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Non-anomalous non-invertible symmetries in 1+1D from gapped boundaries of SymTFTs

Pavel Putrov[a] **and Rajath Radhakrishnan**[a]

[a] *International Centre for Theoretical Physics, Strada Costiera 11, Trieste 34151, Italy*

E-mail: putrov@ictp.it, rajath.radhakrishnan@maths.ox.ac.uk

Abstract: We study the anomalies of non-invertible symmetries in 1+1D QFTs using gapped boundaries of its SymTFT. We establish the explicit relation between Lagrangian algebras which determine gapped boundaries of the SymTFT, and algebras which determine non-anomalous/gaugeable topological line operators in the 1+1D QFT. If the Lagrangian algebras in the SymTFT are known, this provides a method to compute algebras in all fusion categories that share the same SymTFT. We find necessary conditions that a line operator in the SymTFT must satisfy for the corresponding line operator in the 1+1D QFT to be non-anomalous. We use this constraint to show that a non-invertible symmetry admits a 1+1D trivially gapped phase if and only if the SymTFT admits a magnetic Lagrangian algebra. We define a process of transporting non-anomalous line operators between fusion categories which share the same SymTFT and apply this method to the three Haagerup fusion categories.

Keywords: Anomalies in Field and String Theories, Discrete Symmetries, Field Theories in Lower Dimensions, Topological Field Theories

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1 Introduction

A quantum field theory contains a rich collection of extended operators of various dimensions. Among them, topological operators are a particularly rich, and yet relatively simple set of

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operators which can be often analysed explicitly. These operators play a very important role in constraining the dynamics of a QFT as they implement generalized symmetries [1]. The topological nature of these operators puts strong constraints on them to the extent that a "topological bootstrap" approach can be used to classify them. This reveals that such operators are captured by an algebraic structure called a higher-fusion category [2, 3]. In a Topological Quantum Field Theory, which is a QFT in which all operators are topological, the topological operators have to obey additional constraints like modularity. This has led to the (partial) classification of TQFTs in various dimensions [4–10].

While the topological operators that can appear in a d dimensional QFT are not as constrained as those in a TQFT, it is remarkable that their properties can be captured by a $d+1$ dimensional TQFT called the SymTFT [11–18]. The SymTFT allows us to study topological properties of a QFT with symmetry $\mathcal{C}$ and other QFTs obtained from gauging "sub-symmetries" of $\mathcal{C}$ in a unified setting.

Given a 1+1D QFT with a finite invertible symmetry G and anomaly given by ω in $H^3(G, \mathrm{U}(1))$, a natural question to consider is whether G contains non-anomalous subgroups. A subgroup $H \subset G$ is non-anomalous iff the anomaly ω trivializes on the subgroup. More precisely, H is non-anomalous iff $\omega|_H$ is trivial in $H^3(H, \mathrm{U}(1))$. An alternate approach to this problem is to use the SymTFT of the symmetry G with anomaly ω. This is the twisted Dijkgraaf-Witten theory $\mathrm{DW}(G,\omega)$ determined by the gauge group G and 3-cocycle ω [12, 19]. Non-anomalous subgroups of G are in correspondence with gapped boundaries of this SymTFT. In 1+1D, gauging symmetries can produce dual QFTs. For example, gauging non-anomalous subgroups $K, H \subseteq G$ such that K and H are conjugate to each other are physically equivalent. This defines an equivalence relation on the gaugeable subgroups of G. It is then natural to ask:

> *Given a gapped boundary of the 2+1D SymTFT DW(G,ω) how do we determine the equivalence class of physically equivalent gaugings in 1+1D?*

More generally, if a 1+1D QFT has symmetries described by a fusion category $\mathcal{C}$, then the appropriate generalization of "non-anomalous subgroup" is a line operator A (generically non-simple) which admits the structure of an algebra [20, 21]. This constraint is an algebraic generalization of the anomalies for invertible symmetries. For a 1+1D QFT on a spacetime 2-manfiold Σ gauging a line operator A involves constructing a mesh of A lines on Σ. Since this mesh involves trivalent junctions, we have to make a choice of a point operator at a trivalent junction of A lines (see figure 1). We want to choose μ such that the resulting theory is independent of the choice of the mesh. This is guaranteed if the conditions in figure 2 are satisfied. If there is at least one μ which satisfies these constraints, we can regard A as a non-anomalous line operator as it can be consistently gauged. If there are multiple μ which satisfy these constraints, then the choice of μ can be thought of as a choice of "generalized discrete torsion" on A. In this case the object A admits an algebra structure provided by $\mu : A \times A \to A$. Once again, gauging two algebras A_1 and A_2 might be physically equivalent. In fact, in some cases gauging A might be a self-duality of the QFT. For example, in a 1+1D QFT with symmetries described by the Ising fusion category, gauging $1+\psi$, where ψ is the non-trivial order two line is a self-duality [22, 23]. While gauging $1+\psi$ produces the same

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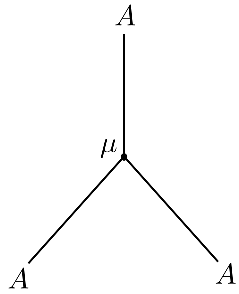


Figure 1. Gauging A involves choosing a point operator μ at a trivalent junction of A lines.

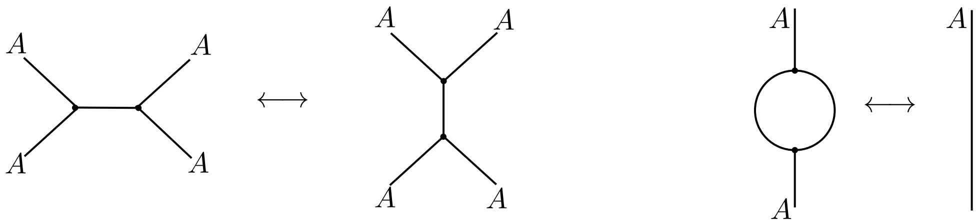


Figure 2. Figure: conditions on μ for the gauging to be consistent.

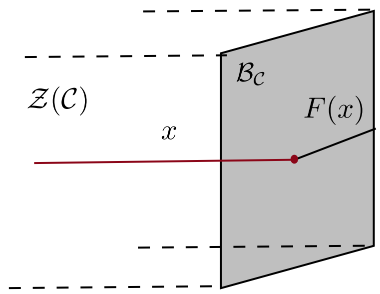


Figure 3. The bulk-to-boundary map F determines the outcome of fusing a bulk line operator on the boundary $\mathcal{B}_{\mathcal{C}}$.

1+1 QFT, it is not a trivial operation. Indeed, gauging $1+\psi$ in the Ising CFT is the famous Krammers-Wannier duality which non-trivially exchanges local operators with twisted-sector operators [22–24]. Therefore, it is important to keep track of physically equivalent gaugings. In this general setting, it is natural to ask:

> *Given a gapped boundary $\mathcal{B}$ of the SymTFT, $\mathcal{Z}(\mathcal{C})$, of a fusion category $\mathcal{C}$ how do we determine the corresponding equivalence class of physically equivalent gaugings in $\mathcal{C}$?*

In this paper, we answer this question by studying the bulk-to-boundary map which tells us how to map line operators in $\mathcal{Z}(\mathcal{C})$ to line operators in $\mathcal{C}$, $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. Physically, $F(x)$ for some line operator $x \in \mathcal{C}$ is the result of perpendicular fusion of x on the gapped boundary $\mathcal{B}_{\mathcal{C}}$ of $\mathcal{Z}(\mathcal{C})$ corresponding to the fusion category $\mathcal{C}$ (see figure 3). We also study a boundary-to-bulk map $K : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ which can be used to determine the gapped boundary of $\mathcal{Z}(\mathcal{C})$ corresponding to a non-anomalous line operator in $\mathcal{C}$.

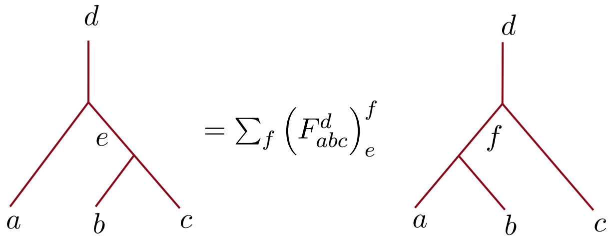
A gapped boundary of $\mathcal{Z}(\mathcal{C})$ is almost completely determined by the line operators which can end on it. In fact, the lines which can end on a gapped boundary form a Lagrangian algebra L [25–27]. Conversely, the Lagrangian algebra L completely determines the gapped boundary. Ending the full Lagrangian algebra object L on the gapped boundary $\mathcal{B}_{\mathcal{C}}$, produces certain line operator $F(L) \in \mathcal{C}$. When $\mathcal{C}$ is a modular tensor category, in [28][29, Proposition 4.3], the authors show that $F(L)$ is an algebra in $\mathcal{C}$. Therefore, $F(L)$ is a non-anomalous line operator in $\mathcal{C}$. We generalize this result and show that for a general fusion category $\mathcal{C}$:

- $F(L)$ is a non-anomalous line operator in $\mathcal{C}$. That is, it can be gauged. (Theorem 4.1)
- The equivalence class of non-anomalous line operators whose gauging is physically equivalent is contained in $F(L)$. (Theorem 4.1)
- The multiplication on an algebra A in $\mathcal{C}$ can be determined from the multiplication on the corresponding Lagrangian algebra L through an explicit formula. (Equation (4.15))

This has various applications:

- If all Lagrangian algebras of a SymTFT are known, then the bulk-to-boundary map F determines all algebras in the fusion category describing the line operators on the chosen boundary. This is an alternative to using NIM-reps to classify algebra objects in a fusion category [30–33].
- The map between Lagrangian algebras in the SymTFT $\mathcal{Z}(\mathcal{C})$ and non-anomalous line operators in $\mathcal{C}$ can be used to determine necessary conditions for a line operator in $\mathcal{C}$ to be non-anomalous. We use this condition to show that $\mathcal{C}$-symmetric trivially gapped phases exist if and only if $\mathcal{Z}(\mathcal{C})$ contains a magnetic Lagrangian algebra with respect to $\mathcal{B}_{\mathcal{C}}$. This relation was first established in [34] through interval compactification of the SymTFT.
- We show that in many cases, the explicit structure of $F(L)$ also allows us to classify line operators in $\mathcal{Z}(\mathcal{C})$ admitting the structure of a Lagrangian algebra in terms of algebras in $\mathcal{C}$. We use this to define the notion of transporting non-anomalous line operators from one fusion category to another which share the same SymTFT.
- When $\mathcal{C}$ is a modular tensor category, both a Lagrangian algebra L in the SymTFT $\mathcal{Z}(\mathcal{C})$ and $F(L)$ have a purely bulk interpretation. In this case, L determines the action of a topological surface operator, say S_L, on the line operators of $\mathcal{C}$. Then $F(L)$ is the equivalence class of algebras which can be higher-gauged to construct the surface S_L.

The plan of the paper is as follows. In section 2 we will briefly review non-invertible symmetries in 1+1D and their gauging. In section 3 we will introduce 2+1D SymTFTs $\mathcal{Z}(\mathcal{C})$ and describe their gapped boundaries in terms of Lagrangian algebras. We will review $\mathcal{C}$-symmetric TQFTs and explain the relation between 1D gapped boundaries of a $\mathcal{C}$-symmetric TQFT and gapped interfaces between gapped boundaries of $\mathcal{Z}(\mathcal{C})$. In section 4 we will explain the explicit map between non-anomalous line operators in $\mathcal{C}$ and gapped-boundaries $\mathcal{Z}(\mathcal{C})$. We begin by proving Theorem 4.1. Then we provide an explicit formula relating the multiplication



$$= \sum_f \left(F^d_{abc}\right)^f_e$$

Figure 4. The associativity of fusion is captured by the F matrices.

on a Lagrangian algebra of $\mathcal{Z}(\mathcal{C})$ and the generalized discrete torsion of the corresponding non-anomalous line operator in $\mathcal{C}$. We then study a boundary-to-bulk map K and give a physical picture of how it can be used to determine the Lagrangian algebra corresponding to an algebra in the boundary. Finally, we will discuss the special case of invertible symmetries in detail and prove Theorem 4.2. In section 5 we discuss various examples involving invertible and non-invertible symmetries. We end this section with an example illustrating how Theorem 4.1 can be used to find Lagrangian algebra objects in $\mathcal{Z}(\mathcal{C})$ using non-anomalous line operators in $\mathcal{C}$. Finally, in section 6 we discuss various applications of our results. We introduce the notion of "transporting algebra objects" between fusion categories and illustrate it using the Haagerup fusion categories. We conclude with some interesting open problems.

2 Review: non-invertible symmetries in 1+1D and their gauging

In this section, we will briefly review non-invertible symmetries of 1+1D QFTs and their gauging. We refer the reader to the excellent exposition of this topic in the references [35, section 5], [36] and [33] for more details.

Symmetries of a 1+1D QFT $\mathcal{Q}$ are implemented by topological line operators.[1] The structure of a finite set of line operators is described by a fusion category $\mathcal{C}$.[2] The simple objects in the category $a, b, c, \ldots$ label simple line operators of the QFT.[3] Their fusion rules are captured by the non-negative integers N_{ab}^c as follows.

$$a \times b = \sum_{c \in \mathcal{C}} N_{ab}^c \; c \,. \tag{2.1}$$

Throughout this paper, the sum $\sum_{c\in\mathcal{C}}$ denotes a sum over the simple objects in $\mathcal{C}$. The fusion of four line operators obeys the Pentagon equations whose solutions are given in terms of the F matrices (see figure 4).

A line operator A in $\mathcal{C}$ is *non-anomalous* if it can be gauged to produce a new QFT $\mathcal{Q}/A$. For this gauging to be consistent, the line operator A must admit the structure of a symmetric separable Frobenius algebra [20, 21]. To explain this structure, suppose A admits the decomposition

$$A = \sum_a N_A^a \; a \,, \tag{2.2}$$

[1] We assume that $\mathcal{Q}$ has a unique vacuum. In this case, there are no non-trivial topological local operators.
[2] See the reviews [37–39].
[3] A line operator is simple if it cannot be decomposed into a sum of other line operators. Equivalently, the only point operator that a simple line operator hosts on it is the identity point operator and its complex multiples.

into simple line operators $a \in \mathcal{C}$, where N_A^a are non-negative integers. The object A admits the structure of an associative algebra if there exist complex numbers

$$\mu_{(a,\alpha),(b,\beta)}^{(c,\gamma),\delta}\,, \tag{2.3}$$

where $\alpha, \beta, \gamma, \delta$ denote indices running from $1, \ldots, N_A^a$; $1, \ldots, N_A^b$; $1, \ldots, N_A^c$ and $1, \ldots, N_{ab}^c$, respectively. These complex numbers should satisfy the following constraint involving the F-symbol:

$$\mu_{ab}^c \mu_{de}^f = \mu_{ag}^f \mu_{be}^g F_{abe}^f\,. \tag{2.4}$$

In writing the above constraint, we have assumed that both the fusion coefficients and the algebra A are multiplicity-free. If there is multiplicity, extra indices need to be added. We will assume that $N_A^{\mathbf{1}} = 1$, where $\mathbf{1}$ is the trivial line operator. In this case, A is called a haploid algebra. A haploid algebra admits a unique symmetric separable algebra structure [40] (see also [36, Footnote 20] and references therein). Therefore, we will not discuss these extra conditions in this review.

Note that a line operator A might admit distinct algebra structures. In other words, there may be more than one inequivalent solution to the constraints (2.4). We will call this a choice of *generalized discrete torsion* for gauging A. This is a generalization of the fact that when a non-anomalous line operator A is of the form

$$A = \sum_{g \in G} g\,, \tag{2.5}$$

for some group G, then the distinct multiplications on A are classified by the discrete torsion $H^2(G, \mathrm{U}(1))$.

Two distinct algebras A_1 and A_2 might correspond to physically equivalent gauging procedures leading to dual QFTs $\mathcal{Q}/A_1$ and $\mathcal{Q}/A_2$. This is captured by Morita equivalence of algebras.[4] We will denote a Morita equivalence class with a representative algebra A as $[A]$. Note that every Morita equivalent class has a haploid representative [41, section 3.3]. Therefore, the condition $N_A^{\mathbf{1}} = 1$ can be imposed without loss of generality. In the following sections, we will use the terms "non-anomalous line operator" and "algebra object" interchangeably.

3 SymTFTs and $\mathcal{C}$-symmetric TQFTs

3.1 SymTFTs

Let $\mathcal{C}$ be a finite subcategory of symmetries of a 1+1D QFT $\mathcal{Q}$. The SymTFT of $\mathcal{C}$ is the Turaev-Viro-Barrett-Westbury 2+1D TQFT described by the Drinfeld centre $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ [11–18, 42, 43]. The simple line operators in $\mathcal{Z}(\mathcal{C})$ can be written in the form

$$(a, e_a)\,, \tag{3.1}$$

where a is a (generically non-simple) line operator in $\mathcal{C}$ and e_a are isomorphisms called half-braidings

$$e_a(b) : a \times b \xrightarrow{\sim} b \times a\,,\ \forall\, b \in \mathcal{C}\,, \tag{3.2}$$

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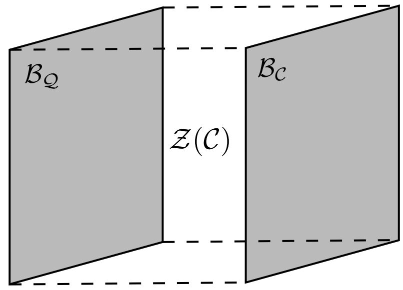

Figure 5. The SymTFT $\mathcal{Z}(\mathcal{C})$ with two boundary conditions. The boundary $\mathcal{B}_{\mathcal{C}}$ is gapped while the $\mathcal{B}_{\mathcal{Q}}$ is generically gappless. The category of line operators on $\mathcal{B}_{\mathcal{C}}$ is $\mathcal{C}$.

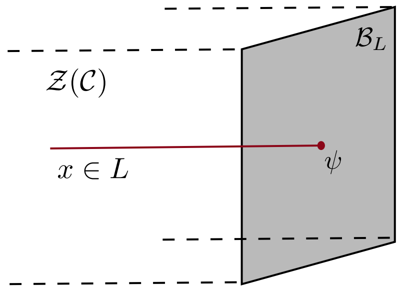

Figure 6. A simple line operator $x \in L$ can end on the gapped boundary $\mathcal{B}_L$.

satisfying several consistency conditions (see, for example, [44, 45]). The SymTFT allows us to separate the data of the symmetry $\mathcal{C}$ from the 1+1D QFT $\mathcal{Q}$ on which it is acting (see figure 5).

Given the full data of the fusion category $\mathcal{C}$, the line operators and modular data of $\mathcal{Z}(\mathcal{C})$ can be computed using the string-net model [46, 47]. We refer the readers to the review in [48, appendix A] for more details.

3.1.1 Lagrangian algebras and gapped boundaries

A gapped boundary of the SymTFT $\mathcal{Z}(\mathcal{C})$ is completely determined by a Lagrangian algebra L [25–27]. Physically, a Lagrangian algebra structure on L can be understood as follows. L is a line operator in $\mathcal{Z}(\mathcal{C})$ and can be written as

$$L = \sum_x N_L^x \, x \,, \tag{3.3}$$

where x are simple line operators and N_L^x are non-negative integers. If $N_L^x \neq 0$, then x can end on the gapped boundary $\mathcal{B}_L$ (see figure 6).

We will focus on simple gapped boundaries on which the only non-trivial point operator is the identity operator and its complex multiples.[5] Therefore, the trivial line operator in the bulk should end on the gapped boundary in a unique way. In other words, we require that L has exactly one copy of the trivial line, $N_L^{\mathbf{1}} = 1$, where $\mathbf{1}$ is the trivial line operator. The line operator L admits the structure of a commutative and associative algebra. This

[4]See [20, 29, 33] for more details.

[5]A general gapped boundary can be written as a sum of simple gapped boundaries.

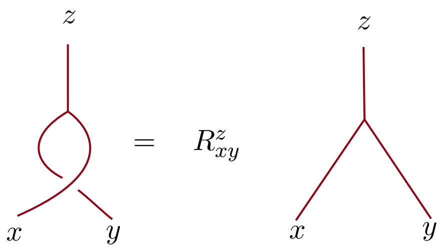

Figure 7. Braiding of line operators is captured by the R matrix.

algebra structure is determined by the following complex numbers

$$m^{(z,k),l}_{(x,i),(y,j)} \tag{3.4}$$

where i, j, k, l denote indices running from $1, \ldots, N^x_L$; $1, \ldots, N^y_L$; $1, \ldots, N^z_L$ and $1, \ldots, N^z_{xy}$, respectively. When the SymTFT $\mathcal{Z}(\mathcal{C})$ has fusion coefficients valued in $N^z_{xy} \in \{0, 1\}$, we will denote the multiplication coefficients in the algebra L as $m^{(z,k)}_{(x,i),(y,j)}$. Moreover, if $N^x_L \in \{0, 1\}$, we will denote it as m^z_{xy}. Since L is an associative algebra, the complex numbers m^z_{xy} must satisfy the constraint (2.4). Recall that a haploid algebra admits a unique symmetric separable Frobenius algebra structure on it. The line operators in $\mathcal{Z}(\mathcal{C})$ can braid with each other. It is given by the R matrix (see figure 7).

For L to be a commutative algebra, the multiplication on L must be compatible with the R matrix. We have the constraint

$$m^z_{xy} = m^z_{yx} R^z_{xy} \,. \tag{3.5}$$

In writing this equation, we have assumed that $\mathcal{Z}(\mathcal{C})$ and L are multiplicity-free. Finally, a haploid associative and commutative algebra is called Lagrangian if it satisfies

$$\dim(L) = \sum_x N^x_L d_x = \sqrt{\dim(\mathcal{Z}(\mathcal{C}))} = \dim(\mathcal{C}) \,. \tag{3.6}$$

It means that the boundary condition $\mathcal{B}_L$ is defined by gauging a maximal set of lines in $\mathcal{Z}(\mathcal{C})$. This condition ensures that all simple line operators which are not in L are confined by the gauging process, rendering the gauged theory trivial. Mathematically, the above condition ensures that the category of local modules of L in $\mathcal{Z}(\mathcal{C})$ is Vec. A line operator L admitting the structure of a Lagrangian algebra satisfies [49, 50]

$$N^x_L \, N^y_L \leq \sum_z N^z_{xy} \, N^z_L, \quad \text{and} \tag{3.7}$$

In fact, a line operator L with multiplication specified by the set of complex numbers $m^{(z,k),l}_{(x,i),(y,j)}$ is a Lagrangian algebra if and only if the constraints (2.4), (3.5), (3.6) and (3.7) are satisfied [49, Corollary 3.8].[6] Note that the conditions (3.7) depend only on the fusion rules of $\mathcal{Z}(\mathcal{C})$ and a choice of the line operator L. However, they are not a set of sufficient conditions for L to be Lagrangian. Indeed, in [51], the author finds that the object $L = \sum_{a \in \mathcal{C}} (a, a)$

[6]Strictly speaking, this result requires unitarity. We will assume that the fusion category is unitary throughout this paper.

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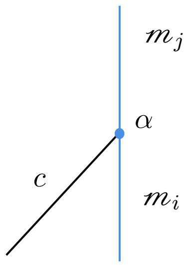

Figure 8. Action of $\mathcal{C}$ on 1D gapped boundaries.

in $\mathcal{C} \times \bar{\mathcal{C}}$ does not always admit the structure of a Lagrangian algebra. However, it clearly satisfies the contraints in (3.7). (3.7) can be used to enumerate a set of candidate Lagrangian algebras for which we can try to solve (2.4), (3.7). However, this requires the knowledge of the full F and R matrices of the SymTFT $\mathcal{Z}(\mathcal{C})$. In sections 5.4 and 6, as a consequence of our main result, we will consider an alternate method to determine line operators in the bulk SymTFT which are guaranteed to admit a Lagrangian algebra structure using non-anomalous line operators on a gapped boundary.

3.2 $\mathcal{C}$-symmetric TQFTs

In this section, we will briefly review the classification of $1+1$D $\mathcal{C}$-symmetric TQFTs in terms of module categories over $\mathcal{C}$. The topological local operators in a 1+1D TQFT form an algebra, E, where the multiplication on E captures the O.P.E. of the local operators. In fact, the local operators form a commutative Frobenius algebra [52–54]. Arbitrary correlation functions of local operators on any closed 2-manifold can be determined using E. In the following, we will assume that E is a semi-simple algebra. The topological line operators in the TQFT are labeled by bimodules of the Frobenius algebra, $\text{BiMod}(E)$ (for example, see [55, section 2.4.1]).[7]

A $\mathcal{C}$-symmetric TQFT is determined by a functor

$$\mathcal{C} \to \text{BiMod}(E)\,, \tag{3.8}$$

which assigns line operators in $\mathcal{C}$ to line operators in $\text{BiMod}(E)$ in a consistent way [57]. In other words, we must be able to identify line operators in $\text{BiMod}(E)$ which form the chosen fusion category $\mathcal{C}$. Let $\mathcal{M}$ be the category of boundary conditions of a 1+1D TQFT with simple objects $\{m_1, \ldots m_n\} \in \mathcal{M}$ labelling the distinct boundary conditions. Then there is an action of the fusion category $\mathcal{C}$ on $\mathcal{M}$ given by

$$c \times m_i = \sum_j N_{ci}^j \; m_j\,, \tag{3.9}$$

where N_{ci}^j are non-negative integers. If $N_{ci}^j \neq 0$, we get figure 8 where α labels a basis for the local operators at the junction of c, m_i and m_j.

