Noname manuscript No.
(will be inserted by the editor)

Cauchy Fluxes and Gauss-Green Formulas for Divergence-Measure Fields over General Open Sets

Gui-Qiang G. Chen ·
Giovanni E. Comi ·
Monica Torres

In memoriam William P. Ziemer

Received: September 4, 2018 / Accepted: December 8, 2018

Abstract We establish the interior and exterior Gauss-Green formulas for divergence-measure fields in L^p over general open sets, motivated by the rigorous mathematical formulation of the physical principle of balance law via the Cauchy flux in the axiomatic foundation, for continuum mechanics allowing discontinuities and singularities. The method, based on a distance function, allows to give a representation of the interior (resp. exterior) normal trace of the field on the boundary of any given open set as the limit of classical normal traces over the boundaries of interior (resp. exterior) smooth approximations of the open set. In the particular case of open sets with continuous boundary, the approximating smooth sets can explicitly be characterized by using a regularized distance. We also show that any open set with Lipschitz boundary has a regular Lipschitz deformable boundary from the interior. In addition, some new product rules for divergence-measure fields and suitable scalar functions are presented, and the connection between these product rules and the representation of the normal trace of the field as a Radon measure is explored. With these formulas at hand, we introduce the notion of Cauchy fluxes as functionals defined on the boundaries of general bounded open sets for the rigorous mathematical formulation of the physical principle of balance law, and show that the Cauchy fluxes can be represented by corresponding divergence-measure fields.

Gui-Qiang G. Chen
Mathematical Institute, University of Oxford, Oxford, OX2 6GG, UK
E-mail: chengq@maths.ox.ac.uk

Giovanni E. Comi
Scuola Normale Superiore, 56126 Pisa, Italy
E-mail: giovanni.comi@sns.it

Monica Torres
Department of Mathematics, Purdue University, West Lafayette, IN 47907-2067, USA
E-mail: torres@math.purdue.edu

Keywords Divergence-measure fields · Cauchy fluxes · Gauss-Green formula · normal traces · general open sets · approximation of sets · distance function · product rules · Green's identities · balance law · axiomatic foundation · continuum mechanics · discontinuity · singularity

Mathematics Subject Classification (2010) Primary: 28C05 · 26B20 · 28A05 · 26B12 · 35L65 · 35L67 · 76A02; Secondary: 28A75 · 28A25 · 26B05 · 26B30 · 26B40 · 35D30

Contents

1 Introduction 2
2 Basic Notations and Divergence-Measure Fields 7
3 Product Rules between Divergence-Measure Fields and Suitable Scalar Functions 10
4 Regularity of Normal Traces of Divergence-Measure Fields 16
5 The Gauss-Green Formula on General Open Sets 27
6 Other Classes of Divergence-Measure Fields with Normal Trace Measure 41
7 Normal Traces for Open Sets as the Limits of the Classical Normal Traces for Smooth Sets 48
8 The Gauss-Green Formula on Lipschitz Domains 52
9 Cauchy Fluxes and Divergence-Measure Fields 66

1 Introduction

We are concerned with the interior and exterior Gauss-Green formula for unbounded divergence-measure fields over general open sets, motivated by the rigorous mathematical formulation of the physical principle of balance law via the Cauchy flux in the axiomatic foundation, for continuum mechanics allowing discontinuities and singularities. The divergence-measure fields are vector fields $\boldsymbol{F} \in L^p$ for $1 \leqslant p \leqslant \infty$, whose distributional divergence is a Radon measure. These vector fields form a Banach space that is denoted by $\mathcal{DM}^p$. Even though the definitions of normal traces for unbounded divergence-measure fields have been given in Chen-Frid [12] and Šilhavý [57] (see also [32]), the objective of this paper is to give a representation of the *interior* (resp. *exterior*) normal trace on the boundary of any given open set and to prove that these normal traces can be computed as the limit of classical normal traces over the boundaries of *interior* (resp. *exterior*) smooth approximations of the open set. In particular, this implies analogous results on general domains (*i.e.* open connected sets).

The approximation of domains is a fundamental problem that has many applications in several fields of analysis. The answer to this question depends on both the regularity of the domain and the type of approximation that is needed. Our interest in this problem is motivated from the field of hyperbolic conservation laws. It is important to approximate the surface of a *discontinuity wave* (such as a shock wave, vortex sheet, and entropy wave) by smooth surfaces from *one side* of the surface so that the *interior* and *exterior* traces of

the solutions can be defined on such a *discontinuity wave* as the limit of classical traces on the smooth approximating surfaces. Furthermore, the physically meaningful notion of Cauchy fluxes as functionals defined on the boundaries of general bounded open sets requires the understanding of the flow behavior in both the interior and exterior neighborhoods of each boundary.

In this paper, we consider arbitrary open sets, which especially include domains with finite perimeter. The sets of finite perimeter are relevant in the field of hyperbolic conservation laws, since the reduced boundaries of sets of finite perimeter are rectifiable sets, while the shock surfaces are often rectifiable, at least for multidimensional scalar conservation laws (*cf.* De Lellis-Otto-Westickenberg [24]). Moreover, one advantage for the sets of finite perimeter is that the normal to these sets can be well defined almost everywhere on the boundaries.

A first natural approach to produce a smooth approximation of a domain is via the convolution with some mollifiers η_ε. Indeed, it is a classical result in geometric measure theory (see the classical monographs of Ambrosio-Fusco-Pallara [2, Theorem 3.42] and Maggi [43, Theorem 13.8]) that any set of finite perimeter E can be approximated with a suitable family of smooth sets E_k such that

$$\mathcal{L}^n(E_k \Delta E) \to 0, \quad \mathscr{H}^{n-1}(\partial^* E_k) \to \mathscr{H}^{n-1}(\partial^* E) \qquad \text{as } k \to \infty, \tag{1.1}$$

where $\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$, $\partial^* E$ is the reduced boundary of E, and Δ denotes the symmetric difference of sets (*i.e.* $A \Delta B := (A \backslash B) \cup (B \backslash A)$).

The approximating smooth sets E_k are the superlevel sets $A_{k;t} := \{u_k > t\}$, for *a.e.* $t \in (0,1)$, of the convolutions $u_k := \chi_E * \eta_{\varepsilon_k}$, for some suitable subsequence $\varepsilon_k \to 0$ as $k \to \infty$. The main difficulty with the convolution approach is that the approximating surfaces $u_k^{-1}(t)$ do not provide an *interior* approximation in general, since portions of $u_k^{-1}(t)$ might intersect the exterior of the set. This problem was solved by Chen-Torres-Ziemer [14] and Comi-Torres [17] by improving the classical result and proving an almost *one-sided* approximation that distinguishes the superlevel sets for *a.e.* $t \in (\frac{1}{2}, 1)$ from the ones corresponding to *a.e.* $t \in (0, \frac{1}{2})$, thus providing an *interior* and an *exterior* approximation of the set with

$$\begin{aligned} &\mathscr{H}^{n-1}(u_k^{-1}(t) \cap E^0) \to 0 \qquad \text{for } a.e.\ t \in (\tfrac{1}{2}, 1), \\ &\mathscr{H}^{n-1}(u_k^{-1}(t) \cap E^1) \to 0 \qquad \text{for } a.e.\ t \in (0, \tfrac{1}{2}), \end{aligned}$$

where E^0 and E^1 are the measure-theoretic exterior and interior of the set, respectively. Moreover, for any measure $|\mu| \ll \mathscr{H}^{n-1}$, the classical result (1.1) was improved to

$$\begin{aligned} &|\mu|(A_{k;t} \Delta E^1) \to 0,\ \mathscr{H}^{n-1}(\partial A_{k;t}) \to \mathscr{H}^{n-1}(\partial^* E) && \text{for } a.e.\ t \in (\frac{1}{2}, 1), \\ &|\mu|(A_{k;t} \Delta (E^1 \cup \partial^* E)) \to 0,\ \mathscr{H}^{n-1}(\partial A_{k;t}) \to \mathscr{H}^{n-1}(\partial^* E) && \text{for } a.e.\ t \in (0, \frac{1}{2}). \end{aligned}$$

This new one-sided approximation for sets of finite perimeter is sufficient to obtain the Gauss-Green formula for vector fields $\boldsymbol{F} \in \mathcal{DM}^{\infty}_{\mathrm{loc}}$. Indeed, we have

$$|div\boldsymbol{F}| \ll \mathscr{H}^{n-1},$$

as first observed by Chen-Frid [11] (also see [14, 56]), which implies

$$\begin{aligned} &div\boldsymbol{F}(A_{k;t}) \to div\boldsymbol{F}(E^1) && \text{for a.e. } t \in (\tfrac{1}{2}, 1),\\ &div\boldsymbol{F}(A_{k;t}) \to div\boldsymbol{F}(E^1 \cup \partial^* E) && \text{for a.e. } t \in (0, \tfrac{1}{2}). \end{aligned}$$

This allows us to obtain the interior and exterior Gauss-Green formulas over sets of finite perimeters (see [14, Theorem 5.2]).

Our focus in this paper is on the Gauss-Green formulas for $\mathcal{DM}^p$ fields, *i.e.*, unbounded weakly differentiable vector fields in L^p whose distributional divergence is a Radon measure. It has been shown that, for $\boldsymbol{F} \in \mathcal{DM}^p$ with $1 \leqslant p < \infty$, the Radon measure $div\boldsymbol{F}$ is no longer absolutely continuous with respect to $\mathscr{H}^{n-1}$ in general. Indeed, it is absolutely continuous with respect to the Sobolev and relative p'–capacities if $p \geqslant \frac{n}{n-1}$, and can be even a Dirac measure if $1 \leqslant p < \frac{n}{n-1}$ (see [56, Theorem 3.2, Example 3.3], [14, Lemma 2.25], and [48, Theorem 2.8]). Thus, a new way of approximating the integration domains entirely from the interior and the exterior separately is required, since we cannot rely anymore on the approximation described above, as in [14].

A second approach to approximate a domain U is to employ the standard distance function and define

$$U^{\varepsilon} := \{x \in U : \mathrm{dist}(x, \partial U) > \varepsilon\};$$

see [57, Theorem 2.4]. In this case, since $\mathrm{dist}(x, \partial U)$ is only Lipschitz continuous for the domains with less than the C^2–regularity, the coarea formula implies that $\{x \in U : \mathrm{dist}(x, \partial U) = \varepsilon\}$ is just a set of finite perimeter, for almost every $\varepsilon > 0$; see §5. In §7, we also use a regularized distance ρ, which is C^{∞}, introduced by Lieberman [41] for the Lipschitz domains and developed further to the C^0–domains by Ball-Zarnescu [7]. For these domains, smooth approximations are obtained, since $\rho^{-1}(\varepsilon)$ is smooth for any $\varepsilon > 0$. Thus, the use of the distance functions provides an interior smooth approximation satisfying $div\boldsymbol{F}(U^{\varepsilon}) \to div\boldsymbol{F}(U)$ even for unbounded divergence-measure fields.

As for the exterior approximation, we consider the sets:

$$U_{\varepsilon} := \{x \in \mathbb{R}^n : \mathrm{dist}(x, U) < \varepsilon\},$$

which clearly satisfy similar properties as U^{ε}. Indeed, we will unify the exposition by defining the signed distance d from ∂U and its regularized version analogously in §5.

Another motivation of this paper is from a result of Schuricht [53, Theorem 5.20], where it is proved that, for any $\boldsymbol{F} \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$ and any compact set

$K \Subset \Omega$, the normal trace functional can be represented as an average on the one-sided tubular neighborhoods of ∂K in the sense that

$$div\boldsymbol{F}(K) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{K_\varepsilon \setminus K} \boldsymbol{F} \cdot \nu_K^d \, \mathrm{d}x, \tag{1.2}$$

where $K_\varepsilon = \{x \in \Omega : \mathrm{dist}(x, K) \leqslant \varepsilon\}$, and $\nu_K^d(x) = \nabla_x \mathrm{dist}(x, K)$ is a unit vector for $\mathscr{L}^n$–*a.e.* $x \in \Omega$ such that $\mathrm{dist}(x, K) > 0$. This last property says that ν_K^d is a sort of generalization of the exterior normal. It is clear that $K_\varepsilon \subset K_{\varepsilon'}$ if $\varepsilon < \varepsilon'$ and that $\bigcap_{\varepsilon>0} K_\varepsilon = K$, which implies that $div\boldsymbol{F}(K_\varepsilon) \to div\boldsymbol{F}(K)$. Therefore, this approach is similar to the one of the exterior approximation U_ε of a bounded open set U. In §5, we use this approach as a starting point by differentiating under the integral sign before passing to the limit in ε, so that we can obtain a boundary integral on the right-hand side.

The classical Gauss-Green formula for Lipschitz vector fields $\boldsymbol{F}$ over sets of finite perimeter was proved first by De Giorgi [22, 23] and Federer [29, 30], and by Burago-Maz'ya [8, 44] and Vol'pert [62, 63] for $\boldsymbol{F}$ in the class of functions of bounded variation (BV). The Gauss-Green formula for vector fields $\boldsymbol{F} \in L^\infty$ with $div\boldsymbol{F} \in \mathcal{M}$ was first investigated by Anzellotti in [4, Theorem 1.9] and [5] on bounded Lipschitz domains, and his methods were then exploited by Ambrosio-Crippa-Maniglia [1], Kawohl-Schuricht [38], Leondardi-Saracco [40], and Scheven-Schmidt [50–52]. Independently, motivated by the problems arising from the theory of hyperbolic conservation laws, Chen-Frid [11] first introduced the approach of defining the interior normal traces on the boundary of a Lipschitz deformable set as the limits of the classical normal traces over the boundaries of the interior approximations of the set, in which the Gauss-Green formulas hold. One of the main objectives of this paper is to develop this approach further for unbounded vector fields to understand the interior normal traces of divergence-measure fields on the boundary of general open sets, and to show the existence of regular Lipschitz deformations introduced in [11]. Even though, locally, we always have the natural regular Lipschitz deformation: $\Psi(\hat{y}, t) = (\hat{y}, \gamma(\hat{y}) + t)$ for γ as in Definition 2.5 and $\hat{y} = (y_1, \cdots, y_{n-1})$, it may not be possible to extend this deformation globally to ∂U in such a way to satisfy Definition 2.5 in general.

Later, the Gauss-Green formulas over sets of finite perimeter for $\mathcal{DM}^\infty$–fields were proved in Chen-Torres [13], Šilhavý [56], and Chen-Torres-Ziemer [14]. Subsequent generalizations of these formulas were given by Comi-Payne [16], Comi-Magnani [15], and Crasta-De Cicco [18, 19]. We refer [16, 20] for a more detailed exposition of the history of Gauss-Green formulas.

The case of divergence-measure vector fields in L^p, $p \neq \infty$, has been studied in Chen-Frid [12] over Lipschitz deformable boundaries and in Šilhavý [57] for open sets. The main focus of this paper is to obtain the Gauss-Green formulas by using the limit of the classical traces over appropriate approximations of the domain, instead of representing it as the averaging over neighborhoods of the boundaries of the domain as in Chen-Frid [12] and Šilhavý [57]. Even though a representation of the normal trace similar to the one in this paper

can also be found in Frid [31], it is required in [31] that the boundary of the domain is Lipschitz deformable. In §8, we show that this last condition can actually be removed.

Degiovanni-Marzocchi-Musesti in [25] and later Schuricht in [53] sought to prove the existence of normal traces under weak regularity hypotheses in order to achieve a representation formula for Cauchy fluxes, contact interactions, and forces in the context of the foundation of continuum physics. The Gauss-Green formulas obtained in [25, 53] are valid for $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for any $p \geqslant 1$, but are applicable only to sets $U \subset \Omega$ which lie in a suitable subalgebra of sets of finite perimeter related to the particular representative of $\boldsymbol{F}$. One of our objectives in this paper is to use the representation of the normal traces as the limits of classical normal traces on smooth boundaries to obtain an analogous representation for the contact interactions and the Cauchy fluxes on the boundaries of any general open set.

This paper is organized in the following way: In §2, some basic notions and facts on the BV theory and $\mathcal{DM}^p$–fields are recalled. In §3, we establish some product rules between $\mathcal{DM}^p$–fields and suitable scalar functions, including continuous bounded scalar functions with gradient in $L^{p'}$ for any $1 \leqslant p \leqslant \infty$, which has not been stated explicitly in the literature to the best of our knowledge. In §4, we investigate the distributional definition of the normal trace functional and its relation with the product rule between $\mathcal{DM}^p$–fields and characteristic functions of Borel measurable sets. We also provide some necessary and sufficient conditions under which the normal trace of a $\mathcal{DM}^p$–field can be represented by a Radon measure. In §5, we describe the properties of the level sets of the signed distance function from a closed set and their applications in the proof of the Gauss-Green formulas for general open sets. As a byproduct, we obtain generalized Green's identities and other sufficient conditions under which the normal trace of a divergence-measure field can be represented by a Radon measure on the boundary of an open set in §5–6. In §7, we show the existence of interior and exterior smooth approximations for U and $\overline{U}$ respectively, where U is a general open set, together with their corresponding Gauss-Green formulas. In the case of C^0 domains U, we employ the results of Ball-Zarnescu [7] to find smooth interior and exterior approximations of U and $\overline{U}$ in an explicit way. Indeed, we are able to write the interior and exterior normal traces as the limits of the classical normal traces on the superlevel sets of a regularized distance introduced in [7, 41]. In §8, we employ Ball-Zarnescu's theorem [7, Theorem 5.1] to show that any Lipschitz domain U is actually Lipschitz deformable in the sense of Chen-Frid (*cf.* Definition 2.5). In addition, we recall the previous approximation theory for open sets with Lipschitz boundary developed by Nečas [45, 46] and Verchota [60, 61] to give a more explicit representation of a particular bi-Lipschitz deformation $\Psi(x,t)$, which is also regular in the sense that

$$\nabla^{\partial U}\Psi(x,t) \to \mathrm{I}_{n-1} \qquad \text{as } t \to 0^+ \text{ for } \mathscr{H}^{n-1}\text{–a.e. } x \in \partial U,$$

where $\nabla^{\partial U}$ denotes the tangential gradient. Finally, in §9, based on the theory of normal traces for $\mathcal{DM}^p$–fields obtained as the limit of classical normal

traces on smooth approximations or deformations, we introduce the notion of Cauchy fluxes as functionals defined on the boundaries of general bounded open sets for the rigorous mathematical formulation of the physical principle of balance law involving discontinuities and singularities, and show that the Cauchy fluxes can be represented by corresponding divergence-measure fields.

2 Basic Notations and Divergence-Measure Fields

In this section, for self-containedness, we first present some basic notations and known facts in geometric measure theory and elementary properties of divergence-measure fields.

In what follows, Ω is an open set in $\mathbb{R}^n$, which is called a *domain* if it is also connected, and $\mathcal{M}(\Omega)$ is the space of all Radon measures in Ω. Unless otherwise stated, $\subset$ and $\subseteq$ are equivalent. We denote by $E \Subset \Omega$ a set E whose closure $\overline{E}$ is a compact set inside Ω, by $\mathring{E}$ the topological interior of E, and by ∂E its topological boundary.

To establish the interior and exterior normal traces in §5 later, we need to use the following classical coarea formula (*cf.* [27, §3.4, Theorem 1 and Proposition 3]):

Theorem 2.1 *Let $u : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz. Then*

$$\int_A |\nabla u| \, \mathrm{d}x = \int_{\mathbb{R}} \mathscr{H}^{n-1}(A \cap u^{-1}(t)) \, \mathrm{d}t \qquad \text{for any } \mathcal{L}^n\text{–measurable set } A. \tag{2.1}$$

In addition, if $\operatorname{essinf}|\nabla u| > 0$, *and $g : \mathbb{R}^n \to \mathbb{R}$ is $\mathscr{L}^n$–summable, then $g|_{u^{-1}(t)}$ is $\mathscr{H}^{n-1}$–summable for $\mathscr{L}^1$–a.e. $t \in \mathbb{R}$ and*

$$\int_{\{u>t\}} g \, \mathrm{d}x = \int_t^\infty \int_{\{u=s\}} \frac{g}{|\nabla u|} \, \mathrm{d}\mathscr{H}^{n-1} \, \mathrm{d}s \qquad \text{for any } t \in \mathbb{R}.$$

In particular, for any $t \in \mathbb{R}$ and $h \geqslant 0$ such that set $\{u = t + h\}$ is negligible with respect to the measure $g \, \mathrm{d}x$,

$$\int_{\{t<u<t+h\}} g \, \mathrm{d}x = \int_t^{t+h} \int_{\{u=s\}} \frac{g}{|\nabla u|} \, \mathrm{d}\mathscr{H}^{n-1} \, \mathrm{d}s. \tag{2.2}$$

In the case that $g : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathscr{L}^n$–summable, the same results follow for each component g_i, $i = 1, \ldots, n$.

The notions of functions of bounded variation (BV) and sets of finite perimeter will also be used.

Definition 2.1 A function $u \in L^1(\Omega)$ is a *function of bounded variation in Ω*, written as $u \in BV(\Omega)$, if its distributional gradient Du is a finite $\mathbb{R}^n$–vector valued Radon measure on Ω. We say that u is of *locally bounded variation* in Ω, written as $u \in BV_{\mathrm{loc}}(\Omega)$, if the restriction of u to every open set $U \Subset \Omega$ is in $BV(U)$. A measurable set $E \subset \Omega$ is said to be a *set of finite perimeter in Ω* if $\chi_E \in BV(\Omega)$ and said to be of *locally finite perimeter* in Ω if $\chi_E \in BV_{\mathrm{loc}}(\Omega)$.

It is well known that the topological boundary of a set of finite perimeter E can be very irregular, since it may even have positive Lebesgue measure. On the other hand, De Giorgi [23] discovered a suitable subset of ∂E of finite $\mathscr{H}^{n-1}$–measure on which $|D\chi_E|$ is concentrated.

Definition 2.2 Let E be a set of locally finite perimeter in Ω. The *reduced boundary* of E, denoted by $\partial^* E$, is defined as the set of all $x \in \operatorname{supp}(|D\chi_E|) \cap \Omega$ such that the limit:

$$\nu_E(x) := \lim_{r \to 0} \frac{D\chi_E(B(x,r))}{|D\chi_E|(B(x,r))}$$

exists in $\mathbb{R}^n$ and satisfies

$$|\nu_E(x)| = 1.$$

The function $\nu_E : \partial^* E \to \mathbb{S}^{n-1}$ is called the *measure-theoretic unit interior normal* to E.

The reason for which ν_E is seen as a generalized interior normal lies in the approximate tangential properties of the reduced boundary (*cf.* [2, Theorem 3.59]). Indeed, $E \cap B(x,\varepsilon)$ is asymptotically close to the half ball $\{y : (y-x) \cdot \nu_E(x) \geqslant 0\} \cap B(x,\varepsilon)$ as $\varepsilon \to 0$, and

$$|D\chi_E| = \mathscr{H}^{n-1} \llcorner \partial^* E. \tag{2.3}$$

It is a well-known result from the BV theory (*cf.* [2, Corollary 3.80]) that every function u of bounded variation admits a representative that is the pointwise limit $\mathscr{H}^{n-1}$–*a.e.* of any mollification of u and coincides $\mathscr{H}^{n-1}$–*a.e.* with the precise representative u^*:

$$u^*(x) := \begin{cases} \lim\limits_{r \to 0} \dfrac{1}{|B(x,r)|} \displaystyle\int_{B(x,r)} u(y)\,\mathrm{d}y & \text{if this limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $u = \chi_E$ for some set of finite perimeter E, then $\chi_E^* = \frac{1}{2}$ on $\partial^* E$ $\mathscr{H}^{n-1}$–*a.e.*

We state now the generalization of the coarea formula for functions of bounded variation, which indeed shows an important connection between BV functions and sets of finite perimeter; see [2, Theorem 3.40] for a more detailed statement and proof.

Theorem 2.2 *If $u \in BV(\Omega)$, then, for $\mathscr{L}^1$–a.e. $s \in \mathbb{R}$, set $\{u > s\}$ is of finite perimeter in Ω and*

$$|Du|(\Omega) = \int_{-\infty}^{\infty} |D\chi_{\{u>s\}}|(\Omega)\mathrm{d}s.$$

We recall now the definition of divergence-measure fields, the main object of study of this paper.

Definition 2.3 A vector field $\boldsymbol{F} \in L^p(\Omega;\mathbb{R}^n)$ for some $1 \leqslant p \leqslant \infty$ is called a *divergence-measure field*, denoted as $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$, if its distributional divergence $div\boldsymbol{F}$ is a real finite Radon measure on Ω. A vector field $\boldsymbol{F}$ is a *locally divergence-measure field*, denoted as $\boldsymbol{F} \in \mathcal{DM}^p_{\rm loc}(\Omega)$, if the restriction of $\boldsymbol{F}$ to U is in $\mathcal{DM}^p(U)$ for any $U \Subset \Omega$ open.

These vector fields have been widely studied in the last two decades; for a general theory, we refer mainly to [1, 11–14, 16, 31, 32, 53, 56, 57] and the references cited therein.

We recall that Lipschitz functions with compact support can be used as test functions in the definition of distributional divergence, since $C_c^\infty(\Omega)$ functions are dense in $\mathrm{Lip}_c(\Omega)$, the space of Lipschitz functions with compact support in Ω.

Finally, we introduce two definitions, which are required in §8, in order that the results on the smooth approximation of domains of class C^0 by Ball-Zarnescu [7] can be employed to show that the boundary of any bounded Lipschitz domain is Lipschitz deformable in the sense of Chen-Frid [11, 12].

Definition 2.4 Let $\Omega \subset \mathbb{R}^n$ be a domain of class C^0. For a point $P \in \mathbb{R}^n$, define a good direction at P, with respect to a ball $B(P,\delta)$ with $\delta > 0$ and $B(P,\delta) \cap \partial\Omega \neq \varnothing$, to be a vector $\nu \in \mathbb{S}^{n-1}$ such that there is an orthonormal coordinate system $Y = (y', y_n) = (y_1, y_2, ...y_{n-1}, y_n)$ with origin at point P so that $\nu = e_n$ is the unit vector in the y_n–direction which, together with a continuous function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ (depending on P, ν, and δ), satisfies

$$\Omega \cap B(P,\delta) = \{y \in \mathbb{R}^n : y_n > f(y'), |y| < \delta\}.$$

We say that ν is a good direction at P if it is a good direction with respect to some ball $B(P,\delta)$ with $B(P,\delta) \cap \partial\Omega \neq 0$. If $P \in \partial\Omega$, then a good direction ν at P is called a pseudonormal at P.

Definition 2.5 Let Ω be an open subset in $\mathbb{R}^n$. We say that $\partial\Omega$ is a deformable Lipschitz boundary, provided that the following hold:

(i). For each $x \in \partial\Omega$, there exist $r > 0$ and a Lipschitz mapping $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, upon rotating and relabeling the coordinate axis if necessary,

$$\Omega \cap Q(x,r) = \{y \in \mathbb{R}^n : y_n > \gamma(y_1, ..., y_{n-1})\} \cap Q(x,r),$$

where $Q(x,r) = \{y \in \mathbb{R}^n : |y_i - x_i| \leqslant r, i = 1, ..., n\}$.

(ii). There exists a map $\Psi : \partial\Omega \times [0,1] \to \overline{\Omega}$ such that Ψ is a bi-Lipschitz homeomorphism over its image and $\Psi(\cdot, 0) \equiv \mathrm{Id}$, where Id is the identity map over $\partial\Omega$. Denote $\partial\Omega_\tau = \Psi(\partial\Omega \times \{\tau\})$ for $\tau \in (0,1]$, and denote Ω_τ the open subset of Ω whose boundary is $\partial\Omega_\tau$. We call Ψ a Lipschitz deformation of $\partial\Omega$.

The Lipschitz deformation is *regular* if

$$\lim_{\tau \to 0^+} J^{\partial\Omega}\Psi_\tau = 1 \qquad \text{in } L^1(\partial\Omega; \mathscr{H}^{n-1}), \tag{2.4}$$

where $\Psi_\tau(x) = \Psi(x,\tau)$, and $J^{\partial\Omega}$ denotes the tangential Jacobian.

3 Product Rules between Divergence-Measure Fields and Suitable Scalar Functions

In this section, we give some new product rules between $\mathcal{DM}^p$–fields and suitable scalar functions. We start by proving a product rule for vector fields in $\mathcal{DM}^p$ for any $1 \leqslant p \leqslant \infty$, which is the explicit formulation of a particular case of the product rule for $\mathcal{DM}^p$–fields stated in [12, Theorem 3.2].

From now on, as customary, we always use a standard mollifier:

$$\eta \in C_c^\infty(B(0,1)) \text{ radially symmetric, with } \eta \geqslant 0 \text{ and } \textstyle\int_{B(0,1)} \eta(x)\,\mathrm{d}x = 1, \tag{3.1}$$

and

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta(\frac{x}{\varepsilon}). \tag{3.2}$$

Proposition 3.1 *If $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$, and $\phi \in C^0(\Omega) \cap L^\infty(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega;\mathbb{R}^n)$ for $p' = \frac{p}{p-1}$, then*

$$\phi\boldsymbol{F} \in \mathcal{DM}^p(\Omega),$$

and

$$div(\phi\boldsymbol{F}) = \phi\, div\boldsymbol{F} + \boldsymbol{F}\cdot\nabla\phi. \tag{3.3}$$

Proof It is clear that $\phi\boldsymbol{F} \in L^p(\Omega;\mathbb{R}^n)$.

We first consider the case $1 < p \leqslant \infty$. Take $\phi_\varepsilon := \phi * \eta_\varepsilon$, where η_ε is defined in (3.2). Then $\phi_\varepsilon \to \phi$ uniformly on compact subsets of Ω, and $\nabla\phi_\varepsilon \to \nabla\phi$ in $L^{p'}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$.

For any test function $\psi \in C_c^1(\Omega)$, we have

$$\begin{aligned}\int_\Omega \phi_\varepsilon \boldsymbol{F}\cdot\nabla\psi\,\mathrm{d}x &= \int_\Omega \boldsymbol{F}\cdot\nabla(\phi_\varepsilon\psi)\,\mathrm{d}x - \int_\Omega \psi\boldsymbol{F}\cdot\nabla\phi_\varepsilon\,\mathrm{d}x \\ &= -\int_\Omega \psi\phi_\varepsilon\,\mathrm{d}div\boldsymbol{F} - \int_\Omega \psi\boldsymbol{F}\cdot\nabla\phi_\varepsilon\,\mathrm{d}x.\end{aligned} \tag{3.4}$$

We can now pass to the limit as $\varepsilon \to 0$ to obtain (3.3) in the sense of distributions. On the other hand, it follows that

$$\left|\int_\Omega \phi\boldsymbol{F}\cdot\nabla\psi\,\mathrm{d}x\right| \leqslant \big(\|\phi\|_\infty |div\boldsymbol{F}|(\Omega) + \|\boldsymbol{F}\|_{L^p(\Omega;\mathbb{R}^n)}\|\nabla\phi\|_{L^{p'}(\Omega;\mathbb{R}^n)}\big)\|\psi\|_\infty.$$

This shows that $div(\phi\boldsymbol{F})$ is a finite Radon measure on Ω, by the density of $C_c^1(\Omega)$ in $C_c(\Omega)$ with respect to the sup norm, and that (3.3) holds in the sense of Radon measures.

For the case $p = 1$, we mollify $\boldsymbol{F}$ instead, since $\phi \in W^{1,\infty}(\Omega) \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega)$. For any $\psi \in C_c^1(\Omega)$, we obtain

$$\int_\Omega \phi\,\boldsymbol{F}_\varepsilon\cdot\nabla\psi\,\mathrm{d}x = \int_\Omega \boldsymbol{F}_\varepsilon\cdot\nabla(\phi\psi)\,\mathrm{d}x - \int_\Omega \psi\,\boldsymbol{F}_\varepsilon\cdot\nabla\phi\,\mathrm{d}x.$$

By passing to the limit as $\varepsilon \to 0$, the L^1–convergence of $\boldsymbol{F}_\varepsilon$ to $\boldsymbol{F}$ implies

$$\begin{aligned}\int_\Omega \phi\, \boldsymbol{F} \cdot \nabla\psi \,\mathrm{d}x &= \int_\Omega \boldsymbol{F} \cdot \nabla(\phi\psi)\,\mathrm{d}x - \int_\Omega \psi\, \boldsymbol{F} \cdot \nabla\phi \,\mathrm{d}x \\ &= -\int_\Omega \psi\phi \,\mathrm{d}div\boldsymbol{F} - \int_\Omega \psi\, \boldsymbol{F} \cdot \nabla\phi \,\mathrm{d}x.\end{aligned}$$

This shows (3.3) in the sense of distributions for $p = 1$. Then we can conclude by arguing as before.

Remark 3.1 Notice that, if $\phi \in L^\infty(\Omega)$ and $\nabla\phi \in L^{p'}(\Omega;\mathbb{R}^n)$, then $\phi \in W^{1,p'}_{\mathrm{loc}}(\Omega)$. Thus, if $p' > n$ (*i.e.* $1 \leqslant p < \frac{n}{n-1}$), we do not have to require that $\phi \in C^0(\Omega)$ in Proposition 3.1, since this follows by Morrey's inequality (see [27, Theorem 3, §4.5.3]).

Proposition 3.1 can be extended to the case $p = \infty$, by taking $g \in BV(\Omega) \cap L^\infty(\Omega)$. Indeed, a product rule between essentially bounded divergence-measure fields and scalar functions of bounded variation was first proved by Chen-Frid [11, Theorem 3.1] (also see [31]).

Theorem 3.1 (Chen-Frid [11]) *Let $g \in BV(\Omega)\cap L^\infty(\Omega)$ and $\boldsymbol{F} \in \mathcal{DM}^\infty(\Omega)$. Then $g\boldsymbol{F} \in \mathcal{DM}^\infty(\Omega)$ and*

$$\mathrm{div}(g\boldsymbol{F}) = g^*\mathrm{div}\boldsymbol{F} + \overline{\boldsymbol{F}\cdot Dg} \tag{3.5}$$

in the sense of Radon measures on Ω, where g^* is the precise representative of g, and $\overline{\boldsymbol{F}\cdot Dg}$ is a Radon measure, which is the weak-star limit of $\boldsymbol{F}\cdot\nabla g_\varepsilon$ for the mollification $g_\varepsilon := g * \eta_\varepsilon$, and is absolutely continuous with respect to $|Dg|$. In addition,

$$|\overline{\boldsymbol{F}\cdot Dg}| \leqslant \|\boldsymbol{F}\|_{L^\infty(\Omega;\mathbb{R}^n)}|Dg|.$$

One could ask whether it would be possible to obtain a similar result also for $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$, $1 \leqslant p < \infty$, by imposing some other assumptions on $\boldsymbol{F}$ weaker than the essential boundedness. It is obvious that $g\boldsymbol{F} \in L^p(\Omega)$ and that $div(g\boldsymbol{F})$ is a distribution of order 1, by definition; hence one should look for conditions under which it can be extended to a linear continuous functional on $C_c(\Omega)$.

Our investigation is motivated by the following example, where g is the characteristic function of a set of finite perimeter E, and $\boldsymbol{F}$ is a vector field in $\mathcal{DM}^p_{\mathrm{loc}}, 1 \leqslant p < 2$, which is unbounded on $\partial^* E$.

Example 3.1 Let $n = 2$, $g = \chi_{(0,1)^2}$, and

$$\boldsymbol{F}(x_1,x_2) = \frac{1}{2\pi}\frac{(x_1,x_2)}{x_1^2+x_2^2},$$

which implies that $div\boldsymbol{F} = \delta_{(0,0)}$. Then $g\boldsymbol{F} \in \mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}^2)$ for any $1 \leqslant p < 2$ with

$$div(g\boldsymbol{F}) = \frac{1}{4}\delta_{(0,0)} + (\boldsymbol{F}, Dg), \tag{3.6}$$

where

$$(\boldsymbol{F}, Dg)(\phi) := -\frac{1}{2\pi}\left(\int_0^1 \frac{\phi(x_1,1)}{1+x_1^2}\,\mathrm{d}x_1 + \int_0^1 \frac{\phi(1,x_2)}{1+x_2^2}\,\mathrm{d}x_2\right) \text{ for any } \phi \in C_c(\mathbb{R}^2). \tag{3.7}$$

This pairing functional can also be regarded as a principal value in the sense that

$$(\boldsymbol{F}, Dg)(\phi) = \lim_{\varepsilon\to 0}\int_{\partial(0,1)^2\setminus B(0,\varepsilon)} \phi\,\boldsymbol{F}\cdot\nu_{(0,1)^2}\,\mathrm{d}\mathscr{H}^1,$$

so that $|(\boldsymbol{F}, Dg)| \ll |Dg| = \mathscr{H}^1 \llcorner \partial^*(0,1)^2$. Moreover, the term, $\frac{1}{4}\delta_{(0,0)}$, comes from the fact that $g^*(0) = \frac{1}{4}$ (*i.e.* the value of the precise representative of g at the origin).

In order to prove these claims, we take $\phi \in C_c^1(\mathbb{R}^2)$ to see

$$\begin{aligned}
&\int_{\mathbb{R}^2} g\boldsymbol{F}\cdot\nabla\phi\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_{(0,1)^2\setminus B(0,\varepsilon)} \boldsymbol{F}\cdot\nabla\phi\,\mathrm{d}x\\
&= \lim_{\varepsilon\to 0}\Bigg(-\int_{\partial(0,1)^2\setminus B(0,\varepsilon)} \phi\boldsymbol{F}\cdot\nu_{(0,1)^2}\,\mathrm{d}\mathscr{H}^1\\
&\qquad -\int_{\partial B(0,\varepsilon)\cap\{x_1>0,x_2>0\}} \phi(x_1,x_2)\frac{1}{2\pi}\frac{(x_1,x_2)}{x_1^2+x_2^2}\cdot\frac{(x_1,x_2)}{\sqrt{x_1^2+x_2^2}}\,\mathrm{d}\mathscr{H}^1\Bigg)\\
&= -\frac{1}{2\pi}\lim_{\varepsilon\to 0}\Bigg(\int_\varepsilon^1 \phi(x_1,0)\frac{(x_1,0)}{x_1^2}\cdot(0,1)\,\mathrm{d}x_1 + \int_\varepsilon^1 \phi(0,x_2)\frac{(0,x_2)}{x_2^2}\cdot(1,0)\,\mathrm{d}x_2\\
&\qquad + \int_0^1 \phi(1,x_2)\frac{(1,x_2)}{1+x_2^2}\cdot(-1,0)\,\mathrm{d}x_2 + \int_0^1 \phi(x_1,1)\frac{(x_1,1)}{x_1^2+1}\cdot(0,-1)\,\mathrm{d}x_1\\
&\qquad + \int_0^{\frac{\pi}{2}} \phi(\varepsilon\cos\theta,\varepsilon\sin\theta)\,\mathrm{d}\theta\Bigg)\\
&= \frac{1}{2\pi}\left(\int_0^1 \frac{\phi(x_1,1)}{1+x_1^2}\,\mathrm{d}x_1 + \int_0^1 \frac{\phi(1,x_2)}{1+x_2^2}\,dx_2\right) - \frac{1}{4}\phi(0,0).
\end{aligned}$$

This shows that $div(g\boldsymbol{F})$ is a distribution of order 0, so that it is a measure, since it can be uniquely extended to a functional on $C_c(\mathbb{R}^2)$ by density.

