SPHERICALLY SYMMETRIC MONOPOLE SOLUTIONS

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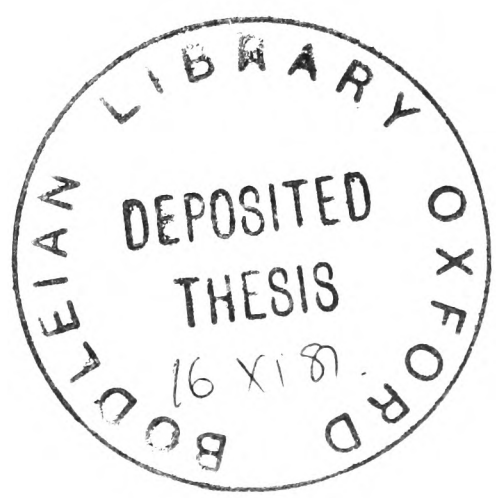
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Abstract

Classical gauge theories are studied for spherically symmetric monopole solutions. The Higgs field is taken in the adjoint representation and in the limit of vanishing self-interaction. The equations of motion can be represented by a Lax pair.

Using techniques from group representation theory, explicit solutions are obtained in the case of the principal embedding of the symmetry group SU(2) in the bigger gauge group. The parameters of the solutions can be chosen to give finite fields everywhere, and the large r behaviour of the Higgs field, which determines the symmetry breaking, is discussed. Some low rank groups are studied as examples.

Next, some properties of the non-principal SU(2) embeddings and their classification are discussed. These are then used to obtain more solutions, but the problem has not been solved for the most general case.

Acknowledgements

I became interested in the problems discussed in this thesis after a seminar given at Oxford by David Olive. Since then, I have been receiving his help and advice; many ideas in this work are due to him. I thank him for everything. I thank my supervisor Robert Heimann for his constant encouragement over the past three years, for many long discussions, and for his careful reading of the manuscript. I also wish to thank Peter Goddard for helpful discussions during the last few months.

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Most of all thanks to my mother for her support and to my wife for my happiness.

To Chrysoula, to L.f.K.

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Section I

I.1 Introduction

Gauge theories form the basis of almost all contemporary discussions of elementary particle theory. Of course experiment is the criterion by which they must be judged, but they are also attractive because they emphasize a symmetry principle as the origin of all forces.

It is also clear that gauge theories are not well understood. Most quantitative predictions come from perturbative calcualtions and these are of limited applicability, especially if we are interested in strong interactions.

This thesis is about classical solutions of gauge theories. It is uncertain whether any results from the classical theory will be relevant to the full quantum theory. One hopes that it can only improve our understanding.

Monopoles are particular examples of such classical solutions. Of course, since Dirac's original work [1] monopoles have been studied in various contexts. It was realised however [2], that in gauge theories they most easily arise as "extended objects", solitons. These are finite energy, non-dissipative structures. Their stability arises for topological reasons, related to the way they realise the boundary conditions at infinity. Their magnetic charge is quantized with respect to their electric charge. It is thought that quantum fluctuations do not alter these results.

It is not clear if monopoles are physically important. They seem to arise naturally under some general assumptions. Their rôle is sometimes emphasized in connection with the structure of the vacuum and the confinement of quarks. One would certainly like to understand better if there is any truth in these arguments.

On the other hand solitons are known to be important in other branches of physics. There are some well studied non-linear systems in one and two dimensions, and we hope that we can make some connection with those.

One remarkable property that some of these systems have, is a large number of conserved quantities. This often underlies the fact that they can be solved exactly. We would now like to describe a fairly general method for constructing non-linear systems with many constants of the motion, due to P.D. Lax [3].

Suppose that we have a one parameter set of operators L(t) that are similar to each other, i.e. they can be written

$$L(t) = U^{-1}(t)\, L(0)\, U(t)$$

with L and U differentiably dependent on t. Introduce

$$B(t) = U^{-1}(t)\, \frac{d}{dt}\, U(t)$$

Then differentiating L(t), we find

$$\frac{d}{dt} L(t) = \frac{d}{dt}(U^{-1})\ L(0)\ U + U^{-1} L(0) \frac{d}{dt} U =$$

$$= - U^{-1} \left(\frac{d}{dt} U\right) U^{-1} L(0)\ U + U^{-1} L(0)\ U\, U^{-1} \frac{d}{dt} U = L(t)\, B(t) - B(t) L(t)$$

In other words we have an equation: $\frac{dL}{dt} = [L, B]$

In any specific example L and B are differential or integral operators, or finite dimensional matrices, etc.; the Lax equation then becomes a non-linear differential equation between the coefficients of the operators. Now L(t) is given by a similarity transformation, and we know that similar operators have the same spectrum. Therefore, the eigenvalues of L(t) are time independent; they are our constants of the motion.

It is also easy to see this, if we can have finite matrix representations for L and B. When we take the trace of a commutator it vanishes. Therefore

$$\frac{d}{dt} Tr \left[L(t) \right]^n = 0$$

Examples of such systems are the Korteweg-de Vries equation, the Sine-Gordon model, the Liouville equation, the Toda lattice.

As we shall see, when we impose certain symmetry requirements, the gauge field equations can be represented in the Lax form. The next two parts of this section are dovoted to showing that.

I.2 B.P.S. monopoles

In this and the next subsection we will set up the problem that we wish to study. We are looking for classical solutions of gauge theories with "monopole like" properties. There are two conditions that enormously simplify the field equations: self-duality and spherical symmetry. Since these two symmetries are so important, we will repeat here some of the arguments that are used to get reduced the equations. For a general review about monopoles, see ref. [4,5].

Firstly one chooses to study a gauge theory with a Higgs field in the adjoint representation. The gauge fields are of course always in the adjoint, and our assumption puts the Higgs field on an equal footing. This is geometrically "natural" and may also be important for physical applications.

With a suitable normalization, the Lagrangian can be written:

$$\mathcal{L} = -\frac{1}{4} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \mathrm{Tr}\, D_\mu \Phi D^\mu \Phi - U(\Phi)$$

We use matrix notation for the fields, and the covariant derivative is

$$D_\mu \Phi = \partial_\mu \Phi - ie [A_\mu, \Phi]$$

The potential $U(\Phi)$ is gauge invariant and provides the symmetry breaking. One usually considers a term

$$U = \frac{1}{4}\lambda(Tr\,\Phi^2 - \frac{m^2}{\lambda})^2$$

We are interested in static solutions with $\partial_0 = 0$. If we further assume that there is a frame in which $A_0=0$, we are only left with the space components. Defining the "chromo-magnetic" field $B_i = \frac{1}{2}\varepsilon_{ijk} F_{jk}$, the energy is given by

$$E = \frac{1}{2}\, Tr \int d^3x \quad [(B_i)^2 + (D_i\Phi)^2 + U(\Phi)]$$

The Bogomolny [6] bound is obtained by writing the above expression in the form

$$E = \frac{1}{2}\, Tr \int d^3x \quad [B_i \mp D_i\Phi]^2 \pm 2\, B_i D_i \Phi + \quad U(\Phi)$$

The first term and $U(\Phi)$ are non-negative and we see that we have a lower bound for the energy:

$$E \geqslant \pm\, Tr \int d^3x \quad B_i D_i \Phi$$

This turns out to be a useful bound because the right hand side depends on the asymptotic values of the fields only. By using the covariant Maxwell's equation $D_i B_i = 0$ and Gauss's law it can be rewritten as

$$E \geqslant \pm\, Tr \int_{S_\infty} dS^i \quad (B_i \Phi)$$

It is proportional to the "flux" coming out of a large sphere. This, suitably normalised, is the magnetic charge. The $\pm$ sign corresponds to a monopole or antimonopole.

The Bogomolny-Prasad-Sommerfield limit [6,7] is obtained when we let $U(\Phi) \to 0$, but keeping asymptotically a non-zero value for the Higgs field. Formally this is done by taking for the self-interaction $m, \lambda \to 0$ with $\frac{m^2}{\lambda}$ finite . In this way we retain the symmetry breaking but the specific form of the potential plays no rôle. In the B.P.S. limit the lower bound for the energy is saturated when

$$B_i = \pm D_i \Phi \tag{I.1}$$

A few general remarks follow:

i) In the B.P.S. limit the Higgs field has $m \to o$, and is therefore long range. The "Higgs vacuum" classically is realised as $r \to \infty$.

ii) The solutions that satisfy (I.1) are local minima of the action. Assuming that there are no singularities, the energy is finite; since we have a magnetic field we interpret the solutions as monopoles with finite mass.

iii) Equation (I.1) can be interpreted as a self-duality condition, if we identify Φ with an imaginary time componenent of the gauge field: $\Phi \to i A_0$; then $D_i \Phi$ corresponds to the "electric field and (I.1.) reads

$$B_i = \pm i E_i$$

As in other cases, the self-duality condition reduces the second order equations to simpler first order ones.

iv) In the case of SU(2), the original ansatz of t'Hooft is

$$e\vec{A} = [K(r) - 1]\ \vec{T} \times \hat{r}/r \qquad (I.2)$$

$$e\Phi = H(r)\ \vec{T}.\hat{r}/r \qquad (I.2')$$

K,H are scalar functions and $\vec{T}$ are constant SU(2) generators, $[T_i , T_j] = i\,\varepsilon_{ijk} T_k$. Note the coupling between space and "isospin" variables. The B.P.S. equation (I.1) reduces to

$$r\frac{dH}{dr} = H \pm (K^2 - 1) \quad , \qquad r\frac{dK}{dr} = \pm HK$$

This system can be solved in terms of elementary functions[7]

$$H = \mp (br \coth br - 1) \quad , \qquad K = \frac{br}{\sinh br}$$

The asymptotic value of the Higgs field is $e\Phi \simeq b\,\vec{T}.\hat{r}$, and the mass $M = \frac{b}{e^2}\,4\pi$.

Note that the soultion is regular at the origin, since both $\Phi, \vec{A} \xrightarrow[r\to 0]{} 0$. There is no need for a "string".

The SU(2) solutions are our prototype and we will be trying to construct this type of solution for an arbitrary gauge group. The problem is of significant complexity and we will be turning our attention to some elegant mathematical methods in order to solve it. In the process we hope that we will also understand better this relatively easy case of SU(2).

I.3 Spherical symmetry and the Lax pair.

One expects that the field configurations that possess a high degree of symmetry will minimize the energy. Of course these configurations are also the simplest ones to solve for, since they involve a smaller number of variables.

The equations for SU(2) presented in the last section, could be written in terms of a radial variable only, but the solutions are not seen to be explicitly spherically symmetric. The Higgs field Φ in (I.2') contains a term proportional to $\vec{T}.\vec{r}$ and this is not manifestly invariant. When we rotate by a small angle $\delta\vec{\theta}$:

$$\vec{T}.\vec{r} \longrightarrow \vec{T}.\vec{r} + \delta\vec{\theta}.(\vec{r}\times\vec{\nabla}(\vec{T}.\vec{r})) = \vec{T}.\vec{r} + \delta\vec{\theta}.(\vec{r}\times\vec{T})$$

It was noticed however, that the difference can be compensated by a gauge transformation:

$$e^{i\delta\vec{\theta}.\vec{T}}\ \vec{T}.\vec{r}\ e^{-i\delta\vec{\theta}.\vec{T}} \approx \vec{T}.\vec{r} + i\delta\vec{\theta}.[\vec{T},\vec{T}.\vec{r}] = \vec{T}.\vec{r} - \delta\vec{\theta}.(\vec{r}\times\vec{T})$$

A "gauge rotated" Φ is physically the same as the original one; in this wider context Φ is spherically symmetric. Since Φ is invariant under the combined action of space and gauge rotation, we conclude that this is the symmetry principle that we would like to generalize.

For an arbitrary group, define combined generators for rotations

$$\vec{J} = \vec{L} + \vec{T} = -i\,\vec{r}\times\vec{\nabla} + \vec{T} \qquad (I.3)$$

There is the usual space part, plus a term that generates an SU(2)

(or SO(3)) subgroup. Then $\vec{J}$ obeys angular momentum commutation relations:

$$[\, J_i \,,\, J_j \,] \;=\; i\, \varepsilon_{ijk}\, J_k$$

$\vec{J}$ determines the transformation properties of the fields. If we want the Higgs field to be a rotational scalar, we demand that it satisfies:

$$[\, J_i \,,\, \Phi \,] \;=\; 0 \qquad \text{(I.4)}$$

For the gauge field to transform like a rotational vector:

$$[\, J_i \,,\, A_j \,] \;=\; i\, \varepsilon_{ijk}\, A_k \qquad \text{(I.5)}$$

We want to find solutions of the B.P.S. equations for fields Φ, $\vec{A}$, that satisfy conditions (I.4) and (I.5). These conditions essentially determine the angular dependence of the fields and leave us with a radial variable only, as may be expected. We briefly indicate how this can be done by adopting the formalism of Goldhaber and Wilkinson [8].

These authors show that it is possible to pick a gauge in which the radial component of $\vec{A}$ vanishes, i.e. $\vec{r}.\vec{A} = 0$. Then the gauge field can be written in terms of a new variable $\vec{M}(\vec{r})$;

$$e\,\vec{A} \;=\; (\,\vec{M}(\vec{r}) - \vec{T}\,) \times \hat{r}/r \qquad \text{(I.6)}$$

$\vec{M}$ also transforms like a rotational vector

$$[\, J_i \,,\, M_j \,] \;=\; i\, \varepsilon_{ijk}\, M_k \qquad \text{(I.7)}$$

We can make use of this and (I.4) to calculate the magnetic field and its "dual" $D_i\Phi$ in terms of the new variable:

$$e\,\vec{D}\Phi \;=\; -\,ie\,[\,\vec{M},\,\Phi\,] \times \hat{r}/r \;+\; e\,(\hat{r}.\,\vec{\nabla}\,\Phi)\,\hat{r} \qquad \text{(I.8)}$$

$$e\vec{B} \;=\; (\hat{r} \times (\hat{r}.\,\vec{\nabla})\vec{M}) \times \hat{r}/r \;-\; i\,(\hat{r}.\,(\vec{M}\times\vec{M} - \vec{T}\times\vec{T}))\,\hat{r}/r^2 \qquad \text{(I.8')}$$

The vector product between generators denotes:

$$(\vec{M} \times \vec{M})_i = \frac{1}{2} \varepsilon_{ijk} [M_j, M_k]$$

Note that $\vec{B}$ and $\vec{D}\Phi$ have transverse as well as radial components, corresponding to the fact that our generalized angular momentum possesses a radial part.

If we can solve for $\vec{B}$ and Φ along the $\hat{z}$ axis, we can rewrite (I.8) in terms of r only, in a spherical basis with $M^{\pm} = M_1 \pm iM_2$:

$$eB_3 = \left(\frac{1}{2}[M^+, M^-] - T_3\right)/r^2 \quad , \quad eB^{\pm} = \frac{1}{r}\frac{d}{dr}M^{\pm}$$

$$eD_3\Phi = e\frac{d}{dr}\Phi \quad , \quad eD^{\pm}\Phi = \pm e[\Phi, M^{\pm}]/r \qquad (I.9)$$

The equations corresponding to (I.4) and (I.7) are

$$[T_3, M^{\pm}] = \pm M^{\pm} \quad , \quad [T_3, \Phi] = 0 \qquad (I.10)$$

$M^{\pm}$ and Φ must be determined from the equations of motion, subject to the constraints (I.10). It is useful to redifine our unknowns as in ref.[9]

$$M^{\pm} = r N^{\pm}$$

$$e\Phi = \left(\Psi + \frac{T_3}{r}\right) \qquad (I.11)$$

In terms of $N^{\pm}$, Ψ , substituting first in (I.9) and then in the self-duality B.P.S. equation (I.1), we arrive at the final form:

$$\frac{d\Psi}{dr} = \frac{1}{2}[N^+, N^-]$$

$$\frac{dN^{\pm}}{dr} = \pm[\Psi, N^{\pm}] \qquad (I.12)$$

These equations must be considered together with the constraints

$$[T_3, N^{\pm}] = \pm N^{\pm} \quad , \quad [T_3, \Psi] = 0 \qquad (I.13)$$

System (I.12) with (I.13) will be the object of study for the rest of this thesis. It was derived here for a classical dynamical system based on an arbitrary gauge group, under the assumptions of spherical symmetry and self-duality, but as we shall see system (I.12) is interesting on its own right. It has a rich structure and hidden symmetries and we will be exploring some of it's properties.

Up to now we have not said anything about the SU(2) generators $\vec{T}$ (or $T_3, T^{\pm}$) which come into the generalized angular momentum. Clearly there are many ways in which we can embed SU(2) in a bigger group. We will be interested only in those that are inequivalent by a gauge transformation (and have T_3 diagonal); this still leaves us with a finite number of possibilities.

The commutation relations of T_3 with the rest of the generators of the group are important as we see from (I.13), since this puts a restriction on Ψ and $N^{\pm}$. Following Olive [9] we define the grade of a generator A by

$$[T_3, A] = n A \qquad grd\, A = n \qquad (I.14)$$

A is like an eigenvector of T_3, with grdA the corresponding eigenvalue. From angular momentum theory we know that grdA can only be an integer or half-integer. The grade acts like an additive quantum number:

$$grd\,[A, B] = grd\, A + grd\, B$$

With these definitions (I.13) reads

$$grd\, N^{\pm} = \pm 1 \quad , \quad grd\, \Psi = 0$$

To get the Lax pair representation of system (I.12) consider

$$L = N^- - 2\Psi - N^+ \quad , \quad R = N^- - \Psi \tag{I.15}$$

Then if we differentiate and use (I.12) we find

$$\frac{dL}{dr} = [L, R] \tag{I.16}$$

The important thing to notice is that L and R have mixed grades, since they are not T_3 "eigenvectors". If we start from (I.16) and decompose both sides of the equation into T_3 eigenspaces we get back to the system (I.12). We have considered here a definition for the Lax pair slightly different from [9]. Note that R contains grades 0, -1 only; this will turn out to be important.

Equation (I.16) implies of course that we have constants of the motion. We cannot tell however how many of these are independent and if the system is in general soluble. In the next section we will be studying a particular case, which is best understood and defines clearly what will have to be done for all other cases.

Section II

In this section we define T_3, which corresponds to a particular embedding, known as the principle embedding. T_3 plays an important rôle in developing the Lax pair equations and intergrating for the solutions. The eigenspaces of T_3 in this case correspond to the root-space decomposition, and that gives all of the interesting properties. For notation and some general results used throughout the text see Appendix A. For spherically symmetric monopoles in SU(N) see also [10].

II.1 Principal embedding and reduction of equations of motion.

Consider a simple (or semisimple) Lie algebra $\mathcal{G}$ with Cartan subalgebra $\mathcal{H}$ and generators corresponding to the roots E_α, $\alpha \in \Phi(\mathcal{G})$; Φ is the root system corresponding to $\mathcal{G}$. Then we can always find a unique element T_3 such that the following commutators are satisfied:

$$[T_3, h] = 0 \quad , \quad [T_3, E_{\pm\alpha}] = \pm E_{\pm\alpha} \quad , \quad \alpha \in \Delta$$

Δ is the system of simple roots of Φ. The above condition implies that generators corresponding to simple roots have grade 1. Therefore the generators all have integer grades only e.g.
$[T_3, E_{\alpha+\beta}] = 2E_{\alpha+\beta}$, $\alpha, \beta \in \Delta$, since by using the Jacobi identity we have

$$[T_3, [E_\alpha, E_\beta]] = -[E_\alpha, [E_\beta, T_3]] - [E_\beta, [T_3, E_\alpha]] = [E_\alpha, E_\beta] - [E_\beta, E_\alpha] = 2[E_\alpha, E_\beta]$$

and $[E_\alpha, E_\beta]$ is proportional to $E_{\alpha+\beta}$.

Let us see how to get T_3 in the Chevalley basis.

Write $T_3 = \sum_{\alpha\in\Delta} t_\alpha H_\alpha$, H_α the commuting generators in this basis. We must have for every simple root $\gamma\in\Delta$:

$$[T_3, E_\gamma] = E_\gamma \Rightarrow \sum_{\alpha\in\Delta} t_\alpha [H_\alpha, E_\gamma] = \sum_{\alpha} t_\alpha \frac{2(\alpha,\gamma)}{(\alpha,\alpha)} E_\gamma = \sum_{\alpha\in\Delta} t_\alpha K_{\alpha\gamma} E_\gamma = E_\gamma$$

where $K_{\alpha\gamma}$ is the Cartan matrix. This condition is therefore equivalent to the system of linear equations

$$\sum_{\alpha\in\Delta} t_\alpha K_{\alpha\gamma} = 1 \qquad \text{for every} \quad \gamma\in\Delta$$

The Cartan matrix for a simple (or semisimple) group is non-degenerate, $\det K_{\alpha\gamma} \neq 0$. We can invert it and solve

$$t_\alpha = \sum_{\beta\in\Delta} K^{-1}_{\beta\alpha}$$

There is an alternative way of describing T_3 , which corresponds to the Cartan-Weyl basis. It may be easier to use it in explicit calculations. We associate with T_3 an element $2\delta^\vee$ of the Euclidean space E, which is spanned by the roots, such that $(T_3, \gamma) = (2\delta^\vee, \gamma)$ for every root γ. $2\delta^\vee$ itself is not a root. We see that it equals $2\delta^\vee = \sum_{\alpha\in\Delta} \frac{2\lambda_\alpha}{(\alpha,\alpha)}$ with λ_α the fundamental weights (see Appendix A3) since:

$$(2\delta^\vee, \gamma) = \sum_{\alpha\in\Delta} \frac{2(\lambda_\alpha,\gamma)}{(\alpha,\alpha)} = \sum_{\alpha\in\Delta} \frac{2(\lambda_\alpha,\gamma)}{(\gamma,\gamma)} \frac{(\gamma,\gamma)}{(\alpha,\alpha)} = \sum_{\alpha} \delta_{\alpha\gamma} \frac{(\gamma,\gamma)}{(\alpha,\alpha)} = 1$$

The two definitions of T_3 are of course equivalent.

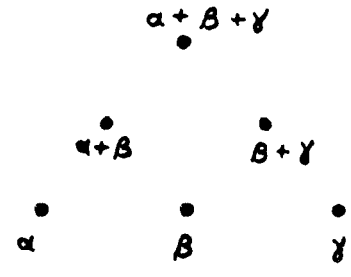
For an arbitrary root $\beta\in\Phi$, the grade of E_β is equal to the height of β , where the height is defined $ht\beta = \sum_\alpha k_\alpha$, where $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$. In short we have

$$grd\, E_\beta = (T_3, \beta) = \sum_{\alpha\in\Delta} k_\alpha (T_3, \alpha) = \sum_\alpha k_\alpha = ht\,\beta$$

Therefore, T_3 separates the roots

in "levels" according to their height, e.g. we have for

$A_3 = SU(4)$ ht 3 2 1



$\{\alpha, \beta, \gamma\} = \Delta$

It is a property of the principal embedding only, that the grade of a root is equal to its height.

