that, for example, any sub-representation with dimension vector $(1, 0, 0)$ is destabilising (that is, disproving stability), so the framing map satisfies $\iota \neq 0$. Indeed, according to Proposition 2.3.27 there are only six allowed dimension vectors, as in Figure 4.1:

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & (1,2,1) \arrow[dr, dash] \arrow[dd, dash] & \\
 & & (0,2,1) \arrow[dd, dash] \\
 & (1,2,0) \arrow[dl, dash] \arrow[dr, dash] & \\
(1,1,0) \arrow[dr, dash] & & (0,2,0) \\
 & (0,1,0) \arrow[ur, dash] &
\end{tikzcd}.$$

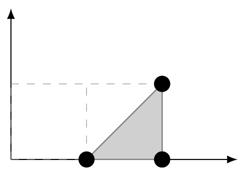


Figure 4.1: Allowed dimension vector for submodules of an $\tilde{A}$-module in $\mathcal{N}_{1,2}$.

Consider $V_b := \langle \operatorname{Im} b_1, \operatorname{Im} b_2 \rangle$: if $\dim V_b < 2$, then $(V_b, V_1, 0)$ is a destabilising subrepresentation of $\tilde{V}$. Therefore $\dim V_b = 2$, and $b_1 \oplus b_2$ is an isomorphism. Choosing a basis, and up to $\mathrm{GL}(2, \mathbb{C})$-action on V_0, we can assume that

$$b_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad a_1 = \begin{pmatrix} \alpha_1 & \beta_1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} \alpha_2 & \beta_2 \end{pmatrix}.$$

By considering the quiver potential W, we have that

$$\partial_{a_1} W = b_1 a_2 b_2 - b_2 a_2 b_1 = \left(\begin{smallmatrix} \beta_2 \\ -\alpha_2 \end{smallmatrix} \right) = 0\,,$$

and similarly for $\partial_{a_2} W$: we deduce that $a_1 = a_2 = 0$. This statement is analogous to Lemma 2.3.16, here obtained by explicit computation. The situation is shown in Figure 4.2.

The only undetermined datum is the framing map $\iota = (e_1, e_2)$, up to $\mathbb{C}^*$-action on V_∞. In conclusion, $\mathcal{N}_{1,2} \simeq \mathbb{P}^1$.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\iota"] & \mathbb{C}^2 & \mathbb{C} \arrow[l, shift right, "\binom{1}{0}"'] \arrow[l, shift left, "\binom{0}{1}"] \end{tikzcd}$$

Figure 4.2: A representation corresponding to a stable pair in $\mathcal{N}_{1,2}$.

4.1.2 General n

We consider now ζ-stable $\tilde{A}$-modules $\tilde{V}$ with dimension vector $\mathbf{v} = (n, n-1, 1)$. As in the previous special case, by Theorem 2.3.22 we have that $\zeta_\infty = -n\zeta_0 - (n-1)\zeta_1 < 0$: therefore $\iota \neq 0$, for otherwise the subrepresentation $\tilde{U} = (0, 0, V_\infty)$ would destabilise $\tilde{V}$.

Let $\tilde{U}$ be a subrepresentation of $\tilde{V}$, having dimension vector $\mathbf{u} = (d_0, d_1, 1)$: recall that we call such a subrepresentation *framed*. Proposition 2.3.27 shows that $\tilde{U}$ is allowed precisely when $d_1 \leq d_0 - 1$. Similarly, an *unframed* subrepresentation with dimension vector $(d_0, d_1, 0)$ is allowed under the same condition..

Let $\tilde{V}$ be a stable framed representation equivalent to a degree-1 stable pair on X. We can treat it as an unframed stable representation V and apply Lemma 2.3.16: clearly $V_0 \simeq V_1$ is not possible, and $b_1 = b_2$ would produce a subrepresentation with dimension vector $(0, n-1, 0)$, which is not allowed.

We are left with $a_1 = a_2 = 0$, and therefore V can be seen as a representation of the Kronecker quiver. A stable A-module is in particular indecomposable, and indecomposable representations of the Kronecker quiver are completely determined [GR97, §1.8]: of the four possibilities, only one matches our dimension vector. We summarise below.

Proposition 4.1.1. *Let $\tilde{V}$ be a $\tilde{\zeta}$-stable $\tilde{A}$-module with dimension vector $\mathbf{v} = (n, n-1, 1)$, that is, corresponding to a stable pair with reduced support.*

Up to the action of $\mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \simeq \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n-1, \mathbb{C})$, $\tilde{V}$ is isomorphic to the representation in Figure 4.3, where Id_{n-1} is the identity matrix and 0 is a zero-matrix of appropriate dimension.

We see that the only information left is carried by the framing map $\iota = (e_1, \ldots, e_n)$, up to $\mathbb{C}^*$-action, and there is an isomorphism $\mathcal{N}_{1,n} \simeq \mathbb{P}^{n-1}$.

Remark 4.1.2. With this choice of basis, we should consider the framing map ι as carrying the information on the *zero locus* of the stable pair, that is, the information on its *section*. We will see in the following that the maps $a_{1,2}$ correspond to the thickening C of the zero section Z. ♠

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\iota"] & \mathbb{C}^n & \mathbb{C}^{n-1} \arrow[l, shift right, "\begin{pmatrix} \mathrm{Id}_{n-1} \\ 0 \end{pmatrix}"'] \arrow[l, shift left, "\begin{pmatrix} 0 \\ \mathrm{Id}_{n-1} \end{pmatrix}"] \end{tikzcd}$$

Figure 4.3: A representation corresponding to a stable pair in $\mathcal{N}_{1,n}$.

4.2 $\mathcal{N}_{2,n}$, general considerations on the quiver description

The dimension vector in this case must be $\mathbf{v} = (n, n-2, 1)$. The computations needed in the preceding section can be repeated with minimal modification, yielding the following. For our convenience, we can specialise Proposition 2.3.27 below.

Corollary 4.2.1. *Let $\tilde{V}$ be a stable $\tilde{A}$-module with dimension vector as above. A subrepresentation $\tilde{U}$ with dimension vector (d_0, d_1, d_∞) is allowed precisely in the following cases:*

- *$d_\infty = 0$ (unframed) and $d_1 < d_0$,*
- *$d_\infty = 1$ (framed) and $d_1 \leq d_0 - 2$.*

Remark 4.2.2. Clearly we could just require $d_1 + d_\infty < d_0$. This is more elegant, but less immediate when graphically checking dimension vectors. Moreover, it follows the same notation as Proposition 2.3.27. ♠

We summarise some consequences below.

Proposition 4.2.3. *Let us define $V_b := \langle \operatorname{Im} b_1, \operatorname{Im} b_2 \rangle \subset V_0$. One of the following must hold:*

Case 1) $V_0 \simeq V_b$,

Case 2) $V_0 \simeq V_b \oplus \operatorname{Im} \iota$.

Proof. We have the following simple statements.

- It must be that $\iota \neq 0$, for otherwise $\tilde{U} = (0, 0, V_\infty)$ would destabilise $\tilde{V}$.

- It cannot happen that $b_1 = b_2 = 0$, since $\tilde{U} = (0, V_1, V_\infty)$ would destabilise.

- If $\dim V_b < n$, then $\operatorname{Im} \iota \subset V_b$ cannot happen, for otherwise $\tilde{U} = (V_b, V_1, V_\infty)$ would be destabilising.

- If we define $V_{\iota,b} := \langle \operatorname{Im} \iota, \operatorname{Im} b_1, \operatorname{Im} b_2 \rangle \subset V_0$; then $V_{\iota,b} \simeq V_0$, otherwise we would have $\tilde{U} = (V_{\iota,b}, V_1, V_\infty)$ destabilising.

$\square$

We will also need the following.

Proposition 4.2.4. $\operatorname{Im} \iota \not\subseteq \ker a_1 \cap \ker a_2$.

Proof. Otherwise we would have a destabilising submodule $(\operatorname{Im} \iota, 0, V_\infty)$. $\square$

4.3 Quiver description of $\mathcal{N}_{2,4}$

We work out the details of the particular value $l = 4$, without relying on the general theory, and proceeding by cases. Let $\tilde{V} = (V_0, V_1, V_\infty)$ where $V_0 \simeq \mathbb{C}^4$ and $V_1 \simeq \mathbb{C}^2$. Allowed dimension vectors are as in Figure 4.4, which follows the same conventions as Figure 2.9.

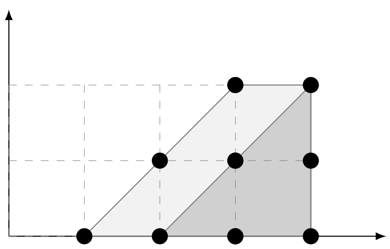


Figure 4.4: Allowed dimension vectors for submodules of an $\tilde{A}$-module in $\mathcal{N}_{2,4}$. In dark grey *framed* submodules; in light grey *unframed* submodules.

First of all, let us assume that $b_1 \oplus b_2 \colon V_1 \oplus V_1 \to V_0$ is an isomorphism. Up to $\mathrm{GL}(4, \mathbb{C})$-action on V_0, we can assume that in a suitable basis b_1 and b_2 are given by matrices

$$B_1 = \begin{pmatrix} \mathrm{Id}_2 \\ \mathbf{0} \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} \mathbf{0} \\ \mathrm{Id}_2 \end{pmatrix},$$

respectively, where Id_2 is the identity matrix and $\mathbf{0}$ is the zero matrix of suitable dimensions. With respect to the same basis, let a_1, a_2 be given, for $i = 1, 2$ by

$$A_i = \begin{pmatrix} M_i & N_i \end{pmatrix},$$

for some 2×2 matrices M_i and N_i. We now determine those matrices by imposing the condition given by the quiver potential W. Evaluating ${}^{\partial}\!/\!{}_{\partial a_1} W = b_1 a_2 b_2 - b_2 a_2 b_1$ we get

$$\begin{pmatrix} N_2 \\ \mathbf{0} \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ M_2 \end{pmatrix} = \mathbf{0}\,,$$

that is, $a_2 = 0$. Similarly, we obtain that $a_1 = 0$. However, the condition $a_1 = a_2 = 0$ is not acceptable, for otherwise the subrepresentation $\tilde{U} = (\mathrm{Im}(\iota), 0, V_\infty)$, with dimension vector $\mathbf{u} = (1, 0, 1)$, would destabilise $\tilde{V}$ according to Proposition 2.3.27, as easily seen in Figure 4.4.

Therefore the initial assumption of $b_1 \oplus b_2$ being an isomorphism leads to a contradiction, and it must be the case that $\mathrm{Im}(b_1 \oplus b_2) \subsetneq V_0$. In other words, we fall under Case 2) of Proposition 4.2.3.

Indeed, it is elementary to see that $\mathrm{Im}(\iota) \notin \mathrm{Im}(b_1 \oplus b_2)$, for otherwise $\tilde{V}$ would be destabilised by the subrepresentation $\tilde{U} = (\mathrm{Im}(b_1 \oplus b_2), V_1, V_\infty)$ with dimension vector $(k, 2, 1)$, $k \leq 3$. Finally, if $V_{\iota,b} := \mathrm{Im}(\iota) \oplus \mathrm{Im}(b_1 \oplus b_2) \subsetneq V_0$, then $(V_{\iota,b}, V_1, V_\infty)$ would be a destabilising subrepresentation. Therefore, we must have $V_{\iota,b} \simeq V_0$.

4.3.1 Matrices of the reduced locus

The discussion of the previous section allows us to find a suitable basis of V_0 and V_1. such that the expected matrices for a stable representation of $\tilde{A}$ have the form below.

$$\begin{aligned}
a_1 &= \begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & y \end{pmatrix} & a_2 &= \begin{pmatrix} 0 & 0 & 0 & z \\ 0 & 0 & 0 & w \end{pmatrix} \\
b_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & b_2 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \\
\iota &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\,.
\end{aligned} \tag{4.1}$$

Notice that the conditions on the maps b_1 and b_2 are a consequence of stability, while the statement that $a_j b_i = 0$ for all i, j comes from the potential W.

Remark 4.3.1. The matrices above should not be surprising. From the geometric discussion in Section 3.3.1 we know there is an open stratum parametrising double curves whose structure sheaf $\mathcal{O}_C$ sits in a non-split extension (3.4). Using the correspondence between quiver modules and sheaves, we can write the same extension in the category $\mathsf{Mod}\text{-}\mathbb{C}\tilde{Q}$; Proposition 2.3.5 carries over to framed representation, and provides a construction for such extensions as given by a collection of block matrices $E(V, W, v)$.

We know the modules corresponding to line bundles $\mathcal{O}_Z(m)$ for $m \in \mathbb{Z}$, as detailed in [NN11, Rem. 4.6]. In particular, the representation for $\mathcal{O}_Z$ has dimension vector $(1, 0, 1)$, and we can take the framing map to be the identity $\mathbb{C} \xrightarrow{\mathrm{Id}} \mathbb{C}$. The module corresponding to $\mathcal{O}_C$ is then written in block form, yielding[1] the matrices appearing in (4.1).

♠

We proceeded by examining individual representations, that is looking at closed points in our moduli scheme. Hence, the discussion above only applies to $(\mathcal{N}_{2,4})_{\mathrm{red}}$, and fails to capture the non-reduced geometry: to achieve that, we write our moduli scheme as a critical locus.

4.3.2 Critical locus description

We clearly have that $\mathcal{N}_{2,4} \simeq \mathcal{R}^{\zeta}_{(4,2,1)}(\tilde{A})$ is a subscheme of $\mathcal{R}^{\zeta}_{(4,2,1)}(\mathbb{C}\tilde{Q})$, and we aim to refine this statement. We will sometimes simplify notation and denote $\mathcal{R}^{\zeta}_{(4,2,1)}(-)$ by $\mathcal{R}(-)$.