In addition, for any $\phi \in C^0([0,1]^2)$ with $\nabla\phi \in L^{p'}((0,1)^2)$ for some $p \in [1,2)$, the following integration by parts formula holds:

$$\begin{aligned}
&\int_{(0,1)^2} \frac{(x_1,x_2)}{x_1^2+x_2^2}\cdot\nabla\phi(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 + \frac{\pi}{2}\phi(0,0)\\
&= \int_0^1 \frac{\phi(x_1,1)}{1+x_1^2}\,\mathrm{d}x_1 + \int_0^1 \frac{\phi(1,x_2)}{1+x_2^2}\,\mathrm{d}x_2.
\end{aligned} \tag{3.8}$$

Indeed, since $\chi_{(0,1)^2}\boldsymbol{F} \in \mathcal{DM}^p(\mathbb{R}^2)$ for any $p \in [1,2)$, then (3.3) yields

$$div(\phi\chi_{(0,1)^2}\boldsymbol{F}) = \phi\, div(\chi_{(0,1)^2}\boldsymbol{F}) + \chi_{(0,1)^2}\boldsymbol{F}\cdot\nabla\phi,$$

which, by (3.6), implies

$$div(\phi\chi_{(0,1)^2}\boldsymbol{F}) = \frac{1}{4}\phi(0,0)\delta_{(0,0)} + \phi(\boldsymbol{F}, D\chi_{(0,1)^2}) + \chi_{(0,1)^2}\boldsymbol{F}\cdot\nabla\phi. \tag{3.9}$$

Finally, (3.8) follows by evaluating (3.9) over $\mathbb{R}^2$, using the fact that $\phi\chi_{(0,1)^2}\boldsymbol{F}$ has compact support to obtain $div(\phi\chi_{(0,1)^2}\boldsymbol{F})(\mathbb{R}^2) = 0$ by [16, Lemma 3.1], and employing (3.7).

In this example, the cancellations between $\boldsymbol{F}$ and $\nu_{(0,1)^2}$ play a crucial role in order to ensure the existence of a measure given by the pairing of $\boldsymbol{F}$ and Dg.

Indeed, we can impose the existence of such a measure in order to achieve a more general product rule.

Theorem 3.2 *Let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$, and let $g \in L^\infty(\Omega)\cap BV(\Omega)$. Assume that there exists a measure $(\boldsymbol{F}, Dg) \in \mathcal{M}(\Omega)$ such that $\boldsymbol{F}_\varepsilon \cdot Dg \rightharpoonup (\boldsymbol{F}, Dg)$ for any mollification $\boldsymbol{F}_\varepsilon$ of $\boldsymbol{F}$. Then*

$$g\boldsymbol{F} \in \mathcal{DM}^p(\Omega),$$

and

$$div(g\boldsymbol{F}) = \widetilde{g}\,div\boldsymbol{F} + (\boldsymbol{F}, Dg), \tag{3.10}$$

where $\widetilde{g} \in L^\infty(\Omega, |div\boldsymbol{F}|)$ is the weak–limit of a suitable subsequence of mollified functions g_ε of g, which satisfies $\widetilde{g}(x) = g^*(x)$ whenever g^* is well defined. In addition,*

$$|(\boldsymbol{F}, Dg)| \ll \mathscr{H}^{n-1} \qquad \text{if } p = \infty,$$

and, if $p \in \left[\frac{n}{n-1}, \infty\right)$,

$$|(\boldsymbol{F}, Dg)|(B) = 0 \qquad \text{for any Borel set } B \text{ with } \sigma\text{–finite } \mathscr{H}^{n-p'} \text{ measure.}$$

Proof It is clear that $g\boldsymbol{F} \in L^p(\Omega;\mathbb{R}^n)$. We now divide the remaining proof into two steps.

1. In order to show (3.10), we take any mollification $g_\varepsilon = g * \eta_\varepsilon$ with η_ε defined in (3.2). Then we select $\phi \in \mathrm{Lip}_c(\Omega)$ to obtain

$$\int_\Omega g_\varepsilon\boldsymbol{F}\cdot\nabla\phi\,\mathrm{d}x = -\int_\Omega \phi g_\varepsilon\,\mathrm{d}div\boldsymbol{F} - \int_\Omega \phi\boldsymbol{F}\cdot\nabla g_\varepsilon\,\mathrm{d}x. \tag{3.11}$$

Since $g_\varepsilon \to g$ in $L^{p'}_{\mathrm{loc}}(\Omega)$, we have

$$\int_\Omega g_\varepsilon\boldsymbol{F}\cdot\nabla\phi\,\mathrm{d}x \to \int_\Omega g\boldsymbol{F}\cdot\nabla\phi\,\mathrm{d}x \qquad \text{as } \varepsilon \to 0. \tag{3.12}$$

Notice that $|g_\varepsilon(x)| \leqslant \|g\|_{L^\infty(\Omega)}$ for any $x \in \Omega$. Then there exists a weak*–limit $\widetilde{g} \in L^\infty(\Omega, |div\boldsymbol{F}|)$ for a suitable subsequence $\{g_{\varepsilon_k}\}$ so that $\widetilde{g}$ coincides with the precise representative g^* whenever this is well defined. Therefore, we obtain

$$\int_\Omega \phi g_\varepsilon\,\mathrm{d}div\boldsymbol{F} \to \int_\Omega \phi\widetilde{g}\,\mathrm{d}div\boldsymbol{F} \tag{3.13}$$

up to a subsequence. As for the last term, we have

$$\int_\Omega \phi(x)\boldsymbol{F}(x)\cdot\nabla g_\varepsilon(x)\,\mathrm{d}x = \int_\Omega (\phi\boldsymbol{F})_\varepsilon(y)\cdot\,\mathrm{d}Dg(y). \tag{3.14}$$

By the uniform continuity of ϕ, for any $\delta>0$ and $x\in\Omega$, there exists $\varepsilon_0>0$ such that $|\phi(y)-\phi(x)|<\eta$ for any $y\in B(x,\varepsilon)$ and $\varepsilon\in(0,\varepsilon_0)$. Since ϕ has compact support in Ω, we can also assume that $B(x,\varepsilon)\subset\Omega$ without loss of generality. This implies

$$\begin{aligned}\big|(\phi\boldsymbol{F})_\varepsilon(x)-\phi(x)\boldsymbol{F}_\varepsilon(x)\big| &= \left|\int_\Omega \big(\phi(y)-\phi(x)\big)\boldsymbol{F}(y)\eta_\varepsilon(x-y)\,\mathrm{d}y\right|\\ &\leqslant \delta\int_{B(0,1)}|\boldsymbol{F}(x+\varepsilon z)|\eta(z)\,\mathrm{d}z\\ &\leqslant \delta\|\eta\|_{L^{p'}(B(0,1))}\|\boldsymbol{F}\|_{L^p(\Omega;\mathbb{R}^n)}.\end{aligned}$$

Hence, it follows that

$$\int_\Omega (\phi\boldsymbol{F})_\varepsilon(y)\cdot\,\mathrm{d}Dg(y) = \int_\Omega \phi(y)\,\boldsymbol{F}_\varepsilon(y)\cdot\,\mathrm{d}Dg(y)+o_\varepsilon(1). \tag{3.15}$$

Now we use our assumption on sequence $\boldsymbol{F}_\varepsilon$ to obtain

$$\int_\Omega \phi(y)\,\boldsymbol{F}_\varepsilon(y)\cdot\,\mathrm{d}Dg(y)\ \to\int_\Omega \phi(y)\,\mathrm{d}(\boldsymbol{F},Dg)(y). \tag{3.16}$$

Combining (3.11)–(3.16), we conclude that $\widetilde{g}$ is actually unique and that (3.10) holds. In particular, we see that $div(g\boldsymbol{F})\in\mathcal{M}(\Omega)$, which implies that $g\boldsymbol{F}\in\mathcal{DM}^p(\Omega)$.

2. As for the absolute continuity property of $(\boldsymbol{F},Dg)$, we notice that

$$(\boldsymbol{F},Dg) = div(g\boldsymbol{F})-\widetilde{g}\,div\boldsymbol{F}$$

and $\boldsymbol{F},g\boldsymbol{F}\in\mathcal{DM}^p(\Omega)$. We recall now that $|div\boldsymbol{F}|+|div(g\boldsymbol{F})|\ll\mathscr{H}^{n-1}$ if $p=\infty$ (see [11, Proposition 3.1] and [56, Theorem 3.2]) and that, if $p\in[\frac{n}{n-1},\infty)$, $|div\boldsymbol{F}|(B)=|div(g\boldsymbol{F})|(B)=0$ for any Borel set B with σ–finite $\mathscr{H}^{n-p'}$ measure, by [56, Theorem 3.2]. This concludes the proof.

It seems to be delicate to characterize the cases in which measure $(\boldsymbol{F},Dg)$ does exist and the absolute continuity, *i.e.* $|(\boldsymbol{F},Dg)|\ll|Dg|$ holds as in Example 3.1. We give here a partial result.

Corollary 3.1 *Let $\boldsymbol{F}\in\mathcal{DM}^p(\Omega)$ for $1\leqslant p\leqslant\infty$, and let $g\in L^\infty(\Omega)\cap BV(\Omega)$. Assume that there exists $\widetilde{\boldsymbol{F}}\in L^\infty_{\mathrm{loc}}(\Omega,|Dg|;\mathbb{R}^n)$ such that $\boldsymbol{F}_\varepsilon\overset{*}{\rightharpoonup}\widetilde{\boldsymbol{F}}$ in $L^\infty_{\mathrm{loc}}(\Omega,|Dg|;\mathbb{R}^n)$, where $\boldsymbol{F}_\varepsilon=\boldsymbol{F}*\eta_\varepsilon$ is the mollification of $\boldsymbol{F}$. Then $g\boldsymbol{F}\in\mathcal{DM}^p_{\mathrm{loc}}(\Omega)$ and*

$$div(g\boldsymbol{F}) = \widetilde{g}\,div\boldsymbol{F}+\widetilde{\boldsymbol{F}}\cdot Dg,$$

where $\widetilde{g}\in L^\infty(\Omega;|div\boldsymbol{F}|)$ is the weak–limit of a subsequence of g_ε so that $\widetilde{g}(x)=g^*(x)$, whenever g^* is well defined. In addition, for any open set $U\Subset\Omega$,*

$$|\widetilde{\boldsymbol{F}}\cdot Dg|\llcorner U \leqslant \inf_{\substack{U\Subset U'\Subset\Omega\\ U'\text{ open}}}\|\boldsymbol{F}\|_{L^\infty(U';\mathbb{R}^n)}\,|Dg|\llcorner U. \tag{3.17}$$

Proof The first part of the result follows directly from Theorem 3.2, since the assumptions imply that $\boldsymbol{F}_\varepsilon \cdot Dg \rightharpoonup (\boldsymbol{F}, Dg) = \widetilde{\boldsymbol{F}} \cdot Dg$. Moreover, since $\widetilde{\boldsymbol{F}} \in L^\infty_{\mathrm{loc}}(\Omega, |Dg|; \mathbb{R}^n)$, we have

$$|\widetilde{\boldsymbol{F}} \cdot Dg| \leqslant \|\widetilde{\boldsymbol{F}}\|_{L^\infty(U,|Dg|;\mathbb{R}^n)}|Dg| \qquad \text{on any open set } U \Subset \Omega.$$

Finally, since $|\boldsymbol{F}_\varepsilon(x)| \leqslant \|\boldsymbol{F}\|_{L^\infty(U+B(0,\varepsilon);\mathbb{R}^n)}$ for any $x \in U$, then, for any open set U' satisfying $U \Subset U' \Subset \Omega$, the lower semicontinuity of the L^∞–norm with respect to the weak*–convergence implies

$$\|\widetilde{\boldsymbol{F}}\|_{L^\infty(U,|Dg|;\mathbb{R}^n)} \leqslant \liminf_{\varepsilon\to 0} \sup_{x\in U} |\boldsymbol{F}_\varepsilon(x)| \leqslant \|\boldsymbol{F}\|_{L^\infty(U';\mathbb{R}^n)}.$$

By taking the infimum over U', we obtain (3.17).

Remark 3.2 The assumptions on $\boldsymbol{F}$ are satisfied in the case $\boldsymbol{F} \in C^0(\Omega; \mathbb{R}^n)$, for which $\widetilde{\boldsymbol{F}} = \boldsymbol{F}$.

If $\boldsymbol{F} \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$, then, for any open set $\Omega' \Subset \Omega$,

$$|\boldsymbol{F}_\varepsilon(x)| \leqslant \|\boldsymbol{F}\|_{L^\infty(\Omega'+B(0,\varepsilon);\mathbb{R}^n)} \qquad \text{for any } x \in \Omega'.$$

Thus, by weak*–compactness, there exists $\widetilde{\boldsymbol{F}} \in L^\infty_{\mathrm{loc}}(\Omega, |Dg|; \mathbb{R}^n)$ such that $\boldsymbol{F}_\varepsilon \overset{*}{\rightharpoonup} \widetilde{\boldsymbol{F}}$ in $L^\infty_{\mathrm{loc}}(\Omega, |Dg|; \mathbb{R}^n)$, up to a subsequence. This implies the result of Corollary 3.1 again. Moreover, since $|div\boldsymbol{F}| \ll \mathscr{H}^{n-1}$, by [56, Theorem 3.2], we can conclude

$$\widetilde{g} = g^* \qquad |div\boldsymbol{F}|\text{–}a.e.$$

since the precise representative of a BV function g exists $\mathscr{H}^{n-1}$–$a.e.$ In addition, by the product rule established in Theorem 3.1, we obtain the identity:

$$\widetilde{\boldsymbol{F}} \cdot Dg = \overline{\boldsymbol{F} \cdot Dg}.$$

Thus, if ν_g is the Borel vector field such that $Dg = \nu_g|Dg|$, then

$$\overline{\boldsymbol{F} \cdot Dg} = (\widetilde{\boldsymbol{F}} \cdot \nu_g)|Dg|.$$

That is, $\widetilde{\boldsymbol{F}} \cdot \nu_g$ is the density of measure $\overline{\boldsymbol{F} \cdot Dg}$ with respect to $|Dg|$.

Finally, the assumption that $\boldsymbol{F} \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ can be relaxed to $\boldsymbol{F} \in L^\infty_{\mathrm{loc}}(U; \mathbb{R}^n)$, for some open set $U \supset \mathrm{supp}(|Dg|)$. Indeed, this implies that $\boldsymbol{F}_\varepsilon$ is uniformly bounded in $L^\infty(U', |Dg|; \mathbb{R}^n)$, for any open set $U' \Subset U$ and ε small enough, which ensures the existence of a weak*–limit $\widetilde{\boldsymbol{F}} \in L^\infty_{\mathrm{loc}}(U, |Dg|; \mathbb{R}^n)$, up to a subsequence.

On the other hand, we have seen that there are some examples of unbounded and discontinuous $\mathcal{DM}^p$–fields which admit a product rule of this type, as in Example 3.1. Moreover, there exists an unbounded $\mathcal{DM}^p$–field $\boldsymbol{G}$ and a set of finite perimeter E for which a product rule holds, but $|(\boldsymbol{G}, D\chi_E)|$ is not absolutely continuous with respect to $|D\chi_E|$, as shown in the following example.

Example 3.2 Let $n = 2$, $E = (0,1)^2$, and $\boldsymbol{F}$ as in Example 3.1. We have shown that $\chi_E \boldsymbol{F} \in \mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}^2)$ for any $p \in [1,2)$ and that

$$div(\chi_E \boldsymbol{F}) = \frac{1}{4}\delta_{(0,0)} + (\boldsymbol{F}, D\chi_E), \tag{3.18}$$

by (3.6). Let now $\boldsymbol{G} := \chi_E \boldsymbol{F}$. It is clear that $\chi_E \boldsymbol{G} = \boldsymbol{G}$ so that $div(\chi_E \boldsymbol{G}) \in \mathcal{M}(\mathbb{R}^2)$. Let $\eta_\varepsilon(x)$ be the mollifiers defined in (3.1)–(3.2), and let $\phi \in C^1_c(\mathbb{R}^2)$. A simple calculation shows that

$$\int_{\mathbb{R}^2} (\eta_\varepsilon * \chi_E)\boldsymbol{G} \cdot \nabla\phi \,\mathrm{d}x = -\int_{\mathbb{R}^2} \phi(\eta_\varepsilon * \chi_E)\,\mathrm{d}\,div\,\boldsymbol{G} - \int_{\mathbb{R}^2} \phi \boldsymbol{G} \cdot \nabla(\eta_\varepsilon * \chi_E)\,\mathrm{d}x.$$

By Lebesgue's dominated convergence theorem, we have

$$\int_{\mathbb{R}^2} (\eta_\varepsilon * \chi_E)\boldsymbol{G} \cdot \nabla\phi \,\mathrm{d}x \to \int_{\mathbb{R}^2} \chi_E \boldsymbol{G} \cdot \nabla\phi \,\mathrm{d}x = -\int_{\mathbb{R}^2} \phi \,\mathrm{d}\,div(\chi_E \boldsymbol{G}),$$

and

$$\begin{aligned}\int_{\mathbb{R}^2} \phi(\eta_\varepsilon * \chi_E)\,\mathrm{d}\,div\,\boldsymbol{G} &= \int_{\mathbb{R}^2} \phi(\eta_\varepsilon * \chi_E)\,\mathrm{d}\big(\frac{1}{4}\delta_{(0,0)} + (\boldsymbol{F}, D\chi_E)\big) \\ &\to \frac{1}{16}\phi(0,0) + \int_{\partial^* E} \frac{1}{2}\phi\,\mathrm{d}(\boldsymbol{F}, D\chi_E),\end{aligned}$$

since $|(\boldsymbol{F}, D\chi_E)| \ll |D\chi_E|$ and $\chi^*_E(0,0) = \frac{1}{4}$. This and the density of $C^1_c(\mathbb{R}^2)$ in $C^0_c(\mathbb{R}^2)$ show that $\boldsymbol{G} \cdot \nabla(\eta_\varepsilon * \chi_E)$ is weakly converging to some measure $(\boldsymbol{G}, D\chi_E)$ that satisfies

$$div(\chi_E \boldsymbol{G}) = \frac{1}{16}\delta_{(0,0)} + \frac{1}{2}(\boldsymbol{F}, D\chi_E) + (\boldsymbol{G}, D\chi_E). \tag{3.19}$$

However, it is clear that $div(\chi_E \boldsymbol{G}) = div\,\boldsymbol{G} = div(\chi_E \boldsymbol{F})$. Therefore, (3.18)–(3.19) imply

$$(\boldsymbol{G}, D\chi_E) = \frac{3}{16}\delta_{(0,0)} + \frac{1}{2}(\boldsymbol{F}, D\chi_E). \tag{3.20}$$

Therefore, $|(\boldsymbol{G}, D\chi_E)| \ll |D\chi_E| = \mathscr{H}^1 \llcorner \partial^* E$ does not hold, since there is a concentration at $(0,0)$.

4 Regularity of Normal Traces of Divergence-Measure Fields

In this section, we investigate the connection between these product rules and the representation of the normal trace of the $\mathcal{DM}^p$–field as a Radon measure.

We first introduce the notion of generalized normal traces of a $\mathcal{DM}^p$–field $\boldsymbol{F}$ on the boundary of a Borel set E, which has indeed a close relation with the product rule between $\boldsymbol{F}$ and χ_E.

Definition 4.1 Given $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$, and a bounded Borel set $E \subset \Omega$, define the normal trace of $\boldsymbol{F}$ on ∂E as

$$\langle \boldsymbol{F} \cdot \nu, \phi \rangle_{\partial E} := \int_E \phi \,\mathrm{d}\,div\boldsymbol{F} + \int_E \boldsymbol{F} \cdot \nabla\phi \,\mathrm{d}x \qquad \text{for any } \phi \in \mathrm{Lip}_c(\mathbb{R}^n). \tag{4.1}$$

Remark 4.1 Since $div\boldsymbol{F}$ is a Radon measure, any Borel set E is $|div\boldsymbol{F}|$–measurable. Moreover, for any $|div\boldsymbol{F}|$–measurable set E, there is a Borel set $B \supset E$ such that $|div\boldsymbol{F}|(B\backslash E) = 0$, so that there exists a $|div\boldsymbol{F}|$–negligible set $\mathcal{N}_E$ with $\mathcal{N}_E = B\backslash E$. Therefore, if $\mathcal{N}_E$ is Lebesgue measurable, then E is admissible for the definition of normal traces.

Furthermore, by the definition, the normal trace of $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ on the boundary of a bounded Borel set $E \subset \Omega$ is a distribution of order 1 on $\mathbb{R}^n$, since

$$|\langle \boldsymbol{F}\cdot\nu, \phi\rangle_{\partial E}| \leqslant \|\phi\|_{L^\infty(\mathbb{R}^n)}|div\boldsymbol{F}|(E) + \|\nabla\phi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)}|E|^{1-\frac{1}{p}}\|\boldsymbol{F}\|_{L^p(E;\mathbb{R}^n)}$$

for any $\phi \in C^1_c(\mathbb{R}^n)$. Moreover, the normal trace is not stable *a priori* under the modifications of E by Lebesgue negligible sets. Indeed, if $\widetilde{E}$ is any Borel set such that $|E\Delta\widetilde{E}| = 0$, then, unless $|div\boldsymbol{F}| \ll \mathscr{L}^n$, we may obtain that $|div\boldsymbol{F}|(E\Delta\widetilde{E}) \neq 0$, even though the second terms in (4.1) are equal.

Therefore, the normal trace depends on the particular Borel representative of set E, not even only on ∂E. Indeed, if $U \subset \Omega$ is an open set with smooth boundary, then $\partial U = \partial\overline{U}$; however, when $|div\boldsymbol{F}|(\partial U) \neq 0$, the normal traces of $\boldsymbol{F}$ on the boundary of U and $\overline{U}$ are different in general.

Remark 4.2 By the definition of normal traces, we have

$$\langle \boldsymbol{F}\cdot\nu, \phi\rangle_{\partial E} = div(\phi\boldsymbol{F})(E).$$

Therefore, Theorem 3.1 implies that, if $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$, functional $\langle \boldsymbol{F}\cdot\nu, \cdot\rangle_{\partial E}$ can be extended to the space of test functions $\phi \in C^0(\Omega) \cap L^\infty(\Omega)$ such that $\nabla\phi \in L^{p'}(\Omega;\mathbb{R}^n)$. Under such conditions, we can also take any Borel set $E \subset \Omega$, since

$$|\langle \boldsymbol{F}\cdot\nu, \phi\rangle_{\partial E}| \leqslant \|\phi\|_{L^\infty(\Omega)}|div\boldsymbol{F}|(E) + \|\nabla\phi\|_{L^{p'}(\Omega;\mathbb{R}^n)}\|\boldsymbol{F}\|_{L^p(E;\mathbb{R}^n)}.$$

Therefore, if $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$, and E is a Borel set in Ω, then the normal trace $\langle \boldsymbol{F}\cdot\nu, \cdot\rangle_{\partial E}$ can be extended to a functional in the dual of

$$\{\phi \in C^0(\Omega) \cap L^\infty(\Omega) \,:\, \nabla\phi \in L^{p'}(\Omega;\mathbb{R}^n)\}.$$

Proposition 4.1 *Let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$. Then the normal trace of $\boldsymbol{F}$ on the boundary of a bounded Borel set $E \subset \Omega$ is a distribution of order* 1 *supported on ∂E.*

Proof Let $V \Subset \Omega\backslash\partial E$ and $\phi \in C^1_c(V)$. We need to show that $\langle \boldsymbol{F}\cdot\nu, \phi\rangle_{\partial E} = 0$.

Since $\phi\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ and $\mathrm{supp}(\phi\boldsymbol{F}) \subset V$, then $\mathrm{supp}(div(\phi\boldsymbol{F})) \subset V$. From this, it follows that

$$\langle \boldsymbol{F}\cdot\nu, \phi\rangle_{\partial E} = div(\phi\boldsymbol{F})(E) = div(\phi\boldsymbol{F})(V \cap \mathring{E}).$$

We may assume that $\mathring{E} \neq \varnothing$ (otherwise, there is nothing to prove) and $V \subset \mathring{E}$, without loss of generality. Then Lemma 3.1 of [16] implies that $div(\phi\boldsymbol{F})(V) = 0$ so that $\langle \boldsymbol{F}\cdot\nu, \phi\rangle_{\partial E} = div(\phi\boldsymbol{F})(V) = 0$.

Remark 4.3 Given $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$, then, for any Borel set E in Ω, the following locality property for the normal trace functional holds:

$$\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial E} = -\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial(\Omega \setminus E)}$$

in the sense of distributions on Ω.

Indeed, given any $\phi \in C_c^1(\Omega)$, $\phi \boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ by Proposition 3.1, and

$$\int_\Omega \phi \, \mathrm{d} div \boldsymbol{F} + \int_\Omega \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d}x = div(\phi \boldsymbol{F})(\Omega) = 0$$

by [16, Lemma 3.1], since $\mathrm{supp}(\phi \boldsymbol{F})$ is compact in Ω. Then

$$\int_E \phi \, \mathrm{d} div \boldsymbol{F} + \int_E \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d}x = -\int_{\Omega \setminus E} \phi \, \mathrm{d} div \boldsymbol{F} - \int_{\Omega \setminus E} \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d}x.$$

Theorem 4.1 *Let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$, and let $E \subset \Omega$ be a bounded Borel set. Then*

$$\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial E} = \chi_E div \boldsymbol{F} - div(\chi_E \boldsymbol{F}) \tag{4.2}$$

in the sense of distributions on Ω. Thus, $\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial E} \in \mathcal{M}(\partial E)$ if and only if $div(\chi_E \boldsymbol{F}) \in \mathcal{M}(\Omega)$; that is, $\chi_E \boldsymbol{F} \in \mathcal{DM}^p(\Omega)$. In addition, if $\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial E}$ is a measure, then

(i) $|\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial E}| \ll \mathscr{H}^{n-1} \llcorner \partial E$, *if* $p = \infty$;
(ii) $|\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial E}|(B) = 0$ *for any Borel set* $B \subset \partial E$ *with* σ*–finite* $\mathscr{H}^{n-p'}$ *measure, if* $\frac{n}{n-1} \leqslant p < \infty$.

Proof By Proposition 4.1, the support of distribution $\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial E}$ is ∂E. As for the equivalence, we notice that

$$\langle \boldsymbol{F} \cdot \nu, \phi \rangle_{\partial E} - \int_E \phi \, \mathrm{d} div \boldsymbol{F} = \int_E \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d}x = \int_\Omega \chi_E \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d}x$$

for any $\phi \in \mathrm{Lip}_c(\Omega)$. This implies (4.2) in the sense of distributions. Since $div \boldsymbol{F} \in \mathcal{M}(\Omega)$, it follows that $\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial E} \in \mathcal{M}(\partial E)$ if and only if $div(\chi_E \boldsymbol{F}) \in \mathcal{M}(\Omega)$, by the density of $\mathrm{Lip}_c(\Omega)$ in $C_c(\Omega)$ with respect to the supremum norm. Since $\chi_E \boldsymbol{F} \in L^p(\Omega; \mathbb{R}^n)$, then $div(\chi_E \boldsymbol{F}) \in \mathcal{M}(\Omega)$ implies that $\chi_E \boldsymbol{F} \in \mathcal{DM}^p(\Omega)$. As for the absolute continuity properties of the normal trace measure, we argue as those in the end of the proof of Theorem 3.2, by employing (4.2) and [56, Theorem 3.2].

We now employ (4.2) to show the relation between $\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial E}$ and $\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial \widetilde{E}}$ for any another Borel representative $\widetilde{E}$, with respect to the Lebesgue measure, of a given bounded Borel set E.

Proposition 4.2 *Let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$, and let $E, \widetilde{E} \subset \Omega$ be bounded Borel sets such that $|E \Delta \widetilde{E}| = 0$. Then*

$$\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial E} - \langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial \widetilde{E}} = (\chi_{E \setminus \widetilde{E}} - \chi_{\widetilde{E} \setminus E}) div \boldsymbol{F}, \tag{4.3}$$

which means that $\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial E} - \langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial\widetilde{E}} \in \mathcal{M}(\Omega)$, *and*

$$|\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial E} - \langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial\widetilde{E}}| = \chi_{E\Delta\widetilde{E}}|div\boldsymbol{F}|. \tag{4.4}$$

In particular, if U *is an open bounded set in* Ω *with* $|\partial U| = 0$, *then*

$$\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial\overline{U}} - \langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial U} = \chi_{\partial U}\, div\boldsymbol{F}. \tag{4.5}$$

Proof Since $|E\Delta\widetilde{E}| = 0$, $div(\chi_E\boldsymbol{F}) = div(\chi_{\widetilde{E}}\boldsymbol{F})$ in the sense of distributions. Thus, by subtracting (4.2) for $\widetilde{E}$ from the same identity with E, we obtain (4.3). Then we see that $\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial E} - \langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial\widetilde{E}} \in \mathcal{M}(\Omega)$ and (4.4). Finally, if U is open bounded set with $|\partial U| = 0$, (4.5) follows from (4.3) with $E = \overline{U}$ and $\widetilde{E} = U$.

Remark 4.4 While $div(\chi_E\boldsymbol{F})$ is not a Radon measure in general, we can employ (4.2) to obtain some information on its restriction to some particular sets. Indeed, since $\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial E}$ is supported on ∂E, by Proposition 4.1, it suffices to restrict (4.2) to ∂E and $\mathring{E}$ to obtain

$$\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial E} = \chi_{E\cap\partial E} div\boldsymbol{F} - div(\chi_E\boldsymbol{F})\llcorner\partial E, \quad div\boldsymbol{F}\llcorner\mathring{E} - div(\chi_E\boldsymbol{F})\llcorner\mathring{E} = 0.$$

In particular, this means that $div(\chi_E\boldsymbol{F})\llcorner\mathring{E} = div\boldsymbol{F}\llcorner\mathring{E}$, so that this restriction is a Radon measure for any $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ and bounded Borel set E in Ω. In addition, if U is an open bounded set in Ω, then

$$\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial U} = -div(\chi_U\boldsymbol{F})\llcorner\partial U.$$

We now state a particular result concerning the sets of finite perimeter and the case $p = \infty$, which gathers much of the known theory (see [14, 16]). It also provides a generalization of the Gauss-Green formulas by allowing for scalar functions $\phi \in C^0(\Omega)$ with $\nabla\phi \in L^1_{\rm loc}(\Omega;\mathbb{R}^n)$. Such a result can be seen as a particular case of [18, Theorem 5.1], when $\Omega = \mathbb{R}^n$.

First, we need to recall the definitions of both *measure-theoretic interior* and *measure-theoretic boundary* of a measurable set E:

$$E^1 := \Big\{x \in \mathbb{R}^n : \lim_{r\to 0}\frac{|B(x,r)\cap E|}{|B(x,r)|} = 1\Big\}, \quad \partial^m E := \mathbb{R}^n\backslash(E^1 \cup (\mathbb{R}^n\backslash E)^1).$$

By Lebesgue's differentiation theorem, it follows that $|E\Delta E^1| = 0$ and $|\partial^m E| = 0$. By [27, Lemma 5.9, §5.11], E^1 and $\partial^m E$ are Borel measurable sets.

We notice that, if $\boldsymbol{F} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$, and $E \subset \Omega$ is a set of locally finite perimeter, then $\partial^* E$ is a $|div\boldsymbol{F}|$–measurable set. Indeed, $\partial^* E \subset \partial^m E$ and $\mathscr{H}^{n-1}(\partial^m E\backslash\partial^* E) = 0$ by [27, Lemma 5.5, §5.8]. This means that $\partial^m E = \partial^* E \cup \mathcal{N}_E$ for some set $\mathcal{N}_E$ satisfying $\mathscr{H}^{n-1}(\mathcal{N}_E) = 0$. Since $|div\boldsymbol{F}| \ll \mathscr{H}^{n-1}$ by [11, Proposition 3.1], $\partial^* E$ is $|div\boldsymbol{F}|$–measurable, because it is the difference between the Borel set $\partial^m E$ and the $|div\boldsymbol{F}|$–negligible set $\mathcal{N}_E$. This means that, if $\boldsymbol{F} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$, and $E \subset \Omega$ is a set of locally finite perimeter, then $\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial E^1}$ and $\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial(E^1\cup\partial^* E)}$ are well defined.

Proposition 4.3 *Let $\boldsymbol{F} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$, and let $E \Subset \Omega$ be a set of finite perimeter. Then the normal trace of $\boldsymbol{F}$ on the boundary of any Borel representative $\widetilde{E}$ of set E is a Radon measure supported on $\partial^* E \cup (\widetilde{E} \Delta E^1) \subset \partial \widetilde{E}$. In particular, if $\widetilde{E} = E^1$ or $\widetilde{E} = E^1 \cup \partial^* E$ up to $\mathscr{H}^{n-1}$–negligible sets, then*

$$|\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial \widetilde{E}}| \ll \mathscr{H}^{n-1} \llcorner \partial^* E$$

with density in $L^\infty(\partial^* E; \mathscr{H}^{n-1})$. More precisely, for any set E of locally finite perimeter in Ω and $\phi \in C^0(\Omega)$ such that $\nabla \phi \in L^1_{\rm loc}(\Omega; \mathbb{R}^n)$ and $\chi_E \phi$ has compact support in Ω, then

$$\int_{E^1} \phi \, \mathrm{d} div \boldsymbol{F} + \int_E \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d}x = - \int_{\partial^* E} \phi \, (\mathfrak{F}_{\rm i} \cdot \nu_E) \, \mathrm{d}\mathscr{H}^{n-1}, \tag{4.6}$$

$$\int_{E^1 \cup \partial^* E} \phi \, \mathrm{d} div \boldsymbol{F} + \int_E \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d}x = - \int_{\partial^* E} \phi \, (\mathfrak{F}_{\rm e} \cdot \nu_E) \, \mathrm{d}\mathscr{H}^{n-1}, \tag{4.7}$$

where $(\mathfrak{F}_{\rm i} \cdot \nu_E), (\mathfrak{F}_{\rm e} \cdot \nu_E) \in L^\infty_{\rm loc}(\partial^* E; \mathscr{H}^{n-1})$ are the interior and exterior normal traces of $\boldsymbol{F}$, respectively, as introduced in [14, Theorem 5.3].

Proof Assume first that $E \Subset \Omega$. By [14, Theorem 5.3] and [16, Theorem 4.2], it follows that the normal traces on the boundaries of E^1 and $E^1 \cup \partial^* E$ are Radon measures. They are indeed absolutely continuous with respect to $\mathscr{H}^{n-1} \llcorner \partial^* E$ and with densities given by essentially bounded interior and exterior normal traces: For any $\phi \in \mathrm{Lip}_c(\Omega)$,

$$\langle \boldsymbol{F} \cdot \nu, \phi \rangle_{\partial E^1} = - \int_{\partial^* E} \phi \, (\mathfrak{F}_{\rm i} \cdot \nu_E) \, \mathrm{d}\mathscr{H}^{n-1},$$

$$\langle \boldsymbol{F} \cdot \nu, \phi \rangle_{\partial (E^1 \cup \partial^* E)} = - \int_{\partial^* E} \phi \, (\mathfrak{F}_{\rm e} \cdot \nu_E) \, \mathrm{d}\mathscr{H}^{n-1}.$$

These two formulas hold also for any set $\widetilde{E}$ with $\mathscr{H}^{n-1}(\widetilde{E} \Delta E^1) = 0$ or $\mathscr{H}^{n-1}(\widetilde{E} \Delta (E^1 \cup \partial^* E)) = 0$, respectively, since $|div \boldsymbol{F}| \ll \mathscr{H}^{n-1}$ if $\boldsymbol{F} \in \mathcal{DM}^\infty(\Omega)$, by [56, Theorem 3.2].

Let $\widetilde{E}$ be any Borel representative of E with respect to the Lebesgue measure so that $|E \Delta \widetilde{E}| = 0$, which implies that $\widetilde{E}^1 = E^1$. By (4.3), we have

$$\langle \boldsymbol{F} \cdot \nu, \phi \rangle_{\partial \widetilde{E}} = \langle \boldsymbol{F} \cdot \nu, \phi \rangle_{\partial E^1} + \int_\Omega \phi \, (\chi_{\widetilde{E} \setminus E^1} - \chi_{E^1 \setminus \widetilde{E}}) \, \mathrm{d} div \boldsymbol{F} \ \text{ for any } \phi \in \mathrm{Lip}_c(\Omega).$$

This shows that $\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial \widetilde{E}}$ is a Radon measure on $\partial^* E \cup (\widetilde{E} \Delta E^1)$, while this set is contained in $\partial \widetilde{E}$, since $\mathring{\widetilde{E}} \subset E^1 \subset \overline{\widetilde{E}}$, coherently with Proposition 4.1.

Finally, let E be a set of locally finite perimeter in Ω, and let $\phi \in C^0(\Omega)$ such that $\nabla \phi \in L^1_{\rm loc}(\Omega)$ and $\mathrm{supp}(\chi_E \phi) \Subset \Omega$. Then (4.6)–(4.7) follow from [16, Theorem 4.2]. Indeed, such equations hold for $\phi \in \mathrm{Lip}_{\rm loc}(\Omega)$ such that $\mathrm{supp}(\chi_E \phi) \subset V \Subset \Omega$ for some open set V. Thus, we can take any mollification ϕ_ε of ϕ, with $\varepsilon > 0$ small enough, such that $\mathrm{supp}(\chi_E \phi_\varepsilon) \subset V$. Then we pass to the limit as $\varepsilon \to 0$ by employing the fact that $\phi_\varepsilon \to \phi$ uniformly on V and $\nabla \phi_\varepsilon \to \nabla \phi$ in $L^1(V; \mathbb{R}^n)$. This completes the proof.

Remark 4.5 Given $\boldsymbol{F} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and a set of locally finite perimeter $E \subset \Omega$, (4.6)–(4.7) hold for any $\phi \in \mathrm{Lip}_c(\Omega)$. This shows that the normal traces of $\boldsymbol{F}$ on the portion of the boundaries $\partial E^1 \cap \Omega$ and $\partial(E^1 \cup \partial^* E) \cap \Omega$ are locally represented by measures $(\mathfrak{F}_i \cdot \nu_E)\,\mathscr{H}^{n-1} \llcorner \partial^* E$ and $(\mathfrak{F}_e \cdot \nu_E)\,\mathscr{H}^{n-1} \llcorner \partial^* E$, respectively.

Remark 4.6 Proposition 4.3 can be seen as a special case of Theorem 4.1, because of Theorem 3.1. In addition, it shows that the normal trace measures of $\boldsymbol{F} \in \mathcal{DM}^\infty(\Omega)$ on ∂E^1 and $\partial(E^1 \cup \partial^* E)$ are actually concentrated on $\partial^* E = \partial^* E^1 = \partial^*(E^1 \cup \partial^* E)$, for any set of finite perimeter $E \Subset \Omega$.

