

# Distributed Design for Decentralized Control Using Chordal Decomposition and ADMM

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**Abstract**—We propose a distributed design method for decentralized control by exploiting the underlying sparsity properties of the problem. Our method is based on the chordal decomposition of sparse block matrices and the alternating direction method of multipliers (ADMM). We first apply a classical parameterization technique to restrict the optimal decentralized control into a convex problem that inherits the sparsity pattern of the original problem. The parameterization relies on a notion of strongly decentralized stabilization, and sufficient conditions are discussed to guarantee this notion. Then, chordal decomposition allows us to decompose the convex restriction into a problem with partially coupled constraints, and the framework of ADMM enables us to solve the decomposed problem in a distributed fashion. Consequently, the subsystems only need to share their model data with their direct neighbors, without needing central computation. Numerical experiments demonstrate the effectiveness of the proposed method.

**Index Terms**—Chordal decomposition, distributed design, optimal decentralized control.

## I. INTRODUCTION

**M**ANY REAL-WORLD complex systems, such as aircraft formation, automated highways, and power systems, consist of a large number of interconnected subsystems. Often in these interconnected systems, the controllers have only access to each subsystem's state information. The problem of design of stabilizing and optimal controllers based on only each subsystem's state information is referred to as *decentralized control*.

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*control*. Due to its importance, this problem has attracted research attention since the late 1970s [1], [2].

Early efforts have centered on decentralized stabilization and its algebraic characterization through the concept of *decentralized fixed modes* [3]. These are the set of eigenvalues that remain unchanged under any decentralized feedback. One seminal result is that a system is stabilizable by a decentralized controller if and only if its decentralized fixed modes have negative real parts [3]. Since then, a wide range of extensions of decentralized control has been investigated, either by considering various types of performance guarantees in addition to stability [4], or by taking into account neighboring information for feedback, known as distributed control [5]. Several classes of systems have been identified, which allow convex formulations for the design of distributed  $\mathcal{H}_\infty$  and  $\mathcal{H}_2$  controllers, for example, quadratically invariant systems [6]. Also, some numerical approaches have been proposed to find an approximate solution to the optimal distributed control problem [7], [8]. The case of decentralized control in the presence of input and state constraints is addressed in [9].

A common assumption made in these papers is that a central model of the global plant is available, indicating that the design is performed in a *centralized* fashion even though the implementation of controllers is *decentralized*. However, this may be impractical for certain complex systems that are shared between private individuals, such as transportation systems and power grids. In this case, a complete model may not be available due to privacy concerns of model information for the subsystems. While discussions on distributed design relying on limited model information can be traced back to [2, Ch. 9], practical approaches to this problem are an active topic of current research. For example, performance bounds of designing linear quadratic regulators distributedly were discussed for systems with an invertible input matrix in [10]. The distributed design framework of [10] has been used to discuss the best closed-loop performance achievable by distributed design strategies for a class of fully actuated discrete-time systems [11]. In [12], independent decoupled problems were derived for optimal decentralized control by utilizing the properties of posets. Recent work has started to use distributed optimization techniques to realize distributed synthesis in the dissipative framework [13]. Furthermore, the recently proposed system-level approach has been promoted to address distributed design of dynamic distributed controllers [14].

In this article, we propose a new distributed design method for optimal decentralized control by exploiting the sparsity

structure of the system. Our method uses local information on the system model to design controllers that rely on subsystems' state measurements. The idea originates from the connection between sparse positive semidefinite (PSD) matrices and chordal graphs [15], [16]. The celebrated chordal decomposition in graph theory [15], [16] allows us to decompose a large sparse PSD cone into a set of smaller and coupled ones, and has been successfully applied to decompose sparse semidefinite programs [17], [18]. These results have recently been used for performance analysis of sparse linear systems [19]–[21], leading to significantly faster solutions than using standard dense methods. Despite the scalability of these approaches, they all required global model information for centralized computation. Zheng *et al.* [22] proposed a sequential approach to improve the scalability of solving a stabilization problem of networked systems, where model privacy can be maintained as a byproduct.

This article extends the scope of exploiting chordal decomposition to distributed design of optimal decentralized control. By using a classical parameterization technique that relies on a notion of *strongly decentralized stabilization* [4], the optimal decentralized control can be restricted to a convex problem that inherits the original sparsity pattern in the system. The convex restriction can be equivalently decomposed into a problem with partially coupled constraints, and we introduce a distributed algorithm to solve the decomposed problem based on the framework of *alternative direction method of multipliers* (ADMM). Precisely, the main contributions of this article are as follows.

- 1) We provide sufficient conditions to guarantee the feasibility of the proposed convex restriction. These conditions are based on characterizing the cases in which the closed-loop system with decentralized feedback admits a block-diagonal Lyapunov function.<sup>1</sup> In particular, we identify two classes of networked systems admitting strongly decentralized stabilization.
- 2) One notable feature of the convex restriction is that the original sparsity pattern of the system is inherited in the resulting convex optimization problem. We combine chordal decomposition with ADMM to solve the convex problem in a distributed fashion. In our algorithm, no central model of the global plant is required and the subsystems only need to share their model data with their neighbors, which helps preserve the privacy of model data.

The rest of this article is organized as follows. We present the problem formulation in Section II. In Section III, we discuss sufficient conditions on strongly decentralized stabilization. Section IV applies a chordal decomposition technique to derive a decomposed problem, and a distributed algorithm is introduced to solve the decomposed problem in Section V. Numerical examples are given in Section VI. We conclude this article in Section VII.

<sup>1</sup>The authors have summarized some preliminary results in an unpublished technical report [23, Sec. 3]. The current manuscript serves as the official version of the report [23], and we do not consider [23] for publication.

## II. BACKGROUND AND PROBLEM STATEMENT

### A. Optimal Decentralized Control

A directed graph  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  is defined by a set of nodes  $\mathcal{V} = \{1, 2, \dots, N\}$  and a set of edges  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ . We consider a complex system consisting of  $N$  subsystems. The interactions between subsystems are modeled by a plant graph  $\mathcal{G}_p(\mathcal{V}, \mathcal{E}_p)$ , in which each node in  $\mathcal{V}$  denotes a subsystem, and the edge  $(i, j) \in \mathcal{E}_p$  means that subsystem  $i$  has dynamical influence on subsystem  $j$ . The dynamics of each subsystem  $i \in \mathcal{V}$  are

$$\dot{x}_i(t) = A_{ii}x_i(t) + \sum_{j \in \mathbb{N}_i} A_{ij}x_j(t) + B_i u_i(t) + M_i d_i(t) \quad (1)$$

where  $x_i \in \mathbb{R}^{n_i}$ ,  $u_i \in \mathbb{R}^{m_i}$ ,  $d_i \in \mathbb{R}^{q_i}$  denote the local state, input and disturbance of subsystem  $i$ , respectively, and  $\mathbb{N}_i$  denotes the set of neighboring nodes that influence node  $i$ , that is,  $\mathbb{N}_i = \{j \mid (j, i) \in \mathcal{E}\}$ . In (1),  $A_{ii} \in \mathbb{R}^{n_i \times n_i}$ ,  $B_i \in \mathbb{R}^{n_i \times m_i}$ ,  $M_i \in \mathbb{R}^{n_i \times q_i}$  represent local dynamics, and  $A_{ij} \in \mathbb{R}^{n_i \times n_j}$  represents the interaction with neighbors. In this article, we refer to  $A_{ii}$ ,  $B_i$ ,  $M_i$ ,  $A_{ij}$  as *model data* of the system.

By collecting the subsystems' states, the overall system can be described compactly as

$$\dot{x}(t) = Ax(t) + Bu(t) + Md(t) \quad (2)$$

where  $x := [x_1^\top, x_2^\top, \dots, x_N^\top]^\top$ , and the vectors  $u, d$  are defined similarly. The matrix  $A$  is composed of blocks  $A_{ij}$ , which has a block sparsity pattern, that is,  $A \in \mathbb{R}^{n \times n}(\mathcal{E}_p, 0)$  with a partition  $\{n_1, \dots, n_N\}$  corresponding to the dimension of each subsystem's state. The matrices  $B, M$  are of the forms  $B = \text{diag}(B_1, \dots, B_N)$  and  $M = \text{diag}(M_1, \dots, M_N)$ . Our goal is to design a decentralized static state feedback

$$u_i(t) = -K_{ii}x_i(t), i = 1, \dots, N \quad (3)$$

such that the  $\mathcal{H}_2$  norm of the transfer function  $T_{zd}$  from disturbance  $d$  to the desired performance output  $z$  is minimized. In (3), the global  $K$  has a decentralized structure  $\mathcal{K}$  as

$$K \in \mathcal{K} := \{K \in \mathbb{R}^{m \times n} \mid K_{ij} = 0 \text{ if } i \neq j\}$$

where  $m = \sum_{i=1}^N m_i$ ,  $n = \sum_{i=1}^N n_i$ , and each entry  $K_{ij}$  is a block of dimension  $m_i \times n_j$ .

