Profinite rigidity of group extensions

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Abstract

This thesis investigates the extent to which finite quotients can distinguish between non-isomorphic groups G sharing a fixed normal subgroup N and a fixed quotient $Q \cong G/N$. We generally assume that Q and N are finitely generated and that Q is good in the sense of Serre. Two broad themes in this study are *profinite conjugacy of outer actions* and *collapsing of cohomology orbits*, which we explore across various contexts.

First, we construct large families of profinitely conjugate outer actions $\varphi_i : F_2 \to \mathrm{Out}(N)$ for N a free group, a surface group, or a free abelian group of sufficiently large finite rank. These lead to infinite families of groups of the form $N \rtimes F_2$ sharing the same finite quotients.

Second, we investigate both themes in crystallographic groups, providing a detailed analysis of examples from the literature and presenting new ones, including a pair of non-isomorphic crystallographic groups in dimension 8 that share the same finite quotients.

Third, we extend Wilkes' results on the profinite rigidity of Seifert fibre spaces to central extensions of 2-orbifold groups with higher-rank centre. We prove that both rigid and non-rigid phenomena occur and that within this family G_1 and G_2 share the same finite quotients if and only if $G_1 \times \mathbb{Z} \cong G_2 \times \mathbb{Z}$.

Fourth, we prove that finitely generated free-by-cyclic groups with centre are all distinguished from each other by their finite quotients. We do this proving a new result that F_n-by-$(\mathbb{Z}/m)$ groups are uniquely determined by a poset that records the conjugacy classes of their finite subgroups, the sizes of these subgroups and the first Betti number of their centralisers.

Statement of originality

I declare that the work in this thesis is, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known.

Chapters 3 and 5 are my own original work, based on my articles [Piw24, Piw23]. Chapter 6 is based on a joint article with Martin Bridson [BP24]. Chapters 1, 2 and 4 are primarily of explanatory and expository nature; the few original proofs are clearly indicated.

Paweł Piwek, Oxford, February 8, 2025.

Contents

Notation

The following conventions are used throughout this thesis. Certain symbols and font families are reserved for specific objects and relations.

- Capital Greek letters, such as Γ and Δ, are reserved for profinite groups. The profinite completion of a group G is denoted as $\widehat{G}$.
- The symbols $<, \subset, \prec$ are used for subgroup, subset, and partial order relations, respectively; **equality is allowed** for each of these symbols. We use $\triangleleft$ to denote normal subgroups and occasionally use the subscript $<_{\mathrm{fi}}$ to denote that a subgroup is of finite index.
- Conjugation is written on the left and is sometimes denoted by $\mathrm{Ad}_g(x) = {}^g x = gxg^{-1}$; commutators are $[x, y] := xyx^{-1}y^{-1}$.
- The letters N and Q are generally used for the kernel and quotient in an N-by-Q group extension, with the inclusion and quotient maps denoted as $\iota\colon N \to G$ and $\pi\colon G \to Q$ in $\left(1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1\right)$.
- A tilde, $\tilde{\phi}$, denotes a lift of a map, while a bar, $\overline{\phi}$, denotes an induced map.
- Covariant induced maps are marked with an asterisk subscript, ϕ_*, whereas contravariant ones are marked with an asterisk superscript, ϕ^*.
- Named families of finite groups are printed as Sym_n, Alt_n, Dih_n, Frob_p.
- Graphs are represented by $\mathcal{G}$ and $\mathcal{H}$, while group extensions are generally denoted by $\mathcal{D}$ and $\mathcal{E}$. Families of N-by-Q extensions are printed as $\mathscr{E}(Q; N)$.
- The center of a group G is denoted by $Z(G)$, while the normalizer and centralizer of a subgroup $H < G$ are denoted by $N_G(H)$ and $C_G(H)$.
- Categories are written as Modules, Groups, AbelianGroups, and GroupActions.
- The commutator subgroup is denoted by $[G, G]$ (and not by G').
- Subscripts are left outside of 'hats'; for instance, $\widehat{F}_m$ denotes the profinite completion of the free group on m letters, $\widehat{F_m}$.
- Elements of a semidirect product $N \rtimes H$ are written as pairs (n, h), where $n \in N$ and $h \in H$.
- Brackets $[x]$ are used for equivalence classes, e.g. orbits of group actions.
- We do not always distinguish between standard cocycles $(\zeta\colon Q^n \to M) \in Z^n(Q; M)$ and their cohomology classes $[\zeta]$ in $H^n(Q; M)$.

Chapter 1

Introduction

Profinite rigidity is the study of how much information about a group G is retained in (the system of) its finite quotients G/N. More formally, the object of study is the *profinite completions functor* which maps a group G to the inverse limit $\widehat{G} = \varprojlim_{N \lhd_{\mathrm{fi}} G} G/N$.

Group extensions are short exact sequences of groups where we (usually) fix the second and fourth terms, the kernel N and the quotient Q in $1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$. Depending on the context, we may study them up to different equivalence relations, or impose restrictions on their structure.

This thesis explores the interplay between these two concepts. It investigates contexts in which the group-extension structure arises naturally and utilizes the machinery of group extensions to produce or understand specific examples.

The leitmotif is that many examples of profinitely isomorphic group extensions N-by-Q come from *profinitely conjugate outer actions* and, for a fixed outer action $\varphi\colon Q \to \mathrm{Out}(N)$, from *cohomology orbits collapsing*.

We start this introduction with Section 1.1 listing the original contributions of this thesis with minimal context. We then give in Section 1.2 the general context of the main directions and important results in the study of profinite rigidity. Next, in Section 1.3 we propose the two main sources of lack of rigidity among group extensions: the mentioned profinitely conjugate outer actions, and cohomology orbits collapsing. We exemplify these paradigms in Section 1.4 with two explanations of the classical Baumslag's examples. Finally, Section 1.5 outlines the rest of this thesis.

1.1 Original contributions

The main original contributions to the study of profinite rigidity of groups in this thesis are the following. We list them together with a minimal description of their context.

Contributions of [Piw24] – Chapter 3

The two results show that the phenomenon of profinitely conjugate actions can lead to infinite families of profinitely isomorphic N-by-F_2 extensions. This contrasts with the case of virtually-polycyclic groups which by [GPS79] have finite genus.

Theorem A gives a general construction and Theorem B carries it out in the case of N being free-abelian, surface group or a non-abelian free group.

Theorem A. *Let N be a finitely generated, residually finite group such that the centralisers $C_N(M)$ of non-trivial normal subgroups $M \triangleleft N$ don't contain a non-abelian free group.*

Let $T < \mathrm{Aut}(N)$ be a subgroup that isn't free. Let $m \geq 2$ and $\{\varphi_i \colon F_m \twoheadrightarrow T\}$ be a family of surjective homomorphisms such that, setting $\overline{\varphi_i} \colon F_m \twoheadrightarrow \overline{T}$ to be the composition of φ_i with $\big(\nu \colon \mathrm{Aut}(N) \twoheadrightarrow \mathrm{Out}(N)\big)|_T$ and $\overline{T} = \nu(T)$, we have

$$\delta \circ \overline{\varphi_i} = \overline{\varphi_j} \circ \varepsilon \text{ for some } \delta \in \mathrm{Aut}(\overline{T}), \varepsilon \in \mathrm{Aut}(F_m) \iff i = j.$$

Then the groups $G_i := N \rtimes_{\varphi_i} F_m$ are residually finite and pairwise non-isomorphic, while having isomorphic profinite completions.

Theorem B. *Let N be one of the following.*

- *Free group F_n with $n \geq 10$.*
- *Fundamental group $\pi_1(S_n)$ of a closed orientable surface with genus $n \geq 5$.*
- *Free abelian group $\mathbb{Z}^n$ with $n \geq 12$.*

Then there exists an infinite family of non-isomorphic groups $G_i := N \rtimes_{\varphi_i} F_2$ having isomorphic profinite completions. These groups are of type F.

Contributions of Chapter 4

Crystallographic groups provide examples of profinitely isomorphic group extensions both through the phenomenon of profinitely conjugate actions and through cohomology orbits collapsing.

The contributions of this chapter are primarily explanatory. Theorem C gives the lowest-dimensional example of profinitely isomorphic crystallographic groups known to the author, discovered through private communication with Gabriele Nebe. Its verification is original, however. Additionally, Proposition D gives a new construction of a pair of isomorphic crystallographic groups in dimension 11.

Theorem C. *Let* $Q_1 = \langle X_1, Y_1 \rangle$ *and* $Q_2 = \langle X_2, Y_2 \rangle$ *be subgroups of* $\mathrm{GL}_8(\mathbb{Z})$ *generated by matrices* X_1, Y_1, X_2 *and* Y_2 *given below. Then* Q_1 *and* Q_2 *are not conjugate in* $\mathrm{GL}_8(\mathbb{Z})$*, but they are conjugate as subgroups of* $\mathrm{GL}_8(\widehat{\mathbb{Z}})$*. The subgroups are of size* 24*, isomorphic to the dicyclic group* Dic_6*.*

$$\underbrace{\begin{pmatrix} 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}}_{X_1}, \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}}_{Y_1}$$

$$\underbrace{\begin{pmatrix} 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -2 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 \end{pmatrix}}_{X_2}, \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & 0 & 0 & -1 & -1 \\ -1 & 1 & -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & -2 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & -1 & -1 \end{pmatrix}}_{Y_2}$$

Proposition D. *Let groups* $G_1 = (\mathbb{Z}/11) \rtimes_{.2} \mathbb{Z}$ *and* $G_2 = (\mathbb{Z}/11) \rtimes_{.2^3} \mathbb{Z}$ *be central extensions of* $M = \mathbb{Z}$ *by the Frobenius group* $\mathrm{Frob}_{11} = (\mathbb{Z}/11) \rtimes (\mathbb{Z}/10)$. *Let* $M' \cong \mathbb{Z}^{11}$ *be the permutation* $\mathbb{Z}$*-module for the transitive action of* Frob_{11} *on* $\{1, 2, \ldots, 11\}$ *and let* $f \colon M \to M'$ *be the diagonal embedding. Let* $G_i' = M' \rtimes G_i / \langle (f(1), -1) \rangle$ *be the pushout extensions along* f. *Then the following hold.*

1. $\widehat{G}_1' \cong \widehat{G}_2'$.
2. G_1' *and* G_2' *are crystallographic groups of dimension* 11.
3. G_1' *and* G_2' *aren't isomorphic.*

Contributions of [Piw23] – Chapter 5

Hempel [Hem14] and Wilkes [Wil17] determined the profinite rigidity of central extensions of $\mathbb{Z}$ by 2-orbifold groups. This chapter generalises their work to the setting of higher-rank kernels $\mathbb{Z}^n$.

Theorem F determines—apart from a single family of exceptions—for a fixed $n > 1$ and a fixed infinite closed orientable 2-orbifold group Q whether the central extensions $\mathbb{Z}^n$-by-Q are distinguished from each other by their profinite completions. Theorem G then shows that all the pairs (G_1, G_2) of such extensions with $\widehat{G}_1 \cong \widehat{G}_2$ come from the phenomenon of Baumslag [Bau74] – i.e. $G_1 \times \mathbb{Z} \cong G_2 \times \mathbb{Z}$.

Theorem E. *Let* n_1, n_2 *be natural numbers and* Q_1, Q_2 *be infinite fundamental groups of closed orientable* 2*-orbifolds. Let* G_1 *and* G_2 *be central extensions* $\mathbb{Z}^{n_1}$*-by-*Q_1 *and* $\mathbb{Z}^{n_2}$*-by-*Q_2 *respectively. If* $\widehat{G}_1 \cong \widehat{G}_2$ *then* $n_1 = n_2$ *and* $Q_1 \cong Q_2$.

Theorem F. *Let* Q *be an infinite fundamental group of a closed orientable* 2*-orbifold with* $m \geq 0$ *cone points of orders* $p_1, \ldots, p_m$, *and let* $d_1, \ldots, d_m$ *be the Smith coefficients associated to* $(p_1, \ldots, p_m)$. *For* $n > 1$ *the following hold.*

1. *If* $n > m$, *or* $d_{m-(n-1)} \in \{1, 2, 3, 4, 6\}$, *then the non-isomorphic central extensions of* $\mathbb{Z}^n$ *by* Q *are distinguished from each other by their profinite completions.*

2. *If* $n \leq m$ *and* $d_{m-(n-1)} \notin \{1, 2, 3, 4, 6, 12\}$*, then there exist non-isomorphic central extensions* G_1, G_2 *of* $\mathbb{Z}^n$ *by* Q *with* $\widehat{G}_1 \cong \widehat{G}_2$*.*

The case not covered by Theorem F is that of $n \leq m$ and $d_{m-(n-1)} = 12$.

Theorem G. *Let* Q *be an infinite fundamental group of a closed orientable* 2*-orbifold. Let* $n > 1$ *and* G_1, G_2 *be central extensions of* $\mathbb{Z}^n$ *by* Q *such that* $\widehat{G}_1 \cong \widehat{G}_2$*. Then* $G_1 \times \mathbb{Z} \cong G_2 \times \mathbb{Z}$*.*

Contributions of [BP24] – Chapter 6

Free-by-cyclic groups constitute an actively studied and rich family of groups. Their profinite rigidity was studied in [BRW17, JZ20, HK23].

Theorem H proves that free-by-cyclic groups with centre are profinitely distinguished from each other. An important feature of this class of extensions playing a part in this rigidity is that the isomorphisms between them don't have to preserve the free kernel. The proof reduces the problem to Theorem J proving rigidity of free-by-(finite cyclic) groups. This stands in contrast with the lack of profinite rigidity among free-by-finite groups [Bau74, GZ11]. Theorem I is the crucial ingredient in this proof, but is of independent interest as a complete isomorphism invariant of free-by-(finite cyclic) groups.

Theorem H. *Free-by-cyclic groups with non-trivial centre are distinguished from each other by their finite quotients.*

Theorem I. *Let* G *and* G' *be* F_n*-by-*$(\mathbb{Z}/m)$ *extensions. Then they are isomorphic if and only if there is a* data-preserving *isomorphism between their finite subgroups conjugacy posets* $\mathsf{FSC}(G)$ *and* $\mathsf{FSC}(G')$*.*

Furthermore, the isomorphism between G *and* G' *can be realised through sliding moves on any of their realisations as reduced graphs of finite cyclic groups and these moves can be constructed explicitly.*

Theorem J. *Groups which are extensions of a finitely generated free group by a finite cyclic group are distinguished from each other by their finite quotients.*

1.2 Profinite rigidity

This section has drawn on surveys in [GZ11, Aka12, Rei18, Bri22].

The modern study of the information contained in the system of finite quotients of a group has received great impetus from the work of Grothendieck [RG71] who noted that for a connected smooth projective algebraic variety X defined over $\overline{\mathbb{Q}}$ the algebraic fundamental group $\pi_1^{\text{alg}}(X)$ is isomorphic with the profinite completion $\widehat{\pi_1(X(\mathbb{C}))}$ of the fundamental group of the corresponding manifold of complex points. A key observation is that, for an automorphism $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\,\mathbb{Q})$ the algebraic fundamental groups $\pi_1^{\text{alg}}(X)$ and $\pi_1^{\text{alg}}(X^\sigma)$ are isomorphic, while the topological fundamental groups $\pi_1(X(\mathbb{C}))$ and $\pi_1(X^\sigma(\mathbb{C}))$ need not be—indeed Serre gave such examples in [Ser64]. This can be generalised to the following meta-question.

Meta-question 1.1. *Which finitely generated residually finite groups G are such that for any finitely generated residually finite group H*

$$\widehat{G} \cong \widehat{H} \iff G \cong H?$$

The assumption of residual-finiteness removes some 'obvious' examples: for any G we have $\widehat{G} \cong \widehat{G/G_0}$ where G_0 is the intersection of all finite index subgroups of G and if $G_0 \neq 1$ we have $G \ncong G/G_0$.

Borrowing from the terminology of integral quadratic forms, the set of isomorphism classes of groups (within a specified family of groups $\mathscr{F}$) whose profinite completions are isomorphic to that of G is called the *genus* of G (in $\mathscr{F}$).

Profinitely isomorphic families

One way of studying Meta-question 1.1 is searching for 'large' families of pairwise non-isomorphic groups which have isomorphic profinite completions.

Simplifying Serre's earlier work, in [Bau74] Baumslag gave examples of non-isomorphic semidirect products $(\mathbb{Z}/11) \rtimes \mathbb{Z}$ which have the same

finite quotients. Many other families of non-isomorphic residually finite groups sharing profinite completions were discovered at about the same time, but these early examples were polycyclic and the families were finite. Indeed, Grunewald–Pickel–Segal showed in [GPS79] that virtually-polycyclic groups have finite genus.

Pickel gave in [Pic74] an infinite family of non-isomorphic finitely presented metabelian groups which all shared profinite completions. His construction involved finding infinitely many R-modules M_i, where $R = \mathbb{Z}\,D[x, x^{-1}, (x+1)^{-1}]$ for a certain finite abelian group D, and where M_i have the same finite quotients (as R-modules). He then formed $G_i := M_i \rtimes A$ for a subgroup $A < R^\times$.

Grothendieck pairs

An important direction of research was finding so called *Grothendieck pairs*: pairs $H \overset{\iota}{\hookrightarrow} G$ of residually finite groups such that $\widehat{H} \overset{\widehat{\iota}}{\to} \widehat{G}$ is an isomorphism, but ι itself isn't. In 1990 Platonov and Tavgen—building on the earlier work [BR84] of Baumslag and Roseblade—showed in [PT90] that a product of two nonabelian free groups contains a subgroup giving a Grothendieck pair; in fact it contains uncountably many such pairwise non-isomorphic subgroups, infinitely many of which are finitely generated, but none is finitely presented – see [Bri22]. Grothendieck pairs of finitely presented groups were later constructed by Bridson–Grunewald in [BG04]. Subsequently, Bridson constructed in [Bri16] an infinite family of Grothendieck pairs $H_i \hookrightarrow \Gamma \times \Gamma$ with H_i finitely presented and Γ hyperbolic.

Most recently, in [Bri24] Bridson showed that for any finitely presented residually finite group G one can construct a finitely generated residually finite group $M_G \cong F_\infty \rtimes F_4$ and an embedding $M_G \hookrightarrow (F_4 * G) \times F_4$ which gives a Grothendieck pair.

For a modern discussion of Grothendieck pairs see Bridson's essay [Bri22] and a recent article of Jaikin-Zapirain–Lubotzky [JZL24] on *Grothendieck rigidity*.

Profinite invariants

A closely related direction of research is the investigation of which properties are shared by groups with isomorphic profinite completions.

Meta-question 1.2. *Which group properties $\mathcal{P}$ are invariant under profinite isomorphism (within a fixed family of groups $\mathscr{F}$)?*

Properties $\mathcal{P}$ which are profinite invariants of all finitely generated groups seem rather difficult to find; they include the following.

1. Being abelian, or more generally: satisfying a given group law.
2. The isomorphism type of abelianisation.
3. The lattice of finite index subgroups.

On the other hand, there is a plethora of properties $\mathcal{P}$ which are **not** profinite invariants of finitely generated groups, including the following.

1. Finite presentability. Indeed Platonov–Tavgen give a general construction for such examples in [PT90]. More generally, Lubotzky showed in [Lub14] that for every $s, r \geq 1$ there exist finitely generated residually finite groups G_s and G_r such that G_s and G_r have finiteness length s and r respectively and $\widehat{G}_s \cong \widehat{G}_r$.
2. The growth type. Nekrashevych gave in [Nek14] a family of finitely generated residually finite branch groups which all share profinite completions, but this family also contains uncountably many word-growth types.
3. Property (FA), as shown by Cheetham-West–Lubotzky–Reid–Spitler in [CWLRS22]. More generally, fixed point properties when acting on complete CAT(0) spaces of dimension at most d, see Bridson's article [Bri23].
4. Cohomological dimension, as showed by Lubotzky in [Lub14], or by Bridson's construction of [Bri24].

5. Kazhdan's property (T), as shown by Aka in [Aka12].

6. Vanishing of bounded cohomology, as shown by Echtler–Kammeyer in [EK23].

7. Amenability, as shown by Kionke–Schesler in [KS23]: they gave a family $\{G_i\}_{i\in I}$ of pairwise non-isomorphic branch groups, all containing F_2 and an amenable branch group A such that $\iota_i\colon A \hookrightarrow G_i$ give Grothendieck pairs.

8. Being a non-trivial direct product, as shown by Nikolov–Segal in [NS07].

9. Being torsion-free – showed by Lubotzky in [Lub14] or by the construction of Bridson in [Bri24]. Relatedly, earlier Lubotzky in [Lub93] gave an example of a finitely generated torsion-free linear group G whose profinite completion $\widehat{G}$ has uncountably many classes of elements of order r for each $r \geq 2$.

The situation becomes more interesting when we restrict the family $\mathscr{F}$ of groups for which the property in question is to be a profinite invariant. Finitely presented groups in particular seem much more rigid.

1. Bridson–Reid–Spitler constructed in [BRS23] finitely presented residually finite groups which are profinitely rigid among finitely presented groups, while having infinite genus among finitely generated groups.

2. For finitely presented residually finite groups Lück's Approximation Theorem (see [Lüc94]) proves that the first L^2-Betti number is a profinite invariant. As a consequence, being large is detected by profinite completions of these groups – see Lackenby's article [Lac10].

3. Wilton–Zalesskii showed in [WZ19] that among the family of fundamental groups of closed oriented 3-manifolds, profinite completions detect both the Kneser–Milnor and Jaco–Shalen–Johannson decompositions, as well as the geometry of the components – see [WZ17].

4. Jaikin-Zapirain showed in [JZ20] that being fibered with a compact fibre is a profinite invariant of the fundamental group of a compact orientable 3-manifold. Later, Hughes–Kielak showed in [HK22] that being algebraically fibred is a profinite invariant of finitely presented LERF groups.

5. In contrast to [Lub93], Kropholler–Wilson showed in [KW93] that in the family of abelian-by-nilpotent groups (which are residually finite by Corollary 3.3 of [CR06]) the set $t(\widehat{G})$ of torsion elements in $\widehat{G}$ is equal to the closure $\overline{t(G)}$ of the set of torsion elements in G. In particular G is torsion-free if and only if $\widehat{G}$ is.

Profinitely distinguished families

Another way of investigating Meta-question 1.1 is to look for families of groups whose profinite completions are not isomorphic. The most desirable property is being distinguished from *all* the residually finite groups: we call this *absolute profinite rigidity*.

It isn't difficult to see that if a group satisfies a *group law* then so does its profinite completion. This, together with structural results for the groups satisfying a given law (if such exist), can be used to show e.g. that finitely generated abelian groups and finitely generated free nilpotent groups are absolutely profinitely rigid. These can be then used to give many more examples, e.g. by considering their group extensions.

It has been much more difficult to give examples of absolutely profinitely rigid *full-sized* groups—ones containing F_2 and thus not satisfying any group law. The first such were given by Bridson–McReynolds–Reid–Spitler in [BMRS20] and they include the Bianchi group $\mathrm{PSL}_2(\mathbb{Z}[\omega])$ for $\omega^2 + \omega + 1 = 0$, and the fundamental group of the Weeks manifold. Twelve more full-sized examples of absolutely rigid triangle groups were given later in [BMRS21].

One of the biggest questions of this fields posed by Remeslennikov (Question 5.48. in [KM14]) remains open.

Question 1.3 (Remeslennikov's Question)**.** *Are finitely generated free groups F_n absolutely profinitely rigid?*

On the other hand, the relative rigidity results have been more abundant. They include the following.

1. [BCR16] showed that lattices in $\mathrm{PSL}_2(\mathbb{R})$ are profinitely rigid among the class of lattices in connected Lie groups.

2. In [Wil17] Wilkes showed that the fundamental groups of Seifert fibre spaces are distinguished from each other by their profinite completions, apart from the examples of Hempel. This is the basis of the work in Chapter 5.

3. Wilton showed in [Wil18, Wil21] that every finitely generated free group and surface group is distinguished by profinite completion from all other limit groups. This was recently extended by Fruchter–Morales in [FM22] to the class of finitely generated residually free groups with the use of virtual second Betti number.

4. Corson–Hughes–Möller–Varghese showed in [CHMV23] that graph products of finite groups are profinitely distinguished from each other.

5. Joint work of Bridson–Piwek [BP24] presented in Chapter 6 shows that finitely generated free-by-(finite cyclic) groups are profinitely distinguished from each other. This extends an earlier result of Grunewald–Zalesskii of [GZ11], who showed it in the case the quotient is a finite cyclic p-group.

1.3 Profinite properties of group extensions

In this section, we synthesize the various phenomena that arise when investigating the profinite properties of a specific class of group extensions.

Two phenomena that seem to explain most examples of profinitely isomorphic families of group extensions—these are also the focus of this thesis—are captured by the orbit maps induced by the following two morphisms of group actions.

The first one comes from the action of $\mathrm{Out}(N) \times \mathrm{Aut}(Q)$ on the set of outer actions $\varphi\colon Q \to \mathrm{Out}(N)$, where $(\overline{\alpha}, \gamma) \cdot \varphi := \mathrm{Ad}_{\overline{\alpha}} \circ \varphi \circ \gamma^{-1}$. If the orbits of the first action are mapped to the same orbit under the second action, we refer to this phenomenon as *profinitely conjugate actions*.

$$\begin{array}{c} \mathrm{Out}(N) \times \mathrm{Aut}(Q) \curvearrowright \{Q \to \mathrm{Out}(N)\} \\ \downarrow \\ \mathrm{Out}(\widehat{N}) \times \mathrm{Aut}(\widehat{Q}) \curvearrowright \{\widehat{Q} \to \mathrm{Out}(\widehat{N})\} \end{array}$$

The second morphism arises from the group action of the stabiliser of φ on the cohomology groups classifying the extensions with a given action $\varphi\colon Q \to \mathrm{Out}(M)$—here we assume the kernel is an abelian group M. Again, orbits of the first action can be mapped to the same orbit in the second action. We refer to this phenomenon as *cohomology orbits collapsing*.

$$\begin{array}{c} \mathrm{Stab}_{\mathrm{Out}(M)\times\mathrm{Aut}(Q)}(\varphi) \curvearrowright H^2(Q; M) \\ \downarrow \\ \mathrm{Stab}_{\mathrm{Out}(\widehat{M})\times\mathrm{Aut}(\widehat{Q})}(\varphi) \curvearrowright H^2(\widehat{Q}; \widehat{M}). \end{array}$$

1.3.1 Preliminary stages

Two phenomena may make it difficult, if not impossible, to use the machinery for classifying group extensions, which is why here we gather them under the name of *preliminary stages*.

Topology induced on N

The profinite completions functor doesn't preserve kernels and isn't exact; the extension $\big(1 \to N \to G \to Q \to 1\big)$ turns into $\big(1 \to \overline{N} \to \widehat{G} \to \widehat{Q} \to 1\big)$, where the kernel $\overline{N}$ depends on the profinite topology induced by G on N. We are interested in contexts where N and Q are finitely generated

and Q has separable cohomology (is good in the sense of Serre), so in our case it turns out that $\overline{N} = \widehat{N}$.

Kernel-detecting

It is much more manageable to classify N-by-Q extensions up to the relation of *similarity* (kernel-preserving isomorphism) rather than up to (unrestricted) isomorphism. Thus we will generally work in regimes of families of group extensions $\big(1 \to N \to G_i \to Q \to 1\big)$ with N and Q fixed where any group isomophism $\beta\colon G_i \to G_j$ restricts to a map $\alpha\colon N \to N$ on the kernels and induces a map $\gamma\colon Q \to Q$ on the quotients. We call such families of extensions *kernel-detecting*.

An important family that does not enjoy this property is that of free-by-cyclic groups: if they have first Betti number greater than 1, then the family of extensions $\big(1 \to F \to F \rtimes \mathbb{Z} \to \mathbb{Z} \to 1\big)$ isn't kernel detecting.

1.3.2 Profinitely conjugate outer actions

Every N-by-Q extension induces an *outer action* $\varphi\colon Q \to \mathrm{Out}(N)$ and the outer actions coming from similar extensions are in the same orbit of $\mathrm{Out}(N) \times \mathrm{Aut}(Q)$ acting on $\{Q \to \mathrm{Out}(N)\}$. Because of this the first stage of classifying N-by-Q extensions is classifying the outer actions modulo the action.

Each such outer action $\varphi\colon Q \to \mathrm{Out}(N)$ extends naturally to a continuous homomorphism $\varphi\colon \widehat{Q} \to \mathrm{Out}(\widehat{N})$ and this assignment gives a morphism of group actions shown in Diagram (1.1).

$$\begin{array}{ccc} \mathrm{Out}(N) \times \mathrm{Aut}(Q) & \curvearrowright & \{Q \to \mathrm{Out}(N)\} \\ \big\downarrow & & \\ \mathrm{Out}(\widehat{N}) \times \mathrm{Aut}(\widehat{Q}) & \curvearrowright & \{\widehat{Q} \to \mathrm{Out}(\widehat{N})\}. \end{array} \tag{1.1}$$

It may happen that outer actions $\varphi\colon Q \to \mathrm{Out}(N)$ lying in different orbits lie in the same orbit after completion, or they even become the very same homomorphism. This in turn may result in non-isomorphic N-by-Q extensions G_1, G_2, whose completions $\widehat{G}_1, \widehat{G}_2$ are isomorphic $\widehat{N}$-by-$\widehat{Q}$ extensions (we assume that Q has separable cohomology).

It should be said that not all of homomorphisms $\varphi\colon Q \to \mathrm{Out}(N)$ are induced from actual extensions. For a detailed discussion see Section 2.3.5.

Injectivity of $\{Q \to \mathrm{Out}(N)\} \to \{\widehat{Q} \to \mathrm{Out}(\widehat{N})\}$

Some examples of profinitely isomorphic pairs come from situations where the map $\{Q \to \mathrm{Out}(N)\} \to \{\widehat{Q} \to \mathrm{Out}(\widehat{N})\}$ itself isn't injective; they include the following.

1. An example of the form $N \rtimes \mathbb{Z}$ where N is a finitely generated residually finite group such that $\mathrm{Out}(N) \to \mathrm{Out}(\widehat{N})$ isn't injective. This gives $\widehat{N \rtimes \mathbb{Z}} \cong \widehat{N \times \mathbb{Z}}$, but $N \rtimes \mathbb{Z} \ncong N \times \mathbb{Z}$. See the article [NS07] of Nikolov and Segal.

2. More examples of the form $N \rtimes \mathbb{Z}$ coming from Grothendieck pairs. If $1 \to N \to G \to Q \to 1$ is such that $\widehat{Q} = 1$ and $\overline{N} \cong \widehat{N}$ then for any $g \in G$ with $\overline{g}$ of infinite order in Q we get $\widehat{N \rtimes \langle g \rangle} \cong \widehat{N} \times \widehat{\mathbb{Z}}$, but with some additional assumptions we can ensure $N \rtimes \langle g \rangle \ncong N \times \mathbb{Z}$. See Section 1.3 of Bridson's article [Bri09].

Action on $\{Q \to \mathrm{Out}(N)\}$

The following results rely on whether the map in Diagram (1.1) is injective.

1. Baumslag's seminal profinitely isomorphic examples of $(\mathbb{Z}/11) \rtimes \mathbb{Z}$ given in [Bau74].

2. Many examples of profinitely isomorphic families and profinitely distinguished families of virtually-free and virtually-free abelian groups given by Grunewald and Zalesskii in [GZ11].

3. Examples of extensions with free quotient given by the author in [Piw24] and discussed in Chapter 3.

4. Examples of infinite families of profinitely isomorphic finitely presented metabelian groups, which all are of the form $B \rtimes D$ where B is a fixed metabelian group and D is a fixed finite abelian group, given by Pickel in [Pic74].

5. SOL manifolds with fundamental groups $\mathbb{Z}^2 \rtimes \mathbb{Z}$ given by Funar in [Fun13], extended to more examples by de Jesus Nery in [dJN20].

6. Groups of the form $F_2 \rtimes \mathbb{Z}$ are profinitely distinguished from each other, see [BRW17].

1.3.3 Cohomology orbits collapsing

The second phenomenon comes from the classification of N-by-Q extensions with a fixed outer action $\varphi\colon Q \to \mathrm{Out}(N)$ where we assume that N is abelian—and to reflect it we rename it to M. The *equivalence* classes are now classified by $H^2(Q; M)$, while their similarity classes are classified by the orbits of the action $\mathrm{Stab}_{\mathrm{Out}(M)\times\mathrm{Aut}(Q)}(\varphi) \curvearrowright H^2(Q; M)$.

Since the quotient groups Q we consider are assumed to have separable cohomology, there is a natural map $H^2(Q; M) \to H^2_{\mathrm{cts}}(\widehat{Q}; \widehat{M})$ – see Proposition 2.59 in Section 2.2.6 for a detailed explanation.

The phenomenon is captured by the orbit map of

$$\begin{array}{c} \mathrm{Stab}_{\mathrm{Out}(M)\times\mathrm{Aut}(Q)}(\varphi) \curvearrowright H^2(Q; M) \\ \downarrow \\ \mathrm{Stab}_{\mathrm{Out}(\widehat{M})\times\mathrm{Aut}(\widehat{Q})}(\varphi) \curvearrowright H^2_{\mathrm{cts}}(\widehat{Q}; \widehat{M}). \end{array}$$

Cohomology orbits collapsing

The cohomology classes lying in different orbits can become identified, for example due to the stabiliser being much larger in $\mathrm{Out}(\widehat{M}) \times \mathrm{Aut}(\widehat{Q})$, then it is in $\mathrm{Out}(M) \times \mathrm{Aut}(Q)$.

This phenomenon is present in the following results.

1. Baumslag's examples of $(\mathbb{Z}/11) \rtimes_{.2} \mathbb{Z}$ and $(\mathbb{Z}/11) \rtimes_{.2^3} \mathbb{Z}$, when treated as central extensions of $\mathbb{Z}$ by the Frobenius group Frob_{11} of size 110.

2. Hempel's examples of Seifert fibre spaces – see [Hem14]. Later Wilkes showed in [Wil17] that the fundamental groups of all SFSs are distinguished apart from the examples of Hempel.

3. Profinite rigidity of central extensions of $\mathbb{Z}^n$ by 2-orbifold groups was studied and largely deteremined by Piwek in [Piw23] and is discussed here in Chapter 5.

4. Crystallographic groups in dimensions up to 4 were shown in [PPW21] by Piwek–Popović–Wilkes to be profinitely distinguished. This is discussed here in Section 4.3.

5. A pair of crystallographic groups in dimension 11 discussed in Section 4.2.2.

1.4 Prototype: Baumslag's example

What is in all likelihood the simplest example of a pair of non-isomorphic groups with isomorphic profinite completions was described by Baumslag in [Bau74]. We present it here, as it serves as an example of both of the proposed sources of lack of profinite rigidity among group extensions: profinitely conjugate outer actions, and cohomology orbits collapsing.

Additionally, the examples described below are used again in Section 4.2.2 to build profinitely isomorphic pairs of crystallographic groups.

Unavoidably, we won't explain all of the notions involved here, and for details we refer the reader to Chapter 2.

Theorem 1.4. *Let* $G_1 = (\mathbb{Z}/11) \rtimes_{\cdot 2} \mathbb{Z}$ *and* $G_2 = (\mathbb{Z}/11) \rtimes_{\cdot 2^3} \mathbb{Z}$. *Then* $G_1 \ncong G_2$, *but* $\widehat{G}_1 \cong \widehat{G}_2$.

Proof using induced actions. In both G_1 and G_2 the torsion elements form a subgroup $(\mathbb{Z}/11)$, so any isomorphism $\beta\colon G_1 \to G_2$ would extend to a similarity of group extensions.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/11 & \longrightarrow & G_1 & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & \mathbb{Z}/11 & \longrightarrow & G_2 & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \end{array}$$

This implies that the actions $\varphi_1, \varphi_2 \colon \mathbb{Z} \to \mathrm{Out}(\mathbb{Z}/11) = (\mathbb{Z}/11)^\times$, which send the generator t of $\mathbb{Z}$ to 2 and 2^3 respectively would satisfy $\mathrm{Ad}_\alpha \circ \varphi_1 = \varphi_2 \circ \gamma$. Since $(\mathbb{Z}/11)^\times$ is abelian the conjugation Ad_α is equal to the identity map, giving $\varphi_1 = \varphi_2 \circ \gamma$, where $\gamma \colon \mathbb{Z} \to \mathbb{Z}$ is an automorphism of $\mathbb{Z}$. This would mean that in $(\mathbb{Z}/11)^\times$ we have $2 = 2^{\pm 3}$, which doesn't hold. Thus $G_1 \ncong G_2$.

On the other hand, there does exist $\kappa \in \widehat{\mathbb{Z}}^\times$ which is congruent to 3 modulo 10, so we can choose an automorphism $\widehat{\gamma} \colon \widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}$ such that $\varphi_1 = \varphi_2 \circ \widehat{\gamma}$, thereby showing that $\widehat{G}_1 \cong \widehat{G}_2$. For details see Proposition 2.79. □

Proof using cohomology. The centre of $G_i = (\mathbb{Z}/11) \rtimes_{\varphi_i} \mathbb{Z}$ is $1 \rtimes 10\,\mathbb{Z}$ for both $i = 1$ and $i = 2$ – see Lemma 6.1. Thus we get central extensions

$$\mathcal{E}_i = \big(0 \to \mathbb{Z} \to G_i \to \underbrace{(\mathbb{Z}/11) \rtimes (\mathbb{Z}/10)}_{\mathrm{Frob}_{11}} \to 1\big),$$

where $\mathrm{Frob}_{11} \cong \mathbb{F}_{11} \rtimes \mathbb{F}_{11}^\times$ is the Frobenius group on 11 elements.

$H^2(\mathrm{Frob}_{11}; \mathbb{Z}) \cong \mathbb{Z}/10$ – see Proposition 2.42. The cohomology classes ζ_1 and ζ_2 representing $\mathcal{E}_1$ and $\mathcal{E}_2$ both generate $H^2(\mathrm{Frob}_{11}; \mathbb{Z})$ and they satisfy $\zeta_2 = 3\zeta_1$ – see Proposition 2.92.

Any isomorphism $\beta \colon G_1 \to G_2$ would again extend to a similarity of extensions $\mathcal{E}_1$ and $\mathcal{E}_2$, so for some $\alpha \in \mathrm{Aut}(\mathbb{Z})$ and $\gamma \in \mathrm{Aut}(\mathrm{Frob}_{11})$ we would have $(\alpha, \gamma) \cdot \zeta_1 = \zeta_2$.

But $\mathrm{Out}(\mathrm{Frob}_{11}) = 1$—see Proposition 2.89—all of its automorphisms are inner, so they act trivially on $H^2(\mathrm{Frob}_{11}; \mathbb{Z})$. The automorphisms α of $\mathbb{Z}$ are just $\cdot(\pm 1)$, so we would get $\pm\zeta_1 = \zeta_2$, which can't hold simultaneously with $\zeta_2 = 3\zeta_1$. Thus $G_1 \ncong G_2$.

On the other hand $\widehat{\mathbb{Z}}^\times$ has many more automorphisms, indeed it has an element $\kappa \equiv 3 \mod 10$. Thus we get that $(\kappa, \mathrm{Id}) \cdot \zeta_1 = \zeta_2$ as elements of $H^2_{\mathrm{cts}}(\mathrm{Frob}_{11}; \widehat{\mathbb{Z}})$ and hence the isomorphism $\widehat{G}_1 \cong \widehat{G}_2$. □

1.5 How this thesis is structured

Chapter 2 covers essential preliminaries and a synthesis of the 'machinery for classifying extensions'.

Specifically, Section 2.1 gives a very short introduction to profinite groups and profinite completions of groups, also in the context of group extensions. Section 2.2 introduces group cohomology and some of its properties, and proceeds to discuss the role of $H^2(Q; M)$ in classifying M-by-Q extensions for abelian M, including the various functoriality properties of this classification. Finally, it discusses two approaches to cohomology of profinite groups, and how they are equivalent in the cases of our interest. Section 2.3 introduces a framework for studying group extensions and classifying them, including some special cases used in the later chapters.

The following Chapters 3, 4, 5 and 6 study the profinite rigidity of different families of extensions exemplifying the different phenomena described in Chapter 1.

Chapter 3 studies N-by-Q extensions with Q being a non-abelian free group. It first builds up a general method for constructing families of pairwise non-isomorphic by-free extensions in Section 3.1, and then in Section 3.2 it applies it to give examples of such families of infinite size.

Chapter 4 discusses distinguishing crystallographic groups by their profinite completions. In Section 4.1 it introduces the main auxiliary results, in Section 4.2 it discusses examples of profinitely isomorphic pairs of crystallographic groups coming from both phenomena described in Chapter 1, and in Section 4.3 it discusses some known results on families of crystallographic groups which are profinitely rigid in the class of crystallographic groups.

Chapter 5 studies which central extensions of $\mathbb{Z}^n$ by closed orientable 2-orbifold groups Q are profinitely distinguished from each other. In Section 5.2 it introduces auxiliary results on 2-orbifold groups, their profinite completions and their second cohomology groups, in Section 5.3 it provides a useful interpretation of the action of $\mathrm{Out}(\mathbb{Z}^n) \times \mathrm{Aut}(Q)$ on

$H^2(Q;\mathbb{Z}^n)$ in terms of a certain action of a matrix group. Section 5.4 then studies this action in detail, the results of which are used in the final section to deduce the main theorems.

Chapter 6 provides an example of a study of profinite rigidity of a class of group extensions which is not done through the machinery of Section 2.3. It shows that within the class of free-by-cyclic groups with centre, and the class of finitely generated free-by-(finite cyclic) groups, the groups are profintiely distinguished from each other. It first reduces in Section 6.2 distinguishing of free-by-cyclic groups with centre to that of free-by-(finite cyclic) groups, then introduces in Section 6.3 the main device for distinguishing these: the *finite subgroups conjugacy poset*, and it it then shows in Section 6.4 that the profinite completions 'detect' the data carried in this poset.

Finally, Appendix A lists `MAGMA` code for verifying two of the examples given in Section 4.2.

Chapter 2

Preliminaries

This chapter forms a reference for background results that the later chapters make use of.

2.1 Profinite groups and profinite completions

This chapter isn't intended as a thorough introduction to the subject of profinite groups and profinite completions and only the basic definitions are given here. The reader is referred to the lecture notes of Reid [Rei13] and the recent book of Wilkes [Wil24] for a proper introduction to the topic, and to the book of Ribes and Zalesskii [RZ00] for a comprehensive reference.

2.1.1 Profinite groups

We need to start by defining what an inverse system is.

Definition 2.1. Let I be a partially ordered set, C be any category and $\{I, C_i, f_{ij}\}$ be a diagram of shape I in C, i.e. a collection of objects C_i in C indexed by the elements of I with a morphism $f_{ij}\colon C_i \to C_j$ for any $i \prec j$, such that for $i \prec j \prec k$ we have $f_{jk} \circ f_{ij} = f_{ik}$.

If I has the property that for any $i_1, i_2 \in I$ there exists $j \in I$ such that $j \prec i_1$ and $j \prec i_2$, we call the diagram $\{C_i\}$ an *inverse system* in C.

Definition 2.2. A *profinite group* is a group isomorphic to the inverse limit $\varprojlim_{i \in I} G_i$ (in the category of groups) of an inverse system of finite groups G_i.

Example 2.3. Groups $\mathbb{Z}/p^i$ indexed by $i \in \mathbb{N}$ naturally form the inverse system

$$\ldots \longrightarrow \mathbb{Z}/p^3 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \ .$$

The group $\mathbb{Z}_p$ of p-adic integers, defined as

$$\mathbb{Z}_p := \left\{ a_0 + a_1 p + a_2 p^2 + \ldots \mid a_i \in \{0, 1, \ldots, p-1\} \right\}$$

with the 'carry over' addition is its inverse limit and thus a profinite group.

In fact the inverse limit can be given in general somewhat explicitly. This is thanks to the fact that in categories with products and equalizers limits of diagrams can be given as equalizers of a collection of maps between the products of objects involved – for details see Lemma 002N of The Stacks Project [Sta24].

Proposition 2.4. *Let $\{I, G_i, f_{ij}\}$ be an inverse system of finite groups. Then the inverse limit exists and is given by*

$$\varprojlim_{i \in I} G_i = \{(g_i)_{i \in I} \in \Pi_{i \in I} G_i \mid f_{ij}(g_i) = g_j \text{ for all } i \prec j\}.$$

Giving the finite groups G_i the discrete topology makes $\Pi_{i \in I} G_i$ a compact, totally disconnected topological group, and since $\varprojlim G_i$ is a closed subset of this topological space, it is a compact totally disconnected topological group.

Conversely, profinite groups also have a natural description in the category of topological groups.

Proposition 2.5 (Implied by Theorem 2.1.3 of [RZ00])**.** *Any topological group Γ which is compact and totally disconnected is a profinite group.*

The topology has some important consequences for subgroups and quotients of profinite groups.

Proposition 2.6. *Let Γ be a profinite group.*

1. *Any closed subgroup $\Delta < \Gamma$ is also a profinite group.*
2. *The quotient Γ/Δ by a normal closed group $\Delta \triangleleft \Gamma$ is a profinite group.*
3. *A subgroup $\Delta < \Gamma$ is open if and only if it is closed of finite index.*

Definition 2.7. A topological group Γ is called *topologically finitely generated* if there are elements $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that the subgroup $\langle \gamma_1, \ldots, \gamma_n \rangle$ is dense in Γ, i.e. $\Gamma = \overline{\langle \gamma_1, \ldots, \gamma_n \rangle}$.

It is a remarkable result that for topologically finitely generated profinite groups in fact *all* finite index subgroups are open. As a consequence, we generally won't need to distinguish homomorphisms and continuous homomorphisms of profinite groups.

Theorem 2.8 (Nikolov–Segal [NS03])**.** *Let Γ be a topologically finitely generated profinite group. Then every finite index subgroup of Γ is open.*

Corollary 2.9. *Let Γ and Δ be profinite groups where Γ is topologically finitely generated. Then any group homomorphism $\phi\colon \Gamma \to \Delta$ is continuous.*

Gaschütz' Lemma for profinite groups will be crucial for us ensuring in Corollary 2.81 that there are isomorphisms between the profinite completions of the groups in the families we construct. Lemma 2.10 and Corollary 2.11 are respectively Lemmas 4.2 and 4.3 of [JK75]

Lemma 2.10 (Gaschütz' Lemma)**.** *Let Γ and Δ be profinite groups such that $\Gamma = \overline{\langle \gamma_1, \ldots, \gamma_d \rangle}$ for some $\gamma_1, \ldots, \gamma_d \in \Gamma$, and let $p\colon \Gamma \twoheadrightarrow \Delta$ be a surjective homomorphism of profinite groups.*

Then given any $(\delta_1, \ldots, \delta_d) \in \Delta^d$ such that $\overline{\langle \delta_1, \ldots, \delta_d \rangle} = \Delta$, there exists $(\gamma_1', \ldots, \gamma_d') \in \Gamma^d$ such that $p(\gamma_i') = \delta_i$ for $i = 1, \ldots, d$ and such that $\overline{\langle \gamma_1', \ldots, \gamma_d' \rangle} = \Gamma$.

Corollary 2.11. *Let Δ be a profinite group, and $p_i\colon \widehat{F}_m \twoheadrightarrow \Delta$ be a surjective homomorphism for $i = 1, 2$.*

Then there exists an automorphism $\varepsilon\colon \widehat{F}_m \to \widehat{F}_m$ such that $p_1 = p_2 \circ \varepsilon$.

Remark 2.12. The existence of *some* homomorphism $\varepsilon\colon \widehat{F}_m \to \widehat{F}_m$ such that $p_1 = p_2 \circ \varepsilon$ is guaranteed from the universal property of $\widehat{F}_m$. The main content of Corollary 2.11 (for which we need Gaschütz' Lemma) is that there exists a *surjective* homomorphism $\varepsilon\colon \widehat{F}_m \to \widehat{F}_m$—which then has to be a bijection.

2.1.2 Profinite completions

The *profinite completion* of a group organises the data of 'all finite quotients of a group' into an algebraic object which also 'remembers' how the different quotients relate to each other.

For each group G we can form an inverse system $\{G/N \mid N \lhd_{\mathrm{fi}} G\}$ of its canonical finite quotients. It is indeed an inverse system since for any finite index normal subgroups $N < M$ of G we have a canonical map $\varphi_{NM}\colon G/N \to G/M$ and given any $N, M \lhd_{\mathrm{fi}} G$, the intersection $N \cap M$ is also a finite index normal subgroup of G.

Definition 2.13. Given a group G together with the inverse system of its canonical finite quotients G/N, we define its *profinite completion* to be the inverse limit

$$\widehat{G} := \varprojlim_{N \lhd_{\mathrm{fi}} G} G/N.$$

Just like we have seen in Proposition 2.4 in the case of (general) profinite groups, the limit $\widehat{G}$ may be realised as a subgroup of the direct product $\Pi_{N \lhd_{\mathrm{fi}} G}\, G/N$.

It comes as no surprise that profinite completion is functorial.

Proposition 2.14. *Given a group homomorphism $\phi\colon G \to H$ the inverse image $\phi^{-1}(M)$ of a finite-index normal subgroup $M \lhd_{\mathrm{fi}} H$ is a finite-index normal subgroup. Thus, we can define a function*

$$\widehat{\phi}\colon (g_N N)_{N \lhd_{\mathrm{fi}} G} \mapsto \left(\phi(g_{\phi^{-1}(M)})M\right)_{M \lhd_{\mathrm{fi}} H}$$

which is a homomorphism. Furthermore, for $\phi_1\colon G \to H$ and $\phi_2\colon H \to K$ we get $\widehat{\phi_2 \circ \phi_1} = \widehat{\phi}_2 \circ \widehat{\phi}_1$, so profinite completion is a functor

$$\mathsf{H}\colon \mathsf{Groups} \to \mathsf{ProfiniteGroups}\,.$$

One additional feature of profinite completions is that from the canonical maps $v_N \colon G \to G/N$ we get a canonical map $\mathsf{h} \colon G \to \widehat{G}$.

Proposition 2.15. *Let G be any group and $\mathsf{h}_G \colon G \to \widehat{G}$ be the canonical map to its profinite completion.*

1. $\ker \mathsf{h}_G = G_0$, *where G_0 is the intersection of all finite-index subgroups of G. In particular, h_G is injective if and only if G is residually finite.*
2. *The image $\mathsf{h}_G(G) < \widehat{G}$ is a dense subgroup.*
3. *Given a group homomorphism $f \colon G \to H$ we have $\mathsf{h}_H \circ f = \widehat{f} \circ \mathsf{h}_G$.*

The following striking result has become standard – see [DFPR82] for the original paper or Corollary 3.2.8 of [RZ00] for a more general version. It shows that for finitely generated groups G the set $\mathcal{C}(G)$ of isomorphism classes of finite quotients of G carries exactly the same information as the profinite completion $\widehat{G}$.

Theorem 2.16. *Let G and H be finitely generated. Then $\mathcal{C}(G) = \mathcal{C}(H)$ if and only if $\widehat{G} \cong \widehat{H}$.*

One remarkable source of isomorphic profinite completions, connected to our examples in Chapter 5, was given by Baumslag in his seminal paper [Bau74]. It follows from a theorem of Remak—see [Rem11]—which proves that finite groups decompose into direct products in a canonical manner.

Theorem 2.17. *Let G_1, G_2, H_1 and H_2 be finitely generated groups such that $\widehat{G}_1 \cong \widehat{G}_2$ and $G_1 \times H_1 \cong G_2 \times H_2$. Then also $\widehat{H}_1 \cong \widehat{H}_2$.*

2.1.3 Profinite completions and extensions of groups

We will need to know what happens to an extension of groups under profinite completion. The following is taken from Proposition 3.2.5 of [RZ00]. Here and throughout $\overline{H}$ denotes the closure of $\mathsf{h}(H)$ in $\widehat{G}$ for $H < G$.

Proposition 2.18. *Given a short exact sequence* $1 \to N \to G \to Q \to 1$ *of any groups, the following diagram commutes and its bottom row is exact.*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \xrightarrow{\pi} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathsf{h}_G|_N} & & \downarrow{\scriptstyle \mathsf{h}_G} & & \downarrow{\scriptstyle \mathsf{h}_Q} & & \\ 1 & \longrightarrow & \overline{N} & \longrightarrow & \widehat{G} & \xrightarrow{\widehat{\pi}} & \widehat{Q} & \longrightarrow & 1 \end{array}$$

Proposition 2.19 shows that (in the case of finitely generated kernel) a profinite completion of a semidirect product is the semidirect product of the profinite completions.

Proposition 2.19. *Let* N *and* Q *be any groups. Then* $\widehat{N \rtimes Q} \cong \overline{N} \rtimes \widehat{Q}$*, where* $\overline{N}$ *is the closure of the image of* N *in* $\widehat{N \rtimes Q}$*. Furthermore, if* N *is finitely generated, then* $\overline{N} \cong \widehat{N}$*, and if* N *and* Q *are residually finite, then so is* $N \rtimes Q$*.*

Proof. Let $1 \longrightarrow N \longrightarrow N \rtimes Q \overset{\pi}{\underset{s}{\rightleftarrows}} Q \longrightarrow 1$ be a short exact sequence with $\pi \circ s = \mathrm{Id}_Q$. Profinite completion turns this into another short exact sequence

$$1 \longrightarrow \overline{N} \longrightarrow \widehat{N \rtimes Q} \overset{\widehat{\pi}}{\underset{\widehat{s}}{\rightleftarrows}} \widehat{Q} \longrightarrow 1$$

with $\widehat{\pi} \circ \widehat{s} = \widehat{\mathrm{Id}_Q} = \mathrm{Id}_{\widehat{Q}}$ – for details see Proposition 3.2.5 of [RZ00]. This is a split short exact sequence, so $\widehat{N \rtimes Q} \cong \overline{N} \rtimes \widehat{Q}$.

Now, if N is finitely generated, then it has a cofinal sequence $N_1 > N_2 > \ldots$ of finite-index characteristic subgroups. We have a sequence of subgroups $N_i \rtimes Q$ of finite index in $N \rtimes Q$, such that for any $M < N$ of finite index, $(N_i \rtimes Q) \cap N < M$ for some i. Thus, the topology induced by $\widehat{N \rtimes Q}$ on the image of N is the full profinite topology and so $\overline{N} \cong \widehat{N}$.

Finally, if N and Q are residually finite, then they canonically embed in their profinite completions and so also $N \rtimes Q$ embeds into $\widehat{N \rtimes Q} \cong \widehat{N} \rtimes \widehat{Q}$ via the natural map, so $N \rtimes Q$ is residually finite. □

Another general context in which the closure $\overline{N}$ has to have the full profinite topology of $\widehat{N}$ is related to the notion *separable cohomology* (Serre's *goodness*) introduced by Serre in [Ser79] which we define later in Section 2.2.6 in Definition 2.58. Remarkably this is a necessary condition for having separable cohomology in dimension 2.

Proposition 2.20 is adapted from Proposition 2.4 in [Lor08] while Corollary 2.21 is adapted from Corollary 6.2 of [GJZZ08].

Proposition 2.20. *Let Q be a finitely generated group. The following are equivalent.*

1. *Q has separable cohomology in dimension 2.*
2. *For any finitely generated group N and for any extension of N by Q $\left(1 \to N \to G \to Q \to 1\right)$, the closure $\overline{N}$ of the image of N in $\widehat{G}$ has the full profinite topology of $\widehat{N}$, i.e. $\overline{N} = \widehat{N}$.*

Corollary 2.21. *Let N be a residually finite, finitely generated group and Q be a residually finite, finitely generated group with separable cohomology in dimension 2. Then any extension of N by Q is residually finite and induces the full profinite topology on N.*

There is another important immediate use case for groups with separable cohomology. It was used, though not in exactly in this form, by [Bri09] to give an abundance of examples of Grothendieck pairs.

Corollary 2.22. *Let Q be a finitely generated group with $\widehat{Q} = 1$ and with separable cohomology in dimension 2, and $\left(1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1\right)$ be a short exact sequence of groups with N finitely generated. Then $\iota\colon N \hookrightarrow G$ is a Grothendieck pair, i.e. $\widehat{\iota}\colon \widehat{N} \to \widehat{G}$ is an isomorphism.*

The above corollary isn't stated in its most natural manner, as for Q with no non-trivial finite quotients having separable cohomology in dimension 2 is equivalent to the vanishing of the second cohomology group $H^2(Q; M)$ for all finite modules M.

2.2 Group cohomology

Group cohomology (and homology) is a very important area of group theory in its own right. For us its importance lies in the way it is involved in classification of group extensions – the topic discussed first in Section 2.2.5 in the context of group extensions with abelian kernel, and then treated in fullness in Section 2.3.5.