Moreover, if $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p < \infty$, the normal trace on ∂E is not a measure that is absolutely continuous with respect to $\mathscr{H}^{n-1}$ in general, as shown in [16, Example 6.1]. However, as we will see in §7, the normal trace on the boundary of open and closed sets can still be represented as the limit of the classical normal traces on an approximating family of smooth sets.

Remark 4.7 Theorem 4.1 shows that, in the case of Example 3.1, the normal trace is a Radon measure on ∂E, since a product rule holds between

$$\boldsymbol{F}(x_1, x_2) = \frac{1}{2\pi} \frac{(x_1, x_2)}{x_1^2 + x_2^2} \quad \text{and} \quad \chi_E \text{ for } E = (0,1)^2.$$

Indeed, we have

$$div(\chi_E \boldsymbol{F}) = \frac{1}{4}\delta_{(0,0)} + (\boldsymbol{F}, D\chi_E)$$

with

$$(\boldsymbol{F}, D\chi_E)(\phi) := -\frac{1}{2\pi}\left(\int_0^1 \frac{\phi(x_1, 1)}{1 + x_1^2}\,\mathrm{d}x_1 + \int_0^1 \frac{\phi(1, x_2)}{1 + x_2^2}\,\mathrm{d}x_2\right).$$

Using (4.2) and $(0,0) \notin E$, it follows that, for any $\phi \in \mathrm{Lip}_c(\mathbb{R}^2)$,

$$\begin{aligned}\langle \boldsymbol{F} \cdot \nu, \phi \rangle_{\partial E} &= \int_{\mathbb{R}^2} \phi \chi_E \,\mathrm{d}div\boldsymbol{F} - \int_{\mathbb{R}^2} \phi \,\mathrm{d}div(\chi_E \boldsymbol{F}) = -\int_{\mathbb{R}^2} \phi \,\mathrm{d}div(\chi_E \boldsymbol{F}) \\ &= -\frac{1}{4}\phi(0,0) - (\boldsymbol{F}, D\chi_E)(\phi).\end{aligned}$$

Therefore, $\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial E}$ is a Radon measure on ∂E.

In this example, E is also a set of finite perimeter with $E = E^1$, but the normal trace is supported on ∂E, not only on $\partial^* E$, since $(0,0) \notin \partial^* E$.

Remark 4.8 Theorem 4.1 implies that, if $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ does not admit a normal trace on ∂E representable by a Radon measure, then $div(\chi_E \boldsymbol{F}) \notin \mathcal{DM}^p(\Omega)$, even for a set of locally finite perimeter E.

An example of such a vector field was provided by [57, Example 2.5] and [32, Remark 2.2] as follows:

$$\boldsymbol{F}(x_1, x_2) := \frac{(-x_2, x_1)}{x_1^2 + x_2^2}.$$

Then $\boldsymbol{F} \in \mathcal{DM}^p_{\rm loc}(\mathbb{R}^2)$ for any $1 \leqslant p < 2$, $div\boldsymbol{F} = 0$ on $E = (-1,1)\times(-1,0)$. For any $\phi \in {\rm Lip}_c((-1,1)^2)$, we have

$$
\begin{aligned}
&\int_{(-1,1)^2} \chi_E \boldsymbol{F}\cdot\nabla\phi \,{\rm d}x_1\,{\rm d}x_2\\
&= \int_{-1}^1\int_{-1}^0 \frac{1}{x_1^2+x_2^2}\left(-x_2\frac{\partial\phi}{\partial x_1}+x_1\frac{\partial\phi}{\partial x_2}\right){\rm d}x_2\,{\rm d}x_1\\
&= \lim_{\varepsilon\to 0}\left(\int_{-1}^{-\varepsilon}+\int_{\varepsilon}^1\right)\int_{-1}^0 \frac{1}{x_1^2+x_2^2}\left(-x_2\frac{\partial\phi}{\partial x_1}+x_1\frac{\partial\phi}{\partial x_2}\right){\rm d}x_2\,{\rm d}x_1\\
&= \lim_{\varepsilon\to 0}\left\{\int_{-1}^0 \frac{x_2}{\varepsilon^2+x_2^2}\big(-\phi(-\varepsilon,x_2)+\phi(\varepsilon,x_2)\big)\,{\rm d}x_2+\left(\int_{-1}^{-\varepsilon}+\int_{\varepsilon}^1\right)\frac{\phi(x_1,0)}{x_1}\,{\rm d}x_1\right\}\\
&= {\rm P.V.}\int_{-1}^1 \frac{\phi(x_1,0)}{x_1}\,{\rm d}x_1,
\end{aligned}
$$

since the two area integrals are simplified and

$$
\begin{aligned}
&\left|\lim_{\varepsilon\to 0}\int_{-1}^0 \frac{x_2}{\varepsilon^2+x_2^2}\big(-\phi(-\varepsilon,x_2)+\phi(\varepsilon,x_2)\big)\,{\rm d}x_2\right|\\
&\leqslant 2L\lim_{\varepsilon\to 0}\int_0^1 \frac{\varepsilon x_2}{\varepsilon^2+x_2^2}\,{\rm d}x_2 = L\lim_{\varepsilon\to 0}\varepsilon\log(1+\frac{1}{\varepsilon^2}) = 0,
\end{aligned}
$$

where L is the Lipschitz constant of ϕ. This shows

$$
div(\chi_E\boldsymbol{F}) = {\rm P.V.}(\frac{1}{x_1})\llcorner(-1,1)\otimes\delta_0,
$$

so that $div(\chi_E\boldsymbol{F}) \notin \mathcal{M}((-1,1)^2)$, which means that $\chi_E\boldsymbol{F} \notin \mathcal{DM}^p((-1,1)^2)$ for any $1 \leqslant p < 2$.

The argument can be generalized to

$$
\boldsymbol{F}(x_1,x_2) = \frac{(-x_2,x_1)}{(x_1^2+x_2^2)^{\frac{\alpha}{2}}} \qquad \text{for } 2 \leqslant \alpha < 3
$$

to obtain

$$
div(\chi_E\boldsymbol{F}) = ({\rm P.V.}\,{\rm sgn}(x_1)\,|x_1|^{1-\alpha})\llcorner(-1,1)\otimes\delta_0.
$$

Remark 4.9 By Theorem 4.1, $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ admits a normal trace on the boundary of a Borel set $E \Subset \Omega$ representable by a Radon measure if and only if $div(\chi_E\boldsymbol{F}) \in \mathcal{DM}^p(\Omega)$. This condition is generally weaker than the requirement of E to be a set of locally finite perimeter in Ω. Indeed, there exist a set $E \subset \mathbb{R}^2$ with $\chi_E \notin BV_{\rm loc}(\mathbb{R}^2)$ and a field $\boldsymbol{F} \in \mathcal{DM}^p(\mathbb{R}^2)$ for any $p \in [1,\infty]$ with $\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial E} \in \mathcal{M}(\partial E)$. The key observation in the construction of such a set E is that, given a constant vector field $\boldsymbol{F} \equiv \boldsymbol{v} \in \mathbb{R}^n$, (4.2) implies that the normal trace is given by

$$
\langle \boldsymbol{v}\cdot\nu,\cdot\rangle_{\partial E} = -div(\chi_E\boldsymbol{v}) = -\sum_{j=1}^n v_j D_{x_j}\chi_E.
$$

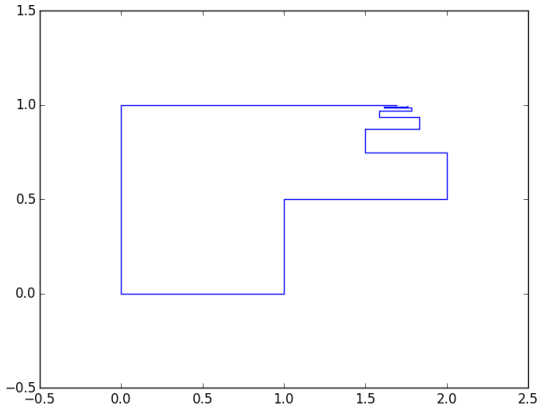


Fig. 4.1 The open bounded set E

Clearly, the requirement that $\sum_{j=1}^{n} v_j D_{x_j}\chi_E \in \mathcal{M}(\Omega)$ is weaker than the requirement that $\chi_E \in BV(\Omega)$, since there may be some cancellations.

We choose E as the open bounded set whose boundary is given by

$$\partial E = \big(\{0\}\times[0,1]\big) \cup \big([0,1]\times\{0\}\big) \cup \big([0,1+\log 2]\times\{1\}\big) \cup S,$$

as shown in Figure 4.9, where

$$\begin{aligned} S = &\Big(\{1\}\times[0,\frac{1}{2}]\Big)\bigcup\Big([1,2]\times\{\frac{1}{2}\}\Big)\\ &\bigcup\Big(\bigcup_{n\geqslant 1}\{1+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\}\times[1-\frac{1}{2^n},1-\frac{1}{2^{n+1}}]\Big)\\ &\bigcup\Big(\bigcup_{n\geqslant 1}[1+\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k},1+\sum_{k=1}^{2n+1}\frac{(-1)^{k-1}}{k}]\times\{1-\frac{1}{2^{2n+1}}\}\Big)\\ &\bigcup\Big(\bigcup_{n\geqslant 1}[1+\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k},1+\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}]\times\{1-\frac{1}{2^{2n}}\}\Big). \end{aligned}$$

Then $\chi_E \notin BV_{\mathrm{loc}}(\mathbb{R}^2)$, since $\mathscr{H}^1(S)=\infty$. However, we can show that $D_{x_1}\chi_E \in \mathcal{M}(\mathbb{R}^2)$.

Indeed, given any $\phi \in C_c^1(\mathbb{R}^2)$, we have

$$
\begin{aligned}
&\int_E \frac{\partial \phi}{\partial x_1}\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&= \int_0^{\frac{1}{2}}\int_0^1 \frac{\partial \phi}{\partial x_1}\,\mathrm{d}x_1\,\mathrm{d}x_2 + \sum_{n=1}^{\infty}\int_{1-\frac{1}{2^n}}^{1-\frac{1}{2^{n+1}}}\int_0^{1+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}} \frac{\partial \phi}{\partial x_1}\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&= \int_0^{\frac{1}{2}} \big(\phi(1,x_2)-\phi(0,x_2)\big)\,\mathrm{d}x_2\\
&\quad + \sum_{n=1}^{\infty}\int_{1-\frac{1}{2^n}}^{1-\frac{1}{2^{n+1}}} \big(\phi(1+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k},x_2)-\phi(0,x_2)\big)\,\mathrm{d}x_2\\
&= -\int_0^1 \phi(0,x_2)\,dx_2 + \int_0^{\frac{1}{2}}\phi(1,x_2)\,\mathrm{d}x_2 + \sum_{n=1}^{\infty}\int_{1-\frac{1}{2^n}}^{1-\frac{1}{2^{n+1}}}\phi(1+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k},x_2)\,\mathrm{d}x_2.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&D_{x_1}\chi_E\\
&= \mathscr{H}^1 \llcorner \big(\{0\}\times(0,1)\big)\\
&\quad - \mathscr{H}^1 \llcorner \Big(\big(\{1\}\times(0,\frac{1}{2})\big)\bigcup\big(\bigcup_{n\geqslant 1}\{1+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\}\times(1-\frac{1}{2^n},1-\frac{1}{2^{n+1}})\big)\Big),
\end{aligned}
$$

which is clearly a finite Radon measure on $\mathbb{R}^2$.

Now we observe that, if $\boldsymbol{F}(x_1,x_2) = f(x_2)g(x_1)(1,0)$ for some $f \in L^p(\mathbb{R})$ and $g \in C_c^1(\mathbb{R})$, then $\boldsymbol{F} \in \mathcal{DM}^p(\mathbb{R}^2)$,

$$
div\boldsymbol{F} = f(x_2)g'(x_1)\mathscr{L}^2,
$$

and

$$
div(\chi_E\boldsymbol{F}) = f(x_2)g(x_1)D_{x_1}\chi_E + \chi_E(x_1,x_2)f(x_2)g'(x_1)\mathscr{L}^2. \tag{4.8}
$$

Indeed, for any $\phi \in C_c^1(\mathbb{R}^2)$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^2}\chi_E\boldsymbol{F}\cdot\nabla\phi\,\mathrm{d}x_1\mathrm{d}x_2 &= \int_{\mathbb{R}^2}\chi_E(x_1,x_2)f(x_2)g(x_1)\frac{\partial\phi(x_1,x_2)}{\partial x_1}\,\mathrm{d}x_1\mathrm{d}x_2\\
&= \int_{\mathbb{R}^2}\chi_E(x_1,x_2)f(x_2)\frac{\partial(g(x_1)\phi(x_1,x_2))}{\partial x_1}\,\mathrm{d}x_1\mathrm{d}x_2\\
&\quad - \int_{\mathbb{R}^2}\chi_E(x_1,x_2)f(x_2)\phi(x_1,x_2)g'(x_1)\,\mathrm{d}x_1\mathrm{d}x_2\\
&= -\int_{\mathbb{R}^2} f(x_2)g(x_1)\phi(x_1,x_2)\,\mathrm{d}D_{x_1}\chi_E\\
&\quad - \int_{\mathbb{R}^2}\chi_E(x_1,x_2)f(x_2)\phi(x_1,x_2)g'(x_1)\,\mathrm{d}x_1\mathrm{d}x_2.
\end{aligned}
$$

Thus, by (4.8), $div(\chi_E \boldsymbol{F}) \in \mathcal{M}(\mathbb{R}^2)$ so that $\langle \boldsymbol{F}\cdot\nu, \cdot\rangle_{\partial E} \in \mathcal{M}(\partial E)$, by Theorem 4.1, even if E is not a set of locally finite perimeter in $\mathbb{R}^2$. In addition, by (4.2), we have

$$\langle \boldsymbol{F}\cdot\nu, \cdot\rangle_{\partial E} = \chi_E div\boldsymbol{F} - div(\chi_E \boldsymbol{F}) = -f(x_2)g(x_1)D_{x_1}\chi_E,$$

from which the following is deduced:

$$|\langle \boldsymbol{F}\cdot\nu, \cdot\rangle_{\partial E}| \ll \mathscr{H}^1 \llcorner \Big(\big(\{0\}\times(0,1)\big) \bigcup \big(\{1\}\times(0,\frac{1}{2})\big) \bigcup \big(\bigcup_{n\geqslant 1} \{1+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\}\times(1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}})\big)\Big).$$

On the other hand, as we will show, whether $\langle \boldsymbol{F}\cdot\nu, \cdot\rangle_{\partial E}$ is a Radon measure on ∂E or not does not play any role in the representation of the normal trace of $\boldsymbol{F}$ on the boundary of an open or closed set as the limit of classical normal traces on the boundaries of a sequence of approximating smooth sets.

We provide now a necessary condition for the normal trace to be a Radon measure.

Proposition 4.4 *Let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$, and let $E \subset \Omega$ be a Borel set such that there exists $\sigma \in \mathcal{M}(\partial E)$ satisfying*

$$\langle \boldsymbol{F}\cdot\nu, \phi\rangle_{\partial E} = \int_{\partial E} \phi \,\mathrm{d}\sigma \qquad \text{for any } \phi \in \mathrm{Lip}_{\mathrm{c}}(\mathbb{R}^n).$$

Then, if $1 \leqslant p < \frac{n}{n-1}$, for any $x \in \partial E$ and $r > 0$, there exists a constant $C > 0$ such that $|\sigma|(\partial E) + |div\boldsymbol{F}|(E) \geqslant C$ and

$$\Big| \int_{B(x,r)\cap E} \boldsymbol{F}(y)\cdot\frac{(y-x)}{|y-x|}\,\mathrm{d}y\Big| \leqslant Cr. \tag{4.9}$$

If $p \geqslant \frac{n}{n-1}$, for any $x \in \partial E$ and $r > 0$,

$$\Big| \int_{B(x,r)\cap E} \boldsymbol{F}(y)\cdot\frac{(y-x)}{|y-x|}\,\mathrm{d}y\Big| = o(r). \tag{4.10}$$

Moreover, given any $\alpha \in (0, n]$, for $\mathscr{H}^\alpha$–a.e. $x \in \partial E$ and $r > 0$, there exists a constant $C = C_{E,\boldsymbol{F},x} > 0$ such that

$$\Big| \int_{B(x,r)\cap E} \boldsymbol{F}(y)\cdot\frac{(y-x)}{|y-x|}\,\mathrm{d}y\Big| \leqslant Cr^{\alpha+1}. \tag{4.11}$$

Proof We just need to choose $\phi(y) := (r - |y-x|)\chi_{B(x,r)}(y)$ so that, by (4.1),

$$\begin{aligned}&\int_{B(x,r)\cap\partial E} (r-|y-x|)\,\mathrm{d}\sigma(y)\\ &= \int_{B(x,r)\cap E} (r-|y-x|)\,\mathrm{d}div\boldsymbol{F} - \int_{B(x,r)\cap E} \boldsymbol{F}(y)\cdot\frac{(y-x)}{|y-x|}\,\mathrm{d}y.\end{aligned}$$

Then we obtain

$$\left|\int_{B(x,r)\cap E} \boldsymbol{F}(y)\cdot\frac{(y-x)}{|y-x|}\,\mathrm{d}y\right| \leqslant r\Big(|\sigma|(B(x,r)\cap\partial E)+|div\boldsymbol{F}|(B(x,r)\cap E)\Big). \tag{4.12}$$

Now, if $1\leqslant p<\frac{n}{n-1}$, then $divF$ and $\sigma=\chi_E divF-div(\chi_E F)$ do not enjoy any absolute continuity property in general, by [56, Example 3.3, Proposition 6.1], so that (4.9) holds from (4.12).

If $p\geqslant\frac{n}{n-1}$, then $|divF|(\{x\})=|\sigma|(\{x\})=0$, by [56, Theorem 3.2] and Theorem 4.1. Therefore, (4.12) implies (4.10). Finally, a consequence of [2, Theorem 2.56] is that, given a positive Radon measure μ on Ω, its α–dimensional upper density $\Theta^*_\alpha(\mu,x)$ satisfies the property:

$$\Theta^*_\alpha(\mu,x)<\infty \qquad \text{for } \mathscr{H}^\alpha\text{–a.e. } x\in\Omega.$$

This means that, for $\mathscr{H}^\alpha$–*a.e.* $x\in\Omega$, there exists a constant $C=C_{\mu,x}$ such that

$$\mu(B(x,r))\leqslant Cr^\alpha.$$

Therefore, this argument holds for both measures $|divF|\llcorner E$ and $|\sigma|\llcorner\partial E$. Then, from (4.12), we achieve (4.11).

Remark 4.10 The result of Proposition 4.4 does not seem to be very restrictive, since the example in Remark 4.8 satisfies all the three conditions at any point on $(-1,1)\times\{0\}$.

Indeed, consider points $(t,0)$ for some $t\in(-1,1)$, and $r>0$ small enough so that $B((t,0),r)\cap\{x_2<0\}\subset E=(-1,1)\times(-1,0)$. Since $(-x_2,x_1)\cdot(x_1-t,x_2)=x_2t$, we have

$$\begin{aligned}
&\int_{B((t,0),r)\cap E}\boldsymbol{F}(x_1,x_2)\cdot\frac{(x_1-t,x_2)}{|(x_1-t,x_2)|}\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&=\int_{B((t,0),r)\cap\{x_2<0\}}\frac{x_2t}{(x_1^2+x_2^2)\sqrt{(x_1-t)^2+x_2^2}}\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&=\int_{B((0,0),1)\cap\{u<0\}}\frac{tu}{((t+rv)^2+r^2u^2)\sqrt{v^2+u^2}}r^2\,\mathrm{d}v\,\mathrm{d}u.
\end{aligned}$$

Therefore, for $t\neq0$, we have

$$\begin{aligned}
&\left|\int_{B((t,0),r)\cap E}\boldsymbol{F}(x_1,x_2)\cdot\frac{(x_1-t,x_2)}{|(x_1-t,x_2)|}\,\mathrm{d}x_1\,\mathrm{d}x_2\right|\\
&=\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\rho|t|\cos\theta}{r^2\rho^2+t^2+2tr\rho\sin\theta}r^2\,\mathrm{d}\theta\,\mathrm{d}\rho\\
&=\int_0^1 r\,\mathrm{sgn}(t)\log\left|\frac{r\rho+t}{r\rho-t}\right|\mathrm{d}\rho=\frac{r^2}{|t|}+o(r^2)
\end{aligned}$$

for any sufficiently small r; while, if $t = 0$, we just have

$$\int_{B((0,0),r)\cap E} \boldsymbol{F}(x_1, x_2) \cdot \frac{(x_1, x_2)}{|(x_1, x_2)|}\, \mathrm{d}x_1\, \mathrm{d}x_2 = 0 \qquad \text{for any } r > 0.$$

These calculations also show that this $\boldsymbol{F}$ satisfies (4.11) for any $\alpha \in (0, 1]$, which is sufficient, since the Hausdorff dimension of ∂E is 1.

Moreover, for any $\boldsymbol{F} \in L^p(\Omega; \mathbb{R}^n)$,

$$\Big| \int_{B(x,r)\cap E} \boldsymbol{F}(y) \cdot \frac{(y-x)}{|y-x|}\, \mathrm{d}y \Big| \leqslant \Big(\int_{B(x,r)} |\boldsymbol{F}|^p\, \mathrm{d}y \Big)^{\frac{1}{p}} (\omega_n r^n)^{1-\frac{1}{p}}.$$

Then condition (4.10) is satisfied for any $r \in (0, 1]$ if $n - \dfrac{n}{p} > 1$, that is, $p > \frac{n}{n-1}$.

On the other hand, we obtain a better decay estimate for $\mathscr{H}^\alpha$–*a.e.* $x \in \partial E$, for any $\alpha \in (0, n]$. Indeed, $\boldsymbol{F} \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ so that measure $\mu = |\boldsymbol{F}|\mathscr{L}^n$ satisfies $\mu(B(x, r)) \leqslant Cr^\alpha$ for $\mathscr{H}^\alpha$–*a.e.* $x \in \Omega$. This implies

$$\Big| \int_{B(x,r)\cap E} \boldsymbol{F}(y) \cdot \frac{(y-x)}{|y-x|}\, \mathrm{d}y \Big| \leqslant Cr^\alpha \qquad \text{for } \mathscr{H}^\alpha\text{–a.e. } x \in \partial E,$$

while we obtain the higher exponent $\alpha + 1$ in (4.11).

5 The Gauss-Green Formula on General Open Sets

We now consider a general open set $U \subset \mathbb{R}^n$ and provide a way to construct its interior and exterior approximations via the signed distance function, suitable for the derivation of the Gauss-Green formula for $\boldsymbol{F} \in \mathcal{DM}^p$ for $1 \leqslant p \leqslant \infty$.

For the given open set U in $\mathbb{R}^n$, we consider the signed distance from ∂U:

$$d(x) = \begin{cases} \operatorname{dist}(x, \partial U) & \text{for } \ x \in U, \\ -\operatorname{dist}(x, \partial U) & \text{for } \ x \notin U. \end{cases} \tag{5.1}$$

We summarize some known results on the signed distance function in the following lemma.

Lemma 5.1 *The distance function $d(x)$ is Lipschitz in $\mathbb{R}^n$ with Lipschitz constant equal to 1 and satisfies*

$$|\nabla d(x)| = 1 \qquad \textit{for } \mathscr{L}^n\textit{–a.e. } x \notin \partial U.$$

In addition, $\nabla d = 0$ $\mathscr{L}^n$–a.e. on sets $\{d = t\}$ for any $t \in \mathbb{R}$.

Proof The elementary properties of the distance show that d is Lipschitz with Lipschitz constant $L \leqslant 1$, and hence differentiable $\mathscr{L}^n$–*a.e.* Then it is clear that $|\nabla d(x)| \leqslant 1$.

Let now $x \in U$ such that d is differentiable at x and $|\nabla d(x)| < 1$. Then there exists a point $y \in \partial U$, depending on x, such that $d(x) = |x - y|$. Indeed, given $z \in \partial U$ such that $d(x) \leqslant |x - z|$, then we can look for $y \in \overline{B}(x, |x - z|) \cap \partial U$, which is a compact set.

Setting $x_r := x + r(y - x)$, we see that $d(x_r) = (1 - r)|x - y|$ for any $r \in [0, 1]$. Otherwise, if there would exist $z \in \partial U$ such that $|x_r - z| < |x_r - y|$, then we would obtain

$$|x - z| \leqslant |x - x_r| + |x_r - z| < r|y - x| + |x_r - y| = |x - y|,$$

which contradicts the assumption that y realizes the minimum distance from x.

Since d is differentiable at x, then

$$d(x_r) - d(x) = \nabla d(x) \cdot (y - x)\, r + o(r),$$

that is,

$$|y - x| = \nabla d(x) \cdot (x - y) + o(1),$$

which yields a contradiction with the assumption that $|\nabla d(x)| < 1$.

Similarly, we also obtain a contradiction, provided that d is differentiable at $x \in \mathbb{R}^b \backslash \overline{U}$ and $|\nabla d(x)| < 1$. Since the Lipschitz constant L satisfies $L \geqslant \|\nabla d\|_{L^\infty(\mathbb{R}^n, \mathbb{R}^n)}$, we conclude that $L = 1$.

As for the second part of the statement, we refer to [3, Theorem 3.2.3].

For any $\varepsilon > 0$, denote

$$U^\varepsilon := \{x \in \mathbb{R}^n : d(x) > \varepsilon\}, \tag{5.2}$$

and

$$U_\varepsilon := \{x \in \mathbb{R}^n : d(x) > -\varepsilon\}. \tag{5.3}$$

Then $U^{\varepsilon'} \subset U^\varepsilon$ when $\varepsilon' > \varepsilon$, and

$$\bigcup_{\varepsilon > 0} U^\varepsilon = U.$$

Similarly, $U_{\varepsilon'} \subset U_\varepsilon$ when $\varepsilon' < \varepsilon$, and

$$\bigcap_{\varepsilon > 0} U_\varepsilon = \overline{U}.$$

It is clear that, for $K := \overline{U}$ and $K_\varepsilon := \overline{U_\varepsilon}$, we recover the same setting of Schuricht's result as in (1.2).

Remark 5.1 By Lemma 5.1, we can integrate indifferently on $\{d \geqslant t\}$ and $\{d > t\}$ for any $t \in \mathbb{R}$ with respect to $\nabla d\, \mathrm{d}x$ (or, analogously, on $\{d \leqslant t\}$ and $\{d < t\}$). This means that ∂U^ε and ∂U_ε are negligible for measure $\nabla d\, dx$ for any $\varepsilon \geqslant 0$ (with $U^0 = U_0 = U$). In particular, it follows that (2.2) holds for any $t \in \mathbb{R}$ and $h \geqslant 0$, if $u = d$ and $g = f\, \nabla d$ for some $f : \mathbb{R}^n \to \mathbb{R}$ $\mathscr{L}^n$–summable.

We can say more on the regularity of sets U^ε and U_ε. Indeed, since d is a Lipschitz function, which is particularly in $BV_{\mathrm{loc}}(\mathbb{R}^n)$, the coarea formula (Theorem 2.2) implies that the superlevel and sublevel sets of d are almost all sets of locally finite perimeter. Thus, we can conclude that U^ε and U_ε are sets of locally finite perimeter for $\mathscr{L}^1$–*a.e.* $\varepsilon > 0$. In fact, we can show the following slightly stronger result.

Lemma 5.2 *For any open set U in $\mathbb{R}^n$, for $\mathscr{L}^1$–a.e. $\varepsilon > 0$,*

$$\mathscr{H}^{n-1}(\partial U^\varepsilon \backslash \partial^* U^\varepsilon) = 0, \qquad \nabla d(x) = \nu_{U^\varepsilon}(x) \quad \text{for } \mathscr{H}^{n-1}\text{–a.e. } x \in \partial U^\varepsilon,$$

where ν_{U^ε} is the measure-theoretic interior normal to U^ε. Analogously, for $\mathscr{L}^1$–a.e. $\varepsilon > 0$,

$$\mathscr{H}^{n-1}(\partial U_\varepsilon \backslash \partial^* U_\varepsilon) = 0, \qquad \nabla d(x) = \nu_{U_\varepsilon}(x) \quad \text{for } \mathscr{H}^{n-1}\text{–a.e. } x \in \partial U_\varepsilon.$$

Proof By the previous remarks, U^ε is a set of locally finite perimeter for $\mathscr{L}^1$–*a.e.* $\varepsilon > 0$. Then, for any smooth vector field $\varphi \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$,

$$\int_{U^\varepsilon} div\, \varphi \, \mathrm{d}x = -\int_{\partial^* U^\varepsilon} \varphi \cdot \nu_{U^\varepsilon} \, \mathrm{d}\mathscr{H}^{n-1} \qquad \text{for } \mathscr{L}^1\text{–a.e. } \varepsilon > 0. \tag{5.4}$$

Consider now the functions:

$$\psi^U_\varepsilon(x) := \begin{cases} \varepsilon & \text{if } x \in U^\varepsilon, \\ d(x) & \text{if } x \in U \backslash U^\varepsilon, \\ 0 & \text{if } x \notin U. \end{cases}$$

Then

$$\begin{aligned} \int_U \psi^U_\varepsilon \, div\, \varphi \, \mathrm{d}x &= \varepsilon \int_{U^\varepsilon} div\, \varphi \, \mathrm{d}x + \int_{U \backslash U^\varepsilon} d(x)\, div\, \varphi \, \mathrm{d}x \\ &= -\int_{U \backslash U^\varepsilon} \varphi \cdot \nabla d \, \mathrm{d}x \\ &= -\int_{U \backslash \overline{U^\varepsilon}} \varphi \cdot \nabla d \, |\nabla d| \, \mathrm{d}x \\ &= -\int_0^\varepsilon \int_{\partial U^t} \varphi \cdot \nabla d \, \mathrm{d}\mathscr{H}^{n-1} \, \mathrm{d}t, \end{aligned}$$

since $|\nabla d(x)| = 1$ for $\mathscr{L}^n$–*a.e.* $x \notin \partial U$ and $\nabla d(x) = 0$ for $\mathscr{L}^n$–*a.e.* $x \in \partial U^\varepsilon$ (Lemma 5.1 and Remark 5.1), and by the coarea formula (2.2) with $u = d$ and

$g = \chi_U \varphi \cdot \nabla d\, |\nabla d|$. Indeed, using (2.1), we have

$$\begin{aligned}\int_{\mathbb{R}^n \setminus \overline{U^\varepsilon}} \chi_U \varphi \cdot \nabla d\, |\nabla d| \,\mathrm{d}x &= \int_{\{d<\varepsilon\}} \chi_U \varphi \cdot \nabla d\, |\nabla d| \,\mathrm{d}x \\ &= \int_{\{-d>-\varepsilon\}} \chi_U \varphi \cdot \nabla d\, |\nabla d| \,\mathrm{d}x \\ &= \int_{-\varepsilon}^{\infty} \int_{\{-d=t\}} \chi_U \varphi \cdot \nabla d \,\mathrm{d}\mathscr{H}^{n-1} \,\mathrm{d}t \\ &= \int_{-\infty}^{\varepsilon} \int_{\{d=t\}} \chi_U \varphi \cdot \nabla d \,\mathrm{d}\mathscr{H}^{n-1} \,\mathrm{d}t \\ &= \int_0^{\varepsilon} \int_{\partial U^t} \varphi \cdot \nabla d \,\mathrm{d}\mathscr{H}^{n-1} \,\mathrm{d}t,\end{aligned}$$

since $d > 0$ in U and $|\nabla d(x)| = 1$ for $\mathscr{L}^n$–*a.e.* $x \in \partial U$.

We can repeat the same calculation with $\psi_{\varepsilon+h}$ for $h > 0$, and subtract the two resultant equations to obtain

$$h \int_{U^{\varepsilon+h}} \mathrm{d}div\varphi + \int_{U^\varepsilon \setminus U^{\varepsilon+h}} (d(x) - \varepsilon)\, \mathrm{d}div\varphi = -\int_{\varepsilon}^{\varepsilon+h} \int_{\partial U^t} \varphi \cdot \nabla d \,\mathrm{d}\mathscr{H}^{n-1} \,\mathrm{d}t.$$

We now divide by h and use the fact that $0 \leqslant d(x) - \varepsilon \leqslant h$ in $U^\varepsilon \setminus U^{\varepsilon+h}$ and $|U^\varepsilon \setminus U^{\varepsilon+h}| \to 0$ as $h \to 0$ to conclude

$$\int_{U^\varepsilon} div\varphi \,\mathrm{d}x = -\int_{\partial U^\varepsilon} \varphi \cdot \nabla d \,\mathrm{d}\mathscr{H}^{n-1} \qquad \text{for } \mathscr{L}^1\text{–}a.e.\ \varepsilon > 0. \tag{5.5}$$

Notice that, for any $R > 0$,

$$\begin{aligned}&\mathscr{H}^{n-1}(B(0,R) \cap \partial^* U^\varepsilon) \\ &= \sup\Big\{ \int_{U^\varepsilon} div(-\varphi) \,\mathrm{d}x \,:\, \varphi \in C_c^1(B(0,R); \mathbb{R}^n), \|\varphi\|_\infty \leqslant 1 \Big\}.\end{aligned} \tag{5.6}$$

Now, we can take a double index sequence of fields $\varphi_{k,m}$ in $C_c^1(B(0,R); \mathbb{R}^n)$ such that

$$\varphi_{k,m} \to \chi_{B(0,R-\frac{1}{m})} \nabla d \qquad \text{in } L^1(\mathbb{R}^n; \mathbb{R}^n) \text{ as } k \to \infty, \text{ for any fixed } m \in \mathbb{N}.$$

For each k and m, there is a set $\mathcal{N}_{k,m} \subset \mathbb{R}$ with $\mathscr{L}^1(\mathcal{N}_{k,m}) = 0$ such that (5.5) holds for any $\varepsilon \notin \mathcal{N}_{k,m}$. Set $\mathcal{N} := \bigcup_{(k,m)\in\mathbb{N}^2} \mathcal{N}_{k,m}$. Then, for any $\varepsilon \notin \mathcal{N}$, we obtain

$$\begin{aligned}&\sup\Big\{ \int_{U^\varepsilon} div(-\varphi) \,\mathrm{d}x \,:\, \varphi \in C_c^1(B(0,R); \mathbb{R}^n), \|\varphi\|_\infty \leqslant 1 \Big\} \\ &\geqslant \int_{U^\varepsilon} div(-\varphi_{k,m}) \,\mathrm{d}x = \int_{\partial U^\varepsilon} \varphi_{k,m} \cdot \nabla d \,\mathrm{d}\mathscr{H}^{n-1}.\end{aligned}$$

Now we let $k \to \infty$ and employ (5.6) to obtain

$$\mathscr{H}^{n-1}(B(0,R) \cap \partial^* U^\varepsilon) \geqslant \mathscr{H}^{n-1}(B(0, R - \frac{1}{m}) \cap \partial U^\varepsilon),$$

and the arbitrariness of $m \in \mathbb{N}$ yields

$$\mathscr{H}^{n-1}(B(0,R) \cap \partial^* U^\varepsilon) \geqslant \mathscr{H}^{n-1}(B(0,R) \cap \partial U^\varepsilon). \tag{5.7}$$

Combining (5.7) with the well-known fact that $\partial^* U^\varepsilon \subset \partial U^\varepsilon$, we obtain

$$\mathscr{H}^{n-1}(B(0,R) \cap (\partial U^\varepsilon \backslash \partial^* U^\varepsilon)) = 0,$$

which implies that $\mathscr{H}^{n-1}(\partial U^\varepsilon \backslash \partial^* U^\varepsilon) = 0$, by the arbitrariness of $R > 0$.

Therefore, from (5.4)–(5.5), we have

$$\int_{\partial^* U^\varepsilon} \varphi \cdot (\nu_{U^\varepsilon} - \nabla d)\, \mathrm{d}\mathscr{H}^{n-1} = 0 \qquad \text{for any } \varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n),$$

which implies our assertion.

The second part of the statement is proved in a similar way by considering the following functions instead:

$$\xi_\varepsilon^U(x) := \begin{cases} \varepsilon & \text{if } x \in U, \\ d(x) + \varepsilon & \text{if } x \in U_\varepsilon \backslash U, \\ 0 & \text{if } x \notin U_\varepsilon. \end{cases}$$

Using similar techniques as in the proof of Lemma 5.2, we are able to show the following Gauss-Green formulas.

Theorem 5.1 (Interior normal trace) *Let $U \subset \Omega$ be a bounded open set, and let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$. Then, for any $\phi \in C^0(\Omega) \cap L^\infty(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega; \mathbb{R}^n)$, there exists a set $\mathcal{N} \subset \mathbb{R}$ with $\mathcal{L}^1(\mathcal{N}) = 0$ such that, for every nonnegative sequence $\{\varepsilon_k\}$ satisfying $\varepsilon_k \notin \mathcal{N}$ for any k and $\varepsilon_k \to 0$, the following representation for the interior normal trace on ∂U holds*:

$$\langle \boldsymbol{F} \cdot \nu, \phi \rangle_{\partial U} = \int_U \phi\, \mathrm{d}\mathrm{div}\boldsymbol{F} + \int_U \boldsymbol{F} \cdot \nabla\phi\, \mathrm{d}x = -\lim_{k\to\infty} \int_{\partial^* U^{\varepsilon_k}} \phi \boldsymbol{F} \cdot \nu_{U^{\varepsilon_k}}\, \mathrm{d}\mathscr{H}^{n-1}, \tag{5.8}$$

where $\nu_{U^{\varepsilon_k}}$ is the inner unit normal to U^{ε_k} on $\partial^* U^{\varepsilon_k}$. In addition, (5.8) holds also for any open set $U \subset \Omega$, provided that $\mathrm{supp}(\phi) \cap U^\delta \Subset \Omega$ *for any $\delta > 0$.*

Proof We divide the proof into three steps.

