

The thermal shallow water equations,
their quasi-geostrophic limit, and
equatorial super-rotation in Jovian
atmospheres



Emma S. Warneford
Corpus Christi College
University of Oxford

A thesis submitted for the degree of
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To my parents and my husband, Stuart

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Observations of Jupiter show a super-rotating (prograde) equatorial jet that has persisted for decades. Shallow water simulations run in the Jovian parameter regime reproduce the mixture of robust vortices and alternating zonal jets observed on Jupiter, but the equatorial jet is invariably sub-rotating (retrograde). Recent work has obtained super-rotating equatorial jets by extending the standard shallow water equations to relax the height field towards its mean value. This Newtonian cooling-like term is intended to model radiative cooling to space, but its addition breaks key conservation properties for mass and momentum.

In this thesis the radiatively damped thermal shallow water equations are proposed as an alternative model for Jovian atmospheres. They extend standard shallow water theory by permitting horizontal variations of the thermodynamic properties of the fluid. The additional temperature equation allows a Newtonian cooling term to be included while conserving mass and momentum. Simulations reproduce equatorial jets in the correct directions for both Jupiter and Neptune (which sub-rotates).

Quasi-geostrophic theory filters out rapidly moving inertia-gravity waves. A local quasi-geostrophic theory of the radiatively damped thermal shallow water equations is derived, and then extended to cover whole planets. Simulations of this global thermal quasi-geostrophic theory show the same transition, from sub- to super-rotating equatorial jets, seen in simulations of the original thermal shallow water model as the radiative time scale is decreased. Thus the mechanism responsible for setting the direction of the equatorial jet must exist within quasi-geostrophic theory.

Such a mechanism is developed by calculating the competing effects of Newtonian cooling and Rayleigh friction upon the zonal mean zonal acceleration induced by equatorially trapped Rossby waves. These waves transport no momentum in the absence of dissipation. Dissipation by Newtonian cooling creates an eastward zonal mean zonal acceleration, consistent with the formation of super-rotating equatorial jets in simulations, while the corresponding acceleration is westward for dissipation by Rayleigh friction.

Statement of Originality

I confirm that this thesis is wholly my own original work, and that no part of my thesis has been accepted or is currently being submitted for any degree, diploma, certificate or other qualification at the University of Oxford or elsewhere. All methods and techniques used in this thesis that have been developed by other authors have been acknowledged by citation of the relevant publication.

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Chapter 1

Introduction

Gas giant planets are fluid objects, characterised by their massive fluid envelopes rather than their relatively small solid cores. There are four gas giant planets in our solar system: Jupiter, Saturn, Uranus and Neptune. Jupiter is the largest planet by diameter, followed by Saturn, Uranus and finally Neptune (see table 1.1). Their atmospheres are primarily composed of hydrogen and helium, e.g. Jupiter is composed of more than 87% hydrogen and helium by mass (Guillot et al., 2007). The different colours of the planets are believed to be related to the chemical composition of the particles; traces of methane in the outermost regions of Neptune are believed, at least in part, to account for the planet's blue appearance. The winds are powered by absorbed sunlight along with internal heat. Jupiter is contracting by ~ 3 cm per terrestrial year, and the release of gravitational potential energy is the source of its internal heat flux (Bagenal et al., 2007). The total power convected up from the interior is roughly comparable to that absorbed from the sun (see table 1.1). All the gas giant planets contain zonal jets, which are large-scale elongated structures that travel in the east-west direction. The equator-to-pole temperature gradient is close to zero for these planets. Stable vortices come in all size ranges, as an example the Great Red Spot on Jupiter is a giant storm which is large enough to contain two to three planets the size of Earth.

1.1 Observations of Jupiter by Voyager, Galileo and Cassini

Vasavada and Showman (2005) recently reviewed the current state of observational data, theory, experiments, and simulations related to Jupiter's atmosphere. The first missions to encounter Jupiter were Pioneer 10 and 11 in 1973 and 1974 respectively.

Table 1.1: Physical parameter values for the gas giant planets (the data is taken from Beebe, 1994, Cho and Polvani, 1996*b*, Ingersoll, 1990, and the planetary fact sheets compiled by D. R. Williams at the NASA Goddard Space Flight Center).

Parameter	Jupiter	Saturn	Uranus	Neptune
Equatorial radius (10^7m)	7.1	6.0	2.6	2.5
Emitted / absorbed power	1.7	1.8	1.0	2.7
Characteristic velocity scale (ms^{-1})	50	300	300	300
Ellipticity	0.065	0.098	0.023	0.017

Both Voyager spacecraft visited in 1979, the Galileo spacecraft spent nearly eight years in orbit between 1995 and 2003, while the Cassini spacecraft flew by Jupiter in 2000 on its way to Saturn. Almost all data comes from these remote observations, which reveal a highly turbulent cloud deck containing long-lived coherent vortices, which are transported around the planet by an alternating pattern of around 30 zonal jets. Within and between these zonal jets are hundreds of vortices of varying sizes in each hemisphere. The famous Great Red Spot is the largest vortex and has been observed for hundreds of years. Since the gas giant planets have no solid surfaces, the winds are measured relative to the magnetic field, which is tied to the interior and rotates with it. A prograde jet moves faster than the internal rotation, while a retrograde jet moves slower. Jupiter has the weakest winds of the gas giant planets (see table 1.1). Figure 1.1a shows Jupiter’s mean zonal wind profile, as derived from feature tracking in images taken by the Voyager 2 mission (Limaye, 1986) and the Cassini mission (Porco et al., 2003). The similarity of these two profiles, taken from missions over 20 years apart, demonstrates the remarkable stability of Jupiter’s zonal winds, and the presence of a broad super-rotating (prograde) equatorial jet (as is also observed on Saturn). According to Hide’s (1969) theorem, this jet must be maintained by an up-gradient eddy flux of angular momentum. Cassini revealed that zonal jets extend to $\sim \pm 80^\circ$ despite the lack of a banded appearance at high latitudes (see the blue dashed line in figure 1.1a). Figure 1.1b shows the corresponding mean zonal wind profile for Neptune taken from Hubble Space Telescope observations (Sromovsky et al., 2001), and reveals fewer mid-latitude jets and a broad sub-rotating (retrograde) equatorial jet (as is also observed on Uranus).

The vertical structure of the jets below the clouds is unknown. The only data from below the Jovian cloud deck comes from the descent by the Galileo probe at 7.3° North, within the equatorial jet. The Galileo mission took measurements of the gases in Jupiter’s atmosphere from the 0.7 bar level down to the 21 bar level before it failed due to the high temperatures. This is well below the estimated base of the water

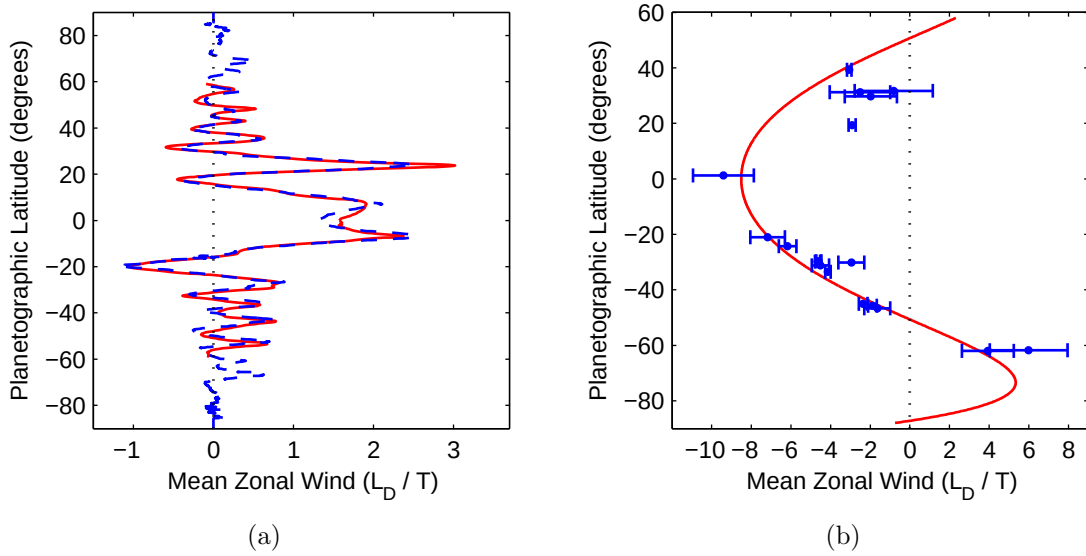


Figure 1.1: Mean zonal wind profiles for (a) Jupiter and (b) Neptune, in units of deformation radii per planetary day (L_D/T). Jovian data are from the Voyager 2 mission (red solid line; Limaye, 1986) and the Cassini mission (blue dashed line; Porco et al., 2003). The Voyager 2 data are restricted to mid and low latitudes. Neptunian data points are from the Hubble Space Telescope (Sromovsky et al., 2001). The red solid line is an empirical fit to Voyager 2 data (Sromovsky et al., 1993).

clouds at 5–6 bars, well below the depth of sunlight penetration, and much deeper than it was designed to go. The (prograde) zonal wind speed increased from around 90 m s^{-1} at 0.7 bars to around 170 m s^{-1} at 4 bars, then remained roughly uniform to the last received data at a depth of 21 bars (Atkinson et al., 1998). This indicates that winds increase below the cloud level, however the Galileo probe only covered an altitude span of $\sim 150 \text{ km}$, which is a tiny fraction of Jupiter’s radius of $7.1 \times 10^4 \text{ km}$ (see table 1.1). Thus these measurements aren’t deep enough to reveal the full vertical structure of Jupiter’s jets. Galileo entered Jupiter’s atmosphere through a $5\text{-}\mu\text{m}$ hot spot (Orton et al., 1998), a region of unusual clarity and dryness, thus the measurements obtained may not represent Jupiter as a whole but merely the conditions within hot spots.

Jupiter’s atmosphere contains a plethora of circulating features ranging from compact vortices to filamentary structures. There are on the order of hundreds of vortices present among Jupiter’s zonal winds at any instant of time. Cyclones rotate in the same sense as the planet (i.e. anticlockwise in the northern hemisphere, clockwise in the southern hemisphere) while anticyclones rotate the opposite way. Voyager and Cassini images show that 90% of Jupiter’s compact vortices are anticyclonic at the

cloud top levels (Li et al., 2004, Mac Low and Ingersoll, 1986) and that nearly all anticyclonic structures are oval-shaped. Small cyclones often appear as irregular ovals, while larger cyclonic regions may extend tens of degrees in longitude but remain confined in latitude by the zonal jets. Features in cyclonic regions tend to be chaotic and filamentary, with lifetimes of days or less. Voyager, Galileo, Cassini and the Hubble Space Telescope all show that long-lived Jovian vortices have relative vorticity of the same sign as the jet in which they are embedded. It is believed that the number of spots on Jupiter remains roughly constant with time (Li et al., 2004). However, these features may be altitude-dependent. Vortices exhibit a variety of behaviours beyond merging, including ejecting filament, orbiting each other, and oscillating in shape and position.

Galileo took images on the day and the night sides of Jupiter, and made many lightning detections as it conducted multiple surveys of the planet’s night side at high spatial resolution. Although lightning can only be seen on the night side, the orbiter was able to follow storms from the day to the night side of the planet, a time interval of 2–3 hours. The camera’s sensitivity to near-infrared radiation made it possible to assess vertical cloud structure at the spatial resolution of the images. While bright expanding clouds observed by Voyager had been noted as candidates for moist convection, Galileo revealed them to be uniquely tall structures with roots at levels where the most probable condensate is water (Banfield et al., 1998, Gierasch et al., 2000). Galileo also provided confirmation that they have lightning in them (Gierasch et al., 2000, Little et al., 1999). These bright, white, rapidly expanding clouds were found chiefly in areas of cyclonic shear and near the centres of westward jets (Little et al., 1999). Galileo’s near-infrared mapping spectrometer (NIMS) found that relative humidity reaches saturation near the location of a storm (Roos-Serote et al., 2000). The combination of the vigorous production of cloud material, the vertical extent from the water condensation level to the 0.5 bar cloud level, the lightning, and association with high relative humidity all suggest the presence of moist convection within regions of cyclonic shear in Jupiter’s zonal winds (Vasavada and Showman, 2005). Gierasch et al. (2000) estimate that the total vertical transport of heat by such storms is of the same order as the planet’s internal heat source.

1.2 Shallow versus deep

The motions in the deep interior of Jupiter’s atmosphere are entirely unknown, and it remains a matter of conjecture whether the surface winds and jets are confined

to a shallow weather layer, or extend deep into the interior. The shallow structure hypothesis (Hess and Panofsky, 1951, Ingersoll and Cuzzi, 1969) assumes that the winds are confined to a layer within several scale heights below the cloud level. Latent heat release associated with condensation of water vapour above the 5–10 bar pressure level could produce thermal contrasts of 5–10 K, which would cause vertical wind shear by the thermal wind equation (Gierasch, 1976, Ingersoll and Cuzzi, 1969). The thermal winds arising from such temperature contrasts are sufficient to explain the approximate magnitudes of Jupiter’s jets even if the interior deeper than ~ 10 bars has no winds (Ingersoll and Cuzzi, 1969, Vasavada and Showman, 2005).

In the deep structure hypothesis, the observed cloud level jets extend throughout the $\sim 10^4$ km thick molecular envelope. This hypothesis is motivated by the fact that the deep interior is almost certainly convecting since Jupiter radiates substantial internal energy (cf. page 1 and see Gierasch, 1976, Ingersoll, 1990, Vasavada and Showman, 2005). On this basis, Busse (1976) suggested that the Taylor–Proudman theorem holds throughout the deep interior. This implies that the jets are constant on cylinders parallel to the rotation axis, called Taylor columns. However, the zonal jets observed at the cloud level are not expected to penetrate into Jupiter’s metallic region that exists at pressures exceeding ~ 1 –2 Mbar (i.e. at depths of $\sim 10^4$ km), since the magnetohydrodynamic Lorentz force would act to brake any strong zonal flows in the electrically conducting metallic region (Busse, 2002, Grote et al., 2000, Kirk and Stevenson, 1987).

It is possible to compute the curvature of the zonal velocity profile for Jupiter’s zonal winds. Jupiter’s absolute vorticity gradient is not monotonic but changes sign at the latitudes of westward jets, and the curvature exceeds the gradient of the planetary vorticity β by a factor of 2.5–3 at each of these westward jets (Ingersoll et al., 1981, Limaye, 1986). This violates the barotropic stability criterion, that a two-dimensional fluid is stable provided that

$$\beta - \bar{u}_{yy} > 0, \quad (1.2.1)$$

where $\bar{u}_{yy}$ is the second derivative of the zonally averaged zonal velocity with respect to the northward distance y . However, numerical simulations of shallow or two-dimensional systems always produce jets with curvature smaller than β (e.g. Huang and Robinson, 1998, Nozawa and Yoden, 1997*a*, Vallis and Maltrud, 1993). However, if an appropriate flow in a deep lower layer under the observed cloud-level upper weather layer is considered, the cloud-level winds become stable (e.g. Dowling, 1995*a,b*, Ingersoll and Cuong, 1981), which perhaps suggests that Jupiter’s winds have

a deep structure. Vasavada and Showman (2005) believe that the vertical structure of Jupiter’s winds implied by the observed cloud-level value for $\bar{u}_{yy}$ remains obscure.

Vasavada and Showman (2005) suggest that Jupiter’s jet structure probably lies between the two extremes of shallow and deep. Jupiter could consist of Taylor columns in the deep interior, and a baroclinic thermal wind region within a few scale heights below the clouds. The deep Jovian interior is almost certainly convecting, and hence largely unstratified (Guillot et al., 2007, Stevenson and Salpeter, 1976). Thus the conclusion that the Taylor–Proudman theorem holds throughout the interior is strong. However, Galileo probe data suggests at least some stable stratification down to a depth of 22 bars (Magalhães et al., 2002). Thus the tops of the Taylor columns must lie below the clouds by at least a scale height. Observations of waves propagating away from the impact points of Shoemaker–Levy comet debris also provide evidence of stable stratification in the visible uppermost atmosphere (Dowling, 1995*b*, Ingersoll et al., 2007). Further support for the shallow structure hypothesis is provided by fully three-dimensional simulations using general circulation models adapted for Jovian parameter regimes. Simulations in spherical annuli occupying the top 10%, or even 6%, of the planet by radius yield plausible results; simulations in annuli with larger radial extents produce jets that are too closely spaced (Heimpel and Aurnou, 2007, Liu and Schneider, 2010).

Vasavada and Showman (2005) and Showman et al. (2006) emphasise that a clear distinction must be made between deep versus shallow models for the structure of the jets, and deep versus shallow models for the forcing that drives the jets. In the shallow forcing scenario, the jets are pumped by turbulence injected at the cloud layer e.g. by thunderstorms, baroclinic instabilities, or other weather layer processes within the cloud layer. In the deep forcing scenario, the jets are pumped due to convection that extends into the deep molecular interior and drives differential rotation which manifests as jets at the cloud level. Jet pumping then results from Reynolds stresses associated with convection within the molecular envelope. Shallow forcing need not imply a shallow structure for the jets, while deep forcing need not imply a deep structure. Showman et al. (2006) and Lian and Showman (2008) constructed a model which showed that deep jets can result from shallow forcing as well as deep forcing (see §1.3.5). Thus the fast winds observed by the Galileo probe down to a depth of 21 bars cannot be construed to imply that the Jovian zonal jets have a deep convective origin, especially since the Galileo probe only covered an altitude span of ~ 150 km when Jupiter has a radius of 7.1×10^4 km.

1.3 Mechanisms for the generation of zonal jets on Jupiter

Local regions of ascent and horizontal divergence have been observed in Jupiter's cloud layer (see the last paragraph of §1.1). However, Jupiter's cloud top motions appear to have a largely two-dimensional character (Gierasch, 1976). Energy cascades from small to large-scales in two-dimensional flow. Many theories for jet formation on Jupiter assume that the jets result from this inverse energy cascade modified by the variation of the Coriolis parameter with latitude, which is characterised by the parameter β . Williams (1978) first demonstrated the relevance of the β -effect and inverse cascade for reproducing the Jovian zonal jets.

The Coriolis force, caused by a planet's rotation, dominates the large-scale motions of its atmosphere. The latitude-dependence of the Coriolis force resists the displacement of fluid parcels in latitude. In the framework mentioned above, interactions between the waves supported by this restoring force (Rossby waves) and atmospheric turbulence are believed to explain the formation of zonal jets on Jupiter. To elucidate this consider the dispersion relation for a Rossby wave,

$$\omega = -\frac{\beta k}{k^2 + l^2}, \quad (1.3.1)$$

where a mid-latitude β -plane is considered to avoid the complexity of working in spherical geometry (see §1.4.1) and (k, l) are wavenumbers in the (x, y) directions respectively. Rossby waves propagating purely in the x direction ($l = 0$) thus have frequency $\omega = -\beta/k$, while $\omega = Uk$ is the typical frequency associated with turbulent advection (where U is the root-mean-square velocity). These two frequencies coincide at the Rhines wavenumber $k_R = \sqrt{\beta/U}$, whose inverse defines the Rhines scale (Rhines, 1975)

$$L_R = \sqrt{U/\beta}. \quad (1.3.2)$$

The Rhines scale is a transition scale: length scales larger than L_R are dominated by Rossby waves, while turbulence dominates at length scales smaller than L_R .

A more accurate estimate would take account of the anisotropy of the Rossby wave dispersion relation. Equating the Rossby wave frequency (1.3.1) with $\omega = U|\mathbf{k}|$ we obtain the anisotropic Rhines wavenumber

$$k_\beta = \left(\frac{\beta}{U}\right)^{1/2} \cos^{3/2} \theta, \quad (1.3.3a)$$

$$l_\beta = \left(\frac{\beta}{U}\right)^{1/2} \cos^{1/2} \theta \sin \theta, \quad (1.3.3b)$$

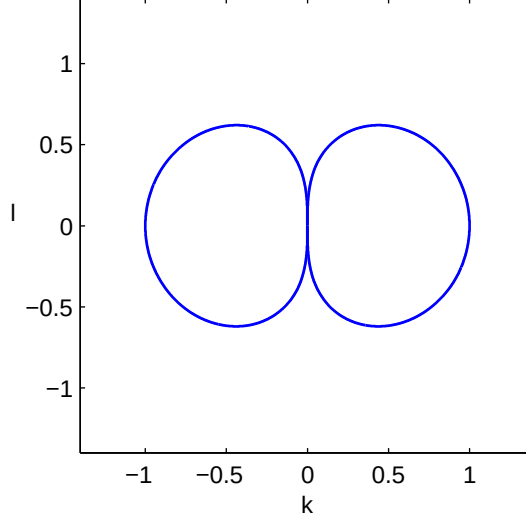


Figure 1.2: The dimensionless transition wavenumber $\mathbf{k}_\beta/k_R$, where $\mathbf{k}_\beta$ is defined by (1.3.3) and $k_R = \sqrt{\beta/U}$.

as a function of the polar angle $\theta = \tan^{-1}(l/k)$. The dimensionless transition wavenumber $\mathbf{k}_\beta/k_R$ is plotted in figure 1.2. Rossby waves dominate within the dumbbell (shown by the blue line) since the Rossby wave frequency for these wavenumbers is higher than the natural turbulent frequency, whereas turbulence dominates outside. If a flow is stirred at a wavenumber higher than this, the energy will cascade to larger scales until it reaches the dumbbell. The energy cannot penetrate the dumbbell, however, because the Rossby waves inside the dumbbell have faster frequencies than the turbulence, so the turbulent flow is unable to efficiently excite modes within the dumbbell. There is still a natural tendency, however, for the energy to seek the largest scales, and it does this by cascading to the y -axis. The corresponding flows have nearly zero k , and small but finite l , which correspond to zonally elongated structures. Thus the combination of Rossby waves and turbulence leads to the formation of zonal flow, and in many cases zonal jets. For more details see Rhines (1975), Vallis (2006), Vallis and Maltrud (1993), Vasavada and Showman (2005).

Figure 1.3 illustrates this mechanism for a Jovian simulation of the global thermal quasi-geostrophic equations, which are derived and numerically solved in chapter 5. The energy spectrum is defined to be the Fourier transform in x and y of the kinetic energy $\frac{1}{2}|\mathbf{u}|^2$. Figure 1.3 shows the two-dimensional energy spectrum in wavenumber space (k, l) at successive times. The spectrum is initially isotropic and concentrated at $|\mathbf{k}| = 42$, the energy then cascades to larger scales, but avoids the region inside the dumbbell, and piles up along the y -axis. In physical space this mechanism leads

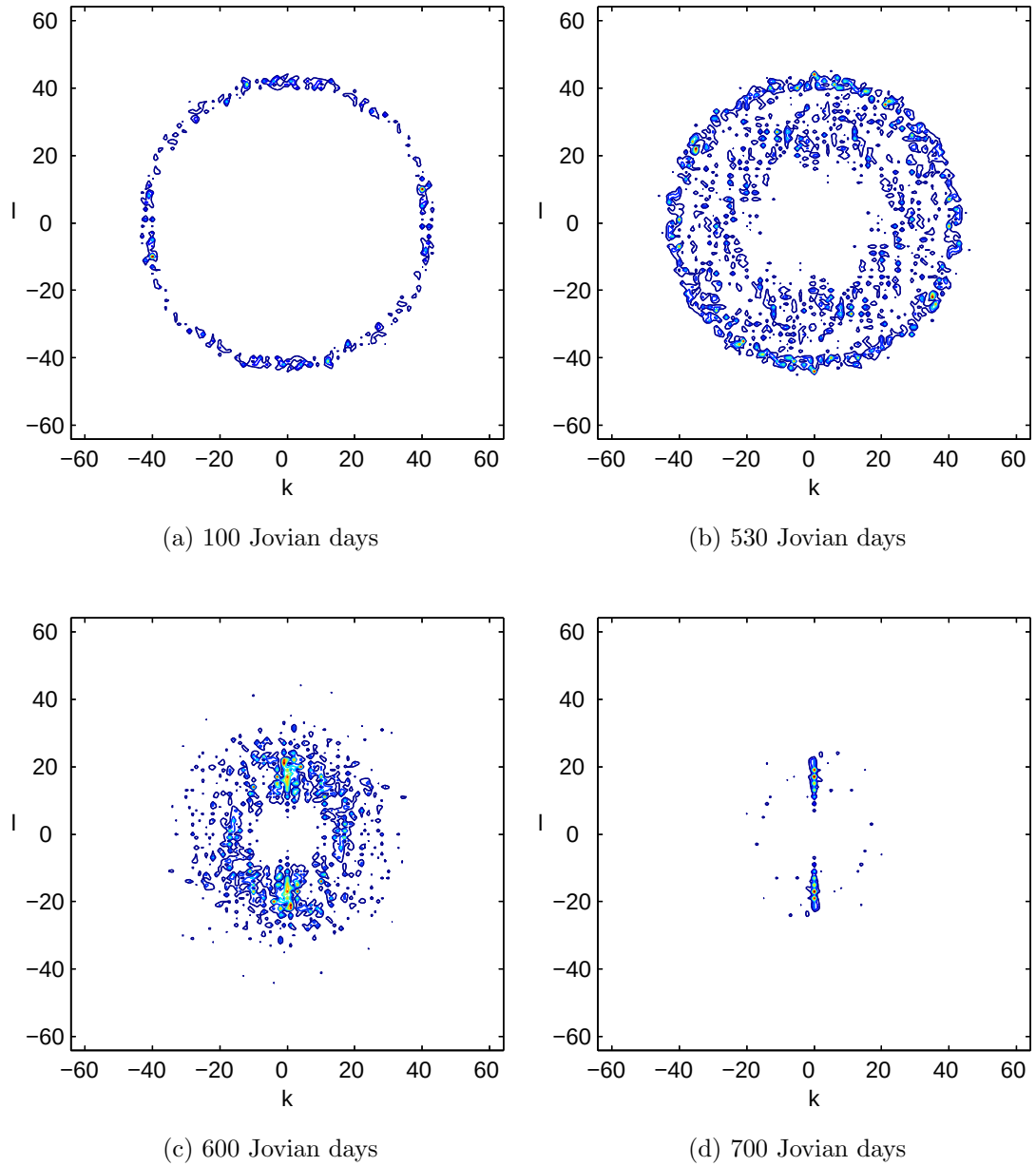


Figure 1.3: Evolution of the two-dimensional energy spectrum for a Jovian global thermal quasi-geostrophic simulation from chapter 5. Wavenumbers are counted from the longest sinusoidal mode in the domain being wavenumber 1.

to the formation of zonal flow and jets, with the typical jet spacing being related to the Rhines scale, which is defined in (1.3.2).

