

Constructing Subordinated Diffusions Calibrated To A Finite Call Price Surface

XIN ZHANG

Supervisor: Dr. Martin Klimmek

New College

University of Oxford



Msc in Mathematical and Computational Finance

Trinity 2013

Acknowledgements

First of all, I would like to express my heartfelt gratitude to my thesis supervisor, Dr. Martin Klimmek, for his encouragement, guidance, strong support and patience for my thesis. Every time I meet him, I learn a lot. I sincerely believe that this thesis would not have been possible without his inspiring ideas and pointing out the right direction when I encounter difficulties.

I would also like to thank Dr Lajos Gergely Gyurkó for the help and efforts he provided through out the course.

Contents

1	Introduction	1
2	Derivation of Carr-Cousot Dupire type formula without jump	3
2.1	Preliminaries	3
2.2	Derivation using Kolmogorov equations	5
2.3	Derivation without using Kolmogorov equations	6
2.4	Carr and Cousot's Dupire-type formula	7
2.5	Connections between Theorem 1 and the Kolmogorov forward equation	10
2.6	The classical Dupire's formula	11
2.7	Construction of time consistent martingale and Calibration	13
2.8	Lévy process	16
2.8.1	Definitions and properties	16
2.8.2	Time changed process	17
2.9	Essential conditions for our Dupire type Carr-Cousot formula	19
3	Simulation for stock paths	21
3.1	Algorithm to generate paths for Gamma process	21
3.2	Simulated paths for time changes	23
3.3	Algorithm for stock path simulation	24
3.4	Analysis of numerical results	28
4	Further generalization using speed measure	34
4.1	Irregular volatility function	34
4.2	Speed measure	35
4.3	Generalization	39
5	Conclusion	40
A	Main MATLAB Code	43

1 Introduction

The observation of an implied volatility smile shows that the Black-Scholes model does not work in general when calibrated to market price data. From a practitioner's point of view, a benchmark method is the local volatility model. Dupire (1994) [6] introduced it and had shown a tractable way to recover the implied volatility surface $\sigma(T, K)$ from call prices for all maturity T and strike K . Unfortunately, in practice, we can only observe a finite number of strikes and maturities and therefore they are discrete. But the local volatility is continuous for both variables. Therefore, this makes it difficult to calibrate with finite calls.

An alternative approach is given by Carr and Cousot, [1] presented an explicit construction of diffusions sampled at independent and exponentially distributed random times and deduced an expression for the implied volatility surface which can be interpreted as the discrete time counterparts of local volatility models. So we call it Carr-Cousot Dupire type formula in this paper (we will use "CCD" for short later). A key ingredient in the construction is we can recover the distribution of the stock price process at time T via call prices due to Breeden-Litzenberger [8]. The idea of Carr & Cousot is to model the stock process as a time changed driftless diffusion (so a local martingale) calibrated to calls with maturity T_1 , $S_t = X_{\gamma(t)}$ for $t < T_1$, where X_t is a diffusion and $\gamma(t)$ is a time change process with $\gamma(T_1) \sim \exp(1)$. In [1], they suggest two time change candidates, one is the gamma process, the other is a linear interpolation of the random time at maturity. Then the volatility between 0 and T_1 , $\sigma_{0,1}(K)$, can be recovered from calls mature at T_1 . If calibrating to calls mature at T_2 , then define a new time change running from T_1 to T_2 such that $\tilde{\gamma}(T_2) \sim \exp(1)$. Similarly the volatility between T_1 and T_2 , $\sigma_{1,2}(K)$, can be recovered when calibrating to call prices with these two maturities. The volatility before T_1 can be treated as taking the payoff as the other call option (i.e. the measure recovered from it is Dirac measure at S_0). Thus this construction can recover the volatility surface piecewise which only requires the strike to be continuous with one (or two) maturity. However, in practice, option prices are listed with multiple maturities and only a finite number of strikes in the market. Therefore, the objective of this paper is to investigate Carr and Cousot's approach, the theory as well as its practical implementation and consider generalizing it by weakening some of the assumption.¹

¹Carr and Cousot did a jump diffusion model, here we concentrate on time changed diffusion with no jump component to avoid the problem caused by the double tail associated to it. However in the non-jump case we generalise the treatment of Carr and Cousot by dropping some of the key assumptions.

In the second section, we start off by analysing the derivation in [1] with a general exponential random time $\tau \sim \exp(\lambda)$ instead. Then based on the remarks made by Carr and Cousot [1], using the theory behind it to provide an alternative derivation and discuss their connection. Furthermore, we extend it to the forward starting case with finite maturities (greater than two) and present the essential conditions that are needed when applying the formula. In addition, by analyzing one popular model in finance, the Variance Gamma model, our model can be fitted into the theory of Lévy processes.

In the third section, we simulate trajectories of the time change candidates suggested in [1] with an efficient algorithm to generate gamma process. Then we calibrate our forward starting model piecewise linearly to call prices (using Black-Scholes price) with different maturities and generate paths of subordinated diffusions with different time changes. Also, we investigate the Monte Carlo price for some simple exotic options through simulation (for single maturity and the forward starting case).

At the end, when we can only observe a finite number of call prices so that we do not have continuity in strike, Carr-Cousot's approach will not work. As linear interpolation of observed data gives the most expensive calls among all interpolation methods. We generalize their approach for this worst case. Throughout this paper, we assume the interest rate be zero, for the case with non-zero interest rate it can be generalised easily by modeling the discounted stock process as a time changed driftless diffusion.

2 Derivation of Carr-Cousot Dupire type formula without jump

2.1 Preliminaries

In this section, we provide the background materials (definitions and theorems) that will be needed to derive the Carr-Cousot Dupire type formula. The following overview is based on the assumptions also made by [1].

In Carr, Cousot [1], for positive time homogeneous Markov process (F_t) with generator $\mathcal{G}$. $\mathcal{G}$ is defined for $f \in C_0^\infty(\mathbb{R}_+)$, the space of infinitely differentiable functions with compact support:

$$\mathcal{G}f(x) = \frac{1}{2}x^2\sigma(x)^2f''(x)$$

Here the SDE for the underlying process is:

$$\frac{dF_t}{F_t} = \sigma(F_t)dW_t \tag{1}$$

Remarks 2.1: In Carr, Cousot [1], they start with the underlying dynamics satisfying equation (1), *i.e.* $dX_t = \sigma(X_t) \cdot X_t dW_t$. Mathematically, a more *canonical* way is to combine the volatility term and express the dynamic as

$$dX_t = \sigma(X_t)dW_t \tag{2}$$

However, in the next two sections, we use dynamic (1) temporally to investigate their approach for CCD formula. It is then easily to be adapted to a version of CCD formula (equation (4)) for dynamic (2). And we will use them for the rest of the thesis. (see Remarks 2.4)

Goal: Constructing the distribution of $X_2|X_1$

Use random time τ which is exponentially distributed as $\exp(\lambda)$ and independent of F_t via

$$[X_2|X_1 = x_1] \stackrel{d}{=} [F_\tau|F_0 = x_1]$$

Remarks 2.2: In Carr, Cousot [1], they use random $\tau \sim \exp(1)$, here we generalise it to an exponential random variable with parameter λ , $\tau \sim \exp(\lambda)$.

Definition 1: A one parameter family $\{T(t) : t \geq 0\}$ of bounded linear operators on a Banach space L is called a semigroup if

$$T(0) = 1, \text{ and } T(s+t) = T(s)T(t) \text{ for all } s, t \geq 0$$

Definition 2: A semigroup $\{T(t)\}$ on L is said to be strongly continuous if

$$\lim_{t \rightarrow 0} T(t)f = f, \forall f \in L.$$

Definition 3: A semigroup $\{T(t)\}$ on L is a contraction semigroup if $\|T(t)\| \leq 1$ for all $t \geq 0$.

Definition 4: The generator of a semigroup $\{T(t)\}$ on L is the linear operator $\mathcal{G}$ defined by

$$\mathcal{G}f = \lim_{t \rightarrow 0} \frac{1}{t} [T(t)f - f]$$

For $\lambda \in \mathbb{R}$, the resolvent of generator $\mathcal{G}$ is defined by $R_\lambda = (\lambda - \mathcal{G})^{-1}$

Theorem 1 (see Ethier, Kurtz [7]): Let $\{T(t)\}$ be a strongly continuous contraction semigroup on L with generator $\mathcal{G}$. Then

$$R_\lambda = (\lambda - \mathcal{G})^{-1}g = \int_0^\infty e^{-\lambda t} T(t)g dt$$

Theorem 2 (Strassen's theorem 1965):

If two probability distributions μ_1 and μ_2 on $\mathbb{R}^+ \equiv \{x \in \mathbb{R} | x > 0\}$ have the same finite first moment, X_0 , and satisfy:

$$\int_0^\infty f(x) \mu_1(dx) \leq \int_0^\infty f(x) \mu_2(dx),$$

for every convex function f on $\mathbb{R}^+$ (where the integrals are possibly infinite) which is known as the “convex order assumption”(see section 2.9), then there exist two random variables, X_1 and X_2 , defined on the same probability space, such that the probability distribution of X_i is μ_i for $i = 1, 2$, and the process (X_i) is a martingale.

2.2 Derivation using Kolmogorov equations

In [1], Carr and Cousot use a heuristic non-rigorous argument based on the forward and backward Kolmogorov equations to derive a key step of CCD formula (for a jump diffusion consistent with call options):

$$\begin{aligned} \text{i.e. } \frac{\partial}{\partial t}(\mathbb{E}[P(F_t)|F_0]) &= \mathbb{E}[\mathcal{G}P(F_t)|F_0] \\ \implies \mathbb{E}[(I - \mathcal{G})P](F_\tau) | F_0 &= P(F_0), \tau \sim \exp(1) \end{aligned}$$

Then they make a remark saying the above equations are based on the fact that the resolvent of the generator $\mathcal{G}$ is linked to the Laplace transform of the associated transition semigroup.

We will now investigate their approach in detail but with general exponential time $\tau \sim \exp(\lambda)$ here. Also we want to explore the remark made by Carr, Cousot about the resolvent, and then provide an alternative approach in next section.

For payoff $P \in C_0^\infty(\mathbb{R}_+)$, by Kolmogorov forward equation

$$\frac{\partial}{\partial t}(\mathbb{E}[P(F_t)|F_0]) = \mathbb{E}[\mathcal{G}P(F_t)|F_0].$$

Multiply both sides by $\lambda e^{-\lambda t}$ and integrating over $\mathbb{R}_+$ and using integration by parts on the LHS

$$\begin{aligned} \int_0^\infty \lambda e^{-\lambda t} \cdot \frac{\partial}{\partial t}(\mathbb{E}[P(F_t)|F_0]) dt &= \lambda e^{-\lambda t} \mathbb{E}[P(F_t)|F_0] \Big|_0^\infty + \int_0^\infty \lambda^2 e^{-\lambda t} \mathbb{E}[P(F_t)|F_0] dt \\ &= -\lambda P(F_0) + \int_0^\infty \lambda^2 e^{-\lambda t} \mathbb{E}[P(F_t)|F_0] dt \\ &= -\lambda P(F_0) + \mathbb{E} \left[\int_0^\infty \lambda^2 e^{-\lambda t} P(F_t) dt \Big| F_0 \right], \end{aligned}$$

and on the other hand, we have

$$\begin{aligned}
RHS &= \int_0^\infty \lambda e^{-\lambda t} \mathbb{E}[\mathcal{G}P(F_t)|F_0] dt \\
&= \mathbb{E} \left[\int_0^\infty \lambda e^{-\lambda t} \mathcal{G}P(F_t) dt | F_0 \right].
\end{aligned}$$

Putting these together we get

$$\begin{aligned}
\lambda P(F_0) &= \mathbb{E} \left[\int_0^\infty \lambda e^{-\lambda t} ([\lambda - \mathcal{G}]P)(F_t) dt | F_0 \right] \\
&= \mathbb{E} [\mathbb{E}[(\lambda - \mathcal{G})P](F_\tau) | F_0] \\
&= \mathbb{E}[(\lambda - \mathcal{G})P](F_\tau) | F_0
\end{aligned} \tag{3}$$

where we used the assumption $\tau \sim \exp(\lambda)$ ² and the tower property.

2.3 Derivation without using Kolmogorov equations

Here we want to understand the remark made by Carr, Cousot [1] about the role of resolvent of the generator in their derivation. Based on the theory behind it, we provide an alternative derivation in the following.

Now we are using Theorem 1 to derive equation (3). By Theorem 1,

$$R_\lambda P = \int_0^\infty e^{-\lambda t} T(t) P dt,$$

by definition $R_\lambda(\lambda - \mathcal{G})P = P$. Thus,

$$\begin{aligned}
P(x) &= (\lambda - \mathcal{G}) \mathbb{E} \left[\int_0^\infty e^{-\lambda t} T(t) P dt | F_0 = x \right] \\
&= (\lambda - \mathcal{G}) \mathbb{E} \left[\int_0^\infty e^{-\lambda t} P(F_t) dt | F_0 = x \right]
\end{aligned}$$

Multiply both sides by λ , we get

²Exponential random variable τ has density $f(t) = \lambda e^{-\lambda t}$.

$$\begin{aligned}
\lambda P(x) &= (\lambda - \mathcal{G}) \mathbb{E} \left[\int_0^\infty \lambda e^{-\lambda t} P(F_t) dt | F_0 = x \right] \\
&= (\lambda - \mathcal{G}) \mathbb{E} [\mathbb{E}[P(F_\tau)] | F_0 = x] \\
&= (\lambda - \mathcal{G}) \mathbb{E} [P(F_\tau) | F_0 = x]
\end{aligned}$$

Then taking expectation over all possible values of F_0 we get equation (3).