1. Suppose first that $U \Subset \Omega$. Then $U^\varepsilon \Subset \Omega$ for any small $\varepsilon > 0$. Recall that $U^{\varepsilon'} \subset U^\varepsilon$ if $\varepsilon' > \varepsilon$ and $\bigcup_{\varepsilon>0} U^\varepsilon = U$. Define

$$\psi_\varepsilon^U(x) := \begin{cases} \varepsilon & \text{if } x \in U^\varepsilon, \\ d(x) & \text{if } x \in U \backslash U^\varepsilon, \\ 0 & \text{if } x \notin U. \end{cases}$$

Since $\psi_\varepsilon^U \in \mathrm{Lip}_c(\Omega)$, we can use it as a test function. In addition, for any $\phi \in C^0(\Omega) \cap L^\infty(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega;\mathbb{R}^n)$, $\phi\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ by Proposition 3.1. Then

$$\begin{aligned}\int_U \psi_\varepsilon^U \,\mathrm{d}div(\phi\boldsymbol{F}) &= -\int_{U\setminus U^\varepsilon} \phi\boldsymbol{F}\cdot\nabla d\,\mathrm{d}x = -\int_{U\setminus U^\varepsilon} \phi\boldsymbol{F}\cdot\nabla d\,|\nabla d|\,\mathrm{d}x\\ &= -\int_0^\varepsilon \int_{\partial U^t} \phi\boldsymbol{F}\cdot\nabla d\,\mathrm{d}\mathscr{H}^{n-1}\,\mathrm{d}t, \end{aligned} \tag{5.9}$$

by the coarea formula (2.2) with $u = d$ and $g = \chi_U \phi\boldsymbol{F}\cdot\nabla d\,|\nabla d|$, by Lemma 5.1 and Remark 5.1. Thus, we use test functions ψ_ε^U and $\psi_{\varepsilon+h}^U$ with $h > 0$ to obtain

$$\int_{U^\varepsilon} \varepsilon\,\mathrm{d}div(\phi\boldsymbol{F}) + \int_{U\setminus U^\varepsilon} d(x)\,\mathrm{d}div(\phi\boldsymbol{F}) = -\int_0^\varepsilon\int_{\partial U^t} \phi\boldsymbol{F}\cdot\nabla d\,\mathrm{d}\mathscr{H}^{n-1}\,\mathrm{d}t,$$

and

$$\begin{aligned}&\int_{U^{\varepsilon+h}} (\varepsilon+h)\,\mathrm{d}div(\phi\boldsymbol{F}) + \int_{U\setminus U^{\varepsilon+h}} d(x)\,\mathrm{d}div(\phi\boldsymbol{F})\\ &= -\int_0^{\varepsilon+h}\int_{\partial U^t} \phi\boldsymbol{F}\cdot\nabla d\,\mathrm{d}\mathscr{H}^{n-1}\,\mathrm{d}t.\end{aligned}$$

Subtracting the first equation from the second one, we have

$$\begin{aligned}&h\int_{U^{\varepsilon+h}} \mathrm{d}div(\boldsymbol{F}\phi) + \int_{U^\varepsilon\setminus U^{\varepsilon+h}} (d(x)-\varepsilon)\,\mathrm{d}div(\boldsymbol{F}\phi)\\ &= -\int_\varepsilon^{\varepsilon+h}\int_{\partial U^t} \phi\boldsymbol{F}\cdot\nabla d\,\mathrm{d}\mathscr{H}^{n-1}\,\mathrm{d}t.\end{aligned}$$

We now divide by h and use the fact that

$$0 \leqslant d(x)-\varepsilon \leqslant h \quad \text{in } U^\varepsilon\setminus U^{\varepsilon+h}, \qquad |div(\boldsymbol{F}\phi)|(U^\varepsilon\setminus U^{\varepsilon+h}) \to 0 \quad \text{as } h\to 0$$

to conclude

$$\int_{U^\varepsilon} \mathrm{d}div(\boldsymbol{F}\phi) = -\int_{\partial U^\varepsilon} \phi\boldsymbol{F}\cdot\nabla d\;\mathrm{d}\mathscr{H}^{n-1} \qquad \text{for } \mathscr{L}^1\text{–a.e. } \varepsilon > 0. \tag{5.10}$$

We can take any sequence $\varepsilon_k \to 0$ of such good values to obtain

$$\int_U \phi\,\mathrm{d}div\boldsymbol{F} + \int_U \boldsymbol{F}\cdot\nabla\phi\,\mathrm{d}x = -\lim_{k\to\infty}\int_{\partial U^{\varepsilon_k}} \phi\boldsymbol{F}\cdot\nabla d\;\mathrm{d}\mathscr{H}^{n-1}. \tag{5.11}$$

By Lemma 5.2, such a sequence can be chosen so that $\mathscr{H}^{n-1}(\partial U^{\varepsilon_k}\setminus\partial^* U^{\varepsilon_k}) = 0$ and ∇d is the inner normal to U^{ε_k} at $\mathscr{H}^{n-1}$–almost every point of $\partial^* U^{\varepsilon_k}$. Then the result follows.

2. Let now $U \subset \Omega$ be bounded. Since $U^\delta \Subset \Omega$ for any $\delta > 0$, we can consider the test functions:

$$\psi_\varepsilon^{U^\delta}(x) := \begin{cases} \varepsilon & \text{if } x \in U^{\varepsilon+\delta}, \\ d(x) - \delta & \text{if } x \in U^\delta \backslash U^{\varepsilon+\delta}, \\ 0 & \text{if } x \notin U^\delta. \end{cases}$$

Clearly, $\psi_\varepsilon^{U^\delta} \in \mathrm{Lip}_c(\Omega)$ for any $\delta, \varepsilon > 0$. Arguing as before, identity (5.9) becomes

$$\int_{U^\delta} \psi_\varepsilon^{U^\delta} \, \mathrm{d}div(\phi \boldsymbol{F}) = -\int_\delta^{\varepsilon+\delta} \int_{\partial U^t} \phi \boldsymbol{F} \cdot \nabla d \, \mathrm{d}\mathscr{H}^{n-1} \, \mathrm{d}t.$$

We use the test functions $\psi_\varepsilon^{U^\delta}$ and $\psi_{\varepsilon+h}^{U^\delta}$ for any $h > 0$, and then subtract the equation involving $\psi_{\varepsilon+h}^{U^\delta}$ from the one involving $\psi_\varepsilon^{U^\delta}$ to obtain

$$\begin{aligned} &h \int_{U^{\varepsilon+h+\delta}} \mathrm{d}div(\phi \boldsymbol{F}) + \int_{U^{\varepsilon+\delta} \backslash U^{\varepsilon+h+\delta}} \big(d(x) - \delta - \varepsilon\big) \, \mathrm{d}div(\phi \boldsymbol{F}) \\ &= -\int_{\varepsilon+\delta}^{\varepsilon+h+\delta} \int_{\partial U^t} \phi \boldsymbol{F} \cdot \nabla d \, \mathrm{d}\mathscr{H}^{n-1} \, \mathrm{d}t. \end{aligned}$$

We can divide by h and send $h \to 0$ to obtain

$$\int_{U^{\varepsilon+\delta}} \mathrm{d}div(\phi \boldsymbol{F}) = -\int_{\partial U^{\varepsilon+\delta}} \phi \boldsymbol{F} \cdot \nabla d \, \mathrm{d}\mathscr{H}^{n-1} \qquad \text{for } \mathscr{L}^1\text{–a.e. } \varepsilon, \delta > 0.$$

Now set $\varepsilon' := \varepsilon + \delta$. We choose a suitable sequence $\varepsilon'_k \to 0$ for which Lemma 5.2 applies so that (5.8) holds by (3.3).

3. Consider the case that $U \subset \Omega$ is not bounded. Then we take ϕ with bounded support in Ω. Thus, we can choose test functions $\eta \psi_\varepsilon^{U^\delta}$ for some $\varepsilon, \delta > 0$ and $\eta \in C_c^\infty(\Omega)$ satisfying $\eta \equiv 1$ on an open set V such that $\mathrm{supp}(\phi) \cap U^\delta \subset V \Subset \Omega$. Indeed, $\eta \psi_\varepsilon^{U^\delta} \in \mathrm{Lip}_c(\Omega)$ and $\phi \eta \psi_\varepsilon^{U^\delta} = \phi \psi_\varepsilon^{U^\delta}$. By the product rule (3.3), we have

$$\int_\Omega \phi \psi_\varepsilon^{U^\delta} \, \mathrm{d}div \boldsymbol{F} = \int_\Omega \eta \psi_\varepsilon^{U^\delta} \, \mathrm{d}div(\phi \boldsymbol{F}) - \int_\Omega \psi_\varepsilon^{U^\delta} \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d}x.$$

Again, by the product rule, we obtain that $\mathrm{supp}(div(\phi \boldsymbol{F})) \subset \mathrm{supp}\,(\phi)$, which implies

$$\int_\Omega \eta \psi_\varepsilon^{U^\delta} \, \mathrm{d}div(\phi \boldsymbol{F}) = \int_\Omega \psi_\varepsilon^{U^\delta} \, \mathrm{d}div(\phi \boldsymbol{F}),$$

since $\eta \equiv 1$ on $\mathrm{supp}(\phi) \cap U^\delta \supset \mathrm{supp}(div(\phi \boldsymbol{F})) \cap \mathrm{supp}(\psi_\varepsilon^{U^\delta})$. Therefore, from this point, we can repeat the same steps as before to conclude the proof.

Remark 5.2 Theorem 5.1 implies that we may take $U = \Omega$ in (5.8) to obtain the Gauss-Green formula up to the boundary of the open set where $\boldsymbol{F}$ is defined.

As an immediate consequence of Theorem 5.1, we obtain approximations of the classical Green's identities for scalar functions with gradients in $\mathcal{DM}^p(\Omega)$.

Theorem 5.2 (First Green's identity) *Let $u \in W^{1,p}(\Omega)$ for $1 \leqslant p \leqslant \infty$ be such that $\Delta u \in \mathcal{M}(\Omega)$, and let $U \subset \Omega$ be a bounded open set. Then, for any $\phi \in C^0(\Omega) \cap L^\infty(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega;\mathbb{R}^n)$, there exists a set $\mathcal{N} \subset \mathbb{R}$ with $\mathcal{L}^1(\mathcal{N}) = 0$ such that, for every nonnegative sequence $\{\varepsilon_k\}$ satisfying $\varepsilon_k \notin \mathcal{N}$ for any k and $\varepsilon_k \to 0$,*

$$\int_U \phi \,\mathrm{d}\Delta u + \int_U \nabla u \cdot \nabla\phi \,\mathrm{d}x = -\lim_{k\to\infty} \int_{\partial^* U^{\varepsilon_k}} \phi \nabla u \cdot \nu_{U^{\varepsilon_k}} \,\mathrm{d}\mathscr{H}^{n-1}, \tag{5.12}$$

where $\nu_{U^{\varepsilon_k}}$ is the inner unit normal to U^{ε_k} on $\partial^* U^{\varepsilon_k}$.

In particular, if $u \in W^{1,2}(\Omega) \cap C^0(\Omega) \cap L^\infty(\Omega)$ with $\Delta u \in \mathcal{M}(\Omega)$,

$$\int_U u \,\mathrm{d}\Delta u + \int_U |\nabla u|^2 \,\mathrm{d}x = -\lim_{k\to\infty} \int_{\partial^* U^{\varepsilon_k}} u \nabla u \cdot \nu_{U^{\varepsilon_k}} \,\mathrm{d}\mathscr{H}^{n-1}. \tag{5.13}$$

In addition, (5.12) *holds also for any open set $U \subset \Omega$, provided that* $\mathrm{supp}(\phi) \cap U^\delta \Subset \Omega$ *for any small $\delta > 0$. Analogously,* (5.13) *holds for any open set $U \subset \Omega$, provided that* $\mathrm{supp}(u) \cap U^\delta \Subset \Omega$ *for any $\delta > 0$.*

Proof In order to obtain (5.12), it suffices to apply Theorem 5.1 to the vector field $\boldsymbol{F} = \nabla u$, which clearly belongs to $\mathcal{DM}^p(\Omega)$. Then, if $u \in W^{1,2}(\Omega) \cap C^0(\Omega) \cap L^\infty(\Omega)$, we can take $\phi = u$ to obtain (5.13).

Corollary 5.1 (Second Green's identity) *Let $u \in W^{1,p}(\Omega) \cap C^0(\Omega) \cap L^\infty(\Omega)$ and $v \in W^{1,p'}(\Omega) \cap C^0(\Omega) \cap L^\infty(\Omega)$ for $1 \leqslant p \leqslant \infty$ be such that $\Delta u, \Delta v \in \mathcal{M}(\Omega)$, and let $U \subset \Omega$ be a bounded open set. Then there exists a set $\mathcal{N} \subset \mathbb{R}$ with $\mathcal{L}^1(\mathcal{N}) = 0$ such that, for every nonnegative sequence $\{\varepsilon_k\}$ satisfying $\varepsilon_k \notin \mathcal{N}$ for any k and $\varepsilon_k \to 0$,*

$$\int_U v \,\mathrm{d}\Delta u - u \,\mathrm{d}\Delta v = -\lim_{k\to\infty} \int_{\partial^* U^{\varepsilon_k}} (v\nabla u - u\nabla v) \cdot \nu_{U^{\varepsilon_k}} \,\mathrm{d}\mathscr{H}^{n-1}, \tag{5.14}$$

where $\nu_{U^{\varepsilon_k}}$ is the inner unit normal to U^{ε_k} on $\partial^* U^{\varepsilon_k}$. In addition, (5.14) *holds also for any open set $U \subset \Omega$, provided that* $\mathrm{supp}(u), \mathrm{supp}(v) \cap U^\delta \Subset \Omega$ *for any small $\delta > 0$.*

Proof We just need to apply Theorem 5.2 to the vector field ∇u, by using v as scalar function, and vice versa. Then we can obtain (5.12) for the vector fields ∇u and ∇v with the same sequence U^{ε_k}, since it is enough to select one sequence suitable for ∇u and then extract a subsequence for ∇v. Thus, we have

$$\int_U v \,\mathrm{d}\Delta u + \int_U \nabla u \cdot \nabla v \,\mathrm{d}x = -\lim_{k\to\infty} \int_{\partial^* U^{\varepsilon_k}} v \nabla u \cdot \nu_{U^{\varepsilon_k}} \,\mathrm{d}\mathscr{H}^{n-1},$$
$$\int_U u \,\mathrm{d}\Delta v + \int_U \nabla u \cdot \nabla v \,\mathrm{d}x = -\lim_{k\to\infty} \int_{\partial^* U^{\varepsilon_k}} u \nabla v \cdot \nu_{U^{\varepsilon_k}} \,\mathrm{d}\mathscr{H}^{n-1},$$

and subtracting the second equation from the first yields (5.14).

Theorem 5.3 (Exterior normal trace) *Let $U \Subset \Omega$ be an open set, and let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$. Then, for any $\phi \in C^0(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega;\mathbb{R}^n)$, there exists a set $\mathcal{N} \subset \mathbb{R}$ with $\mathcal{L}^1(\mathcal{N}) = 0$ such that, for every nonnegative sequence $\{\varepsilon_k\}$ satisfying $\varepsilon_k \notin \mathcal{N}$ for any k and $\varepsilon_k \to 0$, the following representation for the exterior normal trace on ∂U holds*:

$$\langle \boldsymbol{F}\cdot\nu, \phi\rangle_{\partial\overline{U}} = \int_{\overline{U}} \phi \,\mathrm{d}div\boldsymbol{F} + \int_{\overline{U}} \boldsymbol{F}\cdot\nabla\phi \,\mathrm{d}x = -\lim_{k\to\infty} \int_{\partial^* U_{\varepsilon_k}} \phi\boldsymbol{F}\cdot\nu_{U_{\varepsilon_k}} \,\mathrm{d}\mathscr{H}^{n-1}, \tag{5.15}$$

where $\nu_{U_{\varepsilon_k}}$ is the inner unit normal to U_{ε_k} on $\partial^* U_{\varepsilon_k}$. In addition, (5.15) *holds also for any open set U satisfying $\overline{U} \subset \Omega$, provided that* $\mathrm{supp}(\phi)$ *is compact in Ω.*

Proof We start with the case $U \Subset \Omega$. Then $U_\varepsilon \Subset \Omega$ for any $\varepsilon > 0$ small enough. We consider the Lipschitz functions:

$$\xi_\varepsilon^U(x) := \begin{cases} \varepsilon & \text{if } x \in U, \\ d(x) + \varepsilon & \text{if } x \in U_\varepsilon \backslash U, \\ 0 & \text{if } x \notin U_\varepsilon. \end{cases}$$

By Proposition 3.1, $\phi\boldsymbol{F} \in DM^p(\Omega')$ for any open set Ω' satisfying $U_\varepsilon \Subset \Omega' \Subset \Omega$ for any $\varepsilon > 0$ small enough. Thus, we can use ξ_ε^U as test functions to obtain

$$\begin{aligned} \int_{\Omega'} \xi_\varepsilon^U \,\mathrm{d}div(\phi\boldsymbol{F}) &= -\int_{\Omega'} \phi\boldsymbol{F}\cdot\nabla\xi_\varepsilon^U \,\mathrm{d}x = -\int_{U_\varepsilon\backslash U} \phi\boldsymbol{F}\cdot\nabla d \,\mathrm{d}x \\ &= -\int_0^\varepsilon \int_{\partial U_t} \phi\boldsymbol{F}\cdot\nabla d \,\mathrm{d}\mathscr{H}^{n-1}\mathrm{d}t, \end{aligned} \tag{5.16}$$

by the coarea formula (2.2) with $u = d$ and $g = \chi_{\Omega'\backslash U}\phi\boldsymbol{F}\cdot\nabla d|\nabla d|$, by Lemma 5.1 and Remark 5.1.

Now we proceed as in the proof of Theorem 5.1: Take ξ_ε^U and $\xi_{\varepsilon+h}^U$ for some $h > 0$ small enough as test functions so that

$$\int_{U_\varepsilon\backslash\overline{U}} \big(d(x)+\varepsilon\big)\,\mathrm{d}div(\phi\boldsymbol{F}) + \int_{\overline{U}} \varepsilon\,\mathrm{d}div(\phi\boldsymbol{F}) = -\int_0^\varepsilon \int_{\partial U_t} \phi\boldsymbol{F}\cdot\nabla d\,\mathrm{d}\mathscr{H}^{n-1}\,dt,$$

and

$$\begin{aligned} &\int_{U_{\varepsilon+h}\backslash\overline{U}} \big(d(x)+\varepsilon+h\big)\,\mathrm{d}div(\phi\boldsymbol{F}) + \int_{\overline{U}} (\varepsilon+h)\,\mathrm{d}div(\phi\boldsymbol{F}) \\ &= -\int_0^{\varepsilon+h} \int_{\partial U_t} \phi\boldsymbol{F}\cdot\nabla d\,\mathrm{d}\mathscr{H}^{n-1}\,\mathrm{d}t. \end{aligned}$$

By subtracting the first equation from the second one, we have

$$\begin{aligned} &\int_{U_{\varepsilon+h}\backslash\overline{U_\varepsilon}} \big(d(x)+\varepsilon+h\big)\,\mathrm{d}div(\phi\boldsymbol{F}) + \int_{\overline{U_\varepsilon}} h\,\mathrm{d}div(\phi\boldsymbol{F}) \\ &= -\int_\varepsilon^{\varepsilon+h} \int_{\partial U_t} \phi\boldsymbol{F}\cdot\nabla d\,\mathrm{d}\mathscr{H}^{n-1}\,\mathrm{d}t. \end{aligned}$$

It is clear that $0 \leqslant d(x) + \varepsilon \leqslant h$ on $U_{\varepsilon+h} \backslash \overline{U_\varepsilon}$ and that $\bigcap_{h>0} U_{\varepsilon+h} = \overline{U_\varepsilon}$ for any $\varepsilon > 0$ implies

$$|div(\phi \boldsymbol{F})|(U_{\varepsilon+h} \backslash \overline{U_\varepsilon}) \to 0 \qquad \text{as } h \to 0.$$

Then we can divide by h and let $h \to 0$, by applying the Lebesgue theorem, to obtain that, for $\mathscr{L}^1$–*a.e.* $\varepsilon > 0$,

$$\int_{\overline{U_\varepsilon}} \phi \, \mathrm{d} div \boldsymbol{F} + \int_{\overline{U_\varepsilon}} \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d}x = - \int_{\partial U_\varepsilon} \phi \boldsymbol{F} \cdot \nabla d \, \mathrm{d}\mathscr{H}^{n-1}, \tag{5.17}$$

by the product rule (3.3). We now choose a sequence $\varepsilon_k \to 0$ such that (5.17) holds and pass to the limit to obtain

$$\int_{\overline{U}} \phi \, \mathrm{d} div \boldsymbol{F} + \int_{\overline{U}} \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d}x = - \lim_{k \to \infty} \int_{\partial U_{\varepsilon_k}} \phi \boldsymbol{F} \cdot \nabla d \, \mathrm{d}\mathscr{H}^{n-1}. \tag{5.18}$$

As in the proof of Theorem 5.1, we can choose the sequence in such a way that the assertion in Lemma 5.2 also holds. Thus, we obtain the result.

In the general case, $\overline{U} \subset \Omega$, and $\mathrm{supp}(\phi)$ is a compact subset of Ω. Hence, we can argue as in the last part of the proof of Theorem 5.1, by taking a smooth cutoff function $\eta \in C_c^\infty(\Omega)$ such that $\eta \equiv 1$ on an open neighborhood V of $\mathrm{supp}(\phi)$. Following the same steps and replacing ξ_ε^U as test functions, we obtain the desired result.

Remark 5.3 The previous results apply in particular to the case when U is an open set of finite perimeter.

Remark 5.4 As a byproduct of the proofs of Theorems 5.1 and 5.3, we obtain the Gauss-Green formulas for almost every set that is approximating a given open set U from the interior and the exterior. More precisely, if $U \Subset \Omega$ is an open set, $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$, and $\phi \in C^0(\Omega)$ with $\nabla \phi \in L^{p'}(\Omega; \mathbb{R}^n)$, then, for $\mathscr{L}^1$–*a.e.* $\varepsilon > 0$,

$$\int_{U^\varepsilon} \phi \, \mathrm{d} div \boldsymbol{F} + \int_{U^\varepsilon} \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d}x = - \int_{\partial^* U^\varepsilon} \phi \boldsymbol{F} \cdot \nu_{U^\varepsilon} \, \mathrm{d}\mathscr{H}^{n-1}, \tag{5.19}$$

$$\int_{\overline{U_\varepsilon}} \phi \, d div \boldsymbol{F} + \int_{U_\varepsilon} \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d}x = - \int_{\partial^* U_\varepsilon} \phi \boldsymbol{F} \cdot \nu_{U_\varepsilon} \, \mathrm{d}\mathscr{H}^{n-1}. \tag{5.20}$$

This follows from (5.10) and (5.17) and by taking $\varepsilon > 0$ (up to another negligible set) such that Lemma 5.2 holds, so that $\mathscr{H}^{n-1}(\partial U^\varepsilon \backslash \partial^* U^\varepsilon) = 0$, $\mathscr{H}^{n-1}(\partial U_\varepsilon \backslash \partial^* U_\varepsilon) = 0$ (which implies that $|\partial U^\varepsilon| = 0$), $\nabla d = \nu_{U^\varepsilon}$ $\mathscr{H}^{n-1}$–*a.e.* on $\partial^* U^\varepsilon$, and $\nabla d = \nu_{U_\varepsilon}$ $\mathscr{H}^{n-1}$–*a.e.* on $\partial^* U_\varepsilon$.

In addition, (5.19)–(5.20) also hold for any open set U satisfying $\overline{U} \subset \Omega$, provided that $\mathrm{supp}(\phi)$ is compact in Ω. In general, this statement is valid for $\mathscr{L}^1$–*a.e.* $\varepsilon > 0$ because we need to apply Lemma 5.2 and to derive the integrals in $h > 0$:

$$\int_\varepsilon^{\varepsilon+h} \int_{\partial U^t} \phi \boldsymbol{F} \cdot \nabla d \, \mathrm{d}\mathscr{H}^{n-1} \, \mathrm{d}t, \qquad \int_\varepsilon^{\varepsilon+h} \int_{\partial U_t} \phi \boldsymbol{F} \cdot \nabla d \, \mathrm{d}\mathscr{H}^{n-1} \, \mathrm{d}t.$$

Therefore, such a condition may be removed as long as the conclusions of Lemma 5.2 hold for any $\varepsilon > 0$, and $\int_{\partial U^t} \phi \boldsymbol{F} \cdot \nabla d \, \mathrm{d}\mathscr{H}^{n-1}$ and $\int_{\partial U_t} \phi \boldsymbol{F} \cdot \nabla d \, \mathrm{d}\mathscr{H}^{n-1}$ are continuous functions of $t > 0$.

Remark 5.5 It is not necessary to use the signed distance function to construct a family of approximating sets suitable for Theorems 5.1 and 5.3. Such an argument is related to the one in [57, Theorem 2.4].

If, for a given open set $U \Subset \Omega$, there exists a function $m \in \mathrm{Lip}(\Omega)$ satisfying $m > 0$ in U, $m = 0$ on ∂U, and $\mathrm{essinf}(|\nabla m|) > 0$ in U, then sets $\{m > \varepsilon\}, \varepsilon \in \mathbb{R}$, can be used for the approximation. In fact, sets $\{m > \varepsilon\}$ are of finite perimeter for $\mathscr{L}^1$–*a.e.* $\varepsilon > 0$ and, for such good values of ε, the measure-theoretic unit interior normals satisfy

$$\nu_{\{m>\varepsilon\}} = \frac{\nabla m}{|\nabla m|} \qquad \mathscr{H}^{n-1}\text{–a.e. on } \partial^*\{m > \varepsilon\}.$$

In addition, if there exists such a function m with $C^k, k \geqslant 2$, or C^∞ regularity, then $\{m > \varepsilon\}$ has a C^k or smooth boundary.

As we will see in §7, if U is an open bounded set with C^0 boundary, then there exists a smooth regularized distance ρ satisfying the previously mentioned properties. For a general open set U, this may be false.

We can extend Theorem 5.3 to any compact set $K \subset \Omega$, in the spirit of [53, Theorem 5.20]. Indeed, we just need to choose the following Lipschitz functions as test functions:

$$\varphi_K^\varepsilon(x) := \begin{cases} \varepsilon & \text{if } \mathrm{dist}(x, K) = 0, \\ \varepsilon - \mathrm{dist}(x, K) & \text{if } 0 < \mathrm{dist}(x, K) < \varepsilon, \\ 0 & \text{if } \mathrm{dist}(x, K) \geqslant \varepsilon, \end{cases}$$

and then argue as in the proof of Theorem 5.3 to achieve the following result.

Corollary 5.2 *Let $K \subset \Omega$ be a compact set, and let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$. Then, for any $\phi \in C^0(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega; \mathbb{R}^n)$, there exists a set $\mathcal{N} \subset \mathbb{R}$ with $\mathcal{L}^1(\mathcal{N}) = 0$ such that, for every nonnegative sequence $\{\varepsilon_k\}$ satisfying $\varepsilon_k \notin \mathcal{N}$ for any k and $\varepsilon_k \to 0$,*

$$\int_K \phi \, \mathrm{d}div\boldsymbol{F} + \int_K \boldsymbol{F} \cdot \nabla\phi \, \mathrm{d}x = \lim_{k\to\infty} \int_{\partial K_{\varepsilon_k}} \phi \boldsymbol{F} \cdot \nabla \mathrm{dist}(x, K) \, \mathrm{d}\mathscr{H}^{n-1}, \quad (5.21)$$

where $K_\varepsilon := \{x \in \Omega : \mathrm{dist}(x, K) \leqslant \varepsilon\}$. In addition, (5.21) holds also for any closed set $C \subset \Omega$, provided that $\mathrm{supp}(\phi)$ *is compact in Ω.*

The right-hand side of (5.21) can be seen as the definition of the generalized normal trace functional related to $\boldsymbol{F}$ on ∂K, where $\nabla \mathrm{dist}(x, K)$ plays the role of a generalized unit exterior normal, even in the case $\mathring{K} = \varnothing$.

Remark 5.6 The results of Schuricht [53, Theorem 5.20] and Šilhavý [57, Theorem 2.4] can be recovered by (5.16) and (5.9), respectively.

Indeed, under the same assumptions of Theorem 5.1, we divide by ε in (5.9), use the product rule (3.3), and send $\varepsilon \to 0$ to obtain

$$\int_U \phi \,\mathrm{d}div\boldsymbol{F} + \int_U \boldsymbol{F}\cdot\nabla\phi\,\mathrm{d}x = -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{U\setminus U^\varepsilon}\phi\boldsymbol{F}\cdot\nabla d\,\mathrm{d}x, \tag{5.22}$$

since $0 \leqslant \frac{\psi_\varepsilon^U}{\varepsilon} \leqslant 1$ on $U\setminus U^\varepsilon$ and $|div(\boldsymbol{F}\phi)|(U\setminus U^\varepsilon) \to 0$ as $\varepsilon \to 0$. On the other hand, applying the same steps to (5.16) yields

$$\int_{\overline{U}} \phi \,\mathrm{d}div\boldsymbol{F} + \int_{\overline{U}} \boldsymbol{F}\cdot\nabla\phi\,\mathrm{d}x = -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{U_\varepsilon\setminus U}\phi\boldsymbol{F}\cdot\nabla d\,\mathrm{d}x, \tag{5.23}$$

if $\boldsymbol{F}, \phi$, and U satisfy the conditions of Theorem 5.3. In particular, this works for any compact set $K \subset \Omega$, as in Corollary 5.2:

$$\int_K \phi \,\mathrm{d}div\boldsymbol{F} + \int_K \boldsymbol{F}\cdot\nabla\phi\,\mathrm{d}x = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{K_\varepsilon\setminus K}\phi\boldsymbol{F}\cdot\nabla\mathrm{dist}(x,K)\,\mathrm{d}x. \tag{5.24}$$

Remark 5.7 Formulas (5.8) and (5.15) can be used to obtain the Gauss-Green formula on the boundary of $U \Subset \Omega$:

$$\begin{aligned} div(\boldsymbol{F}\phi)(\partial U) &= div(\boldsymbol{F}\phi)(\overline{U}) - div(\boldsymbol{F}\phi)(U) \\ &= \lim_{k\to\infty}\Big(\int_{\partial^* U^{\varepsilon_k}} \phi\boldsymbol{F}\cdot\nu_{U^{\varepsilon_k}}\,\mathrm{d}\mathscr{H}^{n-1} - \int_{\partial^* U_{\varepsilon_k}} \phi\boldsymbol{F}\cdot\nu_{U_{\varepsilon_k}}\,\mathrm{d}\mathscr{H}^{n-1}\Big), \end{aligned} \tag{5.25}$$

since we can extract the same subsequence ε_k for $\overline{U}$ and U. The same result holds for U such that $\overline{U} \subset \Omega$ if ϕ has compact support in Ω.

Remark 5.8 If $U = B(x_0, r)$, we obtain the Gauss-Green formula for $\mathscr{L}^1$–*a.e.* $r > 0$.

Indeed, $\mathrm{dist}(x, \partial B(x_0, r)) = r - |x - x_0|$ for any $x \in B(x_0, r)$ so that (5.19) implies that, for $\mathscr{L}^1$–*a.e.* $\varepsilon \in (0, r)$,

$$\begin{aligned} \int_{B(x_0,r-\varepsilon)} \mathrm{d}div(\phi\boldsymbol{F}) &= \int_{\partial B(x_0,r-\varepsilon)} \phi(x)\boldsymbol{F}(x)\cdot\frac{(x-x_0)}{|x-x_0|}\,\mathrm{d}\mathscr{H}^{n-1}(x) \\ &= -\int_{\partial B(x_0,r-\varepsilon)} \phi\boldsymbol{F}\cdot\nu_{B(x_0,r-\varepsilon)}\,\mathrm{d}\mathscr{H}^{n-1}. \end{aligned}$$

Since the initial choice of r is arbitrary, we conclude

$$\int_{B(x_0,r)} \mathrm{d}div(\phi\boldsymbol{F}) = -\int_{\partial B(x_0,r)} \phi\boldsymbol{F}\cdot\nu_{B(x_0,r)}\,\mathrm{d}\mathscr{H}^{n-1} \qquad \text{for } \mathscr{L}^1\text{–a.e. } r > 0. \tag{5.26}$$

Moreover, the same argument works with closed balls so that, by (5.20),

$$\int_{\overline{B(x_0,r)}} \mathrm{d}div(\phi\boldsymbol{F}) = -\int_{\partial B(x_0,r)} \phi\boldsymbol{F}\cdot\nu_{B(x_0,r)}\,\mathrm{d}\mathscr{H}^{n-1} \qquad \text{for } \mathscr{L}^1\text{–a.e. } r>0, \tag{5.27}$$

since $\mathrm{dist}(x,\partial B(x_0,r)) = |x-x_0|-r$ for any $x\notin B(x_0,r)$.

This can also be seen as a consequence of the fact:

$$|div(\phi\boldsymbol{F})|(\partial B(x_0,r)) = 0 \qquad \text{for } \mathscr{L}^1\text{–a.e. } r>0,$$

since $div(\phi\boldsymbol{F})$ is a Radon measure.

We now present a concrete example of applications of (5.8) and (5.15) to a $\mathcal{DM}^p$–field whose norm blows up on the boundary of the integration domains.

Example 5.1 Let $\boldsymbol{F}:\mathbb{R}^2\backslash\{(0,0)\}\to\mathbb{R}^2$ be the vector field:

$$\boldsymbol{F}(x_1,x_2) := \frac{(x_1,x_2)}{x_1^2+x_2^2}. \tag{5.28}$$

This is the particular case for $n=2$ of the vector field $\boldsymbol{F}:\mathbb{R}^n\backslash\{\mathbf{0}\}\to\mathbb{R}^n$ given by

$$\boldsymbol{F}(x) := \frac{x}{|x|^n} \qquad \text{for } x\neq\mathbf{0}.$$

Then $\boldsymbol{F}\in\mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}^n)$ for $1\leqslant p<\frac{n}{n-1}$ and

$$div\boldsymbol{F} = n\omega_n\delta_{\mathbf{0}}, \tag{5.29}$$

where $\omega_n = |B(\mathbf{0},1)|$. In particular, if $n=2$, $\boldsymbol{F}\in\mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}^2)$ for $1\leqslant p<2$, and $div\boldsymbol{F} = 2\pi\delta_{(0,0)}$.

Consider $U=(0,1)^2$. Chen-Frid [12, Example 1.1] observed that

$$0 = div\boldsymbol{F}(U) \neq -\int_{\partial U}\boldsymbol{F}\cdot\nu_U\,\mathrm{d}\mathscr{H}^1 = \frac{\pi}{2},$$

since $\boldsymbol{F}\cdot\nu_U = 0$ on $\big(\{0\}\times(0,1)\big)\cup\big((0,1)\times\{0\}\big)$ and $\displaystyle\int_0^1\frac{1}{1+x_1^2}\,\mathrm{d}x_1 = \frac{\pi}{4}$.

The approach employed in the proof of Theorems 5.1 and 5.3 enable us to solve this apparent contradiction, by showing that

$$0 = div\boldsymbol{F}(U) = -\lim_{\varepsilon\to0}\int_{\partial U^\varepsilon}\boldsymbol{F}\cdot\nu_{U^\varepsilon}\,\mathrm{d}\mathscr{H}^1,$$
$$2\pi = div\boldsymbol{F}(\overline{U}) = -\lim_{\varepsilon\to0}\int_{\partial U_\varepsilon}\boldsymbol{F}\cdot\nu_{U_\varepsilon}\,\mathrm{d}\mathscr{H}^1,$$

where U^ε and U_ε are given by (5.2) and (5.3), respectively.

In this case, we do not have to select a suitable sequence $\varepsilon_k\to0$. Indeed, $\boldsymbol{F}$ is smooth away from the origin, and U^ε and U_ε are sets of finite perimeter for any $\varepsilon>0$. Moreover, for this choice of U, Lemma 5.2 is valid for any

$\varepsilon \in (0,1)$. Also, the continuity condition mentioned in Remark 5.4 can be checked. Therefore, by (5.19)–(5.20), we obtain that, for any $\varepsilon > 0$,

$$0 = div\boldsymbol{F}(U^\varepsilon) = -\int_{\partial U^\varepsilon} \boldsymbol{F}\cdot\nu_{U^\varepsilon}\,\mathrm{d}\mathscr{H}^1, \tag{5.30}$$

$$2\pi = div\boldsymbol{F}(\overline{U_\varepsilon}) = -\int_{\partial U_\varepsilon} \boldsymbol{F}\cdot\nu_{U_\varepsilon}\,\mathrm{d}\mathscr{H}^1. \tag{5.31}$$

Passing to the limit verifies our assertion.

We may also verify this statement by hand. Observe that $U^\varepsilon = (\varepsilon, 1-\varepsilon)^2$ for any $\varepsilon \in (0,1)$. Therefore, we have

$$\begin{aligned}\int_{\partial U^\varepsilon} \boldsymbol{F}\cdot\nu_{U^\varepsilon}\,\mathrm{d}\mathscr{H}^1 &= \int_\varepsilon^{1-\varepsilon} \frac{\varepsilon}{\varepsilon^2+x_1^2}\,\mathrm{d}x_1 - \int_\varepsilon^{1-\varepsilon} \frac{1-\varepsilon}{(1-\varepsilon)^2+x_1^2}\,\mathrm{d}x_1\\ &\quad + \int_\varepsilon^{1-\varepsilon} \frac{\varepsilon}{\varepsilon^2+x_2^2}\,\mathrm{d}x_2 - \int_\varepsilon^{1-\varepsilon} \frac{1-\varepsilon}{(1-\varepsilon)^2+x_2^2}\,\mathrm{d}x_2\\ &= 2\Big(\arctan\big(\frac{1-\varepsilon}{\varepsilon}\big) - \frac{\pi}{4} - \frac{\pi}{4} + \arctan\big(\frac{\varepsilon}{1-\varepsilon}\big)\Big)\\ &= 0 = -div\boldsymbol{F}(U^\varepsilon)\end{aligned}$$

for any $\varepsilon > 0$, which is (5.30). As for (5.31), ∂U_ε is the union of four segments:

$$(0,1)\times\{-\varepsilon\},\quad \{1+\varepsilon\}\times(0,1),\quad (0,1)\times\{1+\varepsilon\},\quad \{-\varepsilon\}\times(0,1),$$

and of four circumference arcs of angle $\frac{\pi}{2}$ and radius ε centered at the corners of square U. Therefore, these terms give

$$\begin{aligned}&\int_{\partial U_\varepsilon} \boldsymbol{F}\cdot\nu_{U_\varepsilon}\,\mathrm{d}\mathscr{H}^1\\ &= -\int_0^1 \frac{\varepsilon}{\varepsilon^2+x_1^2}\,\mathrm{d}x_1 - \int_0^1 \frac{1+\varepsilon}{(1+\varepsilon)^2+x_2^2}\,\mathrm{d}x_2 - \int_0^1 \frac{1+\varepsilon}{(1+\varepsilon)^2+x_1^2}\,\mathrm{d}x_1\\ &\quad - \int_0^1 \frac{\varepsilon}{\varepsilon^2+x_2^2}\,\mathrm{d}x_2 + \int_\pi^{\frac{3\pi}{2}} \big(-\frac{1}{\varepsilon}\big)\varepsilon\,\mathrm{d}\theta - \int_{\frac{\pi}{2}}^{\pi} \frac{\varepsilon(\varepsilon+\sin\theta)}{1+\varepsilon^2+2\varepsilon\sin\theta}\,\mathrm{d}\theta\\ &\quad - \int_{\frac{3\pi}{2}}^{2\pi} \frac{\varepsilon(\varepsilon+\cos\theta)}{1+\varepsilon^2+2\varepsilon\cos\theta}\,\mathrm{d}\theta - \int_0^{\frac{\pi}{2}} \frac{\varepsilon(\varepsilon+\cos\theta+\sin\theta)}{2+\varepsilon^2+2\varepsilon(\cos\theta+\sin\theta)}\,\mathrm{d}\theta\\ &= -2\arctan\big(\frac{1}{\varepsilon}\big) - 2\arctan\big(\frac{1}{1+\varepsilon}\big) - \frac{\pi}{2} - \frac{\pi}{4} + \arctan\big(\frac{1-\varepsilon}{1+\varepsilon}\big)\\ &\quad - \frac{\pi}{2} + \arctan\big(\frac{1}{\varepsilon}\big) - \frac{\pi}{4} - \arctan\big(\frac{\varepsilon}{\varepsilon+2}\big) + \arctan\big(\frac{1}{1+\varepsilon}\big)\\ &= -\frac{3}{2}\pi - \arctan\big(\frac{1}{\varepsilon}\big) - \arctan\big(\frac{1}{1+\varepsilon}\big) + \frac{\pi}{2} - \arctan\varepsilon - \arctan(1+\varepsilon)\\ &= -\frac{3}{2}\pi - \frac{\pi}{2} = -2\pi = -div\boldsymbol{F}(U_\varepsilon) \qquad \text{for any } \varepsilon > 0.\end{aligned}$$

6 Other Classes of Divergence-Measure Fields with Normal Trace Measure

In this section, as a result of the construction in §5, we characterize a class of $\mathcal{DM}^p$–fields whose normal trace on ∂U is represented by a Radon measure.