However in what is termed the zonostrophic regime (Galperin et al., 2006), the wave to turbulence transition occurs at a wavelength $L_\beta \sim (\epsilon/\beta^3)^{1/5}$, where ϵ is the small-scale isotropic energy injection rate. This scale may be significantly smaller than the Rhines scale (Sukoriansky et al., 2007). This zonostrophic regime is characterised by slowly evolving sets of alternating zonal jets, and steep zonal energy spectra $E_z(k) \propto k^{-5}$, while the classic Kolmogorov scaling $E(k) \propto k^{-5/3}$ holds for the non-zonal modes (cf. §3.3.2). It appears that Jupiter lies within this zonostrophic regime (Galperin et al., 2006).

Observational evidence that an inverse cascade is occurring in Jupiter's cloud layer includes observations of vortex mergers (Li et al., 2004, Mac Low and Ingersoll, 1986), and the approximate compatibility of the jet width with the Rhines scale (Cho and Polvani, 1996*a*). Vortices with diameters smaller than ~ 2000 km tend to be circular, while larger vortices are elongated in the zonal direction (Mac Low and Ingersoll, 1986). This confirms the influence of β at scales exceeding ~ 2000 km, and supports the conclusion that β can modify the inverse cascade to produce jets (Vasavada and Showman, 2005). It is not obvious, however, that mergers between individual vortices play a role in pumping the jets; no one has yet witnessed a vortex expanding longitudinally and merging with a jet. The largest vortices (e.g. the Great Red Spot) have never been observed to elongate into jets, or merge with existing jets. In fact they appear to have become more compact over time, and it is possible that these large vortices may result from the jets via barotropic or baroclinic instabilities, hence robbing the jets of energy rather than pumping them up (Dowling and Ingersoll, 1989, Williams, 1996, Williams and Wilson, 1988). However, it is possible that the Great Red Spot, and other large vortices, could have resulted from a gradual growth due to mergers of smaller vortices (Huang and Robinson, 1998).

Cloud tracking has enabled the estimation of the horizontal eddy momentum fluxes on Jupiter, $\overline{u'v'}$, where u' and v' are deviations from the zonal averages of the zonal and meridional winds respectively, and $\overline{(\cdot)}$ denotes a zonal average (cf. chapter 6). Salyk et al. (2006) used images of Jupiter from the Cassini flyby, an automatic feature tracker, and over 200,000 velocity vectors to calculate $\overline{u'v'}$. Previous analyses had used just the human eye and an order of magnitude fewer velocity vectors (e.g. Ingersoll et al., 1981). Salyk et al. (2006) found a positive correlation between $\overline{u'v'}$ and $d\bar{u}/dy$, including in the equatorial region, which implies that eddies in the visible Jovian atmosphere are supplying energy to the zonal flow (cf. chapter 6 and see Hide's

(1969) theorem on page 2). The rate of energy conversion was estimated to be $\sim 0.7\text{--}1.2 \text{ Wm}^{-2}$, which is about 2–4 times less than that found by Ingersoll et al. (1981). They estimate the power transfer between eddies and zonal winds to be in the range 4–8% of the total thermal energy emitted by Jupiter. This correlation is broadly consistent with the concept of an anisotropic inverse cascade in a geostrophically turbulent flow of the kind described above (Del Genio et al., 2009).

1.3.1 Two-dimensional non-divergent models

In the remainder of this section a hierarchy of models are described that have been used to understand the generation of Jupiter’s zonal jets. The simplest model for Jupiter assumes purely two-dimensional non-divergent motion for the horizontal flow at Jupiter’s cloud level. This model may be expressed by the vorticity evolution equation

$$\frac{d(\omega + f)}{dt} = F - D. \quad (1.3.4)$$

Here $\omega = \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u}$ is the relative vorticity, $f = 2\Omega \sin \phi$ is the traditional Coriolis parameter at latitude ϕ on a planet rotating with angular velocity Ω , and $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the material derivative, where $\mathbf{u}$ and ∇ are the two-dimensional fluid velocity and gradient operators respectively. This equation is written for a frame rotating about the vertical axis, with $\hat{\mathbf{z}}$ being a unit vector in the vertical direction. F represents a vorticity source (forcing), while D represents a vorticity sink (dissipation). Jupiter-like jets have not resulted from freely evolving (unforced) two-dimensional non-divergent turbulence (Cho and Polvani, 1996*a*, Williams, 1978, Yoden et al., 1999, Yoden and Yamada, 1993). However, many investigations have shown that robust zonal jets can result from two-dimensional non-divergent forced turbulence, the first of which was documented in Williams (1978). However, Williams (1978) found that his simulations generated Jupiter-like jets only when the forcing scale was synonymous with the jet scale. In fact, small-scale forcing did not produce Jupiter-like jets, implying that his Jupiter-like simulations lacked any significant amount of vortex mergers at scales smaller than the jets. When jets were forming in Nozawa and Yoden’s (1997*a*, 1997*b*) simulations, eddy energy at large wavenumbers cascaded to eddy energy at small wavenumbers. The eddy energy then arrested at the Rhines wavenumber and was converted into zonal energy and jets. However, once the jets had developed, eddy energy at the forcing scale was directly converted to zonal energy. In Huang and Robinson’s (1998) simulations, the jets were pumped directly by the small-scale vortices produced by the forcing. The energy jumped discontinuously

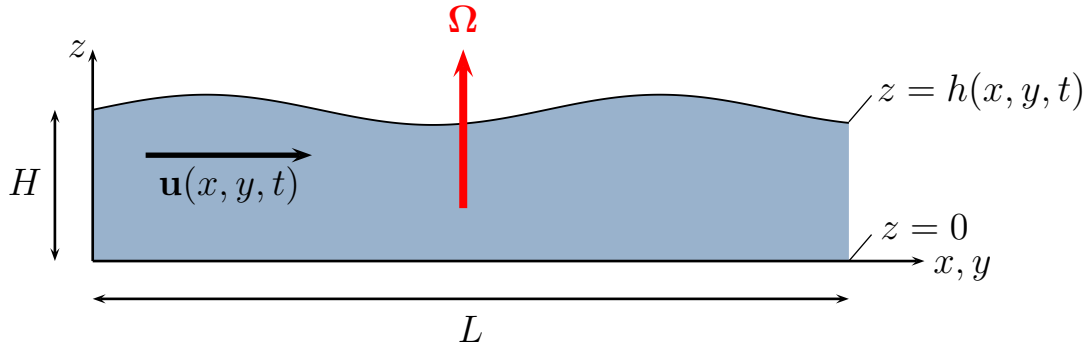


Figure 1.4: The shallow water model with a free surface at $z = h(x, y, t)$.

from the small forcing scale to the large jet scale, as opposed to forming a continuous cascade of energy to larger and larger vortices and finally zonal jets. An inverse cascade produced large vortices in their simulations, however they contributed little to the maintenance of the zonal jets.

These forced two-dimensional non-divergent models have provided a simple framework in which to study zonal jets. Jets have been shown to form over a wide range of parameter values for forcing and friction, and have approximately the correct width. However, simulations often develop polar vortices, which do not exist on Jupiter, and they fail to reproduce the correct direction for the Jovian equatorial jet. Simulations also fail to capture the fact that the equatorial jet is wider and faster than the mid-latitude jets on Jupiter. The cyclone-anticyclone asymmetry on Jupiter is also not captured by these models.

The aforementioned forced simulations generally included a friction term to allow the flow to equilibrate to a statistical steady state, and dissipate the slow leakage of large-scale energy past the Rhines scale. Without such a term, the jets themselves merge over time until the surviving jets reach the largest possible scale, e.g. the domain size or radius of the planet. The role of friction is to remove large-scale energy before it can reach the largest scales, leading to a statistical steady state with jets smaller than the domain size. However, given a specified energy injection rate, jets will form only if the drag is weaker than a specified value.

1.3.2 The standard shallow water model

The two-dimensional non-divergent vorticity equation (1.3.4) lacks column stretching, which excludes gravity waves and an array of other phenomena. The simplest example of a model which includes column stretching is the shallow water model. The shallow water equations are widely used in geophysical and planetary fluid dynamics

as conceptual models for rotating stratified fluids. Shallow water theory describes one or more layers of inviscid fluid, where the horizontal velocity within each layer is taken to be depth-independent, so the fluid moves in columns. They offer the simplest models for interactions between waves and vortical motions, and may be derived from the three-dimensional Euler equations for a Boussinesq fluid by depth-averaging across each layer, followed by posing asymptotic expansions of the velocity and pressure in a small ratio between vertical and horizontal length scales (for further details see §2.2 and Vallis, 2006). They may also be derived as amplitude equations for the vertical normal modes in continuously stratified fluids.

Consider the situation in figure 1.4, where a thin layer of fluid with a free surface at $z = h(x, y, t)$ flows over a rigid bottom at $z = 0$, in a frame rotating with angular velocity $f/2$ about the vertical axis, where $\hat{\mathbf{z}}$ is a unit vector in the vertical z -direction. In terms of the depth of the layer h and the depth-averaged horizontal velocity $\mathbf{u}$, the shallow water equations are given by

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0, \quad (1.3.5a)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} = -g\nabla h, \quad (1.3.5b)$$

where g denotes gravity, f is the traditional Coriolis parameter, and $\nabla = (\partial_x, \partial_y)$ is the horizontal gradient operator. The first equation (1.3.5a) represents conservation of mass, while the second (1.3.5b) represents conservation of horizontal momentum. Shallow water theory is valid when the typical depth scale H is much smaller than the typical length scale L , i.e. when the aspect ratio $\delta = H/L \ll 1$. While the shallow water equations are still a two-dimensional model, the horizontal divergence is non-zero and can cause changes to the fluid depth h through (1.3.5a).

There have been many shallow water studies of the Jovian atmosphere (e.g. see Cho and Polvani, 1996*a,b*, Dowling and Ingersoll, 1989, Iacono, Struglia and Ronchi, 1999, Iacono, Struglia, Ronchi and Nicastro, 1999, Ingersoll, 1990, Marcus, 1988, Scott and Polvani, 2007, 2008, Showman, 2007, Williams and Wilson, 1988, Williams and Yamagata, 1984). Shallow water theory typically arises in a Jovian context under the equivalent barotropic, reduced gravity, or $1\frac{1}{2}$ -layer approximation (Dowling and Ingersoll, 1989, Gill, 1982, Showman, 2007, Vallis, 2006). The scenario sketched in figure 1.5 contains two distinct layers of unequal depths separated by a sharp density contrast. The upper layer may be identified with a low-density weather layer, or cloud deck, which extends upwards to negligibly low pressures, while the lower layer represents the denser deep interior. The interface between the two layers may

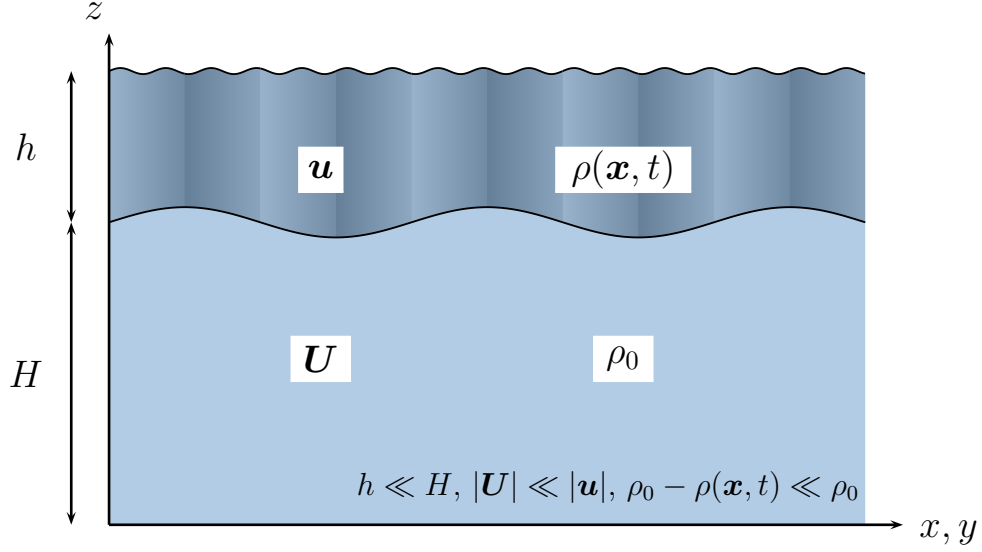


Figure 1.5: A stably stratified two-layer system with density $\rho(\mathbf{x}, t) < \rho_0$ in the upper layer. The equivalent barotropic approximation allows the lower layer to be treated as quiescent when $h \ll H$, $|\mathbf{U}| \ll |\mathbf{u}|$, and $\rho_0 - \rho(\mathbf{x}, t) \ll \rho_0$. The standard shallow water equations take ρ to be constant in the upper layer, while thermal shallow water theory allows ρ to vary with horizontal position and time.

be plausibly related to a change in stratification due to the localised release of latent heat from water vapour condensing at the 5–10 bar pressure level (see §1.2 and Gierasch, 1976, Ingersoll and Cuzzi, 1969). The deeper atmosphere may be convective, and thus largely unstratified (Guillot et al., 2007, Stevenson and Salpeter, 1976), but some density contrast must exist to support the internal gravity waves observed propagating outwards from the impact points of Shoemaker–Levy comet debris (Dowling, 1995b, Ingersoll et al., 2007). The equivalent barotropic approximation provides a closed set of shallow water equations for the upper layer alone when the lower layer is much deeper and relatively quiescent. The sole effect of the lower layer is to replace the actual gravitational acceleration g by the reduced gravity $g' = g\Delta\rho/\rho_0$ in (1.3.5b), where $\Delta\rho$ is the density contrast between the two layers. The shallow water equations contain an additional length scale called the deformation radius $L_D = \sqrt{g'h_0}/2\Omega$, which results from the interaction between the planetary rotation and the gravitational relaxation of thickness variations. The two-dimensional non-divergent vorticity equation (1.3.4) has an infinite deformation radius.

Cho and Polvani (1996a,b) performed freely decaying simulations of the shallow

water equations on a smooth sphere:

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0, \quad (1.3.6a)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} = -g'\nabla h. \quad (1.3.6b)$$

These equations are referred to as the “standard” shallow water equations for the remainder of this thesis, to distinguish them from the other shallow water models introduced below. Initialised with a random turbulent flow, these freely decaying simulations captured many qualitative features of the gas giant atmospheres. Zonal jets formed when the ratio of L_D to the planetary radius a was less than a third, where the zonal jets appeared as the turbulent inverse cascade to larger scales was arrested at the Rhines scale through coupling to Rossby waves. This was the first unforced study to produce multiple bands and jets from isotropic turbulence. Cho and Polvani (1996*b*) claim that the gas giant planets’ banded appearance “is a direct consequence of the intrinsic shallow-water dynamics they all share.” Although the number of zonal jets scaled with the Rhines scale, simulations with Jovian parameters produced ~ 10 – 15 jets instead of the observed ~ 30 jets. Despite the initial vorticity field containing equal numbers of cyclones and anticyclones, as the flow evolved anticyclones grew in preference to cyclones. Thus this simulation, along with subsequent standard shallow water simulations, was able to pick up the predominance of anticyclones relative to cyclones on Jupiter. The equatorial jet is found to dominate the mid-latitude jets in standard shallow water simulations (which was not the case for the two-dimensional non-divergent simulations discussed in §1.3.1). However, Cho and Polvani’s (1996*a,b*) simulations, and subsequent forced-dissipative simulations, invariably produce retrograde jets at the equator when run with Jovian parameter values, in contrast to the prograde super-rotating equatorial jet observed on Jupiter (e.g. see Cho and Polvani, 1996*a,b*, Iacono, Struglia and Ronchi, 1999, Iacono, Struglia, Ronchi and Nicastro, 1999, Scott and Polvani, 2007, Showman, 2007).

Showman (2007) assumed that cloud layer turbulence resulted from the thunderstorms observed by Voyager, Galileo and Cassini (see §1.1). Based on the idea that storms transport mass from the deep atmosphere into the weather layer, these episodic storms were represented by episodically injecting isolated, circular mass pulses into the shallow weather layer. At mid-latitude deformation radii less than 2000km, Showman’s simulations produced broad jets near the equator, but regions polewards of 15° – 30° latitude were dominated by vortices. However at mid-latitude deformation radii greater than 4000km, Showman’s simulations became dominated by zonal jets

and vortices were weak. The lack of mid-latitude jets for small deformation radii is consistent with the idea that the finite deformation radius in shallow water equations inhibits the formation of jets. Under the influence of a finite deformation radius, the dispersion relation for Rossby waves becomes

$$\omega = -\frac{\beta k}{k^2 + l^2 + k_D^2}, \quad (1.3.7)$$

on a mid-latitude β -plane, where k_D is the wavenumber that corresponds to the deformation radius L_D . Equating this Rossby wave frequency with the turbulence frequency $\omega = U|\mathbf{k}|$, we obtain the modified anisotropic Rhines-scale wavenumber

$$|\mathbf{k}_\beta|^2 = \frac{\beta}{U} |\cos \theta| - k_D^2, \quad (1.3.8)$$

as a function of the polar angle $\theta = \tan^{-1}(l/k)$. Thus the finite deformation radius decreases the Rossby wave frequency, and moves the Rhines-scale wavenumber closer to the origin (compare (1.3.3) with (1.3.8)). Thus flows with a finite deformation radius have wider jets than two-dimensional non-divergent forced flows with the same values for U and β . If k_D is large enough, the Rhines-scale wavenumber can shrink to zero, which implies that all wavenumbers are dominated by turbulence. In this case the Rhines scale does not exist, and the flow is governed by turbulence in the presence of a finite deformation radius. Thus small deformation radii can suppress the β -effect and the formation of zonal jets.

1.3.3 Height-damped shallow water models

Scott and Polvani (2007, 2008) addressed the issue that the jet at the equator invariably goes the wrong way for standard shallow water simulations with Jovian parameter values. They introduced dissipation into the shallow water height equation to produce the model

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = -(h - h_0) / \tau_{\text{rad}}, \quad (1.3.9a)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + f \hat{\mathbf{z}} \times \mathbf{u} = -g' \nabla h, \quad (1.3.9b)$$

where they also included a narrow-band isotropic random forcing $\mathbf{F}$ of the kind described in §3.2.4. The Newtonian cooling term on the right-hand side of (1.3.9a) relaxes h towards a mean layer depth h_0 with time scale τ_{rad} . This relaxation is intended to model heating or cooling by radiation, following earlier models of the terrestrial stratosphere (Jukes, 1989, Polvani et al., 1995, Shell and Held, 2004). Simulations

with $\tau_{\text{rad}} = \pi(a/L_D)^2/(2\Omega)$ robustly produced prograde equatorial jets for several different ratios of the deformation radius $L_D = \sqrt{g'h_0}/(2\Omega)$ to the planetary radius a .

Considering the two-layer model in figure 1.5, the right-hand side of (1.3.9a) arises from imagining that radiative heating or cooling causes fluid parcels to cross the interface between the upper weather layer and the deep quiescent layer below. Setting aside the question of how well this picture describes the Jovian atmosphere, there is some uncertainty over how the mass exchange across the interface should contribute to the momentum equation. The Newtonian cooling term in the Scott and Polvani (2007, 2008) model does not conserve momentum, because (1.3.9a) and (1.3.9b) together imply

$$(h\mathbf{u})_t + \nabla \cdot (h\mathbf{u}\mathbf{u} + \tfrac{1}{2}g'h^2\mathbf{I}) + f\hat{\mathbf{z}} \times (h\mathbf{u}) = -(h - h_0)\mathbf{u}/\tau_{\text{rad}}. \quad (1.3.10)$$

Shell and Held (2004) considered the slightly different model

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = -(h - h_0)/\tau_{\text{rad}}, \quad (1.3.11a)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} = -g'\nabla h + (1 - h_0/h)\mathcal{H}(h_0 - h)\mathbf{u}/\tau_{\text{rad}}, \quad (1.3.11b)$$

where $\mathcal{H}(h_0 - h)$ is the Heaviside step function. The asymmetry arises from the expectation that when $h > h_0$ fluid parcels leave the active layer without changing the mean velocity of that layer. Conversely, when $h < h_0$, fluid from the quiescent lower layer enters and mixes with the active layer while conserving total momentum, thus lowering the mean velocity of the active layer. These difficulties originate from the fixed relation between the layer depth h and the pressure $(1/2)g'h^2$ in the standard shallow water equations. In reality, a warmer column of gas should exert a larger pressure than a cooler column with the same mass. Radiative cooling should affect the difference in temperature, but not the mass.

1.3.4 A radiatively damped thermal shallow water model

This thesis presents the thermal shallow water equations as an alternative model for gas giant planets. They enable the incorporation of a consistent treatment of radiative cooling to space in a temperature equation, thus preserving the key conservation properties for mass and momentum.

Standard shallow water theory applies to one or more layers of homogeneous fluid, while thermal shallow water theory permits horizontal variations of the thermodynamic properties of the fluid within each layer. These models were introduced

by Lavoie (1972) to describe atmospheric mixed layers over frozen lakes, and later adopted for tropical and coastal oceans, and the upper ocean mixed layer (McCreary et al., 1991, McCreary and Kundu, 1988, McCreary and Yu, 1992, R  ed and Shi, 1999, Schopf and Cane, 1983) and atmospheric currents (Anderson, 1984). They are commonly employed for the uppermost layer of predictive numerical ocean models using Lagrangian discretisations in the vertical, such as the Miami Isopycnic Coordinate Ocean Model (MICOM) (Bleck et al., 1992) and the Generalized Ocean Layered Model (GOLD) (Adcroft and Hallberg, 2006). Their theoretical properties were initially developed by Ripa (1993, 1995, 1996a).

While standard shallow water theory assumes a fixed density contrast $\Delta\rho$, thermal shallow water theory extends this by relaxing the homogeneity assumption for the upper layer, which is then characterised by a spatially varying density contrast $\Delta\rho = \rho_0 - \rho(\mathbf{x}, t)$ from the lower layer with constant density ρ_0 . Introducing $\Theta = g\Delta\rho/\rho_0$ to represent this density contrast, using a different symbol to emphasise that Θ is a function of $\mathbf{x}$ and t , the equivalent barotropic approximation then leads to the radiatively damped thermal shallow water equations:

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0, \quad (1.3.12a)$$

$$\frac{\partial \Theta}{\partial t} + (\mathbf{u} \cdot \nabla) \Theta = -(\Theta h/h_0 - \Theta_0)/\tau_{\text{rad}}, \quad (1.3.12b)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla(\Theta h) + \frac{1}{2}h\nabla\Theta, \quad (1.3.12c)$$

where a radiative damping term has been added to form a model for gas giant planets. The Newtonian cooling term $-(\Theta h/h_0 - \Theta_0)/\tau_{\text{rad}}$ on the right-hand side of (1.3.12b) represents radiative relaxation towards a temperature $\Theta_0 h_0/h$ with time scale τ_{rad} . The h -dependence represents the idea that columns that extend higher are able to radiate more effectively. In this respect, one should think of the upper layer extending upwards indefinitely with an exponentially decaying density, rather than terminating at a sharp upper boundary as in the terrestrial oceanic context more applicable to figure 1.5. The momentum equation (1.3.12c) reduces to the standard shallow water momentum equation for a homogeneous layer when Θ is constant. The radiatively damped thermal shallow water model conserves both mass and momentum within the active layer,

$$(h\mathbf{u})_t + \nabla \cdot (h\mathbf{u}\mathbf{u} + \frac{1}{2}\Theta h^2\mathbf{I}) + f\hat{\mathbf{z}} \times (h\mathbf{u}) = 0, \quad (1.3.13)$$

because the Newtonian cooling term is included in a separate temperature equation as opposed to the continuity equation. The thermal shallow water equations resemble

shallow water models of moist convection (Bouchut et al., 2009, Lambaerts et al., 2011, Zeitlin, 2007), Young’s depth-averaged description of the upper oceanic mixed layer (Young, 1994), and the shallow water magnetohydrodynamic equations (Dellar, 2003, Gilman, 2000).

1.3.5 Multi-layer models

Multi-layer studies of shallow jet forcing have focused on the hypothesis that the zonal jets emerge from baroclinic instabilities due to horizontal temperature contrasts at cloud-level. Williams (1979) performed a two-layer local quasi-geostrophic study on a β -plane (see chapter 4 for an introduction to the local quasi-geostrophic limit, and §1.4.1 for an introduction to β -planes). The forcing consisted of a latitudinally varying heating that produced a temperature contrast between the top and bottom latitudinal boundaries of between 10 K and over 60 K depending on the simulation. During the simulations, a two-dimensional inverse cascade was energised by the baroclinic instabilities and arrested at the Rhines scale, leading to multiple zonal jets. No additional (random) forcing was included in these simulations; the turbulence was generated by the baroclinic instabilities, which were in turn driven by the imposed latitudinally varying heating.

Williams (2003*a,b,c*) considered whether three-dimensional shallow flows driven by baroclinic instabilities can reproduce the super-rotating Jovian equatorial jet. Williams found that heating profiles with weak latitudinal temperature gradients near the equator produced a sub-rotating equatorial jet. In contrast, heating profiles with large latitudinal temperature gradients near the equator produced a super-rotating equatorial jet. When the jets closest to the equator formed at a latitude less than a critical latitude, they experienced a barotropic instability that drove eddy energy and momentum equatorward, and produced a prograde equatorial jet. It is possible that latent heating in the equatorial zone could produce sufficiently large latitudinal temperature gradients to drive the prograde equatorial jet, although it is not currently known if Jupiter behaves this way near the equator.