This section introduces the main results of group cohomology which we use later. It is not intended as an introduction to group cohomology – for this the reader is referred to Brown's book [Bro12] or to the lecture notes of Löh [Löh19] and the book of Wilkes [Wil24].

2.2.1 Definitions and examples

Group cohomology (and homology) can be defined in many equivalent ways, which gives multiple approaches and interpretations. Here we choose to introduce it from the homological-algebra perspective.

Definition 2.23. Let Q be a group and M be a Q-module. The *n-th cohomology group of Q with coefficients in M* is defined as

$$H^n(Q; M) := \mathrm{Ext}^n(\mathbb{Z}, M),$$

i.e. it is the value of the n-th right derived functor $\mathrm{Ext}^n(\mathbb{Z}, -)$ of the left-exact hom-functor $\mathrm{Hom}_{\mathbb{Z}Q}(\mathbb{Z}, -)\colon \mathbb{Z}Q\text{-}\mathsf{Modules} \to \mathsf{AbelianGroups}$.

This definition comes with functoriality 'for free', but using it directly would involve having to find an *injective* resolution for M in the category of $\mathbb{Z}Q$-modules, which is far from easy. Fortunately, Ext^n can also be treated as a bi-functor

$$\mathrm{Ext}^n\colon (\mathbb{Z}Q\text{-}\mathsf{Modules})^{\mathrm{op}} \times \mathbb{Z}Q\text{-}\mathsf{Modules} \to \mathsf{AbelianGroups}$$

and then $\mathrm{Ext}^n(-, M)\colon (\mathbb{Z}Q\text{-}\mathsf{Modules})^{\mathrm{op}} \to \mathsf{AbelianGroups}$ is the n-th right derived functor for $\mathrm{Hom}_{\mathbb{Z}Q}(-, M)\colon (\mathbb{Z}Q\text{-}\mathsf{Modules})^{\mathrm{op}} \to \mathsf{AbelianGroups}$. This gives us a more practical way to compute the cohomology groups with projective resolutions of $\mathbb{Z}$ explained in Proposition 2.24.

For details the reader is refered to Theorem 2.7.6 of [Wei94].

Proposition 2.24. *Let Q be a group. Let $P_{n+1} \to P_n \to \ldots \to P_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ be a partial projective resolution of the trivial $\mathbb{Z}Q$-module $\mathbb{Z}$ by $\mathbb{Z}Q$-modules, i.e. a chain homotopy equivalence as follows.*

$$\begin{array}{ccccccccccccc}
P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & \ldots & \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & 0 & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \varepsilon} & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \ldots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \ldots
\end{array}$$

Then for any Q-module M the n-th cohomology group $H^n(Q; M)$ can be computed as follows.

$$\begin{aligned}
H^n(Q; M) &\cong H^n\left(\mathrm{Hom}_{\mathbb{Z}Q}(P_\bullet, M)\right) \\
&= \frac{\ker\left(\mathrm{Hom}_{\mathbb{Z}Q}(P_n, M) \xrightarrow{d_{n+1}^*} \mathrm{Hom}_{\mathbb{Z}Q}(P_{n+1}, M)\right)}{\mathrm{im}\left(\mathrm{Hom}_{\mathbb{Z}Q}(P_{n-1}, M) \xrightarrow{d_n^*} \mathrm{Hom}_{\mathbb{Z}Q}(P_n, M)\right)}
\end{aligned}$$

The general results of homological algebra ensure that this definition doesn't depend on the choice of projective resolution. Indeed, given two projective resolutions $P_\bullet$ and $P'_\bullet$ there exist chain homotopy equivalences $f_\bullet \colon P_\bullet \to P'_\bullet$ and $g_\bullet \colon P'_\bullet \to P_\bullet$ which induce maps $f^* \colon H^n(\mathrm{Hom}(P'_\bullet, M)) \to H^n(\mathrm{Hom}(P_\bullet, M))$ and $g^* \colon H^n(\mathrm{Hom}(P_\bullet, M)) \to H^n(\mathrm{Hom}(P'_\bullet, M))$ such that $f^* \circ g^*$ and $g^* \circ f^*$ are identity maps. For details see for example Lemma 00LT and the relevant section at Stacks Project [Sta24].

There are two kinds of resolutions we will see: ones coming from group presentations and the standard (bar) resolutions. Let's now introduce the latter following Section 1.2.1 of [Löh19].

Definition 2.25. Let Q be a group. The group algebra $\mathbb{Z}Q^k$ can be treated as a $\mathbb{Z}Q$-module through the diagonal action

$$q \cdot (q_1, \ldots, q_k) := (qq_1, \ldots, qq_k).$$

The $\mathbb{Z}Q$-module maps $d \colon \mathbb{Z}Q^k \to \mathbb{Z}Q^{k-1}$ sending $(q_1, \ldots, q_k)$ to

$$(q_2, q_3, \ldots, q_k) - (q_1, q_3, \ldots, q_k) + (q_1, q_2, \ldots, q_k) - \ldots + (-1)^{k-1}(q_1, q_2, \ldots, q_{k-1})$$

make $\mathbb{Z}Q^k \xrightarrow{d} \mathbb{Z}Q^{k-1} \xrightarrow{d} \dots \xrightarrow{d} \mathbb{Z}Q \xrightarrow{d} \underbrace{\mathbb{Z}Q^0}_{\cong\mathbb{Z}} \to 0$ into an acyclic chain complex called the *simplicial resolution* of $\mathbb{Z}$.

For computations it is convenient to identify $\mathbb{Z}Q^k$ with the free $\mathbb{Z}Q$-module $\mathbb{Z}Q\{Q^{k-1}\}$ where we denote basis elements by $[q_1 \mid q_2 \mid \dots \mid q_{k-1}]$. This identification is given by

$$(q_1, q_2, q_3, \dots, q_k) \mapsto q_1 \cdot [q_1^{-1}q_2 \mid q_2^{-1}q_3 \mid \dots \mid q_{k-1}^{-1}q_k]$$
$$(1, q_1, q_1q_2, \dots, q_1q_2 \dots q_{k-1}) \leftarrow\!\shortmid [q_1 \mid q_2 \mid \dots \mid q_{k-1}].$$

Definition 2.26. Let Q be a group. The *bar resolution* for $\mathbb{Z}$ is the free resolution

$$\mathbb{Z}Q\{Q^k\} \xrightarrow{d_k} \mathbb{Z}Q\{Q^{k-1}\} \xrightarrow{d_{k-1}} \dots \mathbb{Z}Q\{Q^1\} \xrightarrow{d_1} \mathbb{Z}Q\{Q^0\} \xrightarrow{d_0} \mathbb{Z} \to 0$$

with boundary map d_k sending the basis elements $[q_1 \mid q_2 \mid q_3 \mid \dots \mid q_k]$ to

$$q_1 \cdot [q_2 \mid q_3 \mid \dots \mid q_k] - [q_1q_2 \mid q_3 \mid \dots \mid q_k] + [q_1 \mid q_2q_3 \mid \dots \mid q_k] - \dots$$
$$+(-1)^{k-1}[q_1 \mid q_2 \mid \dots \mid q_{k-1}q_k] + (-1)^k[q_1 \mid q_2 \mid \dots \mid q_{k-1}].$$

As mentioned before, for calculations we will always use resolutions coming from group presentations.

Definition 2.27. Let Q be a group with presentation $\langle x_1, \dots, x_i \mid r_1, \dots, r_j \rangle$. The *presentation (partial) resolution* is any extension of the $\mathbb{Z}Q$-chain complex of the universal cover of the presentation 2-complex.

$$\mathbb{Z}Q\{\mathbf{R}_1, \dots, \mathbf{R}_j\} \xrightarrow{d_2} \mathbb{Z}Q\{\mathbf{x}_1, \dots, \mathbf{x}_i\} \xrightarrow{d_1} \mathbb{Z}Q \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0$$

To finish this section, let's compute two simple examples of cohomology groups.

Proposition 2.28. *Let $F = F(x_1, x_2, \dots, x_n)$ be the free group on n letters. Then the presentation resolution for the trivial $\mathbb{Z}F$-module $\mathbb{Z}$ is as follows.*

$$\dots \longrightarrow 0 \longrightarrow \mathbb{Z}F\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\} \xrightarrow{d_1} \mathbb{Z}F \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where $d_1(\mathbf{x}_i) = x_i - 1$.

In particular we obtain for any $\mathbb{Z}F$-module M

$$\begin{aligned} H^0(F;M) &= M^F, \\ H^1(F;M) &= M/(x_1-1)M \oplus \ldots \oplus M/(x_n-1)M, \\ H^k(F;M) &= 0 \text{ for } k \geq 2. \end{aligned}$$

Proposition 2.29. *Let Q be a finite cyclic group of order n with a generator x. Then the following is a free resolution of the trivial $\mathbb{Z}Q$-module $\mathbb{Z}$.*

$$\ldots \xrightarrow{\cdot\overline{x}} \mathbb{Z}Q \xrightarrow{\cdot(x-1)} \mathbb{Z}Q \xrightarrow{\cdot\overline{x}} \mathbb{Z}Q \xrightarrow{\cdot(x-1)} \mathbb{Z}Q \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where $\overline{x} = (1 + x + \ldots + x^{n-1})$.

Thus, for any $\mathbb{Z}Q$-module M and any $k \geq 1$ we get

$$\begin{aligned} H^0(Q;M) &= M^Q, \\ H^{2k-1}(Q;M) &= \{m \in M \mid \overline{x} \cdot m = 0\}/(x-1)M, \\ H^{2k}(Q;M) &= M^Q/\overline{x}M. \end{aligned}$$

The presentation resolutions will also be used in Section 5.2.6 to compute the second cohomology group of certain 2-orbifold groups.

2.2.2 Functoriality of group cohomology

This section explains in some detail the way in which the cohomology is functorial. While this is somewhat obvious in the case of functoriality induced by module maps, it is much less obvious and delicate in the case of functoriality induced by group maps, especially in the case of using a resolution different than the bar resolution.

This section is greatly influenced by Section II.6 of [Bro12] and by Section 1.1.3 of the lecture notes of Löh [Löh19].

We explore this functoriality in the context of $H^2(Q;M)$ classifying Q-by-M extensions in more depth in Section 2.2.5.

Proposition 2.30. *Given a group Q and Q-modules M and M', any map $\alpha\colon M \to M'$ of $\mathbb{Z}Q$-modules induces a* covariant *map on cohomology groups*

$$\alpha_*\colon H^\bullet(Q;M) \to H^\bullet(Q;M').$$

Proof. If $P_\bullet$ is a projective $\mathbb{Z}Q$-resolution of $\mathbb{Z}$, then $H^\bullet(Q;M)$ is the cohomology of the chain complex $\mathrm{Hom}_{\mathbb{Z}Q}(P_\bullet, M)$. The post-composition with α

$$(\alpha \circ -)\colon \mathrm{Hom}_{\mathbb{Z}Q}(P_\bullet, M) \to \mathrm{Hom}_{\mathbb{Z}Q}(P_\bullet, M')$$

is a chain map, so it induces a covariant map α_* on cohomology. □

Proposition 2.31. *Let $\gamma\colon Q' \to Q$ be a group homomorphism and M be a Q-module, treated as a Q'-module $\gamma^\# M$ through $q' \cdot m := \gamma(q') \cdot m$. Then γ induces a* contravariant *map on cohomology groups*

$$\gamma^*\colon H^\bullet(Q;M) \to H^\bullet(Q';\gamma^\# M). \tag{2.1}$$

Proof. Let $P'_\bullet$ and $P_\bullet$ be projective resolutions of $\mathbb{Z}$ by $\mathbb{Z}Q'$ and $\mathbb{Z}Q$ modules respectively. The map γ induces a map on rings $\gamma_\#\colon \mathbb{Z}Q' \to \mathbb{Z}Q$. While $P_\bullet$ is not necessarily a projective $\mathbb{Z}Q'$ resolution, it is an acyclic chain complex of $\mathbb{Z}Q'$-modules. Thus by the Fundamental Lemma of Homological Algebra (see Lemma I.7.4 of [Bro12]) there exists a (unique up to homotopy) chain map $\gamma_\#\colon P'_\bullet \to P_\bullet$ extending the identity map on $\mathbb{Z}$. This map induces a contravariant map $\gamma^*\colon H^\bullet(Q;M) \to H^\bullet(Q';\gamma^\# M)$ on cohomology groups. □

One of the great features of the bar resolution is that the extension $\gamma_\#\colon P'_\bullet \to P_\bullet$ is extremely easy to find: indeed, we may send $[q'_1 \mid q'_2 \mid \ldots \mid q'_k]$ to $[\gamma(q'_1) \mid \gamma(q'_2) \mid \ldots \mid \gamma(q'_k)]$. An example of how to do it in using a presentation resolution is given in Example 2.33.

It is possible to give a unified treatment of Propositions 2.30 and 2.31 – Löh discusses in Section 1.1.3 of [Löh19] the category $\mathsf{GroupMod}^*$ whose objects are pairs (Q, M) where Q is a group and M is a $\mathbb{Z}Q$-module. The morphisms $(Q, M) \xrightarrow{(\gamma,\alpha)} (Q', M')$ are the pairs (γ, α) such that $\gamma\colon Q \to Q'$ is a group homomorphism and $\alpha\colon \gamma^\# M' \to M$ is a map of $\mathbb{Z}Q$ modules. Cohomology is then a functor

$$H^k(-;-)\colon \left(\mathsf{GroupMod}^*\right)^{\mathrm{op}} \to \mathsf{AbelianGroups},$$

i.e. we get $(\gamma, \alpha)^*\colon H^k(Q';M') \to H^k(Q;M)$.

Example 2.32. Let N be a normal subgroup of a group G, $g \in G$ be any element and M be a $\mathbb{Z}G$-module treated as a $\mathbb{Z}N$ module. The inner automorphism $\mathrm{Ad}_g \colon N \to N$ defined by $\mathrm{Ad}_g(n) := gxq^{-1}$ induces a (new) $\mathbb{Z}N$-module structure $\mathrm{Ad}_g^{\#} M$ on M through $n * m := gng^{-1} \cdot m$. However, the map $\mu_{g^{-1}} \colon m \mapsto g^{-1} \cdot m$ is an isomorphism $\mu_{g^{-1}} \colon \mathrm{Ad}_g^{\#} M \to M$ as

$$\mu_{g^{-1}}(n * m) = g^{-1} \cdot gng^{-1} \cdot m = ng^{-1} \cdot m = n \cdot \mu_{g^{-1}}(m).$$

Thus we get a map $(\mathrm{Ad}_g, \mu_{g^{-1}}) \colon H^k(N; M) \to H^k(N; M)$. We can make it into a *left* action of G on $H^k(N; M)$ by setting

$$g \cdot \zeta := (\mathrm{Ad}_{g^{-1}}, \mu_g)^*(\zeta),$$

which at the level of cochains $C^k(N; M)$ of the bar resolution is defined by

$$(g \cdot \zeta)(n_1, \ldots, n_k) := g \cdot \zeta(g^{-1}n_1g,\ g^{-1}n_2g,\ \ldots,\ g^{-1}n_kg).$$

It is important (and interesting) that in fact this action of N is trivial, see e.g. Section III.8 of [Bro12].

Let's conduct a similar computation using a presentation resolution.

Example 2.33. Let $Q = \langle x \mid x^m = 1 \rangle \cong \mathbb{Z}/m$ and M be a trivial Q-module. Let $\mu \colon x \mapsto x^k$ be an endomorphism of Q. We will compute the effect of μ on the following free partial resolution of $\mathbb{Z}$

$$\mathbb{Z}Q\{\mathbf{Z}\} \xrightarrow{\mathbf{Z} \mapsto (x-1)\mathbf{R}} \mathbb{Z}Q\{\mathbf{R}\} \xrightarrow{\mathbf{R} \mapsto \overline{x} \cdot \mathbf{X}} \mathbb{Z}Q\{\mathbf{X}\} \xrightarrow{\mathbf{X} \mapsto (x-1)} \mathbb{Z}Q \xrightarrow{\quad\varepsilon\quad} \mathbb{Z}\ .$$

The fact that this is a free resolution helps us in that we only need to define the map on the basis elements.

First, we need to choose $(\mu_{\#})_0(1)$ – let's set it to $\mu_{\#}(1) = 1$. The base element $\mathbf{X}$ is then sent to $(\mu_{\#} \circ d_1)(\mathbf{X}) = \mu_{\#}(x - 1) = x^k - 1$, so $(\mu_{\#})_1$ has to send $\mathbf{X}$ to an element whose boundary is $x^k - 1$. We can take this to be $(1 + x + \ldots + x^{k-1}) \cdot \mathbf{X}$. Continuing this,

$$\begin{aligned}
((\mu_{\#})_1 \circ d_2)(\mathbf{R}) &= (\mu_{\#})_1(\overline{x} \cdot \mathbf{X}) = \mu_{\#}(\overline{x})(1 + \ldots + x^{k-1}) \cdot \mathbf{X} \\
&= (1 + x^k + x^{2k} + \ldots + x^{(m-1)k})(1 + \ldots + x^{k-1}) \cdot \mathbf{X} \\
&= (1 + x + \ldots + x^{k-1} + x^k + x^{k+1} + \ldots + x^{mk-1}) \cdot \mathbf{X} \\
&= (1 + x^m + x^{2m} + \ldots + x^{(k-1)m})(1 + x + x^2 + \ldots + x^{m-1}) \cdot \mathbf{X} \\
&= k\overline{x} \cdot \mathbf{X}
\end{aligned}$$

so we can set $(\mu_\#)_2(\mathbf{R}) = k\mathbf{R}$ and then $(\mu_\#)_3(\mathbf{Z}) = k(1 + \ldots + x^{k-1}) \cdot \mathbf{Z}$.

$$\begin{array}{ccccccccc}
\mathbb{Z}\,Q\{\mathbf{Z}\} & \xrightarrow{\mathbf{Z}\mapsto(x-1)\mathbf{R}} & \mathbb{Z}\,Q\{\mathbf{R}\} & \xrightarrow{\mathbf{R}\mapsto\overline{x}\cdot\mathbf{X}} & \mathbb{Z}\,Q\{\mathbf{X}\} & \xrightarrow{\mathbf{X}\mapsto(x-1)} & \mathbb{Z}\,Q & \xrightarrow{\varepsilon} & \mathbb{Z} \\
\downarrow{\scriptstyle \mathbf{Z}\mapsto k(1+\ldots+x^{k-1})\cdot\mathbf{Z}} & & \downarrow{\scriptstyle \mathbf{R}\mapsto k\mathbf{R}} & & \downarrow{\scriptstyle \mathbf{X}\mapsto(1+\ldots+x^{k-1})\cdot\mathbf{X}} & & \downarrow{\scriptstyle \mu_\#} & & \| \\
\mathbb{Z}\,Q\{\mathbf{Z}\} & \xrightarrow{\mathbf{Z}\mapsto(x-1)\mathbf{R}} & \mathbb{Z}\,Q\{\mathbf{R}\} & \xrightarrow{\mathbf{R}\mapsto\overline{x}\cdot\mathbf{X}} & \mathbb{Z}\,Q\{\mathbf{X}\} & \xrightarrow{\mathbf{X}\mapsto(x-1)} & \mathbb{Z}\,Q & \xrightarrow{\varepsilon} & \mathbb{Z}
\end{array}$$

As M is a trivial module, applying $\mathrm{Hom}_{\mathbb{Z}\,Q}(-, M)$ to this diagram factors through applying first $\mathbb{Z} \otimes_{\mathbb{Z}\,Q}(-)$ giving the following diagram. The different font for base elements stresses these are not $\mathbb{Z}\,Q$-modules.

$$\begin{array}{ccccccc}
\mathbb{Z}\{\mathbf{z}\} & \xrightarrow{0} & \mathbb{Z}\{\mathbf{r}\} & \xrightarrow{\mathbf{r}\mapsto m\mathbf{x}} & \mathbb{Z}\{\mathbf{x}\} & \xrightarrow{0} & \mathbb{Z} \\
\downarrow{\scriptstyle \mathbf{z}\mapsto k^2\mathbf{z}} & & \downarrow{\scriptstyle \mathbf{r}\mapsto k\mathbf{r}} & & \downarrow{\scriptstyle \mathbf{x}\mapsto k\mathbf{x}} & & \| \\
\mathbb{Z}\{\mathbf{z}\} & \xrightarrow{0} & \mathbb{Z}\{\mathbf{r}\} & \xrightarrow{\mathbf{r}\mapsto m\mathbf{x}} & \mathbb{Z}\{\mathbf{x}\} & \xrightarrow{0} & \mathbb{Z}
\end{array}.$$

This means that $(\mu^*)_i \colon H^i(Q; M) \to H^i(Q; M)$ is multiplication by $k^{\lceil \frac{i}{2} \rceil}$. Compare this to Example 6.7.10 of [Wei94] where this calculation was done for multiplication by $k = (-1)$.

We can do the computation of Example 2.33 for presentation (partial) resolutions in general. To keep things in order we need a few lemmas first.

Definition 2.34. Let G be a group and M a $\mathbb{Z}\,G$-module. A function $\nu \colon G \to M$ is called a *derivation* if for all $x, y \in G$

$$\nu(xy) = x \cdot \nu(y) + \nu(x).$$

Proposition 2.35. *Let $F = F(S)$ be a free group on a set S and M be a $\mathbb{Z}\,F$-module. Then any function $f \colon S \to M$ extends uniquely to a derivation $\tilde{f} \colon F \to M$.*

Proof. $\tilde{f}(1) = 0$ since

$$\tilde{f}(1) = \tilde{f}(1 \cdot 1) = 1 \cdot \tilde{f}(1) + \tilde{f}(1) = 2\tilde{f}(1),$$

and

$$0 = \tilde{f}(s^{-1} \cdot s) = s^{-1}\tilde{f}(s) + \tilde{f}(s^{-1}),$$

so we necessarily set $\tilde{f}(s^{-1}) := -s^{-1}f(s)$ and extend it to

$$\tilde{f}\big(a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \ldots a_{i_k}^{\epsilon_k}\big) = \tilde{f}(a_{i_1}^{\epsilon_1}) + a_{i_1}^{\epsilon_1} \cdot \tilde{f}(a_{i_2}^{\epsilon_2}) + \ldots + a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \ldots a_{i_{k-1}}^{\epsilon_{k-1}} \cdot \tilde{f}(a_{i_k}^{\epsilon_k}).$$

It is clear that the extension is unique and that it is a derivation. □

Proposition 2.36. *Let Q be a group with a generating set $\{a_i : i \in I\}$ and $F = F(a_i : i \in I)$ be the free group on this set. Let M be a $\mathbb{Z}Q$-module, treated as a $\mathbb{Z}F$-module through $\pi\colon F \twoheadrightarrow Q$. Let $\nu\colon F \to M$ be a derivation.*

1. *For any $r \in \ker \pi$ and any $x \in F$*

$$\nu(xrx^{-1}) = x \cdot \nu(r).$$

2. *For any $r_1, \ldots, r_k \in \ker \pi$ we have*

$$\nu(r_1 r_2 \ldots r_k) = \nu(r_1) + \nu(r_2) + \ldots + \nu(r_k).$$

Proof. For 1. we compute

$$\nu(x \cdot rx^{-1}) = xr \cdot \nu(x^{-1}) + \nu(xr) = x \cdot (-x^{-1}\nu(x)) + x \cdot \nu(r) + \nu(x) = x \cdot \nu(r).$$

For 2. we iteratively apply

$$\nu(r_1 \cdot r_2) = r_1 \cdot \nu(r_2) + \nu(r_1) = \nu(r_2) + \nu(r_1). \qquad \square$$

Lemma 2.37. *Let Q be a group with presentation*

$$\langle a_i : i \in I \mid r_j : j \in J \rangle.$$

Let F be the free group on the set $\{a_i : i \in I\}$ and $R = \langle\langle r_j : j \in J\rangle\rangle_F \triangleleft F$.

Let $C_2 \to C_1 \to C_0 \to \mathbb{Z}$ be the presentation (partial) resolution of $\mathbb{Z}$ with $C_2 = \bigoplus_{j\in J} \mathbb{Z}Q \cdot \mathbf{R}_j$, $\quad C_1 = \bigoplus_{i\in I} \mathbb{Z}Q \cdot \mathbf{A}_i$ and $C_0 = \mathbb{Z}Q$. Let $\nu\colon F \to C_1$ be the unique derivation such that $\nu(a_i) = \mathbf{A}_i$.

Finally, let $\gamma\colon Q \to Q$ be any endomorphism of Q. Choose endomorphisms $\alpha\colon R \to R$ and $\beta\colon F \to F$ so that the following diagram commutes.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R & \xrightarrow{\iota} & F & \xrightarrow{\pi} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 1 & \longrightarrow & R & \xrightarrow{\iota} & F & \xrightarrow{\pi} & Q & \longrightarrow & 1 \end{array}$$

Then we can build maps $\gamma_\#\colon C_\bullet \to C_\bullet$ extending $(\gamma_\#)_0 = \gamma_\#\colon \mathbb{Z}Q \to \mathbb{Z}Q$ as follows.

1. *For $i \in I$*

$$\begin{aligned}(\gamma_\#)_1(\mathbf{A}_i) &= \nu\big(\beta(a_i)\big)\\ &= \nu(a_{i_1}^{\epsilon_1}) + a_{i_1}^{\epsilon_1}\cdot\nu(a_{i_2}^{\epsilon_2}) + \ldots + a_{i_1}^{\epsilon_1}a_{i_2}^{\epsilon_2}\ldots a_{i_{k-1}}^{\epsilon_{k-1}}\cdot\nu(a_{i_k}^{\epsilon_k})\\ &= \overline{\mathbf{A}_{i_1}^{\epsilon_1}} + a_{i_1}^{\epsilon_1}\cdot\overline{\mathbf{A}_{i_2}^{\epsilon_2}} + \ldots + a_{i_1}^{\epsilon_1}a_{i_2}^{\epsilon_2}\ldots a_{i_{k-1}}^{\epsilon_{k-1}}\cdot\overline{\mathbf{A}_{i_k}^{\epsilon_k}}\end{aligned}$$

where $\beta(a_i) = a_{i_1}^{\epsilon_1}a_{i_2}^{\epsilon_2}\ldots a_{i_k}^{\epsilon_k}$ *and* $\overline{\mathbf{A}_i^{\epsilon}} := \begin{cases}\mathbf{A}_i & \text{for } \epsilon = 1,\\ -a_i^{-1}\mathbf{A}_i & \text{for } \epsilon = -1,\end{cases}$.

2. *For $j \in J$*

$$(\gamma_\#)_2(\mathbf{R}_j) = \epsilon_1 x_1\cdot\mathbf{R}_{j_1} + \epsilon_2 x_2\cdot\mathbf{R}_{j_2} + \ldots + \epsilon_l x_l\cdot\mathbf{R}_{j_l},$$

where $\alpha(r_j) = x_1 r_{j_1}^{\epsilon_1}x_1^{-1}\cdot x_2 r_{j_2}^{\epsilon_2}x_2^{-1}\cdot\ldots\cdot x_l r_{j_l}^{\epsilon_l}x_l^{-1}$.

Proof. There are two conditions we need to check: $d_1\circ(\gamma_\#)_1 = (\gamma_\#)_0\circ d_1$ and $d_2\circ(\gamma_\#)_2 = (\gamma_\#)_1\circ d_2$. From $d_1(\overline{\mathbf{A}_i^{\epsilon}}) = a_i^{\epsilon} - 1$ we get

$$\begin{aligned}d_1\big((\gamma_\#)_1(\mathbf{A}_i)\big) &= d_1\big(\overline{\mathbf{A}_{i_1}^{\epsilon_1}} + a_{i_1}^{\epsilon_1}\cdot\overline{\mathbf{A}_{i_2}^{\epsilon_2}} + \ldots + a_{i_1}^{\epsilon_1}a_{i_2}^{\epsilon_2}\ldots a_{i_{k-1}}^{\epsilon_{k-1}}\cdot\overline{\mathbf{A}_{i_k}^{\epsilon_k}}\big)\\ &= (a_{i_1}^{\epsilon_1}a_{i_2}^{\epsilon_2}\ldots a_{i_k}^{\epsilon_k} - 1) = \beta(a_i) - 1 = (\gamma_\#)_0\big(d_1(\mathbf{A}_i)\big).\end{aligned}$$

For the second condition note that the boundary map $d_2\colon C_2\to C_1$ sends the base element $\mathbf{R}_j$ to $\nu(r_j)\in C_1$. Applying Proposition 2.36 to the factorisation $\alpha(r_j) = x_1 r_{j_1}^{\epsilon_1}x_1^{-1}\cdot x_2 r_{j_2}^{\epsilon_2}x_2^{-1}\cdot\ldots\cdot x_l r_{j_l}^{\epsilon_l}x_l^{-1}$ gives

$$\begin{aligned}\nu(\alpha(r_j)) &= \nu\big(x_1 r_{j_1}^{\epsilon_1}x_1^{-1}\cdot x_2 r_{j_2}^{\epsilon_2}x_2^{-1}\cdot\ldots\cdot x_l r_{j_l}^{\epsilon_l}x_l^{-1}\big)\\ &= \epsilon_1 x_1\cdot\nu(r_{j_1}) + \epsilon_2 x_2\cdot\nu(r_{j_2}) + \ldots + \epsilon_l x_l\cdot\nu(r_{j_l})\\ &= d_2\big(\epsilon_1 x_1\cdot\mathbf{R}_{j_1} + \epsilon_2 x_2\cdot\mathbf{R}_{j_2} + \ldots + \epsilon_l x_l\cdot\mathbf{R}_{j_l}\big) = d_2\big((\gamma_\#)_2(\mathbf{R}_j)\big).\end{aligned}$$

To conclude this proof we will show that $\nu(\alpha(x)) = (\gamma_\#)_1\big(\nu(x)\big)$ for any $x\in F$, so in particular for r_j. The way we defined $(\gamma_\#)_1$ ensures this for $x = a_i$. For $x = a_i^{-1}$ we get

$$\begin{aligned}(\gamma_\#)_1(\nu(a_i^{-1})) &= (\gamma_\#)_1(-a_i^{-1}\cdot\mathbf{A}_i)\\ &= \gamma_\#(-a_i^{-1})\cdot\nu(\alpha(a_i))\\ &= -\alpha(a_i)^{-1}\cdot\nu(\alpha(a_i)) = \nu(\alpha(a_i^{-1})).\end{aligned}$$

Similar calculation holds for xy assuming it holds for $x, y \in F$

$$\begin{aligned}(\gamma_\#)_1(\nu(xy)) &= (\gamma_\#)_1(x \cdot \nu(y) + \nu(x)) \\ &= \gamma_\#(x) \cdot \nu(\alpha(y)) + \nu(\alpha(x)) \\ &= \alpha(x) \cdot \nu(\alpha(y)) + \nu(\alpha(x)) = \nu(\alpha(xy))\end{aligned}$$

so inductively $(\gamma_\#)_1(\nu(x)) = \nu(\alpha(x))$ for all $x \in F$. □

2.2.3 Properties and auxiliary results

In this section we give a number of results related to group cohomology – some of them quite foundational, some of them merely auxiliary. These results are used in the later sections in various proofs and computations, but due to their general nature we isolate them here.

Proposition 2.38 (Long Exact Sequence, Theorem 1.3.2 in [NSW13])**.** *Let Q be a group and*

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

be a short exact sequence of Q-modules. Then for $n = 1, 2, \ldots$ there exist maps $\delta\colon H^{n-1}(Q; M_3) \to H^n(Q; M_1)$ such that Diagram (2.2) is an exact sequence.

$$\begin{array}{l} 0 \longrightarrow H^0(Q;M_1) \xrightarrow{f_*} H^0(Q;M_2) \xrightarrow{g_*} H^0(Q;M_3) \\ \qquad \xrightarrow{\delta} H^1(Q;M_1) \xrightarrow{f_*} \ldots \xrightarrow{g_*} H^{n-1}(Q;M_3) \\ \qquad \xrightarrow{\delta} H^n(Q;M_1) \xrightarrow{f_*} H^n(Q;M_2) \xrightarrow{g_*} H^n(Q;M_3) \\ \qquad \xrightarrow{\delta} H^{n+1}(Q;M_1) \xrightarrow{f_*} \ldots \end{array} \tag{2.2}$$

Proposition 2.39 (Corollary 10.2 of [Bro12])**.** *Let Q be a finite group and M a Q-module. Then the map $\mu\colon m \mapsto |Q| \cdot m$ induces $\mu_*\colon H^k(Q; M) \to H^k(Q; M)$ equal to the zero map for $k \geq 1$. In particular, if M is such that the map $\mu\colon M \to M$ is invertible, then $H^k(Q; M) = 0$ for $k \geq 1$.*

The following results explain what the functoriality of group cohomology does to a short exact sequence $1 \to N \to G \to Q \to 1$ of groups. The picture here is much more complicated than in the case of a short exact sequence of modules giving the Long Exact Sequence of Proposition 2.38.

Theorem 2.40 (Inflation-Restriction Sequence, Propositions 1.6.6 and 1.6.7 in [NSW13])**.** *Given a short exact sequence of groups* $1 \to N \to G \to Q \to 1$ *and a* G*-module* M*, there is an exact sequence*

$$0 \longrightarrow H^1(Q; M^N) \longrightarrow H^1(G; M) \longrightarrow H^1(N; M)^Q \xrightarrow{\text{Tr}} H^2(Q; M^N) \longrightarrow H^2(G; M)$$

where the maps $H^k(Q; M^N) \to H^k(G; M)$ *are the* inflation *maps, the maps* $H^k(G; M) \to H^k(N; M)^Q$ *are the* restriction *maps and the map*

$$\text{Tr}\colon H^1(N; M)^Q \to H^2(Q; M^N)$$

is the transgression *map given by the following procedure.*

Given a 1*-cocycle* $\zeta\colon N \to M$ *whose cohomology class is* Q*-invariant, there is a* 1*-cochain* $\xi\colon G \to M$ *such that*

1. $\xi|_N = \zeta$,
2. $(d^*\xi)\colon G \times G \to M$ *is constant on the cosets of* $N \times N$,
3. $\text{im}(d^*\xi)$ *lies in* M^N.

Since $(d^*\xi)$ *is constant on the cosets of* $N \times N$*, we can consider it as a function* $Q \times Q \to M^N$ *and set*

$$\text{Tr}(\zeta) := ((d^*\xi)\colon Q \times Q \to M^N) \in H^2(Q; M^N).$$

There is also a higher version of this exact sequence which holds in the case when a number of initial cohomology groups $H^\bullet(N; M)$ vanish. It can be seen as coming from the Lyndon-Hochschild-Serre Spectral Sequence – see Theorem 2.4.3 in [NSW13].

Theorem 2.41 (Higher Inflation-Restriction Sequence, Proposition 1.6.7 in [NSW13]). *Given a short exact sequence of groups* $1 \to N \to G \to Q \to 1$ *and a* G*-module* M*, such that* $H^i(N; M) = 0$ *for* $i = 1, 2, \ldots, n-1$*, there is an exact sequence*

$$0 \longrightarrow H^n(Q; M^N) \longrightarrow H^n(G; M) \longrightarrow H^n(N; M)^Q \longrightarrow H^{n+1}(Q; M^N) \longrightarrow H^{n+1}(G; M)$$

2.2.4 Example: cohomology of Frobenius group Frob_{11}

In this section we actually compute some cohomology groups, specifically for the Frobenius group $\mathrm{Frob}_{11} = \mathbb{F}_{11} \rtimes \mathbb{F}_{11}^\times \cong (\mathbb{Z}/11) \rtimes (\mathbb{Z}/10)$, with three different quite natural choices of modules. One of them, M_2, is a permutation module coming from interpreting Frob_{11} as a group of affine bijections $\mathbb{F}_{11} \to \mathbb{F}_{11}$, where an element $(b, a) \in \mathbb{F}_{11} \rtimes \mathbb{F}_{11}^\times$ acts by $x \mapsto ax + b$.

We do so with the Higher Inflation-Restriction sequence.

Proposition 2.42. *Let* $G \cong \mathrm{Frob}_{11}$ *be the Frobenius group of size* 110*. Let* $M_1 \cong \mathbb{Z}$ *be the trivial* G*-module,* $M_2 \cong \mathbb{Z}^{11}$ *the permutation module given by the transitive permutation action of* Frob_{11} *on the standard basis for* $\mathbb{Z}^{11}$*. Let* $f\colon M_1 \to M_2$ *be the diagonal embedding, sending* 1 *to the vector* $(1, 1, \ldots, 1)$*. Finally, let* $M_3 = M_2/f(M_1)$*. Then the following hold.*

1. $H^1(G; M_3) \cong H^2(G; M_3) \cong 0$*.*
2. $H^2(G; M_2) \cong \mathbb{Z}/10$*.*
3. *The induced map* $f_*\colon H^2(G; M_1) \to H^2(G; M_2)$ *is an isomorphism.*

Proof. We start by noticing that G fits into a short exact sequence with cyclic kernel and quotient.

$$0 \longrightarrow \underbrace{\mathbb{Z}/11}_{N} \longrightarrow \underbrace{\mathrm{Frob}_{11}}_{G} \longrightarrow \underbrace{\mathbb{Z}/10}_{Q} \longrightarrow 1$$

Step 1. Computing $H^\bullet(N;M)$. Choosing a generator x for $N \cong \mathbb{Z}/11$, and setting $\overline{x} = (1 + x + x^2 + \ldots + x^{10})$ we computed in Proposition 2.29 that for any N-module M

$$H^1(\mathbb{Z}/11; M) \cong \{m \in M \mid \overline{x} \cdot m = 0\}/(x-1)M,$$
$$H^2(\mathbb{Z}/11; M) \cong M^{\langle x \rangle}/\overline{x}M.$$

Let's compute these groups for $M = M_1, M_2$ and M_3.

1. For $M = M_1 \cong \mathbb{Z}$ we get $H^1(\mathbb{Z}/11; M_1) \cong 0$ and $H^2(\mathbb{Z}/11; M_1) \cong M_1/11M_1 = \mathbb{Z}/11$.

2. For $M = M_2 \cong \mathbb{Z}^{11}$ with cyclic permutation of coordinates, we get that the element $\overline{x}$ sends a vector $v = (v_1, \ldots, v_{11})$ to

$$\overline{x}v = (v_1 + v_2 + \ldots + v_{11}) \cdot (1, 1, \ldots, 1).$$

   On the other hand, if $v_1 + \ldots + v_{11} = 0$ then

$$\begin{aligned}(v_1, \ldots, v_{11}) &= (v_1, v_1 + v_2, \ldots, v_1 + \ldots + v_{10}, v_1 + \ldots + v_{11}) \\ &\quad - (0, v_1, \ldots, v_1 + \ldots + v_9, v_1 + \ldots + v_{10}) \\ &= (x-1) \cdot (0, v_1, \ldots, v_1 + \ldots + v_9, v_1 + \ldots + v_{10}).\end{aligned}$$

   Thus $H^1(\mathbb{Z}/11; M_2) \cong 0$. Since $M_2^{\langle x \rangle} = \mathbb{Z}(1, 1, \ldots, 1) = \overline{x}M_2$ we also get $H^2(\mathbb{Z}/11; M_2) = 0$.

3. For $M = M_3 \cong M_2/\overline{x}M_2$ we get $\overline{x}M_3 = 0$ and $M_3^{\langle x \rangle} = 0$ so $H^1(\mathbb{Z}/11; M_3) = H^2(\mathbb{Z}/11; M_3) = 0$.

Step 2. $H^1(G; M_3) \cong H^2(G; M_3) \cong 0$. By the Inflation-Restriction Sequence (see Theorem 2.40) the diagram (2.3) is an exact sequence for any G-module M.

$$0 \longrightarrow H^1(Q; M^N) \longrightarrow H^1(G; M) \longrightarrow H^1(N; M)^Q \tag{2.3}$$

By the Higher Inflation-Restriction Sequence (Theorem 2.41) the diagram (2.4) is an exact sequence for any G-module M such that $H^1(N; M) = 0$ – in particular for $M = M_3$.

$$0 \longrightarrow H^2(Q; M^N) \longrightarrow H^2(G; M) \longrightarrow H^2(N; M)^Q \tag{2.4}$$

In this case we computed that $H^1(N; M_3) = H^2(N; M_3) = 0$, but also $M_3^N = 0$, so $H^1(Q; M_3^N) = H^2(Q; M_3^N) = 0$. By the exactness of diagrams (2.3) and (2.4), we get $H^1(G; M_3) = H^2(G; M_3) = 0$.

Step 3. Computing $H^2(G; M_2)$. Since $H^1(N; M_2) = 0$, the Higher Inflation-Restriction Sequence in diagram (2.4) is exact for $M = M_2$ too. Here also $H^2(N; M_2) = 0$. However, $M_2^N \cong \mathbb{Z}$ with trivial action of Q, so the sequence becomes

$$0 \longrightarrow \underbrace{H^2(Q; M_2^N)}_{\cong \mathbb{Z}/10} \longrightarrow H^2(G; M_2) \longrightarrow 0$$

showing that $H^2(G; M_2) \cong \mathbb{Z}/10$.

Step 4. The isomorphism $f_*\colon H^2(G; M_1) \to H^2(G; M_2)$. In this final step we apply the Long Exact Sequence (see Proposition 2.38) to the exact sequence of diagrams $0 \to M_1 \xrightarrow{f} M_2 \to M_3 \to 0$ getting an exact sequence in Diagram (2.5).

$$\begin{array}{rl} & \ldots \longrightarrow H^1(G; M_3) \longrightarrow \\ \longrightarrow H^2(G; M_1) \xrightarrow{f_*} & H^2(G; M_2) \longrightarrow H^2(G; M_3) \end{array} \tag{2.5}$$

From $H^1(G; M_3) = H^2(G; M_3) = 0$ we get that f_* is an isomorphism. □

2.2.5 $H^2(Q; M)$ classifies extensions

The most important result for us is how the second cohomology group classifies the equivalence classes of group extensions with given kernel and quotient. The reader is referred to Section IV.3 of [Bro12] for a detailed treatment.

Proposition 2.43. *Let Q be a group and let M be a Q-module, i.e. an abelian group with an action $\varphi\colon Q \to \mathrm{Aut}(M)$. Then there is a natural correspondence between the set $\mathscr{E}_\varphi(Q; M)$ of equivalence classes of extensions $\mathcal{E} = \big(0 \to M \to G \to Q \to 1\big)$ inducing the action φ and the group $H^2(Q; M)$.*

Sketch proof. Given a group extension $\mathcal{E} = \big(0 \to M \to G \xrightarrow{\pi} Q \to 1\big)$ we can choose a section $s\colon Q \to G$ to $\pi\colon G \to Q$ and measure how much s 'fails to be a homomorphism' by the function $\zeta_s\colon Q \times Q \to M$ defined as

$$\zeta_s(q_1, q_2) := s(q_1)s(q_2)s(q_1q_2)^{-1}.$$

The associativity of the operation in G has a consequence that

$$(d_3^*\zeta_s)(q_0, q_1, q_2) := q_0 \cdot \zeta_s(q_1, q_2) - \zeta_s(q_0q_1, q_2) + \zeta_s(q_0, q_1q_2) - \zeta_s(q_0, q_1)$$

is zero for any q_0, q_1 and q_2 in Q. Thus, ζ_s is a 2-cocycle in the sense of the bar resolution (see Definition 2.26). Choosing a different section $s'\colon Q \to G$ does change this cocycle by a coboundary.

Starting with a cohomology class representative $\zeta\colon Q \times Q \to M$, we can use it to define an operation on the *set* $M \times Q$ as follows.

$$(m_1, q_1) \cdot (m_2, q_2) := (m_1 + q_1 \cdot m_2 + \zeta(q_1, q_2),\ q_1q_2).$$

What needs to be checked is that this defines a group operation.

2.2.5.1 Functoriality of the classification

The functoriality of the classification of Proposition 2.43 will be used multiple times in this thesis, which is why we make an effort to explain it in detail, putting emphasis on the group-theoretical constructions corresponding to the various maps induced on the cohomology groups $H^2(-;-)$.

A similar account can be found in Section I.1 of [BT06].

We start with an explanation of how a map $\alpha\colon M \to M'$ of Q-modules induces a covariant map $\alpha_*\colon H^2(Q; M) \to H^2(Q; M')$ on the cohomology groups and on the corresponding extensions.

Proposition 2.44 (Exercise 1.b in Section IV.3 of [Bro12])**.** *Let Q be a group, let M and M' be Q-modules and $\alpha\colon M \to M'$ be a map of Q-modules. Then there is an induced map on cohomology groups*

$$\alpha_*\colon H^2(Q; M) \to H^2(Q; M')$$

sending a cohomology class $\zeta\colon Q \times Q \to M$ *representing the extension* $\mathcal{E}_\zeta = \big(0 \to M \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1\big)$ *to the cohomology class* $\alpha_*(\zeta)\colon Q \times Q \to M'$ *given by*

$$\alpha_*(\zeta)\colon (q_1, q_2) \mapsto \alpha\big(\zeta(q_1, q_2)\big)$$

representing the pushout extension *class*

$$\mathcal{E}_{\alpha_*(\zeta)} = \big(0 \to M' \xrightarrow{\iota'} G' \xrightarrow{\pi'} Q \to 1\big)$$

where

$$G' = M' \rtimes G / \{(\alpha(m), \iota(-m)) \mid m \in M\}.$$

Together with map $\beta\colon G \to G'$ *given by* $\beta(g) = \overline{(0, g)}$ *they form the commutative diagram (2.6).*

Moreover, for any extension $\mathcal{E}'' = \big(0 \to M' \xrightarrow{\iota''} G'' \xrightarrow{\pi''} Q \to 1\big)$ *with the same induced action of* Q *on* M' *and a map* $\beta''\colon G \to G''$ *there exists a unique map* $G' \to G''$ *making Diagram (2.6) commute. Thus the extension* $\mathcal{E}_{\alpha_*(\zeta)}$ *is unique up to equivalence.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\iota} & G & \xrightarrow{\pi} & Q & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \| & & \\
0 & \longrightarrow & M' & \xrightarrow{\iota'} & G' & \xrightarrow{\pi'} & Q & \longrightarrow & 1 \\
 & & \| & \swarrow{\scriptstyle \beta''} & \downarrow & & \| & & \\
0 & \longrightarrow & M' & \xrightarrow{\iota''} & G'' & \xrightarrow{\pi''} & Q & \longrightarrow & 1
\end{array} \tag{2.6}$$

Next we explain how a group homomorphism $\gamma\colon Q' \to Q$ induces a *contravariant* map $\gamma^*\colon H^2(Q; M) \to H^2(Q'; M)$ on the cohomology groups, where M is treated as a Q'-module. Again, this gives a map on the corresponding extensions.

Proposition 2.45 (Exercise 1.a in Section IV.3 of [Bro12])**.** *Let* Q *and* Q' *be groups,* $\gamma\colon Q' \to Q$ *be a homomorphism and* M *be a* Q*-module, treated as a* Q'*-module through* $q' \cdot m := \gamma(q') \cdot m$*. Then there is an induced map on cohomology groups*

$$\gamma^*\colon H^2(Q; M) \to H^2(Q'; M)$$

sending a cohomology class $\zeta\colon Q \times Q \to M$ *representing the extension class* $\mathcal{E}_\zeta = \big(0 \to M \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1\big)$ *to the cohomology class* $\gamma^*(\zeta)\colon Q' \times Q' \to M$ *given by*

$$\gamma^*(\zeta)(q_1, q_2) = \zeta\big(\gamma(q_1), \gamma(q_2)\big)$$

representing the pullback extension *class*

$$\mathcal{E}_{\gamma^*(\zeta)} = \big(0 \to M \xrightarrow{\iota'} G' \xrightarrow{\pi'} Q' \to 1\big)$$

where the group G' *with maps* $\pi'\colon G' \to Q'$ *and* $\beta\colon G' \to G$ *is the pullback of the diagram* $\begin{array}{ccc} & & Q' \\ & & \downarrow\gamma \\ G & \xrightarrow{\pi} & Q \end{array}$*. Together they form the commutative diagram (2.7).*

Moreover, for any extension $\mathcal{E}'' = \big(0 \to M \xrightarrow{\iota''} G'' \xrightarrow{\pi''} Q' \to 1\big)$ *with an appropriate map* $\beta''\colon G'' \to G$*, there exists a unique map* $G'' \to G'$ *making Diagram (2.7) commute. Thus the extension* $\mathcal{E}_{\gamma^*(\zeta)}$ *is unique up to equivalence.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\iota''} & G'' & \xrightarrow{\pi''} & Q' & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \xrightarrow{\iota'} & G' & \xrightarrow{\pi'} & Q' & \longrightarrow & 1 \\
 & & \| & \beta'' \swarrow & \downarrow\beta & & \downarrow\gamma & & \\
0 & \longrightarrow & M & \xrightarrow{\iota} & G & \xrightarrow{\pi} & Q & \longrightarrow & 1
\end{array} \tag{2.7}$$

Remark 2.46. The reader is warned that the terms 'pushout extension' and 'pullback extension' aren't standard and that the group G' in the pushout extension is generally not isomorphic to the pushout of the diagram $\begin{array}{ccc} M & \xrightarrow{\iota} & G \\ \downarrow\alpha & & \\ M' & & \end{array}$ in the category of groups.

2.2.5.2 Functoriality and extension similarity

It is convenient to know how the functoriality behaves when we allow *both* a map on modules and a map on groups. This result is adapted from Theorem 1.5.13 of [Löh19], but can also be found as Theorem 1.10 of [BT06].

Proposition 2.47. *Let M_1 be a Q_1-module, let M_2 be a Q_2-module, and $\gamma\colon Q_1 \to Q_2$ and $\alpha\colon M_1 \to M_2$ be homomorphisms such that $\alpha : M_1 \to \gamma^{\#}M_2$ is a map of Q_1-modules, i.e. for any $q \in Q_1$ and $m \in M_1$*

$$\alpha(q \cdot m) = \gamma(q) \cdot \alpha(m).$$

Given cohomology classes $\zeta_i \in H^2(Q_i; M_i)$ for $i = 1, 2$ and the extension equivalence classes

$$\mathcal{E}_{\zeta_i} = \left(0 \to M_i \xrightarrow{\iota_i} G_i \xrightarrow{\pi_i} Q_i \to 1\right)$$

which they correspond to, the following are equivalent.

1. *$\alpha_*(\zeta_1) = \gamma^*(\zeta_2)$ as elements of $H^2(Q_1; \gamma^{\#}M_2)$.*
2. *There is a homomorphism $\beta\colon G_1 \to G_2$ making Diagram (2.8) commute.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{\iota_1} & G_1 & \xrightarrow{\pi_1} & Q_1 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & M_2 & \xrightarrow{\iota_2} & G_2 & \xrightarrow{\pi_2} & Q_2 & \longrightarrow & 1 \end{array} \tag{2.8}$$

This naturality has an important consequence for us.

Corollary 2.48. *Let Q be a group and M be a Q-module with the induced action $\varphi\colon Q \to \mathrm{Aut}(M)$. The stabilizer*

$$\mathrm{Stab}_{\mathrm{Aut}(M)\times\mathrm{Aut}(Q)}(\varphi) = \{(\alpha, \gamma) \in \mathrm{Aut}(Q) \times \mathrm{Aut}(M) \mid \mathrm{Ad}_\alpha \circ \varphi = \varphi \circ \gamma\}$$

acts on the cohomology group $H^2(Q; M)$ on the left through the action defined on standard 2-cocycles $Q \times Q \to M$ by

$$((\alpha, \gamma) \cdot \zeta)(q_1, q_2) := \alpha\left(\zeta\left(\gamma^{-1}(q_1), \gamma^{-1}(q_2)\right)\right).$$

Furthermore, the action of the following image of Q is trivial.

$$\underbrace{\{(\varphi(q), \mathrm{Ad}_q) \mid q \in Q\}}_{\cong \mathrm{Inn}(Q)} < \mathrm{Stab}_{\mathrm{Aut}(M)\times\mathrm{Aut}(Q)}(\varphi)$$

The orbits of this action $\mathrm{Stab}_{\mathrm{Aut}(M)\times\mathrm{Aut}(Q)}(\varphi) \curvearrowright H^2(Q; M)$ are in bijection with the similarity classes of extensions inducing φ, i.e. the orbits of $\mathrm{Stab}_{\mathrm{Aut}(M)\times\mathrm{Aut}(Q)}(\varphi) \curvearrowright \mathscr{E}_\varphi(Q; M)$.

Remark 2.49. The reader is warned against the temptation of trying to define an action of the whole $\mathrm{Aut}(M) \times \mathrm{Aut}(Q)$ on $H^2(Q;M)$. Following Proposition 2.47, for any $\zeta \in H^2(Q;M)$ representing $\mathcal{E} = \big(0 \to M \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1\big)$ and elements $(\alpha, \gamma) \in \mathrm{Aut}(M) \times \mathrm{Aut}(Q)$ we do get a cohomology class $(\alpha_* \circ (\gamma^*)^{-1})(\zeta) \in H^2(Q;M)$ representing

$$\mathcal{E}' = \big(\ 0 \longrightarrow M \xrightarrow{\iota\circ\alpha^{-1}} G \xrightarrow{\gamma\circ\pi} Q \longrightarrow 1\ \big).$$

However, the induced action $\varphi_{\mathcal{E}'} = \mathrm{Ad}_{\alpha^{-1}} \circ \varphi_{\mathcal{E}} \circ \gamma$, is the same map as $\varphi_{\mathcal{E}}$ precisely when (α, γ) is in the stabilizer $\mathrm{Stab}_{\mathrm{Aut}(M)\times\mathrm{Aut}(Q)}(\varphi_{\mathcal{E}})$.

In short, the notation $H^2(Q;M)$ can cause confusion if we forget that the M has a fixed Q-module structure.

There is a complication for using in practice the action of the stabilizer $\mathrm{Stab}(\varphi)$ on $H^2(Q;M)$, as it is defined in terms of the standard bar resolution, while most calculation are carried with the presentation resolution. An example of computing this action with presentation resolution was done in Example 2.33 while a general method was given in Lemma 2.37.

2.2.5.3 Functoriality and Baer sum

A remaining piece of understanding the functoriality of the classification $\mathscr{E}_\varphi(Q;M) \leftrightarrow H^2(Q;M)$ is the addition operation in the cohomology group – it should correspond to a commutative operation on the set $\mathscr{E}_\varphi(Q;M)$.

Proposition 2.50. *Given a pair of extensions*

$$\mathcal{E}_1 = \big(0 \to M_1 \xrightarrow{\iota_1} G_1 \xrightarrow{\pi_1} Q \to 1\big) \in \mathscr{E}_{\varphi_1}(Q;M_1)$$
$$\mathcal{E}_2 = \big(0 \to M_2 \xrightarrow{\iota_2} G_2 \xrightarrow{\pi_2} Q \to 1\big) \in \mathscr{E}_{\varphi_2}(Q;M_2)$$

the pullback $G_1 \times_Q G_2$ *of the diagram* $\begin{array}{ccc} & & G_1 \\ & & \downarrow{\scriptstyle \pi_1} \\ G_2 & \xrightarrow{\pi_2} & Q \end{array}$ *fits into the extension*

$$\mathcal{E}_1 \tilde{\oplus} \mathcal{E}_2 = \big(0 \to M_1 \times M_2 \xrightarrow{(\iota_1,\iota_2)} G_1 \times_Q G_2 \xrightarrow{\pi} Q \to 1\big) \in \mathscr{E}_{\varphi_1\times\varphi_2}(Q; M_1 \times M_2).$$

For $M_1 = M_2 = M$ *we can compose the map thus defined*

$$\mathscr{E}_\varphi(Q;M) \times \mathscr{E}_\varphi(Q;M) \to \mathscr{E}_{\varphi\times\varphi}(Q; M \times M)$$

with the map $\mu_*\colon \mathscr{E}_{\varphi\times\varphi}(Q;M\times M) \to \mathscr{E}_\varphi(Q;M)$ *induced by the* Q*-module map* $\mu\colon M\times M \to M$ *defined by* $\mu(m_1,m_2) = m_1+m_2$*. Thus we get an operation*

$$\oplus\colon \mathscr{E}_\varphi(Q;M)\times\mathscr{E}_\varphi(Q;M)\to\mathscr{E}_\varphi(Q;M)$$

which is a commutative group operation with the identity element

$$\mathcal{E}_0 = \big(0\to M\to M\rtimes Q\to Q\to 1\big)$$

and where the inverse of the extension $\mathcal{E} = \big(0\to M\xrightarrow{\iota} G\to Q\xrightarrow{\pi} 1\big)$ *is given by* $\mathcal{E}^- = \big(0\to M\xrightarrow{-\iota} G\to Q\xrightarrow{\pi} 1\big)$.