The design objective is

$$\begin{aligned} \min_K \quad & \|T_{zd}\|^2 \\ \text{s.t.} \quad & (A - BK) \text{ is Hurwitz} \\ & K \in \mathcal{K} \end{aligned} \quad (4)$$

where  $\|\cdot\|$  is the  $\mathcal{H}_2$  norm of a transfer function. In this article, the performance output  $z$  is chosen as

$$z = \begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix} u$$

where  $Q := \text{diag}(Q_1, \dots, Q_N)$  and  $R := \text{diag}(R_1, \dots, R_N)$  denote the state and control performance weights, respectively, and diagonal block  $Q_i, R_i$  correspond to the subsystem  $i$ . Adopting the same terminology in [1] and [4], we refer to (4) as the *optimal decentralized control* problem.

The constraint  $\mathcal{K}$  does not allow any equivalent convex reformulation of the optimal decentralized problem (4) in general. Hence, problem (4) is challenging to solve exactly. Previous work either imposed special structures on system dynamics [6], [12], [24]; used certain relaxation/restriction techniques [4], [8]; or applied nonconvex optimization directly [7] to address this problem.

### B. Convex Restriction via Block-Diagonal Lyapunov Functions

It is well known that the  $\mathcal{H}_2$  norm of a stable linear system can be calculated using a linear matrix inequality [25].

**Lemma 1 ([25]):** Consider a stable linear system  $\dot{x}(t) = Ax(t) + Md(t)$ ,  $z(t) = Cx(t)$ . The  $\mathcal{H}_2$  norm of the transfer function from  $d$  to  $z$  can be computed by

$$\|T_{zd}\|^2 = \inf_{X \succ 0} \{ \text{Tr}(CXC^T) \mid AX + XA^T + MM^T \preceq 0 \}$$

where  $\text{Tr}(\cdot)$  denotes the trace of a symmetric matrix.

According to Lemma 1, the optimal decentralized control problem (4) can be equivalently reformulated as

$$\begin{aligned} \min_{X, K} \quad & \text{Tr}((Q + K^T RK)X) \\ \text{s.t.} \quad & (A - BK)X + X(A - BK)^T + MM^T \preceq 0 \\ & X \succ 0, K \in \mathcal{K}. \end{aligned} \quad (5)$$

The first inequality in (5) does not depend linearly on  $X$  and  $K$ . A standard change of variables  $Z = KX$  leads to

$$\begin{aligned} \min_{X, Z} \quad & \text{Tr}(QX) + \text{Tr}(RZX^{-1}Z^T) \\ \text{s.t.} \quad & (AX - BZ) + (AX - BZ)^T + MM^T \preceq 0 \\ & X \succ 0, ZX^{-1} \in \mathcal{K}. \end{aligned}$$

To handle the nonlinear constraint  $ZX^{-1} \in \mathcal{K}$ , a classical parameterization idea [4] is to assume a *block diagonal*  $X = \text{diag}(X_1, \dots, X_N)$  with block size compatible to the subsystem's dimensions, which leads to  $ZX^{-1} \in \mathcal{K} \Leftrightarrow Z \in \mathcal{K}$ . Considering the block-diagonal structures of  $Q$ ,  $R$ , we have

$$\begin{aligned} \text{Tr}(QX) &= \sum_{i=1}^N \text{Tr}(Q_i X_i) \\ \text{Tr}(RZX^{-1}Z^T) &= \sum_{i=1}^N \text{Tr}(R_i Z_i X_i^{-1} Z_i^T). \end{aligned}$$

By introducing  $Y_i \succeq Z_i X_i^{-1} Z_i^T$  and using the Schur complement [25], a convex restriction to (4) is derived

$$\begin{aligned} \min_{X_i, Y_i, Z_i} \quad & \sum_{i=1}^N \text{Tr}(Q_i X_i) + \text{Tr}(R_i Y_i) \\ \text{s.t.} \quad & (AX - BZ) + (AX - BZ)^T + MM^T \preceq 0 \\ & \begin{bmatrix} Y_i & Z_i \\ Z_i^T & X_i \end{bmatrix} \succeq 0, X_i \succ 0, i = 1, \dots, N. \end{aligned} \quad (6a)$$

$$(6b)$$

Problem (6) is convex and ready to be solved using general conic solvers, and the decentralized controller is recovered as  $K_{ii} = Z_i X_i^{-1}$ ,  $i = 1, \dots, N$ . In this article, we make the following assumption.

**Assumption 1:** Problem (6) is feasible or, equivalently, system (2) is strongly decentralized stabilizable (see Definition 3).

**Remark 1:** The block-diagonal strategy was formally discussed in the early 1990s [4], which was later implicitly or explicitly used in the field of decentralized stabilization [1], [13], [22]. This strategy requires the closed-loop system to admit a block-diagonal Lyapunov function  $V(x) = x^T P x = \sum_{i=1}^N x_i^T P_i x_i$ , where  $P_i = X_i^{-1}$ ,  $i \in \mathcal{V}$ . Problem (6) is a convex restriction of the original decentralized control problem (4), and allows computing an upper bound of the optimal cost. However, quantifying the gap between the solution to (6) and the optimal solution to (4) is a challenging open problem, which is beyond the scope of this article. Indeed, problem (6) might be infeasible even for the cases in which problem (4) is feasible.

### C. Problem Statement

To connect the block-diagonal strategy with past work on decentralized control, we first present three classical definitions.

**Definition 1 (Stabilization):** System (2) is called *stabilizable*, if there exists a centralized controller  $u = -Kx$  such that the closed-loop system  $\dot{x} = (A - BK)x$  is asymptotically stable.

**Definition 2 (Decentralized stabilization [3]):** System (2) is called *decentralized stabilizable*, if there exists a decentralized controller  $u_i = -K_{ii}x_i$ ,  $i \in \mathcal{V}$  such that the closed-loop system  $\dot{x} = (A - BK)x$  is asymptotically stable.

**Definition 3 (Strongly decentralized stabilization [4]):** System (2) is called *strongly decentralized stabilizable* if there exists a decentralized  $u_i = -K_{ii}x_i$ ,  $i \in \mathcal{V}$  such that the closed-loop system  $\dot{x} = (A - BK)x$  admits a block-diagonal Lyapunov function  $V(x) = \sum_{i=1}^N x_i^T P_i x_i$ .

Then, we define three classes of complex systems:

$$\Sigma_0 = \{(A, B) \mid (2) \text{ is stabilizable}\}$$

$$\Sigma_1 = \{(A, B) \mid (2) \text{ is decentralized stabilizable}\}$$

$$\Sigma_2 = \{(A, B) \mid (2) \text{ is strongly decentralized stabilizable}\}.$$

It is easy to see  $\Sigma_2 \subseteq \Sigma_1 \subseteq \Sigma_0$ . In fact, the inclusion relationship is strict (see counterexamples in Appendix A)

$$\Sigma_2 \subset \Sigma_1 \subset \Sigma_0. \quad (7)$$

The sets  $\Sigma_0$  and  $\Sigma_1$  can be algebraically characterized by centralized fixed modes and decentralized fixed modes [3], [26]. The class  $\Sigma_2$  is useful for synthesizing decentralized controllers as discussed in Section II-B, but has been less studied before. This motivates the first objective of our article.

**Problem 1 (Explicit characterizations):** Derive sufficient conditions to characterize  $\Sigma_2$ .

Solving (6) directly requires global model knowledge, implicitly assuming the existence of a central entity to collect the complete model data. This performs a *centralized design*



of the decentralized controllers. We note that the problem of distributed design using limited model information has received increasing attention [10], [11], [13], [14]. In this article, we partition the subsystems into clusters to solve (6) in a distributed fashion. The second objective is as follows.

**Problem 2 (Distributed computation):** Given system (2) in  $\Sigma_2$  with a plant graph  $\mathcal{G}_p(\mathcal{V}, \mathcal{E}_p)$ , we aim 1) to partition the subsystems into  $t$  clusters  $\mathcal{C}_1, \dots, \mathcal{C}_t$ , where  $\mathcal{V} = \cup_{i=1}^t \mathcal{C}_i$ , and 2) to design a distributed algorithm to solve (6), where the model data of subsystem  $i$  are only shared within the clusters that contain it.

We show that a chordal decomposition technique can be naturally used for the partition  $\mathcal{C}_1, \dots, \mathcal{C}_t$ , and that the number of clusters depends on the sparsity of  $\mathcal{G}_p(\mathcal{V}, \mathcal{E}_p)$ . We address Problem 1 in Section III and Problem 2 in Sections IV and V.

### III. SUFFICIENT CONDITIONS ON STRONGLY DECENTRALIZED STABILIZATION

In this section, we discuss two classes of systems in  $\Sigma_2$ : 1) fully actuated systems, and 2) weakly coupled systems.

#### A. Fully Actuated Systems

**Definition 4 (Fully actuated systems):** System (2) is called fully actuated, if each input matrix  $B_i$  has full row rank,  $i \in \mathcal{V}$ .