Lian and Showman (2008) performed three-dimensional Jovian numerical simulations of the hydrostatic primitive equations in pressure coordinates on a sphere (with 22–30 vertical layers). The simulations were driven by shallow thermal forcing confined to pressures near the cloud tops (above ~ 2 –10 bars). This forcing consisted of latitudinal temperature gradients, to represent latitudinal variations of e.g. latent heat release or solar energy absorption. These gradients were included via the introduction of a Newtonian cooling term in the potential temperature equation. The resulting

thermal contrasts drove baroclinic instabilities which generated eddies, and pumped the jets in the shallow forced layer. This led to multiple mid-latitude jets, where the simulated jet width and total number of jets (25–30 for Jovian simulations) were in approximate agreement with Jovian observations. These shallow jets gradually developed deep barotropic components that extended to the bottom of the model at 100 bars, and reached speeds comparable to the jet speeds in the shallow forced layer. These deep jets were pumped by the Coriolis acceleration acting on deep meridional circulations driven by the eddies in the shallow forced layer. The simulated jet widths in the barotropic region corresponded to the Rhines scale, suggesting that a quasi-two-dimensional inverse cascade modified by the β -effect was energised by baroclinic eddies. When the latitudinal temperature gradients near the equator were steep, simulations reproduced stable equatorial super-rotation, where diagnostics showed that barotropic eddies at low latitudes drove this equatorial super-rotation (cf. Williams, 2003*c*). These simulations support the idea that zonal jets may penetrate deeply on Jupiter even if the forcing is shallow (cf. §1.2).

Lian and Showman (2010) extended this work by explicitly including a simple representation of moist convection, and removing arbitrary heating functions. Water vapour was explicitly included, as was its condensation and the resulting latent heating. Cloud microphysics were neglected; all the condensate was assumed to instantaneously rain downward to the bottom of the domain. A Newtonian cooling scheme was included in the energy equation, where the equilibrium potential temperature profile was independent of latitude (in contrast to Lian and Showman, 2008). Ascending motion led to supersaturation which triggered condensation and released latent heat that was added to the energy equation, this caused a warming that enhanced the ascending motion. A self-sustaining circulation with horizontal temperature gradients induced by the latent heating developed (cf. Lian and Showman, 2008, where temperature gradients resulted from an ad hoc Newtonian cooling profile). Baroclinic instabilities produced eddies, which interacted with the planetary rotation to generate large-scale zonal flows. For plausible water abundances (3–5 times the solar water abundance for Jupiter/Saturn, and 30 times for Uranus/Neptune) these simulations produced ~ 20 zonal jets for Jupiter and Saturn and ~ 3 zonal jets for Uranus and Neptune (similar to the observed number of jets on these planets). Jupiter/Saturn simulations reproduced equatorial super-rotation, while Uranus/Neptune simulations reproduced equatorial sub-rotation. This is the first study to naturally produce super-rotation in a Jupiter/Saturn regime and sub-rotation in a Uranus/Neptune regime, without

tuning and within a single model. Lian and Showman found that equatorial super-rotation preferably formed in simulations with modest water vapour abundance, and is promoted by large planetary radii and fast rotation rates. In contrast, they found that high water vapour abundance leads to equatorial sub-rotation, regardless of the planetary radius and rotation rate. Given the larger radii, faster rotation rates, and probable lower water abundances of Jupiter and Saturn relative to Uranus and Neptune, these simulations provide a possible mechanism for the existence of equatorial super-rotation on Jupiter and Saturn, and equatorial sub-rotation on Uranus and Neptune.

There are few three-dimensional studies that investigate whether jets can result from small-scale turbulence in Jupiter’s shallow weather layer, as opposed to resulting from baroclinic instabilities. Kitamura and Matsuda (2004) considered two-level primitive equation simulations of freely evolving turbulence in a stably stratified fluid on a sphere. Multiple jets emerged at the Rhines scale when the Rossby number was small, and when L_D was small compared to the planetary radius. A strong circumpolar vortex developed at each pole, and the jets were barotropic except at the equator (the top layer developed a prograde jet, while the bottom layer developed a retrograde jet). This model again demonstrates that multi-layer models show promise in understanding the super-rotating Jovian equatorial jet.

Sayanagi et al. (2008) extended Kitamura and Matsuda’s (2004) study to include more vertical layers (between 12 and 36), thus permitting a much wider range of three-dimensional effects. They performed fully three-dimensional primitive equation simulations of freely evolving stratified geostrophic turbulence on a β -plane. They found that robust zonal jets emerged from initial small-scale random turbulence through the upscale redistribution of the kinetic energy in spectral space. The flow was found to be in a jet dominated regime if $L_R \lesssim L_D$, where $L_D = NH/f_0$ is the baroclinic deformation radius (where N is the Brunt–Väisälä frequency, H is the characteristic vertical length scale, and f_0 is the Coriolis parameter). The flow was instead found to be vortex dominated if $L_R \gtrsim L_D$.

1.3.6 Deep models

The alternative theory for the formation of Jupiter’s zonal jets is that they result from deep convection throughout Jupiter’s interior. This hypothesis originated with Busse (1970, 1976), who claimed that surface zonal flows are the surface manifestation of concentric rotating cylinders (Taylor columns) penetrating the depth of the planet, and that their interactions create the zonal jets.

The deep simulations mentioned below are not in a Jovian regime, so their jet profiles must be viewed with caution. The Ekman number (the ratio of viscous to Coriolis forces) of Jupiter’s molecular envelope is $\sim 10^{-15}$, whereas the following simulations use Ekman numbers greater than 10^{-6} (this is a computational limitation which would require major advances in computational methods and hardware to overcome). Previous three-dimensional convection models adopted relatively thick shells, with inner to outer radius ratios typically less than ~ 0.7 . While these simulations achieved speeds similar to Jupiter’s, they generally produced jets that were too few and too wide (Aurnou and Olson, 2001, Christensen, 2001). Only convection in an extremely thin shell (with an inner to outer radius ratio $\sim 0.85\text{--}0.9$) appears capable of producing Jupiter-like zonal jets, with a super-rotating equatorial jet and numerous (~ 20) weaker mid-latitude jets (Heimpel and Aurnou, 2007, Heimpel et al., 2005). However, these simulations require intrinsic heat fluxes several orders of magnitude larger than Jupiter’s. The above deep convection simulations robustly produce a super-rotating equatorial jet at the equator due to the interaction of the columnar convective rolls with the curving spherical planetary surface (Busse, 2002, Vasavada and Showman, 2005). However, the above deep convection simulations make the Boussinesq approximation. Density increases by several orders of magnitude with depth below Jupiter’s troposphere, hence it is unclear whether the jet pumping mechanisms in the above models are relevant for the compressible case.

The models previously mentioned in this section have advanced either differential radiative heating of the upper atmosphere, or intrinsic convective heat fluxes emanating from the deep interior, as the driver of Jupiter’s general circulation. Schneider and Liu (2009) instead considered these mechanisms together. They performed three-dimensional simulations of Jupiter’s upper atmosphere by solving dry hydrostatic primitive equations in a thin shell. This model extended from the top of the atmosphere to a pressure of ~ 3 bars. It adopted a simple grey radiative-transfer scheme, and imposed an intrinsic heat flux at the lower boundary. A convective parametrisation was included that relaxed temperatures in statically unstable parts of atmospheric columns to a profile with neutral static stability. Rayleigh friction was included at the base of the model, as a linear representation of the effects of the magnetohydrodynamic drag deep in Jupiter’s atmosphere on the flow in the upper atmosphere (see page 5 for more details). To represent the downward projection along cylinders in a deep atmosphere, the Rayleigh friction coefficient was constant along the lower boundary polewards of 16.38° latitude and rapidly decreased to zero at lower latitudes (the magnetohydrodynamic drag experienced by Taylor columns that

extend into the electrically conducting inner region is confined within 0.96 Jupiter radii). Taylor columns near the equator cross into the other hemisphere without penetrating the deep interior, and so experience no magnetohydrodynamic drag in this model. Jovian simulations reproduced a broad stable super-rotating equatorial jet and multiple east-west mid-latitude jets, with speeds and widths similar to those observed on Jupiter. Schneider and Liu (2009) propose that baroclinic eddies generated by differential radiative heating of the upper atmosphere are responsible for multiple off-equatorial jets, while prograde equatorial jets occur when intrinsic heat fluxes are strong enough that Rossby waves generated convectively in the equatorial region transport angular momentum toward the equator.

Liu and Schneider (2010) use the above model to simulate all four of the gas giant planets. They demonstrate that the different flow configurations on the giant planets can be reproduced through differences in their radiative heating and intrinsic heat fluxes. Their simulations reproduce many features of the observed flows, including equatorial super-rotation for Jupiter and Saturn and equatorial sub-rotation for Uranus and Neptune. The direction of the equatorial jet is found to depend on whether the deep intrinsic heat fluxes are sufficiently strong so that convection penetrates into the upper troposphere and generates strong equatorial Rossby waves there. Super-rotating equatorial jets result if convective Rossby wave generation is strong and low latitude angular momentum flux divergence owing to baroclinic eddies generated off the equator is sufficiently weak (Jupiter and Saturn). Sub-rotating equatorial jets result if either convective Rossby wave generation is weak (Uranus) or low latitude angular momentum flux divergence owing to baroclinic eddies is sufficiently strong (Neptune). Liu and Schneider (2011) carried out a systematic study to investigate how baroclinicity, heating from below, and bottom drag determine when equatorial super-rotation occurs in rapidly rotating atmospheres. As the heating at the bottom boundary strengthened, a transition to equatorial super-rotation occurred at a threshold heat flux. However, they found that this threshold depended on the baroclinicity of the flow and on the bottom drag. The greater the off-equatorial baroclinicity, and the stronger the bottom drag in the equatorial region, the greater the threshold heat flux at which super-rotation occurred.

1.3.7 Summary

Simulations of the one-layer standard shallow water equations invariably produce a sub-rotating equatorial jet for the gas giant planets (in agreement with Uranus and Neptune, but contrary to Jupiter and Saturn). However, simulations which

additionally incorporate a Newtonian cooling term in the continuity equation produce prograde equatorial jets for several different ratios of the deformation radius to the planetary radius (see Scott and Polvani, 2008, and §1.3.3). This model has not yet yielded a theory to explain why equatorial super-rotation occurs on Jupiter and Saturn but not on Uranus and Neptune. Some recent three-dimensional shallow atmosphere models also reproduce equatorial super-rotation under specific conditions (e.g. Lian and Showman, 2008, Williams, 2003*a,b,c*), but this generally requires the addition of an ad hoc forcing function. Even if such forcing were plausible, it is unclear why it would occur on Jupiter and Saturn but not on Uranus and Neptune. Boussinesq deep convection models in a thin shell (Heimpel and Aurnou, 2007, Heimpel et al., 2005) produce equatorial super-rotation and alternating mid-latitude jets (however they assume excessively viscous flow, and heat fluxes more than 10^6 times larger than those observed). The three-dimensional primitive equation models of Lian and Showman (2010) and Liu and Schneider (2010) are the first studies to naturally produce equatorial super-rotation in a Jupiter/Saturn regime, and equatorial sub-rotation in a Uranus/Neptune regime.

This thesis considers a one-layer radiatively damped thermal shallow water model for gas giant planets, which includes a Newtonian cooling term in a separate temperature equation (see §1.3.4). Chapter 3 demonstrates that this model produces multiple mid-latitude jets and a super-rotating equatorial jet for Jovian parameter values (when τ_{rad} is sufficiently short), while it produces a sub-rotating equatorial jet for Neptunian parameter values. This is the first shallow water model to naturally produce equatorial super-rotation in a Jovian regime and equatorial sub-rotation in a Neptunian regime, whilst also preserving the key conservation properties of mass and momentum (which the Scott and Polvani, 2008, model in §1.3.3 breaks). Chapter 6 provides a possible explanation for the equatorial jet direction in Jovian simulations of the radiatively damped thermal shallow water model and the Scott and Polvani (2008) model.

1.4 Spheres, beta-planes and square planets

The gas giant planets are oblate spheroids. They all have a much larger ellipticity than Earth due to their large radii and relatively rapid rotation (see table 1.1, where Earth's ellipticity is 0.0034 for comparison). However, a very common approximation uses a modified set of spherical polar coordinates in which r labels surfaces of constant geopotential, and the angles ϕ and λ parametrise these surfaces (see van der Toorn

and Zimmerman, 2008). It is important that surfaces of constant geopotential be horizontal to prevent an otherwise large term appearing in the horizontal momentum equations. If the geopotential surfaces are treated as being geometrically spherical, it is thus necessary to omit the centrifugal force, since its potential is a function of distance from the rotation axis, rather than distance from the centre of the sphere.

Standard discretisations on spheres use a truncated expansion in spherical harmonics. Spherical harmonic codes for solving equations on spheres are desirable due to their uniform resolution on the sphere, highly accurate computation of derivatives, absence of problems from clustering of points at the poles, and stability (Temperton, 2000). Standard spherical harmonic codes combine Fourier transforms in the longitude direction with slow Legendre transforms in the latitude direction. However, for a horizontal spectral truncation at total wavenumber N , the number of grid point computations per time step is $O(N^2)$, the work involved due to Fourier transforms is $O(N^2 \log N)$, while the computation of Legendre transforms scales as $O(N^3)$. There is as yet no known fast Legendre transform as there is for the Fourier transform. This is an issue for the simulations in this thesis, since turbulent self-organisation into zonal jets takes many eddy turn-over times. It is desirable to perform simulations on graphical processing units (GPUs), since they offer high memory bandwidth and floating point performance for many scientific algorithms, with performance characteristics resembling the vector machines widely used to simulate atmospheric flows in the 1990s. However spherical harmonic transform implementations for GPUs are only beginning to appear, and their performance remains comparable with conventional microprocessors (Hupca et al., 2012).

1.4.1 The beta-plane approximation

A β -plane approximates a portion of a spherical surface using Cartesian coordinates, whilst still capturing the latitude-dependence of the Coriolis parameter in spherical geometry (Pedlosky, 1987, Salmon, 1998, Vallis, 2006). They were first proposed by Rossby et al. (1939) as a purely conceptual model for motion on a sphere. The β -plane approximation is used to make analysis of the equations more tractable. The traditional Coriolis parameter $f = 2\Omega \sin \phi$ is replaced with a truncated Taylor series expansion in the variable $y = a(\phi - \phi_0)$, where ϕ_0 is a reference latitude and a is the planetary radius. Hence

$$f = 2\Omega \sin \left(\phi_0 + \frac{y}{a} \right) \approx f_0 + \beta^* y, \quad (1.4.1)$$

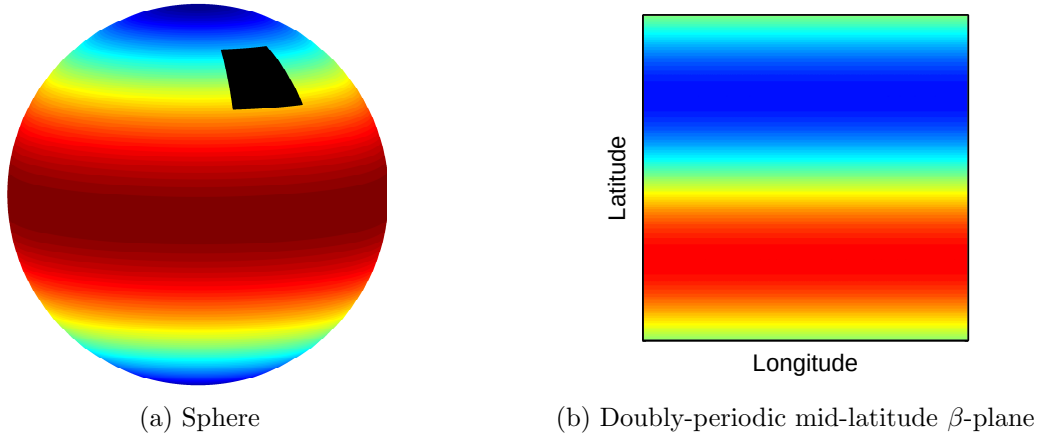


Figure 1.6: The doubly-periodic mid-latitude β -plane approximation.

where $f_0 = 2\Omega \sin \phi_0$ and $\beta^* = 2\Omega \cos \phi_0/a$. The f -plane approximation only keeps the first term f_0 in the Taylor expansion (1.4.1). In chapter 4 the mid-latitude β -plane approximation is used, while in chapter 6 the equatorial β -plane approximation is used (where $\phi_0 = 0$ and thus $f_0 = 0$). For a recent review of the β -plane approximation see Dellar (2011). Simulations performed on β -planes often make the additional assumption that the β -plane is doubly-periodic as in figure 1.6, and in the simulations on a doubly-periodic mid-latitude β -plane in chapter 7. These doubly-periodic boundary conditions allow the use of the GPU fast Fourier transform library.

1.4.2 Square planet domain

To retain the performance benefits of the GPU fast Fourier transform library, without the restriction to a small portion of a sphere, numerical simulations of the equation sets in chapters 3 and 5 are computed in the doubly-periodic Cartesian domain sketched in figure 1.7. This square planet domain can be thought of as a “global β -plane”, where the horizontal x coordinate denotes longitude and the vertical y coordinate denotes latitude. Starting from the north pole at the top of the domain and decreasing y , we reach the equator a quarter of the way down, and the south pole half way down. We then continue to reach the equator again, before returning to the north pole, now at the bottom of the domain. In other words, imagine following a meridian (constant longitude line) from the north pole to the equator to the south pole, and then following a second meridian with longitude offset by 180° back across the equator to the north pole. This domain allows the use of Fourier spectral methods based on the GPU fast Fourier transform implementation in both the longitude and

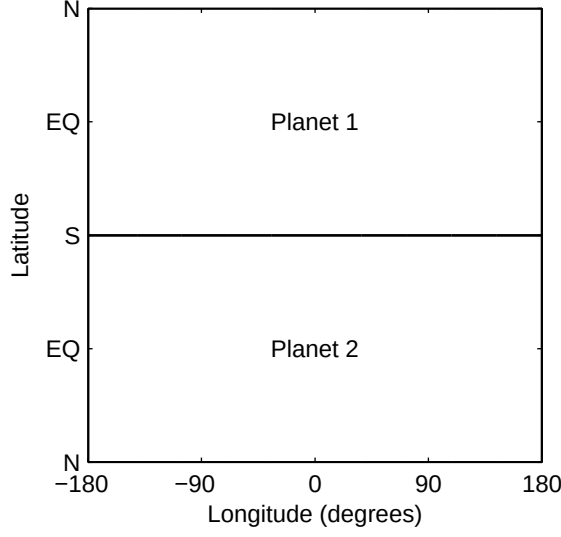


Figure 1.7: Schematic of the square planet domain. The x axis denotes the full range of longitude, and the y axis denotes latitude (where starting at the top of the y axis and moving down we have the north pole, then the equator, then the south pole, we then reach the equator again, and finally return to the north pole). Each half of the domain denotes a full copy of the planet.

latitude directions.

The dimensionless traditional Coriolis parameter in this geometry is taken to be the periodic function $f(y) = \cos(2\pi y/y_{\max})$, where $y_{\max}$ is the dimensionless side length of the domain, which is related to the circumference of the planet in question. Thus the Coriolis parameter is fully varying in latitude, and all its derivatives are doubly periodic. The poles do not consist of a single point in this domain, but each is a line in longitude (although the north pole horizontal lines at the bottom and top of the square planet domain are the same due to the periodic boundary condition). Each half of the domain corresponds to a complete planet, so simulation results are plotted for only the top planet (planet 1 in figure 1.7).

1.4.3 The double Fourier series method on a sphere

This double Fourier series representation can be adapted to the sphere by including the metric coefficients. These are only appreciably different from unity away from the equator. Merilees (1973) developed a fast pseudo-spectral method on the sphere where one-dimensional fast Fourier transforms were used to compute the spatial derivatives in both the longitude and latitude directions (in contrast to standard spherical harmonic codes which use slow Legendre transforms in the latitude direc-

tion). Computing derivatives in both directions on the surface of the sphere is an $O(N^2 \log N)$ operation. However for this double Fourier series method, the grid spacing near the poles places an unacceptable restriction on the time step unless further steps are taken. Filters must thus be used to alleviate these prohibitive time step restrictions and maintain stability, though Merilees' (1973) original filter was found to be unstable for long runs.

Attempts have been made to overcome this problem. Spatz et al. (1998) and Spatz (2000) project the prognostic variables onto the space of spherical harmonics at the end of each time step using Legendre transforms. The numerical results have the same accuracy and stability as a conventional spherical harmonic code. However, this method reintroduces Legendre transforms (an $O(N^3)$ algorithm), although it reduces the number of required transforms. The algorithm which combines the double Fourier series method with a projection onto the space of spherical harmonics is thus only an $O(1)$ numerical factor cheaper than a conventional spherical harmonic code, and is $O(N/\log N)$ more expensive than simulations in the square planet domain detailed above. Additional savings are possible by making the projection more efficient. For instance, Jakob-Chien and Alpert (1997) have developed a fast multipole spherical harmonic filter, with accuracy comparable to the aforementioned projection, which requires $O(N^2 \log N)$ operations.

1.5 Overview

In chapter 2 three derivations of the thermal shallow water equations are presented, where dissipation by Newtonian cooling is neglected for the majority of this chapter (which equates to setting τ_{rad} to be infinite). The first derivation considers a single layer (as depicted in figure 1.4) and rescales the three-dimensional Boussinesq equations with a small aspect ratio between vertical and horizontal length scales, followed by depth-averaging across the fluid layer. A similar derivation is then sketched for the two-layer system depicted in figure 1.5. The third derivation again considers a single layer of fluid, and projects the three-dimensional Lagrangian for a Boussinesq fluid onto columnar fluid motions, followed by applying Hamilton's principle of least action. The potential vorticity equation is discussed, as is the non-canonical Hamiltonian structure of the thermal shallow water equations. The group velocity is shown to be the velocity with which the wave energy is transported. Finally the Newtonian cooling term is reintroduced into the temperature equation of the thermal shallow

water equations. The particular form of the Newtonian cooling term is motivated, and the effect of the Newtonian cooling term on linear plane waves is considered.

In chapter 3, simulations of the radiatively damped thermal shallow water equations (1.3.12) are compared to simulations of the standard shallow water equations (1.3.6), the height-damped shallow water equations (1.3.9), and the momentum-conserving height-damped shallow water equations (1.3.11, with $\mathcal{H} \equiv 1$). Simulations are performed in a doubly-periodic Cartesian geometry, where Rayleigh friction is included to dissipate energy at large-scales, and turbulence is driven by applying a divergence-free isotropic random forcing. Simulations are performed for Jovian parameter values. While simulations of the standard shallow water equations produce a sub-rotating equatorial jet, simulations of the three radiatively damped models all reproduce a strong super-rotating equatorial jet when τ_{rad} is sufficiently short, in line with Jovian observations. Simulations of the thermal model are also performed for Neptunian parameters. These reproduce the broad sub-rotating equatorial jet and prograde mid-latitude jets observed on Neptune.

Chapter 4 formalises a previous derivation of the local quasi-geostrophic (QG) limit of the radiatively damped thermal shallow water equations on a mid-latitude β -plane, by performing a direct asymptotic expansion for small Rossby number. A second derivation is then presented as the small Rossby number limit of a balanced model that projects out high-frequency dynamics due to inertia-gravity waves. The effect of Newtonian cooling on the Rossby waves in the local thermal quasi-geostrophic equations is considered, and waves in the thermal shallow water equations are compared to those in their local quasi-geostrophic limit. The group velocity is shown to be the velocity with which the wave energy is transported for the local thermal quasi-geostrophic equations. The local thermal quasi-geostrophic energy conservation equation is derived by applying the local quasi-geostrophic limit to the pseudo-energy conservation equation for the radiatively damped thermal shallow water equations. This derivation involves the ageostrophic correction to the leading order geostrophic velocity, which is eliminated in the usual derivation for a closed evolution equation for the quasi-geostrophic potential vorticity. Finally, the non-canonical Hamiltonian structure of the local thermal quasi-geostrophic equations is derived from a decomposition in Rossby number of a pseudo-energy and Poisson bracket for the thermal shallow water equations.

Chapter 5 derives the global quasi-geostrophic form of the radiatively damped thermal shallow water equations and the height-damped shallow water equations. These global quasi-geostrophic models are valid for the entire planet, as opposed to

the usual restriction to a mid-latitude β -plane (cf. chapter 4). The effect of radiative cooling on equatorially trapped waves in the global thermal and height-damped quasi-geostrophic equations is considered. The group velocity is shown to be the velocity with which the wave energy is transported for these global quasi-geostrophic models. The Hamiltonian and Poisson bracket for the global thermal quasi-geostrophic equations is derived from a small Rossby number decomposition of a pseudo-energy and Poisson bracket for the thermal shallow water equations. Jovian numerical simulations of the different shallow water models from §3.3 are compared to numerical simulations of the corresponding global quasi-geostrophic forms of these models. Simulations of the global thermal and height-damped quasi-geostrophic models demonstrate a transition from a sub- to super-rotating equatorial jet as the radiative time scale is decreased (this same transition is observed in simulations of the height-damped shallow water equations, and the radiatively damped thermal shallow water equations). The mechanism responsible for this switch in equatorial jet direction must therefore lie within quasi-geostrophic theory.

Equatorially trapped waves play an important role in near-equatorial dynamics, and the equatorial β -plane approximation offers a convenient theoretical model (see §1.4.1). Chapter 6 investigates the role of these waves in establishing the direction of the equatorial jet in Jovian simulations of the standard shallow water model, the height-damped shallow water model, and the radiatively damped thermal shallow water model. The mechanism responsible for the switch in equatorial jet direction from sub- to super-rotating as the radiative time scale is decreased in height-damped and radiatively damped thermal shallow water simulations is known to lie within quasi-geostrophic theory from chapter 5, thus the effects of equatorially trapped gravity, Yanai and Kelvin waves are not considered in the explanation. The shallow water models are considered on an equatorial β -plane, and Rayleigh friction and Newtonian cooling are shown to damp the equatorially trapped Rossby waves. This alters the phase relation between the various terms in the shallow water zonal mean zonal momentum equation, creating non-zero contributions to the zonal mean zonal acceleration. Dissipation solely by Newtonian cooling leads to a positive zonal mean zonal acceleration in the equatorial region, consistent with the formation of a super-rotating equatorial jet. Dissipation solely by Rayleigh friction leads to a negative zonal mean zonal acceleration in the equatorial region, consistent with the formation of a sub-rotating equatorial jet. The Rayleigh friction time scale is much longer than the Newtonian cooling time scale in the Jovian simulations of the height-damped and radiatively damped thermal shallow water models in §3.3. This suggests that the

effect of Newtonian cooling on the zonal mean zonal acceleration outweighs the effect of Rayleigh friction for both shallow water models, consistent with the formation of a super-rotating equatorial jet.