The principal SU(2) subalgebra of $\mathcal{G}$ has T_3 as a diagonal generator, the other ones are $T^{\pm}$ with commutators

$$[T_3, T^{\pm}] = \pm T^{\pm} \qquad [T^+, T^-] = 2T_3$$

Expanding in the Chevalley basis, as before, we write $T^{\pm} = \sum_{\alpha \in \Delta} c_\alpha E_{\pm\alpha}$

$$[T^+, T^-] = \sum_{\alpha,\beta} c_\alpha c_\beta [E_\alpha, E_{-\beta}] = \sum_{\alpha,\beta} c_\alpha c_\beta \delta_{\alpha\beta} H_\alpha = \sum_\alpha c_\alpha^2 H_\alpha = 2 \sum_\alpha t_\alpha H_\alpha$$

so $c_\alpha^2 = 2t_\alpha$ and we have indeed an SU(2) group.

We can now use the above results to expand our variables in the Chevalley basis. The equations of motion from section I.3 are

$$2\frac{d\Psi}{dr} = [N^+, N^-] \quad , \quad \frac{dN^{\pm}}{dr} = \pm [\Psi, N^{\pm}] \qquad \text{(II.1)}$$

It is understood that Ψ is hermitian, $\Psi^\dagger = \Psi$, and $(N^+)^\dagger = N^-$. Remember that Ψ is associated with the Higgs field and $N^{\pm}$ with the transverse components of the gauge fields. We also have the constraint conditions coming from the spherical symmetry requirement:

$$[T_3, \Psi] = 0 \qquad [T_3, N^{\pm}] = \pm N^{\pm}$$

T_3 is the principal embedding generator derived above and we can expand:

$$N^+ = \sum_{\alpha\in\Delta} f_\alpha(r)\, E_\alpha \quad , \quad N^- = \sum_{\alpha\in\Delta} f_\alpha^*(r)\, E_{-\alpha} \quad , \quad \Psi = \sum_{\alpha\in\Delta} \psi_\alpha(r) H_\alpha \tag{II.2}$$

so the constraints are automatically satisfied. Substituting we get the system of non-linear equations:

$$2\frac{d\psi_\alpha}{dr} = f_\alpha f_\alpha^* \quad , \quad \frac{df_\alpha}{dr} = \sum_{\beta\in\Delta} \psi_\beta K_{\beta\alpha} f_\alpha \quad , \quad \frac{df_\alpha^*}{dr} = \sum_{\beta\in\Delta} \psi_\beta K_{\beta\alpha} f_\alpha^* \tag{II.3}$$

$K_{\beta\alpha} = \frac{2(\beta,\alpha)}{(\beta,\beta)}$ is the Cartan matrix, it has integer entries only. The phase of f_α is constant since $\frac{d}{dr}\ln\frac{f_\alpha}{f_\alpha^*} = 0$. It is possible to remove the phase by a global gauge transformation. Consider the action of a constant element Ω of the C.S.A. on one of the generators:

$$e^{\Omega} E_\alpha e^{-\Omega} = E_\alpha + [\Omega, E_\alpha] + \tfrac{1}{2}[\Omega,[\Omega,E_\alpha]] + \ldots = e^{(\Omega,\alpha)} E_\alpha$$

Similarly $e^{\Omega} E_{-\alpha} e^{-\Omega} = e^{-(\Omega,\alpha)} E_{-\alpha}$ and $e^{\Omega} H_\alpha e^{-\Omega} = H_\alpha$. Then the system of equations (II.1) is invariant under this gauge transformation. We can arrange for (Ω,α) to give us the constant phases of f_α , and finally we can take f_α real.

We will rewrite the system of first order equations (II.3) as a second order equation in terms of single variables ρ_α defined by *

$$\frac{d\rho_\alpha}{dr} = 2\psi_\alpha \quad \Rightarrow$$

$$\frac{d^2\rho_\alpha}{dr^2} = f_\alpha^2 \quad , \quad \frac{df_\alpha}{dr} = \frac{1}{2} f_\alpha \sum_\beta \frac{d\rho_\beta}{dr} K_{\beta\alpha} \tag{II.4}$$

and therefore

$$f_\alpha(r) = \exp\left(\tfrac{1}{2}\sum_\beta \rho_\beta(r) K_{\beta\alpha}\right) \tag{II.5}$$

We have chosen $f_\alpha(0) = \exp(\frac{1}{2}\sum_\beta \rho_\beta(0) K_{\beta\alpha})$, since our definition

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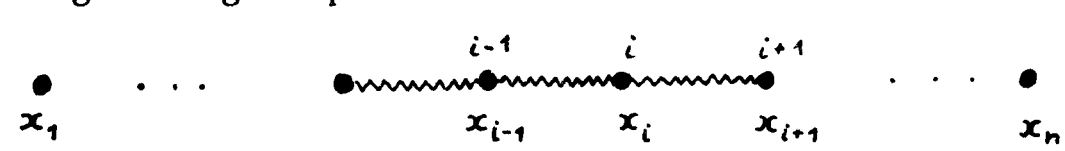
*This definition differs from the one in ref. [9]

of $\frac{d\rho_\alpha}{dr}$ leaves $\rho_\alpha(o)$ arbitrary. Finally substituting for f_α in the first equation we get

$$\frac{d^2\rho_\alpha(r)}{dr^2} = \exp\left(\sum_\beta \rho_\beta(r) K_{\beta\alpha}\right) \qquad \text{(II.6)}$$

If we can solve for ρ_α we can get back and find ψ_α and f_α, and subsequently the Higgs, gauge fields.

System (II.6) for particular values of the matrix K is similar to what is known as the Toda lattice. This is an example of an intergrable non-linear system, first introduced by M. Toda [11] and has been much studied [12,13,16]. There are various generalizations of the Toda lattice and it is related in the continuum limit to another known system, the Korteweg-de Vries equation. We briefly describe the finite non-periodic Toda lattice: it is a system of particles on a line interacting through exponential forces.



With each particle of unit mass, each one at position x_i, the equations of motion are

$$\frac{d^2x_1}{dt^2} = e^{x_2 - x_1}$$

$$\frac{d^2x_i}{dt^2} = e^{x_{i+1} - x_i} - e^{x_i - x_{i-1}}$$

$$\frac{d^2x_n}{dt^2} = - e^{x_n - x_{n-1}}$$

Introducing a new variable $y_i = x_{i+1} - x_i$ we have

$$\ddot{y}_1 = e^{y_2} - 2e^{y_1}$$

$$\ddot{y}_i = e^{y_{i+1}} - 2e^{y_i} + e^{y_{i-1}}$$

$$\ddot{y}_{n-1} = -2e^{y_{n-1}} + e^{y_{n-2}}$$

Therefore $\ddot{y}_i = -\sum_{j=1}^{n-1} K_{ij} e^{y_j}$, with

$$K_{ij} = \begin{pmatrix} 2 & -1 & 0 & \cdot & \cdot & 0 \\ -1 & 2 & -1 & 0 & \cdot & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & -1 & 2 \end{pmatrix}$$

But this is just the Cartan matrix for SU(n). Finally by redefining $y_i = \sum_m K_{im} \rho_m$ we get:

$$\frac{d^2}{dt^2} \rho_m = - \exp\left(\sum_j K_{mj} \rho_j \right) \qquad \text{(II.7)}$$

This is the same as (II.6) above, but with (-) sign. This corresponds to $r \rightarrow it$. However this sign difference is not trivial since we want real solutions to both equation (II.6) and (II.7).

The mathematical theory underlying the integrability of the Toda lattice and its generalizations is discussed extensively by B. Konstant [14]. Also J. Moser [13] gives the relation with inverse scattering methods. We shall not be interested in any of the other aspects of this problem, and we will concentrate on solving exactly equations (II.6) above.

II.2. Integration of the Lax pair.

We now come to one of our main results. We will obtain the solution of the system of equations (II.6) by integrating the corresponding Lax pair. For we saw in section I.3 that there is a Lax pair representation of the classical equations.

Our method is based essentially on the work of A.N. Leznov and M.V. Saveliev [15]. They consider a two dimensional "wave equation"

$$\left(\frac{d^2}{dx^2} - \frac{d^2}{dt^2}\right) \rho_\alpha = e^{K_{\alpha\beta}\rho_\beta}$$

Using the Lax representation they can give the general form of the solutions, which depend on two sets of arbitrary functions; these of course arise because we have a system of second order differential equations in two variables. Their solution involves time ordered exponentials of these arbitrary functions. One way to proceed would be to find some particular functions, such that when we take the static limit we recover solutions of equation (II.6) (one variable only). We would like to stress that in Leznov's and Saveliev's method it is important to have two variables, because there is a special interplay between "space" and "time" components of generators and use is made of that in the integration; with one variable only, this cannot be done.

We try to take a slightly different approach. We have been motivated by the work of M.A. Olshanetsky and A.M. Perelomov [16] on the Toda lattice, to try and integrate the Lax pair with one variable directly. This has led us to expressions on which we could use a similar method to Leznov and Saveliev, taking the expectation value between eigenvectors with fundamental highest weight. The final

expressions for the solutions of (II.6) involve 2ℓ (ℓ =rank of group) arbitrary constants. Although the intermediate arguments are complicated, our final expressions are simple and we can calculate various coefficients in the solutions explicitly. In order to help the reader to follow the separate stages of the argument, we divide the rest of this part in three subsections (a),(b), and (c) each dealing with a specific part of the problem.

II.2(a) The Lax variables in terms of a group element geG.

The Lax pair we are considering is $\frac{dL}{dr} = [L,R]$ with $L = N^- - N^+ - 2\Psi$ and $R = N^- - \Psi$, as defined in section I.3.

For any generator of the group we need to define an "involutive automorphism" of the algebra: $\tilde{X} = e^{i\pi T_3} X^+ e^{-i\pi T_3}$. T_3 is the unique element of the C.S.A. that corresponds to the principal embedding, as defined before. For a given basis

$$\tilde{E}_\alpha = e^{i\pi T_3} E_\alpha^+ e^{-i\pi T_3} = e^{i\pi T_3} E_{-\alpha} e^{-i\pi T_3} = e^{i\pi (T_3, \alpha)} E_{-\alpha} = (-1)^{ht\alpha} E_{-\alpha}$$

for every $\alpha \in \Phi$. We see that it gives a Hermitian conjugation and a (-) sign whenever we have an odd grade generator. Applying the operation twice takes one back to the same element. This holds for the principal embedding where all grades are integers (no half integer). For a product $\widetilde{(AB)} = e^{i\pi T_3} B^+ A^+ e^{-i\pi T_3} = \tilde{B}\,\tilde{A}$, and we can extend $\sim$ for any group element. With these definitions $R = N^- - \Psi$, $\tilde{R} = -N^+ - \Psi$ and $L = R + \tilde{R}$, since N is of grade 1, and Ψ grade 0. We now see that we have got L expressed in terms of R.

Since R is an element of the Lie algebra we can always write it as $R = g^{-1}\left(\frac{d}{dr}\, g\right)$ with g e G is some element of the group. Clearly, g is defined only up to a constant. Now consider the derivative of the Lie algebra element gLg^{-1} .

$$\frac{d}{dr}\left(gLg^{-1}\right) = \dot{g}Lg^{-1} + g\dot{L}g^{-1} + gL\left(g^{-1}\right)^{\cdot}$$

denoting by . differentiation with respect to r. But $\left(g^{-1}\right)^{\cdot} = -g^{-1}\dot{g}\,g^{-1}$, so

$$\frac{d}{dr}\left(gLg^{-1}\right) = g\left(\dot{L} - \left[L, g^{-1}\dot{g}\right]\right)g^{-1} = g\left(\dot{L} - [L,R]\right)g^{-1} = 0.$$

The last relation comes from the Lax equation $\dot{L} = [L,R]$. So gLg^{-1} is constant:

$$g(r)\,L(r)\,g^{-1}(r) = g(r')\,L(r')\,g^{-1}(r') = \ldots = g_0 L_0 g_0^{-1} \qquad \text{(II.8)}$$

where $_0$ denotes the value at some point $r=r_0$.

Also from the definition of L and g we have

$$g(r)\,L(r)\,g^{-1}(r) = g\left(R + \tilde{R}\right)g^{-1} = g\left(g^{-1}\dot{g} + \tilde{\dot{g}}\,\tilde{g}^{-1}\right)g^{-1} =$$

$$= \dot{g}\,g^{-1} + g\,\tilde{\dot{g}}\,\tilde{g}^{-1}\,g^{-1} = \left[\frac{d}{dr}\left(g\tilde{g}\right)\right]\left(g\tilde{g}\right)^{-1} \qquad \text{(II.9)}$$

This last step is important. Note that we have a $\sim$ invariant combination gg. Combining (II.8) and (II.9) we can integrate once more

$$g\tilde{g} = \exp\left[(r-r_0)\, g_0 L_0 g_0^{-1}\right] g_0\tilde{g}_0 = g_0 \exp\left[(r-r_0)L_0\right]\tilde{g}_0$$

(II.10)

On the right side all r dependence is explicit. If we could get g separately we could then find R and L.

II.2(b) The decomposition of g.

Any regular element of the group G can be decomposed (see Appendix B) as

$$g = \exp(X)\exp(Y)\exp(S) \quad , \quad X \in \sum_{\alpha\in\Phi^+} \mathfrak{g}^{\alpha} \quad , \quad Y \in \sum_{\alpha\in\Phi^+} \mathfrak{g}^{-\alpha} \quad , \quad S \in \mathfrak{h}$$

$\mathfrak{g}^{\alpha}$ are the eigenspaces of the Lie algebra $\mathfrak{g}$ that correspond to positive grade generators with root α . Similarly Y is a negative grade generator and S is of zero grade (belongs to the C.S.A.). We will assume that our g(r) is regular and can be decomposed as above. Then, X,Y,S are functions of r, and

$$R = g^{-1}\left(\frac{d}{dr}g\right) = e^{-S}e^{-Y}e^{-X}\left(\frac{d}{dr}e^{X}\right)e^{Y}e^{S} + e^{-S}e^{-Y}\left(\frac{d}{dr}e^{Y}\right)e^{S} + e^{-S}\frac{d}{dr}e^{S} \tag{II.11}$$

Now comes a crucial step from the definition of R into play. R does not contain any positive grade generators and therefore the first term must be zero, i.e. X = constant, independent or r. Equation (II.10) becomes

$$e^{Y(r)}\ e^{S(r)}\ e^{\tilde{S}(r)}\ e^{\tilde{Y}(r)} = e^{Y_0}\ e^{S_0}\ e^{(r-r_0)L_0}\ e^{\tilde{S}_0}\ e^{\tilde{Y}_0} \tag{II.12}$$

on cancelling X and $\tilde{X}$ from both sides.

We assume that we are always referring to a simple Lie algebra $\mathfrak{g}$ so that we can always find its fundamental representations $R(\lambda_\alpha)$.

There are ℓ (=rank) of those and each one has a highest eigenvector $|u_{\lambda_\alpha}\rangle$ of the C.S.A. corresponding to the highest fundamental weight λ_α (see Appendix A3). All raising generators annihilate $|u_{\lambda_\alpha}\rangle$, i.e. $E_\beta |u_{\lambda_\alpha}\rangle = 0$, $\beta \in \Phi^+$. By conjugation it is also true that $\langle u_{\lambda_\alpha}| E_{-\beta} = 0$. We now take the "expectation value" of (II.12) between highest vectors $|u_{\lambda_\alpha}\rangle$:

$$\langle u_{\lambda_\alpha}| e^{\Upsilon} e^{S} e^{\tilde{S}} e^{\tilde{\Upsilon}} |u_{\lambda_\alpha}\rangle = \langle u_{\lambda_\alpha}| e^{S} e^{\tilde{S}} |u_{\lambda_\alpha}\rangle = \langle u_{\lambda_\alpha}| e^{S_0} e^{L_0 (r-r_0)} e^{\tilde{\xi}_0} |u_{\lambda_\alpha}\rangle$$

We have used the fact that Υ contains only lowering generators (- grade) and $\tilde{\Upsilon}$ only positive grade. $S \in \mathfrak{h}$ and therefore are diagonal on the basis of $|u\rangle$; also since we are interested in real coefficients only, so $S = \tilde{S}$. This gives us

$$\exp \langle u_{\lambda_\alpha}| 2S |u_{\lambda_\alpha}\rangle = \exp \langle u_{\lambda_\alpha}| 2S_0 |u_{\lambda_\alpha}\rangle \langle u_{\lambda_\alpha}| \exp L_0 (r-r_0) |u_{\lambda_\alpha}\rangle \quad \text{(II.13)}$$

This equation expresses S(r) in terms of constant generators S_0 (in the C.S.A.) and L_0 $(= R_0 + \tilde{R}_0)$. But how is S(r) related to our original unknowns $\Psi(r)$, $N^{\pm}(r)$?

From (II.11) and the fact that $\frac{d}{dr} X = 0$ we have:

$$-\Psi + N^- \equiv R = e^{-S} e^{-\Upsilon} \left(\frac{d}{dr} e^{\Upsilon}\right) e^{S} + e^{-S} \frac{d}{dr} e^{S}$$

Then by equating the generators of different grades we have

$$-\Psi = e^{-S} \left(\frac{d}{dr} e^{S}\right) = \frac{dS}{dr} \quad , \quad \text{since } \Psi, S \in \mathfrak{h} \text{ (C.S.A.)}$$

Then in the Chevalley basis $-\psi_\alpha = \frac{d}{dr} s_\alpha$. Also from (II.4), $\psi_\alpha = \frac{1}{2} \frac{d}{dr} \rho_\alpha$. So in fact solving for S we have also solved for ρ_α . Using the fact that $H_\beta |u_{\lambda_\alpha}\rangle = \frac{2(\beta, \lambda_\alpha)}{(\beta, \beta)} |u_{\lambda_\alpha}\rangle = \delta_{\alpha\beta} |u_{\lambda_\alpha}\rangle$ (see Appendix A3) we

can rewrite (II.13) in the form

$$e^{-\rho_a(r)} = e^{-\rho_a(0)} \langle u_{\lambda a} | \exp L_0 (r-r_0) | u_{\lambda a} \rangle \qquad \text{(II.14)}$$

The solution for $\rho_a(r)$ directly expresses the solutions for $\Psi(r)$ and $N^{\pm}(r)$ that we seek. Note that we need 2ℓ parameters, say

$$\rho_a(0) = 2 \sum_{\beta \in \Delta} K^{-1}_{\beta a} \ln f_\beta(0) \quad \text{and} \quad \dot{\rho}_a(0) = 2\psi_a(0)$$

From now on put $r_0 = 0$ (choice of origin) ; then

$$L_0 = N_0^- - N_0^+ - 2\Psi_0 = \sum_{\alpha \in \Delta} \exp\left(\tfrac{1}{2} \sum_\beta \rho_\beta(0) K_{\beta a}\right) [E_{-\alpha} - E_a] - \dot{\rho}_\alpha(0) H_\alpha$$

(see (II.4) and (II.5)). So we have integrated system (II.6) by using its Lax pair representation. However we still have to obtain the matrix elements $\langle u_{\lambda a} | e^{L_0 r} | u_{\lambda a} \rangle$, if we want explicit solutions, and these can be calculated if we can "diagonalise" the constant generator L_0 .

II.2(c) Diagonalisation of L

We have found that $g(r)L(r)g^{-1}(r)$ is a constant (II.8). This constant plays an important role in our solutions. We will see that it is related to the asymptotic value of the Higgs field, the "symmetry breaking" term. Note that

$$Tr\left(g(r)\, L(r)\, g^{-1}(r)\right)^n = Tr\left(L(r)\right)^n$$

i.e. the eigenvalues of L(r) are constants of the motion. This of course follows directly from the Lax equation $\dot{L} = [L,R]$.

One can "rotate" a constant generator to lie in the Cartan subalgebra, i.e. one can diagonalise it. This is true, for a connected

complex (or a real compact) group. See [17] and references therein. So we can quite generally have for our constant

$$g(r)\, L(r)\, g^{-1}(r) = g_0 L_0 g_0^{-1} = g_s V g_s^{-1} \qquad \text{where} \quad V \in \mathfrak{h}$$

In any representation L_0 and V have the same set of eigenvalues. In fact this does not fix V completely, we still have the freedom to rotate it inside the C.S.A. using the finite group of "reflections", the Weyl group. In fact we can arrange $(V, \alpha) \geqslant 0$ for any $\alpha \in \Delta$. This then fixes V completely. This property will prove to be useful later on. So we can write for our constant generator:

$$t L_0 t^{-1} = V \qquad \text{for some} \quad t \in G \qquad \text{(II.15)}$$

V is independent of the origin $r_0 = 0$. Consider the decomposition of t in two different ways:

$$t = \exp Z_1 \exp W_1 \exp T_1 = \exp W_2 \exp Z_2 \exp T_2$$

with

$$Z_1, Z_2 \in \sum_{\alpha \in \Phi^+} \mathfrak{g}^{\alpha}, \quad W_1, W_2 \in \sum_{\alpha \in \Phi^+} \mathfrak{g}^{-\alpha} \quad \text{and} \quad T_1, T_2 \in \mathfrak{h}$$

Using the first decompostion we write

$$\begin{array}{lccccc} & e^{-Z_1} V e^{Z_1} & = & e^{W_1} e^{T_1} L_0 e^{-T_1} e^{-W_1} & & \\ \text{grd} & \geqslant 0 & & \leqslant 1 & & \text{(II.16)} \end{array}$$

V is grade 0 (C.S.A.) and therefore the left hand side which is $V - [Z_1, V] + \tfrac{1}{2}, [Z_1, [Z_1, V]] + \ldots$ contains generators with positive or zero grade only. On the right L_0 contains grades $0, \pm 1$, and therefore the whole expression can only contain grades less than or equal to 1. Since the grades must balance on both sides of the equation, conclude

that we can have grades 0 and 1 only. Consider the expansion generated by the outer-most expontentials $e^{\pm W_1}$ only,

$$e^{W_1} e^{T_1} L_0 e^{-T_1} e^{-W_1} \equiv e^{W_1} e^{T_1} (-N_0^+ - 2\Psi_0 + N_0^-) e^{-T_1} e^{-W_1} =$$

$$- e^{T_1} N_0^+ e^{-T_1} - 2\Psi_0 - [W_1, e^{T_1} N_0^+ e^{-T_1}] + \ldots$$

$$\text{grd} \quad 1 \qquad 0 \qquad \leq 0 \qquad \leq -1 \qquad \text{etc.}$$

The only piece that has grade 1 in this expansion is the first term. We can therefore rewrite equation (II.16) as

$$e^{-Z_1} V e^{Z_1} = V - e^{T_1} N_0^+ e^{-T_1}$$

But

$$N_0^+ = \sum_{\alpha \in \Delta} \exp\left(\tfrac{1}{2} \sum_{\beta} \rho_\beta^{(0)} K_{\beta\alpha}\right) E_\alpha$$

can be written as

$$N_0^+ = e^{\frac{1}{2} P_0} \sum_{\alpha \in \Delta} E_\alpha \; e^{-\frac{1}{2} P_0}$$

where $P_0 = \sum_{\beta \in \Delta} \rho_\alpha^{(0)} H_\alpha$ is in the C.S.A. So in fact we have

$$e^{-T_1} e^{-\frac{1}{2} P_0} e^{-Z_1} V e^{Z_1} e^{\frac{1}{2} P_0} e^{T_1} = V - \sum_{\alpha \in \Delta} E_\alpha$$

T_1 and P_0 commute with V as they all belong to the C.S.A.(η). We can define a new element $X \in \sum_{\alpha \in \phi^+} \mathfrak{g}^\alpha$ (positive grades) with $e^X = e^{-T_1} e^{-\frac{1}{2} P_0} e^{-Z_1} e^{\frac{1}{2} P_0} e^{T_1}$ .Then

$$e^X V e^{-X} = V - \sum_{\alpha \in \Delta} E_\alpha \qquad \text{(II.17)}$$

To summarize the above argument: we started from a relation between L_0 and V (II.15) and we have seen that the grade structure of L_0 implies a condition that the parameters of t must satisfy, in the form (II.17).