It is known that the full defining data of a 1+1D $\mathcal{C}$-symmetric TQFT can be obtained from how $\mathcal{C}$ acts on the set of boundary conditions [32, 35, 54, 58]. In fact, $\mathcal{M}$ is a module

[7] $\text{BiMod}(E)$ is a multifusion category [56] which captures the structure of both line operators and local operators of the TQFT.

category over $\mathcal{C}$. All physical data in the $\mathcal{C}$-symmetric TQFT can be computed using the data of this module category.

$$\mathcal{C}\text{-module category} \leftrightarrow 1{+}1\text{D } \mathcal{C}\text{-symmetric TQFT}\,. \tag{3.10}$$

In fact, a $\mathcal{C}$-symmetric 1+1D TQFT which cannot be written as a direct sum of $\mathcal{C}$-symmetric TQFTs are determined by an indecomposable module category. Since we will only consider semi-simple TQFTs, the $\mathcal{C}$-module category must also be finitely semi-simple. An indecomposable finitely semi-simple $\mathcal{C}$-module category is in turn completely determined by a simple algebra A in $\mathcal{C}$ [41] (see also [33, 36]).

$$\text{algebra } A \text{ in } \mathcal{C} \longleftrightarrow \mathcal{C}\text{-module category } \mathcal{M} \longleftrightarrow 1+1\text{D } \mathcal{C}\text{-symmetric TQFT}\,. \tag{3.11}$$

In other words, every gaugeable line operator A in $\mathcal{C}$ determines a $\mathcal{C}$-module category which in turn determines a 1+1D $\mathcal{C}$-symmetric TQFT. Given an algebra A, the module category $\mathcal{M}$ is the category of modules of A in $\mathcal{C}$. Note that the module category only depends on the Morita equivalence class $[A]$ of the algebra A. Conversely, given a module category $\mathcal{M}$, $[A]$ can be determined as follows. Choose some simple object $m \in \mathcal{M}$, then a line operator a can form a junction on the gapped boundary m if and only if $a \times m$ contains m. Let us define the non-simple line operator

$$A_m := \sum_{a\in\mathcal{C}} N^a_m \; a\,, \tag{3.12}$$

where the sum is over the simple objects in $\mathcal{C}$ and N^a_m is the dimension of the Hilbert space of operators at the junction of a with the gapped boundary m. The object A_m is called the Internal Hom, and it admits the structure of a haploid symmetric separable Frobenius algebra [41]. In other words, A_m is non-anomalous and can be gauged. If we choose a different boundary condition in $\mathcal{M}$, we get a Morita equivalent haploid symmetric separable Frobenius algebra.

3.2.1 $\mathcal{C}$-symmetric TQFTs from SymTFTs

In this section, we will review the construction of 1+1D $\mathcal{C}$-symmetric TQFTs from the SymTFT $\mathcal{Z}(\mathcal{C})$. Most of this section is based on the papers [34, 59–61], except for the last part, where we explain how to obtain the gapped boundaries of the 1+1D TQFT from the SymTFT.

Consider the sandwich picture in figure 9. Interval compactification of this picture results in a 1+1D TQFT $\mathcal{T}_L$ with $\mathcal{C}$-symmetry.[8] When $\mathcal{B}_L = \mathcal{B}_\mathcal{C}$, we get the regular $\mathcal{C}$-symmetric TQFT defined in [35] in which the symmetry $\mathcal{C}$ is completely spontaneously broken. For other choices of $\mathcal{B}_L$ we may get TQFTs where $\mathcal{C}$ is partially or fully preserved.

Various crucial properties of the TQFT can be deduced from this picture [34, 59, 60]. For example, the number of local operators in the resulting TQFT can be counted by looking at

[8] Similar procedures can be used to obtain TQFTs in higher-dimensions [48, 62]. For example, interval compactification of the Crane-Yetter TQFT with two fixed gapped boundaries results in a 2+1D TQFT. This played a crucial role in proving that all Spin TQFTs admit a modular extension [63]. See also [64].

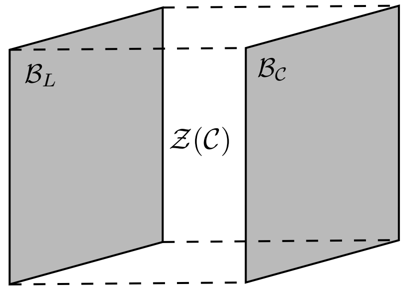

Figure 9. SymTFT with two gapped boundaries $\mathcal{B}_L$ and $\mathcal{B}_\mathcal{C}$ can be compactified to get a $\mathcal{C}$-symmetric TQFT.

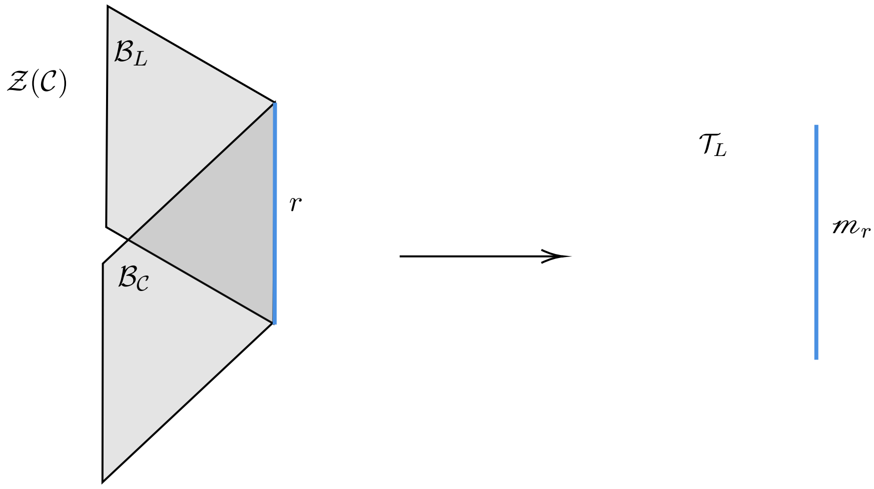

Figure 10. A 1D gapped interface between the 2D gapped boundaries $\mathcal{B}_L$ and $\mathcal{B}_\mathcal{C}$ turns into a 1D gapped boundary of the 1+1D $\mathcal{C}$-symmetric TQFT $\mathcal{T}_L$. Mathematically, the boundary conditions of a 2+1D TQFT of the form $\mathcal{Z}(\mathcal{C})$ form a 2-category. In [65], it is shown that this is the 2-category of $\mathcal{C}$-modules.

the line operators in $\mathcal{Z}(\mathcal{C})$ which can end on both $\mathcal{B}_L$ and $\mathcal{B}_\mathcal{C}$. More precisely, the number of local operators in the 1+1D TQFT is given by

$$\sum_{x\in\mathcal{Z}(\mathcal{C})} N_\mathcal{C}^x\ N_L^x\,, \tag{3.13}$$

where $N_\mathcal{C}^x$ counts the number of copies of x in the canonical Lagrangian algebra of $\mathcal{Z}(\mathcal{C})$ corresponding to the gapped boundary $\mathcal{B}_\mathcal{C}$. Moreover, further details of $\mathcal{T}_L$ including how the symmetry $\mathcal{C}$ is spontaneously broken, the action of the symmetry on the vacua and order parameters can be determined using this picture [59, 60].

The gapped boundaries of $\mathcal{T}_L$ can also be determined from the sandwich picture. To that end, we consider the configuration in figure 10 where r is a simple gapped interface between the gapped boundaries $\mathcal{B}_L$ and $\mathcal{B}_\mathcal{C}$. Compactifying this diagram gives $\mathcal{T}_L$ where the gapped interface r becomes a 1D gapped boundary m_r of $\mathcal{T}_L$.

In section 3.2, we described how the line operators in $\mathcal{C}$ act on the 1D gapped boundaries of $\mathcal{C}$. This can be determined from the SymTFT as follows. Consider a bulk line operator x in the Lagrangian algebra L. The line x can end trivially on the gapped boundary $\mathcal{B}_L$.

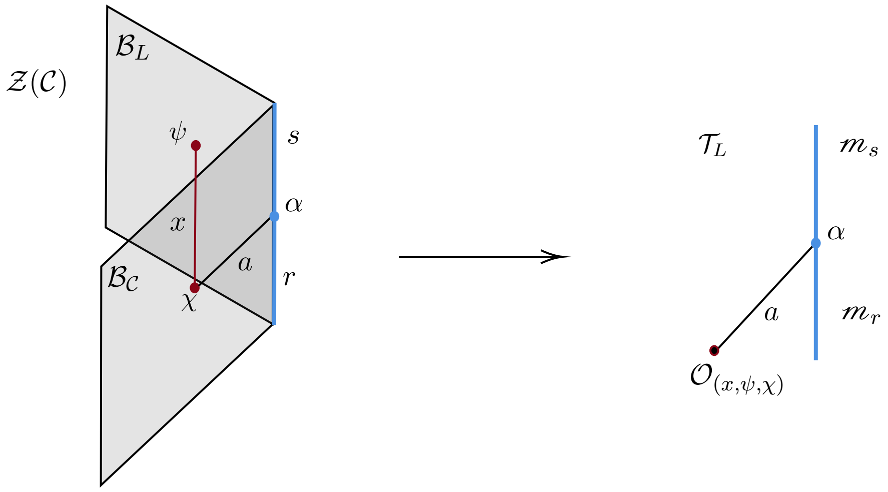

Figure 11. The action of $a \in \mathcal{C}$ on the 1D gapped boundaries of $\mathcal{T}_L$ can be determined by the action of $a \in \mathcal{C}$ on gapped interfaces in the SymTFT. Upon collapsing the wedge on the left, the line operator x becomes a twisted-sector local operator $\mathcal{O}_{(x,\psi,\chi)}$.

Generically, x cannot end on the gapped boundary $\mathcal{B}_\mathcal{C}$ but forms a junction with a line operator $a \in \mathcal{C}$. The action of a on the gapped boundary m_r of the 1+1D TQFT $\mathcal{T}_L$ is determined by the action of a on the corresponding gapped interface r in the SymTFT as in figure 11.

4 Gaugeable line operators in 1+1D and SymTFTs

Consider the symmetries of a 1+1D QFT described by a fusion category $\mathcal{C}$. Given a line operator A in the category $\mathcal{C}$, how do we determine if it is anomalous? One approach is to determine whether A admits the structure of an algebra. This involves solving the constraints (2.4) which requires the F matrices of the fusion category $\mathcal{C}$.

In this section, we will consider an alternate approach by studying gapped boundaries of the SymTFT $\mathcal{Z}(\mathcal{C})$. An important defining property of the SymTFT is that physically distinct gaugings in $\mathcal{C}$ are in one-to-one correspondence with gapped boundaries of $\mathcal{Z}(\mathcal{C})$ [65][66, Proposition 4.8].

$$[A] \in \mathcal{C} \longleftrightarrow L \in \mathcal{Z}(\mathcal{C})\,. \tag{4.1}$$

In the rest of this section, we will study this relation explicitly.

4.1 Non-anomalous line operators from SymTFT

In this section, we will explore the one-to-one correspondence between gapped boundaries of $\mathcal{Z}(\mathcal{C})$ and non-anomalous line operators in $\mathcal{C}$ from a physical perspective. We will describe various properties of this correspondence, which can be used to answer the following question: given a Lagrangian algebra L of $\mathcal{Z}(\mathcal{C})$, which Morita equivalence class of gaugeable algebra $[A]$ in $\mathcal{C}$ does it correspond to? To answer this question, let us consider the perpendicular fusion of a line operator l on a gapped boundary $\mathcal{B}$

$$F_\mathcal{B}(x) = \sum_a N^a_{F_\mathcal{B}(x)}\, a\,, \tag{4.2}$$

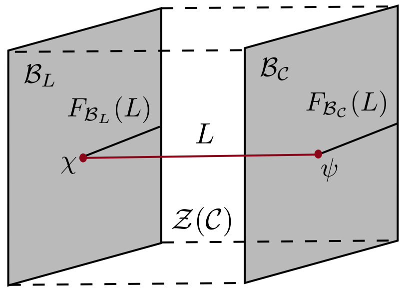

Figure 12. Fusion of line operators in $\mathcal{Z}(\mathcal{C})$ on gapped boundaries.

where the line operators a live on the gapped boundary $\mathcal{B}$ and N_l^a are non-negative integers. In general, $F_\mathcal{B}(x)$ is part of the data of a bulk-to-boundary map which relates the line operators and fusion spaces of the bulk TQFT $\mathcal{Z}(\mathcal{C})$ to those of the category of line operators on the gapped boundary $\mathcal{B}$. Consider the canonical gapped boundary $\mathcal{B}_\mathcal{C}$ of $\mathcal{Z}(\mathcal{C})$ on which the line operators form the fusion category $\mathcal{C}$. In this case, we will denote $F_{\mathcal{B}_\mathcal{C}}$ as F to simplify notation.[9] The action of F on the simple line operators of $\mathcal{Z}(\mathcal{C})$ has a simple description. Recall that simple line operators in $\mathcal{Z}(\mathcal{C})$ can be written as

$$(a, e_a)\,, \;\; a \in \mathcal{C} \tag{4.3}$$

where e_a are half-braidings. In this notation, F is given by[10]

$$F((a, e_a)) = a\,. \tag{4.4}$$

Since F determines the relation between bulk and boundary line operators, it is perhaps unsurprising that it can be used to determine the relation between Lagrangian algebras in $\mathcal{Z}(\mathcal{C})$ and non-anomalous line operators in $\mathcal{C}$. Indeed, consider a Lagrangian algebra L in $\mathcal{Z}(\mathcal{C})$. Consider the perpendicular fusion of L with the gapped boundary $\mathcal{B}_\mathcal{C}$, given by

$$F(L) = \sum_a N^a_{F(L)}\; a\,, \quad a \in \mathcal{C}\,. \tag{4.5}$$

We will show that $F(L)$ is a sum of algebra objects in $\mathcal{C}$ which correspond to physically equivalent gaugings. In other words, $F(L)$ is a sum of Morita-equivalent algebras. To see this recall that $\mathcal{T}_L$ is the 1+1D TQFT obtained from the interval compactification of the SymTFT $\mathcal{Z}(\mathcal{C})$ with gapped boundaries $\mathcal{B}_\mathcal{C}$ and $\mathcal{B}_L$. Consider the $\mathcal{C}$-module category $\mathcal{M}_L$ describing the 1D gapped boundaries of $\mathcal{T}_L$. Recall the definition of the algebra object A_m in section 3.2. We will prove the following theorem.

Theorem 4.1. *Let L be a Lagrangian algebra in the SymTFT $\mathcal{Z}(\mathcal{C})$. Assume that interval compactification of the configuration in figure 13 for different ψ produces all twisted sector local operators at the end of the line operator a in $\mathcal{T}_L$. As an object in $\mathcal{C}$*

$$F(L) = \bigoplus_{m \in \mathcal{M}_L} A_m\,. \tag{4.6}$$

[9]This notation is not to be confused with the associator of $\mathcal{C}$ given by the F matrices.
[10]This is called a forgetful functor from $\mathcal{Z}(\mathcal{C})$ to $\mathcal{C}$ since it forgets about the half-braidings e_a. More generally, for any gapped boundary $\mathcal{B}$, $F_\mathcal{B}$ is a tensor functor [67, section V. B].

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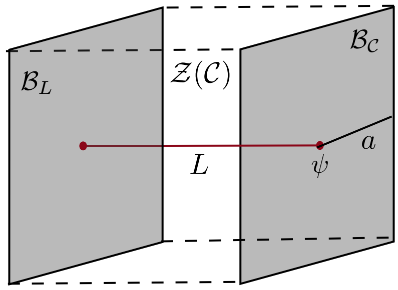


Figure 13. Junctions between L and a on the gapped boundary $\mathcal{B}_\mathcal{C}$ correspond to twisted sector states in the Hilbert space $\mathcal{H}_a$ of the TQFT $\mathcal{T}_L$.

Proof: consider some simple line operator a such that $N^a_{F(L)} \neq 0$. Then, the configuration in figure 13, is non-trivial. Compactifying this diagram gives $N^a_{F(L)}$ operators at the end of the line operator a in the TQFT $\mathcal{T}_L$.[11] Now, using the state-operator correspondence we find that $\dim(\mathcal{H}_a) = N^a_{F(L)}$, where $\mathcal{H}_a$ is the twisted Hilbert space for the line operator a. The Hilbert space $\mathcal{H}_a$ can be determined from the data of the boundary conditions of the TQFT given by the module category $\mathcal{M}_L$. We have the following isomorphism [35, section 3.1]

$$\mathcal{H}_a \cong \bigoplus_{m \in \mathcal{M}_L} \mathrm{Hom}_{\mathcal{M}_L}(m, a \times m)\,. \tag{4.7}$$

This isomorphism stems from introducing a disc labelled by the boundary m in the TQFT. Assuming that $m \in a \times m$, we can attach the line a to this boundary. Shrinking this disc to a point gives a twisted-sector local operator corresponding to a state in $\mathcal{H}_a$. This defines a map from $\bigoplus_{m \in \mathcal{M}_L} \mathrm{Hom}_{\mathcal{M}_L}(m, a \times m)$ to $\mathcal{H}_a$. The authors of [35] show that this map is a bijection. Note that $\mathrm{Hom}_{\mathcal{M}_L}(m, a \times m)$ is non-empty if and only if $a \in A_m$. Indeed,

$$|\mathrm{Hom}_{\mathcal{M}_L}(m, a \times m)| = N^a_m \tag{4.8}$$

where, N^a_m is the number of copies of a in A_m (see definition (3.12)). Therefore,

$$\sum_{m \in \mathcal{M}_L} N^a_m = \sum_{m \in \mathcal{M}_L} |\mathrm{Hom}_{\mathcal{M}_L}(m, a \times m)| = \dim(\mathcal{H}_a) = N^a_{F(L)}\,. \tag{4.9}$$

Therefore, we find that there are $N^a_{F(L)}$ copies of the line operator a in $\bigoplus_{m \in \mathcal{M}_L} A_m$. □

When $\mathcal{C}$ is a modular tensor category, Theorem 4.1 follows from [28][29, Proposition 4.3]. Recall from the discussion in section 3.2 that each A_m admits the structure of a haploid symmetric separable Frobenius algebra for each m. Therefore, their direct sum also admits the structure of a symmetric separable Frobenius algebra (for example, see [40, section 3.5]). Hence, $F(L)$ admits the structure of a non-anomalous line operator in $\mathcal{C}$. It is clear from (4.6) that, in general, the number of copies of the identity line $\mathbf{1}$ in $F(L)$ is equal to the number of distinct boundary conditions of the TQFT $\mathcal{T}_L$ since A_m has exactly one identity operator for every $m \in \mathcal{M}_L$. Therefore, in general, the algebra $F(L)$ is not haploid. In order to

[11] One could wonder whether the choice of the junction of L with the gapped boundary $\mathcal{B}_L$ affects this counting. Note that this is already taken into account in the argument since the number of possible point junctions between a line operator $x \in L$ and the gapped boundary $\mathcal{B}_L$ is precisely N^x_L. The Lagrangian algebra object L contains the term $N^x_L\, x$, and on acting with F we get $N^x_L F(x)$.

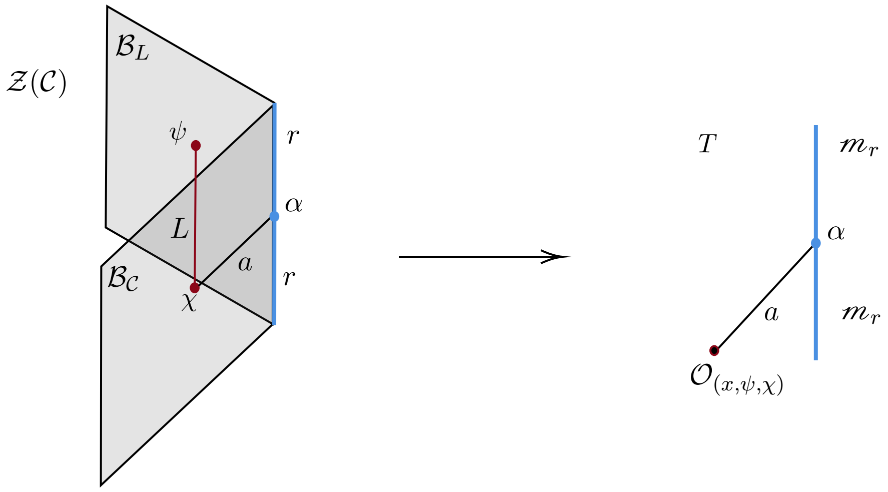


Figure 14. a is a line operator in $\mathcal{C}$ which can form a non-trivial junction on the gapped interface r. On closing the wedge, we find that a can form a junction on the corresponding gapped boundary $m_r \in \mathcal{M}_L$.

identify the haploid subalgebras in $F(L)$, we have to consider the action of the simple line operators in $F(L)$ on gapped interfaces between gapped boundaries $\mathcal{B}_\mathcal{C}$ and $\mathcal{B}_L$ as in figure 11. Choose a simple gapped interface r between the gapped boundaries $\mathcal{B}_\mathcal{C}$ and $\mathcal{B}_L$.[12] Define A_r to be the sub-object/line operator of $F(L)$

$$A_r := \sum_{a \,\in\, F(L)} N_r^a \; a \,, \tag{4.10}$$

where N_r^a is the dimension of the Hilbert space of operators at the junction of a with the gapped interface r (see figure 14). Compactifying this diagram, we clearly have that

$$A_r = A_{m_r} \,, \tag{4.11}$$

where $m_r \in \mathcal{M}_L$ is the 1D gapped boundary of the 1+1D TQFT $\mathcal{T}_L$ corresponding to the gapped interface r.

Suppose $L_1, \ldots, L_n$ is the full set of Lagrangian algebras in the SymTFT $\mathcal{Z}(\mathcal{C})$. Then, $F(L_1), \ldots, F(L_n)$ is the full set of equivalence classes of algebras in $\mathcal{C}$. Therefore, Theorem 4.1 provides an alternative method to using NIM-reps to classify algebra objects in a fusion category [30–33]. In fact, as will explore in detail in section 6.3, if the SymTFT $\mathcal{Z}(\mathcal{C})$ and its Lagrangian algebras are known, then Theorem 4.1 can be used to find algebra objects in all fusion categories which share the same SymTFT $\mathcal{Z}(\mathcal{C})$.

4.2 Generalized discrete torsion from SymTFT

In the previous section, we studied the structure of $F_{\mathcal{B}_\mathcal{C}}(L)$ in terms of the algebras A_m, $m \in \mathcal{M}_L$. This discussion only used the fact that L is a line operator in the SymTFT $\mathcal{Z}(\mathcal{C})$ which admits the structure of a Lagrangian algebra. In particular, so far, we have not used the explicit algebra structure on L. In this section, we will discuss how to determine the explicit algebra structure on $F_{\mathcal{B}_\mathcal{C}}(L)$ from that on L.

[12] A gapped interface is simple if it does not host any non-trivial point operators on it.

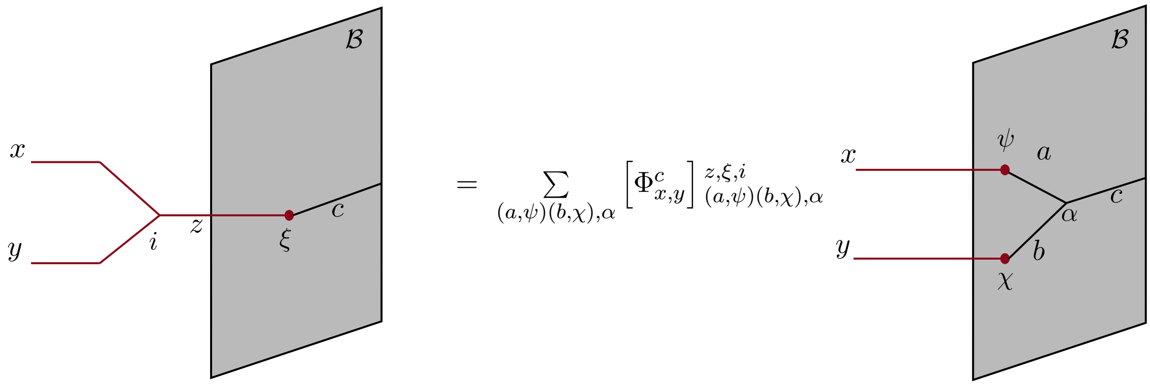


Figure 15. The complex numbers $\left[\Phi^{c}_{x,y}\right]^{z,\xi,i}_{(a,\psi)(b,\chi),\alpha}$ relate the fusion spaces of $\mathcal{Z}(\mathcal{C})$ and $\mathcal{B}$.