2.4 Carr and Cousot's Dupire-type formula

Having equation (2), despite the conditions for the domain of the generator $\mathcal{G}$ for now (solution to this problem is discussed in Remarks 2.6), taking call payoff function $P(x) = (x - K)^+$ and assuming $\mathcal{G}$ is such that calls C_i are of class $C^2(\mathbb{R})$, then we get

$$\mathbb{E}[(\lambda - \mathcal{G})(x - K)^+(X_2) | X_1] = \lambda(X_1 - K)^+ \text{ with fixed marginals } X_1 \text{ and } X_2.$$

Since $P''(x) = \delta(x - K)$, plugging in the expression of our generator $\mathcal{G}$:

$$\begin{aligned}
\lambda(X_1 - K)^+ &= \mathbb{E}[(\lambda - \mathcal{G})(x - K)^+(X_2) | X_1] \\
&= \mathbb{E}[\lambda(X_2 - K)^+ - \frac{1}{2} X_2^2 \sigma^2(X_2) \delta(X_2 - K)] \\
&= \lambda C_2(K) - \frac{1}{2} K^2 \sigma^2(K) \mathbb{P}[X_2 = K] \\
&= \lambda C_2(K) - \frac{1}{2} K^2 \sigma^2(K) \frac{\partial^2 C_2}{\partial K^2}(K)
\end{aligned}$$

Now take expectation both sides again,

$$\lambda C_1(K) = \lambda C_2(K) - \frac{1}{2} K^2 \sigma^2(K) \frac{\partial^2 C_2}{\partial K^2}(K)$$

$$\implies \sigma^2(K) = \frac{\lambda C_2(K) - \lambda C_1(K)}{\frac{1}{2} K^2 \frac{\partial^2 C_2}{\partial K^2}(K)} \quad (4)$$

Remarks 2.3: For maturity T_i , $i = 1, 2, \dots$, we can recover measure μ_i from $C_i(K)$.

$$\begin{aligned} C_i(K) &= \int_0^\infty (x - K)^+ \mu_i(dx) \\ &= \int_K^\infty (x - K) \mu_i(dx) \end{aligned}$$

Differentiate above with respect to K , $\frac{\partial C_i}{\partial K}(K) = \mu_i(K) - 1$. Then differentiate w.r.t. K again, we recover the density of the general measure μ_i by

$$\mu_i(dK) = \frac{\partial^2 C_i}{\partial K^2}(K) dK$$

where μ_i is the probability distribution of S_{T_i} and $\mu_i(dK)$ is the probability density of S_{T_i} . This fact is well known and due to Breeden and Litzenberger [8]. Thus (2) can be represented as

$$\begin{aligned} \frac{dK}{\sigma^2(K)} &= \frac{\frac{1}{2}K^2 \frac{\partial^2 C_2}{\partial K^2}(K) dK}{\lambda(C_2(K) - C_1(K))} \\ &= \frac{\frac{1}{2}K^2 \mu_2(dK)}{\lambda(C_2(K) - C_1(K))} \\ \Rightarrow \sigma^2(K) \cdot K^2 &= \frac{\lambda(C_2(K) - C_1(K)) dK}{\frac{1}{2}\mu_2(dK)} \end{aligned}$$

Remarks 2.4: In Carr, Cousot [1], they start with the underlying dynamic satisfies equation (1), *i.e.* $dX_t = \sigma(X_t) \cdot X_t dW_t$. As mentioned in Remarks 2.1, a more *canonical* way is to use dynamic (2) for the underlying process

$$dX_t = \sigma(X_t) dW_t \text{ with generator } \mathcal{G}f(x) = \frac{1}{2}\sigma(x)^2 f''(x)$$

where our Carr-Cousot Dupire type formula becomes

$$\sigma^2(K) = \frac{\lambda C_2(K) - \lambda C_1(K)}{\frac{1}{2} \frac{\partial^2 C_2}{\partial K^2}(K)},$$

$$\Rightarrow \frac{dK}{\sigma^2(K)} = \frac{\frac{1}{2}\mu_2(dK)}{\lambda(C_2(K) - C_1(K))} \quad (5)$$

(Also details see section 5.4 and Appendix A of [1] for our general driftless diffusion)

For the rest of the thesis, we will use (2), (5) as the expressions for the underlying dynamic and Carr-Cousot Dupire type formula.

Remarks 2.5: Simplest case for $\mu_1 = \delta_{S_0}$, the point mass measure, our Dupire formula is

$$\begin{aligned} \sigma^2(K) &= \frac{\lambda C_2(K) - \lambda(S_0 - K)^+}{\frac{1}{2} \frac{\partial^2 C_2}{\partial K^2}(K)} \\ &= \frac{\lambda [C_2(K) - (S_0 - K)^+] dK}{\frac{1}{2} \mu_2(dK)} \end{aligned}$$

Then assuming that μ_2 does not have an atom at S_0 , by the Put-Call parity, we have

$$\sigma^2(K) = \begin{cases} \frac{2\lambda \cdot P_2(K) dK}{\mu_2(dK)}, & \text{for } K \leq S_0 \\ \frac{2\lambda \cdot C_2(K) dK}{\mu_2(dK)}, & \text{for } K \geq S_0. \end{cases}$$

Remarks 2.6: The call payoff function $P(x) = (x - K)^+$ is not in the domain of $\mathcal{G}$, as $(x - K)^+$ only continuous but not differentiable and it is unbounded, we need our candidate payoffs be at least C^2 functions or C_b bounded functions (i.e. Oksendal [14] Theorem 8.1.5). Possible solutions:

1. We can always take a sequence of bounded functions increasing to $P(x)$, e.g. a sequence of continuous functions f_n with increasing compact domain $[0, a_n]$ where $a_i > a_j, \forall i > j$, thus f_n are bounded functions increase to $P(x)$, and Weierstrass Approximation Theorem can also provide smoothed sequence of approximations (by polynomials) if we need smoothness. Then we can apply Monotone Convergence Theorem.
2. Using smoothed Heaviside function to approximate $\mathbb{I}_{\{x \geq K\}}$ first,

$$H_\varepsilon(S - K) = \Phi\left(\frac{S - K}{\varepsilon}\right),$$

where $\Phi(x)$ is the c.d.f. of normal r.v., $\varepsilon \ll K$

Then integrating it to approximate $P(x)$

$$P_\varepsilon(x) = (S - K)\Phi\left(\frac{S - K}{\varepsilon}\right) + \frac{\varepsilon}{\sqrt{2\pi}} \exp\left(-\frac{(S - K)^2}{2\varepsilon^2}\right)$$

Taking $\varepsilon_n \downarrow 0$ so that $P_{\varepsilon_n}(x) \rightarrow P(x)$, then apply Dominated Convergence Theorem.

2.5 Connections between Theorem 1 and the Kolmogorov forward equation

As we have discussed in previous sections, the CCD formula can be derived through either the KFE approach or the resolvent method. Furthermore, the remark made by Carr, Cousot [1] says their derivation is based on the fact about resolvent of the generator. It would be interesting to analyze how these two approaches connect with each other. Thus we are going to investigate their relation and the preference here.

For the KFE, if we introduce the operator $\mathcal{A}_t : P \rightarrow \mathbb{E}[P(F_t)|F_0]$, then the equation becomes

$$\begin{aligned} \frac{\partial}{\partial t}(\mathcal{A}_t P) &= \mathcal{A}_t(\mathcal{G}P), \\ \mathcal{A}_0 P &= P(F_0) \end{aligned}$$

Similarly, for the KBE we have

$$\frac{\partial}{\partial t}(\mathcal{A}_t P) = \mathcal{G}(\mathcal{A}_t P)$$

Thus the above system of equations say that the operators $\mathcal{A}_t$ and $\mathcal{G}$ commute, and solution of the above equations is $\mathcal{A}_t = e^{t\mathcal{G}}$. (see Morris, Stephen & Robert [19] for details) Obviously, $\mathcal{A}_0 = e^{0 \times \mathcal{G}} = 1$ and

$$\mathcal{A}_{t+s} = e^{(t+s)\mathcal{G}} = e^{t\mathcal{G}} e^{s\mathcal{G}} = \mathcal{A}_t \mathcal{A}_s, \forall s, t \geq 0$$

where $e^{t\mathcal{G}} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \mathcal{G}^k, t \geq 0$

Thus, $\mathcal{A}_t$ is a semigroup and it can easily be seen to be strong continuous etc. The Kolmogorov forward/backward equations are the special cases (results) of our resolvent method (i.e. KFE/KBE satisfy the conditions of Theorem 1). Therefore, I prefer the derivation using Theorem 1 as it is more generalized.

2.6 The classical Dupire's formula

To see why Carr and Cousot's Dupire-type formula is useful, we need to compare and contrast it with the classical Dupire's formula. Thus we recall and present the derivation and conditions of the classical Dupire's formula in this section.

Define the transition probability density be $p(t, x; T, y) = \frac{\partial}{\partial y} \mathbb{P}[S_T \leq y | S_t = x]$, by Kolmogorov forward equation it satisfies

$$\frac{\partial p(t, x; T, y)}{\partial t} = -\frac{\partial}{\partial y}[ryp(t, x; T, y)] + \frac{1}{2} \frac{\partial^2}{\partial y^2}[\sigma(T, y)^2 y^2 p(t, x; T, y)]$$

Thus for call option, we have $C(0, x; T, K) = e^{-rT} \int_K^\infty (y - K)p(t, x; T, y)dy$. Differentiating with respect to T , we get

$$\begin{aligned} \frac{\partial C}{\partial T} &= -rC + e^{-rT} \int_K^\infty (y - K) \frac{\partial p(t, x; T, y)}{\partial T} dy \\ &\stackrel{KFE}{=} -rC + e^{-rT} \int_K^\infty (y - K) \left[-\frac{\partial}{\partial y}[ryp(t, x; T, y)] + \frac{1}{2} \frac{\partial^2}{\partial y^2}[\sigma(T, y)^2 y^2 p(t, x; T, y)] \right] dy \end{aligned}$$

Assume (1) $\lim_{y \rightarrow \infty} (y - K)ryp(t, x; T, y) = 0$, we have

$$\begin{aligned} & - \int_K^\infty (y - K) \frac{\partial}{\partial y}[ryp(t, x; T, y)] dy \\ &= - \int_K^\infty (y - K) d(ryp(t, x; T, y)) \\ &\stackrel{by\ parts}{=} \int_K^\infty ryp(t, x; T, y) dy - \lim_{y \rightarrow \infty} (y - K)ryp(t, x; T, y) \\ &= \int_K^\infty ryp(t, x; T, y) dy \end{aligned}$$

Furthermore, assume (2) $\lim_{y \rightarrow \infty} \frac{\partial}{\partial y}[\sigma^2(T, y)y^2 p(t, x; T, y)](y - K) = 0$ and
 (3) $\lim_{y \rightarrow \infty} \sigma^2(T, y)y^2 p(t, x; T, y) = 0$, we get

$$\begin{aligned}
& \int_K^\infty (y-K) \frac{\partial^2}{\partial y^2} [\sigma(T, y)^2 y^2 p(t, x; T, y)] dy \\
&= \int_K^\infty (y-K) d\left(\frac{\partial}{\partial y} [\sigma(T, y)^2 y^2 p(t, x; T, y)]\right) \\
&\stackrel{\text{by parts}}{=} \lim_{y \rightarrow \infty} \frac{\partial}{\partial y} [\sigma^2(T, y) y^2 p(t, x; T, y)] (y-K) - \int_K^\infty d(\sigma(T, y)^2 y^2 p(t, x; T, y)) \\
&= \lim_{y \rightarrow \infty} \frac{\partial}{\partial y} [\sigma^2(T, y) \cdot y^2 p(t, x; T, y)] (y-K) - \lim_{y \rightarrow \infty} \sigma^2(T, y) \cdot y^2 p(t, x; T, y) \\
&+ \sigma^2(T, K) \cdot K^2 \cdot p(t, x; T, y) \\
&= \sigma^2(T, K) \cdot K^2 \cdot p(t, x; T, y)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial C}{\partial T} &= -rC + e^{-rT} \int_K^\infty r y p(t, x; T, y) dy + \frac{1}{2} e^{-rT} \sigma^2(T, K) \cdot K^2 \cdot p(t, x; T, y) \\
&= -r e^{-rT} \int_K^\infty (y-K) p(t, x; T, y) dy + e^{-rT} \int_K^\infty r y p(t, x; T, y) dy \\
&\quad + \frac{1}{2} e^{-rT} \sigma^2(T, K) \cdot K^2 \cdot p(t, x; T, y) \\
&= e^{-rT} \int_K^\infty r K p(t, x; T, y) dy + \frac{1}{2} e^{-rT} \sigma^2(T, K) \cdot K^2 \cdot p(t, x; T, y)
\end{aligned}$$

Since

$$\frac{\partial C}{\partial K} = -e^{-rT} \int_K^\infty p(t, x; T, y) dy,$$

$$\frac{\partial^2 C}{\partial K^2} = e^{-rT} p(t, x; T, y)$$

we arrive the Dupire's equation:

$$\frac{\partial C}{\partial T} = -rK \frac{\partial C}{\partial K} + \frac{1}{2} \sigma^2(T, K) \cdot K^2 \frac{\partial^2 C}{\partial K^2}$$

$$\text{i.e. } \frac{\sigma^2(T, K)}{2} = \frac{\frac{\partial C_0}{\partial T} + rK \frac{\partial C_0}{\partial K}}{K^2 \frac{\partial^2 C_0}{\partial K^2}}$$

Remarks 2.7: From the above derivation, assumptions (1), (2), (3) are essential. In Carr and Cousot [1], there is a similar one in the sufficient conditions for the existence of centered transition densities of the generalized Laplace distributions (see Lemma 21 of [1]) that ensures $C(K) \rightarrow 0$, as $K \rightarrow \infty$.

Remarks 2.8: In the classical Dupire's formula, if we want to fully recover the volatility surface, we need call option prices for all maturities T and strikes K (as $\sigma(T, K)$ is a function of T). It requires a bi-continuum of maturities and strikes. In contrast to this, Carr and Cousot's Dupire-type formula can easily recover the volatility surface between two fixed maturity times. Thus if we have a finite number of call option prices with fixed maturities, we can fully recover the volatility surface piece-wise linearly using Carr and Cousot's formula. The construction will be discussed in the next section.

2.7 Construction of time consistent martingale and Calibration

If we have a finite number of call option prices with fixed maturities $T_1, T_2, \dots, T_n$, we can recover the marginal distributions $\mu_1, \mu_2, \dots, \mu_n$ from the call prices. Then we can construct a continuous time martingale X_t with marginal densities f_i at time $T_i, \forall i = 1, \dots, n$:

$$X_t \stackrel{d}{=} (F_{\gamma^i(t-T_{i-1})} | F_0^i = X_{T_{i-1}}), \text{ for } t \in (T_{i-1}, T_i]$$

where $(\gamma^i(t))$ is a subordinator (a Levy process whose trajectories are almost surely increasing) starting at 0, independent of (F_t^i) , and with marginal distribution at time $T_i - T_{i-1}$ be an exponential distribution with parameter λ . In addition, $(\gamma^i(t))$ is independent of all the other time changes $\gamma^j(t) \forall j \neq i$.

Here we assumed the conditions for Strassen Theorem in Carr & Cousot [1]:

1. $\mu_0 \equiv \delta_{x_0}$ with $x_0 \in \mathbb{R}^+$
2. μ_i admits a positive and continuous density f_i and has *finite first moment* x_0 .
3. $(x_0 - x)^+ < \int_0^\infty (y - x)^+ f_i(y) dy < \int_0^\infty (y - x)^+ f_{i+1}(y) dy, \forall x > 0, i = 1, 2, \dots, n$.