Definition 4.3.2. Let $\mathcal{U} \subset \mathcal{R}^{\zeta}_{(4,2,1)}(\mathbb{C}\tilde{Q})$ be the open subscheme parametrising isomorphism classes of $\mathbb{C}\tilde{Q}$-modules $\tilde{V} = (V_0, V_1, V_\infty) = (\mathbb{C}^4, \mathbb{C}^2, \mathbb{C})$ whose maps $(\phi_e)_{e \in Q_1}$ satisfy the following conditions:

1. ϕ_{b_1} and ϕ_{b_2} have full rank, that is 2;
2. $\operatorname{Im} \phi_\iota \not\subset \operatorname{Im} \phi_{b_i}$ for $i = 1, 2$;
3. There is no nonzero $f \in V_1$ with $\phi_{b_1}(f)$ and $\phi_{b_2}(f)$ proportional.

♢

Clearly we have that $\mathcal{U}$ contains all closed points of $\mathcal{N}_{2,4}$. We proceed to examining the two cases allowed by Proposition 4.2.3. Assume that $(\tilde{V}, \phi) \in \mathcal{U}$ falls under Case 1), which is to say that $\operatorname{Im} \phi_{b_1} \oplus \operatorname{Im} \phi_{b_2} \simeq V_0$. Set $e_4 := \phi_\iota(1) \in V_0$. Under these hypotheses we find $e_2 \in \operatorname{Im} \phi_{b_1}$ and $e_2' \in \operatorname{Im} \phi_{b_2}$ such that

$$e_4 = e_2' - e_2 \,. \tag{4.2}$$

[1]After discarding zero-dimentional vector spaces.

A consequence of the second condition in Definition 4.3.2 is that both e_2 and e_2' are non-zero. Moreover, e_2 and e_2' are uniquely determined, as $\operatorname{Im}\phi_{b_1}$ and $\operatorname{Im}\phi_{b_2}$ have trivial intersection. Consider $f_1, f_2 \in V_1$ such that $e_2 = \phi_{b_1}(f_2)$ and $e_2' = \phi_{b_2}(f_1)$. If f_1 and f_2 were linearly dependent, their image would be in $\operatorname{Im}\phi_{b_1} \cap \operatorname{Im}\phi_{b_2}$, hence they form a basis of V_1. We also obtain a basis $\{e_1, e_2, e_3, e_4\}$ of V_0, where $e_1 := \phi_{b_1}(e_1)$ and $e_3 := \phi_{b_2}(e_2)$. With respect to this basis, there is the following matrix presentation:

$$\phi_{b_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \phi_{b_2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \phi_\iota = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.3}$$

For a given $t \in \mathbb{C}^*$ we can find, in the $\mathrm{GL}_{(4,2)}(Q)$-orbit of (4.3), the representation with matrices

$$\phi_{b_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \phi_{b_2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ t & 0 \end{pmatrix} \quad \phi_\iota = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.4}$$

In other words, we can repeat the construction above, replacing the condition (4.2) with $te_4 = e_2' - e_2$. The same argument applies, leading to a basis with respect to which $(\tilde{V}, \phi)$ has matrices (4.4).

It remains to consider Case 2) of Proposition 4.2.3. If we allow $t = 0$ in the discussion above, we are simply stating that there are vectors $e_2 = e_2' \in \operatorname{Im}\phi_{b_1} \cap \operatorname{Im}\phi_{b_2}$, which is one-dimensional: this is precisely the situation we want to examine. Notice that we can use the same construction for bases: indeed, the third condition of Definition 4.3.2 ensures that f_1, f_2 are linearly independent, and the same holds for e_1, e_3. In conclusion, in Case 2) we obtain again the matrices in (4.4), but with $t = 0$.

We must account for the fact that we are considering isomorphism classes of representations, and that there were choices made when constructing the bases described: in the first case, the parameter $t \neq 0$ was arbitrary, and in the second case $e_2 = e_2'$ is defined up to scalar. Therefore, we can address both problems together, as changing t will rescale e_2, e_1, and consequently also f_1, f_2, e_1, e_3.

Fix $\mu \in \mathbb{C}^*$, and consider the element $g \in \mathrm{GL}_{(4,2)}(Q) \simeq \mathrm{GL}(4,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C})$ given by $g = (M_0, M_1)$, where the matrices are below:

$$M_0 = \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad M_1 = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}. \tag{4.5}$$

The change of bases $\tilde{V}' := g \cdot \tilde{V}$ is summarised in diagram (2.6). If the module $\tilde{V}$ has matrices[2]

$$\phi_{a_1} = \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & x \\ \varepsilon_4 & \varepsilon_5 & \varepsilon_6 & y \end{pmatrix} \quad \phi_{a_2} = \begin{pmatrix} \varepsilon_7 & \varepsilon_8 & \varepsilon_9 & z \\ \varepsilon_{10} & \varepsilon_{11} & \varepsilon_{12} & w \end{pmatrix},$$

then an immediate computation shows that $\tilde{V}'$ has maps given by

$$\begin{aligned} &\phi'_{a_1} = \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \mu x \\ \varepsilon_4 & \varepsilon_5 & \varepsilon_6 & \mu y \end{pmatrix} \quad \phi'_{a_2} = \begin{pmatrix} \varepsilon_7 & \varepsilon_8 & \varepsilon_9 & \mu z \\ \varepsilon_{10} & \varepsilon_{11} & \varepsilon_{12} & \mu w \end{pmatrix} \\ &\phi'_{b_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \phi'_{b_2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ \mu^{-1}t & 0 \end{pmatrix} \qquad \phi'_{\iota} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{4.6}$$

We only require that the ε_i's are sufficiently small so that $\tilde{V}$ is ζ-stable. A further restriction arises, as Proposition 4.2.4 states that $\phi_{a_1} \circ \phi_\iota$ and $\phi_{a_2} \circ \phi_\iota$ cannot simultaneously vanish, that is $(x, y, z, w) \neq (0, 0, 0, 0)$.

Proposition 4.3.3. *Let $\mathcal{U} \subset \mathcal{R}^{\zeta}_{(4,2,1)}(\mathbb{C}\tilde{Q})$ be as in Definition 4.3.2. There is an isomorphism of schemes*

$$\mathcal{U} \simeq \operatorname{Tot}(\mathcal{O}_{\mathbb{P}^3}(-1)) \times \mathbb{A}^{12}.$$

Proof. It is a standard fact that $\operatorname{Tot}(\mathcal{O}_{\mathbb{P}^3}(-1))$ is the quotient of $(\mathbb{C}^4 \setminus \{(0,0,0,0)\}) \times \mathbb{C}$ by the $\mathbb{C}^*$-action $\mu \cdot (x, y, z, w, t) \sim (\mu x, \mu y, \mu z, \mu w, \mu^{-1} t)$. □

Remark 4.3.4. Notice that, a priori, we expect $\mathcal{U}$ to have the same dimension as $\mathcal{R}^{\zeta}_{\mathbf{d}}(\mathbb{C}\tilde{Q})$, which for $\mathbf{d} = (4, 2, 1)$ is $d_0 - q_Q(d_0, d_1) = d_0 + 4d_0d_1 - d_0^2 - d_1^2 = 16$. ♠

Remark 4.3.5. The construction described above is dependent on a choice of *stability condition*, which is essential to provide a canonical basis for the space of representations. It is not known to the author which global geometry one would obtain, starting from other stability conditions away from the PT-chamber. ♠

We can now plug in the machinery of Theorem 2.3.11 to explicitly write $\mathcal{N}_{2,4}$ as a d-critical locus $\operatorname{Crit}(\operatorname{tr} W, \mathcal{U})$, obtaining equations in the variables $x, y, z, w, \varepsilon_i$. By an explicit computation, easily performed with computer algebra software as in Appendix A.2, we have

$$\begin{aligned} (\operatorname{tr} W)(\tilde{V}) = \varepsilon_{11}\varepsilon_2 - \varepsilon_{10}\varepsilon_3 + \varepsilon_{12}\varepsilon_5 - \varepsilon_{11}\varepsilon_6 - \varepsilon_2\varepsilon_7 + \varepsilon_1\varepsilon_8 - \varepsilon_5\varepsilon_8 + \varepsilon_4\varepsilon_9 + \\ + \varepsilon_2 tw - \varepsilon_7 tx - \varepsilon_8 ty + \varepsilon_1 tz. \end{aligned} \tag{4.7}$$

[2] The reason why the last column has a different notation will become clear momentarily.

Taking derivatives, we see that several variables must vanish: $\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_9 = \varepsilon_{10} = \varepsilon_{11} = 0$. The remaining equations are

$$\begin{gathered}\varepsilon_1 = ty\\ \varepsilon_2 = \varepsilon_6 = -tx\\ \varepsilon_7 = tw\\ \varepsilon_8 = \varepsilon_{12} = -tz\\ t^2x = t^2y = t^2z = t^2w = t(xw - yz) = 0\,.\end{gathered}$$

We wish to simplify the potential (4.7), and reduce the number of dimensions of the ambient variety, in the spirit of [DS09]. Write

$$\begin{gathered}(\operatorname{tr} W)(\tilde{V}) = \varepsilon_{11}(\varepsilon_2 - \varepsilon_6) - \varepsilon_{10}\varepsilon_3 + \varepsilon_5(\varepsilon_{12} - \varepsilon_8) + \varepsilon_4\varepsilon_9 + \\ + f(x, y, z, w, t, \varepsilon_1, \varepsilon_2, \varepsilon_7, \varepsilon_8)\,,\end{gathered} \tag{4.8}$$

with the polynomial f implicitly defined. We can firstly apply Lemma 1.4.8 twice to the expression (4.8) above, removing the pairs of variables $\varepsilon_{10}, \varepsilon_3$ and $\varepsilon_4, \varepsilon_9$. Then, applying Lemma 1.4.9 twice to the remaining affine variables, we are left with the potential

$$f(x, y, z, w, t, \varepsilon_1, \varepsilon_2, \varepsilon_7, \varepsilon_8) = (\varepsilon_2 + tx)(tw - \varepsilon_7) + (\varepsilon_1 - ty)(\varepsilon_8 + tz) + t^2(yz - xw)\,, \tag{4.9}$$

giving an isomorphism of critical charts

$$\operatorname{Crit}(\operatorname{tr}(W), \operatorname{Tot}(\mathcal{O}_{\mathbb{P}^3}(-1)) \times \mathbb{A}^{12}) \quad \simeq \quad \operatorname{Crit}(f, \operatorname{Tot}(\mathcal{O}_{\mathbb{P}^3}(-1)) \times \mathbb{A}^4)\,.$$

Finally, consider without loss of generality the open chart $\{w \neq 0\} \subset \mathbb{P}^3$, and restrict the function (4.9) in affine coordinates, which we denote by the corresponding capital letters. We obtain

$$F = (\varepsilon_2 + TX)(TW - \varepsilon_7) + Q(X, Y, Z, T, \varepsilon_1, \varepsilon_8)$$

where Q is a polynomial. Notice that, due to the linear terms $\varepsilon_2, \varepsilon_7$, we have that both $(\varepsilon_2 + TX)$ and $(TW - \varepsilon_7)$ belong to the ideal $I_{\mathrm{d}F}$. Therefore, F and Q differ by an element of $I_{\mathrm{d}F}^2$ and so they define isomorphic critical structures. Repeating the procedure for different open charts of $\mathbb{P}^3$, we remove the last two variables $\varepsilon_1, \varepsilon_8$. We have thus completed the proof of the statement below.

Theorem 4.3.6. *The moduli scheme* $\mathcal{N}_{2,4}$ *is isomorphic to* $\operatorname{Crit}(q) \subset \operatorname{Tot}(\mathcal{O}_{\mathbb{P}^3}(-1))$, *where*

$$q = t^2 \begin{vmatrix} x & z \\ y & w \end{vmatrix}\,.$$

Remark 4.3.7. This characterisation agrees with the geometric description of Proposition 3.3.1, as it can be checked in Appendix A.3.1. In particular, we can read off the non-reduced structure along the quadric surface given by $xw - yz = 0$. ♠

Remark 4.3.8. Szendrői [Sze12, §4] studied the moduli scheme $\mathcal{N}_{2,4}$, by conjecturing that locally the potential expressing the critical locus structure has the form

$$x_3 x_4^2 + \sum_{i=5,\dots,2n} x_i^2 \,.$$

If, without loss of generality, we consider the open patch $\{w \neq 0\}$, with affine coordinates (X, Y, Z, T), we retrieve Szendrői's potential after the change of coordinates $x_1 = Y, x_2 = Z, x_3 = X - YZ, x_4 = T$, confirming the conjecture. ♠

4.4 Quiver description of $\mathcal{N}_{2,5}$

We complete now the proof of Theorem 3.4.1, providing a critical locus description. Consider $\tilde{V}$, a stable $\tilde{A}$-module of dimension $\mathbf{d} = (d_0, d_1, d_\infty) = (5, 3, 1)$, stable with respect to a parameter corresponding to PT-moduli spaces, as in Theorem 2.3.22. Recall that, with abuse of notation, we will denote by $a_i \colon V_0 \to V_1$ and $b_i \colon V_1 \to V_0$, for $i = 1, 2$, and $\iota \colon V_\infty \to V_0$ the corresponding maps. Allowed dimension vectors of submodules $\tilde{U}$ are represented in Figure 4.5, following Proposition 2.3.27. This will be useful in the next sections, as we will routinely check that certain submodules are destabilising.

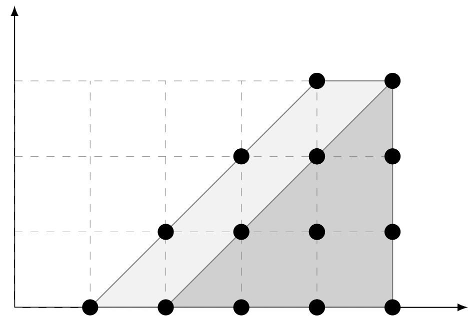


Figure 4.5: Allowed dimension vectors for submodules of an $\tilde{A}$-module in $\mathcal{N}_{2,5}$. Light gray *unframed*; dark grey *framed*.

The next sections will constitute a proof of the statement below.