**Proposition 1:** If system (2) is fully actuated, then we have  $(A, B) \in \Sigma_2$ .

**Proof:** Consider the singular value decomposition of the input matrix  $B_i$

$$B_i = U_i [\Gamma_i \quad 0] V_i^T$$

where 0 is a zero block of appropriate size, and  $\Gamma_i \in \mathbb{R}^{n_i \times n_i}$  is invertible since  $B_i$  has full row rank. We then consider a decentralized feedback controller

$$K_{ii} = V_i \begin{bmatrix} \Gamma_i^{-1} \\ 0 \end{bmatrix} U_i^T (A_{ii} + \alpha_i I_{n_i}), i \in \mathcal{V} \quad (8)$$

where  $\alpha_i \in \mathbb{R}$ . This choice leads to

$$A_{ii} - B_i K_{ii} = -\alpha_i I_{n_i}, i \in \mathcal{V}.$$

Using the decentralized controller (8), the closed-loop system matrix becomes

$$A - BK = \begin{bmatrix} -\alpha_1 I_{n_1} & A_{12} & \dots & A_{1N} \\ A_{21} & -\alpha_2 I_{n_2} & \dots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \dots & -\alpha_N I_{n_N} \end{bmatrix}. \quad (9)$$

By choosing an appropriate  $\alpha_i > 0$ , we can always make  $A - BK$  diagonally dominant with negative diagonal elements. Therefore,  $A - BK$  is diagonally stable, i.e., there exists a diagonal Lyapunov function to certify the stability of (9). Therefore, we have  $(A, B) \in \Sigma_2$ . ■

In essence, a fully actuated system is able to actuate each individual state directly. If each subsystem is of dimension one, i.e.,  $n_i = 1$ , then the condition in Proposition 1 means that the system pair of  $(A_i, B_i)$  is controllable. For general

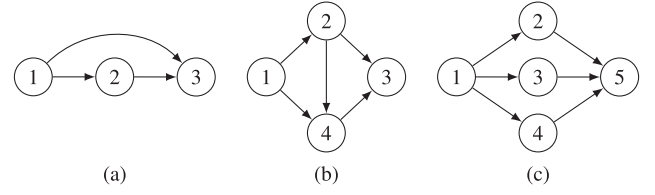


Fig. 1. Examples of directed acyclic graphs.

subsystems, the condition that  $B_i$  has full row rank is stronger than the controllability of  $(A_i, B_i)$ . Note that fully actuated systems have been used in some work on distributed design [10], [11], where it required the input matrix  $B$  to be invertible. Here, we show that a fully actuated system is indeed strongly decentralized stabilizable and suitable for the later development of the distributed algorithm.

#### B. Weakly Coupled Systems

Here, we discuss two types of weakly coupled systems: topologically weakly coupled systems and dynamically weakly coupled systems. A directed graph  $\mathcal{G}$  is called *acyclic* if there exist no directed cycles in  $\mathcal{G}$ . Fig. 1 shows some examples. A complex system with an acyclic  $\mathcal{G}_p$  means that the dynamical influence among subsystems is unidirectional.

**Definition 5 (Topologically weakly coupled system):** System (2) is called weakly coupled in terms of topological connections, if the plant graph  $\mathcal{G}_p$  is acyclic.

**Proposition 2:** For the class of topologically weakly coupled systems, we have

$$\Sigma_1 = \Sigma_2 = \{(A, B) \mid (A_{ii}, B_i) \text{ is stabilizable}, i \in \mathcal{V}\}.$$

**Proof:** This result is a simple consequence of [27] and [28]. If  $\mathcal{G}_p$  is acyclic, then there exists an ordering of the nodes such that for every edge  $(v_1, v_2)$ , node  $v_1$  precedes node  $v_2$  in the ordering. For this ordering, the resulting system matrix  $A$  is block lower triangular. Thus, without loss of generality, for a topologically weakly coupled system (2), the closed-loop system with a decentralized controller remains block lower triangular. It is known that a block triangular matrix is stable if and only if it is block-diagonally stable [27], [28], i.e., there exists a block-diagonal Lyapunov function to certify the stability of the closed-loop system. Therefore, for the class of topologically weakly coupled systems, we have  $(A, B) \in \Sigma_1 \Leftrightarrow (A, B) \in \Sigma_2$ . Meanwhile, considering the block triangular structure, the overall closed-loop system is stable if and only if each isolated closed-loop subsystem  $A_{ii} - B_i K_{ii}$  is stable,  $i \in \mathcal{V}$ . ■

We note that the class of topologically weakly coupled systems is also known as *hierarchical systems*; see [2, Ch. 10]. Hierarchical systems have useful properties, e.g., the equivalence between stability and block-diagonal stability [27], [28]. Proposition 2 further shows that for this type of systems, decentralized stabilization is equivalent to strongly decentralized stabilization ( $\Sigma_1 = \Sigma_2$ ).

Next, we consider dynamically weakly coupled systems. If each pair  $(A_{ii}, B_i)$  is stabilizable, then there exists a local

feedback  $K_{ii}$  such that  $A_{ii} - B_i K_{ii}$  is stable. Consequently, given any  $Q_i \succ 0$ , there exists a  $P_i \succ 0$ , such that

$$(A_{ii} - B_i K_{ii})^\top P_i + P_i (A_{ii} - B_i K_{ii}) + Q_i \prec 0.$$

In some cases, e.g., the singular values of  $A_{ij}$  are small (the strength of interactions is low), there may still exist a solution  $P_i \succ 0$  for the following inequality:

$$(A_{ii} - B_i K_{ii})^\top P_i + P_i (A_{ii} - B_i K_{ii}) + P_i \left( \sum_{j \in \mathbb{N}_i} A_{ij} A_{ij}^\top \right) P_i + Q_i \prec 0. \quad (10)$$

In (10), recall that  $\mathbb{N}_i$  denotes the set of neighboring nodes of node  $i$ . This observation leads to a concept of dynamically weakly coupled systems.

**Definition 6 (Dynamically weakly coupled systems):** System (2) is weakly coupled in terms of dynamical interactions, if there exists a local feedback  $K_{ii}$  such that the following inequality holds:

$$(A_{ii} - B_i K_{ii})^\top P_i + P_i (A_{ii} - B_i K_{ii}) + P_i \left( \sum_{j \in \mathbb{N}_i} A_{ij} W_{ij}^{-1} A_{ij}^\top \right) P_i + \sum_{j \in \mathbb{N}_i} W_{ji} \prec 0 \quad (11)$$

for some  $W_{ij} \succ 0, j \in \mathbb{N}_i, P_i \succ 0, i \in \mathcal{V}$ , where  $\hat{\mathbb{N}}_i$  denotes the set of nodes coming out of node  $i$  in  $\mathcal{G}_p$ .

Definition 6 is more general than condition (10), since inequality (11) is reduced to (10) when setting  $W_{ij} = I_{n_j}, j \in \mathbb{N}_i$ , and  $Q_i = \sigma_i I_{n_i}$ , where  $\sigma_i$  denotes the number of nodes in  $\hat{\mathbb{N}}_i$ .

**Proposition 3:** For a dynamically weakly coupled system (2), i.e., (11) holds, we have  $(A, B) \in \Sigma_2$ .

The proof utilizes the following Lemma.

**Lemma 2:** Given two matrices  $X, Y$  of appropriate dimensions, we have

$$X^\top W X + Y^\top W^{-1} Y \succeq X^\top Y + Y^\top X \quad (12)$$

for any  $W \succ 0$  of appropriate dimension.

**Proof:** Observe that

$$\begin{aligned} X^\top W X + Y^\top W^{-1} Y - (X^\top Y + Y^\top X) \\ = (W X - Y)^\top W^{-1} (W X - Y) \succeq 0. \end{aligned}$$

This means (12) holds. ■

**Proof of Proposition 3:** Consider a decentralized controller  $K = \text{diag}(K_{11}, \dots, K_{NN})$ . Upon defining  $\hat{A}_{ii} = A_{ii} - B_i K_{ii}$  and ignoring the disturbance, the closed-loop dynamics for each subsystem become

$$\dot{x}_i(t) = \hat{A}_{ii} x_i(t) + \sum_{j \in \mathbb{N}_i} A_{ij} x_j(t) \quad \forall i \in \mathcal{V}. \quad (13)$$

Next, we consider a block-diagonal Lyapunov function  $V(x) = \sum_{i=1}^N x_i^\top(t) P_i x_i(t)$ . The derivative of  $V(x)$  along the

closed-loop trajectory (13) is

$$\begin{aligned} \dot{V}(x) &= \sum_{i=1}^N (\dot{x}_i^\top P_i x_i + x_i^\top P_i \dot{x}_i) \\ &= \sum_{i=1}^N \left( x_i^\top \left( \hat{A}_{ii}^\top P_i + P_i \hat{A}_{ii} \right) x_i \right. \\ &\quad \left. + \underbrace{\left( \sum_{j \in \mathbb{N}_i} A_{ij} x_j \right)^\top P_i x_i + x_i^\top P_i \left( \sum_{j \in \mathbb{N}_i} A_{ij} x_j \right)}_{\text{coupling term}} \right). \end{aligned} \quad (14)$$