Remark 5.6 allows us to find a new sufficient condition under which the normal trace functional on an open or closed set can be represented by a Radon measure on the boundary. Such a condition requires a particular representation for the vector field $\boldsymbol{F}$, first introduced by Šilhavý [56, Proposition 6.1]. We also need to recall the notion of lower $(n-1)$–dimensional Minkowski content.

Definition 6.1 Given a closed set K in $\mathbb{R}^n$, the $(n-1)$–dimensional Minkowski content is defined as

$$\mathscr{M}_*^{n-1}(K) := \liminf_{\varepsilon \to 0} \frac{|K + B(0,\varepsilon)|}{2\varepsilon}.$$

Proposition 6.1 *Let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$, and let $U \subset \Omega$ be a bounded open set such that $\mathscr{M}_*^{n-1}(\partial U) < \infty$. Assume that div$\boldsymbol{F}$ has compact support in U and that*

$$\boldsymbol{F}(x) = \frac{1}{n\omega_n} \int_\Omega \frac{(x-y)}{|x-y|^n} \, \mathrm{d}\mathrm{div}\boldsymbol{F}(y) \qquad \text{for } \mathscr{L}^n\text{–a.e. } x \in \Omega. \tag{6.1}$$

Then

$$\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial U} \in \mathcal{M}(\partial U).$$

Similarly, if $K \subset \Omega$ is a compact set such that $\mathscr{M}_*^{n-1}(\partial K) < \infty$, then

$$\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial K} \in \mathcal{M}(\partial K),$$

which is in particular true for $K = \overline{U}$ if $U \Subset \Omega$.

Proof The normal trace $\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial U}$ has the following representation:

$$\langle \boldsymbol{F} \cdot \nu, \varphi \rangle_{\partial U} = -\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{U \setminus U^\varepsilon} \varphi(x) \boldsymbol{F}(x) \cdot \nabla d(x) \, \mathrm{d}x \qquad \text{for any } \varphi \in \mathrm{Lip}_c(\mathbb{R}^n); \tag{6.2}$$

see (5.22) and the observations in Remark 5.6. Thus, in order to prove that $\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial U} \in \mathcal{M}(\partial U)$, it suffices to show

$$|\langle \boldsymbol{F} \cdot \nu, \varphi \rangle_{\partial U}| \leqslant C \, \|\varphi\|_{L^\infty(\partial U)} \tag{6.3}$$

for a constant C independent of φ. Let $V \subset U$ be a compact set such that $\operatorname{supp}(|div\boldsymbol{F}|) \subset V$. Then it follows that

$$\begin{aligned}
\big|\langle \boldsymbol{F}\cdot\nu,\varphi\rangle_{\partial U}\big| &= \lim_{\varepsilon\to 0}\Big|\frac{1}{\varepsilon}\int_{U\setminus U^\varepsilon}\varphi(x)\boldsymbol{F}(x)\cdot\nabla d(x)\,\mathrm{d}x\Big| \\
&= \lim_{\varepsilon\to 0}\Big|\frac{1}{\varepsilon}\int_{U\setminus U^\varepsilon}\varphi(x)\Big(\int_\Omega \frac{(x-y)}{|x-y|^n}\,\mathrm{d}div\boldsymbol{F}(y)\cdot\nabla d(x)\Big)\,\mathrm{d}x\Big| \\
&\leqslant \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\|\varphi\|_{L^\infty(U\setminus U^\varepsilon)}\int_{U\setminus U^\varepsilon}\int_V \frac{1}{|x-y|^{n-1}}\,\mathrm{d}|div\boldsymbol{F}|(y)\,\mathrm{d}x \\
&= \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\|\varphi\|_{L^\infty(U\setminus U^\varepsilon)}\int_V\int_{U\setminus U^\varepsilon} \frac{1}{|x-y|^{n-1}}\,\mathrm{d}x\,\mathrm{d}|div\boldsymbol{F}|(y),
\end{aligned} \tag{6.4}$$

where we have used Fubini's theorem and the fact that $|\nabla d| = 1$ $\mathscr{L}^n$–*a.e.* (Lemma 5.1). Moreover, $\lim_{\varepsilon\to 0}\sup_{U\setminus U^\varepsilon}|\varphi| = \|\varphi\|_{L^\infty(\partial U)}$, by the continuity of φ.

Since $V \subset U$ is a compact set, there exists $k = k(V)$ such that, for small enough ε,

$$|x-y| \geqslant k \qquad \text{for any } x \in U\setminus U^\varepsilon \text{ and } y \in V.$$

Then it follows that

$$\begin{aligned}
|\langle \boldsymbol{F}\cdot\nu,\varphi\rangle_{\partial U}| &\leqslant \|\varphi\|_{L^\infty(\partial U)}\,|div\boldsymbol{F}|(V)k^{1-n}\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}|U\setminus U^\varepsilon| \\
&\leqslant 2k^{1-n}|div\boldsymbol{F}|(V)\,\|\varphi\|_{L^\infty(\partial U)}\liminf_{\varepsilon\to 0}\frac{1}{2\varepsilon}|\partial U + B(0,\varepsilon)| \\
&\leqslant 2k^{1-n}|div\boldsymbol{F}|(V)\mathscr{M}_*^{n-1}(\partial U)\,\|\varphi\|_{L^\infty(\partial U)}\,.
\end{aligned}$$

This proves (6.3), since ∂U is of finite lower $(n-1)$–dimensional Minkowski content.

In the same way, using (5.24), we can show that $\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial K} \in \mathcal{M}(\partial K)$ if $K \subset \Omega$ is a compact set, especially when $K = \overline{U}$ for some bounded open set U.

Remark 6.1 Condition (6.1) is not strongly restrictive in the sense that $\boldsymbol{F}$ may not be compactly supported and unbounded. Indeed, let $\boldsymbol{F}(x) = \frac{x}{|x|^n}$ as in Example 5.1. Then, by (5.29), $\boldsymbol{F}$ satisfies (6.1), even though $\boldsymbol{F}$ is unbounded and supported on the whole $\mathbb{R}^n$.

Moreover, (6.1) is satisfied by a large class of vector fields $\boldsymbol{F}$, as shown in [56, Proposition 6.1]. Indeed, given any $\mu \in \mathcal{M}(\Omega)$ with compact support in Ω, the vector field

$$\boldsymbol{F}(x) = \frac{1}{n\omega_n}\int_\Omega \frac{(x-y)}{|x-y|^n}\,\mathrm{d}\mu(y)$$

satisfies $div\boldsymbol{F} = \mu$ in $\mathcal{M}(\Omega)$, and $\boldsymbol{F} \in L^p_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ for any $1 \leqslant p < \frac{n}{n-1}$. In addition, if $\frac{n}{n-1} \leqslant p \leqslant \infty$, then $\boldsymbol{F} \in L^p_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ if $|\mu|(B(x,r)) \leqslant cr^m$ for any $x \in \mathbb{R}^n$ and $r \in (0,a)$, for some $m > n - \frac{p}{p-1}$, $a > 0$, and $c > 0$.

Remark 6.2 Proposition 6.1 applies to a particular subfamily of sets of finite perimeter. Indeed, any bounded open set U with $\mathscr{M}_*^{n-1}(\partial U) < \infty$ is a set of finite perimeter in $\mathbb{R}^n$.

Even though the result is well known, we give here a short proof for the ease of the reader. Let

$$g_\varepsilon(x) := \max\{0, 1 - \frac{\operatorname{dist}(x, U^\varepsilon)}{\varepsilon}\}.$$

Then $g_\varepsilon \to \chi_U$ in $L^1(\mathbb{R}^n)$, and $|\nabla g_\varepsilon| = \frac{1}{\varepsilon}\chi_{U\setminus U^\varepsilon}$. Thus, for any $\phi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} \chi_U \operatorname{div}\phi \,\mathrm{d}x\Big| &= \lim_{\varepsilon\to 0}\Big|\int_{\mathbb{R}^n} g_\varepsilon \operatorname{div}\phi \,\mathrm{d}x\Big| = \lim_{\varepsilon\to 0}\Big|\int_{\mathbb{R}^n} \phi\cdot\nabla g_\varepsilon \,\mathrm{d}x\Big| \\ &\leqslant \|\phi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \liminf_{\varepsilon\to 0}\frac{|U\setminus U^\varepsilon|}{\varepsilon} \leqslant 2\mathscr{M}_*^{n-1}(\partial U)\|\phi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)}.\end{aligned}$$

This implies that U is a set of finite perimeter with $|D\chi_U|(\mathbb{R}^n) \leqslant 2\mathscr{M}_*^{n-1}(\partial U)$.

Arguing analogously, we can also show that any compact set K with $\mathscr{M}_*^{n-1}(\partial K) < \infty$ is a set of finite perimeter in $\mathbb{R}^n$, with $|D\chi_K|(\mathbb{R}^n) \leqslant 2\mathscr{M}_*^{n-1}(\partial K)$. This can be shown by considering the functions:

$$f_\varepsilon(x) := \max\{0, 1 - \frac{d(x, K)}{\varepsilon}\},$$

which satisfy that $f_\varepsilon \to \chi_K$ in $L^1(\mathbb{R}^n)$ and $|\nabla f_\varepsilon| = \frac{1}{\varepsilon}\chi_{K_\varepsilon\setminus K}$.

In the case $p = \infty$, assumption (6.1) is superfluous, as shown in the following proposition, which can be seen as a particular case of [57, Theorem 2.4] and [32, Theorem 2.4].

Proposition 6.2 *Let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$, and let $U \subset \Omega$ be a bounded open set such that $\mathscr{M}_*^{n-1}(\partial U) < \infty$. If $p = \infty$, or $1 \leqslant p < \infty$ and $\boldsymbol{F}$ satisfies*

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{U\setminus U^\varepsilon} |\boldsymbol{F}|^p \,\mathrm{d}x < \infty, \tag{6.5}$$

then

$$\langle \boldsymbol{F}\cdot\nu, \cdot\rangle_{\partial U} \in \mathcal{M}(\partial U).$$

Analogously, let $K \subset \Omega$ be a compact set such that $\mathscr{M}_*^{n-1}(\partial K) < \infty$. If $p = \infty$, or $1 \leqslant p < \infty$ and $\boldsymbol{F}$ satisfies

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{K_\varepsilon\setminus K} |\boldsymbol{F}|^p \,\mathrm{d}x < \infty, \tag{6.6}$$

then

$$\langle \boldsymbol{F}\cdot\nu, \cdot\rangle_{\partial K} \in \mathcal{M}(\partial K).$$

In particular, this implies that, if $U \Subset \Omega$, $\mathscr{M}_*^{n-1}(\partial U) < \infty$, and the same assumption on $\boldsymbol{F}$ is made with $K = \overline{U}$, then

$$\langle \boldsymbol{F}\cdot\nu, \cdot\rangle_{\partial \overline{U}} \in \mathcal{M}(\partial \overline{U}).$$

Proof Arguing as in the proof of Proposition 6.1, we see that, for any $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n)$,

$$\begin{aligned}\big|\langle \boldsymbol{F}\cdot\nu,\varphi\rangle_{\partial U}\big| &= \lim_{\varepsilon\to 0}\Big|\frac{1}{\varepsilon}\int_{U\setminus U^\varepsilon}\varphi(x)\boldsymbol{F}(x)\cdot\nabla d(x)\,\mathrm{d}x\Big|\\ &\leqslant \liminf_{\varepsilon\to 0}\|\varphi\|_{L^\infty(U\setminus U^\varepsilon)}\,\frac{1}{\varepsilon}|U\setminus U^\varepsilon|^{\frac{1}{p'}}\|\boldsymbol{F}\|_{L^p(U\setminus U^\varepsilon;\mathbb{R}^n)}.\end{aligned}$$

If $p=\infty$, we have

$$|\langle \boldsymbol{F}\cdot\nu,\varphi\rangle_{\partial U}| \leqslant 2\|\boldsymbol{F}\|_{L^\infty(U;\mathbb{R}^n)}\mathscr{M}_*^{n-1}(\partial U)\,\|\varphi\|_{L^\infty(\partial U)}.$$

If $1\leqslant p<\infty$, then

$$|\langle \boldsymbol{F}\cdot\nu,\varphi\rangle_{\partial U}| \leqslant \big(2\mathscr{M}_*^{n-1}(\partial U)\big)^{\frac{1}{p'}}\limsup_{\varepsilon\to 0}\varepsilon^{-\frac{1}{p}}\|\boldsymbol{F}\|_{L^p(U\setminus U^\varepsilon;\mathbb{R}^n)}\ \|\varphi\|_{L^\infty(\partial U)},$$

from which $\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial U}\in\mathcal{M}(\partial U)$, because of (6.5). The case of the compact set follows analogously from (5.24), by employing (6.6), if $\boldsymbol{F}\notin\mathcal{DM}^\infty(\Omega)$.

This proposition may also be seen as an alternative way of obtaining a part of the results of Proposition 4.3 in the case that E is an open or compact set whose boundary has finite Minkowski content.

Remark 6.3 In particular, Propositions 6.1 and 6.2 hold also for $U=\Omega$, when Ω is an open bounded set such that $\mathscr{M}_*^{n-1}(\partial\Omega)<\infty$.

Remark 6.4 We can reinterpret Theorems 5.1 and 5.3 in the distributional sense. Indeed, given $\boldsymbol{F}\in\mathcal{DM}^p(\Omega)$ for $1\leqslant p\leqslant\infty$, and an open bounded set $U\subset\Omega$, (5.8) is equivalent to the following: There exists a set $\mathcal{N}\subset\mathbb{R}$ with $\mathcal{L}^1(\mathcal{N})=0$ such that, for every nonnegative sequence $\{\varepsilon_k\}$ satisfying $\varepsilon_k\notin\mathcal{N}$ for any k and $\varepsilon_k\to 0$,

$$\boldsymbol{F}\cdot\nu_{U^{\varepsilon_k}}\,\mathscr{H}^{n-1}\llcorner\partial U^{\varepsilon_k} = \boldsymbol{F}\cdot D\chi_{U^{\varepsilon_k}} \overset{*}{\rightharpoonup}\langle F\cdot\nu,\cdot\rangle_{\partial U} \tag{6.7}$$

in the distributional sense on $\mathbb{R}^n$; that is, testing the traces against $\phi\in\mathrm{Lip}_c(\mathbb{R}^n)$.

Analogously, if $U\Subset\Omega$, (5.15) implies that there exists a set $\mathcal{N}'\subset\mathbb{R}$ with $\mathcal{L}^1(\mathcal{N}')=0$ such that, for every nonnegative sequence $\{\varepsilon_k\}$ satisfying $\varepsilon_k\notin\mathcal{N}$ for any k and $\varepsilon_k\to 0$,

$$\boldsymbol{F}\cdot\nu_{U_{\varepsilon_k}}\,\mathscr{H}^{n-1}\llcorner\partial U_{\varepsilon_k} = \boldsymbol{F}\cdot D\chi_{U_{\varepsilon_k}} \overset{*}{\rightharpoonup}\langle F\cdot\nu,\cdot\rangle_{\partial\overline{U}} \tag{6.8}$$

in the distributional sense on $\mathbb{R}^n$. In particular, this means that, if $\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial U}\in\mathcal{M}(\Omega)$, then, by the uniform boundedness principle, we have

$$\limsup_{k\to\infty}\|\boldsymbol{F}\cdot\nu_{U^{\varepsilon_k}}\|_{L^1(\partial^* U^{\varepsilon_k};\mathscr{H}^{n-1})}<\infty.$$

Analogously, if $\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial\overline{U}}\in\mathcal{M}(\Omega)$, then

$$\limsup_{k\to\infty}\|\boldsymbol{F}\cdot\nu_{U_{\varepsilon_k}}\|_{L^1(\partial^* U_{\varepsilon_k};\mathscr{H}^{n-1})}<\infty.$$

Furthermore, if U is an open set of finite perimeter in Ω, then

$$D\chi_{U^{\varepsilon_k}} \rightharpoonup D\chi_U$$

in the sense of Radon measures, where ε_k is a vanishing sequence for which the conclusions of Lemma 5.2 hold. Indeed, for any $\phi \in C_c^1(\Omega;\mathbb{R}^n)$,

$$-\int_\Omega \phi \cdot \mathrm{d}D\chi_{U^{\varepsilon_k}} = \int_\Omega \chi_{U^{\varepsilon_k}}\, div\phi\, \mathrm{d}x \longrightarrow \int_\Omega \chi_U\, div\phi\, \mathrm{d}x = -\int_\Omega \phi \cdot \mathrm{d}D\chi_U,$$

and the assertion follows by the density of $C_c^1(\Omega;\mathbb{R}^n)$ in $C_c(\Omega;\mathbb{R}^n)$ with respect to the supremum norm. If $\overline{U}$ is also a set of finite perimeter in Ω, then $D\chi_{U_{\varepsilon_k}} \rightharpoonup D\chi_{\overline{U}}$ analogously.

Thanks to Remark 6.4, we can show that the normal traces on open and closed sets of finite perimeter agree with the classical dot product, provided that $\boldsymbol{F}$ is continuous. It is true that $\boldsymbol{F} \in C^0(\Omega;\mathbb{R}^n) \cap \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$ implies that $\boldsymbol{F} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$. Thus, we may expect the existence of normal traces as locally bounded function by the known theory ([14, 16]). Through (6.7)–(6.8), we now give a more direct proof.

Proposition 6.3 *Let $\boldsymbol{F} \in C^0(\Omega;\mathbb{R}^n) \cap \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$, and let $U \subset \Omega$ be an open set of finite perimeter. Then*

$$\langle \boldsymbol{F}\cdot\nu, \cdot\rangle_{\partial U} = \boldsymbol{F}\cdot\nu_U \mathscr{H}^{n-1}\llcorner \partial^* U \qquad \text{in } \mathcal{M}_{\mathrm{loc}}(\Omega).$$

Similarly, if $\overline{U} \subset \Omega$ is a set of finite perimeter, then

$$\langle \boldsymbol{F}\cdot\nu, \cdot\rangle_{\partial \overline{U}} = \boldsymbol{F}\cdot\nu_{\overline{U}} \mathscr{H}^{n-1}\llcorner \partial^* \overline{U} \qquad \text{in } \mathcal{M}_{\mathrm{loc}}(\Omega).$$

In addition, if $U \Subset \Omega$, the previous identities hold in $\mathcal{M}(\Omega)$.

Proof Let $\varepsilon_k \to 0$ be a sequence such that both (6.7) and the conclusions of Lemma 5.2 hold. By Remark 6.4, we obtain

$$\int_{\partial U^{\varepsilon_k}} \phi\boldsymbol{F}\cdot\nu_{U^{\varepsilon_k}}\, \mathrm{d}\mathscr{H}^{n-1} \longrightarrow \int_{\partial^* U} \phi\boldsymbol{F}\cdot\nu_U\, \mathrm{d}\mathscr{H}^{n-1} \qquad \text{for any } \phi \in C_c(\Omega),$$

since $\phi\boldsymbol{F} \in C_c(\Omega;\mathbb{R}^n)$ and

$$\nu_{U^{\varepsilon_k}}\, \mathscr{H}^{n-1}\llcorner \partial^* U^{\varepsilon_k} = D\chi_{U^{\varepsilon_k}} \rightharpoonup D\chi_U = \nu_U\, \mathscr{H}^{n-1}\llcorner \partial^* U \qquad \text{in } \mathcal{M}(\Omega).$$

This implies

$$\langle \boldsymbol{F}\cdot\nu, \phi\rangle_{\partial U} = \int_{\partial^* U} \phi\boldsymbol{F}\cdot\nu_U\, \mathrm{d}\mathscr{H}^{n-1} \qquad \text{for any } \phi \in C_c(\Omega),$$

which means that $\langle \boldsymbol{F}\cdot\nu, \cdot\rangle_{\partial U} = \boldsymbol{F}\cdot\nu_U \mathscr{H}^{n-1}\llcorner \partial^* U$ in $\mathcal{M}_{\mathrm{loc}}(\Omega)$.

We can argue in a similar way with (6.8) and the fact that $D\chi_{U_{\varepsilon_k}} \rightharpoonup D\chi_{\overline{U}}$ in $\mathcal{M}(\Omega)$ to prove the second part of the statement.

Finally, if $U \Subset \Omega$, there exists $\eta \in C_c(\Omega)$ such that $\eta \equiv 1$ on $\overline{U}$. Hence, for any $\phi \in C^0(\Omega)$, $\eta\phi F \in C_c(\Omega;\mathbb{R}^n)$ so that

$$\begin{aligned}\int_{\partial U^{\varepsilon_k}} \phi \boldsymbol{F}\cdot\nu_{U^{\varepsilon_k}}\,\mathrm{d}\mathscr{H}^{n-1} &= \int_{\partial U^{\varepsilon_k}} \eta\phi \boldsymbol{F}\cdot\nu_{U^{\varepsilon_k}}\,\mathrm{d}\mathscr{H}^{n-1}\\ &\to \int_{\partial^* U} \eta\phi \boldsymbol{F}\cdot\nu_U\,\mathrm{d}\mathscr{H}^{n-1} = \int_{\partial^* U} \phi \boldsymbol{F}\cdot\nu_U\,\mathrm{d}\mathscr{H}^{n-1},\end{aligned}$$

which implies that $\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial U} = \boldsymbol{F}\cdot\nu_U\mathscr{H}^{n-1}\llcorner\partial^* U$ in $\mathcal{M}(\Omega)$. Arguing similarly for $\overline{U}$, we complete the proof.

This result also allows us to obtain Green's identities for scalar functions in $C^1(\Omega)$ with gradient in $\mathcal{DM}^p(\Omega)$ and open sets of finite perimeter.

Proposition 6.4 *Let $u \in C^1(\Omega)\cap W^{1,p}(\Omega)$ for $1 \leqslant p \leqslant \infty$ be such that $\Delta u \in \mathcal{M}(\Omega)$, and let $U \Subset \Omega$ be an open set of finite perimeter. Then, for any $\phi \in C^0(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega;\mathbb{R}^n)$,*

$$\int_U \phi\,\mathrm{d}\Delta u + \int_U \nabla u\cdot\nabla\phi\,\mathrm{d}x = -\int_{\partial^* U}\phi\nabla u\cdot\nu_U\,\mathrm{d}\mathscr{H}^{n-1}. \tag{6.9}$$

In particular, if $u \in C^1(\Omega)\cap W^{1,2}(\Omega)$ with $\Delta u \in \mathcal{M}(\Omega)$, then

$$\int_U u\,\mathrm{d}\Delta u + \int_U |\nabla u|^2\,\mathrm{d}x = -\int_{\partial^* U} u\nabla u\cdot\nu_U\,\mathrm{d}\mathscr{H}^{n-1}. \tag{6.10}$$

In addition, if $u \in C^1(\Omega)\cap W^{1,p}(\Omega)$ and $v \in C^1(\Omega)\cap W^{1,p'}(\Omega)$ for $1 \leqslant p \leqslant \infty$ with $\Delta u, \Delta v \in \mathcal{M}(\Omega)$, then

$$\int_U v\,\mathrm{d}\Delta u - u\,\mathrm{d}\Delta v = -\int_{\partial^* U}(v\nabla u - u\nabla v)\cdot\nu_U\,\mathrm{d}\mathscr{H}^{n-1}. \tag{6.11}$$

Finally, we can also consider open sets of finite perimeter $U \subset \Omega$, if the supports of ϕ, u, and v are required to be compact in Ω.

Proof Clearly, $\nabla u \in C^0(\Omega)\cap\mathcal{DM}^p(\Omega)$ and $\nabla v \in C^0(\Omega)\cap\mathcal{DM}^{p'}(\Omega)$. Thus, it suffices to combine the results of Theorem 5.2, Corollary 5.1, and Proposition 6.3 to complete the proof.

We notice that (6.9) and (6.11) are closely related to the results of Comi-Payne [16, Proposition 4.5], where Green's identities are achieved for C^1 functions (whose gradients are essentially bounded $\mathcal{DM}$–fields) and sets of finite perimeter.

Arguing in a similar way and employing the refinement of the Gauss-Green formula for $\mathcal{DM}^\infty$–fields given in Proposition 4.3, we now achieve all Green's identities for Lipschitz functions with Laplacian measure and sets of finite perimeter.

Proposition 6.5 *Let $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ be such that $\Delta u \in \mathcal{M}_{\mathrm{loc}}(\Omega)$, and let $E \subset \Omega$ be a set of locally finite perimeter. Then there exist interior and exterior normal traces of ∇u: $(\nabla u_{\mathrm{i}} \cdot \nu_E), (\nabla u_{\mathrm{e}} \cdot \nu_E) \in L^\infty_{\mathrm{loc}}(\partial^* E; \mathscr{H}^{n-1})$ such that, for any $v \in C^0(\Omega)$ satisfying $\nabla v \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and $\mathrm{supp}(\chi_E v) \Subset \Omega$,*

$$\int_{E^1} v \,\mathrm{d}\Delta u + \int_E \nabla v \cdot \nabla u \,\mathrm{d}x = -\int_{\partial^* E} v(\nabla u_{\mathrm{i}} \cdot \nu_E)\,\mathrm{d}\mathscr{H}^{n-1}, \tag{6.12}$$

$$\int_{E^1 \cup \partial^* E} v \,\mathrm{d}\Delta u + \int_E \nabla v \cdot \nabla u \,\mathrm{d}x = -\int_{\partial^* E} v(\nabla u_{\mathrm{e}} \cdot \nu_E)\,\mathrm{d}\mathscr{H}^{n-1}. \tag{6.13}$$

For any open set $U \Subset \Omega$, the following estimates hold:

$$\|\nabla u_{\mathrm{i}} \cdot \nu_E\|_{L^\infty(\partial^* E \cap U; \mathscr{H}^{n-1})} \leqslant \|\nabla u\|_{L^\infty(U \cap E; \mathbb{R}^n)}, \tag{6.14}$$

$$\|\nabla u_{\mathrm{e}} \cdot \nu_E\|_{L^\infty(\partial^* E \cap U; \mathscr{H}^{n-1})} \leqslant \|\nabla u\|_{L^\infty(U \setminus E; \mathbb{R}^n)}. \tag{6.15}$$

In addition, if $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ with $\Delta v \in \mathcal{M}_{\mathrm{loc}}(\Omega)$, and $\mathrm{supp}(\chi_E v), \mathrm{supp}(\chi_E u) \Subset \Omega$, then the following formulas hold:

$$\int_{E^1} v \,\mathrm{d}\Delta u - u \,\mathrm{d}\Delta v = -\int_{\partial^* E} \big(v(\nabla u_{\mathrm{i}} \cdot \nu_E) - u(\nabla v_{\mathrm{i}} \cdot \nu_E)\big)\,\mathrm{d}\mathscr{H}^{n-1}, \tag{6.16}$$

$$\int_{E^1 \cup \partial^* E} v \,\mathrm{d}\Delta u - u \,\mathrm{d}\Delta v = -\int_{\partial^* E} \big(v(\nabla u_{\mathrm{e}} \cdot \nu_E) - u(\nabla v_{\mathrm{e}} \cdot \nu_E)\big)\,\mathrm{d}\mathscr{H}^{n-1}. \tag{6.17}$$

In particular, if $\mathrm{supp}(\chi_E u) \Subset \Omega$, then

$$\int_{E^1} u \,\mathrm{d}\Delta u + \int_E |\nabla u|^2 \,\mathrm{d}x = -\int_{\partial^* E} u(\nabla u_{\mathrm{i}} \cdot \nu_E)\,\mathrm{d}\mathscr{H}^{n-1}, \tag{6.18}$$

$$\int_{E^1 \cup \partial^* E} u \,\mathrm{d}\Delta u + \int_E |\nabla u|^2 \,\mathrm{d}x = -\int_{\partial^* E} u(\nabla u_{\mathrm{e}} \cdot \nu_E)\,\mathrm{d}\mathscr{H}^{n-1}. \tag{6.19}$$

Proof Since $\nabla u \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$, the existence of interior and exterior normal traces in $L^\infty_{\mathrm{loc}}(\partial^* E; \mathscr{H}^{n-1})$ and estimates (6.14)–(6.15) follow from [16, Theorem 4.2] and Proposition 4.3. Analogously, (6.12)–(6.13) are an immediate consequence of (4.6)–(4.7), with $\boldsymbol{F} = \nabla u$ and $\phi = v$.

In addition, if $\mathrm{supp}(\chi_E u) \Subset \Omega$ and $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ with $\Delta v \in \mathcal{M}_{\mathrm{loc}}(\Omega)$, then we can exchange the role of u and v in (6.12) and (6.13):

$$\int_{E^1} u \,\mathrm{d}\Delta v + \int_E \nabla v \cdot \nabla u \,\mathrm{d}x = -\int_{\partial^* E} u(\nabla v_{\mathrm{i}} \cdot \nu_E)\,\mathrm{d}\mathscr{H}^{n-1}, \tag{6.20}$$

$$\int_{E^1 \cup \partial^* E} v \,\mathrm{d}\Delta v + \int_E \nabla v \cdot \nabla u \,\mathrm{d}x = -\int_{\partial^* E} u(\nabla v_{\mathrm{e}} \cdot \nu_E)\,\mathrm{d}\mathscr{H}^{n-1}. \tag{6.21}$$

Thus, it suffices to subtract (6.20) from (6.12) to obtain (6.16), and to subtract (6.21) from (6.13) to obtain (6.17). Finally, choosing $u = v$ in (6.12)–(6.13), we obtain (6.18)–(6.19).

7 Normal Traces for Open Sets as the Limits of the Classical Normal Traces for Smooth Sets

In this section, we show that the approximations of a general open set U can be refined in such a way that $\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial U}$ and $\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial\overline{U}}$ can be regarded as the limits of the classical normal traces on the boundaries of smooth sets. In the case that the open set U has continuous boundary, we can exhibit explicit approximating families of open sets with smooth boundary as deformations to the open set U.

7.1 The general case

In order to achieve the smooth approximation, we recall another remarkable result concerning the approximation of any open set by an increasing sequence of open sets with smooth boundary, a very simple proof of which was given by Daners [21, Proposition 8.2.1].

Proposition 7.1 *Let $U\subset\mathbb{R}^n$ be an open set. Then there exists a sequence of bounded open sets U_k with boundary of class C^∞ such that $U_k \Subset U_{k+1}\Subset U$ and $\bigcup_k U_k = U$.*

We can use this result to extend Theorem 5.1, via showing that the normal trace can be approximated by a sequence of the classical normal traces on smooth boundaries.

Theorem 7.1 *Let $U\subset\Omega$ be a bounded open set, and let $\boldsymbol{F}\in\mathcal{DM}^p(\Omega)$. Then, for any $\phi\in C^0(\Omega)\cap L^\infty(\Omega)$ with $\nabla\phi\in L^{p'}(\Omega;\mathbb{R}^n)$, there exists a sequence of bounded open sets U_k with boundary of class C^∞ such that $U_k\Subset U$, $\bigcup_k U_k = U$, and*

$$\langle \boldsymbol{F}\cdot\nu,\phi\rangle_{\partial U} = -\lim_{k\to\infty}\int_{\partial U_k}\phi\boldsymbol{F}\cdot\nu_{U_k}\,\mathrm{d}\mathscr{H}^{n-1}, \tag{7.1}$$

where ν_{U_k} is the inner unit normal to U_k. In addition, (7.1) holds also for any open set $U\subset\Omega$, provided that $\operatorname{supp}(\phi)\cap U^\delta\Subset\Omega$ *for any $\delta>0$.*

Proof We just need to apply Proposition 7.1 to U in order to obtain an approximating sequence of smooth sets U_m, and then argue as in the proof of Theorem 5.1 with respect to any U_m.

We note that sets U_m^ε have smooth boundaries, for any $0<\varepsilon<\delta_m$, for some δ_m sufficiently small. Indeed, the (signed) distance function d_m from ∂U_m is smooth in $U_m\backslash\overline{U_m^{\delta_m}}$ and satisfies $\nabla d_m(x)=\nu_{U_m^\varepsilon}(x)$ for any $x\in\partial U_m^\varepsilon$, which implies that $|\nabla d_m(x)|=1$ for any $x\in U_m\backslash\overline{U_m^{\delta_m}}$ (for a proof of these facts, we refer to [35, Appendix B] and [34, Lemma 14.16]). Therefore, the level sets $\{d_m=\varepsilon\}=\partial U_m^\varepsilon$ are smooth for any $\varepsilon\in[0,\delta_m)$.

Then we obtain a sequence of open bounded sets $U_m^{\varepsilon_j}$ with smooth boundary satisfying

$$\int_{U_m^{\varepsilon_j}} \phi \, \mathrm{d}div\boldsymbol{F} + \int_{U_m^{\varepsilon_j}} \boldsymbol{F} \cdot \nabla\phi \, \mathrm{d}x = -\int_{\partial U_m^{\varepsilon_j}} \phi \boldsymbol{F} \cdot \nu_{U_m^{\varepsilon_j}} \, \mathrm{d}\mathscr{H}^{n-1} \tag{7.2}$$

for some decreasing sequence $\varepsilon_j \to 0$ and any $m, j \in \mathbb{N}$.

Clearly, $U_m^{\varepsilon_j} \Subset U_{m+1}^{\varepsilon_j}$ and $U_m^{\varepsilon_j} \Subset U_m^{\varepsilon_{j+1}}$, so that we can find a subsequence $U_k^{\varepsilon_k} =: U_k$ satisfying $U_k \Subset U_{k+1}$, $U_k \Subset U$, and $\bigcup_k U_k = U$. Therefore, we can pass to the limit on the left-hand side of (7.2) by the Lebesgue theorem to obtain (7.1) (in the case that U is not bounded, we employ the condition on the support of ϕ).

Similarly, an analogous kind of approximation can also be shown for closed sets.

Proposition 7.2 *Let $C \subset \mathbb{R}^n$ be a closed set. Then there exists a sequence of closed sets C_k with boundary of class C^∞ such that $C_k \supset \overset{\circ}{C}_k \supset C_{k+1} \supset \overset{\circ}{C}_{k+1} \supset C$ and $\bigcap_k C_k = C$. In addition, if C is bounded, then the closed sets C_k can be chosen to be bounded.*

Proof Let $U := \mathbb{R}^n \backslash C$. Then it suffices to define $C_k := \mathbb{R}^n \backslash U_k$ and apply Proposition 7.1 to U. The result follows easily.

In the case that C is bounded, then, for any $\delta > 0$, there exists k_0 large enough such that $\partial U_k \cap \partial C_\delta = \varnothing$ for any $k \geqslant k_0$, where $C_\delta = \{x \in \mathbb{R}^n : \mathrm{dist}(x, C) < \delta\}$. Then we set $C_k = \overline{C_\delta} \backslash U_k$, up to relabeling the sequence U_k in such a way that it starts from k_0.

Arguing similarly as before, Proposition 7.2 can be used to represent the exterior normal trace as the limit of the classical normal traces on smooth boundaries, thus improving the result of Theorem 5.3.

Theorem 7.2 *Let $U \Subset \Omega$ be an open set, and let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$. Then, for any $\phi \in C^0(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega; \mathbb{R}^n)$, there exists a sequence of bounded open sets V_k with boundary of class C^∞ such that $U \Subset V_k \subset \Omega$, $\bigcap_k V_k = \overline{U}$, and*

$$\langle \boldsymbol{F} \cdot \nu, \phi \rangle_{\partial\overline{U}} = -\lim_{k\to\infty} \int_{\partial V_k} \phi \boldsymbol{F} \cdot \nu_{V_k} \, \mathrm{d}\mathscr{H}^{n-1}, \tag{7.3}$$

where ν_{V_k} is the classical inner unit normal to V_k. In addition, (7.3) holds also for any open set U satisfying $\overline{U} \subset \Omega$, provided that $\mathrm{supp}(\phi)$ *is compact in Ω.*

Proof It suffices to define $V_k := \overset{\circ}{C}_k$ and apply Proposition 7.2 to $C = \overline{U}$. Then the result follows in an analogous way as in the proof of Theorem 7.1.

7.2 The case of C^0 open sets

We now consider the question of constructing the interior and exterior normal traces as the limit of classical normal traces over *smooth* approximations of the open bounded set U with C^0 boundary. In general, as it has been explained in the introduction, it is a challenging question to approximate an open (bounded) set U with smooth domains U^ε essentially from the *inside* in such a way that $div\boldsymbol{F}(U^\varepsilon) \to div\boldsymbol{F}(U)$ and an *interior* Gauss-Green formula holds for unbounded $\mathcal{DM}^p$–fields. Indeed, in §5, we have used the standard signed distance d to obtain Theorems 5.1 and 5.3, but the approximating sets are not smooth. We have shown that such results in Theorems 7.1 and 7.2 can be improved. On the other hand, such an approximation is quite abstract, and it gives little insight in the actual shape of the approximating sets.

Remark 7.1 We observe that an open bounded set U with C^0 boundary might not have finite perimeter; such examples include von Koch's snowflake [64]. Also, we do not have a notion of unit normals to a C^0 open set. Thus, such a type of sets is more general in this sense. On the other hand, an open set of finite perimeter may have really wild topological boundary even with full Lebesgue measure (see *e.g.* [43, Example 12.25]) so that it is not a C^0 open set in general, since it is well know that, if ∂U can be seen locally as the graph of a continuous function, then $|\partial U| = 0$.

We now exhibit here a rather explicit family of open smooth sets approximating a given bounded open set with C^0 boundary from both the interior and the exterior. To this purpose, we consider a different type of distance, the regularized distance ρ, which was introduced in Lieberman [41].

Definition 7.1 ρ is a regularized distance for U if the following conditions hold:

(i) $\rho \in C^2(\mathbb{R}^n \backslash \partial U) \cap \mathrm{Lip}(\mathbb{R}^n)$;
(ii) The ratios $\frac{\rho(x)}{d(x)}$ and $\frac{d(x)}{\rho(x)}$ are positive and uniformly bounded for all $x \in \mathbb{R}^n \backslash \partial U$, where d is the signed distance introduced in §5.