Proof. For associativity it is enough to note that for any $\mathcal{E}_1\in\mathscr{E}_{\varphi_1}(Q;M_1)$, $\mathcal{E}_2\in\mathscr{E}_{\varphi_2}(Q;M_2)$ and $\mathcal{E}_3\in\mathscr{E}_{\varphi_3}(Q;M_3)$ we get the equivalence of extensions

$$(\mathcal{E}_1\tilde{\oplus}\mathcal{E}_2)\tilde{\oplus}\mathcal{E}_3 = \mathcal{E}_1\tilde{\oplus}(\mathcal{E}_2\tilde{\oplus}\mathcal{E}_3)$$

which is even stronger than the associativity of $\oplus$.

For commutativity notice that for any $\mathcal{E}_1$ and $\mathcal{E}_2\in\mathscr{E}_\varphi(Q;M)$ the isomorphism $\beta\colon G_1\times_Q G_2\to G_2\times_Q G_1$ shows equivalence of $\mathcal{E}_1\tilde{\oplus}\mathcal{E}_2$ and $\mathcal{E}_2\tilde{\oplus}\mathcal{E}_1$, which is again stronger than the associativity of $\oplus$.

For the identity element note that $(M\rtimes Q)\times_Q G\cong M\rtimes G$, so

$$\mathcal{E}_0\tilde{\oplus}\mathcal{E} = \big(0\to M\times M\to M\rtimes G\to Q\to 1\big).$$

Now, the assignment $(m_1,m_2,g)\mapsto \iota(m_1+m_2)\cdot g$ defines a homomorphism from $(M\times M)\rtimes G\cong M\rtimes(M\rtimes G)$ to G whose kernel is precisely

$$\{(m_1+m_2,-m_1,\iota(-m_2))\mid m_1,m_2\in M\} < M\rtimes(M\rtimes G),$$

and thus $\mathcal{E}_0\oplus\mathcal{E} = \mathcal{E}$.

Finally, for the inverses, let's first write down the extension $\mathcal{E}\tilde{\oplus}\mathcal{E}^-$.

$$\mathcal{E}\tilde{\oplus}\mathcal{E}^- = \big(0\to M\times M\xrightarrow{(\iota,-\iota)} G\times_Q G\to Q\to 1\big)$$

Let's define a function $\tilde{\beta}\colon M\rtimes(G\times_Q G)\to M\rtimes Q$ by

$$\tilde{\beta}(m,g,h) = (m+gh^{-1},\overline{g}).$$

We will show that it is a homomorphism whose kernel is

$$\{\big(m_1+m_2,\iota(-m_1),\iota(m_2)\big)\mid m_1,m_2\in M\}<M\rtimes(G\times_Q G)$$

thereby showing that $\mathcal{E}\oplus\mathcal{E}^-=\mathcal{E}_0$. First note that

$$\begin{aligned}\tilde{\beta}\big((m_1,g_1,h_1)(m_2,g_2,h_2)\big)&=\tilde{\beta}\big((m_1+g_1\cdot m_2,\ g_1g_2,\ h_1h_2)\big)\\&=(m_1+g_1\cdot m_2+g_1g_2h_2^{-1}h_1^{-1},\ \overline{g_1g_2}),\end{aligned}$$

while on the other hand

$$\begin{aligned}\tilde{\beta}\big((m_1,g_1,h_1)\big)\tilde{\beta}\big((m_2,g_2,h_2)\big)&=(m_1+g_1h_1^{-1},\overline{g_1})(m_2+g_2h_2^{-1},\overline{g_2})\\&=(m_1+g_1h_1^{-1}+g_1\cdot m_2+g_1\cdot g_2^{-1}h_2,\overline{g_1g_2}),\end{aligned}$$

so we need to show that $g_1h_1^{-1}+g_1\cdot g_2^{-1}h_2=g_1g_2h_2^{-1}h_1^{-1}$ as elements of M. This is true because since $(g_1,h_1)\in G\times_Q G$ then also $g_1\cdot m=h_1\cdot m$ for any $m\in M$, so we have

$$\begin{aligned}g_1g_2h_2^{-1}h_1^{-1}&=g_1h_1^{-1}h_1g_2h_2^{-1}h_1^{-1}\\&=g_1h_1^{-1}+h_1\cdot g_2h_2^{-1}\\&=g_1h_1^{-1}+g_1\cdot g_2h_2^{-1}.\end{aligned}$$

□

Let's see that this operation is indeed what we were looking for.

Proposition 2.51. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be extensions corresponding to cohomology classes ζ_1 and $\zeta_2\in H^2(Q;M)$ respectively. Then $\mathcal{E}_1\oplus\mathcal{E}_2$ corresponds to the cohomology class $\zeta_1+\zeta_2$.*

Proof. The cohomology class of $\mathcal{E}_1\tilde{\oplus}\mathcal{E}_2$ is

$$Q\times Q\ni(q_1,q_2)\mapsto\big(\zeta_1(q_1,q_2),\zeta_2(q_1,q_2)\big)\in M\times M,$$

so after composing this with $\mu(m_1,m_2)=m_1+m_2$ we get that the cohomology class of $\mathcal{E}_1\oplus\mathcal{E}_2$ is precisely $\zeta_1+\zeta_2$. □

2.2.6 Cohomology of profinite groups

This section is based on Sections 2.6 and 2.7 of [NSW13], where the reader will find a detailed treatment.

For profinite groups the usual group cohomology doesn't capture the topological data, so we need a version of it which does. A difficulty arises, however, from two possible 'good' choices for how to define it.

First, let us define the categories of modules of interest to us and the relation between them.

Definition 2.52. Let Γ be a topological group. A Γ-*module* is an abelian topological group M with a continuous action $\Gamma \times M \to M$. We call M *discrete* or *profinite* if it is, respectively, discrete or profinite as a topological space.

Definition 2.53. Let Γ be a profinite group. The *completed group algebra* $\widehat{\mathbb{Z}}[[\Gamma]]$ for Γ is the profinite ring obtained by taking the inverse limit over natural numbers N and open normal subgroups $\Delta \triangleleft_o \Gamma$ as follows.

$$\widehat{\mathbb{Z}}[[\Gamma]] := \varprojlim_{N \in \mathbb{N},\ \Delta \triangleleft_o \Gamma} (\mathbb{Z}/N)[\Gamma/\Delta].$$

Proposition 2.54 (Proposition 5.3.6 of [RZ00])**.** *Let Γ be a profinite group.*

1. *Every $\widehat{\mathbb{Z}}[[\Gamma]]$-module is naturally a Γ-module.*
2. *The category of profinite $\widehat{\mathbb{Z}}[[\Gamma]]$-modules coincides with the category of profinite Γ-modules.*
3. *The category of discrete $\widehat{\mathbb{Z}}[[\Gamma]]$-modules coincides with the category of discrete torsion Γ-modules.*

The first way to define cohomology of profinite groups is to follow homological algebra and define the *profinite cohomology* $H^n_{\text{prof}}(\Gamma; M)$ of a profinite group Γ with coefficients in a *discrete* Γ-module M by setting

$$H^n_{\text{prof}}(\Gamma; M) := \text{Ext}^n(\widehat{\mathbb{Z}}, M)$$

using the Ext bifunctor

$$\text{Ext}^n : (\mathsf{profinite\ }\Gamma\text{-}\mathsf{\ Modules})^{\text{op}} \times (\mathsf{discrete\ }\Gamma\text{-}\mathsf{\ Modules}) \to \mathsf{AbelianGroups}\,.$$

This approach benefits from the fact that we can use any projective resolution to compute the cohomology groups—indeed we will show how to do so in Lemma 2.61.

The downside, however, is that we can't use this definition for *profinite* modules M, as that category doesn't have enough injective objects, and the derived functors don't exist. What one can do instead for a profinite Γ-module $M = \varprojlim_{i \in I} M_i$ is to consider the limit

$$\varprojlim_{i \in I} H^n_{\text{prof}}(\Gamma; M_i).$$

The other possibility is to use an analogue of the bar resolution to define the *continuous cochain cohomology* $H^n_{\text{cts}}(\Gamma; M)$. Specifically, for a topological group Γ and a topological Γ-module M, let $C^n_{\text{cts}}(\Gamma, M)$ denote the group of all continuous functions $f\colon \Gamma^n \to M$ with the boundary operator $d^*_{n+1}\colon C^n_{\text{cts}}(\Gamma, M) \to C^{n+1}_{\text{cts}}(\Gamma, M)$ defined in the standard way by

$$\begin{aligned}(d^*_{n+1}f)(\gamma_0, \gamma_1, \ldots, \gamma_n) := {} & \gamma_0 \cdot f(\gamma_1, \gamma_2, \ldots, \gamma_n) - f(\gamma_0\gamma_1, \gamma_2, \ldots, \gamma_n) \\ & + f(\gamma_0, \gamma_1\gamma_2, \ldots, \gamma_n) - \ldots + (-1)^n f(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\gamma_n) \\ & + (-1)^{n+1} f(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}).\end{aligned}$$

Setting $Z^n_{\text{cts}}(\Gamma, M) := \ker d^*_{n+1}$ and $B^n_{\text{cts}}(\Gamma, M) := \operatorname{im} d^*_n$ we define

$$H^n_{\text{cts}}(\Gamma; M) := \frac{Z^n_{\text{cts}}(\Gamma, M)}{B^n_{\text{cts}}(\Gamma, M)}.$$

The benefit of this paradigm is that the definition works for *any* topological group Γ and *any* topological Γ-module. Also the second cohomology group $H^2_{\text{cts}}(\Gamma; M)$ classifies the equivalence classes of *continuous* extensions of M by Γ admitting a continuous section.

Theorem 2.55 (Theorem 5.3 of [Hu52])**.** *Let Γ be a topological group and M a topological Γ-module. Then the group $H^2_{\text{cts}}(\Gamma; M)$ is in bijection with the equivalence classes of topological group extensions of M by Γ which admit a continuous section.*

Another advantage is that for profinite groups Γ and discrete modules M it has a clear relation to the standard group cohomology

$$H^n_{\text{cts}}(\Gamma; M) = \varinjlim_{\Delta \lhd_{\text{fi}} \Gamma} H^n(\Gamma/\Delta; M)$$

and—through a profinite version of bar resolution—to the first notion of cohomology of profinite groups we introduced.

Proposition 2.56 (Theorem 6.2.4 of [RZ00]). *Let Γ be a profinite group and M be a discrete $\widehat{\mathbb{Z}}[[\Gamma]]$-module. Then there is a canonical isomorphism*

$$H^n_{\text{prof}}(\Gamma; M) \cong H^n_{\text{cts}}(\Gamma; M).$$

The downside is that it doesn't 'automatically' come with many of the properties we normally expect from cohomology – [Min13] gives an example of a short exact sequence of trivial $(\mathbb{R} / \mathbb{Z})$-modules $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R} / \mathbb{Z} \to 0$ which doesn't give a long exact sequence on continuous cohomology groups; for that the existence for continuous sections of quotients is crucial. These always exist for epimorphisms between profinite groups – see Proposition 2.2.2 of [RZ00]. Also, similarly to the case of bar resolution, it isn't clear how to compute $H^n_{\text{cts}}(\Gamma; M)$ in practice.

Fortunately, in the cases of our interest the two notions of cohomology discussed are naturally isomorphic even for (certain) profinite modules M; this is explained in Theorem 2.57.

Theorem 2.57 (Corollary 2.7.6 in [NSW13]). *Let Γ be a profinite group and M be a compact Γ-module such that M is a countable inverse limit*

$$M = \varprojlim_{i \in \mathbb{N}} M_i$$

of finite discrete Γ-modules M_i with $\psi_i \colon M \to M_i$ being the natural maps.

If $H^k_{\text{prof}}(\Gamma, M_i)$ is finite for all i, then the map

$$\psi \colon H^{k+1}_{\text{cts}}(\Gamma; M) \cong \varprojlim_{i \in \mathbb{N}} H^{k+1}_{\text{prof}}(\Gamma; M_i)$$

coming from the limit of maps ψ_i is an isomorphism.

This always applies to $H^1_{\text{cts}}(\Gamma; M)$, and—if the group Γ is topologically finitely generated—also to $H^2_{\text{cts}}(\Gamma; M)$.

Another topic important for us is how the (standard) cohomology $H^\bullet(G; -)$ of G relates to the cohomology $H^\bullet_{\text{cts}}(\widehat{G}; -)$ of its profinite completion $\widehat{G}$. This motivates the following important definition.

Definition 2.58. A group G has *separable cohomology* (also: is *good* in the sense of Serre) in dimension k if the natural restriction map

$$\mathsf{h}^*\colon H^k_{\mathrm{cts}}(\widehat{G};M) \to H^k(G;M)$$

is an isomorphism for any finite module M.

For groups with separable cohomology we can define an equivariant map $H^2(Q;M) \to H^2_{\mathrm{cts}}(\widehat{Q};\widehat{M})$ whose orbits capture the phenomenon of *collapsing cohomology orbits* described in Section 1.3.3.

Proposition 2.59. *Let Q be a finitely generated group with separable cohomology in dimension 2. Let M be a Q-module, finitely generated as an abelian group, which the action $\varphi\colon Q \to \mathrm{Out}(M)$. Then there is a morphism of group actions*

$$\mathsf{Stab}_{\mathrm{Out}(M)\times\mathrm{Aut}(Q)}(\varphi) \curvearrowright H^2(Q;M) \xrightarrow{\Theta} \mathsf{Stab}_{\mathrm{Out}(\widehat{M})\times\mathrm{Aut}(\widehat{Q})}(\varphi) \curvearrowright H^2_{\mathrm{cts}}(\widehat{Q};\widehat{M})$$

which sends a cohomology class ζ representing $\left(0 \to M \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1\right)$ to the cohomology class in $H^2_{\mathrm{cts}}(\widehat{Q};\widehat{M})$ representing $\left(0 \to \widehat{M} \xrightarrow{\widehat{\iota}} \widehat{G} \xrightarrow{\widehat{\pi}} \widehat{Q} \to 1\right)$.

Proof. The reduction modulo n map $m_n\colon M \to M/nM$ induces maps labelled $(m_n)_*$ in Diagram 2.9. The group Q has separable cohomology in dimension 2 so the map $\mathsf{h}^*\colon H^2_{\mathrm{cts}}(\widehat{Q};M/nM) \to H^2(Q;M/nM)$ is an isomorphism, which means that we can invert it. Thus we get maps

$$\phi_n = (\mathsf{h}^*)^{-1} \circ (m_n)_*\colon H^2(Q;M) \to H^2_{\mathrm{cts}}(\widehat{Q};M/nM),$$

which are consistent with the maps induced from $M/nM \to M/n'M$ for $n' \mid n$, so we get a map

$$\phi\colon H^2(Q;M) \to \varprojlim_{n\in\mathbb{N}} H^2_{\mathrm{cts}}(\widehat{Q};M/nM),$$

which we can compose with the inverse of the isomorphism

$$\psi\colon H^2_{\mathrm{cts}}(\widehat{Q};\widehat{M}) \to \varprojlim_{n\in\mathbb{N}} H^2_{\mathrm{cts}}(\widehat{Q};M/nM)$$

coming from Theorem 2.57 to get the required map Θ.

$$\begin{array}{ccc} H^2(Q;M) & & H^2_{\mathrm{cts}}(\widehat{Q};\widehat{M}) \\ \downarrow{\scriptstyle (m_n)_*} & & \downarrow{\scriptstyle (m_n)_*} \\ H^2(Q;M/nM) & \xleftarrow{\mathsf{h}^*} & H^2_{\mathrm{cts}}(\widehat{Q};M/nM) \end{array} \tag{2.9}$$

The maps $(m_n)_*, \mathsf{h}^*$ and ψ are equivariant with respect to the action of $\mathrm{Stab}_{\mathrm{Out}(M)\times\mathrm{Out}(Q)}(\varphi)$, so the map $\Theta = \psi^{-1} \circ \phi$ is also equivariant and thus a morphism of group actions.

Finally, from Section 2.2.5 the image $(m_n)_*(\zeta) \in H^2(Q; M/nM)$ corresponds to the extension class $\big(0 \to M/nM \xrightarrow{\iota_n} G/nM \xrightarrow{\pi_n} Q \to 1\big)$, which also corresponds to the image $\mathsf{h}^*(\widehat{\zeta}_n)$ corresponding to the extension class $\big(0 \to M/nM \xrightarrow{\widehat{\iota}_n} \widehat{G}/n\widehat{M} \xrightarrow{\widehat{\pi}_n} \widehat{Q} \to 1\big)$. This consistent system of such extension classes is the image of the class $\big(0 \to \widehat{M} \xrightarrow{\widehat{\iota}} \widehat{G} \xrightarrow{\widehat{\pi}} \widehat{Q} \to 1\big)$ and since the map ψ is an isomorphism we get that the last extension class must indeed correspond to $\Theta(\zeta)$. □

In Section 4.2.2 we will also need to know how the cohomology $H^\bullet(Q; M)$ of a finite group Q with coefficients in a Q-module M relates to its cohomology $H^\bullet_{\mathrm{cts}}(Q; \widehat{M})$ with coefficients in $\widehat{M}$.

Lemma 2.60. *Let Q be a finite group and M a finitely generated Q-module. Then the map $\mathsf{h}_*\colon H^k(Q; M) \to H^k_{\mathrm{cts}}(Q; \widehat{M})$ induced by $\mathsf{h}\colon M \to \widehat{M}$ is an isomorphism for all $k \geq 1$.*

A proof of Lemma 2.60 can be found in Section 5.3 of [PPW21] and relies on the naturality of Long Exact Sequence for cohomology giving the following commutative diagram, the fact that $M/qM \cong \widehat{M}/q\widehat{M}$, and that multiplication by $q = |Q|$ gives a trivial map on $H^k(Q; -)$ for $k \geq 1$ – see Proposition 2.39.

$$\begin{array}{ccccccc}
H^{k-1}(Q; M/qM) & \twoheadrightarrow & H^k(Q; M) & \xrightarrow[0]{\cdot q} & H^k(Q; M) & \hookrightarrow & H^k(Q; M/qM) \\
\| & & \downarrow \mathsf{h}_* & & \downarrow \mathsf{h}_* & & \| \\
H^{k-1}_{\mathrm{cts}}(Q; \widehat{M}/q\widehat{M}) & \twoheadrightarrow & H^k_{\mathrm{cts}}(Q; \widehat{M}) & \xrightarrow[0]{\cdot q} & H^k_{\mathrm{cts}}(Q; \widehat{M}) & \hookrightarrow & H^k_{\mathrm{cts}}(Q; \widehat{M}/q\widehat{M})
\end{array}$$

Last, we give a tool crucial for computing the cohomology groups $H^\bullet_{\mathrm{prof}}(\widehat{G}; -)$ in practice, assuming that G has separable cohomology. Lemma 2.61 was stated and proved by Wilkes as Proposition 3.14 of [Wil17].

Lemma 2.61. *Let G be a group with separable cohomology, with a partial resolution $(C_i)_{0\leq i\leq n}$ of $\mathbb{Z}$ by finitely generated free $\mathbb{Z}G$-modules $C_i = \bigoplus_{j=1}^{l_i} \mathbb{Z}G \cdot \mathbf{x}_{ij}$. Then $(\widehat{C}_i)_{0\leq i\leq n}$ is a partial resolution of $\widehat{\mathbb{Z}}$ by free $\widehat{\mathbb{Z}}[[\widehat{G}]]$-modules, where*

$$\widehat{C}_i := \bigoplus_{j=1}^{l_i} \widehat{\mathbb{Z}}[[\widehat{G}]] \cdot \mathbf{x}_{ij}.$$

2.3 Group extensions

The *extension problem* formulated as early as the late 19th century by Otto Hölder seeks to describe all groups G with a prescribed normal subgroup N and quotient $G/N \cong Q$. Together with classification of finite simple groups this was one part of the Jordan-Hölder Program, which in light of the Jordan-Hölder Theorem sought to classify all finite groups.

The extension problem was studied and solved in the first half of 20th century by efforts of Schur, Schreier, Baer, Turing, Eilenberg and MacLane – see [Sch04, Sch24, Bae34, Tur38, EM47]. The modern treatment of group extensions developed alongside.

This study of group extensions is interesting in its own right, but it is also useful for building understanding of many 'naturally occuring' groups in general. Particularly relevant to this thesis, many examples of profinitely isomorphic pairs of groups are group extensions with the same kernel and quotient.

Studies of group extensions similar to ours can be found in the books of Beyl–Tappe [BT06] and Passi–Singh–Yadav [PSY18].

2.3.1 Definitions

2.3.1.1 Equivalent, similar and isomorphic extensions

The topic of group extensions can be confusing due to unclear definitions and a number of different notions of being 'the same'. We try to avoid this confusion here.

Definition 2.62. An *N-by-Q extension* is a short exact sequence

$$1 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} Q \longrightarrow 1.$$

Definition 2.63. For $j = 1, 2$ the extensions $\mathcal{E}_j = \left(1 \to N \xrightarrow{\iota_j} G_j \xrightarrow{\pi_j} Q \to 1\right)$ are called

- *isomorphic as groups* if there exists an isomorphism $\beta\colon G_1 \to G_2$;
- *similar* if there exist isomorphisms $\alpha\colon N \to N$, $\beta\colon G_1 \to G_2$ and $\gamma\colon Q \to Q$ such that the diagram 2.10 commutes;

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \xrightarrow{\iota_1} & G_1 & \xrightarrow{\pi_1} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 1 & \longrightarrow & N & \xrightarrow{\iota_2} & G_2 & \xrightarrow{\pi_2} & Q & \longrightarrow & 1 \end{array} \tag{2.10}$$

- *equivalent* if there exists an isomorphism $\beta\colon G_1 \to G_2$ such that the diagram 2.11 commutes.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \xrightarrow{\iota_1} & G_1 & \xrightarrow{\pi_1} & Q & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \beta} & & \| & & \\ 1 & \longrightarrow & N & \xrightarrow{\iota_2} & G_2 & \xrightarrow{\pi_2} & Q & \longrightarrow & 1 \end{array} \tag{2.11}$$

It is clear that these are equivalence relations.

We denote the set of equivalence classes of N-by-Q extensions by $\mathscr{E}(Q; N)$ and the set of similarity classes by $\overline{\mathscr{E}}(Q; N)$.

It is clear that equivalent extensions are in particular similar, but the converse isn't true.

Example 2.64. The extension

$$\mathcal{E}_1 = \left(\; 0 \longrightarrow \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \xrightarrow{\text{mod } 3} \mathbb{Z}/3 \longrightarrow 1 \; \right)$$

isn't equivalent to

$$\mathcal{E}_2 = \left(\; 0 \longrightarrow \mathbb{Z} \xrightarrow{\times(-3)} \mathbb{Z} \xrightarrow{\text{mod } 3} \mathbb{Z}/3 \longrightarrow 1 \; \right)$$

as a map $\beta\colon \mathbb{Z} \to \mathbb{Z}$ in the diagram 2.12 would satisfy $\beta(3) = -3$, but then also $\beta(1) = -1$, which is in violation of the right hand square commuting, since $1 \not\equiv (-1) \mod 3$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 3} & \mathbb{Z} & \xrightarrow{\text{mod } 3} & \mathbb{Z}/3 & \longrightarrow & 1 \\ & & \| & & \vdots{\scriptstyle \beta} & & \| & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times(-3)} & \mathbb{Z} & \xrightarrow{\text{mod } 3} & \mathbb{Z}/3 & \longrightarrow & 1 \end{array} \tag{2.12}$$

The extensions $\mathcal{E}_1$ and $\mathcal{E}_2$ are similar though, as the shown by the diagram 2.13.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 3} & \mathbb{Z} & \xrightarrow{\text{mod } 3} & \mathbb{Z}/3 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \times(-1)} & & \| & & \| & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times(-3)} & \mathbb{Z} & \xrightarrow{\text{mod } 3} & \mathbb{Z}/3 & \longrightarrow & 1 \end{array} \tag{2.13}$$

2.3.1.2 Kernel-detecting families

Not every class of group extensions is equally useful: the ones that enable us to 'upgrade' group isomorphisms to similarities of extensions will be particularly important.

Definition 2.65 (Kernel-detecting families)**.** Let $\mathscr{E}$ be a class of N-by-Q extensions $\left(1 \to N \xrightarrow{\iota_i} G_i \xrightarrow{\pi_i} Q \to 1\right)$. We say that $\mathscr{E}$ *is kernel-detecting* if each group isomorphism $\beta\colon G_i \to G_j$ extends to a similarity of extensions $\left(1 \to N \xrightarrow{\iota_i} G_i \xrightarrow{\pi_i} Q \to 1\right)$ and $\left(1 \to N \xrightarrow{\iota_j} G_j \xrightarrow{\pi_j} Q \to 1\right)$.

Equivalently, $\beta(\iota_i(N)) = \iota_j(N)$.

Let's see some examples of classes which are kernel-detecting. These will be of use in later chapters.

Proposition 2.66. *Let N be an abelian group and Q be a centerless group, i.e. $Z(Q) = 1$. Let $\mathscr{E}_{\text{cent}}(Q;N)$ be the class of central N-by-Q extensions, i.e. extensions $\left(0 \to N \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1\right)$ such that $\iota(N) < Z(G)$.*

Then $\mathscr{E}_{\text{cent}}(Q;N)$ is kernel-detecting.

Proof. Given a central extension $\left(0 \to N \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1\right) \in \mathscr{E}_{\text{cent}}(Q;N)$, we get that not only $\iota(N) < Z(G)$, but also $\pi(Z(G)) < Z(Q) = 1$, so in fact $Z(G) = \iota(N)$. Since any isomorphism between two groups $\beta\colon G_1 \to G_2$ must map $Z(G_1)$ to $Z(G_2)$, so the class indeed is kernel-detecting. □

Crystallographic groups are extensions of $\mathbb{Z}^n$ by a finite group Q, where the induced action $Q \curvearrowright \mathbb{Z}^n$ (see Section 2.3.2) is faithful. We study them in detail in Chapter 4. One of the main foundational results there—the Bieberbach Theorem—is that the kernel $\mathbb{Z}^n$ is characterised

as the unique finite-index maximal free-abelian normal subgroup – see Theorem 4.2. This implies the following.

Proposition 2.67. *Let Q be a finite group and let $\varphi\colon Q \to \mathrm{GL}_n(\mathbb{Z})$ be a faithful action.*

Then the class $\mathscr{E}_\varphi(Q;\mathbb{Z}^n)$ is kernel-detecting.

We can also cook-up kernel-detecting families by imposing that no non-trivial quotients of the kernel N appear as normal subgroups of Q. This gives a variant of Proposition 2.68 which appears later as Proposition 3.1. Pairs (N, Q) to which the result applies include the following.

- N a torsion group and Q a torsion-free group.
- N an abelian group and $Q = F_2$.
- $N = \mathrm{SL}_2(\mathbb{F}_9)$ and $Q = C_2$ – see Proposition 2.93.

Proposition 2.68. *Let Q and N be groups such that among the normal subgroups of Q only the trivial subgroup 1 is a quotient of N.*

Then the family $\mathscr{E}(Q;N)$ of all N-by-Q extensions is kernel-detecting.

Proof. Let $1 \to N \xrightarrow{\iota} G \to Q \to 1$ be any such extension. We will show that $\iota(N)$ is the only normal subgroup of G isomorphic to N with the quotient $G/\iota(N)$ isomorphic to Q.

Assume that N' is another such normal subgroup. The subgroup $\iota(N)N'/N' \lhd G/N' \cong Q$ is normal and a quotient of $N' \cong N$, so it must be trivial, so $\iota(N) < N'$.

The argument is symmetric with respect to $\iota(N)$ and N' so we get $\iota(N) = N'$, and the family $\mathscr{E}(Q;N)$ is indeed kernel-detecting. □

One should note that if a family $\mathscr{E}$ of N-by-Q extensions is kernel-detecting then the images $\iota_i(N)$ are characteristic in G_i, but the reverse implication doesn't hold as the next example shows.

The following example was found and verified with the `GroupNames` database [Dok20] with `SmallGroupId` (32, 12).

Example 2.69. Let $G = (\mathbb{Z}/4) \rtimes (\mathbb{Z}/8)$ be the non-trivial semidirect product of $\mathbb{Z}/4 = \langle x \rangle$ with $\mathbb{Z}/8 = \langle y \rangle$. Let $N_1 = \langle x, y^4 \rangle$ and $N_2 = \langle xy^2, y^4 \rangle$.

Then both subgroups N_i are isomorphic to $(\mathbb{Z}/4) \times (\mathbb{Z}/2)$, both are characteristic in G, and $G/N_1 \cong G/N_2 \cong \mathbb{Z}/4$, but there is no automorphism $\beta \in \mathrm{Aut}(G)$ such that $\beta(N_1) = N_2$.

Thus the family of two $\big((\mathbb{Z}/4) \times (\mathbb{Z}/2)\big)$-by-$(\mathbb{Z}/4)$ extensions

$$\mathscr{E} = \Big\{ \big(1 \to N_1 \to G \to G/N_1 \to 1\big), \big(1 \to N_2 \to G \to G/N_2 \to 1\big) \Big\}$$

with the natural inclusion and quotient maps, is not kernel-detecting.

2.3.1.3 Morphisms of group actions

In order to state some of the results of Sections 1.3 and 2.3 it is convenient to introduce the language of the category of group actions.

Definition 2.70. The *category of group actions* GroupActions is defined as a category in which

- the objects are triples $(G, X, \circ)$, where G is a group, X is a set, and $\circ \colon G \times X \to X$ is a group action of G on X;
- the morphisms between $(G, X, \circ)$ and $(H, Y, *)$ are pairs (ϕ, f) where $\phi \colon G \to H$ is a group homomorphism, $f \colon X \to Y$ is a function such that for any $g \in G$ and $x \in X$:

$$\phi(g) * f(x) = f(g \circ x).$$

When the action is clear from the context we will suppress it in the notation.

To each group action (G, X) we can associate the set $G\backslash X$ of orbits. It turns out that the assignment is functorial and it has a right adjoint.

Proposition 2.71. *The assignment $(G, X) \mapsto G\backslash X$ is a functor*

$$\mathsf{GroupActions} \to \mathsf{Sets}\,.$$

Furthermore, it is left adjoint to the assignment $X \mapsto (1, X)$.

Proof. Any morphism $(G, X) \xrightarrow{(\phi, f)} (H, Y)$ of group actions induces a map of sets $G\backslash X \xrightarrow{f_*} H\backslash Y$ through $f_*([x]) := [f(x)]$. This is well-defined since $f([g \cdot x]) = [f(g \cdot x)] = [\phi(g) \cdot f(x)] = [f(x)]$.

Now, any morphism $(G, X) \xrightarrow{(1,f)} (1, Y)$ factors uniquely through the morphism $(G, X) \longrightarrow (1, G\backslash X)$ and likewise for any function $f\colon G\backslash X \to Y$ there is a corresponding morphism $(G, X) \xrightarrow{(1,\tilde{f})} (1, Y)$ of group actions. □

The next lemma will be important to our classification of extensions.

Lemma 2.72. *Let* $(G, X) \xrightarrow{(\mathrm{Id}, f)} (G, Y)$ *be a morphism of group actions. Then*

1. *for each* $y \in Y$ *there is an induced action* $(\mathrm{Stab}_G(y), f^{-1}(y))$ *of the stabilizer* $\mathrm{Stab}_G(y)$ *on the preimage* $f^{-1}(y) \subset X$*;*

2. *the orbits of* (G, X) *decompose as*

$$G\backslash X \longleftrightarrow \bigsqcup_{Gy \in G\backslash Y} \mathrm{Stab}_G(y)\backslash f^{-1}(y).$$

Proof. If $g \in \mathrm{Stab}_G(y)$ and $x \in f^{-1}(y)$, then also $f(g \cdot x) = g \cdot f(x) = g \cdot y = y$, so $g \cdot x \in f^{-1}(y)$. Thus the stabilizer $\mathrm{Stab}_G(y)$ indeed acts on $f^{-1}(y)$.

The orbit space $G\backslash X$ decomposes as

$$G\backslash X = \bigsqcup_{Gy \in G\backslash Y} G\backslash f^{-1}(Gy).$$

Consider the group actions inclusion morphism

$$h\colon (\mathrm{Stab}_G(y), f^{-1}(y)) \to (G, f^{-1}(Gy)).$$

We will show that the induced orbit map h_* is bijective. Surjectivity comes from the fact that $f^{-1}(Gy) = Gf^{-1}(y)$, so for each $x \in f^{-1}(Gy)$ there exists $x' \in f^{-1}(y)$ such that $x = gx'$ and so that $[x] = h_*([x'])$. Now, for injectivity, let $x_1, x_2 \in f^{-1}(y)$ be such that $h_*([x_1]) = h_*([x_2])$, i.e. $x_2 = gx_1$ for some $g \in G$. But then $y = f(x_2) = f(gx_1) = gf(x_1) = gy$, so $g \in \mathrm{Stab}_G(y)$ and $[x_1] = [x_2]$.

Thus we get $\mathrm{Stab}_G(y)\backslash f^{-1}(y) \longleftrightarrow G\backslash f^{-1}(Gy)$ and also

$$G\backslash X \longleftrightarrow \bigsqcup_{Gy\in G\backslash Y} \mathrm{Stab}_G(y)\backslash f^{-1}(y). \qquad \square$$

2.3.2 Induced outer actions

An important characteristic of an N-by-Q extension is the *outer action* $\varphi\colon Q \to \mathrm{Out}(N)$ it induces.

Definition 2.73. Given an N-by-Q extension $\mathcal{E} = \left(1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1\right)$ the conjugation action of G on itself induces an action by automorphisms on N, we denote it $\tilde{\varphi}_{\mathcal{E}}\colon G \to \mathrm{Aut}(N)$. Elements of N induce inner automorphisms, so $\tilde{\varphi}_{\mathcal{E}}(N) < \mathrm{Inn}(N)$ and $\tilde{\varphi}_{\mathcal{E}}$ induces a homomorphism $\varphi_{\mathcal{E}}\colon Q \to \mathrm{Out}(N)$ called *the induced homomorphism* or *the induced outer action* of $\mathcal{E}$.

Remark 2.74. When N is abelian $\mathrm{Out}(N) = \mathrm{Aut}(N)$ and $\varphi_{\mathcal{E}}\colon Q \to \mathrm{Out}(N)$ gives an actual *action* of Q on N.

In fact equivalent extensions give the same induced homomorphism. For similar extensions it can change though, both of which are described in Proposition 2.75.

Proposition 2.75. *For $i = 1, 2$, let $\left(1 \to N \xrightarrow{\iota_i} G_i \xrightarrow{\pi_i} Q \to 1\right)$ be similar extensions, with $\alpha\colon N \to N$ and $\gamma\colon Q \to Q$ as in diagram 2.10. Then, letting $\overline{\alpha}$ denote the image of α in $\mathrm{Out}(N)$, the induced homomorphisms $\varphi_i\colon Q \to \mathrm{Out}(N)$ satisfy*

$$\mathrm{Ad}_{\overline{\alpha}} \circ \varphi_1 = \varphi_2 \circ \gamma \quad \text{as maps} \quad Q \to \mathrm{Out}(N). \tag{2.14}$$

In particular,

- *the images $\varphi_1(Q)$ and $\varphi_2(Q)$ are conjugate in $\mathrm{Out}(N)$;*
- *the induced homomorphisms of equivalent extensions are equal.*

Proof. Let's denote by $\tilde{\varphi}_i \colon G_i \to \mathrm{Aut}(N)$ the conjugation actions.

$$\begin{aligned}(\tilde{\varphi}_2 \circ \beta)(g_1) \colon \alpha(n) \mapsto &\quad \beta(g_1)\alpha(n)\beta(g_1)^{-1} \\ &= \beta(g_1 n g_1^{-1}) = \alpha(g_1 n g_1^{-1}) \\ (\mathrm{Ad}_\alpha \circ \tilde{\varphi}_1)(g_1) \colon \alpha(n) \mapsto &\quad \alpha(\tilde{\varphi}_1(g_1)(n)) = \alpha(g_1 n g_1^{-1})\end{aligned}$$

so we get an equality

$$\mathrm{Ad}_\alpha \circ \tilde{\varphi}_1 = \tilde{\varphi}_2 \circ \beta \quad \text{as maps} \quad G_1 \to \mathrm{Aut}(N).$$

Post-composing with the map $\mathrm{Aut}(N) \twoheadrightarrow \mathrm{Out}(N)$ and factoring through $\pi_1 \colon G_1 \twoheadrightarrow Q$ we get the required $\mathrm{Ad}_{\overline{\alpha}} \circ \varphi_1 = \varphi_2 \circ \gamma$. □

We will call outer actions $\varphi_1, \varphi_2 \colon Q \to \mathrm{Out}(N)$ which satisfy Equation (2.14) *conjugate*.

2.3.3 The actions of $\mathrm{Out}(N) \times \mathrm{Aut}(Q)$

The classification of N-by-Q extensions is closely related to two actions of the group $\mathrm{Out}(N) \times \mathrm{Aut}(Q)$: one of them is on the set $\mathscr{E}(Q;N)$ of equivalence classes of N-by-Q extensions; the other is on the set $\{Q \to \mathrm{Out}(N)\}$ of possible outer actions. These actions are related through a morphism of group actions, which is to say that the map $\mathcal{E} \mapsto \varphi_{\mathcal{E}}$ is equivariant.

The actions are also discussed in Section 2.4 of [PSY18].

Proposition 2.76. *The group* $\mathrm{Aut}(N) \times \mathrm{Aut}(Q)$ *acts on the set* $\mathscr{E}(Q;N)$ *of equivalence classes of N-by-Q extensions by*

$$(\alpha, \gamma) \cdot \big(1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1\big) = \big(1 \to N \xrightarrow{\iota \circ \alpha^{-1}} G \xrightarrow{\gamma \circ \pi} Q \to 1\big). \quad (2.15)$$

Two extensions are in the same orbit if and only if they are similar. Furthermore, the action of $\mathrm{Inn}(N)$ *is trivial, so we can consider this to be an action of* $\mathrm{Out}(N) \times \mathrm{Aut}(Q)$.

Proof. The maps $\iota \circ \alpha^{-1}$ and $\gamma \circ \pi$ are still respectively injective and surjective. Also, $\mathrm{im}(\iota \circ \alpha^{-1}) = \mathrm{im}(\iota) = \ker(\pi) = \ker(\gamma \circ \pi)$, so

$$\big(1 \to N \xrightarrow{\iota \circ \alpha^{-1}} G \xrightarrow{\gamma \circ \pi} Q \to 1\big)$$

really is an N-by-Q extension. Thus, the formula 2.15 defines a group action; the usage of α^{-1} ensures that this is a left action.

Any similarity of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \xrightarrow{\iota_1} & G_1 & \xrightarrow{\pi_1} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 1 & \longrightarrow & N & \xrightarrow{\iota_2} & G_2 & \xrightarrow{\pi_2} & Q & \longrightarrow & 1 \end{array}$$

can be extended to a larger diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \xrightarrow{\iota_1} & G_1 & \xrightarrow{\pi_1} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha} & & \| & & \downarrow{\scriptstyle \gamma} & & \\ 1 & \longrightarrow & N & \xrightarrow{\iota_1 \circ \alpha^{-1}} & G_1 & \xrightarrow{\gamma \circ \pi_1} & Q & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \beta} & & \| & & \\ 1 & \longrightarrow & N & \xrightarrow{\iota_2} & G_2 & \xrightarrow{\pi_2} & Q & \longrightarrow & 1 \end{array}$$

in which the two bottom rows form an equivalence of extensions, showing that indeed the bottom extension $\left(1 \to N \xrightarrow{\iota_2} G \xrightarrow{\pi_2} Q \to 1\right)$ is equivalent to the extension $(\alpha, \gamma) \cdot \left(1 \to N \xrightarrow{\iota_1} G \xrightarrow{\pi_1} Q \to 1\right)$. The other direction is immediate, too.

Finally, if $\alpha^{-1} = \mathrm{Ad}_n$ is the conjugation by some $n \in N$, then we can extend it to a conjugation by $\iota(n)$ in G, and $\iota \circ \mathrm{Ad}_n = \mathrm{Ad}_{\iota(n)} \circ \iota$ gives the equivalence of extensions in Diagram 2.16.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \xrightarrow{\iota} & G & \xrightarrow{\pi} & Q & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \mathrm{Ad}_{\iota(n)}} & & \| & & \\ 1 & \longrightarrow & N & \xrightarrow{\iota \circ \mathrm{Ad}_n} & G & \xrightarrow{\pi} & Q & \longrightarrow & 1 \end{array} \tag{2.16}$$

Thus, the action of $\mathrm{Inn}(N)$ is trivial, and we can factor the action through $\mathrm{Aut}(N) \times \mathrm{Aut}(Q) \to \mathrm{Out}(N) \times \mathrm{Aut}(Q)$. □

The second action deals with the set of outer actions.

Proposition 2.77. *The group* $\mathrm{Out}(N) \times \mathrm{Aut}(Q)$ *acts on the set of outer actions* $\{Q \to \mathrm{Out}(N)\}$ *by*

$$(\overline{\alpha}, \gamma) \cdot \varphi = \mathrm{Ad}_{\overline{\alpha}} \circ \varphi \circ \gamma^{-1}.$$

With this action, the assignment $\mathcal{E} \mapsto \varphi_{\mathcal{E}}$ is a morphism of group actions

$$\big(\mathrm{Out}(N) \times \mathrm{Aut}(Q), \mathscr{E}(Q;N)\big) \longrightarrow \big(\mathrm{Out}(N) \times \mathrm{Aut}(Q), \{Q \to \mathrm{Out}(N)\}\big).$$

Proof. Let $\mathcal{E} = \big(1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1\big)$ be an extension with an induced action $\varphi_{\mathcal{E}} \colon Q \to \mathrm{Out}(N)$. In Proposition 2.75 we computed that for a pair $(\alpha, \gamma) \in \mathrm{Aut}(N) \times \mathrm{Aut}(Q)$ the extension

$$(\alpha, \gamma) \cdot \mathcal{E} = \big(1 \to N \xrightarrow{\iota \circ \alpha^{-1}} G \xrightarrow{\gamma \circ \pi} Q \to 1\big)$$

induces the action $\mathrm{Ad}_{\overline{\alpha}} \circ \varphi_{\mathcal{E}} \circ \gamma^{-1}$. □

By Lemma 2.72 the orbits of $(\mathrm{Out}(N) \times \mathrm{Aut}(Q), \mathscr{E}(Q;N))$ can be understood through the orbits of $(\mathrm{Out}(N) \times \mathrm{Aut}(Q), \{Q \to \mathrm{Out}(N)\})$ and, for each orbit $[\varphi \colon Q \to \mathrm{Out}(N)]$, the orbits of

$$\big(\mathrm{Stab}_{\mathrm{Out}(N)\times\mathrm{Aut}(Q)}(\varphi), \quad \{\mathcal{E} \in \mathscr{E}(Q;N) \mid \varphi_{\mathcal{E}} = \varphi\}\big).$$

In case the induced homomorphism $\varphi \colon Q \to \mathrm{Out}(N)$ is injective this stabilizer is easy to compute.

Proposition 2.78. *Let $\varphi \colon Q \to \mathrm{Out}(N)$ be any homomorphism. Then*

$$\mathrm{Stab}_{\mathrm{Out}(N)\times\mathrm{Aut}(Q)}(\varphi) = \{(\overline{\alpha}, \gamma) \in \mathrm{Out}(N) \times \mathrm{Aut}(Q) \mid \varphi \circ \gamma = \mathrm{Ad}_{\overline{\alpha}} \circ \varphi\}.$$

Furthermore, if $\varphi \colon Q \to \mathrm{Out}(N)$ is injective, then

$$\begin{aligned}\mathrm{Stab}_{\mathrm{Out}(N)\times\mathrm{Aut}(Q)}(\varphi) &= \{(\overline{\alpha}, \varphi^{-1} \circ \mathrm{Ad}_{\overline{\alpha}} \circ \varphi) \mid \overline{\alpha} \in N_{\mathrm{Out}(N)}(\mathrm{im}\,\varphi)\} \\ &\cong N_{\mathrm{Out}(N)}(\mathrm{im}\,\varphi).\end{aligned}$$

Proof. By the assumption

$$\varphi = (\overline{\alpha}, \gamma) \cdot \varphi = \mathrm{Ad}_{\overline{\alpha}} \circ \varphi \circ \gamma^{-1},$$

so in particular $\mathrm{im}\,\varphi = \mathrm{Ad}_{\overline{\alpha}}(\mathrm{im}\,\varphi)$ and thus $\overline{\alpha} \in N_{\mathrm{Out}(N)}(\mathrm{im}\,\varphi)$ and γ is invertible, which gives $\varphi \circ \gamma = \mathrm{Ad}_{\overline{\alpha}} \circ \varphi$. If additionally φ is injective, it is invertible on its image, so $\gamma \in \mathrm{Aut}(Q)$ is determined by

$$\gamma = \varphi^{-1} \circ \mathrm{Ad}_{\overline{\alpha}} \circ \varphi.$$ □

The computation of the stabilizer was also done in [PSY18] – see Equation 2.35 in Section 2.4 therein.

2.3.4 Some special cases of classification

We established that having conjugate outer actions is a prerequisite for similarity of extensions. In general, it isn't enough, but a partial converse holds for split extensions.

Proposition 2.79. *For $i = 1, 2$, let $G_i = N \rtimes_{\tilde{\varphi}_i} Q$ be semidirect products defined by homomorphisms $\tilde{\varphi}_i \colon Q \to \mathrm{Aut}(N)$. Assume that there exist automorphisms $\alpha \in \mathrm{Aut}(N)$ and $\gamma \in \mathrm{Aut}(Q)$ such that*

$$\mathrm{Ad}_\alpha \circ \tilde{\varphi}_1 = \tilde{\varphi}_2 \circ \gamma \quad \text{as maps} \quad Q \to \mathrm{Aut}(N).$$

Then the map $\beta \colon (n, q) \mapsto (\alpha(n), \gamma(q))$ defines an isomorphism $\beta \colon G_1 \to G_2$, which gives a similarity of extensions.

Proof. We need to check that $\beta\big((n_1, q_1)(n_2, q_2)\big) = \beta\big((n_1, q_1)\big)\beta\big((n_2, q_2)\big)$.

$$\begin{aligned}
\beta\big((n_1, q_1)\big) \cdot \beta((n_2, q_2)) &= (\alpha(n_1), \gamma(q_1)) \cdot (\alpha(n_2), \gamma(q_2)) \\
&= (\alpha(n_1) \cdot \tilde{\varphi}_2(\gamma(q_1))(\alpha(n_2)), \gamma(q_1) \cdot \gamma(q_2)) \\
&= (\alpha(n_1) \cdot \alpha(\tilde{\varphi}_1(q_1)(n_2)), \gamma(q_1 q_2)) \\
&= \beta\big((n_1 \cdot \tilde{\varphi}_1(q_1)(n_2), q_1 q_2)\big) \\
&= \beta\big((n_1, q_1)(n_2, q_2)\big).
\end{aligned}$$

Thus β is a homomorphism. It is clear that it makes the diagram commute, and by the Five Lemma it must be an isomorphism. □

There is a special case of Proposition 2.79 which is actually a converse to Proposition 2.75. Namely, when the quotient Q is a free group.

Proposition 2.80. *For $i = 1, 2$, let $\mathcal{E}_i = \big(1 \to N \to G_i \to F \to 1\big)$ be extensions of a group N by a free group F, inducing homomorphisms $\varphi_i \colon F \to \mathrm{Out}(N)$, which are conjugate. Then the extensions $\mathcal{E}_1$ and $\mathcal{E}_2$ are similar.*

Proof. Extensions 'by-free' are split, so we can assume that $G_i \cong N \rtimes_{\tilde{\varphi}_i} F$. We will show that we can choose the lifts $\tilde{\varphi}_i \colon F \to \mathrm{Aut}(N)$ so that they satisfy the conditions of Proposition 2.79.

Choose any lift $\tilde{\varphi}_2 \colon F \to \mathrm{Aut}(N)$. Then choose a basis X for F and define a function $f \colon X \to \mathrm{Aut}(N)$ by setting $f(x) = \alpha^{-1}\tilde{\varphi}_2(\gamma(x))\alpha$. This function extends uniquely to a homomorphism $\tilde{\varphi}_1 \colon F \to \mathrm{Aut}(N)$ which satisfies $\alpha \cdot \tilde{\varphi}_1(q) \cdot \alpha^{-1} = \tilde{\varphi}_2(\gamma(q))$. This implies that $\tilde{\varphi}_1$ is indeed a lift of $\varphi_1 \colon F \to \mathrm{Out}(N)$.

The similarity follows from Proposition 2.79. □

The situation simplifies further in the case of free profinite groups, thanks to Corollary 2.11.

Corollary 2.81. *For $i = 1, 2$, let $\mathcal{E}_i = \big(1 \to \Delta \to \Gamma_i \to \widehat{F}_n \to 1\big)$ be extensions of a topologically finitely generated profinite group Δ by a free profinite group $\widehat{F}_n$, inducing homomorphisms $\varphi_i \colon \widehat{F}_n \to \mathrm{Out}(\Delta)$, whose images are conjugate. Then the extensions $\mathcal{E}_1$ and $\mathcal{E}_2$ are similar.*

2.3.5 Classification of N-by-Q extensions with general N

The following account of how to study group extensions has been greatly inspired and informed by Huebschmann's essay [Hue23].

The reader should be forewarned that the general classification of all N-by-Q extensions is difficult and much less common than that in the case of abelian N. This is also true in this thesis, where none of Chapters 3, 4, 5 and 6 relies on it. However, the classification is an important and not well-known piece of mathematics, and the author deemed it appropriate to include it. Furthermore, the examples of the computation done in practice are scarce in the literature, which is why we carry out a detailed example of such classification in Section 2.3.7.

2.3.5.1 Classification up to equivalence

Theorem 2.82. *Let $\varphi \colon Q \to \mathrm{Out}(N)$ be an outer action induced by an extension $\mathcal{E}_1 = \big(1 \to N \xrightarrow{\iota_1} G_1 \to Q \to 1\big)$. Then the set $\mathscr{E}_\varphi(Q; N)$ of equivalence classes of N-by-Q extensions inducing φ is acted upon transitively and faithfully by the group $H^2(Q; Z(N))$ as follows.*

Given an extension $\mathcal{D} = \big(0 \to Z(N) \xrightarrow{\iota} E \to Q \to 1\big)$ *we define* $\mathcal{D} \cdot \mathcal{E}_1$ *to be the extension* $\mathcal{E}_2 = \big(1 \to N \to G_2 \to Q \to 1\big)$*, where*

$$G_2 := G_1 \times_Q E / \{(z, z^{-1}) \mid z \in Z(N)\}.$$

Remark 2.83. The fact that the action $H^2(Q; Z(N)) \curvearrowright \mathscr{E}_\varphi(Q; N)$ is free and transitive means that there is a bijection

$$\mathscr{E}_\varphi(Q; N) \leftrightarrow H^2(Q; Z(N)).$$

This bijection however is in general *not* canonical: it depends on the choice of an existing extension $\mathcal{E}_1$. When N is abelian, there is a canonical choice though: the split extension $N \rtimes Q$; in this case the bijection is canonical.

The reader is invited to compare Theorem 2.82 with Theorem 2.26 of [PSY18] where $H^2(Q; Z(N)) \curvearrowright \mathscr{E}_\varphi(Q; N)$ is defined in terms of 2-cocycles.

2.3.5.2 Classification up to similarity

The classification $\mathscr{E}_\varphi(Q; N) \leftrightarrow H^2\big(Q; Z(N)\big)$ of Remark 2.83 turns out to **not** be $\mathrm{Stab}_{\mathrm{Out}(N)\times\mathrm{Aut}(Q)}(\varphi)$-equivariant, at least not necessarily. Following notation of Theorem 2.82, we have

$$(\alpha, \gamma) \cdot (\mathcal{D} \cdot \mathcal{E}_1) = \big((\alpha, \gamma) \cdot \mathcal{D}\big) \cdot \big((\alpha, \gamma) \cdot \mathcal{E}_1\big),$$

which is not the same as $\big((\alpha, \gamma) \cdot \mathcal{D}\big) \cdot \mathcal{E}_1$, unless $(\alpha, \gamma) \cdot \mathcal{E}_1 = \mathcal{E}_1$. In Proposition 2.96 we will see the case of $N = \mathrm{SL}_2(\mathbb{F}_9)$ and $Q = \mathbb{Z}/2$, where $H^2(Q; Z(N)) \cong \mathbb{Z}/2$, but one of the outer actions $\varphi\colon Q \to \mathrm{Out}(N)$ has only *one* similarity class of extensions.

Notably, in the case of abelian N the canonical choice of $\mathcal{E}_1$, i.e. $\big(0 \to N \to N \rtimes_\varphi Q \to Q \to 1\big)$ is fixed by all $(\alpha, \gamma) \in \mathrm{Stab}_{\mathrm{Out}(N)\times\mathrm{Aut}(Q)}(\varphi)$.

Finally, there is a way to combine the two actions of the stabilizer and of $H^2(Q; Z(N))$: to act by $H^2(Q; Z(N)) \rtimes \mathrm{Stab}_{\mathrm{Out}(N)\times\mathrm{Aut}(Q)}(\varphi)$ instead – for details see Section 2.5 of [PSY18].

2.3.5.3 Extendibility of outer actions

The classification we've done so far leaves the following question unanswered: which outer actions $\varphi\colon Q \to \mathrm{Out}(N)$ (also called *abstract kernels* in the literature) are *extendible*, i.e. they are induced by an actual N-by-Q extension? This connects to the topic of *crossed modules*, which is a very rich area of mathematics. Necessarily, we give only a little motivation and some statements here.

Definition 2.84. A *crossed module* (C, G, ∂) is a pair of groups C, G with a (left) action $G \curvearrowright C$ denoted $g \cdot c := {}^g c$, together with a homomorphism $\partial\colon C \to G$ which satisfies the following conditions.

1. For $c \in C$ and $g \in G$: $\partial({}^g c) = g\partial(c)g^{-1}$ in G, i.e. ∂ is a morphism of actions $(G, C) \to (G, G)$, where G acts on itself by conjugation.
2. For $c, d \in C$: $cdc^{-1} = {}^{\partial(c)}d$.

Examples 2.85. Crossed modules generalise the notions of group modules and also group extensions.

1. If Q is a group, and M is a Q-module, then (M, Q, Triv) is a crossed module with $\mathrm{Triv}\colon M \to Q$ being the trivial homomorphism.
2. The action $\mathrm{Aut}(G) \curvearrowright G$ makes $(G, \mathrm{Aut}(G), \partial)$ a crossed module, with $\partial(g) := \mathrm{Ad}_g$.
3. Let N and Q be any groups, and $\varphi\colon Q \to \mathrm{Out}(N)$ be an outer action. Then the pullback $G^\varphi := \mathrm{Aut}(N) \times_{\mathrm{Out}(N)} Q$ acts on N through $\mathrm{Aut}(N) \times_{\mathrm{Out}(N)} Q \twoheadrightarrow \mathrm{Aut}(N)$. Together with $\partial^\varphi\colon N \to G^\varphi$, this makes $(N, G^\varphi, \partial^\varphi)$ a crossed module.

In fact point 3 of Examples 2.85 shows something that happens for all crossed modules.

Proposition 2.86. *Let (C, G, ∂) be a crossed module. Then*

1. $\mathrm{im}\,\partial$ *is a normal subgroup of G; denote $G/\mathrm{im}\,\partial$ by Q;*

2. $Z := \ker \partial$ *is a central subgroup of* C*; it is naturally a* Q*-module.*

We can summarise this in an exact sequence of groups

$$0 \longrightarrow Z \longrightarrow C \xrightarrow{\partial} G \longrightarrow Q \longrightarrow 1 \ .$$

For an outer action $\varphi\colon Q \to \mathrm{Out}(N)$ the associated exact sequence is

$$\mathcal{M}_\varphi := \big(\ 0 \longrightarrow Z(N) \longrightarrow N \xrightarrow{\partial^\varphi} G^\varphi \longrightarrow Q \longrightarrow 1 \ \big).$$

It turns out that the family of crossed modules (C, G, ∂) with *fixed* $Q \cong G/\operatorname{im}\partial$ and a Q-module $Z \cong \ker\partial$, with an appropriate notion of *equivalence* (similar to equivalence of group extensions of Definition 2.63, but not requiring that the vertical arrows in the middle are isomorphisms) together with an operation similar to Baer sum of Section 2.2.5.3 form a group isomorphic to the *third* cohomology group $H^3(Q; Z)$. Under this correspondence, the identity element is

$$\mathcal{M}_0 := \big(\ 0 \longrightarrow Z \longrightarrow Z \xrightarrow{\mathrm{Triv}} Q \longrightarrow Q \longrightarrow 1 \ \big),$$

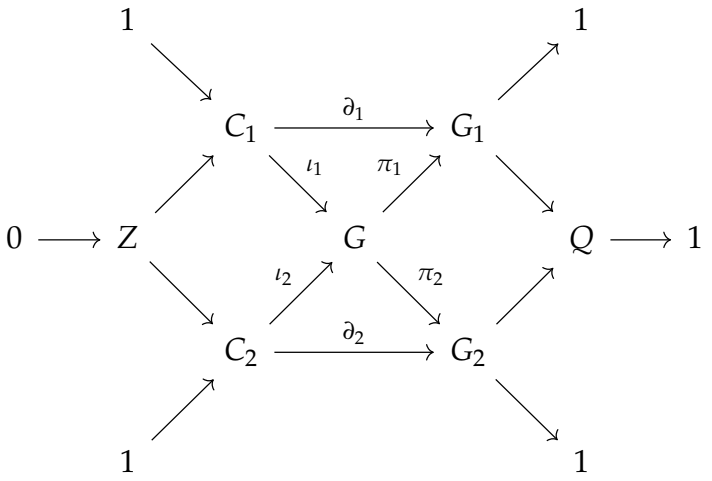
with $\mathrm{Triv}\colon Z \to Q$ being the trivial homomorphism.