Consequently, X_t is calibrated to the marginal distributions μ_i which are recovered from call prices $C_i(K)$. Finally, the volatility surface is calibrated piece-wise linearly by:

$$\begin{aligned}\frac{dK}{\sigma_{i-1,i}^2(K)} &= \frac{\frac{1}{2}\mu_i(dK)}{\lambda(C_i(K) - C_{i-1}(K))} \\ &= \frac{\frac{1}{2}f_i(K)dK}{\lambda(C_i(K) - C_{i-1}(K))}\end{aligned}$$

where f_i are the densities of μ_i .

Following are plots for $\sigma_{0,1}(K)$ and $\sigma_{1,2}(K)$ derived from Black-Scholes models with different Black-Scholes volatilities.

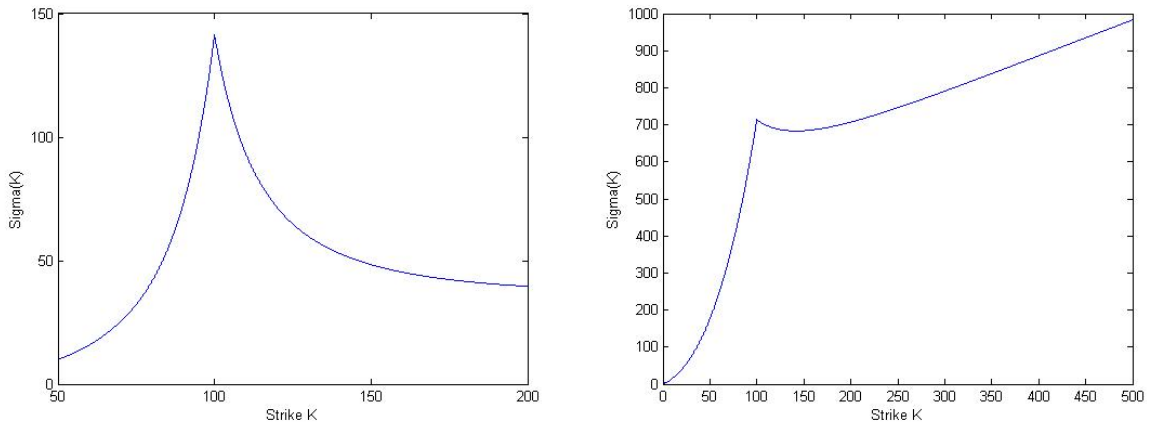
Note the density under this model is log-normal:

$$f_i(K) = \frac{1}{\sigma K \sqrt{2\pi T_i}} e^{-\frac{(\log(K) - \log(S_0) + \frac{1}{2}\sigma^2 T_i))^2}{2\sigma^2 T_i}}$$

Therefore, we could plot the graph of volatility function with different maturities and volatility for Black-Scholes model.

Remarks 2.9: From these figures, we can easily see that the Black-Scholes volatility controls the value of our CCD function. The range of CCD $\sigma(K)$ becomes larger with larger Black-Scholes volatility σ we chosed. We will use $\sigma = 0.1$ for our later simulations.

Figure 1: $\sigma_{0,1}(K)$ derived from Black-Scholes call prices with $r = 0$, $T_1 = 0$, $T_2 = 1$, $S_0 = 100$, $\sigma = 0.1$ (upper left), $\sigma = 0.5$ (upper right) and $\sigma = 0.05$ (bottom).



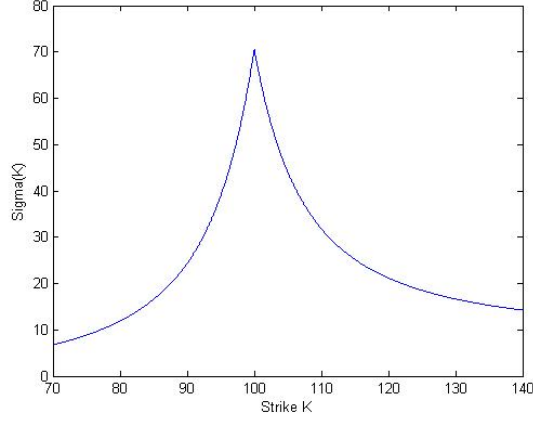
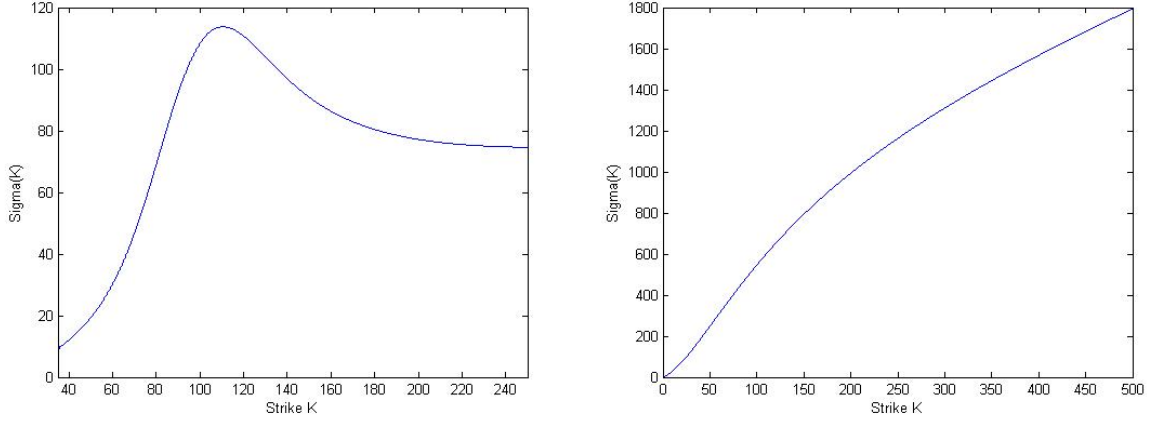


Figure 2: $\sigma_{1,2}(K)$ derived from Black-Scholes call prices with $r = 0$, $T_1 = 1$, $T_2 = 2$, $S_0 = 100$, $\sigma = 0.1$ (left) and $\sigma = 0.5$ (right).



Remarks 2.10: We can expect the behaviour seen in Figure 1 and 2 tends to infinity (when using Black-Scholes calls) as $K \rightarrow \infty$ due to the asymptotic analysis shown in [20]³. Also, we can expect another key feature from these figures, Figure 2 shows that the CCD volatility function is more smooth when calibrated to two maturities. The reason is that we are taking two smooth call function in CCD in the forward starting case (from equation (4)). But for the case calibrated only to one maturity, we take the payoff of call as the first call price (see Remarks 2.5) and it has non-differentiability at point S_0 , thus the CCD function in this case has a kink at $S_0 = 100$ as shown in Figure 1.

³In [20], it showed $\sigma(K) \sim C \frac{K^2}{\log^2(K)} \rightarrow \infty$ if $K \rightarrow \infty$, as K grows faster than $\log(K)$.

2.8 Lévy process

In last section, our approach is to construct time changed diffusions consistent with market call options. Here we discuss how our model fit into the general theory of Lévy processes and why Lévy process is important for this thesis.

2.8.1 Definitions and properties

Definition 5 : A $\mathbb{R}^d$ valued Càdlàg⁴ stochastic process $(X_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with $X_0 = 0$ is called a Lévy process if it satisfies the following properties:

1. *Independent increments:* for any sequence of time $0 \leq t_0 < t_1 < t_2 < \dots < t_n < \infty$, $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.
2. *Stationary increments:* $\forall s, t \geq 0$, the distribution of $X_{t+s} - X_t$ is independent of t , $X_{t+s} - X_t \stackrel{d}{=} X_s$.
3. *Stochastic continuity:* $\forall \varepsilon > 0, \lim_{h \rightarrow 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0$.

It can be checked easily that the simplest examples of Lévy process are Brownian motion and Compound Poisson process. Thus Lévy processes can jump. The following definition is an important characterization of Lévy process which measures the jump.

Definition 6: Let $(X_t)_{t \geq 0}$ be a Lévy process on $\mathbb{R}^d$. The measure ν on $\mathbb{R}^d$ defined by:

$$\nu(A) = \mathbb{E}[\#\{t \in [0, 1] : \Delta X_t \neq 0, \Delta X_t \in A\}], \quad A \in \mathcal{B}(\mathbb{R}^d)$$

is called the Lévy measure of X . (i.e. $\nu(A)$ is the expected number, per unit time, of jumps whose size belongs to A)

Another characterization of Lévy process X_t is its characteristic function which is determined by the characteristic triplet (Σ, ν, γ) ⁵ via Lévy-Khinchin representation (see Cont, Tankov [5]):

$$\mathbb{E}[e^{iz \cdot X_t}] = e^{t\psi(z)}, \quad z \in \mathbb{R}^d$$

$$\text{where } \psi(z) = i\gamma \cdot z - \frac{1}{2}z^\top \Sigma z + \int_{\mathbb{R}^d} (e^{iz \cdot x} - 1 - iz \cdot x \mathbb{I}_{\{|x| \leq 1\}}) \nu(dx)$$

⁴A stochastic process is called Càdlàg if $\lim_{s \downarrow t} X_s(\omega) = X_t(\omega)$ and $\lim_{s \uparrow t} X_s(\omega)$ exists for a.e. ω (i.e. it is right continuous with left limit, see Karatzas, Shreve [11]). Although this condition may not always be included, one can construct a version of it that satisfies Càdlàg.

⁵The characteristic triplet/Lévy triplet are given by Lévy-Itô decomposition, details see Cont, Tankov [5].

is called the characteristic exponent of X_t .

Example 2.1: Here we provide two important examples for Lévy process:

- Gamma process: This process is very important for our thesis as it is the candidate for the time change process.

Gamma process is a stochastic process with independent gamma distributed increments, $X_{t+s} - X_t \sim \text{Gamma}(t; c, \lambda)$. It is an increasing pure jump Lévy process with Lévy measure

$$\nu(x) = \frac{c \cdot e^{-\lambda x}}{x} \mathbb{I}_{\{x>0\}}$$

and characteristic function

$$\mathbb{E} [e^{iz \cdot X_t}] = \left(1 - \frac{iz}{\lambda}\right)^{-ct}$$

The mean and variance of a gamma process at time t are:

$$\mu = \frac{ct}{\lambda}, \sigma^2 = \frac{ct}{\lambda^2}.$$

- Cauchy process: It is a pure jump Lévy process with infinite variation which has the symmetric Cauchy distribution⁶ with Lévy measure

$$\nu(dx) = \frac{dx}{\pi x^2}$$

and characteristic function

$$\mathbb{E} [e^{iz \cdot X_t}] = e^{-t|z|}.$$

2.8.2 Time changed process

The subordinator is a non-decreasing real valued Lévy process starting from zero, thus it can be used as time changes for other processes. In financial modeling, a natural and simple candidate is Brownian subordination (a time changed Brownian motion).

Remarks 2.11: Note that Brownian motion itself is a Brownian subordination. By Dubins-Schwarz Theorem⁷, $W_t = B_{\langle W \rangle_t} = B_t$, thus it is a Brownian motion with quadratic variation

⁶Cauchy distribution has density function $f(t, x) = \frac{t}{\pi(x^2 + t^2)}$.

⁷Dubins-Schwarz Theorem (see Karatzas, Shreve [11]): Let $M \in \mathcal{M}^{c,loc}$ satisfy $\lim_{t \rightarrow \infty} \langle M \rangle_t = \infty$ a.s. $\mathbb{P}$. Define, for each $0 \leq s < \infty$, the stopping time $\tau(s) = \inf \{t \geq 0; \langle M \rangle_t > s\}$. Then the time changed process $B_s = M_{\tau(s)}$, $\mathcal{G}_s = \mathcal{F}_{\tau(s)}$ is a standard Brownian motion and $M_t = B_{\langle M \rangle_t}$; $0 \leq t < \infty$.

time changes. Also our time changed diffusion using speed measure in section 4.2 is a special case of a time changed Brownian motion.

For the construction of the time consistent processes we discussed in the last section, one good example for it (Lévy process) is the variance gamma process, a Brownian subordination time changed by a gamma process. This Variance Gamma model is first proposed by Madan and Senata (1990) [18] for stock price model. And it is very popular in finance (see Carr, Madan et al [4] for details).

Economic interpretation: The economic interpretation of it is that our model is under the business clock which includes news and shocks rather than the calendar time. Each unit of calendar time has an economically relevant time length. For instance, although we have fixed length calendar year, its economic values are different which are depending on the state of current whole (marco) economy or economic situation. Thus you could have an economic year greater or less than the real calendar year. In our model with gamma time change, the units of business time in calender time with fixed length is modeled by a random variable with gamma distribution (so the business time is random).

Note: The subordinated diffusion that are treated in this thesis can be seen as a generalisation of the Variance Gamma model.

Let $W_t = B_t + \mu t$ be a standard Brownian motion with drift μt and Γ_t be an independent gamma process. Choosing $c = \lambda^8$, so our gamma process has mean $\mu_\Gamma = 1$ and variance $\sigma^2 = \frac{1}{c} = \frac{1}{\lambda}$ per unit time. As we know, Γ_t is a pure jump process, thus $X_t = W_{\Gamma_t}$ is also a jump process and $X_t = \sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})$, where t_i is a partition of time. Since we have discussed the properties of gamma process in last subsection, the characteristic function of variance gamma process can be done by a direct calculation:

$$\begin{aligned} \mathbb{E} [e^{iz \cdot X_t}] &= \mathbb{E} \left[\mathbb{E} \left[e^{iz \cdot (B_{\Gamma_t} + \mu \Gamma_t)} \mid \Gamma_t \right] \right] = \mathbb{E} \left[e^{-\frac{1}{2} z^2 \cdot \Gamma_t + i \mu \Gamma_t} \right] \\ &= \mathbb{E} \left[e^{i \cdot (\mu z + \frac{1}{2} i \cdot z^2) \Gamma_t} \right] \\ &= \left(1 - \left(\frac{i \mu z - \frac{1}{2} z^2}{\lambda} \right) \right)^{-\lambda t} \end{aligned}$$

And the Lévy measure is

⁸This choice of parameters is very important, it is discussed in the Remarks 3.1.

$$\nu(x) = \frac{\lambda}{|x|} (\mathbb{I}_{\{x>0\}} \cdot e^{-\lambda_+ x} + \mathbb{I}_{\{x<0\}} \cdot e^{-\lambda_- x}),$$

where $\lambda_+ = \sqrt{\mu^2 + 2\lambda} - \mu$ and $\lambda_- = \sqrt{\mu^2 + 2\lambda} + \mu$. (A general form of it with different parameters λ and c see section 4.5 of Cont, Tankov [5].)