It was proved by Lieberman [41, Lemma 1.1] that any open set U has a regularized distance, since the signed distance d is a 1-Lipschitz function (Lemma 5.1). Indeed, given any $\eta \in C^2(\mathbb{R}^n)$, supp $\eta \subset B(0,1)$, and $\int_{\mathbb{R}^n} \eta(z)\mathrm{d}z = 1$, we can define

$$G(x,\tau) = \int_{B(0,1)} d(x - \frac{\tau}{2}z)\eta(z)\,\mathrm{d}z. \tag{7.4}$$

The regularized distance ρ is then given by the equation:

$$\rho(x) = G(x, \rho(x)), \tag{7.5}$$

which has a unique solution for every $x \in \mathbb{R}^n$. Moreover,

$$\frac{1}{2} \leqslant \frac{\rho(x)}{d(x)} \leqslant 2 \qquad \text{for all } x \in \mathbb{R}^n \backslash \partial U. \tag{7.6}$$

Then the following result holds, for which we refer to Lieberman [41, Lemma 1.1, Corollary 1.2] and the comments before it.

Lemma 7.1 *Every open set U has a regularized distance ρ. Moreover, if $\eta \in C^\infty(\mathbb{R}^n)$ is chosen for* (7.4)*, then $\rho \in C^\infty(\mathbb{R}^n \backslash \partial U)$.*

Even though every open set U has a regularized distance, it is important to obtain properties concerning the non-degeneracy of gradient $\nabla\rho$. Indeed, if the gradient of ρ does not vanish in a neighborhood of ∂U, then we can apply the techniques in §5 to obtain the *interior* and *exterior* Gauss-Green formulas for $\mathcal{DM}^p$–fields.

The non-degeneracy of $\nabla\rho$ might not be true for general open sets U (see [41, Corollary 1.2] and the comments following it). However, it was proved by Ball-Zarnescu [7, Proposition 3.1] that this property holds for C^0 domains, which yields the following result.

Theorem 7.3 (Ball-Zarnescu) *If U is an open bounded set with C^0 boundary, then $|\nabla\rho(x)| \neq 0$ for all x in a neighborhood of ∂U but $x \notin \partial U$.*

Remark 7.2 The argument of the proof of Theorem 7.3 relies on both the construction of a suitable good neighborhood for any point of ∂U and the use of the compactness assumption. Hence, we see that the local version of this result also applies to general open sets with C^0 boundary: For any compact subset $K \subset \partial U$, there exists a suitable neighborhood V of K such that $|\nabla\rho(x)| \neq 0$ for any $x \in V \backslash \partial U$.

Thanks to the Ball-Zarnescu theorem (Theorem 7.3), we can proceed as in §5 to obtain an analogous statement, by approximating U and $\overline{U}$ in Ω with the following smooth sets:

$$U^{\varepsilon,\rho} := \{x \in \mathbb{R}^n : \rho(x) > \varepsilon\}, \quad U_{\varepsilon,\rho} := \{x \in \mathbb{R}^n : \rho(x) > -\varepsilon\} \qquad \text{for } \varepsilon > 0.$$

Theorem 7.4 (Interior normal trace via smooth approximations) *Let $U \subset \Omega$ be a bounded open set with C^0 boundary, and let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$. Then, for any $\phi \in C^0(\Omega) \cap L^\infty(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega;\mathbb{R}^n)$, there exists a set $\mathcal{N} \subset \mathbb{R}$ with $\mathcal{L}^1(\mathcal{N}) = 0$ such that, for every nonnegative sequence $\{\varepsilon_k\}$ satisfying $\varepsilon_k \notin \mathcal{N}$ for any k and $\varepsilon_k \to 0$,*

$$\langle \boldsymbol{F}\cdot\nu, \phi\rangle_{\partial U} = \int_U \phi \,\mathrm{d}\mathrm{div}\boldsymbol{F} + \int_U \boldsymbol{F}\cdot\nabla\phi \,\mathrm{d}x = -\lim_{k\to\infty} \int_{\partial U^{\varepsilon_k,\rho}} \phi \boldsymbol{F}\cdot\nu_{U^{\varepsilon_k,\rho}} \,\mathrm{d}\mathscr{H}^{n-1}, \tag{7.7}$$

where $\nu_{U^{\varepsilon_k,\rho}}$ is the inner unit normal to the smooth sets $U^{\varepsilon_k,\rho}$. In addition, (7.7) *holds also for any open set $U \subset \Omega$, provided that* $\mathrm{supp}(\phi) \cap U^\delta \Subset \Omega$ *for any small $\delta > 0$.*

Proof Since ρ is smooth and $|\nabla\rho(x)| \neq 0$ for any $x \in U \backslash U^{\varepsilon,\rho}$ for small enough ε, it follows that $\{x \in \mathbb{R}^n : \rho(x) = \varepsilon\}$ is a smooth hypersurface in $\mathbb{R}^n$. Therefore, $\partial U^{\varepsilon,\rho} = \partial^* U^{\varepsilon,\rho}$ and

$$\frac{\nabla\rho}{|\nabla\rho|}(x) = \nu_{U^{\varepsilon,\rho}}(x) \qquad \text{for every } x \in \partial U^{\varepsilon,\rho}.$$

We can now proceed in the same way as in the second step of the proof of Theorem 5.1 by noticing that the only difference is in the application of the coarea formula and in the use of ρ, instead of d, in the definition of $\psi_{\varepsilon,\rho}^{U^{\delta,\rho}}$ for $\delta, \varepsilon > 0$. Indeed, using Theorem 7.3, we rewrite (5.9) in the case $U \Subset \Omega$ as follows:

$$\begin{aligned}\int_U \psi_{\varepsilon,\rho}^{U^{\delta,\rho}} \,\mathrm{d}div(\phi \boldsymbol{F}) &= -\int_{U^{\delta,\rho}\setminus U^{\delta+\varepsilon,\rho}} \phi \boldsymbol{F}\cdot\nabla\rho\,\mathrm{d}x \\ &= -\int_{U^{\delta,\rho}\setminus U^{\delta+\varepsilon,\rho}} \phi \boldsymbol{F}\cdot\frac{\nabla\rho}{|\nabla\rho|}|\nabla\rho|\,\mathrm{d}x \\ &= -\int_{\delta}^{\delta+\varepsilon}\int_{\partial U^{t,\rho}} \phi \boldsymbol{F}\cdot\frac{\nabla\rho}{|\nabla\rho|}\,d\mathscr{H}^{n-1}\,\mathrm{d}t,\end{aligned}$$

by the coarea formula (2.2) with $u = \rho$ and $g = \chi_{U^{\delta,\rho}}\phi\boldsymbol{F}\cdot\frac{\nabla\rho}{|\nabla\rho|}\,|\nabla\rho|$, since

$$\operatorname{essinf}|\nabla\rho| > 0 \qquad \text{on } U^{\delta,\rho}\setminus U^{\delta+\varepsilon,\rho} \text{ for any } \delta,\varepsilon > 0.$$

Then we can proceed as in Steps 2–3 of the proof of Theorem 5.1. Finally, in the case that U is not bounded, we employ Remark 7.2 to obtain the desired result.

Remark 7.3 In particular, Theorem 7.4 implies that, if Ω is of C^0 boundary, the Gauss-Green formula up to the boundary by approximating $\partial\Omega$ with a sequence of smooth sets. This can be seen by taking $U = \Omega$ in (7.7).

Analogously, we also have a smooth version of Theorem 5.3, in which we employ the fact that $|\partial U| = 0$ if U has a C^0 boundary, by Remark 7.1, in order to integrate $\boldsymbol{F}\cdot\nabla\phi$ only on U.

Theorem 7.5 (Exterior normal traces via smooth approximations) *Let $U \Subset \Omega$ be a C^0 open set, and let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$. Then, for any $\phi \in C^0(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega;\mathbb{R}^n)$, there exists a set $\mathcal{N} \subset \mathbb{R}$ with $\mathcal{L}^1(\mathcal{N}) = 0$ such that, for every nonnegative sequence $\{\varepsilon_k\}$ satisfying $\varepsilon_k \notin \mathcal{N}$ for any k and $\varepsilon_k \to 0$,*

$$\langle \boldsymbol{F}\cdot\nu, \phi\rangle_{\partial\overline{U}} = \int_{\overline{U}} \phi \,\mathrm{d}div\boldsymbol{F} + \int_U \boldsymbol{F}\cdot\nabla\phi\,\mathrm{d}x = -\lim_{k\to\infty}\int_{\partial U_{\varepsilon_k,\rho}} \phi\boldsymbol{F}\cdot\nu_{U_{\varepsilon_k,\rho}}\,\mathrm{d}\mathscr{H}^{n-1}, \tag{7.8}$$

where $\nu_{U_{\varepsilon_k,\rho}}$ is the inner unit normal to the smooth sets $U_{\varepsilon_k,\rho}$. In addition, (7.8) holds also for any open set U satisfying $\overline{U} \subset \Omega$, provided that $\operatorname{supp}(\phi)$ *is compact in Ω.*

8 The Gauss-Green Formula on Lipschitz Domains

In Ball-Zarnescu [7], it is shown that, if a domain U is of class C^0, then there is a canonical smooth field of good directions defined in a suitable neighborhood

of ∂U, in terms of which a corresponding flow can be defined. By means of this flow, U can be approximated from both the interior and the exterior by diffeomorphic domains of class C^∞; see also Hofmann-Mitrea-Taylor [37] for the definition of a continuous vector field transversal to the boundary of an open set of locally finite perimeter.

In a related issue, Chen-Frid in [11, 12] introduced the notion of *regular Lipschitz deformable boundary* (see Definition 2.5). Then, for a bounded open set U satisfying this condition, they proceeded to obtain the Gauss-Green formulas for $\mathcal{DM}^p$–fields $\boldsymbol{F}$. For the case $p \neq \infty$, the normal trace of $\boldsymbol{F}$ is defined as a distribution which is expressed as an average over a neighborhood of Lipschitz deformable boundary ∂U determined by the Lipschitz deformations. However, as explained in the introduction, the main goal of this paper is to present the Gauss-Green formula for the case $p \neq \infty$, by using the classical normal traces $\boldsymbol{F} \cdot \nu$ which are defined on almost every surface that approximates ∂U. Thus, the present paper aligns with the later work by Chen-Torres-Ziemer [14], in which the Gauss-Green formula for bounded $\mathcal{DM}$–fields has been established over arbitrary sets of finite perimeter E via the normal trace on $\partial^* E$ as the limit of classical normal traces on smooth approximations of E. The goal of this section is to show that the main result in Ball-Zarnescu [7] implies that any Lipschitz domain satisfies condition (ii) of Definition 2.5, which indicates that condition (ii) holds automatically for a Lipschitz domain.

For a domain U in the class C^0, the concept of *a good direction* (see Definition 2.4) has been introduced, and the following result has been established in [7].

Proposition 8.1 [7, Proposition 2.1] *Let $U \subset \mathbb{R}^n$ be a bounded open set with boundary of class C^0. Then there exist a neighborhood V of ∂U and a smooth function $G : V \to \mathbb{S}^{n-1}$ so that, for each $P \in V$, the unit vector $G(P)$ is a good direction.*

Remark 8.1 Proposition 8.1 can be localized. Indeed, if U is an unbounded open set with boundary of class C^0, then $U \cap B(0, R)$ is a bounded open set with boundary of class C^0 for any $R > 0$. Therefore, there exist a neighborhood V of any compact set $K \subset \partial U$ and a smooth function $G : V \to \mathbb{S}^{n-1}$ so that, for each $P \in V$, the unit vector $G(P)$ is a good direction.

We recall that $\nu = G(P)$ is a good direction if ∂U is the graph of a continuous function in a small neighborhood $B(P, \delta)$ and in some system of coordinates (y', y_n), where ν is a unit vector in the direction of y_n. Using the field of good directions $G(p)$, a flow $S(\cdot)(\cdot) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ can be defined through:

$$\dot{x}(t) = \gamma(x(t))G(x(t)), \quad x(0) = x_0, \tag{8.1}$$

with $x(t) = S(t)(x_0)$ as the solution of the initial value problem (8.1) for the differential equation at time t, where γ is an appropriate smooth function (see [7, §4] for the details on the construction of the flow).

By exploiting the properties of the flow of good directions, the following theorem has been proved in Ball-Zarnescu [7], which is very helpful in the rest of this section.

Theorem 8.1 [7, Theorem 5.1] *Let $U \subset \mathbb{R}^n, n \geqslant 2$, be a bounded domain of class C^0. Let ρ be a regularized distance defined in §7. For $\varepsilon \in \mathbb{R}$, define*

$$U^{\varepsilon,\rho} = \{x \in \mathbb{R}^n : \rho(x) > \varepsilon\}, \qquad U_{\varepsilon,\rho} = \{x \in \mathbb{R}^n : \rho(x) > -\varepsilon\}.$$

Then there exists $\varepsilon_0 = \varepsilon_0(U) > 0$ such that, if $0 < \varepsilon < \varepsilon_0$, $U_{\varepsilon,\rho}$ and $U^{\varepsilon,\rho}$ are bounded domains of class C^∞ and satisfy the following:

(i) $\bigcap_{0<\varepsilon<\varepsilon_0} U_{\varepsilon,\rho} = \overline{U}$, $\bigcup_{0<\varepsilon<\varepsilon_0} U^{\varepsilon,\rho} = U$, *and* $\overline{U}^{\varepsilon',\rho} \subset U^{\varepsilon,\rho}$ *and* $U_{\varepsilon',\rho} \supset \overline{U}_{\varepsilon,\rho}$ *if* $0 < \varepsilon < \varepsilon' < \varepsilon_0$.

(ii) *For $-\varepsilon_0 < \varepsilon < \varepsilon_0$, there is a homeomorphism $f(\varepsilon,\cdot)$ of $\mathbb{R}^n$ onto $\mathbb{R}^n$ with inverse denoted $f^{-1}(\varepsilon,\cdot)$ so that*

- $f(\varepsilon,\overline{U}) = \overline{U}^{\varepsilon,\rho}$ *and* $f(\varepsilon,\partial U) = \partial U^{\varepsilon,\rho}$ *for* $\varepsilon > 0$, $f(\varepsilon,\overline{U}) = \overline{U}_{-\varepsilon,\rho}$ *and* $f(\varepsilon,\partial U) = \partial U_{-\varepsilon,\rho}$ *for* $\varepsilon < 0$;
- $f(\varepsilon,x) = x$ *for* $|\rho(x)| \geqslant 3|\varepsilon|$, *so that* $f(0,\cdot)$ *is the identity*;
- $f(\varepsilon,\cdot) : \mathbb{R}^n \backslash \partial U \to \mathbb{R}^n \backslash \partial U^{\varepsilon,\rho}$ *for* $\varepsilon > 0$, *and* $f(\varepsilon,\cdot) : \mathbb{R}^n \backslash \partial U \to \mathbb{R}^n \backslash \partial U_{-\varepsilon,\rho}$ *for* $\varepsilon < 0$ *are both* C^∞ *diffeomorphisms. In addition,*

$$f, f^{-1} \in C^0((-\varepsilon_0,\varepsilon_0) \times \mathbb{R}^n; \mathbb{R}^n).$$

(iii) *There is a map $g : (0,\varepsilon_0) \times (-\varepsilon_0, 0) \times \mathbb{R}^n \to \mathbb{R}^n$ such that, if $\varepsilon \in (0,\varepsilon_0)$ and $\varepsilon' \in (-\varepsilon_0, 0)$, then*

- $g(\varepsilon,\varepsilon',\cdot)$ *is a C^∞ diffeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$ with inverse $g^{-1}(\varepsilon,\varepsilon',\cdot) : \mathbb{R}^n \to \mathbb{R}^n$*;
- $g(\varepsilon,\varepsilon',U^{\varepsilon,\rho}) = U_{-\varepsilon',\rho}$, $g(\varepsilon,\varepsilon',\partial U^{\varepsilon,\rho}) = \partial U_{-\varepsilon',\rho}$;
- $g(\varepsilon,\varepsilon',x) = x$ *for* $3\varepsilon \leqslant \rho(x) \leqslant 3\varepsilon'$.

Remark 8.2 An easy consequence of Theorem 8.1(ii) is that $f(\varepsilon,\cdot)$ converges uniformly to the identity. Indeed,

$$|f(\varepsilon,x) - x| = 0 \qquad \text{in } \{x : |\rho(x)| \geqslant 3|\varepsilon|\},$$

and

$$|f(\varepsilon,x) - x| \leqslant \operatorname{diam}(\{|\rho| < 3|\varepsilon|\}) \qquad \text{in } \{x : |\rho(x)| < 3|\varepsilon|\},$$

since $f(\varepsilon,\cdot)$ is injective so that, for any x such that $|\rho(x)| < 3|\varepsilon|$, $f(\varepsilon,x) - x$ cannot belong to $\{|\rho| \geqslant 3|\varepsilon|\}$. Then this shows that

$$\sup_{x\in\mathbb{R}^n} |f(\varepsilon,x) - x| \to 0 \qquad \text{as } \varepsilon \to 0.$$

We now proceed to show that condition (ii) of Definition 2.5 is not necessary; that is, any Lipschitz domain admits a bi-Lipschitz deformation. Even though the proof of Theorem 8.2 is outlined in [7, Remark 5.3], we present a detailed proof for our purpose of the subsequent developments. First we recall a result of Lieberman [42, Lemma A.1] and an extension theorem for Sobolev functions.

Lemma 8.1 *If U is a Lipschitz domain with Lipschitz constant of the local parameterization of ∂U uniformly bounded, then there exists $\delta > 0$ such that, for any x with $0 < |\rho(x)| < \delta$,*

$$\frac{\partial \rho}{\partial x_n}(x) \geqslant \frac{2}{3\sqrt{1+L^2}}, \tag{8.2}$$

where $x = (x', x_n)$ is an orthonormal coordinate system for which ∂U is locally parameterized by a Lipschitz function with Lipschitz constant less or equal to L, and e_n is a good direction.

Remark 8.3 Lemma 8.1 applies to the case that U is a bounded Lipschitz domain, since, by compactness, ∂U can be covered with a finite number of charts of the local Lipschitz parameterization. However, there are cases of unbounded U which still satisfy the assumption, such as the half-spaces. In addition, since (8.2) is a local result, it holds for any open set U with Lipschitz boundary, up to the localization to a bounded subset of ∂U.

Lemma 8.2 *Let U be a domain satisfying the minimal smoothness conditions; that is, there exist $\varepsilon > 0$, $N \in \mathbb{N}$, $M > 0$, and a sequence of open sets $\{V_i\}$ such that:*

(i) *If $x \in \partial U$, then $B(x, \varepsilon) \subset V_i$ for some i;*

(ii) *No point in $\mathbb{R}^n$ is contained in more than N of $\{V_i\}$;*

(iii) *For each i, there exists a Lipschitz function ψ_i with Lipschitz constant less or equal to M such that $V_i \cap U$ is the subgraph of ψ_i inside V_i.*

Then there exists a continuous linear operator $\mathcal{E} : W^{k,p}(U) \to W^{k,p}(\mathbb{R}^n)$, for any $k \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$, such that $\mathcal{E}(f) = f$ on U.

For the proof of this result, we refer to Stein [58, Chapter VI, §3, Theorem 5]. These conditions are satisfied for any bounded open Lipschitz domain U. However, they may fail in the general case of an unbounded open Lipschitz domain.

Theorem 8.2 *If U is a bounded Lipschitz domain, then $f_\varepsilon(\cdot) : \overline{U} \to \overline{U}^{\varepsilon,\rho}$ is bi-Lipschitz with Lipschitz constants uniformly bounded in $\varepsilon > 0$. If $\varepsilon < 0$, the corresponding result is also true.*

Proof Unless otherwise stated, in this proof, ∇ stands for the gradient with respect to the spatial variable, denoted by x, y, or z. We divide the proof into five steps.

1. We first consider $0 < \varepsilon < \varepsilon_0$, for some $\varepsilon_0 > 0$ sufficiently small to be assigned. From Theorem 8.1, $f(\varepsilon, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is continuous with continuous inverse f^{-1}. Thus, the restriction:

$$f(\varepsilon, \cdot) : \overline{U} \to \overline{U}^{\varepsilon,\rho}$$

is also continuous with continuous inverse. We need to show that $f(\varepsilon, \cdot) : \overline{U} \to \overline{U}^{\varepsilon,\rho}$ is Lipschitz with the Lipschitz constant uniformly bounded in $\varepsilon \in (0, \varepsilon_0)$.

Following the proof of [7, Theorem 5.1], the continuous map $f(\varepsilon,\cdot)$ is defined as

$$f(\varepsilon,x)=\begin{cases}S(t(\varepsilon,x))x & \text{for } x\in\overline{U}\backslash U^{3\varepsilon,\rho},\\ x & \text{for } x\in U^{3\varepsilon,\rho},\end{cases} \tag{8.3}$$

where $t(\varepsilon,x)$ is the unique $t\geqslant 0$ such that

$$\rho(S(t(\varepsilon,x))x)=\rho(x)+h(\varepsilon,\rho(x)), \tag{8.4}$$

and $h(\varepsilon,\cdot):\mathbb{R}_+\to[0,\varepsilon]$ is smooth with value ε on $[0,\varepsilon]$ and 0 on $[\frac{5\varepsilon}{2},\infty)$ and, for some $\sigma>0$, $-1+\sigma<\frac{\partial h}{\partial r}(\varepsilon,r)\leqslant 0$ for any $r\geqslant 0$ and ε sufficiently small.

2. In view of (8.3), it suffices to show that $t(\varepsilon,x)$ is Lipschitz, with Lipschitz constant uniformly bounded in ε. Define

$$F(t,\varepsilon,x):=\rho(S(t)x)-\rho(x)-h(\varepsilon,\rho(x)). \tag{8.5}$$

Then, from (8.4),

$$F(t(\varepsilon,x),\varepsilon,x)=0. \tag{8.6}$$

We take derivatives in (8.6) with respect to x_i to obtain $\frac{\partial F}{\partial t}\frac{\partial t}{\partial x_i}+\frac{\partial F}{\partial x_i}=0$; that is, $\frac{\partial t}{\partial x_i}=-\frac{\partial F}{\partial x_i}\big(\frac{\partial F}{\partial t}\big)^{-1}$. We need to show that $|\frac{\partial F}{\partial x_i}|$ is bounded from above. By definition (8.5), it follows that

$$\frac{\partial F}{\partial x_i}=\nabla\rho(S(t)x)\cdot\frac{\partial(S(t)x)}{\partial x_i}-\frac{\partial\rho}{\partial x_i}-\frac{\partial h}{\partial r}(\varepsilon,\rho(x))\frac{\partial\rho}{\partial x_i}. \tag{8.7}$$

Notice that $|\frac{\partial h}{\partial r}(\varepsilon,\rho(x))|<1$ by the definition of h, and $|\nabla(S(t)x)|\leqslant M=M_{\varepsilon_0}$ uniformly on $x\in\overline{U}\backslash U^{3\varepsilon_0,\rho}$ for some $\varepsilon_0>0$, since it is the flow of the smooth compactly supported vector field γG; see (8.1). Thus, it suffices to show that $|\nabla\rho(x)|$ is bounded above. Relation (7.5) implies

$$\frac{\partial\rho}{\partial x_i}=\frac{\partial G}{\partial x_i}+\frac{\partial G}{\partial\tau}\frac{\partial\rho}{\partial x_i}\quad\Longrightarrow\quad\frac{\partial\rho}{\partial x_i}=\frac{\frac{\partial G}{\partial x_i}}{1-\frac{\partial G}{\partial\tau}}\quad\Longrightarrow\quad\nabla\rho(x)=\frac{\nabla G}{1-\frac{\partial G}{\partial\tau}}. \tag{8.8}$$

From (7.4), we obtain

$$\begin{aligned}\nabla G(x,\tau)&=\int_{B(0,1)}\nabla d(x-\frac{\tau}{2}z)\eta(z)\,\mathrm{d}z,\\ \frac{\partial G}{\partial\tau}(x,\tau)&=-\frac{1}{2}\int_{B(0,1)}\nabla d(x-\frac{\tau}{2}z)z\eta(z)\,\mathrm{d}z.\end{aligned}$$

In turn, it follows that

$$|\nabla G(x,\tau)|\leqslant 1,\quad\Big|\frac{\partial G}{\partial\tau}(x,\tau)\Big|\leqslant\frac{1}{2}\qquad\text{for any } x\in\mathbb{R}^n \text{ and } \tau=\rho(x), \tag{8.9}$$

so that

$$|\nabla\rho(x)|\leqslant 2\qquad\text{for any } x\in\mathbb{R}^n, \tag{8.10}$$

since $|\nabla d| = 1$ $\mathscr{L}^n$–*a.e.* by Lemma 5.1. Therefore, from (8.7) and (8.9), we have

$$|\nabla F| \leqslant 2(M+2).$$

It remains to show that $|\frac{\partial F}{\partial t}|$ is bounded away from zero. From (8.5), it follows by [7, Remark 3.1] that

$$\frac{\partial F}{\partial t} = \gamma(x(t))\,(\nabla\rho \cdot G)(S(t)x) > 0$$

for any t small enough and x in a suitable neighborhood of ∂U.

However, in order to prove the uniformity of this estimate in $x \in \overline{U}\backslash U^{3\varepsilon_0,\rho}$, independent of ε, we need to employ the compactness of ∂U. As recalled in [7, Remark 5.3], in the case that ∂U is Lipschitz, Lemma 8.1 can be applied. In this way, since $G(x)$ is a field of good directions, we use (8.2) to show that there exists $\delta > 0$ such that, for any $x \in U$ with $d(x) < \delta$,

$$\nabla\rho(x) \cdot G(x) \geqslant \frac{2}{3\sqrt{1+L^2}}, \tag{8.11}$$

where L is the maximal Lipschitz constant of the Lipschitz parameterizations of ∂U. This implies that there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that (8.11) holds for any $x \in U\backslash U^{3\varepsilon,\rho}$ and any $\varepsilon \in (0, \varepsilon_0)$. Since γ can be chosen in such a way that $\gamma \equiv 1$ in $\overline{U}\backslash U^{3\varepsilon_0,\rho}$ (see [7, §4 and Theorem 5.1]), we obtain

$$\frac{\partial F}{\partial t} \geqslant \frac{2}{3\sqrt{1+L^2}}.$$

Thus, this implies

$$|\nabla t(\varepsilon, x)| \leqslant 3(M+2)\sqrt{1+L^2} \qquad \text{for any } (\varepsilon, x) \in (0,\varepsilon_0) \times U\backslash U^{3\varepsilon_0,\rho}. \tag{8.12}$$

From the proof of [7, Theorem 5.1], we also know that $t \in C^\infty((0,\varepsilon_0) \times U) \cap C^0([0,\varepsilon_0) \times \overline{U})$ and $t(\varepsilon, x) = 0$ for any $x \in U^{3\varepsilon,\rho}$. From (8.12), it follows that $f(\varepsilon, \cdot) \in W^{1,\infty}(U; \mathbb{R}^n)$. Since ∂U is only Lipschitz, the classical extension theorems for Sobolev mappings (*cf.* [27, Theorem 1, §4.4] and [28, Theorem 1, §5.4]) do not apply, and we know only that $f(\varepsilon, \cdot) \in \mathrm{Lip}_{\mathrm{loc}}(U; \mathbb{R}^n)$ (*cf.* [27, Theorem 5, §4.2.3] and [28, Theorem 4, §5.8.2]). However, Lemma 8.2 can be applied in the case $p = \infty$.

Therefore, there exists a function $\bar{f}(\varepsilon, \cdot) \in W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^n)$, uniformly in $\varepsilon \in [0, \varepsilon_0)$, such that $f(\varepsilon, \cdot) = \bar{f}(\varepsilon, \cdot)$ on U. Moreover, $f(\varepsilon, \cdot)$ and $\bar{f}(\varepsilon, \cdot)$ also agree on ∂U since, for any $x \in \partial U$, there exists a sequence $\{x_j\} \subset U$ with $x_j \to x$, so that $f(\varepsilon, x) = \bar{f}(\varepsilon, x)$ because both $f(\varepsilon, \cdot)$ and $\bar{f}(\varepsilon, \cdot)$ are continuous in $\overline{U}$. Clearly, $\bar{f}(\varepsilon, \cdot)$ is Lipschitz in $\mathbb{R}^n$ uniformly in $\varepsilon \in [0, \varepsilon_0)$ so that $f(\varepsilon, \cdot) \in \mathrm{Lip}(\overline{U}; \mathbb{R}^n)$ uniformly in $\varepsilon \in [0, \varepsilon_0)$.

3. Let us now consider the inverse map $f^{-1}(\varepsilon, \cdot) : \overline{U}^{\varepsilon,\rho} \to \overline{U}$. From the proof of [7, Theorem 5.1], we know that it can be defined as

$$f^{-1}(\varepsilon, z) := \begin{cases} S(\beta(\varepsilon, z))z & \text{if } z \in \overline{U}^{\varepsilon,\rho}\backslash U^{3\varepsilon,\rho}, \\ z & \text{if } z \in U^{3\varepsilon,\rho}, \end{cases}$$

where $\beta(\varepsilon, z)$ is the unique solution to the equation:

$$g(\varepsilon, z, \beta(\varepsilon, z)) = 0,$$

where

$$g(\varepsilon, z, \tau) := \rho(z) - \rho(S(\tau)z) - h(\varepsilon, \rho(S(\tau)z)).$$

By the implicit function theorem, $\beta(\varepsilon, \cdot) \in C^\infty(U^{\varepsilon,\rho}) \cap C^0(\overline{U}^{\varepsilon,\rho})$. It is also clear from the definition of h that $\beta(\varepsilon, z) = 0$ for any $z \in U^{3\varepsilon,\rho}$. In addition, we have

$$\nabla\beta(\varepsilon, z) = -\frac{\nabla g(\varepsilon, z, \beta(\varepsilon, z))}{\frac{\partial g}{\partial \tau}(\varepsilon, z, \beta(\varepsilon, z))},$$

and

$$\begin{aligned}\nabla g(\varepsilon, z, \tau) &= \nabla\rho(z) - \nabla(S(\tau)z) \cdot \nabla\rho(S(\tau)z) \\ &\quad - \frac{\partial h}{\partial r}(\varepsilon, \rho(S(\tau)z))\nabla(S(\tau)z) \cdot \nabla\rho(S(\tau)z), \qquad (8.13)\\ \frac{\partial g}{\partial \tau}(\varepsilon, z, \tau) &= -\nabla\rho(S(\tau)z) \cdot (\gamma G)(S(\tau)z)\big(1 + \frac{\partial h}{\partial r}(\varepsilon, \rho(S(\tau)z))\big). \qquad (8.14)\end{aligned}$$

Arguing as above and using (8.9), we obtain

$$|\nabla g(\varepsilon, z, \beta(\varepsilon, z))| \leqslant 2(M_{\varepsilon_0} + 2)$$

for any $\varepsilon \in [0, \varepsilon_0)$ with any fixed $\varepsilon_0 > 0$, and $z \in \overline{U}^{\varepsilon,\rho}$.

As for the estimate on the denominator, we can use (8.2) and the fact that $\frac{\partial h}{\partial r}(\varepsilon, r) \geqslant -1 + \sigma > -1$, for some $\sigma > 0$ independent on ε, in order to show that there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that

$$\left|\frac{\partial g}{\partial \tau}(\varepsilon, z, \beta(\varepsilon, z))\right| > \frac{2}{3\sqrt{1 + L^2}}\sigma \qquad \text{for any } \varepsilon \in [0, \varepsilon_0) \text{ and } x \in U^{\varepsilon,\rho}\backslash U^{3\varepsilon,\rho}.$$

Therefore, for any $\varepsilon \in [0, \varepsilon_0)$ and $x \in U^{\varepsilon,\rho}\backslash U^{3\varepsilon,\rho}$,

$$|\nabla\beta(\varepsilon, z)| \leqslant \frac{3}{\sigma}(M + 2)\sqrt{1 + L^2}$$

for some η depending on the choice of h, $M = M_{\varepsilon_0}$ depending on domain U, and L depending on the Lipschitz parameterization of ∂U.

This implies that $f^{-1}(\varepsilon, \cdot) \in W^{1,\infty}(U^{\varepsilon,\rho}; \mathbb{R}^n)$ uniformly in $\varepsilon \in (0, \varepsilon_0)$. Thus, arguing as before, there exists an extension $\bar{f}^{-1}(\varepsilon, \cdot) \in W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^n)$ uniformly in $\varepsilon \in (0, \varepsilon_0)$, which coincides with $f^{-1}(\varepsilon, \cdot)$ on $\overline{U}^{\varepsilon,\rho}$ by the uniform continuity, and it is Lipschitz uniformly in $\varepsilon \in (0, \varepsilon_0)$. Thus, we have proved that $f^{-1}(\varepsilon, \cdot) \in \mathrm{Lip}(\overline{U}^{\varepsilon,\rho}; \mathbb{R}^n)$ uniformly in $\varepsilon \in (0, \varepsilon_0)$; that is, the Lipschitz constant of $f^{-1}(\varepsilon, \cdot)$ on $\overline{U}^{\varepsilon,\rho}$ is uniformly bounded in $\varepsilon \in (0, \varepsilon_0)$.

4. Let now $\varepsilon < 0$. By the proof of [7, Theorem 5.1], we know that, for $\varepsilon \in (-\varepsilon_0, 0)$ and $x \in \overline{U}$, f is defined as

$$f(\varepsilon, x) := g(-\varepsilon, \varepsilon, f(-\varepsilon, x)),$$

where $g(-\varepsilon,\varepsilon,\cdot)$ is the diffeomorphism introduced in Theorem 8.1 (iii), and f is the map defined in the first part of the proof; see also definition (5.16) in the proof of [7, Theorem 5.1].

Since $f(-\varepsilon,\cdot)$ is uniformly Lipschitz as proved in the first part of the proof, we need to check the same property for $g(-\varepsilon,\varepsilon,\cdot)$. In order to do so, we need to introduce some auxiliary functions from the proof of [7, Theorem 5.1].

Let $\theta=\theta(\tau,y)$ be the unique solution to the equation: $\rho(S(\theta)y)=\tau$. By [7, Lemma 4.1], such a function is well defined for $|\tau|<3\varepsilon_0$ and $|\rho(y)|<3\varepsilon_0$, is smooth for $\tau\neq 0$, and

$$\nabla\theta(\tau,y)=-\frac{\nabla(S(\theta)y)\cdot\nabla\rho(S(\theta)y)}{\nabla\rho(S(\theta)y)\cdot(\gamma G)(S(\theta)y)}(\tau,y) \tag{8.15}$$

by the implicit function theorem, since $\partial_\theta\big(S(\theta)y\big)=(\gamma G)(S(\theta)y)$. This implies that, using (8.2) if $0<|d(S(\theta)y)|<\delta$ for some $\delta>0$ sufficiently small, the denominator of $\nabla\theta(\tau,y)$ is bounded away from zero uniformly in τ. Arguing in a similar way as above, we can show that there exists $\varepsilon_0=\varepsilon_0(\delta)>0$ such that

$$|\nabla\theta(\tau,y)|\leqslant 3M_{\varepsilon_0}\sqrt{1+L^2} \tag{8.16}$$

for any $\tau\in(-3\varepsilon_0,3\varepsilon_0)$, $\tau\neq 0$, and $y\in U_{3\varepsilon_0,\rho}\backslash\overline{U}^{3\varepsilon_0,\rho}$, since $d(S(\theta(\tau,y))y)=0$ if and only if $\rho(S(\theta(\tau,y))y)=0$, which means that $\tau=0$. On the other hand, for our purposes, $\tau=0$ if and only if $\varepsilon=0$; in such a case, $f(0,x)=x$ for any $x\in\mathbb{R}^n$, which is clearly Lipschitz.

Furthermore, $\theta(\tau,y)$ is increasing in τ for fixed y, and

$$\frac{\partial\theta}{\partial\tau}(\tau,y)=\frac{1}{(\nabla\rho\cdot(\gamma G))(S(\theta(\tau,y))y)}>\frac{3}{2}\sqrt{1+L^2},$$

which is uniformly strictly positive in $U_{3\varepsilon_0,\rho}\backslash\overline{U}^{3\varepsilon_0,\rho}$. Given $\tilde{\varepsilon}\in(0,\varepsilon_0)$ and $\varepsilon'\in(-\varepsilon_0,0)$, we obtain

$$q(\tilde{\varepsilon},\varepsilon',y):=-\frac{\theta(2\varepsilon',y)}{\theta(2\tilde{\varepsilon},y)-\theta(2\varepsilon',y)}, \tag{8.17}$$

$$r(\tilde{\varepsilon},\varepsilon',y):=\frac{\theta(\tilde{\varepsilon},y)-\theta(2\varepsilon',y)}{\theta(2\tilde{\varepsilon},y)-\theta(2\varepsilon',y)}, \tag{8.18}$$

$$s(\tilde{\varepsilon},\varepsilon',y):=\frac{\theta(\varepsilon',y)-\theta(2\varepsilon',y)}{\theta(2\tilde{\varepsilon},y)-\theta(2\varepsilon',y)}. \tag{8.19}$$

By the monotonicity property of θ, it is clear that $r,s\in(0,1)$.

Let $u:(0,1)\times(0,1)\times\mathbb{R}\to\mathbb{R}$ be the smooth function described in [7, Lemma 5.1], which particularly satisfies the property that $u(a,b,c)=c$ for any $(a,b)\in(0,1)\times(0,1)$ and $c\notin[0,1]$.

Finally, we define g by

$$g(\tilde{\varepsilon},\varepsilon',y):=\begin{cases}S(w(\tilde{\varepsilon},\varepsilon',y))y & \text{if } \rho(y)\in(3\varepsilon',3\tilde{\varepsilon}),\\ y & \text{otherwise,}\end{cases} \tag{8.20}$$

where

$$w(\tilde{\varepsilon},\varepsilon',y) := (\theta(2\tilde{\varepsilon},y)-\theta(2\varepsilon',y))u(r(\tilde{\varepsilon},\varepsilon',y),s(\tilde{\varepsilon},\varepsilon',y),q(\tilde{\varepsilon},\varepsilon',y))+\theta(2\varepsilon',y). \tag{8.21}$$

We observe that, if $y \in U$, then $\theta(2\varepsilon',y) > 0$ so that $q(\tilde{\varepsilon},\varepsilon',y) > 0$. Moreover, if $q(\tilde{\varepsilon},\varepsilon',y) > 1$, then $w(\tilde{\varepsilon},\varepsilon',y) = 0$ by the property of u. Hence, we can restrict ourselves to the case that $0 < q(\tilde{\varepsilon},\varepsilon',y) \leqslant 1$, without loss of generality.

In our case, we are dealing with $g(-\varepsilon,\varepsilon,y)$ for $\varepsilon < 0$, and $y = f(-\varepsilon,x) \in U^{-\varepsilon,\rho}$, so that we can select $\tilde{\varepsilon} = -\varepsilon$ and $\varepsilon' = \varepsilon$.