Chapter 2

The thermal shallow water equations

This chapter is partially based upon Warneford and Dellar (2013).

2.1 Introduction

Shallow water theory describes one or more layers of inviscid fluid. Within each layer the horizontal velocity is assumed to be depth-independent, so the fluid moves in columns. In thermal shallow water theory, the homogeneity assumption of standard shallow water theory is relaxed, and the thermodynamic fluid properties within each layer are allowed to vary in the horizontal. This scenario arises most naturally in an equivalent barotropic or $1\frac{1}{2}$ -layer configuration, as sketched in figure 1.5. The active upper layer is characterised by a spatially varying, but bounded from below, density contrast $\Delta\rho$ from a much deeper, quiescent lower layer with constant density ρ_0 . The variable $\Theta = g\Delta\rho/\rho_0$ is introduced to represent this density contrast, in the form commonly called the reduced gravity g' for spatially homogeneous layers. This chapter provides three derivations of the thermal shallow water equations (1.3.12), although dissipation due to Newtonian cooling is neglected until §2.7. Since the equivalent barotropic approximation (see §2.3) leads to a closed set of thermal shallow water equations for only the active upper layer in figure 1.5, for two of these derivations the geometry in figure 1.4 is instead considered, which consists of a thin layer of fluid between a rigid surface at $z = 0$ and a free surface at $z = h(x, y, t)$. This is the configuration often used in previous shallow water derivations. The non-canonical Hamiltonian structure of the thermal shallow water equations is discussed. Energy is also discussed, along with the form of the Newtonian cooling term in the temperature

equation (1.3.12b) and its effect on linear plane waves in the thermal shallow water equations.

2.2 Derivation by layer-averaging for a single layer

First consider the simplest case of one active layer, as depicted in figure 1.4. The thermal shallow water equations are derived by layer-averaging the Boussinesq approximation to the three-dimensional Euler fluid equations in a rotating frame. In the Boussinesq approximation, density differences are considered to be sufficiently small to be neglected except in terms multiplied by gravity g (Corfield, 1984, Spiegel and Weiss, 1982, Vallis, 2006). This eliminates compressible effects such as sound waves. Two-dimensional equations for the depth-averaged horizontal velocities, the layer depth and the temperature are obtained.

2.2.1 The three-dimensional fluid equations

The governing equations for a three-dimensional rotating Boussinesq fluid, between a rigid surface at $z = 0$ and a free surface at $z = h(x, y, t)$, are given by

$$\nabla_3 \cdot \mathbf{u}_3 = 0, \quad (2.2.1a)$$

$$\frac{\partial \Theta}{\partial t} + (\mathbf{u}_3 \cdot \nabla_3) \Theta = 0, \quad (2.2.1b)$$

$$\frac{\partial \mathbf{u}_3}{\partial t} + (\mathbf{u}_3 \cdot \nabla_3) \mathbf{u}_3 + 2\boldsymbol{\Omega}_3 \times \mathbf{u}_3 = -\frac{1}{\rho_0} \nabla_3 p - \Theta \hat{\mathbf{z}}. \quad (2.2.1c)$$

Here $\mathbf{u}_3$ is the fluid velocity, h is the layer depth, p is the pressure and Θ is some advected scalar such as temperature (or salinity in the ocean). Gravity g has been included in Θ so that Θ becomes equivalent to the reduced gravity $g(\rho - \rho_0)/\rho_0$, where ρ_0 is a constant reference density. The system is assumed to rotate with angular velocity vector $\boldsymbol{\Omega}_3 = (\Omega_x, \Omega_y, \Omega_z)$, which gives rise to the Coriolis force $2\boldsymbol{\Omega}_3 \times \mathbf{u}_3$. A coordinate system is considered where $\nabla_3 \Phi = \Theta \hat{\mathbf{z}}$, where Φ is the geopotential. Then z is in the direction that is locally upward as defined by $\nabla_3 \Phi$, and the horizontal directions are tangent to the surfaces of constant geopotential, with $\hat{\mathbf{z}}$ denoting a unit vector in the z direction. The Θ term thus includes both the gravitational force and the centrifugal force due to rotation. A subscript 3 denotes a three-component vector e.g. $\mathbf{u}_3 = (u, v, w)$, while un-subscripted vectors are taken to be purely horizontal e.g. $\mathbf{u} = (u, v)$.

The pressure is assumed to vanish at the free surface, so

$$p = 0 \quad \text{on} \quad z = h(x, y, t). \quad (2.2.2a)$$

The kinematic boundary conditions are also made on the two surfaces,

$$w = 0 \quad \text{on} \quad z = 0, \quad (2.2.2b)$$

$$w = \left(\frac{\partial h}{\partial t} + (\mathbf{u} \cdot \nabla) h \right) \quad \text{on} \quad z = h(x, y, t). \quad (2.2.2c)$$

2.2.2 The shallow water scaling

A non-dimensionalisation is applied:

$$\begin{aligned} x = L\tilde{x}, \quad y = L\tilde{y}, \quad z = \delta L\tilde{z}, \quad u = U\tilde{u}, \quad v = U\tilde{v}, \quad w = \delta U\tilde{w}, \quad t = (L/U)\tilde{t}, \\ p = 2\Omega UL\rho_0\tilde{p}, \quad \Theta = \Theta_0\tilde{\Theta}, \quad \mathbf{\Omega}_3 = \Omega\tilde{\mathbf{\Omega}}_3, \quad h = \delta L\tilde{h}. \end{aligned} \quad (2.2.3)$$

Here, L is the horizontal length scale, U is the horizontal velocity scale, H is the vertical length scale, Ω is the magnitude of the angular velocity, Θ_0 is a typical value for Θ , and $\delta = H/L$ is the aspect ratio, i.e. the small parameter that characterises the shallowness of the layer. The non-dimensionalisation for w is chosen so that δ does not enter the dimensionless continuity equation, and the non-dimensionalisation for t is chosen so that $\partial \mathbf{u}_3 / \partial t$ is comparable to $(\mathbf{u}_3 \cdot \nabla_3) \mathbf{u}_3$ in the dimensionless momentum equations. The dimensionless equation set is then given by

$$\tilde{\nabla} \cdot \tilde{\mathbf{u}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0, \quad (2.2.4a)$$

$$\frac{\partial \tilde{\Theta}}{\partial \tilde{t}} + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) \tilde{\Theta} + \tilde{w} \frac{\partial \tilde{\Theta}}{\partial \tilde{z}} = 0, \quad (2.2.4b)$$

$$Ro \left(\frac{\partial \tilde{\mathbf{u}}}{\partial \tilde{t}} + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) \tilde{\mathbf{u}} + \tilde{w} \frac{\partial \tilde{\mathbf{u}}}{\partial \tilde{z}} \right) + \tilde{\Omega}_z (\hat{\mathbf{z}} \times \tilde{\mathbf{u}}) + \delta (\tilde{\mathbf{\Omega}} \times \hat{\mathbf{z}}) \tilde{w} = -\tilde{\nabla} \tilde{p}, \quad (2.2.4c)$$

$$\delta^2 Ro \left(\frac{\partial \tilde{w}}{\partial \tilde{t}} + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) \tilde{w} + \tilde{w} \frac{\partial \tilde{w}}{\partial \tilde{z}} \right) + \delta (\tilde{\Omega}_x \tilde{v} - \tilde{\Omega}_y \tilde{u}) = -\frac{\partial \tilde{p}}{\partial \tilde{z}} - \frac{Bu}{Ro} \tilde{\Theta}, \quad (2.2.4d)$$

where $\tilde{\nabla} = (\partial_{\tilde{x}}, \partial_{\tilde{y}})$, and the Rossby and Burger numbers are given respectively by $Ro = U / (2\Omega L)$ and $Bu = \Theta_0 H / (4\Omega^2 L^2)$. In the following derivation, Ro and Bu are assumed to remain $O(1)$ as $\delta \rightarrow 0$ (the smallness of the Rossby number with respect to the Burger number is considered in chapters 4 and 5). The tilde notation is dropped with the understanding that all variables and operators are now dimensionless.

2.2.3 Layer-averaged equations

The equations governing the dynamics of the depth-averaged horizontal velocity in the layer are now derived. To perform this averaging, a result from Wu (1981) is used:

$$\int_{\eta_1(x,y,t)}^{\eta_2(x,y,t)} \left\{ \frac{\partial F}{\partial t} + (\mathbf{u} \cdot \nabla) F + w \frac{\partial F}{\partial z} \right\} dz = \frac{\partial}{\partial t} (h\bar{F}) + \nabla \cdot (h\bar{\mathbf{u}}F), \quad (2.2.5)$$

for a general $F(x, y, z, t)$, and for any two material surfaces $\eta_1(x, y, t)$ and $\eta_2(x, y, t)$, with $h(x, y, t) = \eta_2(x, y, t) - \eta_1(x, y, t)$. The overbar denotes a layer-averaged quantity

$$\bar{F} = \frac{1}{\eta_2(x, y, t) - \eta_1(x, y, t)} \int_{\eta_1(x, y, t)}^{\eta_2(x, y, t)} F(x, y, z, t) dz. \quad (2.2.6)$$

In this section $\eta_1(x, y, t) = 0$ and $\eta_2(x, y, t) = h(x, y, t)$ in (2.2.5) and (2.2.6).

Setting $F = 1$ in (2.2.5) gives the layer-averaged continuity equation

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\bar{\mathbf{u}}) = 0. \quad (2.2.7a)$$

Setting $F = \Theta$ in (2.2.5) enables the integration of (2.2.4b) over the layer:

$$\frac{\partial (h\bar{\Theta})}{\partial t} + \nabla \cdot (h\bar{\mathbf{u}}\bar{\Theta}) = 0. \quad (2.2.7b)$$

Similarly, setting $F = u$ and $F = v$ in (2.2.5) enables the integration of (2.2.4c) over the layer, as described in Dellar and Salmon (2005) and Stewart and Dellar (2010):

$$\begin{aligned} Ro \left(\frac{\partial (h\bar{\mathbf{u}})}{\partial t} + \nabla \cdot (h\bar{\mathbf{u}}\bar{\mathbf{u}}) \right) + \Omega_z h (\hat{\mathbf{z}} \times \bar{\mathbf{u}}) + \delta (\mathbf{\Omega} \times \hat{\mathbf{z}}) \int_0^{h(x,y,t)} w dz \\ = - \int_0^{h(x,y,t)} \nabla p dz, \end{aligned} \quad (2.2.7c)$$

where $\mathbf{\Omega}$ has been assumed to be independent of z . Setting $F = w$ in (2.2.5) gives the layer-averaged vertical momentum equation

$$\begin{aligned} \delta^2 Ro \left(\frac{\partial (h\bar{w})}{\partial t} + \nabla \cdot (h\bar{\mathbf{u}}\bar{w}) \right) + \delta \int_0^{h(x,y,t)} (\Omega_x v - \Omega_y u) dz = - \int_0^{h(x,y,t)} \frac{\partial p}{\partial z} dz \\ - \int_0^{h(x,y,t)} \frac{Bu}{Ro} \Theta dz. \end{aligned} \quad (2.2.7d)$$

2.2.4 Asymptotic expansion for small aspect ratio

The four equations (2.2.7a)–(2.2.7d) do not form a closed system for h and the averaged quantities $\bar{\mathbf{u}}$, $\bar{w}$, $\bar{p}$ and $\bar{\Theta}$. To obtain a closed system, an asymptotic expansion of the dependent variables $\mathbf{u}$, w , p and Θ is posed in the aspect ratio δ :

$$\begin{aligned}\mathbf{u} &= \mathbf{u}^0 + \delta \mathbf{u}^1 + \cdots, \quad w = w^0 + \delta w^1 + \cdots, \quad p = p^0 + \delta p^1 + \cdots, \\ \Theta &= \Theta^0 + \delta \Theta^1 + \cdots,\end{aligned}\tag{2.2.8}$$

where $\mathbf{u}^0$ and Θ^0 are assumed to be independent of z . Then $\overline{\mathbf{u}\Theta} = \bar{\mathbf{u}}\bar{\Theta} + O(\delta^2)$, $\overline{\mathbf{u}\mathbf{u}} = \bar{\mathbf{u}}\bar{\mathbf{u}} + O(\delta^2)$, and at leading order (2.2.7) becomes

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\bar{\mathbf{u}}) = 0,\tag{2.2.9a}$$

$$\frac{\partial (h\bar{\Theta})}{\partial t} + \nabla \cdot (h\bar{\mathbf{u}}\bar{\Theta}) = 0,\tag{2.2.9b}$$

$$Ro \left(\frac{\partial (h\bar{\mathbf{u}})}{\partial t} + \nabla \cdot (h\bar{\mathbf{u}}\bar{\mathbf{u}}) \right) + \Omega_z h (\hat{\mathbf{z}} \times \bar{\mathbf{u}}) = - \int_0^{h(x,y,t)} \nabla p^0 \, dz,\tag{2.2.9c}$$

$$0 = - \int_0^{h(x,y,t)} \frac{\partial p^0}{\partial z} \, dz - \frac{Bu}{Ro} h \bar{\Theta},\tag{2.2.9d}$$

since $a = \bar{a} = a^0$ at leading order in δ for $a = u, v, \Theta$. Substituting the asymptotic expansions (2.2.8) into (2.2.4d) gives

$$0 = - \frac{\partial p^0}{\partial z} - \frac{Bu}{Ro} \bar{\Theta},\tag{2.2.10}$$

at leading order. If the boundary condition (2.2.2a) is non-dimensionalised with the scalings (2.2.3), the tilde notation is dropped, and the asymptotic expansions (2.2.8) are introduced, then at leading order the boundary condition on p^0 is,

$$p^0 = 0 \quad \text{on} \quad z = h(x, y, t).\tag{2.2.11}$$

Integrating (2.2.10) with respect to z , and using the boundary condition (2.2.11) gives

$$p^0 = \frac{Bu}{Ro} (h - z) \bar{\Theta},\tag{2.2.12}$$

thus

$$\nabla p^0 = \frac{Bu}{Ro} (\bar{\Theta} \nabla h + (h - z) \nabla \bar{\Theta}),\tag{2.2.13}$$

and

$$\int_0^{h(x,y,t)} \nabla p^0 \, dz = h \frac{Bu}{Ro} \left(\nabla (\bar{\Theta} h) - \frac{h}{2} \nabla \bar{\Theta} \right).\tag{2.2.14}$$

Finally (2.2.9a) and (2.2.14) are used to obtain the simplified form of (2.2.9), where the overbar notation is dropped for convenience,

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0, \quad (2.2.15a)$$

$$\frac{\partial \Theta}{\partial t} + (\mathbf{u} \cdot \nabla) \Theta = 0, \quad (2.2.15b)$$

$$Ro \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) + \Omega_z (\hat{\mathbf{z}} \times \mathbf{u}) = -\frac{Bu}{Ro} (\nabla (\Theta h) - \frac{1}{2} h \nabla \Theta). \quad (2.2.15c)$$

Shallow water equations governing the depth-averaged horizontal fluid velocities, the layer depth, and the temperature have thus been obtained. For later convenience (2.2.15) is re-dimensionalised using (2.2.3), with the understanding that all variables and operators below are dimensional,

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0, \quad (2.2.16a)$$

$$\frac{\partial \Theta}{\partial t} + (\mathbf{u} \cdot \nabla) \Theta = 0, \quad (2.2.16b)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + f \hat{\mathbf{z}} \times \mathbf{u} = -\nabla (\Theta h) + \frac{1}{2} h \nabla \Theta, \quad (2.2.16c)$$

where $f = 2\Omega_z$. These are the same as the equations given in Dellar (2003), where Ripa's (1993) multi-layered model reduces to Lavoie's (1972) rotating shallow water equations with a horizontally varying temperature field Θ . If $\Theta = g' = \text{constant}$, the standard shallow water equations (1.3.6) are obtained. The further condition

$$\mathbf{n} \times \nabla \Theta = 0, \quad (2.2.17)$$

on the boundary of the domain is required in order that the circulation around each connected component of the boundary be constant in time (Ripa, 1993).

2.3 Derivation by layer-averaging for a two-layer system

As shown in (2.2.10), the dominant balance in the vertical momentum equation for a shallow layer is hydrostatic balance $\partial_z p = -g\rho$. This determines the pressure p in terms of the density distribution within each fluid column. In this section a derivation of the thermal shallow water momentum equations for the two-layer system depicted in figure 1.5 is sketched. It is assumed that $\Delta\rho \ll \rho_0$, where $\Delta\rho$ is now defined by $\Delta\rho = \rho_0 - \rho(\mathbf{x}, t)$. Deviations in the position of the upper free surface are then much smaller than deviations in the position of the interface between the two layers, so the

upper free surface may be approximated by a rigid lid at $z = 0$ that applies a pressure $P(x, y, t)$ (see Vallis, 2006). The pressure $p(x, y, z, t)$ in the upper layer is then given by hydrostatic balance as

$$p = P(x, y, t) - g(\rho_0 - \Delta\rho(x, y, t))z. \quad (2.3.1)$$

The density contrast $\Delta\rho(x, y, t)$ is assumed to be independent of depth in the upper layer, as sketched in figure 1.5. Hydrostatic balance, together with continuity of pressure across the interface at $z = -h(x, y, t)$, gives the pressure p_2 in the lower layer as

$$p_2 = P - \rho_0 g z - \Delta\rho g h. \quad (2.3.2)$$

The equivalent barotropic approximation applies to a lower layer that is deep and quiescent, so a vanishing horizontal pressure gradient $\nabla p_2 = \mathbf{0}$ is imposed. This determines the pressure $P = gh\Delta\rho$ at the rigid lid. The pressure in the upper layer is thus

$$p = \rho_0[-gz + \Theta(z + h)], \quad (2.3.3)$$

where $\Theta = g\Delta\rho/\rho_0$ as before.

In standard shallow water theory, with spatially uniform Θ , the horizontal pressure gradient $\nabla p = \rho_0\Theta\nabla h$ is independent of z . This is compatible with the assumption that the fluid moves predominantly in columns, $\mathbf{u}(x, y, z, t) = \mathbf{u}^0(x, y, t) + O(\delta)$ for the horizontal velocity. This property is lost when Θ varies horizontally, since now $\nabla p = \rho_0[\Theta\nabla h + (h + z)\nabla\Theta]$ varies with z , cf. (2.2.13). However, one may regard the thermal shallow water equations as a projection onto columnar motions that results from integrating the three-dimensional equations in the vertical. Inserting the above expression for ∇p , for the case where Θ varies horizontally, into Wu's (1981) formula for the depth-averaged horizontal momentum equation gives

$$\partial_t(h\bar{\mathbf{u}}) + \nabla \cdot (h\bar{\mathbf{u}}\bar{\mathbf{u}}) + hf\hat{\mathbf{z}} \times \bar{\mathbf{u}} = - \int_{-h}^0 \Theta\nabla h + (h + z)\nabla\Theta \, dz = -\frac{1}{2}\nabla(h^2\Theta), \quad (2.3.4)$$

assuming f is constant, and after making the Boussinesq approximation and dividing through by ρ_0 . An overbar denotes a vertical average, which is defined by $\bar{u}(x, y, t) = h^{-1} \int_{-h}^0 u(x, y, z, t) \, dz$ for this geometry. The only questionable step in the derivation of the thermal shallow water momentum equations (1.3.12c) is the use of the closure relation $\overline{\mathbf{u}\mathbf{u}} = \bar{\mathbf{u}}\bar{\mathbf{u}} + O(\delta^2)$ that holds for near-columnar motions (Camassa et al., 1996, Stewart and Dellar, 2010, Su and Gardner, 1969), cf. §2.2.4.

2.3.1 Justification for vertical averaging

McCreary and Kundu (1988), Fukamachi et al. (1995) and Eldevik (2002) postulated turbulent eddy stresses to balance the z -dependent part of the horizontal gradient of pressure, while Young (1994) included an explicit relaxation of the horizontal velocities towards their vertical averages in the starting point for deriving his three-dimensional subinertial model of the oceanic mixed layer. Young's (1994) goal was to develop approximate equations to describe the subinertial motions in a vertically homogeneous and rapidly rotating layer of fluid with a free lower boundary, as an idealised model of the oceanic mixed layer. The mixed layer is the uppermost layer of the ocean in which recent mixing events have created almost vertically homogeneous temperature and salinity fields. Young (1994) obtained his subinertial mixed layer (SML) approximation from the primitive equations by assuming that the dynamical time scale is long compared with both the inertial time scale and the time scale over which tracers (temperature and salinity) mix vertically. However, the time scale τ_u over which momentum mixes vertically was taken to be comparable to the inertial time scale $1/f$. In the additional limit of rapid vertical mixing, $\mu = 1/(f\tau_u) \rightarrow \infty$, Young's (1994) equations (4.1)–(4.6) reduce to the $\lambda = \beta = 0$ version of the local thermal quasi-geostrophic equations (4.2.12) derived in chapter 4. This can be seen after identifying Young's (1994) buoyancy B with θ from chapter 4, after noting that $\psi = \eta + \theta$ for the local thermal quasi-geostrophic equations (4.2.12), and after a suitable rescaling and neglecting the pressure force at the base of the mixed layer in Young's (1994) SML approximation (this is appropriate for a $1\frac{1}{2}$ -layer model in which the pressure gradient in the lower layer is assumed to be zero, cf. §2.3). Note that the fourth term in Young's (1994) equation (4.5) contains an error and should read as $-J(\phi, \eta)$ (W. R. Young, 2014, personal communication).

Fukamachi et al. (1995) showed that the thermal shallow water equations capture an analogue of the ageostrophic baroclinic instability studied by Stone (1966) for continuously stratified fluids (see also Barth, 1994). In suitable dimensionless variables, the two dispersion relations for the growth rate ω may be written as

$$(1 + \kappa^2)\omega^2 - \kappa^2 l \omega + \alpha \kappa^2 l^2 = 0, \quad (2.3.5)$$

where k and l are wavenumbers parallel and perpendicular to the background temperature gradient and $\kappa^2 = k^2 + l^2$. The constant α takes the value $\alpha = 1/3$ for a continuously stratified fluid, and $\alpha = 1/4$ for the thermal shallow water equations. The two dispersion relations are otherwise identical, showing that the thermal shallow

water equations offer a single layer model that captures some aspects of baroclinic instability in continuously stratified fluids. The change in α may be attributed to the vertical mixing implicit in the derivation of the thermal shallow water equations. Numerical simulations of the nonlinear development of these instabilities in the thermal shallow water equations also show reasonable agreement with three-dimensional simulations, and with observations of small-scale frontal instabilities in the ocean (see Barth, 1994, Fukamachi et al., 1995, McCreary et al., 1991, Shi and Røed, 1999).

The assumption of large vertical mixing seems reasonable for applications of the thermal shallow water equations to the upper oceanic mixed layer, or to the cloud decks of gas giant planets, as clouds are indicative of vertical mixing through convection (Gierasch, 1976, Gill, 1982, Houghton, 2002, Ingersoll et al., 2000). Gierasch et al. (2000) report observations of strong moist convection in Jupiter's atmosphere made by the Galileo spacecraft (see §1.1). The combination of Jupiter's observed surface heat flux with the opacity of its atmosphere at depth indicates substantial convective heat transfer takes place through much of its interior (Guillot, 2005). In addition, rotation about a vertical axis will encourage columnar motion through the propagation of internal waves parallel to the rotation axis.

The projection approach was also used by Charney (1949) in his equivalent barotropic model for a continuously stratified atmosphere (see also Eliassen and Kleinschmidt, 1957, Holton, 2004). The horizontal components of the three-dimensional velocity field are assumed to take the separable form

$$\mathbf{u}(x, y, z, t) = A(z) \mathbf{u}_\perp(x, y, t) \quad (2.3.6)$$

for some prescribed vertical structure function $A(z)$. Evolution equations for $\mathbf{u}_\perp$ are derived by vertically averaging the three-dimensional equations. This approximation was subsequently applied to the ocean by Neumann (1960) and Salby (1989). It has been widely used to model the Antarctic Circumpolar Current (e.g. Ivchenko et al., 1999, Killworth and Hughes, 2002, Krupitsky et al., 1996, LaCasce and Isachsen, 2010) following Killworth's (1992) observation that the time-averaged output of the Fine Resolution Antarctic Model (FRAM Group, 1991) fits the equivalent barotropic form (2.3.6).

2.3.2 Potential vorticity

The thermal shallow water equations imply the evolution equation

$$\frac{\partial \Pi}{\partial t} + (\mathbf{u} \cdot \nabla) \Pi = \frac{1}{2h} \hat{\mathbf{z}} \cdot (\nabla h \times \nabla \Theta) \quad (2.3.7)$$

for the shallow water potential vorticity $\Pi = h^{-1}(f + v_x - u_y)$. The source term on the right-hand side may be derived from the depth average of the baroclinic torque in a continuously stratified fluid (Ripa, 1995).