Remarks 2.12: Above example of variance gamma process is just a time changed diffusion for classical financial models which use Brownian motion as the driving noise. In fact, a more general financial model using Lévy process is useful and becoming more and more popular. Since for a Lévy process Y_t , if it is time changed by another independent subordinator γ_t , then the resulting process $X_t = Y_{\gamma_t}$ is still a Lévy process. Therefore, our construction of time consistent subordinated process can also be applied to general financial models which use Lévy processes as the random components. The theoretic foundation of time changed Lévy process is presented in the following theorem.

Theorem 3: (Subordination of Lévy process, Cont, Tankov [5])

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(Y_t)_{t \geq 0}$ be a Lévy process on $\mathbb{R}^d$ with characteristic exponent $\psi(t)$ and triplet (Σ, ν, γ) . Let $(\Gamma_t)_{t \geq 0}$ be a subordinator with Laplace exponent $\ell(t)$ ⁹ and triplet $(0, \rho, b)$ ¹⁰. Then the process $(X_t)_{t \geq 0}$ defined for each $\omega \in \Omega$ by $X_t(\omega) = Y_{\Gamma_t}(\omega)$ is a Lévy process. And its characteristic function is

$$\mathbb{E} [e^{iz \cdot X_t}] = e^{t \cdot \ell(\psi(z))}.$$

2.9 Essential conditions for our Dupire type Carr-Cousot formula

In equation (2), our Carr-Cousot Dupire-type formula for the diffusion coefficient $\sigma^2(K) \geq 0$, hence we are implicitly assuming that $C_i(K) \geq C_{i-1}(K), \forall K \geq 0$, This “Convex order” condition is essential for our formula. As we can see, it is the assumption (3) we used in section 1.7 (Note: the version used in Strassen Theorem is the case for $n = 2$). Also, see Hobson [2] and Oblój [21] for an overview of the Skorokhod embedding problem and the importance of the “Convex order condition” (details see Cox, Klimmek [9] Minimal embeddings to minimal diffusions). The convex order condition for Skorokhod embedding problem is provided by Rost Theorem.

Remarks 2.13: The Skorokhod embedding problem for (X, μ) is , given a distribution μ on state space I , to find a stopping time τ such that the law of the stopped process is μ . In Hobson [2], for $W_0 \sim \mu_0$, given centred probability measures $(\mu_i)_{0 \leq i \leq n}$, where μ_i correspond to given

⁹The Laplace exponent of X_t is $\ell(z)$ such that $\mathbb{E} [e^{z X_t}] = e^{t \cdot \ell(z)}$.

¹⁰independent subordinator

call prices, we want to find an increasing sequence of stopping times $0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_n$ such that $W_{\tau_i} \sim \mu_i$ and $(W_{t \wedge \tau_n})$ is uniformly integrable. The necessity of our “Convex order” condition is provided by the following theorem

Theorem (Rost): *A necessary and sufficient condition for the existence of a solution to the above problem is $C_{\mu_{i-1}}(K) \leq C_{\mu_i}(K)$ for all K .*

Proof of the necessity: Since $(x - K)^+$ is a convex function of x ,

$$\begin{aligned} C_{\mu_i}(K) &= \mathbb{E}[(W_\tau - K)^+] = \mathbb{E}[\mathbb{E}[(W_\tau - K)^+ | \mathcal{F}_0]] \\ &\geq \mathbb{E}[(W_0 - K)^+] \text{ by Jensen's inequality} \\ &= C_{\mu_{i-1}}(K) \end{aligned}$$

All in all, the conditions for our Carr-Cousot Dupire-type formula are the following:

1. μ_i admits a continuous and positive density f_i and have the same finite first moment X_0 , $\forall i \geq 1$, μ_0 with the Dirac measure δ_{x_0} ,
 2. C_μ are in convex order, i.e. $C_{\mu_i}(K) \leq C_{\mu_{i+1}}(K)$, $\forall K > 0$,
 3. $C_\mu(K)$ are decreasing with K , and $\lim_{K \rightarrow \infty} C_{\mu_i}(K) = 0$,
 4. $\frac{\sigma_{i-1,i}^2(K)}{K^2} = \frac{\lambda(C_i(K) - C_{i-1}(K))}{\frac{1}{2}f_i(K)}$ defines a bounded function σ^2 on $\mathbb{R}^+$,
- where $f_i(K) = \frac{\partial^2 C_i}{\partial K^2}$.

3 Simulation for stock paths

Now, we are ready to simulate the stock trajectories by using our subordinated diffusions. In Carr and Cousot [1], they suggest two simple candidates for the random time change process $\gamma^i(t)$:

- $\gamma^i(t)$ is a Gamma process $\Gamma_t^{\mu_i, \nu_i}$ with Lévy measure

$$\nu(x) = \frac{\mu_i^2 \exp\left(-\frac{\mu_i}{\nu_i}x\right)}{\nu_i x} \mathbb{I}_{x>0}, \mu_i = \frac{1}{\theta(T_i - T_{i-1})}, \nu_i = \frac{1}{\theta^2(T_i - T_{i-1})}$$

- $\gamma^i(t)$ is a linear interpolation of the random time τ_i :

$$\gamma^i(t) = \frac{t}{T_i - T_{i-1}} \tau_i, \tau_i \sim \exp(\theta)$$

3.1 Algorithm to generate paths for Gamma process

Our time change process should be a subordinator (the trajectories are almost surely increasing), one simple candidate is a Gamma process. The economic interpretation of this particular time change see section 2.8.2.

For $\Gamma_t \sim \text{Gamma}(c, \lambda)$, it has Levy measure

$$\nu(x) = \frac{ce^{-\lambda x}}{x} \mathbb{I}_{x>0} \quad (6)$$

and probability density $p_t(x) = \frac{\lambda^{ct}}{\Gamma(ct)} x^{ct-1} e^{-\lambda x}$. (See Remarks 3.1 on how to choose parameters for our simulation)

Fact: if $S_t \sim \text{Gamma}(c, \lambda)$, then $\lambda S_t \sim \text{Gamma}(c, 1)$. Thus it is enough to simulate Gamma random variable with density

$$p(x) = \frac{x^{a-1}}{\Gamma(a)} e^{-x}, \text{ with } a = ct.$$

If the parameters are $c = \frac{1}{T^\star}$ and λ , then $a = \frac{t}{T^\star}$.

We need to generate $\Gamma_t \sim \text{Gamma}(ct, \lambda)$ at time points $n\Delta t$, so we can generate $g_n \sim \text{Gamma}(c\Delta t, \lambda)$, and

$$\Gamma_{n\Delta t} = \Gamma_{(n-1)\Delta t} + g_n$$

Thus just need to generate $Y \sim \text{Gamma}(c\Delta t, 1)$, then $\frac{Y}{\lambda} \sim \text{Gamma}(c\Delta t, \lambda)$.

In general $a = c\Delta t \leq 1$, so we can use Johnk's generator (see Cont, Tankov [5]):

Repeat generating i.i.d uniform[0,1] random variables U, V

Set $X = U^{1/a}$, $Y = V^{1/(1-a)}$ until $X + Y \leq 1$.

Then generate an exponential $\exp \sim (1)$ random variable E .

Return $\frac{XE}{X + Y}$.

Remarks 3.1: By the definition of Lévy measure, $\nu(x)$ behaves like the instantaneous intensity at x . For gamma process, from (6) we can see when parameter $c = \frac{1}{T^\star}$ becomes smaller, the expected number of jumps happened per unit time is smaller, thus one needs to wait a longer time on average for a fixed number of jumps to occur. Furthermore, if λ becomes larger, then Γ_t needs to have more jumps to reach a certain positive value. Therefore, it means that parameter $c = \frac{1}{T^\star}$ controls the jump arrival rate. On the other hand, given a jump has occurred, parameter λ controls the mean jump size. Moreover, as we mentioned in 2.8.1, $\mathbb{E}[\Gamma_t] = \frac{ct}{\lambda}$. If we choose $\lambda = c = \frac{1}{T^\star}$, then our gamma time change is unbiased, $\mathbb{E}[\Gamma_t] = t$. Thus this choice of parameter for the time changed diffusion is fairly important and the motivation is more natural. So for the path simulation of the forward starting case, we use this particular choice of parameters for gamma time changes (Figure 8, 9).

3.2 Simulated paths for time changes

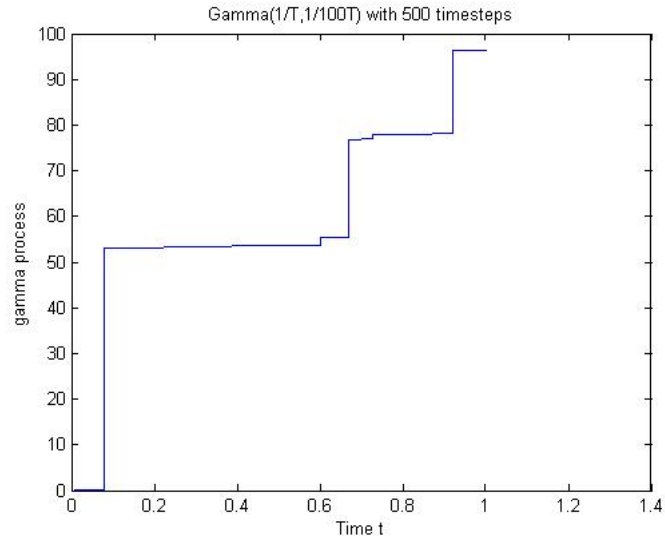


Figure 3: Simulated path of Gamma process with $c = T^* = 1$, $\lambda = 0.01$, $T_0 = 0$, $T_1 = 1$.

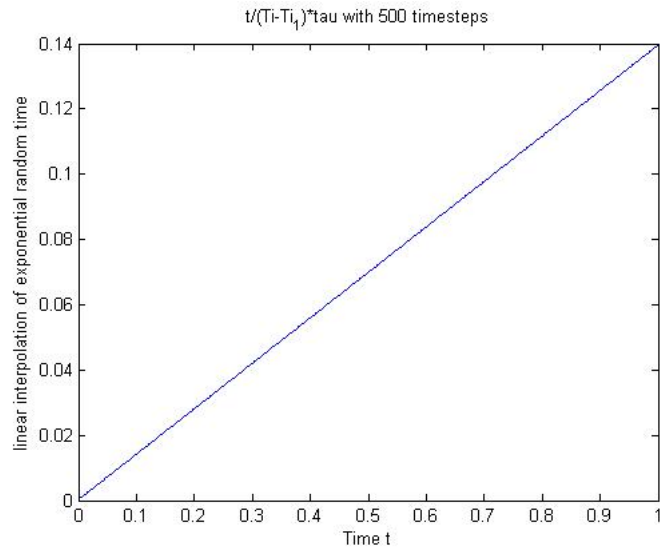


Figure 4: Simulated path of linear interpolation of exponential random time between 0 and 1 with $\lambda = 1$

3.3 Algorithm for stock path simulation

By Carr and Cousot's construction, the stock process between each maturities T_{i-1} and T_i is the time changed martingale $S_t \stackrel{d}{=} F_{\gamma^i(t)}$, $dF_t = \sigma(F_t)dW_t$

Thus for each maturity interval $[T_{i-1}, T_i]$, introduce a partition $t_0 = T_{i-1}, t_1, t_2, \dots, t_n = T_i$ and we construct the stock path through the following steps:

1. Build our Dupire-type volatility function $\sigma_{i-1,i}(K)$ for each K .
2. Simulate the increments of our random time changes (the subordinator) $\Delta\gamma_k^i = \gamma^i(t_k) - \gamma^i(t_{k-1})$.
3. Simulate the increments of the stock process, $\Delta S_k = \sigma(S_{k-1})\Delta W_k$, where $\Delta W_k \stackrel{d}{=} \Delta\gamma_k^i \cdot N_k$, $N_k \sim N(0, 1)$.
4. Sum the increments to get the stock trajectory: $S_{t_k} = \sum_{j=1}^k \Delta S_j$.

(Within each time step $[t_{k-1}, t_k]$, we use the left endpoint as the approximation of volatility $\sigma(S_{k-1})$.)

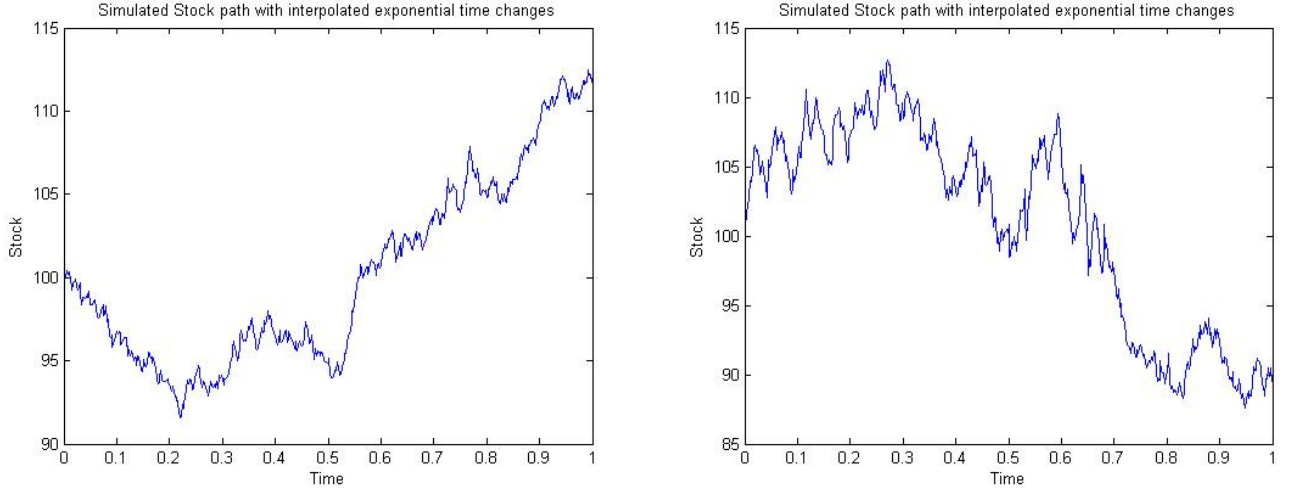


Figure 5: Simulated stock path with one maturity using linear interpolated time changes for $\lambda = 0.01$ (right), $\lambda = 100$ (left).