Proposition 4.4.1. *Let $\tilde{V}$ be a stable $\tilde{A}$-module as above. Then b_1, b_2 are injective, and one of the following two cases must occur:*

1.
- $\dim(\operatorname{Im} b_1 \cap \operatorname{Im} b_2) = 1$,
- $\operatorname{Im} \iota \subset \langle \operatorname{Im} b_1, \operatorname{Im} b_2 \rangle$,
- $b_1^{-1}(\operatorname{Im} b_1 \cap \operatorname{Im} b_2) \neq b_2^{-1}(\operatorname{Im} b_1 \cap \operatorname{Im} b_2)$,
- $\langle \operatorname{Im} b_1, \operatorname{Im} b_2 \rangle = V_0$;

2.
- $\dim(\operatorname{Im} b_1 \cap \operatorname{Im} b_2) = 2$,
- $\operatorname{Im} \iota \not\subset \langle \operatorname{Im} b_1, \operatorname{Im} b_2 \rangle$,
- $\dim(U) = 1$, *where* $U := b_1^{-1}(\operatorname{Im} b_1 \cap \operatorname{Im} b_2) \cap b_2^{-1}(\operatorname{Im} b_1 \cap \operatorname{Im} b_2)$,
- $b_1(U) \neq b_2(U)$,
- $\langle \operatorname{Im} b_1, \operatorname{Im} b_2 \rangle \oplus \operatorname{Im} \iota = V_0$.

Remark 4.4.2. It is a well known fact that the stability conditions corresponding to the DT-chamber have a simple algebraic interpretation: the framing vertex has to be *cyclic*, that is a representation $\tilde{V}$ is DT-stable if $1 \in V_\infty \simeq \mathbb{C}$ is a cyclic vector with respect to the $\mathbb{C}\tilde{Q}$-module structure. It is not known to the author whether a similarly simple algebraic characterisation of PT-stability exists, which would unify all conditions appearing in Proposition 4.4.1. ♠

Remark 4.4.3. Each case of Proposition 4.4.1 corresponds to one component of the moduli scheme $\mathcal{N}_{2,5}$, as in the statement of Theorem 3.4.1. In particular, the correspondence is as follows.

1. This is Case 2γa-ii below. It represents closed points in the extra three-fold component of $\mathcal{N}_{2,5}$, but not on the $\mathbb{P}^5$ component; up to gauge action the matrices of the representation are as in (4.16) and (4.23).

2. This is Case 2γb-ii below, corresponding to the $\mathbb{P}^5$ component; matrices are as in (4.18).

Notice that, while the two components intersect, we ascribe the intersection to the second case. ♠

Overview of the proof. We proceed by exhaustion. Since the cases listed above correspond to Case 2γa-ii and Case 2γb-ii below, respectively, we want to show that all other cases cannot occur. To complete the proof, we consider each of the possible configurations of the b_i maps, finding in each case suitable bases $e_1, \dots, e_5$ and $f_1, \dots, f_3$ of $V_0 \simeq \mathbb{C}^5$ and $V_1 \simeq \mathbb{C}^3$, respectively; we then use the quiver potential to impose restrictions on the a_i maps. These computations can be automated using computer algebra software, showed in Appendix A.2. In most of the cases, we can reach a precise description of the general representation, which enables us to find destabilising subrepresentations, ruling out the particular case under exam. This is seen in the next sections. $\square$

We start with the observation below.

Proposition 4.4.4. *The maps b_1, b_2 have rank at least 2.*

Proof. From Proposition 4.2.3 we know that each b_i cannot be zero. Suppose by contradiction that $\operatorname{rk} b_1 = 1$, without loss of generality. Again Proposition 4.2.3 ensures that $\operatorname{rk} b_2 = 3$ and that ι, b_1, b_2 have disjoint images. Let f be an element of $V_1 \setminus \ker b_1$, and $e := b_1(f)$. For $j = 1, 2$, the potential relation ensures that $b_2 a_j b_1 f = b_1 a_j b_2 f$. This means that $(\langle e \rangle, \langle f \rangle, 0)$ is a submodule, whose dimension vector $(1, 1, 0)$ is not allowed. $\square$

4.4.1 Case 1: neither b_1,b_2 injective

We assume that neither b_1 nor b_2 is injective, hence $(\operatorname{nul} b_1, \operatorname{nul} b_2, \operatorname{rk} b_1, \operatorname{rk} b_2) = (1, 1, 2, 2)$. First of all, we have that $\ker b_1 \neq \ker b_2$, for otherwise we would produce a destabilising submodule $(0, \ker b_1, 0)$. Let $\langle f_1 \rangle = \ker b_2$ and $\langle f_3 \rangle = \ker b_1$, with f_2 completing a basis of V_1. By the general argument in Section 4.2, we know that $\langle \operatorname{Im} b_1, \operatorname{Im} b_2 \rangle$ must have dimension at least, so exactly, 4. Let e_5 generate $\operatorname{Im} \iota$, and set $e_1 = b_1(f_1)$, $e_2 = b_1(f_2)$, $e_3 = b_2(f_2)$, and $e_4 = b_2(f_3)$. In this basis, the matrices corresponding to b_1, b_2 have a simple form, and we can use the quiver potential relation to deduce the form of the a_1, a_2 matrices, employing for example the routine in Appendix A.2. We obtain

$$\begin{aligned} b_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & b_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ a_1 &= \begin{pmatrix} * & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & * & * \end{pmatrix} & a_2 &= \begin{pmatrix} * & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & * & * \end{pmatrix}, \end{aligned} \tag{4.10}$$

where $*$ denotes a possibly non-zero entry. The situation is represented in Figure 4.6, where each dot represents a basis element: the arrows are color-coded, and a multi-headed arrow means that it has image in the span of the targets. It is easy to see that $\tilde{U} := (\langle e_1 \rangle, \langle f_1 \rangle, 0)$ is destabilising, therefore this case cannot occur.

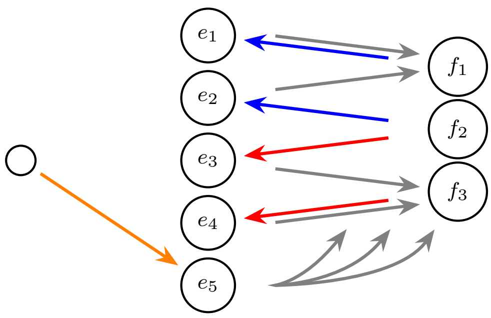


Figure 4.6: Case 1: neither b_1,b_2 injective: b_1, b_2, a_i, ι.

4.4.2 Case 2: b_1 injective

In Case 2, we assume b_1 injective, and consider subcases α, β, γ according to $\operatorname{rk} b_2 = 1, 2, 3$.

4.4.2.1 Case 2α

Assume $(\operatorname{nul} b_1, \operatorname{nul} b_2, \operatorname{rk} b_1, \operatorname{rk} b_2) = (0, 2, 3, 1)$.

Again, we must have that $\operatorname{Im} \iota \nsubseteq \langle \operatorname{Im} b_1, \operatorname{Im} b_2 \rangle$, and $\operatorname{Im} b_1 \cap \operatorname{Im} b_2 = 0$. Let f_1, f_2 be a basis of $\ker b_2$, completed by f_3 to a basis of V_1. For $i = 1, 2, 3$, let $e_i := b_1(f_i)$, $e_3 := b_2(f_3)$, and e_5 generating $\operatorname{Im} \iota$. We have hence

$$\begin{aligned} b_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & b_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ a_1 &= \begin{pmatrix} * & * & * & 0 & * \\ * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix} & a_2 &= \begin{pmatrix} * & * & * & 0 & * \\ * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}. \end{aligned} \tag{4.11}$$

Looking at the depiction in Figure 4.7, where we omitted to label basis elements and where the curly brace denotes the span, we see that $\tilde{U} = (\langle e_1, e_2 \rangle, \langle f_1, f_2 \rangle, 0)$ is a submodule, and it has dimension vector $(2, 2, 0)$, which is not allowed. Therefore also this case cannot occur.

4.4.2.2 Case 2β

Let $\operatorname{rk} b_2 = 2$, so $(\operatorname{nul} b_1, \operatorname{nul} b_2, \operatorname{rk} b_1, \operatorname{rk} b_2) = (0, 1, 3, 2)$. We have subcases **a** and **b** according to whether $\operatorname{Im} \iota$ is contained or not in $\langle \operatorname{Im} b_1, \operatorname{Im} b_2 \rangle$, respectively.

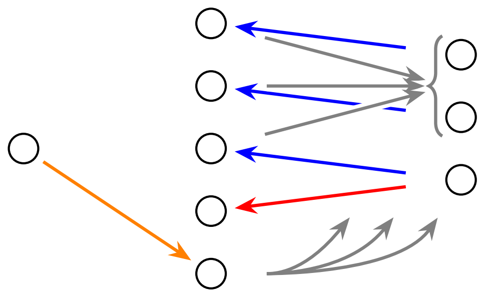


Figure 4.7: Case 2α: b_1, b_2, a_i, ι.

Case 2βa Let $\operatorname{Im}\iota \subseteq \langle \operatorname{Im} b_1, \operatorname{Im} b_2\rangle$, so in particular $\operatorname{Im} b_1$ and $\operatorname{Im} b_2$ are disjoint and generate V_0. Let f_1 be a generator of $\ker b_2$, completed by f_2, f_3 to a basis of V_1. Let $e_i := b_1(f_i)$ for $i = 1, 2, 3$ and $e_j := b_2(f_{j-2})$ for $j = 4, 5$. This yields

$$\begin{aligned} b_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & b_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ a_1 &= \begin{pmatrix} * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & a_2 &= \begin{pmatrix} * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.12}$$

This is represented in Figure 4.8. We have that $\tilde{U} = (\langle e_1\rangle, \langle f_1\rangle, 0)$ is a destabilising submodule with dimension vector $(1, 1, 0)$. Therefore this case cannot occur.

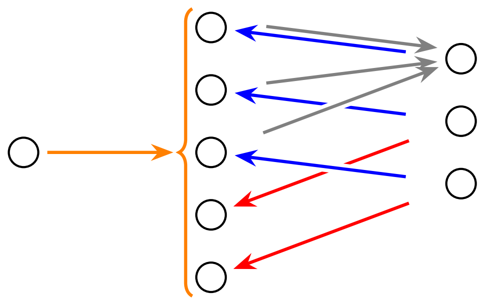


Figure 4.8: Case 2βa: b_1, b_2, a_i, ι.

Case 2βb Let $\operatorname{Im}\iota \not\subseteq \langle \operatorname{Im} b_1, \operatorname{Im} b_2\rangle$, implying that V_0 is spanned by the images of ι, b_1, b_2 and that $\operatorname{Im}(b_1) \cap \operatorname{Im}(b_2)$ is 1-dimensional. Let f_1 be a generator for $\ker b_2$, $e_1 := b_1(f_1)$, let e_3 generate $\operatorname{Im}(b_1) \cap \operatorname{Im}(b_2)$, and e_5 generate $\operatorname{Im}(\iota)$; finally, we set $f_3 := b_1^{-1}(e_3)$. We have to consider two further sub-cases, according to whether $\langle e_3\rangle$ has different preimages along b_1 and b_2.

Case 2βb-i Assume that $b_1^{-1}\langle e_3\rangle = b_2^{-1}\langle e_3\rangle$, let f_2 complete a basis of V_1, set $e_2 := b_1(f_2)$ and $e_4 := b_2(f_2)$. Now the matrices are

$$\begin{aligned} b_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & b_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ a_1 &= \begin{pmatrix} * & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & \varepsilon_1 & * & \delta\varepsilon_1 & * \end{pmatrix} & a_2 &= \begin{pmatrix} * & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & \varepsilon_2 & * & \delta\varepsilon_2 & * \end{pmatrix}, \end{aligned} \tag{4.13}$$

where $\delta \neq 0$, and ε_i is used to highlight symmetries in the a_i matrix. From Figure 4.9 we easily see that again $\tilde{U} = (\langle e_1\rangle, \langle f_1\rangle, 0)$ is a destabilising submodule. Therefore, it is impossible for this case to occur.

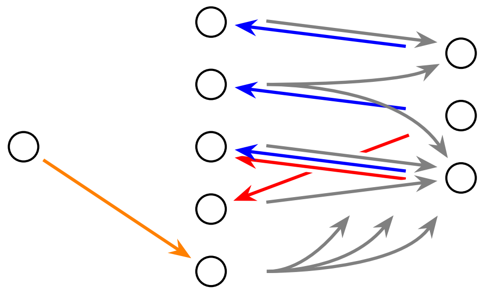


Figure 4.9: Case 2βb-i: b_1, b_2, a_i, ι.

Case 2βb-ii Assume now that $b_1^{-1}\langle e_3\rangle \neq b_2^{-1}\langle e_3\rangle$, and let $f_2 := b_2^{-1}(e_3)$ complete a basis of V_1; set $e_2 := b_1(f_2)$ and $e_4 := b_2(f_2)$. In this basis,

$$\begin{aligned} b_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & b_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ a_1 &= \begin{pmatrix} * & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix} & a_2 &= \begin{pmatrix} * & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}, \end{aligned} \tag{4.14}$$

as displayed in Figure 4.10. We have that $\tilde{U} = (\langle e_1\rangle, \langle f_1\rangle, 0)$ is a destabilising submodule, hence this case does not occur.

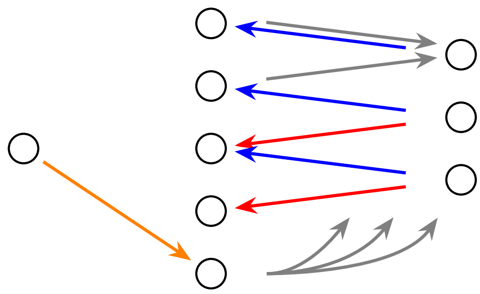


Figure 4.10: Case 2βb-ii: b_1, b_2, a_i, ι.

4.4.2.3 Case 2γ

Let us now assume that both b_1 and b_2 are injective. We consider again subcases **a** and **b** according to whether $\operatorname{Im}\iota$ is contained or not in $\langle \operatorname{Im} b_1, \operatorname{Im} b_2 \rangle$, respectively.

Case 2γa Let $\operatorname{Im}\iota \subseteq \langle \operatorname{Im} b_1, \operatorname{Im} b_2 \rangle$, so $\operatorname{Im} b_1$ and $\operatorname{Im} b_2$ generate V_0 and have 1-dimensional intersection. Let e_3 be a generator of $\operatorname{Im} b_1 \cap \operatorname{Im} b_2$, and consider two sub-cases according to whether e_3 has distinct preimages along b_1 and b_2.