For the coupling term in (14), according to Lemma 2, we have

$$\begin{aligned} &\left( \sum_{j \in \mathbb{N}_i} A_{ij} x_j \right)^\top P_i x_i + x_i^\top P_i \left( \sum_{j \in \mathbb{N}_i} A_{ij} x_j \right) \\ &= \sum_{j \in \mathbb{N}_i} (x_j^\top A_{ij}^\top P_i x_i + x_i^\top P_i A_{ij} x_j) \\ &\leq \sum_{j \in \mathbb{N}_i} (x_i^\top P_i A_{ij} W_{ij}^{-1} A_{ij}^\top P_i x_i + x_j^\top W_{ij} x_j) \end{aligned} \quad (15)$$

for any  $W_{ij} \succ 0, j \in \mathbb{N}_i$ . Substituting (15) into (14), we get

$$\begin{aligned} \dot{V}(x) &\leq \sum_{i=1}^N \left( x_i^\top \left( \hat{A}_{ii}^\top P_i + P_i \hat{A}_{ii} \right) x_i \right. \\ &\quad \left. + P_i \left( \sum_{j \in \mathbb{N}_i} A_{ij} W_{ij}^{-1} A_{ij}^\top \right) P_i x_i + \sum_{j \in \mathbb{N}_i} x_j^\top W_{ij} x_j \right) \\ &= \sum_{i=1}^N x_i^\top \left( \hat{A}_{ii}^\top P_i + P_i \hat{A}_{ii} \right. \\ &\quad \left. + P_i \left( \sum_{j \in \mathbb{N}_i} A_{ij} W_{ij}^{-1} A_{ij}^\top \right) P_i + \sum_{j \in \hat{\mathbb{N}}_i} W_{ji} \right) x_i. \end{aligned}$$

If condition (11) holds for some  $W_{ij} \succ 0, j \in \mathbb{N}_i, P_i \succ 0, i \in \mathcal{V}$ , then,  $\dot{V}(x)$  is negative definite. Thus,  $V(s)$  is a block-diagonal Lyapunov function for the closed-loop system. ■

Note that condition (11) can be equivalently formulated into the following problem: we aim to find static scaling matrices  $W_{ij} \succ 0$  such that there exist  $K_{ii}, i \in \mathcal{V}$  satisfying

$$\|\hat{W}_i (sI - A_{ii} + B_i K_{ii})^{-1} \hat{A}_i W_i\|_\infty < 1, \quad i \in \mathcal{V} \quad (16)$$

where  $\hat{A}_i = [A_{ij_1} \quad A_{ij_2} \quad \dots \quad A_{ij_s}]$ ,  $W_i = \text{diag}\{W_{ij_1}^{-\frac{1}{2}}, W_{ij_2}^{-\frac{1}{2}}, \dots, W_{ij_s}^{-\frac{1}{2}}\}$ ,  $\hat{W}_i = (\sum_{j \in \mathbb{N}_i} W_{ji})^{\frac{1}{2}}$ , and  $\mathbb{N}_i = \{j_1, \dots, j_s\}$ . It is clear that both (11) and (16) are coupled between

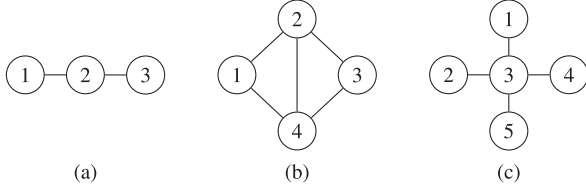


Fig. 2. Examples of chordal graphs. (a) Line graph. (b) Triangulated graph. (c) Star graph.

subsystems due to the scaling matrices  $W_{ij} \succ 0$ . If we *a priori* fix the weights  $W_{ij}$ , then the constraints in (11) and (16) are decoupled. This leads to a set of localized conditions to certify the dynamically weakly coupled condition (11). The sufficient conditions for block-diagonal stability based on scaled diagonal dominance in [27] may be good choices for choosing the weights  $W_{ij}$ .

#### IV. CHORDAL DECOMPOSITION IN OPTIMAL DECENTRALIZED CONTROL

In this section, by assuming that an undirected version of the plant graph  $\mathcal{G}_p(\mathcal{V}, \mathcal{E}_p)$  is chordal, we derive a decomposed version of problem (6), leading to multiple local subproblems. The chordal structure provides a way to define local computing agents or clusters of subsystems. This facilitates us to develop a distributed algorithm to solve (6) in Section V.

##### A. Chordal Graphs and Sparse Matrices

For completeness, we first review some graph-theoretic notion, and refer the interested reader to [29], [30] for details. Graph  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  is called undirected if  $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$ . A *clique*  $\mathcal{C}$  is a subset of nodes in  $\mathcal{V}$  where any pair of distinct nodes has an edge, i.e.,  $(i, j) \in \mathcal{E}, \forall i, j \in \mathcal{C}, i \neq j$ . If a clique  $\mathcal{C}$  is not included in any other clique, then it is called a *maximal clique*. A *cycle* of length  $k$  is a sequence of nodes  $\{v_1, v_2, \dots, v_k\} \subseteq \mathcal{V}$  with  $(v_k, v_1) \in \mathcal{E}$  and  $(v_i, v_{i+1}) \in \mathcal{E} \quad \forall i = 1, \dots, k-1$ . A *chord* in a cycle  $\{v_1, v_2, \dots, v_k\}$  is an edge  $(v_i, v_j)$  that joins two nonadjacent nodes in the cycle.

An undirected graph  $\mathcal{G}$  is called *chordal* if every cycle of length at least four has one chord [29]. Note that the set of maximal cliques is unique in a chordal graph, and the graph decomposition based on the maximal cliques is unique accordingly [30]. Fig. 2 illustrates some examples, and there are two maximal cliques,  $\mathcal{C}_1 = \{1, 2\}$  and  $\mathcal{C}_2 = \{2, 3\}$  for the chordal graph shown in Fig. 2(a). We highlight that maximal cliques can serve as computing agents and the overlapping elements, e.g., node 2 in Fig. 2(a), will play a role of coordination among maximal cliques. This feature enables preserving model data privacy (see Remarks 3 and 4).

Given a sequence of integers  $\{n_1, \dots, n_N\}$ , and an undirected graph  $\mathcal{G}(\mathcal{V}, \mathcal{E})$ , we define the space of symmetric block matrices with a particular sparsity pattern as

$$\mathbb{S}^n(\mathcal{E}, 0) := \{X \in \mathbb{S}^n | X_{ij} = X_{ji}^\top = 0 \text{ if } (j, i) \notin \hat{\mathcal{E}}\}$$

where  $n = \sum_{i=1}^N n_i$ ,  $X_{ij}$  is a block of dimension  $n_i \times n_j$  and  $\hat{\mathcal{E}} = \mathcal{E} \cup \{(i, i), \forall i \in \mathcal{V}\}$ . The cone of sparse block PSD matrices is defined as

$$\mathbb{S}_+^n(\mathcal{E}, 0) := \{X \in \mathbb{S}^n(\mathcal{E}, 0) | X \succeq 0\}.$$

Given a partition  $\{n_1, \dots, n_N\}$  and a maximal clique  $\mathcal{C}_k$  of  $\mathcal{G}$ , we define a block index matrix  $E_{\mathcal{C}_k} \in \mathbb{R}^{|\mathcal{C}_k| \times n}$  with  $|\mathcal{C}_k| = \sum_{j \in \mathcal{C}_k} n_j$  and  $n = \sum_{i=1}^N n_i$  as

$$(E_{\mathcal{C}_k})_{ij} := \begin{cases} I_{n_i}, & \text{if } \mathcal{C}_k(i) = j \\ 0_{n_i \times n_j}, & \text{otherwise} \end{cases}$$

where  $\mathcal{C}_k(i)$  denotes the  $i$ th node in  $\mathcal{C}_k$ , sorted in the natural ordering,  $I_{n_i}$  denotes an identity matrix of size  $n_i \times n_i$ , and  $0_{n_i \times n_j}$  denotes a zero matrix of size  $n_i \times n_j$ . Note that  $X_k = E_{\mathcal{C}_k} X E_{\mathcal{C}_k}^\top \in \mathbb{S}^{|\mathcal{C}_k|}$  extracts a principal submatrix according to clique  $\mathcal{C}_k$ , and the operation  $E_{\mathcal{C}_k}^\top X_k E_{\mathcal{C}_k}$  inflates a  $|\mathcal{C}_k| \times |\mathcal{C}_k|$  matrix into a sparse  $n \times n$  matrix. Then, we have the following result.