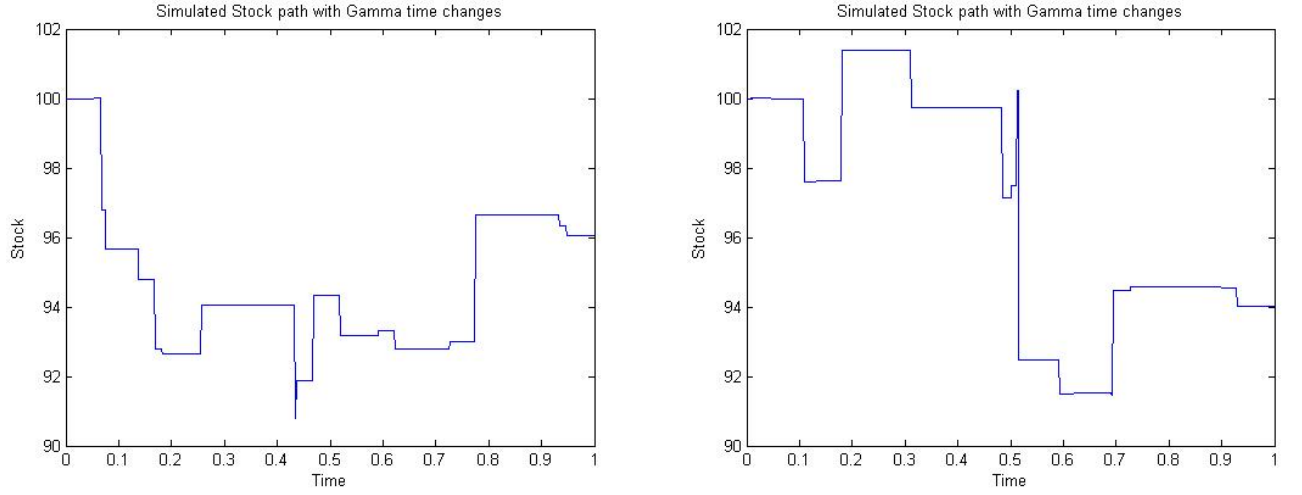
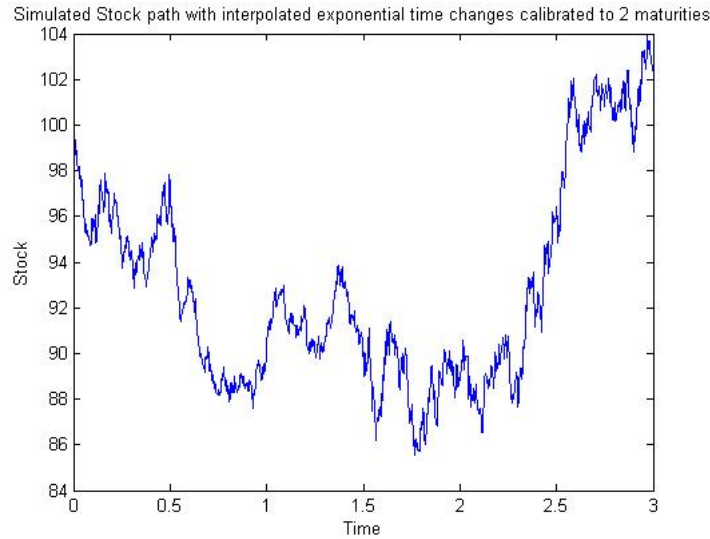


Figure 6: Simulated stock path with one maturity using Gamma time changes for $\lambda = 100$ (left), $\lambda = 0.01$ (right)

Remarks 3.2: From Figure 5 and 6, we see that different choice of λ does not influence our time changed process much. The reason is when generating stock trajectories, both the increment of subordinator and CCD volatility contribute to stock path increments. Therefore, in the following we choose parameters $\lambda = c = 1$ so that we have unbiased gamma time changes.

Then we can generalize it to iterate over finite maturities. i.e. Construct our Dupire-type volatility function piece-wise linearly then following the above procedure within each maturity interval $[T_{i-1}, T_i]$.



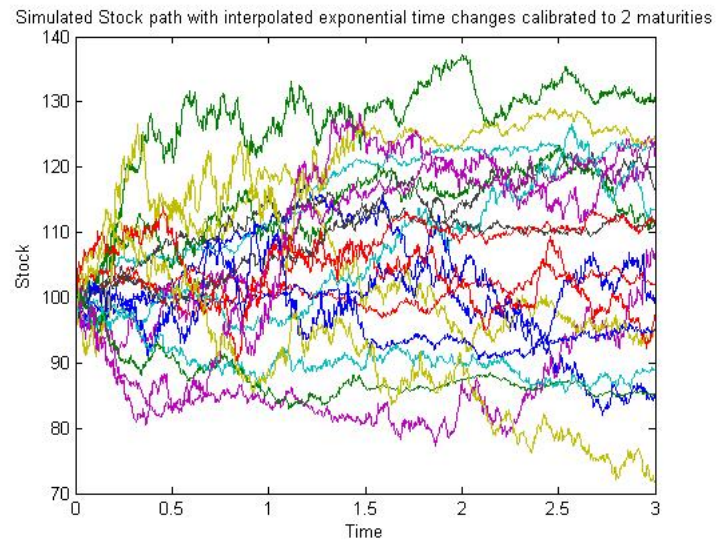
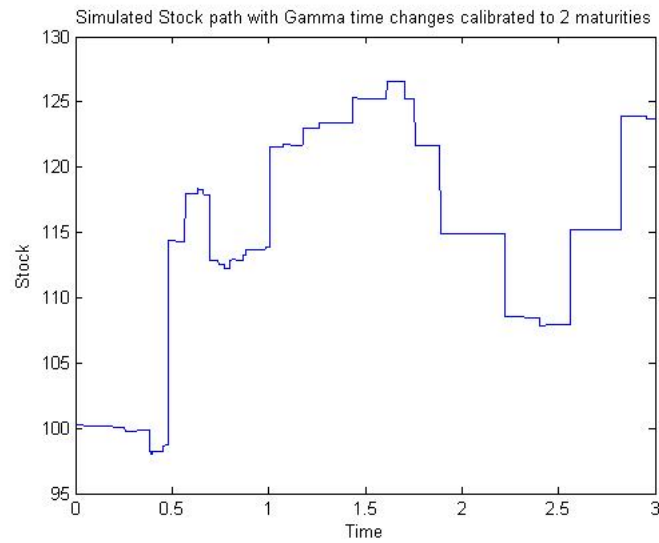


Figure 7: Simulated stock path with two maturities using linear interpolated time changes
(upper: single path, lower: 20 paths)



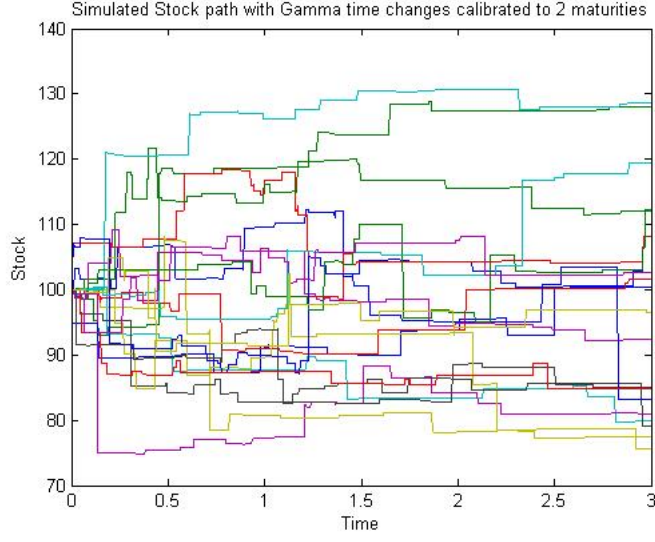


Figure 8: Simulated stock path with two maturities ($T_1 = 2$, $T_2 = 3$) using Gamma time changes (upper: single path, lower: 20 paths)

Now we are going to generate the gamma time change random variable $(\Gamma)_t$ which has an exponential distribution at $T_1 = 2$. Then linear interpolate Γ_{T_1} , and use this as the second time change. Next, using these two as subordinators to generate our time changed diffusion (the stock paths). Although, these two random time end at the same point, the stock paths with different time changes should not be ended with the same value. Figure 9 illustrates the results. For the forward starting case, the two time random variables can be generated piece-wisely using the same procedure and the ending points of stock paths with different time changes should also not be the same. (e.g. generate Γ_t twice between $[0, T_1]$ and $[T_1, T_2]$ with $\Gamma_{T_1} \sim \exp(\lambda)$ and $\Gamma_{T_2-T_1} \sim \exp(\lambda)$ respectively)

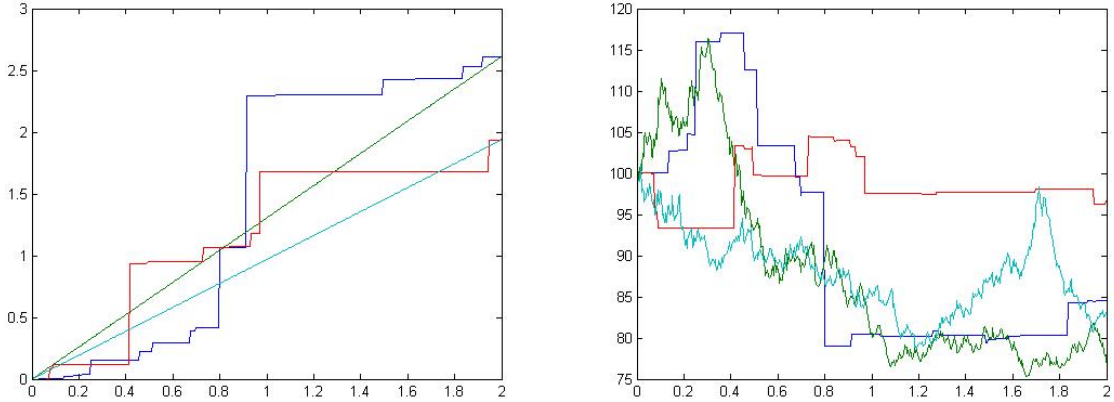


Figure 9: Simulated trajectories for different time changes (right) and stock processes (left)

Remarks 3.3: From above figures, although different time change processes arrive the same maturity (i.e. they have the same distributions at maturity, $\tau_i \sim \exp(\lambda)$), the time changed stock paths using different time changes behave differently and do not reach the same spot at maturity.

3.4 Analysis of numerical results

In this section, we will use our time changed stock process which has been calibrated to european calls with different maturities to price the familiar digital call option and barrier option. Then we examine the numerical results with the market prices (the analytical solution under Black-Scholes model).

A. Down-and-out Cash-or-nothing Options

First, let's examine the down-and-out cash-or-nothing option which has payoff $\mathbb{I}_{\{\min_t S_t \geq B\}}$, where B is the barrier. Since we are using Black-Scholes model for the stock path simulation, by the so-called Reflection principle¹¹, the market price (analytical reference price) is :

$$Price = C^{digital}(0, S_0; T, B) - \left(\frac{S_0}{B}\right) C^{digital}\left(0, \frac{B^2}{S_0}; T, B\right)^{12}$$

where in the current case $K = B$ and $r = 0$ (as the dynamic is driftless).

Now we are using Monte Carlo method to get the numerical value of the option price. The stock process is simulated via the algorithm in section 3.3. Since Euler-Maruyama method $\hat{S}_{n+1} = \hat{S}_n + \sigma_{i-1,i}(\hat{S}_n)\Delta W_n$ only gives $O(\sqrt{h})$ weak convergence for barrier type options. Thus we use the Brownian bridge technique to recover $O(h)$ weak convergence. The probability of having crossed the barrier within timestep $[t_n, t_{n+1}]$ which falls between maturities T_{i-1} and T_i is

$$P_n^i = \exp\left(-\frac{2\left(\hat{S}_{t_{n+1}} - B\right)^+ \left(\hat{S}_{t_n} - B\right)^+}{\sigma_{i-1,i}^2\left(\hat{S}_{t_n}\right) \cdot \left(\gamma^i(t_{n+1}) - \gamma^i(t_n)\right)}\right) \quad (7)$$

¹¹Reflection principle: Denote $\alpha = 1 - \frac{2r}{\sigma^2}$. If $v(t, s)$ satisfies the BSPDE, so does $\hat{v}(t, s) = \left(\frac{s}{B}\right)^\alpha \cdot v(t, \frac{B^2}{s})$ for any constant $B > 0$.

¹²see [17]

Then the payoff which is also the probability at end of not having crossed barrier is (see [16])

$$\hat{f}(\hat{S}) = \prod_i \prod_n (1 - P_n^i), \text{ where } r = 0$$

Therefore, the numerical value is $\hat{C} = \frac{1}{M} \sum_{j=1}^M \hat{f}_j$.

Due to the program run time and the computational complexity, we choose 500 time steps and 1000 simulations. And we only present the numerical result for one maturity and the forward starting case with two maturities. Prices calibrated to more maturities can be iterated easily using codes in Appendix.

Results for down-and-out cash-or-nothing option with $r = 0$, $T = 1$, $S_0 = 100$, $B = 90$,
 $\sigma_{BS} = 0.1$.

time change	MC price	BS market price	error
gamma ($\lambda = 1$)	0.6891	0.6923	-0.0032
linear interpolated <i>exp</i> (1)	0.7112	0.6923	0.0189

For price calibrated to one maturity, numerical result for gamma time changed process shows a better accuracy than a linear interpolation of random time.

Next, we are going to calculate the Monte Carlo price for the forward starting case, i.e. the subordinated diffusions are calibrated to european call prices with maturities T_1 and T_2 , then comparing it with the Monte Carlo price which only calibrated to one maturity T_2 .

Results for down-and-out cash-or-nothing option calibrated to calls with maturities $T_1 = 1$, $T_2 = 2$ versus the one calibrated to only $T_2 = 2$, parameters are $r = 0$, $S_0 = 100$, $B = 85$,
 $\sigma_{BS} = 0.1$.

time change	call maturities for calibration	MC price	BS market price	error
gamma($\lambda = 1$)	T_2	0.5429	0.7287	-0.1858
gamma ($\lambda = 1$)	T_1 and T_2	0.8246	0.7287	0.0959
linear interpolated <i>exp</i> (1)	T_1 and T_2	0.9003	0.7287	0.1716

Remarks 3.4: Above numerical results show that the price calibrated to one maturity is lower than the BS price, while the one calibrated to two maturity is higher than BS price. The reason is the volatility for the one maturity case is larger (as we only calibrated to calls at T_2 which has the largest implied volatility, so it is more volatile during $[0, T_2]$) and therefore

bigger probability of hitting the barrier. Similarly, the one calibrated to two maturities is lower (as for $\sigma_{0,1}$ we calibrated to calls at T_1 which gives lower volatility. Furthermore, $\sigma_{1,2}$ is calibrated to calls both at T_1 and T_2 . Thus, on average it gives a lower volatility) and smaller probability of hitting the barrier. In addition, the result shows that the price calibrated to calls with two maturities performs better than the one calibrated to just one maturity. Therefore, we could expect that if we calibrate to call prices with more maturities, the MC price will converge to the market price for barriers that are likely to be observed in the market (e.g. $B \in [70, 130]$ for $S_0 = 100$ in this example).