Case 2γa-i Assume that $b_1^{-1}\langle e_3 \rangle = b_2^{-1}\langle e_3 \rangle$, and let $f_3 := b_1^{-1}(e_3)$, completed by f_1, f_2 to a basis of V_1; let $e_i := b_1(f_i)$ and $e_{i+3} := b_2(f_i)$, for $i = 1, 2$. The matrices are

$$\begin{aligned} b_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & b_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ a_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \varepsilon_1 & \varepsilon_2 & * & \delta\varepsilon_1 & \delta\varepsilon_2 \end{pmatrix} & a_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \varepsilon_3 & \varepsilon_4 & * & \delta\varepsilon_3 & \delta\varepsilon_4 \end{pmatrix}, \end{aligned} \tag{4.15}$$

where again $*$ and $\varepsilon_1, \ldots, \varepsilon_4$ denote possibly-zero entries, and $\delta \neq 0$. Figure 4.11 shows the situation, and we see that $\tilde{U} = (\langle e_3 \rangle, \langle f_3 \rangle)$ is destabilising.

Case 2γa-ii Let as before e_3 be a generator for $\operatorname{Im} b_1 \cap \operatorname{Im} b_2$, chosen up to rescaling, and assume that $b_1^{-1}\langle e_3 \rangle \neq b_2^{-1}\langle e_3 \rangle$. Set $f_1 := b_2^{-1}(e_3)$ and $f_3 := b_1^{-1}(e_3)$.

Proposition 4.4.5. *Under the hypotheses of Case 2γa-ii, we have that* $\operatorname{Im} b_1 \cap \operatorname{Im} b_2 \subseteq \ker a_1 \cap \ker a_2$.

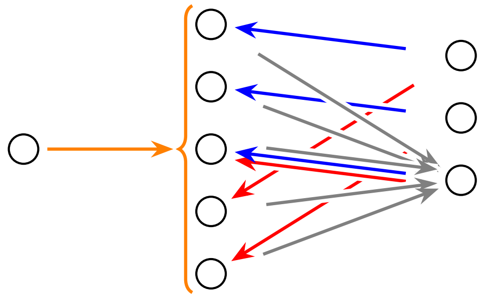


Figure 4.11: Case 2γa-i: b_1, b_2, a_i, ι.

Proof. Recall that we are assuming b_1, b_2 injective. Apply the map a_1 to $e_3 = b_2 f_1 = b_1 f_3$, and call f its image; notice that, by the potential relations, $b_2 f = b_2 a_1 b_1 f_3 = b_1 a_1 b_2 f_1 = b_1 f \in \operatorname{Im} b_1 \cap \operatorname{Im} b_2$. This would be a contradiction unless $f = 0$, that is $e_3 \in \ker a_1$. An identical argument shows that $e_3 \in \ker a_2$. $\square$

Corollary 4.4.6. *Combining Proposition 4.2.4 with Proposition 4.4.2.3 gives* $\operatorname{Im} \iota \neq \langle e_3 \rangle$.

Corollary 4.4.7. *Set* $e_1 := b_1(f_1)$, *and* $e_5 := b_2(f_3)$. *Then* $\langle e_1, e_5 \rangle \subseteq \ker a_1 \cap \ker a_2$.

Proof. Call $f := a_1 e_1$. We have that $b_2 f = b_2 a_1 b_1 f_1 = b_2 a_1 b_2 f_1 = b_2 a_1 e_3 = 0$, so $f = 0$ by injectivity of b_2. Analogous arguments show that $a_2 e_1 = a_1 e_5 = a_2 e_5 = 0$. $\square$

We conclude that $\operatorname{Im} \iota \subset V_0 \setminus \langle e_1, e_3, e_5 \rangle$. Let e_4 be a generator of $\operatorname{Im} \iota$, and assume that e_4 is contained in the image of either b_1 or b_2. Say $e_4 \in \operatorname{Im} b_2$, up to swapping b_1 and b_2. Consider $f_2 := b_2^{-1}(e_4)$, noticing that f_1, f_2, f_3 form a basis of V_1. Finally, let $e_2 := b_1(f_2)$; clearly $e_2 \notin \langle e_3, e_4, e_5 \rangle$, for otherwise $\operatorname{Im} b_1 \cap \operatorname{Im} b_2$ would be 2-dimensional. The situation is represented in Figure 4.12 , with matrices as below:

$$\begin{aligned} b_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & b_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ a_1 &= \begin{pmatrix} 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 \end{pmatrix} & a_2 &= \begin{pmatrix} 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 \end{pmatrix}. \end{aligned} \tag{4.16}$$

In case we do not assume that $\operatorname{Im}(\iota) = e_4$ is contained in either $\operatorname{Im}(b_1)$ or $\operatorname{Im}(b_2)$, the situation is more complicated. In order to avoid duplicates, we refer to Section 4.4.3, where different bases are used.

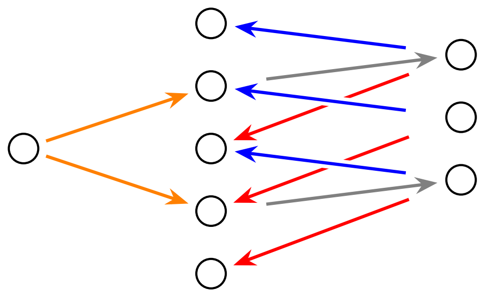


Figure 4.12: Case 2γa-ii: b_1, b_2, a_i, ι.

Case 2γb Let $\operatorname{Im}\iota \not\subseteq \langle \operatorname{Im} b_1, \operatorname{Im} b_2\rangle$: in this case, $\langle \operatorname{Im} b_1, \operatorname{Im} b_2\rangle$ is 4-dimensional and $\operatorname{Im} b_1 \cap \operatorname{Im} b_2$ is 2-dimensional, with e_2, e_3 a basis. Call e_5 a generator of $\operatorname{Im}\iota$.

Case 2γb-i Assume that $b_1^{-1}\langle e_2, e_3\rangle = b_2^{-1}\langle e_2, e_3\rangle$. Their preimages $f_2 := b_1^{-1}(e_2)$ and $f_3 := b_1^{-1}(e_3)$ can be completed by f_1 to a basis of V_1. Finally, call $e_1 := b_1(f_1)$ and $e_4 := b_2(f_1)$. The corresponding matrices are given by

$$\begin{aligned} b_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & b_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta_1 & \delta_2 \\ 0 & \delta_3 & \delta_4 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ a_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & x \\ * & * & * & * & y \\ * & * & * & * & z \end{pmatrix} & a_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & u \\ * & * & * & * & v \\ * & * & * & * & w \end{pmatrix}. \end{aligned} \tag{4.17}$$

We notice, by considering the maps displayed in Figure 4.13, that $\tilde{U} = (\langle e_2, e_3\rangle, \langle f_2, f_3\rangle, 0)$ is a destabilising submodule. This case does not occur.

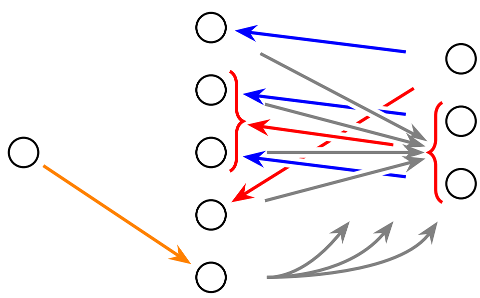


Figure 4.13: Case 2γb-i: b_1, b_2, a_i, ι.

Case 2γb-ii Assume now that $b_1^1(\operatorname{Im} b_1 \cap \operatorname{Im} b_2) \neq b_2^{-1}(\operatorname{Im} b_1 \cap \operatorname{Im} b_2)$, that is $b_1^{-1}(\operatorname{Im} b_1 \cap \operatorname{Im} b_2) \cap b_2^{-1}(\operatorname{Im} b_1 \cap \operatorname{Im} b_2)$ is 1-dimensional, generated by f_2. Let as before e_5 generate $\operatorname{Im}\iota$.

Proposition 4.4.8. *Under the hypotheses of Case 2γb-ii,* $\langle b_1(f_2)\rangle \neq \langle b_2(f_2)\rangle$.

Proof. Assume by contradiction that the two spaces coincide, with e_2 a generator. Complete it to a basis e_2, e_3 of $\operatorname{Im} b_1 \cap \operatorname{Im} b_2$. Let $f_1 := b_2^{-1}(e_3)$, $f_3 := b_1^{-1}(e_3)$, $e_1 := b_1(f_1)$ and $e_4 := b_2(f_3)$. Notice that $e_1, \ldots, e_5$ and $f_1, \ldots, f_3$ are bases of V_0 and V_1, respectively. The maps ι, b_1 and b_2 are represented in Figure 4.14. Consider now $a_1 e_2$: since $b_1 a_1 b_2 = b_2 a_1 b_2$, we have that $a_1 e_2 \in \langle f_2\rangle$. Analogously, $a_1 e_2 \in \langle f_2\rangle$ as well.[3] It follows that $\tilde{U} = (\langle e_2\rangle, \langle f_2\rangle, 0)$ is a destabilising submodule. □

We can now start from f_2, which is a generator of $b_1^{-1}(\operatorname{Im} b_1 \cap \operatorname{Im} b_2) \cap b_2^{-1}(\operatorname{Im} b_1 \cap \operatorname{Im} b_2)$, and use a zig-zag procedure to construct bases of V_0 and V_1. Explicitly, $e_2 := b_1(f_2)$ and $e_3 := b_2(f_2)$, $f_1 := b_2^{-1}(e_2)$ and $f_3 := b_1^{-1}(e_3)$, $e_1 := b_1(f_1)$ and $e_4 := b_2(f_3)$. Figure 4.15 depicts the situation, where the maps a_1, a_2 are described by the proposition below.

Proposition 4.4.9. *Under the hypotheses of Case 2γb-ii,* $\operatorname{Im} b_i \subseteq \ker a_j$*, for all* $i, j = 1, 2$.

Proof. Checked using the quiver potential relation. □

Remark 4.4.10. We should consider Proposition 4.4.9 as an analogous to Corollary 2.3.17, that is part of [NN11, Lemma 4.4], by regarding $(V_0 \setminus \operatorname{Im} \iota, V_1)$ as an *unframed* stable A-module. ♠

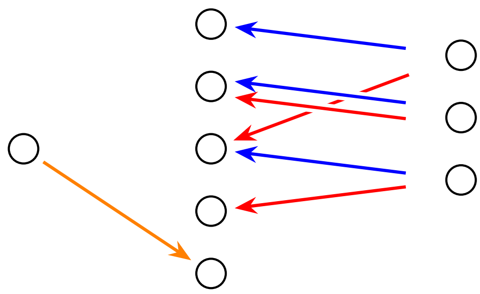


Figure 4.14: Proposition 4.4.8: b_1, b_2, ι.

In conclusion, in the given basis, the maps have matrices

[3] In fact, $\forall i = 1, 2$ and $\forall j = 1, \ldots, 4$ we have $a_i e_j \in \langle f_2\rangle$.

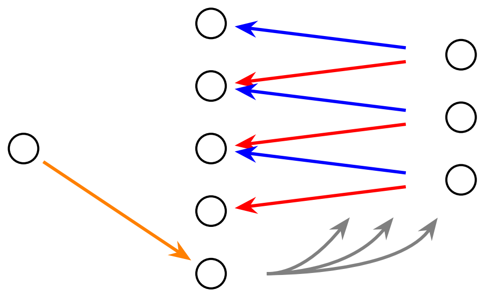


Figure 4.15: Case 2γb-ii: b_1, b_2, a_i, ι.

$$b_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad b_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$a_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & z \end{pmatrix} \quad a_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 & w \end{pmatrix}. \tag{4.18}$$

This completes the proof of Proposition 4.4.1.

Remark 4.4.11. Again, the matrices above can be interpreted in terms of extensions of framed quiver representations, following the same considerations of Remark 4.3.1. ♠

4.4.3 Critical locus description

Recall that that $\mathcal{N}_{2,5} \simeq \mathcal{R}^{\zeta}_{(5,3,1)}(\tilde{A})$ is a subscheme of $\mathcal{R}^{\zeta}_{(5,3,1)}(\mathbb{C}\tilde{Q})$, and that we denote for simplicity $\mathcal{R}^{\zeta}_{(5,3,1)}$ by $\mathcal{R}$. We proceed similarly to Section 4.3.2.

Definition 4.4.12. Let $\mathcal{U} \subset \mathcal{R}^{\zeta}_{(5,3,1)}(\mathbb{C}\tilde{Q})$ be the open subscheme parametrising isomorphism classes of $\mathbb{C}\tilde{Q}$-modules $\tilde{V} = (V_0, V_1, V_\infty) = (\mathbb{C}^5, \mathbb{C}^3, \mathbb{C})$ whose maps $(\phi_e)_{e\in Q_1}$ satisfy the following conditions:

1. ϕ_{b_1} and ϕ_{b_2} have full rank, that is 3;
2. $\operatorname{Im}\phi_\iota \not\subset \operatorname{Im}\phi_{b_i}$ for $i = 1, 2$;
3. $\operatorname{Im}\phi_\iota \not\subset \left\langle V_{\cap b}\, , \phi_{b_1}(\phi_{b_2}^{-1}(V_{\cap b}))\, , \phi_{b_2}(\phi_{b_1}^{-1}(V_{\cap b}))\right\rangle$, where $V_{\cap b} := \operatorname{Im}\phi_{b_1} \cap \operatorname{Im}\phi_{b_2}$;
4. $\dim(U) < 2$, where $U := b_1^{-1}(V_{\cap b}) \cap b_2^{-1}(V_{\cap b})$.
5. There is no nonzero $f \in V_1$ with $\phi_{b_1}(f)$ and $\phi_{b_2}(f)$ proportional.

We denote $\mathcal{N}^{\mathcal{U}}_{2,5} \simeq \mathcal{R}(\tilde{A})^{\mathcal{U}} := \mathcal{R}^{\zeta}_{(5,3,1)}(\tilde{A}) \cap \mathcal{U}$. ◊

A linear algebra computation shows that $\mathcal{U}$ contains all $\tilde{V}$ which fall under Case 2γb-ii. Suppose first that $(\tilde{V}\,,\phi)\in\mathcal{U}$ satisfies Case 1) in Proposition 4.2.3, which is to say that $\operatorname{Im}\phi_{b_1}\oplus\operatorname{Im}\phi_{b_2}\simeq V_0$. Set now $e_5:=\phi_\iota(1)\in V_0$. We wish to determine vectors $u\in\operatorname{Im}(\phi_{b_1})$ and $u'\in\operatorname{Im}(\phi_{b_2})$ such that e_5 is in their span.