**Lemma 3 ([15], [20], [31]):** Let  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  be a chordal graph with maximal cliques  $\{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_t\}$ . Given a partition  $\{n_1, n_2, \dots, n_N\}$ , we have  $X \in \mathbb{S}_+^n(\mathcal{E}, 0)$  if and only if there exist matrices  $X_k \in \mathbb{S}_+^{|\mathcal{C}_k|}, k = 1, \dots, t$ , such that

$$X = \sum_{k=1}^t E_{\mathcal{C}_k}^\top X_k E_{\mathcal{C}_k}.$$

**Example 1:** Consider the following positive semidefinite matrix with a trivial partition  $\{1, 1, 1\}$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \succeq 0$$

which has a chordal sparsity pattern corresponding to Fig. 2(a) with maximal cliques  $\mathcal{C}_1 = \{1, 2\}$  and  $\mathcal{C}_2 = \{2, 3\}$ . Then, Lemma 3 guarantees the following decomposition:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = E_{\mathcal{C}_1}^\top \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 0.5 \end{bmatrix}}_{\succeq 0} E_{\mathcal{C}_1} + E_{\mathcal{C}_2}^\top \underbrace{\begin{bmatrix} 0.5 & 1 \\ 1 & 2 \end{bmatrix}}_{\succeq 0} E_{\mathcal{C}_2}$$

where

$$E_{\mathcal{C}_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, E_{\mathcal{C}_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Indeed, for any PSD matrix with partition  $\{2, 1, 1\}$  corresponding to Fig. 2(a), Lemma 3 guarantees a blockwise decomposition as follows (\* denotes a real number):

$$\underbrace{\begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & * \end{bmatrix}}_{\succeq 0} = \underbrace{\begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\succeq 0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}}_{\succeq 0}.$$

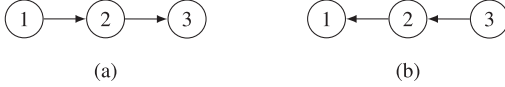


Fig. 3. Constructing the transpose graph. (a) Directed plant graph  $\mathcal{G}_p(\mathcal{V}, \mathcal{E}_p)$ . (b) Transpose graph of  $\mathcal{G}_p$ .

### B. Chordal Decomposition of Problem (6)

In (6), the variables  $X_i, Y_i, Z_i$  are coupled by the inequality (6a) only, while the rest of the constraints and the objective function are naturally separable due to the separable performance weights  $Q, R$ . Meanwhile, thanks to the block-diagonal assumption on  $X$ , the coupled linear matrix inequality has a structured sparsity pattern characterized by an undirected version of  $\mathcal{G}_p(\mathcal{V}, \mathcal{E}_p)$ . Precisely, we define an undirected graph  $\mathcal{G}_u(\mathcal{V}, \mathcal{E}_u)$  with  $\mathcal{E}_u = \mathcal{E}_p \cup \mathcal{E}_p^\top$ , where  $\mathcal{E}_p^\top$  denotes the edge set of the transpose graph of  $\mathcal{G}_p$ , i.e., the graph associated to the transpose of the adjacency matrix of  $\mathcal{G}_p$ .

**Assumption 2:** Graph  $\mathcal{G}_u$  is chordal with maximal cliques  $\mathcal{C}_1, \dots, \mathcal{C}_t$ .

**Remark 2:** The undirected graph  $\mathcal{G}_u$  will be used in the development of distributed computation using ADMM. For example, consider an interconnected system with a directed line graph in Fig. 3(a). Its transpose graph is shown in Fig. 3(b), and the resulting undirected graph  $\mathcal{G}_u$  is the same as that in Fig. 2(a). If  $\mathcal{G}_u$  is not chordal, we can add suitable edges to  $\mathcal{E}_u$  to obtain a chordal graph [30]. In this case, sharing model data with directed neighbors in  $\mathcal{G}_p$  is not sufficient for the proposed distributed solution. Still, privacy of model data are maintained within each maximal clique in  $\mathcal{G}_u$ . For simplicity, we assume that  $\mathcal{G}_u$  is chordal. As shown in Fig. 2, some graphs, such as chains, trees and stars, are already chordal.

Considering the inherent structure of system (1), it is straightforward to see that  $(AX - BZ) + (AX - BZ)^\top + MM^\top \in \mathbb{S}^n(\mathcal{E}_u, 0)$ . To ease the exposition, we define

$$F(X, Z) := -(AX - BZ) - (AX - BZ)^\top - MM^\top.$$

According to Lemma 3,  $F(X, Z) \succeq 0$  is equivalent to the condition that there exist  $J_k \in \mathbb{S}_+^{|\mathcal{C}_k|}, k = 1, \dots, t$ , such that

$$F(X, Z) = \sum_{k=1}^t E_{\mathcal{C}_k}^\top J_k E_{\mathcal{C}_k}. \quad (17)$$

Therefore, (6) can be equivalently decomposed into

$$\begin{aligned} \min_{X_i, Y_i, Z_i, J_k} \quad & \sum_{i=1}^N \text{Tr}(Q_i X_i) + \text{Tr}(R_i Y_i) \\ \text{s.t.} \quad & \sum_{k=1}^t E_{\mathcal{C}_k}^\top J_k E_{\mathcal{C}_k} = F(X, Z) \\ & \begin{bmatrix} Y_i & Z_i \\ Z_i^\top & X_i \end{bmatrix} \succeq 0, X_i \succ 0, i = 1, \dots, N \\ & J_k \succeq 0, k = 1, \dots, t. \end{aligned} \quad (18)$$

One notable feature of (18) is that the global constraint (6a) is replaced by a set of small coupled constraints (17). In other words, (18) has partially coupled constraints, which can be solved in a distributed way by introducing consensus variables.

The cliques  $\mathcal{C}_1, \dots, \mathcal{C}_t$  give a partition of subsystems, and will serve as local computing agents. If there is no overlap among the cliques  $\mathcal{C}_1, \dots, \mathcal{C}_t$  (i.e., the system (2) is composed by dynamically disjoint components), then (18) is trivially decomposed into  $t$  decoupled subproblems of decentralized optimal control, which can be solved by cliques  $\mathcal{C}_1, \dots, \mathcal{C}_t$  independently. In the case where different cliques share some common nodes with each other, we can introduce appropriate auxiliary variables to achieve a distributed solution using ADMM.

### V. DISTRIBUTED DESIGN BASED ON ADMM

To formulate our distributed approach to solve the decomposed problem (18) [equivalent to problem (6)], we briefly review the basic setup of ADMM; see [32] for a comprehensive review. ADMM is a first-order method that solves a convex optimization problem of the form

$$\begin{aligned} \min_{x, y} \quad & f(x) + g(y) \\ \text{s.t.} \quad & Ex + Fy = c \end{aligned} \quad (19)$$

where  $x \in \mathbb{R}^{n_x}, y \in \mathbb{R}^{n_y}$  are decision variables,  $f$  and  $g$  are convex functions, and  $E \in \mathbb{R}^{n_c \times n_x}, F \in \mathbb{R}^{n_c \times n_y}$ , and  $c \in \mathbb{R}^{n_c}$  are problem data. Given a penalty parameter  $\rho > 0$ , the scaled ADMM algorithm solves (19) using the following iterations:

$$\begin{aligned} x^{h+1} &= \arg \min_x f(x) + \frac{\rho}{2} \|Ex + Fy^h - c + \lambda^h\|^2 \\ y^{h+1} &= \arg \min_y g(y) + \frac{\rho}{2} \|Ex^{h+1} + Fy - c + \lambda^h\|^2 \\ \lambda^{h+1} &= \lambda^h + Ex^{h+1} + Fy^{h+1} - c \end{aligned}$$

where  $\lambda \in \mathbb{R}^{n_c}$  is a scaled dual variable, and  $h$  denotes the iteration index. In many applications, splitting the minimization over  $x$  and  $y$  often leads to multiple subproblems, allowing distributed computation; see [32] for detailed discussions.

#### A. Simple Example

To illustrate the approach, we first consider an interconnected system characterized by a chain of three nodes, as shown in Fig. 2(a). In this case, the model data are  $B = \text{diag}\{B_1, B_2, B_3\}, M = \text{diag}\{M_1, M_2, M_3\}$ , and

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix}.$$

Note that the following illustration is directly suitable for systems with a directed graph. For example, if the plant graph is a directed line as in Fig. 3(a), then we have  $A_{12} = 0, A_{23} = 0$  in matrix  $A$  and need to construct the same undirected graph in Fig. 2(a) for the distributed computation.

In this case, there are two cliques  $\mathcal{C}_1 = \{1, 2\}, \mathcal{C}_2 = \{2, 3\}$ , and  $J_k, k = 1, 2$  in (17) are in the following form:

$$J_1(X_1, X_2, Z_1, J_{22,1}) := - \begin{bmatrix} J_{11} & A_{12}X_2 + X_1A_{21}^\top \\ * & J_{22,1} \end{bmatrix}$$

$$J_2(X_2, X_3, Z_3, J_{22,2}) := - \begin{bmatrix} J_{22,2} & A_{23}X_3 + X_2A_{32}^\top \\ * & J_{33} \end{bmatrix}$$

where  $*$  denotes the corresponding symmetric part and

$$J_{11} := A_{11}X_1 - B_1Z_1 + (A_{11}X_1 - B_1Z_1)^\top + M_1M_1^\top$$

$$J_{33} := A_{33}X_3 - B_3Z_3 + (A_{33}X_3 - B_3Z_3)^\top + M_3M_3^\top.$$

The coupling effect is imposed on the overlapping node 2

$$J_{22,1} + J_{22,2} = A_{22}X_2 - B_2Z_2$$

$$+ (A_{22}X_2 - B_2Z_2)^\top + M_2M_2^\top.$$

For any coupling variables that appear in two cliques, we introduce auxiliary variables. For this case, we introduce auxiliary variables for node 2

$$J_{22,1} = \hat{J}_{22,1}, J_{22,2} = \hat{J}_{22,2}$$

$$X_2 = X_{2,1}, \quad X_2 = X_{2,2}. \quad (20)$$

Also, we split the variables according to the cliques and the overlapping node

$$\text{Node 2, } y := \{X_2, Y_2, Z_2, \hat{J}_{22,1}, \hat{J}_{22,2}\}$$

$$\text{Clique } \mathcal{C}_1, \quad x_{\mathcal{C}_1} := \{X_1, Y_1, Z_1, X_{2,1}, J_{22,1}\}$$

$$\text{Clique } \mathcal{C}_2, \quad x_{\mathcal{C}_2} := \{X_3, Y_3, Z_3, X_{2,2}, J_{22,2}\}.$$