The equivalence relation on this family of crossed modules has a different, equivalent, interpretation.

Proposition 2.87 (Lemma 2.3 of [Hol79])**.** *Let* Q *be a group, and* Z *be a* Q*-module. Let* (C_1, G_1, ∂_1) *and* (C_2, G_2, ∂_2) *be crossed modules with* $\ker\partial_1 \cong \ker\partial_2 \cong Z$ *and* $G_1/\operatorname{im}\partial_1 \cong G_2/\operatorname{im}\partial_2 \cong Q$*.*

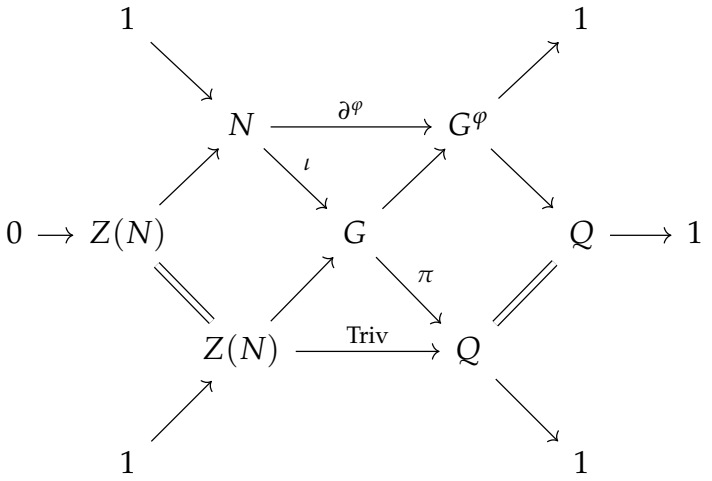
Then (C_1, G_1, ∂_1) *and* (C_2, G_2, ∂_2) *are equivalent if and only if there exist a group* G *and homomorphisms* $\iota_i\colon C_i \to G$ *and* $\pi_i\colon G \to G_i$ *for* $i = 1, 2$ *such that the diagram 2.17 commutes and additionally:*

1. *the diagonal sequences are exact,*
2. $\iota_1(C_1) \cap \iota_2(C_2) = \iota_1(Z) = \iota_2(Z)$*,*
3. *the conjugation in* G *corresponds to the given actions* $G_i \curvearrowright C_i$*.*



(2.17)

In the case the of the crossed module associated to an outer action $\varphi\colon Q \to \mathrm{Out}(N)$ which is induced by a genuine N-by-Q extension $\left(1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1\right)$ we get Diagram 2.18, which is a special case of Diagram 2.17. This is part of the proof of Theorem 2.88.



(2.18)

Theorem 2.88. *Let N, Q be groups and $\varphi\colon Q \to \mathrm{Out}(N)$ be an outer action. Then φ is extendable if and only if the crossed module $(N, G^\varphi, \partial^\varphi)$ is equivalent to the crossed module $\mathcal{M}_0$, i.e. it corresponds to the zero element of $H^3(Q; Z(N))$.*

For the details we refer the reader to the excellent article [Hue23] of Huebschmann, his older paper [Hue80], and to a paper [Hol79] of Holt.

2.3.6 Example: central extensions of $\mathbb{Z}$ by Frob_{11}

In this section we show how the classification of extensions with an abelian kernel is done in practice. This explains some of the details of the cohomological proof of the classical Baumslag's examples $(\mathbb{Z}/11) \rtimes \mathbb{Z}$ done in Section 1.4.

The choice of the quotient $Q = \mathrm{Frob}_{11}$ is not an arbitrary one – thanks to the fact that $\mathrm{Out}(\mathrm{Frob}_{11}) = 1$ the action of $\mathrm{Out}(M) \times \mathrm{Aut}(Q) \curvearrowright H^2(Q; M)$ is particularly easy to compute.

Proposition 2.89. *Let p be a prime and $\mathrm{Frob}_p = \mathbb{F}_p \rtimes \mathbb{F}_p^\times$ be the Frobenius group acting on p points. Then $\mathrm{Out}(\mathrm{Frob}_p) = 1$.*

Proof. Since $\mathbb{F}_p^\times$ is of size coprime to p, the set of elements of Frob_p whose p-th power is trivial is precisely the subgroup $\mathbb{F}_p \cong \mathbb{Z}/p$, which thus must be fixed by every automorphism. Thus, for any $\beta \in \mathrm{Aut}(\mathrm{Frob}_p)$ we get a similarity of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{F}_p & \longrightarrow & \mathrm{Frob}_p & \longrightarrow & \mathbb{F}_p^\times & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 1 & \longrightarrow & \mathbb{F}_p & \longrightarrow & \mathrm{Frob}_p & \longrightarrow & \mathbb{F}_p^\times & \longrightarrow & 1 \end{array} .$$

The action $\mathbb{F}_p^\times \curvearrowright \mathbb{F}_p$ gives $\varphi\colon \mathbb{F}_p^\times \to \mathrm{Out}(\mathbb{F}_p)$ which is actually an isomorphism. By Proposition 2.75 we get $\varphi = \mathrm{Ad}_{\overline{\alpha}} \circ \varphi = \varphi \circ \gamma$, but since $\varphi\colon \mathbb{F}_p^\times \to \mathrm{Out}(\mathbb{F}_p) = \mathbb{F}_p^\times$ is an isomorphism, this means that $\gamma = \mathrm{Id}$.

Now, let $\sigma \in \mathbb{F}_p^\times$ be a (multiplicative) generator and let's assume that

$$(0, \sigma) \overset{\beta}{\mapsto} (n_0, \sigma), \quad (1, 1) \overset{\beta}{\mapsto} (n_1, 1).$$

Since $(1, 1)$ is of order p we have $n_1 \neq 0$ in $\mathbb{F}_p$; similarly, $\sigma - 1 \neq 0$, so we can set $a := (\sigma - 1)^{-1} n_1^{-1} n_0$. We will show that $\mathrm{Ad}_{(a, n_1^{-1})} \circ \beta = \mathrm{Id}$. Indeed

$$\begin{aligned} (a, n_1^{-1}) \cdot (n_1, 1) &= (a + 1, n_1^{-1}) \\ (1, 1) \cdot (a, n_1^{-1}) &= (1 + a, n_1^{-1}) \\ (a, n_1^{-1}) \cdot (n_0, \sigma) &= (a + n_1^{-1} n_0, n_1^{-1} \sigma) \\ (0, \sigma) \cdot (a, n_1^{-1}) &= (\sigma \cdot a, \sigma n_1^{-1}), \end{aligned}$$

and we chose a to satisfy $(\sigma - 1)a = n_1^{-1} n_0$. Thus $\mathrm{Ad}_{(a, n_1^{-1})} \circ \beta$ is identity on the two elements $(1,1)$ and $(0,\sigma)$, but $\langle (1,1), (0,\sigma) \rangle = \mathrm{Frob}_p$. □

Remark 2.90. For a group G, the semidirect product $G \rtimes \mathrm{Aut}(G)$, known as the *holomorph* of G and denoted $\mathrm{Hol}(G)$, was already observed in 1903 (see [Mil03]) to be a *complete* group for $G = \mathbb{Z}/n$ where n is odd. This means that it has trivial centre and no non-trivial outer automorphisms.

Now we are ready to compute all the extensions up to isomorphism.

Proposition 2.91. *Let $Q = \mathrm{Frob}_{11}$ be the Frobenius group $\mathbb{F}_{11} \rtimes \mathbb{F}_{11}^\times$, described as $Q = \langle x, y \mid x^{11} = y^{10} = 1,\ yxy^{-1} = x^2 \rangle$. The equivalence classes $\mathscr{E}_{\mathrm{cent}}(Q; \mathbb{Z})$ of central extensions of $\mathbb{Z}$ by Q are*

$$\mathcal{E}_n := \left(0 \to \langle z \rangle \xrightarrow{\iota} G_n \xrightarrow{\pi} Q \to 1\right)$$

where $n = 0, 1, \ldots, 9$ and

$$G_n := \langle x, y, z \mid x^{11} = 1, y^{10} = z^n, yxy^{-1} = x^2, [x,z] = [y,z] = 1 \rangle.$$

Furthermore, the family $\mathscr{E}_{\mathrm{cent}}(Q; \mathbb{Z})$ is kernel-detecting and the extension classes $\mathcal{E}_n$ and $\mathcal{E}_m$ are similar if and only if $n \equiv \pm m$ modulo 10.

Proof. Since $H^1(\mathbb{Z}/11; \mathbb{Z}) \cong 0$, the term $H^1(\langle x \rangle; \mathbb{Z})^{\langle y \rangle}$ is trivial in the Inflation-Restriction Sequence in Diagram (2.19). Hence the inflation map $\mathrm{Inf}\colon H^2(\langle y \rangle; \mathbb{Z}) \to H^2(\mathrm{Frob}_{11}; \mathbb{Z})$ is injective and thus an isomorphism, since it is a map between two groups isomorphic to $(\mathbb{Z}/10)$, as we computed in Proposition 2.42. Thus, all of the 2-cohomology classes of Q are inflated from $\langle y \rangle$.

$$0 \longrightarrow H^1(\langle y \rangle; \mathbb{Z}) \longrightarrow H^1(\mathrm{Frob}_{11}; \mathbb{Z}) \longrightarrow H^1(\langle x \rangle; \mathbb{Z})^{\langle y \rangle} \xrightarrow{\mathrm{Tr}} \underbrace{H^2(\langle y \rangle; \mathbb{Z})}_{\cong(\mathbb{Z}/10)} \xrightarrow{\mathrm{Inf}} \underbrace{H^2(\mathrm{Frob}_{11}; \mathbb{Z})}_{\cong(\mathbb{Z}/10)} \tag{2.19}$$

The equivalence classes of central $\mathbb{Z}$-by-$(\mathbb{Z}/10)$ extensions are

$$\mathcal{D}_n := \left(0 \to \langle z \rangle \xrightarrow{\iota} Z_n \xrightarrow{\pi} \langle y \rangle \to 1\right)$$

where $n = 0, 1, \ldots, 9$ and $Z_n = \langle y, z \mid [y, z] = 1, y^{10} = z^n \rangle$. Since $\mathcal{D}_n$ fit in a pull-back diagram of extensions with $\mathcal{E}_n$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \langle z \rangle & \longrightarrow & Z_n & \longrightarrow & \langle y \rangle & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \langle z \rangle & \longrightarrow & G_n & \longrightarrow & Q & \longrightarrow & 1 \end{array} \tag{2.20}$$

by Proposition 2.45 we get that $\mathcal{E}_n$ corresponds to the cohomology class being the inflation of the cohomology class corresponding to $\mathcal{D}_n$ and thus $\{\mathcal{E}_n \mid n = 0, 1, \ldots, 9\}$ gives a complete set of representatives of the equivalence classes of central $\mathbb{Z}$-by-Frob_{11} extensions.

The Frobenius group has trivial centre, so $\mathscr{E}_{\mathrm{cent}}(Q; \mathbb{Z})$ is indeed kernel-detecting.

Finally, the similarity classes of central $\mathbb{Z}$-by-Q extensions correspond to the orbits of the action $\mathrm{Out}(\mathbb{Z}) \times \mathrm{Aut}(Q) \curvearrowright H^2(Q; \mathbb{Z})$, where the action of $\mathrm{Inn}(Q)$ is trivial. But $\mathrm{Out}(\mathrm{Frob}_{11}) = 1$, so all automorphisms of Frob_{11} are inner, and thus we are left with the action of $\mathrm{Out}(\mathbb{Z}) = \{\pm 1\}$. This gives the condition that $\mathcal{E}_n$ and $\mathcal{E}_m$ are similar precisely when $n \equiv \pm m$ modulo 10. □

We need one minor result more to relate it to the Baumslag's examples.

Proposition 2.92. *In the notation of Proposition 2.91,* $G_1 \cong (\mathbb{Z}/11) \rtimes_{.2} \mathbb{Z}$ *and* $G_3 \cong (\mathbb{Z}/11) \rtimes_{.2^3} \mathbb{Z}$.

Proof. The group G_1 has presentation

$$G_1 = \langle x, y, z \mid x^{11} = 1, y^{10} = z, yxy^{-1} = x^2, [x, z] = [y, z] = 1 \rangle$$

and it is quite obvious that $\langle y, z \rangle = \langle y \rangle \cong \mathbb{Z}$ is a complement to the normal subgroup $\langle x \rangle \cong (\mathbb{Z}/11)$, and that the action of y on $\langle x \rangle$ is precisely by $x \mapsto x^2$, so $G_1 \cong \langle x \rangle \rtimes_{.2} \langle y \rangle$.

On the other hand G_3 has presentation

$$G_3 = \langle x, y, z \mid x^{11} = 1, y^{10} = z^3, yxy^{-1} = x^2, [x, z] = [y, z] = 1 \rangle.$$

The subgroup $\langle y, z\rangle$ is actually cyclic, generated by y^3z^{-1}, since

$$z = z^{-9} \cdot z^{10} = y^{-30} \cdot z^{10} = (y^3z^{-1})^{-10}$$
$$y = y^{10} \cdot y^{-9} = z^3 \cdot y^{-3\cdot 3} = (y^3z^{-1})^{-3},$$

and y^3z^{-1} acts on $\langle x\rangle \cong (\mathbb{Z}/11)$ by $x \mapsto x^{2^3}$, so $G_3 \cong \langle x\rangle \rtimes_{\cdot 2^3} \langle y^3z^{-1}\rangle$. □

2.3.7 Example: a non-extendible outer action

The author learnt of this example from Derek Holt who gave it in a reply to a Math Stack Exchange post [Hol22]. The verification and remarks are original though.

Proposition 2.93. *Let $\mathbb{F}_9$ be the field with 9 elements, $N = \mathrm{SL}_2(\mathbb{F}_9)$ and $Q = (\mathbb{Z}/2)$. Then*

$$\begin{aligned} Z(N) &\cong (\mathbb{Z}/2), \\ \mathrm{Inn}(N) &\cong \mathrm{PSL}_2(\mathbb{F}_9) \cong \mathrm{Alt}_6, \\ \mathrm{Aut}(N) &\cong \mathrm{P\Gamma L}_2(\mathbb{F}_9) = \mathrm{PGL}_2(\mathbb{F}_9) \rtimes \mathrm{Gal}(\mathbb{F}_9 / \mathbb{F}_3) \cong \mathrm{PGL}_2(\mathbb{F}_9) \rtimes (\mathbb{Z}/2), \\ \mathrm{Out}(N) &\cong (\mathbb{Z}/2) \times (\mathbb{Z}/2) \end{aligned}$$

and there are four $\mathrm{Out}(N) \times \mathrm{Aut}(Q)$-orbits of homomorphisms $\varphi\colon Q \to \mathrm{Out}(N)$. Three of these are induced by extensions, but one of them isn't.

Proof. First, let's note that $\mathbb{F}_9 = \mathbb{F}_3[i]$ where $i^2 = -1$. Additionally, it will be useful to also establish that $\mathbb{F}_{81} = \mathbb{F}_3[\varepsilon]$ where $\varepsilon^2 = 1 + i$—then ε is a root of $X^4 + X^2 - 1$, which is irreducible over $\mathbb{F}_3$. The Galois action of $\mathrm{Gal}(\mathbb{F}_9 / \mathbb{F}_3)$ on $\mathbb{F}_9$ maps $i \mapsto \pm i$, let's denote by σ the non-trivial automorphism. Note that this is the Frobenius automorphism. Also, denote the non-trivial element of Q by q.

Second, we want to find an automorphism generating the first factor of $\mathrm{Out}(N) = (\mathbb{Z}/2) \times \mathrm{Gal}(\mathbb{F}_9 / \mathbb{F}_3)$. For the matrix $D \in \mathrm{GL}_2(\mathbb{F}_{81})$ shown in Equation (2.21) the automorphism $\mathrm{Ad}_D|_N \in \mathrm{Aut}(N)$ is not inner.

$$D := \begin{pmatrix} 0 & \varepsilon^5 \\ \varepsilon^{-5} & 0 \end{pmatrix} \tag{2.21}$$

This is because any matrix commuting with matrices $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ is a scalar matrix zI, and $z \cdot D \in \mathrm{SL}_2(\mathbb{F}_9)$ for $z \in \mathbb{F}_{81}^{\times}$ implies $z = \pm i = \pm\varepsilon^4$, but $\pm\varepsilon^4 D = \pm\left(\begin{smallmatrix}0&\varepsilon^9\\\varepsilon^{-1}&0\end{smallmatrix}\right) \notin \mathrm{SL}_2(\mathbb{F}_9)$ as $\varepsilon \notin \mathbb{F}_9$. Thus $\mathrm{Ad}_D|_N \notin \mathrm{Inn}(N)$. Also, $(\mathrm{Ad}_D|_N) \circ \sigma \notin \mathrm{Inn}(N)$, since it changes the trace of some matrices in $\mathrm{SL}_2(\mathbb{F}_9)$.

Third, notice that the outer actions $\varphi_\sigma, \varphi_D \colon Q \to \mathrm{Out}(N)$ sending q to σ and $(\mathrm{Ad}_D|_N)$ respectively lift to actual actions $\tilde{\varphi}_\sigma, \tilde{\varphi}_D \colon Q \to \mathrm{Aut}(N)$ since $\sigma^2 = \mathrm{Id}$ and $D^2 = I$. This means that φ_σ and φ_D are induced by semidirect products. The one corresponding to $\tilde{\varphi}_\sigma$ is $\Sigma\mathrm{L}_2(\mathbb{F}_9) := \mathrm{SL}_2(\mathbb{F}_9) \rtimes \mathrm{Gal}(\mathbb{F}_9 \,/\, \mathbb{F}_3)$ with $\mathrm{P}\Sigma\mathrm{L}_2(\mathbb{F}_9) := \mathrm{PSL}_2(\mathbb{F}_9) \rtimes \mathrm{Gal}(\mathbb{F}_9 \,/\, \mathbb{F}_3) \cong \mathrm{Sym}_6$ – see Section 3.3.7 p.49 of [Wil09].

Fourth, we will show that the outer action $\varphi_{D\sigma} \colon Q \to \mathrm{Out}(N)$ sending the generator to $(\mathrm{Ad}_D|_N) \circ \sigma$ **does not** arise from any extension. By contradiction assume that it did, and that $1 \to N \to G \to Q \to 1$ were such an extension. Then we can choose $g \in G$ such that $\mathrm{Ad}_g|_N = (\mathrm{Ad}_D|_N) \circ \sigma$. Thus $g^2 := C \in \mathrm{SL}_2(\mathbb{F}_9)$ and $\mathrm{Ad}_C = \mathrm{Ad}_{g^2} = \mathrm{Ad}_{D\sigma(D)}$, where $\sigma(D)$ is defined through the Frobenius automorphism extended to $\mathrm{GL}_2(\mathbb{F}_{81})$. C induces the same inner automorphism as

$$D\sigma(D) = \underbrace{\begin{pmatrix}0&\varepsilon^5\\\varepsilon^{-5}&0\end{pmatrix}}_{D} \cdot \underbrace{\begin{pmatrix}0&\varepsilon^{15}\\\varepsilon^{-15}&0\end{pmatrix}}_{\sigma(D)} = \begin{pmatrix}\varepsilon^{-10}&0\\0&\varepsilon^{10}\end{pmatrix} = \begin{pmatrix}1-i&0\\0&-1-i\end{pmatrix}.$$

Hence $g^2 = C = \pm D\sigma(D)$. On the other hand $\mathrm{Ad}_g(g^2) = g^2 = C$, so $D\sigma(D)$ should be invariant under $\mathrm{Ad}_g|_N = (\mathrm{Ad}_D|_N) \circ \sigma$. However

$$\begin{aligned}\big((\mathrm{Ad}_D|_N) \circ \sigma\big)\big(D\sigma(D)\big) &= \begin{pmatrix}0&\varepsilon^5\\\varepsilon^{-5}&0\end{pmatrix} \cdot \underbrace{\begin{pmatrix}1+i&0\\0&-1+i\end{pmatrix}}_{\sigma(D\sigma(D))} \cdot \begin{pmatrix}0&\varepsilon^5\\\varepsilon^{-5}&0\end{pmatrix} \\ &= \begin{pmatrix}-1+i&0\\0&1+i\end{pmatrix} = -D\sigma(D),\end{aligned}$$

so neither $D\sigma(D)$ nor $-D\sigma(D)$ is invariant under $\mathrm{Ad}_C = \mathrm{Ad}_{g^2}$. Hence we get a contradiction: there is no extension inducing $\varphi_{D\sigma} \colon Q \to \mathrm{Out}(N)$. $\square$

Remark 2.94. The proof that $\varphi_{D\sigma}$ is not extendible could have been done with a direct computation of the corresponding cohomology class in $H^3(Q; Z(N))$ following [EM47]. We briefly describe this approach below.

We can choose a (set-theoretical) lift $\zeta_1 \colon Q \to \mathrm{Aut}(N)$ of the outer action $\varphi_{D\sigma}$ by sending the generator to $(\mathrm{Ad}_D \,|_N) \circ \sigma$ and 1 to Id_N. This is *not* a homomorphism, as $((\mathrm{Ad}_D \,|_N) \circ \sigma)^2 = \mathrm{Ad}_{D\sigma(D)} \neq \mathrm{Id}_N$.

The map $(\partial\zeta_1) \colon Q \times Q \to \mathrm{Aut}(N)$ is given by

$$(\partial\zeta_1)(q_1, q_2) = \zeta_1(q_1)\zeta_1(q_2)\zeta_1(q_1q_2)^{-1} = \begin{cases} \mathrm{Ad}_{D\sigma(D)} & \text{for } q_1 = q_2 = q \\ \mathrm{Id}_N & \text{otherwise.} \end{cases}$$

and we can lift it to map $\zeta_2 \colon Q \times Q \to N$ in a similar manner:

$$\zeta_2(q_1, q_2) = \begin{cases} D\sigma(D) & \text{for } q_1 = q_2 = q, \\ I & \text{otherwise.} \end{cases}$$

The map $\zeta_3 = (\partial\zeta_2) \colon Q \times Q \times Q \to Z(N)$ computes the 3-cocycle as

$$\begin{aligned} (\partial\zeta_2)(q_1, q_2, q_3) &= \zeta_1(q_1)\big(\zeta_2(q_2, q_3)\big) \cdot \zeta_2(q_1, q_2q_3) \cdot \zeta_2(q_1q_2, q_3)^{-1} \cdot \zeta_2(q_1, q_2)^{-1} \\ &= \begin{cases} -I & \text{for } q_1 = q_2 = q_3 = q, \\ I & \text{otherwise.} \end{cases} \end{aligned}$$

It can be shown that this is not a 3-coboundary in $H^3(Q; Z(N))$.

Using Theorem 2.82 we will give a full classification of equivalence classes of $\mathrm{SL}_2(\mathbb{F}_9)$-by-$(\mathbb{Z}/2)$ extensions in Proposition 2.96. Before that though, we need Lemma 2.95 which will make the calculations easier.

Lemma 2.95. *Let $\mathcal{E} = \big(1 \to N \to G \to Q \to 1\big)$ be a split N-by-Q extension with $G = N \rtimes_{\tilde{\varphi}} Q$ and $\gamma \colon Q' \to Q$ be any map. Then in the pullback extension diagram (2.22) we can identify the pullback $G \times_Q Q'$ with the semidirect product $N \rtimes_{\tilde{\varphi}\circ\gamma} Q'$.*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \xrightarrow{\iota'} & G \times_Q Q' & \xrightarrow{\pi'} & Q' & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 1 & \longrightarrow & N & \xrightarrow{\iota} & G & \xrightarrow{\pi} & Q & \longrightarrow & 1 \end{array} \tag{2.22}$$

Proof. The semidirect product $N \rtimes_{\tilde{\varphi}\circ\gamma} Q'$ maps to $N \rtimes_{\tilde{\varphi}} Q = G$ and Q' satisfying a commutative pullback square, so we get a map $\phi\colon N \rtimes_{\tilde{\varphi}\circ\gamma} Q' \to G \times_Q Q'$. The kernels of maps $N \rtimes_{\tilde{\varphi}\circ\gamma} Q' \to N \rtimes_{\tilde{\varphi}} Q$ and $N \rtimes_{\tilde{\varphi}\circ\gamma} Q' \to Q'$ intersect trivially, so ϕ is injective. Also, for any $(n, q, q') \in (N \rtimes Q) \times_Q Q'$ we have $q = \gamma(q')$, so (n, q, q') is the image of $(n, q') \in N \rtimes Q'$ under ϕ. Hence the identification. □

With that we are ready to classify all equivalence classes.

φ	G^φ	extension eq. classes	`SmallGroupId`
φ_0	$\mathrm{PSL}_2(\mathbb{F}_9)$	$\mathrm{SL}_2(\mathbb{F}_9) \times (\mathbb{Z}/2)$	$(1440, 4698)$
		$\mathrm{SL}_2^{\pm}(\mathbb{F}_9)$	$(1440, 4597)$
φ_D	$\mathrm{PGL}_2(\mathbb{F}_9)$	$\langle \mathrm{SL}_2(\mathbb{F}_9), D\rangle < \mathrm{SL}_2^{\pm}(\mathbb{F}_{81})$	$(1440, 4593)$
		$\langle \mathrm{SL}_2(\mathbb{F}_9), iD\rangle < \mathrm{SL}_2(\mathbb{F}_{81})$	$(1440, 4594)$
φ_σ	$\mathrm{P\Sigma L}_2(\mathbb{F}_9)$ $\cong \mathrm{Sym}_6$	$\langle \mathrm{SL}_2(\mathbb{F}_9), (I, \sigma)\rangle = \Sigma\mathrm{L}_2(\mathbb{F}_9)$	$(1440, 4591)$
		$\langle \mathrm{SL}_2(\mathbb{F}_9), ((1+i)I, \sigma)\rangle < \Gamma\mathrm{L}_2(\mathbb{F}_9)$	
$\varphi_{D\sigma}$	M_{10}	—	—

Table 2.1: Equivalence classes of $\mathrm{SL}_2(\mathbb{F}_9)$-by-$(\mathbb{Z}/2)$ extensions.

Proposition 2.96. *The equivalence classes of* $\mathrm{SL}_2(\mathbb{F}_9)$*-by-*$(\mathbb{Z}/2)$ *extensions are summarised in Table 2.1; we list the groups* G *as matrix groups containing* $\mathrm{SL}_2(\mathbb{F}_9)$ *as an index* 2 *subgroup, which is assumed to be the image of* $\iota\colon N \to G$*.*

Proof. First, note that the extensions presented in the proof of Proposition 2.93 all are semidirect products: they are $\mathrm{SL}_2(\mathbb{F}_9) \times (\mathbb{Z}/2)$, $\mathrm{SL}_2(\mathbb{F}_9) \rtimes \langle D\rangle$ and $\mathrm{SL}_2(\mathbb{F}_9) \rtimes \langle\sigma\rangle$.

The non-trivial element of $H^2(\mathbb{Z}/2; \mathbb{Z}/2)$ corresponds to the extension

$$\mathcal{D} = \Big(\ 0 \longrightarrow \underbrace{\mathbb{Z}/2}_{Z(N)} \longrightarrow \underbrace{\mathbb{Z}/4}_{E} \xrightarrow{\gamma} \underbrace{\mathbb{Z}/2}_{Q} \longrightarrow 1\ \Big). \tag{2.23}$$

By Theorem 2.82 the second extension class in each case comes from the quotient of $G \times_Q E$ by a 'diagonal' image of $Z(N)$. By Lemma 2.95 we can identify the pullback $G \times_Q E$ with $N \rtimes E$; what is left to find surjections from $N \rtimes E$ to the proposed second extension whose kernel is precisely

the image $\langle(-I,\tau^2)\rangle$ of $Z(N)$, where we denote a generator of E by τ.

$$\begin{aligned}
\beta_0\colon\ &\mathrm{SL}_2(\mathbb{F}_9)\times\langle\tau\rangle \to \mathrm{SL}_2^{\pm}(\mathbb{F}_9) && \text{maps } (X,\tau^j)\mapsto X\cdot(iI)^j\\
\beta_D\colon\ &\mathrm{SL}_2(\mathbb{F}_9)\rtimes_D\langle\tau\rangle \to \mathrm{SL}_2(\mathbb{F}_{81}) && \text{maps } (X,\tau^j)\mapsto X\cdot(iD)^j\\
\beta_\sigma\colon\ &\mathrm{SL}_2(\mathbb{F}_9)\rtimes_\sigma\langle\tau\rangle \to \underbrace{\Gamma\mathrm{L}_2(\mathbb{F}_9)}_{\mathrm{GL}_2(\mathbb{F}_9)\rtimes\langle\sigma\rangle} && \text{maps } (X,\tau^j)\mapsto (X,\mathrm{Id})\cdot\big((1+i)I,\ \sigma\big)^j
\end{aligned}$$

It is clear that *if* β_0, β_D and β_σ are homomorphisms, then their images are respectively $\mathrm{SL}_2^{\pm}(\mathbb{F}_9)$, $\langle\mathrm{SL}_2(\mathbb{F}_9), iD\rangle$ and $\langle\mathrm{SL}_2(\mathbb{F}_9), ((1+i)I,\sigma)\rangle$ as claimed in Table 2.1. Therefore let's prove that they are homomorphisms.

The case of β_0 is clear as the scalar matrix iI commutes with all elements of $\mathrm{SL}_2(\mathbb{F}_9)$. The non-trivial element of $\ker\beta_0$ is $(-I,\tau^2)$, as required.

In the case of β_D, we have $\mathrm{Ad}_{iD}=\mathrm{Ad}_D$, so

$$X_1\cdot(iD)^{j_1}\cdot X_2\cdot(iD)^{j_2} = X_1\cdot D^{j_1}X_2D^{-j_1}\cdot(iD)^{j_1+j_2};$$

since $(iD)^j$ has entries in $\mathbb{F}_9$ if and only if $j=0$ or 2, we get that the non-trivial element of $\ker\beta_D$ is $(-I,\tau^2)$, too.

Similarly in the case of β_σ, we have $\mathrm{Ad}_{((1+i)I,\sigma)}=\mathrm{Ad}_{(I,\sigma)}$, so

$$(X_1,\mathrm{Id})\cdot\big((1+i)I,\sigma\big)^{j_1}\cdot(X_2,\mathrm{Id})\cdot\big((1+i)I,\sigma\big)^{j_2} = (X_1\sigma^{j_1}(X_2),\mathrm{Id})\cdot\big((1+i)I,\sigma\big)^{j_1+j_2}.$$

Finally, $\ker\beta_\sigma$ again has just one non-trivial element, $(-I,\tau^2)$. □

Remark 2.97. We can verify the findings with a query of The L-Functions and Modular Forms Database [LMF24], which allows dynamic search of databases of finite groups. The query `https://beta.lmfdb.org/Groups/Abstract/?subgroup=720.409&normal=yes&quotient=2.1&search_type=Subgroups` searches for *all* isomorphism types of groups which are $\mathrm{SL}_2(\mathbb{F}_9)$-by-$(\mathbb{Z}/2)$ extensions and it gives a curious result: there are only 5 isomorphism classes of such groups. Some further work with the extensions in Table 2.1 allows identification of the `SmallGroupId` of each – the two equivalence classes inducing φ_σ are actually isomorphic! Since $\mathrm{SL}_2(\mathbb{F}_9)$ has trivial abelianisation, the isomorphism comes from a similarity of extensions.

We can construct this similarity directly! Both of the extension classes for φ_σ are in fact semidirect products:

$$G_1 = \langle \mathrm{SL}_2(\mathbb{F}_9), (I, \sigma) \rangle = \mathrm{SL}_2(\mathbb{F}_9) \rtimes \langle \sigma \rangle,$$
$$G_2 = \langle \mathrm{SL}_2(\mathbb{F}_9), \big((1+i)I,\ \sigma\big) \rangle = \mathrm{SL}_2(\mathbb{F}_9) \rtimes \langle (B, \sigma) \rangle$$

where we use (B, σ) defined below instead of $\big((1+i)I,\ \sigma\big)$ for the splitting as $\big((1+i)I,\ \sigma\big)^2 = (-I, \mathrm{Id})$, but $(B, \sigma)^2 = (I, \mathrm{Id})$ for

$$B := \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1+i & 0 \\ 0 & -1+i \end{pmatrix}}_{\in \mathrm{SL}_2(\mathbb{F}_9)} \cdot \underbrace{\begin{pmatrix} 1+i & 0 \\ 0 & 1+i \end{pmatrix}}_{(1+i)I}.$$

To build an isomorphism $\beta\colon G_1 \to G_2$ with Proposition 2.79 we need an automorphism $\alpha \in \mathrm{Aut}(N)$ such that $\alpha \circ \sigma \circ \alpha^{-1} = \mathrm{Ad}_B \circ \sigma$. For matrix $A := \left(\begin{smallmatrix} -1+i & 0 \\ 0 & 1 \end{smallmatrix}\right)$ we get $A^{-1} := \left(\begin{smallmatrix} 1+i & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\sigma(A^{-1}) := \left(\begin{smallmatrix} 1-i & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so that

$$A\sigma(A^{-1}) = \begin{pmatrix} -1+i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1-i & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix} = B'$$

so $\alpha = \mathrm{Ad}_A$ satisfies $\alpha \circ \sigma \circ \alpha^{-1} = \mathrm{Ad}_B \circ \sigma$..

This shows that $\beta\colon (X, \sigma^j) \mapsto (\alpha(X), \mathrm{Id}) \cdot (B', \sigma)^j$ is an isomorphism.

Remark 2.98. The direct proof given here that one of the outer actions in Proposition 2.93 is not extendible takes its main idea from a paper of Kovács [Kov03]. Therein another such example is given.

$$N = \langle a, b, c \mid a^{16} = b^2 = c^2 = 1,\ bab^{-1} = a^{-1},\ cac^{-1} = a^9,\ [b, c] = 1 \rangle$$
$$\cong \langle a \rangle \rtimes \langle b, c \rangle \text{ where } \langle a \rangle \cong \mathbb{Z}/16,\ \langle b, c \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$$

We can compute the centre of N with Lemma 6.1 as

$$Z(N) = \langle a^8 \rangle \cong \mathbb{Z}/2.$$

Let $\alpha \in \mathrm{Aut}(N)$ be the automorphism defined by

$$\alpha(a) = a^3,\ \alpha(b) = b,\ \alpha(c) = a^8 c.$$

Then $\alpha^2 = \mathrm{Ad}_c$ and the elements of N inducing α^2 are precisely c and $a^8 c$. On the other hand $\alpha(c) = a^8 c$ and $\alpha(a^8 c) = c$, so for $Q = \mathbb{Z}/2$ there is no extension $1 \to N \to G \to Q \to 1$ which induces the outer action $\varphi\colon Q \to \mathrm{Out}(N)$ which sends the generator of Q to $\overline{\alpha} \in \mathrm{Out}(N)$.

Chapter 3

Lack of profinite rigidity among 'by-free' extensions

The purpose of this chapter is to showcase one extreme example of lack of profinite rigidity arising from profinitely conjugate outer actions. Namely, for N being either a free group, a surface group, or a free abelian group of sufficient complexity, we produce infinite families of split extensions $N \rtimes F_2$ which aren't isomorphic, but which are profinitely isomorphic.

The main 'ingredient' is the fact that finitely generated profinite groups have exactly one Nielsen class of generating sets, which allows us to easily construct profinitely isomorphic families. The more difficult part is showing that the groups in question are pairwise non-isomorphic.

3.1 Kernel-detecting families of split extensions with free quotient

In proving Theorem A we will need to show that the class of split extensions we consider $\big(1 \to N \to N \rtimes F_2 \to F_2 \to 1\big)$ is kernel-detecting.

In some situations (such as when $N \cong \mathbb{Z}^n$) a simple criterion of Proposition 3.1 (being a variation on Proposition 2.68) inspired by Remark 3.17 of [GZ11] is enough. However, since we want to treat the cases of N being free or a surface group, a more complicated criterion of Theorem 3.2 will be necessary.

Proposition 3.1. *Let N be a group having no non-abelian free quotients and $m \geq 2$. Then the subgroup $N = N \rtimes 1$ of $G = N \rtimes_\varphi F_m$ is the unique normal subgroup of G such that $G/N \cong F_m$.*

Proof. Let $M \triangleleft G$ be another such normal subgroup. Then $\overline{N} = NM/M$ is a normal subgroup of F_m, so it is either trivial, or non-abelian free.

Since N has no non-abelian free quotients, $\overline{N} = 1$ and so $N < M$, but this gives a surjection

$$F_m \cong G/N \twoheadrightarrow G/M \cong F_m$$

which must be an isomorphism by the Hopf property of finitely generated free groups. □

Theorem 3.2. *Let N be a group in which the centralisers of non-trivial normal subgroups don't contain a non-abelian free group.*

Let $m \geq 2$, let F_m be a finitely generated free group of rank m and let T be a subgroup of $\mathrm{Aut}(N)$ which isn't free. Suppose that $\varphi : F_m \twoheadrightarrow T$ is a surjection.

Then the subgroup $N = N \rtimes 1$ of $G = N \rtimes_\varphi F_m$ is characterised as the only subgroup M of G satisfying the following conditions.

- *M is isomorphic to N,*
- *M is normal in G,*
- *$G/M \cong F_m$,*
- *$C_G(M)$ contains a non-abelian free group.*

In particular, N is characteristic in G.

Furthermore, the family of such extensions $\left(1 \to N \to N \rtimes_\varphi F_m \to F_m \to 1\right)$ is kernel-detecting, i.e. given any isomorphism $\beta\colon G_1 \to G_2$ of two such groups $G_i := N \rtimes_{\varphi_i} F_m$, we have $\beta(N) = N$ and there exist $\alpha \in \mathrm{Aut}(N)$ and $\gamma \in \mathrm{Aut}(F_m)$ such that $\mathrm{Ad}_\alpha \circ \varphi_1 = \varphi_2 \circ \gamma$.

3.1.1 Notation and lemmas

For the sake of clarity we fix some notation for the proof of Theorem 3.2.

$$\begin{aligned}
N &:= \text{a group st. for non-trivial } O \triangleleft N,\ F_2 \not\leq C_N(O),\\
G &:= N \rtimes_\varphi F_m,\\
N &:= N \rtimes 1\\
H &:= 1 \rtimes F_m\\
K &:= 1 \rtimes (\ker \varphi),\\
T &:= H/K,\\
M &:= \text{a subgroup of } G \text{ satisfying the four conditions,}\\
\overline{G} &:= G/M,\\
\overline{N}, \overline{H}, \overline{K} &:= \text{images of } N, H, K \text{ in } G \twoheadrightarrow \overline{G},\\
L &:= \ker(H \twoheadrightarrow \overline{H}) = H \cap M,\\
D &:= \langle\langle n \cdot \varphi(l)(n^{-1}) \mid l \in L, n \in N\rangle\rangle_N.
\end{aligned}$$

We will be proving that $M = N$.

Lemma 3.3. *Let $G = N \rtimes_\varphi H$ be a semidirect product and $L \triangleleft H$. Then*

$$\langle\langle L\rangle\rangle_G = \langle\langle n \cdot \varphi(l)(n^{-1}) \mid l \in L, n \in N\rangle\rangle_N \cdot L.$$

Proof. Take $(n,1) \in N$ and $(1,l) \in L$. Then

$$(n,1)(1,l)(n^{-1},1) = \left(n \cdot \varphi(l)(n^{-1}), l\right) = \left(n \cdot \varphi(l)(n^{-1}), 1\right)(1,l).$$

Thus the subgroup $D := \langle\langle n \cdot \varphi(l)(n^{-1}) \mid l \in L\rangle\rangle_N$ is contained in $\langle\langle L\rangle\rangle_G$. It is enough to show that $D \cdot L \triangleleft G$.

$D \triangleleft N$ and $(n,1)(1,l)(n,1)^{-1} \in D \cdot L$ so N normalises $D \cdot L$. Also, $L \triangleleft H$ and

$$\begin{aligned}
(1,h)\left(n \cdot \varphi(l)(n^{-1}), 1\right)(1,h)^{-1} &= \left(\varphi(h)\left(n \cdot \varphi(l)(n^{-1})\right), 1\right)\\
&= \left(\varphi(h)(n) \cdot \varphi(hl)(n^{-1}), 1\right)\\
&= \left(\varphi(h)(n) \cdot \varphi(hlh^{-1})\left(\varphi(h)(n^{-1})\right), 1\right)\\
&= \left(n' \cdot \varphi(l')(n'^{-1}), 1\right)
\end{aligned}$$

for $n' := \varphi(h)(n)$, which is an element of N and for $l' := hlh^{-1}$, which is an element of L since $L \triangleleft H$. □

Proposition 3.4. *N and K are normal subgroups of G.*

Proof. The normality of N is immediate; $\varphi(k)(n) = n$ for any $k \in K$ and $n \in N$, so by the calculation in Lemma 3.3,

$$\langle\langle K \rangle\rangle_G = \langle\langle 1 \rangle\rangle_N \cdot K = K$$

and so K is normal in G. □

3.1.2 Proof of Theorem 3.2

The proof is admittedly complicated, so we give a plan for it first.

Overview of the proof of Theorem 3.2. The main steps are as follows.

1. Notice that one of the images $\overline{N}$ and $\overline{K}$ must be trivial.
2. With the Hopf property of F_m show that $\overline{N} = 1$ implies $M = N$.
3. Show that $L = H \cap M$ contains K properly using the fact that T isn't free and thus $H \twoheadrightarrow T \twoheadrightarrow \overline{H}$ has kernel larger than $\ker(H \twoheadrightarrow T) = K$.
4. Deduce that the subgroup $D \triangleleft N$ in the formula $\langle\langle L \rangle\rangle_G = D \cdot L$ from Lemma 3.3 is non-trivial.
5. Use it to show $C_G(M) < C_N(D)$ and get a contradiction with assumptions that $F_2 \hookrightarrow C_G(M)$ and $F_2 \not\hookrightarrow C_N(D)$.

Proof of Theorem 3.2. We follow the overview indicating where steps start.

Step 1. Consider the images $\overline{N}$ and $\overline{K}$ of N and K in the quotient $\overline{G} \cong F_m$, which is non-abelian free. Since $[N, K] = 1$, then also in the quotient $[\overline{N}, \overline{K}] = 1$.

Now, $\overline{N}$ and $\overline{K}$ are normal in $\overline{G} \cong F_m$, so they are either trivial or non-abelian. However, non-abelian subgroups of free groups have trivial centralisers. Thus $\overline{N} = 1$, or $\overline{K} = 1$.

Step 2. In the first case the implication is that $N < M$, but then there would be a surjection

$$F_m \cong G/N \twoheadrightarrow G/M \cong F_m,$$

but finitely generated free groups are Hopfian, implying that $N = M$, which is what we wanted to prove.

Step 3. Otherwise $K < M$ and so $\overline{H}$ is a quotient of $H/K = T$. Let

$$L := \ker(H \twoheadrightarrow \overline{H}) = H \cap M.$$

Since $T = H/K$ isn't free, the map

$$H \twoheadrightarrow T \twoheadrightarrow \overline{H} < \overline{G} \cong F_m$$

must have a non-trivial kernel at the step $T \twoheadrightarrow \overline{H}$ and so L is properly larger than K.

Step 4. M is a normal subgroup of G and $L < M$, so M must also contain the normal closure $\langle\langle L \rangle\rangle_G$, which by Lemma 3.3 is equal to

$$\langle\langle L \rangle\rangle_G = \underbrace{\langle\langle n \cdot \varphi(l)(n^{-1}) \mid l \in L, n \in N \rangle\rangle_N}_{=:D} \cdot L.$$

Notably, since L is properly larger than $K = 1 \rtimes (\ker \varphi)$ it contains some element $l_0 \notin K$ acting non-trivially, i.e. such that there is some $n_0 \in N$ for which $\varphi(l_0)(n_0) \neq n_0$. Thus the subgroup D is non-trivial.

Step 5. Finally, since $\langle\langle L \rangle\rangle_G < M$, then also $C_G(M) < C_G\big(\langle\langle L \rangle\rangle_G\big)$, but

$$C_G\big(\langle\langle L \rangle\rangle_G\big) = C_G(DL) = C_G(D) \cap C_G(L).$$

G retracts onto H with kernel N, H is non-abelian free and $L \lhd H$ is a non-trivial normal subgroup, so $C_H(L) = 1$ and the centraliser $C_G(L)$ is contained in N. Thus

$$C_G(M) < C_G(\langle\langle L \rangle\rangle_G) = C_G(D) \cap C_G(L) = C_N(D) \cap C_N(L) < C_N(D).$$

D is a non-trivial non-abelian normal subgroup of N and $C_N(D)$ can't contain a non-abelian free group, and thus $C_G(M)$ also doesn't contain one.

This is a contradiction to the assumption that $C_G(M)$ contains an F_2, so $N = M$.

Now, given two groups $G_i = N \rtimes_{\varphi_i} F_m$ for $i = 1, 2$ an isomorphism $\beta\colon G_1 \to G_2$ must satisfy $\beta(N) = N$ since the four conditions characterising N are isomorphism-invariant. Now, taking $\alpha = \beta|_N \in \mathrm{Aut}(N)$ and $\gamma \in \mathrm{Aut}(F_m)$ induced on $G_1/N \to G_2/N$ we get $\mathrm{Ad}_\alpha \circ \varphi_1 = \varphi_2 \circ \gamma$. □

3.2 Groups with infinitely many T-systems

The previous section and Proposition 2.80 showed that the question of isomorphism of the specific groups $N \rtimes_{\varphi_1} F_m$ and $N \rtimes_{\varphi_2} F_m$ which we are constructing is equivalent to the existence of $\alpha \in \mathrm{Aut}(N)$ and $\gamma \in \mathrm{Aut}(F_m)$ such that $\mathrm{Ad}_\alpha \circ \varphi_1 = \varphi_2 \circ \gamma$. We want the groups to be pairwise non-isomorphic, so we need to show that $\mathrm{Ad}_\alpha \circ \varphi_1 \neq \varphi_2 \circ \gamma$ for all $\alpha \in \mathrm{Aut}(N)$ and $\gamma \in \mathrm{Aut}(F_m)$.

We will, in fact, prove a stronger condition. The homomorphisms φ_i all have the same image in $\mathrm{Aut}(N)$, let's call it T. We will show that for any $\delta \in \mathrm{Aut}(T)$ and $\varepsilon \in \mathrm{Aut}(F_m)$

$$\delta \circ \varphi_i = \varphi_j \circ \varepsilon \quad \iff \quad i = j.$$

We start by giving this relation a name.

Definition 3.5. Two surjective homomorphisms $\varphi_i\colon F_m \twoheadrightarrow T$ are *Nielsen equivalent* if there exists $\varepsilon \in \mathrm{Aut}(F_m)$ such that $\varphi_1 = \varphi_2 \circ \varepsilon$, and *T-equivalent* if there exist $\delta \in \mathrm{Aut}(T)$ and $\varepsilon \in \mathrm{Aut}(F_m)$ such that $\delta \circ \varphi_1 = \varphi_2 \circ \varepsilon$.

The study of Nielsen equivalence and of so-called *T-systems* has been an area of active research. Notably, though not of direct relevance to us, [Lou10] showed that all of the surjections from a finitely generated free group to a surface group $\pi_1(S)$ with $\chi(S) \leq 0$ are Nielsen equivalent.

3.2.1 Torus link groups have infinitely many T-systems

Most notably for us, there exist rather easy examples of groups which have infinitely many non-T-equivalent surjections from F_2. The examples we

give—*torus link groups*—are constructed by amalgamating two copies of $\mathbb{Z}$ along an infinite subgroup.

An alternative family we could have used is hyperbolic 2-bridge knot groups. They have infinitely many Nielsen systems of 2 generators, as shown in [HW10], but on the other hand the fundamental groups of their complements in S^3 have finite outer automorphism groups, implying that they have infinitely many non-equivalent T-systems. Moreover, they are virtually special—see Lemma 17.20 and Corollary 15.3 of [Wis21]—and techniques similar to those used here can show that they embed into $\mathrm{GL}_n(\mathbb{Z}), \mathrm{Out}(F_n)$ and $\mathrm{MCG}(S_n)$—see [Bri13]—for sufficiently large n.

Proposition 3.6. *Let $p, q \in \mathbb{N}$ be such that $p, q \geq 2$ and $p + q \geq 5$, and T be the one-relator group given by the presentation*

$$T(p,q) := \langle x, y \mid x^p = y^q \rangle. \tag{3.1}$$

Then, there are infinitely many non-T-equivalent surjective maps $F_2 \twoheadrightarrow T$.

Proof. [Zie77] proves that any m-tuple of generators of T is Nielsen equivalent to one of the form $(x^a, y^b, 1, \ldots, 1)$ where

- $\gcd(a, b) = \gcd(a, p) = \gcd(b, q) = 1$, and
- $0 < 2a \leq pb$, and $0 < 2b \leq qa$.

It also proves that any such m-tuple generates T, and that such pairs (x^a, y^b) and (x^c, y^d) are Nielsen equivalent if and only if $a = c$ and $b = d$.

Additionally, [Sch24] showed that an automorphism of T sends

$$x \mapsto t\, \sigma(x)^\varepsilon t^{-1}, \quad y \mapsto t\, \sigma(y)^\varepsilon t^{-1},$$

for $\varepsilon = \pm 1$, $t \in T$ and σ being either identity or the swap $x \leftrightarrow y$, (with the latter possible only if $a = b$). None of these maps change the Nielsen class of a pair, so this implies that T has indeed infinitely many non-T-equivalent generating pairs. □

3.2.2 Finding torus link groups in $\mathrm{Out}(N)$

In order to be able to construct our examples of infinite families of extensions N-by-F_2 which are non-isomorphic, we will need to find a torus link group $T = T(p,q)$ with $p,q \geq 2$ and $p+q \geq 5$ in the outer automorphism group $\mathrm{Out}(N)$.

In all of the cases we consider the embedding $T \hookrightarrow \mathrm{Out}(N)$ factors through the map $\mathrm{Aut}(N) \twoheadrightarrow \mathrm{Out}(N)$

Proposition 3.7. *Let $T = T(p,q)$ be the torus link group defined by Presentation (3.1). Let $\pi\colon T \to (\mathbb{Z}\,/\,\mathrm{lcm}(p,q))$ be defined by $\pi(x) = \mathrm{lcm}(p,q)/p$ and $\pi(y) = \mathrm{lcm}(p,q)/q$.*

Then $\ker\pi \cong F_k \times \mathbb{Z}$ where $k = (pq-p-q)/\gcd(p,q)+1$.

Proof. The centre of T is the subgroup $Z = \langle x^p\rangle = \langle y^q\rangle$, which lies in the kernel of π, so π factors through $G \twoheadrightarrow G/Z$, which is isomorphic to $(\mathbb{Z}\,/\,p) * (\mathbb{Z}\,/\,q)$.

All torsion elements of G/Z inject under $G/Z \twoheadrightarrow (\mathbb{Z}\,/\,\mathrm{lcm}(p,q))$, so $(\ker\pi)/Z$ is a torsion-free virtually free group and thus it must be free. We can compute its Euler characteristic

$$\begin{aligned}\chi\left((\ker\pi)/Z\right) &= \mathrm{lcm}(p,q)\cdot\chi\left((\mathbb{Z}\,/\,p)*(\mathbb{Z}\,/\,q)\right)\\ &= \mathrm{lcm}(p,q)\cdot(1/p+1/q-1)\,.\end{aligned}$$

Since $pq = \mathrm{lcm}(p,q)\cdot\gcd(p,q)$ for $k = (pq-p-q)/\gcd(p,q)+1$ we get $(\ker\pi)/Z \cong F_k$.

Finally, $\ker\pi$ is a central extension of $\mathbb{Z}$ by F_k and any such extension is isomorphic to $F_k \times \mathbb{Z}$. □

As a result, we got that $T = T(3,3)$ is a $(F_2 \times \mathbb{Z})$-by-$(\mathbb{Z}\,/3)$ extension.

For embedding T into outer automorphism groups we will need the Universal Embedding Theorem, likely proved first in [KK51].

Lemma 3.8 (Universal Embedding Theorem)**.** *Let N and Q be groups, and G be an N-by-Q extension. Then G embeds into the wreath product $N \wr Q$.*

Thus, in order to ensure an embedding $T \hookrightarrow \mathrm{Out}(N)$ it is enough to embed $T' := (F_2 \times \mathbb{Z}) \wr (\mathbb{Z}/3)$ in $\mathrm{Out}(N)$. Note that T' is

$$((F_2 \times \mathbb{Z}) \times (F_2 \times \mathbb{Z}) \times (F_2 \times \mathbb{Z})) \rtimes (\mathbb{Z}/3)$$

with $(\mathbb{Z}/3)$ permuting the factors cyclically.

Proposition 3.9. *The group $T' := (F_2 \times \mathbb{Z}) \wr (\mathbb{Z}/3)$ embeds into $\mathrm{Aut}(F_9)$ and thus also into $\mathrm{Out}(F_n)$ for any $n \geq 10$.*

Proof. We need to find three copies of $F_2 \times \mathbb{Z}$ commuting with each other and an automorphism σ which permutes them.

Let's fix a free basis $x_0, \ldots, x_8$ and define homomorphisms f_i, g_i, h_i for $i = 0, 1, 2$, and σ as follows.

- f_i fixes all basis elements except x_{3i} and sends $x_{3i} \mapsto x_{3i} \cdot x_{3i+1}$,
- g_i fixes all basis elements except x_{3i} and sends $x_{3i} \mapsto x_{3i} \cdot x_{3i+2}$,
- h_i fixes all basis elements except x_{3i} and sends $x_{3i} \mapsto x_{3i+1} \cdot x_{3i}$,
- $\sigma(x_i) := x_{i+3}$, for $i = 0, \ldots, 8$,

where the indices are taken modulo 9. Then $\langle f_i, g_i \rangle \cong F_2$, $\langle h_i \rangle \cong \mathbb{Z}$, these factors all commute and σ permutes them as needed.

Finally, for $n \geq 10$ we have $F_n \cong F_9 * F_{n-9}$ and we can extend any automorphism of F_9 by identity on the factor F_{n-9}. Since F_{n-9} is non-trivial, none of these extended automorphisms is inner, so T' indeed embeds into $\mathrm{Out}(F_n)$. □

Proposition 3.10. *For $n \geq 5$, the group $T' := (F_2 \times \mathbb{Z}) \wr (\mathbb{Z}/3)$ embeds into $\mathrm{MCG}(S_n)$—the mapping class group of the closed orientable surface S_n of genus n—which is an index 2 subgroup of $\mathrm{Out}(\pi_1(S_n))$. Furthermore, the embedding factors through $\mathrm{Aut}(\pi_1(S_n))) \twoheadrightarrow \mathrm{Out}(\pi_1(S_n))$.*

Proof. Figure 3.1 shows decompositions with a symmetry of order 3 which extends to a homeomorphism R of the surface $S := S_n$. We give simple closed curves $\alpha_i, \beta_i, \gamma_i \subset S$ for $i = 0, 1, 2$ such that

$$R\colon \quad \alpha_i \mapsto \alpha_{i+1}, \quad \beta_i \mapsto \beta_{i+1}, \quad \gamma_i \mapsto \gamma_{i+1}$$

and such that all all are disjoint apart from α_i and β_i for $i = 0, 1, 2$, which have one intersection.

This implies—according to the classification of [Ish96] of subgroups of $\mathrm{MCG}(S)$ generated by two Dehn twists—that $\langle T_{\alpha_i}^2, T_{\beta_i} \rangle \cong F_2$ and $\langle T_{\gamma_i} \rangle \cong \mathbb{Z}$, and that these all groups commute with each other, while being permuted in a cyclic manner by R.

This gives a copy of T' inside $\mathrm{MCG}(S_n)$, which is isomorphic to an index 2 subgroup of $\mathrm{Out}(\pi_1(S_n))$ by Dehn–Nielsen–Baer Theorem—see Theorem 8.1 of [FM11].