B. Down-and-out Barrier Call Options

Now, let's examine the down-and-out barrier call option which has payoff $(S_T - K)^+ \cdot \mathbb{I}_{\{\min_t S_t \geq B\}}$, where B is the barrier. Here, the relationship between strike K and barrier B matters. When $K \geq B$, the payoff is the same as before, while if $K < B$, then the payoff can be decomposed as:

$$(S_T - K)^+ \cdot \mathbb{I}_{\{\min_t S_t \geq B\}} = (S_T - B)^+ \cdot \mathbb{I}_{\{\min_t S_t \geq B\}} + (B - K) \cdot \mathbb{I}_{\{\min_t S_t \geq B\}}$$

This is the sum of a down-and-out call with strike B and $(B - K)$ units of down-and-out cash-or-nothing call with strike B .

Since we are using Black-Scholes model for the stock path simulation, again by the Reflection principle the market price (analytical reference price) is :

$$Price = \begin{cases} C^{euro}(0, S_0; T, K) - \left(\frac{S_0}{B}\right) C^{euro}\left(0, \frac{B^2}{S_0}; T, K\right), & K \geq B \\ C^{euro}(0, S_0; T, B) + (B - K) C^{digital}(0, S_0; T, B) \\ - \left(\frac{S_0}{B}\right) \left[C^{euro}\left(0, \frac{B^2}{S_0}; T, B\right) + (B - K) C^{digital}\left(0, \frac{B^2}{S_0}; T, B\right) \right], & K < B \end{cases}$$

13

Similarly, using the Brownian bridge technique to recover $O(h)$ weak convergence. The probability of having crossed the barrier within timestep $[t_n, t_{n+1}]$ which falls between maturities T_{i-1} and T_i is still (7). And the payoff is

$$\hat{f}(\hat{S}) = (\hat{S}_{t_N} - K)^+ \cdot \prod_{i=1}^N (1 - P_n^i), \text{ where } r = 0$$

¹³see [17]

Therefore, the numerical value is $\hat{C} = \frac{1}{M} \sum_{j=1}^M \hat{f}_j$.

The following figures are plots of Monte Carlo down-and-out barrier call prices with Black-Scholes market prices and the errors for our numerical approximation (using 1000 and 5000 simulated paths, Figure 10, 11 & 12). It can be seen that both of them demonstrate a quite good accuracy when calibrated to one maturity ($T = 1$). As shown when $K \rightarrow \infty$, the payoff tends to zero and the barrier call prices tend to zero. This is because the Black-Scholes price tends to zero as strike goes to infinity. So the numerical error also tends to zero. Although, Figure 12 shows a much better accuracy, the program run time is much longer. Thus, due to computational complexity, we will use 1000 simulated trajectories when calibrating to two maturities.

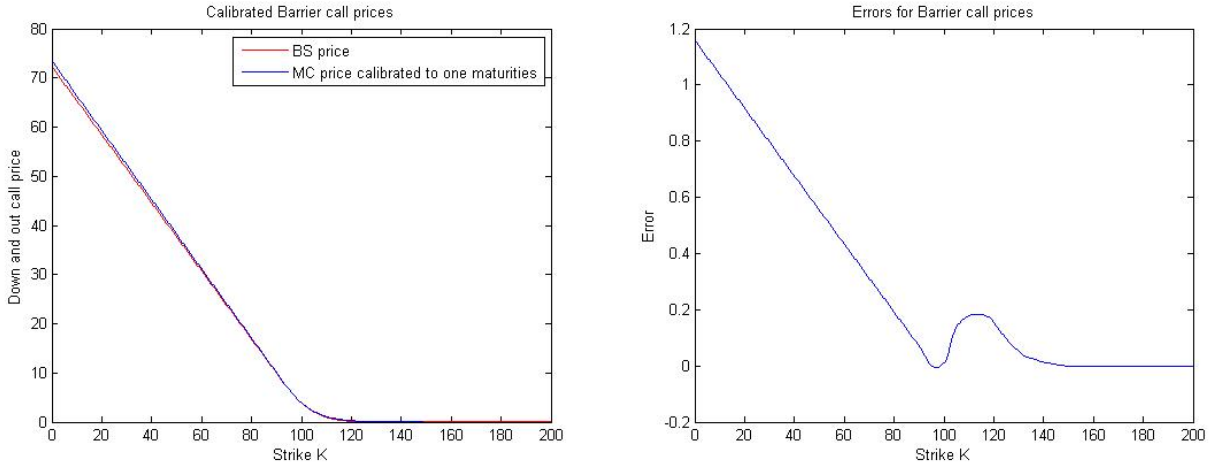


Figure 10: Monte Carlo down-and-out barrier call prices (calibrated to single maturity using gamma time changes) with Black-Scholes market prices (left) and errors (right) with $r = 0$, $T = 1$, $S_0 = 100$, $B = 90$, $\sigma_{BS} = 0.1$, 500 time steps and 1000 simulated paths.

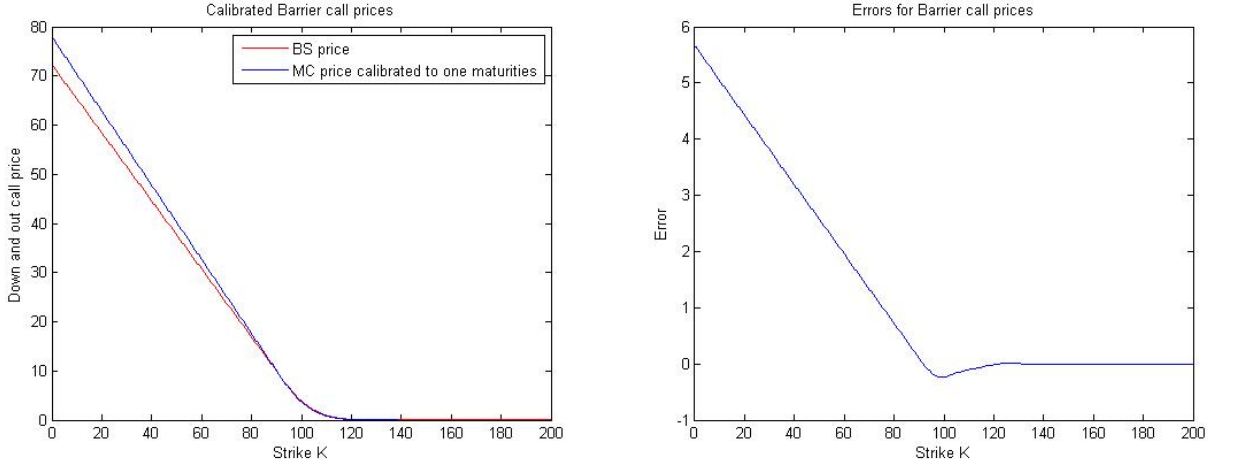


Figure 11: Monte Carlo down-and-out barrier call prices (calibrated to single maturity using linear interpolation of random time changes) with Black-Scholes market prices (left) and errors (right) with $r = 0$, $T = 1$, $S_0 = 100$, $B = 90$, $\sigma_{BS} = 0.1$, 500 time steps and 1000 simulated paths.

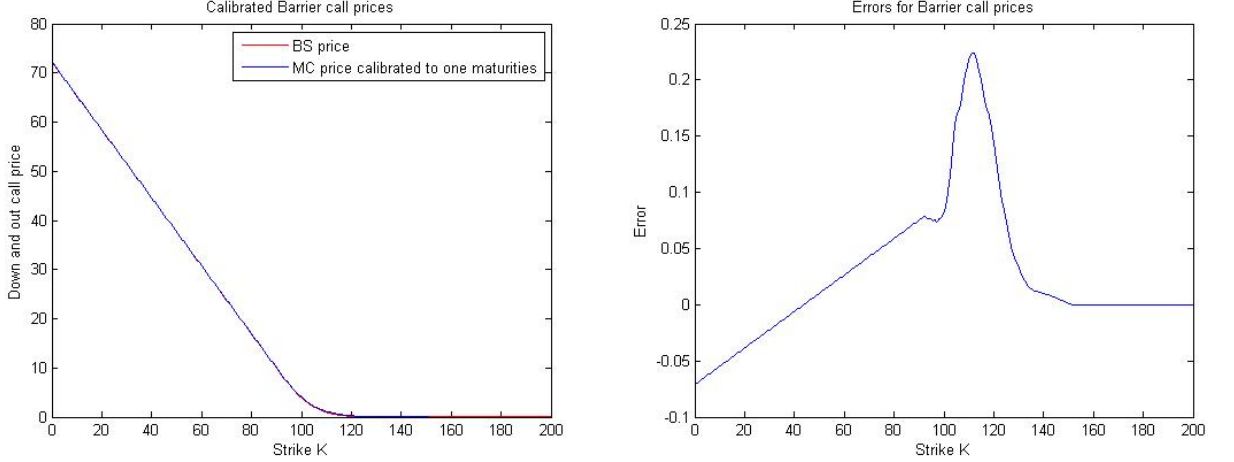


Figure 12: Monte Carlo down-and-out barrier call prices (calibrated to single maturity using gamma time changes) with Black-Scholes market prices (left) and errors (right) with $r = 0$, $T = 1$, $S_0 = 100$, $B = 90$, $\sigma_{BS} = 0.1$, 500 time steps and 5000 simulated paths.

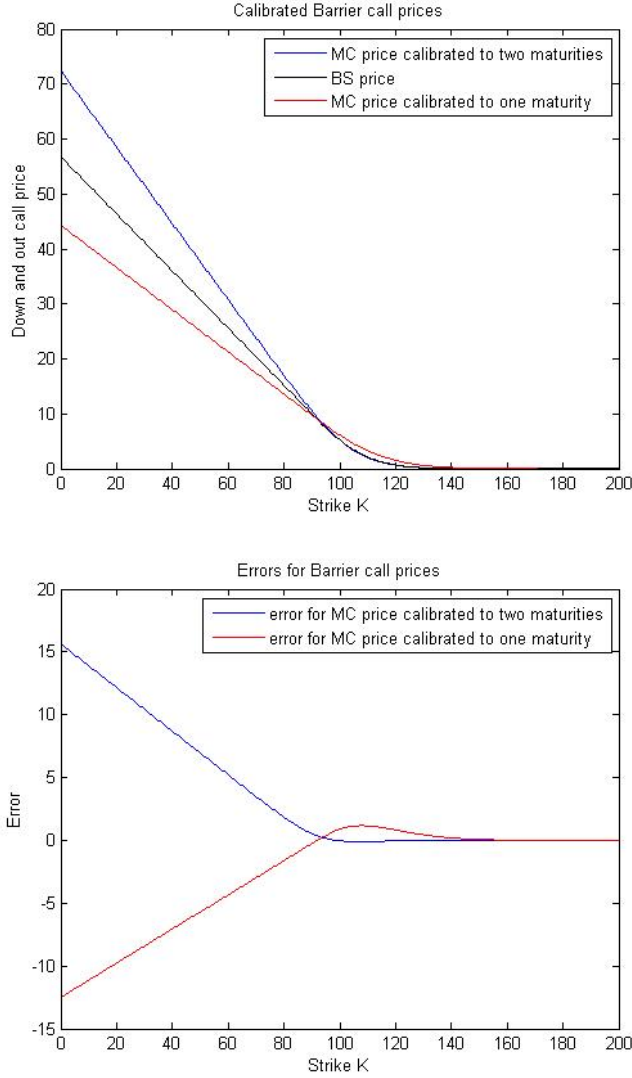


Figure 13: Monte Carlo down-and-out barrier call prices (calibrated to two maturities using gamma time changes) with Black-Scholes market prices (upper) and errors (lower) with $r = 0$, $T = 1$, $S_0 = 100$, $B = 90$, $\sigma_{BS} = 0.1$.

In Figure 13, for $K \rightarrow 0$ Monte Carlo price calibrated to one maturity shows a smaller error, but as K increases, the Monte Carlo price that calibrated to two maturities gives a much better accuracy than the one with one maturity (especially for strikes likely to be observed in the market, e.g. $K \in [80, 120]$ with $S_0 = 100$). Therefore, our forward starting model constructed in previous sections provides a better approximation for option prices when calibrating to standard european calls.

4 Further generalization using speed measure

4.1 Irregular volatility function

The discussion in previous sections all assumed that our volatility function $\sigma^2(K)$ is nice and regular. But in practice, we could only collect finite number of call option prices which means for each maturity we only get call values for finite strikes $K_j, j = 1, \dots, m$. To get the whole surface of $C_{\mu_i}(K)$, the worst benchmark case (see Remarks 4.1) is to use simple linear interpolation for call prices which gives us the most expensive call option prices among all interpolation methods.

Remarks 4.1: Linear interpolation is only one way of interpolating (other methods can see [22]). But it gives us the highest call prices and can therefore be interpreted as a prudent method from the seller's point of view. Also linear interpolation gives the most irregular call curve which lose the smoothness, therefore, it is interesting to generalise CCD under this worst benchmark case.

Thus for each i , $C_{\mu_i}(K)$ is not a smooth function (only a piece-wise linear one).

$$C'_{\mu_i}(K_{j-}) \neq C'_{\mu_i}(K_{j+}) \text{ and } C''_{\mu_i}(K_j) \text{ does not exist } \forall j = 1, \dots, m$$

In this case, $f_i(K)$ does not exist and $\frac{\sigma_{i-1,i}^2(K)}{K^2}$ can not define a bounded function. So assumption 4 of section 2.9 does not hold, and we can not use Carr-Cousot Dupire type formula. The following example motivates a more general treatment than that of Carr-Cousot.