Since S is the flow of the smooth vector field γG, it suffices to show uniform bounds on the gradient of w for $y \in U\backslash\overline{U}^{3\varepsilon_0,\rho}$. We have

$$\begin{aligned}
&\nabla w(-\varepsilon,\varepsilon,y)\\
&= \nabla\theta(2\varepsilon,y) + \big(\nabla\theta(-2\varepsilon,y)-\nabla\theta(2\varepsilon,y)\big)u(r(-\varepsilon,\varepsilon,y),s(-\varepsilon,\varepsilon,y),\beta(-\varepsilon,\varepsilon,y))\\
&\quad + \big(\theta(-2\varepsilon,y)-\theta(2\varepsilon,y)\big)\Big(\frac{\partial u}{\partial r}(r(-\varepsilon,\varepsilon,y),s(-\varepsilon,\varepsilon,y),q(-\varepsilon,\varepsilon,y))\nabla r(-\varepsilon,\varepsilon,y)\\
&\qquad\qquad + \frac{\partial u}{\partial s}(r(-\varepsilon,\varepsilon,y),s(-\varepsilon,\varepsilon,y),q(-\varepsilon,\varepsilon,y))\nabla s(-\varepsilon,\varepsilon,y)\\
&\qquad\qquad + \frac{\partial u}{\partial q}(r(-\varepsilon,\varepsilon,y),s(-\varepsilon,\varepsilon,y),q(-\varepsilon,\varepsilon,y))\nabla q(-\varepsilon,\varepsilon,y)\Big).
\end{aligned} \tag{8.22}$$

Observe that h is smooth, and $r(-\varepsilon,\varepsilon,y), s(-\varepsilon,\varepsilon,y) \in (0,1)$ for any $\varepsilon \in (-\varepsilon_0,0)$ and $y \in U\backslash\overline{U}^{3\varepsilon_0,\rho}$ by the properties of θ. In addition, only the intersection with set $\{(\varepsilon,y) : 0 < q(-\varepsilon,\varepsilon,y) \leqslant 1\}$ is relevant to us, since w vanishes on the outside of the intersection. Therefore, h and all its derivatives are uniformly bounded in $U\backslash\overline{U}^{3\varepsilon_0,\rho}$ for any $\varepsilon \in (-\varepsilon_0,0)$.

Moreover, by standard calculations, we have

$$\begin{aligned}
&\big(\theta(-2\varepsilon,y)-\theta(2\varepsilon,y)\big)\nabla q(-\varepsilon,\varepsilon,y)\\
&\quad = -\nabla\theta(2\varepsilon,y) - q(-\varepsilon,\varepsilon,y)\big(\nabla\theta(-2\varepsilon,y)-\nabla\theta(2\varepsilon,y)\big),\\
&\big(\theta(-2\varepsilon,y)-\theta(2\varepsilon,y)\big)\nabla r(-\varepsilon,\varepsilon,y)\\
&\quad = \nabla\theta(-\varepsilon,y) - \nabla\theta(2\varepsilon,y) - r(-\varepsilon,\varepsilon,y)\big(\nabla\theta(-2\varepsilon,y)-\nabla\theta(2\varepsilon,y)\big),\\
&\big(\theta(-2\varepsilon,y)-\theta(2\varepsilon,y)\big)\nabla s(-\varepsilon,\varepsilon,y)\\
&\quad = \nabla\theta(\varepsilon,y) - \nabla\theta(2\varepsilon,y) - s(-\varepsilon,\varepsilon,y)\big(\nabla\theta(-2\varepsilon,y)-\nabla\theta(2\varepsilon,y)\big).
\end{aligned}$$

From these formulas, the bounds on (q,r,s), and (8.16), we conclude that $\nabla w(-\varepsilon,\varepsilon,\cdot) \in L^\infty(U\backslash U^{3\varepsilon_0,\rho};\mathbb{R}^n)$ uniformly in $\varepsilon \in (-\varepsilon_0,0)$.

Arguing now as in the previous two cases, we can extend $f(\varepsilon,\cdot)$ for $\varepsilon < 0$ to a $W^{1,\infty}$–map on the whole $\mathbb{R}^n$, whose restriction on $\overline{U}$ coincides with $f(\varepsilon,\cdot)$; thus proving that $f(\varepsilon,\cdot) \in \mathrm{Lip}(\overline{U};\mathbb{R}^n)$ uniformly in $\varepsilon \in (-\varepsilon_0,0]$.

5. Finally, the inverse map for $\varepsilon < 0$ is given by

$$f^{-1}(\varepsilon,x) := f^{-1}(-\varepsilon,g^{-1}(-\varepsilon,\varepsilon,x)) \qquad \text{for } x \in \overline{U}_{\varepsilon,\rho} \text{ and } \varepsilon \in (-\varepsilon_0,0), \tag{8.23}$$

for some $\varepsilon_0 > 0$ sufficiently small (*cf.* definition (5.17) in the proof of [7, Theorem 5.1]).

The inverse map $g^{-1}(-\varepsilon, \varepsilon, \cdot)$ is defined in a similar way to (8.20), by using $h^{-1}(a, b, \cdot)$ instead of h; that is, the inverse function of $h(a, b, \cdot)$. Since $h^{-1}(a, b, d) = d$ for any $(a, b) \in (0, 1) \times (0, 1)$ and $d \notin [0, 1]$, then we can argue as before to obtain the uniform essential boundedness of $\nabla g^{-1}(-\varepsilon, \varepsilon, \cdot)$, which concludes that $f(\varepsilon, \cdot)$ for $\varepsilon < 0$ is a uniform bi-Lipschitz function.

Remark 8.4 As a consequence of Theorem 8.2, we can show that, if U has Lipschitz boundary, then $\nabla f(\varepsilon, \cdot) \to \mathrm{I}_n$ in $L^p(\mathbb{R}^n; \mathbb{R}^{n\times n})$ for any $1 \leqslant p < \infty$.

Indeed, by Theorem 8.1 (ii), $\nabla f(\varepsilon, x) = \mathrm{I}_n$ for any x such that $|\rho(x)| > 3|\varepsilon|$. This implies that $\nabla f(\varepsilon, x) \to \mathrm{I}_n$ for any $x \in \mathbb{R}^n \backslash \partial U$, and

$$\int_{\mathbb{R}^n} |\nabla f(\varepsilon, x) - \mathrm{I}_n|^p \, \mathrm{d}x \leqslant C^p \mathscr{L}^n(\{|\rho(x)| \leqslant 3|\varepsilon|\}), \tag{8.24}$$

where C is a constant depending only on U and n, since the Lipschitz constants of $f(\varepsilon, \cdot)$ are uniformly bounded for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, by Theorem 8.2. This implies the convergence, since $\mathscr{L}^n(\partial U) = 0$.

As an immediate consequence, we have the following result.

Theorem 8.3 *The boundary of any Lipschitz domain is Lipschitz deformable in the sense of Definition* 2.5.

Proof Indeed, we can employ Theorem 8.2 to construct a Lipschitz deformation Ψ as in Definition 2.5. It suffices to set

$$\Psi(x, \tau) := f(\tau\varepsilon_1, x) \qquad \text{for any } 0 < \varepsilon_1 < \varepsilon_0,$$

where f is given in Theorem 8.1. By the properties of $f(\varepsilon, \cdot)$, $\Psi(\cdot, \tau)$ is a bi-Lipschitz homeomorphism over its image uniformly in $\tau \in [0, 1]$ and $\Psi(\cdot, 0) = \mathrm{Id}$.

Remark 8.5 In fact, Definition 2.5 refers to open sets with Lipschitz boundary, while, thanks to Theorem 8.2, we are able to deal with open bounded Lipschitz domains. However, the connectedness assumption is not relevant, since one can work separately with each connected component of a bounded open set with Lipschitz boundary to achieve Theorems 8.2 and 8.3 for each component. In a similar way, one can also consider an unbounded open set with Lipschitz boundary U, and then localize the problem by considering, for instance, $U \cap B(0, R)$ for $R > 0$, which are open bounded sets with Lipschitz boundary. It is then clear that Theorems 8.2 and 8.3 apply to $U \cap B(0, R)$ for any $R > 0$. Thus, we can conclude that any open set with Lipschitz boundary has a regular Lipschitz deformable boundary, at least locally.

An immediate consequence of the existence of such Lipschitz diffeomorphism between ∂U and $\partial U^{\varepsilon,\rho}$ or $\partial U_{\varepsilon,\rho}$ is that the area formula can be employed in order to consider only integrals on ∂U.

Theorem 8.4 *Let $U \Subset \Omega$ be an open set with Lipschitz boundary, let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \leqslant p \leqslant \infty$, and let $\phi \in C^0(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega;\mathbb{R}^n)$. Then there exists a set $\mathcal{N} \subset \mathbb{R}$ with $\mathcal{L}^1(\mathcal{N}) = 0$ such that, for every nonnegative sequence $\{\varepsilon_k\}$ satisfying $\varepsilon_k \notin \mathcal{N}$ for any k and $\varepsilon_k \to 0$,*

$$\begin{aligned}&\int_U \phi \,\mathrm{d}div\boldsymbol{F} + \int_U \boldsymbol{F}\cdot\nabla\phi\,\mathrm{d}x\\ &= -\lim_{k\to\infty}\int_{\partial U}\Big(\phi\boldsymbol{F}\cdot\frac{\nabla\rho}{|\nabla\rho|}\Big)(f(\varepsilon_k,x))J^{\partial U}f(\varepsilon_k,x)\,\mathrm{d}\mathscr{H}^{n-1},\end{aligned}\tag{8.25}$$

and

$$\begin{aligned}&\int_{\overline{U}} \phi \,\mathrm{d}div\boldsymbol{F} + \int_U \boldsymbol{F}\cdot\nabla\phi\,\mathrm{d}x\\ &= -\lim_{k\to\infty}\int_{\partial U}\Big(\phi\boldsymbol{F}\cdot\frac{\nabla\rho}{|\nabla\rho|}\Big)(f(-\varepsilon_k,x))J^{\partial U}f(-\varepsilon_k,x)\,\mathrm{d}\mathscr{H}^{n-1},\end{aligned}\tag{8.26}$$

where $f(\pm\varepsilon,\cdot)$ is the bi-Lipschitz diffeomorphism introduced in Theorem 8.1.

In addition, (8.25) *holds also for any bounded open set U with Lipschitz boundary if $\phi \in L^\infty(\Omega)$, and even for an unbounded open set U with Lipschitz boundary if* $\mathrm{supp}(\phi) \cap U^\delta \Subset \Omega$ *for any $\delta > 0$. Similarly,* (8.26) *also holds for any open set U satisfying $\overline{U} \subset \Omega$, provided that* $\mathrm{supp}(\phi)$ *is compact in Ω.*

Proof We need to apply the area formula to the Lipschitz maps $f(\varepsilon,\cdot) : \partial U \to \partial U^{\varepsilon,\rho}$ in (7.7)–(7.8).

We denote by $J^{\partial U}f(\varepsilon,\cdot)$ the $(n-1)$–dimensional Jacobian of $f(\varepsilon,\cdot)$ on ∂U, and recall that the inner unit normal to $\partial U^{\varepsilon,\rho}$ is given by $\frac{\nabla\rho}{|\nabla\rho|}$ from Theorem 7.4. Then

$$\int_{\partial U^{\varepsilon_k,\rho}}\Big(\phi\boldsymbol{F}\cdot\frac{\nabla\rho}{|\nabla\rho|}\Big)(x)\,\mathrm{d}\mathscr{H}^{n-1} = \int_{\partial U}\Big(\phi\boldsymbol{F}\cdot\frac{\nabla\rho}{|\nabla\rho|}\Big)(f(\varepsilon_k,x))J^{\partial U}f(\varepsilon_k,x)\,\mathrm{d}\mathscr{H}^{n-1}.$$

We can argue analogously with $\partial U_{\varepsilon_k,\rho}$. Therefore, we can rewrite (7.7)–(7.8) as (8.25)–(8.26).

From this result, we can deduce some known facts again from the theory of $\mathcal{DM}$–fields.

Corollary 8.1 *Let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)\cap C^0(\Omega;\mathbb{R}^n)$ for $1 \leqslant p \leqslant \infty$, let $\phi \in C^0(\Omega)\cap L^\infty(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega;\mathbb{R}^n)$, and let $U \subset \Omega$ be a bounded Lipschitz domain. Then*

$$\langle \boldsymbol{F}\cdot\nu,\phi\rangle_{\partial U} = \int_U \phi\, d div\boldsymbol{F} + \int_U \boldsymbol{F}\cdot\nabla\phi\,\mathrm{d}x = -\int_{\partial U}\phi\boldsymbol{F}\cdot\nu_U\,\mathrm{d}\mathscr{H}^{n-1}.\tag{8.27}$$

If $\boldsymbol{F} \in \mathcal{DM}^\infty(\Omega)$, then the normal trace functional on U is indeed a Radon measure, absolutely continuous with respect to $\mathscr{H}^{n-1}\llcorner\partial U$, with essentially bounded density function $\mathfrak{F}_{\mathrm{i}}\cdot\nu_U \in L^\infty(\partial U;\mathscr{H}^{n-1})$.

Proof In order to prove the first statement, we consider $\psi \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$.

Notice that

$$\int_U div\,\psi\,\mathrm{d}x = -\int_{\partial U} \psi\cdot\nu_U\,\mathrm{d}\mathscr{H}^{n-1}, \quad \int_{U^{\varepsilon,\rho}} div\,\psi\,\mathrm{d}x = -\int_{\partial U^{\varepsilon,\rho}} \psi\cdot\frac{\nabla\rho}{|\nabla\rho|}\,\mathrm{d}\mathscr{H}^{n-1}.$$

Arguing as in the proof of Theorem 8.4, we obtain

$$\int_{\partial U^{\varepsilon,\rho}} \psi\cdot\frac{\nabla\rho}{|\nabla\rho|}\,\mathrm{d}\mathscr{H}^{n-1} = \int_{\partial U} (\psi\circ f)(\varepsilon,\cdot)\cdot\Big(\frac{\nabla\rho}{|\nabla\rho|}\circ f\Big)(\varepsilon,\cdot)\,J^{\partial U}f(\varepsilon,\cdot)\,\mathrm{d}\mathscr{H}^{n-1}.$$

Since

$$\lim_{\varepsilon\to 0}\int_{U^{\varepsilon,\rho}} div\,\psi\,\mathrm{d}x = \int_U div\,\psi\,\mathrm{d}x,$$

we conclude

$$\lim_{\varepsilon\to 0}\int_{\partial U} (\psi\circ f)(\varepsilon,\cdot)\cdot\Big(\frac{\nabla\rho}{|\nabla\rho|}\circ f\Big)(\varepsilon,\cdot)\,J^{\partial U}f(\varepsilon,\cdot)\,\mathrm{d}\mathscr{H}^{n-1} = \int_{\partial U}\psi\cdot\nu_U\,\mathrm{d}\mathscr{H}^{n-1} \tag{8.28}$$

for any $\psi \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$. By the density of $C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ in $C_c(\mathbb{R}^n;\mathbb{R}^n)$ with respect to the supremum norm, we can deduce that (8.28) holds also for any $\psi \in C_c(\mathbb{R}^n;\mathbb{R}^n)$. Thus, by (8.25), we conclude that (8.27) holds.

As for the second part of the statement, we can argue as in the proof of [11, Theorem 2.2], since U has a Lipschitz deformable boundary, by Theorem 8.3.

Remark 8.6 Corollary 8.1 can also be regarded as a consequence of Proposition 6.3, together with the well-known fact that $\mathscr{H}^{n-1}(\partial U\backslash\partial^* U) = 0$ for any open set U with Lipschitz boundary. In addition, this implies that, in the case that ∂U is Lipschitz regular, $\partial^* U$ can be substituted with ∂U in (6.9)–(6.11).

We end this section by recalling an alternative result concerning the approximation of open bounded sets with Lipschitz boundary which has been proved by Nečas in [45]. For this exposition, we refer mostly to the paper of Verchota [60], in which the result in [45] is extended and applied.

Definition 8.1 We denote by $Z(P,r)$ the truncated cylinder centered at point P and with basis radius r. Given a Lipschitz domain U and a point $P \in \partial U$, we say that $Z(P,r)$ is a coordinate cylinder if

(i) The bases of $Z(P,r)$ have a positive distance from ∂U;
(ii) There exists a coordinate system $(\hat{x}_n, x_n)$ such that $\{\hat{x}_n = 0\}$ is the axis of $Z(P,r)$, and there exists a Lipschitz function $\varphi = \varphi_Z : \mathbb{R}^{n-1} \to \mathbb{R}$ such that
$$Z(P,r)\cap U = Z(P,r)\cap\{(\hat{x}_n, x_n) : x_n > \varphi(\hat{x}_n)\};$$
(iii) $P = (0,\varphi(0))$ or, equivalently, P is the origin of the coordinate system and $\varphi(0) = 0$.

The pair (Z,φ) is called a coordinate pair.

Remark 8.7 If the Lipschitz domain U is bounded, then ∂U can be covered by a finite number of coordinate cylinders $\{Z_j\}_{j=1}^N$, to which corresponds a finite number of coordinate pairs. In addition, cylinders Z_j can be selected in such a way that some dilation $Z_j^* = \lambda_j Z_j^*$, $\lambda_j > 1$, still gives a coordinate pair (Z_j^*, φ_j). We denote by L the maximum of the Lipschitz constants of functions φ_j. Also we may assume that $\varphi_j \in \mathrm{Lip}_c(\mathbb{R}^{n-1})$ without loss of generality.

Remark 8.8 Given $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n)$, there exists a sequence $\psi_k \in C_c^\infty(\mathbb{R}^n)$ such that $\psi_k \to \psi$ uniformly, $\|\nabla\psi_k\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \leqslant \|\nabla\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)}$, and $\nabla\psi_k \to \nabla\varphi$ in $L^q(\mathbb{R}^n;\mathbb{R}^n)$ for any $1 \leqslant q < \infty$. This can achieved by taking the convolution of φ with a smooth mollifier.

The following approximation results hold, for which we refer to [45, Theorem 1.1], [46, Lemma 1.1], [59, Appendix], and [60, Theorem 1.12]; see also the alternative proof given in [26]. For the self-containedness, we also give here a sketch of the proof.

Proposition 8.2 *Let U be a bounded Lipschitz domain. Then the following statements hold*:

(i) *There exists a sequence of open sets U_k satisfying that ∂U_k is of class C^∞, $U_k \Subset U_{k+1} \Subset U$, and $\bigcup_k U_k = U$;*

(ii) *There exists a covering of ∂U by coordinate cylinders such that, for any coordinate pair (Z, φ) with $\varphi \in \mathrm{Lip}_c(\mathbb{R}^{n-1})$, $Z^* \cap \partial U_k$ is the graph of a function $\varphi_k \in C_c^\infty(\mathbb{R}^{n-1})$ satisfying that $\varphi_k \to \varphi$ uniformly, $\|\nabla\varphi_k\|_{L^\infty(\mathbb{R}^{n-1};\mathbb{R}^{n-1})} \leqslant \|\nabla\varphi\|_{L^\infty(\mathbb{R}^{n-1};\mathbb{R}^{n-1})}$, and $\nabla\varphi_k \to \nabla\varphi$ $\mathscr{L}^{n-1}$–a.e. and in $L^q(\mathbb{R}^{n-1};\mathbb{R}^{n-1})$ for any $1 \leqslant q < \infty$;*

(iii) *There exists a sequence of Lipschitz diffeomorphisms $f_k : \mathbb{R}^n \to \mathbb{R}^n$ such that $f_k(\partial U) = \partial U_k$, the Lipschitz constants are uniformly bounded in k, $f_k \to \mathrm{Id}$ uniformly on ∂U, and $\nabla f_k \to \mathrm{I}_n$ for $\mathscr{H}^{n-1}$–a.e. $x \in \partial U$;*

(iv) *There exists a sequence of nonnegative functions $\omega_k = J^{\partial U} f_k$ uniformly bounded and bounded away from zero such that $(f_k^{-1})_\#(\mathscr{H}^{n-1} \llcorner \partial U_k) = \omega_k \mathscr{H}^{n-1} \llcorner \partial U$:*

$$\mathscr{H}^{n-1}(\partial U_k \cap f_k(E)) = \int_E \omega_k \,\mathrm{d}\mathscr{H}^{n-1} \qquad \text{for any Borel set } E \subset \partial U,$$

and that $\omega_k \to 1$ $\mathscr{H}^{n-1}$–a.e. on ∂U and in $L^q(\partial U;\mathscr{H}^{n-1})$ for any $1 \leqslant q < \infty$;

(v) *The normal vector to U_k satisfies that $\nu_{U_k} \circ f_k \to \nu_U$ for $\mathscr{H}^{n-1}$–a.e. $x \in \partial U$ and in $L^q(\partial U;\mathscr{H}^{n-1})$ for any $1 \leqslant q < \infty$, and an analogous statement holds for the tangent vectors*;

(vi) *There exists a C^∞ vector field H in $\mathbb{R}^n$ such that*

$$H(f_k(P)) \cdot \nu_{U_k}(f_k(P)) \geqslant C > 0 \qquad \text{for any } P \in \partial U,$$

where $C = C(H, L)$, and L is the maximal Lipschitz constant of the parameterization of ∂U.

Sketch of Proof. Results (i)–(ii) have been proved by Nečas in [45] (see also [59, Appendix]); while the others follows from the first two.

Indeed, we can define the homeomorphisms f_k in each coordinate cylinder Z_j by

$$f_k(x) = (\hat{x}_n, x_n + \varphi_k(\hat{x}_n) - \varphi(\hat{x}_n))$$

for the coordinate system $(\hat{x}_n, x_n)$ related to Z_j, and then glue these definitions together with the aid of some cutoff functions, by using the fact that the same coordinate pair can also be used in the larger cylinder Z_j^*. In this way, the uniform convergence follows immediately.

As for result (iv), we can find that

$$f_k^{-1}(y) = (\hat{y}_n, y_n - \varphi_k(\hat{y}_n) + \varphi(\hat{y}_n)),$$

where $(\hat{y}_n, y_n)$ is the coordinate system related to some Z_j. This also shows that f_k is invertible with continuous inverse, so that it is indeed a homeomorphism. In fact, since $\varphi \in \mathrm{Lip}_c(\mathbb{R}^{n-1})$ and $\varphi_k \in C_c^\infty(\mathbb{R}^{n-1})$, we can conclude that f_k is a Lipschitz diffeomorphism, with Lipschitz constants uniformly bounded in k, by using that $\nabla\varphi_k \to \nabla\varphi$ for $\mathscr{L}^{n-1}$–*a.e.* $\hat{x} \in \mathbb{R}^{n-1}$.

Moreover, by employing the area formula, it follows that ω_k is exactly the $(n-1)$–dimensional Jacobian of f_k on ∂U, $J^{\partial U} f_k$. Notice that

$$\nabla f_k = \left(\begin{array}{c|c} \mathrm{I}_{n-1} & 0 \\ \hline (\nabla(\varphi_k - \varphi))^\top & 1 \end{array} \right),$$

where I_{n-1} is the $(n-1)\times(n-1)$ identity matrix. Therefore, the convergence of $\nabla\varphi_k(x)$ to $\nabla\varphi(x)$ for $\mathscr{L}^{n-1} = \mathscr{H}^{n-1}$–*a.e.* $x \in \partial U$ implies that $\nabla f_k \to \mathrm{I}_n$ for $\mathscr{H}^{n-1}$–*a.e.* $x \in \partial U$, which in turn implies that $J^{\partial U} f_k \to 1$ for $\mathscr{H}^{n-1}$–*a.e.* $x \in \partial U$. Then the L^q–convergence follows by the Lebesgue theorem and the boundedness properties, which can be shown by calculating the Jacobian explicitly. □

Proposition 8.2 allows us to refine Theorem 8.3, by showing that any open bounded set with Lipschitz boundary admits a *regular* Lipschitz deformation in the sense of Definition 2.5. Analogously, if the set is unbounded, such a statement should hold locally.

Theorem 8.5 *If U is a bounded open set with Lipschitz boundary in $\mathbb{R}^n$, then there exists a regular Lipschitz deformation $\Psi(x,\tau) = \Psi_\tau(x)$ of ∂U satisfying*

$$\lim_{\tau \to 0^+} J^{\partial U} \Psi_\tau = 1 \qquad \text{in } L^1(\partial U; \mathscr{H}^{n-1}). \tag{8.29}$$

Proof Set

$$\Psi(x,\tau) := \big(k+1-k(k+1)\tau\big) f_{k+1}(x) + \big(k(k+1)\tau - k\big) f_k(x) \quad \text{if } \tau \in (\frac{1}{k+1}, \frac{1}{k}],$$

where the functions f_k are given by Proposition 8.2. It is clear that $\Psi(\cdot,\tau)$ is a bi-Lipschitz diffeomorphism from U over its image, by Proposition 8.2(iii),

with Lipschitz constants uniformly bounded in $\tau > 0$. Since $J^{\partial U} f_k \to 1$ in $L^q(\partial U; \mathscr{H}^{n-1})$ for any $1 \leqslant q < \infty$, by Proposition 8.2 (iv), and

$$0 \leqslant k(k+1)\tau - k \leqslant 1, \quad 0 \leqslant k+1-k(k+1)\tau \leqslant 1 \qquad \text{for } \tau \in \left(\tfrac{1}{k+1}, \tfrac{1}{k}\right],$$

we conclude that $J^{\partial U}\Psi(x,\tau) \to 1$ in $L^q(\partial U; \mathscr{H}^{n-1})$ for $1 \leqslant q < \infty$, which implies (8.29).

Remark 8.9 Hofmann-Mitrea-Taylor in [37, Proposition 4.19] worked with a strongly Lipschitz domain U in $\mathbb{R}^n$ such that there exists a C^1–vector field h satisfying

$$|h(x)| = 1, \quad h(x) \cdot \nu_U(x) \geqslant \kappa \qquad \text{for } \mathscr{H}^{n-1}\text{–a.e. } x \in \partial U$$

for some $\kappa \in (0,1)$. In the literature, a domain is said to be strongly Lipschitz if the Lipschitz constants of the parameterization of ∂U are uniformly bounded, so that any open bounded set U with Lipschitz boundary is a strongly Lipschitz domain, by compactness. For a more detailed exposition, we refer to [6, Appendix B]. Then, if $U_\tau := \{x - \tau h(x) : x \in U\}$, there exists $\tau_0 > 0$ such that, for any $\tau \in (0, \tau_0)$, U_τ is a strongly Lipschitz domain satisfying $\overline{U_\tau} \subset U$ and $\partial U_\tau = \{x - \tau h(x) : x \in \partial U\}$. In addition, the results of Proposition 8.2(ii)–(vi) hold with Lipschitz regularity, instead of C^∞. However, it is clear that more regularity on the vector field h would imply more regularity of ∂U_τ.

Moreover, a similar approximation holds from the exterior of U, if we consider $U_{-\tau}$ for $\tau \in (-\tau_1, 0]$, for some $\tau_1 > 0$.

Following Theorem 8.5, we see that the assumptions of Hofmann-Mitrea-Taylor [37, Proposition 4.19] are not strictly necessary; however, they allow to have this particular representation of the approximating sets.

9 Cauchy Fluxes and Divergence-Measure Fields

In Continuum Physics, the fundamental principle of balance law can be stated in the most general terms (*cf.* Dafermos [20] and Lax [39]): *A balance law in an open set Ω of $\mathbb{R}^n$ postulates that the production of a vector-valued "extensive" quantity in any bounded open subset $U \Subset \Omega$ is balanced by the Cauchy flux of this quantity through the boundary of U.*

For smooth continuum media, the physical principle of balance law can be formulated in the classical form:

$$\int_U b(y)\,\mathrm{d}y = \int_{\partial U} f(y)\,\mathrm{d}\mathscr{H}^{n-1}(y) \tag{9.1}$$

for any given open set U that is of smooth boundary, where f is a density function of the Cauchy flux, and b is a production density function. In mechanics, f represents the surface force per unit area on ∂U, while f gives the heat flow per unit area across the boundary ∂E in thermodynamics.

In 1823, Cauchy [9] (also see [10]) established the *stress theorem*, which states that, if $f(y) := f(y, \nu(y))$, defined for each y in an open region Ω and every unit vector ν, is continuous in y, and $b(y)$ is uniformly bounded on Ω, and if (9.1) is satisfied for every smooth region $U \Subset \Omega$, then $f(y, \nu)$ must be linear in ν; that is, there exists a vector field $\boldsymbol{F}$ such that

$$f(y, \nu) = \boldsymbol{F}(y) \cdot \nu.$$

The *Cauchy postulate* states that the density flux f through a surface depends on the surface solely through the normal at that point. Since the time of Cauchy's stress result [9, 10], many efforts have been made to generalize his ideas and remove some of his hypotheses. The first results in this direction were obtained by Noll [47] in 1959, who set up a basis for an axiomatic foundation for continuum thermodynamics. In particular, Noll [47] showed that the Cauchy postulate may directly follow from the balance law. In [36], Gurtin-Martins introduced the concept of Cauchy flux and removed the continuity assumption on f. They represented the Cauchy flux as an additive mapping $\mathcal{F}$ on surfaces S such that there exists a constant $C > 0$ so that

$$|\mathcal{F}(S)| \leqslant C\mathscr{H}^{n-1}(S), \quad |\mathcal{F}(\partial B)| \leqslant C\mathscr{L}^n(B) \tag{9.2}$$

for any surface S and subbody B.

In 1983, Ziemer [65] proved Noll's theorem in the context of geometric measure theory, in which the Cauchy fluxes were first formulated via employing sets E of finite perimeter to represent the bodies and $\partial^* E$ to represent the surfaces. His formulation of the balance law for the flux function yields the existence of a vector field $\boldsymbol{F} \in L^\infty$ with $div\boldsymbol{F} \in L^\infty$. The papers by Šilhavý [54, 55] extended definition (9.2) by requiring

$$|\mathcal{F}(S)| \leqslant \int_S h \, \mathrm{d}\mathscr{H}^{n-1}, \quad |\mathcal{F}(\partial B)| \leqslant \int_B g \, dx \tag{9.3}$$

for suitable functions g and h in L^p for $p \geqslant 1$ and almost every surface S. The vector fields obtained under these conditions have distributional divergences that are integrable; that is, $\boldsymbol{F} \in L^1$ and $div\boldsymbol{F} \in L^1$. However, all the previous formulations of Cauchy fluxes do not allow the presence of "shock waves" since $div\boldsymbol{F}$ is absolutely continuous with respect to $\mathcal{L}^n$.

Degiovanni-Marzocchi-Musesti [25] further generalized conditions (9.2) and considered the Cauchy fluxes defined on *almost* every surface and satisfying

$$|\mathcal{F}(S)| \leqslant \int_S h \, \mathrm{d}\mathscr{H}^{n-1}, \quad |\mathcal{F}(\partial B)| \leqslant \sigma(B) \tag{9.4}$$

for a suitable function $h \in L^1_{\mathrm{loc}}$ and a nonnegative Radon measure σ. This definition of Cauchy fluxes induced the existence of a vector field $\boldsymbol{F} \in \mathcal{DM}^1_{\mathrm{loc}}$. Schuricht [53] studied an alternative formulation to (9.2), which consists in considering the contact interactions f as maps on pairs of disjoint subbodies (instead of surfaces). Thus, $f(B, A)$ is the resultant force exerted on B by A. The function f is assumed to be countable additive in the first argument (*i.e.*,

a measure) and finitely additive with respect to the second argument. This alternative formulation also implies the existence of $\boldsymbol{F} \in \mathcal{DM}^1_{\mathrm{loc}}$, depending on A, such that $div\boldsymbol{F} = f(\cdot, A)$. The Gauss-Green formulas obtained in [25] and [53] are valid for $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for any $p \geqslant 1$, but only on the sets of finite perimeter, $E \subset \Omega$, which lie in a suitable subalgebra related to the particular representative of $\boldsymbol{F}$. In other words, these Gauss-Green formulas are valid only on *almost every* set, thus missing the exceptional surfaces or "shock waves". In order to recover the flux on *every surface*, it is necessary to develop a theory of normal traces for divergence-measure fields.

In Chen-Torres-Ziemer [14], such a theory of normal traces on reduced boundaries of sets of finite perimeter has been established for $\mathcal{DM}^\infty(\Omega)$–fields. The method in [14] consists in constructing the normal trace as the limit of the classical normal traces over smooth approximations of the set of finite perimeter. This approach requires a new approximation theorem of sets of perimeter that can distinguish between the measure-theoretic interior and exterior of the set. The Cauchy flux introduced in [14] is defined on *every* set of finite perimeter, $E \Subset \Omega$, and on *every* $\mathscr{H}^{n-1}$–rectifiable surface $S \subset \partial^* E$ (so that S is oriented with the normal to the set). The conditions that there exists a nonnegative Radon measure σ such that

$$|\mathcal{F}(\partial^* E)| \leqslant \sigma(E^1), \qquad |\mathcal{F}(S)| \leqslant C\mathscr{H}^{n-1}(S) \tag{9.5}$$

imply the existence of a $\mathcal{DM}^\infty$–field $\boldsymbol{F}$ so that the Cauchy flux over every surface can be recovered through the normal traces of $\boldsymbol{F}$ on the oriented surface (see Remark 9.1(ii) below).

The Cauchy fluxes in the sense of Chen-Torres-Ziemer [14] allow the presence of exceptional surfaces (*i.e.* shock waves) in the formulation of the axioms. In this setting, $\mathrm{div}\boldsymbol{F} = g\sigma$ for some function g and Radon measure σ, and hence $\mathrm{div}\boldsymbol{F}$ is not in general absolutely continuous with respect to $\mathcal{L}^n$. The measure σ does not vanish on the exceptional surfaces and the Cauchy flux $\mathcal{F}$ has a discontinuity since $\mathcal{F}(S) \neq -\mathcal{F}(-S)$. Formulations (9.2)–(9.3) deal with the particular case $\sigma = \mathcal{L}^n$, and hence the measure vanishes on any $\mathscr{H}^{n-1}$–dimensional surface, excluding the shock waves.

Example 9.1 Let

$$\boldsymbol{F}(x_1, x_2) = f(x_2)g(x_1)(1, 0) \qquad \text{for some } f \in L^\infty(\mathbb{R}) \text{ and } g \in C^1_c(\mathbb{R}),$$

so that $\boldsymbol{F} \in \mathcal{DM}^\infty(\mathbb{R}^2)$, and let E be as in Remark 4.9. Then, by (4.8), $\boldsymbol{G} = \chi_E \boldsymbol{F}$ is also in $\mathcal{DM}^\infty(\mathbb{R}^2)$, while $\boldsymbol{G} \notin BV_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$, since E is not a set of locally finite perimeter.

If $U := (0, 2)^2 = U^1$, then $E \cap U = E$ and $\chi_U \boldsymbol{G} = \chi_E \boldsymbol{F}$. Hence, applying (4.8) again, we obtain

$$\begin{aligned}
&\langle \boldsymbol{G} \cdot \nu, \cdot \rangle_{\partial U} \\
&= \chi_U div\, \boldsymbol{G} - div(\chi_U \boldsymbol{G}) = (\chi_{U^1} - 1) div\, \boldsymbol{G} = (\chi_{U^1} - 1) f(x_2) g(x_1) D_{x_1} \chi_E \\
&= -f(x_2) g(x_1) \mathscr{H}^1 \llcorner \big(\{0\} \times (0, 1)\big) + f(x_2) g(x_1) \mathscr{H}^1 \llcorner \big(\{2\} \times (\tfrac{1}{2}, \tfrac{3}{4})\big).
\end{aligned} \tag{9.6}$$

On the other hand, if $L := U \cup \partial^* U$ instead, we still obtain that $\chi_L \boldsymbol{G} = \chi_E \boldsymbol{F} = \boldsymbol{G}$ and

$$\begin{aligned} \langle \boldsymbol{G} \cdot \nu, \cdot \rangle_{\partial L} &= \chi_L div\, \boldsymbol{G} - div(\chi_L \boldsymbol{G}) = (\chi_L - 1) div\, \boldsymbol{G} \\ &= (\chi_{U^1 \cup \partial^* U}(x_1, x_2) - 1) f(x_2) g(x_1) D_{x_1} \chi_E = 0, \end{aligned} \tag{9.7}$$

since $\mathrm{supp}(|D_{x_1} \chi_E|) \subset U^1 \cup \partial^* U$. Hence, it follows that

$$\langle \boldsymbol{G} \cdot \nu, \cdot \rangle_{\partial U} \neq \langle \boldsymbol{G} \cdot \nu, \cdot \rangle_{\partial (U \cup \partial^* U)} \tag{9.8}$$

in general. This condition is satisfied, for instance, if $g(0) < 0$, $g(2) = 0$, and $f > 0$ in $(0, 1)$, since $\langle \boldsymbol{G} \cdot \nu, \cdot \rangle_{\partial U}$ is a nontrivial nonnegative Radon measure in this case.

Thanks to the fact that $\boldsymbol{G} \in \mathcal{DM}^\infty(\mathbb{R}^2)$, we can define an associated Cauchy flux $\mathcal{F}$ by using the theory developed in [14]. Given a bounded set of finite perimeter M, there exist the interior and exterior normal traces of $\boldsymbol{G}$: $(\mathfrak{G}_{\mathrm{i}} \cdot \nu_M)$ and $(\mathfrak{G}_{\mathrm{e}} \cdot \nu_M) \in L^\infty(\partial^* M; \mathscr{H}^1)$ (see Proposition 4.3). Then we define

$$\mathcal{F}(S) := -\int_S \mathfrak{G}_{\mathrm{i}} \cdot \nu_M \, \mathrm{d}\mathscr{H}^1, \tag{9.9}$$

if S is an $\mathscr{H}^1$–rectifiable surface such that $S \subset \partial^* M$ which is oriented by ν_M, for some bounded set of finite perimeter M; and

$$\mathcal{F}(S) := -\int_S \mathfrak{G}_{\mathrm{e}} \cdot \nu_M \, \mathrm{d}\mathscr{H}^1, \tag{9.10}$$

if S is an $\mathscr{H}^1$–rectifiable surface such that $S \subset \partial^* M$ which is oriented by $-\nu_M$, for some bounded set of finite perimeter M.

It is not difficult to check that $\mathcal{F}$ is a Cauchy flux in the sense of [14]. Indeed, by definition, $\mathcal{F}$ is a finitely additive functional on disjoint surfaces. Since the normal traces are essentially bounded, from (9.9)–(9.10), we obtain

$$|\mathcal{F}(S)| \leqslant C \mathscr{H}^1(S).$$

Then $|\mathcal{F}(\partial^* M)| \leqslant \sigma(M^1)$ for any bounded set M of finite perimeter, if $\sigma = |div\, \boldsymbol{G}|$ is chosen. Indeed, we need just to employ the Gauss-Green formulas and the fact that $\boldsymbol{G}$ has compact support.

