An efficient sum of squares nonnegativity certificate for quaternary quartic

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Let $f \in \mathbb{R}[X_0, \dots, X_n]$ be homogeneous of degree $2d$ and nonnegative on $\mathbb{R}^{n+1}$. While f can be represented as a sum of squares of rational functions, only for $d = 1$, or $n = 2$, reasonably good bounds on degrees of such functions are known, see e.g. [6]. A polynomial g is called *sos* polynomial if $g = \sum_{k=1}^{N} h_k^2$, for some polynomials h_k.

The aim of this note is to show the following.

Theorem. Let f be a homogeneous 4-variate nonnegative polynomial of degree 4. Then $f = p/q$, where p and q are homogeneous sos polynomials of degrees 8 and 4, respectively. The identity $f = p/q$ holds almost everywhere in the usual measure-theoretic sense.

In what follows, we denote $X := (X_1, \dots, X_n)$. First, we recall a well-known lemma (e.g. similar observations are made in [4]).

Lemma 1. Let $f \in \mathbb{R}[X_0, \dots, X_n]$ be homogeneous of degree $2d$ and nonnegative on $\mathbb{R}^{n+1}$, and $S := \{x \in \mathbb{R}^{n+1} \mid \sum_{k=0}^{n} x_k^2 = 1\}$ be the unit sphere in $\mathbb{R}^{n+1}$.
(i) Let f have a zero on S. An orthogonal change of coordinates bringing the zero to $(1 : 0 : \dots : 0)$ transforms f into the form

$$f(X_0, X) = \sum_{k=2}^{2d} f_k(X) X_0^{2d-k}, \quad f_k \in \mathbb{R}[X_1, \dots, X_n]. \tag{1}$$

(ii) Let f be strictly positive on S, with a minimum on S reached at $x* \in S$. An orthogonal change of coordinates bringing x^* to $(1 : 0 : \dots : 0)$ transforms f into the form

$$f(X_0, X) = X_0^{2d} + \sum_{k=2}^{2d} f_k(X) X_0^{2d-k}, \quad f_k \in \mathbb{R}[X_1, \dots, X_n], \tag{2}$$

so that $f(X_0, X) - X_0^{2d}$ is nonnegative.
(iii) Polynomials f_2 and f_{2d} are nonnegative. Moreover, f_2 is an sos of linear forms.

Proof. (i). After the transformation, f cannot have a term X_0^{2d}, as otherwise it cannot vanish on $(1 : 0 : \dots : 0)$. It cannot have a term $f_1(X) X_0^{2d-1}$, as it is nonnegative. Thus we have the required form.

(ii). We may assume without loss of generality that after the transformation we have

$$f(X_0, X) = X_0^{2d} + g(X_0, X), \quad g(X_0, X) = \sum_{k=1}^{2d} f_k(X) X_0^{2d-k}.$$

Note that on S one has $f(X_0, X) \geq 1$, as the minimum of f on S equals 1. Thus on S one has $g(X_0, X) = f(X_0, X) - X_0^{2d} \geq f(X_0, X) - 1 \geq 0$. Hence $g(X_0, X)$ is (globally) nonnegative, as it is homogeneous. Moreover, it has a zero, $(1 : 0 \dots : 0)$, on S, and this (i) applies, ensuring $f_1 = 0$, as required.

(iii). Nonnegativity of f_2 and f_{2d} follows from nonnegativity of f in the case (i), and nonnegativity of $f(X_0, X) - X_0^{2d}$ in the case (ii). As f_2 is quadratic, it is an sos. $\square$

From now on, consider the case $d = 2$. By Lemma 1, we have

$$g(X_0, X) := f_2(X) X_0^2 + f_3(X) X_0 + f_4(X) \geq 0, \quad f_2(X) \geq 0, \; f_4(X) \geq 0.$$

Note that the nonnegativity of g implies that $4 f_2(X) f_4(X) - f_3(X)^2 \geq 0$. Write, omitting X for brevity:

$$4 f_2 g(X_0) = 4 f_2^2 X_0^2 + 4 f_2 f_3 X_0 + 4 f_2 f_4 = (2 f_2 X_0 + f_3)^2 - f_3^2 + 4 f_2 f_4.$$

The latter is the sum of a square and a nonnegative homogenous degree 6 polynomial $h(X) := 4f_2(X)f_4(X) - f_3(X)^2$ in n variables.

In particular, for $n = 3$, by [3] (or by a recent [1, Sect. 5-6]), $h(X) = u(X)/q(X)$, with u and q sos polynomials of degrees 8 and 2, respectively. To summarise, we state

Lemma 2. Let $g(X_0, X) := f_2(X)X_0^2 + f_3(X)X_0 + f_4(X)$ be a nonnegative degree 4 homogeneous polynomial in X_0, X, with $X = (X_1, X_2, X_3)$. Then there exists a homogeneous degree 2 sos polynomial $q(X)$ such that

$$q(X)f_2(X)g(X_0, X) = \sum_{k=1}^{N} r_k(X_0, X)^2, \tag{3}$$

i.e. it is an sos polynomial. □

As an orthogonal change of coordinates (e.g. the inverse G of the one in Lemma 1 (i)) respects sos decompositions, and as $g(1, 0, 0, 0) = 0$, we obtain, applying G to the both sizes of (3), the following.

Lemma 3. Let f be a nonnegative degree 4 homogeneous polynomial in $X_0, \dots, X_3$ with a zero on S. Then there are degree 2 homogeneous sos polynomials $q_t(X_0, \dots, X_3)$, $(t = 1, 2)$ such that $q_1q_2f = s$ is an sos polynomial, and $f = s/(q_1q_2)$. □

To complete the proof of the Theorem, it remains to observe that in the case of strictly positive $f(X_0) = X_0^4 + f_2X_0^2 + f_3X_0 + f_4$ in the form (2), by Lemma 2 and Lemma 1 (ii) we have $f(X_0) = X_0^4 + \frac{\sum_{k=1}^{N} r_k^2}{qf_2}$, i.e.

$$q(X)f_2(X)f(X_0, X) = q(X)f_2(X)X_0^4 + \sum_{k=1}^{N} r_k(X)^2.$$

The 1st term on the RHS of the latter is an sos, and the argument used to establish Lemma 3 applies here, too. □

Remarks

(a) It can be seen from the proof that q in Theorem is the product of two quadrics, each of which is a sum of at most 3 squares.

(b) The decomposition $qf = p$ can be used to apply semidefinite programming to compute q and p in terms of their *Gram matrices* (cf. e.g. [5]), following the approach described in [2].

(c) The degree of q is 4. Is this the best possible?

(d) What can be said about f_2 in Lemma 1? For instance, one can see that if f_2 is a square, $f_2 = \ell^2$, then f is sos. Indeed, in this case $4f_2f_4 - f_3^2$ factors: $4f_2f_4 - f_3^2 = (2\ell)^2f_4 - f_3^3 \geq 0$ shows that ℓ divides f_3. This implies that f is sos polynomial (cf. [4, Sect. 2.2]).

(f) It is easy to show, using the decomposition of qf into a sum of squares, that f can be written as a sum of at most 8 squares of of rational functions; this is the same number of terms as given by the classical result of Pfister.

References

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