Since R and the $T_{\alpha_i}, T_{\beta_i}, T_{\gamma_i}$ for $i = 0, 1$ and 2 fix points of the surface we can set the basepoint to be one of them and factor the embedding through $\mathrm{Aut}(\pi_1(S_n)) \twoheadrightarrow \mathrm{Out}(\pi_1(S_n))$. This is also why we don't use a version of this construction with a 'hole' in the middle, which the axis of rotation would 'go through': in this case there would be no fixed point on the surface.

Accordingly, the resulting genus of the surface is $3 + 3k, 4 + 3k$ or $2 + 3k$ respectively for the top, middle and bottom surfaces in Figure 3.1, with $k \geq 1$ in all cases. This means that all genera starting from 5 can be realised in this construction, hence the assumption of $n \geq 5$ in the statement. □

Proposition 3.11. *For $n \geq 12$, the group $T' := (F_2 \times \mathbb{Z}) \wr (\mathbb{Z}/3)$ embeds into $\mathrm{GL}_n(\mathbb{Z}) \cong \mathrm{Out}(\mathbb{Z}^n)$.*

Proof. The matrices

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generate a subgroup of $\mathrm{GL}_2(\mathbb{Z})$ isomorphic to F_2. Thus, the subgroup generated by 12×12 block-diagonal matrices

$$\begin{aligned}
a_0 &= \mathrm{Diag}(A, I, I, I, I, I), \quad a_1 = \mathrm{Diag}(I, I, A, I, I, I), \quad a_2 = \mathrm{Diag}(I, I, I, I, A, I), \\
b_0 &= \mathrm{Diag}(B, I, I, I, I, I), \quad b_1 = \mathrm{Diag}(I, I, B, I, I, I), \quad b_2 = \mathrm{Diag}(I, I, I, I, B, I), \\
c_0 &= \mathrm{Diag}(I, A, I, I, I, I), \quad c_1 = \mathrm{Diag}(I, I, I, A, I, I), \quad c_2 = \mathrm{Diag}(I, I, I, I, I, A).
\end{aligned}$$

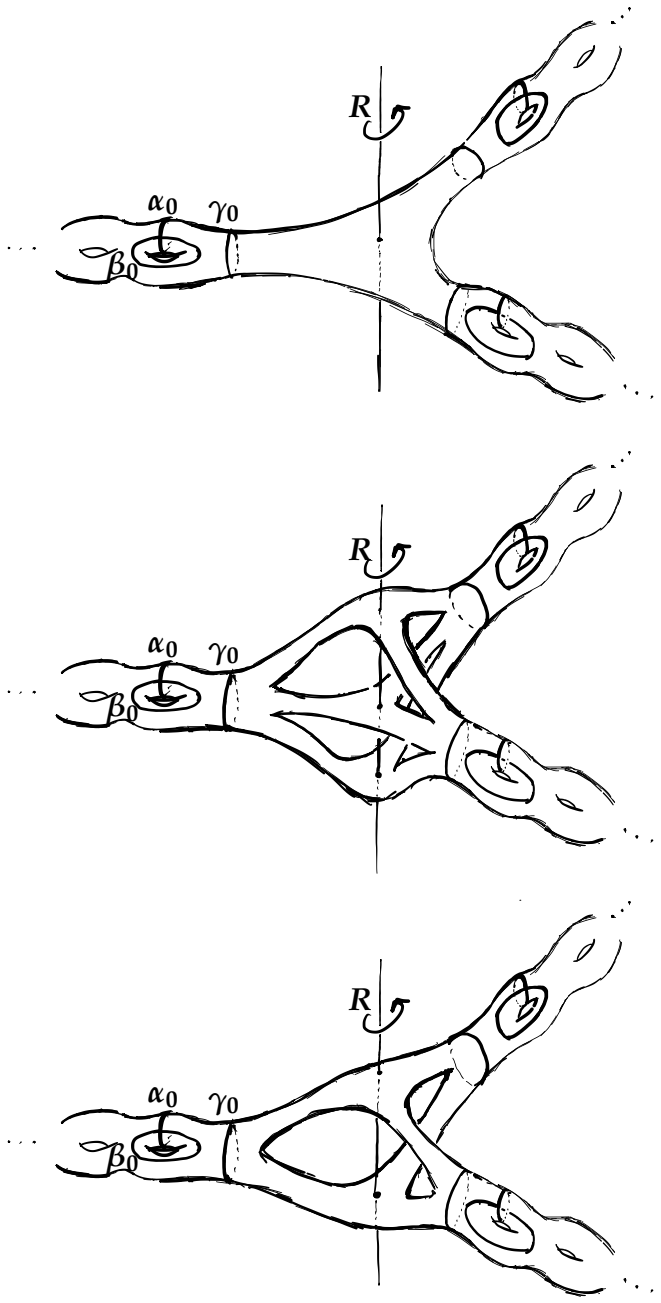


Figure 3.1: Decomposition of S_g depending on $g \mod 3$.

is actually isomorphic to $(F_2 \times \mathbb{Z}) \times (F_2 \times \mathbb{Z}) \times (F_2 \times \mathbb{Z})$ and if we additionally include the matrix σ, which permutes the blocks by 2 to the right, we get a subgroup isomorphic to T'. □

3.3 Final results

In this section we combine the results proven in previous sections to prove Theorem A – a general tool for producing families of non-isomorphic semidirect products $N \rtimes F_m$ which share profinite completions.

This, together with the constructions done in Section 3.2, allows us to produce in Theorem B three infinite families of non-isomorphic groups of type F sharing the same profinite completion. One should note that the assumptions that $n \geq 10$ for $N = F_n$, $n \geq 5$ for $N = \pi_1(S_n)$, and $n \geq 12$ for $N = \mathbb{Z}^n$ are chosen for convenience only and aren't implied to be minimal for which the phenomenon occurs.

Theorem A. *Let N be a finitely generated, residually finite group such that the centralisers $C_N(M)$ of non-trivial normal subgroups $M \triangleleft N$ don't contain a non-abelian free group.*

Let $T < \mathrm{Aut}(N)$ be a subgroup that isn't free. Let $m \geq 2$ and $\{\varphi_i \colon F_m \twoheadrightarrow T\}$ be a family of surjective homomorphisms such that, setting $\overline{\varphi_i} \colon F_m \twoheadrightarrow \overline{T}$ to be the composition of φ_i with $\big(\nu \colon \mathrm{Aut}(N) \twoheadrightarrow \mathrm{Out}(N)\big)|_T$ and $\overline{T} = \nu(T)$, we have

$$\delta \circ \overline{\varphi_i} = \overline{\varphi_j} \circ \varepsilon \text{ for some } \delta \in \mathrm{Aut}(\overline{T}), \varepsilon \in \mathrm{Aut}(F_m) \iff i = j.$$

Then the groups $G_i := N \rtimes_{\varphi_i} F_m$ are residually finite and pairwise non-isomorphic, while having isomorphic profinite completions.

Proof. Group N together with homomorphisms $\varphi_i \colon F_m \to T < \mathrm{Aut}(N)$ satisfy the assumptions of Theorem 3.2 and thus the family $\mathcal{D}$ of extensions

$$\Big(1 \longrightarrow \underbrace{N \rtimes 1}_{N} \xrightarrow{\iota_i} \underbrace{N \rtimes_{\varphi_i} F_m}_{G_i} \xrightarrow{\pi_i} F_m \longrightarrow 1 \Big)$$

is kernel-detecting. Proposition 2.75 then shows that the groups G_i are pairwise non-isomorphic.

On the other hand, the fact that φ_i all surject the same group $T < \mathrm{Aut}(N)$ proves that $\widehat{G}_i$ are all isomorphic, as shown in Corollary 2.81. □

Theorem B. *Let N be one of the following.*

- *Free group F_n with $n \geq 10$.*
- *Fundamental group $\pi_1(S_n)$ of a closed orientable surface with genus $n \geq 5$.*
- *Free abelian group $\mathbb{Z}^n$ with $n \geq 12$.*

Then there exists an infinite family of non-isomorphic groups $G_i := N \rtimes_{\varphi_i} F_2$ having isomorphic profinite completions. These groups are of type F.

Proof. The groups N satisfying the assumptions are finitely generated and residually finite; their centralisers don't contain non-abelian free groups.

In Section 3.2 we proved that the torus link group $T = T(3,3)$—which isn't free—embeds into $\mathrm{Out}(N)$ in all of the cases covered by the assumptions; furthermore, the embedding factors through $\mathrm{Aut}(N) \twoheadrightarrow \mathrm{Out}(N)$.

Since T has infinitely many non-T-equivalent surjections $\varphi_i \colon F_2 \twoheadrightarrow T$, all of the assumptions of Theorem A are satisfied, and so the groups $N \rtimes_{\varphi_i} F_2$ form an infinite family of non-isomorphic groups which share the profinite completion.

[Wal61] shows that given finite resolutions of $\mathbb{Z}$ by finitely generated free $\mathbb{Z}N$- and $\mathbb{Z}Q$-modules respectively, we can construct a finite resolution of $\mathbb{Z}$ by finitely generated free $\mathbb{Z}G$-modules. This shows in particular that any extension of N by Q is of type FL. Finally, a finitely presented group of type FL is also of type F – see e.g Theorem VIII.7.1 of [Bro12]. □

Chapter 4

Profinite rigidity of crystallographic groups

Crystallographic groups are (free abelian)-by-finite groups where the induced action of the finite group is faithful. One remarkable property which distinguishes them from other virtually free abelian groups is that they have a *unique* maximal free abelian normal subgroup. This makes their classification much easier, as the subgroup can be used for kernel-detecting in the sense of Section 2.3 and thus to promote isomorphisms between the groups to similarities of certain group extensions. Furthermore, this ease of kernel-detecting carries over to the profinite completions of crystallographic groups.

4.1 Background on crystallographic groups

We will start with a—perhaps more natural—geometric definition.

Definition 4.1. A discrete and co-compact subgroup of $\mathrm{Isom}(\mathbb{E}^n)$ is called a *crystallographic group*.

The isometry group $\mathrm{Isom}(\mathbb{E}^n)$ decomposes as $\mathbb{R}^n \rtimes \mathrm{O}(n)$ where the first component corresponds to translations and the second to symmetries fixing any chosen origin.

While it isn't immediate that there are any pure translations in a crystallographic group, it turns out that they form a finite index subgroup isomorphic to $\mathbb{Z}^n$.

Theorem 4.2 (First Bieberbach Theorem)**.** *Let G be a crystallographic group of dimension n with translation subgroup $N < G$. Then:*

1. *N is isomorphic to $\mathbb{Z}^n$;*
2. *N is maximal abelian; and*
3. *N is normal in G with finite index.*

These properties characterise the translation subgroup N as the unique finite-index maximal free-abelian normal subgroup.

Definition 4.3. Let $G < \mathbb{R}^n \rtimes \mathrm{O}(n)$ be a crystallographic group in dimension n. We define its *point group* to be the isomorphism class of the image of G in $\mathrm{O}(n)$.

Let's denote the point group of G by Q. The intersection of G with $\ker(\mathrm{Isom}(\mathbb{E}^n) \to \mathrm{O}(n)) = \mathbb{R}^n \rtimes 1$ is the translation subgroup N, which immediately gives us a short exact sequence $0 \to N \to G \to Q \to 1$. Since $N \cong \mathbb{Z}^n$ is a full rank lattice in $\mathbb{R}^n$ and non-trivial matrix in Q must act on it non-trivially, so the induced homomorphism $Q \to \mathrm{Out}(N) = \mathrm{Aut}(N)$ is injective.

This turns out to have a converse, which is also a convenient algebraic characterisation of crystallographic groups.

Proposition 4.4. *Let $0 \to \mathbb{Z}^n \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$ be an extension with Q a finite group such that the action of Q on $\mathbb{Z}^n$ is faithful. Then G is a crystallographic group.*

Proof. The proof proceeds in a couple of steps.

Step 1. Injection into $\mathbb{R}^n \rtimes Q$. Let $G' = \mathbb{R}^n \rtimes G / \{(-v, \iota(v)) \mid v \in \mathbb{Z}^n\}$ and $\mathcal{E}' = \big(0 \to \mathbb{R}^n \xrightarrow{\iota'} G' \xrightarrow{\pi'} Q \to 1\big)$ be the $\mathbb{R}^n$-by-Q pushout extension induced by the standard injection $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$. Since the $\mathbb{R}^n \xrightarrow{\cdot|Q|} \mathbb{R}^n$ map is an invertible Q-homomorphism, by Proposition 2.39, we get that $H^2(Q; \mathbb{R}^n) = 0$ and also that the extension $\mathcal{E}'$ is equivalent to the split extension $\big(0 \to \mathbb{R}^n \to \mathbb{R}^n \rtimes Q \to Q \to 1\big)$. The induced map $G \to G'$ is injective by Proposition 2.44.

Step 2. Injection into $\mathrm{Isom}(\mathbb{E}^n)$**.** Since $Q < \mathrm{GL}_n(\mathbb{R})$, by Weyl's Unitary Trick, there exists a matrix $p \in \mathrm{GL}_n(\mathbb{R})$ such that $pQp^{-1} < \mathrm{O}(n) < \mathrm{GL}_n(\mathbb{R})$. The group $\mathbb{R}^n \rtimes Q$ can be thus identified with the following group of block matrices

$$\left\{ \begin{pmatrix} pqp^{-1} & pv \\ 0^T & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{R}) \middle| q \in Q < \mathrm{GL}_n(\mathbb{R}), v \in \mathbb{R}^n \right\}$$

which is inside a similarily constructed copy of $\mathrm{Isom}(\mathbb{E}^n) \cong \mathbb{R}^n \rtimes \mathrm{O}(n) < \mathrm{GL}_{n+1}(\mathbb{R})$.

Step 3. Discreteness and cocompactness. The previous steps identified G with a group of matrices of the form

$$G \cong \left\{ \begin{pmatrix} pqp^{-1} & pv \\ 0^T & 1 \end{pmatrix} \middle| q \in Q < \mathrm{GL}_n(\mathbb{R}), v \in \mathbb{Z}^n \right\} < \mathbb{R}^n \rtimes \mathrm{O}(n) \cong \mathrm{Isom}(\mathbb{E}^n).$$

The set of matrices $p\,\mathbb{Z}^n < \mathbb{R}^n$ is discrete and there are just $|Q|$ values which pqp^{-1} takes, so G is a discrete subgroup of $\mathrm{GL}_{n+1}(\mathbb{R})$. For the cocompactness, the quotient $p\,\mathbb{Z}^n \setminus \mathbb{R}^n$ is already compact, so $G \setminus \mathbb{R}^n$ is compact as it is a further quotient. □

Another characterisation of crystallographic groups among the virtually free abelian groups is the following.

Proposition 4.5 (Proposition 2.6 in [CHMV24])**.** *Let G be a $\mathbb{Z}^n$-by-finite group, for $n \geq 1$. Then G is crystallographic if and only if it doesn't have non-trivial finite normal subgroups.*

4.1.1 Classifying crystallographic groups

Theorem 4.2 shows that in a crystallographic group the translations form a characteristic subgroup. We will first explore some isomorphism invariants which are consequences of this.

Corollary 4.6. *The following are isomorphism invariants of a crystallographic group G:*

1. *the dimension n of its translation subgroup,*

2. *the isomorphism type Q of its point group,*

3. *the conjugacy class of $\varphi(Q)$, where $\varphi\colon Q \to \mathrm{GL}_n(\mathbb{Z})$ is the induced action, called the* $\mathbb{Z}$-class *of G,*

4. *the conjugacy class of $\varphi(Q)$ in $\mathrm{GL}_n(\mathbb{Q})$, called the* $\mathbb{Q}$-class *of G.*

Proof. The fact that the translation subgroup N is isomorphic to $\mathbb{Z}^n$—where n is the 'ambient' dimension such that G is a crystallographic group in dimension n—was proved in Theorem 4.2. The isomorphism type Q of its point group is the isomorphism type of the quotient G/N.

The class of extensions $\big(0 \to N \to G \to Q \to 1\big)$, where G is a crystallographic group, N is its translation subgroup and $Q = G/N$ is kernel-detecting in the sense of Definition 2.65, i.e. any group isomorphism 'upgrades' to a similarity of extensions, so by Proposition 2.75 their induced actions are conjugate, hence the invariance of the $\mathbb{Z}$-class, which also implies invariance of the $\mathbb{Q}$-class. □

Theorem 4.7 (Fundamental Theorem of Mathematical Crystallography)**.** *The isomorphism classes of crystallographic groups in dimension n whose point group is $Q < \mathrm{GL}_n(\mathbb{Z})$ are in bijection with the orbits of $N_{\mathrm{GL}_n(\mathbb{Z})}(Q) \curvearrowright H^2\big(Q;\mathbb{Z}^n\big)$. Furthermore, the action of $Q < N_{\mathrm{GL}_n(\mathbb{Z})}(Q)$ is trivial.*

Proof. First, the class of extensions $\big(0 \to \mathbb{Z}^n \to G \to Q \to 1\big)$ is kernel-detecting: indeed the translation subgroup $\mathbb{Z}^n$ is characterised as the unique maximal normal free abelian subgroup.

Now, we need to compute the stabilizer of the induced homomorphism $\varphi\colon Q \to \mathrm{Out}(\mathbb{Z}^n)$, which in this case is injective, since the action of Q is faithful. Conveniently, Proposition 2.78 tells us that the stabilizer is the normalizer in $\mathrm{Out}(N) = \mathrm{GL}_n(\mathbb{Z})$ of the image of φ.

By Corollary 2.48 in $\mathscr{E}(Q;N) \to \{Q \to \mathrm{Out}(N)\}$ the orbits in the preimage of φ are in bijection with the orbits of the stabilizer of φ acting on $H^2(Q;\mathbb{Z}^n)$. We know that the stabilizer is $N_{\mathrm{GL}_n(\mathbb{Z})}(Q)$; therein it is also showed that Q acts trivially. □

Finally, it is convenient to be able to construct for a cohomology class $\zeta \in H^2(Q;\mathbb{Z}^n)$ an explicit representative of the corresponding crystallographic group given as a subgroup of the affine group $\mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R}) < \mathrm{GL}_{n+1}(\mathbb{R})$. Proposition 4.8 provides exactly that; for details see page 19 of [Szc12], or Proposition 4.2.1 of [HP89].

Proposition 4.8. *Let $Q < \mathrm{GL}_n(\mathbb{Z})$ be a finite group and $\mathbb{Z}^n$ be the corresponding Q-module. Then there is a natural isomorphism*

$$H^2\big(Q;\mathbb{Z}^n\big) \cong H^1\big(Q;(\mathbb{Q}\,/\,\mathbb{Z})^n\big).$$

In particular, given a group extension $0 \to \mathbb{Z}^n \to G \to Q \to 1$ with the prescribed action, the map $G \hookrightarrow \mathrm{GL}_{n+1}(\mathbb{Q}) < \mathrm{Isom}(\mathbb{E}^n)$ given by

$$v \cdot s(q) \mapsto \begin{pmatrix} q & v+\zeta(q) \\ 0 & 1 \end{pmatrix},$$

is a homomorphism, where $s\colon Q \to G$ is a choice of section and $\zeta\colon Q \to \mathbb{Z}^n$ is a representative of the cohomology class in $H^1\big(Q;(\mathbb{Q}\,/\,\mathbb{Z})^n\big)$.

4.1.2 Profinite completions of crystallographic groups

The structure of crystallographic groups works particularly nicely with profinite completions.

Proposition 4.9 (Proposition 3.8. in [PPW21])**.** *Let G be a crystallographic group with N its subgroup of translations. Then $\overline{N} \cong \widehat{N}$ is the unique finite-index normal torsion-free maximal abelian subgroup.*

In particular, the class of extensions $\big(0 \to \widehat{N}_i \to \widehat{G}_i \to Q \to 1\big)$ for G_i any crystallographic groups and N_i its translation subgroup is kernel-detecting.

The next step towards the classification of isomorphism types, according to Section 2.3, is determining which of the induced actions $\widehat{\varphi}\colon Q \to \mathrm{GL}_n(\widehat{\mathbb{Z}})$ lie in the same orbit under the action of $\mathrm{GL}_n(\widehat{\mathbb{Z}}) \times \mathrm{Aut}(Q)$. This is equivalent to asking which non-conjugate subgroups $Q_1, Q_2 < \mathrm{GL}_n(\mathbb{Z})$ isomorphic to Q are conjugate in $\mathrm{GL}_n(\widehat{\mathbb{Z}})$.

It turns out that there are examples of such pairs (Q_1, Q_2); they always come from non-isomorphic pairs (G_1, G_2) of crystallographic groups, whose profinite completions may be isomorphic. This is the subject of Section 4.2.1.

For the remainder of this section we are interested in the profinite completions of crystallographic groups with a fixed $\mathbb{Z}$-class $Q < \mathrm{GL}_n(\mathbb{Z})$, i.e. with similar actions in the sense of Section 2.3.5.

Proposition 4.10 (Proposition 4.2 in [PPW21])**.** *Let $Q < \mathrm{GL}_n(\mathbb{Z})$ be a finite subgroup. The isomorphism types of profinite completions of crystallographic groups with $\mathbb{Z}$-class Q are in bijection with the orbits of $N_{\mathrm{GL}_n(\widehat{\mathbb{Z}})}(Q) \curvearrowright H^2(Q; \widehat{\mathbb{Z}}^n)$.*

Again, following the 'general machinery' we get the following summary result.

Corollary 4.11. *Let G_1 and G_2 be crystallographic groups with $\mathbb{Z}$-class $Q < \mathrm{GL}_n(\mathbb{Z})$. Let ζ_i and $\mathsf{h}_*(\zeta_i)$ be their corresponding cohomology classes in $H^2(Q; \mathbb{Z}^n)$ and $H^2(Q; \widehat{\mathbb{Z}}^n)$ respectively.*

1. *$G_1 \cong G_2$ if and only if $[\zeta_1] = [\zeta_2]$ in the orbit space $N_{\mathrm{GL}_n(\mathbb{Z})}(Q) \backslash H^2(Q; \mathbb{Z}^n)$.*
2. *$\widehat{G}_1 \cong \widehat{G}_2$ if and only if we have $[\mathsf{h}_*(\zeta_1)] = [\mathsf{h}_*(\zeta_2)]$ in the orbit space $N_{\mathrm{GL}_n(\widehat{\mathbb{Z}})}(Q) \backslash H^2(Q; \widehat{\mathbb{Z}}^n)$.*

Remarkably, the classification can be improved upon, with a result of Finken–Neubüser–Plesken [FNP80], building on the work of Maranda [Mar53]. It turns out that for G_1 and G_2 to be profinitely isomorphic it is enough if they have a pair of *sufficiently large* isomorphic finite quotients.

The proof relies on the fact that the image of translation subgroup can be 'characterised' in a sufficiently large finite quotient, and on Maranda's work on the p-genus of $\mathbb{Z}\,Q$-lattices.

Theorem 4.12 (Theorem 4.1.11 in [HP89], Theorem II.1 in [FNP80])**.** *Let G_1 and G_2 be crystallographic groups with points group Q of size d. Let N_1 and N_2 denote their respective translation subgroups. The following conditions are equivalent.*

1. *There exist normal subgroups* $M_i \lhd G_i$ *with* $G_1/M_1 \cong G_2/M_2$ *and* $M_i < q^2 N_i$ *where* $q = 4$ *if* $d = 2$ *and* $q = d$ *if* $d \neq 2$.

2. $(G_1)_{(d)} \cong (G_2)_{(d)}$ *where* $(G_i)_{(d)}$ *is the pushout extension induced by the module map* $\mathbb{Z}^n \to \mathbb{Z}^n_{(d)}$, *where* $\mathbb{Z}_{(d)}$ *denotes the subring of fractions in* $\mathbb{Q}$ *whose denominators are coprime to* d.

3. *There exists a subgroup* $H < G_1$ *isomorphic to* G_2 *with* $[G_1 : H]$ *finite and coprime to* d.

4. $\widehat{G}_1 \cong \widehat{G}_2$.

The theorem has the following immediate corollaries.

Corollary 4.13. *Let* G_1 *and* G_2 *be crystallographic groups with point groups of size* d *and translation subgroups* N_1 *and* N_2. *Let* $q = 4$ *if* $d = 2$, *and* $q = d$ *if* $d \neq 2$. *Then*

$$\widehat{G}_1 \cong \widehat{G}_2 \iff G_1/q^2N_1 \cong G_2/q^2N_2.$$

Proof. The closures $\overline{N}_i$ are characteristic subgroups, so $\widehat{G}_1 \cong \widehat{G}_2$ implies that

$$G_1/mN_1 \cong \widehat{G}_1/m\overline{N}_1 \cong \widehat{G}_2/m\overline{N}_2 \cong G_2/mN_2$$

for any natural number m. For the other direction choosing $M_i := q^2N_i$ satisfies the condition 1. of Theorem 4.12, so it gives $\widehat{G}_1 \cong \widehat{G}_2$. □

Corollary 4.14. *Let* $Q_1, Q_2 < \mathrm{GL}_n(\mathbb{Z})$ *be subgroups of size* d *and let* $q = 4$ *if* $d = 2$, *and* $q = d$ *if* $d \neq 2$. *Let* $\overline{Q}_1$ *and* $\overline{Q}_2$ *denote images of* Q_1 *and* Q_2 *in* $\mathrm{GL}_n(\mathbb{Z}/q^2)$. *Then, the following are equivalent.*

1. Q_1 *and* Q_2 *are conjugate as subgroups of* $\mathrm{GL}_n(\widehat{\mathbb{Z}})$.

2. Q_1 *and* Q_2 *are conjugate as subgroups of* $\mathrm{GL}_n(\mathbb{Z}_{(d)})$, *where* $\mathbb{Z}_{(d)}$ *denotes the subring of reduced fractions in* $\mathbb{Q}$ *whose denominators are coprime to* d.

3. *There exists a matrix* $X \in M_{n\times n}(\mathbb{Z})$ *such that* $\det(X)$ *is non-zero and coprime to* d *and* $XQ_1X^{-1} = Q_2$.

4. $\overline{Q}_1$ *and* $\overline{Q}_2$ *are conjugate as subgroups of* $\mathrm{GL}_n(\mathbb{Z}/q^2)$.

Proof. Let's set $G_i := \mathbb{Z}^n \rtimes Q_i$, $(G_i)_{(d)} := \mathbb{Z}^n_{(d)} \rtimes Q_i$ and $\overline{G}_i := G_i/q^2N_i \cong (\mathbb{Z}/q^2)^n \rtimes \overline{Q}_i$. $\widehat{G}_1 \cong \widehat{G}_2$ is equivalent to $(G_1)_{(d)} \cong (G_2)_{(d)}$ and also to $\overline{G}_1 \cong \overline{G}_2$ by Theorem 4.12.

Since the subgroup $(\mathbb{Z}^n_{(d)} \rtimes 1) \lhd (G_i)_{(d)}$ is the maximal normal abelian subgroup, the isomorphism implies by Proposition 2.75 that the induced actions $\varphi_i \colon Q_i \to \mathrm{GL}_n(\mathbb{Z}_{(d)})$ have conjugate images, i.e. Q_1 and Q_2 are conjugate in $\mathrm{GL}_n(\mathbb{Z}_{(d)})$, thus proving [1. $\implies$ 2.].

For [2. $\implies$ 3.], let $X \in \mathrm{GL}_n(\mathbb{Z}_{(d)})$ be such that $XQ_1X^{-1} = Q_2$. Let k be the least common multiple of the denominators of the entries, when written in lowest terms. Since these denominators are coprime to d then so is k. The matrix kX has integer entries, its determinant is $k^n \det(X)$ which is coprime to d and $(kX)Q_1(kX)^{-1} = XQ_1X^{-1} = Q_2$.

[3. $\implies$ 4.] holds because such a matrix X with determinant coprime to d and thus to q^2 its reduction mod q^2 is invertible in $\mathrm{GL}_n(\mathbb{Z}/q^2)$.

Finally, for [4. $\implies$ 1.] note that in either group $\overline{G}_i$ we can characterise

$$\overline{M}_i := q(\mathbb{Z}/q^2)^n \rtimes 1 \cong (\mathbb{Z}/q)^n$$

as the set of elements in $\overline{G}_i$ which are q-th powers themselves and their q-th power is trivial. The map $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/q)$ is injective on Q_i, so the action of $1 \rtimes \overline{Q}_i$ on $\overline{M}_i$ is faithful. Thus, we get $\overline{N}_i = (\mathbb{Z}/q^2)^n \rtimes 1 = C_{\overline{G}_i}(\overline{M}_i)$. In particular, any isomorphism $\overline{\beta} \colon \overline{G}_1 \to \overline{G}_2$ must map $\overline{N}_1$ to $\overline{N}_2$ and so induce a similarity of extensions $\left(0 \to \overline{N}_i \to \overline{G}_i \to \overline{Q}_i \to 1\right)$ so by Proposition 2.75,

$$\overline{G}_1 \cong \overline{G}_2 \iff \exists \overline{X} \in \mathrm{GL}_n(\mathbb{Z}/q^2) : \quad \overline{Q}_2 = \overline{X}\,\overline{Q}_1\overline{X}^{-1}. \qquad \square$$

Corollary 4.15. *Let G_1 and G_2 be crystallographic groups and let $Q_1, Q_2 < \mathrm{GL}_n(\mathbb{Z})$ be their respective $\mathbb{Z}$-classes.*

Then

$$\widehat{G}_1 \cong \widehat{G}_2 \implies Q_1 \text{ and } Q_2 \text{ are conjugate in } \mathrm{GL}_n(\mathbb{Q}),$$

i.e. G_1 and G_2 have the same $\mathbb{Q}$-class.

4.2 Examples of profinitely isomorphic crystallographic groups

In this section we use the theory built in the previous sections to construct non-isomorphic pairs of profinitely isomorphic crystallographic groups in two ways: using profinitely conjugate actions, and using collapsing cohomology orbits.

4.2.1 Profinitely conjugate actions

The first kind of pairs of non-isomorphic crystallographic groups whose profinite completions are isomorphic comes from their actions being profinitely, but not discretely, conjugate.

This is similar to the methods used in Chapter 3 for constructing infinite families of non-conjugate actions $F_2 \to \mathrm{Out}(N)$, which were conjugate as actions $F_2 \to \mathrm{Out}(\widehat{N})$. There however, the maps had the same images, and the non-conjugacy appeared at the level of maps. Here, the maps $Q \to \mathrm{Out}(\mathbb{Z}^n)$ are injective, and the non-conjugacy happens at the level of their images.

Proposition 4.16. *Let Q be a finite group and $\varphi_1, \varphi_2 \colon Q \hookrightarrow \mathrm{GL}_n(\mathbb{Z})$ be two injections such that $\varphi_1(Q)$ and $\varphi_2(Q)$ are not conjugate in $\mathrm{GL}_n(\mathbb{Z})$, but they are conjugate in $\mathrm{GL}_n(\widehat{\mathbb{Z}})$. Then the crystallographic groups $G_i = \mathbb{Z}^n \rtimes_{\varphi_i} Q$ are not isomorphic, but $\widehat{G}_1 \cong \widehat{G}_2$.*

Proof. First, the groups $G_i = \mathbb{Z}^n \rtimes_{\varphi_i} Q$ are $\mathbb{Z}^n$-by-Q with the action of Q being faithful, so by Proposition 4.4 they are actually crystallographic groups.

Second, the class of extensions $\big(0 \to N \to G \to Q \to 1\big)$ with N being the translation subgroup of the crystallographic group G is kernel-detecting; so any group isomorphism $G_1 \to G_2$ gives similarity of the induced actions: $\mathrm{Ad}_{\overline{\alpha}} \circ \varphi_1 = \varphi_2 \circ \gamma$ which implies that $\varphi_1(Q)$ and $\varphi_2(Q)$ are conjugate subgroups of $\mathrm{Out}(N) \cong \mathrm{GL}_n(\mathbb{Z})$. They aren't by assumption, so $G_1 \ncong G_2$.

On the other hand, the conjugacy of $\varphi_1(Q)$ and $\varphi_2(Q)$ in $\mathrm{GL}_n(\widehat{\mathbb{Z}})$ together with the injectivity of φ_1 and φ_2 implies that there are maps $\alpha \in \mathrm{Aut}(\widehat{\mathbb{Z}}^n)$ and $\gamma \in \mathrm{Aut}(Q)$ such that $\mathrm{Ad}_{\overline{\alpha}} \circ \varphi_1 = \varphi_2 \circ \gamma$. Now, since $\overline{N} = \widehat{\mathbb{Z}}^n$ is abelian we get that $\mathrm{Ad}_\alpha \circ \tilde{\varphi}_1 = \tilde{\varphi}_2 \circ \gamma$, which by Proposition 2.79 implies that $\widehat{G}_1 \cong \widehat{G}_2$. □

One source of such groups is, perhaps unexpectedly, algebraic number theory. A nice introduction to these concepts are the notes [Con12] of Keith Conrad. We start by quoting a theorem of Latimer and MacDuffee.

Theorem 4.17 (Theorem III.13 of [New72])**.** *Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial of degree m that is irreducible over $\mathbb{Q}$ and let λ be a root of $f(x)$.*

Then there is a one-to-one correspondence between the conjugacy classes of matrices in $M_{m\times m}(\mathbb{Z})$ with characteristic polynomial f and the ideal classes of the order $\mathbb{Z}[\lambda]$.

Corollary 4.18. *Let $\zeta = \zeta_n$ be n-th primitive root of unity such that the ring $\mathbb{Z}[\zeta]$ of integers in the cyclotomic number field $\mathbb{Q}(\zeta)$ has non-trivial class group.*

Then there exists a pair of crystallographic groups $G_i = \mathbb{Z}^{\phi(n)} \rtimes (\mathbb{Z}/n)$ for $i = 1, 2$ such that $G_1 \ncong G_2$, but $\widehat{G}_1 \cong \widehat{G}_2$.

Proof. Let $C \in \mathrm{GL}_{\phi(n)}(\mathbb{Z})$ be the companion matrix for the n-th cyclotomic polynomial and $D \in \mathrm{GL}_{\phi(n)}(\mathbb{Z})$ be any matrix whose conjugacy class corresponds to a non-principle ideal class of $\mathbb{Z}[\zeta]$. We will show that all of the primitive powers C^k are conjugate to C and thus that $\langle C\rangle$ and $\langle D\rangle$ are not conjugate in $\mathrm{GL}_{\phi(n)}(\mathbb{Z})$.

Any field automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ induces an automorphism of $\mathbb{Z}[\zeta]$, so $\left(1, \sigma(\zeta), \ldots, \sigma(\zeta^{\phi(n)-1})\right)$ is a $\mathbb{Z}$-basis for $\mathbb{Z}[\zeta]$. Now, any primitive power ζ^k equals $\sigma(\zeta)$ for a unique $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$, and then the multiplication by ζ^k written in the basis of $\left(1, \sigma(\zeta), \ldots, \sigma(\zeta^{\phi(n)-1})\right)$ has matrix equal to the companion matrix C, and thus C^k is conjugate to C. □

Remark 4.19. Latimer and MacDuffee's theorem was extended to an 'efficient' algorithm for deciding conjugacy of finite order matrices in $\mathrm{GL}_n(\mathbb{Z})$

by Eick–Hofmann–O'Brien in [EHO19], and this algorithm is now implemented in MAGMA.

Similarly, the full classification of the genus of a crystallographic group of the form $\mathbb{Z}^n \rtimes (\mathbb{Z}/p)$ was also done in Section 2.3 of [GZ11]. It used the fact that for $Q = \mathbb{Z}/p$ each finitely generated $\mathbb{Z}Q$-lattice decomposes as a direct sum of three types of indecomposable $\mathbb{Z}Q$-modules.

Proposition 4.20 (Theorem 74.3 of Curtis-Reiner)**.** *Let $Q = \mathbb{Z}/p$. Then every torsion-free $\mathbb{Z}Q$-module M is isomorphic to the direct sum*

$$M \cong M(r,s,t,\mathfrak{A},\mathfrak{B}) = M_0^r \oplus \bigoplus_{i=1}^{s} M_1(\mathfrak{a}_i) \oplus \bigoplus_{i=1}^{t} M_2(\mathfrak{b}_i),$$

where $\mathfrak{A} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_s)$ and $\mathfrak{B} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_t)$ are tuples of non-zero ideals in $\mathbb{Z}[\zeta_p]$ and

- *$M_0 = \mathbb{Z}$ is the trivial $\mathbb{Z}Q$-module,*
- *$M_1(\mathfrak{a}) = \mathfrak{a} \cong \mathbb{Z}^{p-1}$ with the action being the multiplication by ζ_p,*
- *$M_2(\mathfrak{b}) = \mathfrak{b} \oplus \mathbb{Z} \cong \mathbb{Z}^p$ with the action of $Q = \langle\sigma\rangle$ given by*

  $$\sigma \cdot (b, m) = (\zeta_p b + m b_0, m),$$

  where $b_0 \in \mathfrak{b} \setminus (1-\zeta_p)\mathfrak{b}$ is fixed.

It turns out that isomorphisms classes and profinite isomorphisms classes for these modules are easy to determine.

Proposition 4.21 (Proposition 2.22 of [GZ11] rephrased)**.** *Let $M = M(r,s,t,\mathfrak{A},\mathfrak{B})$ and $M' = M(r',s',t',\mathfrak{A}',\mathfrak{B}')$ be finitely generated $\mathbb{Z}Q$ lattices for $Q = (\mathbb{Z}/p)$ as described in Proposition 4.20. Then*

- *$\widehat{M} \cong \widehat{M}'$ as $\widehat{\mathbb{Z}}Q$-lattices if and only if $(r,s,t) = (r',s',t')$.*
- *$M \cong M'$ as $\mathbb{Z}Q$-lattices if and only if $(r,s,t) = (r',s',t')$ and the ideals*

  $$\mathfrak{a}_1 \cdot \ldots \cdot \mathfrak{a}_s \cdot \mathfrak{b}_1 \cdot \ldots \cdot \mathfrak{b}_t \qquad \text{and} \qquad \mathfrak{a}'_1 \cdot \ldots \cdot \mathfrak{a}'_s \cdot \mathfrak{b}'_1 \cdot \ldots \cdot \mathfrak{b}'_t$$

  lie in the same orbit of the action $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \curvearrowright H(\mathbb{Z}[\zeta_p])$ of the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong \mathbb{Z}/(p-1)$ on the ideal class group $H(\mathbb{Z}[\zeta_p])$.

Furthermore, assuming $(s,t),(s',t') \neq (0,0)$*, the symmorphic crystallographic groups* $G = M \rtimes (\mathbb{Z}/p)$ *and* $G' = M' \rtimes (\mathbb{Z}/p)$ *satisfy*

- $\widehat{G} \cong \widehat{G}'$ *if and only if* $\widehat{M} \cong \widehat{M}'$ *as* $\widehat{\mathbb{Z}}Q$*-lattices,*
- $G \cong G'$ *if and only if* $M \cong M'$ *as* $\mathbb{Z}\,Q$*-lattices.*

Due to number-theoretical constraints—namely that $H(\mathbb{Z}[\zeta_p])$ is trivial for $p < 23$—profinitely isomorphic pairs of crystallographic groups with point group $\mathbb{Z}/p$ don't arise below dimension 22. However, thanks to a private correspondence with Gabriele Nebe the author became aware of a much lower dimensional example of such pair.

Theorem C. *Let* $Q_1 = \langle X_1, Y_1 \rangle$ *and* $Q_2 = \langle X_2, Y_2 \rangle$ *be subgroups of* $\mathrm{GL}_8(\mathbb{Z})$ *generated by matrices* X_1, Y_1, X_2 *and* Y_2 *given below. Then* Q_1 *and* Q_2 *are not conjugate in* $\mathrm{GL}_8(\mathbb{Z})$*, but they are conjugate as subgroups of* $\mathrm{GL}_8(\widehat{\mathbb{Z}})$*. The subgroups are of size* 24*, isomorphic to the dicyclic group* Dic_6*.*

$$\underbrace{\begin{pmatrix} 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}}_{X_1}, \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}}_{Y_1}$$

$$\underbrace{\begin{pmatrix} 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -2 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 \end{pmatrix}}_{X_2}, \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & 0 & 0 & -1 & -1 \\ -1 & 1 & -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & -2 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & -1 & -1 \end{pmatrix}}_{Y_2}$$

Proof. It can be checked that the matrices satisfy the following relations, which are the defining relations of the dicyclic group Dic_6.

$$X_i^{12} = Y_i^4 = \mathrm{Id}, \quad X_i^6 = Y_i^2, \quad Y_i X_i Y_i^{-1} = X_i^{-1}$$

Together with the fact that the groups are of size 24 (which can be checked with `MAGMA`) this shows that $Q_1 \cong Q_2 \cong \mathrm{Dic}_6$. The fact that they are not conjugate in $\mathrm{GL}_8(\mathbb{Z})$ can also be checked in `MAGMA` using the function `IsGLZConjugate`.

On the other hand, for the following matrix A with determinant 71^6

$$A := \begin{pmatrix} 8 & 23 & 7 & -9 & -2 & 0 & -5 & 10 \\ 13 & 28 & 5 & -7 & -9 & -2 & 3 & -21 \\ -3 & 21 & 9 & 2 & -18 & 12 & 15 & -11 \\ 15 & 15 & -6 & 6 & -21 & -6 & 24 & -22 \\ 20 & 1 & 16 & -4 & 2 & -13 & 11 & -13 \\ -1 & 21 & 4 & 12 & 13 & -11 & 13 & -2 \\ 36 & -3 & -4 & -5 & -9 & 0 & 13 & -26 \\ 3 & 33 & 5 & -9 & 0 & -9 & 26 & -13 \end{pmatrix}$$

we get relations

$$AX_1A^{-1} = X_2, \quad AY_1A^{-1} = X_2^{-2}Y_2,$$

showing that $AQ_1A^{-1} = Q_2$. By Corollary 4.14 this shows that Q_1 and Q_2 are conjugate in $\mathrm{GL}_8(\widehat{\mathbb{Z}})$.

The code to conduct the computations described in this proof is available in Listing A.2. □

4.2.2 Examples coming from action on H^2

Another source of lack of profinite rigidity in crystallographic groups is the collapsing of cohomology orbits under the action of relevant normalisers. It turns out that many such examples can be constructed in a simple way starting with the classical Baumslag's examples and embedding them into crystallographic groups.

We start with Proposition 4.22 giving some conditions for profinite isomorphism of pushout extensions.

Proposition 4.22. *For $i = 1, 2$, let G_i be extensions of an abelian group M by a finite group Q and $\beta\colon \widehat{G}_1 \to \widehat{G}_2$ be an isomorphism inducing a similarity of*

extensions with identity map on Q as follows.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widehat{M} & \longrightarrow & \widehat{G}_1 & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \| & & \\ 1 & \longrightarrow & \widehat{M} & \longrightarrow & \widehat{G}_2 & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

Let $f\colon M \to M'$ be a $\mathbb{Z}Q$-module homomorphism and $\alpha'\colon \widehat{M}' \to \widehat{M}'$ a $\widehat{\mathbb{Z}}Q$-module homomorphism such that the diagram (4.1) of $\widehat{\mathbb{Z}}Q$-modules commutes.

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\alpha} & \widehat{M} \\ \downarrow{\scriptstyle \widehat{f}} & & \downarrow{\scriptstyle \widehat{f}} \\ \widehat{M}' & \xrightarrow{\alpha'} & \widehat{M}' \end{array} \tag{4.1}$$

Then the pushout extensions G_i' of $\big(1 \to M \to G_i \to Q \to 1\big)$ along f satisfy $\widehat{G}_1' \cong \widehat{G}_2'$.

Proof. Let's denote the subgroup $\{(f(m), -m) \mid m \in M\}$ of $M' \rtimes G_i$ by N_i so that $\big(0 \to M' \to G_i' \to Q \to 1\big)$ with $G_i' \cong (M' \rtimes G_i)/N_i$ is the pushout extension, as described in Proposition 2.44. N_i is then isomorphic to M. By Proposition 2.19, $\widehat{M' \rtimes G_i} \cong \widehat{M}' \rtimes \widehat{G}_i$. We will exhibit a specific isomorphism $\tilde{\beta}'\colon \widehat{M}' \rtimes \widehat{G}_1 \to \widehat{M}' \rtimes \widehat{G}_2$ which maps $\overline{N}_1$ to $\overline{N}_2$ inducing an isomorphism $\beta'\colon \widehat{G}_1' \to \widehat{G}_2'$.

The map

$$\tilde{\beta}'\colon \big(m', g\big) \mapsto \big(\alpha'(m'), \beta(g)\big)$$

is an isomorphism from $\widehat{M}' \rtimes \widehat{G}_1'$ to $\widehat{M}' \rtimes \widehat{G}_2'$ since the actions are through $\widehat{G}_i' \twoheadrightarrow Q$, we required $\beta\colon \widehat{G}_1 \to \widehat{G}_2$ to induce the identity on Q and $\alpha'\colon \widehat{M}' \to \widehat{M}'$ is a map of Q-modules.

$$\tilde{\beta}'\big((f(m), -m)\big) = \big(\alpha'(f(m)), \beta(-m)\big) = \big(f(\alpha(m)), -\alpha(m)\big),$$

so $\tilde{\beta}'$ maps $\overline{N}_1$ isomorphically to $\overline{N}_2$ and so it induces an isomorphism $\beta'\colon \widehat{G}_1' \to \widehat{G}_2'$ between the quotients $\widehat{G}_i' = \widehat{M' \rtimes G_i}/\overline{N}_i$. □

We can now apply this general method to Baumslag's examples of $G_i = (\mathbb{Z}/11) \rtimes \mathbb{Z}$ to produce a pair of non-isomorphic crystallographic groups in dimension 11 with isomorphic profinite completions.

Proposition D. *Let groups* $G_1 = (\mathbb{Z}/11) \rtimes_{.2} \mathbb{Z}$ *and* $G_2 = (\mathbb{Z}/11) \rtimes_{.2^3} \mathbb{Z}$ *be central extensions of* $M = \mathbb{Z}$ *by the Frobenius group* $\mathrm{Frob}_{11} = (\mathbb{Z}/11) \rtimes (\mathbb{Z}/10)$. *Let* $M' \cong \mathbb{Z}^{11}$ *be the permutation* $\mathbb{Z}$*-module for the transitive action of* Frob_{11} *on* $\{1, 2, \ldots, 11\}$ *and let* $f\colon M \to M'$ *be the diagonal embedding. Let* $G_i' = M' \rtimes G_i / \langle (f(1), -1) \rangle$ *be the pushout extensions along* f. *Then the following hold.*

1. $\widehat{G}_1' \cong \widehat{G}_2'$.
2. G_1' *and* G_2' *are crystallographic groups of dimension* 11.
3. G_1' *and* G_2' *aren't isomorphic.*

Proof. As described in Section 1.4, there is an isomorphism $\beta\colon \widehat{G}_1 \to \widehat{G}_2$ which induces the identity on $\widehat{G}_i/\widehat{M} \cong \mathrm{Frob}_{11}$ and multiplication by $\kappa \in \widehat{\mathbb{Z}}^\times$ on $\widehat{M}$ – this is the map $\alpha\colon \widehat{M} \to \widehat{M}$ from Proposition 4.22.

The scalar map $\kappa\,\mathrm{Id}$ extends any $\widehat{\mathbb{Z}}Q$-module map $\widehat{f}\colon \widehat{M} \to \widehat{M}'$, thus we do get $\alpha'\colon \widehat{M}' \to \widehat{M}'$ satisfying the conditions of Proposition 4.22 so indeed $\widehat{G}_1' \cong \widehat{G}_2'$.

The groups G_i' are extensions of $M' \cong \mathbb{Z}^{11}$ by Frob_{11}, with a faithful induced action of Frob_{11} on $\mathbb{Z}^{11}$, so by Proposition 4.4, they are crystallographic groups in dimension 11.

Now, the translation subgroup in a crystallographic group is characteristic, and so is it submodule of invariants, and thus the images $\alpha(M)$ are characteristic in each of G_i'. Furthermore, they are generated by vector $(1, 1, \ldots, 1)$, which is primitive in $M' \cong \mathbb{Z}^{11}$, so any subgroup H_i of G_i' which contains $\alpha(M)$ as a finite-index subgroup must satisfy $H_i \cap M' = \alpha(M)$. G_i is a subgroup of G_i' containing $\alpha(M)$ at index $110 = |\mathrm{Frob}_{11}|$. We claim that it is the only such subgroup. Indeed, any other H_i satisfying these conditions would have to contain $\alpha(M)$ and map onto the whole Frob_{11} under the quotient $G_i' \twoheadrightarrow G_i'/M' \cong \mathrm{Frob}_{11}$. Thus it must be G_i.

Finally, we managed to identify G_1 and G_2 as subgroups of G_1' and G_2' given by the same characteristic description, so if $G_1' \cong G_2'$, then $G_1 \cong G_2$, which we know doesn't hold. □

Remark 4.23. The examples of Proposition D aren't the only ones which we can construct using this method. Instead of using the permutation $\mathbb{Z}$-module M' for Frob_{11}, we could have used $M'' = M \oplus (M'/M)$.

The profinite isomorphism of part 1 again follows from the fact that we can extend a scalar map of Q-modules. The groups $G_i'' = M'' \rtimes G_i / \langle (\alpha(1), -1) \rangle$ are crystallographic, because the action on M'/M is faithful. Finally, the non-isomorphism of G_1'' and G_2'' follows from the same proof and the fact that $\alpha(M) = M \oplus 0 < M''$ is again the invariants submodule.

The groups G_i'' aren't isomorphic to any of G_i' as an isomorphism between them would induce an isomorphism of the translation subgroups treated as Q-modules. M'' decomposes as a direct sum, but M' doesn't, so they aren't isomorphic.

Since pushout extensions are closely related to the functoriality of group cohomology, it comes as no surprise that there is a cohomological explanation for Proposition D which we give in Proposition 4.26. Before that though we need Lemma 4.24 which computes normalizers of certain permutation groups in $\mathrm{GL}_n(\mathbb{Z})$. With it, we will compute the normalizer $N_{\mathrm{GL}_{11}(\mathbb{Z})}(\mathrm{Frob}_{11})$ in Corollary 4.25.

Lemma 4.24. *Let $n \geq 3$ and let $Q < \mathrm{Sym}_n$ be a permutation group, treated as a subgroup of $\mathrm{GL}_n(\mathbb{Z})$ through permuting the basis vectors. Assume also that*

1. $\mathrm{Out}(Q) = 1$,
2. $Q \curvearrowright \{1, \ldots, n\}$ *is* 2-transitive, *i.e. for all $i, i', j, j' \in \{1, \ldots, n\}$ such that $i \neq j$ and $i' \neq j'$ there exists $\sigma \in Q$ such that $\sigma(i) = i'$ and $\sigma(j) = j'$.*

Then the normalizer $N_{\mathrm{GL}_n(\mathbb{Z})}(Q)$ is equal to $Q \cdot \{\pm I\} < \mathrm{GL}_n(\mathbb{Z})$.

Proof. Since $\mathrm{Out}(Q) = 1$, any matrix $N \in N_{\mathrm{GL}_n(\mathbb{Z})}(Q)$ induces an inner automorphism on Q, so it can be written as $N = C \cdot X$ where $C \in C_{\mathrm{GL}_{11}(\mathbb{Z})}(Q)$ and $X \in Q$. The real work in this proof is to show that the centraliser $C_{\mathrm{GL}_n(\mathbb{Z})}(Q)$ consists of just two matrices: I and $-I$.

The permutation matrix corresponding to $\sigma \in \mathrm{Sym}_n$ permutes column vectors according to σ, i.e. $S_\sigma = (S_{ij})$ where

$$S_{ij} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{otherwise.} \end{cases}$$

Conjugating a matrix $A \in M_{n\times n}(\mathbb{Z})$ by $S_\sigma^{-1} = S_\sigma^T$ gives

$$(S_\sigma^{-1} A S_\sigma)_{ij} = \sum_{k,l} S_{ik}^T A_{kl} S_{lj} = \sum_{k,l} S_{ki} A_{kl} S_{lj} = S_{\sigma(i)i} A_{\sigma(i)\sigma(j)} S_{\sigma(j)j} = A_{\sigma(i)\sigma(j)}.$$

Thus, if C is in the centralizer $C_{\mathrm{GL}_n(\mathbb{Z})}(Q)$, then $C = S_\sigma^{-1} C S_\sigma$ for all permutations $\sigma \in Q$ and so $C_{ij} = C_{\sigma(i)\sigma(j)}$. The transitivity of the action means that all of the diagonal entries are equal to some $a \in \mathbb{Z}$, and the 2-transitivity implies that all of the off-diagonal entries are equal to some $n \in \mathbb{Z}$. Such a matrix has an eigenvector $(1, 1, \ldots, 1)^T$ with eigenvalue $(a - b) + nb$ and a $(n-1)$-dimensional eigenspace

$$\{(v_1, \ldots, v_n) \in \mathbb{Z}^n \mid v_1 + \ldots + v_n = 0\}$$

with eigenvalue $(a - b)$. The only possibility for $(a - b) + nb$ and $(a - b)$ to simultaneously be in $\mathbb{Z}^\times$ is $b = 0$ and $a = \pm 1$. □

Corollary 4.25. *Let $Q = \mathrm{Frob}_{11} < \mathrm{GL}_{11}(\mathbb{Z})$ be the permutation representation corresponding to the standard action of the Frobenius group $\mathrm{Frob}_{11} = \mathbb{F}_{11} \rtimes \mathbb{F}_{11}^\times$.*

Then $N_{\mathrm{GL}_{11}(\mathbb{Z})}(Q) = Q \cdot \{\pm I\}$.

We are ready for a cohomological explanation for Proposition D. We use the continuous cohomology in the statement despite the fact that for a finite group Q (which has the discrete topology) and any topological Q-module M the groups $H^k(Q; M)$ and $H^k_{\mathrm{cts}}(Q; M)$ are *equal*, as every cocycle $\zeta \colon Q^k \to M$ is continuous.

Proposition 4.26. *Let Q be the Frobenius group Frob_{11}. The diagram of Q-modules*

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{f} & \mathbb{Z}^{11} \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ \widehat{\mathbb{Z}} & \xrightarrow{\widehat{f}} & \widehat{\mathbb{Z}}^{11} \end{array}$$

induces the following commutative diagram of group actions.

$$\begin{array}{ccccc} \mathbb{Z}^\times \curvearrowright H^2(Q;\mathbb{Z}) & \xrightarrow{f_*} & N_{\mathrm{GL}_{11}(\mathbb{Z})}(Q) \curvearrowright H^2(Q;\mathbb{Z}^{11}) \\ \downarrow \mathsf{h}_* & & \downarrow \mathsf{h}_* \\ \widehat{\mathbb{Z}}^\times \curvearrowright H^2_{\mathrm{cts}}(Q;\widehat{\mathbb{Z}}) & \xrightarrow{\widehat{f}_*} & N_{\mathrm{GL}_{11}(\widehat{\mathbb{Z}})}(Q) \curvearrowright H^2_{\mathrm{cts}}(Q;\widehat{\mathbb{Z}}^{11}) \end{array} \tag{4.2}$$

Baumslag's examples $\big(0 \to \mathbb{Z} \to G_i \to Q \to 1\big)$ *correspond to cohomology classes* $\zeta_i \in H^2(Q;\mathbb{Z})$ *where*

- *the orbits* $[\zeta_1]$ *and* $[\zeta_2]$ *are distinct,*
- *but* $[\mathsf{h}_*(\zeta_1)] = [\mathsf{h}_*(\zeta_2)]$.