Recall the bulk-to-boundary map

$$F_{\mathcal{B}} : \mathcal{Z}(\mathcal{C}) \to \mathcal{B}\,, \tag{4.12}$$

introduced in (4.2) where we will use $\mathcal{B}$ to denote both a gapped boundary as well as the fusion category of line operators living on the gapped boundary. In order to describe how the full data of an algebra in $\mathcal{Z}(\mathcal{C})$ is mapped to $\mathcal{B}$ under $F_{\mathcal{B}}$, we have to determine how the fusion spaces in $\mathcal{Z}(\mathcal{C})$ get mapped to fusion spaces in $\mathcal{B}$. This is determined by an isomorphism

$$\Phi_{x,y} : F_{\mathcal{B}}(x) \times F_{\mathcal{B}}(y) \to F_{\mathcal{B}}(x \times y)\,. \tag{4.13}$$

By choosing a basis, we can write these isomorphisms explicitly in terms of some complex numbers as given in figure 15. The choice of isomorphisms $\Phi_{x,y}$ must be compatible with the associativity of fusion rules in both $\mathcal{Z}(\mathcal{C})$ and $\mathcal{B}$. This implies that $\Phi_{x,y}$ must satisfy the constraints in figure 16. The isomorphisms Φ make $F_{\mathcal{B}}$ a strong monoidal functor (see, for example [68, 69]).

Consider a Lagrangian algebra L in $\mathcal{Z}(\mathcal{C})$. Let $m : L \times L \to L$ be the multiplication on L. By choosing a basis, we can write the multiplication m explicitly as in figure 17. A multiplication on $F(L)$ is determined by the complex numbers as defined in figure 18. In fact, $F(L)$ is an associative algebra in $\mathcal{B}$ given by the multiplication

$$\mu : m \circ \Phi\,. \tag{4.14}$$

By choosing a basis, this can be explicitly depicted as in the figure 19. Comparing figure 18 and figure 19, we get the equation

$$\mu^{(c,(\xi,z,k))}_{(a,(\psi,x,i)),(b,(\chi,y,j)),\delta} = \sum_{l} m^{(z,k)}_{(x,i),(y,j),l} \left[\Phi^{c}_{x,y}\right]^{z,\xi,l}_{(a,\psi)(b,\chi),\delta}\,. \tag{4.15}$$

We are in particular interested in the perpendicular fusion of line operators on the canonical gapped boundary $\mathcal{B}_{\mathcal{C}}$. In this case, we have the forgetful functor $F \equiv F_{\mathcal{B}_{\mathcal{C}}}$

$$F((a, e_a)) = a. \tag{4.16}$$

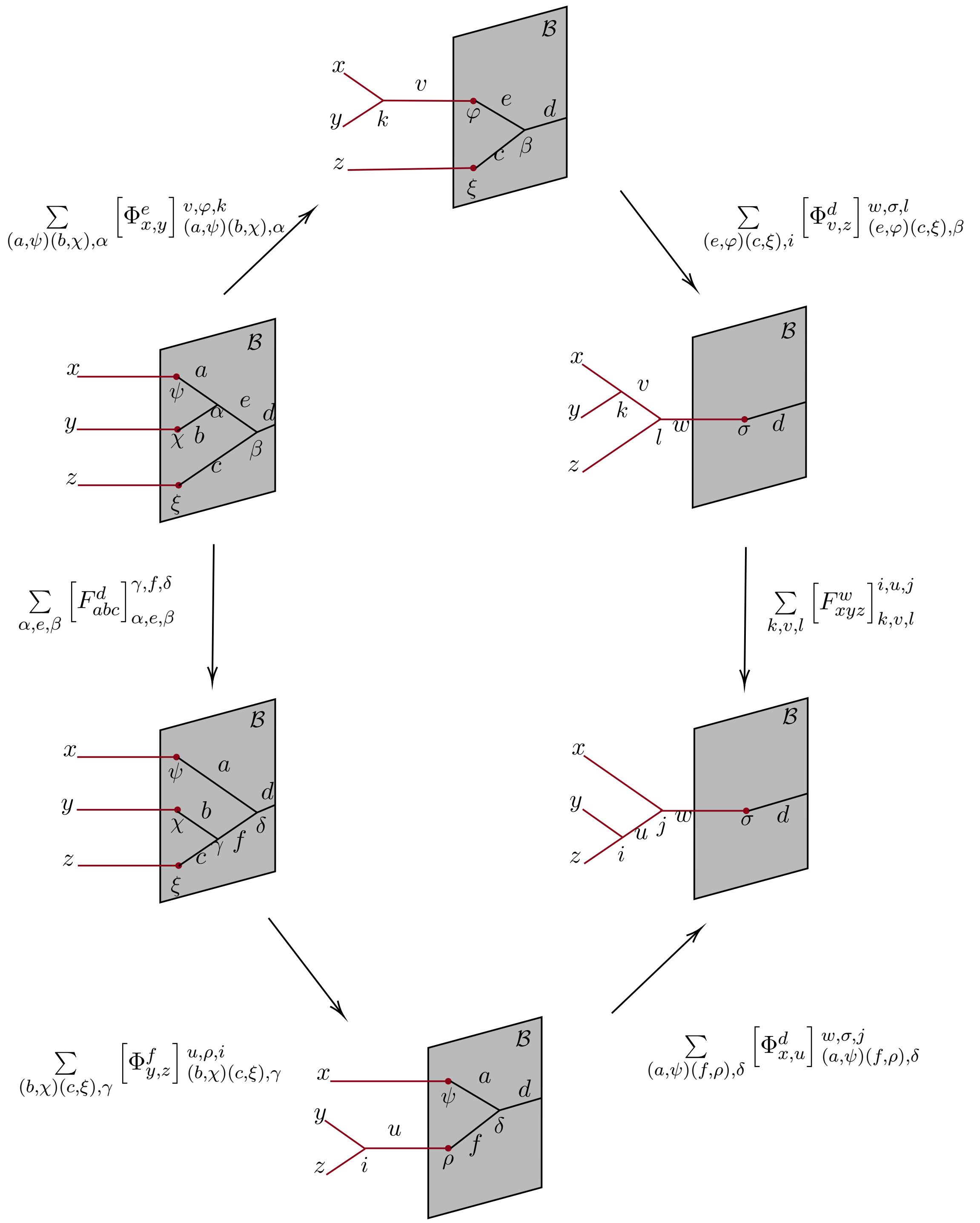


Figure 16. Associativity of fusion rules in $\mathcal{Z}(\mathcal{C})$ and $\mathcal{B}_L$ implies that the above diagram must commute.

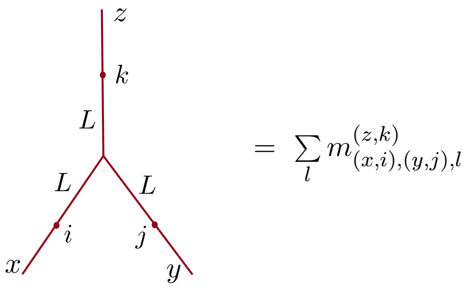


Figure 17. The multiplication on L is determined by the complex numbers $m^{(z,k)}_{(x,i),(y,j),l}$.

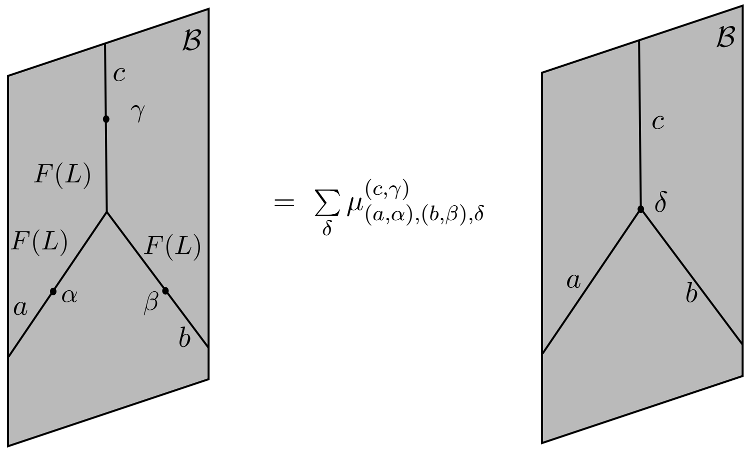


Figure 18. A multiplication on $F(L)$ is determined by the complex numbers $\mu^{(c,\gamma)}_{(a,\alpha),(b,\beta),\delta}$.

Therefore, we have

$$F((a,e_a)) \times F((b,e_b)) = a \times b\,, \tag{4.17}$$

$$F((a,e_a) \times (b,e_b)) = F((a \times b, e_{a\times b})) = a \times b\,, \tag{4.18}$$

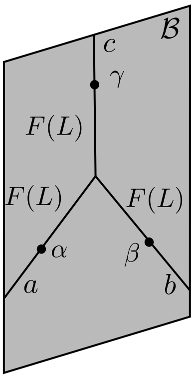
and the isomorphisms $\Phi_{x,y}$ thus can be chosen to be trivial. In other words, there exists a basis in which the complex numbers $\left[\Phi^c_{x,y}\right]^{z,\xi,k}_{(a,\psi)(b,\chi),\alpha}$ are all equal to 1. This significantly simplifies the expression (4.15).

Remark: the discussion above implies that the bulk-to-boundary map F can also be used to transport non-Lagrangian algebras in $\mathcal{Z}(\mathcal{C})$ to $\mathcal{C}$. If E is a general algebra in $\mathcal{Z}(\mathcal{C})$, then (4.15) specifies a multiplication on $F(E)$. In general, $F(E)$ is not haploid.

4.3 Boundary-to-bulk map: from $A \in \mathcal{C}$ to $L \in \mathcal{Z}(\mathcal{C})$

Given an algebra A with multiplication μ in $\mathcal{C}$ the corresponding Lagrangian algebra with object L and multiplication m in $\mathcal{Z}(\mathcal{C})$ can be found through a two-step process. Let $K : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ be the boundary-to-bulk map whose action on the line operators is given by

$$K(a) := \bigoplus_{(x,e_x)\in\mathcal{Z}(\mathcal{C})} N^x_a(x,e_x) \tag{4.19}$$



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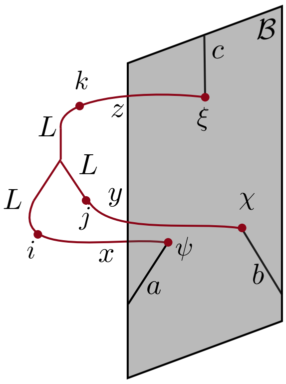
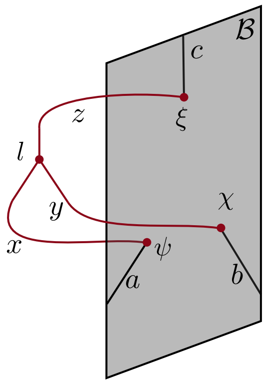


$= \sum_l m^{(z,k)}_{(x,i),(y,j),l}$



$$= \sum_l m^{(z,k)}_{(x,i),(y,j),l} \sum_\delta \left[\Phi^c_{x,y}\right]^{z,\xi,l}_{(a,\psi),(b,\chi),\delta}$$

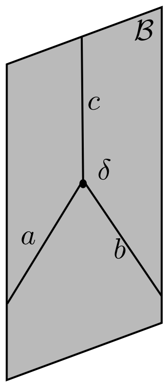


Figure 19. The multiplication on $F(L)$ in terms of m and Φ. In the first equality we used $\mathrm{Hom}(F(L),a) = \bigoplus_{x\in\mathcal{Z}(\mathcal{C})} \mathrm{Hom}(L,x)\times \mathrm{Hom}(x,a)$, so that $\alpha = (\psi,x,i)$, $\beta = (x,y,j)$, $\gamma = (\xi,z,k)$. In the second equality we used figure 17 and in the third equality we used figure 15.

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where N_a^x counts the number of point operators at the junction of a and x in $\mathcal{C}$ [44, section 8.1].[13] $K(a)$ has the interpretation as the sum of line operators of the SymTFT $\mathcal{Z}(\mathcal{C})$ which can form a junction with $a \in \mathcal{C}$ on the gapped boundary $\mathcal{B}_\mathcal{C}$. Given an algebra A in $\mathcal{C}$, consider the line operator $K(A)$ in $\mathcal{C}$. In general, $K(A)$ does not admit the structure of a commutative algebra. This is in particular because $K(A)$ may contain non-bosonic simple line operators in $\mathcal{Z}(\mathcal{C})$. However, since A is an algebra in $\mathcal{C}$, $K(A)$ does admit the structure of an algebra in $\mathcal{Z}(\mathcal{C})$. Indeed, the map K gives an algebra structure on $K(A)$. Similar to the case of F, to determine how K transports algebras from $\mathcal{C}$ to $\mathcal{Z}(\mathcal{C})$, we have to determine how K relates the fusion spaces of $\mathcal{C}$ and $\mathcal{Z}(\mathcal{C})$. We need to specify maps

$$\Xi_{a,b} : K(a) \times K(b) \to K(a \times b)\,. \tag{4.20}$$

$\Xi_{a,b}$ can be determined as follows. The bulk-to-boundary map F and boundary-to-bulk map K together satisfies [44, section 8.1]

$$\mathrm{Hom}_\mathcal{C}(F((a,e_a)),b) \simeq \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}((a,e_a),K(b))\,. \tag{4.21}$$

Let $\Theta_{(a,e_a),b}$ be this isomorphism. Let $\mathrm{id}_{\mathcal{Z}(\mathcal{C})}$ be the identity map on $\mathcal{Z}(\mathcal{C})$. Consider the natural transformation

$$\delta : \mathrm{id}_{\mathcal{Z}(\mathcal{C})} \to K \circ F\,, \tag{4.22}$$

specified by

$$\delta_{(a,e_a)} := \Theta_{(a,e_a),F((a,e_a))}(\mathrm{id}_{F(a,e_a)}) : (a,e_a) \to K \circ F((a,e_a))\,. \tag{4.23}$$

$\delta_{(a,e_a)}$ is non-trivial because the action of the composition $K \circ F$ on $\mathcal{Z}(\mathcal{C})$ is not the same as the identity map $\mathrm{id}_{\mathcal{Z}(\mathcal{C})}$. Similarly, we have

$$\rho : F \circ K \to \mathrm{id}_\mathcal{C}\,, \tag{4.24}$$

specified by

$$\rho_a := \Theta^{-1}_{K(a),a}(\mathrm{id}_{K(a)}) : F \circ K(a) \to a\,. \tag{4.25}$$

$\rho_{(a,e_a)}$ is non-trivial because the action of the composition $F \circ K$ on $\mathcal{C}$ is not the same as the identity map $\mathrm{id}_\mathcal{C}$. Using the data introduced above, we can write

$$\begin{aligned}\Xi_{a,b} : K(a) \times K(b) \xrightarrow{\delta} K \circ F(K(a) \times K(b)) &\xrightarrow{\Phi^{-1}} K \circ (F \circ K(a) \times F \circ K(a)) \\ &\xrightarrow{K(\rho_a \times \rho_b)} K(a \times b)\,,\end{aligned} \tag{4.26}$$

where Φ is the map introduced in (4.13) which relates the fusion spaces of $\mathcal{Z}(\mathcal{C})$ and $\mathcal{C}$ under F.[14,15] Therefore, Ξ can be used to determine the multiplication on $K(A)$ given the

[13] K is sometimes called the induction functor [45, section 9.2].

[14] In other words, K is the adjoint of the forgetful functor F, and therefore K itself is a lax monoidal functor (see, for example, [69, Lemma 2.7], [70]).

[15] Since K is a two-sided adjoint of $F_{\mathcal{B}_\mathcal{C}}$, it is in fact both a lax and colax monoidal functor (see, for example, [69]). K is generally not strong monoidal ([71] (see also [69])). However, K still maps Frobenius algebras to Frobenius algebras. This latter statement can be shown using [72, Theorem 3.3].

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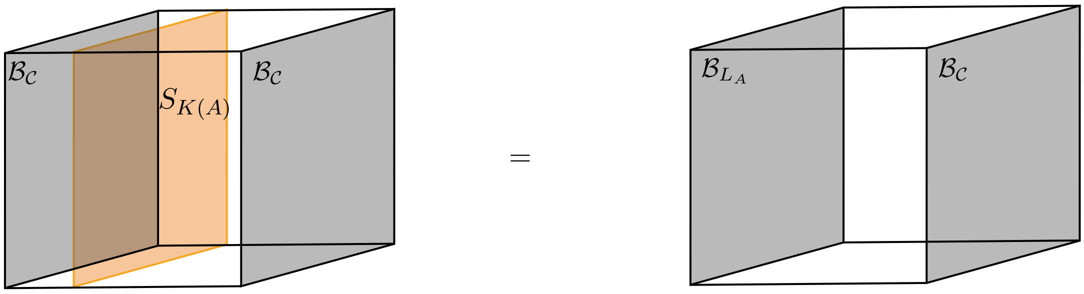

Figure 20. Parallel fusion of the surface operator $S_{K(A)}$ on the canonical gapped boundary $\mathcal{B}_{\mathcal{C}}$ produces a new gapped boundary $\mathcal{B}_{L_A}$.

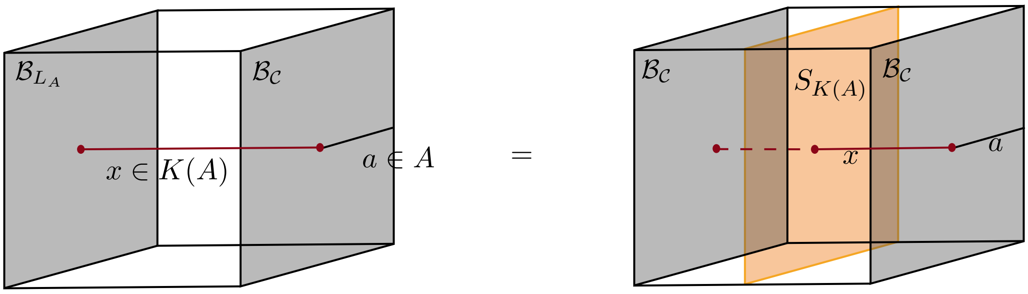

Figure 21. The line operator $x \in K(A)$ can end on the gapped boundary $\mathcal{B}_{L_A}$ if the action of $S_{K(A)}$ on x produces the trivial line operator.

multiplication on A precisely as in (4.15). However, note that unlike Φ, Ξ is not trivial in general. Therefore, finding the multiplication on $K(A)$ requires finding Ξ explicitly.

As we discussed above, though the map K specifies a multiplication on $K(A)$, it is not the Lagrangian algebra that we are looking for since $K(A)$ is not even commutative in general. However, $K(A)$ can be higher-gauged to obtain a surface operator $S_{K(A)}$ of the SymTFT [73, 74]. Consider the gapped boundary $\mathcal{B}_{L_A}$ of the SymTFT obtained from fusing the surface $S_{K(A)}$ on the canonical gapped boundary $\mathcal{B}_{\mathcal{C}}$ as in figure 20.

The Lagrangian algebra object L_A corresponding to A is given by the set of line operators in $K(A)$ that can end on $\mathcal{B}_{L_A}$. From figure 21 it is clear that a line operator $x \in K(A)$ can end on $\mathcal{B}_{L_A}$ if

$$S_{K(A)} \cdot x = 1 + \dots \tag{4.27}$$

In other words, the action of the surface operator $S_{K(A)}$ on x produces the trivial line. As discussed in [75, section 5.2.2], the set of line operators which satisfy (4.27) is given by the left center $C_L(K(A))$ of $K(A)$. The left center, $C_L(K(A))$ of $K(A)$, is the maximal sub-object of $K(A)$ such that

$$m^y_{x,i}\, R^y_{i,x} = m^y_{i,x}\,, \tag{4.28}$$

where m is the multiplication in $K(A)$, and $R^y_{i,x}$ is the braiding of the lines $x \in A$ and $i \in C_L(K(A))$ in $\mathcal{Z}(\mathcal{C})$. Diagrammatically, this is figure 22. Therefore, the Lagrangian algebra L_A corresponding to the algebra A contains the left center of $K(A)$. In fact, it is known

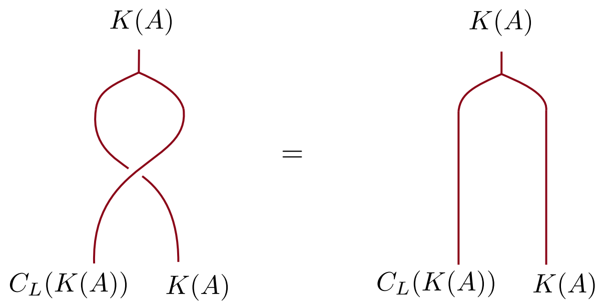

Figure 22. Diagrammatic definition of left center of $K(A)$.

that $C_L(K(A))$ is Lagrangian [29, 76–78]. Therefore, $C_L(K(A))$ is the Lagrangian algebra corresponding to A. Mathematically, $C_L(K(A))$ is called the full-center of the algebra A.

To make the discussion above more transparent, let us consider the example $\mathcal{C} = \mathrm{Vec}_{\mathbb{Z}_2}$. Suppose the two line operators in $\mathcal{C}$ are labeled by e, g, where g is of order two. We have two haploid algebra objects in this case,

$$A_1 = e\,, \ \ A_2 = e + g\,. \tag{4.29}$$

The SymTFT $\mathcal{Z}(\mathcal{C})$ is the $\mathbb{Z}_2$ Dijkgraaf-Witten theory with simple line operators

$$([e], \mathbb{1})\,, \ \ ([e], \pi)\,, \ \ ([g], \mathbb{1})\,, \ \ ([g], \pi)\,, \tag{4.30}$$

where $\mathbb{1}, \pi$ are the two irreducible representations of $\mathbb{Z}_2$. $([e], \mathbb{1}), ([e], \pi)$ and $([g], \mathbb{1})$ are bosons while $([g], \pi)$ is a fermion. The boundary-to-bulk map K is given by

$$K(e) = ([e], \mathbb{1}) + ([e], \pi)\,, \ \ K(g) = ([g], \mathbb{1}) + ([g], \pi)\,. \tag{4.31}$$

Therefore, we get

$$K(A_1) = ([e], \mathbb{1}) + ([e], \pi)\,. \tag{4.32}$$

This is already a commutative algebra because both $([e], \mathbb{1})$ and $([e], \pi)$ are bosons, and they are closed under fusion. Indeed, $K(A_1)$ is precisely the Lagrangian algebra corresponding to A_1.

Now, consider

$$K(A_2) = ([e], \mathbb{1}) + ([e], \pi) + ([g], \mathbb{1}) + ([g], \pi)\,. \tag{4.33}$$

Clearly, $K(A_2)$ is not a commutative algebra because of the fermion $([g], \pi)$. However, it admits the structure of an algebra and we can define a surface operator $S_{K(A_2)}$ by higher-gauging A_2 on a 2-manifold. The action of $S_{K(A_2)}$ on the line operators is given by [74, section 6.3.1]

$$S_{K(A_2)} \cdot ([e], \mathbb{1}) = S_{K(A_2)} \cdot ([g], \mathbb{1}) = ([e], \mathbb{1}) + ([e], \pi)\,, \tag{4.34}$$

$$S_{K(A_2)} \cdot ([e], \pi) = S_{K(A_2)} \cdot ([g], \pi) = 0\,. \tag{4.35}$$

The set of lines on which the surface operator acts to produce the trivial line is

$$([e], \mathbb{1}) \text{ and } ([g], \mathbb{1})\,. \tag{4.36}$$

Therefore, we get a commutative algebra $A = ([e], \mathbb{1}) + ([g], \mathbb{1})$ which is precisely the Lagrangian algebra object corresponding to A_2.

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4.4 Invertible symmetry

When the symmetry is invertible, the category $\mathcal{C}$ is equivalent to Vec_G^ω for some group G and anomaly given by a 3-cocycle $\omega \in H^3(G, \mathrm{U}(1))$. When ω is non-trivial, the symmetry G cannot be gauged. Given a subgroup $H \subseteq G$, we can ask whether H is non-anomalous. This is true if and only if

$$\omega|_H = 1 \in H^3(H, \mathrm{U}(1))\,. \tag{4.37}$$

Given a non-anomalous group H, we can gauge it with a choice of discrete torsion $\sigma \in H^2(H, \mathrm{U}(1))$. The algebra objects in this case will be denoted as $A(H,\sigma)$. Clearly, if H is non-anomalous, so is the subgroup gHg^{-1} for any $g \in G$. In fact, the algebras $A(H,\sigma)$ and $A(gHg^{-1}, g\sigma g^{-1})$ are Morita equivalent for any $g \in G$ where $g\sigma g^{-1}(h,k) := \sigma(ghg^{-1}, gkg^{-1})$. Therefore, the equivalence classes $[A(H,\sigma)]$ are given by conjugacy classes of subgroups of G on which ω trivializes and corresponding 2-cocycles [41].

So far we discussed how to determine whether $H \subset G$ is anomalous purely from a 1+1D perspective. Given $H \subseteq G$, how do we determine whether H is anomalous from the SymTFT $\mathcal{Z}(\mathrm{Vec}_G^\omega) \equiv \mathrm{DW}(G,\omega)$? We learned that for $H \subseteq G$ to be non-anomalous there must be a Lagrangian algebra $L_{A(H,\sigma)}$ in $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ corresponding to it. To determine this Lagrangian algebra object in terms of the simple objects of $\mathcal{Z}(\mathrm{Vec}_G^\omega)$, we can use the theory of characters for objects in $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ [79, 80]. Let us define

$$\omega(g,h|k) := \omega(g,h,k)^{-1}\omega(g,hgh^{-1},h)\omega(ghk(gh)^{-1},g,h)^{-1}\,. \tag{4.38}$$

Let $([g],\pi_g)$ be a simple object in $\mathcal{Z}(\mathrm{Vec}_G^\omega)$, where $[g]$ is a conjugacy class in G and π_g is a projective representation of the centralizer C_g of g satisfying

$$\pi_g(h)\pi_g(k) = \omega(h,k|g)\pi_g(hk) \quad \forall h,k \in C_g\,. \tag{4.39}$$

Note that for $h,k \in C_g$, the phases $\omega(h,k|f)$ form a 2-cocycle in $H^2(C_g, \mathrm{U}(1))$. The character of $([g],\pi_g)$ is defined as

$$\chi_{([g],\pi_g)}(h,k) := \begin{cases} \chi_{\pi_h}(k) & \text{if } h \in [g]\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{4.40}$$

where h,k is a pair of commuting elements in G and $\chi_{\pi_g}(k)$ is the character of the representation π_h of the centralizer C_h of h evaluated on $k \in C_h$. See appendix B for the definition of the character of a general line operator in the SymTFT and its properties.