We can now repeat the whole argument for the second decomposition of t. We have negative grades in the place of positive grades and $+N_0^-$

in place of $-N_0^+$. The final condition is

$$e^{Y}\, V\, e^{-Y} = V + \sum_{\alpha\in\Delta} E_{-\alpha} \qquad \text{(II.18)}$$

with

$$e^{-Y} = e^{-T_2}\, e^{\frac{1}{2}P_0}\, e^{W_2}\, e^{-\frac{1}{2}P_0}\, e^{T_2} \quad , \qquad Y \in \sum_{\alpha\in\Phi^+} g^{-\alpha}$$

In this way we have solved indirectly for the parameters of t. We can now substitute in solution (II.14).

$$e^{-P_a(r)} = \langle u_{\lambda_a} |\, e^{-P_0}\, e^{L_0 r}\, | u_{\lambda_a}\rangle = \langle u_{\lambda_a} |\, e^{-P_0}\, t^{-1}\, e^{Vr}\, t\, | u_{\lambda_a}\rangle$$

$$= \langle u_{\lambda_a} |\, e^{-P_0}\, e^{-T_1}\, e^{-W_1}\, e^{-Z_1}\, e^{Vr}\, e^{W_2}\, e^{Z_2}\, e^{T_2}\, | u_{\lambda_a}\rangle$$

Using the fact that Z_2 is a raising generator, W_1 a lowering generator, and that P_0 , T_1, T_2, V are all diagonal generators, we rewrite it as

$$e^{-P_a(r)} = \langle u_{\lambda_a} |\, e^{-\frac{1}{2}P_0}\, e^{-T_1}\, e^{-Z_1}\, e^{Vr}\, e^{W_2}\, e^{T_2}\, e^{-\frac{1}{2}P_0}\, | u_{\lambda_a}\rangle =$$

$$= \langle u_{\lambda_a} |\, e^{X}\, e^{-\frac{1}{2}P_0}\, e^{-T_1}\, e^{Vr}\, e^{T_2}\, e^{-\frac{1}{2}P_0}\, e^{-Y}\, | u_{\lambda_a}\rangle$$

and finally get

$$e^{-P_a(r)} = \langle u_{\lambda_a} |\, e^{X}\, e^{-P_0 - T_1 + T_2}\, e^{Vr}\, e^{-Y}\, | u_{\lambda_a}\rangle$$

This is effectively the diagonalisation of L_0 . We can see that all the r dependence comes from the term e^{Vr} which belongs to the C.S.A.

In order to clarify the meaning of the above expression it helps to introduce a new constant generator $C=-P_0-T_1+T_2$. The constants P_0 and $\dot{P}_0$ are related through L_0 to C and V in a complicated way that involves the parameters of the group element t. We can regard C and V as our independent constants and write the solution as

$$e^{-P_a(r)} = \langle u_{\lambda_a} |\, e^{X}\, e^{-C}\, e^{Vr}\, e^{-Y}\, | u_{\lambda_a}\rangle \qquad \text{(II.19)}$$

This is our central result. X an Y satisfy conditions (II.17) and (II.18) (X and Y depend on V only). There are clearly seen to be ℓ parameters in C and ℓ parameters in V, since they are generators of the C.S.A. This is of course what we expect, a total of 2ℓ constants, corresponding to the $\rho_\alpha(o)$ and $\dot{\rho}_\alpha(o)$ occuring in the expression (II.14) above (there are ℓ simple roots $\alpha \in \Delta$). The constants $\rho_\alpha(o)$, $\dot{\rho}_\alpha(o)$ are given by

$$e^{-\rho_\alpha(0)} = \langle u_{\lambda_\alpha} | e^{X} e^{-C} e^{-Y} | u_{\lambda_\alpha} \rangle \quad , \quad e^{-\rho_\alpha(0)} \dot{\rho}_\alpha(0) = - \langle u_{\lambda_\alpha} | e^{X} e^{-C} V e^{-Y} | u_{\lambda_\alpha} \rangle$$

We now have an expression for getting solutions of $\ddot{\rho}_\alpha = e^{\rho_\beta K_{\beta\alpha}}$. We have started by defining L(r) which was essential for developing the underlying group structure and integrating, but our final expression (II.19) does not involve L. It has been "exchanged" for V which is diagonal. T_3 , the diagonal generator of SU(2) subgroup, has played an important rôle all along, since it defined the grades of the different generators and the decompositions.

We would like to point out that it was assumed, that the various constant group elements were regular, in order to do the decompositions. This means that there are no specific relations between constant parameters that can make various sub-determinants equal to zero. (see Appendix B). Regular elements form an open set in G, non-regular elements have measure zero.

Section III

In this section we further study equation (II.19). We use representation theory to get the exact coefficients of the solutions. These solutions are to be interpreted as monopoles and we consider the boundary conditions necessary for this. Finally we give some examples.

III.1 Weights and matrix elements

Equation (II.19) involves constant elements V and C in the C.S.A. and nilpotent elements X,Y which depend on V and are determined from (II.17) and (II.18). Consider a representation $R(\lambda_\alpha)$ of the group, with highest weight λ_α. There is a complete set of eigenvectors of the Cartan subalgebra, each eigenvector with weight μ, and they span the space R (See appendix A3). We can take this set to be orthonormal. Each weight μ may correspond to several eigenvectors which denote $|\mu_{(i)}\rangle$, (i.e. μ has the multiplicity >1). We insert a complete set of states in (II.19):

$$e^{-\rho_\alpha(r)} = \sum_{|\mu_i\rangle} \langle\lambda_\alpha| e^{X} e^{-C} e^{Vr} |\mu_{(i)}\rangle \langle\mu_{(i)}| e^{-Y} |\lambda_\alpha\rangle$$

$$= \sum_{|\mu_i\rangle} \langle\lambda_\alpha| e^{X} |\mu_{(i)}\rangle \langle\mu_{(i)}| e^{-Y} |\lambda_\alpha\rangle \; e^{-(C,\mu)} \; e^{r(V,\mu)} \tag{III.1}$$

Where (C,μ) and (V,μ) are the eigenvalues of C and V on vectors $|\mu_{(i)}\rangle$ This expression gives the solution as a sum of exponential terms, all the r dependence is explicit, and we have constant coefficients which are matrix elements. The sum is over a (finite) set of weights, so the

number of terms depends on the dimension of the particular representation that corresponds to λ_α. We will be concerned with the constant coefficients, how to generate them, and in the process we will understand better which weights μ correspond to each representation. Consider (II.17)

$$e^X V e^{-X} = V - \sum_\alpha E_\alpha$$

X consists of positive grade generators $X = X^{(1)} + X^{(2)} + X^{(3)} + \ldots$

$X^{(i)}$ has grade i. Then

$$\begin{array}{lccccc} e^X V e^{-X} = & V & + [X^{(1)}, V] & + [X^{(2)}, V] & + \tfrac{1}{2}[X^{(1)}, [X^{(1)}, V]] & + \cdots \\ \text{grd} & 0 & 1 & 2 & 2 & >2 \end{array}$$

When we balance the different grades on both sides of equation (II.17), we see that we must have $(V, \alpha) \neq 0$, $\alpha \in \Delta$, in order that no component of $[X^{(1)}, V]$ vanishes. We have already said that it is possible to choose in general $(V, \alpha) \geqslant 0$ (we say that V lies in the fundamental Weyl chamber). The stricter condition $(V, \alpha) > 0$ is necessary, in order to satisfy (II.17). This also guarantees that X is uniquely defined. We will see later what happens when we let some of the $(V, \alpha) \to 0$.

The same holds for Y, $Y \in \sum_{\alpha \in \Phi^+} g^{-\alpha}$, in (II.18). In fact X and Y are related. Consider acting on (II.18) with the $\sim$ operation:

$$e^Y \widetilde{V} e^{-Y} = \tilde{V} + \sum_{\alpha \in \Delta} \widetilde{E}_{-\alpha} \Rightarrow e^{i\pi T_3} e^{-Y^+} V e^{Y^+} e^{-i\pi T_3} = V - \sum_{\alpha \in \Delta} E_\alpha$$

Comparing with (II.17) we see that $e^{-Y} = e^{-i\pi T_3} e^{X^+} e^{i\pi T_3}$ (III.2)

We are interested in matrix elements of X in the form $\langle \lambda_\alpha | e^X | \mu \rangle$ If we consider a particular matrix representation of equation (II.17), then it can be interpreted as the diagonalisation of $V - \sum_\alpha E_\alpha$ by a similarity transformation. If we can find the eigenvectors of $V - \sum_\alpha E_\alpha$

we can find the required matrix elements. We proceed to construct these directly.

The weight μ is related to the highest weight λ_α of the representation, $\mu = \lambda_\alpha - \sum_{\beta\in\Delta} k_\beta \beta$, $k_\beta \geqslant 0$. We can get from $|\mu\rangle$ to $|\lambda_\alpha\rangle$ by acting with the corresponding set of raising generators $|\lambda_\alpha\rangle = E_\alpha E_\beta \ldots E_\sigma |\mu\rangle$ The level of μ is defined as $s = \sum_\beta k_\beta$; it corresponds to the "distance" of μ from the highest weight λ_α and it is also equal to $(T_3, \lambda_\alpha - \mu)$. So the different weights fall into layers corresponding to the different integers. $(T_3, \lambda_\alpha - \mu) = 0,1,2,\ldots$ (see also section I.1). Now consider

$$|u_\mu\rangle = |\mu\rangle + \sum_{\alpha_1} \alpha_{\alpha_1} E_{\alpha_1}|\mu\rangle + \sum_{\alpha_1,\alpha_2} \alpha_{\alpha_1} b_{\alpha_2} E_{\alpha_1} E_{\alpha_2}|\mu\rangle + \ldots + \sum_{\alpha_1 \ldots \alpha_s} \alpha_{\alpha_1} \ldots y_{\alpha_{s-1}} z_{\alpha_s} E_{\alpha_1} \ldots |\mu\rangle$$

a,b, ... y, z $\in \mathbb{R}$. The sum is independently over all simple roots. Each E_{α_i} changes the level of the weight by 1. The last term has s raising generators where $s = (T_3, \lambda_\alpha - \mu)$ is the level of the weight we start off with. So $|u_\mu\rangle$ is a linear combination of vectors with weights $\nu > \mu$. We will now determine a_{α_1} , b_{α_2} ... y_{α_s} so that the expression above is an eigenvector of $V - \sum_{\alpha\in\Delta} E_\alpha$.

We make use of the commutator $[V, E_{\alpha_i}] = (V, \alpha_i) E_{\alpha_i}$ and the eigenvalues of $V|\mu\rangle = (V,\mu)|\mu\rangle$. When $V - \sum_{\alpha\in\Delta} E_\alpha$ acts on $|u_\mu\rangle$ we get:

$$\left(V - \sum_{\alpha\in\Delta} E_\alpha\right)|u_\mu\rangle = (V,\mu)|\mu\rangle + \sum_{\alpha_1\in\Delta} \alpha_{\alpha_1}\left[(V,\mu) + (V,\alpha_1)\right] E_{\alpha_1}|\mu\rangle +$$

$$+ \sum_{\alpha_1,\alpha_2} \alpha_{\alpha_1} b_{\alpha_2}\left[(V,\alpha_1) + (V,\alpha_2) + (V,\mu)\right] E_{\alpha_1} E_{\alpha_2}|\mu\rangle + \quad . \quad . \quad . \quad +$$

$$+ \sum_{\alpha_1 \ldots \alpha_s} \alpha_{\alpha_1} \ldots y_{\alpha_{s-1}} z_{\alpha_s}\left[(V,\alpha_1) + \ldots (V,\alpha_s) + (V,\mu)\right] E_{\alpha_1} \ldots E_{\alpha_s}|\mu\rangle \quad - \sum_{\alpha_1\in\Delta} E_{\alpha_1}|\mu\rangle$$

$$- \sum_{\alpha_1,\alpha_2} \alpha_{\alpha_2} E_{\alpha_1} E_{\alpha_2}|\mu\rangle - \quad . \quad . \quad . \quad - \sum_{\alpha_1 \ldots \alpha_s} \alpha_{\alpha_2} b_{\alpha_3} \ldots y_{\alpha_s} E_{\alpha_1} E_{\alpha_2} \ldots E_{\alpha_s}|\mu\rangle$$

We have permuted the dummy variables α_i , on the last s terms which are given by the action of $-\sum_\alpha E_\alpha$. Now if the expression above is an eigenvector we must have

$$\alpha_{\alpha_1}\left[(V,\mu)+(V,\alpha_1)\right]-1=\alpha_{\alpha_1}(V,\mu)\Rightarrow\qquad \alpha_{\alpha_1}=\frac{1}{(V,\alpha_1)}$$

$$\alpha_{\alpha_1}b_{\alpha_2}\left[(V,\mu)+(V,\alpha_1)+(V,\alpha_2)\right]-\alpha_{\alpha_2}=\alpha_{\alpha_1}b_{\alpha_2}(V,\mu)\Rightarrow\quad \alpha_{\alpha_1}b_{\alpha_2}=\frac{\alpha_{\alpha_2}}{(V,\alpha_1+\alpha_2)}$$

$$\alpha_{\alpha_1}b_{\alpha_2}\ldots y_{\alpha_{s-1}}z_{\alpha_s}\left[(V,\mu)+(V,\alpha_1)+\ldots+(V,\alpha_s)\right]-\alpha_{\alpha_2}b_{\alpha_3}\ldots y_{\alpha_s}=\alpha_{\alpha_1}\ldots y_{\alpha_{s-1}}z_{\alpha_s}(V,\mu)\Rightarrow$$

$$\Rightarrow \alpha_{\alpha_1}b_{\alpha_2}\ldots z_{\alpha_s}=\frac{\alpha_{\alpha_2}b_{\alpha_3}\ldots y_{\alpha_s}}{(V,\alpha_1+\alpha_2+\ldots\alpha_s)}=\frac{\alpha_{\alpha_3}b_{\alpha_4}\ldots}{(V,\alpha_1+\alpha_2+\ldots\alpha_s)(V,\alpha_2+\ldots\alpha_s)}=\ldots=\frac{1}{(V,\alpha_1+\ldots\alpha_s)\ldots(V,\alpha_s)}$$

All denominators are positive since we have the requirement $(V,\alpha)>0$, $\alpha\in\Delta$

Then with these values for $\alpha, b, \ldots$, $|u_\mu\rangle$ is by construction an eigenvector of $V-\sum_\alpha E_\alpha$ with eigenvalue (V,μ):

$$\left.\begin{aligned}&\left(V-\sum_\alpha E_\alpha\right)|u_\mu\rangle=(V,\mu)\,|u_\mu\rangle\\ &e^{X}\,V\,e^{-X}=V-\sum_\alpha E_\alpha\end{aligned}\right\}\Rightarrow\qquad V e^{-X}|u_\mu\rangle=(V,\mu)\,e^{-X}|u_\mu\rangle$$

But we already have $V|\mu\rangle=(V,\mu)|\mu\rangle$. Then we have two sets of eigenvectors with the same eigenvalues and therefore $e^{-X}|u_\mu\rangle=|\mu\rangle$. (In general if there are many vectors $|\mu_{(i)}\rangle$ corresponding to the same weight we would have $e^{-X}|u_{\mu_{(i)}}\rangle=\sum_j\lambda_{ij}|\mu_{(j)}\rangle$, but then again, because of our chosen normalisation of $|u_{\mu_{(i)}}\rangle$ we have $\lambda_{ij}=\delta_{ij}$, even if there is degeneracy). Finally the required matrix elements:

$$\langle\lambda_a|e^{X}|\mu_{(i)}\rangle=\langle\lambda_a|u_{\mu_{(i)}}\rangle=\langle\lambda_a|\sum_{\alpha_1\ldots\alpha_s}\alpha_{\alpha_1}b_{\alpha_2}\ldots z_{\alpha_s}E_{\alpha_1}\ldots E_{\alpha_s}|\mu_{(i)}\rangle=$$

$$=\sum_{\alpha_1,\alpha_2\ldots\alpha_s}\frac{\langle\lambda_a|E_{\alpha_1}E_{\alpha_2}\ldots E_{\alpha_s}|\mu_{(i)}\rangle}{(V,\alpha_1+\alpha_2+\ldots\alpha_s)(V,\alpha_2+\alpha_3+\ldots\alpha_s)\ldots(V,\alpha_{s-1}+\alpha_s)(V,\alpha_s)}\qquad\text{(III.3)}$$

Combining (III. 1-3) we can write for the solutions

$$e^{-\rho_\alpha(r)} = \sum_{|\mu_{(i)}\rangle} \langle \lambda_\alpha | e^{X} | \mu_{(i)} \rangle \langle \mu_{(i)} | e^{X^+} | \lambda_\alpha \rangle \; (-1)^{(T_3, \lambda_\alpha - \mu)} \; e^{-(C,\mu)} e^{r(V,\mu)} =$$

$$= \sum_{|\mu_{(i)}\rangle} \left\{ \sum_{\alpha_1 \ldots \alpha_s} \frac{\langle \lambda_\alpha | E_{\alpha_1} \ldots E_{\alpha_s} | \mu_{(i)} \rangle}{(V, \alpha_1 + \alpha_2 + \ldots \alpha_s) \ldots (V, \alpha_s)} \right\}^2 (-1)^{(T_3, \lambda_\alpha - \mu)} \; e^{-(C,\mu)} e^{r(V,\mu)}$$

Notice there is a (-) sign whenever the level of μ is odd. As it stands this expression depends on the multiplicity of each weight. We can rearrange it:

$$e^{-\rho_\alpha(r)} = \sum_{|\mu_i\rangle} \left(\sum_{\substack{\alpha_1 \ldots \alpha_s \\ \alpha'_1 \ldots \alpha'_s}} \frac{\langle \lambda_\alpha | (-1)^{T_3} E_{\alpha_1} \ldots E_{\alpha_s} \; e^{-C} e^{rV} (-1)^{T_3} | \mu_{(i)} \rangle \langle \mu_{(i)} | E_{-\alpha'_s} \ldots E_{-\alpha'_1} | \lambda \rangle}{(V, \alpha_1 + \ldots \alpha_s)(V, \alpha_2 + \ldots \alpha_s) \ldots (V, \alpha_s)(V, \alpha'_1 + \ldots \alpha'_s) \ldots (V, \alpha'_s)} \right)$$

We can split $\sum_{|\mu\rangle} = \sum_s \sum_{\substack{\text{states} \\ \text{of level } s}}$. All states $|\nu\rangle$ with a level different than s have $\langle \lambda_\alpha | E_{\alpha_1} \ldots E_{\alpha_s} | \nu \rangle = 0$. Using again the completeness of $|\mu_{(i)}\rangle$ we end up with

$$e^{-\rho_\alpha(r)} = \sum_{s=0}^{(T_3, \lambda_\alpha - \bar{\lambda}_\alpha)} (-1)^s \sum_{\substack{\alpha_1 \ldots \alpha_s \\ \alpha'_1 \ldots \alpha'_s}} \frac{e^{(rV - C, \lambda_\alpha - \alpha_1 - \ldots \alpha_s)} \langle \lambda_\alpha | E_{\alpha_1} \ldots E_{\alpha_s} E_{-\alpha'_s} \ldots E_{-\alpha'_1} | \lambda_\alpha \rangle}{(V, \alpha_1 + \ldots \alpha_s) \ldots (V, \alpha_s)(V, \alpha'_1 + \ldots \alpha'_s) \ldots (V, \alpha'_s)}$$

(III.4)

The sum is over all possible levels. $\bar{\lambda}_\alpha$ is the lowest weight of the representation with highest weight λ_α . $(T_3, \lambda_\alpha - \bar{\lambda}_\alpha)$ corresponds to the maximum possible level. The sum over $\alpha_1 \ldots \alpha_s, \alpha'_1 \ldots \alpha'_s$ is over all simple roots independently. But note that whenever $\lambda_\alpha - \alpha_1 - \alpha_2 - \ldots - \alpha_s$ is not a weight the term $\langle \lambda_\alpha | \ldots . | \lambda_\alpha \rangle$ will be zero. So again we

have a sum over weights only, as in (III.1). However now we do not have to worry about the different multiplicities.

In order to check (III.4), apply it to the simplest example, SU(2). The reduced equation for the monopole is $\frac{d^2\rho}{dr^2} = e^{2\rho}$. The fundamental representation of SU(2) has just two weights, highest $\lambda = \frac{1}{2}\alpha$ and lowest $\bar{\lambda} = -\frac{1}{2}\alpha$, with α the simple root. Then

$$e^{-\rho(r)} = e^{(rV-C,\lambda)} - \frac{e^{(rV-C,\lambda-\alpha)}\langle\lambda|E_\alpha E_{-\alpha}|\lambda\rangle}{(V,\alpha)^2}$$

Also $\langle\lambda|E_\alpha E_{-\alpha}|\lambda\rangle = \langle\lambda|H_\alpha|\lambda\rangle = 1$. With the particular choice of $e^{(C,\lambda)} = (V,\alpha)$ (it makes $e^{-\rho(0)} \sim 0$), we have:

$$e^{-\rho(r)} = \frac{e^{r\frac{1}{2}(V,\alpha)}}{(V,\alpha)} - \frac{e^{-r\frac{1}{2}(V,\alpha)}}{(V,\alpha)} = \frac{\sinh br}{b}, \quad b = \frac{1}{2}(V,\alpha)$$

This is the expected solution.

For a general group, (III.4) depends on the structure of the weights of the fundamental representations, which in turn are determined from the simple roots. So in principle we can calculate everything starting from the Dynkin diagram.