The pushout extensions $\big(0 \to \mathbb{Z}^{11} \to G_i' \to Q \to 1\big)$ *as defined in Proposition D correspond to cohomology classes* $f_*(\zeta_i) \in H^2(Q;\mathbb{Z}^{11})$ *such that*

- *the orbits* $[f_*(\zeta_1)]$ *and* $[f_*(\zeta_2)]$ *are distinct,*
- *but* $[\mathsf{h}_*(f_*(\zeta_1))] = [\mathsf{h}_*(f_*(\zeta_2))]$ *in the lower row of the diagram.*

Proof. As shown in Corollary 2.48 the similarity classes of extensions of M by Q inducing φ are in a canonical bijection with the orbits of

$$\mathrm{Stab}_{\mathrm{Out}(M)\times\mathrm{Aut}(Q)}(\varphi) \curvearrowright H^2(Q;M).$$

What we need to do first is to compute the stabilizers and to show that their actions on appropriate cohomology groups extend to the diagram of actions (4.2). In Proposition 2.78 the stabilizer $\mathrm{Stab}_{\mathrm{Out}(M)\times\mathrm{Aut}(Q)}(\varphi)$ was computed; in the case of a faithful action the stabilizer was shown to be isomorphic to $N_{\mathrm{Out}(M)}(\mathrm{im}\,\varphi)$. As stated therein, the inner automorphisms of Q act trivially, and in the case of Frobenius group Frob_{11}, all automorphisms are inner – see Proposition 2.89. This allows us to compute the action of stabilizers to be that of $\mathbb{Z}^\times, \widehat{\mathbb{Z}}^\times, N_{\mathrm{GL}_{11}(\mathbb{Z})}(Q)$ and $N_{\mathrm{GL}_{11}(\widehat{\mathbb{Z}})}(Q)$ as in diagram (4.2), with the maps $\mathbb{Z}^\times \to N_{\mathrm{GL}_{11}(\mathbb{Z})}(Q)$ and $\widehat{\mathbb{Z}}^\times \to N_{\mathrm{GL}_{11}(\widehat{\mathbb{Z}})}(Q)$ being scalar embeddings $\kappa \mapsto \kappa\,\mathrm{Id}$ giving the commutative diagram of embeddings below; the actions of these stabilizers are equivariant with respect to the maps f_* and h^* between the respective cohomology groups.

$$\begin{array}{ccc} \mathbb{Z}^\times & \longrightarrow & N_{\mathrm{GL}_{11}(\mathbb{Z})}(Q) \\ \downarrow & & \downarrow \\ \widehat{\mathbb{Z}}^\times & \longrightarrow & N_{\mathrm{GL}_{11}(\widehat{\mathbb{Z}})}(Q) \end{array}$$

Now, as computed in Section 2.2.4, for $Q = \mathrm{Frob}_{11}$ we have $H^2(Q;\mathbb{Z}) \cong H^2_{\mathrm{cts}}(Q;\widehat{\mathbb{Z}}) \cong (\mathbb{Z}/10)$. Proposition 2.42 showed that the map $H^2(Q;\mathbb{Z}) \to H^2(Q;\mathbb{Z}^{11}) \cong H^2_{\mathrm{cts}}(Q;\widehat{\mathbb{Z}}^{11})$ is the identity.

Now, as before, the conjugation action of $Q < N_{\mathrm{GL}_{11}(\mathbb{Z})}(Q)$ on $H^2(Q;\mathbb{Z}^{11})$ is trivial, but $N_{\mathrm{GL}_{11}(\mathbb{Z})}(Q)/Q \cong \mathbb{Z}^\times$ as computed in Lemma 4.24, acting by scalar multiplication.

The cohomology classes ζ_1 and ζ_2 in $H^2(Q;\mathbb{Z})$ are in distinct orbits under the action of $\mathbb{Z}^\times$, as discussed in Section 1.4 and shown in Section 2.3.6. The orbits of the actions in the top row of Diagram (4.2) are the same, so $f_*(\zeta_1)$ and $f_*(\zeta_2)$ are in distinct orbits.

On the other hand, as discussed in Section 1.4, $\mathsf{h}_*(\zeta_2) = \kappa \cdot \mathsf{h}_*(\zeta_1)$ for $\kappa \in \widehat{\mathbb{Z}}^\times$ congruent to 3 modulo 10. Thus, $\mathsf{h}_*(\zeta_1)$ and $\mathsf{h}_*(\zeta_2)$ are in the same orbit of $\widehat{\mathbb{Z}}^\times \curvearrowright H^2(Q;\widehat{\mathbb{Z}})$ and their images in $N_{\mathrm{GL}_{11}(\widehat{\mathbb{Z}})}(Q) \curvearrowright H^2(Q;\widehat{\mathbb{Z}}^{11})$ are also in the same orbit. □

4.3 Results on profinite distinguishing crystallographic groups

Several results explored classes of crystallographic groups which are distinguished by their profinite completions.

Theorem 4.27 (Piwek-Popović-Wilkes)**.** *Crystallographic groups of dimensions at most* 4 *are distinguished by their profinite completions.*

For distinguishing the $\mathbb{Z}$-classes $Q_i < \mathrm{GL}_n(\mathbb{Z})$ within a given $\mathbb{Q}$-class $Q < \mathrm{GL}_n(\mathbb{Q})$ of size q the invariants used are the following.

- Comparing sizes of respective cohomology groups $H^1(Q_i;(\mathbb{Z}/q)^n)$ and $H^2(Q_i;(\mathbb{Z}/q)^n)$ – this computationally inexpensive invariant

separates 51 out of 64 $\mathbb{Z}$-class pairs in dimension 3, and 1105 out of 1429 pairs in dimension 4.

- Checking conjugacy in finite quotients $\mathrm{GL}_n(\mathbb{Z}/p)$ for primes $p \mid q$.
- Checking conjugacy in $\mathrm{GL}_n(\mathbb{Z}_p)$ for primes $p \mid q$.

All of the pairs of $\mathbb{Z}$-classes apart from one in dimension 4 can be distinguished this way. Interestingly the pair (Q_1, Q_2) which passes all the tests above turns out to not be conjugate in $\mathrm{GL}_4(\mathbb{Z}_{\{2,3\}}) \cong \mathrm{GL}_4(\mathbb{Z}_2) \times \mathrm{GL}_4(\mathbb{Z}_3)$, despite being conjugate in $\mathrm{GL}_4(\mathbb{Z}_2)$ and in $\mathrm{GL}_4(\mathbb{Z}_3)$. We take a close look at this pair in Example 4.28.

For distinguishing crystallographic groups with the same $\mathbb{Z}$-class $Q < \mathrm{GL}_n(\mathbb{Z})$ three checks are performed.

- If $H^2(Q; \mathbb{Z}^n)$ is of size 1 or 2, then no orbits can collapse in

$$N_{\mathrm{GL}_n(\mathbb{Z})}(Q) \curvearrowright H^2(Q; \mathbb{Z}^n) \to N_{\mathrm{GL}_n(\widehat{\mathbb{Z}})}(Q) \curvearrowright H^2(Q; \widehat{\mathbb{Z}}^n).$$

  This alone separates groups in 47 out of 73 $\mathbb{Z}$-classes in dimension 3, and 340 out of 710 $\mathbb{Z}$-classes in dimension 4.

- Let e' be the exponent of $H^2(Q; \mathbb{Z}^n)$, and $e = e'$ unless $e' = 2$, in which case $e = 4$. Then the action of the normalizer $N_{\mathrm{GL}_n(\widehat{\mathbb{Z}})}(Q)$ factors through the action of the normalizer $N_{\mathrm{GL}_n(\mathbb{Z}/e)}(Q)$. If no orbits collapse in

$$N_{\mathrm{GL}_n(\mathbb{Z}/e)}(Q) \curvearrowright H^2(Q; \mathbb{Z}^n) \to N_{\mathrm{GL}_n(\widehat{\mathbb{Z}})}(Q) \curvearrowright H^2(Q; \widehat{\mathbb{Z}}^n),$$

  then all groups in the $\mathbb{Z}$-class are separated.

- Otherwise, the image of $N_{\mathrm{GL}_n(\widehat{\mathbb{Z}})}(Q)$ in $N_{\mathrm{GL}_n(\mathbb{Z}/e)}(Q)$ is computed, and the orbits of its action on $H^2(Q; \widehat{\mathbb{Z}}^n)$ are computed too.

It may seem concerning that a pair of groups $Q_1, Q_2 < \mathrm{GL}_4(\mathbb{Z})$ be conjugate in $\mathrm{GL}_4(\mathbb{Z}_2)$ and $\mathrm{GL}_4(\mathbb{Z}_3)$ but not in $\mathrm{GL}_4(\mathbb{Z}_2) \times \mathrm{GL}_4(\mathbb{Z}_3)$.

Example 4.28. The following pair is precisely the pair from [PPW21]. Let $Q_1 = \langle X_1, Y \rangle$ and $Q_2 = \langle X_2, Y \rangle$ be subgroups of $\mathrm{GL}_4(\mathbb{Z})$ isomorphic to the dihedral group Dih_{12} of size 24 representing the two $\mathbb{Z}$-classes in the $\mathbb{Q}$-class with symbol $(4, 20, 11)$ in `CrystCat` package, where X_1, X_2 and Y are the following matrices satisfying $X_1^2 = X_2^2 = Y^{12} = \mathrm{Id}$ and $X_1 Y X_1^{-1} = X_2 Y X_2^{-1} = Y^{-1}$.

$$X_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Let's consider the following matrices $A, B \in \mathrm{GL}_4(\mathbb{Q})$. Their determinants are $\det(A) = -1/2$ and $\det(B) = -1/3$.

$$A = \begin{pmatrix} 1/2 & -1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2/3 & 1/3 \\ 0 & 0 & -1/3 & -2/3 \end{pmatrix}$$

We get the following equalities showing $AQ_1A^{-1} = BQ_1B^{-1} = Q_2$.

$$\begin{aligned} AYA^{-1} = BYB^{-1} &= Y^5 \\ AX_1A^{-1} &= X_2Y^{-1} \\ BX_1B^{-1} &= X_2Y^2 \end{aligned}$$

Since $A \in \mathrm{GL}_4(\mathbb{Z}_p)$ for all $p \neq 2$ and $B \in \mathrm{GL}_4(\mathbb{Z}_p)$ for all $p \neq 3$ this shows that Q_1 and Q_2 are conjugate in $\mathrm{GL}_4(\mathbb{Z}_p)$ for *all* primes p. As a consequence their images in $\mathrm{GL}_4(\mathbb{Z}/p^\alpha)$ are also conjugate for any prime power p^α.

However, the images of Q_1 and Q_2 turn out to *not* be conjugate in $\mathrm{GL}_4(\mathbb{Z}/6)$, so they are clearly not conjugate in $\mathrm{GL}_4(\widehat{\mathbb{Z}})$.

These computations are verified in `MAGMA` in Listing A.1.

Beyond the determination of the genus of $\mathbb{Z}^n \rtimes (\mathbb{Z}/p)$ groups done by Grunewald–Zalesskii in [GZ11] and described here in Proposition 4.21, the case of *non-split* $\mathbb{Z}^n$-by-$(\mathbb{Z}/p)$ extensions was investigated in the same paper, and extended by Genildo de Jesus Nery in [Ner21].

These non-split extensions of $\mathbb{Z}^n$ by $(\mathbb{Z}/p)$, also called *Bieberbach groups*, are the torsion-free crystallographic groups, which are exactly the fundamental groups of compact flat manifolds.

Theorem 4.29 (Corollary 1.2 in [Ner21] rephrased)**.** *Let G be a Bieberbach group with point group is isomorphic to $\mathbb{Z}/p$ for p a prime. Then G is profinitely rigid if and only if $p < 23$.*

Theorem 4.30 (Corollary 1.3 in [Ner21])**.** *Let X be an n-dimensional compact flat manifold with holonomy of prime order. If $n \leq 21$, then X is determined among all n-dimensional compact flat manifolds by the profinite completion of its fundamental group.*

De Jesus Nery recently extended this in [Ner24]: he determined the genus of Bieberbach groups with point groups $(\mathbb{Z}/m)$ for m square-free.

Finally, many profinite invariants of crystallographic groups were studied by the collaboration of Corson–Hughes–Möller–Varghese in [CHMV24]—mostly in the context of profinite rigidity of irreducible affine Coxeter groups, which they proved to be profinitely rigid in the absolute sense.

Chapter 5

Profinite rigidity of central extensions of 2-orbifold groups

5.1 Introduction

In [Hem14] Hempel constructed pairs of finite order homeomorphisms of surfaces so that the fundamental groups of their mapping tori are not isomorphic, but their profinite completions are isomorphic. In each case, the 4-manifolds obtained by taking a product with a circle become homeomorphic.

Later Wilkes showed in [Wil17] that the fundamental groups of Seifert fibre spaces are distinguished from each other apart from the examples given by Hempel; this used an earlier result of Bridson–Conder–Reid [BCR16] that Fuchsian groups are distinguished from each other (and other lattices in connected Lie groups) by their profinite completions.

This chapter, based on the author's article [Piw23], extends a part of this result by considering group extensions with centre $\mathbb{Z}^n$ for $n > 1$ and the quotient being a 2-orbifold group.

A related question was also studied by Ma–Wang in [MW23] – the authors were interested in extending Wilkes' results to the setting of 4-dimensional Seifert fibre spaces with the fibre a 2-torus and the base a closed orientable 2-orbifold. Their results contradict our Theorem F, but their Theorem 2 is incorrect, as we discuss in Section 5.6.

Original results

The first result of this chapter restricts the study of profinite rigidity within the class of groups with free abelian centre where the quotient is an infinite closed orientable 2-orbifold group: the only case we need to consider is when the kernel and the quotient are fixed.

Theorem E. *Let n_1, n_2 be natural numbers and Q_1, Q_2 be infinite fundamental groups of closed orientable 2-orbifolds. Let G_1 and G_2 be central extensions $\mathbb{Z}^{n_1}$-by-Q_1 and $\mathbb{Z}^{n_2}$-by-Q_2 respectively. If $\widehat{G}_1 \cong \widehat{G}_2$ then $n_1 = n_2$ and $Q_1 \cong Q_2$.*

Proposition 2.9 of [BRS23] is a similar result obtained independently.

The main result in this chapter is Theorem F which—apart from a single family of exceptions—determines for a fixed $n > 1$ and a fixed infinite closed orientable 2-orbifold group Q whether the central extensions $\mathbb{Z}^n$-by-Q are distinguished from each other by their profinite completions.

The result depends on the following definition of *Smith coefficients* d_j associated to the m-tuple of orders of cone points of the base orbifold. The exceptional family which Theorem F doesn't classify is characterised by $d_{m-(n-1)} = 12$.

Definition 5.1 (Smith coefficients)**.** Let $P = (p_1, \ldots, p_m)$ be an m-tuple of natural numbers. We define the *Smith coefficients* $d_1, \ldots, d_m$ associated to P as follows.

$$d_j := \frac{\gcd(p_{i_1} p_{i_2} \ldots p_{i_j} \text{ for } 1 \leq i_1 < i_2 < \ldots < i_j \leq m)}{\gcd(p_{i_1} p_{i_2} \ldots p_{i_{j-1}} \text{ for } 1 \leq i_1 < i_2 < \ldots < i_{j-1} \leq m)} \tag{5.1}$$

where for $j = 1$ the denominator is defined to be 1.

Theorem F. *Let Q be an infinite fundamental group of a closed orientable 2-orbifold with $m \geq 0$ cone points of orders $p_1, \ldots, p_m$, and let $d_1, \ldots, d_m$ be the Smith coefficients associated to $(p_1, \ldots, p_m)$. For $n > 1$ the following hold.*

1. *If $n > m$, or $d_{m-(n-1)} \in \{1, 2, 3, 4, 6\}$, then the non-isomorphic central extensions of $\mathbb{Z}^n$ by Q are distinguished from each other by their profinite completions.*

2. *If* $n \leq m$ *and* $d_{m-(n-1)} \notin \{1, 2, 3, 4, 6, 12\}$*, then there exist non-isomorphic central extensions* G_1, G_2 *of* $\mathbb{Z}^n$ *by* Q *with* $\widehat{G}_1 \cong \widehat{G}_2$*.*

The third result shows that in fact all examples of lack of rigidity come from the phenomenon described by Baumslag in [Bau74] – that $\widehat{G}_1 \cong \widehat{G}_2$ implies in this context that $G_1 \times \mathbb{Z} \cong G_2 \times \mathbb{Z}$.

Theorem G. *Let* Q *be an infinite fundamental group of a closed orientable* 2*-orbifold. Let* $n > 1$ *and* G_1, G_2 *be central extensions of* $\mathbb{Z}^n$ *by* Q *such that* $\widehat{G}_1 \cong \widehat{G}_2$*. Then* $G_1 \times \mathbb{Z} \cong G_2 \times \mathbb{Z}$*.*

Note that in Theorem F and Theorem G we made the assumption that $n > 1$. The theorems could be also stated for $n = 1$, but this case has already been covered by [Hem14, Wil17].

Structure of this chapter

Section 5.2 discusses 2-orbifold groups, their automorphisms and centres. In particular, in Section 5.2.2 we show that the 2-orbifold groups we consider and their profinite completions are centreless. In Section 5.2.5 we cite important results on automorphisms of 2-orbifold groups and their profinite completions, specifying how they act on the maximal torsion elements. Finally, Section 5.2.6 computes the cohomology groups of the 2-orbifold groups we consider and also of their profinite completions, as well as the actions of their automorphism groups on the cohomology groups.

Section 5.3 translates the questions of isomorphism of central extensions of $\mathbb{Z}^n$ by 2-orbifold groups and their profinite isomorphism to questions about orbits of certain matrix actions on sets of matrices.

Section 5.4 explores the translated question and gives a classification of when distinct orbits of the first action collapse to one orbit under the second action.

Section 5.5 states the final results back in the setting of distinguishing non-isomorphic central extensions of $\mathbb{Z}^n$ by 2-orbifold groups and proves them.

Finally, in Section 5.6 we relate our results to the preprint [MW23] and explain the contradiction between them.

5.2 2-orbifold groups and their properties

5.2.1 2-orbifold groups

When a group G acts on a n-manifold M properly discontinuously and freely, the quotient space $G\backslash M$ is naturally a manifold and we get a covering $M \twoheadrightarrow G\backslash M$. *Orbifolds* are objects that describe the quotient accurately when we allow the action to have non-trivial finite isotropy groups.

It is natural for us to set aside the technicalities associated to orbifolds, and restrict ourselves to only the geometric ones, and more precisely, to their fundamental groups.

Definition 5.2. Let M be one of $\mathbb{E}^n, \mathbb{H}^n$, or $\mathbb{S}^n$. A group $G < \mathrm{Isom}(M)$ is called an *n-orbifold group* if its action on M is properly discontinuous. We say that an n-orbifold group G is *orientable* if $G < \mathrm{Isom}^+(M)$. We call a 2-orbifold group *closed* if the quotient topological space $G\backslash M$ is homeomorphic to a closed surface.

Examples 5.3.

1. Any surface group $\pi_1 S$ is a closed 2-orbifold group.
2. $\mathrm{PSL}_2(\mathbb{Z}) < \mathrm{Isom}^+(\mathbb{H}^2)$ is an orientable 2-orbifold group, but it isn't a *closed* 2-orbifold group, as the quotient $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2$ is topologically a punctured sphere.
3. Any *branched cover* of surfaces of degree d gives rise to a pair $H < G$ where H is the fundamental group of the covering surface, G is a 2-orbifold group (which is in general *not* the fundamental group of the underlying surface of the quotient) and $[G : H] = d$.
4. Every discrete lattice in $\mathrm{PSL}_2(\mathbb{R})$ (i.e. a *Fuchsian group*) is a 2-orbifold group, but only the *cocompact* lattices are closed 2-orbifold groups.

The closed orientable 2-orbifold groups have presentations which are convenient to work with, which were known already by Klein [Kle83], and derived from Poincaré's work on fundamental convex polygons for Fuchsian groups – see [PS85].

Proposition 5.4. *A closed orientable* 2*-orbifold group G has a presentation*

$$\begin{aligned}\langle x_1, \dots, x_g, y_1, \dots, y_g, a_1, \dots, a_m \mid \Pi_{i=1}^{g}[x_i, y_i] \cdot a_1 \dots a_m = 1, \\ a_i^{p_i} = 1 \text{ for } i = 1, \dots, m\rangle,\end{aligned} \tag{5.2}$$

where g is the genus of the quotient surface $G\backslash M$, while m is the number of orbits of singular points and p_i is the size of the stabilizer of the i-th orbit.

To save some space in later sections, we introduce the following definition. The reader should be forewarned that the term "*nice* 2-orbifold group" is by no means standard.

Definition 5.5. A 2-orbifold group is called *nice* if it is closed and orientable, infinite and not isomorphic to $\mathbb{Z}^2$.

In fact, almost all of the closed orientable 2-orbifold groups are nice.

Proposition 5.6. *A closed orientable* 2*-orbifold group $G < \mathrm{Isom}^+(M)$ with an underlying surface of genus g and $m \geq 0$ cone points of orders $p_1, \dots, p_m$ is nice if and only if one of the following conditions holds.*

- $g > 1$.
- $g = 1$, *and* $m \geq 1$.
- $g = 0$, *and* $m \geq 4$.
- $g = 0$, *and* $m = 3$ *and* $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \leq 1$.

Proof. If G is infinite if and only if the orbifold Euler characteristic is non-positive, which is $\chi = 2 - 2g - \sum_{i=1}^{m}(1 - \frac{1}{p_i})$. The only case that needs to be excluded is $G \cong \mathbb{Z}^2$ which is $g = 1$ and $m = 0$. □

We will later use the following fact, which tells us that 2-orbifold groups are virtually surface groups. It was proved in [EEK82, BS83] for Fuchsian groups, (or [Fox52] in the spherical case corrected by [Cha83]); it is also a consequence of Theorem 2.5 of [Sco83].

Proposition 5.7. *Let G be a 2-orbifold group. Then, there is a finite index subgroup $H < G$ isomorphic to a surface group.*

We will also need to know the torsion elements of nice 2-orbifold groups.

Proposition 5.8. *Let G be a nice 2-orbifold group with presentation*

$$\langle x_1, \dots, x_g, y_1, \dots, y_g, a_1, \dots, a_m \mid \Pi_{i=1}^{g}[x_i, y_i] \cdot a_1 \dots a_m = 1, \\ a_i^{p_i} = 1 \text{ for } i = 1, \dots, m \rangle,$$

Then any torsion element of G is conjugate to a power of one of the a_i.

Proof. As an oriented 2-orbifold group, G is a subgroup of $\mathrm{Isom}^+(M)$ and since it's infinite $M = \mathbb{E}^2$ or $\mathbb{H}^2$. Any torsion element of $\mathrm{Isom}^+(M)$ must fix a point in M. Since the action away from the lifts of cone points is free, the fixed point of the torsion element must be one of the lifts of cone points and thus the torsion element is conjugate to a power of one of a_i. □

5.2.2 Trivial centres

The fact that nice 2-orbifolds have trivial centres is well known. It is more difficult though to extend this to the profinite completion, which is the content of Proposition 5.10.

Proposition 5.9. *Let G be a nice 2-orbifold group. Then $Z(G) = 1$.*

Proof. Consider G as a subgroup of $\mathrm{Isom}^+(M)$ where $M = \mathbb{E}^2$ or $\mathbb{H}^2$. Then, take $z \in Z(G)$, assume that $z \neq 1$. If z has a fixed point, then it acts as a non-identity rotation and has only one fixed point, and the whole group G has to fix it too. This would mean that G is finite cyclic, which is a contradiction.

If z has no fixed points, it must have a translation axis, which would be fixed by G (not necessarily pointwise). This would mean that G is virtually-cyclic, which is also excluded. □

Proposition 5.10. *Let G be a nice 2-orbifold group. Then $Z(\widehat{G}) = 1$.*

Proof. First, if $z \in Z(\widehat{G})$ is a torsion element, then it is conjugate to a torsion element in G, which we showed in Proposition 5.9 to be centerless. Thus z is of infinite order.

By Proposition 5.7 any 2-orbifold group G contains a surface group H as a finite-index subgroup. Since $\widehat{H}$ is a finite index subgroup of $\widehat{G}$, then for any $g \in \widehat{G}$ there is some $k \in \mathbb{N}$ such that $g^k \in \widehat{H}$. In particular, if $z \in Z(\widehat{G})$ and $z^k \in \widehat{H}$, we have $z^k \in Z(\widehat{H})$.

Now, if H is a fundamental group of a surface of genus $g > 1$, then $\widehat{H}$ is centerless as shown in Proposition 18 of [And74]. This would imply that $z^k = 1$, which is a contradiction.

If H is of genus 0, then G is a finite group, which can't be the case as G is nice and thus infinite.

If H is of genus 1, then G is a 2-dimensional crystallographic group – a discrete, cocompact subgroup of $\mathrm{Isom}^+(\mathbb{E}^2)$. There are exactly five orientation-preserving crystallographic groups in dimension 2 and they are split extensions $\mathbb{Z}^2 \rtimes (\mathbb{Z}/d)$ for $d = 1, 2, 3, 4, 6$ where the action is by rotation of square or hexagonal grid – see Section 3.2 of [Szc12]. The semidirect product structure carries over to the profinite completion (see Proposition 2.19) where a direct computation shows that the completions are centerless unless $d = 1$. □

The only infinite closed orientable 2-orbifold group which isn't nice, i.e. $\mathbb{Z}^2$, obviously doesn't satisfy the previous propositions. However, it turns out that for a central $\mathbb{Z}^n$-by-$\mathbb{Z}^2$ extension G unless $G \cong \mathbb{Z}^{n+2}$ the centre is the kernel $\mathbb{Z}^n$ of the extension. The same holds for the profinite completions of these extensions.

Proposition 5.11. *Any central extension of $\mathbb{Z}^n$ by $\mathbb{Z}^2$ is isomorphic to $H_k \times \mathbb{Z}^{n-1}$ for some integer $k \geq 0$, where H_k has presentation*

$$\langle a, b, c \mid [a, c] = [b, c] = 1, [a, b] = c^k \rangle.$$

Furthermore, unless $k = 0$, we have

$$Z(H_k) = \langle c \rangle, \qquad Z(\widehat{H}_k) = \overline{\langle c \rangle}.$$

Proof. The fact that any central $\mathbb{Z}^n$-by-$\mathbb{Z}^2$ extension admits such a form is a consequence of the Cohomological Hopf's Formula—see page 229 of [Wil24]—and the fact that $H^2(\mathbb{Z}^2; \mathbb{Z}^n) \cong \mathbb{Z}^n$. The number k is such that the cohomology class in $\mathbb{Z}^n$ is a k-th multiple of a primitive vector in $\mathbb{Z}^n$.

H_k can also be described as the set $\mathbb{Z}^3$ with multiplication given by

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + kx_1y_2).$$

Similarily, $\widehat{H}_k$ is isomorphic to the set $\widehat{\mathbb{Z}}^3$ with the same multiplication formula. In both cases

$$[(1, 0, 0), (x, y, z)] = (0, 0, ky)$$
$$[(0, 1, 0), (x, y, z)] = (0, 0, -kx).$$

This implies that unless $k = 0$, the centre is generated by the element $(0, 0, 1)$, as both $\mathbb{Z}$ and $\widehat{\mathbb{Z}}$ are torsion-free. □

5.2.3 Profinite rigidity of 2-orbifold groups

Here we cite a strong result which we will later make use of. It was stated as Corollary 4.2 of [Wil17], and proved using Theorem 1.1 of [BCR16].

Theorem 5.12. *Let O_1, O_2 be closed 2-orbifolds. If there is an isomorphism $\widehat{\pi_1(O_1)} \cong \widehat{\pi_1(O_2)}$, then $\pi_1(O_1) \cong \pi_1(O_2)$.*

Notice that our definition of a nice 2-orbifold requires being closed, so Theorem 5.12 applies to it.

5.2.4 Residual finiteness of central extensions of 2-orbifold groups

In order to study profinite rigidity of extensions $\mathbb{Z}^n$-by-Q we need to know that they are residually finite, and that they induce the full topology on the kernel. This is a consequence of Corollary 2.21 and the following facts: 2-orbifold groups have separable cohomology in dimension 2 – see Proposition 3.6 of [GJZZ08]; they are residually finite as a consequence of Mal'cev's Theorem [Mal40] – see Section III.7 of [LS77].

Proposition 5.13. *Let $1 \to N \to G \to Q \to 1$ be a short exact sequence such that $N = \mathbb{Z}^n$ and Q is a closed orientable 2-orbifold group. Then G is residually finite and $\overline{N} = \widehat{N}$ in $\widehat{G}$.*

5.2.5 Automorphisms and torsion elements

It will be important for us to study the automorphisms of nice 2-orbifold groups and their profinite completions. A result of Theorem 5.1 in [BCR16], which we state as Theorem 5.14, makes it viable for Fuchsian groups. To include the nice 2-orbifold groups which aren't Fuchsian groups, we rephrase Proposition 4.3 of [Wil17] as Proposition 5.15.

Theorem 5.14. *Let Q be a finitely generated Fuchsian group, then every finite subgroup of $\widehat{Q}$ is conjugate to a subgroup of Q and if two maximal finite subgroups of Q are conjugate in $\widehat{Q}$, then they are also conjugate in Q.*

Proposition 5.15. *Let Q be a nice 2-orbifold group isomorphic to a subgroup of $\mathrm{Isom}^+(\mathbb{E}^2)$. Then every torsion element of $\widehat{Q}$ is conjugate to a torsion element of Q, and if two torsion elements of Q are conjugate in $\widehat{Q}$, then they are already conjugate in Q.*

Theorem 5.14 and Proposition 5.15 together with Proposition 5.8 show that the generators a_i of a nice 2-orbifold group Q (as in presentation 5.2) are sent under any automorphism $\gamma\colon Q \to Q$ (respectively: under any automorphism $\gamma\colon \widehat{Q} \to \widehat{Q}$) to

$$g_i(a_{\sigma(i)}^{k_i}) := g_i a_{\sigma(i)}^{k_i} g_i^{-1}$$

where $g_i \in Q$ (respectively: $g_i \in \widehat{Q}$) and σ is a permutation in Sym_m.

Finally, it will prove useful to show that any permutation of a_i which respects the orders p_i is actually possible.

Proposition 5.16. *Let Q be a closed orientable 2-orbifold group with m cone points of orders $p_1, \dots, p_m$, and with presentation (5.2). Let $\sigma \in \mathrm{Sym}_m$ be a permutation of $\{1, 2, \dots, m\}$ such that $p_i = p_{\sigma(i)}$ for all $i = 1, \dots, m$. Then there exists an automorphism $\gamma \in \mathrm{Aut}(Q)$ such that $\gamma(a_i) = {}^{g_i}(a_{\sigma(i)})$ for some $g_i \in Q$.*

Proof. It is enough to prove the claim in the case where σ is a transposition, say $\sigma = (ij)$. Then

$$\begin{aligned} & a_1 \dots a_i \dots a_j \dots a_m \\ &= a_1 \dots a_{i-1} \cdot {}^{a_i}\big(a_{i+1} \dots a_j\big) \cdot a_i \cdot a_{j+1} \dots a_m \\ &= a_1 \dots a_{i-1} \cdot {}^{a_i}a_j \cdot {}^{a_i a_j^{-1}}\big(a_{i+1} \dots a_{j-1}\big) \cdot a_i \cdot a_{j+1} \dots a_m \end{aligned}$$

so we can define a homomorphism of Q by setting the following images of generators a_i (and keeping generators x_i, y_i fixed)

$$\begin{array}{lll} a_1 \mapsto a_1 & a_{i+1} \mapsto {}^{a_i a_j^{-1}}(a_{i+1}) & a_{j+1} \mapsto a_{j+1} \\ \vdots & \vdots & \vdots \\ a_{i-1} \mapsto a_{i-1} & a_{j-1} \mapsto {}^{a_i a_j^{-1}}(a_{j-1}) & a_m \mapsto a_m \\ a_i \mapsto {}^{a_i}a_j & a_j \mapsto a_i & \end{array}$$

All of the relators are sent to relators and each of the generators is in the image, so the map is surjective. Since Q is finitely generated residually finite, it is Hopfian and so γ is an automorphism. □

5.2.6 Cohomology of nice 2-orbifold groups

In this section we compute the cohomology groups for nice 2-orbifold groups, taking coefficient modules with trivial action.

We base our computation on a partial free resolution of length 3 given in Proposition 6.6 of [Wil17]. It should be noted that these results were known much earlier: Majumdar [Maj70] gives this resolution and computes all integral homology and cohomology groups, Patterson [Pat75] computes the second cohomology groups of Fuchsian groups and i.a. uses it to study certain extensions of them, and Ellis–Williams [EW05] study the cohomology groups and extensions of generalised triangle groups. The cohomology rings of Fuchsian groups were also computed in [Hug23] using a spectral sequence argument.

While we could quote these results together with the Universal Coefficient Theorem to compute the cohomology groups $H^2(Q;\mathbb{Z}^n)$, doing it in a concrete manner makes it easier to later compute the actions $\mathrm{Out}(\mathbb{Z}^n) \times \mathrm{Aut}(Q) \curvearrowright H^2(Q;\mathbb{Z}^n)$ and its profinite counterpart $\mathrm{Out}(\widehat{\mathbb{Z}}^n) \times \mathrm{Aut}(\widehat{Q}) \curvearrowright H^2_{\mathrm{prof}}\big(\widehat{Q};(\mathbb{Z}/N)^n\big)$, which we need for classifying group extensions up to isomorphism.

Proposition 5.17. *The presentation complex of the presentation (5.2) for a nice 2-orbifold group Q*

$$\underbrace{\mathbb{Z}Q\{\mathbf{R}_0,\dots,\mathbf{R}_m\}}_{C_2} \xrightarrow{d} \underbrace{\mathbb{Z}Q\{\mathbf{X}_1,\dots,\mathbf{Y}_g,\mathbf{A}_1,\dots,\mathbf{A}_m\}}_{C_1} \xrightarrow{d} \underbrace{\mathbb{Z}Q}_{C_0} \xrightarrow{d} \mathbb{Z}$$

can be resolved one dimension further by setting $C_3 = \mathbb{Z}Q\{\mathbf{Z}_1,\dots,\mathbf{Z}_m\}$ where the map $d\colon C_3 \to C_2$ is defined as

$$d(\mathbf{Z}_i) = (a_i - 1)\cdot \mathbf{R}_i.$$

Since the extensions we study are central, it will be useful to study the chain complex in Proposition 5.17 after applying the functor $\mathbb{Z} \otimes_{\mathbb{Z}Q} (-)$. Indeed any homomorphism $\phi\colon C \to M$ from a $\mathbb{Z}Q$-module C to a *trivial* $\mathbb{Z}Q$-module M must satisfy $\phi(g\cdot c) = \phi(c)$ for any $g \in Q$ and $c \in C$. Thus—again assuming that M is a trivial Q-module—the functor $\mathrm{Hom}_{\mathbb{Z}Q}(-,M)\colon \mathbb{Z}Q\text{-}\mathsf{Modules} \to \mathsf{AbelianGroups}^{\mathrm{op}}$ factors as in Diagram (5.3).

$$\begin{array}{ccc} \mathbb{Z}\,Q\text{-Modules} & \xrightarrow{\quad\mathrm{Hom}_{\mathbb{Z}\,Q}(-,M)\quad} & \mathsf{AbelianGroups}^{\mathrm{op}} \\ & {\scriptstyle \mathbb{Z}\otimes_{\mathbb{Z}\,Q}(-)} \searrow \quad \nearrow {\scriptstyle \mathrm{Hom}_{\mathbb{Z}}(-,M)} & \\ & \mathsf{AbelianGroups} & \end{array} \tag{5.3}$$

Proposition 5.18. *Functor* $\mathbb{Z}\otimes_{\mathbb{Z}\,\Delta}(-)\colon \mathbb{Z}\,\Delta\text{-Modules} \to \mathsf{AbelianGroups}$ *applied to the complex* $C_3 \to C_2 \to C_1 \to C_0$ *from Proposition 5.17 gives a complex* $D_3 \to D_2 \to D_1 \to D_0$ *which together with maps* $1\otimes(-)\colon C_i \to \mathbb{Z}\otimes_{\mathbb{Z}\,Q} C_i$ *form Diagram (5.4) which commutes. The maps* $D_3 \to D_2$ *and* $D_1 \to D_0$ *are zero, and the map* $d\colon D_2 \to D_1$ *is given by*

$$\mathbf{r}_0 \mapsto \sum_{i=1}^{m} \mathbf{a}_i \quad \textit{and} \quad \mathbf{r}_i \mapsto p_i \cdot \mathbf{a}_i.$$

$$\begin{array}{ccccccc} \underbrace{\mathbb{Z}Q\{\mathbf{z}_1,\dots,\mathbf{z}_m\}}_{C_3} & \xrightarrow{d} & \underbrace{\mathbb{Z}Q\{\mathbf{R}_0,\dots,\mathbf{R}_m\}}_{C_2} & \xrightarrow{d} & \underbrace{\mathbb{Z}Q\{\mathbf{x}_1,\dots,\mathbf{A}_m\}}_{C_1} & \xrightarrow{d} & \underbrace{\mathbb{Z}Q}_{C_0} \\ {\scriptstyle 1\otimes(-)}\downarrow & & {\scriptstyle 1\otimes(-)}\downarrow & & {\scriptstyle 1\otimes(-)}\downarrow & & {\scriptstyle 1\otimes(-)}\downarrow \\ \underbrace{\mathbb{Z}\{\mathbf{z}_1,\dots,\mathbf{z}_m\}}_{D_3} & \xrightarrow{0} & \underbrace{\mathbb{Z}\{\mathbf{r}_0,\dots,\mathbf{r}_m\}}_{D_2} & \xrightarrow{d} & \underbrace{\mathbb{Z}\{\mathbf{x}_1,\dots,\mathbf{a}_m\}}_{D_1} & \xrightarrow{0} & \underbrace{\mathbb{Z}}_{D_0} \end{array} \tag{5.4}$$

Proof. The boundary maps $D_3 \to D_2$ and $D_1 \to D_0$ in the bottom row became zero maps, because $d(\mathbf{z}_i) = (a_i - 1)\cdot \mathbf{R}_i$ and $1\otimes(a_i-1)\cdot\mathbf{r}_i = 0\cdot\mathbf{r}_i$, and similarly for the map in degree 1. □

It is worth noting that the map $d\colon \mathbb{Z}\{\mathbf{r}_0,\dots,\mathbf{r}_m\} \to \mathbb{Z}\{\mathbf{x}_1,\dots,\mathbf{a}_m\}$ is described by Matrix (5.5).

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 1 & p_1 & 0 & \dots & 0 \\ 1 & 0 & p_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & p_m \end{pmatrix} \tag{5.5}$$

Proposition 5.19. *Given a nice 2-orbifold group Q with presentation (5.2), the second cohomology group $H^2(Q;M)$ of Q with trivial coefficient module M is*

$$H^2(Q;M) \cong \bigoplus_{i=1}^{m} M \cdot \mathbf{r}_i^* \,/\, (\mathbf{r}_0^* + p_i \cdot \mathbf{r}_i^* \text{ for } i = 1, \ldots, m). \tag{5.6}$$

Proof. M is a trivial $\mathbb{Z}Q$-module, so applying the functor $\mathrm{Hom}_{\mathbb{Z}Q}(-, M)$ to the partial resolution

$$C_3 \to C_2 \to C_1 \to C_0$$

from Proposition 5.17 factors through the functor $\mathbb{Z} \otimes_{\mathbb{Z}Q} (-)$ as shown in Diagram (5.3). Thus we only need to apply the functor $\mathrm{Hom}_{\mathbb{Z}}(-, M)$ to the chain complex $D_3 \to D_2 \to D_1 \to D_0$ of Proposition 5.18, which gives us the following chain complex

$$\underbrace{M\{\mathbf{z}_1^*, \ldots, \mathbf{z}_m^*\}}_{\mathrm{Hom}(D_3,M)} \xleftarrow{0} \underbrace{M\{\mathbf{r}_0^*, \ldots, \mathbf{r}_m^*\}}_{\mathrm{Hom}(D_2,M)} \xleftarrow{d^*} \underbrace{M\{\mathbf{x}_1^*, \ldots, \mathbf{a}_m^*\}}_{\mathrm{Hom}(D_1,M)} \xleftarrow{0} \underbrace{M}_{\mathrm{Hom}(D_0,M)}$$

where $m \cdot \mathbf{b}^*$ denotes a map sending a basis element $\mathbf{b}$ to $m \in M$ and sending all the other basis elements to 0. Now, the kernel of the map $d^*\colon \mathrm{Hom}(D_2, M) \to \mathrm{Hom}(D_3, M)$ is the whole domain $\mathrm{Hom}(D_2, M)$ and so $H^2(Q;M)$ is the cokernel of d^*. Since all of $\mathbf{x}_i^*$ and $\mathbf{y}_i^*$ are mapped to zero, this cokernel is precisely

$$\bigoplus_{i=1}^{m} M \cdot \mathbf{r}_i^* \,/\, (\mathbf{r}_0^* + p_i \cdot \mathbf{r}_i^* \text{ for } i = 1, \ldots, m). \qquad \square$$

Our goal is classifying central extensions of $\mathbb{Z}^n$ by Q up to group isomorphism, which we showed is equivalent to classifying them up to similarity, so we need to compute the action $\mathrm{Out}(M) \times \mathrm{Aut}(Q) \curvearrowright H^2(Q;M)$, as introduced in Corollary 2.48.

Proposition 5.20. *Let $\alpha \in \mathrm{Out}(M)$ and $\gamma \in \mathrm{Aut}(Q)$ be automorphisms, and $\zeta = [\sum_{i=0}^{m} m_i \cdot \mathbf{r}_i^*]$ be a cohomology class in $H^2(Q;M)$. Then*

$$\alpha \cdot \zeta = \alpha_*(\zeta) = [\sum_{i=0}^{m} \alpha(m_i) \cdot \mathbf{r}_i^*]$$

and

$$\gamma^{-1} \cdot \zeta = \gamma^*(\zeta) = \kappa \cdot [m_0 \cdot \mathbf{r}_0^* + \sum m_i \cdot \mathbf{r}_{\sigma(i)}],$$

where $\gamma(a_i) = g_i a_{\sigma(i)}^{k_i} g_i^{-1}$ *and* $\kappa \in \mathbb{Z}$ *is such that* $\kappa \equiv k_i \mod p_i$. *Furthermore if* $M \cong \mathbb{Z}^r \oplus T$ *with* $r \geq 1$ *and* T *a finite abelian group, then* $\kappa \in \mathbb{Z}^\times = \{\pm 1\}$.

The proof of this theorem follows the general method of Lemma 2.37 as well as the proof of Proposition 6.6 of [Wil17], with the exception of changing $\mathbb{Z}$ to M.

Proof. The action of α is the usual action of the automorphism group of the coefficient module.

For computing γ^*, we follow Lemma 2.37 to first compute the chain map $\gamma_\#\colon C_\bullet \to C_\bullet$. Let's denote by F the free group generated by the letters $x_1, \ldots, y_g, a_1, \ldots, a_m$ and by R the kernel of $\pi\colon F \to Q$.

We need to choose a map $\beta\colon F \to F$ such that $\pi \circ \beta = \gamma \circ \pi$ as maps $F \to Q$. For easy calculation we choose $\beta(a_i) = g_i a_{\sigma(i)}^{k_i} g_i^{-1}$ on the generators a_i and extend it to the other generators in any way which makes the diagram commute. This is a valid choice as $(\pi \circ \beta)(a_i) = g_i a_{\sigma(i)}^{k_i} g_i^{-1} = (\gamma \circ \pi)(a_i)$.

Now, for $i = 1, \ldots, m$ we can easily compute the effect of β on the r_i.

$$\beta(r_i) = \beta(a_i)^{p_i} = (g_i a_{\sigma(i)}^{k_i} g_i^{-1})^{p_i} = (g_i a_{\sigma(i)}^{p_i} g_i^{-1})^{k_i} = (g_i r_{\sigma(i)} g_i^{-1})^{k_i}.$$

Just like in Lemma 2.37 we get a derivation $\nu\colon F \to C_1$ extending $a_i \mapsto \mathbf{A}_i$ and also for $i = 1, \ldots, m$

$$\begin{aligned}
\gamma_\#(\mathbf{A}_i) &= \nu(\beta(a_i)) = \nu(g_i) + g_i \nu(a_{\sigma(i)}^{k_i}) + g_i a_{\sigma(i)}^{k_i} \nu(g_i^{-1}) \\
&= (1 - g_i a_{\sigma(i)}^{k_i} g_i^{-1}) \cdot \nu(g_i) + (1 + a_{\sigma(i)} + \ldots + a_{\sigma(i)}^{k_i - 1}) \cdot \mathbf{A}_{\sigma(i)}, \\
\gamma_\#(\mathbf{R}_i) &= k_i g_i \cdot \mathbf{R}_{\sigma(i)}.
\end{aligned}$$

Since our module M is trivial the $\mathrm{Hom}_{\mathbb{Z}Q}(-, M)$ functor factors through $\mathbb{Z} \otimes_{\mathbb{Z}Q} (-)$, so we can compute

$$\begin{aligned}
(\mathrm{Id} \otimes \gamma_\#)(\mathbf{a}_i) &= k_i \cdot \mathbf{a}_{\sigma(i)} \text{ for } i = 1, \ldots, m \\
(\mathrm{Id} \otimes \gamma_\#)(\mathbf{r}_i) &= k_i \cdot \mathbf{r}_{\sigma(i)} \text{ for } i = 1, \ldots, m \\
(\mathrm{Id} \otimes \gamma_\#)(\mathbf{r}_0) &= \kappa \cdot \mathbf{r}_0 + \sum_{i=1}^{m} \mu_i \cdot \mathbf{r}_{\sigma(i)} \text{ for some } \kappa, \mu_i \in \mathbb{Z}.
\end{aligned}$$

Since the maps $(\mathrm{Id} \otimes \gamma_\#)$ and $(\mathrm{Id} \otimes d)$ commute, we get a relationship between κ and μ_i.

$$\big(\mathrm{Id} \otimes (\gamma_\# \circ d)\big)\,(\mathbf{r}_0) = (\mathrm{Id} \otimes \gamma_\#)(\mathbf{a}_1 + \ldots + \mathbf{a}_m) = \sum_{i=1}^{m} k_i \cdot \mathbf{a}_{\sigma(i)}$$

$$\big(\mathrm{Id} \otimes (d \circ \gamma_\#)\big)\,(\mathbf{r}_0) = (\mathrm{Id} \otimes d)(\kappa \cdot \mathbf{r}_0 + \sum_{i=1}^{m} \mu_i \cdot \mathbf{r}_{\sigma(i)}) = \sum_{i=1}^{m} (\kappa + p_i \mu_i) \cdot \mathbf{a}_{\sigma(i)}$$

so we get $k_i = \kappa + p_i \mu_i$ for $i = 1, \ldots, m$. To see what happens at the cohomology level, let's first apply $\mathrm{Hom}_{\mathbb{Z}}(-, M)$. We will write $m \cdot \mathbf{r}_j^*$ for the unique map $C_2 \to M$ which sends $\mathbf{r}_j$ to m and all other $\mathbf{r}_{j'}$ to 0, and similarly $m \cdot \mathbf{a}_i$ for an appropriate map $C_1 \to M$.

$$\begin{aligned}
\gamma^\#(m \cdot \mathbf{r}_0^*) &= \kappa \cdot m \cdot \mathbf{r}_0^* \\
\gamma^\#(m \cdot \mathbf{r}_i^*) &= \mu_i \cdot m \cdot \mathbf{r}_0^* + k_i \cdot m \cdot \mathbf{r}_{\sigma(i)}^* \\
d^*(m \cdot \mathbf{a}_i^*) &= m \cdot \mathbf{r}_0^* + p_i \cdot m \cdot \mathbf{r}_i^*
\end{aligned}$$

Finally, let's compute the action of γ^* on cohomology.

$$\begin{aligned}
\gamma^*([m \cdot \mathbf{r}_0^*]) &= \kappa \cdot [m \cdot \mathbf{r}_0^*] \\
\gamma^*([m \cdot \mathbf{r}_i^*]) &= \mu_i \cdot [m \cdot \mathbf{r}_0^*] + k_i \cdot [m \cdot \mathbf{r}_{\sigma(i)}^*] \\
&= \mu_i \cdot [m \cdot \mathbf{r}_0^*] + (\kappa + p_i \mu_i) \cdot [m \cdot \mathbf{r}_{\sigma(i)}^*] \\
&= \mu_i \cdot [m \cdot \mathbf{r}_0^*] + \kappa \cdot [m \cdot \mathbf{r}_{\sigma(i)}^*] + \mu_i \cdot [p_i \cdot m \cdot \mathbf{r}_{\sigma(i)}^*] \\
&= \kappa \cdot [m \cdot \mathbf{r}_{\sigma(i)}^*] + \mu_i \cdot [m \cdot \mathbf{r}_0^* + p_i \cdot m \cdot \mathbf{r}_{\sigma(i)}^*] \\
&= \kappa \cdot [m \cdot \mathbf{r}_{\sigma(i)}^*] + \mu_i \cdot [m \cdot \mathbf{r}_0^* + p_{\sigma(i)} \cdot m \cdot \mathbf{r}_{\sigma(i)}^*] \\
&= \kappa \cdot [m \cdot \mathbf{r}_{\sigma(i)}^*] + \mu_i \cdot [d^*(m \cdot \mathbf{a}_{\sigma(i)}^*)] = \kappa \cdot [m \cdot \mathbf{r}_{\sigma(i)}^*]
\end{aligned}$$

Thus, the action is by multiplication by $\kappa \in \mathbb{Z}$ together with permutation of the generators $\mathbf{r}_i^*$ for the cocycles $Z^2(Q; M)$.

Taking the $(m!)$-th power of γ^* to get rid of the permutation σ, we get for any $\zeta \in H^2(Q; M)$

$$\big(\gamma^*\big)^{m!}(\zeta) = \kappa^{m!} \cdot (\zeta)$$

and if $M \cong \mathbb{Z}^r \oplus T$, then $H^2(Q; M)$ contains a copy of $\mathbb{Z}^r$ too and so $\kappa = \pm 1$ since $(\gamma^{m!})^*$ is an automorphism. $\square$

Now, we can do the very similar calculations for the cohomology of profinite groups defined in Section 2.2.6.

Proposition 5.21. *Given a nice* 2*-orbifold group Q with presentation (5.2) and a finite trivial $\widehat{Q}$-module M, the second cohomology group is*

$$H^2_{\mathrm{prof}}(\widehat{Q}; M) = \bigoplus_{i=1}^{m} M \cdot \mathbf{r}_i^* \ / (\mathbf{r}_0^* + p_i \cdot \mathbf{r}_i^* \text{ for } i = 1, \dots, m). \tag{5.7}$$

Proof. The partial free resolution from Proposition 5.17 gives rise to a partial free resolution 5.8 of $\widehat{\mathbb{Z}}$ by $\widehat{\mathbb{Z}}[[\widehat{Q}]]$-modules by Lemma 2.61 – a crucial ingredient is that 2-orbifold groups have separable cohomology. We can use it to compute the $H^2_{\mathrm{prof}}(\widehat{Q}; \widehat{\mathbb{Z}}^n)$.

$$\widehat{C}_3 \xrightarrow{\widehat{d}_3} \widehat{C}_2 \xrightarrow{\widehat{d}_2} \widehat{C}_1 \xrightarrow{\widehat{d}_1} \widehat{C}_0 \xrightarrow{\widehat{d}_0} \widehat{\mathbb{Z}} \tag{5.8}$$

Applying the functor $\mathrm{Hom}_{\widehat{\mathbb{Z}}[[\widehat{Q}]]}(-, M)$ to the resolution 5.8 factors through the functor $\widehat{\mathbb{Z}} \otimes_{\widehat{\mathbb{Z}}[[\widehat{Q}]]} (-)$, similarly to Proposition 5.18.

Thus, we only need to apply the functor $\mathrm{Hom}_{\widehat{\mathbb{Z}}}(-, M)$ to the chain complex $\widehat{D}_3 \to \widehat{D}_2 \to \widehat{D}_1 \to \widehat{D}_0$ of $\widehat{D}_i = \widehat{\mathbb{Z}} \otimes_{\widehat{\mathbb{Z}}[[\widehat{Q}]]} \widehat{C}_i$, where incidently $\widehat{D}_i$ is the profinite completion of the abelian group D_i from Proposition 5.18 treated as a trivial $\widehat{\mathbb{Z}}[[\widehat{Q}]]$-module.

In particular, we get the commutative diagram 5.9, where $R := \widehat{\mathbb{Z}}[[\widehat{Q}]]$.

$$\begin{array}{ccccccc}
\underbrace{R\{\mathbf{z}_1, \dots, \mathbf{z}_m\}}_{\widehat{C}_3} & \xrightarrow{\widehat{d}} & \underbrace{R\{\mathbf{R}_0, \dots, \mathbf{R}_m\}}_{\widehat{C}_2} & \xrightarrow{\widehat{d}} & \underbrace{R\{\mathbf{x}_1, \dots, \mathbf{A}_m\}}_{\widehat{C}_1} & \xrightarrow{\widehat{d}} & \underbrace{R}_{\widehat{C}_0} \\
{\scriptstyle 1\otimes(-)}\downarrow & & {\scriptstyle 1\otimes(-)}\downarrow & & {\scriptstyle 1\otimes(-)}\downarrow & & {\scriptstyle 1\otimes(-)}\downarrow \\
\underbrace{\widehat{\mathbb{Z}}\{\mathbf{z}_1, \dots, \mathbf{z}_m\}}_{\widehat{D}_3} & \xrightarrow{0} & \underbrace{\widehat{\mathbb{Z}}\{\mathbf{r}_0, \dots, \mathbf{r}_m\}}_{\widehat{D}_2} & \xrightarrow{\widehat{d}} & \underbrace{\widehat{\mathbb{Z}}\{\mathbf{x}_1, \dots, \mathbf{a}_m\}}_{\widehat{D}_1} & \xrightarrow{0} & \underbrace{\widehat{\mathbb{Z}}}_{\widehat{D}_0}
\end{array} \tag{5.9}$$

Applying the functor $\mathrm{Hom}_{\widehat{\mathbb{Z}}}(-, M)$ to the bottom row of Diagram (5.9) gives us chain complex in Diagram (5.10).

$$\underbrace{M\{\mathbf{z}_1^*, \dots, \mathbf{z}_m^*\}}_{\mathrm{Hom}(\widehat{D}_3, M)} \xleftarrow{0} \underbrace{M\{\mathbf{r}_0^*, \dots, \mathbf{r}_m^*\}}_{\mathrm{Hom}(\widehat{D}_2, M)} \xleftarrow[\widehat{d}^*]{} \underbrace{M\{\mathbf{x}_1^*, \dots, \mathbf{a}_m^*\}}_{\mathrm{Hom}(\widehat{D}_1, M)} \xleftarrow{0} \underbrace{M}_{\mathrm{Hom}(\widehat{D}_0, M)} \tag{5.10}$$

Now, the kernel of the map $\mathrm{Hom}(\widehat{D}_2, M) \to \mathrm{Hom}(\widehat{D}_3, M)$ is the whole $\mathrm{Hom}(\widehat{D}_2, M)$ and so $H^2_{\mathrm{prof}}(\widehat{Q}; M)$ is the cokernel of $\widehat{d}^*$. Since all of $\mathbf{x}_i^*$ and $\mathbf{y}_i^*$ are mapped to zero by $\widehat{d}^*$ this cokernel is precisely

$$\bigoplus_{i=1}^{m} M \cdot \mathbf{r}_i^* / (\mathbf{r}_0^* + p_i \cdot \mathbf{r}_i^* \text{ for } i = 1, \ldots, m). \qquad \square$$

Proposition 5.22. *Let $\alpha \in \mathrm{Out}(M)$ and $\gamma \in \mathrm{Aut}(\widehat{Q})$ be automorphisms, and $\zeta = [\sum_{i=0}^{m} m_i \cdot \mathbf{r}_i^*]$ be an element of $H^2_{\mathrm{prof}}(\widehat{Q}; M)$. Then*

$$\alpha \cdot \zeta = [\sum_{i=0}^{m} \alpha(m_i) \cdot \mathbf{r}_i^*]$$

and

$$\gamma^{-1} \cdot \zeta = \gamma^*(\zeta) = \kappa \cdot [m_0 \cdot \mathbf{r}_0^* + \sum m_i \cdot \mathbf{r}_{\sigma(i)}^*],$$

where $\gamma(a_i) = {}^{g_i}\left(a_{\sigma(i)}^{k_i}\right)$ and $\kappa \in \widehat{\mathbb{Z}}^\times$ is such that $\kappa \equiv k_i \mod p_i$.

The proof of Proposition 5.22 is a calculation following that in the proof of Proposition 5.20 with two differences. A general difference in the profinite setting is that in the profinite analogue of Lemma 2.37 the images $\beta(a_i)$ and $\gamma(r_j)$ aren't necessarily products of the a_i's and conjugates of r_j's respectively, but rather, they are *limits* of such products. A difference specific to the cohomology of $\widehat{Q}$ is that $\widehat{\mathbb{Z}}^\times$ is now a much larger group than $\mathbb{Z}^\times = \{\pm 1\}$.

5.3 Matrix Correspondence

In this section we introduce the Matrix Correspondence – a conglomerate of a few minor results and our main tool for studying the central extensions of $\mathbb{Z}^n$ by nice 2-orbifold groups. Essentially, it translates the problem of profinite rigidity among these extensions to a question regarding multiplying matrices and permuting columns.