The Lagrangian algebra object $L_{A(H,\sigma)}$ can be determined by first computing the character

$$\chi_{\mathcal{Z}(A(H,\sigma))}(k,l) := \sum_{y\in Y} \frac{\omega(y^{-1}ly, y^{-1}|k)\sigma(y^{-1}ky, y^{-1}ly)}{\omega(y^{-1}, l|k)\sigma(y^{-1}ly, y^{-1}ky)}\,, \tag{4.41}$$

where

$$Y := \{y \in G, y^{-1}ky \in H, y^{-1}ly \in H\}/H \subset G/H\,, \tag{4.42}$$

and then decomposing it in terms of the characters $\chi_{([g],\pi_g)}$.

So far, we discussed how to get the Lagrangian algebra object $L_{A(H,\sigma)}$ corresponding to an algebra $A(H,\sigma)$ in Vec_G^ω. Given a subgroup $H \subseteq G$, we would like to determine if H is non-anomalous using the SymTFT $\mathcal{Z}(\mathrm{Vec}_G^\omega)$. To this end, let us consider the following set

$$C(H) := \{[h], h \in H\}\,, \tag{4.43}$$

where $[h]$ is the conjugacy class in G with representative h. As a set, $C(H)$ contains all elements in the subgroup H as well as elements of any subgroup conjugate to H.

Theorem 4.2. *$H \subseteq G$ is non-anomalous if and only if there exists a Lagrangian algebra L in $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ such that*

$$([h], \pi_h) \in L \quad \forall\ [h] \in C(H)\,, \tag{4.44}$$

for some projective representation π_h of the centralizer of h.

Proof: see appendix B.

The above theorem shows that from a given object L of $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ that admits the structure of a Lagrangian algebra we can identify a conjugacy class of subgroups $[H]$ such that any element of $[H]$ is a non-anomalous subgroup of G.

4.4.1 Lagrangian subcategory

In the previous section, we proved a theorem which can be used to determine whether a subgroup $H \subseteq G$ is non-anomalous. To completely specify the equivalence class of algebras $[A(H,\sigma)]$, we also need to find the 2-cocycle σ. In general, this requires not just the decomposition of L into simple objects, but also the multiplication of the algebra L. Indeed, the same object L can admit distinct Lagrangian algebra structures [81] which then correspond to two non-equivalent algebra structures on some subgroup $H \subseteq G$. In general, the 2-cocycle σ can be determined using (4.15). However, when the simple line operators which form L are closed under fusion, the 2-cocycle σ can be easily obtained. In this case, these simple line operators form a fusion subcategory of $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ with total quantum dimension $|G|$. Such a fusion subcategory is called a Lagrangian subcategory of $\mathcal{Z}(\mathrm{Vec}_G^\omega)$. Suppose we have

$$L = \sum_{([g],\pi_g)\in\mathcal{Z}(\mathrm{Vec}_G^\omega)} N_L^{([g],\pi_g)}\ ([g],\pi_g)\,. \tag{4.45}$$

Since the non-trivial simple line operators in L are closed under fusion, the conjugacy classes $[g]$ such that $N_L^{([g],\pi_g)} \neq 0$ must all be closed under fusion. The Lagrangian algebra L corresponds to an algebra object

$$A = \sum_{([g]\pi_g)\in\mathcal{Z}(\mathrm{Vec}_G^\omega)} N_L^{([g],\pi_g)}\ A_g\,, \tag{4.46}$$

where $A_g = \sum_{h\in[g]} h$. Since the non-trivial conjugacy classes of simple objects in L are closed under fusion, we find that A is in fact a sum over elements of a normal subgroup of G given by

$$N = \bigcup_{N_L^{([g],\pi_g)}\neq 0} [g]\,. \tag{4.47}$$

Moreover, since the fusion of line operators in $\mathcal{Z}(\text{Vec}_G^\omega)$ is commutative, N is an abelian normal subgroup. Therefore, the subset of gapped boundaries of the SymTFT $\mathcal{Z}(\text{Vec}_G^\omega)$ determined by Lagrangian subcategories correspond to non-anomalous abelian normal subgroups of G. In fact, there is a one-to-one correspondence between Lagrangian subcategories of $\mathcal{Z}(\text{Vec}_G^\omega)$ and algebra objects $A(N,\sigma)$ in Vec_G^ω where $N \subseteq G$ is an abelian normal subgroup and $\sigma \in H^2(N, \text{U}(1))$ [82].

The 2-cocycle $\sigma \in H^2(N, \text{U}(1))$ corresponding to the Lagrangian subcategory (4.45) can be determined using the relation

$$\frac{\sigma(h_1,h_2)}{\sigma(h_2,h_1)} = \frac{\beta_g(l,h_2)\beta_g(lh_2,l^{-1})}{\beta_g(l,l^{-1})} \, \frac{\chi_{\pi_g}(l^{-1}h_2 l)}{\dim(\pi_g)} \tag{4.48}$$

where

$$\beta_g(h,k) := \omega(g,h,k)\omega(h,h^{-1}gh,k)^{-1}\omega(h,k,(hk)^{-1}ghk)\,, \tag{4.49}$$

h_1, l and g satisfies $h_1 = lgl^{-1}$ for some $l \in G$ and $g \in N$. (See [82, Theorem 4.12] for more details)

5 Examples

In this section, we will go through various explicit examples to understand the general discussion in the previous sections. We will start with the familiar case of invertible symmetries. This is a good starting point to understand Theorems 4.1 and 4.2. Then we will consider the case of a modular tensor category $\mathcal{C}$, before moving on to the case of fusion categories.

5.1 Invertible symmetries

5.1.1 $G = \mathbb{Z}_4$

When $G = \mathbb{Z}_4$ the anomaly is classified by $H^3(\mathbb{Z}_4, \text{U}(1)) \cong \mathbb{Z}_4$. We will consider the non-anomalous case described by the fusion category $\text{Vec}_{\mathbb{Z}_4}$ whose SymTFT is the $\mathbb{Z}_4$ Dijkgraaf-Witten theory. It has three Lagrangian algebras given by

$$L_1 = \{(e,\mathbb{1}),(e,\pi_1),(e,\pi_2),(e,\pi_3)\}\,, \tag{5.1}$$

$$L_2 = \{(e,\mathbb{1}),(g,\mathbb{1}),(g^2,\mathbb{1}),(g^3,\mathbb{1})\}\,, \tag{5.2}$$

$$L_3 = \{(e,\mathbb{1}),(g^2,\mathbb{1}),(e,\pi_2),(g^2,\pi_2)\}\,, \tag{5.3}$$

where g is the generator of $\mathbb{Z}_4$ and $g^4 = e$, and π_j are the representations of $\mathbb{Z}_4$ given by $\pi_j(g) = e^{\frac{2\pi i j}{4}}$. Using Theorem 4.2, we can read off the non-anomalous subgroups of $\mathbb{Z}_4$ from the simple objects in the Lagrangian algebras above. We get

$$L_1 \to \mathbb{Z}_1,\; L_2 \to \mathbb{Z}_4,\; L_3 \to \mathbb{Z}_2\,. \tag{5.4}$$

Alternatively, using Theorem 4.1, the algebra objects which can be gauged in the $\mathbb{Z}_4$ group can be obtained by fusing the Lagrangian algebra objects L_i in the bulk TQFT with the gapped boundary $\mathcal{B}_{L_1}$ corresponding to L_1. Fusing L_1 with the gapped boundary $\mathcal{B}_{L_1}$, we get

$$e + e + e + e\,. \tag{5.5}$$

We see that the resulting algebra is not haploid because of multiple copies of the identity line in it. Fusing L_2 with the gapped boundary $\mathcal{B}_{L_1}$, we get

$$e + g + g^2 + g^3 \,. \tag{5.6}$$

In this case, we get a haploid algebra. Fusing L_3 with the gapped boundary $\mathcal{B}_{L_1}$, we get

$$e + g^2 + e + g^2 \,. \tag{5.7}$$

In this case we get a non-haploid algebra. A haploid subalgebra is $e + g^2$.

5.1.2 $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

Consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry without anomaly. The SymTFT is given by the $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle g_1, g_2 \rangle$ Dijkgraaf-Witten theory. The line operators in this SymTFT can be labeled as

$$\begin{aligned}
&([e], \mathbb{1}),\ ([e], \omega_1),\ ([e], \omega_2),\ ([e], \omega_1\omega_2)\,,\\
&([g_1], \mathbb{1}),\ ([g_1], \omega_1),\ ([g_1], \omega_2),\ ([g_1], \omega_1\omega_2)\,,\\
&([g_2], \mathbb{1}),\ ([g_2], \omega_1),\ ([g_2], \omega_2),\ ([g_2], \omega_1\omega_2)\,,\\
&([g_1g_2], \mathbb{1}),\ ([g_1g_2], \omega_1),\ ([g_1g_2], \omega_2),\ ([g_1g_2], \omega_1\omega_2)\,,
\end{aligned} \tag{5.8}$$

where $\mathbb{1}, \omega_1, \omega_2$ and $\omega_1\omega_2$ label the four irreducible representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$. This SymTFT has six gapped boundaries determined by the Lagrangian algebra objects

$$\begin{aligned}
L_1 &= ([e], \mathbb{1}),\ ([e], \omega_1),\ ([e], \omega_2),\ ([e], \omega_1\omega_2)\,,\\
L_2 &= ([e], \mathbb{1}) + ([g_1], \mathbb{1}) + ([g_2], \mathbb{1}) + ([g_1g_2], \mathbb{1})\,,\\
L_3 &= ([e], \mathbb{1}) + ([g_1], \omega_2) + ([g_2], \omega_1) + ([g_1g_2], \omega_1\omega_2)\,,\\
L_4 &= ([e], \mathbb{1}) + ([e], \omega_2) + ([g_1], \mathbb{1}) + ([g_1], \omega_2)\,,\\
L_5 &= ([e], \mathbb{1}) + ([e], \omega_1)([g_2], \mathbb{1}) + ([g_2], \omega_1)\,,\\
L_6 &= ([e], \mathbb{1}) + ([e], \omega_1\omega_2)([g_1g_2], \mathbb{1}) + ([g_1g_2], \omega_1\omega_2)\,.
\end{aligned} \tag{5.9}$$

The canonical gapped boundary corresponding to the symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$ is $\mathcal{B}_{L_1}$. Fusing the eight Lagrangian algebra objects on the gapped boundary $\mathcal{B}_{L_1}$, we get

$$F_{\mathcal{B}_{L_1}}(L_1) = e + e + e + e\,, \tag{5.10}$$
$$F_{\mathcal{B}_{L_1}}(L_2) = e + g_1 + g_2 + g_1g_2\,, \tag{5.11}$$
$$F_{\mathcal{B}_{L_1}}(L_3) = e + g_1 + g_2 + g_1g_2\,, \tag{5.12}$$
$$F_{\mathcal{B}_{L_1}}(L_4) = e + e + g_1 + g_1\,, \tag{5.13}$$
$$F_{\mathcal{B}_{L_1}}(L_5) = e + e + g_2 + g_2\,, \tag{5.14}$$
$$F_{\mathcal{B}_{L_1}}(L_6) = e + e + g_1g_2 + g_1g_2\,. \tag{5.15}$$

Clearly, both Lagrangian algebras L_2 and L_3 correspond to the same algebra object

$$A = e + g_1 + g_2 + g_1g_2\,. \tag{5.16}$$

Since distinct Lagrangian algebras must correspond to non-equivalent gaugings in 1+1D, the algebra object A has two distinct multiplications on it. These multiplications can be determined using the multiplications of the Lagrangian algebras L_2 and L_3.

Recall that in the multiplicity-free case, the multiplication in a Lagrangian algebra L is determined by the complex numbers

$$m_{xy}^{z}\,, \tag{5.17}$$

where x, y, z are three simple line operators in L. These have to satisfy the constraints (2.4) and (3.5). For the $\mathbb{Z}_2 \times \mathbb{Z}_2$ discrete gauge theory, there is a gauge in which the F symbols are all trivial. The R matrices can be written as

$$R^{([gh],\pi_g\times\pi_h)}_{([g],\pi_g),([h],\pi_h)} = \chi_{\pi_h}(g)\,, \tag{5.18}$$

where χ_{π_g} is the character of the irreducible representation π_g. Using this expression for the R matrix and constraint (3.5), we find that for the Lagrangian algebra L_2, we get

$$m_{xy}^{z} = 1 \;\forall\; x, y, z \in L_2\,. \tag{5.19}$$

Therefore, using (4.15), we find that the multiplication on the algebra A corresponding to L_2 is trivial.

Using the constraint (3.5), we find that for the Lagrangian algebra L_3, we get

$$m^{([g_1g_2],\omega_1\omega_2)}_{([g_1],\omega_2),([g_2],\omega_1)} = -1 \tag{5.20}$$

and all other m_{xy}^{z} are trivial. Therefore, using (4.15), we find that the multiplication on the algebra A corresponding to L_3 is given by

$$m^{g_1g_2}_{g_1,g_2} = -1\,, \tag{5.21}$$

and all the other m_{xy}^{z} are trivial. Therefore, the non-trivial multiplication on the algebra object A corresponds to the Lagrangian algebra L_3.

5.1.3 $G = S_3$

Consider the group

$$S_3 = \{r, s | r^3 = s^2 = e; srs = r^{-1}\}\,. \tag{5.22}$$

The conjugacy classes in this group are

$$[e], \quad [r] = \{r, r^2\}, \quad [s] = \{s, rs, r^2s\}\,. \tag{5.23}$$

$[e]$ has centralizer isomorphic to S_3 with the trivial representation $\mathbb{1}_e$, non-trivial 1-dimensional representation π_1 and non-trivial two dimensional irreducible representation π_2. $[r]$ has a centralizer isomorphic to $\mathbb{Z}_3$ with irreducible representations $\mathbb{1}_r, \omega, \omega^2$. Finally, $[s]$ has a centralizer isomorphic to $\mathbb{Z}_2$ which has the trivial representation $\mathbb{1}_s$ and non-trivial representation γ. The full set of line operators in the S_3 DW theory are given by

$$([e], \mathbb{1}_e), ([e], \pi_1), ([e], \pi_2), ([r], \mathbb{1}_r), ([r], \omega), ([r], \omega^2), ([s], \mathbb{1}_s), ([s], \gamma)\,. \tag{5.24}$$

The 1+1D gapped and gapless phases in this case and their relation to SymTFTs were studied in detail in [59–61, 83].

There are four Lagrangian algebras given by [49, 80]

$$L_1 := ([e], \mathbb{1}_e) + ([e], \pi_1) + 2([e], \pi_2)\,, \tag{5.25}$$
$$L_2 := ([e], \mathbb{1}_e) + ([e], \pi_2) + ([s], \mathbb{1}_s)\,, \tag{5.26}$$
$$L_3 := ([e], \mathbb{1}_e) + ([e], \pi_1) + 2([r], \mathbb{1}_r)\,, \tag{5.27}$$
$$L_4 := ([e], \mathbb{1}_e) + ([s], \mathbb{1}_s) + ([r], \mathbb{1}_r)\,. \tag{5.28}$$

Let us consider the fusion of all line operators on the $\mathcal{B}_{L_1}$ gapped boundary given by the map

$$F_{\mathcal{B}_{L_1}} : \mathcal{Z}(\mathrm{Vec}_{S_3}) \to \mathrm{Vec}_{S_3}\,. \tag{5.29}$$

In the following, we will denote $F_{\mathcal{B}_{L_1}}$ simply as F. We get (see, for example, [49, section 2.6])

$$F(([e], \mathbb{1}_e)) = F(([e], \pi_1)) = e\,,\ F(([e], \pi_2)) = 2e\,, \tag{5.30}$$
$$F(([r], \mathbb{1}_r)) = F(([r], \omega)) = e\,,\ F(([r], \omega^2)) = r + r^2\,, \tag{5.31}$$
$$F(([s], \mathbb{1}_s)) = F(([s], \gamma)) = s + sr + sr^2\,. \tag{5.32}$$

Note that F preserves quantum dimensions. Using this, we can find the fusion of all the Lagrangian algebras on $\mathcal{B}_{L_1}$. Fusing L_1 on the gapped boundary $\mathcal{B}_{L_1}$ results in the line operator

$$e + e + 4e\,. \tag{5.33}$$

Fusing L_2 on the gapped boundary $\mathcal{B}_{L_1}$ results in the non-anomalous line operator

$$e + 2e + s + sr + sr^2\,. \tag{5.34}$$

This is a sum of haploid subalgebras

$$A_1 = e + s\,,\ A_2 = e + sr\,,\ A_3 = e + sr^2\,, \tag{5.35}$$

which are all Morita equivalent. Fusing L_3 on the gapped boundary $\mathcal{B}_{L_1}$ results in the non-anomalous line operator

$$e + e + 2r + 2r^2\,. \tag{5.36}$$

This is two copies of the haploid algebra

$$e + r + r^2\,. \tag{5.37}$$

Fusing L_4 on the gapped boundary $\mathcal{B}_{L_1}$ results in the non-anomalous line operator

$$e + s + sr + sr^2 + r + r^2\,. \tag{5.38}$$

5.2 When $\mathcal{C}$ is modular

When $\mathcal{C}$ is a modular tensor category the SymTFT has a particularly simple form [44].

$$\mathcal{Z}(\mathcal{C}) \cong \mathcal{C} \times \bar{\mathcal{C}}\,, \tag{5.39}$$

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where $\bar{\mathcal{C}}$ is the category with the opposite braiding. Therefore, the simple line operators in $\mathcal{Z}(\mathcal{C})$ can be labeled as

$$(a, b),\ \ a, b \in \mathcal{C}\,. \tag{5.40}$$

However, for using the bulk-to-boundary map $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$, it is convenient to relabel the simple line operators in the form (a, e_a) for some object $a \in \mathcal{C}$ and half-braiding e_a. This can be done using the explicit isomorphism between $\mathcal{Z}(\mathcal{C})$ and $\mathcal{C} \times \bar{\mathcal{C}}$. We have [44, Theorem 7.10]

$$(a, b) \mapsto (a \times b, e_{a\times b})\,, \tag{5.41}$$

where

$$e_{a\times b}(c) := (R^{-1}_{a,c} \times \mathrm{id}_b) \circ (\mathrm{id}_a \times R^{-1}_{c,b})\,, \tag{5.42}$$

with $R_{a,b} : a \times b \xrightarrow{\sim} b \times a$ being the braiding on $\mathcal{C}$.

5.2.1 Fibonacci category

Let $\mathcal{C}$ be the Fibonacci MTC with the non-trivial simple line operator τ. The SymTFT in this case is

$$\mathcal{Z}(\mathcal{C}) = \mathcal{C} \times \bar{\mathcal{C}}\,. \tag{5.43}$$

Therefore, the simple line operators in the SymTFT can be labeled as

$$(\mathbf{1}, \mathbf{1}), (\mathbf{1}, \tau), (\tau, \mathbf{1}), (\tau, \tau)\,. \tag{5.44}$$

The line operators $(\mathbf{1}, \mathbf{1})$ and (τ, τ) are bosonic. For identifying gaugeable algebras in $\mathcal{C}$, we have to describe the simple objects in the notation (a, e_a) where $a \in \mathcal{C}$ and e_a is the half-braiding. Using (5.41) we get[16]

$$(\mathbf{1}, e_{\mathbf{1}}), (\tau, e_\tau), (\tau, e'_\tau), (\mathbf{1} + \tau, e_{\mathbf{1}+\tau})\,. \tag{5.45}$$

We have a unique Lagrangian algebra

$$L = (\mathbf{1}, e_{\mathbf{1}}) + (\mathbf{1} + \tau, e_{\mathbf{1}+\tau})\,. \tag{5.46}$$

Fusing L with the gapped boundary $\mathcal{B}_L$, we find the algebra object

$$\mathbf{1} + \mathbf{1} + \tau\,, \tag{5.47}$$

which is non-haploid. We can decompose this algebra into haploid subalgebras

$$A_1 = \mathbf{1}, A_2 = \mathbf{1} + \tau\,. \tag{5.48}$$

The fact that both of these algebras correspond to the same gapped boundary implies that they are Morita equivalent. Indeed, gauging $\mathbf{1} + \tau$ is a self-duality of a 1+1D QFT with $\mathcal{C}$ symmetry [33, 36].

[16]The line operators in $\mathcal{Z}(\mathcal{C})$ can also be directly obtained in the form (a, e_a) using the string-net construction [46, section VI. B].

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5.2.2 Ising category

Let $\mathcal{C}$ be the Ising MTC with non-trivial line operators ψ, σ. The SymTFT in this case is

$$\mathcal{Z}(\mathcal{C}) = \mathcal{C} \times \bar{\mathcal{C}} \,. \tag{5.49}$$

Therefore, the simple line operators in the SymTFT can be labeled as

$$(\mathbf{1},\mathbf{1}), (\mathbf{1},\psi), (\mathbf{1},\sigma)\,, \tag{5.50}$$
$$(\psi,\mathbf{1}), (\psi,\psi), (\psi,\sigma)\,, \tag{5.51}$$
$$(\sigma,\mathbf{1}), (\sigma,\psi), (\sigma,\sigma)\,. \tag{5.52}$$

The line operators $(\mathbf{1},\mathbf{1}), (\psi,\psi)$ and (σ,σ) are bosonic. Using (5.41) we can relabel the simple line operators as

$$(\mathbf{1}, e^{(1)}_{\mathbf{1}}), (\psi, e^{(1)}_{\psi}), (\sigma, e^{(1)}_{\sigma}) \tag{5.53}$$
$$(\psi, e^{(2)}_{\psi}), (\mathbf{1}, e^{(2)}_{\mathbf{1}}), (\sigma, e^{(2)}_{\sigma}) \tag{5.54}$$
$$(\sigma, e^{(2)}_{\sigma}), (\sigma, e^{(3)}_{\sigma}), (\mathbf{1}+\psi, e_{\mathbf{1}+\psi})\,. \tag{5.55}$$

The only Lagrangian algebra is

$$L = (\mathbf{1}, e^{(1)}_{\mathbf{1}}) + (\mathbf{1}, e^{(2)}_{\mathbf{1}}) + (\mathbf{1}+\psi, e_{\mathbf{1}+\psi})\,. \tag{5.56}$$

Fusing L with the gapped boundary $\mathcal{B}_L$ we get the algebra object

$$\mathbf{1}+\mathbf{1}+\mathbf{1}+\psi\,. \tag{5.57}$$

This algebra is a sum of three haploid subalgebras

$$A_1 = A_2 = \mathbf{1}\,,\;\; A_3 = \mathbf{1}+\psi\,. \tag{5.58}$$

This is consistent with the fact that A_3 is a Morita trivial algebra [33, 36, 84].