In order to simplify some of the calculations, we have derived a recursion relation for a matrix element $\langle\lambda|E_{\alpha_1}\ldots E_{-\alpha_1'}|\lambda\rangle$ with 2s raising and lowering generators in terms of one with 2s-2 generators. We use the Chevalley basis. In this basis $[E_\gamma, E_{-\gamma'}] = \delta_{\gamma\gamma'}H_\gamma$; this is because $\gamma-\gamma'$ is not a root. So by commuting E_{α_s} and $E_{-\alpha_s'}$ we have:

$$\langle\lambda|E_{\alpha_1}\ldots E_{\alpha_s}E_{-\alpha_s'}\ldots E_{-\alpha_1'}|\lambda\rangle = \langle\lambda|E_{\alpha_1}\ldots E_{\alpha_{s-1}}E_{-\alpha_s'}E_{\alpha_s}E_{-\alpha_{s-1}'}\ldots E_{-\alpha_1'}|\lambda\rangle +$$

$$+ \quad \delta_{\alpha_s\alpha_s'} \quad \langle\lambda|E_{\alpha_1}\ldots E_{\alpha_{s-1}}H_{\alpha_s}E_{-\alpha_{s-1}'}\ldots E_{-\alpha_1'}|\lambda\rangle$$

We repeatedly commute E_{α_s} until it reaches the right where it annihilates on $|\lambda\rangle$, since this is a highest vector. We also commute all the H to the right, using

$$H_{\alpha_i} E_{-\alpha_\ell'} E_{-\alpha_{\ell-1}'} \cdots E_{-\alpha_1'} |\lambda\rangle = (H_{\alpha_i}, \lambda - \alpha_\ell' - \alpha_{\ell-1}' - \cdots - \alpha_1') |\lambda\rangle$$

In this way we generate s matrix elements of strings of length 2s-2

$$\langle\lambda| E_{\alpha_1} \cdots E_{\alpha_s} E_{-\alpha_s'} \cdots E_{-\alpha_1'} |\lambda\rangle = \delta_{\alpha_s \alpha_s'} (H_{\alpha_s}, \lambda - \alpha_{s-1}' - \cdots - \alpha_1') \langle\lambda| E_{\alpha_1} \cdots E_{\alpha_{s-1}} \cdots E_{-\alpha_1'} |\lambda\rangle$$

$$+ \delta_{\alpha_s \alpha_{s-1}'} (H_{\alpha_s}, \lambda - \alpha_{s-2}' - \cdots \alpha_1') \langle\lambda| E_{\alpha_1} \cdots E_{\alpha_{s-1}} E_{-\alpha_s'} E_{-\alpha_{s-2}'} \cdots E_{-\alpha_1'} |\lambda\rangle + \cdots$$

$$+ \delta_{\alpha_s \alpha_1'} (H_{\alpha_s}, \lambda) \langle\lambda| E_{\alpha_1} \cdots E_{\alpha_{s-1}} E_{-\alpha_s'} E_{-\alpha_{s-1}'} \cdots E_{-\alpha_2'} |\lambda\rangle \qquad \text{(III.5)}$$

We can apply this formula many times, and eventually we will reduce all matrix elements to expressions with two generators only, say $E_{\alpha_i} E_{-\alpha_j'}$ and then we have

$$\langle\lambda| E_{\alpha_i} E_{-\alpha_j'} |\lambda\rangle = \delta_{\alpha_i \alpha_j'} \langle\lambda| H_{\alpha_i} |\lambda\rangle = \delta_{\alpha_i \alpha_j'} \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}$$

However we are summing over many terms and the general expressions are necessarily complicated. It is easier for each particular application to look for the cancellations that occur.

Using the above result we have calculated the first few terms in the sum (III.4), up to s=3, for an arbitrary group:

$$e^{-\rho_{\alpha_i}(r)} = e^{(rV-C,\,\lambda_{\alpha_i})} - \frac{e^{(rV-C,\,\lambda_{\alpha_i}-\alpha_i)}}{(V,\alpha_i)^2} + \sum_{\alpha_j\neq\alpha_i} \frac{e^{(rV-C,\,\lambda_{\alpha_i}-\alpha_i-\alpha_j)}(H_{\alpha_j},-\alpha_i)}{(V,\alpha_j)^2(V,\alpha_i+\alpha_j)^2}$$

$$- \sum_{\alpha_j\neq\alpha_i} \frac{e^{(rV-C,\,\lambda_{\alpha_i}-2\alpha_i-\alpha_j)}\,[1+(H_{\alpha_i},\alpha_j)]\,(H_{\alpha_j},\alpha_i)}{(V,2\alpha_i+\alpha_j)^2(V,\alpha_i+\alpha_j)^2(V,\alpha_i)^2}$$

$$- \frac{1}{2}\sum_{\alpha_j\neq\alpha_i} \frac{e^{(rV-C,\,\lambda_{\alpha_i}-\alpha_i-2\alpha_j)}\,[1+(H_{\alpha_j},\alpha_i)]\,(H_{\alpha_j},\alpha_i)}{(V,\alpha_i+2\alpha_j)^2(V,\alpha_j)^4}$$

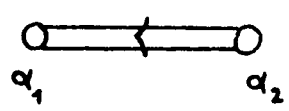
$$- \frac{1}{2}\sum_{\alpha_i\neq\alpha_j\neq\alpha_k} \frac{e^{(rV-C,\,\lambda_{\alpha_i}-\alpha_i-\alpha_j-\alpha_k)}\,[(H_{\alpha_j},\alpha_i+\alpha_k)(H_{\alpha_k},\alpha_i)(V,\alpha_k)^2+(H_{\alpha_k},\alpha_i+\alpha_j)(H_{\alpha_j},\alpha_i)(V,\alpha_j)^2+}{(V,\alpha_i+\alpha_j+\alpha_k)^2(V,\alpha_j+\alpha_k)^2(V,\alpha_j)^2(V,\alpha_k)^2}$$

$$\ldots \underline{+2(H_{\alpha_j},\alpha_i)(H_{\alpha_k},\alpha_i)(V,\alpha_j)(V,\alpha_k)]} \qquad \text{(III.6)}$$

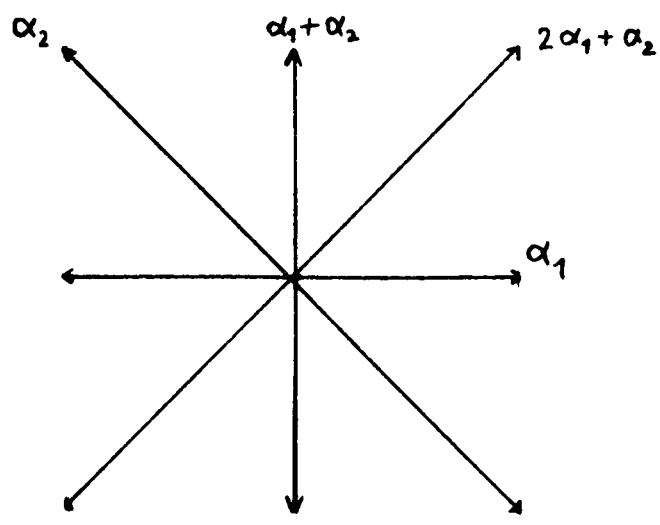
Many terms in the above expression are zero whenever we have orthogonal roots (i.e. roots that are not tied in the Dynkin diagram). Note that $(H_{\alpha_i},\alpha_j) = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ are just the integer entries of the Cartan matrix. If $(\alpha_i,\alpha_j)\neq 0$, $\alpha_i\neq\alpha_j$, then $\alpha_i+\alpha_j$ is a root. So when there is a contribution to the sum we have $(V,\alpha_i+\alpha_j)$ in the denominator and $\alpha_i+\alpha_j$ is a root. Similarly the term $[1+(H_{\alpha_i},\alpha_j)]\cdot(H_{\alpha_j},\alpha_i)$ (for $\alpha_i\neq\alpha_j$) is non-zero only if $(H_{\alpha_i},\alpha_j) < -1$, but then $2\alpha_i+\alpha_j$ is a root. Also in the last term if $\alpha_j+\alpha_k\notin\Phi$ (is not

a root), then $(\alpha_j, \alpha_k)=0$. But then the numerator becomes (H_{α_j}, α_i) (H_{α_k}, α_i) $(V, \alpha_j+\alpha_k)^2$; therefore the factor $(V, \alpha_j+\alpha_k)^2$ cancells from the denominator when $\alpha_j+\alpha_k$ is not a root. It turns out that this is always the case, i.e. we only get terms of the form (V,γ) in the denominator with γ a root.

To illustrate the use of (III.6) we will now give a simple example, which nevertheless is non-trivial to solve by elementary methods only. The algebra of SO(5) is $B_2 \simeq C_2$ and has rank 2. There are two simple roots α_1 and α_2, of unequal length which we normalise $(\alpha_1, \alpha_1)=1$ $(\alpha_2, \alpha_2)=2$. The Dynkin diagram is



and the corresponding root diagram has a 8 non-zero roots:



The Cartan matrix is $K_{ij}=(H_{\alpha_i}, \alpha_j) = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$

and the equations of motion corresponding to (II.6) are

$$\frac{d^2}{dr^2}\rho_1 = e^{2\rho_1} e^{-\rho_2} \quad , \quad \frac{d^2}{dr^2}\rho_2 = e^{2\rho_2} e^{-2\rho_1}$$

There are two fundamental weights $\lambda_1 = \alpha_1 + \frac{1}{2}\alpha_2$, $\lambda_2 = \alpha_1 + \alpha_2$

which satisfy $\frac{2(\alpha_i,\lambda_j)}{(\alpha_i,\alpha_i)} = \delta_{ij}$. The corresponding representations

are of dimension 4 and 5 respectively. There is just one weight of single multiplicity at each level:

$R(\lambda_1)$

- $\lambda_1 = \alpha_1 + \frac{1}{2}\alpha_2$
- $\lambda_1 - \alpha_1 = \frac{1}{2}\alpha_2$
- $\lambda_1 - \alpha_1 - \alpha_2 = -\frac{1}{2}\alpha_2$
- $\lambda_1 - 2\alpha_1 - \alpha_2 = -\alpha_1 - \frac{1}{2}\alpha_2$

$R(\lambda_2)$

- $\lambda_2 = \alpha_1 + \alpha_2$
- $\lambda_2 - \alpha_2 = \alpha_1$
- $\lambda_2 - \alpha_1 - \alpha_2 = 0$
- $\lambda_2 - 2\alpha_1 - \alpha_2 = -\alpha_1$
- $\lambda_2 - 2\alpha_1 - 2\alpha_2 = -\alpha_1 - \alpha_2$

$R(\lambda_1)$ has weights only up to s=3 and we can use the expression (III.6) for $e^{-\rho_1}$ directly. We observe that $-(H_{\alpha_2}, \alpha_1) = 1$ and the only contribution from the level s=3 comes from the term proportional to $[1 + (H_{\alpha_1}, \alpha_2)] = 1-2 = -1$ (cf equ. (III.6))

$$e^{-\rho_1(r)} = e^{(rV-C,\lambda_1)} - \frac{e^{(rV-C,\lambda_1-\alpha_1)}}{(V,\alpha_1)^2} + \frac{e^{(rV-C,\lambda_1-\alpha_1-\alpha_2)}}{(V,\alpha_2)^2(V,\alpha_1+\alpha_2)^2} - \frac{e^{(rV-C,\lambda_1-2\alpha_1-\alpha_2)}}{(V,2\alpha_1+\alpha_2)^2(V,\alpha_1+\alpha_2)^2(V,\alpha_1)^2} \tag{III.7}$$

For $e^{-\rho_2}$ we have to find the coefficient of the s=4 term. From (III.4) this is

$$\frac{\langle\lambda_2|\, E_{\alpha_2} E_{\alpha_1} E_{\alpha_1} E_{\alpha_2} E_{-\alpha_2} E_{-\alpha_1} E_{-\alpha_1} E_{-\alpha_2} \,|\lambda_2\rangle}{4\,(V,\alpha_1+\alpha_2)^4\,(V,2\alpha_1+\alpha_2)^2\,(V,\alpha_2)^2}$$

and using recursion relation (III.5) the numerator is 4. Then

$$e^{-\rho_2(r)} = e^{(rV-C,\,\lambda_2)} - \frac{e^{(rV-C,\,\lambda_2-\alpha_2)}}{(V,\alpha_2)^2} + \frac{2\,e^{(rV-C,\,\lambda_2-\alpha_2-\alpha_1)}}{(V,\alpha_1)^2\,(V,\alpha_1+\alpha_2)^2}$$

$$- \frac{e^{(rV-C,\,\lambda_2-\alpha_2-2\alpha_1)}}{(V,2\alpha_1+\alpha_2)^2\,(V,\alpha_1)^4} + \frac{e^{(rV-C,\,\lambda_2-2\alpha_2-2\alpha_1)}}{(V,\alpha_1+\alpha_2)^4\,(V,2\alpha_1+\alpha_2)^2\,(V,\alpha_2)^2} \qquad \text{(III.7')}$$

The solutions involve combinations of four arbitrary constants, which are the (V, α_i) and (C, α_i) with i=1 and 2. To check directly that (III.7) and (III.7') are solutions requires a long and tedious substitution.

Further applications will be presented at the end of this section.

III.2 Boundary conditions at the origin

We have found solutions with 2ℓ (ℓ=rank) arbitrary parmeters through C and V. We will now see that these constants are determined by physical requirements. We first study the behaviour at the origin, where it is C that must be adjusted to give finite solutions.

As we have seen in section I, the magnetic field obeys an inverse square law:

$$B_3 = \frac{1}{e r^2} \left(r^2 \frac{d\Psi}{dr} - T_3 \right)$$

In order to avoid the singularity at the origin, we must have:

$$\frac{d\Psi}{dr} \longrightarrow \frac{T_3}{r^2} \qquad \text{as} \qquad r \to 0 \tag{III.8}$$

If in addition $\Psi \to -\frac{T_3}{r}$, the Higgs field and the transverse components of B also vanish at the origin (cf. equ. (I.9), (I.11)). This also implies that at the origin $N^{\pm} \longrightarrow \frac{T^{\pm}}{r}$, where $T^{\pm}, T_3$ are the generators of the SU(2) principal subalgebra.

Since $\Psi = \sum_{i=1}^{\ell} \psi_{\alpha_i} H_{\alpha_i} = \sum_i \frac{1}{2} \dot{\rho}_{\alpha_i} H_{\alpha_i}$ in order to satisfy (III.8) we must have near r=0 :

$$\frac{t_{\alpha_i}}{r^2} = \frac{1}{2} \ddot{\rho}_{\alpha_i} = \frac{1}{2} \exp \sum_j \rho_{\alpha_j} K_{ji} = \frac{1}{2} \prod_{j=1}^{\ell} \left(e^{-\rho_{\alpha_j}} \right)^{-K_{ji}} \Rightarrow$$

$$\Rightarrow \quad e^{-\rho_{\alpha_j}} = \prod_{i=1}^{\ell} \left(2 t_{\alpha_i} \right)^{-K^{-1}_{ij}} r^{2 \sum_i K^{-1}_{ij}} = \prod_{i=1}^{\ell} \left(2 t_{\alpha_i} \right)^{-K^{-1}_{ij}} r^{2 t_{\alpha_j}} \tag{III.9}$$

We see that $e^{-\rho_{\alpha_j}}$ must have a zero of order $(2t_{\alpha_j})$. $2 t_{\alpha_j} = 2 \sum_i K^{-1}_{ij}$ is postitive since all entries of K^{-1} are positive, and it is an

integer since T_3 has integer or half integer eigenvalues. If we look at the Taylor expansion of $e^{-\rho_{\alpha_j}} = \sum_n A_n r^n$, condition (III.9) implies that all A_n up to $n = 2t_{\alpha_j} - 1$ must vanish. Since $2t_{\alpha_j} \geqslant \ell$, the number of constraints seems to be greater than the number of parameters. It is remarkable that these constraints can be simultaneously satisfied. We understand only partially why this is possible and we will try to indicate it by an (incomplete) argument. It is a certain mathematical property, namely that the number $2t_{\alpha_j}$ coincides with the level of the lowest weight of the representation $R(\lambda_j)$, that allows us to satisfy these conditions.

We first show that the level of the lowest weight $\bar{\lambda}_i$ is $s=(T_3,\lambda_i-\bar{\lambda}_i)= 2t_{\alpha_i}$. There is a unique element σ of the Weyl group that takes each positive root to some negative root, i.e. $\sigma(\Delta) = -\Delta$. Since $\sigma^2(\Delta)=\Delta$ we also conclude $\sigma^2=1$ [18]. The action of σ on each simple root $\alpha_i \in \Delta$ is $\sigma(\alpha_i) = -\alpha_{i'}$. The indices i' are some permutation of the set of indices $\{i\}$. Let λ_j be the fundamental weights. Then

$$\frac{2(\sigma(\lambda_j),\alpha_i)}{(\alpha_i,\alpha_i)} = \frac{2(\sigma^2(\lambda_j),\sigma(\alpha_i))}{(\sigma(\alpha_i),\sigma(\alpha_i))} = \frac{2(\lambda_j,-\alpha_{i'})}{(\alpha_{i'},\alpha_{i'})} = -\delta_{i'j}$$

We have used the property that the inner product (,) is invariant under simultaneous reflections of both vectors. The relation above shows that $\sigma(\lambda_j)$ has non-positive eigenvalues, and since it is in the same Weyl orbit with λ_j , it is the lowest weight. i.e. $\bar{\lambda}_j = \sigma(\lambda_j)$ We know that $\lambda_j = \sum_{m=1}^{\ell} \alpha_m K^{-1}_{mj}$. The level of $\bar{\lambda}_j$ is

$$s = (T_3,\ \lambda_j - \sigma(\lambda_j)) = \sum_{m=1}^{\ell} K^{-1}_{mj}\ (T_3,\ \alpha_m + \alpha_{m'}) =$$

$$= 2\sum_{m=1}^{\ell} K^{-1}_{mj} = 2t_{\alpha_j}$$

This concludes the proof.

Now consider the solution as represented by equ.(II.19). Because of condition (II.17), we can move e^{X} to the right:

$$e^{-\rho_{\alpha_i}(r)} = \langle \lambda_i | \; e^{r\left(V - \sum_j E_{\alpha_j}\right)} e^{X} e^{-C} e^{-Y} \; | \lambda_i \rangle \qquad \text{(III.10)}$$

Thus

$$\frac{d^n}{dr^n} \; e^{-\rho_{\alpha_i}(r)} \Big|_{r=0} = \langle \lambda_i | \; \left(V - \sum_j E_{\alpha_j}\right)^n \; e^{X} e^{-C} e^{-Y} | \lambda_i \rangle$$

(III.11)

Each time $\sum_j E_{\alpha_j}$ acts on a weight vector $|\nu\rangle$ it decreases its level by 1. If we start with the lowest weight $|\bar{\lambda}_i\rangle$ and act with $\left(\sum_j E_{\alpha_j}\right)^{2t_{\alpha_i}}$ it will take us to the highest vector $|\lambda_i\rangle$ since the level changes from $2t_{\alpha_i}$ to 0. Therefore if we can take C such that $e^{X} e^{-C} e^{-Y} |\lambda_i\rangle = \kappa \, |\bar{\lambda}_i\rangle$ κ constant, we will find that all of the derivatives $\frac{d^n}{dr^n} \; e^{-\rho_{\alpha_i}}\Big|_{r=0} = 0$ for $n=1,2, \ldots, 2t_{\alpha_i}-1$, because of the orthogonality of different states. Only for $n \geq 2t_{\alpha_i}$ we have non-vanishing contributions and therefore $e^{-\rho_{\alpha_i}(r)} \sim r^{2t_{\alpha_i}}$ when $r \sim 0$, as required by (III.9)

So $e^{X} e^{-C} e^{-Y}$ takes us from the highest to the lowest weight for every representation. This is our boundary condition at the origin, now expressed as a condition on the parameters of C. To find such a C consider the matrix element

$$\kappa = \langle \bar{\lambda}_i | \; e^{X} \; e^{-C} \; e^{-Y} \; | \lambda_i \rangle = \langle \bar{\lambda}_i | \; e^{-C} \; e^{-Y} \; | \lambda_i \rangle = e^{-(C, \bar{\lambda}_i)} \langle \bar{\lambda}_i | \; e^{-Y} \; | \lambda_i \rangle$$

$$= e^{-(C, \bar{\lambda}_i)} \langle \bar{\lambda}_i | \; e^{i\pi T_3} \; e^{X^+} \; e^{i\pi T_3} \; | \lambda_i \rangle = e^{-(C, \bar{\lambda}_i)} (-1)^{(T_3, \lambda_i - \bar{\lambda}_i)} \langle \bar{\lambda}_i | e^{X^+} | \lambda_i \rangle$$

$$= e^{-(C, \bar{\lambda}_i)} \; (-1)^{2t_{\alpha_i}} \; \langle \lambda_i | \; e^{X} \; | \bar{\lambda}_i \rangle$$

(III.12)

The last step is valid since the constants are real. Then $e^{(C,\bar{\lambda}_i)} = \frac{(-1)^{2t_{\alpha_i}}}{\kappa} \langle \lambda_i | e^X | \bar{\lambda}_i \rangle$. From (III.3) we know

$$\langle \lambda_i | e^X | \bar{\lambda}_i \rangle = \sum_{\substack{\alpha_1 \cdots \alpha_s \\ s = 2t_{\alpha_i}}} \frac{\langle \lambda_i | E_{\alpha_1} E_{\alpha_2} \cdots E_{\alpha_s} | \bar{\lambda}_i \rangle}{(V, \alpha_1 + \alpha_2 + \cdots \alpha_s) \ \ldots \ (V, \alpha_s)} \qquad \text{(III.13)}$$

We have noticed that one can write this in a compact expression

$$\langle \lambda_i | e^X | \bar{\lambda}_i \rangle = \prod_{\gamma \in \Phi^+} (V, \gamma)^{\frac{2(\gamma, \bar{\lambda}_i)}{(\gamma, \gamma)}} \qquad \text{(III.14)}$$

The product is over all positive roots (not just simple). We have not found a general proof for this statement but it has been verified for many cases for the groups with small rank (A_2 ,B_2 ,G_2 ,A_3 ,A_4 etc) and for all "end point" representations corresponding to λ_1 of the series A_ℓ , B_ℓ ,C_ℓ ,D_ℓ .