Necessarily, $\dim\operatorname{Im}(\phi_{b_1})\cap\operatorname{Im}(\phi_{b_2})=1$, hence let v_3 be a generator of this intersection (defined up to rescaling); set $v_1:=\phi_{b_1}(\phi_{b_2}^{-1}(v_3))$ and $v_5:=\phi_{b_2}(\phi_{b_1}^{-1}(v_3))$. By the assumptions in Definition 4.4.12, we have that $e_5\neq\langle v_1,v_3,v_5\rangle$.

Clearly, u,u' are determined up to $\operatorname{Im}(\phi_{b_1})\cap\operatorname{Im}(\phi_{b_2})$, that is, with obvious notation, if $u_1+u_1'=u_2+u_2'=e_5$ then both u_1-u_1' and u_2-u_2' are multiples of v_3. Moreover, by construction of $\mathcal{U}$ neither u or u' can be zero, nor a multiple of v_3. Fixed $s\,,t\in\mathbb{C}^*$, we can find $e_2\,,e_3\in\operatorname{Im}(\phi_{b_1})$ and $e_2'\,,e_3'\in\operatorname{Im}(\phi_{b_2})$ such that $s\,e_5=-e_2+e_2'$ and $t\,e_5=-e_3+e_3'$, the pairs $\{e_2\,,e_3\}$ and $\{e_2'\,,e_3'\}$ are independent, and we require

$$v_3=-t\,e_2+s\,e_3=-t\,e_2'+s\,e_3'\,.$$

Notice that the requirements above uniquely determine $e_2\,,e_2'$, and hence $e_3\,,e_3'$. Define $f_1\,,f_2\,,f_3\in V_1$ by

$$\phi_{b_2}(f_1)=e_2'\,,\quad \phi_{b_2}(f_2)=e_3'\,,\quad \phi_{b_1}(f_3)=e_3\,,$$

and set moreover $e_1:=\phi_{b_1}(f_1)$ and $e_4:=\phi_{b_2}(f_3)$. One can check that $\{e_1,e_2,e_3,e_4,e_5\}$ and $\{f_1,f_2,f_3\}$ are bases of V_0 and V_1, respectively, and a case-by-case analysis shows that $\phi_{b_1}(f_2)=e_2$. In these bases, the matrices for $\tilde{V}$ have the form below,

$$\phi_{b_1}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0\end{pmatrix}\qquad \phi_{b_2}=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\\0&0&1\\s&t&0\end{pmatrix}\qquad \phi_\iota=\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} \tag{4.19}$$
$$\phi_{a_1}=\begin{pmatrix}\varepsilon_1&\varepsilon_2&\varepsilon_3&\varepsilon_4&x\\\varepsilon_5&\varepsilon_6&\varepsilon_7&\varepsilon_8&y\\\varepsilon_9&\varepsilon_{10}&\varepsilon_{11}&\varepsilon_{12}&z\end{pmatrix}\quad \phi_{a_2}=\begin{pmatrix}\varepsilon_{13}&\varepsilon_{14}&\varepsilon_{15}&\varepsilon_{16}&u\\\varepsilon_{17}&\varepsilon_{18}&\varepsilon_{19}&\varepsilon_{20}&v\\\varepsilon_{21}&\varepsilon_{22}&\varepsilon_{23}&\varepsilon_{24}&w\end{pmatrix}.$$

where the variables ε_k are sufficiently small so that $\tilde{V}$ is ζ-stable, and $(x,y,z,u,v,w)\neq 0$, following Proposition 4.2.4.

Now, assume that $\tilde{V}$ satisfies Case 2) of Proposition 4.2.3, that is $V_0\simeq\langle\operatorname{Im}b_1\,,\operatorname{Im}b_2\rangle\oplus\operatorname{Im}\iota$. If we take $s=t=0$ in the construction above, we are precisely stating that $\dim\operatorname{Im}(\phi_{b_1})\cap\operatorname{Im}(\phi_{b_2})=2$, which is the case under consideration. We can then construct bases $\{e_1,e_2,e_3,e_4,e_5\}$ and $\{f_1,f_2,f_3\}$ carrying out the same procedure described in Section 4.4.2.3: with notation as above, $\{e_2\,, e_3\}$ is simply a basis for $\mathrm{Im}(\phi_{b_1}) \cap \mathrm{Im}(\phi_{b_2})$. In fact, we obtain the same matrices appearing in (4.18).

Remark 4.4.13. Notice that, if *either* s or t is nonzero, we are under Case 1) of Proposition 4.2.3, and in Case 2) otherwise. ♠

We must now account for the gauge action. In Case 1), we chose up to scale a generator $\langle v_3 \rangle = \mathrm{Im}(\phi_{b_1}) \cap \mathrm{Im}(\phi_{b_2})$, and s, t were arbitrary. Similarly, in Case 2), one starts with a generator $\langle f_2 \rangle = b_1^{-1}(\mathrm{Im}\, b_1 \cap \mathrm{Im}\, b_2) \cap b_2^{-1}(\mathrm{Im}\, b_1 \cap \mathrm{Im}\, b_2)$. Different choices will rescale all basis vectors, with the exception of e_5 which is determined entirely by the framing, in both cases.

As in Section 4.3.2, let $\mu \in \mathbb{C}^*$ and take $g = (M_0, M_1)$ inside the gauge group $\mathrm{GL}_{(5,3)}(Q) \simeq \mathrm{GL}(5, \mathbb{C}) \times \mathrm{GL}(3, \mathbb{C})$, given by the matrices

$$M_0 = \begin{pmatrix} \mu & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad M_1 = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} . \tag{4.20}$$

The change of bases $\tilde{V}' := g \cdot \tilde{V}$ produces, via diagram (2.6),

$$\phi'_{b_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \phi'_{b_2} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mu^{-1}s & \mu^{-1}t & 0 \end{pmatrix} \qquad \phi'_{\iota} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$
$$\phi'_{a_1} = \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \mu x \\ \varepsilon_5 & \varepsilon_6 & \varepsilon_7 & \varepsilon_8 & \mu y \\ \varepsilon_9 & \varepsilon_{10} & \varepsilon_{11} & \varepsilon_{12} & \mu z \end{pmatrix} \quad \phi'_{a_2} = \begin{pmatrix} \varepsilon_{13} & \varepsilon_{14} & \varepsilon_{15} & \varepsilon_{16} & \mu u \\ \varepsilon_{17} & \varepsilon_{18} & \varepsilon_{19} & \varepsilon_{20} & \mu v \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & \varepsilon_{24} & \mu w \end{pmatrix} . \tag{4.21}$$

We have the following immediate consequence.

Proposition 4.4.14. *There is an isomorphism of* $\mathbb{C}$*-schemes*

$$\mathcal{U} \simeq \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^5}(-1)^{\oplus 2}) \times \mathbb{A}^{24} ,$$

where $\mathcal{U} \subset \mathcal{R}(\mathbb{C}\tilde{Q})$ *is given in Definition 4.4.12.*

Remark 4.4.15. Clearly $\mathcal{U}$ has the expected dimension. Indeed, $\dim \mathcal{R}^{\zeta}_{\mathbf{d}}(\mathbb{C}\tilde{Q}) = d_0 - q_Q(d_0, d_1) = d_0 + 4d_0 d_1 - d_0^2 - d_1^2 = 31 = 7 + 24$, for $\mathbf{d} = (5, 3, 1)$. ♠

Remark 4.4.16. Notice that the open subscheme $\mathcal{U}$ contains "almost all" of $\mathcal{N}_{2,5} \simeq \mathcal{R}^{\zeta}_{(5,3,1)}(\tilde{A})$. In fact, closed point corresponding to stable $\tilde{A}$-modules falling under Case 2γa-ii will belong to $\mathcal{U}$, *provided that the image* $\operatorname{Im}\iota$ *is not contained in only one among* $\operatorname{Im} b_1$ *and* $\operatorname{Im} b_2$. What we miss out are matrices of the form (4.16), which account for a $\mathbb{P}^1 \times \mathbb{P}^1$ worth of parameters. ♠

We are ready to apply Theorem 2.3.11, evaluating the derivatives of $\operatorname{tr} W$ and obtaining equations in x, y, z, u, v, w, s, t, cutting out $\tilde{A}$-modules of $\mathcal{N}^{\mathcal{U}}_{2,5}$ inside $\mathcal{U} \subset \mathcal{R}^{\zeta}_{(5,3,1)}(\mathbb{C}\tilde{Q})$. We then simplify those equations, applying dimension-reduction methods for critical loci. Let $\tilde{V}$ represent an isomorphism class of modules in $\mathcal{N}^{\mathcal{U}}_{2,5}$. Evaluating $\operatorname{tr} W$ on the matrices (4.19) we have

$$
\begin{aligned}
(\operatorname{tr} W)(\tilde{V}) = {} & \varepsilon_1\varepsilon_{14} + \varepsilon_{18}\varepsilon_2 + \varepsilon_{10}\varepsilon_{20} - \varepsilon_{12}\varepsilon_{23} + \varepsilon_{11}\varepsilon_{24} + \varepsilon_{22}\varepsilon_3 + \\
& - \varepsilon_{21}\varepsilon_4 + \varepsilon_{15}\varepsilon_5 + \varepsilon_{19}\varepsilon_6 + \varepsilon_{23}\varepsilon_7 - \varepsilon_{22}\varepsilon_8 + \varepsilon_{16}\varepsilon_9 + \\
& + s(\varepsilon_1 u + \varepsilon_2 v + \varepsilon_3 w) + t(\varepsilon_5 u + \varepsilon_6 v + \varepsilon_7 w) + \\
& - \varepsilon_{13}(\varepsilon_2 + sx) - \varepsilon_{17}(\varepsilon_3 + tx) - \varepsilon_{14}(\varepsilon_6 + sy) + \\
& - \varepsilon_{18}(\varepsilon_7 + ty) - \varepsilon_{15}(\varepsilon_{10} + sz) - \varepsilon_{19}(\varepsilon_{11} + tz)\,.
\end{aligned}
\tag{4.22}
$$

By considering derivatives, several of the affine variables must vanish

$$\varepsilon_4 = \varepsilon_9 = \varepsilon_{10} = \varepsilon_{11} = \varepsilon_{16} = \varepsilon_{21} = \varepsilon_{22} = \varepsilon_{23} = 0\,,$$

and others can be rewritten as below

$$
\begin{array}{llll}
\varepsilon_1 = sy + tz & \varepsilon_6 = tz & \varepsilon_{13} = sv + tw & \varepsilon_{18} = tw \\
\varepsilon_2 = -sx & \varepsilon_7 = -sx - ty & \varepsilon_{14} = -su & \varepsilon_{19} = -su - tv \\
\varepsilon_3 = -tx & \varepsilon_8 = -tx & \varepsilon_{15} = -tu & \varepsilon_{20} = -tu \\
\varepsilon_5 = sz & \varepsilon_{12} = -sx + ty & \varepsilon_{17} = sw & \varepsilon_{24} = -su - tv\,,
\end{array}
$$

leaving us with the equations

$$
\begin{aligned}
s(sv + 2tw) &= 0 & -s^2x + t^2z &= 0 \\
s^2u - t^2w &= 0 & -t(2sx + ty) &= 0 \\
t(2su + tv) &= 0 & -svx - twx + suy + tuz &= 0 \\
s(sy + 2tz) &= 0 & -swx - twy + suz + tvz &= 0\,.
\end{aligned}
$$

Therefore, the matrices have the following expected form:

$$
\begin{aligned}
a_1 &= \begin{pmatrix} sy+tz & -sx & -tx & 0 & x \\ sz & tz & -sx-ty & -tx & y \\ 0 & 0 & 0 & -sx-ty & z \end{pmatrix} \\
a_2 &= \begin{pmatrix} sv+tw & -su & -tu & 0 & u \\ sw & tw & -su-tv & -tu & v \\ 0 & 0 & 0 & -su-tv & w \end{pmatrix} \\
b_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad b_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ s & t & 0 \end{pmatrix} .
\end{aligned}
\tag{4.23}
$$

Remark 4.4.17. As discussed above, when $s = t = 0$, we obtain modules of the form (4.18), and those stable $\tilde{A}$-modules correspond to stable pairs on the $\mathbb{P}^5$ component of $\mathcal{N}_{2,5}$. ♠

In order to perform dimension reduction on (4.22), we can first of all apply Lemma 1.4.8 multiple times to remove to terms with free affine variables (whose variable appears only once), like $\varepsilon_{21}\varepsilon_4$ and $\varepsilon_{16}\varepsilon_{19}$. Then, for example, we consider terms like

$$-\varepsilon_{12}\varepsilon_{23} + \varepsilon_{23}\varepsilon_7 = (\varepsilon_7 - \varepsilon_{12})\varepsilon_{23} \,.$$

We apply Lemma 1.4.9 whose effect is that we can ignore the entire term, but retain ε_7 as a variable. Continuing in this fashion, a long but straightforward application of Lemma 1.4.8 and Lemma 1.4.9 removes the remaining superfluous terms containing affine variables. We summarise our result in the statement below, which complements Theorem 3.4.1.