5.3 Non-invertible symmetries

5.3.1 Rep(S_3)

For Rep(S_3) the SymTFT is dual to $\mathcal{Z}(\mathrm{Vec}_{S_3})$ for which we have already considered all the Lagrangian algebras. Recall the simple line operators

$$([e],\mathbb{1}_e), ([e],\pi_1), ([e],\pi_2), ([r],\mathbb{1}_r), ([r],\omega), ([r],\omega^2), ([s],\mathbb{1}_s), ([s],\gamma)\,, \tag{5.59}$$

and gapped boundaries

$$L_1 := ([e],\mathbb{1}_e) + ([e],\pi_1) + 2([e],\pi_2)\,, \tag{5.60}$$
$$L_2 := ([e],\mathbb{1}_e) + ([e],\pi_2) + ([s],\mathbb{1}_s)\,, \tag{5.61}$$
$$L_3 := ([e],\mathbb{1}_e) + ([e],\pi_1) + 2([r],\mathbb{1}_r)\,, \tag{5.62}$$
$$L_4 := ([e],\mathbb{1}_e) + ([s],\mathbb{1}_s) + ([r],\mathbb{1}_r)\,. \tag{5.63}$$

For identifying the algebra objects in $\mathrm{Rep}(S_3)$, we have to choose a gapped boundary of $\mathcal{Z}(\mathrm{Vec}_{S_3})$ such that the boundary line operators form $\mathrm{Rep}(S_3)$. For example, we can choose the gapped boundary given by

$$L_2 = ([e], \mathbb{1}_e) + ([e], \pi_2) + ([s], \mathbb{1}_s)\,. \tag{5.64}$$

Let $\mathbb{1}, \pi_1, \pi_2$ be the three irreducible representations of S_3. Fusing the simple line operators in $\mathcal{Z}(\mathrm{Vec}_{S_3})$ on $\mathcal{B}_{L_2}$, we get (see, for example, [49, section 2.6])

$$F_{\mathcal{B}_{L_2}}(([e], \mathbb{1}_e)) = \mathbb{1}\,,\; F_{\mathcal{B}_{L_2}}(([e], \pi_1)) = \pi_1\,,\; F_{\mathcal{B}_{L_2}}(([e], \pi_2)) = \mathbb{1} + \pi_1\,, \tag{5.65}$$
$$F_{\mathcal{B}_{L_2}}(([r], \mathbb{1}_r)) = F_{\mathcal{B}_{L_2}}(([r], \omega)) = F_{\mathcal{B}_{L_2}}(([r], \omega^2)) = \pi_2\,, \tag{5.66}$$
$$F_{\mathcal{B}_{L_2}}(([s], \mathbb{1}_s)) = \mathbb{1} + \pi_2\,,\; F_{\mathcal{B}_{L_2}}(([s], \gamma)) = \pi_1 + \pi_2\,. \tag{5.67}$$

Fusing L_1 on the gapped boundary $\mathcal{B}_{L_2}$ results in the non-anomalous line operator

$$3\mathbb{1} + 3\pi_1\,. \tag{5.68}$$

It decomposes into three copies of the haploid algebra $\mathbb{1} + \pi_1$. Fusing L_2 on the gapped boundary $\mathcal{B}_{L_2}$ results in the non-anomalous line operator

$$2\mathbb{1} + \mathbb{1} + \pi_1 + \pi_2\,. \tag{5.69}$$

It decomposes into the haploid algebras

$$A_1 = \mathbb{1}\,,\; A_2 = \mathbb{1} + \pi_1 + \pi_2\,. \tag{5.70}$$

Note that this implies that gauging A_2 is a self-duality as A_2 is a Morita trivial algebra. Fusing L_3 on the gapped boundary $\mathcal{B}_{L_2}$ results in the non-anomalous line operator

$$\mathbb{1} + \pi_1 + 2\pi_2\,. \tag{5.71}$$

This is a haploid algebra corresponding to the unique fibre functor admitted by $\mathrm{Rep}(S_3)$. Fusing L_4 on the gapped boundary $\mathcal{B}_{L_2}$ results in the non-anomalous line operator

$$2\mathbb{1} + 2\pi_2\,. \tag{5.72}$$

It decomposes into two copies of the haploid algebra object $\mathbb{1} + \pi_2$. The algebra objects obtained above agree with the classification of algebras in $\mathrm{Rep}(S_3)$ given in [85].

5.4 Lagrangian algebras in $\mathcal{Z}(\mathrm{Vec}_{D_8})$ from algebras in Vec_{D_8}

Given the non-anomalous line operators in a fusion category $\mathcal{C}$, we can use Theorem 4.1 to put strong constraints on consistent Lagrangian algebra objects in its SymTFT. In this section, we will show that all Lagrangian algebra objects in the SymTFT $\mathcal{Z}(\mathrm{Vec}_{D_8})$, where D_8 is the dihedral group of 8 elements, can be completely fixed using Theorem 4.1.

Consider the invertible symmetry Vec_{D_8} where

$$D_8 := \langle r, s \mid r^4 = s^2 = e, srs = r^3\rangle\,. \tag{5.73}$$

Since this D_8 is chosen to be non-anomalous, any subgroup of D_8 can be gauged. The physically inequivalent gaugings correspond to conjugacy classes of subgroups of D_8 and a choice of discrete torsion. The conjugacy classes of subgroups are as follows. The normal subgroups are

$$\{e\}\,,\ \{e,r^2\}\,,\ \{e,r,r^2,r^3\}\,,\ \{e,r^2,s,r^2s\}\,,\ \{e,r^2,rs,r^3s\}\,,\ D_8\,. \tag{5.74}$$

We will denote these groups as $\mathbb{Z}_1, \mathbb{Z}_2^{(r^2)}, \mathbb{Z}_4^{(r)}, \mathbb{Z}_2^{(r^2)}\times\mathbb{Z}_2^{(s)}, \mathbb{Z}_2^{(r^2)}\times\mathbb{Z}_2^{(rs)}, D_8$, respectively. There are four non-normal subgroups which form the following two conjugacy classes of subgroups

$$\{\{e,s\},\{e,r^2s\}\}\,,\ \{\{e,rs\},\{e,r^3s\}\}\,. \tag{5.75}$$

All groups in these two conjugacy classes are isomorphic to $\mathbb{Z}_2$, and we will denote them by the representatives $\mathbb{Z}_2^{(s)}$ and $\mathbb{Z}_2^{(rs)}$, respectively. A physically inequivalent gaugings in Vec_{D_8} can be labeled as (G,σ) where G is a representative of a conjugacy class of subgroups in D_8 and $\sigma\in H^2(G,\mathrm{U}(1))$ is the discrete torsion. Using the explicit data above, we get the following 11 physically inequivalent gaugings in Vec_{D_8}

$$(\mathbb{Z}_1,1)\,,\ (\mathbb{Z}_2^{(r^2)},1)\,,\ (\mathbb{Z}_4^{(r)},1)\,,\ (\mathbb{Z}_2^{(r^2)}\times\mathbb{Z}_2^{(s)},1)\,,\ (\mathbb{Z}_2^{(r^2)}\times\mathbb{Z}_2^{(s)},\alpha)\,, \tag{5.76}$$

$$(\mathbb{Z}_2^{(r^2)}\times\mathbb{Z}_2^{(rs)},1)\,,\ (\mathbb{Z}_2^{(r^2)}\times\mathbb{Z}_2^{(rs)},\beta)\,,\ (D_8,1)\,,\ (D_8,\gamma)\,,\ (\mathbb{Z}_2^{(s)},1)\,,\ (\mathbb{Z}_2^{(rs)},1)\,, \tag{5.77}$$

where α,β,γ denote the choice of non-trivial discrete torsion in $H^2(\mathbb{Z}_2^{(r^2)}\times\mathbb{Z}_2^{(s)},\mathrm{U}(1))\cong\mathbb{Z}_2$, $H^2(\mathbb{Z}_2^{(r^2)}\times\mathbb{Z}_2^{(rs)},\mathrm{U}(1))\cong\mathbb{Z}_2$ and $H^2(D_8,\mathrm{U}(1))\cong\mathbb{Z}_2$, respectively. We will relate these 11 algebra objects to gapped boundaries of the SymTFT of Vec_{D_8}.

The SymTFT $\mathcal{Z}(\mathrm{Vec}_{D_8})$ is the Dijkgraaf-Witten theory with gauge group D_8. The conjugacy classes of D_8 are

$$[e]=\{e\}\,,\ [r^2]=\{r^2\}\,,\ [s]=\{s,r^2s\}\,,\ [rs]=\{rs,r^3s\}\,,\ [r]=\{a^2,a^3\}\,. \tag{5.78}$$

The centralizers of the representative of these conjugacy classes are

$$D_8\,,\ D_8\,,\ \{e,r^2,s,r^2s\}\cong\mathbb{Z}_2\times\mathbb{Z}_2\,,\ \{e,r^2,rs,r^3s\}\cong\mathbb{Z}_2\times\mathbb{Z}_2\,,\ \{e,r,r^2,r^3\}\cong\mathbb{Z}_4\,, \tag{5.79}$$

respectively. The simple line operators in $\mathcal{Z}(\mathrm{Vec}_{D_8})$, their quantum dimensions and spins are given by

Line operator	$([e],\mathbb{1})$	$([e],\pi_1)$	$([e],\pi_2),$	$([e],\pi_3)$	$([e],\pi_4)$
d_x	1	1	1	1	2
θ_x	1	1	1	1	1

Line operator	$([r^2],\mathbb{1})$	$([r^2],\pi_1)$	$([r^2],\pi_2),$	$([r^2],\pi_3)$	$([r^2],\pi_4)$
d_x	1	1	1	1	2
θ_x	1	1	1	1	-1

Line operator	$([s],\mathbb{1}_s)$	$([s],\omega_1)$	$([s],\omega_2),$	$([s],\omega_1\omega_2)$
d_x	2	2	2	2
θ_x	1	-1	1	-1

Line operator	$([rs], \mathbb{1}_{rs})$	$([rs], \tilde{\omega}_1)$	$([rs], \tilde{\omega}_2)$,	$([rs], \tilde{\omega}_1\tilde{\omega}_2)$
d_x	2	2	2	2
θ_x	1	-1	1	-1

Line operator	$([r], \mathbb{1}_r)$	$([r], \tilde{\omega})$	$([r], \tilde{\omega}^2)$,	$([r], \tilde{\omega}^3)$
d_x	2	2	2	2
θ_x	1	i	-1	i

where $\mathbb{1}$, π_1, π_2, π_3 are the 1-dimensional irreducible representations of D_8 and π_4 is the 2-dimensional one; π_1, π_2 and π_3 are the irreducible representations with kernels $\mathbb{Z}_4^{(r)}, \mathbb{Z}_2^{(r^2)} \times \mathbb{Z}_2^{(s)}$ and $\mathbb{Z}_2^{(r^2)} \times \mathbb{Z}_2^{(rs)}$, respectively; $\mathbb{1}_s$, ω_1, ω_2 and $\omega_1\omega_2$ are the four irreducible representations of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ centralizer of s; $\mathbb{1}_{rs}$, $\tilde{\omega}_1$, $\tilde{\omega}_2$ and $\tilde{\omega}_1\tilde{\omega}_2$ are the four irreducible representations of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ centralizer of rs; $\mathbb{1}_r$, ω, ω^2 and ω^3 are the four irreducible representations of the $\mathbb{Z}_4$ centralizer of r. The fusion rules of these line operators and their modular S matrix are given in [86, appendices A,B].

A line operator in $\mathcal{Z}(\text{Vec}_{D_8})$ which admits the structure of a Lagrangian algebra must be a direct sum of the bosonic line operators in the list above. In fact, using Theorem 4.2, we can identify all Lagrangian algebra objects in the SymTFT. For example, consider the algebra $(\mathbb{Z}_2^{(rs)}, 1)$. We have two Morita equivalent classes of algebras

$$A_1 := e + rs\,, \ \ A_2 = e + r^3 s. \tag{5.80}$$

We need to choose a Lagrangian algebra object L in the SymTFT such that $F(L)$ produces the above two algebras. Note that the Lagrangian algebra L has quantum dimension 8. Since $F : \mathcal{Z}(\text{Vec}_{D_8}) \to \text{Vec}_{D_8}$ preserves quantum dimensions, we must have

$$F(L) = 2A_1 + 2A_2\,, \ \text{or} \ F(L) = A_1 + 3A_2\,, \ \text{or} \ F(L) = 3A_1 + A_2\,. \tag{5.81}$$

Therefore, L must be of the form

$$\begin{aligned} L = \ & ([e], \mathbb{1}) + N_L^{([e],\pi_1)}([e], \pi_1) + N_L^{([e],\pi_2)}([e], \pi_2) + N_L^{([e],\pi_3)}([e], \pi_3) + N_L^{([e],\pi_4)}([e], \pi_4) + \\ & N_L^{([rs],\mathbb{1}_{rs})}([rs], \mathbb{1}_{rs}) + N_L^{([rs],\tilde{\omega}_2)}([rs], \tilde{\omega}_2)\,, \end{aligned} \tag{5.82}$$

where N_L^* are non-negative integers which need to be determined. Since the conjugacy class $[rs] = \{rs, r^3 s\}$, $F(L)$ contains equal number of rs and $r^3 s$ terms. This implies that $F(L) = A_1 + 3A_2$ and $F(L) = 3A_1 + A_2$ are inconsistent. Therefore, we must have $F(L) = 2A_1 + 2A_2$.

Suppose $N_L^{([rs],\mathbb{1}_{rs})} \neq 0$. We will now show that $N_L^{([rs],\mathbb{1}_{rs})} = 1$. To see this, suppose $N_L^{([rs],\mathbb{1}_{rs})} > 1$. We have the fusion rule [86, appendix B]

$$([rs], \mathbb{1}_{rs}) \times ([rs], \mathbb{1}_{rs}) = ([e], \pi_4) + ([r^2], \pi_4)\,. \tag{5.83}$$

Using the constraint (3.7) we find that

$$N_L^{([e],\pi_4)} \geq 4\,. \tag{5.84}$$

However, this contradicts the requirement that $F(L) = 2A_1 + 2A_2$ contains exactly 4 trivial line operators. Therefore, we must have $N_L^{([rs],\mathbb{1}_{rs})} = 1$. Since $F(L)$ must contain two copies of rs, we also find that $N_L^{([rs],\tilde{\omega}_2)} = 1$. Now, consider the fusion rule

$$([rs], \mathbb{1}_{rs}) \times ([rs], \tilde{\omega}_2) = ([e], \pi_4) + ([r^2], \pi_4)\,. \tag{5.85}$$

Since $([r^2], \pi_4)$ is a fermion, using (3.7) we find that $N_L^{([e],\pi_4)} = 1$. Combining the results above, we have determined that the Lagrangian algebra must be of the form

$$\begin{aligned} L = \; & ([e], \mathbb{1}) + N_L^{([e],\pi_1)}([e], \pi_1) + N_L^{([e],\pi_2)}([e], \pi_2) + N_L^{([e],\pi_3)}([e], \pi_3) + ([e], \pi_4) + \\ & ([rs], \mathbb{1}_{rs}) + ([rs], \tilde{\omega}_2)\,. \end{aligned} \tag{5.86}$$

L must have quantum dimension 8. Therefore, only one among $N_L^{([e],\pi_1)}$, $N_L^{([e],\pi_2)}$, $N_L^{([e],\pi_3)}$ can be non-zero. Using the fusion rules, we know that $([e], \pi_3)$ is the only line operator such that

$$([rs], \tilde{\omega}_2) \times ([e], \pi_2) \text{ contains } ([rs], \mathbb{1}_{rs})\,, \tag{5.87}$$

and

$$([rs], \mathbb{1}_{rs}) \times ([e], \pi_2) \text{ contains } ([rs], \tilde{\omega}_2)\,, \tag{5.88}$$

which implies that $N_L^{([e],\pi_3)} = 1$. Therefore, the Lagrangian algebra object corresponding to the algebra $(\mathbb{Z}_2^{(rs)}, 1)$ is

$$L = ([e], \mathbb{1}) + ([e], \pi_3) + ([e], \pi_4) + ([rs], \mathbb{1}_{rs}) + ([rs], \tilde{\omega}_2)\,. \tag{5.89}$$

Similar analysis can be used to determine all Lagrangian algebra objects in the SymTFT $\mathcal{Z}(\mathrm{Vec}_{D_8})$. They are summarized in the table below.

Algebra objects in Vec_{D_8}	Lagrangian algebra objects in $\mathcal{Z}(\mathrm{Vec}_{D_8})$
$(\mathbb{Z}_1, 1)$	$([e], \mathbb{1}) + ([e], \pi_1) + ([e], \pi_2) + ([e], \pi_3) + 2([e], \pi_4)$
$(\mathbb{Z}_2^{(r^2)}, 1)$	$([e], \mathbb{1}) + ([e], \pi_1) + ([e], \pi_2) + ([e], \pi_3)$ $+ ([r^2], \mathbb{1}) + ([r^2], \pi_1) + ([r^2], \pi_2) + ([r^2], \pi_3)$
$(\mathbb{Z}_4^{(r)}, 1)$	$([e], \mathbb{1}) + ([e], \pi_1) + 2([r], \mathbb{1}_r) + ([r^2], \mathbb{1}) + ([r^2], \pi_1)$
$(\mathbb{Z}_2^{(r^2)} \times \mathbb{Z}_2^{(s)}, 1)$	$([e], \mathbb{1}) + ([e], \pi_2) + 2([s], \mathbb{1}) + ([r^2], \mathbb{1}) + ([r^2], \pi_2)$
$(\mathbb{Z}_2^{(r^2)} \times \mathbb{Z}_2^{(s)}, \alpha)$	$([e], \mathbb{1}) + ([e], \pi_2) + 2([s], \omega_2) + ([r^2], \pi_1) + ([r^2], \pi_3)$
$(\mathbb{Z}_2^{(r^2)} \times \mathbb{Z}_2^{(rs)}, 1)$	$([e], \mathbb{1}) + ([e], \pi_3) + 2([rs], \mathbb{1}_{rs}) + ([r^2], \mathbb{1}) + ([r^2], \pi_3)$
$(\mathbb{Z}_2^{(r^2)} \times \mathbb{Z}_2^{(rs)}, \beta)$	$([e], \mathbb{1}) + ([e], \pi_3) + 2([rs], \tilde{\omega}_2) + ([r^2], \pi_1) + ([r^2], \pi_2)$
$(D_8, 1)$	$([e], \mathbb{1}) + ([r^2], \mathbb{1}) + ([s], \mathbb{1}_s) + ([r], \mathbb{1}_r) + ([rs], \mathbb{1}_{rs})$
(D_8, γ)	$([e], \mathbb{1}) + ([r^2], \pi_1) + ([s], \omega_2) + ([r], \mathbb{1}_r) + ([rs], \tilde{\omega}_2)$
$(\mathbb{Z}_2^{(s)}, 1)$	$([e], \mathbb{1}) + ([e], \pi_2) + ([e], \pi_4) + ([s], \mathbb{1}_s) + ([s], \omega_2)$
$(\mathbb{Z}_2^{(rs)}, 1)$	$([e], \mathbb{1}) + ([e], \pi_3) + ([e], \pi_4) + ([rs], \mathbb{1}_{rs}) + ([rs], \tilde{\omega}_2)$

(5.90)

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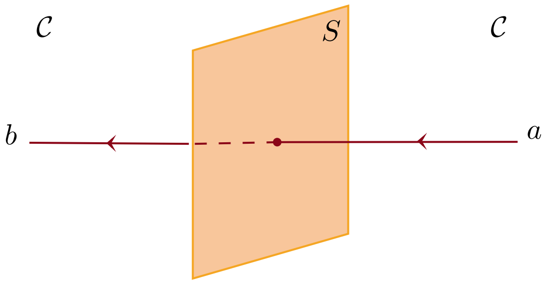

Figure 23. Action of a topological surface operator S on line operators.

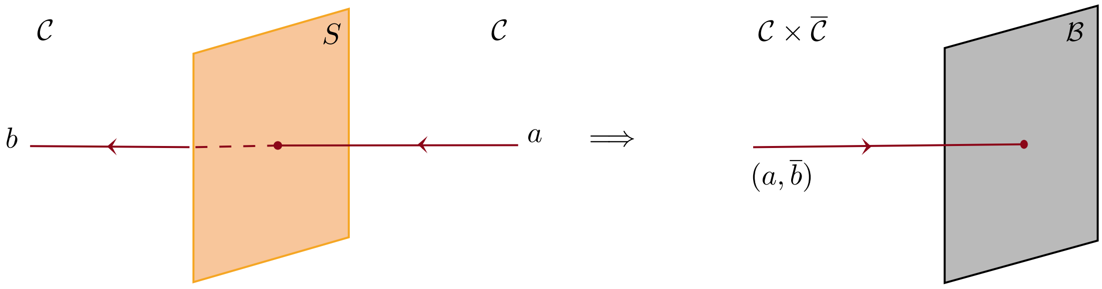

Figure 24. Folding trick to produce a gapped boundary from a surface operator.

6 Applications

6.1 Higher-gauging data from the action of surface operators

Consider a 2+1D TQFT whose line operators are described by a modular tensor category $\mathcal{C}$. Let S be a topological surface operator whose action on the line operators is given by

$$S \cdot a = \sum_{b \in \mathcal{C}} n^{b}_{S\cdot a} \, b \,. \tag{6.1}$$

S implements a non-invertible symmetry if the sum above is non-trivial. All surface operators in a 2+1D TQFT can be constructed through higher-gauging [65, 73, 74, 87]. Given (6.1), we can ask: which algebra $A \in \mathcal{C}$ should be higher-gauged to construct the surface operator S? When the higher-gauging involves only invertible line operators, this was answered in [75] by providing an explicit map between the action of the surface operator and the abelian group of lines + discrete torsion data for higher-gauging. In this section, we give a general answer to this question for arbitrary higher-gaugings.

If $n^{b}_{S\cdot a} \neq 0$, then a and b can form a non-trivial junction on the surface operator (see figure 23). Upon folding figure 23 the surface operator becomes a gapped boundary of the TQFT $\mathcal{C} \times \bar{\mathcal{C}}$ on which the line operator $(a, \bar{b})$ can end (see figure 24). Therefore, the action of the surface operator (6.1) specifies a Lagrangian algebra object

$$L_S = \sum_{a,b \in \mathcal{C}} n^{b}_{S\cdot a} \, (a, \bar{b}) \,. \tag{6.2}$$

In fact, the action of S on the fusion spaces of the line operators also determines the Lagrangian algebra structure on L_S (see [75] for more details). Consider the fusion of L_S

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on the canonical gapped boundary $\mathcal{B}_\mathcal{C}$ of $\mathcal{C}\times\bar{\mathcal{C}}$. From Theorem 4.1, we know that $F(L_S)$ is a sum of Morita equivalent algebras in $\mathcal{C}$. This is precisely the algebra which can be higher-gauged to construct the surface S.

Remark: for a modular tensor category $\mathcal{C}$, one can define the functors $T:\mathcal{C}\times\bar{\mathcal{C}}\to\mathcal{C}$ and $\mathcal{C}\to\mathcal{C}\times\bar{\mathcal{C}}$ as follows [29, 69]

$$T:(a,b)\to a\times b\,, \tag{6.3}$$

$$R:a\to\sum_{c\in\mathcal{C}}(a\times\bar{c},c)\,. \tag{6.4}$$

These functors can also be used to transport algebras between $\mathcal{C}$ and $\mathcal{C}\times\bar{\mathcal{C}}$. In fact, the functor T is the same as the forgetful functor $F:\mathcal{Z}(\mathcal{C})\to\mathcal{C}$ under the equivalence $\mathcal{Z}(\mathcal{C})\cong\mathcal{C}\boxtimes\bar{\mathcal{C}}$. Nevertheless, it maybe useful to use F instead of T since the isomorphisms $\Phi_{x,y}$ studied in section 4.2 are trivial for F while it is non-trivial for T.

6.2 A necessary condition for trivial anomaly

Let A be a non-simple line operator in $\mathcal{C}$. Suppose A is non-anomalous. Then we know that there must be a Lagrangian algebra L corresponding to A. Using Theorem 4.1, we must have

$$A\subseteq F(L)\,. \tag{6.5}$$

More explicitly, if

$$L=\sum_{(a,e_a)\in\mathcal{Z}(\mathcal{C})}N_L^{(a,e_a)}\,(a,e_a)\,, \tag{6.6}$$

then, we must have

$$\sum_{(a,e_a)\in\mathcal{Z}(\mathcal{C})}N_L^{(a,e_a)}N_a^b\neq 0\;\forall\;b\in A\,, \tag{6.7}$$

where N_a^b is non-zero if the line operators a and b can form a topological junction. For an arbitrarily chosen A, we may not have bosonic line operators $(a,e_a)\in\mathcal{Z}(\mathcal{C})$ satisfying (6.7) implying that A must be anomalous. In the next section, we will use this necessary condition to prove a statement about $\mathcal{C}$-symmetric trivially gapped phases.

6.2.1 Trivially gapped phase $\Longleftrightarrow$ magnetic Lagrangian algebra

In [34], the authors argue that a fusion category $\mathcal{C}$ admits a trivially gapped phase only if the SymTFT $\mathcal{Z}(\mathcal{C})$ admits a magnetic Lagrangian algebra L_m defined by the property that the only simple line operator L_m which can also end on the canonical gapped boundary $\mathcal{B}_\mathcal{C}$ is the trivial line operator. This is clear from the sandwich picture which shows that the local operators of the 1+1D TQFT $\mathcal{T}_L$ correspond to the line operators which can end both on $\mathcal{B}_{L_m}$ and $\mathcal{B}_\mathcal{C}$ (see (3.13)). In this section, we will use Theorem 4.1 to show that this is a necessary and sufficient condition for the existence of a $\mathcal{C}$-symmetric trivially gapped phase.

Consider a general haploid algebra object

$$A=\sum_a N_A^a\,a\,, \tag{6.8}$$

in a fusion category $\mathcal{C}$. The non-negative integers N_A^a are bounded by the quantum dimension $N_A^a \leq d_a$ [84, Proposition 3.3]. Therefore, the largest algebra object that a fusion category $\mathcal{C}$ can admit is of the form

$$A_{\max} := \sum_{a \in \mathcal{C}} d_a \ a \,. \tag{6.9}$$

This is consistent only if $d_a \ \forall a \in \mathcal{C}$ are integers. In fact, $A_{\max}$ admits the structure of an algebra (is non-anomalous) if and only if $\mathcal{C}$ admits a trivially gapped phase. To see this, suppose $\mathcal{C}$ admits a $\mathcal{C}$-symmetric TQFT that is trivially gapped. Then this TQFT only has the identity local operator. Correspondingly, there is only one simple 1D gapped boundary, say m. Therefore the module category $\mathcal{M}$ has only one simple object, m. Consider the algebra

$$A_m := \sum_{a \in \mathcal{C}} N_m^a \ a \,, \tag{6.10}$$

where N_m^a is the dimension of Hilbert space of operators at the junction of a with the gapped boundary m. Since m is the only gapped boundary, all line operators in $\mathcal{C}$ must be able to form a junction on it. Moreover, $N_m^a = d_a$ to preserve quantum dimensions. Therefore, we have

$$A_m = \sum_{a \in \mathcal{C}} \ d_a \ a \,. \tag{6.11}$$

Using [41] (see section 3.2), we know that $A_m = A_{\max}$ is non-anomalous.