Note that using this expression the first non-zero term in the Taylor expansion for $e^{-\rho_{\alpha_j}}$ becomes

$$e^{-\rho_{\alpha_j}} \simeq \kappa \frac{r^{2t_{\alpha_j}}}{(2t_{\alpha_j})!} (-1)^{2t_{\alpha_j}} \langle \lambda_j | (\sum_i E_{\alpha_i})^{2t_{\alpha_j}} | \bar{\lambda}_j \rangle + \ldots$$

$$= \kappa \, r^{2t_{\alpha_j}} \, (-1)^{2t_{\alpha_j}} \sum_{\alpha_1 \cdots \alpha_s} \frac{\langle \lambda_j | E_{\alpha_1} E_{\alpha_2} \cdots E_{\alpha_s} | \bar{\lambda}_j \rangle}{(2t_{\alpha_j})!}$$

$$= \kappa \, r^{2t_{\alpha_j}} \, (-1)^{2t_{\alpha_j}} \prod_{\gamma \in \Phi^+} (T_3, \gamma)^{\frac{2(\gamma, \bar{\lambda}_j)}{(\gamma, \gamma)}}$$

The last step follows from (III.13) and (III.14) above, if we substitute $V=T_3$, since $(2t_{\alpha_j})! = (T_3, \alpha_1 + \alpha_2 + \ldots \alpha_s)(T_3, \alpha_2 + \ldots \alpha_s) \ldots (T_3, \alpha_s)$ with $s=2t_{\alpha_j}$. We see that the order of the zero of $e^{-\rho_{\alpha_j}}$ is indeed $2t_{\alpha_j}$.

Comparing with (III.9) we must have $\kappa = (-1)^{2t_{\alpha_j}}$ and

$$\prod_{i=1}^{\ell} (2t_{\alpha_i})^{-K^{-1}_{ij}} = \prod_{\gamma \in \Phi^+} (T_3, \gamma)^{\frac{2(\gamma, \bar{\lambda}_j)}{(\gamma,\gamma)}}$$

But

$$\bar{\lambda}_j = \sigma(\lambda_j) = \sum_{i=1}^{\ell} \sigma(\alpha_i) K^{-1}_{ij}$$

and

$$\frac{2(\gamma, \bar{\lambda}_j)}{(\gamma,\gamma)} = \sum_{i=1}^{\ell} K^{-1}_{ij} \frac{2(\gamma, \sigma(\alpha_i))}{(\gamma,\gamma)} = \sum_i K^{-1}_{ij} \frac{2(\sigma(\gamma), \alpha_i)}{(\sigma(\gamma), \sigma(\gamma))}$$

Therefore we can write

$$\prod_{i=1}^{\ell} (2t_{\alpha_i})^{-K^{-1}_{ij}} = \prod_{i=1}^{\ell} \prod_{\gamma \in \Phi^+} \left[(T_3, -\sigma(\gamma))^{\frac{2(-\sigma(\gamma), \alpha_i)}{(-\sigma(\gamma), -\sigma(\gamma)}} \right]^{-K^{-1}_{ij}}$$

$$\Rightarrow \qquad 2t_{\alpha_i} = \prod_{\gamma \in \Phi^+} (T_3, \gamma)^{\frac{2(\gamma, \alpha_i)}{(\gamma,\gamma)}} \qquad \text{(III.15)}$$

The last expression is independent of the weights of the representation. (T_3, γ) is just the height of the root. We have verified this result for the same cases as (III.14) above.

To summarize, in order to get the correct behaviour at the origin we must choose C as a function of the eigenvalues of V, in the form $e^{(C,\bar{\lambda}_i)} = \langle \lambda_i | e^X | \bar{\lambda}_i \rangle$ (remember X depends on V). It is too complicated to show by direct substitution in the general solution that this indeed gives the required behaviour, but we gave an indirect demonstration. Since the $\bar{\lambda}_i$ span the weight space we can also write for any weight μ:

$$e^{(C,\mu)} = \prod_{\gamma \in \Phi^+} (V, \gamma)^{\frac{2(\gamma,\mu)}{(\gamma,\gamma)}} \qquad \text{(III.15')}$$

(using (III.14)). We will have the chance to use this expression in some examples.

III.3 Symmetry Breaking.

The Higgs field was originally introduced in gauge theories in order to provide the masses of the gauge bosons. In quantum theory this is done through a "non-zero vacuum expectation value" $\langle 0|\Phi|0\rangle$. In the classical theory the vacuum is realised at infinity. The equivalent of $\langle 0|\Phi|0\rangle$ is a non-zero constant value for the Higgs field as $r\to\infty$ Then the symmetry of the solution is "less" than the symmetry of the Lagrangian and we have "spontaneous symmetry breaking". In this section we shall see that our single parameter of the solutions V, is just this constant Higgs field.

In the case of the adjoint representation that we are considering it is easy to find the residual symmetry, the exact symmetry group H. We follow the notation and definitions of the review article [4]. H is the little group of $\Phi_\infty \equiv \Phi(r\to\infty)$ in the big group G. The generators of H are just those elements of the Lie algebra $\mathcal{g}$ = L(G) that commute with the asymptotic Higgs field:

$$L(H) = \{ X \in \mathcal{g} \;/\; [X, \Phi_\infty] = 0 \}$$

Of course $\Phi_\infty \in L(H)$, and since it commutes with all other generators of H it forms an invariant U(1) factor. Next consider all generators of H orthogonal to Φ_∞ :

$$L(K) = \{ X \in L(H) \;/\; Tr(\Phi_\infty X) = 0 \}$$

One can easily verify that $[L(K),L(K)] \subset L(K)$, i.e L(K) is a subalgebra of L(H). Every generator of H can be decomposed in a part parallel plus a part orthogonal to Φ_∞, and therefore the local structure of H is

$$L(H) \simeq U(1) \otimes L(K)$$

From the physical point of view we would require K to be semisimple. Then there are no more invariant U(1) factors and our single U(1) can be identified with electromagnetism. The electric charge is identified with the U(1) generator

$$Q = \frac{1}{\alpha} e \Phi_\infty \qquad \text{(III.16)}$$

where α is the "length" of the Higgs field $\alpha^2 = Tr(\Phi_\infty)^2$. The above correspondence arises naturally when we compare the usual U(1) covariant derivative $\partial_\mu - iQA_\mu$, with the H group covariant derivative, decomposed along the U(1) direction

$$\partial_\mu - ieW_\mu = \partial_\mu - i\frac{e\Phi_\infty}{\alpha} Tr\, W_\mu \frac{\Phi_\infty}{\alpha} - X_\mu , \qquad X_\mu \in L(K)$$

The magnetic charge is similarly defined as the component of the total flux along the Φ direction.

$$g = Tr \int_{S_\infty} (B_i \frac{\Phi}{\alpha}) \, dS^i \qquad \text{(III.17)}$$

It is possible to relate Q and g in one of the different forms of the "Dirac quantisation condition", depending only on some general assumptions about the solutions (see ref.[4] and [21]). We will not repeat these arguments here, but we will later determine Q and g in terms of the parameters of our solutions.

The magnetic charge is also related to the energy of the solution. We saw in section I that in the B.P.S. limit of vanishing self interaction for the Higgs field, the lower bound for the energy of the solution is saturated and

$$E = Tr \int_{S_\infty} (B^i \Phi) \, dS^i = \alpha g \qquad \text{(III.18)}$$

And if there is no angular dependence

$$E = 4\pi \; Tr \left(r^2 B_r \Phi \right)_{r\to\infty} \tag{III.19}$$

We see that the energy only depends on the asymptotic values of the fields.

In terms of the $\Psi, N^{\pm}$ variables and with $\hat{r}=(0,0,1)$ (see section I.3) we have

$$B_3 = \frac{1}{er^2}\left(r^2 \frac{d\Psi}{dr} - T_3\right) \qquad\qquad \Phi = \frac{1}{e}\left(\Psi + \frac{T_3}{r}\right)$$

We define the magnetic charge matrix G(r) such that

$$B_3 = \frac{1}{4\pi r^2} G(r) \tag{III.20}$$

Provided that G(r) tends to be a constant as $r \to \infty$, we have an inverse square law for the magnetic field. The total energy (which is also the mass of the monopole) is finite

$$E = Tr\left(G_\infty \Phi_\infty\right) \tag{III.21}$$

In terms of our variables

$$G_\infty = \frac{4\pi}{e}\left(r^2 \frac{d\Psi}{dr} - T_3\right)_{r\to\infty} \quad , \qquad \Phi_\infty = \frac{1}{e}\Psi_{r\to\infty}$$

and therefore we only need to calculate Ψ and $\frac{d\Psi}{dr}$ as $r \to \infty$. These constants then determine the electric and magnetic charges.

Equation (III.20) also arises in connection with point monopole solutions that satisfy Φ = constant, $D_i\Phi = 0$ everywhere except at the origin. Goldhaber and Wilkinson [8] have shown that point monopole

solutions that are in addition invariant under rotations $\vec{J} = -i\,\vec{r}\times\vec{\nabla} + \vec{T}$ (section I.3), can always be expressed in the form:

$$B_3 = \frac{1}{e r^2}\,(I_3 - T_3)\,, \qquad \Phi = \Phi_\infty = \text{constant}$$

where I_3 is a generator of an SU(2) subalgebra $\vec{I}$ of the group with the property

$$[\,\vec{I}\,,\ I_3 - T_3\,] = 0 \qquad\qquad [\,\vec{I}\,,\ \Phi_\infty\,] = 0 \tag{III.22}$$

We expect that the asymptotic form of our solutions will give some of these point monopoles. We see indeed that we have to identify

$$r^2\,\frac{d\Psi}{dr}\bigg|_{r\to\infty} = I_3 \tag{III.23}$$

I_3 will be determined, and as we shall see, $\vec{I} \in L(K)$ with $[T_3\,,\ I^\pm] = \pm I^\pm$ for our solutions, so that (III.22) is satisfied.

We will now calculate the asymptotic magnetic fields. We need to know the large r behaviour of $e^{-\rho_\alpha(r)}$ in order to determine Ψ_∞, $r^2\dot{\Psi}_\infty$ since these are given in the Chevalley basis by (II.4):

$$\Psi_\alpha\,\big|_{r\to\infty} = -\frac{1}{2}\,\frac{d}{dr}\,(\ln e^{-\rho_\alpha})\,\big|_{r\to\infty} \tag{III.24}$$

$$r^2\frac{d\Psi_\alpha}{dr}\bigg|_{r\to\infty} = -\frac{1}{2}\,r^2\frac{d^2}{dr^2}\,(\ln e^{-\rho_\alpha})\,\big|_{r\to\infty} \tag{III.24'}$$

The asymptotic form of $e^{-\rho_\alpha}$ after having satisfied the boundary conditions at the origin, depends only on (the eigenvalues of) V. So we have established the direct connection between V and the asymptotic value of the Higgs field Φ_∞. We will consider two different cases:

(a) $(V, \alpha_i) > 0$ (as required in section (III.1). Write the solution $e^{-P_{\alpha_i}}$ (III.1) in the form

$$e^{-P_{\alpha_i}} = e^{r(V, \lambda_i)} \sum_{\mu \in \Pi(\lambda_i)} D_\mu \; e^{r(V, \mu - \lambda_i)} \quad , \quad D_\mu \in \mathbb{R}$$

Since, λ_i is the highest weight we know $\mu = \lambda_i - \sum_{\beta \in \Delta} k_\beta \beta$, $k_\beta \geqslant 0$ and therefore all exponents $(V, \mu - \lambda_i)$ are non-positive. When we take $r \to \infty$ the only contribution that survives is from the term $\mu = \lambda_i$.

$$e^{-P_{\alpha_i}} \Big|_{r \to \infty} \simeq e^{r(V, \lambda_i)} \; D_{\lambda_i}$$

Therefore we have from (III.24) and (III.24')

$$\Psi_{\alpha_i} \Big|_{r \to \infty} = -\frac{1}{2} (V, \lambda_i) \quad , \quad r^2 \frac{d\Psi_{\alpha_i}}{dr} \Big|_{r \to \infty} = 0$$

In other words we have $e\Phi_\infty = \Psi_\infty = -\frac{1}{2} V$ and $I_3 = 0$ and the magnetic field at large distance is $B_3 = -\frac{T_3}{e r^2}$.

The only generators that commute with Φ_∞ (the electric charge generator) are in the C.S.A., since for any root γ we have

$$[\Phi_\infty , E_\gamma] = -\frac{1}{2e} [V, E_\gamma] = -\frac{1}{2e} (V, \gamma) E_\gamma \neq 0$$

So the exact symmetry group is just

$$H \simeq U(1) \otimes U(1) \otimes \ldots U(1) \qquad (\ell \text{ factors}).$$

(b) $(V, \alpha_i) \to 0$ for some α_i. The previous example clearly is not very interesting from the physical point of view, becuase of the many U(1) factors. If we want H to contain only one U(1) we must have $(\Phi_\infty, \alpha_i) = 0$ for all $\alpha_i \in \Delta$ except one, say $(\Phi_\infty, \alpha_m) \neq 0$ (see ref. [22]). This implies that Φ_∞ is proportional to $\sum_i \lambda_m^i H_i$ in the

Cartan-Weyl basis (or $\sum_j K^{-1}_{mj} H_{\alpha_j}$ in the Chevalley basis). The invariant subalgebra L(K) consists of all generators that can be obtained from $E_{\pm\alpha_i}$, H_{α_i} , $i \neq m$, since

$$[\Phi_\infty , E_{\pm\alpha_i}] = \pm (\Phi_\infty , \alpha_i) E_{\pm\alpha_i} = 0 \quad , \quad [\Phi_\infty , H_{\alpha_i}] = 0$$

L(K) is indeed seen to be semisimple, it is based on the system of simple roots $\Delta(K) = \{ \alpha_i , i \neq m \}$. The only invariant U(1) factor is generated by Φ_∞ itself, and we have the usual relation

$$H \simeq K \otimes U(1)$$

When we try to apply this, we are faced with a problem: the solutions for $e^{-P\alpha_i}$ were determined in section III.1 under the assumption that $(V, \alpha_j) \neq 0$. It is seen from equation (III.4) that various coefficients diverge as $(V, \alpha_j) \to 0$. However <u>after</u> satisfying the boundary conditions at the origin, in other words writing C in terms of V, it is found that singular denominators are compensated by appropriate powers in the numerator, so that the limit $(V, \alpha_j) \to 0$ exists. This can best be seen from the Taylor expansion of $e^{-P\alpha_i}$ (use equ. (III.10) and $e^{X} e^{-C} e^{-Y} |\lambda_i\rangle \sim |\bar{\lambda}_i\rangle$):

$$e^{-P\alpha_i} = \sum_n \frac{r^n}{n!} \langle \lambda_i | (V - \sum_j E_{\alpha_j})^n | \bar{\lambda}_i \rangle$$

All coefficients are polynomials in V and we can safely take $(V, \alpha_j) \to 0$ This property was also noted earlier for some SU(N) monopole solutions (Bais and Weldon [23]). Unfortunately the expression above is not very useful for studying the $r \to \infty$ behaviour, so we had to use an indirect method; it is interesting to see how the group K is used in the argument. Consider again the solution the form (III.1), as a sum over weights $\mu = \lambda_i - \sum_\beta k_\beta \beta$. Whenever $\mu - \lambda_i$ contains α_m , the

exponent $r(V, \mu-\lambda_i) < 0$, and therefore the term vanishes as $r \to \infty$. Hence we only consider terms with $(\lambda_i - \mu, \lambda_m)=0$ (these do not contain α_m)

$$e^{-\rho_{\alpha_i}} \approx e^{r(V,\lambda_i)} \left[\sum_{\substack{\mu \\ (\lambda_i-\mu,\lambda_m)=0}} (-1)^{s(\mu)} |\langle \lambda_i | e^{X} | \mu \rangle|^2 \; e^{-(C,\mu)} \; e^{r(V,\mu-\lambda_i)} \right]$$

Now put $V = \lambda_m . H + \epsilon T_3$; we will later take $\epsilon \to 0$, so that $(V, \alpha_i) \to 0$ except for α_m, $(V, \alpha_m) \neq 0$. With this substitution the coefficients become (cf. equation (III.3))

$$|\langle \lambda_i | e^{X} | \mu \rangle|^2 \sim \frac{1}{\epsilon^{2s(\mu)}} \cdot \frac{1}{\epsilon^{2(T_3, \lambda_i - \mu)}}$$

We will use (III.15') for the constant $e^{-(C,\mu)}$; we make use of the fact that the roots that do not contain α_m, form a root system $\Phi(K)$:

$$e^{-(C,\mu)} \sim \prod_{\gamma \in \Phi^+(K)} \epsilon^{-\frac{2(\gamma,\mu)}{(\gamma,\gamma)}} \quad \text{const.}$$

We will now need an alternative definition for the principal SU(2) generator, that is derived in Appendix C. We show there (equ. (C.1)) that in the Cartan-Weyl basis we can take

$$T_3 = 2\delta^{\vee} . H = \sum_{\beta \in \Phi^+(G)} \frac{\beta}{(\beta,\beta)} . H$$

Therefore if I_3 is the corresponding principal generator for the group K we have $$I_3 = \sum_{\beta \in \Phi^+(K)} \frac{\beta}{(\beta,\beta)} . H$$

It also follows from the definition of the group K that $(T_3 - I_3, \alpha_i)=0$ for $\alpha_i \neq \alpha_m$, and therefore

$$T_3 - I_3 \propto \lambda_m . H$$

Collecting together the above results we write for the solution:

$$e^{-\rho_{\alpha_i}} \approx e^{r(V,\lambda_i)} \left[\sum_{\substack{\mu \\ (\lambda_i-\mu,\lambda_m)=0}} \frac{e^{r(V,\mu-\lambda_i)}}{\epsilon^{2(T_3,\lambda_i-\mu)}} \frac{\text{const.}}{\epsilon^{2(I_3,\mu)}} \right] =$$

$$= e^{r(V,\lambda_i)} \frac{\sum_{\mu} e^{r(V,\mu-\lambda_i)}}{\epsilon^{2(I_3,\lambda_i)}} \text{const.}$$

The exponents are of course also proportional to ϵ, since they do not contain the root α_m. Now, since the whole term inside the brackets is finite as $\epsilon \to 0$, we conclude that there must be cancellations that leave us only with a polynomial of degree $n_{\alpha_i} = 2(I_3,\lambda_i)$:

$$e^{-\rho_{\alpha_i}} \simeq e^{r(V,\lambda_i)} \left\{ r^{n_{\alpha_i}} + \dots \right\}$$

This last expression tells us what we need to know for calculating the asymptotic behaviour.

$$\rho_{\alpha_i}\Big|_\infty \simeq -r(V,\lambda_i) - n_{\alpha_i} \ln r$$

$$\psi_{\alpha_i}\Big|_\infty \simeq -\tfrac{1}{2}(V,\lambda_i) \qquad \text{and} \qquad r^2 \frac{d\psi_{\alpha_i}}{dr}\Big|_\infty \simeq \tfrac{1}{2} n_{\alpha_i} = I_{\alpha_i}$$

We conclude again that $e\Phi_\infty = \Psi_\infty = -\frac{1}{2}V$, the same as for (a) but we now have $B_3 \sim \frac{1}{e r^2}(I_3 - T_3)$, where I_3 is a principal embedding of the subgroup K.

It is not difficult to see from the two cases considered here, that in general the exact symmetry is

$$H \simeq \underbrace{U(1) \otimes U(1) \dots U(1)}_{\leq \ell \text{ factors}} \otimes K' \qquad , K' \text{ semisimple}$$

and $I_3 \subset L(K')$, always a principal embedding. Then the generators $I^{\pm}$ have grades ± 1, and conditions (III.22) are always satisfied.

Finally let us calculate the magnetic charge. We have $\mathrm{Tr}\, I_3 \Phi_\infty$ since $I_3 \in L(K') \subset L(K)$ and when we use the asymptotic value of the magnetic field we only get a contribution from the T_3 term:

$$g = \frac{4\pi}{e} \mathrm{Tr}\,(I_3 - T_3)\frac{\Phi_\infty}{\alpha} = -\frac{4\pi}{e\alpha}\mathrm{Tr}\, T_3 \Phi_\infty = -4\pi \frac{\mathrm{Tr}\, T_3 Q}{\mathrm{Tr}\, Q Q} \qquad \text{(III.25)}$$

This is precisely what we expect from a known general result about spherically symmetric monopole solutions [4].

III.4 Applications

We will now apply the methods developed above on some cases of low rank groups. This is instructive as we shall see, because the form of the solutions suggests them to have further symmetries not previously expected.

(a) group SU(3) : This is the simplest case and a lot is already known [10, 24] about monopole solutions. We will describe in some detail the successive steps.

With the usual normalisation for the two simple roots:

$$(\alpha_1 , \alpha_1) = (\alpha_2 , \alpha_2) = 1 \qquad (\alpha_1, \alpha_2) = -\frac{1}{2}$$

the Cartan matrix is $K_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and the reduced Bogomolny equations for the monopole (equ. (II.6)) become

$$\ddot{\rho}_{\alpha_1} = e^{2\rho_{\alpha_1}} e^{-\rho_{\alpha_2}} \qquad \ddot{\rho}_{\alpha_2} = e^{-\rho_{\alpha_1}} e^{2\rho_{\alpha_2}} \tag{III.26}$$

The two fundamental representations for SU(3) based on highest weights λ_1 and λ_2 are:

- $\lambda_1 = \frac{1}{3}(2\alpha_1 + \alpha_2)$
- $\lambda_1 - \alpha_1 = \frac{1}{3}(-\alpha_1 + \alpha_2)$
- $\lambda_1 - \alpha_1 - \alpha_2 = \frac{1}{3}(-\alpha_1 - 2\alpha_2)$
- $\lambda_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2)$
- $\lambda_2 - \alpha_2$
- $\lambda_2 - \alpha_2 - \alpha_1$

It is easy to check that $\frac{2(\lambda_i , \alpha_j)}{(\alpha_j , \alpha_j)} = \delta_{ij}$ as required. The two weight systems are identical when we interchange $\alpha_1 \longleftrightarrow \alpha_2$.

Applying (III.4) to find the solution, the only terms that contribute:

$$e^{-\rho_{\alpha_1}} = e^{r(V,\lambda_1)}\, e^{-(C,\lambda_1)} - \frac{e^{(rV-C,\lambda_1-\alpha_1)}\,\langle\lambda_1|E_{\alpha_1}E_{-\alpha_1}|\lambda_1\rangle}{(V,\alpha_1)^2} +$$

$$+ \frac{e^{(rV-C,\lambda_1-\alpha_1-\alpha_2)}\,\langle\lambda_1|E_{\alpha_1}E_{\alpha_2}E_{-\alpha_2}E_{-\alpha_1}|\lambda_1\rangle}{(V,\alpha_2)^2\,(V,\alpha_1+\alpha_2)^2}$$

Using the commutation relations (or expression (III.5)) it is easy to see that the matrix elements are 1. The solution for $e^{-\rho_{\alpha_2}}$ has everywhere α_1 interchanged with α_2.