Proposition 5.23 (Matrix Correspondence)**.** *Let Q be a nice 2-orbifold group with m cone points of order $p_1, \dots, p_m$.*

1. *There is a morphism of group actions*

$$\mathrm{GL}_n(\mathbb{Z}) \times \mathrm{Aut}(Q) \curvearrowright H^2(Q;\mathbb{Z}^n) \xrightarrow{\Psi} \mathrm{GL}_n(\widehat{\mathbb{Z}}) \times \mathrm{Aut}(\widehat{Q}) \curvearrowright H^2_{\mathrm{cts}}(\widehat{Q};\widehat{\mathbb{Z}}^n)$$

   given by sending a cohomology class ζ which represents a central $\mathbb{Z}^n$-by-Q extension $\left(0 \to \mathbb{Z}^n \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1\right)$ to the cohomology class in $H^2_{\mathrm{cts}}(\widehat{Q};\widehat{\mathbb{Z}}^n)$ representing $\left(0 \to \widehat{\mathbb{Z}}^n \xrightarrow{\widehat{\iota}} \widehat{G} \xrightarrow{\widehat{\pi}} \widehat{Q} \to 1\right)$.

2. *Cohomology classes ζ_1 and ζ_2 represent extensions $\left(0 \to \mathbb{Z}^n \to G_i \to Q \to 1\right)$ with $G_1 \cong G_2$ if and only if they lie in the same orbit of the action $\mathrm{GL}_n(\mathbb{Z}) \times \mathrm{Aut}(Q) \curvearrowright H^2(Q;\mathbb{Z}^n)$.*

   Furthermore, $\widehat{G}_1 \cong \widehat{G}_2$ if and only if $\Psi(\zeta_1)$ and $\Psi(\zeta_2)$ lie in the same orbit of $\mathrm{GL}_n(\widehat{\mathbb{Z}}) \times \mathrm{Aut}(\widehat{Q}) \curvearrowright H^2_{\mathrm{cts}}(\widehat{Q};\widehat{\mathbb{Z}}^n)$.

3. *We can identify $H^2(Q;\mathbb{Z}^n)$ with a quotient of $\mathbb{Z}^{n\times(m+1)}$ as follows.*

$$A_Q^{(n)} := \mathbb{Z}^{n\times(m+1)} / (\mathbf{r}_0^* + p_i \mathbf{r}_i^* \text{ for } i = 1, \dots, m)$$

   where $\mathbf{r}_0^*, \mathbf{r}_1^*, \dots, \mathbf{r}_m^*$ denote generators of $\mathbb{Z}^{n\times(m+1)}$ treated as a free $\mathbb{Z}^n$-module, where $x \cdot \mathbf{r}_i^*$ will denote the matrix having vector x as the i-th column and zeros everywhere else.

   Similarly, we can identify $H^2_{\mathrm{cts}}(\widehat{Q};\widehat{\mathbb{Z}}^n)$ with $\widehat{A}_Q^{(n)}$.

   Furthermore, the orbits of the action $\mathrm{GL}_n(\mathbb{Z}) \times \mathrm{Aut}(Q) \curvearrowright A_Q^{(n)}$ and of the action $\mathrm{GL}_n(\widehat{\mathbb{Z}}) \times \mathrm{Aut}(\widehat{Q}) \curvearrowright \widehat{A}_Q^{(n)}$ are the same as, respectively, those of $\mathrm{GL}_n(\mathbb{Z}) \times P \curvearrowright A_Q^{(n)}$, and of $\mathrm{GL}_n(\widehat{\mathbb{Z}}) \times P \curvearrowright \widehat{A}_Q^{(n)}$, where

$$P = \{\sigma \in \mathrm{Sym}_m \mid p_{\sigma(i)} = p_i \text{ for } i = 1, \dots, m\}$$

   acts by permuting the columns by $\sigma(\mathbf{r}_i^*) = \mathbf{r}_{\sigma(i)}^*$.

Proof of part 1. Since the action $\varphi\colon Q \to \mathrm{Out}(\mathbb{Z}^n)$ of Q on $\mathbb{Z}^n$ is trivial, the stabilizer $\mathrm{Stab}_{\mathrm{Out}(\mathbb{Z}^n)\times\mathrm{Aut}(Q)}(\varphi)$ is the whole $\mathrm{Out}(\mathbb{Z}^n) \times \mathrm{Aut}(Q)$. Proposition 2.59 together with $\mathrm{Out}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{Out}(\widehat{\mathbb{Z}}^n) \cong \mathrm{GL}_n(\widehat{\mathbb{Z}})$ complete the proof. □

Proof of part 2. By Proposition 5.9 nice 2-obrifold groups Q are centerless and by Proposition 2.66 for a centreless Q the family $\mathscr{E}_{\text{cent}}(Q;M)$ of central M-by-Q extensions is kernel-detecting (in the sense of Definition 2.65), so any group-isomorphism of such extensions is actually an extension similarity. Now, by Corollary 2.48 the orbits of the action

$$\text{Stab}_{\text{Out}(M)\times\text{Aut}(Q)}(\varphi) \curvearrowright H^2(Q;M)$$

classify precisely the similarity classes of extensions inducing φ.

□

Proof of part 3. Propositions 5.19 and 5.21 computed the second cohomology group $H^2(Q;M)$ for trivial Q-modules M and $H^2_{\text{prof}}(\widehat{Q};M)$ for finite trivial $\widehat{Q}$-modules M. Thus we indeed have $H^2(Q;\mathbb{Z}^n) \cong A_Q^{(n)}$ via a natural isomorphism. Similar computation shows

$$H^2\big(Q;(\mathbb{Z}/N)^n\big) \cong H^2_{\text{prof}}\big(\widehat{Q};(\mathbb{Z}/N)^n\big) \cong A_Q^{(n)}/NA_Q^{(n)} \tag{5.11}$$

and together with Theorem 2.57 this gives a canonical isomorphism

$$H^2_{\text{cts}}(\widehat{Q};\widehat{\mathbb{Z}}^n) \cong \varprojlim_N H^2_{\text{prof}}\big(\widehat{Q};(\mathbb{Z}/N)^n\big) \cong \widehat{A}_Q^{(n)},$$

and the map Ψ can be identified with $\mathsf{h}\colon A_Q^{(n)} \to \widehat{A}_Q^{(n)}$.

Finally, we need to use the explicit formulas from Propositions 5.20 and 5.22 for the actions of $\text{Aut}(Q)$ and $\text{Aut}(\widehat{Q})$ to show that the orbits don't change when we restrict the actions to smaller subgroups—respectively $\text{GL}_n(\mathbb{Z}) \times P$ and $\text{GL}_n(\widehat{\mathbb{Z}}) \times P$. An element $\gamma^{-1} \in \text{Aut}(Q)$ acts on $\zeta = [m_0 \cdot \mathbf{r}_0^* + \sum_{i=1}^m m_i \cdot \mathbf{r}_i^*]$ by

$$\gamma^{-1} \cdot \zeta = \gamma^*(\zeta) = \kappa \cdot [m_0 \cdot \mathbf{r}_0^* + \sum_{i=1}^m m_i \cdot \mathbf{r}^*_{\sigma(i)}],$$

where $\gamma(a_i) = {}^{g_i}\big(a_{\sigma(i)}^{k_i}\big)$ and $\kappa \in \mathbb{Z}^\times$ is such that $\kappa \equiv k_i \mod p_i$. Thus, for all $\zeta \in H^2(Q;\mathbb{Z}^n)$ we have $(\text{Id},\gamma^{-1}) \cdot \zeta = (\kappa \cdot \text{Id},\sigma) \cdot \zeta$, and thus the orbits of $\text{GL}_n(\mathbb{Z}) \times \text{Aut}(Q)$ and of $\text{GL}_n(\widehat{\mathbb{Z}}) \times P$ are indeed the same. Similarly, an automorphism $\gamma^{-1} \in \text{Aut}(\widehat{Q})$ acts by the very same formula with the exception that κ is now allowed to be in a much bigger group $\widehat{\mathbb{Z}}^\times$. Finally, $\kappa \cdot \text{Id}$ is an element of $\text{GL}_n(\widehat{\mathbb{Z}})$ and thus the orbits of $\text{GL}_n(\widehat{\mathbb{Z}}) \times \text{Aut}(\widehat{Q})$ are the same as those of $\text{GL}_n(\widehat{\mathbb{Z}}) \times P$. □

Remark 5.24. For a finitely generated group Q with separable cohomology in dimension 2 and for a finitely generated Q-module M the homomorphism $H^2(Q;M) \to H^2_{\text{cts}}(\widehat{Q};\widehat{M})$ introduced in Proposition 2.59 extends to a homomorphism $\widehat{H^2(Q;M)} \to H^2_{\text{cts}}(\widehat{Q};\widehat{M})$. This homomorphism doesn't have to be bijective in general. The fact that it is an isomorphism in Proposition 5.23 relies heavily on the equality (5.11) stating that for all m we have

$$H^2(Q;(\mathbb{Z}/N)^n) \cong H^2(Q;\mathbb{Z}^n)/\big(N \cdot H^2(Q;\mathbb{Z}^n)\big).$$

5.4 Using the Matrix Correspondence

This section aims at understanding the group $A_Q^{(n)}$ introduced in Proposition 5.23 together with the actions of $\mathrm{GL}_n(\mathbb{Z})$ and P on it. The question we investigate is: how are the orbits of $\mathrm{GL}_n(\mathbb{Z}) \times P \curvearrowright A_Q^{(n)}$ related to the orbits of $\mathrm{GL}_n(\widehat{\mathbb{Z}}) \times P \curvearrowright \widehat{A}_Q^{(n)}$?

Throughout this section Q will be any nice 2-orbifold group with cone points of orders $p_1,\dots,p_m$. Let $A_Q^{(n)}$ be as defined in Proposition 5.23, i.e. a quotient of the additive group $\mathbb{Z}^{n\times(m+1)}$ of $n \times (m+1)$ matrices with integer coefficients by relations $\mathbf{r}_0^* + p_i\mathbf{r}_i^*$ for $i = 1,\dots,m$ on columns.

5.4.1 General structure of $A_Q^{(n)}$

The next proposition enables us to understand better the structure of $A_Q^{(n)}$ and the action of $\mathrm{GL}_n(\mathbb{Z}) \times P$ on it.

Proposition 5.25. *The projection $\mathbb{Z}^{n\times(m+1)} \to \bigoplus_{i=1}^m (\mathbb{Z}/p_i)^n$ descends to a short exact sequence*

$$0 \to \mathbb{Z}^n \to A_Q^{(n)} \xrightarrow{q} \bigoplus (\mathbb{Z}/p_i)^n \to 0,$$

which is equivariant under the action of $\mathrm{GL}_n(\mathbb{Z}) \times P$. Also, P acts trivially on the kernel $\mathbb{Z}^n = \ker\big(q\colon A_Q^{(n)} \to \bigoplus(\mathbb{Z}/p_i)^n\big)$.

Proof. To make following the proof easier, all mentioned groups and maps between them are shown in Diagram (5.12).

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{Z}^n\{\mathbf{r}_0^* + p_i\mathbf{r}_i^*\} & = & \mathbb{Z}^n\{\mathbf{r}_0^* + p_i\mathbf{r}_i^*\} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathbb{Z}^n \oplus \bigoplus_{i=1}^m p_i\mathbb{Z}^n & \longrightarrow & \mathbb{Z}^{n\times(m+1)} & \longrightarrow & \bigoplus_{i=1}^m (\mathbb{Z}/p_i)^n & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & A_Q^{(n)} & \xrightarrow{q} & \bigoplus_{i=1}^m (\mathbb{Z}/p_i)^n & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{5.12}$$

All of the relations $\mathbf{r}_0^* + p_i\mathbf{r}_i^*$ are sent to 0 by the projection map, so we indeed get a map $A_Q^{(n)} \twoheadrightarrow \bigoplus(\mathbb{Z}/p_i)^n$. The kernel of this map is the quotient of $\mathbb{Z}^n \oplus \bigoplus p_i\mathbb{Z}^n$ by the kernel

$$\mathbb{Z}^n\{\mathbf{r}_0^* + p_i\mathbf{r}_i^* \mid i = 1, \ldots, m\} = \ker\left(\mathbb{Z}^{n\times(m+1)} \twoheadrightarrow A_Q^{(n)}\right),$$

which is isomorphic to $\mathbb{Z}^n$. □

The next result computes the group $A_Q^{(n)}$ (which is not an original result – see [Maj70, Hug23]) and relates this computation to the quotient map $q\colon A_Q^{(n)} \to \bigoplus_{i=1}^m \left(\mathbb{Z}/p_i\right)^n$.

Proposition 5.26. *There is an isomorphism*

$$\bigoplus_{i=1}^m \left(\mathbb{Z}/p_i\right)^n \xrightarrow{\alpha} \bigoplus_{j=1}^m \left(\mathbb{Z}/d_j\right)^n$$

where $(d_1, \ldots, d_m)$ are the Smith coefficients of $(p_1, \ldots, p_m)$ (see Definition 5.1).

There is also an isomorphism

$$A_Q^{(n)} \xrightarrow{\beta} \mathbb{Z}^n \oplus \bigoplus_{j=1}^{m-1} \left(\mathbb{Z}/d_j\right)^n.$$

Furthermore, α and β are equivariant with respect to the action of $\mathrm{GL}_n(\mathbb{Z})$ and this identification gives the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & A_Q^{(n)} & \xrightarrow{q} & \bigoplus_{i=1}^m \left(\mathbb{Z}/p_i\right)^n & \longrightarrow & 0 \\ & & \| & & \downarrow \beta & & \downarrow \alpha & & \\ 0 & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\cdot d_m} & \mathbb{Z}^n \oplus \bigoplus_{j=1}^{m-1} \left(\mathbb{Z}/d_j\right)^n & \longrightarrow & \bigoplus_{j=1}^m \left(\mathbb{Z}/d_j\right)^n & \longrightarrow & 0 \end{array}$$

Proof. The fact that α is an isomorphism and that d_j are given by the formula 5.1 comes from the computation of the Smith normal form using matrix minors. $A_Q^{(n)}$ is isomorphic to $\mathbb{Z}^n \otimes_{\mathbb{Z}} \operatorname{coker}\left(\mathbb{Z}^m \to \mathbb{Z}^{m+1}\right)$ where the map is given by the matrix

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ p_1 & 0 & \dots & 0 \\ 0 & p_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_m \end{pmatrix}.$$

We can first use the row and column operations which gave us α to get the diagonal matrix with d_j for $j = 1, 2, \dots, m$ and later do the column operations showed in Equation (5.13).

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ d_1 & 0 & \dots & 0 & 0 \\ 0 & d_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{m-1} & 0 \\ 0 & 0 & \dots & 0 & d_m \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ d_1 & 0 & \dots & 0 & 0 \\ 0 & d_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{m-1} & 0 \\ -d_m & -d_m & \dots & -d_m & d_m \end{pmatrix}. \tag{5.13}$$

Now, since $d_j \mid d_m$ for all $j = 1, 2, \dots, m$ we can fully eliminate the bottom row with row operations affecting only the bottom row, which means that the first m basis elements for $\mathbb{Z}^{m+1}$ weren't changed, whence that the diagram in the statement actually commutes.

Finally, because we only manipulated the $\mathbb{Z}^n$-bases, the maps are indeed $\mathrm{GL}_n(\mathbb{Z})$-equivariant. □

Proposition 5.25 allowed us to understand better the torsion part of $A_Q^{(n)}$, but it will be necessary to understand the torsion-free part too. Proposition 5.27 helps with this. It comes from the notion of *Euler class* defined for Seifert fibre spaces.

Proposition 5.27. *Let* $\tilde{E}\colon \mathbb{Z}^{n\times(m+1)} \to \mathbb{Z}^n$ *be defined by*

$$\tilde{E}\colon \sum_{i=0}^{m} x_i \cdot \mathbf{r}_i^* \mapsto -d_m \cdot x_0 + \sum_{i=1}^{m} \frac{d_m}{p_i} \cdot x_i.$$

Then $\tilde{E}$ *factors through* $\mathbb{Z}^{n\times(m+1)} \twoheadrightarrow A_Q^{(n)}$ *giving* $E\colon A_Q^{(n)} \to \mathbb{Z}^n$. *Furthermore, the map is surjective and equivariant with respect to the action of* $\mathrm{GL}_n(\mathbb{Z}) \times P$ *(with* P *acting trivially).*

Proof. First of all, notice that

$$\begin{aligned} d_m &= p_1 \dots p_m \,/\, \gcd(p_1 \dots p_{m-1},\ p_1 \dots p_{m-2}p_m,\ \dots,\ p_2p_3 \dots p_m) \\ &= \mathrm{lcm}(p_1, \dots, p_m), \end{aligned}$$

so each $\frac{d_m}{p_i}$ is an integer and the map $\tilde{E}\colon \mathbb{Z}^{n\times(m+1)} \to \mathbb{Z}^n$ is actually well-defined. All of the relations are sent to zero by the map

$$E(x \cdot \mathbf{r}_0^* + p_i \cdot x \cdot \mathbf{r}_i^*) = -d_m \cdot x + \frac{d_m}{p_i} \cdot p_i \cdot x = 0,$$

so $\tilde{E}$ factors through $\mathbb{Z}^{n\times(m+1)} \twoheadrightarrow A_Q^{(n)}$ giving $E\colon A_Q^{(n)} \to \mathbb{Z}^n$. $\tilde{E}$ is clearly equivariant with respect to the action of $\mathrm{GL}_n(\mathbb{Z})$ and also with respect to the action of P since elements of P permute the columns which have the same p_i. Since the quotient $\mathbb{Z}^{n\times(m+1)} \twoheadrightarrow A_Q^{(n)}$ is also equivariant, the induced map E is equivariant too. Now, for surjectivity let's notice that $\gcd(d_m/p_1, \dots, d_m/p_m) = 1$ so there exist integers k_i such that $\sum k_i \cdot \frac{d_m}{p_i} = 1$. Hence for any $v \in \mathbb{Z}^n$

$$E\big(\sum_{i=1}^{m} k_i \cdot v \cdot \mathbf{r}_i^*\big) = v$$

so the map is indeed surjective. □

5.4.2 Orbits collapsing

Using the quotient map shown in Proposition 5.25 we can already show that some orbits of $\mathrm{GL}_n(\mathbb{Z}) \times P \curvearrowright A_Q^{(n)}$ become identified in $\mathrm{GL}_n(\widehat{\mathbb{Z}}) \times P \curvearrowright \widehat{A}_Q^{(n)}$. This is the content of Proposition 5.29, but before we can prove it, we need Lemma 5.28 to know when the effect of acting by a profinite matrix $\hat{X} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$ on some $[A] \in A_Q^{(n)} < \widehat{A}_Q^{(n)}$ gives a matrix $\hat{X} \cdot [A] \in \widehat{A}_Q^{(n)}$ which still lies in (the copy of) $A_Q^{(n)}$.

Lemma 5.28. *Let $[A] \in A_Q^{(n)}$ and let $\hat{X}$ be a profinite matrix in $\mathrm{GL}_n(\widehat{\mathbb{Z}})$. Then $\hat{X} \cdot [A]$ is in $A_Q^{(n)}$ in and only if $\hat{X} \cdot E([A]) \in \mathbb{Z}^n$ for the map $E\colon A_Q^{(n)} \to \mathbb{Z}^n$ defined in Proposition 5.27.*

Proof. Consider the composition

$$(E \circ \beta^{-1})\colon \mathbb{Z}^n \oplus \bigoplus_{j=1}^{m-1} \left(\mathbb{Z}/d_j\right)^n \twoheadrightarrow \mathbb{Z}^n$$

as shown in Diagram (5.14).

$$\begin{array}{ccccc} \mathbb{Z}^n \oplus \bigoplus_{j=1}^{m-1} \left(\mathbb{Z}/d_j\right)^n & \xrightarrow[\cong]{\beta^{-1}} & A_Q^{(n)} & \xrightarrow{E} & \mathbb{Z}^n \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\mathbb{Z}}^n \oplus \bigoplus_{j=1}^{m-1} \left(\mathbb{Z}/d_j\right)^n & \xrightarrow[\cong]{\widehat{\beta}^{-1}} & \widehat{A}_Q^{(n)} & \xrightarrow{\widehat{E}} & \widehat{\mathbb{Z}}^n \end{array} \tag{5.14}$$

Its surjectivity implies that the kernel must be precisely the torsion subgroup $\bigoplus_{j=1}^{m-1} \left(\mathbb{Z}/d_j\right)^n$. Thus we can regard $E \circ \beta^{-1}$ as a projection onto the $\mathbb{Z}^n$ direct factor with possibly some automorphism of $\mathbb{Z}^n$.

Now, the bottom row is equivariant with respect to the action of $\mathrm{GL}_n(\widehat{\mathbb{Z}})$, so

$$\hat{X} \cdot E([A]) \in \mathbb{Z}^n \Longrightarrow \widehat{E}(\hat{X} \cdot [A]) \in \mathbb{Z}^n,$$

which means that the $\widehat{\mathbb{Z}}^n$-coordinate of $\hat{X} \cdot \beta^{-1}([A])$ is actually in $\mathbb{Z}^n$, but the set of all such elements is precisely the set $\beta(A_Q^{(n)}) \subset \widehat{\beta}(\widehat{A}_Q^{(n)})$. □

Proposition 5.29. *Assume that $n > 1$ and that there exists a natural number d different from $1, 2, 3, 4$ and 6, such that at least n of the numbers p_i are divisible by d. Then there exist $[A], [B] \in A_{\mathbb{Q}}^{(n)}$ and a matrix $\hat{X} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$ such that $\hat{X} \cdot [A] = [B]$, but there is no integer matrix $X \in \mathrm{GL}_n(\mathbb{Z})$ and $\sigma \in P$ such that $X \cdot [A] = \sigma \cdot [B]$.*

Proof. In Proposition 5.25 we defined the map $q \colon A_{\mathbb{Q}}^{(n)} \twoheadrightarrow \bigoplus_{i=1}^{m} \left(\mathbb{Z}/p_i\right)^n$. Setting l to be the number of p_i divisible by d, we can quotient further to get

$$\mathbb{Z}^{n\times(m+1)} \xrightarrow{p} A_{\mathbb{Q}}^{(n)} \xrightarrow{q} \bigoplus_{i=1}^{m} \left(\mathbb{Z}/p_i\right)^n \xrightarrow{q'} \bigoplus_{i=1}^{l} (\mathbb{Z}/d)^n \ .$$

The map q' is clearly equivariant with respect to the action of $\mathrm{GL}_n(\mathbb{Z})$ and since P can permute columns without changing their respective p_i, we get a quotient action of P on $\bigoplus_{i=1}^{l} \left(\mathbb{Z}/d\right)^n$ by column permutation.

We can assume that the p_i which are divisible by d are $p_1, \ldots, p_l$. Now, let a be an integer coprime to d and such that $a \not\equiv \pm 1 \mod p_n$—here is where we use that $d \neq 1, 2, 3, 4, 6$ and that $d \mid p_n$—and consider the following $n \times (m+1)$ matrix $A \in \mathbb{Z}^{n\times(m+1)}$.

$$A = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{pmatrix}$$

Now, take any matrix $\hat{X} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$ such that

- $\hat{X} \cdot E([A]) = E([A])$ – in particular, $\hat{X} \cdot [A] \in A_{\mathbb{Q}}^{(n)}$;
- $\det \hat{X} \not\equiv \pm 1 \mod d$.

We set $[B] := \hat{X} \cdot [A]$ and show that there is no $X \in \mathrm{GL}_n(\mathbb{Z})$ and $\sigma \in P$ such that

$$X \cdot (q' \circ q)([A]) = \sigma \cdot (q' \circ q)([\hat{X} \cdot [A]]) = \sigma \cdot (q' \circ q)([B]).$$

Then $(q' \circ q)([A])$ is the $n \times n$ identity matrix with $(n-l)$ zero columns appended on the right. Thus $X \cdot (q' \circ q)([A])$ is the matrix $(X \mod d)$ appended by $(n-l)$ zero columns. Note that $(X \mod d)$ can't have any zero columns. In a similar spirit $(q' \circ q)(\hat{X} \cdot [A])$ is the matrix $(\hat{X} \mod d)$—whose columns are all non-zero—followed by $(n-l)$ zero columns. This implies that σ sends the last $(n-l)$ columns to themselves and possibly permutes the first n ones.

Thus we get that $X \equiv \sigma \cdot \hat{X} \mod d$, which in turn implies that also $\det X \equiv \det \sigma \cdot \det \hat{X}$, but $\det X \equiv \pm 1$ and $\det \sigma \equiv \pm 1$, while $\hat{X}$ was chosen specifically so that $\det \hat{X} \not\equiv \pm 1$. □

5.4.3 Orbits separated

Analysing the action of $\mathrm{GL}_n(\widehat{\mathbb{Z}})$ closely we will show that sometimes orbits don't change.

Lemma 5.30. *Given any matrix $A \in \mathbb{Z}^{n \times (m+1)}$, we can find a matrix $X \in \mathrm{GL}_n(\mathbb{Z})$ such that $A' = XA$ is an upper-triangular matrix, i.e. that $A'_{ij} = 0$ for $i > j$.*

Proof. We can multiply by permutation matrices from $\mathrm{GL}_n(\mathbb{Z})$ to permute rows of A and multiply by elementary matrices to perform elementary row operations (adding an integer multiple of a row to another one). Following Euclid's algorithm, we can in this way make the first column have at most one non-zero entry—equal to the gcd of its entries—in the top place. We can do the same for the next column, not doing any operations that would change the first row. Proceeding inductively, we get an upper-triangular matrix. □

In the following result $\mathrm{SL}_n^{\pm 1}(R)$ denotes the group of $n \times n$ matrices with coefficients in a commutative ring R and determinant equal to 1 or -1.

Lemma 5.31. *In the following commuting diagram, with v_1 and v_2 being reductions modulo d*

$$\begin{array}{ccc} \mathrm{GL}_n(\mathbb{Z}) & \longrightarrow & \mathrm{SL}_n^{\pm 1}(\widehat{\mathbb{Z}}) \\ & \searrow^{v_1} & \downarrow v_2 \\ & & \mathrm{GL}_n(\mathbb{Z}/d) \end{array}$$

the image of v_1 is the same as the image of v_2. In particular, given a matrix $\hat{X}$ with profinite entries whose determinant is ± 1, we can find an integer matrix X with determinant ± 1 such that $\hat{X} \equiv X \mod d$.

Furthermore, if the first column of $\hat{X}$ is $(\pm 1, 0, \ldots, 0)^T$, then we can require that the first column of X is also $(\pm 1, 0, \ldots, 0)^T$ (with the same choice of sign).

Proof. The image of $v_2 \colon \mathrm{SL}_n^{\pm 1}(\widehat{\mathbb{Z}}) \to \mathrm{GL}_n(\mathbb{Z}/d)$ is inside $\mathrm{SL}_n^{\pm 1}(\mathbb{Z}/d)$, and the image of $v_1 \colon \mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/d)$ is $\mathrm{SL}_n^{\pm 1}(\mathbb{Z}/d)$, so two images must be the same.

For the second part, let's notice that if $\hat{X}_{1*} = (\pm 1, 0, \ldots, 0)^T$, then $\hat{X}$ is a block matrix $\begin{pmatrix} \pm 1 & (\hat{v}')^T \\ 0 & \hat{X}' \end{pmatrix}$ where $\det \hat{X}' = \pm 1$. Then we take $X' \equiv \hat{X}'$ and any $v' \in \mathbb{Z}^{n-1}$ such that $v' \equiv \hat{v}'$ modulo d. Then the matrix

$$X = \begin{pmatrix} \pm 1 & (v')^T \\ 0 & X' \end{pmatrix}$$

is in $\mathrm{GL}_n(\mathbb{Z})$ and $X \equiv \hat{X}$ modulo d. □

Having proved Lemma 5.30 and Lemma 5.31, we can prove our main result about orbits remaining distinct.

The reason that the statement counts j such that $d_j \notin \{1, 2, 3, 4, 6\}$ is that $1, 2, 3, 4, 6$ are the only numbers t such that $(\mathbb{Z}/t)^\times$ has at most two elements.

Proposition 5.32. *Let Q be a nice 2-orbifold with $m \geq 0$ cone points of order $p_1, \ldots, p_m$. Let $A_Q^{(n)} \cong \mathbb{Z}^n \oplus \bigoplus_{j=1}^{m-1} (\mathbb{Z}/d_j)^n$ as in Proposition 5.26 and let*

$$k = 1 + \left|\left\{1 \leq j \leq m-1 \mid d_j \notin \{1, 2, 3, 4, 6\}\right\}\right|.$$

Let $[A], [B] \in A_Q^{(n)}$ be related by $\hat{X} \cdot [A] = [B]$ for some $\hat{X} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$.

Then, if $n > k$, we can find a matrix $X \in \mathrm{GL}_n(\mathbb{Z})$ with integer coefficients such that $X \cdot [A] = \hat{X} \cdot [A] = [B]$.

Proof. First, let's take A and B to their upper-triangular form with respect to the decomposition $A_Q^{(n)} \cong \mathbb{Z}^n \oplus \bigoplus_{j=1}^{m-1} (\mathbb{Z}/d_{m-j})^n$ – note that here we wrote $A_Q^{(n)}$ as a quotient of $\mathbb{Z}^n \cdot \mathbf{s}_0^* \oplus \mathbb{Z}^n \cdot \mathbf{s}_{m-1}^* \oplus \ldots \oplus \mathbb{Z}^n \cdot \mathbf{s}_1^*$ and the matrices A' and B' were chosen to be upper-triangular with respect to this basis $(\mathbf{s}_0^*, \mathbf{s}_{m-1}^*, \ldots, \mathbf{s}_1^*)$. Let $X_A, X_B \in \mathrm{GL}_n(\mathbb{Z})$ be the matrices used to perform this upper-triangularisation, i.e. such that $X_A \cdot A$ and $X_B \cdot B$ are upper-triangular. With that we get

$$\begin{aligned} \hat{X} \cdot [A] = [B] &\iff \left(X_B \, \hat{X} \, X_A^{-1}\right) \cdot [X_A \cdot A] = [X_B \cdot B] \\ &\iff \hat{X}' \cdot [A'] = [B'], \end{aligned}$$

where $\hat{X}' = \left(X_B^{-1} \hat{X} \, X_A^{-1}\right)$, $A' = X_A \cdot A$ and $B' = X_B \cdot B$.

Now, let $A' = x_0 \cdot \mathbf{s}_0^* + x_{m-1} \cdot \mathbf{s}_{m-1}^* + \ldots + x_1 \cdot \mathbf{s}_1^*$ and similarly $B' = y_0 \cdot \mathbf{s}_0^* + y_{m-1} \cdot \mathbf{s}_{m-1}^* + \ldots + y_1 \cdot \mathbf{s}_1^*$. Then,

$$\hat{X}' \cdot [A'] = [B'] \iff \hat{X}' x_0 = y_0, \hat{X}' x_j \equiv y_j \mod d_j \text{ for } j = 1, \ldots, m-1.$$

Now comes the crucial bit of the proof.

Consider $\hat{X}'' \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$ obtained from $\hat{X}'$ by multiplying the last column by $\kappa \in \widehat{\mathbb{Z}}^\times$. We claim that if $\kappa \equiv 1 \mod d_{m-(n-1)}$, then still $\hat{X}'' \cdot [A'] = [B']$.

There are two cases to check here.

1. For $j = 0,\ m-1, m-2, \ldots, m-(n-2)$, the bottom coordinate of x_j is zero (as A' is upper-triangular). Then, changing the last column of $\hat{X}'$ doesn't affect the result of the multiplication; in fact $\hat{X}'' \cdot x_j = y_j$ with genuine equality, not just congruence.

2. For $j = m-(n-1), m-n, \ldots, 1$, we have $d_j \mid d_{m-(n-1)}$ and so $\hat{X}'' \equiv \hat{X}' \mod d_j$ as $\kappa \equiv 1 \mod d_{m-(n-1)}$. In particular, $\hat{X}'' \cdot x_j \equiv \hat{X}' \cdot x_j = y_j \mod d_j$.

Thus we know that indeed $\hat{X}'' \cdot [A'] = [B']$.

Now, notice that we have some freedom in the choice of κ. We only assumed that $\kappa \in \widehat{\mathbb{Z}}^\times$ and that $\kappa \equiv 1 \mod d_{m-(n-1)}$. Since $d_{m-(n-1)}$ is in the set $\{1,2,3,4,6\}$, the integers coprime to $d_{m-(n-1)}$ are $\pm 1 \mod d_{m-(n-1)}$, so in particular $\det \hat{X}' \equiv \pm 1 \mod d_{m-(n-1)}$. Since $\det \hat{X}'' = \kappa \cdot \det \hat{X}'$, we can choose $\kappa = \pm(\det \hat{X}')^{-1}$ so that $\det \hat{X}'' = \pm 1$, i.e. $\hat{X}'' \in \mathrm{SL}_n^{\pm 1}(\widehat{\mathbb{Z}})$.

At this point we know that $\hat{X}'' \cdot [A'] = [B']$ for some $\hat{X}'' \in \mathrm{SL}_n^{\pm 1}(\widehat{\mathbb{Z}})$. By Lemma 5.31, we can choose $X \in \mathrm{GL}_n(\mathbb{Z})$ such that $X \equiv \hat{X}'' \mod d_{m-1}$ and additionally if the first column of $\hat{X}''$ is $\pm(1,0,\ldots,0)^T$, then we can require the first column of X to be the same.

Now, since $d_j \mid d_{m-1}$ for all $j = 1,\ldots,m-1$, we have

$$X \cdot x_j \equiv \hat{X}'' \cdot x_j = y_j \mod d_j$$

for all $j \neq 0$. For $j = 0$ notice that if $x_0 \neq 0$, then the first column of $\hat{X}''$ must be $\pm(1,0,\ldots,0)^T$ and it is the same as the first column of X, so $X \cdot x_0 = y_0$. If $x_0 = y_0 = 0$, then we get $X \cdot x_0 = y_0$ trivially.

Finally, $X \cdot [A'] = [B']$ gives $\left(X_B^{-1} X\, X_A\right) \cdot [A] = [B]$. □

We can use the same idea of modifying the last column of $\hat{X}$ which was used in Proposition 5.32 to show that in fact the elements of $A_Q^{(n)}$ lying in the same orbit of $\mathrm{GL}_n(\widehat{\mathbb{Z}})$ actually become members of the same orbit of $\mathrm{GL}_{n+1}(\mathbb{Z}) \curvearrowright A_Q^{(n+1)}$ after appending with a row of zeros on the bottom.

Proposition 5.33. *The map $f\colon \mathbb{Z}^{n\times(m+1)} \to \mathbb{Z}^{(n+1)\times(m+1)}$ which appends a row of zeros to a matrix from $\mathbb{Z}^{n\times(m+1)}$ thereby making it an $(n+1)\times(m+1)$ matrix induces a map $\overline{f}\colon A_Q^{(n)} \to A_Q^{(n+1)}$. Furthermore, if $[A],[B] \in A_Q^{(n)}$ are such that $\hat{X} \cdot [A] = [B]$ for some $\hat{X} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$, then there exists $X \in \mathrm{GL}_{n+1}(\mathbb{Z})$ such that $X \cdot \overline{f}([A]) = f([B])$.*

Proof. Take X_A, X_B such that $[A'] = X_A \cdot [A]$ and $[B'] = X_B \cdot [B]$ are upper-triangular as matrices in $\mathbb{Z}^n \oplus \bigoplus_{j=1}^{m-1} \left(\mathbb{Z}/d_j\right)^n$. Then let $\hat{X}' = X_B\, \hat{X}\, X_A^{-1}$ so that $\hat{X}' \cdot [A'] = [B']$. Take $\hat{X}''$ to be the block diagonal matrix

$$\hat{X}'' = \begin{pmatrix} \hat{X} & 0 \\ 0^T & (\det \hat{X})^{-1} \end{pmatrix}.$$

Since the bottom rows of $\overline{f}([A'])$ and $\overline{f}([B'])$ are both 0^T, we get the equality $\hat{X}'' \cdot \overline{f}([A']) = \overline{f}([B'])$. Also, $\det \hat{X}'' = 1$, so we can choose a matrix $X \in \mathrm{GL}_{n+1}(\mathbb{Z})$ such that $X \equiv \hat{X}'' \mod d_{m-1}$ and such that if the first column of $\hat{X}''$ is $\pm(1,0,\ldots,0)^T$, then so is the first column of X. As before, this implies that $X \cdot \overline{f}([A']) = \overline{f}([B'])$ and so

$$\begin{pmatrix} X_B^{-1} & 0 \\ 0^T & 1 \end{pmatrix} \cdot X \cdot \begin{pmatrix} X_A & 0 \\ 0^T & 1 \end{pmatrix} \cdot \overline{f}([A]) = \overline{f}([B]). \qquad \square$$

5.4.4 Summary of results

Now we are ready to summarise our state of knowledge.

Proposition 5.34. *Let $n > 1$. Let Q be a nice 2-orbifold with cone points of orders $p_1, \ldots, p_m$. Let $d_1, \ldots, d_m$ be as in Proposition 5.26 – in particular, $d_j \mid d_{j+1}$. Then, exactly one of the following statements holds.*

1. *$n > m$ or $d_{m-(n-1)} \in \{1,2,3,4,6\}$, and $\hat{X} \cdot [A] = [B]$ for some $\hat{X} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$ implies that $X \cdot [A] = [B]$ for some $X \in \mathrm{GL}_n(\mathbb{Z})$.*

2. *$n \leq m$ and $d_{m-(n-1)} \notin \{1,2,3,4,6,12\}$, and there exist elements $[A], [B] \in A_Q^{(n)}$ such that $\hat{X} \cdot [A] = [B]$ for some $\hat{X} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$, but there are no $X \in \mathrm{GL}_n(\mathbb{Z})$ and $\sigma \in P$ such that $X \cdot [A] = \sigma \cdot [B]$.*

3. *$n \leq m$ and $d_{m-(n-1)} = 12$.*

Proof. Following Proposition 5.32 let

$$k = 1 + |\{j \leq m-1 \mid d_j \notin \{1,2,3,4,6\}\}|.$$

If $n > m$, then clearly $k < n$. If $d_{m-(n-1)} \in \{1,2,3,4,6\}$, then

$$d_1, \ldots, d_{m-(n-1)} \in \{1,2,3,4,6\}$$

and so

$$\{j \leq m-1 \mid d_j \notin \{1,2,3,4,6\}\} \subset \{m-(n-2), \ldots, m-1\},$$

so $k \leq |\{m-(n-2), \ldots, m-1\}| + 1 = n - 1 < n$. Thus, the hypotheses of Proposition 5.32 are satisfied so there indeed does exist $X \in \mathrm{GL}_n(\mathbb{Z})$ such that $X \cdot [A] = [B]$.

Alternatively, if $n \leq m$ and $d_{m-(n-1)} \notin \{1,2,3,4,6,12\}$, then there is some prime power $p^\alpha \notin \{2,3,4\}$ such that $p^\alpha \mid d_{m-(n-1)}$. Since

$$d_{m-(n-1)} \mid d_{m-(n-2)} \mid \ldots \mid d_{m-1} \mid d_m,$$

this implies that p^α divides at least n of $p_1, \ldots, p_m$. By Proposition 5.29, this implies that there are some $[A], [B] \in A_Q^{(n)}$ and $\hat{X} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$ such that $[B] = \hat{X} \cdot [A]$, but for no $X \in \mathrm{GL}_n(\mathbb{Z})$ and $\sigma \in P$ do we have $X \cdot [A] = \sigma \cdot [B]$.

The only number which isn't in $\{1,2,3,4,6\}$ and isn't divisible by a prime power different to 2, 3 and 4 is 12, so if conditions (1) and (2) don't hold, we must have $d_{m-(n-1)} = 12$. □

Remark 5.35. The attentive reader certainly noticed that the 'classification' isn't conclusive if $d_{m-(n-1)} = 12$. The reason for this is that in this case it is possible that there is no $X \in \mathrm{GL}_n(\mathbb{Z})$ such that $X \cdot [A] = \hat{X} \cdot [A] = [B]$, but there is a *pair* $(X, \sigma) \in \mathrm{GL}_n(\mathbb{Z}) \times \mathrm{Aut}(Q)$ such that $X \cdot [A] = \sigma \cdot [B]$. The author spent considerable time trying to understand this case, but wasn't able to arrive at a full classification.

5.5 Profinite rigidity of $\mathbb{Z}^n$-by-Q central extensions

Finally, having done the 'calculations', we can go on to proving our main result – deciding which central extensions of finitely generated free abelian groups by infinite 2-orbifold groups are distinguished from each other by their finite quotients.

Section 5.5.1 reduces the question to distinguishing between central extensions of the same $\mathbb{Z}^n$ by the same 2-orbifold group Q. Section 5.5.2 solves this reduced question.

We restate the results for the reader's convenience.

5.5.1 Kernel and quotient must agree

Theorem E. *Let n_1, n_2 be natural numbers and Q_1, Q_2 be infinite fundamental groups of closed orientable 2-orbifolds. Let G_1 and G_2 be central extensions $\mathbb{Z}^{n_1}$-by-Q_1 and $\mathbb{Z}^{n_2}$-by-Q_2 respectively. If $\widehat{G}_1 \cong \widehat{G}_2$ then $n_1 = n_2$ and $Q_1 \cong Q_2$.*

Proof. In Proposition 5.13 we showed that such central extensions are residually finite and that we get commutative diagrams of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^{n_i} & \longrightarrow & G_i & \longrightarrow & Q_i & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widehat{\mathbb{Z}}^{n_i} & \longrightarrow & \widehat{G}_i & \longrightarrow & \widehat{Q}_i & \longrightarrow & 1 \end{array}$$

where all of the vertical maps are the natural inclusions.

First, let's exclude the case that $Q \cong \mathbb{Z}^2$ and that G_1 is isomorphic to $\mathbb{Z}^{n_1+2}$– then also $G_2 \cong \mathbb{Z}^{n_1+2}$ and so Q is abelian, and thus it is $\mathbb{Z}^2$. The kernel of a surjection from $\mathbb{Z}^{n_1+2}$ to $\mathbb{Z}^2$ is isomorphic to $\mathbb{Z}^{n_1}$, so the theorem statement holds.

Second, in Section 5.2.2 we studied the centres of 2-orbifold groups and their profinite completions, showing in Proposition 5.10 that for any nice 2-orbifold group Q we have $Z(\widehat{Q}) = 1$. Since the extensions

$$0 \to \widehat{\mathbb{Z}}^{n_i} \to \widehat{G}_i \to \widehat{Q}_i \to 1$$

are also central, we get that $Z(\widehat{G}_i) = \widehat{\mathbb{Z}}^{n_i}$. In Proposition 5.11 we showed that in the case of $Q \cong \mathbb{Z}^2$ also $Z(\widehat{G}_i) = \widehat{\mathbb{Z}}^{n_i}$ unless $G_i \cong \mathbb{Z}^{n_i+2}$, which was covered in the previous paragraph. This gives

$$\widehat{\mathbb{Z}}^{n_1} \cong Z(\widehat{G}_1) \cong Z(\widehat{G}_2) \cong \widehat{\mathbb{Z}}^{n_2},$$
$$\widehat{Q}_1 \cong \widehat{G}_1 / Z(\widehat{G}_1) \cong \widehat{G}_2 / Z(\widehat{G}_2) \cong \widehat{Q}_2.$$

Now, finitely generated abelian groups are distinguished by their profinite completions, so we get $n_1 = n_2$.

The isomorphism of Q_1 and Q_2 follows directly from Theorem 5.12. □

5.5.2 Distinguishing $\mathbb{Z}^n$-by-Q central extensions

In this section, we state our main results for distinguishing central extensions of a fixed $\mathbb{Z}^n$ by a fixed infinite closed 2-orbifold group Q.

Theorem F. *Let Q be an infinite fundamental group of a closed orientable 2-orbifold with $m \geq 0$ cone points of orders $p_1, \dots, p_m$, and let $d_1, \dots, d_m$ be the Smith coefficients associated to $(p_1, \dots, p_m)$. For $n > 1$ the following hold.*

1. *If $n > m$, or $d_{m-(n-1)} \in \{1, 2, 3, 4, 6\}$, then the non-isomorphic central extensions of $\mathbb{Z}^n$ by Q are distinguished from each other by their profinite completions.*

2. *If $n \leq m$ and $d_{m-(n-1)} \notin \{1, 2, 3, 4, 6, 12\}$, then there exist non-isomorphic central extensions G_1, G_2 of $\mathbb{Z}^n$ by Q with $\widehat{G}_1 \cong \widehat{G}_2$.*

Proof. First, let's solve the case of $Q \cong \mathbb{Z}^2$. By Proposition 5.11, the central extensions of $\mathbb{Z}^n$ by $\mathbb{Z}^2$ are isomorphic to $H_k \times \mathbb{Z}^{n-1}$, where H_k has presentation $\langle a, b, c \mid [a, c] = [b, c] = 1, [a, b] = c^k \rangle$. The abelianisation of H_k is $\mathbb{Z}^{n+1} \oplus (\mathbb{Z}/k)$, so it is different for different values of k. Abelianisation is detected by profinite completions, so central extensions of $\mathbb{Z}^n$ by $\mathbb{Z}^2$ are distinguished by their profinite completions.

Now, let $Q \ncong \mathbb{Z}^2$. For $j = 1, 2$ let $\left(0 \to \mathbb{Z}^n \xrightarrow{\iota_j} G_j \xrightarrow{\pi_j} Q \to 1\right)$ be central extensions. From Proposition 5.23 we know that $G_1 \cong G_2$ if and only if the representatives $[A_j] \in A_Q^{(n)}$ lie in the same orbit of $\mathrm{GL}_n(\mathbb{Z}) \times P \curvearrowright A_Q^{(n)}$; furthermore, $\widehat{G}_1 \cong \widehat{G}_2$ in and only if $\hat{X} \cdot [A_1] = \sigma \cdot [A_2]$ for some $\hat{X} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$ and $\sigma \in P$.

Proposition 5.34 tells us when these conditions coincide given the hypotheses (1) and (2). □

Finally, we prove Theorem G.

Theorem G. *Let Q be an infinite fundamental group of a closed orientable 2-orbifold. Let $n > 1$ and G_1, G_2 be central extensions of $\mathbb{Z}^n$ by Q such that $\widehat{G}_1 \cong \widehat{G}_2$. Then $G_1 \times \mathbb{Z} \cong G_2 \times \mathbb{Z}$.*

Proof. First, in the case of $Q \cong \mathbb{Z}^2$ we have $\widehat{G}_1 \cong \widehat{G}_2 \iff G_1 \cong G_2$, so the statement holds.

The short exact sequences $0 \to \mathbb{Z}^n \xrightarrow{\iota_j} G_j \xrightarrow{\pi_j} Q \to 1$ stabilise to the sequences

$$0 \longrightarrow \mathbb{Z}^n \times \mathbb{Z} \xrightarrow{\iota_j \times \mathrm{Id}_{\mathbb{Z}}} G_j \times \mathbb{Z} \xrightarrow{\pi_j \times 1} Q \longrightarrow 1$$

represented by $\overline{f}([A_j])$ where $[A_j] \in A_Q^{(n)}$ represents the original extension, and where $\overline{f}\colon A_Q^{(n)} \to A_Q^{(n+1)}$ is the map defined in Proposition 5.33. It is induced from $f\colon \mathbb{Z}^{n\times(m+1)} \to \mathbb{Z}^{(n+1)\times(m+1)}$ by appending an $n \times (m+1)$ matrix with a zero row at the bottom.

Now, $\widehat{G}_1 \cong \widehat{G}_2$, so $\hat{X} \cdot [A_1] = \sigma \cdot [A_2] = [\sigma \cdot A_2]$ for some $\hat{X} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$ and $\sigma \in P$. In Proposition 5.33 it was shown that $\hat{X} \cdot [A] = [B]$ implies that $X \cdot \overline{f}([A]) = \overline{f}([B])$ for some $X \in \mathrm{GL}_{n+1}(\mathbb{Z})$. In the present case, it means that $X \cdot \overline{f}([A_1]) = \overline{f}([\sigma \cdot A_2]) = \sigma \cdot \overline{f}([A_2])$ and hence that $G_1 \times \mathbb{Z} \cong G_2 \times \mathbb{Z}$. □

5.6 Counterexamples to Theorem 2 of [MW23]

Theorem F of this thesis is in contradiction to the Theorem 2 of the preprint [MW23] of Ma–Wang, so the author deemed it appropriate to include examples showing that the mentioned Theorem 2 is in fact incorrect. The authors there did not consider the action on the cohomology group by the automorphism group of the kernel, which collapses cohomology orbits further, beyond the orbits of $\mathrm{Aut}(Q)$ and $\mathrm{Aut}(\widehat{Q})$.

Ma–Wang are aware of the error and are working on a correction.

We include the (edited) statement here for convenience.

Incorrect claim (Theorem 2 of [MW23])**.** *The closed orientable Seifert 4-manifolds with $\widetilde{\mathrm{SL}_2} \times \mathbb{E}$ geometry are profinitely rigid. For closed orientable Seifert 4-manifolds M and N with $\mathbb{H}^2 \times \mathbb{E}^2$ geometry of trivial monodromy, $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$ if and only if there exists a profinite integer $\kappa \in \widehat{\mathbb{Z}}^\times$ such that the Seifert invariants of M and N are (m_i, a_i, b_i) and $(m_i, \kappa a_i, \kappa b_i)$ respectively.*

Relating the claim to this chapter, the fundamental group of a closed orientable Seifert 4-manifold M is an extension of $\mathbb{Z}^2$ by a closed orientable 2-orbifold group Q. It has trivial monodromy if the action of Q on $\mathbb{Z}^2$ is trivial, i.e. the extension is central. Finally, Proposition 5.36 relates the cases of $\widetilde{\mathrm{SL}_2} \times \mathbb{E}$ and $\mathbb{H}^2 \times \mathbb{E}^2$ geometries to the value of the *rational Euler class*, a version of which we defined in Proposition 5.27.

Proposition 5.36. *Let G be a central extension of $\mathbb{Z}^2$ by a* hyperbolic *closed orientable* 2*-orbifold group Q. Then G is isomorphic to the fundamental group $\pi_1(M)$ of a closed orientable Seifert* 4*-manifold M with a geometric structure of one of the following types.*

- $\mathbb{H}^2 \times \mathbb{E}^2$*, if and only if the rational Euler class of G is* $(0,0)^T$.
- $\widetilde{\mathrm{SL}_2} \times \mathbb{E}$*, if and only if the rational Euler class of G is not* $(0,0)^T$.

Proof. We can build M topologically following a construction similar to that showing that any extension of $\mathbb{Z}$ by a geometric 2-orbifold group has fundamental group isomorphic to that of a Seifert fibre space. For details see [Zie69].

Theorem A of [Ue90] showed that a closed orientable Seifert 4-manifold M over a hyperbolic 2-orbifold O is either of geometry $\mathbb{H}^2 \times \mathbb{E}^2$, $\widetilde{\mathrm{SL}_2} \times \mathbb{E}$, or is not geometric. Theorem 4.7 of [Kem08] proved that M is geometric if and only if its monodromies form a finite group, and it distinguished the two geometries by the Euler class of M. □

Proposition 5.37. *Let Q be the* 2*-orbifold group with genus $g = 0$ and* 3 *cone points of orders* $5, 10$ *and* 15*. Then the groups G_1, G_2 and H_1, H_2 defined by the presentations below are fundamental groups of Seifert* 4*-manifolds M_1, M_2 and N_1, N_2 such that*

1. *N_1 and N_2 have geometry $\mathbb{H}^2 \times \mathbb{E}^2$ and $\widehat{H}_1 \cong \widehat{H}_2$, but there is no profinite integer $\kappa \in \widehat{\mathbb{Z}}^\times$ such that the Seifert invariants of N_2 are simultaneously κ-multiples of the Seifert invariants of N_1,*
2. *M_1 and M_2 have geometry $\widetilde{\mathrm{SL}_2} \times \mathbb{E}$ and $\widehat{G}_1 \cong \widehat{G}_2$, but $G_1 \ncong G_2$.*

$$
\begin{aligned}
Q &= \langle a, b, c \mid a^5 = b^{10} = c^{15} = abc = 1 \rangle \\
H_1 &= \langle a, b, c, d, e \mid (d, e \text{ central}), a^5 = d, b^{10} = e^2, c^{15} = d^{-3}e^{-3}, abc = 1 \rangle \\
H_2 &= \langle a, b, c, d, e \mid (d, e \text{ central}), a^5 = d, b^{10} = e^4, c^{15} = d^{-3}e^{-6}, abc = 1 \rangle \\
G_1 &= \langle a, b, c, d, e \mid (d, e \text{ central}), a^5 = d, b^{10} = e^2, c^{15} = e^{-3}, abc = 1 \rangle \\
G_2 &= \langle a, b, c, d, e \mid (d, e \text{ central}), a^5 = d, b^{10} = e^4, c^{15} = e^{-6}, abc = 1 \rangle
\end{aligned}
$$

Proof. We indicate which assertions are proved by each part.

The geometries of N_1, N_2 and M_1, M_2. The base orbifold O of Q is hyperbolic since $\frac{1}{5} + \frac{1}{10} + \frac{1}{15} < 1$. Let's compute the rational Euler classes of the fundamental groups of these manifolds.

$$
\begin{aligned}
e(H_1) &= \frac{1}{5}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{10}\begin{pmatrix}0\\2\end{pmatrix} + \frac{1}{15}\begin{pmatrix}-3\\-3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \\
e(H_2) &= \frac{1}{5}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{10}\begin{pmatrix}0\\4\end{pmatrix} + \frac{1}{15}\begin{pmatrix}-3\\-6\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \\
e(G_1) &= \frac{1}{5}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{10}\begin{pmatrix}0\\2\end{pmatrix} + \frac{1}{15}\begin{pmatrix}0\\-3\end{pmatrix} = \begin{pmatrix}1/5\\0\end{pmatrix} \\
e(G_2) &= \frac{1}{5}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{10}\begin{pmatrix}0\\4\end{pmatrix} + \frac{1}{15}\begin{pmatrix}0\\-6\end{pmatrix} = \begin{pmatrix}1/5\\0\end{pmatrix}
\end{aligned}
$$

Proposition 5.36 shows that the geometries of the Seifert 4-manifolds M_1, M_2, N_1 and N_2 are as claimed.

Isomorphisms $\widehat{G}_1 \cong \widehat{G}_2$ and $\widehat{H}_1 \cong \widehat{H}_2$. We will construct isomorphisms $\phi\colon \widehat{G}_1 \to \widehat{G}_2$ and $\psi\colon \widehat{H}_1 \to \widehat{H}_2$ directly, defining both by

$$
a \mapsto a, \quad b \mapsto be^{\kappa_1}, \quad c \mapsto ce^{-\kappa_1}, \quad d \mapsto d, \quad e \mapsto e^{\kappa_0},
$$

where $\kappa_0 \in \widehat{\mathbb{Z}}^\times$ is such that $\kappa_0 = 2 + 5\kappa_1$ for a profinite integer $\kappa_1 \in \widehat{\mathbb{Z}}$, i.e. $\kappa_0 \equiv 2$ modulo 5.

ϕ and ψ are homomorphisms—this can be checked directly; here we

do the less obvious checks only.

$$b^{10} \cdot e^{-2} \overset{\phi,\psi}{\longmapsto} b^{10} e^{10\kappa_1} \cdot e^{-2\kappa_0} = b^{10} e^{-4}$$
$$a \cdot b \cdot c \overset{\phi,\psi}{\longmapsto} a \cdot b e^{\kappa_1} \cdot c e^{-\kappa_1} = abc$$
$$c^{15} \cdot e^{3} \overset{\phi}{\mapsto} c^{15} e^{-15\kappa_1} \cdot e^{3\kappa_0} = c^{15} e^{6}$$
$$c^{15} \cdot d^{3} \cdot e^{3} \overset{\psi}{\mapsto} c^{15} e^{-15\kappa_1} \cdot d^{3} \cdot e^{3\kappa_0} = c^{15} d^{3} e^{6}$$

We can construct the inverses ϕ^{-1} and ψ^{-1} as follows, proving that ϕ and ψ are actually isomorphisms.

$$a \mapsto a, \quad b \mapsto b e^{-\kappa_1/\kappa_0}, \quad c \mapsto c e^{\kappa_1/\kappa_0}, \quad d \mapsto d, \quad e \mapsto e^{1/\kappa_0}.$$

The invariants of N_1 and N_2 are not multiplies of each other. The Seifert invariants of N_1 and N_2 are respectively $(5,1,0),(10,0,2),(15,-3,-3)$ and $(5,1,0),(10,0,4),(15,-3,-6)$, so the profinite integer κ would have to simultaneously satisfy the congruences

$$\begin{cases} \kappa \cdot 1 \equiv 1 \mod 5, \\ \kappa \cdot 2 \equiv 4 \mod 10, \end{cases}$$

which is not possible.