Conversely, suppose $A_{\max}$ is non-anomalous. Consider the module category $\mathcal{M}$ corresponding to this algebra. Let $m \in \mathcal{M}$ and A_m be the line operator as defined above. We have the relation [88, Proposition 2.2]

$$\sum_m d_m^2 = \dim(\mathcal{C}) \,, \tag{6.12}$$

where

$$d_m^2 := d_{A_m} \,. \tag{6.13}$$

Now, since $\mathcal{M}$ is the module category defined using the algebra $A_{\max}$, there is some $m' \in \mathcal{M}$ such that

$$A_{m'} = A_{\max} \,. \tag{6.14}$$

Then, we have $d_{A_{m'}} = d_{A_{\max}} = \dim(\mathcal{C})$. To be compatible with the equation (6.12), m' must be the only simple object in $\mathcal{M}$. In other words, the $\mathcal{C}$-symmetric TQFT defined using $\mathcal{M}$ has only one simple gapped boundary. Consequently, it is trivially gapped and the whole symmetry $\mathcal{C}$ is preserved. It follows from this argument that $A_{\max}$ is not Morita equivalent to any other haploid algebra.

From Theorem 4.1, we know that

$$F(L) = \sum_{m \in \mathcal{M}_L} A_m \,. \tag{6.15}$$

The number of haploid algebras in the sum above is equal to the number of identity line operators in $F(L)$. Since F preserves quantum dimensions and $d_L = \dim(\mathcal{C})$ by definition,

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the largest haploid algebra object that we can get on the R.H.S of the above equation is precisely $A_{\max}$. Therefore, we find

$$F(L) = A_{\max} \tag{6.16}$$

if and only if $F(L)$ contains a single identity operator. Since $F(L)$ is obtained from the perpendicular fusion of L on the canonical gapped boundary $\mathcal{B}_{\mathcal{C}}$, this implies that L does not contain any non-trivial line operators which can end on $\mathcal{B}_{\mathcal{C}}$. Therefore, L is a magnetic Lagrangian algebra with respect to $\mathcal{B}_{\mathcal{C}}$.

6.3 Transporting non-anomalous line operators between fusion categories

Consider two fusion categories $\mathcal{C}$ and $\mathcal{D}$ which share the same SymTFT. In many cases, $\mathcal{D}$ might be a simpler fusion category in which the non-anomalous line operators can be explicitly classified, while it could be hard to do the same in $\mathcal{C}$. For example, $\mathcal{C}$ could be a fusion category without multiplicity in its fusion coefficients while $\mathcal{D}$ is not multiplicity-free.

Let $[B_1], \dots [B_n]$ be the n Morita equivalence classes of algebra objects/non-anomalous line operators in $\mathcal{D}$ for some integer n. Using the data of this algebra, we can determine all the Lagrangian algebras $L_1, \dots, L_n$ of the SymTFT $\mathcal{Z}(\mathcal{D})$. As we will see in explicit examples below, in many cases, even without knowledge of the multiplication in the algebras in $\mathcal{D}$, we can determine the objects $L_1, \dots, L_n$ which admit the structure of a Lagrangian algebra. Now, since the fusion category $\mathcal{C}$ share the same SymTFT, there must be a boundary condition $\mathcal{B}_{\mathcal{C}}$ of $\mathcal{Z}(\mathcal{D})$ on which the line operators form the fusion category $\mathcal{C}$. The non-anomalous line operators in $\mathcal{C}$ can then be determined by fusing $L_1, \dots, L_n$ with the gapped boundary $\mathcal{B}_{\mathcal{C}}$ to get $F_{\mathcal{B}_{\mathcal{C}}}(L_1), F_{\mathcal{B}_{\mathcal{C}}}(L_2), \dots, F_{\mathcal{B}_{\mathcal{C}}}(L_n)$. In this way, the non-anomalous line operators in $\mathcal{D}$ can be 'transported' through the SymTFTs to non-anomalous line operators in $\mathcal{C}$.

In the following subsections, we will go through three examples in detail to illustrate the idea described above. First we will consider the fusion categories

$$\mathcal{C} = \mathrm{Rep}(D_8), \quad \mathcal{D} = \mathrm{Vec}_{D_8}\,, \tag{6.17}$$

where D_8 is the dihedral group with 8 elements. We will use the results from section 5.4 to determine all algebra objects in $\mathrm{Rep}(D_8)$ recovering the classification of algebra objects of $\mathrm{Rep}(D_8)$ in [33, 85]. In appendix C, we use a similar method to classify the algebra objects in A_4, where A_4 is the alternating group of order 12. This group is the smallest for which the category $\mathrm{Rep}(A_4)$ has multiplicities in its fusion rules. We will show how the non-anomalous line operators in Vec_{A_4} can be used to determine the non-anomalous line operators in $\mathrm{Rep}(A_4)$. The final example that we consider is the three Morita equivalent Haagerup fusion categories.

$$\mathcal{C} = \mathscr{H}_1, \quad \mathcal{D} = \mathscr{H}_2, \quad \mathcal{E} = \mathscr{H}_3\,. \tag{6.18}$$

6.3.1 Rep(D_8) and Vec$_{D_8}$

$\mathrm{Rep}(D_8)$ is a non-invertible symmetry of various CFTs [33, 89] and lattice models [90]. In this section, we will identify the non-anomalous line operators (algebra objects) in $\mathrm{Rep}(D_8)$ using the Lagrangian algebras in $\mathcal{Z}(\mathrm{Rep}(D_8))$ and Theorem 4.1.

Note that the SymTFT $\mathcal{Z}(\mathrm{Vec}_{D_8})$ is the same as $\mathcal{Z}(\mathrm{Rep}(D_8))$ since the symmetries Vec_{D_8} and $\mathrm{Rep}(D_8)$ are dual to each other under gauging. Therefore, the Lagrangian algebras in $\mathcal{Z}(\mathrm{Rep}(D_8))$ are the same as in $\mathcal{Z}(\mathrm{Vec}_{D_8})$ which were constructed explicitly in section 5.4. Recall that the bosonic line operators in $\mathcal{Z}(\mathrm{Vec}_{D_8})$ are

$$([e],\mathbb{1})\,,\ ([e],\pi_1)\,,\ ([e],\pi_2)\,,\ ([e],\pi_3)\,,\ ([e],\pi_4)\,, \tag{6.19}$$
$$([r^2],\mathbb{1})\,,\ ([r^2],\pi_1)\,,\ ([r^2],\pi_2)\,,\ ([r^2],\pi_3)\,,\ ([s],\mathbb{1}_s)\,, \tag{6.20}$$
$$([s],\omega_2)\,,\ ([rs],\mathbb{1}_{rs})\,,\ ([rs],\tilde{\omega}_2)\,,\ ([r],\mathbb{1}_r)\,. \tag{6.21}$$

All Lagrangian algebras in $\mathcal{Z}(\mathrm{Vec}_{D_8})$ are certain linear combinations of these bosons. The Lagrangian algebra

$$L_{\mathrm{Rep}(D_8)} = ([e],\mathbb{1}) + ([r^2],\mathbb{1}) + ([s],\mathbb{1}_s) + ([r],\mathbb{1}_r) + ([rs],\mathbb{1}_{rs})\,, \tag{6.22}$$

specifies a gapped boundary on which the line operators form the fusion category $\mathrm{Rep}(D_8)$. Therefore, in order to determine the algebra objects in $\mathrm{Rep}(D_8)$ we can use the map

$$F_{\mathcal{B}_{\mathrm{Rep}(D_8)}} : \mathcal{Z}(\mathrm{Rep}(D_8)) \to \mathrm{Rep}(D_8)\,, \tag{6.23}$$

and using Theorem 4.1. In order to determine $F_{\mathcal{B}_{\mathrm{Rep}(D_8)}}$ it is useful to write the bosonic line operators in the form (a, e_a) for some $a \in \mathrm{Rep}(D_8)$ and half-braiding e_a. Then the map $F_{\mathcal{B}_{\mathrm{Rep}(D_8)}}$ is just given by

$$F_{\mathcal{B}_{\mathrm{Rep}(D_8)}}((a,e_a)) = a\,. \tag{6.24}$$

Using the construction of $\mathcal{Z}(\mathrm{Rep}(D_8))$ as the Drinfeld center of the Tambara-Yamagami category $\mathrm{Rep}(D_8)$ we have [34, 91]

$$\begin{aligned}
&([e],\mathbb{1}) \to (\mathbb{1}, e_{\mathbb{1}})\,,\ ([e],\pi_1) \to (\pi_1, e_{\pi_1})\,,\ ([e],\pi_2) \to (\pi_2, e_{\pi_2})\,,\ ([e],\pi_3) \to (\pi_3, e_{\pi_3})\,,\\
&([e],\pi_4) \to (\pi_4, e^{(1)}_{\pi_4})\,,\ ([r^2],\mathbb{1}) \to (\mathbb{1}, e'_{\mathbb{1}})\,,\ ([r^2],\pi_1) \to (\pi_1, e'_{\pi_1})\,,\\
&([r^2],\pi_2) \to (\pi_2, e'_{\pi_2})\,,\ ([r^2],\pi_3) \to (\pi_3, e'_{\pi_3})\,,\ ([s],\mathbb{1}_s) \to (\mathbb{1}+\pi_2, e_{\mathbb{1}+\pi_2})\,,\\
&([s],\omega_2) \to (\pi_4, e^{(2)}_{\pi_4})\,,\ ([rs],\mathbb{1}_{rs}) \to (1+\pi_3, e_{1+\pi_3})\,,\ ([rs],\tilde{\omega}_2) \to (\pi_4, e^{(3)}_{\pi_4})\,,\\
&([r],\mathbb{1}_r) \to (\mathbb{1}+\pi_1, e_{\mathbb{1}+\pi_1})\,,
\end{aligned} \tag{6.25}$$

where the explicit form of the half-braidings is not specified as they will not play a role in the following discussion. In general, the line operators in $\mathcal{Z}(\mathrm{Vec}_{D_8})$ and $\mathcal{Z}(\mathrm{Rep}(D_8))$ are related by

$$([g],\pi_g) \to (\mathrm{Ind}^{D_8}_{C_g}(\pi_g), e)\,, \tag{6.26}$$

for some half-braiding e and $\mathrm{Ind}^{D_8}_{C_g}(\pi_g)$ is the induction of the representation π_g of C_g to the full group D_8. The Lagrangian algebra objects in $\mathcal{Z}(\mathrm{Rep}(D_8))$ and the corresponding

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non-anomalous line operators in $\mathrm{Rep}(D_8)$ are given in the following table.

Lagrangian algebra objects in $\mathcal{Z}(\mathrm{Rep}(D_8))$	Non-anomalous lines in $\mathrm{Rep}(D_8)$
$(\mathbb{1}, e_{\mathbb{1}}) + (\pi_1, e_{\pi_1}) + (\pi_2, e_{\pi_2}) + (\pi_3, e_{\pi_3}) + 2(\pi_4, e_{\pi_4})$	$\mathbb{1} + \pi_1 + \pi_2 + \pi_3 + 2\pi_4$
$(\mathbb{1}, e_{\mathbb{1}}) + (\pi_1, e_{\pi_1}) + (\pi_2, e_{\pi_2}) + (\pi_3, e_{\pi_3})$ $+ (\mathbb{1}, e_{\mathbb{1}}) + (\pi_1, e'_{\pi_1}) + (\pi_2, e'_{\pi_2}) + (\pi_3, e'_{\pi_3})$	$2\mathbb{1} + 2\pi_1 + 2\pi_2 + 2\pi_3$
$(\mathbb{1}, e_1) + (\pi_1, e_{\pi_1}) + 2(\mathbb{1} + \pi_1, e_{\mathbb{1}+\pi_1}) + (\mathbb{1}, e'_{\mathbb{1}}) + (\pi_1, e'_{\pi_1})$	$4\mathbb{1} + 4\pi_1$
$(\mathbb{1}, e_1) + (\pi_2, e_{\pi_2}) + 2(\mathbb{1} + \pi_2, e_{\mathbb{1}+\pi_2}) + (\mathbb{1}, e'_{\mathbb{1}}) + (\pi_2, e'_{\pi_2})$	$4\mathbb{1} + 4\pi_2$
$(\mathbb{1}, e_1) + (\pi_2, e_{\pi_2}) + 2(\pi_4, e^{(2)}_{\pi_4}) + (\pi_1, e'_{\pi_1}) + (\pi_3, e'_{\pi_3})$	$\mathbb{1} + \pi_1 + \pi_2 + \pi_3 + 2\pi_4$
$(\mathbb{1}, e_1) + (\pi_3, e_{\pi_3}) + 2(\mathbb{1} + \pi_3, e_{\mathbb{1}+\pi_3}) + (\mathbb{1}, e'_{\mathbb{1}}) + (\pi_3, e_{\pi_3})$	$4\mathbb{1} + 4\pi_3$
$(\mathbb{1}, e_1) + (\pi_3, e_{\pi_3}) + 2(\pi_4, e^{(3)}_{\pi_4}) + (\pi_1, e'_{\pi_1}) + (\pi_2, e'_{\pi_2})$	$\mathbb{1} + \pi_1 + \pi_2 + \pi_3 + 2\pi_4$
$(\mathbb{1}, e_{\mathbb{1}}) + (\mathbb{1}, e'_{\mathbb{1}}) + (\mathbb{1} + \pi_2, e_{\mathbb{1}+\pi_2}) +$ $(\mathbb{1} + \pi_1, e_{\mathbb{1}+\pi_1}) + (\mathbb{1} + \pi_3, e_{\mathbb{1}+\pi_3})$	$5\mathbb{1} + \pi_1 + \pi_2 + \pi_3$
$(\mathbb{1}, e_{\mathbb{1}}) + (\pi_1, e'_{\pi_1}) + (\pi_4, e^{(2)}_{\pi_4}) +$ $(\mathbb{1} + \pi_1, e_{\mathbb{1}+\pi_1}) + (\pi_4, e^{(3)}_{\pi_4})$	$2\mathbb{1} + 2\pi_1 + 2\pi_4$
$(\mathbb{1}, e_{\mathbb{1}}) + (\pi_2, e_{\pi_2}) + (\pi_4, e_{\pi_4}) + (\mathbb{1} + \pi_2, e_{\mathbb{1}+\pi_2}) + (\pi_4, e^{(2)}_{\pi_4})$	$2\mathbb{1} + 2\pi_2 + 2\pi_4$
$(\mathbb{1}, e_{\mathbb{1}}) + (\pi_3, e_{\pi_3}) + (\pi_4, e_{\pi_4}) + (\mathbb{1} + \pi_3, e_{\mathbb{1}+\pi_3}) + (\pi_4, e^{(3)}_{\pi_4})$	$2\mathbb{1} + 2\pi_3 + 2\pi_4$

(6.27)

We find that there are three different Lagrangian algebra objects in the SymTFT which gives the algebra object

$$\mathbb{1} + \pi_1 + \pi_2 + \pi_3 + 2\pi_4 \,. \tag{6.28}$$

Therefore, there are three distinct multiplications on this algebra object. This agrees with the fact that $\mathrm{Rep}(D_8)$ admits three fibre functors. The algebra object

$$2\mathbb{1} + 2\pi_1 + 2\pi_2 + 2\pi_3 \,, \tag{6.29}$$

decomposes into two copies of the algebra object $\mathbb{1} + \pi_1 + \pi_2 + \pi_3$. This corresponds to gauging the $\mathbb{Z}_2 \times \mathbb{Z}_2$ invertible line operators in $\mathrm{Rep}(D_8)$. The algebra object

$$4\mathbb{1} + 4\pi_1 \,, \tag{6.30}$$

decomposes into four copies of the algebra object $\mathbb{1} + \pi_1$. The same is true for the algebra objects $4\mathbb{1} + 4\pi_2$ and $4\mathbb{1} + 4\pi_3$. These again correspond to gauging invertible line operators in $\mathrm{Rep}(D_8)$.

We also have the algebra object

$$5\mathbb{1} + \pi_1 + \pi_2 + \pi_3 \,, \tag{6.31}$$

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which decomposes into $4A_1 + A_2$ where $A_1 := \mathbb{1}, A_2 := \mathbb{1} + \pi_1 + \pi_2 + \pi_3$. This shows that $\mathbb{1} + \pi_1 + \pi_2 + \pi_3$ is a Morita trivial algebra which agrees with the fusion rules of $\mathrm{Rep}(D_8)$. Finally, we have the algebra object

$$2\mathbb{1} + 2\pi_1 + 2\pi_4 \tag{6.32}$$

which decomposes into two copies of the algebra $\mathbb{1} + \pi_1 + \pi_4$.[17] The same is true for the algebra objects $2\mathbb{1} + 2\pi_2 + 2\pi_4$ and $2\mathbb{1} + 2\pi_3 + 2\pi_4$. The haploid algebra objects obtained above from Lagrangian algebra objects of the bulk SymTFT agree with [33, 85].

6.3.2 Haagerup fusion categories

The Haagerup fusion categories $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}_3$ is a Morita equivalence class of three fusion categories [30, 92]. The fusion category $\mathscr{H}_1$ has line operators $\mathbf{1}, \nu, \eta, \mu$ with fusion rules given by

	$\mathbf{1}$	ν	η	μ
$\mathbf{1}$	$\mathbf{1}$	ν	η	μ
ν	ν	$\mathbf{1} + 2\nu + 2\eta + \mu$	$2\nu + \eta + \mu$	μ
η	η	$2\nu + \eta + \mu$	$\mathbf{1} + \nu + \eta + \mu$	$\nu + \eta$
μ	μ	$\nu + \eta + mu$	$\nu + \eta$	$\mathbf{1} + \nu$

(6.33)

These line operators have quantum dimensions given by

$$1, d+1, d, d-1\,, \tag{6.34}$$

respectively where $d = \frac{3+\sqrt{13}}{2}$. The fusion rules are commutative and have multiplicity.

Both $\mathscr{H}_2$ and $\mathscr{H}_3$ have line operators $\mathbf{1}, \alpha, \alpha^2, \rho, \alpha\rho, \alpha^2\rho$ and the same fusion rules given by

	$\mathbf{1}$	α	α^2	ρ	$\alpha\rho$	$\alpha^2\rho$
$\mathbf{1}$	$\mathbf{1}$	α	α^2	ρ	$\alpha\rho$	$\alpha^2\rho$
α	α	α^2	$\mathbf{1}$	$\alpha\rho$	$\alpha^2\rho$	ρ
α^2	α^2	$\mathbf{1}$	α	$\alpha^2\rho$	ρ	$\alpha\rho$
ρ	ρ	$\alpha^2\rho$	$\alpha\rho$	$\mathbf{1} + H$	$\alpha^2 + H$	$\alpha + H$
$\alpha\rho$	$\alpha\rho$	ρ	$\alpha^2\rho$	$\alpha + H$	$\mathbf{1} + H$	$\alpha^2 + H$
$\alpha^2\rho$	$\alpha^2\rho$	$\alpha\rho$	ρ	$\alpha^2 + H$	$\alpha + H$	$\mathbf{1} + H$

(6.35)

where $H = \rho + \alpha\rho + \alpha^2\rho$. The quantum dimensions of the lines are

$$1, 1, 1, d, d, d\,, \tag{6.36}$$

respectively. Note that the fusion rules are non-commutative and without multiplicity.

[17] At the level of the object we can also consider the decomposition of $2\mathbb{1}+2\pi_2+2\pi_4$ into two copies of $\mathbb{1}+\pi_1$ and $\mathbb{1}+\pi_4$. $\mathbb{1}+\pi_1$ admits a unique multiplication on it. Therefore, this decomposition is not consistent with the fact that $\mathbb{1}+\pi_1$ corresponds to the Lagrangian algebra object $(\mathbb{1}, e_1)+(\pi_1, e_{\pi_1})+2(\mathbb{1}+\pi_1, e_{\mathbb{1}+\pi_1})+(\mathbb{1}, e'_{\mathbb{1}})+(\pi_1, e'_{\pi_1})$.

The fusion category $\mathscr{H}_2$ has three module categories $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ and we have [30] (see also [93, section 7], appendix E)

$$\begin{aligned} \bigoplus_{m\in\mathcal{M}_1} A_m &= 4\mathbf{1} + \alpha + \alpha^2 + 3H\,. \\ \bigoplus_{m\in\mathcal{M}_2} A_m &= 6\mathbf{1} + 3H\,. \\ \bigoplus_{m\in\mathcal{M}_3} A_m &= 2\mathbf{1} + 2\alpha + 2\alpha^2 + 3H\,. \end{aligned} \tag{6.37}$$

These three algebras correspond to three Lagrangian algebra objects in the SymTFT $\mathcal{Z}(\mathscr{H}_2)$.

$\mathcal{Z}(\mathscr{H}_2)$ has line operators [92]

$$\mathbf{1}, \pi_1, \pi_2, \sigma_1, \sigma_2, \sigma_3, \mu_1, \mu_2, \ldots \mu_6\,. \tag{6.38}$$

These have quantum dimensions

$$1, 3d+1, 3d+2, 3d+2, 3d+2, 3d+2, 3d, 3d, 3d, 3d, 3d, 3d\,, \tag{6.39}$$

respectively. The fusion rules and modular data of these line operators can be found, for example, in [94]. The only bosonic line operators are $\mathbf{1}, \pi_1, \pi_2, \sigma_1$. All Lagrangian algebras in this SymTFT must be supported on a linear combination of these bosonic line operators. In order to identity the Lagrangian algebra corresponding to the three classes of algebras in $\mathscr{H}_2$, it is useful to write the bosonic line operators as (a, e_a) for some $a \in \mathscr{H}_2$ and half-braidings e_a. This is given explicitly in [95, section 2.2] (see also [96, section 4.4.3]) and is given by

$$\mathbf{1} \to (\mathbf{1}, e_{\mathbf{1}}),\;\; \pi_1 \to (\mathbf{1}+\rho, e_{\mathbf{1}+\rho}),\;\; \pi_2 \to (\alpha+\alpha^2+H, e_{\alpha+\alpha^2+H}),\;\; \sigma_1 \to (\mathbf{1}+\mathbf{1}+H, e_{\mathbf{1}+\mathbf{1}+H}) \tag{6.40}$$

where $H = \rho + \alpha\rho + \alpha^2\rho$. Consider the Lagrangian algebra

$$L = N_L^{\mathbf{1}}\mathbf{1} + N_L^{\pi_1}\pi_1 + N_L^{\pi_2}\pi_2 + N_L^{\sigma_1}\sigma_1. \tag{6.41}$$

Moving L to the canonical gapped boundary $\mathcal{B}_{\mathscr{H}_2}$ corresponding to the fusion category $\mathscr{H}_2$, we get

$$F_{\mathcal{B}_{\mathscr{H}_2}}(L) = N_L^{\mathbf{1}}\mathbf{1} + N_L^{\pi_1}(\mathbf{1}+\rho) + N_L^{\pi_2}(\alpha+\alpha^2+H) + N_L^{\sigma_1}(\mathbf{1}+\mathbf{1}+H)\,, \tag{6.42}$$

where $N_L^{\pi_1}, N_L^{\pi_2}$ and $N_L^{\sigma_1}$ are non-negative integers. Using Theorem 4.1 we know that $F_{\mathcal{B}_{\mathscr{H}_2}}(L)$ must agree with one of the algebra objects in (6.37). Using this, we get the three Lagrangian algebras

$$\mathcal{M}_1 \to L_1 := \mathbf{1} + \pi_1 + \pi_2 + \sigma_1\,, \tag{6.43}$$

$$\mathcal{M}_2 \to L_2 := \mathbf{1} + \pi_1 + 2\sigma_1\,, \tag{6.44}$$

$$\mathcal{M}_3 \to L_3 := \mathbf{1} + \pi_1 + 2\pi_2\,. \tag{6.45}$$

The canonical gapped boundary corresponding to $\mathscr{H}_2$ is given by the Lagrangian algebra L_2. L_3 corresponds to the fusion category $\mathscr{H}_3$ and L_1 corresponds to $\mathscr{H}_1$.

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The algebra objects/non-anomalous line operators in $\mathscr{H}_3$ can be found by computing

$$F_{\mathcal{B}_{L_3}}(L_1),\ F_{\mathcal{B}_{L_3}}(L_2) \text{ and } F_{\mathcal{B}_{L_3}}(L_3). \tag{6.46}$$

Using [96, section 4.4.3], we find

$$F_{\mathcal{B}_{L_3}}(\pi_1) = \mathbf{1} + H,\ F_{\mathcal{B}_{L_3}}(\pi_2) = \mathbf{1} + \mathbf{1} + H \text{ and } F_{\mathcal{B}_{L_3}}(\sigma_1) = \alpha + \alpha^2 + H\,. \tag{6.47}$$

Using this, we get

$$\begin{aligned} F_{\mathcal{B}_{L_3}}(L_1) &= \mathbf{1} + \mathbf{1} + H + \mathbf{1} + \mathbf{1} + H + \alpha + \alpha^2 + H = 4\mathbf{1} + \alpha + \alpha^2 + 3H\,. \\ F_{\mathcal{B}_{L_3}}(L_2) &= \mathbf{1} + \mathbf{1} + H + 2(\alpha^2 + \alpha^2 + H) = 2\mathbf{1} + 2\alpha + 2\alpha^2 + 3H\,, \\ F_{\mathcal{B}_{L_3}}(L_3) &= \mathbf{1} + \mathbf{1} + H + 2(\mathbf{1} + \mathbf{1} + H) = 6\mathbf{1} + 3H\,, \end{aligned} \tag{6.48}$$

These three algebra objects are consistent with the three module categories in $\mathscr{H}_3$ [30] (see appendix E).