The principal SU(2) subalgebra generator is $T_3 = H_{\alpha_1} + H_{\alpha_2}$ and in order to make the magnetic field B_3 zero at the origin, we must have (see (III.9)) $e^{-\rho_{\alpha_1}} \sim 2^{-\frac{2}{3}}\ 2^{-\frac{1}{3}}\ r^2 = \frac{1}{2}\, r^2$. This is satisfied (III.12,15') with

$$e^{(C,\alpha_1)} = (V,\alpha_1)^2\,(V,\alpha_2)^{-1}\,(V,\alpha_1+\alpha_2) \qquad e^{(C,\alpha_2)} = (V,\alpha_1)^{-1}\,(V,\alpha_2)^2\,(V,\alpha_1+\alpha_2)$$

and substituting these values in the solution we get

$$e^{-\rho_{\alpha_1}} = \frac{e^{r(V,\lambda_1)}}{(V,\alpha_1)(V,\alpha_1+\alpha_2)} - \frac{e^{r(V,\lambda_1-\alpha_1)}}{(V,\alpha_1)(V,\alpha_2)} + \frac{e^{r(V,\lambda_1-\alpha_1-\alpha_2)}}{(V,\alpha_2)(V,\alpha_1+\alpha_2)} \quad \text{(III.27)}$$

$$e^{-\rho_{\alpha_2}} = \frac{e^{r(V,\lambda_2)}}{(V,\alpha_2)(V,\alpha_1+\alpha_2)} - \frac{e^{r(V,\lambda_2-\alpha_2)}}{(V,\alpha_1)(V,\alpha_2)} + \frac{e^{r(V,\lambda_2-\alpha_2-\alpha_1)}}{(V,\alpha_1)(V,\alpha_1+\alpha_2)}$$

This is in agreement with the result of Bais and Wilkinson [10].

It was observed* by Olive that the solutions are invariant under

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* Private communication. I also thank him for a conjecture related to (III.28)

the Weyl group. The Weyl group W is a finite group generated by reflections that leave invariant the root system (see the book [18] for more details). A simple reflection in the plane perpendicular to the root γ permutes the roots:

$$\sigma_\gamma(\beta) = \beta - \frac{2(\beta,\gamma)}{(\gamma,\gamma)}\gamma \qquad \beta,\gamma \in \Phi \qquad \sigma_\gamma \in W$$

Since the roots can be associated with the elements of the C.S.A. η we can also think of the elements σ_γ of the Weyl group acting on the C.S.A. In the Chevalley basis:

$$\sigma_\gamma(V) = V - (V,\gamma)\,H_\gamma \qquad V, H_\gamma \in \eta$$

This is just a gauge transformation on V, and it leaves V diagonal, i.e. $\sigma_\gamma(V) = s\,V\,s^{-1}$ with

$$s = \exp\left(\frac{i\pi}{2}(E_\alpha + E_{-\alpha})\right)$$

as it can be easily checked.

In the case of SU(3) the Weyl group consists of six elements: 1, σ_{α_1}, σ_{α_2}, $\sigma_{\alpha_1}\sigma_{\alpha_2}$, $\sigma_{\alpha_2}\sigma_{\alpha_1}$, $\sigma_{\alpha_1}\sigma_{\alpha_2}\sigma_{\alpha_1}$ (The last element is the one denoted σ in section III.2). The effect on the roots, for example of σ_{α_1} is:

$$\sigma_{\alpha_1}(\alpha_1) = -\alpha_1 \qquad \sigma_{\alpha_1}(\alpha_2) = \alpha_2 + \alpha_1$$

If we replace $V \longrightarrow \sigma_{\alpha_1}(V) = V - (V,\alpha_1)H_{\alpha_1}$ the constants in the denominators of our solutions (III.27) become:

$$(V,\alpha_1) \longrightarrow -(V,\alpha_1) \qquad (V,\alpha_2) \rightarrow (V,\alpha_1+\alpha_2) \qquad (V,\alpha_1+\alpha_2) \rightarrow (V,\alpha_2)$$

and the numerators

$$(V,\lambda_1)\rightarrow(V,\lambda_1-\alpha_1)\qquad(V,\lambda_1-\alpha_1)\rightarrow(V,\lambda_1)\qquad(V,\lambda_1-\alpha_1-\alpha_2)\rightarrow(V,\lambda_1-\alpha_1-\alpha_2)$$

and one can see that the expressions remain invariant. This is true for every Weyl group element. This discrete symmetry is not present in the original solution, before satisfying the boundary conditions at the origin. So presumably it is this extra symmetry that allows us to satisfy the constraints at the origin.

Since the terms in the solution (III.27) transform into one another, we can take advantage of the symmetry and write

$$e^{-\rho_{\alpha_1}} = \frac{1}{2}\sum_{w\in W}\frac{e^{r(w(V),\lambda_1)}}{(w(V),\alpha_1)(w(V),\alpha_1+\alpha_2)}$$

The factor $\frac{1}{2}$ is necessary, since the dimension of the representation is 3, and there are 6 W group elements. There is a similar expression for $e^{-\rho_{\alpha_2}}$. In fact both of them can be put in the form:

$$e^{-\rho_{\alpha_i}} = \frac{1}{2}\sum_{w\in W}\prod_{\beta\in\Phi^+}(w(V),\beta)^{-\frac{2(\beta,\lambda_i)}{(\beta,\beta)}}e^{r(w(V),\lambda_i)} \qquad \text{(III.28)}$$

We expect a sum over Weyl group elements whenever we have a Weyl symmetric expression.

We now come to discuss the asymptotic behaviour. It is possible to break the gauge symmetry in two ways:

$$SU(3)\longrightarrow U(1)\otimes U(1)\qquad\text{or}\qquad SU(3)\longrightarrow SU(2)\otimes U(1)$$

The second possibility requires one of the two parameters of V to be zero (section III,2(b)); e.g. $(V,\alpha_2)=0$ and to make things simpler take $(V,\alpha_1)=1$. (These two conditions imply $V=\frac{1}{3}(2H_{\alpha_1}+H_{\alpha_2})$). Then,

taking the limit, equ. (III.27) become:

$$e^{-\rho_{\alpha_1}} = e^{\frac{2}{3}r}\left[1 - e^{-r}(r+1)\right]$$

$$e^{-\rho_{\alpha_2}} = e^{\frac{1}{3}r}\left[(r-1) + e^{-r}\right]$$

(A version of the solution found by Bais and Weldon [23]. For large r we get the expected behaviour

$$\rho_{\alpha_1} \sim -\frac{2}{3}r \qquad\qquad \rho_{\alpha_2} \sim -\frac{1}{3}r - \ln r$$

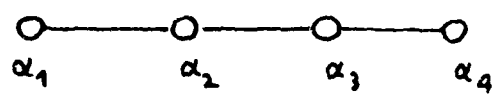
which gives $I_3 = \frac{1}{2}H_{\alpha_2}$. The magnetic charge matrix is then $G = \frac{4\pi}{e}(I_3 - T_3) = -\frac{4\pi}{e}(H_{\alpha_1} + \frac{1}{2}H_{\alpha_2})$; this is proportional to V and therefore the magnetic field is "pure electromagnetic". For the "Dirac quantization condition" use (III.25)

$$gQ = -4\pi \frac{Tr\ V T_3}{Tr\ V V} V = -4\pi \frac{1}{2}(2H_{\alpha_1} + H_{\alpha_2})$$

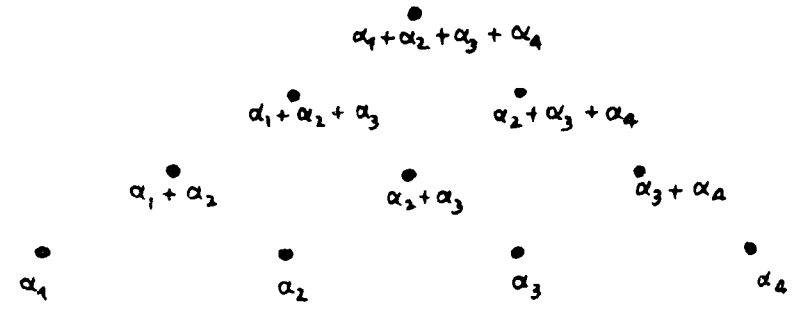
The eigenvalues of $2H_{\alpha_1} + H_{\alpha_2}$ are integers and therefore we have for the smallest charge q of the theory: $\frac{qg}{4\pi} = -\frac{1}{2}$ (This corresponds to solution IV of ref. [24]).

(b) group SU(5) : This is one of the candidate gauge groups for unifying strong and electroweak interactions. A Higgs field in the adjoint is used to break the symmetry (at an enormous energy). Our method can be applied to describe a spherical monopole solution, the general features are the same as for SU(3). There is one complication, present in most other cases, but absent from our previous example, the fact that there can be more than one weight for each level.

The Dynkin diagram for SU(5) (= A_4, rank 4) is



There are ten positive roots arranged by height:



The fundamental weights are:

$$\lambda_1 = \frac{1}{5}(4\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4) \qquad \lambda_2 = \frac{1}{5}(3\alpha_1 + 6\alpha_2 + 4\alpha_3 + 2\alpha_4)$$

$$\lambda_3 = \frac{1}{5}(2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4) \qquad \lambda_4 = \frac{1}{5}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4)$$

To find the system of weights in each representation one uses the following result [18, p.70]: if μ is a weight, then $\mu - i\alpha$ is also a weight for every integer i between 0 and $\frac{2(\alpha,\mu)}{(\alpha,\alpha)}$ and $\alpha \in \Phi$. So starting from the highest weight we can work our way down by subtracting appropriate simple roots. The weight diagrams for λ_1 and λ_2 arranged in levels, is presented in table (1).

We get the solutions using (III.4) and table (1). There are several contributions for each weight in $e^{-\rho_{\alpha_2}}$, because there is more than one "path" that contributes to the matrix element. The final expressions contain 5 terms (= $\dim R(\lambda_1)$) for $e^{-\rho_{\alpha_1}}$ and 10 (= $\dim R(\lambda_2)$) for $e^{-\rho_{\alpha_2}}$; they are presented in table (2). The expressions for $e^{-\rho_{\alpha_3}}$ and $e^{-\rho_{\alpha_4}}$ can be obtained from ρ_{α_2} , ρ_{α_1} by interchanging $\alpha_4 \leftrightarrow \alpha_1$, $\alpha_3 \leftrightarrow \alpha_2$.

SU(5) Weights

$R(\lambda_1)$

- λ_1
- $\lambda_1 - \alpha_1$
- $\lambda_1 - \alpha_1 - \alpha_2$
- $\lambda_1 - \alpha_1 - \alpha_2 - \alpha_3$
- $\lambda_1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$

$R(\lambda_2)$

- λ_2
- $\lambda_2 - \alpha_2$
- $\lambda_2 - \alpha_2 - \alpha_1$
- $\lambda_2 - \alpha_2 - \alpha_3$
- $\lambda_2 - \alpha_2 - \alpha_1 - \alpha_3$
- $\lambda_2 - \alpha_2 - \alpha_3 - \alpha_4$
- $\lambda_2 - 2\alpha_2 - \alpha_1 - \alpha_3$
- $\lambda_2 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_1$
- $\lambda_2 - 2\alpha_2 - \alpha_1 - \alpha_3 - \alpha_4$
- $\lambda_2 - \alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$

TABLE (1)

SU(5) Solutions

$$e^{-\rho_{\alpha_1}} = e^{(rV-C,\,\lambda_1)}\left[\,1 - \frac{e^{-(rV-C,\,\alpha_1)}}{(V,\alpha_1)^2} + \frac{e^{-(rV-C,\,\alpha_1+\alpha_2)}}{(V,\alpha_2)^2(V,\alpha_1+\alpha_2)^2} - \right.$$

$$\left. - \frac{e^{-(rV-C,\,\alpha_1+\alpha_2+\alpha_3)}}{(V,\alpha_3)^2(V,\alpha_2+\alpha_3)^2(V,\alpha_1+\alpha_2+\alpha_3)^2} + \frac{e^{-(rV-C,\,\alpha_1+\alpha_2+\alpha_3+\alpha_4)}}{(V,\alpha_4)^2(V,\alpha_3+\alpha_4)^2(V,\alpha_2+\alpha_3+\alpha_4)^2(V,\alpha_1+\alpha_2+\alpha_3+\alpha_4)^2}\right]$$

$$e^{-\rho_{\alpha_2}} = e^{(rV-C,\,\lambda_2)}\left[\,1 - \frac{e^{-(rV-C,\,\alpha_2}}{(V,\alpha_2)^2} + \frac{e^{-(rV-C,\,\alpha_1+\alpha_2)}}{(V,\alpha_1)^2(V,\alpha_1+\alpha_2)^2} + \right.$$

$$+ \frac{e^{-(rV-C,\,\alpha_2+\alpha_3)}}{(V,\alpha_3)^2(V,\alpha_2+\alpha_3)^2} - \frac{e^{-(rV-C,\,\alpha_2+\alpha_3+\alpha_4)}}{(V,\alpha_2+\alpha_3+\alpha_4)^2(V,\alpha_3+\alpha_4)^2(V,\alpha_4)^2}$$

$$- \frac{e^{-(rV-C,\,\alpha_1+\alpha_2+\alpha_3)}}{(V,\alpha_1)^2(V,\alpha_3)^2(V,\alpha_1+\alpha_2+\alpha_3)^2} + \frac{e^{-(rV-C,\,\alpha_1+2\alpha_2+\alpha_3)}}{(V,\alpha_2)^2(V,\alpha_1+\alpha_2)^2(V,\alpha_2+\alpha_3)^2(V,\alpha_1+\alpha_2+\alpha_3)^2}$$

$$+ \frac{e^{-(rV-C,\,\alpha_1+\alpha_2+\alpha_3+\alpha_4)}}{(V,\alpha_4)^2(V,\alpha_1)^2(V,\alpha_3+\alpha_4)^2(V,\alpha_1+\alpha_2+\alpha_3+\alpha_4)^2}$$

$$- \frac{e^{-(rV-C,\,\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}}{(V,\alpha_4)^2(V,\alpha_2)^2(V,\alpha_1+\alpha_2)^2(V,\alpha_2+\alpha_3+\alpha_4)^2(V,\alpha_1+\alpha_2+\alpha_3+\alpha_4)^2}$$

$$\left. + \frac{e^{-(rV-C,\,\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}}{(V,\alpha_3)^2(V,\alpha_3+\alpha_4)^2(V,\alpha_2+\alpha_3)^2(V,\alpha_2+\alpha_3+\alpha_4)^2(V,\alpha_1+\alpha_2+\alpha_3)^2(V,\alpha_1+\alpha_2+\alpha_3+\alpha_4)^2}\right]$$

TABLE (2)

After satisfying the boundary condition at the origin we have checked that we have invariance under the Weyl group $V \rightarrow w(V)$, $w \in \tilde{W}$, and the solutions can be written in a form like (III.28) but instead of $\frac{1}{2}$ there is a constant:

$$\frac{\dim \Pi(\lambda_i)}{\dim \tilde{W}} \begin{cases} = \frac{5}{5!} & \text{for } \lambda_1, \lambda_4 \\ = \frac{10}{5!} & \text{for } \lambda_2, \lambda_3 \end{cases}$$

One is naturally led to ask whether (III.28) is always valid. It has been verified in many examples; let us note however that if the weights of a representation lie on more than one orbit of the Weyl group, expression (III.28) would certainly have to be modified. This is because if we start with the highest weight λ_i, we only get $w(\lambda_i)$, weights in the same orbit. This does not happen for any of the SU(N) groups, but it occurs for most other cases. We have not been able to prove the invariance under $\tilde{W}$ in general, in order to carry these results further.

Finally let us briefly discuss symmetry breaking in SU(5). If we choose $V = \frac{1}{5}(3H_{\alpha_1} + 6H_{\alpha_2} + 4H_{\alpha_3} + 2H_{\alpha_4})$ in the Chevalley basis ($V = 2\lambda_2 . H$ in the Cartan-Weyl basis), the eigenvalues of V are $(V, \alpha_1)=(V, \alpha_3)=(V, \alpha_4)=0 \quad (V, \alpha_2)=1$ and this determines the exact symmetry group:

$$SU(5) \longrightarrow SU(3) \otimes SU(2) \otimes U(1)$$

The large r behaviour, $\rho(r \rightarrow \infty)$ is calculated to be

$$\rho_{\alpha_1} \sim -\frac{3}{5} r - \ln r \qquad \rho_{\alpha_2} \sim -\frac{6}{5} r$$

$$\rho_{\alpha_3} \sim -\frac{4}{5} r - 2 \ln r \qquad \rho_{\alpha_4} \sim -\frac{2}{5} r - 2 \ln r$$

and therefore $I_3 = \frac{1}{2}H_{\alpha_1} + H_{\alpha_3} + H_{\alpha_4}$, and the magnetic field

$$B_3 \sim \frac{I_3 - T_3}{e\,r^2}$$

where $T_3 = 2H_{\alpha_1} + 3H_{\alpha_2} + 3H_{\alpha_3} + 2H_{\alpha_4}$ for SU(5)

The magnetic charge, as we have seen from (III.25), depends only on T_3 and not on the particular form of the solution:

$$gQ = -4\pi \frac{Tr\, T_3 V}{Tr\, VV} V = -4\pi \frac{1}{2}\left(3H_{\alpha_1} + 6H_{\alpha_2} + 4H_{\alpha_3} + 2H_{\alpha_4}\right)$$

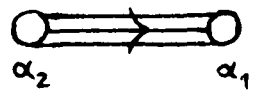
If there are matter fields in the fundamental representation, the smallest charge q corresponds to the state $|\lambda_4\rangle$; then $gq=-4\pi$.

(c) group G_2 : Just as another example we include the solutions for the exceptional group of rank 2, G_2 . the reduced equations are very similar to those of SU(3) (equ. III.26)

$$\ddot{\rho}_{\alpha_1} = e^{2\rho_{\alpha_1}} e^{-\rho_{\alpha_2}} \qquad \ddot{\rho}_{\alpha_2} = e^{2\rho_{\alpha_2}} e^{-3\rho_{\alpha_1}}$$

(There are two roots $(\alpha_1, \alpha_1)=1$ $(\alpha_2, \alpha_2)=3$ $(\alpha_1, \alpha_2)=-\frac{3}{2}$)
The solutions are more complicated then for SU(3), there are 7 and 13 terms for $e^{-\rho_{\alpha_1}}$ and $e^{-\rho_{\alpha_2}}$ respectively. They are presented in table (4), after satisfying the conditions at the origin. Table (3) presents the weight diagrams. Note that there are two Weyl orbits for $\Pi(\lambda_1)$ and three for $\Pi(\lambda_2)$.

G_2 Dynkin diagram:



Weights:

$R(\lambda_1)$

- $\lambda_1 = 2\alpha_1 + \alpha_2$
- $\lambda_1 - \alpha_1 = \alpha_1 + \alpha_2$
- $\lambda_1 - \alpha_1 - \alpha_2 = \alpha_1$
- $\lambda_1 - 2\alpha_1 - \alpha_2 = 0$
- $\lambda_1 - 3\alpha_1 - \alpha_2 = -\alpha_1$
- $\lambda_1 - 3\alpha_1 - 2\alpha_2 = -\alpha_1 - \alpha_2$
- $\lambda_1 - 4\alpha_1 - 2\alpha_2 = -2\alpha_1 - \alpha_2$

$R(\lambda_2)$ (= adjoint)

- $\lambda_2 = 3\alpha_1 + 2\alpha_2$
- $\lambda_2 - \alpha_2$
- $\lambda_2 - \alpha_1 - \alpha_2$
- $\lambda_2 - 2\alpha_1 - \alpha_2$
- $\lambda_2 - 3\alpha_1 - \alpha_2$
- $\lambda_2 - 2\alpha_1 - 2\alpha_2$
- $\lambda_2 - 3\alpha_1 - 2\alpha_2 = 0$
- $\lambda_2 - 3\alpha_1 - 3\alpha_2$
- $\lambda_2 - 4\alpha_1 - 2\alpha_2$
- $\lambda_2 - 4\alpha_1 - 3\alpha_2$
- $\lambda_2 - 5\alpha_1 - 3\alpha_2$
- $\lambda_2 - 6\alpha_1 - 3\alpha_2$
- $\lambda_2 - 6\alpha_1 - 4\alpha_2$

TABLE (3)

G_2 Regular solutions:

Set $\quad V_1 \equiv (V, \alpha_1) \qquad V_2 \equiv (V, \alpha_2)$

$$e^{-P\alpha_1} = \frac{e^{(rV,\,\lambda_1)}}{V_1\,(V_1+V_2)(2V_1+V_2)^2(3V_1+V_2)(3V_1+2V_2)} - \frac{e^{(rV,\,\lambda_1-\alpha_1)}}{V_1(2V_1+V_2)(V_1+V_2)^2\,V_2\,(3V_1+2V_2)}$$

$$+ \frac{e^{(rV,\,\lambda_1-\alpha_1-\alpha_2)}}{V_1^2\,V_2\,(V_1+V_2)(2V_1+V_2)(3V_1+V_2)} - \frac{2}{V_1^2\,(V_1+V_2)^2(2V_1+V_2)^2}$$

$$+ \frac{e^{(rV,\,\lambda_1-3\alpha_1-\alpha_2)}}{V_1^2\,V_2\,(V_1+V_2)(2V_1+V_2)(3V_1+V_2)} - \frac{e^{(rV,\,\lambda_1-3\alpha_1-2\alpha_2)}}{V_1V_2\,(2V_1+V_2)(V_1+V_2)^2(3V_1+2V_2)}$$

$$+ \frac{e^{(rV,\,\lambda_1-4\alpha_1-2\alpha_2)}}{V_1\,(V_1+V_2)(2V_1+V_2)^2(3V_1+V_2)(3V_1+2V_2)}$$

$$e^{-P\alpha_2} = \frac{e^{r(V,\lambda_2)}}{V_2\,(V_1+V_2)^3(2V_1+V_2)^3(3V_1+V_2)(3V_1+2V_2)^2} - \frac{e^{(rV,\,\lambda_2-\alpha_2)}}{V_1^3\,V_2\,(2V_1+V_2)^3(3V_1+V_2)^2(3V_1+2V_2)}$$

$$+ \frac{3\,e^{(rV,\,\lambda_2-\alpha_2-\alpha_1)}}{V_1^3\,(V_1+V_2)^3(2V_1+V_2)^2(3V_1+V_2)(3V_1+2V_2)} - \frac{3\,e^{(rV,\,\lambda_2-2\alpha_1-\alpha_2)}}{V_1^3\,V_2\,(V_1+V_2)^2(2V_1+V_2)^3(3V_1+2V_2)}$$

$$+ \frac{3\,e^{(rV,\,\lambda_2-2\alpha_1-2\alpha_2)}}{V_1^2\,V_2\,(V_1+V_2)^3(2V_1+V_2)^3(3V_1+V_2)} + \frac{e^{(rV,\,\lambda_2-3\alpha_1-\alpha_2)}}{V_1^3\,V_2^2\,(V_1+V_2)^3(3V_1+V_2)(3V_1+2V_2)}$$

$$- \frac{24\,(3V_1^2+V_2^2+3V_1V_2)}{(3V_1+2V_2)^2(3V_1+V_2)^2(2V_1+V_2)^2(V_1+V_2)^2V_1^2V_2^2}$$

$$+ \frac{e^{(rV,\,\lambda_2-3\alpha_1-3\alpha_2)}}{V_1^3V_2^2\,(V_1+V_2)^3(3V_1+V_2)(3V_1+2V_2)} + \frac{e^{(rV,\,\lambda_2-4\alpha_1-2\alpha_2)}\,3}{V_1^2\,V_2\,(V_1+V_2)^3(2V_1+V_2)^3(3V_1+V_2)}$$

$$- \frac{3\,e^{(rV,\,\lambda_2-4\alpha_1-3\alpha_2)}}{V_1^3\,V_2\,(V_1+V_2)^2(2V_1+V_2)^3(3V_1+2V_2)} + \frac{3\,e^{(rV,\,\lambda_2-5\alpha_1-3\alpha_2)}}{V_1^3\,(V_1+V_2)^3(2V_1+V_2)^2(3V_1+V_2)(3V_1+2V_2)}$$

$$- \frac{e^{(rV,\,\lambda_2-6\alpha_1-3\alpha_2)}}{V_1^3V_2\,(2V_1+V_2)^3(3V_1+V_2)^2(3V_1+2V_2)} + \frac{e^{(rV,\,\lambda_2-6\alpha_1-4\alpha_2)}}{V_2\,(V_1+V_2)^3(2V_1+V_2)^3(3V_1+V_2)(3V_1+2V_2)^2}$$

TABLE (4)

Section IV

We have presented the case of principal embedding in great detail; we now come to discuss some results concerning the cases of non-principal SU(2) embeddings in the gauge group G.