Theorem 4.4.18. *The moduli scheme $\mathcal{N}_{2,5}$ contains an open subscheme $\mathcal{N}_{2,5}^{\mathcal{U}}$, as in Definition 4.4.12, and it is described as $\mathcal{N}_{2,5}^{\mathcal{U}} \simeq \mathrm{Crit}(q) \subseteq \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^5}(-1)^{\oplus 2})$, where*

$$q = s^2 \begin{vmatrix} u & x \\ v & y \end{vmatrix} + 2st \begin{vmatrix} u & x \\ w & z \end{vmatrix} + t^2 \begin{vmatrix} v & y \\ w & z \end{vmatrix} .$$

We can explore the geometry of $\mathrm{Crit}\, q$. The full set of equations is below:

$$
\begin{aligned}
s(sv + 2tw) &= 0 \\
s(sy + 2tz) &= 0 \\
t(2su + tv) &= 0 \\
t(2sx + ty) &= 0
\end{aligned}
$$

$$
\begin{aligned}
&s^2u - t^2w = 0 \qquad && svx + twx - suy - tuz = 0 \\
& && swx + twy - suz - tvz = 0 \\
&t^2z - s^2x = 0 && suy + tuz - svx - twx = 0 \\
& && suz + tvz - swx - twy = 0\,.
\end{aligned}
$$

There are two components. One, which we will call A, has ideal (s, t). The other, called U, has ideal generated by

$$
\begin{aligned}
(ys + 2zt, vs + 2wt, yt + 2xs, vt + 2us, y^2 - 4xz \\
v^2 - 4uw, yv - 4xw, xv - yu, xw - zu, zv - yw)\,.
\end{aligned}
$$

Remark 4.4.19. Notice that the equations $xv - yu = xw - zu = zv - yw = 0$, corresponding to the last three terms in the provided list of generators for the ideal above, amount to

$$
\operatorname{rank}\begin{pmatrix} x & y & z \\ u & v & w \end{pmatrix} \leq 1\,. \tag{4.24}
$$

If we consider homogeneous polynomials P and Q, as in (3.17) up to renaming the coefficients, the inequality above corresponds to these polynomials being proportional to each other. The condition (4.24) is locally cut by two equations: for example, away from the line $\{y = 0\}$ it is enough to require $xv - yu = 0 = zv - yw$.

The terms y^2-4xz and v^2-4uw are respectively the discriminants of P and Q, and cut out the locus where each has a double root (unless it is the zero polynomial). Of course, if the two polynomials are proportional, it suffices to enforce the discriminant being zero on any of P or Q which is not the zero polynomial, and we can show this by elementary techniques. Consider the inequality

$$
\operatorname{rank}\begin{pmatrix} y & 2x & v & 2u \\ 2z & y & 2w & v \end{pmatrix} \leq 1\,, \tag{4.25}
$$

from which the equations $y^2 - 4xz = yv - 4xw = v^2 - 4uw = 0$ follow as three adjacent minors. From (4.24) we have vanishing of determinants

$$
\begin{vmatrix} y & v \\ 2z & 2w \end{vmatrix} = \begin{vmatrix} 2x & 2u \\ y & v \end{vmatrix} = 0\,.
$$

Hence, for example away from $\{y = 0\}$, the equation $y^2 - 4xz = 0$ is sufficient to entail condition (4.25) above. ♠

There are embedded components A $\supset$ B $\supset$ C. The first component B is given by

$$(s, t, z^2u^2 - yzuv + xzv^2 + y^2uw - 2xzuw - xyvw + x^2w^2)\,,$$

that is, it is a quartic hypersurface inside A, cut by the determinantal equation

$$\begin{vmatrix} x & y & z & 0 \\ 0 & x & y & z \\ u & v & w & 0 \\ 0 & u & v & w \end{vmatrix} = 0\,.$$

The component C has ideal

$$(s, t, xv - yu, xw - zu, yw - zy)\,,$$

that is it is cut inside A by the condition (4.24). The intersection D := A $\cap$ U has ideal

$$(s, t, y^2 - 4xz, xv - yu, xw - zu, yw - zv)\,,$$

and D is cut inside C by $y^2 - 4xz$. In Appendix A.3.2 we compute the dimensions of the Zariski tangent spaces.

Proposition 4.4.20. *The underlying reduced scheme* $(\mathrm{Crit}(q))_{\mathrm{red}}$ *is reducible, with the embedded subschemes* B*,* C*,* U*, along which the Zariski tangent space has dimension* 6*,* 7 *and* 3 *respectively.*

Remark 4.4.21. By considering the defining equations, similarly to what was done in Section 3.4.9, there is a correspondence between A, B, C, D and $\mathbb{P}^5, \mathbf{H}, \mathbf{G}, \mathbf{F}$ in the decomposition of $\mathcal{N}_{2,5}$ as in the statement of Theorem 3.4.1. ♠

Chapter 5

General insights into $\mathcal{N}_{2,n}$

We can now summarise some expected features of the moduli spaces $\mathcal{N}_{2,n}$ for $n \geq 3$. The present section is conjectural.

5.1 Geometric description

The starting point is attempting to thicken the zero-section Z along a line bundle $\mathcal{L}$, given by the image of a map $\alpha \colon \mathcal{O}_Z(2-n) \longrightarrow \mathcal{O}_Z(-1)^{\oplus 2}$; set-theoretically, such maps are parametrised by $\mathbb{P}^{2n-5}$. Therefore we have an open subscheme of $\mathcal{N}_{2,l}$ whose reduced geometry is that of a projective space. The construction of a family over $\mathbb{P}^{2n-5}$ proceeds as in Section 3.3.2, and we look here at closed points of the moduli space.

If the two polynomials P, Q of degree $n-3$ defining α do not have common factors, we obtain an honest image line bundle $\mathcal{L} \simeq \mathcal{O}_Z(2-n)$, and a stable pair is just the thickening along $\mathcal{L}$, an extension $\mathcal{F} := j_* O_C \in \operatorname{Ext}^1_X(O_Z, \mathcal{O}_Z(n-2))$, which indeed satisfies $\chi(\mathcal{F}) = n$ and admits a unique section making it into a stable pair; we do not get marked points points on the support curve, in this case.

There are various level of degeneracy of the map α, corresponding to possible common factors of the polynomials P and Q. In general, α will factor as

$$\begin{tikzcd} \mathcal{O}_Z(2-n) \arrow[rr, hook, "\alpha"] \arrow[dr, hook, "\alpha'"'] & & \mathcal{O}_Z(-1)^{\oplus 2} \\ & \mathcal{O}_Z(-d)\,, \arrow[ur, hook, "\lambda"'] & \end{tikzcd} \tag{5.1}$$

when the two polynomials have $n-d-2$ common factors, counted with multiplicities: that is, $P = MP'$ and $Q = MQ'$, for M homogeneous of degree $n-d-2$ corresponding to the map α'. This is an immediate generalisation of Lemma 3.3.2. The map λ, obtained from P' and Q', gives a line bundle $\mathcal{L} \simeq \mathcal{O}(-d) \subset \mathcal{N}_{Z/X}$, and α' defines via

M a zero-dimensional subscheme of Z of length $n-d-2$, that is the number of common factors. This led to the constructions in Section 3.3 and Section 3.4.

From now on, for simplicity we will be looking at the innermost stratum, where the two polynomials are proportional. For intermediate cases, the procedure is analogous. In the situation we are considering, α factors as

$$\begin{tikzcd} \mathcal{O}_Z(2-n) \arrow[rr, hook, "\alpha"] \arrow[dr, hook, "\alpha'"'] & & \mathcal{O}_Z(-1)^{\oplus 2} \\ & \mathcal{O}_Z(-1) \arrow[ur, hook, "l"'] \arrow[dr, two heads] & \\ & & \mathcal{O}_{\mathbf{p}}(-1)\,. \end{tikzcd} \tag{5.2}$$

The map $[l] = [l_0, l_1] \in \mathbb{P}^1$ gives a line sub-bundle $\mathcal{L} \simeq \mathcal{O}(-1)$ of $\mathcal{N}_{Z/X}$, along which we can thicken Z obtaining the double curve C with $\mathcal{O}_C \in \mathrm{Ext}^1_X(O_Z, \mathcal{O}_Z(1))$. The map α' defines, after an appropriate twist, an ideal sheaf $\mathcal{I}_{\mathbf{p},Z}$, and we obtain therefore a subscheme of $\mathbf{p} \subset Z$, of length $n-3$.

We use this subscheme $\mathbf{p}$ to define a stable pair, proceeding as in Section 3.3 and Section 3.4. Consider the diagram below,

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_Z(1) \arrow[r] \arrow[d, equal] & \mathcal{I}_{\mathbf{p}/C} \arrow[r] \arrow[d, hook] & \mathcal{I}_{\mathbf{p}/Z} \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathcal{O}_Z(1) \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d, two heads] & \mathcal{O}_Z \arrow[r] \arrow[d, two heads] & 0 \\ & & \mathcal{O}_{\mathbf{p}} \arrow[r, equal] & \mathcal{O}_{\mathbf{p}} & \end{tikzcd} \tag{5.3}$$

and construct a stable pair $0 \to \mathcal{O}_C \to \mathcal{F}$ using Proposition 1.3.5. In particular, $\mathcal{F} \simeq \mathcal{H}om_C(\mathcal{I}_{\mathbf{p}/C}, \mathcal{O}_C)$, and $\chi(\mathcal{F}) = n$.

We call this construction of *type* $\boxed{1}$, or $(n-3)\boxed{1}$ if we need to stress the length of the zero-dimensional subscheme of Z being considered.

There are more stable pairs supported on C, which we are not finding with this approach. Consider $\mathcal{J}_{\mathbf{q}}$ the ideal sheaf of a length-$(n-3)$ zero-dimensional subscheme $\mathbf{q}$ of C, *which is not a subscheme of* Z. In geometrical terms, this means that there is at least one double point $q \subset \mathbf{q}$, such that the Zariski tangent space to p has a component in the direction of the line bundle $\mathcal{L}$ along which the double curve C is constructed from Z.

We have now the diagram

$$\begin{tikzcd}
 & & \mathcal{J}_{\mathbf{q}} \arrow[d, hook] & & \\
0 \arrow[r] & \mathcal{O}_Z(1) \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d, two heads] & \mathcal{O}_Z \arrow[r] & 0 \\
 & & \mathcal{O}_{\mathbf{q}}\,, & &
\end{tikzcd} \tag{5.4}$$

and consider the stable pair constructed from Proposition 1.3.5, with $\mathcal{F} := \mathcal{H}om_C(\mathcal{I}_{\mathbf{q}/C}, \mathcal{O}_C)$, which satisfying $\chi(\mathcal{F}) = n$. We call this construction of *type* $\boxed{2}$. More generally, once the thickening C is fixed, we consider constructions of type $i\boxed{1} + j\boxed{2}$, where $i + 2j = n - 3$ and j is the number of double points not lying on Z.

Remark 5.1.1. Observe that, since we are considering *double* thickenings C of Z, there is no need for construction of type $\boxed{3}$. Indeed, a triple point would yield either a construction of type $\boxed{1} + \boxed{2}$ or $\boxed{1} + \boxed{1} + \boxed{1}$. ♠

We can think about deformations of stable pairs in terms of collision of points in the zero-locus, ignoring points away from those of interest. When a pair of type $2\boxed{1}$ admits a double point $\mathbf{p}$ supported at $p \in Z$, above it we can construct a family of type $\boxed{2}$ pairs, again supported at p, parametrised by an extra $\mathbb{P}^1$: the choice corresponds to choosing a subscheme of C, that is to say determine the normal bundle as a rank-1 sub-bundle

$$\mathcal{N}_{\mathbf{p}/C} \subset \mathrm{T}_{Z,p} \oplus \mathcal{L}_p\,. \tag{5.5}$$

We can at this point perform a naive moduli count of the deformation space to a given stable pair: since the starting point is a map $\alpha\colon \mathcal{O}(2-n) \longrightarrow \mathcal{O}(-1)^{\oplus 2}$, we know that on an open subscheme of the moduli space, stable pairs will correspond to the choice of a point in $\mathbb{P}^{2n-5}$. A contribution of type $\boxed{1}$ will add +1 moduli, corresponding to the infinitesimal deformation of the point in the direction of the thickening line bundle $\mathcal{L}$; this is, for example, the case for the quadric $\mathbf{Q} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathcal{N}_{2,4}$ as described in Section 3.3.

A construction of type $\boxed{2}$ requires more care. We have one positive contributions,

- +1: we can deform the tangent space (5.5).

There are, however, deformations parametrised by the ambient $\mathbb{P}^{2n-5}$ which are no longer admissible:

- -1: we can't deform the two points away from each other;

- -2: each of the two points can't move away from C.

In conclusion, the net influence of a contribution of type $\boxed{2}$ is -2 moduli.

Remark 5.1.2. In evocative terms, we can say that a contribution of type $\boxed{2}$ is "locked". That is, the corresponding stable pair cannot be deformed in such a way as to make the zero-locus into two reduced points. $\spadesuit$

5.2 Quiver description

We consider a $\tilde{A}$-modules $\tilde{V}$, with dimension vector $(n, n-2, 1)$ which is stable with respect to a parameter $\tilde{\zeta}$ in the PT-chamber, as in 2.3.22. Recall that, calling $V_b := \langle \operatorname{Im} b_1, \operatorname{Im} b_2 \rangle$, by Proposition 4.2.3 either $V_0 \simeq V_b$ or $V_0 \simeq V_b \oplus \operatorname{Im} \iota$. We can consider the open subscheme of $\mathcal{N}_{2,n}$ where $\operatorname{Im} \iota \not\subseteq V_b$, that is the second case. We show that this component is isomorphic to a $\mathbb{P}^{2n-5}$. Arguing as in Section 4.4, we can prove that $a_{i|V_b} = 0$ for $i = 1, 2$.

Proposition 5.2.1. *Let $\tilde{V}$ be $\tilde{\zeta}$-stable, with dimension vector $\mathbf{v} = (n, n-2, 1)$, where $\tilde{\zeta}$ is in the PT-chamber. If $\operatorname{Im} \iota \not\subseteq V_b$, then $\iota \neq 0$ and $V_0 \simeq V_b \oplus \operatorname{Im} \iota$. In particular,* $\dim V_b = n - 1$.

Consider the submodule $\tilde{U} := (V_b, V_1, 0)$, with the natural maps, which has dimension vector $(n-1, n-2, 0)$ and is therefore allowed. Since $a_{i|V_b} = 0$, we can regard $\tilde{U}$ as a representation of the Kronecker quiver, repeating the reasoning of Proposition 4.4.9. Therefore, we have the following.