Example 4.1: Consider for $\mu_0 = \delta_{5/2}$, $\mu_1 = \frac{1}{4}\delta_1 + \frac{1}{4}\delta_2 + \frac{1}{4}\delta_3 + \frac{1}{4}\delta_4$. Thus we are in the most extreme model. Suppose that the call price with maturity T_1 is defined as

$$C_{\mu_1}(K) = \begin{cases} \frac{5}{2} - K, & 0 \leq K \leq 1 \\ \frac{3}{2} - \frac{3}{4}(K - 1), & 1 \leq K \leq 2 \\ \frac{3}{4} - \frac{1}{2}(K - 2), & 2 \leq K \leq 3 \\ \frac{1}{4} - \frac{1}{4}(K - 3), & 3 \leq K \leq 4 \\ 0 & K \geq 4 \end{cases}$$

It can be easily checked that

$$\int_0^\infty x \mu_0(dx) = \frac{5}{2},$$

and

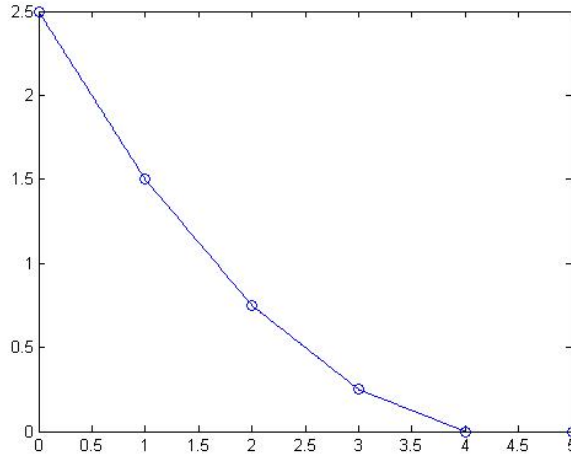
$$\int_0^{\infty} x \mu_1(dx) = \frac{1}{4}(1 + 2 + 3 + 4) = \frac{5}{2},$$

so μ_0 and μ_1 have the same finite first moment. Also $C_{\mu_i}(K)$ is a decreasing function with respect to K (this is first half of assumption 3 in section 2.9, see the following plot). As each upper part of $C_{\mu_1}(K)$ is greater than or equal to the lower part of it (by decreasing property),

$$C_{\mu_1}(K) \geq C_{\mu_0}(K) = \begin{cases} \frac{5}{2} - K, & 0 \leq K \leq \frac{5}{2} \\ 0, & K > \frac{5}{2} \end{cases} = \left(\frac{5}{2} - K \right)^+$$

It satisfies the convex order condition (assumption 2 of section 2.9). Obviously, for $i = 0, 1$, $\lim_{K \rightarrow \infty} C_{\mu_i}(K) = 0$, second half of assumption 3 in section 2.9 is satisfied. Although most essential conditions are satisfied, as $C'_{\mu}(x) = -\mathbb{P}[X \geq x]$, $C'_{\mu_1}(K^-) \neq C'_{\mu_1}(K^+)$ at the points with mass ($K = 1, 2, 3, 4$). Thus $\sigma^2(K)/K^2$ does not define a bounded function, assumption 4 does not hold and we can not use our Carr-Cousot Dupire type formula directly in this case. The following is the plot of $C_{\mu_1}(K)$, as we can see it is piecewise differentiable. i.e. For maturity T_1 , we can only collect finite number of call prices, then recover the curve by simple linear interpolation. We can think of this as reflecting the situation in practice when we can only observe a finite number of calls.

Figure: Plot of $C_{\mu_1}(K)$



4.2 Speed measure

One solution for this problem is to generalize the construction by using *speed measure* rather than our diffusion coefficients.

Fix an interval $I \subseteq \mathbb{R}$, let m be a non-negative Borel measure on I . If $B = (B_t)_{t \geq 0}$ is a Brownian motion with starting point $B_0 \neq 0$ on $(\Omega, \mathcal{F}_t^B, \mathbb{P})$

Definition 5 (Introduced by Paul Lévy): *The local time of B at point x is the stochastic process $\{L_t^x : t \geq 0, x \in \mathbb{R}\}$:*

$$L_t^x = \int_0^t \delta(x - B_s) ds,$$

where $\delta(x)$ is the Dirac delta function. It is also been known as the occupation time which measures how much time B_s has spend at x up to time t (Note: this definition can be generalized to any diffusion X_t).

$$\text{i.e. } L_t^x = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{I}\{|B_s - x| \leq \varepsilon\} ds$$

Now for this local time process L_t^x , we can define a continuous, increasing and additive functional

$$A_t = \int_I L_t^x m(dx)$$

and its right continuous inverse $\Gamma_t = \inf \{s \geq 0 : A_s > t\}$. Then we can express $X_t = B_{\Gamma_t}$ as a time change of Brownian motion, and $X = (X_t)_{t \geq 0}$ is called a one dimensional generalized diffusion in natural scale (i.e. the scale function $s(x) = x$) with speed measure m . The measure m appeared is call the *speed measure* of diffusion X .

Remarks 4.2: Note that above generalized diffusion using speed measure is a special case of Dubins-Schwarz Theorem (DDST see Remarks 2.11). We can understand it in the context of the Occupation times formula.

Occupation times formula (see Klebaner [13] or Roger, Williams [10] 45.4): Let X_t be a continuous semimartingale with local time L_t^x . Then for any bounded measurable function $g(x)$:

$$\int_0^t g(X_s) d\langle X \rangle_s = \int_{-\infty}^{\infty} g(x) L_t^x dx$$

And in particular,

$$\langle X \rangle_t = \int_{-\infty}^{\infty} L_t^x dx = \int_{-\infty}^{\infty} \tilde{L}_t^x m(dx) = A_t$$

where $\tilde{L}_t^x$ is the local time of Brownian motion, L_t^x is the local time process of X and the speed measure m behaves like a change of measure here.

Here, we remind you that for diffusion X_t , a function $s(x)$ is called a scale function if $Y_t = s(X_t)$ is a local martingale. Alternatively, for diffusion X on I , $s(x)$ is a continuous strictly increasing function such that for all $x \in I$,

$$H(x) = \mathbb{P}^x [H_U < H_L] = \frac{s(x) - s(L)}{s(U) - s(L)},$$

where H_x is the hitting time of x , $H(x)$ is the probability that x hits the upper boundary first, L and U are the upper and lower bound in I respectively.

Remarks 4.3: For positions where $m(dx)$ is very large, the time it stays there is larger (e.g. $A_t \uparrow$), our diffusion X moves slower in this region. That's why it is called speed measure, even though "sloth measure" might be a better terminology. (More details of speed measure see Roger, Williams [10])

Back to our problem, although $C''_\mu(K)$ does not exist, it still exists in a distributional sense as

$$C'_\mu(K) = \int_0^K C''_\mu(x) dx = \int_0^K \mu(dx)$$

Therefore, we can generalise it by defining

$$m_{i-1,i}(dx) = \frac{\frac{1}{2}\mu_i(dx)}{\lambda(C_i(x) - C_{i-1}(x))} \quad (8)$$

When $C_{\mu_i}(K)$ are smooth and nice, we get back our Carr-Cousot Dupire Type formula and

$$m_{i-1,i}(dx) = \frac{dx}{\sigma_{i-1,i}^2(x)}.$$

Remarks 4.4: The expression for equation (8) is a direct result of Cox, Hobson, Klimmek (2013) [3] for the non-bubble¹⁴ martingale case ($c = 0$) which is

$$m_{i-1,i}(dx) = \frac{\mu_i(dx)}{2\lambda(C_i(x) - C_{i-1}(x) + c)} \quad (9)$$

Now for example 4.1, as μ only assigns mass to $K = 1, 2, 3, 4$, $m_1(dx) = 0$ for all the other points and (with $\lambda = 1$)

$$m_1(\{x\}) = \begin{cases} \frac{1}{4}/0 = \infty, & x = 1 \\ \frac{1}{4}/\left(\frac{3}{4} - \frac{5}{2} + 2\right) = 1, & x = 2 \\ \frac{1}{4}/\left(\frac{1}{4} - 0\right) = 1, & x = 3 \\ \frac{1}{4}/0 = \infty, & x = 4 \end{cases},$$

The behaviour of our generalised diffusion X_t is, X starts at $\frac{5}{2}$, then it jumps to either 2 or 3 instantly. Since X is a martingale, the transition probabilities are $\frac{1}{2}$ for states 2 and 3. If X jumps to 2 after $t = 0$, then after spending some time at 2 it will jump to either 1 or 3 with probability $\frac{1}{2}$ for each. Similarly, if X jumps to 3 after $t = 0$, then after spending some time at 3 it will jump to either 2 or 4 with probability $\frac{1}{2}$ for each. Notice that $m_1(x) = \infty$, $x = 1, 4$ from the expression, denote t_1 be the first hitting time of 1 for X_t , then

¹⁴The discounted price process S has a bubble if S is a strict local martingale under the risk-neutral measure $\mathbb{Q}$. see Cox, Hobson [15]

$$A_{t_1} = \int_{\mathbb{R}^+} L_{t_1}^x m(dx) = \infty$$

and for all $t \geq t_1$, $A_t = \infty$. Heuristically, when $m(\{x\}) = \infty$, the time it spends at x goes to infinity (similarly for 4). Therefore 1 and 4 are absorbing boundaries. If X jumps to 1 or 4 later, then it will stay there forever. (details see Cox, Klimmek [9] example 3.3 which is a similar example)

Remarks 4.5: By using the speed measure, we do not need the smoothness for the call price surface (C^2 w.r.t K) with each maturity. Therefore, for each given maturity, our generalized diffusion $X_t = B_{\Gamma_t}$ can be calibrated to finite call prices (with finite strikes).

4.3 Generalization

In practice, we are given a finite number of call prices with maturities T_i , $i = 1, 2, \dots, n$ (for each maturity, we have call prices with finite number of strikes). Summing up all the ideas we have discussed in previous sections, our stock process $S = (S_t)_{t \geq 0}$ becomes

$$S_t = X_{\gamma^i(t - T_{i-1})} = B_{\Gamma_{\gamma^i(t - T_{i-1})}}, \quad t \in [T_{i-1}, T_i]$$

where we denote $\gamma^i(t - T_{i-1})$ as the time change subordinator conditioned on T_{i-1} between T_{i-1} and T_i .

$$\Gamma_{\gamma^i(t - T_{i-1})} = \inf \{s : A_s > \gamma^i(t - T_{i-1})\},$$

$$A_t = \int_{\mathbb{R}^+} L_t^x m(dx)$$

with m recovered from the call prices using (8).

Therefore, our generalization of Carr-Cousot Dupire type formula using speed measure gives us a consistent model when calibrating to only a finite number of call prices.

5 Conclusion

In this paper, we have studied the derivation of CCD formula with general exponential random time using two different methods. And iterating it to the forward starting case for finite maturities. The general CCD formula can be constructed piecewise linearly via market call prices:

$$\sigma_{i-1,i}^2(K) = \frac{\lambda(C_i(K) - C_{i-1}(K))}{\frac{1}{2}f_i(K)}$$

Then using the above volatility structure, we model the stock price process as a time changed driftless diffusion sampled at random times such that:

$$S_t = X_{\gamma^i(t-T_{i-1})},$$

$$dX_t = \sigma(X_t) dW_t$$

where $\gamma^i(t - T_{i-1})$ is a time change process (subordinator) and $\gamma(T_i - T_{i-1}) \sim \exp(\lambda)$, $\forall i = 1, \dots, n$.

We have investigated how the CCD approach fits into the theory of Lévy processes. The subordinated diffusion in our thesis can be seen as a generalisation of the popular Variance Gamma model. Using Black-Scholes calls as calibration prices, we have simulated the subordinated diffusions. Next, we have examined the numerical results for some exotic option (Barrier type) via simulation of this construction. Although the numerical error increases when maturity becomes longer, the Monte Carlo prices show good accuracy when comparing with the market price.

By dropping the assumption of having a continuum of strikes, if there is only a finite number of calls, we have generalized the model further via speed measure. This generalisation is done for the worst benchmark case using simple linear interpolation which gives the most expensive call prices. And therefore the stock price process is modeled as

$$S_t = X_{\gamma^i(t-T_{i-1})} = B_{\Gamma_{\gamma^i(t-T_{i-1})}}, t \in [T_{i-1}, T_i]$$

It provides a consistent model when we can only observe a finite number of calls for calibration. All in all, the CCD approach gives a quite easy and explicit way for real life calibration which do not require too much data.

Bibliography

References

- [1] Peter Carr and Laurent Cousot, 2011, *Explicit Constructions of Martingales Calibrated to Given Implied Volatility Smiles*, SIAM J. FINANCIAL MATH.
- [2] David Hobson, 2009, *The Skorokhod Embedding Problem And Model-independent Bounds For Option Prices*.
- [3] A.Cox, D.Hobson, M.Klimmek, 2013, *Consistent models for bubbles: Constructing models to match call prices in a strict local martingale world*, Working paper, June 2013.
- [4] D.Madan, P.Carr, E.Chang, 1998, *The Variance Gamma Process And Option Pricing*, European Finance Review 2, 1998, pages 79- 105.
- [5] Rama Cont and Peter Tankov. *Financial Modelling With Jump Process*. Chapman & Hall / CRC Press, 2003.
- [6] B.Dupire, 1994. *Pricing With A Smile*. Risk 7, 18-20.
- [7] S. Ethier and T. Kurtz, *Markov processes: Characterization and convergence*, Wiley Series in Probability and Statistics, John Wiley and Sons, second ed., 2005.
- [8] D. Breeden and R. Litzenberger, *Prices of state-contingent claims implicit in options prices*, J. Bus, 51 (1978), pp. 621–651.
- [9] A.Cox, M.Klimmek, 2013, *From minimal embeddings to minimal diffusions*, ArXiv preprint.
- [10] L.C.G. Rogers and D. Williams. *Diffusions, Markov Processes and Martingales, volume 2*. Cambridge University Press, 2000.
- [11] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus (GTM 113)*, Springer-Verlag.
- [12] A.M.G. Cox, D. Hobson, and Jan Oblój. *Time-homogeneous diffusions with a given marginal at a random time*. ESAIM: Probability and Statistics, 15:11-24, 2011.
- [13] F. C. Klebaner: *Introduction to Stochastic Calculus with Applications*, Imperial College Press.
- [14] B. Oksendal: *Stochastic Differential Equations: An introduction with applications*, Universitext, Springer (6th edition).

- [15] A. M. G. Cox, David Hobson, 2005, *Local martingales, bubbles and option prices*.
- [16] Mike Giles, 2013, *Lecture notes of Monte Carlo Methods*
- [17] Hanqing Jin, 2013, *Lecture notes of Financial Derivatives II*
- [18] MADAN, D.B. and SENETA, E. (1990): *The variance gamma (V.G.) model for share market returns*. Journal of Business, 63, 511-524.
- [19] Stephen Smale, Morris W. Hirsch, Robert L. Devaney, 2003, *Differential Equations, Dynamical Systems, and an Introduction to Chaos (2ed)*, Academic Press
- [20] Qing Liu, 2011, *Time-Homogeneous Diffusion with a given marginal subordinated by different time changes*.
- [21] Jan Obłój, 2004, *The Skorokhod embedding problem and its offspring*. Probab. Surv., 1: 321-390 (electronic).
- [22] Cassio Neri, Lorenz Schneider, 2011, *A Family of Maximum Entropy Densities Matching Call Option Prices*, ArXivpreprint.