The variable  $y$  corresponds to the same  $y$  in the canonical form (19) and variables  $x_{\mathcal{C}_1}, x_{\mathcal{C}_2}$  corresponds to  $x$  in (19). This can be seen more directly in (21). Next, we show that (18) can be rewritten into the standard ADMM form (19) by defining indicator functions as

$$\mathbb{I}_{\mathcal{S}_k}(x_{\mathcal{C}_k}) := \begin{cases} 0, & x_{\mathcal{C}_k} \in \mathcal{S}_k \\ +\infty, & \text{otherwise} \end{cases}$$

$$\mathbb{I}_{\mathcal{L}}(y) := \begin{cases} 0, & y \in \mathcal{L} \\ +\infty, & \text{otherwise} \end{cases}$$

where sets  $\mathcal{S}_1, \mathcal{S}_2$  are defined as

$$\mathcal{S}_1 := \left\{ x_{\mathcal{C}_1} \mid J_1(X_1, X_{2,1}, Z_1, J_{22,1}) \succeq 0, X_1 \succ 0 \right.$$

$$\left. \begin{bmatrix} Y_1 & Z_1 \\ Z_1^\top & X_1 \end{bmatrix} \succeq 0 \text{ are feasible} \right\}$$

$$\mathcal{S}_2 := \left\{ x_{\mathcal{C}_2} \mid J_2(X_{2,2}, X_3, Z_3, J_{22,2}) \succeq 0, X_3 \succ 0 \right.$$

$$\left. \begin{bmatrix} Y_3 & Z_3 \\ Z_3^\top & X_3 \end{bmatrix} \succeq 0 \text{ are feasible} \right\}$$

$$\mathcal{L} := \left\{ y \mid \hat{J}_{22,1} + \hat{J}_{22,2} = A_{22}X_2 - B_2Z_2 \right.$$

$$+ (A_{22}X_2 - B_2Z_2)^\top + M_2M_2^\top$$

$$\left. X_2 \succ 0, \begin{bmatrix} Y_2 & Z_2 \\ Z_2^\top & X_2 \end{bmatrix} \succeq 0 \text{ are feasible} \right\}.$$

This allows us to rewrite (18) as an optimization problem in the form of (19)

$$\min_{x_{\mathcal{C}_k}, y} \sum_{k=1}^2 f_k(x_{\mathcal{C}_k}) + g(y)$$

$$\text{s.t. } (20) \text{ holds} \quad (21)$$

where  $f_1(x_{\mathcal{C}_1}), f_2(x_{\mathcal{C}_2})$  based on each clique are defined as

$$f_1(x_{\mathcal{C}_1}) := \text{Tr}(Q_1X_1) + \text{Tr}(R_1Y_1) + \mathbb{I}_{\mathcal{S}_1}(x_{\mathcal{C}_1}) \quad (22a)$$

$$f_2(x_{\mathcal{C}_2}) := \text{Tr}(Q_3X_3) + \text{Tr}(R_3Y_3) + \mathbb{I}_{\mathcal{S}_2}(x_{\mathcal{C}_2}) \quad (22b)$$

and  $g(y)$  based on the overlapping node 2 is defined as

$$g(y) := \text{Tr}(Q_2X_2) + \text{Tr}(R_2Y_2) + \mathbb{I}_{\mathcal{L}}(y).$$

Upon denoting  $\hat{x}_{\mathcal{C}_k}$  as the variables in  $x_{\mathcal{C}_k}$  that appears in the consensus constraint (20), and  $y_l(\mathcal{C}_k)$  as the corresponding local copies, e.g.,  $\hat{x}_{\mathcal{C}_1} = \{X_{2,1}, J_{22,1}\}, y(\mathcal{C}_1) = \{X_2, \hat{J}_{22,1}\}$ , the ADMM algorithm for (21) takes a distributed form.

*ADMM Algorithm for the Distributed Design*

1)  $x$ -update: for each clique  $k$ , solve the local problem

$$x_{\mathcal{C}_k}^{h+1} = \arg \min_{x_{\mathcal{C}_k}} f_k(x_{\mathcal{C}_k}) + \frac{\rho}{2} \|\hat{x}_{\mathcal{C}_k} - y^h(\mathcal{C}_k) + \lambda_{\mathcal{C}_k}^h\|^2. \quad (23)$$

2)  $y$ -update: solve the following problem to update local variables

$$y^{h+1} = \arg \min_y g(y) + \frac{\rho}{2} \sum_{k=1}^2 \|\hat{x}_{\mathcal{C}_k}^{h+1} - y(\mathcal{C}_k) + \lambda_{\mathcal{C}_k}^h\|^2. \quad (24)$$

3)  $\lambda$ -update: compute the dual variable

$$\lambda_{\mathcal{C}_k}^{h+1} = \lambda_{\mathcal{C}_k}^h + \hat{x}_{\mathcal{C}_k}^{h+1} - y^{h+1}(\mathcal{C}_k), k = 1, 2. \quad (25)$$

At each iteration  $h$ , subproblem (23) only depends on each clique  $\mathcal{C}_k$ . Consequently, the cliques can serve as computing agents to solve subproblem (23) to update the variable  $x_{\mathcal{C}_k}^{h+1}$  in parallel. For example, clique  $\mathcal{C}_1$  needs to solve the following convex problem:

$$\min_{x_{\mathcal{C}_1}} \text{Tr}(Q_1X_1) + \text{Tr}(R_1Y_1) + \frac{\rho}{2} \|\hat{x}_{\mathcal{C}_1} - y^h(\mathcal{C}_1) + \lambda_{\mathcal{C}_1}^h\|^2$$

$$\text{s.t. } \begin{bmatrix} J_{11} & A_{12}X_{2,1} + X_1A_{21}^\top \\ * & J_{22,1} \end{bmatrix} \preceq 0$$

$$\begin{bmatrix} Y_1 & Z_1 \\ Z_1^\top & X_1 \end{bmatrix} \succeq 0, X_1 \succ 0$$



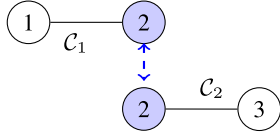


Fig. 4. Illustration of the ADMM algorithm for solving (21): cliques  $\mathcal{C}_1 = \{1, 2\}$  and  $\mathcal{C}_2 = \{2, 3\}$  serve as two computing agents and the overlapping node 2 plays a role of a coordinator.

where the regularization term is

$$\begin{aligned} \|\hat{x}_{\mathcal{C}_1} - y^h(\mathcal{C}_1) + \lambda_{\mathcal{C}_1}^h\|^2 &= \|X_{2,1} - X_2^h + \Lambda_{2,1}^h\|^2 \\ &+ \|J_{22,1} - \hat{J}_{22,1}^h + \Lambda_{22,1}^h\|^2. \end{aligned}$$

The subproblems (24) and (25) deal with the consensus variables  $y^{h+1}$  and multipliers  $\lambda_{\mathcal{C}_k}^{h+1}$ ,  $k = 1, 2$ , which can be computed by node 2. Fig. 4 illustrates the distributed nature of this algorithm.

**Remark 3 (Privacy of model data):** At each iteration, the coordinator (i.e., node 2) only requires model data of itself  $A_{22}, B_2, M_2$  and the local copies  $X_{2,k}^{h+1}, J_{22,k}^{h+1}$  from cliques  $\mathcal{C}_k$ ,  $k = 1, 2$  to update  $y^{h+1}, \lambda_{\mathcal{C}_k}^{h+1}$  by solving (24) and (25). Therefore, the proposed ADMM algorithm for solving (18) has a distributed nature (see Fig. 4 for illustration): cliques  $\mathcal{C}_1$  and  $\mathcal{C}_2$  can solve (23) based on the model data within each clique in parallel, and node 2 plays a role of coordination by updating the auxiliary variables  $y^{h+1}, \lambda_{\mathcal{C}_k}^{h+1}$ . Consequently, the model data of node 1 (i.e.,  $A_{11}, B_1, M_1, A_{12}, A_{21}$ ) are accessible only to clique  $\mathcal{C}_1$  only, while clique  $\mathcal{C}_2$  holds the model data of node 3 (i.e.,  $A_{33}, B_3, M_3, A_{32}, A_{23}$ ), exclusively.

**Remark 4 (Privacy and maximal cliques):** In our ADMM algorithm, the privacy of model data is maintained within each maximal clique of  $\mathcal{G}_u$ . Therefore, the level of privacy depends on the sparsity of  $\mathcal{G}_u$ . For highly interconnected systems with only one maximal clique, the decomposition (17) brings no benefit for privacy, and a global model is still required. In practice, if the plant graph  $\mathcal{G}_p$  is a chain or star graph (see Fig. 2 for illustration), then each maximal clique is of size two only, meaning that each subsystem need to share its model data with its direct neighbors only, and the model data privacy can be, therefore, maintained to a large extent.