Non-isomorphism $G_1 \ncong G_2$. An isomorphism $\beta\colon G_1 \to G_2$ would preserve the fibre $\langle d, e\rangle \cong \mathbb{Z}^2$ (see Proposition 2.66) and thus induce a similarity of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \langle d,e\rangle & \xrightarrow{\iota_1} & G_1 & \xrightarrow{\pi_1} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & \langle d,e\rangle & \xrightarrow{\iota_2} & G_2 & \xrightarrow{\pi_2} & Q & \longrightarrow & 1 \end{array}.$$

By Proposition 5.20 we get that in Q

$$\gamma(a) = g_a a^{\kappa} g_a^{-1}, \quad \gamma(b) = g_b b^{\kappa} g_b^{-1}, \quad \gamma(c) = g_c c^{\kappa} g_c^{-1}$$

for some elements $g_a, g_b, g_c \in \Delta$ and $\kappa = \pm 1$. This means that for some $x, x', y, y', z, z' \in \mathbb{Z}$ we get equalities in G_2

$$\beta(a) = g_a a^{\kappa} g_a^{-1} \cdot d^{x} e^{x'}, \quad \beta(b) = g_b b^{\kappa} g_b^{-1} \cdot d^{y} e^{y'}, \quad \beta(c) = g_c c^{\kappa} g_c^{-1} \cdot d^{z} e^{z'},$$

which in turn give

$$\beta(d) = \beta(a^5) = d^{\kappa+5x}e^{5x'}, \quad \beta(e^2) = \beta(b^{10}) = d^{10y}e^{4\kappa+10y'}.$$

This allows us to compute $\alpha\colon \langle d, e\rangle \to \langle d, e\rangle$ as

$$\alpha(d) = d^{\kappa+5x}e^{5x'}, \quad \alpha(e) = d^{5y}e^{2\kappa+5y'},$$

which means that the matrix of α with respect to basis (d, e) is

$$A = \begin{pmatrix} \kappa + 5x & 5y \\ 5x' & 2\kappa + 5y' \end{pmatrix}.$$

Now, the determinant of A has to be ± 1 as $A \in \mathrm{GL}_2(\mathbb{Z})$, but

$$\det(A) = (\kappa + 5x) \cdot (2\kappa + 5y') - 5x' \cdot 5y \equiv 2\kappa^2 \equiv \pm 2 \mod 5. \qquad \square$$

Chapter 6

Profinite rigidity of free-by-cyclic groups with centre

This chapter differs from Chapters 3, 4 and 5 in that the techniques for studying group extensions developed in Chapter 2 aren't very helpful. The main reason for that is that the class of extensions we consider here, namely $\big(1 \to F_n \rtimes 1 \to F_n \rtimes_\phi \mathbb{Z} \to \mathbb{Z} \to 1\big)$ with $\Phi := [\phi]$ of finite order in $\mathrm{Out}(F_n)$, isn't kernel-detecting – indeed the first Betti number $b_1(F_n \rtimes_\phi \mathbb{Z})$ equals 1 only if the determinant $\det(\phi_{\mathrm{ab}} - \mathrm{Id})$ is non-zero. Because of this, completely different techniques have to be deployed.

6.1 Results and context

Free-by-cyclic groups constitute an immensely rich family of groups, reflecting the complexity of $\mathrm{Out}(F_n)$ itself.

Profinite rigidity of free-by-cyclic groups was explored in [BRW17] where the authors showed that F_2-by-$\mathbb{Z}$ groups are profinitely distinguished, and studied the rigidity of free-by-cyclic groups among the 3-manifold groups, continuing the direction of [BR20]. Later Jaikin-Zapirain extended the fibring results of [BR20]; in [JZ20] he proved that being fibred over the circle is a profinite invariant for compact orientable 3-manifolds.

In [HK23] Hughes–Kudlinska studied profinite rigidity within the class of free-by-cyclic groups: they proved that the irreducible free-by-cyclic groups with Betti number one have finite genus among all

irreducible free-by-cyclic groups. This was motivated by Liu's result of [Liu23] on the finite profinite genus for hyperbolic compact 3-manifolds. They also proved that profinitely isomorphic pairs share a number of invariants: notably, being Gromov-hyperbolic.

Original results

The first original result of this chapter is that F_n-by-$(\mathbb{Z}/m)$ groups are distinguished by their finite quotients. A special case of this—when m is a prime power—was proved by Grunewald and Zalesskii in Theorem 3.12 of [GZ11]. Their method however relied on a certain decomposition for F_n-by-$(\mathbb{Z}/p^\alpha)$ groups, which doesn't carry over to F_n-by-$(\mathbb{Z}/m)$ groups.

It is important to contrast this rigidity phenomenon to the case of $(\mathbb{Z}/m)$-by-free groups. Indeed, Baumslag's examples of the form $(\mathbb{Z}/11) \rtimes \mathbb{Z}$ share profinite completions and although they are $\mathbb{Z}$-by-finite, they are *not* $\mathbb{Z}$-by-$(\mathbb{Z}/m)$.

Theorem J. *Groups which are extensions of a finitely generated free group by a finite cyclic group are distinguished from each other by their finite quotients.*

The main ingredient in the proof of Theorem J is Theorem I, giving a complete isomorphism invariant for F_n-by-$(\mathbb{Z}/m)$ groups, which additionally can be determined from their profinite completion. It is the *finite subgroups conjugacy poset* – the poset of conjugacy classes of finite subgroups, where we also keep the information about the subgroup's size and the first Betti number of its centraliser.

A version of this poset has been seen to play an important role in other results on profinite rigidity. [CHMV23] proves that graph products of finite groups are profinitely distinguished from each other; it first establishes that the poset is a complete invariant and then proves that it can be determined from the profinite completion. There however, it is not necessary for the poset to retain any information about the centralisers. Recently, [BZ24] showed i.a. that under some conditions (which include p-goodness and finite virtual p-cohomological dimension) profinite comple-

tion induces a bijection on the conjugacy classes of finite p-subgroups and furthermore the normalisers and centralisers of such finite p-subgroups in the profinite completion are the closures of the normalisers and centralisers before the profinite completion.

Theorem I. *Let G and G' be F_n-by-$(\mathbb{Z}/m)$ extensions. Then they are isomorphic if and only if there is a* data-preserving *isomorphism between their finite subgroups conjugacy posets $\mathsf{FSC}(G)$ and $\mathsf{FSC}(G')$.*

Furthermore, the isomorphism between G and G' can be realised through sliding moves on any of their realisations as reduced graphs of finite cyclic groups and these moves can be constructed explicitly.

The nature of this isomorphism—the fact that it can be exhibited through *sliding moves* on their graphs of groups—is crucial for the proof of Theorem H, though *a posteriori* it results from the general results about deformation spaces – see [For02] and Theorem 7.2 of [GL07].

It uses the fact that we can use 'matching' graph of groups decompositions for G and $G/Z(G)$ and any sliding moves on the graph for $G/Z(G)$ can be repeated on the graph for G.

Theorem H. *Free-by-cyclic groups with non-trivial centre are distinguished from each other by their finite quotients.*

Outline of this chapter

Section 6.2 starts by describing how every free-by-cyclic group $G = F_n \rtimes_\phi \mathbb{Z}$ with $\Phi := [\phi]$ of finite order in $\mathrm{Out}(F_n)$ can be represented as a fundamental group of a graph of infinite cyclic groups. Then it determines the centre $Z(G)$ and it explains how a special kind of isomorphism between $G_1/Z(G_1)$ and $G_2/Z(G_2)$ implies an isomorphism between G_1 and G_2.

Section 6.3 introduces the notion of finite subgroups conjugacy poset for a group, explains how to find it for a graph of finite cyclic groups, and proves in Theorem I that it is a complete isomorphism invariant of F_n-by-$(\mathbb{Z}/m)$ groups.

Finally, Section 6.4 shows that the profinite completion of an F_n-by-$(\mathbb{Z}/m)$ group detects this poset, thereby proving Theorem J.

6.2 Reduction to the free-by-(finite cyclic) case

Let's start by noting that free-by-cyclic groups with non-trivial centre are precisely those of the form $F_n \rtimes_\phi \mathbb{Z}$ with $[\phi]$ of finite order in $\mathrm{Out}(F_n)$. We first prove a more general result.

Lemma 6.1 (Centre of a Semidirect Product)**.** *Let $G = N \rtimes_{\tilde{\varphi}} Q$ for $\tilde{\varphi}\colon Q \to \mathrm{Aut}(N)$. Then the centre $Z(G)$ consists of elements (n_0, q_0) satisfying the following three conditions.*

1. $q_0 \in Z(Q)$,
2. $n_0 \in \mathrm{Fix}_N(\tilde{\varphi}(Q))$,
3. $\tilde{\varphi}(q_0) = \mathrm{Ad}_{n_0}^{-1}$.

Proof. The first condition is least surprising, since under $G \twoheadrightarrow Q$ the centre $Z(G)$ is sent into $Z(Q)$.

For the second condition let $q \in Q$ be any element, while we assume that $(n_0, q_0) \in Z(G)$.

$$\begin{aligned}(n_0, q_0) &= (1, q) \cdot (n_0, q_0) \cdot (1, q^{-1}) \\ &= \big(\tilde{\varphi}(q)(n_0), q q_0 q^{-1}\big) = \big(\tilde{\varphi}(q)(n_0), q_0\big),\end{aligned}$$

where the last equality comes from the already established $q_0 \in Z(Q)$. Thus we get $n_0 = \tilde{\varphi}(q)(n_0)$, so n_0 is fixed by all of $\tilde{\varphi}(Q)$.

For the third condition take $n \in N$ to be any element.

$$(n n_0, q_0) = (n, 1) \cdot (n_0, q_0) = (n_0, q_0) \cdot (n, 1) = \big(n_0 \tilde{\varphi}(q_0)(n), q_0\big),$$

so $n_0 \tilde{\varphi}(q_0)(n) = n n_0$ and thus $\tilde{\varphi}(q_0) = \mathrm{Ad}_{n_0}^{-1}$.

Since the equations above hold if we assume the three conditions, they show that an element (n_0, q_0) which satisfies them commutes with any element of N and of Q, so it must be central. □

Lemma 6.1 gives us the wanted description of free-by-cyclic groups with centre.

Corollary 6.2. *Let $G = F_n \rtimes_\phi \mathbb{Z}$ be a free-by-cyclic group. Then for $n \geq 2$*

$$Z(G) = \{xt^k \in F_n \rtimes_\phi \mathbb{Z} \mid \phi^k = \mathrm{Ad}_{x^{-1}} \text{ as elements of } \mathrm{Aut}(F_n)\},$$

where $\mathrm{Ad}_x(g) = xgx^{-1}$.

In particular, G has trivial centre if and only $[\phi]$ is of infinite order in $\mathrm{Out}(F_n)$; if $[\phi]$ is of finite order and $G \ncong \mathbb{Z}^2$ then $Z(G) \cong \mathbb{Z}$.

Proof. The first condition of Lemma 6.1 holds as $\mathbb{Z}$ is abelian. The third condition implies that $\phi^k = \mathrm{Ad}_{x^{-1}}$. Finally, the second condition comes from the fact that

$$\mathrm{Ad}_{\phi(x^{-1})} = \phi \cdot \mathrm{Ad}_{x^{-1}} \cdot \phi^{-1} = \phi \cdot \phi^k \cdot \phi^{-1} = \mathrm{Ad}_{x^{-1}},$$

which in a non-abelian F_n implies $\phi(x) = x$. If $n = 1$, then either $\phi = \mathrm{Id}$ and $G \cong \mathbb{Z}^2$, or $\phi = -\mathrm{Id}$ and the centre is $\langle t^2 \rangle \cong \mathbb{Z}$. □

Next we prove that each such group is the fundamental group of a finite graph of $\mathbb{Z}$'s of a special type. Proposition 6.3 is similar to Proposition 4.1 of [Lev15], but we require a more detailed version.

Proposition 6.3. *Let $G = F_n \rtimes_\phi \mathbb{Z}$ be a free-by-cyclic group such that $[\phi]$ is of finite order in $\mathrm{Out}(F_n)$. Then there exists a reduced graph of groups $\mathcal{G} = (V, E)$ such that $G \cong \pi_1(\mathcal{G})$ and*

- *all G_v and G_e are isomorphic to $\mathbb{Z}$,*
- *we can choose the generators for $\mathbb{Z}$'s so that the inclusion maps $\iota_e \colon G_e \to G_{t(e)}$ are $\mathbb{Z} \xrightarrow{\times d_e} \mathbb{Z}$ for positive d_e,*
- *for any circuit $e_1 e_2 \dots e_k$ of edges in $\mathcal{G}$*

$$\frac{[G_{o(e_1)} : G_{e_1}] \cdot \ldots \cdot [G_{o(e_k)} : G_{e_k}]}{[G_{t(e_1)} : G_{e_1}] \cdot \ldots \cdot [G_{t(e_k)} : G_{e_k}]} = 1. \tag{6.1}$$

Proof. Since $[\phi]$ is of finite order in $\mathrm{Out}(F_n)$, by [Cul84] there exists a graph K with $\pi_1(K) \cong F_n$ and an automorphism $f\colon K \to K$ such that $[f_*] = [\phi]$ on $\mathrm{Out}(F_n)$. Doing subdivision if necessary we can assume that f inverts no edges. The mapping torus M_f has $\pi_1(M_f) \cong F_n \rtimes_\phi \mathbb{Z}$. On the other hand M_f is a graph of spaces: $S^1 \times$ (vertex with adjacent open edges) and $S^1 \times$ (open edge), which are $K(\mathbb{Z}, 1)$. Thus G is the fundamental group of a graph of groups whose vertex and edge groups are $\mathbb{Z}$'s. This graph can be made reduced.

The inclusion maps $G_e \to G_{t(e)}$ come from counting the edges in K of a given edge orbit which connect to a specific vertex v, and thus are multiplications by a positive integer.

Finally, G_e is the stabilizer of some edge $\tilde{e}$ under the action $\mathbb{Z} \curvearrowright K$, so $[\mathbb{Z} : G_e]$ is equal to the size of orbit $|\mathbb{Z}\,\tilde{e}|$; the same holds for vertices. Thus the numerator in Equation (6.1) becomes $|\mathbb{Z}\,\tilde{e}_1| \,/\, |\mathbb{Z}\,o(\tilde{e}_1)| \cdot \ldots \cdot |\mathbb{Z}\,\tilde{e}_k| \,/\, |\mathbb{Z}\,o(\tilde{e}_k)|$. Since $e_1 e_2 \ldots e_k$ is a circuit $o(\tilde{e}_{i+1}) = t(\tilde{e}_i)$ so the product is equal to $|\mathbb{Z}\,\tilde{e}_1| \,/\, |\mathbb{Z}\,t(\tilde{e}_1)| \cdot \ldots \cdot |\mathbb{Z}\,\tilde{e}_k| \,/\, |\mathbb{Z}\,t(\tilde{e}_k)|$ which is equal to the denominator. Thus the value of the fraction in Equation (6.1) is 1. □

Remark 6.4. Proposition 6.3 can be phrased in a more natural manner in the language of *Generalised Baumslag-Solitar* groups – fundamental groups of finite graphs of groups with all vertex and edge groups infinite cyclic. There, there is a notion of a *modular homomorphism* $\Delta_G\colon G \to \mathbb{Q}^\times$, which records for an element $g = g_1 e_1 \ldots e_k g_{k+1}$ the ratio in Equation (6.1) with an appropriate sign. For more details, see [Lev15].

Thus, our class of groups can be described as the GBS groups with trivial modular homomorphism.

Our main tool for studying these free-by-cyclic groups G will be considering the quotients $G/Z(G)$ which have a nice induced structure of graphs of finite cyclic groups. This is also discussed in Section 4 of [Lev07].

Proposition 6.5. *Let G be a free-by-cyclic group with centre and $\mathcal{G}$ be an associated reduced graph of $\mathbb{Z}$'s as described in Proposition 6.3, such that $G \cong \pi_1(\mathcal{G})$.*

Then, unless $G \cong \mathbb{Z}^2$,

$$Z(G) = \bigcap_{e \in E} G_e.$$

Furthermore $G/Z(G) \cong \pi_1(\mathcal{G}')$ *where* $\mathcal{G}'$ *is a graph of finite cyclic groups* $G'_v := \mathbb{Z}/d_v$ *and* $G'_e := \mathbb{Z}/d_e$ *which is isomorphic to* $\mathcal{G}$ *as a graph. With the induced choice of generators for local groups the inclusion maps* $\iota_e \colon \mathbb{Z}/d_e \to \mathbb{Z}/d_{t(e)}$ *are* $[1 \mod d_e] \mapsto [d_{t(e)}/d_e \mod d_{t(e)}]$ *with* $d_{t(e)}/d_e = [G_{t(e)} : G_e]$.

Proof. Let z be a nontrivial element of $Z(G)$. There are two cases to consider: in the action on the Bass-Serre tree T of $\mathcal{G}$, either z fixes a point or else it acts as a hyperbolic element.

The second case is impossible: as $\mathcal{G}$ is reduced, the action of G on T is minimal (i.e. there are no proper invariant subtrees), so the axis of $\langle z \rangle$, which is G-invariant, would be the whole tree, which together with assumption that $\mathcal{G}$ satisfies the conditions of Proposition 6.3 implies that $G \cong \mathbb{Z}^2$, which we excluded.

Thus each $z \in Z(G)$ fixes a non-empty subtree $T_0 \subset T$, and since this is G-invariant, we again use minimality to conclude $T_0 = T$, forcing z to be contained in $\bigcap\{G_e^g \mid e \in E, g \in G\}$, the kernel of the action on T. We also need to note that for any subgroup $H < G$ and any element $g \in G$ we have $Z(G) \cap H = Z(G) \cap gHg^{-1}$, hence

$$Z(G) \cap \bigcap_{\substack{e \in E \\ g \in G}} G_e^g = Z(G) \cap \bigcap_{e \in E} G_e,$$

so the choice of representatives of the local groups doesn't matter.

To see that $Z(G)$ is the whole of the kernel, first observe that the latter, being non-trivial and an intersection of infinite cyclic groups, is infinite cyclic. It is also normal, so the action of G by conjugation on it is either trivial or else some element acts as multiplication by -1. But this last case is impossible, because the kernel contains $Z(G) \neq 1$.

Finally, once we know that $Z(G)$ equals the kernel of the action of G on its Bass-Serre tree T, we can factor this action through $G/Z(G)$. The local subgroups in $\mathcal{G}' := (G/Z(G))\backslash T$ are then finite cyclic, with a choice

of generators inherited from $\mathcal{G}$, which proves that they satisfy the required conditions. □

It will be necessary for us to work with *slide moves* on graphs of groups. These were defined first by Forester in [For02].

Definition 6.6 (Sliding move)**.** Let $\mathcal{G}$ be a graph of groups and $e, f \in E(\mathcal{G})$ be edges such that $t(e) = o(f)$. If $\iota_e(G_e) < \iota_{\bar{f}}(G_f)$, then we can do a *slide move* of e along f defined as follows. We replace the edge e with e' such that $o(e') = o(e)$ and $t(e') = t(f)$ with $G_{e'} = G_e$ and inclusions $\iota_{\bar{e}'} = \iota_{\bar{e}}$ and $\iota_{e'} = \iota_f \circ \iota_{\bar{f}}^{-1} \circ \iota_e$.

A sliding move constructs a new graph of groups whose fundamental group is isomorphic to that of $\mathcal{G}$.

Corollary 6.7. *Let G_1 and G_2 be free-by-cyclic groups with centre, being fundamental groups of graphs of groups $\mathcal{G}_1$ and $\mathcal{G}_2$, as shown in Proposition 6.3. If the graphs $\mathcal{G}_1'$ and $\mathcal{G}_2'$, whose fundamental groups are $G_1/Z(G_1)$ and $G_2/Z(G_2)$, are isomorphic via a sequence of slide moves, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are also isomorphic via a sequence of slide moves.*

With Corollary 6.7 our strategy now becomes as follows: given two free-by-cyclic groups G_1 and G_2 with centre such that $\widehat{G}_1 \cong \widehat{G}_2$, show that $G_1/Z(G_1) \cong G_2/Z(G_2)$ via a sliding-moves equivalence of their graphs $\mathcal{G}_1'$ and $\mathcal{G}_2'$.

For that, we need to know that the isomorphic profinite completions $\widehat{G}_1$ and $\widehat{G}_2$ when divided by their centres become the completions of $G_1/Z(G_1)$ and $G_2/Z(G_2)$.

First we need a lemma with the consequence that the order of $[\phi]$ in $\mathrm{Out}(F_n)$ equals the order of $[\widehat{\phi}]$ in $\mathrm{Out}(\widehat{F}_n)$.

Lemma 6.8. *The canonical homomorphism $\mathrm{Out}(F_n) \to \mathrm{Out}(\widehat{F}_n)$ induced by the completion homomorphism $\mathrm{Aut}(F_n) \to \mathrm{Aut}(\widehat{F}_n)$ is injective.*

Proof. The result is clear for $n = 1$. For $n \geq 2$ we need to show that for an automorphism $\phi \in \mathrm{Aut}(F_n)$ such that $\widehat{\phi} = \mathrm{Ad}_g$ in $\mathrm{Aut}(\widehat{F}_n)$ the element $g \in \widehat{F}_n$ actually lies in F_n.

For any $x \in F_n$ the image $\widehat{\phi}(x) = \phi(x)$ is an element of F_n conjugate to x in $\widehat{F}_n$, but since F_n is conjugacy separable (see [Ste70]) $\phi(x)$ must be conjugate to x in F_n. Thus $\phi = \widehat{\phi}|_{F_n}$ fixes all conjugacy classes of F_n. Such an automorphism must be inner – see Lemma 1 in [Gro74]. Finally, since $\widehat{F}_n$ is centerless for $n \geq 2$, we get that $g \in F_n$. □

Proposition 6.9. *Let G_1 and G_2 be free-by-cyclic groups with finite monodromy. Then $Z(\widehat{G}_i) = \overline{Z(G_i)}$ and thus $\widehat{G}_i / Z(\widehat{G}_i) \cong \widehat{G_i / Z(G_i)}$.*

Proof. First, let's notice that $\widehat{F_n \rtimes_\phi \mathbb{Z}} \cong \widehat{F}_n \rtimes_{\widehat{\phi}} \widehat{\mathbb{Z}}$, with $\widehat{\mathbb{Z}}$ acting on $\widehat{F}_n$ through $\kappa \cdot g = \widehat{\phi}^{(\kappa \mod m)}(g)$ for $\widehat{\phi}$ being the extension of ϕ to $\widehat{F}_n$ and m being the order of ϕ in $\mathrm{Out}(F_n)$, which by Lemma 6.8 we know is equal to the order of $\widehat{\phi}$ in $\mathrm{Out}(\widehat{F}_n)$.

Now, let's notice that the computation of the centre of $F_n \rtimes_\phi \mathbb{Z}$ done in Corollary 6.2 works also in the case of $\widehat{G} = \widehat{F}_n \rtimes_{\widehat{\phi}} \widehat{\mathbb{Z}}$ and it yields $Z(\widehat{G}) = \{(x^{-1}t^m)^\kappa \mid \kappa \in \widehat{\mathbb{Z}}\}$, where $\phi^m = \mathrm{Ad}_{x^{-1}}$ so indeed $Z(\widehat{G}) = \overline{Z(G)}$. □

6.3 Finite subgroups and centralisers

The groups of interest in this section will be F_n-by-(Z/m) extensions and it is convenient to know that they too can be decomposed as graphs of cyclic groups in a way analogous to that of Proposition 6.3.

Proposition 6.10. *Let G be an F_n-by-$(\mathbb{Z}/m)$ extension. Then it is a fundamental group of a finite graph $\mathcal{G} = (V, E)$ of finite cyclic subgroups $\mathbb{Z}/d_{v_i}$ and $\mathbb{Z}/d_{e_i}$ and the inclusion maps $\iota_e \colon \mathbb{Z}/d_e \to \mathbb{Z}/d_{t(e)}$ and $\iota_{\bar{e}} \colon \mathbb{Z}/d_e \to \mathbb{Z}/d_{o(e)}$ come from $[1 \mod d_e] \mapsto [d_{t(e)}/d_e \mod d_{t(e)}]$ and $[1 \mod d_e] \mapsto [d_{o(e)}/d_e \mod d_{o(e)}]$.*

Furthermore, given a finite graph of groups of this type, its fundamental group is an extension with finitely generated free kernel and finite cyclic quotient.

Proof. First, let's notice that G is virtually-free, so it is the fundamental group of a finite graph of finite groups; these finite groups don't intersect the torsion-free kernel of $q \colon G \twoheadrightarrow (\mathbb{Z}/m)$, so they embed into $(\mathbb{Z}/m)$ and are thus finite cyclic.

To show that the maps are as described, let's fix a generator $\bar{1}$ for the $(\mathbb{Z}/m)$. Then, for each vertex v let's choose as a generator for $G_v = (\mathbb{Z}/d_v)$ the unique preimage 1_v of $m/d_v \cdot \bar{1}$ under the quotient map q; do the same for the edge groups.

Now, for an edge e with $t(e) = v$, we will show that $\iota_e \colon 1_e \mapsto d_v/d_e \cdot 1_v$.

Indeed, let $\iota_e(1_e) = \alpha_e \cdot d_v/d_e \cdot 1_v$ with $\alpha_e \in \mathbb{Z}$. Then the image in the quotient map q is equal to $q(\iota_e(1_e)) = \alpha_e \cdot d_v/d_e \cdot m/d_v \cdot \bar{1} = \alpha_e \cdot m/d_e \cdot \bar{1}$, but it must be also equal to $q(1_e) = m/d_e \cdot \bar{1}$, so we get $\alpha_e \equiv 1 \mod d_e$. This however means that $\alpha_e \cdot d_v/d_e \equiv d_v/d_e \mod d_v$ and so that $\iota_e(1_e) = d_v/d_e \cdot 1_v$.

Finally, let $\mathcal{G}$ be a finite graph of groups of this type and G be its fundamental group. We can define a map from G to $(\mathbb{Z}/m)$ for $m = \operatorname{lcm}(d_v \mid v \in V(G))$ by defining it on all of G_v and G_e and extending it trivially on the edge-letters. Since G is virtually-free and all of the finite subgroups are mapped isomorphically, the kernel of this map is free and G is indeed a finitely generated free by finite cyclic extension. □

Remark 6.11. Not all of the graphs described in Proposition 6.10 have fundamental groups isomorphic to $G/Z(G)$ for $G \cong F_n \rtimes_\phi \mathbb{Z}$ with $[\phi]$ of finite order in $\mathrm{Out}(F_n)$. Indeed, the commutator subgroup G' lies inside the normal subgroup $F_n \rtimes 1$ while if $G \ncong \mathbb{Z}^2$ then any non-trivial elements (x, y) of the centre $Z(G)$ have $y \neq 0$, as computed in Corollary 6.2. This means that $G' \cap Z(G) = 1$. Whenever $\bar{z} \in Z(G/Z(G))$ i.e. $[x, z] \in Z(G)$ for all $x \in G$, we get that actually $[x, z] = 1$, so $z \in Z(G)$. Thus the quotient $G/Z(G)$ has trivial centre. Together with Proposition 6.5 this implies that as G ranges over the set of free-by-cyclic groups with centre, the groups $G/Z(G)$ are isomorphic to the fundamental groups of precisely those graphs of finite cyclic groups where $|G_e|$ have greatest common divisor equal to 1. In fact, if we consider the quotient $G/kZ(G)$ by a smaller central subgroup $kZ(G)$ for $k \geq 1$, the greatest common divisor of $|G_e|$ will equal k.

We introduce Definition 6.12 for convenience, as all of the information

about a graph of groups of the type described in Proposition 6.10 is captured by the sizes of the vertex and edge groups.

Definition 6.12 (Numbered graph, reduced numbered graph)**.** A graph $\mathcal{G}$—possibly with multi-edges and loops—is called *numbered* if each vertex and edge are labelled with natural numbers so that for any edge e its label d_e divides the labels $d_{o(e)}$ and $d_{t(e)}$ of its endpoints.

A numbered graph $\mathcal{G}$ is *reduced* if $d_{t(e)} = d_e$ or $d_{o(e)} = d_e$ imply that e is a loop.

Definition 6.13 (Associated poset)**.** Let $\mathcal{G}$ be a numbered graph. We will create a poset $\mathsf{P}(\mathcal{G})$, called the *associated poset*, where the elements carry two types of additional numerical data—*levels* and *labels*—in the following manner.

- For $k \in \mathbb{N}$, let $a_{1,k}, a_{2,k}, \ldots$ be the connected components of the subgraph $\mathcal{G}_k \subset \mathcal{G}$ formed by vertices and edges whose labels are divisible by k. These will be the elements of *level* k.
- $\mathsf{P}(\mathcal{G}) = \{a_{i,k} \mid k \in \mathbb{N}\}$.
- The ordering given by $a_{i,k} \prec a_{j,l}$ if and only if $k \mid l$ and $a_{i,k} \supset a_{j,l}$.
- Each element $a_{i,k}$ is *labelled* with its first Betti number $b_1(a_{i,k})$, i.e. the number of edges minus the number of vertices plus one.

Example 6.14. Let $\mathcal{G}$ be any of the graphs in Figure 6.1. The process of constructing its associated poset $\mathsf{P}(\mathcal{G})$ is shown in Figure 6.2.

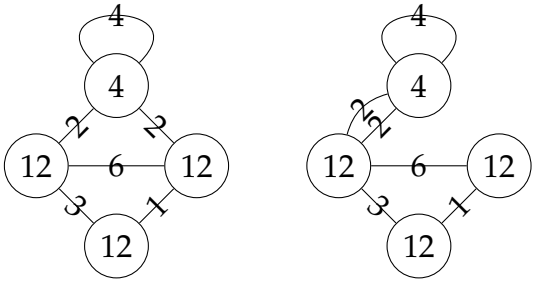


Figure 6.1: An example two numbered graphs which are slide-equivalent.

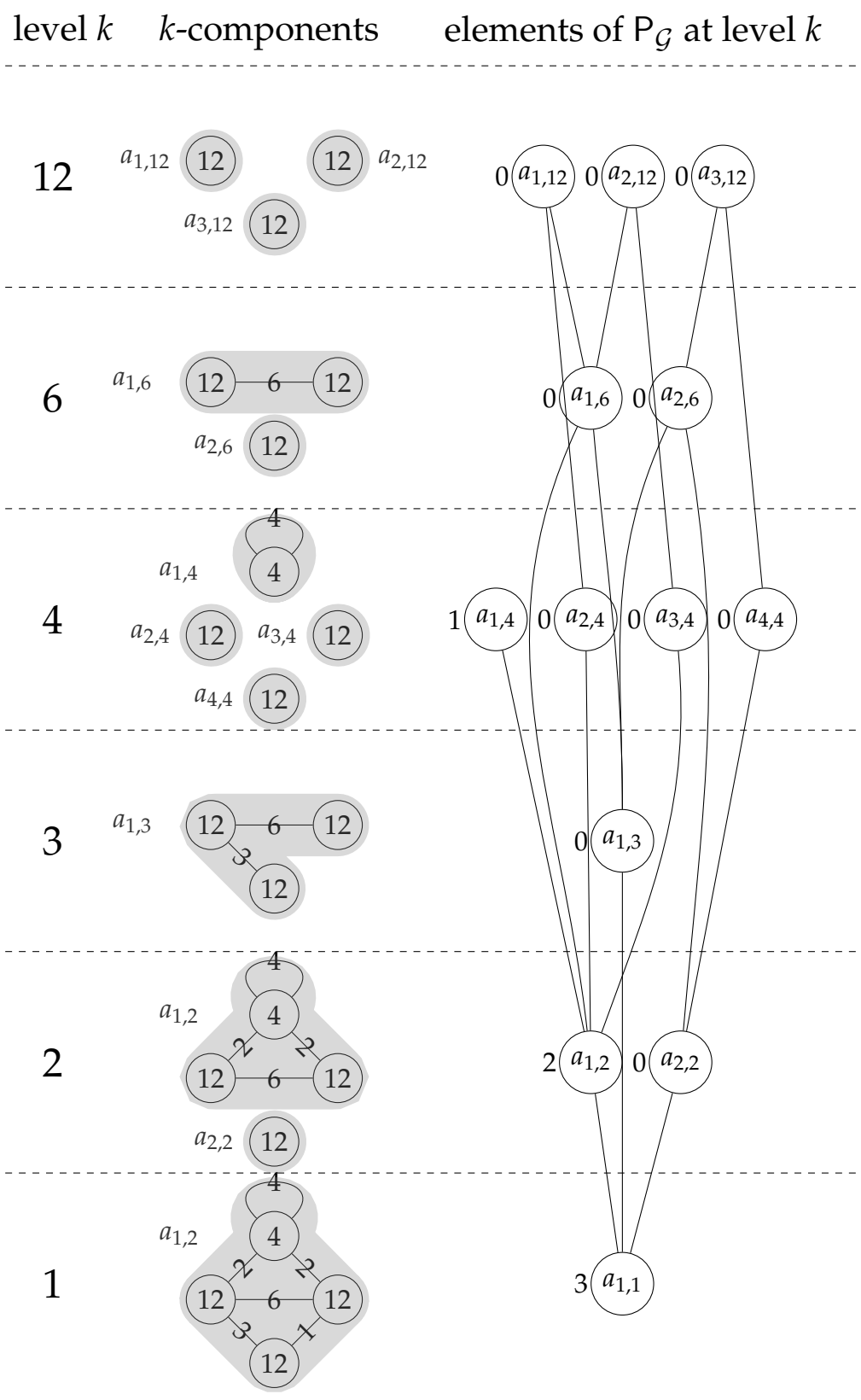


Figure 6.2: An example of the associated poset.

Definition 6.15 (Sliding move)**.** Let $\mathcal{G}$ be a numbered graph. A *sliding move* is a transformation of $\mathcal{G}$ into a graph $\mathcal{G}'$ defined as follows. Given two distinct edges e, f such that $d_e \mid d_f$ and $t(e) = o(f)$, the graph $\mathcal{G}'$ is created from $\mathcal{G}$ by replacing the edge e with a new edge e' such that $o(e') = o(e)$, $t(e') = t(f)$ and $d_{e'} = d_e$.

It is important to notice that applying a sliding move to a *reduced* numbered graph gives another reduced numbered graph; furthermore, it doesn't change the relation between the components $a_{i,k}$ and thus also the associated poset $\mathsf{P}(\mathcal{G})$.

Proposition 6.16. *Let $\mathcal{G}$ and $\mathcal{G}'$ be reduced numbered graphs. Then their associated posets $\mathsf{P}(\mathcal{G})$ and $\mathsf{P}(\mathcal{G}')$ are isomorphic if and only if $\mathcal{G}$ can be transformed into $\mathcal{G}'$ with a sequence of sliding moves.*

Overview of the proof. The proof has the following major steps.

1. There is a level-preserving bijection $V(\mathcal{G}) \leftrightarrow V(\mathcal{G}')$ between the vertex sets.

2. The components $a_{i,k}$ and $a'_{i,k}$ have the same numbers of edges and vertices.

3. For each $l \in \mathbb{N}$, $a_{i,k}$ and $a'_{i,k}$ have the same number of edges of label l.

4. Top-down induction on $\mathsf{P}(\mathcal{G})$: assuming that the vertex-bijection gives graph isomorphism on all components $a_{j,l}$ greater than $a_{i,k}$, do sliding moves with edges labelled k in $a_{i,k}$ to get a graph isomorphism of $a_{i,k}$ to $a'_{i,k}$.

Proof. We will follow the overview and indicate where each step starts and ends.

Step 1. Let's notice that the maximal elements of $\mathsf{P}(\mathcal{G})$ correspond to the vertices of $\mathcal{G}$, as for any reduced numbered graph $\mathcal{H}$ of at least two vertices $\mathcal{H}_k$ becomes disconnected for some k and subgraphs of a reduced graph are reduced. This means that there is a level-preserving bijection (denoted by $v \mapsto v'$) on the vertex sets of $\mathcal{G}$ and $\mathcal{G}'$ such that for any $k \in \mathbb{N}$

and any vertices v, w of $\mathcal{G}$, v and w are connected in $\mathcal{G}_k$ if any only if v' and w' are connected in $\mathcal{G}'_k$.

Step 2. Each connected component $a_{i,k}$ of $\mathcal{G}_k$ has the same number of vertices and edges as the corresponding component $a'_{i,k}$ of $\mathcal{G}'_k$. This is because the isomorphism between the associated posets implies that $a_{i,k}$ and $a'_{i,k}$ have the same number of maximal elements 'above them', which we have seen correspond to single vertices; they also have the same first Betti numbers $b(a_{i,k})$ and $b(a'_{i,k})$ which are equal to the number of edges minus the number of vertices plus one.

Step 3. Now, we can use Möbius inversion to strengthen this claim: for any $l \in \mathbb{N}$ the components $a_{i,k}$ and $a'_{i,k}$ have the same number of edges labelled l.

For l divisible by k any path of edges in $\mathcal{G}_k$ divisible by l with endpoints in $a_{i,k}$ lies entirely in $a_{i,k}$. Thus the associated poset $\mathsf{P}(\mathcal{G})$ includes the associated poset for $a_{i,k}$ treated as a numbered graph: it consists of the elements $a_{j,l}$ which are comparable to $a_{i,k}$. Thus, to prove the claim that $a_{i,k}$ and $a'_{i,k}$ have the same number of edges of label l it is enough to prove the following: given numbered graphs $\mathcal{H}$ and $\mathcal{H}'$ with isomorphic associated posets $\mathsf{P}(\mathcal{H})$ and $\mathsf{P}(\mathcal{H}')$, for each l the graphs $\mathcal{H}$ and $\mathcal{H}'$ have the same number of edges of label l.

To prove this, let's first fix some notation. Let's denote by $e(\mathcal{H})$ the number of edges of a numbered graph $\mathcal{H}$, and by $e_l(\mathcal{H})$ the number of edges of $\mathcal{H}$ labelled l. As before, $\mathcal{H}_m$ denotes the subgraph of $\mathcal{H}$ formed by vertices and edges whose labels are divisible by m. Finally, $\mu(n)$ denotes the standard Möbius function, equal to 0 if n is divisible by a perfect square and $(-1)^{\#\text{prime factors of } n}$ otherwise. We will show that

$$e_l(\mathcal{H}) = \sum_{m \,\text{st.}\, l|m} e(\mathcal{H}_m)\mu(\frac{m}{l}).$$

Since $e(\mathcal{H}_m)$ can be recovered from the poset $\mathsf{P}(\mathcal{H})$, this proves the claim.

Indeed, $e(\mathcal{H}_m) = \sum_{m|n} e_n(\mathcal{H}_m)$, since every edge label in $\mathcal{H}_m$ is divisible by m. Also, for $m|n$ we have $e_n(\mathcal{H}) = e_n(\mathcal{H}_m)$ since all of the edges of $\mathcal{H}$

with label n are present in $\mathcal{H}_m$. Thus

$$\begin{aligned}
\sum_{m \,\mathrm{st.}\, l|m} e(\mathcal{H}_m)\mu(\frac{m}{l}) &= \sum_{m \,\mathrm{st.}\, l|m} \sum_{n \,\mathrm{st.}\, m|n} e_n(\mathcal{H}_m)\mu(\frac{m}{l}) \\
&= \sum_{m \,\mathrm{st.}\, l|m} \sum_{n \,\mathrm{st.}\, m|n} e_n(\mathcal{H})\mu(\frac{m}{l}) \\
&= \sum_{n \,\mathrm{st.}\, l|n} \sum_{m \,\mathrm{st.}\, l|m|n} e_n(\mathcal{H})\mu(\frac{m}{l}) \\
&= \sum_{n \,\mathrm{st.}\, l|n} e_n(\mathcal{H}) \sum_{d \,\mathrm{st.}\, d|\frac{n}{l}} \mu(d) \\
&= \sum_{n \,\mathrm{st.}\, l|n} e_n(\mathcal{H}) \cdot \mathbf{1}(\frac{n}{l} = 1) \\
&= e_l(\mathcal{H}).
\end{aligned}$$

Step 4. We will show that if $a_{i,k}$ and $a'_{i,k}$—treated as graphs on the same set of vertices, with the same levels—have the same edges of labels other than k and the same number of edges of label k, we can perform sliding moves of edges labelled k to turn $a_{i,k}$ into $a'_{i,k}$. This will allow us to inductively transform $\mathcal{G}$ into $\mathcal{G}'$ starting by ensuring that they have the same highest-label edges and then ensuring the same for edges with lower and lower labels.

The base of this induction isn't difficult—the maximal elements of $\mathsf{P}(\mathcal{G})$ have just one vertex and possibly some loops whose label is the same as the level of the vertex; the number of these loops is detected by the recorded first Betti number, so it matches between the two graphs. We don't need any sliding moves here.

For the inductive step, notice that it is enough to prove it if *all* edges of $a_{i,k}$ and $a'_{i,k}$ are labelled k. This is because within the connected components of the subgraph of $a_{i,k}$ formed by throwing away all edges labelled k we can freely slide the endpoints of the edges labelled k, so it is enough to solve the sliding in the graph formed by collapsing these connected components to single vertices.

We will do so by exhibiting a normal form of such graphs under the sliding moves. Let's choose a vertex v_0. For every edge e at least one of $o(e)$ and $t(e)$ is connected to v_0 by a path which doesn't include the edge

e. Thus we can slide this edge along the path until the endpoint becomes v_0. We can continue this procedure until each edge has v_0 as at least one of its endpoints and the vertices other than v_0 are leaves getting an *octopus* – see Figure 6.3. Doing the same procedure with v_0' we get the same graph, as the isomorphism class of an octopus is determined by the number of vertices and edges. □

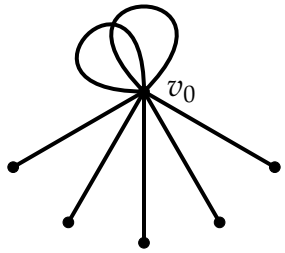


Figure 6.3: An octopus graph.

Now, having understood that we can distinguish which numbered graphs are equivalent via sliding moves by their associated posets, we need to find an analogue on the group-theory side.

Definition 6.17 (Finite subgroups conjugacy poset)**.** Let G be any group. We define the *finite subgroups conjugacy poset* $\mathsf{FSC}(G)$ to have as elements the conjugacy classes $[T]$ of finite subgroups $T < G$ decorated with a pair of numbers: $|T|$ – the *level* of an element, and the first Betti number of the centraliser $C_G(T)$; the ordering $[T_1] \prec [T_2]$ is inclusion up to conjugacy.

Proposition 6.18. *Let G be an F_n-by-$(\mathbb{Z}/m)$ extension, with $\mathcal{G}$ being a labelled graph of finite cyclic groups whose fundamental group is G.*

Then the associated poset $\mathsf{P}(\mathcal{G})$ is isomorphic to the finite subgroups conjugacy poset $\mathsf{FSC}(G)$.

Proof. First, given a natural number k and a connected component $a_{i,k}$ of $\mathcal{G}$ such that $(\mathbb{Z}/k)$ is a subgroup of all of its vertex groups and edge groups—in our case of finite cyclic group this subgroup is uniquely deter-

mined by its size—we do get a subgroup isomorphic to $(\mathbb{Z}/k)$ inside G, as all vertex groups and edge groups are subgroups of G.

Second, a finite subgroup T of G has fixed points in the action on the associated tree Γ. Then the image of $\Delta := \mathrm{Fix}_\Gamma(T)$ under the quotient $\Gamma \to G\backslash\Gamma = \mathcal{G}$ doesn't change if we replace T with xTx^{-1}, and it is a connected subgraph $\mathcal{G}_T$ such that T lies in all vertex groups and edge groups of $\mathcal{G}_T$. Moreover, among such graphs $\mathcal{G}_T$ is a maximal connected one—if $T < G_e$ then by definition e is an edge of $\mathcal{G}_T$.

Furthermore, the normaliser $N_G(T)$ acts on Δ and the quotient $N_G(T)\backslash\Delta$ is a subgraph of $G\backslash\Gamma = \mathcal{G}$. Indeed, if v and gv are vertices in Δ, then both are fixed by T, so also $g^{-1}Tg$ fixes v, so $T^g < G_v$. Since G_v is a finite cyclic group, it has exactly one subgroup of size $|T|$, so we get $T = T^g$ and $g \in N_G(T)$. Finally, $N_G(T) = C_G(T)$ for a finite subgroup T in a (torsion-free)-by-abelian group G and the first Betti number of the fundamental group of a finite graph of finite groups is equal to the first Betti number of the underlying graph.

These two maps are inverses of each other, preserving the levels and labels. □

This allows us to prove one of the main results – Theorem I.

Theorem I. *Let G and G' be F_n-by-$(\mathbb{Z}/m)$ extensions. Then they are isomorphic if and only if there is a* data-preserving *isomorphism between their finite subgroups conjugacy posets $\mathsf{FSC}(G)$ and $\mathsf{FSC}(G')$.*

Furthermore, the isomorphism between G and G' can be realised through sliding moves on any of their realisations as reduced graphs of finite cyclic groups and these moves can be constructed explicitly.

Proof. The 'only if' direction is clear.

Let $\mathcal{G}$ and $\mathcal{G}'$ be graphs of finite cyclic groups whose fundamental groups are respectively G and G', treated as numbered graphs. By Proposition 6.18 we have $\mathsf{FSC}(G) \cong \mathsf{P}(\mathcal{G})$ and $\mathsf{FSC}(G') \cong \mathsf{P}(\mathcal{G}')$, so $\mathsf{FSC}(G) \cong \mathsf{FSC}(G')$ implies that $\mathsf{P}(\mathcal{G}) \cong \mathsf{P}(\mathcal{G}')$. Now, by Proposition 6.16, isomorphism of associated posets implies that $\mathcal{G}$ can be transformed into

$\mathcal{G}'$ with a sequence of sliding moves. Finally, sliding moves of graphs of groups don't change the fundamental group, so we get $G \cong G'$. □

6.4 Completions detect finite subgroups and centralisers

Proposition 6.19. *Let G be a virtually-free group. Then any finite subgroup T of $\widehat{G}$ is conjugate to a finite subgroup of G and any finite subgroups of G which are conjugate in $\widehat{G}$ are already conjugate in G.*

Proof. Let $\mathcal{G}$ be a finite graph of finite groups such that G is its (discrete) fundamental group. By [Rib17, Proposition 8.2.3] $\widehat{G}$ is actually isomorphic to the *profinite* fundamental group of $\mathcal{G}$, which means that there is a profinite tree $\widehat{\Gamma}$ such that $\mathcal{G} = \widehat{G}\backslash\widehat{\Gamma}$.

Now, each finite subgroup T of $\widehat{G}$ fixes some vertex v of $\widehat{\Gamma}$, and thus $gTg^{-1} < G_w < G$ for $w = g \cdot v$ and G_w being one of the vertex groups of $\mathcal{G}$. Thus, finite subgroups of $\widehat{G}$ are conjugate to finite subgroups of G.

The second assertion that non-conjugate finite subgroups of G don't become conjugate in $\widehat{G}$ is implied by the subgroup conjugacy separability of all virtually-free groups proved by Chagas and Zalesskii in [CZ15]. □

Proposition 6.20. *Let G be a free-by-(finite cyclic) group and $T < G$ be a finite subgroup. Then $C_{\widehat{G}}(T) \cong \widehat{C_G(T)}$.*

Proof. First, Theorem 2.6 of [RZ14] proves that for virtually-free groups $N_{\widehat{G}}(T) = \overline{N_G(T)}$. Secondly, for a (torsion-free)-by-abelian group H we have $N_H(T) = C_H(T)$. Thus $C_{\widehat{G}}(T) = \overline{C_G(T)}$. Finally, centralisers of finite subgroups in virtually-(finitely generated free) groups are finitely generated, and virtually-free groups are LERF, so $\overline{C_G(T)} \cong \widehat{C_G(T)}$. □

Now we are ready to prove Theorem J.

Theorem J. *Groups which are extensions of a finitely generated free group by a finite cyclic group are distinguished from each other by their finite quotients.*

Proof of Theorem J. Let G_1 and G_2 be (finitely generated free)-by-(finite cyclic) extensions such that $\widehat{G}_1 \cong \widehat{G}_2$.

We will show that the finite subgroups conjugacy posets $\mathsf{FSC}(G_1)$ and $\mathsf{FSC}(G_2)$ are isomorphic, which thanks to Theorem I implies that $G_1 \cong G_2$.

First, the conjugacy classes of finite subgroups in G_i are the same as those in $\widehat{G}_i$, as shown in Proposition 6.19. The relation of 'being conjugate into' another subgroup is also preserved. Since $\widehat{G}_1 \cong \widehat{G}_2$ the elements and the relations in the posets $\mathsf{FSC}(G_1)$ and $\mathsf{FSC}(G_2)$ agree.

Second, we need to show that the first Betti numbers of centralisers of the corresponding finite subgroups agree. This is true since by Proposition 6.20 the centraliser of a finite subgroup in $\widehat{G}_i$ is isomorphic to profinite completion of the centraliser in G_i. Thus given a finite subgroup $T_1 < G_1$ and a corresponding $T_2 < G_2$ (conjugate in $\widehat{G}_1 = \widehat{G}_2$) we get $\widehat{C_{G_1}(T_1)}_{\mathrm{ab}} = \widehat{C_{G_2}(T_2)}_{\mathrm{ab}}$, which means that $C_{G_1}(T_1)$ and $C_{G_2}(T_2)$ have the same first Betti numbers. □

Theorem H is now a consequence.

Theorem H. *Free-by-cyclic groups with non-trivial centre are distinguished from each other by their finite quotients.*

Proof of Theorem H. Let G_1 and G_2 be (finitely generated free)-by-cyclic groups with finite monodromy such that $\widehat{G}_1 \cong \widehat{G}_2$.

Then also $\widehat{G}_1/Z(\widehat{G}_1) \cong \widehat{G}_2/Z(\widehat{G}_2)$ and by Proposition 6.9 also $\widehat{G_1/Z(G_1)} \cong \widehat{G_2/Z(G_2)}$. Since $G_i/Z(G_i)$ are (finitely generated free)-by-(finite cyclic), by Theorem J they are isomorphic with an isomorphism coming from sliding moves equivalence of their graphs of groups $\mathcal{G}_1'$ and $\mathcal{G}_2'$. By Corollary 6.7 this implies that $G_1 \cong G_2$. □

Appendix A

MAGMA code verifying Theorem C and Example 4.28

Listing A.1: MAGMA code verifying that the subgroups $Q_1, Q_2 < \mathrm{GL}_4(\mathbb{Z})$ are conjugate in $\mathrm{GL}_4(\mathbb{Z}_2)$ and in $\mathrm{GL}_4(\mathbb{Z}_3)$, but not in $\mathrm{GL}_4(\mathbb{Z}/6)$.

```
X1 := Matrix(Rationals(), 4, 4, [
    0,  -1,   0,   0,
   -1,   0,   0,   0,
    0,   0,   1,   1,
    0,   0,   0,  -1
]);
X2 := Matrix(Rationals(), 4, 4, [
    0,  1,  0,  0,
    1,  0,  0,  0,
    0,  0,  0,  1,
    0,  0,  1,  0
]);
Y := Matrix(Rationals(), 4, 4, [
    0,   1,   0,   0,
   -1,   0,   0,   0,
    0,   0,   0,  -1,
    0,   0,   1,   1
]);

print "Determinant(X1), Determinant(X2), Determinant(Y): ",
   Determinant(X1), Determinant(X2), Determinant(Y);

GLQ := GeneralLinearGroup(4, Rationals());
G1Q := sub< GLQ | [X1, Y] >;
G2Q := sub< GLQ | [X2, Y] >;
print "#G1Q, #G2Q: ", #G1Q, #G2Q;

A := Matrix(Rationals(), 4, 4, [
    1/2, -1/2,   0,    0,
    1/2,  1/2,   0,    0,
      0,    0,   0,    1,
      0,    0,   1,    0
]);
```

```
B := Matrix(Rationals(), 4, 4, [
    1,    0,    0,    0,
    0,    1,    0,    0,
    0,    0,  2/3,  1/3,
    0,    0, -1/3, -2/3
]);

print "Det(A), Det(B): ", Determinant(A), Determinant(B);

print "A*Y*A^-1 eq Y^5: ", A*Y*A^-1 eq Y^5;
print "B*Y*B^-1 eq Y^5: ", B*Y*B^-1 eq Y^5;
print "A*X1*A^-1 eq X2*Y^-1: ", A*X1*A^-1 eq X2*Y^-1;
print "B*X1*B^-1 eq X2*Y^2: ", B*X1*B^-1 eq X2*Y^2;

modulus := 6;
GLZmod := GeneralLinearGroup(4, Integers(modulus));
H1 := sub< GLZmod | [X1, Y] >;
H2 := sub< GLZmod | [X2, Y] >;
print "Is H1 conjugate to H2 mod 6: ", IsConjugate(GLZmod, H1
   , H2);
```

Listing A.2: MAGMA code verifying that the subgroups $Q_1, Q_2 < \mathrm{GL}_8(\mathbb{Z})$ are conjugate in $\mathrm{GL}_8(\widehat{\mathbb{Z}})$, but not in $\mathrm{GL}_8(\mathbb{Z})$.

```
Z := Integers();

Q1_1 := Matrix(Z, 8, 8, [
    0,  0, -1,  1,  0,  0,  0,  0,
    0,  0, -1,  0,  0,  0,  0,  0,
    1, -1,  0,  0,  0,  0,  0,  0,
    1,  0,  0,  0,  0,  0,  0,  0,
    0,  0,  0,  0,  0,  0,  1, -1,
    0,  0,  0,  0,  0,  0,  1,  0,
    0,  0,  0,  0, -1,  1,  0,  0,
    0,  0,  0,  0, -1,  0,  0,  0
]);

Q1_2 := Matrix(Z, 8, 8, [
    0,  0,  0,  0,  1, -1,  0,  0,
    0,  0,  0,  0,  0, -1,  0,  0,
    0,  0,  0,  0,  0,  0,  1, -1,
    0,  0,  0,  0,  0,  0,  0, -1,
   -1,  1,  0,  0,  0,  0,  0,  0,
    0,  1,  0,  0,  0,  0,  0,  0,
    0,  0, -1,  1,  0,  0,  0,  0,
    0,  0,  0,  1,  0,  0,  0,  0
]);

Q2_1 := Matrix(Z, 8, 8, [
     0,  0, -1,  1, -1,  0,  0,  0,
     0,  0, -1,  0, -2,  1,  1,  0,
     1, -1,  0,  0, -1,  0,  1,  0,
     1,  0,  0,  0, -2,  1,  1, -1,
```

```
     0,  0,  0,  0, -1,  1,  1, -1,
     0,  0,  0,  0, -1,  0,  1,  0,
     0,  0,  0,  0, -2,  2,  1, -1,
     0,  0,  0,  0, -2,  0,  1,  0
]);

Q2_2 := Matrix(Z, 8, 8, [
     0,  0,  0,  1,  1, -1, -1,  0,
     0,  0, -1,  2,  0, -1, -1,  0,
     0,  0,  0,  1,  0,  0,  0, -1,
     0,  0, -1,  2,  0,  0, -1, -1,
    -1,  1, -1,  1,  0,  0, -1,  0,
     0,  1,  0,  1,  0,  0, -1, -1,
     0,  0, -2,  2,  0,  0, -1,  0,
     0,  0,  0,  2,  0,  0, -1, -1
]);

GL8Z := GeneralLinearGroup(8, Z);

Q1 := sub < GL8Z | [Q1_1, Q1_2] >;
Q2 := sub < GL8Z | [Q2_1, Q2_2] >;

print "Sizes of Q1 and Q2: ", #Q1, #Q2;
iso, map := IsIsomorphic(Q1, Q2);
print "IsIsomorphic(Q1, Q2): ", iso;
conjZ, _ := IsGLZConjugate(Q1, Q2);
print "IsGLZConjugate(Q1, Q2): ", conjZ;

B := Matrix(Rationals(), 8, 8, [
     8,  23,   7,  -9,  -2,   0,  -5,  10,
    13,  28,   5,  -7,  -9,  -2,   3, -21,
    -3,  21,   9,   2, -18,  12,  15, -11,
    15,  15,  -6,   6, -21,  -6,  24, -22,
    20,   1,  16,  -4,   2, -13,  11, -13,
    -1,  21,   4,  12,  13, -11,  13,  -2,
    36,  -3,  -4,  -5,  -9,   0,  13, -26,
     3,  33,   5,  -9,   0,  -9,  26, -13
]);

print "B*Q1_1*B^-1 eq Q2_1: ", B*Q1_1*B^-1 eq Q2_1;
print "B*Q1_2*B^-1 eq Q2_1^-2*Q2_2: ", B*Q1_2*B^-1 eq Q2_1
   ^-2*Q2_2;
detB := Numerator(Determinant(B));
if Denominator(Determinant(B)) ne 1 then
     print "WRONG. The determinant isn't an integer.";
else
     print "Det(B) = ", detB, " = ", Factorisation(detB);
     print "Det(B) mod 6 = ", detB mod 6;
end if;
```

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