Finally, by computing

$$F_{\mathcal{B}_{L_1}}(\pi_1),\ F_{\mathcal{B}_{L_1}}(\pi_2) \text{ and } F_{\mathcal{B}_{L_1}}(\sigma_1)\,, \tag{6.49}$$

we can find the non-anomalous line operators in $\mathscr{H}_1$. This requires finding the bulk-to-boundary map for the boundary condition $\mathcal{B}_{L_1}$. This can be determined by taking the quotient of $\mathcal{Z}(\mathscr{H}_2)$ by the Lagrangian algebra L_1 as described in [67]. The details of this calculation are given in appendix D. We get,

$$F_{\mathcal{B}_{L_1}}(\pi_1) = \mathbf{1} + \nu + \eta + \mu,\ F_{\mathcal{B}_{L_1}}(\pi_2) = \mathbf{1} + 2\eta + \nu \text{ and } F_{\mathcal{B}_{L_1}}(\sigma_1) = \mathbf{1} + 2\nu + \mu\,. \tag{6.50}$$

Using this, we find

$$\begin{aligned} F_{\mathcal{B}_{L_1}}(L_1) &= \mathbf{1} + \mathbf{1} + \nu + \eta + \mu + \mathbf{1} + 2\eta + \nu + \mathbf{1} + 2\nu + \mu = 4\mathbf{1} + 4\nu + 3\eta + 2\mu\,. \\ F_{\mathcal{B}_{L_1}}(L_2) &= \mathbf{1} + \mathbf{1} + \nu + \eta + \mu + 2(\mathbf{1} + 2\nu + \mu) = 4\mathbf{1} + 5\nu + \eta + 3\mu\,, \\ F_{\mathcal{B}_{L_1}}(L_3) &= \mathbf{1} + \mathbf{1} + \nu + \eta + \mu + 2(\mathbf{1} + 2\eta + \nu) = 4\mathbf{1} + 3\nu + 5\eta + \mu\,. \end{aligned} \tag{6.51}$$

These three algebra objects are consistent with the three module categories of $\mathscr{H}_1$ [30] (see appendix E).

7 Conclusion

In this paper, we studied the non-anomalous line operators in a 1+1D QFT with fusion category symmetry $\mathcal{C}$ using gapped boundaries of the SymTFT $\mathcal{Z}(\mathcal{C})$. Physically equivalent gaugings in 1+1D correspond to the same gapped boundary of the SymTFT. Our analysis determines the explicit map between them.

- Given a Lagrangian algebra object L in $\mathcal{Z}(\mathcal{C})$ the corresponding physically equivalent gaugings in $\mathcal{C}$ can be found from $F(L)$, where F is the bulk-to-boundary map. (Theorem 4.1).

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- In fact, F maps the algebra structure of L to the line operator $F(L)$ through equation (4.15).
- The explicit examples considered in this paper show that in some cases Theorems 4.1 and 4.2 can be used to determine all non-anomalous line operators in $\mathcal{C}$ even without the knowledge of the algebra structure on the Lagrangian algebras in $\mathcal{Z}(\mathcal{C})$.
- Conversely, knowledge of non-anomalous line operators in $\mathcal{C}$ puts strong constraints on which line operators in the bulk SymTFT can form a Lagrangian algebra. We used this to find a necessary condition for a line operator $A \in \mathcal{C}$ to be non-anomalous. This was then used to prove that $\mathcal{C}$-symmetric trivially gapped phases exist if and only if the bulk SymTFT admits a magnetic Lagrangian algebra.
- We discussed the notion of transporting non-anomalous line operators between fusion categories which share the same SymTFT. This method can be used to determine non-anomalous line operators in a fusion category from those in a "simpler" fusion category. This provides an alternative to using NIM-reps to determine algebra objects in a fusion category $\mathcal{C}$.

The following are some interesting future directions:

- It will be interesting to generalize the analysis in this paper to SymTFTs for fermionic symmetries [97].
- Generalization to higher dimensions. It will be interesting to apply the ideas presented here to the setting in [48, 62, 98, 99].
- General relation between anomalies and gapped boundaries for SymTFTs of non-finite and/or continuous symmetries [100–105].
- For an invertible non-anomalous symmetry G, the discrete torsion takes values in $H^2(G, \mathrm{U}(1))$. It would be interesting to explore the structure of generalized discrete torsions for a fixed non-anomalous line operator using gapped boundaries of the SymTFT.
- In [106], the authors show that massive 1+1D QFTs with spontaneously broken non-invertible symmetries often have particle degeneracies. A crucial part of finding the degeneracies involves determining the kernel and cokernel of certain maps relating different 1D gapped boundaries of the QFT. It will be interesting to use the map from gapped interfaces between 1+1D gapped boundaries of $\mathcal{Z}(\mathcal{C})$ and 1D gapped boundaries of 1+1D QFTs/TQFTs studied in this work to give a bulk perspective on the results in [106].

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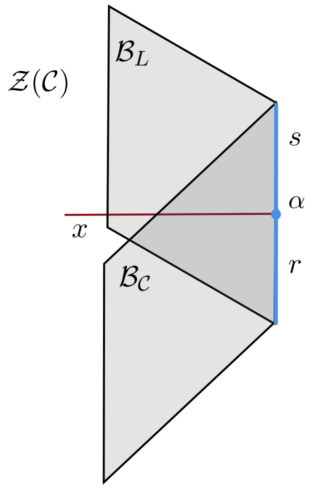

Figure 25. $N_{xr}^s \neq 0$ implies a non-trivial junction between the lines x and the gapped interfaces r and s where α labels a basis of point operators at the junction.

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A Constraints on the action of line operators on gapped interfaces

In Theorem 4.1, we learned that

$$F(L) = \sum_{m \in \mathcal{M}_L} A_m \,. \tag{A.1}$$

From figure 14 and surrounding discussion we know that finding the haploid algebra objects in $F(L)$ directly from the SymTFT requires determining how the line operators on the gapped boundary $\mathcal{B}_\mathcal{C}$ act on the gapped interfaces between $\mathcal{B}_\mathcal{C}$ and $\mathcal{B}_L$. When $\mathcal{Z}(\mathcal{C})$ is a Dijkgraaf-Witten theory, this can be determined using the results in [67].

In this section, we will determine some general constraints on the action of the line operators in $F(L)$ on gapped interfaces which may be used in explicit computations to determine haploid sub-objects of $F(L)$. To this end, let us consider the action of $x \in L$ on the gapped interface r. We get

$$x \times r = \sum_s N_{xr}^s \; s \,, \tag{A.2}$$

where N_{xr}^s are non-negative integers and s labels all the simple gapped interfaces between $\mathcal{B}_\mathcal{C}$ and $\mathcal{B}_L$. If $N_{xr}^s \neq 0$, then we get the configuration in figure 25. The condition $N_{xr}^s \neq 0$ also implies the non-trivial configurations in figure 26. Therefore, we have

$$N_{xr}^s = \sum_{b \in \mathcal{C}_L} N_x^b N_{br}^s = \sum_{a \in \mathcal{C}} N_x^a N_{ar}^s \,, \tag{A.3}$$

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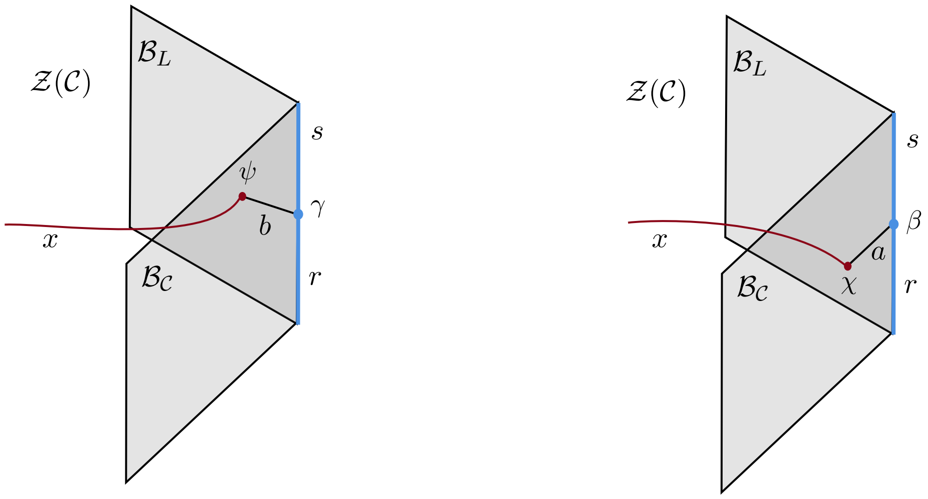

Figure 26. If $N_{xr}^s \neq 0$, then there must be some line operator $b \in F_{\mathcal{B}_L}(L)$ such that the diagram on the left is non-trivial. Similarly, there must be some line operator $a \in F_{\mathcal{B}_\mathcal{C}}(L)$ such that the diagram on the right is non-trivial.

where $\mathcal{C}_L$ is the category of line operators on the gapped boundary $\mathcal{B}_L$. Since $x \in L$, the fusion of x with the gapped boundary $\mathcal{B}_L$ contains the identity line in $\mathcal{C}_L$. In other words,

$$F_{\mathcal{B}_L}(x) = 1 + \dots \,. \tag{A.4}$$

Therefore, we have

$$N_{xr}^r \neq 0 \;\forall\; x \in L\,, \tag{A.5}$$

for any gapped interface r. Using (A.3), we find that

$$\sum_{a \in \mathcal{C}} N_x^a N_{ar}^r \neq 0\,. \tag{A.6}$$

Therefore, for a fixed simple gapped interface r, there is some $a \in F_{\mathcal{B}_\mathcal{C}}(x)$ which acts trivially on it.

B Proof of theorem 4.2

In this section, we will review some properties of characters of simple line operators in $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ and give a proof of Theorem 4.2.

Recall that Vec_G^ω is the category of G-graded vector spaces with associator given by the 3-cocycle ω. An ω-projective G-action on a G-graded vector space $V = \bigoplus_{g\in G} V_g$ is a set of linear maps

$$f_g : V \to V\,,\;\; g \in G\,, \tag{B.1}$$

such that

$$f_g(V_h) = V_{ghg^{-1}}\,, \text{and } (f_g f_h)(v) = \omega(g,h|k)\; f_g(f_h(v))\,,\; \forall\; v \in V_k\,, \tag{B.2}$$

where

$$\omega(g,h|k) := \omega(g,h,k)^{-1}\omega(g,hgh^{-1},h)\omega(ghk(gh)^{-1},g,h)^{-1}\,. \tag{B.3}$$

It is known that any line operator in the SymTFT $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ can be labeled by a G-graded vector space V with α-projective G-action (see [77, section 2.1, 3.1]). Its character is defined as

$$\chi_V(g,h) := \mathrm{Tr}_{V_g}(h)\,, \tag{B.4}$$

where g,h is a pair of commuting elements in G. For any two $x,y \in \mathcal{Z}(\mathrm{Vec}_G^\omega)$, it can be shown that the characters satisfy [80]

$$\chi_x(kgk^{-1},khk^{-1}) = \frac{\omega(k,h|g)}{\omega(khk^{-1},k|g)}\chi_x(g,h)\,, \tag{B.5}$$

$$\chi_{x\times y}(g,h) = \chi_x(g,h)\times\chi_y(g,h)\,, \tag{B.6}$$

$$\chi_{x+y}(g,h) = \chi_x(g,h)+\chi_y(g,h)\,. \tag{B.7}$$

Consider the following inner product on the characters

$$\langle\chi_x,\chi_y\rangle = \frac{1}{|G|}\sum_{g,h\in G,\ gh=hg}\omega(h^{-1},h|g)\chi_x(g,h^{-1})\chi_y(g,h)\,. \tag{B.8}$$

We will use the following important property of this inner product [80]

$$\langle\chi_x,\chi_y\rangle = |\mathrm{Hom}_{\mathcal{Z}(\mathrm{Vec}_G^\omega)}(x,y)|\,. \tag{B.9}$$

It will also be useful to recall some properties of characters of projective representations.

$$\overline{\chi_{\pi_g}(h)} = \omega(h^{-1},h|g)\ \chi_{\pi_g}(h^{-1})\,, \tag{B.10}$$

$$\chi_{\pi_g}(khk^{-1}) = \frac{\omega(k,h|g)}{\omega(khk^{-1},k|f)}\ \chi_{\pi_g}(g)\,, \tag{B.11}$$

where $\overline{\chi_{\pi_g}(h)}$ is the complex conjugate of $\chi_{\pi_g}(h)$ [107, Proposition 2.2]. Now, we are ready to prove Theorem 4.2.

Theorem 4.2: *$H\subseteq G$ is non-anomalous if and only if there exists a Lagrangian algebra L in $\mathcal{Z}(Vec_G^\omega)$ such that*

$$([h],\pi_h)\in L \quad \forall\ [h]\in C(H)\,, \tag{B.12}$$

for some representation π_h of the centralizer of h.

Proof: let H is a non-anomalous group. Then $A(H,1)$ is an algebra object. Using (4.41), we can compute the character of $\mathcal{Z}(A(H,1))$ to get

$$\chi_{\mathcal{Z}(A(H,1))}(k,l) := \sum_{y\in Y}\frac{\omega(y^{-1}ly,y^{-1}|k)}{\omega(y^{-1},l|k)}\,. \tag{B.13}$$

Let $([l],\pi_l)$ be a simple object in $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ such that $[l]\in C(H)$. Consider the inner product

$$\langle\chi_{([l],\pi_l)},\chi_{\mathcal{Z}(A(H,1))}\rangle = \frac{1}{|G|}\sum_{g,h\in G,\ gh=hg}\omega(h^{-1},h|g)\chi_{([l],\pi_l)}(g,h^{-1})\chi_{\mathcal{Z}(A(H,1))}(g,h)\,. \tag{B.14}$$

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Using (4.40), we get

$$\langle \chi_{([l],\pi_l)}, \chi_{\mathcal{Z}(A(H,1))} \rangle = \frac{1}{|G|} \sum_{g\in[l],\ gh=hg} \omega(h^{-1},h|g)\chi_{\pi_g}(h^{-1})\chi_{\mathcal{Z}(A(H,1))}(g,h)\,, \tag{B.15}$$

$$= \frac{1}{|[l]|} \sum_{g\in[l]} \left[\frac{1}{|C_g|} \sum_{h\in C_g} \omega(h^{-1},h|g)\chi_{\pi_g}(h^{-1})\chi_{\mathcal{Z}(A(H,1))}(g,h) \right], \tag{B.16}$$

where we used the fact that $|G| = |[l]||C_g|$ for any $l \in G$ and $g \in [l]$, $|[l]|$ is the size of the conjugacy class $[l]$ and $|C_l|$ is the order of the group C_l. Using (B.10), we can write the above expression as

$$\frac{1}{|[l]|} \sum_{g\in[l]} \left[\frac{1}{|C_g|} \sum_{h\in C_g} \overline{\chi_{\pi_g}(h)}\chi_{\mathcal{Z}(A(H,1))}(g,h) \right]. \tag{B.17}$$

Now, note that for a fixed g, $\chi_{\mathcal{Z}(A(H,1))}(g,h)$ is a function on the group C_g. Moreover, if $k,h \in C_g$, we find that this function satisfies

$$\chi_{\mathcal{Z}(A(H,1))}(g,khk^{-1}) = \chi_{\mathcal{Z}(A(H,1))}(kgk^{-1},khk^{-1}) = \frac{\omega(k,h|g)}{\omega(khk^{-1},k|g)}\chi_{\mathcal{Z}(A(H,1))}(g,h)\,, \tag{B.18}$$

where in the first equality we used the fact that $k \in C_g$ commutes with g and in the second equality we used (B.5). Moreover, since the conjugacy class $[l] \in C(H)$, $\chi_{\mathcal{Z}(A(H,1))}(g,h)$ is non-zero for some g and h. (For example, $\chi_{\mathcal{Z}(A(H,1))}(g,1)$ is non-zero.) Therefore, for a fixed g, $\chi_{\mathcal{Z}(A(H,1))}(g,h)$ is a non-zero projective character on the group C_g and

$$\frac{1}{|C_g|} \sum_{h\in C_g} \overline{\chi_{\pi_g}(h)}\ \chi_{\mathcal{Z}(A(H,1))}(g,h)\,, \tag{B.19}$$

is the inner-product of the characters $\chi_{\pi_g}(h)$ and $\chi_{\mathcal{Z}(A(H,1))}(g,h)$. Since the characters of irreducible projective representations form a basis of projective class functions [107, Theorem 3.1], we can always choose a π_l in $([l],\pi_l)$ such that the inner-product (B.19) is non-zero. Therefore, there is always some π_l for which $\langle \chi_{([l],\pi_l)}, \chi_{\mathcal{Z}(A(H,1))} \rangle \neq 0$.

Conversely, consider some line operator $([l],\pi_l)$ such that $\langle \chi_{([l],\pi_l)}, \chi_{\mathcal{Z}(A(H,\sigma))} \rangle \neq 0$ is non-zero. We have

$$\langle \chi_{([l],\pi_l)}, \chi_{\mathcal{Z}(A(H,\sigma))} \rangle \tag{B.20}$$

$$= \frac{1}{|G|} \sum_{g,h\in G,\ gh=hg} \omega(h^{-1},h|g)\chi_{([l],\pi_l)}(g,h^{-1})\chi_{\mathcal{Z}(A(H,\sigma))}(g,h)\,, \tag{B.21}$$

$$= \frac{1}{|G|} \sum_{g\in[l],\ gh=hg} \omega(h^{-1},h|g)\chi_{\pi_g}(h^{-1})\chi_{\mathcal{Z}(A(H,\sigma))}(g,h)\,, \tag{B.22}$$

$$= \frac{1}{|G|} \sum_{g\in[l],\ gh=hg} \omega(h^{-1},h|g)\chi_{\pi_g}(h^{-1})\chi_{\mathcal{Z}(A(H,\sigma))}(g,h)\,, \tag{B.23}$$

$$= \frac{1}{|G|} \sum_{g\in[l],\ gh=hg} \omega(h^{-1},h|g)\chi_{\pi_l}(h^{-1}) \sum_{y\in Y} \frac{\omega(y^{-1}hy,y^{-1}|g)\sigma(y^{-1}gy,y^{-1}hy)}{\omega(y^{-1},h|g)\sigma(y^{-1}hy,y^{-1}gy)}\,, \tag{B.24}$$

where

$$Y := \{y \in G, y^{-1}gy \in H, y^{-1}hy \in H\}/H \subset G/H\,. \tag{B.25}$$

Since $\langle \chi_{([l],\pi_l)}, \chi_{\mathcal{Z}(A(H,\sigma))} \rangle \neq 0$ is non-zero, the set Y must be non-zero, which implies that $y^{-1}gy$ must be in H for some y where $g \in [l]$. In other words, $[l] \in C(H)$. □

C Algebras in Rep(A_4) from algebras in Vec_{A_4}

Consider the alternating group A_4 of even permutations of four objects.

$$A_4 = \langle (123), (234) \rangle \,. \tag{C.1}$$

The category Vec_{A_4} has 12 line operators labeled by even permutations. The Morita equivalence classes of algebras in Vec_{A_4} are determined by conjugacy classes of subgroups of A_4 and associated 2-coycles. The conjugacy classes of subgroups of A_4 are given by

$$\begin{aligned}&\mathbb{Z}_1,\ \{\langle (12)(34)\rangle, \langle (13)(24)\rangle, \langle (14)(23)\rangle\}\\ &\{\langle (243)\rangle, \langle (123)\rangle, \langle (142)\rangle, \langle (134)\rangle\},\\ &\{\langle (13)(24), (12)(34)\rangle\}, A_4\,.\end{aligned} \tag{C.2}$$

The groups in the second conjugacy class are all isomorphic to $\mathbb{Z}_2$, those in the third conjugacy class are isomorphic to $\mathbb{Z}_3$ and those in the fourth are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, the Morita equivalence classes of algebras in Vec_{A_4} can be labeled as (G, σ) where H is the isomorphism class of the groups in a conjugacy class of subgroups and $\sigma \in H^2(G, \text{U}(1))$. We get

$$(\mathbb{Z}_1, 1), (\mathbb{Z}_2, 1), (\mathbb{Z}_3, 1), (\mathbb{Z}_2 \times \mathbb{Z}_2, 1), (\mathbb{Z}_2 \times \mathbb{Z}_2, \alpha), (A_4, 1), (A_4, \beta) \tag{C.3}$$

where α, β are the non-trivial 2-cocycles corresponding to the non-trivial elements in $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \text{U}(1)) \cong \mathbb{Z}_2$ and $H^2(A_4, \text{U}(1)) \cong \mathbb{Z}_2$, respectively. These seven Morita equivalence classes of algebras correspond to seven Lagrangian algebras in the SymTFT $\mathcal{Z}(\text{Vec}_{A_4})$.

The SymTFT $\mathcal{Z}(\text{Vec}_{A_4})$ is the A_4 Dijkgraaf-Witten theory. The line operators in this theory are labeled by

$$([g], \pi_g) \tag{C.4}$$

where $[g]$ is a conjugacy class of A_4 with representative g and π_g is an irreducible representation of the centralizer C_g of g. The conjugacy classes of A_4 are

$$\begin{aligned}&[()], [(12)(34)] = \{(12)(34), (13)(24), (14)(23)\}, [(123)] = \{(123), (142), (243), (134)\},\\ &[(124)] = \{(124), (234), (132), (143)\}\,,\end{aligned} \tag{C.5}$$

where () denotes the trivial permutation. The centralizers of the representatives of these conjugacy classes are

$$A_4, \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle (13)(24), (12)(34)\rangle, \mathbb{Z}_3 = \langle (123)\rangle, \mathbb{Z}_3 = \langle (124)\rangle\,. \tag{C.6}$$

The line operators in $\mathcal{Z}(\text{Vec}_{A_4})$, their quantum dimensions and topological spins are given by

Line operator	$([()], \mathbb{1})$	$([()], \pi_1)$	$([()], \pi_2)$,	$([()], \pi_3)$	$([(12)(34)], \mathbb{1}_1)$
d_x	1	1	1	3	3
θ_x	1	1	1	1	1

Line operator	$([(12)(34)], \omega_1)$	$([(12)(34)], \omega_2)$	$([(12)(34)], \omega_1\omega_2)$	$([(123), \mathbb{1}_2])$
d_x	3	3	3	4
θ_x	1	-1	-1	1

Line operator	$([(123),\omega])$	$([(123)],\omega^2)$	$([(124)],\mathbb{1}_3)$	$([(124),\tilde{\omega}])$	$([(124),\tilde{\omega}^2])$
d_x	4	4	4	4	4
θ_x	$e^{\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	1	$e^{\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$

where $\mathbb{1}, \pi_1, \pi_2$ are the 1-dimensional representations of A_4 and π_3 is the 3-dimensional irreducible representation. $\mathbb{1}_1, \omega_1, \omega_2, \omega_1\omega_2$ are the four 1-dimensional representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$. $\mathbb{1}_2, \omega, \omega^2$ are the three 1-dimensional representations of $\mathbb{Z}_3 = \langle(123)\rangle$. $\mathbb{1}_3, \tilde{\omega}, \tilde{\omega}^2$ are the three 1-dimensional representations of $\mathbb{Z}_3 = \langle(124)\rangle$.

Now, using the character theory developed in section 4.4, we can find the Lagrangian algebras in this theory from the algebra objects (C.3). In fact, we can completely identify all algebra objects by using just Theorem 4.2. Since $\langle(13)(24),(12)(34)\rangle$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ normal subgroup of A_4, the discussion in section 4.4 and [82] implies that it must correspond to a Lagrangian subcategory of $\mathcal{Z}(\mathrm{Vec}_{A_4})$. The algebra objects and corresponding Lagrangian algebras as summarised in the following table.