Let us first note that we can make some progress towards the solution of (I.12) without specifying T_3 exactly. The first part of the integration of the Lax pair, equation (II.10), is valid (after a suitable re-interpretation of the variables). When we come to decompose $g = e^{X} e^{Y} e^{S}$, the only difference is that S may no longer be diagonal. Equation (II.12) is still true but it is not clear if one ought to take matrix elements between highest weights, since this may "kill" some of the parameters of e^{S}. Because there is this crucial step missing from the argument we cannot find the most general solution. It was possible however to solve for many specific cases, and this gives us hope that the problem can always be solved.

In what follows we first give an example of a non-principal embedding, which for all groups, leads to a familiar equation. Next we discuss some defining properties of SU(2) subgroups and their classification; this then suggests a way for finding solutions of the Lax pair, which however may not be the most general ones.

IV.1 The highest root embedding.

One can associate three generators with every root $\alpha \in \Phi^{+}(G)$: one diagonal H_{α} (or $\alpha.H$) and two nilpotent (raising and lowering generators) $E_{\pm\alpha}$. It is obvious that these can form the basis of an

SU(2) subalgebra. Except when G is itself an SU(2), this embedding is non-principal. However most of these subalgebras, in every group, are equivalent to one another:

In every root system there are at most two root lengths and all roots of the same length are related through the action of the Weyl group. This in turn corresponds to a conjugation (global gauge transformation) of the generators, so there are at most only two classes of inequivalent SU(2) subalgebras corresponding to roots.

We now want to concentrate on the highest root θ of $\Phi(G)$. We choose θ as a representative of its class, since it is uniquely defined and, as we shall see, it is easy to find the corresponding grades.

Define the SU(2) generators in the Chevalley basis

$$T_3 = \frac{1}{2} H_\theta \qquad\qquad T^\pm = E_{\pm\theta}$$

it is obvious that these have the required commutation relations. We will also use the Cartan-Weyl representation

$$T_3 = \sum_i \frac{\theta_i H_i}{(\theta,\theta)} = \frac{\theta . H}{(\theta,\theta)}$$

We need to know the grade 1 generators. Examine the commutator of T_3 with the raising generators:

$$[T_3, E_\beta] = \frac{(\theta,\beta)}{(\theta,\theta)} E_\beta \quad , \qquad \beta \in \Phi^+$$

We will show that $\frac{(\theta,\beta)}{(\theta,\theta)} = 1$ only when $\beta = \theta$. Consider the

reflection of $\beta \in \Phi^+$ in a plane perpendicular to the highest root θ, i.e.

$$\sigma_\theta(\beta) = \beta - \frac{2(\beta,\theta)}{(\theta,\theta)}\theta$$

If we assume $\frac{(\beta,\theta)}{(\theta,\theta)} = 1$, then $\sigma_\theta(\beta) = \beta - 2\theta$. But $\sigma_\theta(\beta)$ must be a root, and this is only possible when $\beta = \theta$ (otherwise $\sigma_\theta(\beta)$ is lower then the lowest root $-\theta$). We therefore conclude that the only generator with the grade 1 is $E_\theta = T^+$. If we now go back to our field equations (I.12), we see that we must expand

$$N^\pm = f\,T^\pm$$

We also decompose Ψ in two parts ("parallel" and "perpendicular" to T_3).

$$\Psi = \psi T_3 + C \quad , \quad C \in \mathcal{G} \qquad \text{(IV.1)}$$

where $\mathrm{Tr}T_3C = 0$. Apart from C, these are the variables that we would get, if we had started with SU(2) as our gauge group. But from (I.12) we see that C is constant, because $\frac{d}{dr}\Psi$ must be parallel to T_3.

C is of grade 0 and in general does not lie entirely in the C.S.A., since there are also raising and lowering generators of grade 0. We can, however, choose the Higgs field to be diagonal at infinity (that means Ψ_∞ in the C.S.A.). Then, since C is constant it is forced to be in the C.S.A. everywhere. Together with the trace condition this means:

$$C = c.H \quad , \quad (C,\theta) = 0 \qquad \text{(IV.2)}$$

It is now easy to see that the equations that we get from (I.12), are the same as for an SU(2) gauge group, and the solution is:

$$f = \frac{b}{\sinh br} \qquad \psi = -\,b\coth br$$

It has the correct behaviour at the origin, $B_3 \longrightarrow 0$ while for large r :

$$e\Phi_\infty = \Psi_\infty = - b T_3 + C$$

We see that the asymptotic value of the Higgs field completely determines the parameters of C and b. The magnetice charge matrix is $G_\infty = -T_3$. It is worth pointing out the following:

i) in general the magnetic field will have a "non-abelian" component (parallel to C). This differs from the case of principal embedding.

ii) For the same symmetry breaking (i.e. with Ψ_∞ "parallel" to a fundamental weight), the magnetic charge is always smaller than in the case of the principal embedding. This follows from (III.25), when we observe that

$$Tr\ Q\, T_3^{(princ.)} \quad > \quad Tr\ Q T_3^{(\theta)}$$

since $T_3^{(princ.)} = 2\,\delta^\vee . H$ and $2\,\delta^\vee = \sum_{\alpha \in \Phi^+} \frac{\alpha}{(\alpha,\alpha)} \quad \rangle \quad \frac{\theta}{(\theta,\theta)}$

(see appendix equ. C.1)

The above are illustrated in the case of SU(5): the highest weight $\theta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. We choose Ψ_∞ as in (III.4(b)), that is $\Psi_\infty = - \lambda_2 . H$ Then we conclude that

$$b = (\lambda_2, \theta) = \frac{3}{5} \qquad\qquad C = - \frac{1}{5} (3\alpha_2 + \alpha_3 - \alpha_4) . H$$

Also, when we calculate the magnetic charge, we find that it comes to one sixth of the value of the principal embedding: $gg = - \frac{4\pi}{6}$.

The solutions that we have just derived must have been expected, since they follow directly from knowledge of the SU(2) results. We have tried to show how this idea can be workerd out in detail and how it is related to our previous results.

IV.2 Other SU(2) embeddings

This section is not complete and is only a first attempt to discuss equations (I.12), when T_3 is a generator of an arbitrary SU(2) subalgebra. There are two difficulties: first, there is no easy description of T_3 and second, it is not clear which are the correct variables for the resulting equations (the equivalent of the ρ_α variables).

In order to find all inequivalent SU(2) subalgebras, we only need to consider diagonal generators for T_3 (in the C.S.A.), that in addition lie in the fundamental Weyl chamber:

$$(T_3, \alpha) \geqslant 0 \quad , \quad \alpha \in \Delta$$

This is always possible by a appropriate choice of our basis of the root system. This requirement implies that all raising generators have positive or zero grade, since

$$[T_3, E_\gamma] = (T_3, \gamma) E_\gamma \Rightarrow \text{grd } E_\gamma \geqslant 0 \quad \text{if} \quad \gamma \in \Phi^+$$

(Similarly, lowering generators have negative or zero grade). The reader can easily check, that both principal and highest root embeddings have this property.

The grades, since they are T_3 eigenvalues, are all integer or half-integer. This means that we can write for T :

$$T_3 = \sum_{\beta \in \Delta} n_\beta \frac{\lambda_\beta \cdot H}{(\beta, \beta)} \quad , \; n_\beta = 0, 1, \ldots \quad \text{(IV.3)}$$

This expression gives the required grades, since for any positive root

$$\gamma = \sum_{\alpha \in \Delta} k_\alpha \alpha \quad , \quad k_\alpha \geqslant 0 \quad \text{we have}$$

$$(T_3 , \gamma) = \sum_{\alpha, \beta \in \Delta} n_\beta \, k_\alpha \, \frac{(\lambda_\beta , \alpha)}{(\beta , \beta)} = \sum_{\alpha, \beta} n_\beta \, k_\alpha \, \frac{1}{2} \, \delta_{\alpha\beta} = \sum_{\beta} \frac{n_\beta \, k_\beta}{2}$$

The right hand side is clearly seen to be a non-negative integer or half-integer. It also follows that the grades of the simple roots are $\frac{n_\alpha}{2}$

The other two generators of SU(2) are

$$T^+ = \sum_{\gamma \in \Gamma} c_\gamma E_\gamma \quad , \quad T^- = \sum_{\gamma \in \Gamma} c_\gamma^* E_{-\gamma} \qquad \text{(IV.4)}$$

where Γ is the set of grade 1 roots, i.e $(T_3 , \gamma) = 1$. The complex constants c_γ must be such, so that

$$[T^+ , T^-] = 2 T_3$$

This means that there are equations that relate n_β and c_γ. Whenever these constraints can be satisfied, we have an SU(2) subalgebra.

The search for suitable T_3 generators is simplified, when we realise that in (IV.3) we can only have $n_\alpha = 0,1,$ or 2. The proof follows: Consider the commutator

$$[E_{-\alpha} , T^+] , \qquad \alpha \in \Delta$$

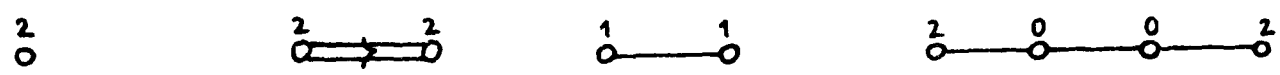
Since T^+ consists of raising generators only, and α is a simple root, the expression does not contain any lowering generators and therefore it has grade positive or zero. On the other hand we can calculate the grade directly, using the Jacobi identity:

$$[T_3 , [E_{-\alpha} , T^+]] = - [E_{-\alpha} , [T^+ , T_3]] - [T^+ , [T_3 , E_{-\alpha}]] =$$

$$= [E_{-\alpha} , T^+] - (T_3 , \alpha) [E_{-\alpha} , T^+] = \left(1 - \frac{n_\alpha}{2}\right) [E_{-\alpha} , T^+]$$

Since the grade must be non-negative we conclude that the only possible values are $n_\alpha = 0,1,2$.

The problem is therefore reduced to finding all posssible sets $\{n_\alpha\}$ that correspond to SU(2) subalgebras. Given such a set, T_3 is determined from (IV.3) and it also means that we can find the constants c_γ in (IV.4). Dynkin [25] defines the characteristic diagram of the subalgebra by attaching the integer n_α to each vertex α of the simple-root (Dynkin) diagram. Examples of characteristic diagrams (c.d.) are:



The c.d.s are what we need to know for our problem, since we can read off from them all grades of the generators, and in particular we can find the generators of grades 0, ± 1, that we need for expanding in equations (I.12).

To make the connection with what we already know, let us note that for the principal embedding, all entries n_α in the c.d. are 2, since

$$n_\alpha = 2\,(T_3\,,\,\alpha\,) = 2$$

i.e. we have the maximum possible eigenvalues. The other "extreme" is the highest root embedding, where most n_α are 0. For example, it is easy to check, that in A_ℓ ($\ell \geqslant 2$) the end roots α_1, α_ℓ have $n_{\alpha_1} = n_{\alpha_\ell} = 1$ and all other $n_{\alpha_i} = 0$:



(All other groups B_ℓ , C_ℓ etc. have only one non-zero entry in their c.d.).

Finding all possible characteristic diagrams for every group is more difficult. This problem is discussed in detail in E.B.Dynkin's

article [25] (but see also [26] for some examples). Dynkin gives a method for constructing all c.d.s but the problem with this is that each group has to be studied separately. This becomes increasingly more complicated for higher ranks. In [25], there is a list of all c.d. in the case of the exceptional groups.

We will now illustrate Dynkin's method for $SU(\ell+1) = A_\ell$. It is based on his result that every SU(2) subalgebra of A_ℓ can be considered as a principal subalgebra of a regular semisimple subalgebra. The inclusion relations are:

$$SU(2) \subset g' \subset g$$

(This result does not hold always for groups other than A_ℓ. See [25]).

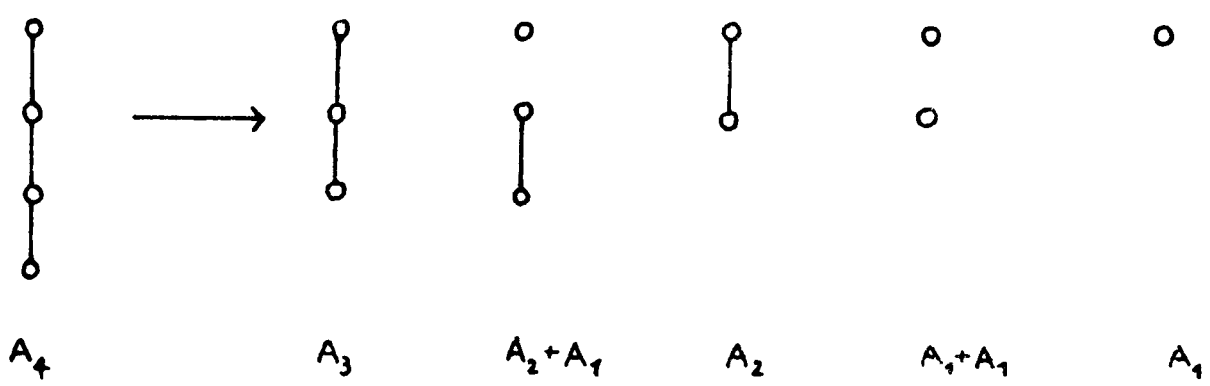
We will have to explain what is a regular subalgebra: g' is regular if it is based on a root system that is a subset of the root system of g :

$$\Phi(g') \subset \Phi(g)$$

One can get all regular subalgebras of A_ℓ by deleting an arbitrary number of vertices and their corresponding bonds from the Dynkin diagram of A_ℓ. For example A_2 has only one proper regular subalgebra,



and in A_4 (=SU(5)) there are five:



There are many possible root systems $\Phi(g') \subset \Phi(g)$ that correspond to each diagram and they are all equivalent. (The subalgebras that they define are all conjugate).

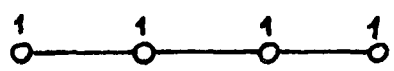
Given a regular subalgebra g' of g, we now consider it's principal SU(2). This is of course also a subalgebra of g. It is best to choose $\Phi(g')$ so that when we calculate T_3, it will lie in the fundamental Weyl chamber. (Otherwise we have to conjugate, i.e. "gauge transform", which is equivalent to picking a different Φ). An example will make clear how this procedure works: Consider the regular subalgebra of A_4, $g' = A_2 + A_1$. One possible set of "simple roots" for this subalgebra is

$$\Delta(g') = \{ \alpha_1 + \alpha_2 , \alpha_3 + \alpha_4 , \alpha_2 + \alpha_3 \} \qquad (IV.5)$$

The first two roots generate the A_2 factor, the last root generates the A_1 factor. The principal T_3 of g' (use (C.1) from appendix) is

$$T_3 = (2\alpha_1 + 3\alpha_2 + 3\alpha_3 + 2\alpha_4).H \qquad (IV.6)$$

Therefore the c.d. of this embedding is given:



This procedure can be repeated for each one of the regular subalgebras of A_4; the results are summarized in table (5). (Note that the first entry in the table is the highest root embedding, and the last entry, the principal).

We have described in some detail the properties of SU(2) subalgebras, because we hope that it will help us to understand better the reduced Bogomolny equations (I.12).

We are now in a position to expand for the field variables Ψ, $N^{\pm}$ in terms of the 0, ± 1 grade generators (found from the c.d.). When we

A_4 (=SU(5))

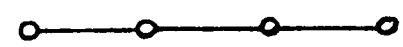
Regular subalgebra	T_3	c.d.
		
A_1	$(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4).H$	1 0 0 1
$A_1 + A_1$	$(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4).H$	0 1 1 0
A_2	$2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4).H$	2 0 0 2
$A_2 + A_1$	$(2\alpha_1 + 3\alpha_2 + 3\alpha_3 + 2\alpha_4).H$	1 1 1 1
A_3	$(3\alpha_1 + 4\alpha_2 + 4\alpha_3 + 3\alpha_4).H$	2 1 1 2
A_4	$(4\alpha_1 + 6\alpha_2 + 6\alpha_3 + 4\alpha_4).H$	2 2 2 2

TABLE (5)

substitute these in (I.12), we get a system of non-linear equations. This system has many features in common with (II.3), and may be considered as an interesting generalization. However, we will not pursue further the full set of these equations, because in general, we do not know how to deal with them. We suggest instead, the following simplification, which will make the system exactly like (II.3) and which can still satisfy the physical boundary conditions.

Set equal to zero all parameters of N^+, which correspond to generators that are not contained in the regular subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$. Since all generators of N^+ are of grade 1, we are left with the "simple roots" of $\mathfrak{g}'$ only, i.e.

$$N^+(r) = \sum_{\alpha \in \Delta(\mathfrak{g}')} f_\alpha(r)\, E_\alpha \qquad \text{(IV.7)}$$

(Of course the roots α may not be simple with respect to the whole algebra $\mathfrak{g}$, as for example in (IV.5)).

Similarly $N^-(r) = \sum_{\alpha \in \Delta(\mathfrak{g}')} f_\alpha^*(r)\, E_{-\alpha}$

The advantage of keeping only generators of $\mathfrak{g}'$ becomes apparent when we substitute $N^\pm$ in the first equation of (I.12); since

$$[E_\alpha, E_{-\beta}] = 0 \qquad \text{for} \quad \alpha, \beta \in \Delta(\mathfrak{g}'), \ \alpha \neq \beta$$

we conclude that $\frac{d\Psi}{dr}$ is in the C.S.A. of $\mathfrak{g}'$. But this is precisely what we had in the previous section IV.1. It should be now clear how we ought to proceed.

We argue in the same way as in IV.1, that Ψ can be taken to lie

in the C.S.A. of $\mathfrak{g}$. Then the analogue of equations (IV.1,2) is

$$\Psi(r) = \sum_{\alpha \in \Delta(\mathfrak{g}')} \psi_\alpha H_\alpha + C \quad , \quad \text{with} \quad \mathrm{Tr}\, C H_\alpha = 0 \tag{IV.8}$$

This expansion is always possible because the H_α are linearly independent and the Cartan matrix of $\mathfrak{g}'$ invertible.

Using (IV. 7,8) we can now see that the equations of motion are those of (II.3), but with the roots running over the set $\Delta(\mathfrak{g}')$. We conclude that we can solve for f_α , ψ_α as in the case of the principal embedding.

What makes these solutions acceptable from the physical point of view is that the finite energy condition at the origin can be satisfied. Remember that this means that

$$\frac{d\Psi}{dr} \xrightarrow[r \to 0]{} \frac{T_3}{r^2}$$

This is consistent with having $\frac{d\Psi}{dr}$ in the C.S.A. of $\mathfrak{g}'$.

The method that we have just described is not special to the A_ℓ groups, it also applies to most other cases. There are some exceptions however, when one can find no regular subalgebra relative to which T_3 is principal. This case has not been studied.

One last point that we wish to make: when there are no grade 1 roots other than the ones in $\Delta(\mathfrak{g}')$, our solutions are of course the most general ones. This, for example, happens for the $A_2 + A_1$ regular subalgebra of A_4 . One can check that the only grade 1 roots

are the ones in (IV.5). Then we expand

$$N^{\pm} = f_1 E_{\pm(\alpha_1+\alpha_2)} + f_2 E_{\pm(\alpha_3+\alpha_4)} + f_3 E_{\pm(\alpha_2+\alpha_3)}$$

$$\Psi = \psi_1 H_{\alpha_1+\alpha_2} + \psi_2 H_{\alpha_3+\alpha_4} + \psi_3 H_{\alpha_2+\alpha_3} + c\,(2H_{\alpha_1} - H_{\alpha_2} + H_{\alpha_3} - 2H_{\alpha_4})$$

and the equations are

c = constant, $\quad 2\dot{\psi}_1 = f_1^2 \quad , 2\dot{\psi}_2 = f_2^2 \quad , 2\psi_3 = f_3^2$

$$\dot{f}_1 = f_1(2\psi_1 - \psi_2) \; , \; \dot{f}_2 = f_2(-\psi_1 + 2\psi_2) \; , \; \dot{f}_3 = f_3\, 2\psi_3$$

These fall into the class described by (II.3), as it was claimed.

Discussion

In this thesis we have been studying monopole solutions that are direct generalizations of known results in SU(2) gauge theory. We have indeed found many new solutions, but we do not seem to have improved much our physical understanding. Then what has been gained?

Firstly, we hope that we have convinced the reader that it is important to use the standard mathematical language (roots, weights, Weyl reflections etc.), when dealing with groups. If we want to explore deeper the structure of gauge theories, we must be equipped with the right tools.

Secondly, we believe that some of the techniques used here are interesting on their own right. Non-linear equations come up in many problems in physics and we must teach ourselves to deal with them.

We now want to enumerate some problems that still have to be done:

i) We have seen that there are solutions corresponding to each embedding, but we have not tried to classify and compare the different charges and masses in any great detail. This can be done separately, since these depend mainly on the asymptotic conditions and not on the specific form of the solution. We would also have to decide which monopoles are expected to be stable.

ii) The solutions corresponding to the principal embedding are interesting mathematically. One should study their properties. Try to get alternative and simpler expressions for (III.4).

iii) The action of the Weyl group on the solutions should be understood better; how is it related to the boundary condition at the origin?

iv) Further studying of the non-principal embeddings is needed.

Finally, we must ask ourselves whether any of the solutions that we have described can be realized in nature. The possibility remains open.

Appendix A

We give some definitions and results (without proofs) from Lie Algebra representation theory. These notes are necessarily brief and non-rigorous, we just want to establish the notation used throughout the text. For more details see the book by B.G. Wybourne [19] (written for physicists), or the mathematics textbook by J.E. Humphreys [18].