Proposition 5.2.2. *Let $\tilde{V}$ be as in Proposition 5.2.1. Up to the action of* $\mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$ *$\tilde{V}$ can be written in the following form:*

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\iota"] & \mathbb{C}^n \arrow[r, bend left=40, "a_1"] \arrow[r, bend left=15, "a_2"'] & \mathbb{C}^{n-2} \arrow[l, bend left=15, "b_1"] \arrow[l, bend left=40, "b_2"'] \end{tikzcd},$$

where

$$\iota = \begin{pmatrix} 0_{1,n-1} & 1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} \mathrm{Id}_{n-2} \\ 0_{1,n-2} \\ 0_{1,n-2} \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0_{1,n-2} \\ \mathrm{Id}_{n-2} \\ 0_{1,n-2} \end{pmatrix},$$

$$a_1 = \begin{pmatrix} 0_{n-1,n-2} & * \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0_{n-1,n-2} & * \end{pmatrix}.$$

We can interpret this result by saying that there is an open component of the moduli scheme where $a_{i|\operatorname{Im}\iota}$ carries all information on the stable pairs. The residual action of $\mathbb{C}^*$ on the framing vertex acts by scaling the $2n-4$ non-zero entries of the a_i-matrices, and proportional a_i maps will produce equivalent modules. This is in accordance with Section 5.1.

5.3 Examples of naive moduli counts

Each moduli scheme $\mathcal{N}_{2,n}$ contains an open subscheme, whose reduced structure is isomorphic to $\mathbb{P}^{2n-5}$, parametrising pairs of polynomials giving a thickening C of the zero section Z. There is an innermost stratum, corresponding to proportional polynomials: we can perform naive moduli counts on this stratum using the rule described above. By looking at the stratum corresponding to proportional polynomials, we can exploit the formalism of the 3-fold vertex described in Section 2.2.2. Indeed, torus-fixed stable pairs must be supported on a curve C thickened by $\mathcal{O}(-1) \xhookrightarrow{l} \mathcal{O}(-1)^{\oplus 2}$, with $l = [0:1]$ or $l = [1:0]$, and their reduced zero-locus must be supported at the points $[0:1]$ and $[1:0]$ in Z. See Section 2.2 and Appendix B for more details.

5.3.1 $\mathcal{N}_{2,3}$

This is just a $\mathbb{P}^1$, and all pairs have form $0\boxed{1} + 0\boxed{2}$. Indeed, the only two torus-fixed point have box configuration $[0\,|\,0]$. They are the structure sheaf of the zero-section Z thickened along one of the two summands in $\mathcal{O}(-1)^{\oplus 2}$.

5.3.2 $\mathcal{N}_{2,4}$

Generically a $\mathbb{P}^3$. The pairs in the degeneraci locus, that is the quadric $\mathbf{Q} \simeq \mathbb{P}^1 \times \mathbb{P}^1$, all have type $1\boxed{1}$, and deformation space indeed of dimension 3+1.

5.3.3 $\mathcal{N}_{2,5}$

Generically a $\mathbb{P}^5$. Where the polynomials P and Q have a common factor, they lead to a stable pair of type $1\boxed{1}$, whose tangent space has dimension $5+1$.

When P and Q are proportional, we have pairs of type $2\boxed{1}$, with deformations of dimension $5+2$. We find among them the six torus-fixed pairs of box configurations $[1,1\,|\,0]$, $[1\,|\,1]$ and $[0\,|\,1,1]$.

When P and Q are proportional and square, we also have pairs of type $\boxed{2}$ with deformations $5-2$. Here are the four torus-fixed points with box configurations $[2\,|\,0]$ and $[0\,|\,2]$.

5.3.4 $\mathcal{N}_{2,6}$

Generically a $\mathbb{P}^7$. We look only at the strata containing the torus fixed points, so that we can compare results with Appendix B. Stable pairs of type $3\boxed{1}$ will have deformation space of dimension $7+3$, and there are eight torus-fixed pairs of this type, with box configurations

$$[1,1,1\,|\,0]\ ,\quad [1,1\,|\,1]\ ,\quad [1\,|\,1,1]\ ,\quad [0\,|\,1,1,1]\ .$$

For type $\boxed{1}+\boxed{2}$, the deformation space has dimension 7-2+1, with eight torus-fixed pairs of shapes

$$[1,2\,|\,0]\ ,\quad [1\,|\,2]\ ,\quad [2\,|\,1]\ ,\quad [0\,|\,2,1]\ .$$

5.3.5 $\mathcal{N}_{2,7}$

Generically a $\mathbb{P}^9$. Again, we only look at the locus where the two polynomials only have common factors. Pairs of type $4\boxed{1}$ have deformation space of dimension $9+4$, with ten torus-fixed pairs of shapes

$$[1,1,1,1\,|\,0]\ ,\quad [1,1,1\,|\,1]\ ,\quad [1,1\,|\,1,1]\ ,\quad [1\,|\,1,1,1]\ ,\quad [0\,|\,1,1,1,1]\ .$$

Pairs of type $2\boxed{1}+\boxed{2}$ have $(9-2+2)$-dimensional deformation space, and include twelve fixed pairs

$$[1,1,2\,|\,0]\ ,\quad [1,2\,|\,1]\ ,\quad [1\,|\,2,1]\ ,\quad [2\,|\,1,1]\ ,\quad [1,1\,|\,2]\ ,\quad [0\,|\,2,1,1]\ .$$

Finally, stable pairs of type $2\boxed{2}$ have deformation space of dimension $9-2-2$, and there are six stable pairs of shapes

$$[2,2\,|\,0]\ ,\quad [2\,|\,2]\ ,\quad [0\,|\,2,2]\ .$$

Appendix A

Code

A.1 `Wolfram Mathematica`: 3-fold Vertex

We use computations similar to [MNOP06, §4.6] and [PT09b, §4.4]. The aim is to find the dimensions of positive and negative summands of the virtual tangent space, which are equal by Serre duality. In the following, V_α and V_β are polynomials corresponding to the vertex data, and $E_{\alpha\beta}$ is the generating function for the edge partition. We use variables (a, b, c): the zero-section of the resolved conifold is given in both charts around the vertices as $b = c = 0$.

```
VirTChar[Valpha_, Vbeta_, Ealphabeta_] := (
  (*VERTEX*)
  Fa = Valpha;
  Fabar = Fa /. {a → a^(-1), b → b^(-1), c → c^(-1)};
  Fb = Vbeta;
  Fbbar = Fb /. {a → a^(-1), b → b^(-1), c → c^(-1)};

  (*Stable Pairs Chi Vertex alpha *)
  PtrChiVa = Fa - Fabar / (a*b*c) + Fa*Fabar*(1 - a) (1 - b) (1 - c) / (a*b*c);
  PtrChiVb = Fb - Fbbar / (a*b*c) + Fb*Fbbar*(1 - a) (1 - b) (1 - c) / (a*b*c);

  (*EDGE*)
  Fab = Ealphabeta;
  Fabbar = Fab /. {a → a^(-1), b → b^(-1), c → c^(-1)};
  delta = 1/(1 - a) + (a^(-1)) / (1 - a^(-1));

  (*Stable Pairs Chi Edge alpha *)
  PtrChiEab = -delta*(-Fab - Fabbar / (b*c) + Fab*Fabbar*(1 - b)*(1 - c) / (b*c));

  (*REDISTRIBUTION*)
  Gab = -Fab - Fabbar / (b*c) + Fab*Fabbar *(1 - b)*(1 - c) / (b*c);
  (* change of coordinates going to the Ubeta patch *)
  Gabev = Gab /. {b → a*b, c → a*c};

  (*New Vertex Character - only one neighboring vertex*)
  Va = PtrChiVa + Gab / (1 - a);
  Vb = PtrChiVb + Gab / (1 - a);

  (*New Edge Character*)
  Eab = a^(-1) *Gab / (1 - a^(-1)) - (Gabev) / (1 - a^(-1));

  (*T-character of virtual tg space*)
  TrT := Va + Vb + Eab;

  Expand[FullSimplify[TrT]]
 )
```

The following routine takes in input the vertex and edge characters, and gives in output the character of the virtual tangent space, counting positive and negative coefficients.

```
ConiVT[VaChar_, VbChar_, Edge_] := (
  Print[{VaChar, VbChar, Edge} // StandardForm];
  VTC = VirTChar[VaChar, VbChar, Edge];
  Print[VTC // StandardForm];
  L = (List @@ VTC) /. {a → 1, b → 1, c → 1};
  Print[L];
  {Select[L, Positive] // Total, Select[L, Negative] // Total}
 )
```

In the following example, we consider the case of a box configuration with shape $[2\,|\,0]$ in $\mathcal{N}_{2,5}$, as in Appendix B.

```
ConiVT[a ^ (-1) * (1 + b) / (1 - a), (1 + b) / (1 - a), 1 + b]
```

$$\left\{\frac{1+b}{(1-a)\,a}, \frac{1+b}{1-a}, 1+b\right\}$$

$$\frac{1}{a} - \frac{1}{a\,b^2} + \frac{b}{a} - \frac{1}{b^2\,c} - \frac{1}{b\,c} + \frac{b}{c}$$

```
{1, -1, 1, -1, -1, 1}
{3, -3}
```

We then read off the coefficients of the *virtual tangent space*, as in [PT09b, §4.3].

A.2 Wolfram Mathematica: Quiver Representations

The routine below takes in input four matrices A_1, A_2, B_1, B_2, corresponding to a $\mathbb{C}Q$-module, as in Section 2.3.4, and a list $\mathbf{e}$ of variables. The routine returns a (not necessarily complete) list of equations corresponding to the conifold quiver potential. In Section 4.4, we deduce the form of the B_1, B_2 matrices, and use this routine to obtain constraints on the A_1, A_2 matrices.

```
CritLocus[A1_, A2_, B1_, B2_, ee_] :=
Module[{q, vars, Dq, Dq1, Dq1eq, eSol, Dq2, eqns},
q = Tr[A1.B1.A2.B2 - A1.B2.A2.B1];
vars = {DeleteDuplicates[
   DeleteCases[Level[{A1, A2, B1, B2}, {-1}], _Integer]]};
Dq = D[q, vars];
Dq1 = Select[Dq, AllTrue[Coefficient[#, ee], IntegerQ] &];
Dq1eq = # == 0 & /@ Dq1 ;
eSol = Solve[Dq1eq, ee] // Flatten;
Dq2 = Complement[Dq, Dq1];
```

```
  eqns = DeleteCases[Dq2 /. eSol // Simplify, 0];
  eqns // TableForm]
```

A.3 Macaulay2: Critical loci

We provide here the code used to prove Theorem 4.3.6 and Theorem 4.4.18.

A.3.1 $\mathcal{N}_{2,4}$

Below the basic setup, where q is the potential obtained from the quiver description.

```
R=QQ[x,y,z,w,t]
q=t^2*(x*w-y*z)
J=ideal(diff(x,q),diff(y,q),diff(z,q),diff(w,q),diff(t,q))
```

The ideal is

$$\left(w\,t^2, -z\,t^2, -y\,t^2, x\,t^2, -2\,y\,z\,t + 2\,x\,w\,t\right) .$$

We find the components with the commands

```
-- components
minimalPrimes J
-- embeddeed components
associatedPrimes J
```

giving output

```
{ideal (y, x, w, z), ideal t}
{ideal t, ideal (t, y*z - x*w), ideal (w, z, y, x)}
```

Remark A.3.1. Recall that we are not including the information that $[x : y : z : w]$ are homogeneous coordinates, therefore the critical locus must satisfy $t = 0$, and t is indeed a *nilpotent* direction. ♠

We check that indeed those components are irreducible.

```
-- irreducibility
primaryDecomposition ideal(t)
primaryDecomposition ideal(x*w-y*z)
primaryDecomposition ideal(x,w,y,z)
```

Finally, we can compute the dimension of the Zariski tangent space.

```
-- Zariski tg spaces
A=R/J
-- point NOT on the quadric
n=ideal (x-1,y,z,w-1,t)
degree Hom(A,n/n^2)
-- point on the quadric
m= ideal (x-1,y,z,w,t)
degree Hom(A,m/m^2)
```

We obtain dimensions 4 and 5, respectively. This is one more than expected, since we are considering *affine* coordinates. Once we include the $\mathbb{C}^*$ action on $\mathbb{C}[x, y, z, w]$ we confirm the expected dimensions.

A.3.2 $\mathcal{N}_{2,5}$

Below, the basic setup.

```
R=QQ[x,y,z,u,v,w,s,t]
q=-s^2*v*x-2*s*t*w*x+s^2*u*y-t^2*w*y+2*s*t*u*z+t^2*v*z
J=ideal(diff(x,q),diff(y,q),diff(z,q),diff(u,q),diff(v,q),
   diff(w,q),diff(s,q),diff(t,q))
```

We see that there are two main components,

```
-- components
minJ=minimalPrimes J
J1=minJ#0;
J2=minJ#1;
```

one of which is given by

$$(t, s) \ .$$

and corresponds to the $\mathbb{P}^5$. The intersection of the two components is given by

```
-- intersection of the two components
intJ1J2=trim(J1+J2)
```

and we find that it has ideal

$$\Big(t, s, v^2 - 4\,u\,w, z\,v - y\,w, y\,v - 4\,x\,w, z\,u - x\,w, y\,u - x\,v, y^2 - 4\,x\,z\Big) ..$$

We proceed to find the embedded components.

```
-- EMBEDDED COMPONENTS
comp=associatedPrimes J
-- The mbient P^5 and its strata {s=t=0}
-- P^5
P=comp#0
P==J2
-- Hypersurface given as resultant (one common root)
C1=comp#1
-- Quadric of rk<2 (two common roots)
C2=comp#3
-- The extra component
G=comp#2
G==J1
```

obtaining the embedded components C_1

$$\left(t, s, z^2u^2 - y\,z\,u\,v + x\,z\,v^2 + y^2u\,w - 2\,x\,z\,u\,w - x\,y\,v\,w + x^2w^2\right) ,$$

and C_2

$$(t, s, z\,v - y\,w, z\,u - x\,w, y\,u - x\,v) .$$

Finally, we compute the dimension of the Zariski tangent space.

```
-- Zariski tg spaces
A=R/J

-- Two generic polynomials (dimension 5+1)
m=ideal(t,s,x-1,y-1,z,u-1,v+3,w+2);degree Hom(A,m/m^2)

-- Two polynomials with one common root (dimension 6+1)
m=ideal(t,s,x-1,y-1,z,u-1,v+1,w);degree Hom(A,m/m^2)

-- Two proportional polynomials (dimension 7+1)
m=ideal(t,s,x,y,z-1,u,v,w-1);degree Hom(A,m/m^2)

-- Two polynomials proportional and square (dimension 7+1)
m=ideal(t,s,x-1,y+2,z-1,u-1,v+2,w-1);degree Hom(A,m/m^2)
m=ideal(t,s,x-1,y-2,z-1,u-1,v-2,w-1);degree Hom(A,m/m^2)

-- We move into the extra component (dimension 3+1)
m=ideal(t-1,s-1,x-1,y+2,z-1,u-1,v+2,w-1);degree Hom(A,m/m^2)
```

We obtain the expected dimensions, augmented by 1, since we are ignoring, in the `Macaulay2` code, the $\mathbb{C}^*$-action on the coordinates.