**Remark 5 (Convergence of the ADMM algorithm):** The general ADMM algorithm is guaranteed to converge for convex problems under very mild conditions [32, Sec. 3.2]. In this case, under the feasibility assumption of (6), the proposed ADMM algorithm (23)–(25) is guaranteed to find a solution asymptotically. In the examples considered in this article, ADMM typically found a solution with moderate accuracy (in the sense of standard stopping criteria [32, Sec. 3.3]) within a few hundred iterations (see Section VI). Note that adjusting the penalty parameter  $\rho$  dynamically may further improve the practical convergence of the ADMM algorithm [32, Sec. 3.4.1]. In our simulations, we used a fixed choice of  $\rho = 5$ , since it led to a satisfactory convergence for our instances.

## B. General Case

The idea abovementioned can be extended to solve (18) with a general chordal graph pattern, and the general

problem (18) shares great similarities with the simple example in Section V-A. First, we define a set  $\mathcal{N}_0 := \{i \in \mathcal{V} \mid \exists q, k = 1, \dots, p, \text{ such that } i \in \mathcal{C}_q \cap \mathcal{C}_k\}$  that contains the overlapping nodes, and a set  $\mathcal{E}_0 := \{(i, j) \in \mathcal{E}_u \mid \exists q, k = 1, \dots, t, \text{ such that } (i, j) \in (\mathcal{C}_q \times \mathcal{C}_q) \cap (\mathcal{C}_k \times \mathcal{C}_k)\}$  that contains the overlapping edges. For the example in Fig. 4, we have  $\mathcal{N}_0 = \{2\}$  and  $\mathcal{E}_0 = \emptyset$ . Also, we define  $\mathcal{N}_i := \{k \mid i \in \mathcal{C}_k, k = 1, \dots, p\}$  that denotes the cliques containing node  $i$ , and  $\mathcal{E}_{ij} := \{(i, j) \in \mathcal{C}_k \times \mathcal{C}_k, k = 1, \dots, t\}$  that denotes the cliques containing edge  $(i, j)$ .

In fact, the elements in  $\mathcal{N}_0$  and  $\mathcal{E}_0$  make the constraint (17) coupled among different maximal cliques. Similar to (20), for each node  $i \in \mathcal{N}_0$ , we introduce local consensus constraints

$$X_i = X_{i,k}, \hat{J}_{ii,k} = J_{ii,k} \quad \forall k \in \mathcal{N}_i. \quad (26)$$

For each overlapping edge  $(i, j) \in \mathcal{E}_0$ , we introduce local consensus constraints

$$X_{ij} = X_{i,k}, \hat{J}_{ij,k} = J_{ij,k} \quad \forall (i, j) \in \mathcal{E}_{ij}. \quad (27)$$

Then, variable  $x_{\mathcal{C}_k}$  for each maximal clique  $k = 1, \dots, t$  includes the following:

- 1)  $X_i, Y_i, Z_i, i \in \mathcal{C}_k \setminus \mathcal{N}_0$  that belongs to clique  $\mathcal{C}_k$  exclusively;
- 2)  $X_{i,k}, J_{ii,k}, i \in \mathcal{C}_k \cap \mathcal{N}_0$  that corresponds to overlapping nodes in  $\mathcal{C}_k$ ;
- 3)  $J_{ij,k}, (i, j) \in (\mathcal{C}_k \times \mathcal{C}_k) \cap \mathcal{E}_0$  that corresponds to overlapping edges in  $\mathcal{C}_k$ .

We also collect the local copies  $X_i, Y_i, Z_i, \hat{J}_{ii,k}, i \in \mathcal{N}_0$  and  $\hat{J}_{ij,k}, X_{ij,k}, (i, j) \in \mathcal{E}_0$  as the consensus variable  $y$ .

Then, (18) can be written into the canonical ADMM form

$$\begin{aligned} \min_{x_{\mathcal{C}_k}, y} \quad & \sum_{k=1}^t f_k(x_{\mathcal{C}_k}) + g(y) \\ \text{s.t.} \quad & (26) \text{ and } (27) \text{ hold} \end{aligned} \quad (28)$$

where the clique function  $f_k(x_{\mathcal{C}_k})$  is defined as

$$f_k(x_{\mathcal{C}_k}) := \sum_{i \in \mathcal{C}_k \setminus \mathcal{N}_0} \text{Tr}(Q_i X_i) + \text{Tr}(R_i Y_i) + \mathbb{I}_{\mathcal{S}_k}(x_{\mathcal{C}_k}) \quad (29)$$

and  $g(y)$  is defined as

$$g(y) := \sum_{i \in \mathcal{N}_0} \text{Tr}(Q_i X_i) + \text{Tr}(R_i Y_i) + \mathbb{I}_{\mathcal{L}}(y). \quad (30)$$

In (29), set  $\mathcal{S}_k$  is defined as

$$\begin{aligned} \mathcal{S}_k := \left\{ x_{\mathcal{C}_k} \mid J_k(x_{\mathcal{C}_k}) \succeq 0, X_i \succ 0 \right. \\ \left. \left[ \begin{array}{cc} Y_i & Z_i \\ Z_i^\top & X_i \end{array} \right] \succeq 0, i \in \mathcal{C}_k \setminus \mathcal{N}_0 \text{ are feasible} \right\} \end{aligned}$$

and in (30), set  $\mathcal{L}$  is defined as

$$\mathcal{L} := \left\{ y \left| \begin{aligned} &\sum_{k \in \mathcal{N}_i} \hat{J}_{ii,k} = A_{ii}X_i - B_iZ_i + (A_{ii}X_i - B_iZ_i)^\top \\ &+ M_i M_i^\top, X_i \succ 0, \begin{bmatrix} Y_i & Z_i \\ Z_i^\top & X_i \end{bmatrix} \succeq 0, i \in \mathcal{N}_0 \\ &\sum_{k \in \mathcal{E}_{ij}} \hat{J}_{ij,k} = A_{ij}X_{ij} + X_{ji}A_{ji}^\top, (i,j) \in \mathcal{E}_0 \text{ are feasible} \end{aligned} \right. \right\}.$$

By applying the ADMM to (28), we obtain iterations that are identical to (23)–(25). Note that the set  $\mathcal{L}$  can be equivalently rewritten as a product of sets defined by  $X_i, Y_i, Z_i, \hat{J}_{ii,k}, i \in \mathcal{N}_0$  and  $\hat{J}_{ij,k}, X_{ij}, (i,j) \in \mathcal{E}_0$ . For each  $i \in \mathcal{N}_0$ , the set for  $X_i, Y_i, Z_i, \hat{J}_{ii,k}$  is defined as

$$\mathcal{L}_i := \left\{ (X_i, Y_i, Z_i, \hat{J}_{ii,k}) \left| \begin{aligned} &\sum_{k \in \mathcal{N}_i} \hat{J}_{ii,k} = A_{ii}X_i - B_iZ_i \\ &+ (A_{ii}X_i - B_iZ_i)^\top + M_i M_i^\top, X_i \succ 0, \begin{bmatrix} Y_i & Z_i \\ Z_i^\top & X_i \end{bmatrix} \succeq 0 \end{aligned} \right. \right\}.$$

This means that  $y$ -update (24) can be distributed among the overlapping nodes  $\mathcal{N}_0$  and overlapping edges  $\mathcal{E}_0$ . Therefore, similar to the example in Section V-A, variables  $x_{C_k}^h$  can be updated on each clique in parallel, and the overlapping elements in  $\mathcal{N}_0$  and  $\mathcal{E}_0$  can update  $y_{C_k}^h, \lambda_{C_k}^h$  individually until convergence.

Here, as stated in Remark 3, we emphasize that the main interest of this algorithm is the ability of distributing the computation to cliques and overlapping elements, thus preserving the privacy of model data in the problem.

## VI. NUMERICAL CASES

This section demonstrates the effectiveness of the proposed distributed design method.<sup>2</sup> For the examples, we ran the ADMM algorithm with termination tolerance  $10^{-3}$  and the number of iterations was limited to 500. In our simulations, SeDuMi [33] and YALMIP [34] were used to solve the subproblems within each clique and overlapping elements.

### A. First-Order Systems With Acyclic Directed Graphs

We first consider a network of four unstable coupled first-order subsystems, where  $\mathcal{G}_p$  is the directed acyclic graph shown in Fig. 1(b). In the experiment, the global dynamics are

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 2 & 3 & 4 \\ 1 & 2 & 0 & 4 \end{bmatrix} x(t) + u(t) + d(t). \quad (31)$$

This system is both fully actuated and topologically weakly coupled according to Section III. We chose  $Q_i = 1$  and  $R_i = 1, i \in \mathcal{V}$  in our simulation. When the global dynamics are available, solving (6) directly returned a decentralized controller

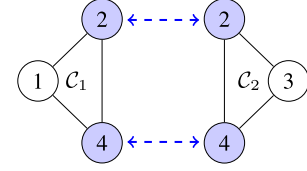


Fig. 5. Illustration of the ADMM algorithm for solving (6) corresponding to the example (31): the cliques  $C_1 = \{1, 2, 4\}$  and  $C_2 = \{2, 3, 4\}$  serve as two computing agents and the overlapping nodes play a role of coordinators by updating the axillary variables.