The curl of the three-dimensional momentum equation gives

$$\frac{D\boldsymbol{\omega}_a}{Dt} - (\boldsymbol{\omega}_a \cdot \nabla_3) \mathbf{u}_3 = -\nabla_3 \times \left(\frac{\nabla_3 p}{\rho} \right) = -\nabla_3 \times \left(\frac{\nabla_3 [\Theta(h+z) - gz]}{1 - \Theta/g} \right) = \mathbf{S}, \quad (2.3.8)$$

where $\mathbf{u}_3$ is the three-dimensional velocity field and $\boldsymbol{\omega}_a = f\hat{\mathbf{z}} + \nabla_3 \times \mathbf{u}_3$ is the absolute vorticity (Vallis, 2006). Here the temperature Θ is assumed to be an advected scalar so $\nabla_3 \cdot \mathbf{u}_3 = 0$, and f is assumed to be constant. There is no need to divide by ρ to eliminate the $\boldsymbol{\omega}_a \nabla_3 \cdot \mathbf{u}_3$ term and obtain a material derivative for $\boldsymbol{\omega}_a/\rho$. However, it is necessary to take spatial derivatives before applying the Boussinesq approximation to $\nabla_3 \times (\rho^{-1} \nabla_3 p)$. The components of the baroclinic torque vector on the right-hand side of (2.3.8) are

$$S_x = \frac{\partial_y \Theta}{1 - \Theta/g}, \quad S_y = -\frac{\partial_x \Theta}{1 - \Theta/g}, \quad S_z = \frac{g\Theta}{(g - \Theta)^2} \hat{\mathbf{z}} \cdot (\nabla h \times \nabla \Theta). \quad (2.3.9)$$

The Boussinesq approximation now corresponds to neglecting $O(\Theta/g)$ terms in the denominators. Moreover, the vertical component S_z is $O(\Theta/g)$ smaller than the two horizontal components.

An evolution equation for a potential vorticity follows from taking the inner product of (2.3.8) with the gradient of an advected scalar. Normally this scalar would be the buoyancy Θ , to form the Ertel potential vorticity, but in this scenario $\nabla_3 \Theta$ has only horizontal components, while the largest component of vorticity is ω_z . Instead the quantity $c = z/h(x, y, t)$ is used, which becomes a Lagrangian label when the fluid is constrained to move in columns (see §2.4.3 and Dellar and Salmon, 2005, Miles and Salmon, 1985, Ripa, 1995). The vertical average for the geometry in figure 1.5 is

$$\frac{1}{h} \int_{-h}^0 \mathbf{S} \cdot \nabla_3 c \, dz = \frac{1}{2h} \hat{\mathbf{z}} \cdot (\nabla h \times \nabla \Theta). \quad (2.3.10)$$

This is the source term that appears on the right-hand side of (2.3.7). The corresponding vertical average of $\boldsymbol{\omega}_a \cdot \nabla_3 c$ gives the standard shallow water potential vorticity Π after neglecting terms of $O(\delta^2)$ (Dellar and Salmon, 2005, Miles and Salmon, 1985).

Equation (2.3.7) becomes easier to interpret if the integral form of Ertel potential vorticity conservation is considered, i.e. the Kelvin theorem for conservation of circulation around any material curve lying within an isentropic surface. The analogues of isentropic surfaces in the thermal shallow water equations are isotherms (lines of

constant Θ) in the horizontal plane, and from (2.3.7) an integral conservation law for the total potential vorticity inside a closed isotherm is obtained (Dellar, 2003),

$$\frac{d}{dt} \int_{S(\Theta_*)} h\Pi \, d\mathbf{x} = 0, \quad (2.3.11)$$

where $S(\Theta_*)$ is the material surface bounded by the $\Theta = \Theta_*$ isotherm. This is the thermal shallow water analogue of the potential vorticity impermeability property of isentropic surfaces in a stratified fluid (e.g. Vallis, 2006). By Stokes' theorem, equation (2.3.11) implies conservation of the circulation of $\mathbf{u} + \mathbf{R}$ around the boundary isotherm of each material surface $S(\Theta_*)$. The vector field $\mathbf{R}$ satisfying $\nabla_3 \times \mathbf{R} = f\hat{\mathbf{z}}$ is the vector potential for twice the angular velocity vector, so $\hat{\mathbf{z}} \cdot \nabla_3 \times (\mathbf{u} + \mathbf{R}) = h\Pi$ is the integrand in (2.3.11). In §2.4.5, equations (2.3.7) and (2.3.11) are related to the particle relabelling symmetry, as modified from standard shallow water theory by the existence of an additional Lagrangian scalar Θ .

2.4 Derivation from a variational principle for a single layer

The thermal shallow water equations (1.3.12, with infinite τ_{rad}) are now derived from the application of Hamilton's principle of least action, by projecting the three-dimensional Lagrangian onto columnar fluid motions. This section follows Miles and Salmon's (1985) re-derivation of the Green and Naghdi (1976) equations for weakly non-hydrostatic motions, and Dellar and Salmon's (2005) derivation of shallow water equations with the complete Coriolis force. This derivation guarantees that the projected equations preserve the expected conservation laws and Hamiltonian structure of the three-dimensional equations.

Hamilton's principle gives the equations of motion for a mechanical system as being those that make the action

$$\mathcal{S} = \int \mathcal{L} \, d\tau \quad (2.4.1)$$

stationary, over variations that vanish at the end points of the integration in τ . An f -plane is considered in this section (Ω_3 is constant).

2.4.1 Formulation of the three-dimensional Lagrangian

The exact Lagrangian for a three-dimensional Boussinesq fluid in a frame rotating with constant angular velocity $\boldsymbol{\Omega}_3$ may be written as

$$\begin{aligned} \mathcal{L} = \int \frac{1}{2} \left| \frac{\partial \mathbf{x}_3}{\partial \tau} + \boldsymbol{\Omega}_3 \times \mathbf{x}_3 \right|^2 - \frac{1}{2} |\boldsymbol{\Omega}_3 \times \mathbf{x}_3|^2 - \Theta(\mathbf{a}_3) z \\ + p(\mathbf{a}_3, \tau) \left(\frac{\partial(x, y, z)}{\partial(a, b, c)} - \frac{1}{\rho_0} \right) da db dc, \end{aligned} \quad (2.4.2)$$

see Eckart (1960) and Salmon (1982, 1983, 1988*a*, 1998), and for the particle mechanics version see Landau and Lifshitz (1976). In this formulation fluid elements are described by their particle positions $\mathbf{x}_3(\mathbf{a}_3, \tau)$ as functions of Lagrangian particle labels $\mathbf{a}_3 = (a, b, c)$ and time τ . The variable τ is used to emphasise that partial time derivatives $\partial/\partial\tau$ are taken at fixed particle labels $\mathbf{a}_3$, instead of at fixed positions $\mathbf{x}_3$. Thus

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + (\mathbf{u}_3 \cdot \nabla_3) \quad (2.4.3)$$

corresponds to a material derivative, and $\mathbf{u}_3 = \partial \mathbf{x}_3 / \partial \tau$ is the Eulerian fluid velocity seen in the rotating frame. Again a coordinate system is used in which the geopotential gradient $\nabla_3 \Phi$ is vertically upward, and the horizontal coordinates x and y are tangent to the surfaces of constant geopotential.

The first term in (2.4.2) is the kinetic energy as seen in an inertial frame. The second term subtracts the contribution to the kinetic energy that gives rise to the centrifugal force, since this is conventionally included in the gravitational potential energy appearing in the third term (see discussion on page 33). The labels $\mathbf{a}_3$ are assigned so that the density ρ is given by the reciprocal of the Jacobian of the label to particle map,

$$\frac{1}{\rho} = \frac{\partial(x, y, z)}{\partial(a, b, c)}. \quad (2.4.4)$$

The fourth term thus enforces incompressibility by imposing $\rho = \rho_0$ is constant (see 2.4.5) using the pressure $p(\mathbf{a}_3, \tau)$ as a Lagrange multiplier (Eckart, 1960). The temperature Θ is an advected scalar, $\partial\Theta/\partial\tau = 0$, so $\Theta = \Theta(\mathbf{a}_3)$ with no τ -dependence, as in Seliger and Whitham's (1968) treatment of the entropy for a compressible fluid. The gravitational acceleration g is absorbed into Θ . Throughout this section, the variations δp and $\delta \mathbf{x}_3$ are assumed to vanish at the endpoints of the τ and $\mathbf{a}_3$ integrations, as in classical particle mechanics (Goldstein, 1980), to allow integrations by parts with respect to τ and $\mathbf{a}_3$. All variables are also assumed to be sufficiently

smooth to allow all required operations to be performed including: differentiation, changing the order of integration and integration by parts.

Taking variations separately with respect to p and $\mathbf{x}_3$, together with the relation that $\partial\Theta/\partial\tau = 0$, yields the three-dimensional rotating Boussinesq equations (2.2.1). Taking variations with respect to p gives the relation between the density ρ_0 and the Jacobian of the label to particle map

$$\frac{\partial(x, y, z)}{\partial(a, b, c)} = \frac{1}{\rho_0}. \quad (2.4.5)$$

Differentiating this with respect to τ gives

$$\frac{\partial(u, y, z)}{\partial(a, b, c)} + \frac{\partial(x, v, z)}{\partial(a, b, c)} + \frac{\partial(x, y, w)}{\partial(a, b, c)} = 0. \quad (2.4.6)$$

Using the chain rule for Jacobians gives

$$\frac{\partial(u, y, z)}{\partial(x, y, z)} \frac{\partial(x, y, z)}{\partial(a, b, c)} + \frac{\partial(x, v, z)}{\partial(x, y, z)} \frac{\partial(x, y, z)}{\partial(a, b, c)} + \frac{\partial(x, y, w)}{\partial(x, y, z)} \frac{\partial(x, y, z)}{\partial(a, b, c)} = 0, \quad (2.4.7)$$

and if (2.4.5) is used we obtain

$$\frac{1}{\rho_0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0, \quad (2.4.8)$$

which gives (2.2.1a). Taking variations of (2.4.2) with respect to x , as defined by (2.5.6), gives

$$\begin{aligned} \delta \int \mathcal{L} d\tau = \int \left\{ \int \frac{\partial x}{\partial \tau} \frac{\partial(\delta x)}{\partial \tau} + \frac{\partial(\delta x)}{\partial \tau} (\Omega_y z - \Omega_z y) + \frac{\partial y}{\partial \tau} \Omega_z \delta x - \frac{\partial z}{\partial \tau} \Omega_y \delta x \right. \\ \left. + p(\mathbf{a}_3, \tau) \frac{\partial(\delta x, y, z)}{\partial(a, b, c)} da db dc \right\} d\tau. \end{aligned} \quad (2.4.9)$$

Integrating by parts with respect to τ and $\mathbf{a}_3$ gives

$$\begin{aligned} \delta \int \mathcal{L} d\tau = \int \left\{ \int \left\{ -\frac{\partial^2 x}{\partial \tau^2} + \left(2\Omega_z \frac{\partial y}{\partial \tau} - 2\Omega_y \frac{\partial z}{\partial \tau} \right) \right. \right. \\ \left. \left. - \frac{\partial(p, y, z)}{\partial(a, b, c)} \right\} \delta x da db dc \right\} d\tau. \end{aligned} \quad (2.4.10)$$

Hamilton's principle of least action thus gives

$$\frac{\partial u}{\partial \tau} + (2\boldsymbol{\Omega}_3 \times \mathbf{u}_3) \cdot \hat{\mathbf{x}} = -\frac{\partial(p, y, z)}{\partial(a, b, c)}. \quad (2.4.11)$$

If (2.4.5) and the chain rule for Jacobians are again invoked, the x -component of (2.2.1c) is obtained. Similarly, taking variations of (2.4.2) with respect to y and z gives the y and z components of (2.2.1c).

2.4.2 The shallow water scaling

As in §2.2, a thin layer of fluid is considered between a rigid surface at $z = 0$ and a free surface at $z = h(x, y, t)$, as depicted in figure 1.4. This is the configuration used in previous Lagrangian derivations of the shallow water equations (Miles and Salmon, 1985, Salmon, 1983, 1988a). The reduced gravity $\Theta = g(\rho(\mathbf{x}, t) - \rho_0) / \rho_0$ in this geometry. The expression (2.4.2) for the Lagrangian implicitly imposes the correct free-surface boundary condition that the external fluid above $z = h(x, y, t)$ does no work on the layer (Miles and Salmon, 1985, Stewart and Dellar, 2010).

As in §2.2.2, dimensionless variables are introduced. The scalings (2.2.3) are used together with

$$\tau = (L/U) \tilde{\tau}, \quad a = \rho_0^{1/3} L \tilde{a}, \quad b = \rho_0^{1/3} L \tilde{b}, \quad c = \rho_0^{1/3} \delta L \tilde{c}, \quad \mathcal{L} = 2\delta\rho_0\Omega UL^4 \tilde{\mathcal{L}}, \quad (2.4.12)$$

where again tildes denote dimensionless variables. Then the incompressibility condition (2.4.5) becomes

$$\frac{\partial(\tilde{x}, \tilde{y}, \tilde{z})}{\partial(\tilde{a}, \tilde{b}, \tilde{c})} = 1, \quad (2.4.13)$$

and the dimensionless Lagrangian is given by

$$\begin{aligned} \tilde{\mathcal{L}} = \int \frac{Ro}{2} \left\{ \left(\frac{\partial \tilde{x}}{\partial \tilde{\tau}} \right)^2 + \left(\frac{\partial \tilde{y}}{\partial \tilde{\tau}} \right)^2 + \delta^2 \left(\frac{\partial \tilde{z}}{\partial \tilde{\tau}} \right)^2 \right\} + \frac{\tilde{\Omega}_z}{2} \left(\frac{\partial \tilde{y}}{\partial \tilde{\tau}} \tilde{x} - \frac{\partial \tilde{x}}{\partial \tilde{\tau}} \tilde{y} \right) \\ + \frac{\delta}{2} \left(\frac{\partial \tilde{x}}{\partial \tilde{\tau}} \tilde{\Omega}_y - \frac{\partial \tilde{y}}{\partial \tilde{\tau}} \tilde{\Omega}_x \right) \tilde{z} + \frac{\delta}{2} \frac{\partial \tilde{z}}{\partial \tilde{\tau}} \left(\tilde{\Omega}_x \tilde{y} - \tilde{\Omega}_y \tilde{x} \right) - \frac{Bu}{Ro} \tilde{\Theta}(\tilde{\mathbf{a}}_3) \tilde{z} \\ + \tilde{p}(\tilde{\mathbf{a}}_3, \tilde{\tau}) \left(\frac{\partial(\tilde{x}, \tilde{y}, \tilde{z})}{\partial(\tilde{a}, \tilde{b}, \tilde{c})} - 1 \right) d\tilde{a} d\tilde{b} d\tilde{c}. \end{aligned} \quad (2.4.14)$$

As before, $Ro = U / (2\Omega L)$ and $Bu = \Theta_0 H / (4\Omega^2 L^2)$; $\delta = H/L$ is the aspect ratio which is assumed to be small to enforce the shallowness of the layer. From now on the $O(\delta)$ terms are dropped, and Ro and Bu are assumed to remain $O(1)$ as $\delta \rightarrow 0$, as in §2.2.4. The tilde notation is also dropped, with the understanding that all variables and operators are dimensionless.

2.4.3 Restriction to columnar motion

The particle labels $\mathbf{a}_3$ can always be assigned to correspond to the Cartesian locations $\mathbf{x}_3$ in a hypothetical initial state in which the fluid is at rest. Following Miles and Salmon (1985) and Salmon (1983, 1988a) the fluid is restricted to columnar motion. That is, the horizontal particle positions and the temperature are rewritten as

$$x = x(a, b, \tau), \quad y = y(a, b, \tau), \quad \Theta = \Theta(a, b), \quad (2.4.15)$$

with no dependence on the vertical Lagrangian label c . Equation (2.4.13) then simplifies to

$$\frac{\partial(x, y)}{\partial(a, b)} \frac{\partial z}{\partial c} = 1. \quad (2.4.16)$$

The labels c are assigned so that $c = 0$ on $z = 0$ and $c = 1$ on the free surface $z = h$. Equation (2.4.16) then gives

$$z = h(a, b, \tau)c, \quad (2.4.17)$$

where h is defined by

$$h(a, b, \tau) = \frac{\partial(a, b)}{\partial(x, y)}. \quad (2.4.18)$$

The expression (2.4.17) is then substituted into the Lagrangian (2.4.14), while remembering that the $O(\delta)$ terms have been dropped. The term multiplied by the pressure p is then discarded since the incompressibility constraint is now automatically satisfied, to give the two-dimensional thermal shallow water Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{SW}} = \int \int \int_0^1 \frac{Ro}{2} \left\{ \left(\frac{\partial x}{\partial \tau} \right)^2 + \left(\frac{\partial y}{\partial \tau} \right)^2 \right\} + \frac{\Omega_z}{2} \left(\frac{\partial y}{\partial \tau} x - \frac{\partial x}{\partial \tau} y \right) \\ - \frac{Bu}{Ro} \Theta(a, b) h c \, dc \, da \, db. \end{aligned} \quad (2.4.19)$$

After performing the c integration in (2.4.19), we obtain

$$\begin{aligned} \mathcal{L}_{\text{SW}} = \frac{1}{2} \int \int Ro \left\{ \left(\frac{\partial x}{\partial \tau} \right)^2 + \left(\frac{\partial y}{\partial \tau} \right)^2 \right\} + \Omega_z \left(\frac{\partial y}{\partial \tau} x - \frac{\partial x}{\partial \tau} y \right) \\ - \frac{Bu}{Ro} \Theta(a, b) h \, da \, db. \end{aligned} \quad (2.4.20)$$

Having integrated over the third direction, two-dimensional vector notation is re-introduced: $\mathbf{x} = (x, y)$, $\mathbf{a} = (a, b)$ and $\nabla = (\partial/\partial x, \partial/\partial y)$. In particular, $\mathbf{u} = (u, v) = (\partial x/\partial \tau, \partial y/\partial \tau)$, thus $\partial/\partial \tau$ is now defined by $\partial/\partial \tau = \partial/\partial t + (\mathbf{u} \cdot \nabla)$.

2.4.4 Derivation of the mass and momentum equations

Differentiating the definition of $h(a, b, \tau)$ in (2.4.18) with respect to τ gives

$$-\frac{1}{h} \frac{\partial h}{\partial \tau} = \frac{\partial(a, b)}{\partial(x, y)} \left(\frac{\partial(u, y)}{\partial(a, b)} + \frac{\partial(x, v)}{\partial(a, b)} \right) = \nabla \cdot \mathbf{u}, \quad (2.4.21)$$

using the chain rule for Jacobians. The definition of $\partial/\partial \tau$ as a Lagrangian derivative then gives the thermal shallow water continuity equation (2.2.15a). The temperature equation (2.2.15b) follows directly from the assumption that $\Theta = \Theta(a, b)$ with no explicit τ -dependence.

To derive the thermal shallow water momentum equations, Hamilton's principle of least action is invoked: the variation of the action integral with respect to $\mathbf{x}(a, b, \tau)$ must vanish. The thermal shallow water Lagrangian (2.4.20) depends on the time-dependent map $\mathbf{a} \rightarrow \mathbf{x}$ not only explicitly, but also implicitly via the Jacobian of this map that appears in h . Following Miles and Salmon (1985), variations of (2.4.18) are taken to give

$$\delta \left(\frac{1}{h} \right) = \frac{\partial(\delta x, y)}{\partial(a, b)} + \frac{\partial(x, \delta y)}{\partial(a, b)}, \quad (2.4.22)$$

and thus,

$$\delta h = -h \left(\frac{\partial(a, b)}{\partial(x, y)} \right) \left\{ \frac{\partial(\delta x, y)}{\partial(a, b)} + \frac{\partial(x, \delta y)}{\partial(a, b)} \right\}, \quad (2.4.23a)$$

$$= -h \left\{ \frac{\partial(\delta x, y)}{\partial(x, y)} + \frac{\partial(x, \delta y)}{\partial(x, y)} \right\}, \quad (2.4.23b)$$

$$= -h \nabla \cdot (\delta \mathbf{x}), \quad (2.4.23c)$$

where the definition of h (2.4.18) and the chain rule for Jacobians have been used. Then (as in Miles and Salmon, 1985),

$$\begin{aligned} \int F \delta h \, da \, db &= \int h F \delta h \, dx \, dy && \text{using (2.4.18),} \\ &= \int - (h^2 F \nabla \cdot (\delta \mathbf{x})) \, dx \, dy && \text{using (2.4.23c),} \\ &= \int (\nabla (h^2 F)) \cdot \delta \mathbf{x} \, dx \, dy && \text{by parts,} \\ &= \int \frac{1}{h} (\nabla (h^2 F)) \cdot \delta \mathbf{x} \, da \, db && \text{using (2.4.18).} \end{aligned} \quad (2.4.24)$$

Taking variations of (2.4.20) with respect to $\mathbf{x}$ gives

$$\begin{aligned} \delta \int \mathcal{L}_{\text{SW}} \, d\tau &= \int \left\{ \int Ro \frac{\partial x}{\partial \tau} \frac{\partial(\delta x)}{\partial \tau} + Ro \frac{\partial y}{\partial \tau} \frac{\partial(\delta y)}{\partial \tau} + \frac{\Omega_z}{2} \left(\frac{\partial y}{\partial \tau} \delta x + \frac{\partial(\delta y)}{\partial \tau} x \right. \right. \\ &\quad \left. \left. - \frac{\partial x}{\partial \tau} \delta y - \frac{\partial(\delta x)}{\partial \tau} y \right) - \frac{Bu}{2Ro} \Theta(a, b) \delta h \, da \, db \right\} d\tau. \end{aligned} \quad (2.4.25)$$

Integration by parts and (2.4.24) then give

$$\begin{aligned} \delta \int \mathcal{L}_{\text{SW}} \, d\tau &= \int \left\{ \int -Ro \frac{\partial^2 x}{\partial \tau^2} \delta x - Ro \frac{\partial^2 y}{\partial \tau^2} \delta y + \Omega_z \left(\frac{\partial y}{\partial \tau} \delta x \right. \right. \\ &\quad \left. \left. - \frac{\partial x}{\partial \tau} \delta y \right) - \frac{1}{2} \frac{Bu}{Ro} \frac{1}{h} \nabla (h^2 \Theta(a, b)) \cdot \delta \mathbf{x} \, da \, db \right\} d\tau. \end{aligned} \quad (2.4.26)$$

Invoking Hamilton's principle gives the equation

$$Ro \frac{\partial \mathbf{u}}{\partial \tau} + \Omega_z (\hat{\mathbf{z}} \times \mathbf{u}) = -\frac{Bu}{Ro} \left(\frac{1}{2h} \nabla (h^2 \Theta) \right), \quad (2.4.27)$$

which coincides with (2.2.15c) after recalling the definition of $\partial/\partial\tau$ as a material time derivative. Thus the thermal shallow water equations (2.2.15) may also be derived from a variational principle.

2.4.5 Particle relabelling symmetry

Following Salmon (1983, 1998), the potential vorticity equation for the thermal shallow water equations (cf. §2.3.2) is now derived from the particle relabelling symmetry of the Lagrangian (2.4.20), for the case where Ω_z is constant. Throughout this section all variables are assumed to be sufficiently smooth to allow all required operations to be performed including: differentiation, integration by parts and changing the order of integration.