Algebra objects in Vec_{A_4}	Lagrangian algebra objects in $\mathcal{Z}(\mathrm{Vec}_{A_4})$
$(\mathbb{Z}_1, 1)$	$([()],\mathbb{1}) + ([()],\pi_1) + ([()],\pi_2) + 3([()],\pi_3)$
$(\mathbb{Z}_2, 1)$	$([()],\mathbb{1}) + ([(12)(34)],\mathbb{1}_1) + ([(12)(34)],\omega_1)$ $+ ([()],\pi_1) + ([()],\pi_2) + ([()],\pi_3)$
$(\mathbb{Z}_3, 1)$	$([()],\mathbb{1}) + ([(123)],\mathbb{1}_2) + ([(124)],\mathbb{1}_3) + ([()],\pi_3)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2, 1)$	$([()],\mathbb{1}) + ([()],\pi_1) + ([()],\pi_2) + 3([(12)(34)],\mathbb{1}_1)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2, \alpha)$	$([()],\mathbb{1}) + ([()],\pi_1) + ([()],\pi_2) + 3([(12)(34)],\omega_1)$
$(A_4, 1)$	$([()],\mathbb{1}) + ([(12)(34)],\mathbb{1}_1) + ([(123)],\mathbb{1}_2) + ([(124)],\mathbb{1}_3)$
(A_4, β)	$([()],\mathbb{1}) + ([(12)(34)],\omega_1) + ([(123)],\mathbb{1}_2) + ([(124)],\mathbb{1}_3)$

(C.7)

The line operators of the gapped boundary of the SymTFT $\mathcal{Z}(\mathrm{Vec}_{A_4})$ specified by the Lagrangian algebra

$$L_6 = ([()],\mathbb{1}) + ([(12)(34)],\mathbb{1}_1) + ([(123)],\mathbb{1}_2) + ([(124)],\mathbb{1}_3)\,, \tag{C.8}$$

form the category $\mathrm{Rep}(A_4)$. The fusion category $\mathrm{Rep}(A_4)$ contains four line operators $\mathbb{1}, \pi_1, \pi_2, \pi_3$ corresponding to the three 1-dimensional representations and one 3-dimensional representation of A_4, respectively. The fusion rules are

$$\pi_1 \times \pi_1 = \pi_2,\;\; \pi_1 \times \pi_2 = \mathbb{1},\;\; \pi_3 \times \pi_3 = \mathbb{1} + \pi_1 + \pi_2 + 2\pi_3\,. \tag{C.9}$$

The non-anomalous line operators in $\mathrm{Rep}(A_4)$ can be identified by applying the map F_{L_6} to all the Lagrangian algebras in the table above. In order to do this, it is convenient to use the isomorphism

$$\mathcal{Z}(\mathrm{Vec}_{A_4}) \cong \mathcal{Z}(\mathrm{Rep}(A_4))\,, \tag{C.10}$$

and write all line operators in the SymTFT in the form (a, e_a) for some $a \in \mathrm{Rep}(A_4)$ and half-braiding e_a. Once we write all line operators in this form, we know that

$$F_{L_6}(a, e_a) = a\,. \tag{C.11}$$

Therefore, in order to find the non-anomalous line operators in $\mathrm{Rep}(A_4)$, we don't have to explicitly determine the half-braidings e_a. Under the isomorphism (C.10), we get

$$([g], \pi_g) \to (\mathrm{Ind}_{C_g}^{A_4}(\pi_g), e)\,, \tag{C.12}$$

for some half-braiding e and $\mathrm{Ind}_{C_g}^{A_4}(\pi_g)$ is the induction of the representation π_g of C_g to the full group A_4. The induction of representations can be found using the character table of A_4 and Frobenius reciprocity. We get

$$\begin{aligned}
&([()], \mathbb{1}) \to (\mathbb{1}, e_1),\ ([()], \pi_1) \to (\pi_1, e_{\pi_1}),\ ([()], \pi_2) \to (\pi_2, e_{\pi_2}),\ ([()], \pi_3) \to (\pi_3, e_{\pi_3}),\\
&([(12)(34)], \mathbb{1}_1) \to (\mathbb{1} + \pi_1 + \pi_2, e_{\mathbb{1}+\pi_1+\pi_2}),\ ([(12)(34)], \omega_1) \to (\pi_3, e_{\pi_3}^{(1)}),\\
&([(12)(34)], \omega_2) \to (\pi_3, e_{\pi_3}^{(2)}),\ ([(12)(34)], \omega_1\omega_2) \to (\pi_3, e_{\pi_3}^{(3)})\,,\\
&([(123), \mathbb{1}_2]) \to (\mathbb{1} + \pi_3, e_{\mathbb{1}+\pi_3})\,, ([(123)], \omega) \to (\pi_2 + \pi_3, e_{\pi_2+\pi_3}),\\
&([(123)], \omega^2) \to (\pi_1 + \pi_3, e_{\pi_1+\pi_3}),\ ([(124)], \mathbb{1}_3) \to (\mathbb{1} + \pi_3, \tilde{e}_{\mathbb{1}+\pi_3}),\\
&([(124)], \tilde{\omega}) \to (\pi_1 + \pi_3, \tilde{e}_{\pi_1+\pi_3})\,, ([(124)], \tilde{\omega}^2) \to (\pi_2 + \pi_3, \tilde{e}_{\pi_1+\pi_3})\,.
\end{aligned} \tag{C.13}$$

Using this isomorphism, the 7 Lagrangian algebras can be written in terms of line operators in $\mathcal{Z}(\mathrm{Rep}(A_4))$. We can apply the map F_{L_6} to these Lagrangian algebras to find all the non-anomalous line operators in $\mathrm{Rep}(A_4)$.

Lagrangian algebra objects in $\mathcal{Z}(\mathrm{Rep}(A_4))$	Non-anomalous lines in $\mathrm{Rep}(A_4)$
$(\mathbb{1}, e_1) + (\pi_1, e_{\pi_1}) + (\pi_2, e_{\pi_2}) + 3(\pi_3, e_{\pi_3})$	$\mathbb{1} + \pi_1 + \pi_2 + 3\pi_3$
$(\mathbb{1}, e_1) + (\mathbb{1} + \pi_1 + \pi_2, e_{\mathbb{1}+\pi_1+\pi_2}) + (\pi_3, e_{\pi_3}^{(1)})$ $+\ (\pi_1, e_{\pi_1}) + (\pi_2, e_{\pi_2}) + (\pi_3, e_{\pi_3})$	$2\mathbb{1} + 2\pi_1 + 2\pi_2 + 2\pi_3$
$(\mathbb{1}, e_1) + (\mathbb{1} + \pi_3, e_{\mathbb{1}+\pi_3}) + (\mathbb{1} + \pi_3, \tilde{e}_{1+\pi_3}) + (\pi_3, e_{\pi_3})$	$3\mathbb{1} + 3\pi_3$
$(\mathbb{1}, e_1) + (\pi_1, e_{\pi_1}) + (\pi_2, e_{\pi_2}) + 3(\mathbb{1} + \pi_1 + \pi_2, e_{\mathbb{1}+\pi_1+\pi_2})$	$4\mathbb{1} + 4\pi_1 + 4\pi_2$
$(\mathbb{1}, e_1) + (\pi_1, e_{\pi_1}) + (\pi_2, e_{\pi_2}) + 3(\pi_3, e_{\pi_3}^{(1)})$	$\mathbb{1} + \pi_1 + \pi_2 + 3\pi_3$
$(\mathbb{1}, e_1) + (\mathbb{1} + \pi_1 + \pi_2, e_{\mathbb{1}+\pi_1+\pi_2})$ $(\mathbb{1} + \pi_3, e_{\mathbb{1}+\pi_3}) + (\mathbb{1} + \pi_3, \tilde{e}_{\mathbb{1}+\pi_3})$	$4\mathbb{1} + \pi_1 + \pi_2 + 2\pi_3$
$(\mathbb{1}, e_1) + (\pi_3, e_{\pi_3}^{(1)}) + (\mathbb{1} + \pi_3, e_{\mathbb{1}+\pi_3}) + (\mathbb{1} + \pi_3, \tilde{e}_{\mathbb{1}+\pi_3})$	$3\mathbb{1} + 3\pi_3$

(C.14)

$A_1 := \mathbb{1} + \pi_1 + \pi_2 + 3\pi_3$ being an algebra object implies the well-known fact that $\mathrm{Rep}(A_4)$ admits a fibre functor. In fact, $\mathrm{Rep}(A_4)$ admits two fibre functors, which is reflected in the above table by another Lagrangian algebra which gives the same algebra object. The algebra object $2\mathbb{1} + 2\pi_1 + 2\pi_2 + 2\pi_3$ can be decomposed into two copies of the haploid algebra object

$$A_2 := \mathbb{1} + \pi_1 + \pi_2 + \pi_3\,. \tag{C.15}$$

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The algebra object $3\mathbb{1} + 3\pi_3$ can be decomposed into three copies of the algebra object

$$A_3 = \mathbb{1} + \pi_3 \,. \tag{C.16}$$

From the table above, it is clear that A_3 admits at least two distinct algebra structures. The algebra object $4\mathbb{1} + 4\pi_1 + 4\pi_2$ can be decomposed into four copies of the algebra

$$A_4 = \mathbb{1} + \pi_1 + \pi_2 \,. \tag{C.17}$$

These lines form a non-anomalous $\mathbb{Z}_3$ group. Finally, from the Lagrangian algebra L_6, we get the algebra object

$$4\mathbb{1} + \pi_1 + \pi_2 + 2\pi_3 = 3A_5 + A_6 \,, \tag{C.18}$$

where $A_5 := \mathbb{1}$ and $A_6 := \mathbb{1} + \pi_1 + \pi_2 + 2\pi_3$. The algebra A_6 is Morita trivial which is consistent with the fusion rules of $\mathrm{Rep}(A_4)$.

D Bulk-to-boundary map from Lagrangian algebra

Given the line operators in the SymTFT $\mathcal{Z}(\mathcal{C})$ in the notation

$$(a, e_a) \,, \tag{D.1}$$

for some $a \in \mathcal{C}$ and half-braiding e_a, recall that the bulk to boundary map $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{B}_{\mathcal{C}}$ is given by the forgetful functor

$$F((a, e_a)) = a \,. \tag{D.2}$$

However, often we may have to study the bulk-to-boundary map to a different gapped boundary, say $\mathcal{B}_L$ determined by the Lagrangian algebra L. This is particularly important for transporting non-anomalous line operators between Morita equivalent fusion categories, as described in section 6.3. In this section, we will describe how to use the Lagrangian algebra L to determine the relationship between bulk and boundary line operators.

Given a Lagrangian algebra L in $\mathcal{Z}(\mathcal{C})$, the fusion category of line operators on the gapped boundary $\mathcal{B}_L$ is given by the quotient category construction in [108, Propositions 2.15, 2.16] (see also [49, 67]). We first define the quotient pre-category $\mathcal{Z}(\mathcal{C})/L$ as follows:

- Objects of $\mathcal{Z}(\mathcal{C})/L$ are objects of $\mathcal{Z}(\mathcal{C})$.
- Morphisms are given by

$$\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})/L}(x, y) := \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(x, L \times y) \,. \tag{D.3}$$

Note that, in general, the category $\mathcal{Z}(\mathcal{C})/L$ is not semi-simple, since $\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(x, x)$ could be higher-dimensional for some simple line operator x while it must be $\mathbb{C}$ in a semi-simple category. Given the pre-quotient category $\mathcal{Z}(\mathcal{C})/L$, the fusion category $\mathcal{C}_L$ of line operators on the gapped boundary is given by the quotient category defined as follows:

- The objects of $\mathcal{C}_L$ are (x, p), where $x \in \mathcal{Z}(\mathcal{C})/L$ and $p = p^2 \in \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(x, x)$.

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- Morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}_L}((x,p),(y,q)) := \{f \in \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})/L}(x,y) \mid f\circ p = p\circ f, f\circ q = q\circ f\}\,. \quad \text{(D.4)}$$

The above definition must be understood as follows. If x is a simple line operator in $\mathcal{Z}(\mathcal{C})$ such that $\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})/L}(x,x)$ is n-dimensional, then x is split into n objects using the idempotents p. This will then result in a semi-simple category $\mathcal{C}_L$. See [49, section 3.6] for an explicit calculation of the bulk to boundary map using the above quotient category construction for gapped boundary of S_3 Dijkgraaf-Witten theory.

Let us now explain the bulk to boundary map (6.50) using the quotient category construction described above. Consider the SymTFT $\mathcal{Z}(\mathscr{H}_2)$ and its gapped boundary $\mathcal{B}_{L_1}$ determined by the Lagrangian algebra object

$$L_1 = \mathbf{1} + \pi_1 + \pi_2 + \sigma_1\,. \quad \text{(D.5)}$$

In order to understand how the bulk line operators are related to boundary lines, we look at the Hom space of the pre-quotient category $\mathcal{Z}(\mathscr{H}_2)/L_1$

$$\mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)/L_1}(x,y) := \mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)}(x, L_1\times y) = \mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)}(x,(\mathbf{1}+\pi_1+\pi_2+\sigma_1)\times y)\,. \quad \text{(D.6)}$$

We will use the fusion rules for $\mathcal{Z}(\mathscr{H}_2)$ given in [94] to determine the above Hom spaces.

- $y=1$: $\mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)/L_1}(x,\mathbf{1}) := \mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)}(x,(\mathbf{1}+\pi_1+\pi_2+\sigma_1)\times\mathbf{1})$. This Hom space is non-zero if and only if $x=\mathbf{1},\pi_1,\pi_2$ or σ_1 and it implies that these line operators get identified with the vacuum in $\mathcal{Z}(\mathscr{H}_2)/L_1$. This is expected as the boundary condition $\mathcal{B}_{L_1}$ is obtained from gauging the line operator L_1.

- $y=\pi_1$: $\mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)/L_1}(x,\pi_1) := \mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)}(x,(\mathbf{1}+\pi_1+\pi_2+\sigma_1)\times\pi_1)$. Using the fusion rules of $\mathcal{Z}(\mathscr{H}_2)$, we get

$$\mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)/L_1}(x,\pi_1) := \mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)}\left(x,\mathbf{1}+4\pi_1+4\pi_2+4\sigma_1+3\sigma_2+3\sigma_3+3\sum_{i=1}^{6}\rho_i\right). \quad \text{(D.7)}$$

- $y=\pi_2$: $\mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)/L_1}(x,\pi_2) := \mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)}(x,(\mathbf{1}+\pi_1+\pi_2+\sigma_1)\times\pi_2)$. Using the fusion rules of $\mathcal{Z}(\mathscr{H}_2)$, we get

$$\mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)/L_1}(x,\pi_2) := \mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)}\left(x,\mathbf{1}+4\pi_1+6\pi_2+3\sigma_1+4\sigma_2+4\sigma_3+3\sum_{i=1}^{6}\rho_i\right). \quad \text{(D.8)}$$

- $y=\sigma_1$: $\mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)/L_1}(x,\sigma_1) := \mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)}(x,(\mathbf{1}+\pi_1+\pi_2+\sigma_1)\times\sigma_1)$. Using the fusion rules of $\mathcal{Z}(\mathscr{H}_2)$, we get

$$\mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)/L_1}(x,\sigma_1) := \mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)}\left(x,\mathbf{1}+4\pi_1+3\pi_2+6\sigma_1+4\sigma_2+4\sigma_3+3\sum_{i=1}^{6}\rho_i\right). \quad \text{(D.9)}$$

- $y=\sigma_2$: $\mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)/L_1}(x,\sigma_2) := \mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)}(x,(\mathbf{1}+\pi_1+\pi_2+\sigma_1)\times\sigma_2)$. Using the fusion rules of $\mathcal{Z}(\mathscr{H}_2)$, we get

$$\mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)/L_1}(x,\sigma_2) := \mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)}\left(x, 3\pi_1+4\pi_2+4\sigma_1+5\sigma_2+5\sigma_3+3\sum_{i=1}^{6}\rho_i\right). \tag{D.10}$$

- $y=\sigma_3$: $\mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)/L_1}(x,\sigma_3) := \mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)}(x,(\mathbf{1}+\pi_1+\pi_2+\sigma_1)\times\sigma_3)$. Using the fusion rules of $\mathcal{Z}(\mathscr{H}_2)$, we get

$$\mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)/L_1}(x,\sigma_3) := \mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)}\left(x, 3\pi_1+4\pi_2+4\sigma_1+5\sigma_2+5\sigma_3+3\sum_{i=1}^{6}\rho_i\right). \tag{D.11}$$

- $y=\rho_1$: $\mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)/L_1}(x,\rho_1) := \mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)}(x,(\mathbf{1}+\pi_1+\pi_2+\sigma_1)\times\rho_1)$. Using the fusion rules of $\mathcal{Z}(\mathscr{H}_2)$, we get

$$\mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)/L_1}(x,\rho_1) := \mathrm{Hom}_{\mathcal{Z}(\mathscr{H}_2)}\left(x, 3\left(\pi_1+\pi_2+\sigma_1+\sigma_2+\sigma_3+\sum_{i=1}^{6}\rho_i\right)\right). \tag{D.12}$$

  We get the same Hom space for all the other ρ_i, $2\le i\le 6$.

Let $\mathbb{1}$,[18] μ,ν,η be the simple line operators on the gapped boundary $\mathcal{B}_{L_1}$. These line operators can be written as combinations of the bulk line operators as follows:

$$\mathbb{1} \to L_3 = \mathbf{1}+\pi_1+\pi_2+\sigma_1\,. \tag{D.13}$$

$$\mu \to \pi_1+\pi_2+\sum_{i=1}^{6}\rho_i\,. \tag{D.14}$$

$$\nu \to \pi_1+2\pi_2+\sigma_1+2\sigma_2+2\sigma_3+\sum_{i=1}^{6}\rho_i\,. \tag{D.15}$$

$$\eta \to \pi_1+2\sigma_1+\sigma_2+\sigma_3+\sum_{i=1}^{6}\rho_i\,. \tag{D.16}$$

The above is precisely the K map introduced in section 4.3. This map satisfies the consistency condition relating quantum dimensions.

$$d_a = \frac{d_{K(a)}}{d_{K(\mathbb{1})}}\,, \quad a\in\{\mathbb{1},\mu,\nu,\eta\}\,, \tag{D.17}$$

and is chosen such that the linear combination of bulk lines in the Hom spaces computed above can be written as linear combinations of the boundary line operators $\mathbb{1},\mu,\nu,\eta$. The

[18] Unlike in other sections, here we denote the identity object of $\mathscr{H}_2$ as $\mathbb{1}$ to avoid confusion with the identity object $\mathbf{1}$ of $\mathcal{Z}(\mathscr{H}_2)$.

bulk-to-boundary map is given by

$$F_{\mathcal{B}_{L_1}}(\mathbf{1}) = \mathbb{1}\,, \tag{D.18}$$

$$F_{\mathcal{B}_{L_1}}(\pi_1) = \mathbb{1} + \mu + \eta + \nu\,, \tag{D.19}$$

$$F_{\mathcal{B}_{L_1}}(\pi_2) = \mathbb{1} + 2\nu + \mu\,, \tag{D.20}$$

$$F_{\mathcal{B}_{L_1}}(\sigma_1) = \mathbb{1} + 2\eta + \nu\,, \tag{D.21}$$

$$F_{\mathcal{B}_{L_1}}(\sigma_2) = 2\nu + \eta\,, \tag{D.22}$$

$$F_{\mathcal{B}_{L_1}}(\sigma_3) = 2\nu + \eta\,, \tag{D.23}$$

$$F_{\mathcal{B}_{L_1}}(\rho_i) = \mu + \eta + \nu\,,\ \forall\, 1 \leq i \leq 6\,. \tag{D.24}$$

Note that the $F_{\mathcal{B}_{L_1}}$ above preserves quantum dimensions.

E Morita equivalence classes of algebras in Haagerup fusion categories

The algebras in three Morita equivalent Haagerup fusion categories have been classified in [30]. In this section, we give a brief review of the results in this paper.

E.1 Module categories over $\mathscr{H}_1$

Consider the Haagerup fusion category $\mathscr{H}_1$ with simple line operators

$$\mathbf{1}, \mu, \nu, \eta\,. \tag{E.1}$$

It has three module categories $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ each with four simple objects, say $m_1, \ldots, m_4$. The Morita equivalent classes of algebras corresponding to these module categories can be read off from the graphs in [30, Theorem 3.25]. Recall the definition of the algebra object

$$A_m := \sum_{a \in \mathscr{H}_1} N_m^a\, a\,, \tag{E.2}$$

where N_m^a is the dimension of Hilbert space of operators at the junction of a with the gapped boundary $m \in \mathcal{M}$.

$\mathcal{M}_1$ is the regular module and we get the algebras

$$A_{m_1} = \mathbf{1}, \tag{E.3}$$

$$A_{m_2} = \mathbf{1} + \nu, \tag{E.4}$$

$$A_{m_3} = \mathbf{1} + \mu + 2\nu + 2\eta, \tag{E.5}$$

$$A_{m_4} = \mathbf{1} + \mu + \nu + \eta\,. \tag{E.6}$$

Therefore, we get

$$\sum_{m \in \mathcal{M}_1} A_m = 4\mathbf{1} + 4\nu + 3\eta + 2\mu\,. \tag{E.7}$$

For the module category $\mathcal{M}_2$, we have

$$A_{m_1} = \mathbf{1} + \mu + \nu, \tag{E.8}$$

$$A_{m_2} = \mathbf{1} + 2\nu + \eta, \tag{E.9}$$

$$A_{m_3} = \mathbf{1} + \mu + \nu, \tag{E.10}$$

$$A_{m_4} = \mathbf{1} + \mu + \nu\,. \tag{E.11}$$

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Therefore, we get

$$\sum_{m\in\mathcal{M}_2} A_m = 4\mathbf{1} + 5\nu + \eta + 3\mu\,. \tag{E.12}$$

Finally, for the module category $\mathcal{M}_3$, we have

$$A_{m_1} = \mathbf{1} + \eta, \tag{E.13}$$
$$A_{m_2} = \mathbf{1} + \eta, \tag{E.14}$$
$$A_{m_3} = \mathbf{1} + \mu + 3\nu + 2\eta, \tag{E.15}$$
$$A_{m_4} = \mathbf{1} + \eta\,. \tag{E.16}$$

Therefore, we get

$$\sum_{m\in\mathcal{M}_3} A_m = 4\mathbf{1} + 3\nu + 5\eta + \mu\,. \tag{E.17}$$

E.2 Module categories over $\mathscr{H}_2$

Consider the Haagerup fusion category $\mathscr{H}_2$ with simple line operators

$$\mathbf{1}, \alpha, \alpha^2, \rho, \alpha\rho, \alpha^2\rho\,. \tag{E.18}$$

There are three module categories $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ over $\mathscr{H}_2$ given by the category of modules over the algebras $\mathbf{1}$, $\mathbf{1}+\rho$ and $\mathbf{1}+\alpha+\alpha^2$, respectively.

The module category $\mathcal{M}_1$ contains 4 objects, say $m_1, \ldots, m_4$. We have

$$A_{m_1} = \mathbf{1} + \rho, \tag{E.19}$$
$$A_{m_2} = \mathbf{1} + \alpha\rho, \tag{E.20}$$
$$A_{m_3} = \mathbf{1} + \alpha^2\rho, \tag{E.21}$$
$$A_{m_4} = \mathbf{1} + \alpha + \alpha^2 + 2H\,. \tag{E.22}$$

Therefore, we get

$$\sum_{m\in\mathcal{M}_1} A_m = 4\mathbf{1} + \alpha + \alpha^2 + 3H\,. \tag{E.23}$$

The module category $\mathcal{M}_2$ is the regular module with 6 objects corresponding to the simple objects in $\mathscr{H}_2$. Using the graphs in [30, Corollary 3.16] (see also [93, section 7]), we can read off the following

$$A_1 = A_\alpha = A_{\alpha^2} = \mathbf{1}\,, \tag{E.24}$$
$$A_\rho = \mathbf{1} + H, \tag{E.25}$$
$$A_{\alpha\rho} = \mathbf{1} + H, \tag{E.26}$$
$$A_{\alpha^2\rho} = \mathbf{1} + H\,, \tag{E.27}$$

where $H = \rho + \alpha\rho + \alpha^2\rho$. Therefore, we get

$$\sum_{m\in\mathcal{M}_2} A_m = 6\mathbf{1} + 3H\,. \tag{E.28}$$

Finally, the module category $\mathcal{M}_3$ contains two simple objects, say m_1 and m_2. We have

$$A_{m_1} = \mathbf{1} + \alpha + \alpha^2, \tag{E.29}$$
$$A_{m_2} = \mathbf{1} + \alpha + \alpha^2 + 3H\,. \tag{E.30}$$

Therefore, we get

$$\sum_{m\in\mathcal{M}_3} A_m = 2\mathbf{1} + 2\alpha + 2\alpha^2 + 3H\,. \tag{E.31}$$

E.3 Module categories over $\mathscr{H}_3$

There are again three module categories $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ over $\mathscr{H}_3$. The Morita equivalent classes of algebras corresponding to these module categories can be read off from the graphs in [30, Theorem 3.25].

$\mathcal{M}_1$ is a module category with four objects say $m_1, \ldots, m_4$. We have

$$A_{m_1} = \mathbf{1} + \alpha\rho + \alpha^2\rho, \tag{E.32}$$
$$A_{m_2} = \mathbf{1} + \alpha + \alpha^2 + H, \tag{E.33}$$
$$A_{m_3} = \mathbf{1} + \rho + \alpha^2\rho, \tag{E.34}$$
$$A_{m_4} = \mathbf{1} + \rho + \alpha\rho\,. \tag{E.35}$$

Therefore, we get

$$\sum_{m\in\mathcal{M}_1} A_m = 4\mathbf{1} + \alpha + \alpha^2 + 3H\,. \tag{E.36}$$

Note that even though this sum is the same as (E.23), the individual algebras A_{m_i} that constitute this sum is different in $\mathscr{H}_2$ and $\mathscr{H}_3$.

The structure of the module category $\mathcal{M}_2$ is the same as in the module category $\mathcal{M}_3$ in $\mathscr{H}_2$. Therefore, we get

$$\sum_{m\in\mathcal{M}_2} A_m = 2\mathbf{1} + 2\alpha + 2\alpha^2 + 3H\,. \tag{E.37}$$

Finally, $\mathcal{M}_3$ is the regular module and the Morita equivalent algebras are the same as in the module category $\mathcal{M}_2$ in $\mathcal{H}_2$. We get

$$\sum_{m\in\mathcal{M}_3} A_m = 6\mathbf{1} + 3H\,. \tag{E.38}$$

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