A Main MATLAB Code

Due to the large number of programmes, we only includes the Matlab codes for key functions in this section.

```
% function for generating Gamma Process
function g = gamma_rv(lambda, n, start_t, end_t, gamma_0)

interval = end_t - start_t;
t = start_t:(interval/n):end_t;
% a = t/T, T=1
path = zeros(1,n + 1);
dt = interval / n;
i = 1;
path(i) = gamma_0;
while (t(i) < end_t)
    U1 = rand(1);
    U2 = rand(1);
    % a = ct = t/T, T=1, c = 1/T
    % but here, path = gamma(ct,lambda) at points n*dt, so generate
    % g = gamma(c*dt,lambda), path(n*dt)=path((n-1)dt)+g, so a = c*dt
    % as a <=1 in this case, use Johnk's generaor
    a = dt;
    x = U1^(1/a);
    y = U2^(1/(1-a));
    i = i + 1;
    path(i) = path(i - 1);
    t(i) = t(i-1) + dt;
    if x + y <= 1
        % generate exponential(1) random variable
        U = rand(1);
        ex_rv = -log(U);
        path(i) = path(i-1) + x*ex_rv / ((x + y)*lambda);
    end
end

g = path;
```

```

end

% function for generating linear interpolated random time

function e = exp_time(lambda,n,start_t,end_t)

interval = end_t - start_t;
t = start_t:(interval/n):end_t;
path = zeros(1,n + 1);
dt = interval / n;
U = rand(1);
tau = -log(U) / lambda;
for i = 2:(n+1)
    path(i) = i*dt*tau / (end_t - start_t);
end
e = path;

% Plot gamma trajectories

st = 0;
et = 1;
n = 500;
dt = (et-st)/n;
t = st:dt:et;
plot(t,gamma_rv(1,n,st,et,1),'b-');
xlabel('Time t'); ylabel('gamma process');
title('Gamma(1/T,1/100T) with 500 timesteps,Ti=0,Ti-1=1');

% function for forward starting Carr-Cousot Dupire type formula
% use Black-Scholes price as market price

function s = sigma(K,lambda,start_t,end_t)

% dupire-type sigma with S0 = 100, B-S sig = 0.1

S0 = 100;
T1 = start_t;
T2 = end_t;

```

```

sig = 0.1;
% denote miu_dK be second derivatives of C w.r.t. K
miu_dK = exp(-(log(K/S0)+0.5*sig^2*T2).^2/(2*sig^2*T2))./(sig*K.*sqrt(2*pi*T2));
C_2 = euro_call(0,sig,T2,S0,K);
if T1 ~= 0
    C_1 = euro_call(0,sig,T1,S0,K);
else
    C_1 = max(S0 - K, 0);
end
% if use generator Gf(x) = 0.5(x*sigma)^2*f''(x), then
% s = sqrt(lambda*(C_2 - C_1)./(0.5*(K.^2).*miu_dK));

s = sqrt(lambda*(C_2 - C_1)./(0.5*miu_dK));

function V = euro_call(r,sigma_1,T,S,K)
% function for european call
d1 = ( log(S) - log(K) + (r+0.5*sigma_1^2)*T ) / (sigma_1*sqrt(T));
d2 = ( log(S) - log(K) + (r-0.5*sigma_1^2)*T ) / (sigma_1*sqrt(T));
V = S.*N(d1) - exp(-r*T)*K.*N(d2);

function ncf = N(x)

%function of normal cumulative function
xr = real(x);
xi = imag(x);

if abs(xi)>1e-10
    error 'imag(x) too large in N(x)'
end

ncf = 0.5*(1+erf(xr/sqrt(2))) ...
    + 1i*xi.*exp(-0.5*xr.^2)/sqrt(2*pi);

% Plot CCD function

k = linspace(0,500,10000);

```

```

plot(k, sigma(k,100,1,2), 'b-');
xlim([35 250]);
xlabel('Strike K'); ylabel('Sigma(K) ');

% function for generating stock path with Gamma time change

function [s,t_change] = stock_gamma(n,lambda,start_t,end_t,S0)

interval = end_t - start_t;
t = start_t:(interval/n):end_t;
S = S0;
path = zeros(1,n + 1);
t_change = gamma_rv(lambda,n,start_t,end_t,start_t);
path(1) = S;
for i = 2:(n+1)
    delta_t = t_change(i) - t_change(i-1);
    U = randn(1);
    % sigma is calculated piece-wise linearly
    delta_s = sigma(S,lambda,start_t,end_t)*sqrt(delta_t)*U;
    S = S + delta_s;
    if S >= 0
        path(i) = S;
    else
        S = 0;
    end
end

s = path;
end

% Plot time changed diffusion path calibrated to one maturity

t = 0:0.001:1;
s = stock_gamma(1000,100,0,1,100)
plot(t,s, 'b-');
xlabel('Time'); ylabel('Stock');
title('Simulated Stock path with Gamma time changes');

```

```

% function for generating stock path with linear interpolation
% of random time as time changes

function [s,t_change] = stock_exp(n,lambda,start_t,end_t,S0)

interval = end_t - start_t;
t = start_t:(interval/n):end_t;
S = S0;
path = zeros(1,n + 1);
t_change = exp_time(lambda,n,start_t,end_t);
path(1) = S;
for i = 2:(n+1)
    delta_t = t_change(i) - t_change(i-1);
    U = randn(1);
    delta_s = sigma(S,lambda,start_t,end_t)*sqrt(delta_t)*U;
    S = S + delta_s;
    path(i) = S;
end

s = path;
end

% Plot stock path
t = 0:0.001:1;
s = stock_exp(1000,100,0,1,100)
Plot time changed diffusion path calibrated to one maturity
plot(t,s,'b-');
xlabel('Time'); ylabel('Stock');
title('Simulated Stock path with interpolated exponential time changes');

% plot stock paths with gamma time change which calibrated to
% two maturities, parameters are
% n = 500, lambda = 1, T1 = 1, T2 = 2, T3 = 3, S0 = 100

n = 500;
lambda = 1;
T1 = 0;
T2 = 2;

```

```

T3 = 3;
S0 = 100;
interval = T3 -T1;
t = T1: interval/(2*n):T3;
for i = 1:20
    S_1 = stock_gamma(n,lambda,T1,T2,S0);
    S_m = S_1(end);
    S_2 = stock_gamma(n,lambda,T2,T3,S_m);
    S_2 = S_2(2:end);
    S = [S_1, S_2];
    plot(t,S);
    hold all;
end

xlabel('Time'); ylabel('Stock');
title('Simulated Stock path with Gamma time changes calibrated to 2 maturities');

% plot stock paths with linear interpolated random time change
% which calibrated to two maturities, parameters are
% n = 500, lambda = 1, T1 = 1, T2 = 2, T3 = 3, S0 = 100

n = 500;
lambda = 1;
T1 = 0;
T2 = 2;
T3 = 3;
S0 = 100;
interval = T3 -T1;
t = T1: interval/(2*n):T3;
figure(1)
for i = 1:20
    S_1 = stock_exp(n,lambda,T1,T2,S0);
    S_m = real(S_1(end));
    S_2 = stock_exp(n,lambda,T2,T3,S_m);
    S_2 = S_2(2:end);
    S = [S_1, S_2];

    plot(t,S);

```

```

        hold all;
end

xlabel('Time'); ylabel('Stock');
title('Simulated Stock path with interpolated ...
      exponential time changes calibrated to 2 maturities');

% Plots of stock trajectories using different time changes together

% Path for Gamma time changes
n = 500;
lambda = 1;
T1 = 0;
T2 = 2;
S0 = 100;
interval = T2 - T1;
t = T1:(interval/n):T2;
path1 = zeros(1,n + 1);
t_change1 = gamma_rv(lambda,n,T1,T2,T1);
S1 = S0;
path1(1) = S1;
for i = 2:(n+1)
    delta_t = t_change1(i) - t_change1(i-1);
    U = randn(1);
    % sigma is calculated piece-wise linearly
    delta_s = sigma(S1,lambda,T1,T2)*sqrt(delta_t)*U;
    S1 = S1 + delta_s;
    path1(i) = S1;
end

% Path for linear interpolated time changes
path2 = zeros(1,n + 1);
tau = t_change1(end);
t_change2 = t*tau / (T2 - T1);
S2 = S0;
path2(1) = S2;
for i = 2:(n+1)
    delta_t = t_change2(i) - t_change2(i-1);

```

```

    U = randn(1);
    delta_s = sigma(S2,lambda,T1,T2)*sqrt(delta_t)*U;
    S2 = S2 + delta_s;
    path2(i) = S2;
end

figure (1)
plot(t,path1);
hold all;
plot(t,path2);
figure (2)
plot(t,t_change1);
hold all;
plot(t,t_change2);

% function for MC price of digital barrier option using
% gamma time changed process calibrated to one maturity
% the other time changed diffusion can be done by simply
% changed the input trajectory function
% so it won't be included here

function c = do_call(n,T,M,B)

sum = 0;
% r = 0
S0 = 100;
lambda = 1;

for j = 1:M
    prob = zeros(1,n);
    p = 1;
    [path, tau] = stock_gamma(n,lambda,0,T,S0);
    for i = 1:n
        dt = tau(i+1) - tau(i);
        if dt ~= 0
            prob(i) = -2*max(path(i+1)-B,0)*max(path(i)-B,0) / ...
                (dt*sigma(path(i),lambda,0,T)^2);
            p_i = 1 - exp(prob(i));

```

```

        else
            p_i = 1;
        end
        p = p * p_i;
    end
    sum = sum + p;
end

c = sum/M;
end

% function for MC price of digital barrier option using
% gamma time changed process calibrated to two maturities

function c = do_call_2(n,T1,T2,M,B)

sum = 0;
% r = 0
S0 = 100;
lambda = 1;

for j = 1:M
    prob = zeros(1,n);
    p = 1;
    [path1, tau1] = stock_gamma(n,lambda,0,T1,S0);
    pathm = path1(end);
    [path2, tau2] = stock_gamma(n,lambda,T1,T2,pathm);
    path2 = path2(2:end);
    tau2 = tau2(2:end);
    path = [path1, path2];
    tau = [tau1, tau2];
    m = length(tau1);
    for i = 1:n
        dt = tau(i+1) - tau(i);
        if dt ~= 0
            if i <= m
                prob(i) = -2*max(path(i+1)-B,0)*max(path(i)-B,0) / ...
                    (dt*sigma(path(i),lambda,0,T1)^2);
            end
        end
    end
end

```

```

        p_i = 1 - exp(prob(i));
    else
        prob(i) = -2*max(path(i+1)-B,0)*max(path(i)-B,0) / ...
                    (dt*sigma(path(i),lambda,T1,T2)^2);
        p_i = 1 - exp(prob(i));
    end
    else
        p_i = 1;
    end
    p = p * p_i;
end
sum = sum + p;
end

c = sum/M;
end

% function for MC price of down-and-out barrier call using
% gamma time changed process calibrated to one maturity
% the other time changed diffusion can be done by simply
% changed the input trajectory function
% so it won't be included here

function c = bdo_call(n,T,M,B,K)

c = zeros(1,length(K));
% r = 0
S0 = 100;
lambda = 1;

for j = 1:M
    prob = zeros(1,n);
    p = 1;
    [path, tau] = stock_gamma(n,lambda,0,T,S0);
    for i = 1:n
        dt = tau(i+1) - tau(i);
        if dt ~= 0
            prob(i) = -2*max(path(i+1)-B,0)*max(path(i)-B,0) / ...

```

```

                                (dt*sigma(path(i),lambda,0,T)^2);
                p_i = 1 - exp(prob(i));
            else
                p_i = 1;
            end
            p = p * p_i;
        end
        for l = 1:length(K)
            c(l) = c(l) + max(path(end)-K(l),0)*p;
        end
    end

c = c/M;
end

% Plots of MC down-and-out call price calibrated to one maturity
% with BS market price and the errors of MC price

K = 0:0.5:200;
c1 = bdo_call(500,1,1000,90,K);
bs = bsDOCallPrice(100, K, 90, 0, 0, 0.1, 1);
% here we used function bsDOCallPrice in Monte Carlo
% lecture to calculate the market price
figure (1)

plot(K, bs, 'r-');
hold all;
plot(K, c1, 'b-');
xlabel('Strike K'); ylabel('Down and out call price');
title('Calibrated Barrier call prices');
legend('BS price', 'MC price calibrated to one maturities');

figure (2)

err1 = c1 - bs;
plot(K, err1);
xlabel('Strike K'); ylabel('Error');
title('Errors for Barrier call prices');

```

```

% function for MC price of down-and-out barrier call using
% gamma time changed process calibrated to two maturities
% the other time changed diffusion can be done by simply
% changed the input trajectory function
% so it won't be included here

function c = bdo_call_2(n,T1,T2,M,B,K)

c = zeros(1,length(K));
% r = 0
S0 = 100;
lambda = 1;

for j = 1:M
    prob = zeros(1,n);
    p = 1;
    [path1, tau1] = stock_gamma(n,lambda,0,T1,S0);
    pathm = path1(end);
    [path2, tau2] = stock_gamma(n,lambda,T1,T2,pathm);
    path2 = path2(2:end);
    tau2 = tau2(2:end);
    path = [path1, path2];
    tau = [tau1, tau2];
    m = length(tau1);
    for i = 1:n
        dt = tau(i+1) - tau(i);
        if dt ~= 0
            if i <= m
                prob(i) = -2*max(path(i+1)-B,0)*max(path(i)-B,0) / ...
                    (dt*sigma(path(i),lambda,0,T1)^2);
                p_i = 1 - exp(prob(i));
            else
                prob(i) = -2*max(path(i+1)-B,0)*max(path(i)-B,0) / ...
                    (dt*sigma(path(i),lambda,T1,T2)^2);
                p_i = 1 - exp(prob(i));
            end
        else
            p_i = 1;
        end
    end
end

```

```

        p_i = 1;
    end
    p = p * p_i;
end
for l = 1:length(K)
    c(l) = c(l) + max(path(end)-K(l),0)*p;
end
end

c = c/M;
end

% Plots of MC down-and-out call price calibrated to two maturities
% with BS market price and the errors of MC price

K = 0:0.5:200;
c = bdo_call_2(500,1,2,1000,90,K);
c1 = bdo_call(500,2,1000,90,K);
bs = bsDOCallPrice(100, K, 90, 0, 0, 0.1, 2);
% here we used function bsDOCallPrice in Monte Carlo
% lecture to calculate the market price
figure (1)
plot(K, c, 'b-');
hold all;
plot(K, bs, 'k-');
hold all;
plot(K, c1, 'r-');

xlabel('Strike K'); ylabel('Down and out call price');
title('Calibrated Barrier call prices');
legend('MC price calibrated to two maturities', 'BS price', ...
    'MC price calibrated to one maturity');

figure (2)
err = c - bs;
err1 = c1 - bs;
plot(K, err, 'b-');
hold all;

```

```

plot(K, err1 , 'r-');

xlabel('Strike K'); ylabel('Error ');
title('Errors for Barrier call prices ');
legend ('error for MC price calibrated to two maturities ',...
        'error for MC price calibrated to one maturity ');

```