According to Hamilton's principle, the evolution equations are such as to make the action stationary. Thus the action must be stationary under infinitesimal relabellings of the form

$$\mathbf{a} \rightarrow \mathbf{a}' = \mathbf{a} + \delta\mathbf{a}(a, b, \tau), \quad (2.4.28)$$

provided that $\delta\mathbf{a}$ vanishes at the end points of the τ integration. Relabellings are considered that leave the layer depth h , as defined by (2.4.18), unchanged. The chain rule for Jacobians then implies

$$\frac{\partial(a', b')}{\partial(a, b)} = \frac{\partial(a', b')}{\partial(x, y)} \frac{\partial(x, y)}{\partial(a, b)} = 1, \quad (2.4.29)$$

so

$$\frac{\partial(\delta a)}{\partial a} + \frac{\partial(\delta b)}{\partial b} = 0, \quad (2.4.30)$$

on neglecting terms of $O(\delta\mathbf{a}^2)$. A streamfunction $\delta\Psi(a, b, \tau)$ is thus introduced such that

$$\delta a = -\frac{\partial(\delta\Psi)}{\partial b}, \quad \delta b = \frac{\partial(\delta\Psi)}{\partial a}. \quad (2.4.31)$$

Putting $\mathbf{a} = \mathbf{a}' - \delta\mathbf{a}$ in the change of variables formula gives

$$\left. \frac{\partial\mathbf{x}}{\partial\tau} \right|_{\mathbf{a}'} = \left. \frac{\partial\mathbf{x}}{\partial\tau} \right|_{\mathbf{a}} + \frac{\partial\mathbf{x}}{\partial a} \frac{\partial a}{\partial\tau} \Big|_{\mathbf{a}'} + \frac{\partial\mathbf{x}}{\partial b} \frac{\partial b}{\partial\tau} \Big|_{\mathbf{a}'}, \quad (2.4.32)$$

so the variation in $\dot{\mathbf{x}} = \partial\mathbf{x}/\partial\tau$ due to relabelling is

$$\delta\dot{\mathbf{x}} = \left. \frac{\partial\mathbf{x}}{\partial\tau} \right|_{\mathbf{a}'} - \left. \frac{\partial\mathbf{x}}{\partial\tau} \right|_{\mathbf{a}} = - \left\{ \frac{\partial\mathbf{x}}{\partial a} \frac{\partial(\delta a)}{\partial\tau} + \frac{\partial\mathbf{x}}{\partial b} \frac{\partial(\delta b)}{\partial\tau} \right\}. \quad (2.4.33)$$

Similarly, by using Taylor's theorem, neglecting terms of $O(\delta\mathbf{a}^2)$, and using (2.4.31), the variation in Θ due to relabelling is found to be

$$\delta\Theta = \Theta|_{\mathbf{a}'} - \Theta|_{\mathbf{a}} = -\frac{\partial(\delta\Psi)}{\partial b} \frac{\partial\Theta}{\partial a} + \frac{\partial(\delta\Psi)}{\partial a} \frac{\partial\Theta}{\partial b} = \frac{\partial(\delta\Psi, \Theta)}{\partial(a, b)}. \quad (2.4.34)$$

Variations of the two-dimensional thermal shallow water Lagrangian (2.4.20) due to this relabelling are now considered. Noting that h is unchanged by relabelling gives

$$\delta \int \mathcal{L}_{\text{SW}} d\tau = \int \left\{ \int p_x \delta \dot{x} + p_y \delta \dot{y} - \frac{1}{2} \frac{Bu}{Ro} h \frac{\partial(\delta \Psi, \Theta)}{\partial(a, b)} da db \right\} d\tau, \quad (2.4.35)$$

where the canonical momenta are

$$p_x = \frac{\delta \mathcal{L}_{\text{SW}}}{\delta \dot{x}} = Ro \frac{\partial x}{\partial \tau} - \frac{\Omega_z}{2} y \quad \text{and} \quad p_y = \frac{\delta \mathcal{L}_{\text{SW}}}{\delta \dot{y}} = Ro \frac{\partial y}{\partial \tau} + \frac{\Omega_z}{2} x. \quad (2.4.36)$$

Substituting for $\delta \dot{\mathbf{x}}$ using (2.4.33) and integrating by parts with respect to τ and $\mathbf{a}$ gives

$$\begin{aligned} \delta \int \mathcal{L}_{\text{SW}} d\tau = \int \left\{ \int \frac{\partial}{\partial \tau} \left(p_x \frac{\partial x}{\partial a} + p_y \frac{\partial y}{\partial a} \right) \delta a + \frac{\partial}{\partial \tau} \left(p_x \frac{\partial x}{\partial b} + p_y \frac{\partial y}{\partial b} \right) \delta b \right. \\ \left. + \frac{1}{2} \frac{Bu}{Ro} \frac{\partial(h, \Theta)}{\partial(a, b)} \delta \Psi da db \right\} d\tau, \end{aligned} \quad (2.4.37)$$

where $\delta \mathbf{a}$ is assumed to vanish at the end points of the τ integration, and $\delta \Psi$ is assumed to vanish at the end points of the $\mathbf{a}$ integration. Substituting for $\delta \mathbf{a}$ using (2.4.31) and integrating by parts with respect to $\mathbf{a}$ gives

$$\begin{aligned} \delta \int \mathcal{L}_{\text{SW}} d\tau = - \int \left\{ \int \left[\frac{\partial}{\partial \tau} \left\{ \frac{\partial}{\partial a} \left(p_x \frac{\partial x}{\partial b} + p_y \frac{\partial y}{\partial b} \right) - \frac{\partial}{\partial b} \left(p_x \frac{\partial x}{\partial a} + p_y \frac{\partial y}{\partial a} \right) \right\} \right. \right. \\ \left. \left. - \frac{1}{2} \frac{Bu}{Ro} \frac{\partial(h, \Theta)}{\partial(a, b)} \right] \delta \Psi da db \right\} d\tau. \end{aligned} \quad (2.4.38)$$

Hamilton's principle thus implies the evolution equation

$$\frac{\partial \Pi}{\partial \tau} = \frac{1}{2} \frac{Bu}{Ro} \frac{\partial(h, \Theta)}{\partial(a, b)}, \quad (2.4.39)$$

where

$$\Pi = \frac{\partial}{\partial a} \left(p_x \frac{\partial x}{\partial b} + p_y \frac{\partial y}{\partial b} \right) - \frac{\partial}{\partial b} \left(p_x \frac{\partial x}{\partial a} + p_y \frac{\partial y}{\partial a} \right). \quad (2.4.40)$$

After expanding out the derivatives and using the chain rule for Jacobians, (2.4.40) simplifies to

$$\Pi = \left(\frac{\partial(p_x, x)}{\partial(x, y)} + \frac{\partial(p_y, y)}{\partial(x, y)} \right) \frac{\partial(x, y)}{\partial(a, b)} = \frac{1}{h} \left(\frac{\partial p_y}{\partial x} - \frac{\partial p_x}{\partial y} \right), \quad (2.4.41)$$

where the definition of h (2.4.18) has been used. Finally, using (2.4.36), the definition of the potential vorticity is obtained

$$\Pi = \frac{1}{h} \left(\Omega_z + Ro \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \right). \quad (2.4.42)$$

The evolution equation for the potential vorticity defined by (2.4.42) may be written as

$$\frac{\partial \Pi}{\partial t} + (\mathbf{u} \cdot \nabla) \Pi = \frac{1}{2} \frac{Bu}{Ro} \frac{1}{h} [h, \Theta], \quad \text{where } [c, d] = \frac{\partial(c, d)}{\partial(x, y)}, \quad (2.4.43)$$

where again the chain rule for Jacobians and (2.4.18) have been used. Re-dimensionalising (2.4.42) and (2.4.43) using (2.2.3), and scaling Π with $2\Omega/H$, gives

$$\Pi = \frac{1}{h} \left(2\Omega_z + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right), \quad (2.4.44)$$

$$\frac{\partial \Pi}{\partial t} + (\mathbf{u} \cdot \nabla) \Pi = \frac{1}{2h} [h, \Theta], \quad (2.4.45)$$

where all variables are now assumed to be dimensional. This is synonymous with (2.3.7) on identifying f with $2\Omega_z$.

2.5 Hamiltonian structure

The concept of a non-canonical Hamiltonian system is introduced in this section, which is a generalisation of classical Hamiltonian particle mechanics. This more general definition is adopted in this section because it enables Hamiltonian descriptions of fluid systems using Eulerian variables. The non-canonical Hamiltonian structure of the thermal shallow water equations (2.2.16) as formulated by Ripa (1993) is described (also see Dellar, 2003).

2.5.1 Finite-dimensional Hamiltonian systems in canonical form

Let $\mathbf{q} = (q_1, q_2, \dots, q_n)$ be a set of n canonical coordinates and $\mathbf{p} = (p_1, p_2, \dots, p_n)$ be the corresponding set of canonical momenta. Hamilton's equations for describing the evolution of a finite-dimensional system, for instance a collection of point particles, are (Goldstein, 1980)

$$\frac{dq_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i}. \quad (2.5.1)$$

Here $\mathcal{H}(\mathbf{q}, \mathbf{p})$ is the scalar Hamiltonian, which frequently corresponds to the total energy of the system.

This system of equations may also be written in matrix notation as

$$\frac{d\boldsymbol{\eta}}{dt} = \mathbf{J}_c \frac{\partial \mathcal{H}}{\partial \boldsymbol{\eta}}, \quad (2.5.2)$$

where

$$\boldsymbol{\eta} = \begin{pmatrix} q_i \\ p_i \end{pmatrix} \quad \text{and} \quad \frac{\partial \mathcal{H}}{\partial \boldsymbol{\eta}} = \begin{pmatrix} \partial \mathcal{H} / \partial q_i \\ \partial \mathcal{H} / \partial p_i \end{pmatrix} \quad (2.5.3)$$

are $2n$ column vectors, and

$$\mathbf{J}_c = \begin{pmatrix} \mathbf{0}_n & \mathbf{I}_n \\ -\mathbf{I}_n & \mathbf{0}_n \end{pmatrix}, \quad (2.5.4)$$

is a $2n \times 2n$ matrix, with $\mathbf{0}_n$ denoting the elements of the $n \times n$ zero matrix, and $\mathbf{I}_n$ denoting the elements of the $n \times n$ identity matrix. Any system of equations (2.5.2) where $\mathbf{J}_c$ takes this particular form (2.5.4) are referred to as a Hamiltonian system in canonical form.

2.5.2 Continuous Hamiltonian systems in canonical form

This section considers a generalisation of this theory to continuous systems such as fluids (Morrison, 1998, Salmon, 1988*a*, Schmid, 2006, Shepherd, 1990). The dependent variables $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_{2n})$ are taken to be functions of spatial coordinates $\mathbf{x}$ and time t , defined over some spatial domain $\mathcal{D}$. The continuous variable $\mathbf{x}$ is the continuous analogue of the discrete index i in the finite-dimensional case. Instead of the system of first-order ordinary differential equations defined in (2.5.2), a system of partial differential equations is considered. The function $\mathcal{H}$ becomes a functional, defined as an integral over the spatial coordinate $\mathbf{x}$ of functions such as $\mathbf{u}(\mathbf{x}, t)$. For instance, the Hamiltonian for a simple fluid system might be the kinetic energy,

$$\mathcal{H} = \int \frac{1}{2} |\mathbf{u}(\mathbf{x}, t)|^2 d\mathbf{x}. \quad (2.5.5)$$

Instead of partial derivatives of functions, functional or variational derivatives $\delta \mathcal{F} / \delta \boldsymbol{\eta}$ of functionals are considered, defined by the relation (see Shepherd, 1990, p.293)

$$\delta \mathcal{F} = \mathcal{F}(\boldsymbol{\eta} + \delta \boldsymbol{\eta}) - \mathcal{F}(\boldsymbol{\eta}) = \left\langle \frac{\delta \mathcal{F}}{\delta \boldsymbol{\eta}}, \delta \boldsymbol{\eta} \right\rangle + O(\delta \boldsymbol{\eta}^2), \quad (2.5.6)$$

for arbitrary variations $\delta \boldsymbol{\eta}$, where $\mathcal{F}$ is a functional and $\langle \cdot, \cdot \rangle$ is some inner product, which is typically an integral of the form

$$\langle f, g \rangle = \int f(\mathbf{x}, t) g(\mathbf{x}, t) d\mathbf{x}. \quad (2.5.7)$$

Sufficient smoothness of the functions within the inner product is assumed. Note that when $\delta \mathcal{F}$ was previously formed for a functional $\mathcal{F}$, terms of $O(\delta \boldsymbol{\eta}^2)$ were neglected. The Hamiltonian dynamics may then be represented by

$$\frac{\partial \boldsymbol{\eta}}{\partial t} = \mathbf{J} \frac{\delta \mathcal{H}}{\delta \boldsymbol{\eta}}, \quad (2.5.8)$$

where $\mathbf{J}$ is a linear (differential) operator which takes the form (2.5.4) for a canonical system.

An example of a canonical, continuous, Hamiltonian system is the Klein–Gordon equation (e.g. see Goldstein, 1980, Schmid, 2006), which is defined by

$$-\frac{\partial^2 \psi}{\partial t^2} + \nabla^2 \psi = m^2 \psi, \quad (2.5.9)$$

where ψ is a wave function and m is a parameter. It is related to the linearised rotating shallow water equations (see Majda, 2003, p.56). Introducing $\pi = \partial\psi/\partial t$, the Klein–Gordon equation may be rewritten as the system

$$\frac{\partial \psi}{\partial t} = \pi, \quad (2.5.10a)$$

$$\frac{\partial \pi}{\partial t} = \nabla^2 \psi - m^2 \psi. \quad (2.5.10b)$$

This system is Hamiltonian with dependent variables $\boldsymbol{\eta} = (\psi, \pi)$ and Hamiltonian functional

$$\mathcal{H}(\psi, \pi) = \int \frac{1}{2} (|\pi|^2 + |\nabla \psi|^2) + \frac{1}{2} m^2 \psi^2 \, d\mathbf{x}. \quad (2.5.11)$$

The variational derivatives of $\mathcal{H}$ are given by

$$\frac{\delta \mathcal{H}}{\delta \psi} = -\nabla^2 \psi + m^2 \psi \quad \text{and} \quad \frac{\delta \mathcal{H}}{\delta \pi} = \pi, \quad (2.5.12)$$

so the Klein–Gordon equation (2.5.9) is represented by

$$\frac{\partial \boldsymbol{\eta}}{\partial t} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\delta \mathcal{H}}{\delta \boldsymbol{\eta}}, \quad (2.5.13)$$

which is in canonical form.

2.5.3 Non-canonical Hamiltonian systems

According to Shepherd (1990), it was “the preoccupation with the canonical equations that for so long held up progress in the fluid-dynamical context.” Non-canonical Hamiltonian systems allow Hamiltonian descriptions of fluid systems using Eulerian variables. A non-canonical Hamiltonian system consists of an equation of the form

$$\dot{\mathcal{F}} = \{\mathcal{F}, \mathcal{H}\}, \quad (2.5.14)$$

that describes the evolution of $\mathcal{F}$ in terms of a Poisson bracket $\{\cdot, \cdot\}$ and a given scalar Hamiltonian $\mathcal{H}$ (see Morrison, 1982, for more details). If a finite-dimensional system is considered then $\mathcal{F}$ and $\mathcal{H}$ are functions; if a continuous system is considered

then $\mathcal{F}$ and $\mathcal{H}$ are functionals. The Poisson bracket is a bilinear, antisymmetric map which satisfies the Jacobi identity, thus

$$\{\mathcal{F}, \mathcal{G} + \lambda \mathcal{K}\} = \{\mathcal{F}, \mathcal{G}\} + \lambda \{\mathcal{F}, \mathcal{K}\}, \quad (2.5.15a)$$

$$\{\mathcal{F}, \mathcal{G}\} = -\{\mathcal{G}, \mathcal{F}\}, \quad (2.5.15b)$$

$$\{\mathcal{F}, \{\mathcal{G}, \mathcal{K}\}\} + \{\mathcal{G}, \{\mathcal{K}, \mathcal{F}\}\} + \{\mathcal{K}, \{\mathcal{F}, \mathcal{G}\}\} = 0, \quad (2.5.15c)$$

for all functions/functionals $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{K}$ (Morrison, 1998). It turns out that these three conditions are precisely those for which the Poisson bracket generates a Lie algebra, and hence a Lie group, on the space of suitable functions/functionals. This abstract approach is used extensively by Arnold (1978).

This definition of a Hamiltonian system reduces to the particle mechanics version when $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are taken to be functions of the finite set of variables $\boldsymbol{\eta} = (q_1, \dots, q_n, p_1, \dots, p_n)$, and the Poisson bracket is expressed in terms of the matrix $\mathbf{J}$ with components J_{ij} ,

$$\{\mathcal{F}, \mathcal{G}\} = \sum_{i,j=1}^n \frac{\partial \mathcal{F}}{\partial \eta_i} J_{ij} \frac{\partial \mathcal{G}}{\partial \eta_j}. \quad (2.5.16)$$

Choosing $\mathcal{F} = \eta_k$ and $\mathcal{G} = \mathcal{H}$ in (2.5.16) shows that the evolution equation (2.5.14) is equivalent to

$$\frac{d\boldsymbol{\eta}}{dt} = \mathbf{J} \frac{\partial \mathcal{H}}{\partial \boldsymbol{\eta}}. \quad (2.5.17)$$

The expression on the right of (2.5.16) is clearly bilinear, and is antisymmetric if J_{ij} is antisymmetric. To satisfy the Jacobi identity requires

$$J_{il} \frac{\partial J_{jk}}{\partial \eta_l} + J_{jl} \frac{\partial J_{ki}}{\partial \eta_l} + J_{kl} \frac{\partial J_{ij}}{\partial \eta_l} = 0. \quad (2.5.18)$$

If $\mathbf{J} = \mathbf{J}_c$ defined by (2.5.4), the matrix $\mathbf{J}$ satisfies all three of the conditions, and the Poisson bracket is given by

$$\{\mathcal{F}, \mathcal{G}\} = \sum_{i=1}^n \frac{\partial \mathcal{F}}{\partial q_i} \frac{\partial \mathcal{G}}{\partial p_i} - \frac{\partial \mathcal{F}}{\partial p_i} \frac{\partial \mathcal{G}}{\partial q_i}. \quad (2.5.19)$$

This definition of a Hamiltonian system also reduces to the continuous Hamiltonian version of §2.5.2 when $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are taken to be functionals of the dependent variables $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_{2n})$ which are in turn functions of spatial coordinates $\mathbf{x}$ and time t , and the Poisson bracket is expressed in terms of the linear operator $\mathbf{J}$ (which is the continuous analogue of the matrix $\mathbf{J}$ in (2.5.17)) as,

$$\{\mathcal{F}, \mathcal{G}\} = \left\langle \frac{\delta \mathcal{F}}{\delta \boldsymbol{\eta}}, \mathbf{J} \frac{\delta \mathcal{G}}{\delta \boldsymbol{\eta}} \right\rangle, \quad (2.5.20)$$

where $\langle \cdot, \cdot \rangle$ is the inner product on the function space $\{\boldsymbol{\eta}\}$. This expression for the bracket is automatically bilinear, and to satisfy the antisymmetry property $\mathbf{J}$ must be anti-selfadjoint with respect to the inner product,

$$\left\langle \frac{\delta \mathcal{F}}{\delta \boldsymbol{\eta}}, \mathbf{J} \frac{\delta \mathcal{G}}{\delta \boldsymbol{\eta}} \right\rangle = - \left\langle \mathbf{J} \frac{\delta \mathcal{F}}{\delta \boldsymbol{\eta}}, \frac{\delta \mathcal{G}}{\delta \boldsymbol{\eta}} \right\rangle. \quad (2.5.21)$$

The evolution equation for continuous non-canonical systems is thus,

$$\frac{\partial \boldsymbol{\eta}}{\partial t} = \mathbf{J} \frac{\delta \mathcal{H}}{\delta \boldsymbol{\eta}}, \quad (2.5.22)$$

which is the generalisation to continuous systems of (2.5.17). Partial derivatives have been replaced by variational derivatives, and matrices have been replaced by linear (differential) operators. For the remainder of this section only continuous Hamiltonian systems will be considered. Particular emphasis will be placed on the class of linear (differential) operators $\mathbf{J}$ with Lie–Poisson form, where the components of $\mathbf{J}$ are such that

$$J_{ij} = C_{ij}^k \eta_k, \quad (2.5.23)$$

and the C_{ij}^k are the structure operators (Morrison, 1998).

2.5.4 Casimirs

Symmetries of the Hamiltonian functional can correspond to invariants of the motion (see Noether’s theorem in Goldstein, 1980, p.588). For instance, if the Hamiltonian is invariant under translations in time, i.e. there is no explicit dependence of $\mathcal{H}$ on t , then

$$\frac{d\mathcal{H}}{dt} = \left\langle \frac{\delta \mathcal{H}}{\delta \boldsymbol{\eta}}, \frac{\partial \boldsymbol{\eta}}{\partial t} \right\rangle = \left\langle \frac{\delta \mathcal{H}}{\delta \boldsymbol{\eta}}, \mathbf{J} \frac{\delta \mathcal{H}}{\delta \boldsymbol{\eta}} \right\rangle = 0, \quad (2.5.24)$$

where (2.5.22) and the fact that $\mathbf{J}$ is anti-selfadjoint have been used. This gives conservation of total energy (when $\mathcal{H}$ corresponds to the total energy).

A special feature of non-canonical representations is that they also possess a class of invariants known as Casimir functionals, $\mathcal{C}$, which are associated with the kernel of $\mathbf{J}$. For a continuous system they are the solutions of

$$\mathbf{J} \frac{\delta \mathcal{C}}{\delta \boldsymbol{\eta}} = 0, \quad (2.5.25)$$

(see Ripa, 1993, Salmon, 1988*a*, Shepherd, 1990). That they are invariants of the dynamics follows from

$$\frac{d\mathcal{C}}{dt} = \left\langle \frac{\delta \mathcal{C}}{\delta \boldsymbol{\eta}}, \frac{\partial \boldsymbol{\eta}}{\partial t} \right\rangle = \left\langle \frac{\delta \mathcal{C}}{\delta \boldsymbol{\eta}}, \mathbf{J} \frac{\delta \mathcal{H}}{\delta \boldsymbol{\eta}} \right\rangle = - \left\langle \mathbf{J} \frac{\delta \mathcal{C}}{\delta \boldsymbol{\eta}}, \frac{\delta \mathcal{H}}{\delta \boldsymbol{\eta}} \right\rangle = 0, \quad (2.5.26)$$

where $\langle \cdot, \cdot \rangle$ is the inner product on the function space $\{\boldsymbol{\eta}\}$. This result used (2.5.22), the fact that $\mathbf{J}$ is anti-selfadjoint with respect to the inner product (2.5.21), and the definition (2.5.25). The equivalent condition in terms of the Poisson brackets is

$$\{\mathcal{C}, \mathcal{F}\} = 0, \quad \text{for all functionals } \mathcal{F}. \quad (2.5.27)$$

It is important to note that the Casimirs are associated with the operator $\mathbf{J}$, and thus with the Poisson bracket and not the Hamiltonian. A canonical system has no nontrivial Casimirs. This is because when $\mathbf{J}$ is invertible, (2.5.25) implies that $\delta\mathcal{C}/\delta\boldsymbol{\eta} = 0$, so $\mathcal{C}$ is simply a constant.

The Korteweg–de Vries (KdV) equation is now considered as an example (see Morrison, 1998),

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \quad (2.5.28)$$

It describes long-wavelength water waves or ion-acoustic waves in plasmas. The Hamiltonian may be taken to be

$$\mathcal{H} = \int \frac{1}{6} u^3 - \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 dx. \quad (2.5.29)$$

Taking variations of $\mathcal{H}$ gives

$$\frac{\delta \mathcal{H}}{\delta u} = \frac{1}{2} u^2 + \frac{\partial^2 u}{\partial x^2}, \quad (2.5.30)$$

hence the linear operator $\mathbf{J}$ is given by

$$\mathbf{J} = -\frac{\partial}{\partial x}, \quad (2.5.31)$$

and the KdV equation may be written as

$$\frac{\partial u}{\partial t} = \mathbf{J} \frac{\delta \mathcal{H}}{\delta u}. \quad (2.5.32)$$

The corresponding Poisson bracket is given by

$$\{\mathcal{F}, \mathcal{G}\} = - \int \frac{\delta \mathcal{F}}{\delta u} \frac{\partial}{\partial x} \frac{\delta \mathcal{G}}{\delta u} dx, \quad (2.5.33)$$

and it possesses one Casimir invariant

$$\mathcal{C} = \int u dx. \quad (2.5.34)$$

The Poisson bracket (2.5.33) is bilinear, and may be shown to be antisymmetric by an integration by parts, provided that the admissible functionals satisfy $\delta\mathcal{F}/\delta u = 0$ on the boundary of the domain. The Jacobi identity is automatic since $\mathbf{J}$ in (2.5.31) is independent of u (Morrison, 1982).

2.5.5 Non-canonical Hamiltonian structure of the thermal shallow water equations

Most equations used in geophysical fluid dynamics have a Hamiltonian structure, which enables their conservation laws to be related to symmetries via Noether's theorem (Morrison, 1998, Salmon, 1988*a*, 1998, Shepherd, 1990). Equations that are derived by making approximations that preserve these symmetries will then inherit equivalent conservation laws. The material in this section provides the key to deriving an energy conservation equation for the local quasi-geostrophic theory in §4.6.

A Hamiltonian formulation of the dimensional thermal shallow water equations is described in this section. Ripa (1993) described the non-canonical Hamiltonian structure of the undamped (infinite τ_{rad}) thermal shallow water equations (1.3.12), but only gave a long “formal proof” of the Jacobi identity. The field variables in this formulation are $\boldsymbol{\eta} = (u, v, h, \Theta)$, and the Hamiltonian is

$$\mathcal{H} = \frac{1}{2} \int h |\mathbf{u}|^2 + \Theta h^2 \, dx \, dy. \quad (2.5.35)$$

This Hamiltonian coincides with the total energy, with the first term denoting the kinetic energy, and the second term the gravitational potential energy for the reduced gravity Θ . Taking variations of (2.5.35) gives

$$\begin{aligned} \delta \mathcal{H} &= \int \left(\frac{1}{2} |\mathbf{u}|^2 + \Theta h \right) \delta h + h \mathbf{u} \cdot \delta \mathbf{u} + \frac{1}{2} h^2 \delta \Theta \, dx \, dy + O(\delta \boldsymbol{\eta}^2), \\ &= \int \frac{\delta \mathcal{H}}{\delta h} \delta h + \frac{\delta \mathcal{H}}{\delta \mathbf{u}} \cdot \delta \mathbf{u} + \frac{\delta \mathcal{H}}{\delta \Theta} \delta \Theta \, dx \, dy + O(\delta \boldsymbol{\eta}^2). \end{aligned} \quad (2.5.36)$$

Hence the variational derivatives of $\mathcal{H}$ are given by

$$\frac{\delta \mathcal{H}}{\delta h} = \frac{1}{2} |\mathbf{u}|^2 + \Theta h, \quad \frac{\delta \mathcal{H}}{\delta \mathbf{u}} = h \mathbf{u}, \quad \frac{\delta \mathcal{H}}{\delta \Theta} = \frac{1}{2} h^2, \quad (2.5.37)$$

and the Poisson tensor is

$$\mathbf{J} = - \begin{pmatrix} 0 & -\Pi & \partial_x & -h^{-1} \Theta_x \\ \Pi & 0 & \partial_y & -h^{-1} \Theta_y \\ \partial_x & \partial_y & 0 & 0 \\ h^{-1} \Theta_x & h^{-1} \Theta_y & 0 & 0 \end{pmatrix}, \quad (2.5.38)$$

where $\Theta_x = \partial_x \Theta$, $\Theta_y = \partial_y \Theta$ and $\Pi = h^{-1}(f + v_x - u_y)$ is the shallow water potential vorticity. With these definitions, Hamilton's equations (2.5.22) are the thermal shallow water equations,

$$\frac{\partial}{\partial t} \begin{pmatrix} \mathbf{u} \\ h \\ \Theta \end{pmatrix} = \mathbf{J} \begin{pmatrix} \delta \mathcal{H} / \delta \mathbf{u} \\ \delta \mathcal{H} / \delta h \\ \delta \mathcal{H} / \delta \Theta \end{pmatrix}. \quad (2.5.39)$$

Defining the inner product by (2.5.7), the corresponding Poisson bracket as defined by (2.5.20) is given explicitly by

$$\begin{aligned} \{\mathcal{F}, \mathcal{G}\} = & \int \Pi \left(\frac{\delta \mathcal{F}}{\delta u} \frac{\delta \mathcal{G}}{\delta v} - \frac{\delta \mathcal{G}}{\delta u} \frac{\delta \mathcal{F}}{\delta v} \right) + \nabla \cdot \left(\frac{\delta \mathcal{F}}{\delta \mathbf{u}} \right) \frac{\delta \mathcal{G}}{\delta h} - \nabla \cdot \left(\frac{\delta \mathcal{G}}{\delta \mathbf{u}} \right) \frac{\delta \mathcal{F}}{\delta h} \\ & + \frac{\nabla \Theta}{h} \cdot \left(\frac{\delta \mathcal{F}}{\delta \mathbf{u}} \frac{\delta \mathcal{G}}{\delta \Theta} - \frac{\delta \mathcal{G}}{\delta \mathbf{u}} \frac{\delta \mathcal{F}}{\delta \Theta} \right) dx dy. \end{aligned} \quad (2.5.40)$$

This Poisson bracket may be shown to be bilinear, antisymmetric, and to satisfy the Jacobi identity, provided that the admissible functionals satisfy

$$\mathbf{n} \cdot \frac{\delta \mathcal{F}}{\delta \mathbf{u}} = 0, \quad (2.5.41)$$

on the boundary of the domain. Ripa (1993) gave a long “formal” proof of the Jacobi identity. Dellar (2003) later gave a much shorter proof of the Jacobi identity for the Poisson bracket (2.5.40) by identifying it with the Poisson bracket for the shallow water magnetohydrodynamic equations (Dellar, 2002, Gilman, 2000) through a change of variables in which Θ plays the role of the magnetic vector potential. An even shorter proof is available by observing that (2.5.40) is the Poisson bracket for non-isentropic gas dynamics written in density, velocity and entropy variables (e.g. Morrison, 2006). The thermal shallow water Poisson tensor and bracket reduce to the standard shallow water Poisson tensor and bracket (e.g. see Shepherd, 1990) on omitting the last row and column containing the $\nabla \Theta$ terms in (2.5.38), and the last term containing $\nabla \Theta$ in (2.5.40).