Appendix B

Box Configurations

Our base Cohen-Macaulay Torus-invariant curve C is the double thickening of the zero-section Z, that is $\mathbb{P}^1 \subset \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})$, with thickening line bundle either of the two $\mathcal{O}_{\mathbb{P}^1}(-1)$ summands. We consider monomial ideals $m_p \subset \mathcal{O}_C$ of varying length, marked with stars, corresponding to zero-dimensional subschemes of C. For each 2d partition, we get two 3d partitions, according to the two possible torus-fixed thickenings of Z.

Let $(\mathcal{F}, s)$ be a $\mathbf{T}$-fixed stable pair, and consider as usual

$$I^\bullet := \{\mathcal{O}_X \xrightarrow{s} \mathcal{F}\}\,.$$

We are interested in computing the virtual tangent space

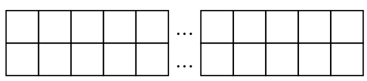
$$\mathcal{T}_{[I^\bullet]} := \mathrm{Ext}^1_X(I^\bullet, I^\bullet) - \mathrm{Ext}^2_X(I^\bullet, I^\bullet)\,.$$

We use a and b, c for the tangent and normal directions to the zero-section Z, respectively: the edge character is always

$$\mathsf{F}_{\alpha\beta} = b + 1\,,$$

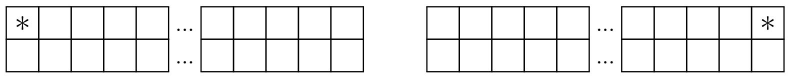
while the vertex characters depend on the shape of the box configuration corresponding to the $\mathbf{T}$-fixed stable pair considered. All the computations of the $\mathbf{T}$-character $\mathrm{tr}_{\mathcal{T}_{[I^\bullet]}}$ were performed using `Wolfram Mathematica`, applying the results on the 3-fold vertex explained in §2.2.2. The explicit code is provided in Appendix A.1.

$\mathcal{N}_{2,3}$ 2 fixed points, length 0



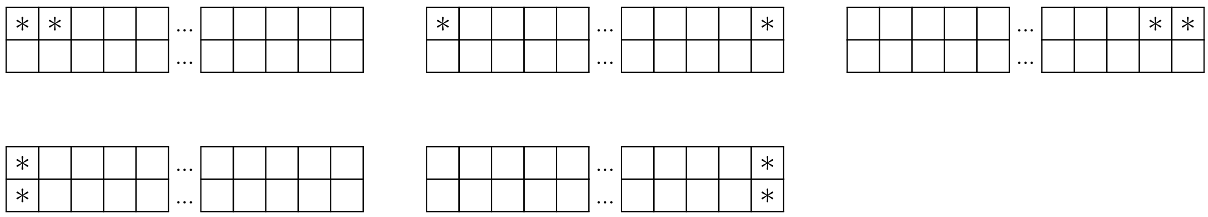
Shape	F_α	F_β	$\mathrm{tr}_{\mathcal{T}_{[I\bullet]}}$	$\dim \mathcal{T}_{[I\bullet]}$
$[0 \mid 0]$	$\frac{b+1}{1-a}$	$\frac{b+1}{1-a}$	$\frac{b}{c} - \frac{1}{ab^2}$	$1-1$

$\mathcal{N}_{2,4}$ 4 fixed points, length 1



Shape	F_α	F_β	$\mathrm{tr}_{\mathcal{T}_{[I\bullet]}}$	$\dim \mathcal{T}_{[I\bullet]}$
$[1 \mid 0]$	$\frac{b}{a} + \frac{b+1}{1-a}$	$\frac{b+1}{1-a}$	$-\frac{1}{ab^2} + \frac{b}{ac} - \frac{1}{ac} + \frac{1}{a} - \frac{1}{b^2} + \frac{b}{c} - \frac{1}{bc} + \frac{1}{b}$	$4-4$

$\mathcal{N}_{2,5}$ 10 fixed points, length 2



Shape	F_α	F_β	$\mathrm{tr}_{\mathcal{T}_{[I\bullet]}}$	$\dim \mathcal{T}_{[I\bullet]}$
$[1,1 \mid 0]$	$\left(\frac{1}{a^2} + \frac{1}{a}\right) b + \frac{b+1}{1-a}$	$\frac{b+1}{1-a}$	$\frac{b}{a^2c} - \frac{1}{a^2c} + \frac{1}{a^2} - \frac{a}{b^2} - \frac{1}{ab^2} - \frac{a}{bc} + \frac{b}{ac} + \frac{a}{b} - \frac{1}{ac} + \frac{1}{a} - \frac{1}{b^2} + \frac{b}{c} - \frac{1}{bc} + \frac{1}{b}$	$7-7$
$[1 \mid 1]$	$\frac{b}{a} + \frac{b+1}{1-a}$	$\frac{b}{a} + \frac{b+1}{1-a}$	$-\frac{1}{ab^2} + \frac{2b}{ac} - \frac{2}{ac} + \frac{2}{a} - \frac{2}{b^2} + \frac{b}{c} - \frac{2}{bc} + \frac{2}{b}$	$7-7$
$[2 \mid 0]$	$\frac{b+1}{(1-a)a}$	$\frac{b+1}{1-a}$	$-\frac{1}{ab^2} + \frac{b}{a} + \frac{1}{a} - \frac{1}{b^2c} + \frac{b}{c} - \frac{1}{bc}$	$3-3$

$\mathcal{N}_{2,6}$ 16 fixed points, length 3

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					...					

Shape	F_α	F_β	$\mathrm{tr}_{\mathcal{T}_{[I\bullet]}}$	$\dim \mathcal{T}_{[I\bullet]}$
$[1,2\,\vert\,0]$	$\frac{b}{a^2}+\frac{b+1}{1-a}+\frac{b+1}{a}$	$\frac{b+1}{1-a}$	$\frac{b}{a^2}-\frac{a}{b^2c}-\frac{1}{ab^2}+\frac{b}{ac}-\frac{1}{ac}+\frac{2}{a}-\frac{1}{b^2}+\frac{b}{c}-\frac{2}{bc}+\frac{1}{b}$	$6-6$
$[2\,\vert\,1]$	$\frac{b+1}{1-a}+\frac{b+1}{a}$	$\frac{b}{a}+\frac{b+1}{1-a}$	$-\frac{1}{ab^2}+\frac{b}{ac}+\frac{b}{a}-\frac{1}{ac}+\frac{2}{a}-\frac{1}{b^2c}-\frac{1}{b^2}+\frac{b}{c}-\frac{2}{bc}+\frac{1}{b}$	$6-6$
$[1,1,1\,\vert\,0]$	$(\frac{1}{a^3}+\frac{1}{a^2}+\frac{1}{a})b+\frac{b+1}{1-a}$	$\frac{b+1}{1-a}$	$\frac{b}{a^3c}-\frac{1}{a^3c}+\frac{1}{a^3}-\frac{a^2}{b^2}-\frac{a^2}{bc}+\frac{b}{a^2c}+\frac{a^2}{b}-\frac{1}{a^2c}+\frac{1}{a^2}-\frac{a}{b^2}-\frac{1}{ab^2}-\frac{a}{bc}+\frac{b}{ac}+\frac{a}{b}-\frac{1}{ac}+\frac{1}{a}-\frac{1}{b^2}+\frac{b}{c}-\frac{1}{bc}+\frac{1}{b}$	$10-10$
$[1,1\,\vert\,1]$	$\left(\frac{1}{a^2}+\frac{1}{a}\right)b+\frac{b+1}{1-a}$	$\frac{b}{a}+\frac{b+1}{1-a}$	$\frac{b}{a^2c}-\frac{1}{a^2c}+\frac{1}{a^2}-\frac{a}{b^2}-\frac{1}{ab^2}-\frac{a}{bc}+\frac{2b}{ac}+\frac{a}{b}-\frac{2}{ac}+\frac{2}{a}-\frac{2}{b^2}+\frac{b}{c}-\frac{2}{bc}+\frac{2}{b}$	$10-10$

$\mathcal{N}_{2,7}$ 28 fixed points, length 4

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Shape	F_α	F_β	$\mathrm{tr}_{\mathcal{T}_{[I\bullet]}}$	$\dim \mathcal{T}_{[I\bullet]}$
$[2,2\,\vert\,0]$	$\left(\frac{1}{a^2}+\frac{1}{a}\right)(b+1)+\frac{b+1}{1-a}$	$\frac{b+1}{1-a}$	$\frac{b}{a^2}+\frac{1}{a^2}-\frac{a}{b^2c}-\frac{1}{ab^2}-\frac{a}{bc}+\frac{b}{a}+\frac{1}{a}-\frac{1}{b^2c}+\frac{b}{c}-\frac{1}{bc}$	$5-5$
$[1,1,2\,\vert\,0]$	$\left(\frac{1}{a^3}+\frac{1}{a^2}\right)b+\frac{b+1}{1-a}+\frac{b+1}{a}$	$\frac{b+1}{1-a}$	$\frac{b}{a^3}-\frac{a^2}{b^2c}+\frac{b}{a^2c}-\frac{1}{a^2c}+\frac{1}{a^2}-\frac{a}{b^2}-\frac{1}{ab^2}-\frac{a}{bc}+\frac{b}{ac}+\frac{a}{b}-\frac{1}{ac}+\frac{2}{a}-\frac{1}{b^2}+\frac{b}{c}-\frac{2}{bc}+\frac{1}{b}$	$9-9$
$[1,2\,\vert\,1]$	$\frac{b}{a^2}+\frac{b+1}{1-a}+\frac{b+1}{a}$	$\frac{b}{a}+\frac{b+1}{1-a}$	$\frac{b}{a^2}-\frac{a}{b^2c}-\frac{1}{ab^2}+\frac{2b}{ac}-\frac{2}{ac}+\frac{3}{a}-\frac{2}{b^2}+\frac{b}{c}-\frac{3}{bc}+\frac{2}{b}$	$9-9$
$[2\,\vert\,1,1]$	$\frac{b+1}{1-a}+\frac{b+1}{a}$	$\left(\frac{1}{a^2}+\frac{1}{a}\right)b+\frac{b+1}{1-a}$	$\frac{b}{a^2c}-\frac{1}{a^2c}+\frac{1}{a^2}-\frac{a}{b^2}-\frac{1}{ab^2}-\frac{a}{bc}+\frac{b}{ac}+\frac{a}{b}+\frac{b}{a}-\frac{1}{ac}+\frac{2}{a}-\frac{1}{b^2c}-\frac{1}{b^2}+\frac{b}{c}-\frac{2}{bc}+\frac{1}{b}$	$9-9$
$[2\,\vert\,2]$	$\frac{b+1}{1-a}+\frac{b+1}{a}$	$\frac{b+1}{1-a}+\frac{b+1}{a}$	$-\frac{1}{ab^2}+\frac{2b}{a}+\frac{2}{a}-\frac{2}{b^2c}+\frac{b}{c}-\frac{2}{bc}$	$5-5$
$[1,1,1,1\,\vert\,0]$	$(\frac{1}{a^4}+\frac{1}{a^3}+\frac{1}{a^2}+\frac{1}{a})b+\frac{b+1}{1-a}$	$\frac{b+1}{1-a}$	$\frac{b}{a^4c}-\frac{1}{a^4c}+\frac{1}{a^4}-\frac{a^3}{b^2}-\frac{a^3}{bc}+\frac{b}{a^3c}+\frac{a^3}{b}-\frac{1}{a^3c}+\frac{1}{a^3}-\frac{a^2}{b^2}-\frac{a^2}{bc}+\frac{b}{a^2c}+\frac{a^2}{b}-\frac{1}{a^2c}+\frac{1}{a^2}-\frac{a}{b^2}-\frac{1}{ab^2}-\frac{a}{bc}+\frac{b}{ac}+\frac{a}{b}-\frac{1}{ac}+\frac{1}{a}-\frac{1}{b^2}+\frac{b}{c}-\frac{1}{bc}+\frac{1}{b}$	$13-13$
$[1,1,1\,\vert\,1]$	$(\frac{1}{a^3}+\frac{1}{a^2}+\frac{1}{a})b+\frac{b+1}{1-a}$	$\frac{b}{a}+\frac{b+1}{1-a}$	$\frac{b}{a^3c}-\frac{1}{a^3c}+\frac{1}{a^3}-\frac{a^2}{b^2}-\frac{a^2}{bc}+\frac{b}{a^2c}+\frac{a^2}{b}-\frac{1}{a^2c}+\frac{1}{a^2}-\frac{a}{b^2}-\frac{1}{ab^2}-\frac{a}{bc}+\frac{2b}{ac}+\frac{a}{b}-\frac{2}{ac}+\frac{2}{a}-\frac{2}{b^2}+\frac{b}{c}-\frac{2}{bc}+\frac{2}{b}$	$13-13$
$[1,1\,\vert\,1,1]$	$\left(\frac{1}{a^2}+\frac{1}{a}\right)b+\frac{b+1}{1-a}$	$\left(\frac{1}{a^2}+\frac{1}{a}\right)b+\frac{b+1}{1-a}$	$\frac{2b}{a^2c}-\frac{2}{a^2c}+\frac{2}{a^2}-\frac{2a}{b^2}-\frac{1}{ab^2}-\frac{2a}{bc}+\frac{2b}{ac}+\frac{2a}{b}-\frac{2}{ac}+\frac{2}{a}-\frac{2}{b^2}+\frac{b}{c}-\frac{2}{bc}+\frac{2}{b}$	$13-13$

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