In particular, if we apply (4.6) to $\boldsymbol{G}$, a bounded set of finite perimeter M and $\phi \in \mathrm{Lip}_c(\mathbb{R}^2)$ with $\phi \equiv 1$ on $\overline{M}$, then

$$\mathcal{F}(\partial^* M) = -\int_{\partial^* M} \mathfrak{G}_{\mathrm{i}} \cdot \nu_M \, \mathrm{d}\mathscr{H}^1 = div\, \boldsymbol{G}(M^1).$$

Arguing analogously, from (4.7), we have

$$\mathcal{F}(-\partial^* M) = -\int_{\partial^* M} \mathfrak{G}_{\mathrm{e}} \cdot \nu_M \, \mathrm{d}\mathscr{H}^1 = div\, \boldsymbol{G}(M^1 \cup \partial^* M).$$

Since $\boldsymbol{G}$ has compact support in $\mathbb{R}^2$, by [16, Lemma 3.1], we obtain

$$0 = div\, \boldsymbol{G}(\mathbb{R}^2) = div\, \boldsymbol{G}(M^1 \cup \partial^* M) + div\, \boldsymbol{G}((\mathbb{R}^2 \backslash M)^1),$$

from which it follows that

$$\mathcal{F}(\partial^*(\mathbb{R}^2\backslash M)) = \mathcal{F}(-\partial^* M) = -div\,\boldsymbol{G}((\mathbb{R}^2\backslash M)^1).$$

Thus, (9.5) is satisfied. Then we have proved that $\mathcal{F}$ is a Cauchy flux in the sense of [14].

Choose $S := \partial^* U$ oriented by ν_U, by (9.10), Proposition 4.3, and (9.6), we have

$$\mathcal{F}(\partial^* U) = \langle \boldsymbol{G}\cdot\nu, \phi\rangle_{\partial U} = -\int_{\partial^* U} \phi(x_1,x_2) f(x_2) g(x_1)\,\mathrm{d}D_{x_1}\chi_E$$

for any $\phi \in \mathrm{Lip}_c(\mathbb{R}^2)$ with $\phi \equiv 1$ on $\overline{U}$. Arguing analogously, by (9.10), Proposition 4.3, and (9.7), we have

$$\mathcal{F}(-\partial^* U) = \langle \boldsymbol{G}\cdot\nu, \phi\rangle_{\partial(U\cup\partial^* U)} = 0,$$

for any $\phi \in \mathrm{Lip}_c(\mathbb{R}^2)$ with $\phi \equiv 1$ on $\overline{U}$. Then, by (9.8), it follows that, in general,

$$\mathcal{F}(\partial^* U) \neq -\mathcal{F}(-\partial^* U),$$

which shows that $\mathcal{F}$ may have a discontinuity on the rectifiable surface $\partial^* U$.

In this paper above, we have developed a more general theory of normal traces for unbounded $\mathcal{DM}^p$ fields. In particular, we have shown that the normal trace can be represented as the limit of the classical normal traces on smooth approximations or deformations. Hence, we can now give a more general definition of Cauchy fluxes over general open sets (not necessarily of finite perimeter).

Definition 9.1 (Side surfaces) A side surface in Ω is a pair (S,U) so that $S \Subset \Omega$ is a Borel set and $U \Subset \Omega$ is a open set such that $S \subset \partial U$. The side surface (S,U) is often written as S for simplicity, when no confusion arises from the context.

Definition 9.2 (Cauchy fluxes) Let Ω be a bounded open set. A Cauchy flux is a functional $\mathcal{F}$ defined on the side surfaces (S,U) such that the following properties hold:

(i) $\mathcal{F}(S_1 \cup S_2) = \mathcal{F}(S_1) + \mathcal{F}(S_2)$ for any pair of disjoint side surfaces S_1 and S_2 in ∂U, for some $U \Subset \Omega$;

(ii) There exists a nonnegative Radon measure σ in Ω such that

$$|\mathcal{F}(\partial U)| \leqslant \sigma(U) \qquad \text{for every open set } U \Subset \Omega;$$

(iii) There exists a nonnegative Borel function $h \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$|\mathcal{F}(S)| \leqslant \int_S h\,\mathrm{d}\mathscr{H}^{n-1}$$

for any side surface $S \subset \partial U$ and any open set $U \Subset \Omega$ (the integral could be ∞, in which case the axiom is also true).

For simplicity, the Cauchy flux is often written as $\mathcal{F}(S)$ as (i)–(iii) above, when no confusion arises from the context.

We state now our main result on the representation of general Cauchy fluxes.

Theorem 9.1 *Let $\mathcal{F}$ be a Cauchy flux in Ω with $h \in L^1_{\mathrm{loc}}(\Omega)$ as in Definition* 9.2. *Then there exists a unique $\boldsymbol{F} \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$ such that, for every open set $U \Subset \Omega$,*

(i) *For any $\phi \in C^1_c(\Omega)$ such that $\phi \equiv 1$ on a neighborhood of ∂U,*

$$\mathcal{F}(\partial U) = \langle \boldsymbol{F} \cdot \nu,\, \phi \rangle_{\partial U}\,, \tag{9.11}$$

and there exists an interior smooth approximation U^ε of U as in Theorem 7.1 *such that, for a suitable subsequence $\varepsilon_k \to 0$ as $k \to \infty$,*

$$\mathcal{F}(\partial U) = -\lim_{\epsilon_k \to 0} \int_{\partial U^{\varepsilon_k}} \boldsymbol{F} \cdot \nu_{U^{\varepsilon_k}}\, \mathrm{d}\mathscr{H}^{n-1}$$

where $\boldsymbol{F} \cdot \nu_{U_{\varepsilon_k}}$ denotes the classical dot product;

(ii) *If $\chi_U \boldsymbol{F} \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$, then there exists $\mu_b \in \mathcal{M}(\partial U)$ such that*

$$\mathcal{F}(\partial U) = \int_{\partial U} \mathrm{d}\mu_b;$$

(iii) *If U is a C^0 domain, then there exists a sequence of smooth set $U^{\varepsilon,\rho}$ as in Theorem* 7.4, *which can be represented as a deformation generated by $f(\varepsilon, x)$ (defined in Theorem* 8.1) *that is C^∞ in x when $\varepsilon > 0$ and C^0 in x when $\varepsilon = 0$, such that, for a suitable subsequence $\varepsilon_k \to 0$,*

$$\mathcal{F}(\partial U) = -\lim_{\varepsilon_k \to 0} \int_{\partial U^{\varepsilon_k,\rho}} \boldsymbol{F} \cdot \nu_{U^{\varepsilon_k,\rho}}\, \mathrm{d}\mathscr{H}^{n-1};$$

(iv) *If U has a Lipschitz boundary, then there exists a regular Lipschitz deformation $\Psi(x, \varepsilon) =: \Psi_\varepsilon(x)$ of ∂U such that, for a suitable subsequence $\varepsilon_k \to 0$ as $k \to \infty$,*

$$\mathcal{F}(\partial U) = -\lim_{\varepsilon_k \to 0} \int_{\partial U} \boldsymbol{F}(\Psi_{\varepsilon_k}(x)) \cdot \nu_U(\Psi_{\varepsilon_k}(x)) J\Psi_{\varepsilon_k}(x) \mathrm{d}\mathscr{H}^{n-1}(x).$$

Proof We divide the proof into four steps.

1. We first show the existence of such an $\boldsymbol{F} \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$. Let $\mathcal{I}_\Omega$ be the collection of all closed cubes in $\mathbb{R}^n$ of the form:

$$I = [a_1, b_1] \times \cdots \times [a_n, b_n],$$

such that $I \Subset \Omega$. For almost every $s \in [a_j, b_j]$, define

$$I_{j,s} := \{y \in I \,:\, y_j = s\}.$$

Let $\{e_1, \dots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. We fix $j \in \{1, \dots, n\}$. For every cube $I \in \mathcal{I}_\Omega$, define

$$\mu^j(I) := \int_{a_j}^{b_j} \mathcal{F}(I_{j,s})\, \mathrm{d}s.$$

From Definition 9.2(iii), we have

$$|\mu^j(I)| \leqslant \int_{a_j}^{b_j} |\mathcal{F}(I_{j,s})|\, \mathrm{d}s \leqslant \int_{a_j}^{b_j} \int_{I_{j,s}} h\, d\mathscr{H}^{n-1} \mathrm{d}s \leqslant \int_I |h|\, \mathrm{d}x = \|h\|_{L^1(I)},$$

where the Fubini theorem has been used. Thus, for any finite collection of disjoint cubes $I_1, \dots, I_K$, we have

$$\sum_{i=1}^K |\mu^j(I_i)| \leqslant \sum_{i=1}^K \|h\|_{L^1(I_i)} \leqslant \|h\|_{L^1(\cup_{i=1}^K I_i)}. \tag{9.12}$$

Since $h \in L^1_{\mathrm{loc}}(\Omega)$, then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mathscr{L}^n(A) < \delta \quad \Longrightarrow \quad \int_A |h|\, \mathrm{d}x < \varepsilon.$$

Hence, if $\{I_i\}_{i=1}^K$ is a finite collection of disjoint cubes $I_1, \dots, I_k$, satisfying $\sum_{i=1}^K \mathcal{L}^n(I_i) = \mathcal{L}^n(\cup_{i=1}^K I_i) < \delta$, then

$$\sum_{i=1}^K |\mu^j(I_i)| \leqslant \|h\|_{L^1(\cup_{i=1}^K I_i)} < \varepsilon. \tag{9.13}$$

Hence, μ^j is an additive set function defined on $\mathcal{I}_\Omega$. We can now apply a generalization of Riesz's theorem, due to Fuglede [33] (see also [14, Theorem 9.5]), to conclude that there exists $f_j \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\mu^j(I) = \int_I f_j\, \mathrm{d}x \qquad \text{for every } I \in \mathcal{I}_\Omega.$$

We take sequences $\alpha_{k,j} \uparrow s$ and $\beta_{k,j} \downarrow s$ as $k \to \infty$. We have

$$\frac{1}{\beta_{k,j} - \alpha_{k,j}} \int_{\alpha_{k,j}}^{\beta_{k,j}} \mathcal{F}(I_{j,s})\, \mathrm{d}s = \frac{1}{\beta_{k,j} - \alpha_{k,j}} \int_{\alpha_{k,j}}^{\beta_{k,j}} \int_{I_{j,s}} f_j\, \mathrm{d}x \mathrm{d}s.$$

Letting $k \to \infty$ yields

$$\mathcal{F}(I_{j,s}) = \int_{I_{j,s}} f_j\, d\mathscr{H}^{n-1} \qquad \text{for } \mathscr{L}^1\text{–a.e. } s.$$

Define

$$\boldsymbol{F} := (f_1, \dots, f_n).$$

We obtain that, for every $j \in \{1, \dots, n\}$,

$$\mathcal{F}(I_{j,s}) = -\int_{I_{j,s}} \boldsymbol{F}(y) \cdot e_j \, \mathrm{d}\mathscr{H}^{n-1}(y) \qquad \text{for } \mathscr{L}^1\text{–a.e. } s. \tag{9.14}$$

From this point on, we say that a statement holds for almost every cube if it holds for all cubes whose side intervals with endpoints in $\mathbb{R}\backslash\mathcal{N}$, for some $\mathscr{L}^1$ negligible set $\mathcal{N}$.

From (9.14), it follows that, for almost every cube $I \in \mathcal{I}_\Omega$,

$$\mathcal{F}(\partial I) = -\int_{\partial I} \boldsymbol{F}(y) \cdot \nu_I(y) \, \mathrm{d}\mathscr{H}^{n-1}(y), \tag{9.15}$$

which, by Definition 9.2(ii), implies

$$\Big| \int_{\partial I} \boldsymbol{F}(y) \cdot \nu_I(y) \, \mathrm{d}\mathscr{H}^{n-1}(y) \Big| = |\mathcal{F}(\partial I)| \leqslant \sigma(\mathring{I}) \leqslant \sigma(I), \tag{9.16}$$

where $\mathring{I}$ denotes the open cube.

Using (9.16), we can now proceed as in [14, Lemma 9.6], or use [25, Theorem 5.3], to conclude that $\boldsymbol{F}$ is a vector field with divergence measure satisfying $|\mathrm{div}\boldsymbol{F}| \leqslant \sigma$, which means that $\boldsymbol{F} \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$.

2. *Uniqueness of the $\mathcal{DM}^1$–field $\boldsymbol{F}$.* Assume now that there exists another vector field $\mathbf{G} = (g_1, \cdots, g_n)$ such that (9.14) holds. For fixed $j \in \{1, \cdots, n\}$, we obtain that, for any cube $I \in \mathcal{I}_\Omega$,

$$\int_I f_j \mathrm{d}x = \int_{a_j}^{b_j} \int_{I_{s,j}} f_j \, d\mathscr{H}^{n-1}(y)\mathrm{d}s = \int_{a_j}^{b_j} \int_{I_{s,j}} g_j \, \mathrm{d}\mathscr{H}^{n-1}(y)ds = \int_I g_j \mathrm{d}x.$$

Hence, $f_j(x) = g_j(x)$ for $\mathcal{L}^n$–*a.e.* x.

3. In order to prove (i), we approximate ∂U with closed cubes in such a way that

$$\partial U = \bigcap_{i=1}^{\infty} J_i,$$

where each J_i is a finite union of closed cubes in $\mathcal{I}_\Omega$, which can be chosen so that (9.15) holds, and $J_{i+1} \subset J_i$. In addition, we can also choose $\{J_i\}$ in such a way that

$$\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial \mathring{J}_i} = -\boldsymbol{F} \cdot \nu_{J_i} \mathscr{H}^{n-1} \llcorner \partial J_i. \tag{9.17}$$

This follows for instance from [25, Theorem 7.2], which states that, for almost every closed cube $I \in \mathcal{I}_\Omega$, we have

$$div\boldsymbol{F}(I) = -\int_{\partial I} \boldsymbol{F} \cdot \nu_I \, \mathrm{d}\mathscr{H}^{n-1}. \tag{9.18}$$

On the other hand, for any $\varphi \in \mathrm{Lip}_c(\Omega)$, $\varphi\boldsymbol{F} \in \mathcal{DM}^1(\Omega)$ by Proposition 3.1. It is clear that, by (3.3),

$$div(\varphi\boldsymbol{F})(\mathring{I}) = div(\varphi\boldsymbol{F})(I) - \int_{\partial I} \varphi \, \mathrm{d}div\boldsymbol{F}.$$

By Remark 4.2 and (9.18), it follows that

$$\langle \boldsymbol{F}\cdot\nu,\varphi\rangle_{\partial \mathring{I}} = div(\varphi\boldsymbol{F})(\mathring{I}) = -\int_{\partial I}\varphi\,\mathrm{d}div\boldsymbol{F} - \int_{\partial I}\varphi\boldsymbol{F}\cdot\nu_I\,\mathrm{d}\mathscr{H}^{n-1}.$$

Then, arguing as in [2, Example 1.63], we obtain that $|div\boldsymbol{F}|(\partial I) = 0$ for almost every $I \in \mathcal{I}_\Omega$, since $div\boldsymbol{F}$ is a Radon measure. All in all, we conclude (9.17) for almost every finite union J of closed cubes; then the sequence $\{J_i\}$ is chosen from these finite unions $\{J\}$ of closed cubes.

Now, from Definition 9.2(ii), we have

$$|\mathcal{F}(\partial(\mathring{J}_i \cap U))| \leqslant \sigma(\mathring{J}_i \cap U).$$

Standard measure theory arguments imply that

$$\lim_{i\to\infty}\sigma(\mathring{J}_i \cap U) = \sigma((\cap_i J_i)\cap U) = \sigma(\partial U \cap U) = 0, \tag{9.19}$$

so that

$$\lim_{i\to\infty}\mathcal{F}(\partial(\mathring{J}_i \cap U)) = 0. \tag{9.20}$$

Now, we consider $\langle \boldsymbol{F}\cdot\nu,\phi\rangle_{\partial(\mathring{J}_i\cap U)}$ for some $\phi \in C_c^1(\Omega)$. We notice by Theorem 5.1 that, for any $\phi \in C_c^1(\Omega)$,

$$\langle \boldsymbol{F}\cdot\nu,\phi\rangle_{\partial U} = -\lim_{k\to\infty}\int_{\partial^* U^{\varepsilon_k}}\phi\boldsymbol{F}\cdot\nu_{U^{\varepsilon_k}}\,\mathrm{d}\mathscr{H}^{n-1}. \tag{9.21}$$

In the same way, we have

$$\langle \boldsymbol{F}\cdot\nu,\phi\rangle_{\partial(\mathring{J}_i\cap U)} = -\lim_{k\to\infty}\int_{\partial^* W^{\varepsilon_k}}\phi\boldsymbol{F}\cdot\nu_{W^{\varepsilon_k}}\,\mathrm{d}\mathscr{H}^{n-1} \qquad \text{for any } \phi \in C_c^1(\Omega), \tag{9.22}$$

where $\partial W^{\varepsilon_k}$ is defined as the superlevel set of the signed distance function associated to $W = \mathring{J}_i \cap U$, as in (5.2).

We now choose a test function $\phi \in C_c^1(\Omega)$ such that $\phi \neq 0$ in a neighborhood of ∂U, but $\phi \equiv 0$ on $\Omega\backslash J_i$. Since J_i is closed, then $\phi = 0$ in a neighborhood of $\partial J_i \cap U$. With this choice of ϕ, (9.22) reduces to

$$\langle \boldsymbol{F}\cdot\nu,\phi\rangle_{\partial(\mathring{J}_i\cap U)} = -\lim_{k\to\infty}\int_{\partial^* U^{\varepsilon_k}}\chi_{\mathring{J}_i}\phi\boldsymbol{F}\cdot\nu_{U^{\varepsilon_k}}\,\mathrm{d}\mathscr{H}^{n-1} \tag{9.23}$$

and, from (9.21) and the fact that $\mathring{J}_i \cap \partial^* U^{\varepsilon_k} = \partial^* U^{\varepsilon_k}$ for ε_k small enough and i fixed, we obtain

$$\langle \boldsymbol{F}\cdot\nu,\phi\rangle_{\partial(\mathring{J}_i\cap U)} = \langle \boldsymbol{F}\cdot\nu,\phi\rangle_{\partial U} \qquad \text{for any such } \phi. \tag{9.24}$$

Therefore, the distribution $\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial(\mathring{J}_i\cap U)}$ coincides with $\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial U}$ on ∂U.

Arguing similarly, we can show

$$\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial(\mathring{J}_i\cap U)} = \langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial \mathring{J}_i} \qquad \text{on } U\cap\partial J_i. \tag{9.25}$$

Therefore, we conclude

$$\begin{aligned}\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial(\mathring{J}_i\cap U)} &= \langle \chi_{J_i}\boldsymbol{F}\cdot\nu,\,\cdot\rangle_{\partial U} + \langle \chi_U\boldsymbol{F}\cdot\nu,\,\cdot\rangle_{\partial\mathring{J}_i}\\ &= \langle \boldsymbol{F}\cdot\nu,\,\cdot\rangle_{\partial U} + \langle \chi_U\boldsymbol{F}\cdot\nu,\,\cdot\rangle_{\partial\mathring{J}_i},\end{aligned} \tag{9.26}$$

since $J_i \supset \partial U$. By (4.2), we have

$$\langle \boldsymbol{F}\cdot\nu,\cdot\rangle_{\partial(\mathring{J}_i\cap U)} = \chi_{\mathring{J}_i\cap U}\,div\boldsymbol{F} - div(\chi_{\mathring{J}_i\cap U}\boldsymbol{F}). \tag{9.27}$$

If we now choose $\phi \in C^1_c(\Omega)$ such that $\phi \equiv 1$ on a neighborhood of ∂U, then $\phi \equiv 1$ on $\partial U \cup \partial J_i$ for any i large enough. Then, from (9.17) and (9.24)–(9.27), we obtain

$$\begin{aligned}\langle \boldsymbol{F}\cdot\nu,\,\phi\rangle_{\partial U} - \int_{U\cap\partial J_i}\boldsymbol{F}\cdot\nu_{J_i}\,\mathrm{d}\mathscr{H}^{n-1} &= \langle \boldsymbol{F}\cdot\nu,\,\phi\rangle_{\partial U} - \int_{\partial J_i}\phi\chi_U\boldsymbol{F}\cdot\nu_{J_i}\,\mathrm{d}\mathscr{H}^{n-1}\\ &= \langle \boldsymbol{F}\cdot\nu,\,\phi\rangle_{\partial U} + \langle \chi_U\boldsymbol{F}\cdot\nu,\,\phi\rangle_{\partial\mathring{J}_i}\\ &= \langle \boldsymbol{F}\cdot\nu,\phi\rangle_{\partial(\mathring{J}_i\cap U)}\\ &= \int_{\mathring{J}_i\cap U}\phi\,\mathrm{d}div\boldsymbol{F} - \int_\Omega \phi\,\mathrm{d}div(\chi_{\mathring{J}_i\cap U}\boldsymbol{F})\\ &= \int_{\mathring{J}_i\cap U}\phi\,\mathrm{d}div\boldsymbol{F} + \int_{J_i\cap U}\boldsymbol{F}\cdot\nabla\phi\,\mathrm{d}x \to 0\end{aligned}$$

as $i\to\infty$, since $|J_i\cap U|\to 0$ and $|div\boldsymbol{F}|(\mathring{J}_i\cap U)\to 0$. This implies

$$\langle \boldsymbol{F}\cdot\nu,\,\phi\rangle_{\partial U} = \lim_{i\to\infty}\int_{U\cap\partial J_i}\boldsymbol{F}(y)\cdot\nu_{J_i}(y)\mathrm{d}\mathscr{H}^{n-1}(y). \tag{9.28}$$

On the other hand, by Definition 9.2(i), we have

$$\mathcal{F}(\partial(\mathring{J}_i\cap U)) = \mathcal{F}(\partial U) + \mathcal{F}(U\cap\partial J_i),$$

so that, from (9.15) and (9.20), we obtain

$$\mathcal{F}(\partial U) = \lim_{i\to\infty}\int_{U\cap\partial J_i}\boldsymbol{F}(y)\cdot\nu_{J_i}(y)\mathrm{d}\mathscr{H}^{n-1}(y). \tag{9.29}$$

Finally, from (9.28)–(9.29), we conclude (9.11). Then Theorem 7.1 implies the second part of point (i) (see also Theorem 5.1).

4. For (ii), we notice that $\chi_U\boldsymbol{F}\in\mathcal{DM}^1(\Omega)$, since $U\Subset\Omega$. Hence, Theorem 4.1 implies the existence of a finite Radon measure μ_b on ∂U such that

$$\langle \boldsymbol{F}\cdot\nu,\phi\rangle_{\partial U} = \int_{\partial U}\phi\,\mathrm{d}\mu_b \qquad \text{for any } \phi\in C^1_c(\Omega).$$

Thus, if we take $\phi\equiv 1$ on a neighborhood of ∂U, result (i) immediately implies (ii). Cases (iii) and (iv) follow analogously from (i): for (iii), we need to apply Theorem 7.4; while (iv) is obtained by employing the bi-Lipschitz regular deformation $\Psi_\varepsilon(x)$ from Theorem 8.5 as in Theorem 8.4.

Remark 9.1 In particular, the following assertions also hold:

(i) If U is an open set of finite perimeter, $\sigma(\partial U) = 0$, and $\int_{\partial^* U} h \, \mathrm{d}\mathscr{H}^{n-1} < \infty$, then

$$\mathcal{F}(\partial U) = -\int_{\partial^* U} \boldsymbol{F} \cdot \nu_U \, \mathrm{d}\mathscr{H}^{n-1},$$

where $\boldsymbol{F} \cdot \nu_U$ is the classical dot product.
To see this, we use the assumption that $\sigma(\partial U) = 0$ and $\int_{\partial^* U} h \, \mathrm{d}\mathscr{H}^{n-1} < \infty$ to apply [25, Theorem 5.4], which shows that $\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial U}$ is represented by the classical dot product between $\boldsymbol{F}$ and ν_U. Hence, from Theorem 9.1(i), we have

$$\mathcal{F}(\partial U) = -\int_{\partial^* U} \boldsymbol{F}(x) \cdot \nu_U(x) \, \mathrm{d}\mathscr{H}^{n-1}(x).$$

(ii) If the Borel function h in Definition 9.2(iii) is constant and U is an open set of finite perimeter, then $\boldsymbol{F} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$, and

$$\mathcal{F}(\partial U) = -\int_{\partial^* U} (\mathfrak{F}_{\mathrm{i}} \cdot \nu_U) \, \mathrm{d}\mathscr{H}^{n-1},$$

where $\mathfrak{F}_{\mathrm{i}} \cdot \nu_U \in L^\infty(\partial^* U; \mathscr{H}^{n-1})$ is the interior normal trace of $\boldsymbol{F}$ on $\partial^* U$. This corresponds to the case already treated in [14, Theorem 9.4], since $U \Subset \Omega$ is a set of finite perimeter and h is constant. Hence, we obtain $\boldsymbol{F} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$, and the normal trace $\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial U}$ is represented by the measure $-(\mathfrak{F}_{\mathrm{i}} \cdot \nu_U) \, \mathscr{H}^{n-1} \llcorner \partial^* U$, for some $(\mathfrak{F}_{\mathrm{i}} \cdot \nu_U) \in L^\infty(\partial^* U; \mathscr{H}^{n-1})$; see also Proposition 4.3.

Remark 9.2 The importance of Theorem 9.1 and Remark 9.1 is that the flux can be recovered on every open set U.

Remark 9.3 It has been discussed above that Schuricht [53] considered an alternative formulation for the axioms, representing the contact interactions as maps $f(B, A)$ on pairs of disjoint subbodies (instead of surfaces). In this formulation, our results on normal traces for $\mathcal{DM}^p$–fields improve those in [53, Theorem 5.20, equation (5.21)], since $f(B, A)$ can be written as the limit of the classical normal traces on the approximations of B, versus an integral average.

Acknowledgements The research of Gui-Qiang G. Chen was supported in part by the UK Engineering and Physical Sciences Research Council Award EP/E035027/1 and EP/L015811/1, and the Royal Society–Wolfson Research Merit Award (UK). The research of Giovanni E. Comi was supported in part by the PRIN2015 MIUR Project "Calcolo delle Variazioni". The research of Monica Torres was supported in part by the Simons Foundation Award No. 524190 and by the National Science Foundation Award #1813695.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Ambrosio, L., Crippa, G., Maniglia, S.: Traces and fine properties of a BD class of vector fields and applications. *Ann. Fac. Sci. Toulouse Math.* (6), 14(4):527–561, 2005. http://dx.doi.org/10.5802/afst.1102
2. Ambrosio, L., Fusco, N., Pallara, D.: *Functions of Bounded Variation and Free Discontinuity Problems.* The Clarendon Press, Oxford University Press: New York, 2000. https://doi.org/10.1017/S0024609301309281
3. Ambrosio, L., Tilli, P.: *Topics on Analysis in Metric Spaces.* Oxford University Press: Oxford, 2004. ISBN: 9780198529385
4. Anzellotti, G.: Pairings between measures and bounded functions and compensated compactness. *Annali di Matematica Pura ed Applicata*, 135(1):293–318, 1983. https://doi.org/10.1007/bf01781073
5. Anzellotti, G.: Traces of bounded vector-fields and the divergence theorem, *Preprint*, 1983.
6. Auscher, P., Russ, E., Tchamitchian, P.: Hardy Sobolev spaces on strongly Lipschitz domains of $\mathbb{R}^n$. *J. Funct. Anal.* 218(1):54–109, 2005. https://doi.org/10.1016/j.jfa.2004.06.005
7. Ball, J. M., Zarnescu, A.: Partial-regularity and smooth topology preserving approximations of rough domains, *Calc. Var. & PDEs.* 56(1), Art. 13, 32pp., 2017. https://doi.org/10.1007/s00526-016-1092-6
8. Burago, Y. D., Maz'ya, V. G.: *Potential Theory and Function Theory for Irregular Regions.* Translated from Russian. Seminars in Mathematics, V. A. Steklov Mathematical Institute, Leningrad, Vol. 3. Consultants Bureau: New York, 1969. ASIN: B01KK6TXLK
9. A. L. Cauchy. Recherches sur l'équilibre et le mouvement intérieur des corps solides ou fluides, élastiques or non élastiques. *Bull. Soc. Philomathique*, 10(2):9–13, 1823. https://doi.org/10.1017/CBO9780511702518.038
10. A. L. Cauchy. Da la pression ou tension dans un corps solide. *Exercises de Matehématiques*, 2(2):42–56, 1827.
11. Chen, G.-Q., Frid, H.: Divergence-measure fields and hyperbolic conservation laws. *Arch. Ration. Mech. Anal.* 147(2):89–118, 1999. https://doi.org/10.1007/s002050050146
12. Chen, G.-Q., Frid, H.: Extended divergence-measure fields and the Euler equations for gas dynamics. *Commun. Math. Phys.* 236(2):251–280, 2003. https://doi.org/10.1007/s00220-003-0823-7
13. Chen, G.-Q., Torres, M.: Divergence-measure fields, sets of finite perimeter, and conservation laws. *Arch. Rational Mech. Anal.* 175(2):245–267, 2005. https://doi.org/10.1007/s00205-004-0346-1
14. Chen, G.-Q., Torres, M., Ziemer, W. P.: Gauss-Green theorem for weakly differentiable vector fields, sets of finite perimeter, and balance laws. *Comm. Pure Appl. Math.* 62(2):242–304, 2009. https://doi.org/10.1002/cpa.20262
15. Comi, G. E., Magnani, V.: The Gauss–Green theorem in stratified groups. *Preprint arXiv:1806.04011*, 2018.
16. Comi, G. E., Payne, K. R.: On locally essentially bounded divergence measure fields and sets of locally finite perimeter. *Adv. Calc. Var.* 2018 (to

appear). https://doi.org/10.1515/acv-2017-0001
17. Comi, G. E., Torres, M.: One sided approximations of sets of finite perimeter. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* 28(1):181–190, 2017. https://doi.org/10.4171/RLM/757
18. Crasta, G., De Cicco, V.: Anzellotti's pairing theory and the Gauss–Green theorem. *Preprint arXiv:1708.00792*, 2017.
19. Crasta, G., De Cicco, V.: An extension of the pairing theory between divergence-measure fields and BV functions. *J. Funct. Anal.* 2018 (to appear). https://doi.org/10.1016/j.jfa.2018.06.007
20. Dafermos, C. M.: *Hyperbolic Conservation Laws in Continuum Physics*, 4th Ed., Springer-Verlag: Berlin, 2016. https://doi.org/10.1007/978-3-662-49451-6
21. Daners, D.: Domain perturbation for linear and semilinear boundary value problems. In: *Handbook of Differential Equations - Stationary Partial Differential Equations*, pp. 1–81. Elsevier, 2008. https://doi.org/10.1016/S1874-5733(08)80018-6
22. De Giorgi, E.: Su una teoria generale della misura $(r-1)$–dimensionale in uno spazio ad r dimensioni. *Ann. Mat. Pura Appl.* 36.1:191–213, 1954. https://doi.org/10.1007/BF02412838
23. De Giorgi, E.: Frontiere orientate di misura minima. *Seminario di Matematica della Scuola Normale Superiore di Pisa*, 1960–61. Editrice Tecnico Scientifica, Pisa, 1961.
24. De Lellis, C., Otto, F., Westdickenberg, M.: Structure of entropy solutions for multi-dimensional scalar conservation laws. *Arch. Ration. Mech. Anal.* 170(2):137–184, 2003. https://doi.org/10.1007/s00205-003-0270-9
25. Degiovanni, M., Marzocchi, A., Musesti, A.: Cauchy fluxes associated with tensor fields having divergence measure. *Arch. Ration. Mech. Anal.* 147(3):197–223, 1999. https://doi.org/10.1007/s002050050149
26. Doktor, P.: Approximation of domains with Lipschitzian boundary. *Časopis pro Pěstováni Matematiky*, 101(3):237–255, 1976. ISSN: 0528-2195
27. Evans, L. C., Gariepy, R. F.: *Measure Theory and Fine Properties of Functions.* CRC Press: Boca Raton, FL, 1992. ISBN: 9780849371578
28. Evans, L. C.: *Partial Differential Equations.* AMS: Providence, RI, 2010. https://doi.org/10.1090/gsm/019
29. Federer, H.: The Gauss-Green theorem. *Trans. Amer. Math. Soc.* 58:44–76, 1945. https://doi.org/10.2307/1990234
30. Federer, H.: A note on the Gauss-Green theorem. *Proc. Amer. Math. Soc.* 9:447–451, 1958. https://doi.org/10.2307/2033002
31. Frid, H.: Remarks on the theory of the divergence-measure fields. *Quart. Appl. Math.* 70(3):579–596, 2012. https://doi.org/10.1090/S0033-569X-2012-01311-5
32. Frid, H.: Divergence-measure fields on domain with Lipschitz boundary. In: *Hyperbolic Conservation Laws and Related Analysis with Applications*, pp. 207–225, G.-Q. Chen, H. Holden & K. Karlsen (Eds.), Springer: Heidelberg, 2014. https://doi.org/10.1007/978-3-642-39007-4_10

33. Fuglede, B.: On a theorem of F. Riesz. *Math. Scand.* 3: 283–302, 1955. https://doi.org/10.7146/math.scand.a-10448
34. Gilbarg, D., Trudinger, N. S.: *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, Berlin, 1983. https://doi.org/10.1007/978-3-642-61798-0
35. Giusti, E.: *Minimal Surfaces and Functions of Bounded Variation.* Birkhäuser-Verlag, Basel, 1984. https://doi.org/10.1007/978-1-4684-9486-0
36. Gurtin, M. E., Martins, L. C.: Cauchy's theorem in classical physics. *Arch. Ration. Mech. Anal.* 60(4):305–324, 1975/76. https://doi.org/10.1007/BF00248882
37. Hofmann, S., Mitrea, M., Taylor, M.: Geometric and transformational properties of Lipschitz domains, Semmes-Kenig-Toro domains, and other classes of finite perimeter domains. *J. Geometric Anal.* 17:593–647, 2007. https://doi.org/10.1007/BF02937431
38. Kawohl, B., Schuricht, F.: Dirichlet problems for the 1-Laplace operator, including the eigenvalue problem. *Commun. Contemp. Math.* 9(4):515–543, 2007. https://doi.org/10.1142/S0219199707002514
39. Lax, P. D.: *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, CBMS-RCSM, SIAM: Philiadelphia, 1973.
40. Leonardi, G. P., Saracco G.: The prescribed mean curvature equation in weakly regular domains. *NoDEA Nonlinear Diff. Eqs. Appl.* 25:9, 29pp., 2018. https://doi.org/10.1007/s00030-018-0500-3
41. Lieberman, G. M.: Regularized distance and its applications. *Pacific J. Math.* 117(2):329–352, 1985. ISSN: 0030-8730
42. Lieberman, G. M.: The conormal derivative problem for equations of variational type in nonsmooth domains. *Trans. Amer. Math. Soc.* 330(1): 41–67, 1992. https://doi.org/10.2307/2154153
43. Maggi, F.: *Sets of Finite Perimeter and Geometric Variational Problems.* Cambridge University Press: Cambridge, 2012. https://doi.org/10.1017/CBO9781139108133
44. Maz'ya, V. G.: *Sobolev Spaces with Applications to Elliptic Partial Differential Equations.* Springer-Verlag: Berlin-Heidelberg, 2011. https://doi.org/10.1007/978-3-642-15564-2
45. Nečas, J.: On domains of type $\mathfrak{N}$ (Russian). *Czech. Math. J.* 12(2):274–287, 1962. ISSN: 0011-4642
46. Nečas, J.: Sur les équations différentielles aux dérivées partielles du type elliptique du deuxième ordre. *Czech. Math. J.* 14(1):125–146, 1964. ISSN: 0011-4642
47. W. Noll. The foundations of classical mechanics in the light of recent advances in continuum mechanics. In: *The Axiomatic Method. With Special Reference to Geometry and Physics*, Eds. L. Henkin, P. Suppes, and A. Tarski, Studies in Logic and the Foundations of Mathematics, 266–281pp., North-Holland Publishing Co.: Amsterdam, 1959.
48. Phuc, N. C., Torres, M.: Characterizations of the existence and removable singularities of divergence-measure vector fields. *Indiana Univ. Math. J.*

57(4):1573–1597, 2008. https://doi.org/10.1512/iumj.2008.57.3312
49. Phuc, N. C., Torres, M.: Characterizations of signed measures in the dual of BV and related isometric isomorphisms. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5), 17(1): 385–417, 2017. ISSN: 0391-173X
50. Scheven, C., Schmidt, T.: BV supersolutions to equations of 1-Laplace and minimal surface type. *J. Diff. Eqs.* 261(3):1904–1932, 2016. https://doi.org/10.1016/j.jde.2016.04.015
51. Scheven, C., Schmidt, T.: An Anzellotti type pairing for divergence-measure fields and a notion of weakly super-1-harmonic functions. *Preprint arXiv:1701.02656*, 2017.
52. Scheven, C., Schmidt, T.: On the dual formulation of obstacle problems for the total variation and the area functional. *Ann. Inst. Henri Poincaré, Anal. Non Lineaire*, 2018 (to appear). https://doi.org/10.1016/j.anihpc.2017.10.003
53. Schuricht, F.: A new mathematical foundation for contact interactions in continuum physics. *Arch. Ration. Mech. Anal.* 184: 495–551, 2007. https://doi.org/10.1007/s00205-006-0032-6
54. M. Šilhavý. The existence of the flux vector and the divergence theorem for general Cauchy fluxes. *Arch. Ration. Mech. Anal.* 90(3):195–212, 1985. https://doi.org/10.1007/BF00251730
55. M. Šilhavý. Cauchy's stress theorem and tensor fields with divergences in L^p. *Arch. Ration. Mech. Anal.* 116(3):223–255, 1991. https://doi.org/10.1007/BF00375122
56. Šilhavý, M.: Divergence-measure fields and Cauchy's stress theorem. *Rend. Sem. Mat Padova*, 113:15–45, 2005. ISSN: 0041-8994
57. Šilhavý, M.: The divergence theorem for divergence measure vectorfields on sets with fractal boundaries. *Math. Mech. Solids*, 14:445–455, 2009. https://doi.org/10.1177/1081286507081960
58. Stein, E. M.: *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press: Princeton, 1970.
59. Verchota, G. C.: *Layer Potentials and Boundary Value Problems for the Laplace Equation in Lipschitz Domains*. PhD thesis, 1982.
60. Verchota, G. C.: Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. *J. Funct. Anal.* 59(3):572–611, 1984. https://doi.org/10.1016/0022-1236(84)90066-1
61. Verchota, G. C.: The biharmonic Neumann problem in Lipschitz domains. *Acta Math.* 194(2):217–279, 2005. https://doi.org/10.1007/BF02393222
62. Vol'pert, A. I.: Spaces BV and quasilinear equations (Russian). *Mat. Sb.* (*N.S.*), 73 (115):255–302, 1967.
63. Vol'pert, A. I., Hudjaev, S. I.: *Analysis in Classes of Discontinuous Functions and Equation of Mathematical Physics*. Martinus Nijhoff Publishers: Dordrecht, 1985. ISBN: 90-247-3109-7
64. von Koch, H.: Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire. *Archiv för Matemat., Astron. och Fys.* 1:681–702, 1904.

65. W. P. Ziemer. Cauchy flux and sets of finite perimeter. *Arch. Ration. Mech. Anal.* 84(3):189–201, 1983. https://doi.org/10.1007/BF00281518