The Casimir functionals for (2.5.38) and (2.5.40) are

$$\mathcal{C} = \int h \Pi F(\Theta) + h G(\Theta) dx dy \quad (2.5.42)$$

for arbitrary functions F and G (Dellar, 2002, 2003, Ripa, 1993). The conservation of the total potential vorticity inside any contour of constant Θ , (2.3.11), follows from this definition of the Casimirs by taking $G(\Theta) = 0$, and $F(\Theta) = 1$ when $\Theta \geq \Theta_*$ and $F(\Theta) = 0$ otherwise (i.e. $F(\Theta) = \mathcal{H}(\Theta - \Theta_*)$, where $\mathcal{H}$ is the Heaviside step function). These results rely upon being able to integrate by parts over the fluid domain $\mathcal{D}$ without acquiring surface terms. The necessary boundary conditions (Ripa, 1993) are $\mathbf{u} \cdot \mathbf{n} = 0$ and $\nabla \Theta \times \mathbf{n} = \mathbf{0}$ on the boundary $\partial \mathcal{D}$, or $|\mathbf{u}| \rightarrow 0$ and $\Theta \rightarrow \Theta_0$ (constant) as $|\mathbf{x}| \rightarrow \infty$.

2.6 Energy and the group velocity property

Waves can either be thought of as transporting their energy with the group velocity, or the wave energy can be thought of as being transported by the energy flux. In this section these two concepts are shown to be synonymous for Rossby waves in the thermal shallow water equations, following the method given by Masuda (1978). In this section f is assumed to be an arbitrary function of y , subject to the condition that it behaves sufficiently well for normal modes to exist. The domain under consideration may be periodic or extend to infinity in the y direction. Thus the boundary condition that v is periodic, or that v tends to zero as $y \rightarrow \pm\infty$ must be imposed, depending on the domain under consideration.

First the expansion $h = h_0 + \eta$ and $\Theta = \Theta_0 + \theta$ is substituted in (2.2.16). Assuming $\eta \ll 1$, $\theta \ll 1$, and $\mathbf{u} \ll 1$ gives the system of linearised equations

$$\frac{\partial \eta}{\partial t} + h_0 \nabla \cdot \mathbf{u} = 0, \quad (2.6.1a)$$

$$\frac{\partial \theta}{\partial t} = 0, \quad (2.6.1b)$$

$$\frac{\partial u}{\partial t} - fv = -\Theta_0 \frac{\partial \eta}{\partial x} - \frac{h_0}{2} \frac{\partial \theta}{\partial x}, \quad (2.6.1c)$$

$$\frac{\partial v}{\partial t} + fu = -\Theta_0 \frac{\partial \eta}{\partial y} - \frac{h_0}{2} \frac{\partial \theta}{\partial y}. \quad (2.6.1d)$$

Consider a normal mode solution of the form

$$\theta = \tilde{\theta}(y) \cos(kx - \omega t), \quad \eta = \tilde{\eta}(y) \cos(kx - \omega t), \quad (2.6.2)$$

where k is the wavenumber and ω the frequency. From (2.6.1) we obtain

$$u = -\frac{\Theta_0}{F} (f\tilde{\eta}' + k\omega\tilde{\eta}) \cos(kx - \omega t), \quad (2.6.3a)$$

$$v = -\frac{\Theta_0}{F} (fk\tilde{\eta} + \omega\tilde{\eta}') \sin(kx - \omega t), \quad (2.6.3b)$$

$$\tilde{\theta} = 0, \quad (2.6.3c)$$

where primes are used to denote d/dy and $F = f^2 - \omega^2$. Since $\tilde{\theta} = 0$, these equations reduce to the standard shallow water equations with $g' = \Theta_0$. Substituting (2.6.3a) and (2.6.3b) into (2.6.1a), and letting $G = h_0\Theta_0/F$, gives the normal mode equation,

$$\{\omega(1 + k^2G) - k(fG)'\} \tilde{\eta} - \omega(G\tilde{\eta}')' = 0. \quad (2.6.4)$$

Following an approach introduced by Chandrasekhar (1961) for hydrodynamic stability calculations, (2.6.4) is multiplied by $\tilde{\eta}$ and integrated over y to obtain the integral formula

$$\begin{aligned} \int \omega (1 + k^2 G) \tilde{\eta}^2 + k (fG) (\tilde{\eta}^2)' + \omega (G\tilde{\eta}') \tilde{\eta}' dy \\ = [G\tilde{\eta} (fk\tilde{\eta} + \omega\tilde{\eta}')] . \end{aligned} \quad (2.6.5)$$

The term in square brackets on the right-hand side of (2.6.5) is proportional to v , which is either periodic or tends to zero as $y \rightarrow \pm\infty$, hence the right-hand side vanishes, giving

$$\int \omega (1 + k^2 G) \tilde{\eta}^2 + k (fG) (\tilde{\eta}^2)' + \omega G (\tilde{\eta}')^2 dy = 0. \quad (2.6.6)$$

An expression for the group velocity is now found by using a variational method following Longuet-Higgins (1968*a*). A small variation of k in (2.6.6) is considered, and using $\delta G = 2\omega(G/F)\delta\omega$ gives

$$\begin{aligned} \delta\omega \int (1 + k^2 G) \tilde{\eta}^2 + G (\tilde{\eta}')^2 + 2\omega^2 \frac{G}{F} k^2 \tilde{\eta}^2 + 2f\omega \frac{G}{F} k (\tilde{\eta}^2)' + 2\omega^2 \frac{G}{F} (\tilde{\eta}')^2 dy \\ + \delta\tilde{\eta} \int 2\omega (1 + k^2 G) \tilde{\eta} + 2fGk\tilde{\eta}' - 2k (fG\tilde{\eta})' - 2\omega (G\tilde{\eta}')' dy \\ + \delta k \int 2\omega G\tilde{\eta}^2 k + fG (\tilde{\eta}^2)' dy + [2G\delta\tilde{\eta} (fk\tilde{\eta} + \omega\tilde{\eta}')] = 0. \end{aligned} \quad (2.6.7)$$

The term in square brackets in (2.6.7) is again proportional to v , and due to the boundary condition that either v is periodic or tends to zero as $y \rightarrow \pm\infty$, the square bracket term vanishes. Using (2.6.4) to simplify (2.6.7) gives

$$\begin{aligned} \frac{\partial\omega}{\partial k} \left(\int \tilde{\eta}^2 + \frac{G}{F} \left\{ (f^2 + \omega^2) \left((\tilde{\eta}')^2 + k^2 \tilde{\eta}^2 \right) + 2f\omega k (\tilde{\eta}^2)' \right\} dy \right) \\ = - \int 2\omega Gk\tilde{\eta}^2 + fG (\tilde{\eta}^2)' dy. \end{aligned} \quad (2.6.8)$$

The energy density, E_L , conserved by the linearised shallow water equations (2.6.1) is defined by

$$E_L = \frac{1}{2} h_0 (u^2 + v^2) + \frac{1}{2} \Theta_0 \eta^2. \quad (2.6.9)$$

Differentiating (2.6.9) with respect to t and using (2.6.1) together with $\tilde{\theta} = 0$, gives the energy equation

$$\frac{\partial E_L}{\partial t} + \nabla \cdot \mathbf{F}_L = 0, \quad (2.6.10)$$

where

$$\mathbf{F}_L = h_0 \Theta_0 \eta \mathbf{u}. \quad (2.6.11)$$

Using (2.6.3a,b) and (2.6.2) gives

$$\int \overline{E_L} dy = \frac{1}{4} \Theta_0 \int \tilde{\eta}^2 + \frac{G}{F} \left\{ (f^2 + \omega^2) \left((\tilde{\eta}')^2 + k^2 \tilde{\eta}^2 \right) + 2f\omega k (\tilde{\eta}^2)' \right\} dy, \quad (2.6.12)$$

where $\bar{a}$ denotes the average of a quantity over one period of a wave. For the eastward energy flux,

$$\int \overline{F_L^x} dy = \int \overline{h_0 \Theta_0 \eta u} dy = -\frac{1}{4} \Theta_0 \int 2\omega G k \tilde{\eta}^2 + f G (\tilde{\eta}^2)' dy, \quad (2.6.13)$$

and using (2.6.8) and (2.6.12), the required result is obtained:

$$C_g^x \langle E_L \rangle = \langle F_L^x \rangle, \quad (2.6.14)$$

where

$$\langle E_L \rangle = \int \overline{E_L} dy, \quad \langle F_L^x \rangle = \int \overline{F_L^x} dy, \quad (2.6.15)$$

and $C_g^x = \partial\omega/\partial k$ is the group velocity in the x direction. Hence the group velocity may be regarded as the velocity at which the wave energy is transported for the thermal shallow water equations.

2.7 The radiatively damped thermal shallow water equations

The remainder of this chapter considers the thermal shallow water equations with radiative damping i.e. (1.3.12) from chapter 1, where the equations are rewritten below for convenience

$$\frac{\partial h}{\partial t} + \nabla \cdot (h \mathbf{u}) = 0, \quad (2.7.1a)$$

$$\frac{\partial \Theta}{\partial t} + (\mathbf{u} \cdot \nabla) \Theta = -(\Theta h/h_0 - \Theta_0) / \tau_{\text{rad}}, \quad (2.7.1b)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + f(\hat{\mathbf{z}} \times \mathbf{u}) = -\nabla(\Theta h) + \frac{1}{2} h \nabla \Theta. \quad (2.7.1c)$$

These are the same as (2.2.16) after adding a Newtonian cooling term to the right-hand side of (2.2.16b), which represents radiative relaxation towards a temperature $\Theta_0 h_0/h$ with time scale τ_{rad} . The h -dependence represents the idea that columns that extend higher are able to radiate more effectively. All variables in (2.7.1) are dimensional.

2.7.1 Linear plane waves

This section investigates the effect of the Newtonian cooling term on the waves in the thermal shallow water equations (2.7.1). Linear plane waves are considered about a rest state. The Coriolis parameter f is assumed to be constant in this section, while §6.2.2 considers equatorially trapped waves in the radiatively damped thermal shallow water equations on an equatorial β -plane. An expansion $\mathbf{u} = \mathbf{0} + \mathbf{u}'$, $h = h_0 + \eta'$ and $\Theta = \Theta_0 + \theta'$ is substituted into (2.7.1), where dashed variables denote small perturbations. The linearised equations are then given by

$$\frac{\partial \eta'}{\partial t} + h_0 \nabla \cdot \mathbf{u}' = 0, \quad (2.7.2a)$$

$$\frac{\partial \theta'}{\partial t} + \frac{1}{h_0 \tau_{\text{rad}}} (h_0 \theta' + \Theta_0 \eta') = 0, \quad (2.7.2b)$$

$$\frac{\partial u'}{\partial t} - f v' + \Theta_0 \frac{\partial \eta'}{\partial x} + \frac{h_0}{2} \frac{\partial \theta'}{\partial x} = 0, \quad (2.7.2c)$$

$$\frac{\partial v'}{\partial t} + f u' + \Theta_0 \frac{\partial \eta'}{\partial y} + \frac{h_0}{2} \frac{\partial \theta'}{\partial y} = 0. \quad (2.7.2d)$$

Solutions of the form $u' = \text{Re}\{\hat{u} \exp(i(kx + ly - \omega t))\}$ etc. are sought, where k is the wavenumber in the x direction, l is the wavenumber in the y direction and ω is the frequency, which gives

$$-i\omega \hat{\eta} + h_0 (ik\hat{u} + il\hat{v}) = 0, \quad (2.7.3a)$$

$$-i\omega \hat{\theta} + \frac{1}{h_0 \tau_{\text{rad}}} (h_0 \hat{\theta} + \Theta_0 \hat{\eta}) = 0, \quad (2.7.3b)$$

$$-i\omega \hat{u} - f \hat{v} + \Theta_0 ik \hat{\eta} + \frac{h_0}{2} ik \hat{\theta} = 0, \quad (2.7.3c)$$

$$-i\omega \hat{v} + f \hat{u} + \Theta_0 il \hat{\eta} + \frac{h_0}{2} il \hat{\theta} = 0, \quad (2.7.3d)$$

or in matrix form

$$\begin{pmatrix} -i\omega & h_0 ik & h_0 il & 0 \\ \Theta_0 ik & -i\omega & -f & h_0 ik/2 \\ \Theta_0 il & f & -i\omega & h_0 il/2 \\ \Theta_0/(h_0 \tau_{\text{rad}}) & 0 & 0 & 1/\tau_{\text{rad}} - i\omega \end{pmatrix} \begin{pmatrix} \hat{\eta} \\ \hat{u} \\ \hat{v} \\ \hat{\theta} \end{pmatrix} = \mathbf{0}. \quad (2.7.4)$$

For a non-trivial solution the determinant of the above matrix must be zero. Non-dimensionalising by setting

$$\omega = f \tilde{\omega}, \quad k = \frac{f}{\sqrt{\Theta_0 h_0}} \tilde{k}, \quad l = \frac{f}{\sqrt{\Theta_0 h_0}} \tilde{l}, \quad \frac{1}{\tau_{\text{rad}}} = f \varepsilon, \quad (2.7.5)$$

gives the dispersion relation

$$\tilde{\omega} \left[\tilde{\omega}^3 + i\varepsilon \tilde{\omega}^2 - (1 + \tilde{k}^2 + \tilde{l}^2) \tilde{\omega} - i\varepsilon (1 + (\tilde{k}^2 + \tilde{l}^2)/2) \right] = 0. \quad (2.7.6)$$

Thus $\tilde{\omega} = 0$ or

$$\tilde{\omega}^3 + i\varepsilon\tilde{\omega}^2 - \left(1 + \tilde{k}^2 + \tilde{l}^2\right)\tilde{\omega} - i\varepsilon\left(1 + \left(\tilde{k}^2 + \tilde{l}^2\right)/2\right) = 0. \quad (2.7.7)$$

Setting $\varepsilon = 0$ gives the standard shallow water dispersion relation, with the zero frequency mode $\tilde{\omega} = 0$ and the two inertia-gravity waves $\tilde{\omega}_{\pm} = \pm\sqrt{1 + \tilde{k}^2 + \tilde{l}^2}$. Perturbing the zero frequency mode from $\tilde{\omega} = 0$ to $O(\varepsilon)$ gives

$$\tilde{\omega} = -i\varepsilon \frac{\left(1 + \left(\tilde{k}^2 + \tilde{l}^2\right)/2\right)}{1 + \tilde{k}^2 + \tilde{l}^2} + O(\varepsilon^2), \quad (2.7.8)$$

so the zero frequency mode is damped by the Newtonian cooling term on the right-hand side of (2.7.1b). Perturbing the inertia-gravity waves from $\tilde{\omega}_{\pm}$ to $O(\varepsilon)$ gives

$$\tilde{\omega}_{\pm} = \pm\sqrt{1 + \tilde{k}^2 + \tilde{l}^2} - i\varepsilon \frac{\left(\tilde{k}^2 + \tilde{l}^2\right)}{4\left(1 + \tilde{k}^2 + \tilde{l}^2\right)} + O(\varepsilon^2), \quad (2.7.9)$$

thus the inertia-gravity waves are also damped by the Newtonian cooling term on the right-hand side of (2.7.1b). Chapter 6 demonstrates that the Newtonian cooling term also damps the equatorially trapped waves in the radiatively damped thermal shallow water equations on an equatorial β -plane.

2.7.2 Motivation for the form of the Newtonian cooling term

This section motivates the form of the radiative damping term on the right-hand side of (2.7.1b).

2.7.2.1 Newton's law of cooling

As an alternative model, first consider the equation set

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0, \quad (2.7.10a)$$

$$\frac{\partial \Theta}{\partial t} + (\mathbf{u} \cdot \nabla) \Theta = -\lambda(\Theta - \Theta_0), \quad (2.7.10b)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + f(\hat{\mathbf{z}} \times \mathbf{u}) = -\nabla(\Theta h) + \frac{1}{2}h\nabla\Theta, \quad (2.7.10c)$$

where the right-hand side of (2.7.1b) has been replaced by the standard form of Newton's law of cooling. Linear plane waves about a rest state are again considered, where f is assumed to be constant throughout §2.7.2. An expansion $\mathbf{u} = \mathbf{0} + \mathbf{u}'$,

$h = h_0 + \eta'$ and $\Theta = \Theta_0 + \theta'$ is substituted into (2.7.10), where dashed variables denote small perturbations, and the resulting linearised equations are given by

$$\frac{\partial \eta'}{\partial t} + h_0 \nabla \cdot \mathbf{u}' = 0, \quad (2.7.11a)$$

$$\frac{\partial \theta'}{\partial t} + \lambda \theta' = 0, \quad (2.7.11b)$$

$$\frac{\partial u'}{\partial t} - f v' + \Theta_0 \frac{\partial \eta'}{\partial x} + \frac{h_0}{2} \frac{\partial \theta'}{\partial x} = 0, \quad (2.7.11c)$$

$$\frac{\partial v'}{\partial t} + f u' + \Theta_0 \frac{\partial \eta'}{\partial y} + \frac{h_0}{2} \frac{\partial \theta'}{\partial y} = 0. \quad (2.7.11d)$$

Solutions of the form $u' = \text{Re}\{\hat{u} \exp(i(kx + ly - \omega t))\}$ etc. are again sought, resulting in the system

$$\begin{pmatrix} -i\omega & h_0 i k & h_0 i l & 0 \\ \Theta_0 i k & -i\omega & -f & h_0 i k/2 \\ \Theta_0 i l & f & -i\omega & h_0 i l/2 \\ 0 & 0 & 0 & \lambda - i\omega \end{pmatrix} \begin{pmatrix} \hat{\eta} \\ \hat{u} \\ \hat{v} \\ \hat{\theta} \end{pmatrix} = \mathbf{0}. \quad (2.7.12)$$

For a non-trivial solution the determinant of the above matrix must be zero, and non-dimensionalising as in (2.7.5) and setting $\lambda = f\varepsilon$ gives the four roots of the dimensionless dispersion relation

$$\tilde{\omega} = 0, \quad \tilde{\omega}_{\pm} = \pm \sqrt{1 + \tilde{k}^2 + \tilde{l}^2}, \quad \tilde{\omega} = -i\varepsilon. \quad (2.7.13)$$

Thus if the standard form of Newton's law of cooling is instead added to the right-hand side of (2.7.1b), the cooling term merely causes the thermal mode to decay and nothing more. If a solution of (2.7.10) is sought with Θ_0 constant and the initial condition for Θ , then Θ will equal Θ_0 for all time (in fact $\Theta = \Theta_0 = \text{constant}$ is used as the initial condition for Θ in the thermal simulations in chapter 3). Thus the form of cooling on the right-hand side of the temperature equation (2.7.10b) does not represent a very interesting model when Θ_0 is taken to be constant and the initial condition for Θ since it reduces to the standard shallow water case.

2.7.2.2 Height-dependent cooling

Next consider the equation set

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0, \quad (2.7.14a)$$

$$\frac{\partial \Theta}{\partial t} + (\mathbf{u} \cdot \nabla) \Theta = -\kappa (h - h_0), \quad (2.7.14b)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + f(\hat{\mathbf{z}} \times \mathbf{u}) = -\nabla(\Theta h) + \frac{1}{2} h \nabla \Theta, \quad (2.7.14c)$$

where the right-hand side of (2.7.1b) has been replaced by a height-dependent cooling term (cf. the height-damped shallow water equations (1.3.9)). The cooling term encourages regions where h is larger than the mean layer depth h_0 to cool, and regions where h is smaller than the mean layer depth h_0 to heat up. Considering linear plane waves again about a rest state by substituting the expansion $\mathbf{u} = \mathbf{0} + \mathbf{u}'$, $h = h_0 + \eta'$ and $\Theta = \Theta_0 + \theta'$ into (2.7.14), linearising the resulting system under the assumption that dashed variables denote small perturbations, and seeking solutions of the form $u' = \text{Re}\{\hat{u} \exp(i(kx + ly - \omega t))\}$ etc. gives the matrix system

$$\begin{pmatrix} -i\omega & h_0 i k & h_0 i l & 0 \\ \Theta_0 i k & -i\omega & -f & h_0 i k/2 \\ \Theta_0 i l & f & -i\omega & h_0 i l/2 \\ \kappa & 0 & 0 & -i\omega \end{pmatrix} \begin{pmatrix} \hat{\eta} \\ \hat{u} \\ \hat{v} \\ \hat{\theta} \end{pmatrix} = \mathbf{0}. \quad (2.7.15)$$

For a non-trivial solution the determinant of the above matrix must be zero. Non-dimensionalising as in (2.7.5) and setting $\kappa = (f\Theta_0/h_0)\varepsilon$ gives the dimensionless dispersion relation $\tilde{\omega} = 0$ or

$$\tilde{\omega}^3 - \tilde{\omega} \left(1 + \tilde{k}^2 + \tilde{l}^2\right) + i\varepsilon \left(\tilde{k}^2 + \tilde{l}^2\right)/2 = 0. \quad (2.7.16)$$

Setting $\varepsilon = 0$ in (2.7.16) gives the standard shallow water dispersion relation with the zero frequency mode $\tilde{\omega} = 0$ and the two inertia-gravity waves $\tilde{\omega}_{\pm} = \pm\sqrt{1 + \tilde{k}^2 + \tilde{l}^2}$. Perturbing the inertia-gravity waves from $\tilde{\omega}_{\pm}$ to $O(\varepsilon)$ gives (2.7.9), thus the inertia-gravity waves are damped by the height-dependent cooling term. However, perturbing the zero frequency mode from $\tilde{\omega} = 0$ to $O(\varepsilon)$ gives

$$\tilde{\omega} = i\varepsilon \frac{(\tilde{k}^2 + \tilde{l}^2)}{2(1 + \tilde{k}^2 + \tilde{l}^2)} + O(\varepsilon^2). \quad (2.7.17)$$

Thus the form of cooling on the right-hand side of the temperature equation (2.7.14b) destabilises the zero frequency mode.

2.7.2.3 Combined source term

Finally consider the equation set

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0, \quad (2.7.18a)$$

$$\frac{\partial \Theta}{\partial t} + (\mathbf{u} \cdot \nabla) \Theta = -\lambda(\Theta - \Theta_0) - \kappa(h - h_0), \quad (2.7.18b)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + f(\hat{\mathbf{z}} \times \mathbf{u}) = -\nabla(\Theta h) + \frac{1}{2}h\nabla\Theta, \quad (2.7.18c)$$

where the standard form of Newton's law of cooling has been combined with the height-dependent cooling term. Performing a linear stability analysis as above yields the dimensionless dispersion relation $\tilde{\omega} = 0$ or

$$(\tilde{\omega} + i\varepsilon) \left(\tilde{\omega}^2 - (1 + \tilde{k}^2 + \tilde{l}^2) \right) + i\delta (\tilde{k}^2 + \tilde{l}^2) / 2 = 0, \quad (2.7.19)$$

where ω and $\mathbf{k}$ have been non-dimensionalised as in (2.7.5), and I have set $\lambda = f\varepsilon$ and $\kappa = (f\Theta_0/h_0)\delta$. Setting $\varepsilon = \delta = 0$ in (2.7.19) gives the standard shallow water dispersion relation with the zero frequency mode $\tilde{\omega} = 0$ and the two inertia-gravity waves $\tilde{\omega}_{\pm} = \pm\sqrt{1 + \tilde{k}^2 + \tilde{l}^2}$. Perturbing the inertia-gravity waves from $\tilde{\omega}_{\pm}$ to $O(\varepsilon, \delta)$ gives

$$\tilde{\omega}_{\pm} = \pm\sqrt{1 + \tilde{k}^2 + \tilde{l}^2} - i\delta \frac{(\tilde{k}^2 + \tilde{l}^2)}{4(1 + \tilde{k}^2 + \tilde{l}^2)}, \quad (2.7.20)$$

which are both stable and independent of ε . However, perturbing the zero frequency mode from $\tilde{\omega} = 0$ to $O(\varepsilon, \delta)$ gives

$$\tilde{\omega} = -i\varepsilon + i\delta \frac{(\tilde{k}^2 + \tilde{l}^2)}{2(1 + \tilde{k}^2 + \tilde{l}^2)}. \quad (2.7.21)$$

The zero frequency mode is thus stable provided that $\delta/2 - \varepsilon < 0$, i.e. provided that $\kappa < 2\lambda\Theta_0/h_0$. Thus by combining the standard form of Newton's law of cooling with a height-dependent cooling term on the right-hand side of the temperature equation, a more interesting model may be obtained for which the zero frequency mode is not destabilised by a solution of the form $\mathbf{u} = \mathbf{0} + \mathbf{u}'$, $h = h_0 + \eta'$ and $\Theta = \Theta_0 + \theta'$. The particular form of the cooling term on the right-hand side of (2.7.1b) was chosen to couple the height and temperature fields while satisfying the stability condition for the zero frequency mode. Comparing (2.7.2) with (2.7.18) gives $\lambda = 1/\tau_{\text{rad}}$ and $\kappa = \Theta_0/(h_0\tau_{\text{rad}})$, so $\kappa < 2\lambda\Theta_0/h_0$ as required for the stability of the zero frequency mode.