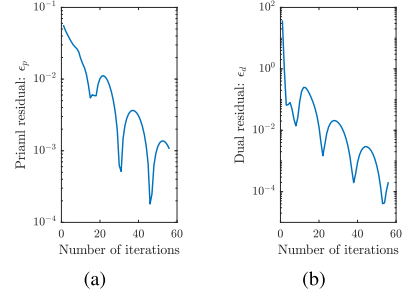


Fig. 6. Primal and dual residuals versus iteration number for the directed graph (42). (a) Primal residual. (b) Dual residual.

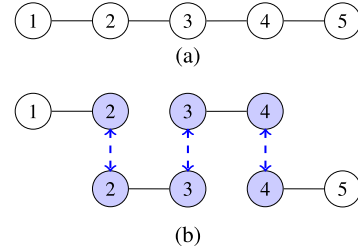


Fig. 7. (a) Chain of five nodes. (b) Four maximal cliques  $C_i = \{i, i+1\}, i = 1, 2, 3, 4$ , which serve as four computing agents relying only on the model data within each clique; the overlapping nodes 2, 3, and 4 play a role of coordinators.

$K_{11} = 7.34; K_{22} = 11.38; K_{33} = 6.16, K_{44} = 13.48$  with an  $\mathcal{H}_2$  performance of 5.36.

Instead, when the privacy of model data is concerned, the proposed ADMM algorithm can solve (6) in a distributed fashion. As shown in Fig. 5, for clique 1, only the model data of nodes 1, 2, and 4 are required, while clique 2 only needs the model data of nodes 2, 3, 4, and the overlapping nodes 2 and 4 play a role of coordinations in the algorithm. In this way, the model of node 1 can be kept private within clique 1 and the model of node 3 is known within clique 2 exclusively. For this instance, after 54 iterations, the ADMM algorithm returned the decentralized controller  $K_{11} = 7.35; K_{22} = 11.41; K_{33} = 6.16, K_{44} = 13.49$  with an  $\mathcal{H}_2$  performance of 5.37. The convergence plot of our algorithm for this instance is given in Fig. 6.

### B. Chain of Unstable Second-Order Coupled Systems

Here, we use a chain of five nodes (see Fig. 7) to provide a comparison between the proposed ADMM algorithm and the following three approaches.

<sup>2</sup>Code is available via [https://github.com/zhengy09/distributed\\_design\\_methods](https://github.com/zhengy09/distributed_design_methods).

TABLE I

COMPARISON OF THE PROPOSED ADMM ALGORITHM, SEQUENTIAL APPROACH [22], LOCALIZED LQR [2, Ch. 7.3], AND TRUNCATED LQR DESIGN FOR THE SYSTEM (32)

	ADMM	Sequential	Localized LQR	Truncated LQR
pct. <sup>‡</sup>	100 %	72 %	54 %	62 %
$\mathcal{H}_2$ <sup>†</sup>	6.06	6.36	6.50	6.49

<sup>‡</sup>: Successful percentage of returning a stabilizing decentralized controller.

<sup>†</sup>: Average  $\mathcal{H}_2$  performance of based on common successful instances.

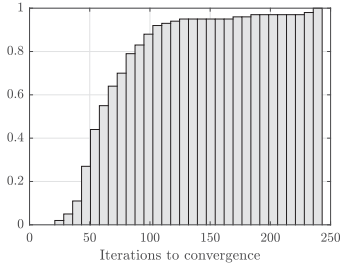


Fig. 8. Cumulative plot of the fraction of 100 random trails of (32) that required a given number of iterations to converge.

- 1) A *sequential approach* [22], which exploits the properties of clique trees in chordal graphs.
- 2) *Localized LQR design* [2, Ch. 7.3], which computes a local LQR controller for each subsystem independently by ignoring the coupling terms  $A_{ij}$ .
- 3) *Truncated LQR design*, which computes a centralized LQR controller using the global model data and only keeps the diagonal blocks for decentralized feedback.

It is assumed that each node is an unstable second order system coupled with its neighboring nodes

$$\dot{x}_i = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x_i + \sum_{j \in \mathbb{N}_i} A_{ij} x_j + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u_i + d_i) \quad (32)$$

where the entries of coupling term  $A_{ij}$  were generated randomly from  $-0.5$  to  $0.5$  to ensure that the numerical examples are strongly decentralized stabilizable. There are four maximal cliques  $\mathcal{C}_i = \{i, i+1\}, i = 1, 2, 3, 4$ . The model data can be kept private within each clique, and the overlapping nodes (i.e., 2, 3, 4) coordinate the consensus variables among maximal cliques. In the simulation, the state and control weights were  $Q_i = I_2$  and  $R_i = 1$  for each subsystem.

We generated 100 random instances of this interconnected system (32). The performance comparison between the four methods is listed in Table I. The proposed ADMM algorithm was able to return stabilizing decentralized controllers for all 100 tests, while the sequential approach, localized LQR and truncated LQR design only succeeded for 72%, 54%, 62% of the tests, respectively. This is expected since the proposed ADMM algorithm only requires the system being strongly decentralized stabilizable. The sequential approach requires an additional equal-splitting assumption among maximal cliques (see [22, Sec. VI-B]), and the localized LQR and truncated LQR design has no guarantees of success in general. Also, the

average  $\mathcal{H}_2$  performance for the common succeeded instances by the ADMM algorithm is the best. Finally, Fig. 8 shows the cumulative plot of convergence performance of our algorithm, where 90% of the tests required less than 150 iterations.

## VII. CONCLUSION

We introduced a distributed design method for decentralized control that relies on local model information only. Our main strategy is consistent with the recent general idea of exploiting sparsity in systems theory via chordal decomposition [19]–[22]. In this article, we further demonstrated the potential of chordal decomposition in distributed design of decentralized controllers, by combining this approach with the ADMM algorithm. Similar to [13], [22], our method relies on a block-diagonal Lyapunov function, which may bring some conservatism in general. Currently, we are studying convex restrictions that are less restrictive than the block-diagonal assumption, while still allowing distributed computation.

## APPENDIX

This appendix shows (7) using counterexamples. Consider the following system with two scalar subsystems

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ a_1 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (33)$$

where the first scalar subsystem is not affected by the first control input, i.e.,  $B_1 = 0$  in (1). Since (33) is controllable  $\forall a_1 \in \mathbb{R}, a_2 \in \mathbb{R}$ , then, we know

$$\left( \begin{bmatrix} 1 & 2 \\ a_1 & a_2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \in \Sigma_0 \quad \forall a_1 \in \mathbb{R}, a_2 \in \mathbb{R}.$$

Next, consider a decentralized controller for (33)  $u_1 = -k_1 x_1, u_2 = -k_2 x_2$ , then the closed-loop system becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ a_1 & a_2 - k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (34)$$

The stability of (34) means that the real parts of its eigenvalues are negative. This requires

$$\begin{cases} a_2 + 1 - k_2 < 0 \\ a_2 - k_2 - 2a_1 > 0 \end{cases}$$

which is equivalent to  $a_2 + 1 < k_2 < a_2 - 2a_1$ . This means

$$\left( \begin{bmatrix} 1 & 2 \\ a_1 & a_2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \in \Sigma_1, \Leftrightarrow a_1 < -0.5, a_2 \in \mathbb{R}.$$

The Lyapunov inequality with a diagonal certificate reads as

$$\begin{aligned} & \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ a_1 & a_2 - k_2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ a_1 & a_2 - k_2 \end{bmatrix}^T \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} \\ &= \begin{bmatrix} 2p_1 & 2p_1 + a_1 p_2 \\ 2p_1 + a_1 p_2 & 2a_2 p_2 - 2k_2 p_2 \end{bmatrix} \prec 0 \end{aligned} \quad (35)$$

where  $p_1 > 0, p_2 > 0$ . Since the first principle minor  $2p_1 > 0$ , we know that (35) is infeasible,  $\forall a_1, a_2, k_2$ . Thus, we have

$$\left( \begin{bmatrix} 1 & 2 \\ a_1 & a_2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \notin \Sigma_2 \quad \forall a_1 \in \mathbb{R}, a_2 \in \mathbb{R}.$$

If both subsystems are fully actuated, i.e.,  $B_1 = 1, B_2 = 1$  in (33), then according to Proposition 1, we know

$$\left( \begin{bmatrix} 1 & 2 \\ a_1 & a_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \in \Sigma_0 \quad \forall a_1 \in \mathbb{R}, a_2 \in \mathbb{R}$$

$$\left( \begin{bmatrix} 1 & 2 \\ a_1 & a_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \in \Sigma_1 \quad \forall a_1 \in \mathbb{R}, a_2 \in \mathbb{R}$$

$$\left( \begin{bmatrix} 1 & 2 \\ a_1 & a_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \in \Sigma_2 \quad \forall a_1 \in \mathbb{R}, a_2 \in \mathbb{R}.$$

This simple example also shows that the ability of actuating the nodes is important for strongly decentralized stabilization.

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