A.1 Roots

This is a finite set Φ of non-zero vectors in the Euclidean space E. There is a scalar product which we denote (α,β) $\alpha,\beta \in \Phi$. The following properties must be satisfied for a root system Φ:

(1) if $\alpha \in \Phi$, the only multiples of α are $\pm\alpha$;

(2) if $\alpha,\beta \in \Phi$, $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ is an integer;

(3) A reflection on a hyperplane perpendicular to any $\alpha \in \Phi$, leaves system Φ invariant.

This reflection acts on every root $\beta \to \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ and we denote it $\sigma_\alpha(\beta)$. All reflections σ_α, $\alpha \in \Phi$ generate a finite group, called the Weyl group.

The rank ℓ of the root system is the dimension of E. It is an important result about root systems that we can find a basis of simple roots Δ,

such that any γ can be written $\gamma = \sum_{\alpha\in\Delta} k_\alpha \alpha$ with k_α integers all nonpositive or all nonnegative. The following results hold for simple roots

(1) $\alpha,\beta \in \Delta$ then $\alpha-\beta$ is not a root;

(2) $(\alpha,\beta) \leqslant 0$ if $\alpha,\beta \in \Delta$, $\alpha \neq \beta$

We can talk about positive roots $\gamma \in \Phi^+$ when all $k_\alpha \geqslant 0$ and

negative roots, $k_\alpha \leq 0$. We also have a partial ordering $\alpha > \beta$ if $\alpha - \beta$ is the sum of positive roots or zero.

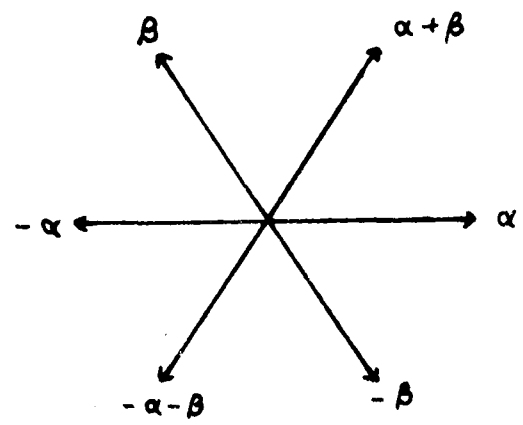
Example

rank 1: There is only one root system, denoted A_1 (it corresponds to the SU(2) algebra).



α is the simple root.

rank 2:



There are two simple roots. The positive roots are $\{\alpha, \beta, \alpha+\beta\}$ This root system is called A_2 and corresponds to SU(3).

The Cartan Matrix $K_{\alpha\beta}$ is defined as $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ for all simple roots α, β. The dimension of the matrix is equal to the rank. Of course all entries are integers (by definition of a root system). There is 2 along the diagonal and negative or zero entries off-diagonal.

$$A_1: \quad 2 \qquad A_2: \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad A_3: \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

There is a useful way to code the relations between simple roots of a root system known as a Dynkin diagram. This is a graph whose vertices are in one to one correspondance with the simple roots,

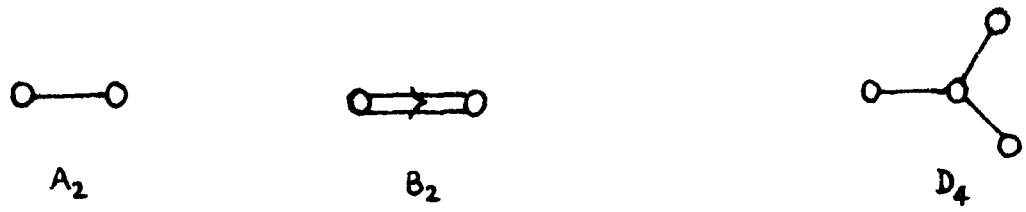
vertices corresponding to α_i and α_j being joined by m_{ij} bonds, where

$$m_{ij} = \frac{4(\alpha_i, \alpha_j)^2}{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)} = 4\cos^2\theta_{ij}$$

These bonds correspond to the angles between simple roots; the only possible angles are

θ_{ij} :	$\frac{\pi}{2}$	m_{ij}	0
	$\frac{2\pi}{3}$		1
	$\frac{3\pi}{4}$		2
	$\frac{5\pi}{6}$		3

An arrow points towards the shorter root (if any):



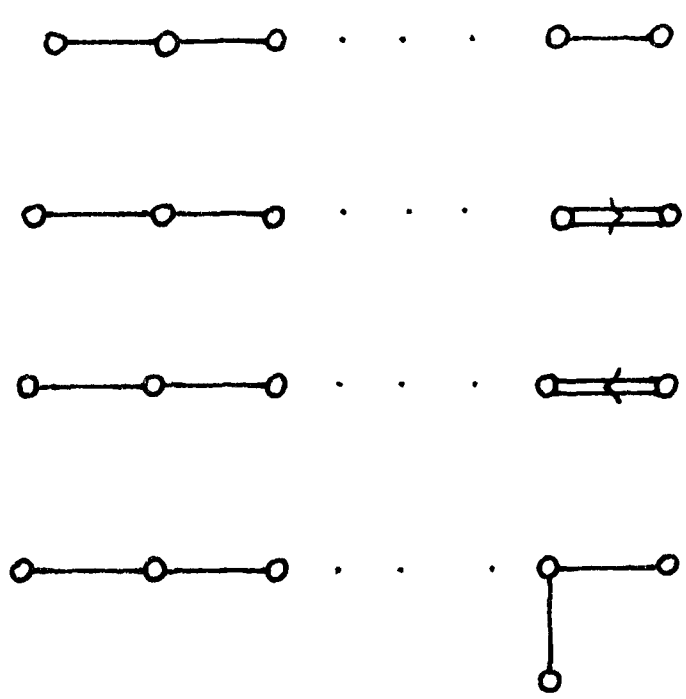
There is a completed classification of the irreducible root systems. We have the series

A_ℓ ($\ell \geq 1$)

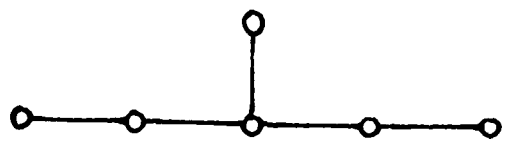
B_ℓ ($\ell \geq 2$)

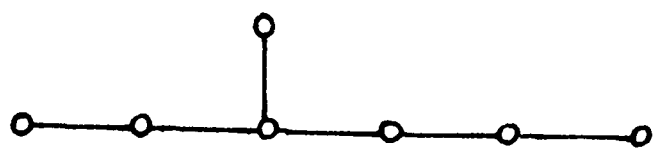
C_ℓ ($\ell \geq 3$)

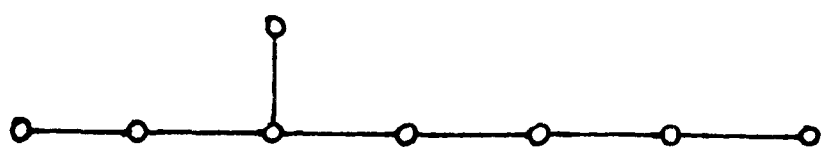
D_ℓ ($\ell \geq 4$)

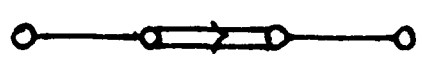


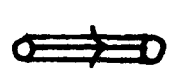
and the exceptional groups

E_6 

E_7 

E_8 

F_4 

G_2 

The Dynkin diagram (or equivalently the Cartan matrix) determine completely each root system.

A.2 Cartan subalgebras and bases.

We will be studying complex simple Lie algebras. Each algebra $\mathfrak{g}$ has a maximal abelian subalgebra (maximum set of commuting generators). This is called the Cartan subalgebra (C.S.A.) $\mathfrak{h}$ of $\mathfrak{g}$. For a complex Lie algebra all C.S.A.'s are conjugate, we can define it once and for all.

Pick as a basis of C.S.A. ℓ (= rank) commuting generators H_i : $[H_i, H_j] = 0$. Those correspond to our diagonal generators. We also have the "step up" and "step down" generators: $[H_i, E_{\pm\alpha}] = \pm \alpha_i E_{\pm\alpha}$ with $\alpha \in \Phi^+$ a "positive root". Each root has ℓ components α_i $i=1,..\ell$ corresponding to the ℓ C.S.A. generators. There is an existence

theorem, that for each abstract root system we have a corresponding Lie algebra.

In the Cartan-Weyl basis the commutation relations of a Lie algebra are:

$$[H_i, H_j] = 0$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha = (H_i, \alpha) E_\alpha \quad \text{for every root} \quad \alpha$$

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \quad \text{if } \alpha+\beta \text{ is a root}$$

$$[E_\alpha, E_\beta] = 0 \quad \text{if } \alpha+\beta \text{ is not a root}$$

$$[E_\alpha, E_{-\alpha}] = \sum_i \alpha_i H_i = \alpha . H$$

The constants $N_{\alpha,\beta}$ and their phases are also determined.

Note that we have introduced the notation(H_i, α) for the component of root α along a certain direction. This corresponds to the scalar product between roots (α, β) $\alpha, \beta \in \Phi$. For an arbitrary element of the C.S.A. Q we have:

$$(Q, \alpha) = (\sum_i Q_i H_i, \alpha) = \sum_i Q_i (H_i, \alpha) = \sum_i Q_i \alpha_i$$

Instead of summing over the rank, we will sometimes also write for an arbitrary element of the C.S.A. $Q = Q.H \equiv \sum_i Q_i H_i$.

There is another basis called the Chevalley basis which has only integer structure constants:

A generator H_α is associated with each simple root α.

We define $H_\alpha = \frac{2\,\alpha . H}{(\alpha, \alpha)}$ .The commutators are

$$[H_\alpha, H_\beta] = 0$$

$$[H_\alpha, E_\beta] = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} E_\beta \quad \text{for all} \quad \beta \in \Phi$$

(Note that if $\alpha, \beta \in \Delta$(simple roots), then we have the Cartan matrix).

$$[E_\beta, E_{-\beta}] = H_\beta$$

$$[E_\beta, E_\gamma] = N_{\beta,\gamma} E_{\beta+\gamma} \quad \text{if} \quad \beta+\gamma \in \Phi$$

(The $N_{\beta,\gamma}$ are not the same as for the Cartan-Weyl basis and must be separately determined; also the E_α in the Chevalley basis have been rescaled).

A.3 Weights and Representations

A linear representation of a Lie algebra $\mathfrak{g}$ is a homomorphism of $\mathfrak{g}$ into the Lie algebra of all linear transformations of the N-dimensional space $R^{(N)}$. Each generator is represented by a NxN matrix. The ℓ (= rank) matrices of the Cartan subalgebra (C.S.A.) are self commuting, so we may construct a set of simultaneous eigenvectors $|u\rangle$ such that

$$H_i |u\rangle = \lambda_i |u\rangle = (H_i, \lambda) |u\rangle$$

The set of ℓ eigenvalues λ_i form a weight vector λ. For an arbitrary element of the C.S.A. Q we write

$$Q|u\rangle = (Q,\lambda) |u\rangle = \sum_i Q_i \lambda_i |u\rangle$$

Given an eigenvector $|u\rangle$ with weight λ, we can find a new eigenvector $E_\beta|u\rangle$ with weight $\lambda+\beta$:

$$H_i E_\beta |u\rangle = ([H_i, E_\beta] + E_\beta H_i) |u\rangle = (\beta_i + \lambda_i) E_\beta |u\rangle$$

In other words E_β, $\beta \in \Phi$, act as raising or lowering operators. (Of course we might have $E_\beta |u\rangle = 0$).

The representation space is decomposed into the direct sum of weight subspaces $R = \sum_{\mu \in \Pi} R^\mu$ where Π is the (finite) set of all weights μ. R^μ, is an eigenspace of H_i with eigenvalue μ_i.
For any vector $|\xi\rangle$ of R^μ, and any root $\beta \in \Phi$

$$E_\beta |\xi\rangle \in R^{\mu+\beta} \quad \text{if} \quad \mu+\beta \in \Pi$$

$$E_\beta |\xi\rangle = 0 \quad \text{if} \quad \mu+\beta \notin \Pi$$

The set of simple roots Δ defines a partial ordering of weights in Π : for $\mu,\nu \in \Pi$, $\mu > \nu$ iff $\mu-\nu$ is a sum of positive roots. There is a highest vector $|\lambda\rangle$ with highest weight λ such that $E_\beta|\lambda\rangle = 0$ for every $\beta \in \Phi^+$. Equivalently $\lambda+\beta \notin \Pi$ for $\beta \in \Phi^+$. The highest weight λ determines the whole representation.

We use the following important results:

(1) For λ to be the highest weight of an irreducible representation, we must have $\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}_0^+$, a nonnegative integer, for all simple α .

(2) For any weight $\mu \in \Pi$, $\frac{2(\mu,\alpha)}{(\alpha,\alpha)}$ an integer. We can get any weight from the highest weight of the representation $\mu = \lambda - \sum_{\beta\in\Delta} k_\beta \beta$ and $k_\beta \geqslant 0$.

(3) Dim R^μ is called the multiplicity of weight μ . For the highest weight the multiplicity is one. The multiplicity tells us the number of independent eigenvectors we need to span the eigenspace R^μ .

(4) if $\mu \in \Pi$, then $\sigma_\alpha(\mu) \in \Pi$, where σ_α an element of the Weyl group, (a reflection), for every root $\alpha \in \Phi$. μ and $\sigma_\alpha(\mu)$ have the same multiplicity.

A very important class of representations are the fundamental representations with corresponding highest fundamental weights. Such weights are reciprocals to the simple roots. Each fundamental weight λ_α is associated with a simple root $\alpha \in \Delta$ such that $\frac{2(\lambda_\alpha,\beta)}{(\beta,\beta)} = \delta_{\alpha\beta}$ for $\alpha,\beta \in \Delta$. There are of course ℓ (= rank) such fundamental representations. (Note that the term fundamental is not used in the same way as in the physics literature, e.g. in SU(4) all 4, 4*, 6 are fundamental representations according to the definition given above).

To express the fundamental weights in terms of the simple roots

write $\lambda_\alpha = \sum_{\beta \in \Delta} \beta K^{-1}_{\beta\alpha}$; one sees that

$$\frac{2(\lambda_\alpha, \beta)}{(\beta,\beta)} = \sum_{\gamma\in\Delta} \frac{2(\gamma,\beta)}{(\beta,\beta)} K^{-1}_{\gamma\alpha} = \sum_\gamma K_{\beta\gamma} K^{-1}_{\gamma\alpha} = \delta_{\beta\alpha}$$

as required.

Note that in the Chevalley basis all eigenvalues are integers, since

$$H_\alpha |\lambda\rangle = \frac{2\,\alpha\cdot H}{(\alpha,\alpha)} |\lambda\rangle = \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} |\lambda\rangle$$

Example

There is one fundamental representation in SU(2) = A_1 . The fundamental weight is $\lambda = \frac{1}{2}\alpha$, and the lowest weight $\bar\lambda = -\frac{1}{2}\alpha$.

- λ
- $\bar\lambda$

We have the familiar relations for the raising generator

$$E_\alpha |\bar\lambda\rangle = |\lambda\rangle \qquad E_\alpha |\lambda\rangle = 0$$

and for the lowering generator:

$$E_{-\alpha} |\lambda\rangle = |\bar\lambda\rangle \qquad E_{-\alpha} |\bar\lambda\rangle = 0$$

Another familiar example is the 3* of SU(3). The highest weight is $\lambda = \frac{1}{3}(\alpha+2\beta)$:

- $\lambda = \frac{1}{3}(\alpha+2\beta)$
- $\mu = \frac{1}{3}(\alpha-\beta)$
- $\bar\lambda = \frac{1}{3}(-2\alpha-\beta)$

Appendix B

Decompositions of group elements

We will consider $G = GL(n, \mathbb{C})$ the group of all complex invertible matrices. All classical groups used in this thesis (A_ℓ , B_ℓ , ...) and their finite dimensional representations are subgroups of G. We are also interested in the following subgroups of G:

B is the group of all upper triangular matrices with elements:

$$b = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdot & \cdot & b_{1n} \\ 0 & b_{22} & b_{23} & \cdot & \cdot & \\ 0 & 0 & b_{33} & & & \\ \cdot & \cdot & \cdot & & & \\ 0 & 0 & \cdot & & 0 & b_{nn} \end{pmatrix}$$

with $b_{11} \neq 0$, $b_{22} \neq 0$ . . $b_{nn} \neq 0$. The diagonal elements are non-zero since we want an invertible matrix, det $b \neq 0$.

There is also the corresponding group of lower triangular matrices $\bar{B}$.

$$\bar{b} = \begin{pmatrix} \bar{b}_{11} & 0 & \cdot & 0 \\ \bar{b}_{21} & \bar{b}_{22} & 0 & \cdot \\ \cdot & \cdot & \cdot & 0 \\ \bar{b}_{n1} & \bar{b}_{n2} & & \bar{b}_{nn} \end{pmatrix} \qquad \bar{b}_{11} \neq 0, \ldots \bar{b}_{nn} \neq 0$$

The subgroup of B with diagonal elements $b_{ii} = 1$ is denoted N, and the subgroup of $\bar{B}$ with $\bar{b}_{ii} = 1$ is $\bar{N}$. We denote with H the group of diagonal matrices with elements

$$h = \mathrm{diag}\,(h_1, h_2, \ldots, h_n) \qquad h_1 \neq 0 , \ldots h_n \neq 0$$

We have the following results.

(1) Each element of B can be uniquely represented in the form b = nh.

This decomposition implies the system of equations

$$b_{ij} = n_{ij} h_j \quad , \quad i \leqslant j$$

with unique solutions $h_j = b_{jj}$ and $n_{ij} = \dfrac{b_{ij}}{b_{jj}} \quad , \quad i < j$

Definition: A <u>regular</u> element of the group G has $\Delta_\ell(g) \neq 0$ where

$$\Delta_\ell(g) = \begin{vmatrix} g_{11} & g_{12} & & g_{1\ell} \\ g_{21} & g_{22} & & g_{2\ell} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \cdot \\ g_{\ell 1} & \cdot & & g_{\ell\ell} \end{vmatrix} \qquad \text{for } \ell = 1, \ldots n$$

These are minors of determinants of the group element g.

(2) Every regular element of G can be represented in the form $g = \bar{n}b$.

Write $b = (\bar{n}^{-1})g$.

Since b is an upper triangular matrix $b_{ij} = 0$ for $i > j$:

$$0 = b_{ij} = \sum_k (\bar{n}^{-1})_{ik} g_{kj} = g_{ij} + \sum_{k>i}^{n} (\bar{n}^{-1})_{ik} g_{kj}$$

$$\text{for} \qquad i, j = 1, 2, \ldots n$$

For a given i this is a system of equations in variables $(\bar{n}^{-1})_{ik}$

k= 1,2, ..., i-1. The corresponding determinant $\det g_{kj} \quad k, j > i$

is $\Delta_{i-1}(g) \neq 0$ by assumption. So we can solve for $\bar{n}$.

Finally combining results (1) and (2) we get:

(3) Every regular element g of the group G can be uniquely decomposed in the form:

$$g = \bar{n}nh \text{ with } \quad \bar{n} \in \bar{N}, \quad n \in N, \quad h \in H$$

This is the decomposition used in the text. Of course this is only one

of many different possible decompositions e.g. $g = n_1 \bar{n}_1 h_1$, $g = h_2 \bar{n}_2 n_2$ etc. The proofs for those are similar with the one presented above.

To make the connection with Lie algebras and their roots system note the following: The Lie algebra $\mathfrak{g}$ of a complex simple group G, can be decomposed relative to its C.S.A. $\mathfrak{h}$ as

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi^+} (\mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{-\alpha})$$

where $\mathfrak{g}^{\pm\alpha}$ are eigenspaces corresponding to roots $\pm\alpha$. Then one <u>may choose a basis</u> such that in a representation, $\mathfrak{h}$ are written as <u>diagonal matrices</u>, $\mathfrak{g}^{\alpha}$ as <u>upper triangular</u> matrices (and $\mathfrak{g}^{-\alpha}$ as <u>lower triangular</u> matrices) with zeros along the diagonal.

Group elements are the exponentials of generators. Therefore, we can rewrite (3) for a regular element

$$g = \exp \bar{M} \exp M \exp T \qquad \text{with} \qquad \bar{M} \in \sum_{\alpha \in \Phi^+} \mathfrak{g}^{-\alpha}, \; M = \sum_{\alpha} \mathfrak{g}^{\alpha}, \; T \in \mathfrak{h}$$

This result can be shown for each of the classical groups.

For more details about decompositions see Gelfand I.M & Naimark M.A. : Unitary representations of the classical groups [20].

Appendix C

We will derive an alternative expression for the principal embedding generator T_3 . We will give first some definitions that will help us to get to the desired result.

i) σ_α , $\alpha \in \Delta$ permutes the positive roots other than α . σ_α is a reflection in the plane perpendicular to α and of course $\sigma_\alpha(\alpha) = -\alpha$. As for the other positive roots $\sigma_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\beta$; now if $\beta \neq \alpha$ it will always have a component along one of the other simple roots, forcing $\sigma_\alpha(\beta)$ to be positive.

ii) Define $\delta = \frac{1}{2} \sum_{\beta>0} \beta$, half the sum of all positive roots. Then for each σ_α , $\alpha \in \Delta$ we have

$$\sigma_\alpha(\delta) = \delta - \alpha$$

This follows from i) since all terms except α are permuted in the sum. On the other hand we know $\sigma_\alpha(\delta) = \delta - \frac{2(\alpha,\delta)}{(\alpha,\alpha)}\alpha$. It follows that

$$\frac{(\alpha, 2\delta)}{(\alpha, \alpha)} = 1 \qquad , \qquad \alpha \in \Delta$$

iii) For each $\beta \in \Phi$ define the coroot $\beta^\vee = \frac{\beta}{(\beta,\beta)}$. Also the coroot system $\Phi^\vee = \{ \beta^\vee / \beta \in \Phi \}$. This is a root system on its own right and its Weyl group is isomorphic to that of Φ . The simple roots are $\Delta^\vee = \{\alpha^\vee / \alpha \in \Phi\}$. We can apply the result of ii) on $\Phi^\vee$. We have

$$\delta^\vee = \frac{1}{2} \sum_{\beta^\vee>0} \beta^\vee = \frac{1}{2} \sum_{\beta>0} \frac{\beta}{(\beta,\beta)} \qquad \text{(C.1)}$$

and

$$1 = \frac{(\alpha^\vee, 2\delta^\vee)}{(\alpha^\vee, \alpha^\vee)} = (\alpha, 2\delta^\vee) \qquad \alpha \in \Delta$$

We now see that if we write in the Cartan-Weyl basis $T_3 = 2\delta^\vee \cdot H$ it has the required property:

$$(T_3, \alpha) = (2\delta^\vee, \alpha) = 1 \qquad , \alpha \in \Delta$$

Expression (C.1) provides an easy method for calculating T_3 .

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