2.8 Summary

Thermal shallow water theory extends standard shallow water theory by allowing horizontal variations of thermodynamic properties within each layer. It offers single active layer descriptions of phenomena arising from differential heating and cooling in winds over lakes (Lavoie, 1972), tropical oceans (McCreary and Yu, 1992, Schopf and Cane, 1983), oceanic mixed layers (Bleck et al., 1992, Young, 1994, Young and Chen,

1995), and coastal or boundary currents and fronts in the atmosphere (Anderson, 1984) and oceans (Carbonel and Galeao, 2007, Eldevik, 2002, McCreary et al., 1991, McCreary and Kundu, 1988, Røed and Shi, 1999). Thermal shallow water theory has close connections with shallow water models of moist convection (Bouchut et al., 2009, Lambaerts et al., 2011, Zeitlin, 2007), and also with shallow water magnetohydrodynamics (Gilman, 2000) in which the advected scalar becomes the flux function for a horizontal magnetic field (Dellar, 2002, 2003).

Three derivations of the thermal shallow water equations have been presented (where dissipation due to radiative damping was neglected in these derivations). The first two derivations involved averaging over the fluid layer, while the third used Hamilton's principle of least action and projected the three-dimensional Lagrangian for a Boussinesq fluid onto columnar fluid motions. The loss of the material conservation of potential vorticity was discussed, where the source term in the thermal shallow water potential vorticity equation was shown to be the depth average of the baroclinic torque in a continuously stratified fluid. However, the thermal shallow water equations were shown to instead possess an integral conservation law for the total potential vorticity inside a closed isotherm. The non-canonical Hamiltonian structure of the thermal shallow water equations was presented, enabling conservation laws to be related to symmetries via Noether's theorem. Equations that are derived by making approximations that preserve these symmetries then inherit equivalent conservation laws. In future chapters, the Hamiltonian structure of reduced equation sets are derived from the Hamiltonian structure of the thermal shallow water equations (notably the local and global thermal quasi-geostrophic equations in chapters 4 and 5). The group velocity was shown to be the velocity at which the wave energy is transported for the thermal shallow water equations. Finally, the form of the Newtonian cooling term was motivated and shown to damp the linear plane waves in the thermal shallow water equations on an f -plane. In §6.2.2, the effect of the Newtonian cooling term on equatorially trapped waves in the radiatively damped thermal shallow water equations on an equatorial β -plane is considered.

Chapter 3

Numerical simulations of the standard shallow water model, the two height-damped shallow water models, and the radiatively damped thermal shallow water model

This chapter is based upon Warneford and Dellar (2014b).

3.1 Introduction

This chapter considers numerical simulations of the radiatively damped thermal shallow water equations (1.3.12). Results are compared to numerical simulations of the standard shallow water equations (1.3.6), the Scott and Polvani (2007, 2008) shallow water equations (1.3.9), and a momentum-conserving variant of the Scott and Polvani (2007, 2008) shallow water equations (similar to (1.3.11), but with the Heaviside step function omitted). The four shallow water models are solved in a doubly-periodic Cartesian geometry (as described in §1.4.2), and stochastic forcing is included in all simulations (as described in §3.2.4). This injection of turbulence can either be thought of as representing deep forcing of the jets (where convective cells in the deep interior drive differential rotation, that results in turbulent motions in the cloud layer), or shallow forcing (where the jets are driven by turbulence injected in the cloud layer by, for instance, moist convection or horizontal contrasts in solar heating). The models are first solved for Jovian parameter values, then a radiatively damped thermal

shallow water simulation of Neptune is considered.

These shallow models require just a few parameters: the planetary radius a , rotation period $T = 2\pi/\Omega$, gravity wave speed $\sqrt{\Theta_0 h_0}$, and the radiative cooling rate. The first two are easily determined from observations, and a good knowledge of the internal gravity wave speed on Jupiter is known from observing waves propagating outwards from the impacts of Shoemaker–Levy comet debris (Dowling, 1995*b*, Ingersoll et al., 2007). These and other data from Ingersoll (1990) and Beebe (1994) are given in table 3.1. The radiative time scale formula from Houghton (2002) is used,

$$\tau_{\text{rad}} = \frac{c_p p}{8\sigma g \mathcal{T}^3}, \quad (3.1.1)$$

where $\mathcal{T}$ and p are the gas temperature and pressure respectively, $\sigma = 5.670 \times 10^{-8} \text{J m}^{-2} \text{s}^{-1} \text{K}^{-4}$ is the Stefan–Boltzmann constant, and $c_p = 1.26 \times 10^4 \text{J kg}^{-1} \text{K}^{-1}$ is the specific heat for a hydrogen atmosphere in the Jovian temperature range (this value is taken from Liu and Schneider, 2010).

Liu and Schneider (2010) compared three-dimensional general circulation model (GCM) simulations with observations at the 25 millibar pressure level for Neptune. This pressure level is used to calculate radiative time scales for both Jupiter and Neptune. For Jupiter, a cloud layer is considered that extends downwards on the order of 100 km from the 25 millibar pressure level, as suggested by observations of moist convection (Ingersoll et al., 2000). The latent heat released by condensation of water vapour suggests a temperature contrast of around $\Delta\mathcal{T} = 10\text{K}$, giving a reduced gravity $\Theta_0 = g\Delta\mathcal{T}/\mathcal{T} \approx 2 \text{m s}^{-2}$. This requires a layer depth $h_0 \approx 200 \text{km}$ to match the observed internal wave speed. At the 25 millibar level, $\mathcal{T} \approx 120 \text{K}$ for Jupiter (Seiff et al., 1998) giving a radiative time scale of 429 hours, or 43 Jovian days. Neptune has a much lower temperature at the 25 millibar level, $\mathcal{T} \approx 60 \text{K}$ (Hammel and Lockwood, 2007) which gives a slightly lower specific heat capacity $c_p = 1.12 \times 10^4 \text{J kg}^{-1} \text{K}^{-1}$, and has a lower gravity, $g = 11.2 \text{m s}^{-2}$ (Liu and Schneider, 2010). Together these differences give a much longer radiative time scale of 7088 hours, or 396 Neptunian days. These and other parameter values for Jupiter and Neptune are summarised in table 3.1. In §3.5, Jovian and Neptunian simulations for alternative values of τ_{rad} are considered.

Table 3.1: Parameter values for Jupiter and Neptune (taken from Beebe, 1994, Hammel and Lockwood, 2007, Ingersoll, 1990, Seiff et al., 1998).

	a (10^7m)	$2\pi/\Omega$ (hours)	$\sqrt{\Theta_0 h_0}$ (m s^{-1})	g (m s^{-2})	$\mathcal{T}$ (K)	τ_{rad} (hours)	τ_{fric} (hours)
Jupiter	7.1	9.9	678	26	120	429	9900
Neptune	2.5	17.9	574	11.2	60	7088	17900

3.2 Numerical formulation

3.2.1 Dimensionless equations

The various shallow water models are non-dimensionalised using the deformation radius $L_D = \sqrt{\Theta_0 h_0}/(2\Omega)$ and the planetary rotation period $T = 2\pi/\Omega$ as length and time scales. The field variables h and Θ are scaled with their equilibrium values h_0 and Θ_0 , where $g' = \Theta_0$ for the non-thermal models. The dimensionless height-damped shallow water model is then defined by

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = -\kappa(h - 1), \quad (3.2.1a)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{1}{Ro} f \hat{\mathbf{z}} \times \mathbf{u} = -\frac{1}{Ro^2} \nabla h + \mathbf{F} - \gamma \mathbf{u}. \quad (3.2.1b)$$

The Rossby number $Ro = 1/(4\pi)$ is fixed due to the velocity scale being $U = L_D/T = L_D\Omega/(2\pi)$. The true Rossby numbers calculated from the root-mean-square (RMS) velocities in the simulations are within 20% of this value, as described below. No Burger number appears because the horizontal lengths have been scaled with the deformation radius L_D . The dimensionless radiative cooling rate is κ . A forcing $\mathbf{F}$ has been added, as described below, as has a Rayleigh friction with rate γ (to dissipate energy at the largest scales). These scales would otherwise grow secularly due to the slow leakage of energy past the Rhines scale in the inverse cascade, and would thus prevent a Jovian-like banded structure from persisting as a statistical steady state (Vasavada and Showman, 2005).

The standard shallow water model sets $\kappa = 0$. A variant that conserves momentum during radiative relaxation may be written as

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = -\kappa(h - 1), \quad (3.2.2a)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{1}{Ro} f \hat{\mathbf{z}} \times \mathbf{u} = -\frac{1}{Ro^2} \nabla h + \kappa \left(1 - \frac{1}{h}\right) \mathbf{u} + \mathbf{F} - \gamma \mathbf{u}, \quad (3.2.2b)$$

where the Heaviside step function has been omitted from the Shell and Held (2004) model. This model is referred to as the momentum-conserving model in this chapter,

while (3.2.1) and (3.2.2) are collectively called the two height-damped models. The dimensionless radiatively damped thermal shallow water model is

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0, \quad (3.2.3a)$$

$$\frac{\partial \Theta}{\partial t} + (\mathbf{u} \cdot \nabla) \Theta = -\kappa (h\Theta - 1), \quad (3.2.3b)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{1}{Ro} f \hat{\mathbf{z}} \times \mathbf{u} = -\frac{1}{Ro^2} (\nabla (\Theta h) - \frac{1}{2} h \nabla \Theta) + \mathbf{F} - \gamma \mathbf{u}. \quad (3.2.3c)$$

This model is referred to as the thermal model for brevity. The control parameters are thus the domain size, in units of deformation radii, the radiative cooling rate κ , the frictional damping rate γ , and the properties of the forcing $\mathbf{F}$. Values of these parameters for Jupiter and Neptune are given in table 3.2.

3.2.2 Square planet domain

The four equation sets were solved in the doubly-periodic Cartesian domain sketched in figure 1.7 (see §1.4.2 for further details of this square planet domain). Each half of the domain corresponds to a complete planet, so for clarity simulation results were only plotted for the top planet (planet 1 in figure 1.7). The dimensionless traditional Coriolis parameter is still fully varying in latitude and is defined in the simulations to be the periodic function $f(y) = \cos(2\pi y/y_{\max})$, where $y_{\max}$ is the dimensionless side length of the domain, which is related to the circumference of Jupiter or Neptune (see table 3.1). Thus the Coriolis parameter and all its derivatives are doubly periodic. Simulations of geostrophic turbulence in gas giant planet regimes require many rotation periods to reach statistical steady states. Cho and Polvani (1996*a*) showed that some early simulations (Williams, 1978, Yoden et al., 1999, Yoden and Yamada, 1993) of non-divergent flows were not integrated for long enough to reach statistically steady states. This doubly-periodic Cartesian geometry enables the exploitation of the superior memory bandwidth and floating point performance of GPUs using a spectral spatial representation based on the GPU fast Fourier transform, while the performance of existing spherical harmonic transforms for GPUs remains comparable with that of conventional microprocessors (Hupca et al., 2012). It is possible that some distortion may exist at high latitudes in the simulations, due to the poles consisting of a line in longitude in this domain as opposed to a single point. However, the results for simulations in this domain (see §3.3) are in good qualitative agreement with spherical simulations (Cho and Polvani, 1996*a,b*, Scott and Polvani, 2007, 2008), and the zonal energy spectra $E(k_y)$ are consistent with observations of Jupiter (see §3.3.2).

3.2.3 Numerical method

The four dimensionless models were discretised in space using a grid of 1024×1024 Fourier collocation points, where the nonlinear terms were computed pseudo-spectrally. The resulting large system of ordinary differential equations was integrated using the standard fourth-order Runge–Kutta scheme, where the time step was determined dynamically from the Courant–Friedrichs–Lewy condition $k_{\max} c_{\max} \Delta t < 2.4$, a slightly reduced threshold, with $k_{\max} = 1024\sqrt{2\pi}/y_{\max}$ the maximum of $|\mathbf{k}|$. The dimensionless maximum wave speed is $c_{\max} = \max(\sqrt{\Theta h}/Ro + |\mathbf{u}|)$ for the thermal model, and $c_{\max} = \max(\sqrt{h}/Ro + |\mathbf{u}|)$ for the other three models. The maxima were taken over all grid points in the domain. To prevent the build-up of enstrophy at the highest resolved wavenumbers, the tensor product form of the one-dimensional Hou and Li (2007) spectral filter was used. It multiplies the Fourier coefficient with wave vector (k_x, k_y) by $\exp\{-36[(k_x/k_{\max})^{36} + (k_y/k_{\max})^{36}]\}$ after each Runge–Kutta time step. The energy spectra shown in §3.3.2 confirm that this filter provides sufficient dissipation at the finest resolved scales without the parameter adjustment required by more conventional hyperviscous dissipation. The initial conditions were $h = 1$ and $u = v = 0$ for all four models, with $\Theta = 1$ for the thermal model.

3.2.4 Forcing

The turbulence was driven by applying a divergence-free isotropic random forcing to $\mathbf{u}$. The forcing was inside a narrow annulus of wavenumbers $|\mathbf{k}| \in [k_c - 2, k_c + 2]$ centred around a wavenumber k_c . These wavenumbers were counted from the longest sinusoidal mode in the domain being wavenumber 1. Each mode was forced with amplitude ϵ_f , using random phases that were δ -correlated (white) in time. This type of forcing corresponds to the zero-correlation-time limit of the forcing introduced by Lilly (1969). It has been widely used in numerical studies of zonal jet formation (e.g. see Scott and Polvani, 2007, Smith, 2004, Smith et al., 2002, Srinivasan and Young, 2012), and may be interpreted as a model for small-scale three-dimensional convection. Three-dimensional geostrophic turbulence theory suggests an injection of energy into the lowest vertical mode at horizontal length scales comparable to the deformation radius (Salmon, 1980). The value $k_c = 42$ was used for the Jovian simulations in §3.3 to follow Scott and Polvani (2007, 2008). Setting $k_c = k_{\max}/4$ leaves sufficiently many larger wavenumbers for a forward enstrophy cascade (Scott, 2007), and gives $k_c = 42$ for the T170 spherical harmonic truncation used in their lower resolution simulations (Scott and Polvani, 2007). Choi and Showman (2011) showed that the eddy

kinetic energy power spectra reconstructed from Cassini observations of Jupiter show a change in slope around wavenumber 135. The location of this change in slope from an inverse to a forward cascade is commonly interpreted as the length scale at which energy is injected into large-scale quasi-two-dimensional motions. Section 3.3.2 shows some numerical spectra resulting from forcing centred around wavenumber 100, and an area for future work would be to run further simulations with the forcing centred around wavenumber 135.

There is no analytical theory for the energetics of forcing combined with Newtonian cooling, as there is for Rayleigh friction alone (Scott and Polvani, 2007). Instead, the forcing amplitude ϵ_f was adjusted at each time step so that the kinetic energy roughly followed the prescribed profile $\mathcal{E}(t) = \mathcal{E}_\infty \tanh(t/t_{\text{ramp}})$ with $t_{\text{ramp}} = 1000$ planetary days. If the kinetic energy exceeded $\mathcal{E}(t)$ at the start of the time step, the forcing strength ϵ_f was decreased by a factor of $\exp(-\Delta t/t_{\text{adj}})$. If instead the kinetic energy was less than $\mathcal{E}(t)$, ϵ_f was increased by a factor of $\exp(\Delta t/t_{\text{adj}})$. In all cases t_{adj} was set as 50 planetary days. Setting $\mathcal{E}_\infty = 2 \times 10^4$, or about 0.37 per unit area, for Jovian simulations gave zonal jet speeds comparable to those shown in figure 1.1a for Jupiter. Using $\mathcal{E}_\infty$ to determine a velocity scale $u_{\text{RMS}} \approx 0.86 L_D/T$ leads to the Jovian simulations having Rossby number $u_{\text{RMS}}/(2\Omega L_D) \approx 0.069$ in their eventual statistical steady states.

This method of forcing enables fair comparisons to be drawn between solutions of the four models, each with the same kinetic energy. Controlling the kinetic energy rather than the total energy is similar to the “thermostatting” approach widely used to control the kinetic energy in molecular dynamics simulations (Hünenberger, 2005). If a constant forcing strength was used throughout, it would be very difficult to determine values of ϵ_f to give comparable kinetic energies in the four different models. I also used the calculated average of the forcing strength from a self-adjusting simulation that had reached a statistical steady state to provide a value of ϵ_f for a second run with constant forcing amplitude. The two solutions were very similar.

The MT19937 variant of the Mersenne Twister algorithm (Matsumoto and Nishimura, 1998) was used to generate uniformly distributed random phases. Detailed simulation results for one particular value of the seed are presented, but simulations with a different initial seed gave very similar results, as quantified by the error indicators on the diagnostics in table 3.3 and table 3.4, as did simulations with the Mersenne Twister replaced by the ACORN random number generator (Wikramaratna, 1989). In both cases the generator was executed on the host microprocessor, and the resulting sequence of random numbers was copied to the GPU. A simple linear congruential

generator running directly on the GPU produced noticeably different results from these two much more sophisticated generators.

Table 3.2: Dimensionless parameter values for Jupiter and Neptune.

	$y_{\max} = 2\pi a/L_D$	$\kappa^{-1} = \tau_{\text{rad}}/T$	$\mathcal{E}_{\infty}/y_{\max}^2$	k_c	γ^{-1}
Jupiter	232	43	0.37	42	10^3
Neptune	53	396	0.71	10	10^3

3.2.5 Performance

The fast Fourier transforms accounted for 66% of the computational time in the GPU code, which performed about 22 time steps per second at 1024×1024 resolution. Most of the rest of the time was taken up with computing the nonlinear terms. Each calculation of the four time derivatives of h , u , v , Θ required sixteen fast Fourier transformations of single fields. These together took 7.4 milliseconds on a Nvidia K20c GPU, and 38 milliseconds using FFTW 3.2.2 with 8 threads (the optimal number) across two Intel E5-2640 CPUs. This is a factor of five speed-up. Even assuming the rest of the CPU code took no time, the GPU code would be 3.5 times faster. This is consistent with the GPU code being about 24 times faster than the original single-threaded CPU code.

3.3 Jovian simulations

Figure 3.1 shows the instantaneous relative vorticity $\omega = \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u}$ after 2×10^4 Jovian days for the four different shallow water models, where (a) is the standard shallow water, (b) the height-damped, (c) the momentum-conserving height-damped, and (d) the thermal model. Figure 3.2 shows the corresponding zonal velocity and temperature fields for the thermal model. Only the top half of the domain, planet 1 in figure 1.7, is shown for ease of interpretation. All four simulations show a mixture of vortices and zonally elongated structures. The temperature fluctuations in the northern hemisphere of figure 3.2a have a correlation coefficient of 0.29 with respect to the relative vorticity fluctuations shown in the northern hemisphere of figure 3.1d at 2×10^4 Jovian days. Two otherwise identical thermal simulations initialised with different seeds in the random number generator gave correlations of 0.32 and 0.33. This suggests that cyclonic regions are typically warmer than anticyclonic regions for these Jovian thermal simulations.

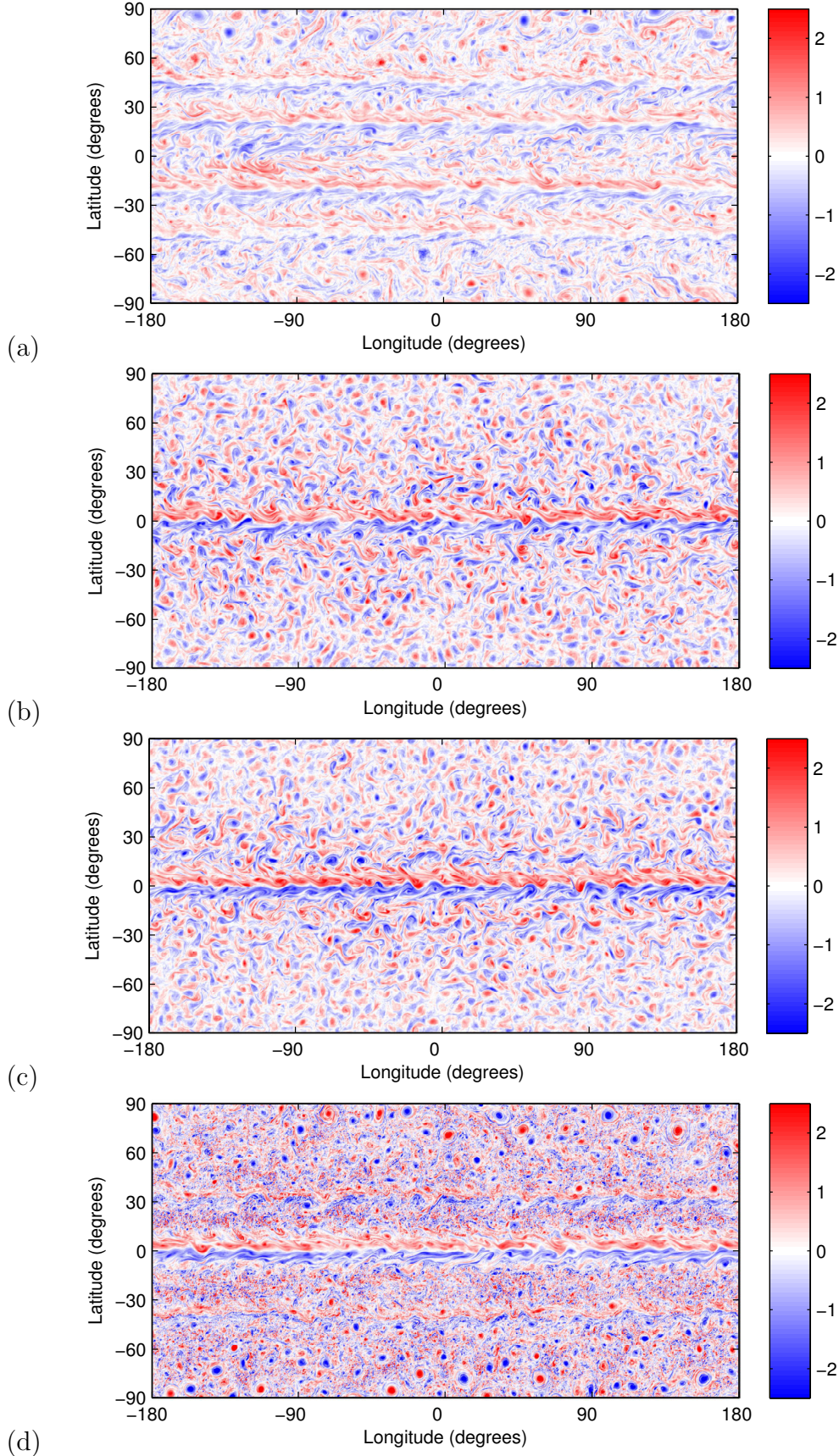


Figure 3.1: Dimensionless relative vorticity after 2×10^4 Jovian days for the four shallow water models: (a) standard shallow water, (b) height-damped, (c) momentum-conserving height-damped, (d) thermal.

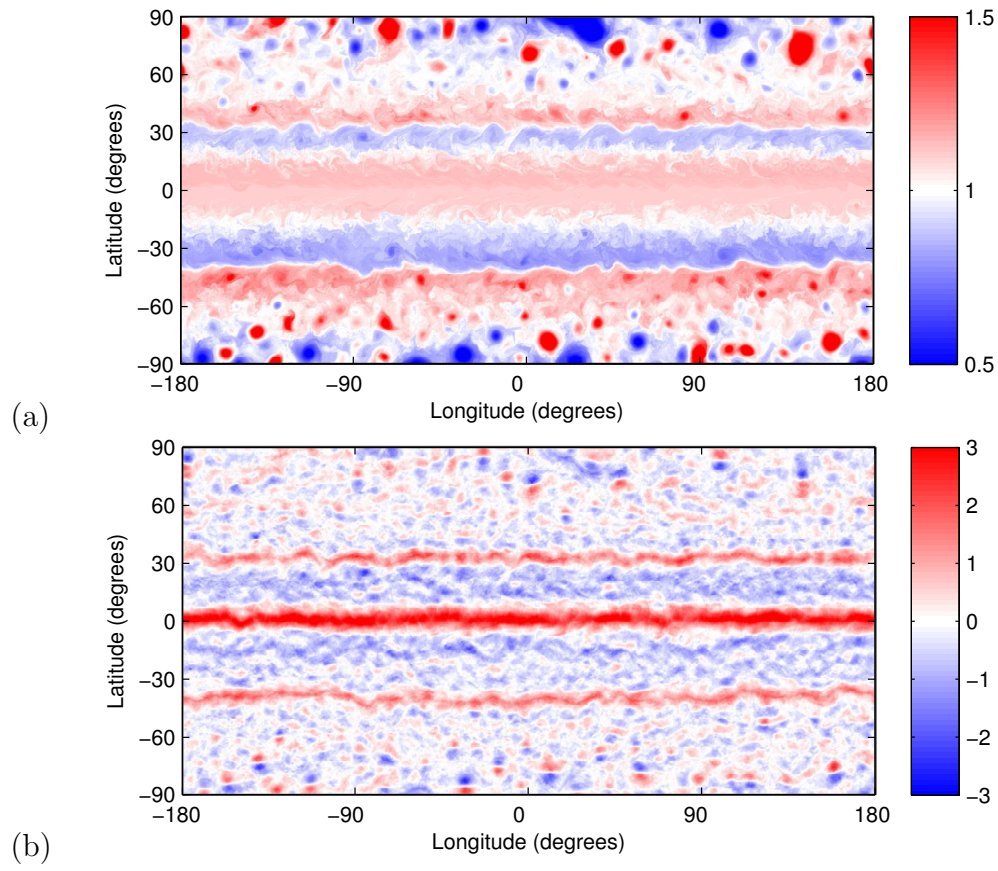


Figure 3.2: Dimensionless (a) temperature and (b) zonal velocity from a Jovian thermal simulation after 2×10^4 Jovian days.

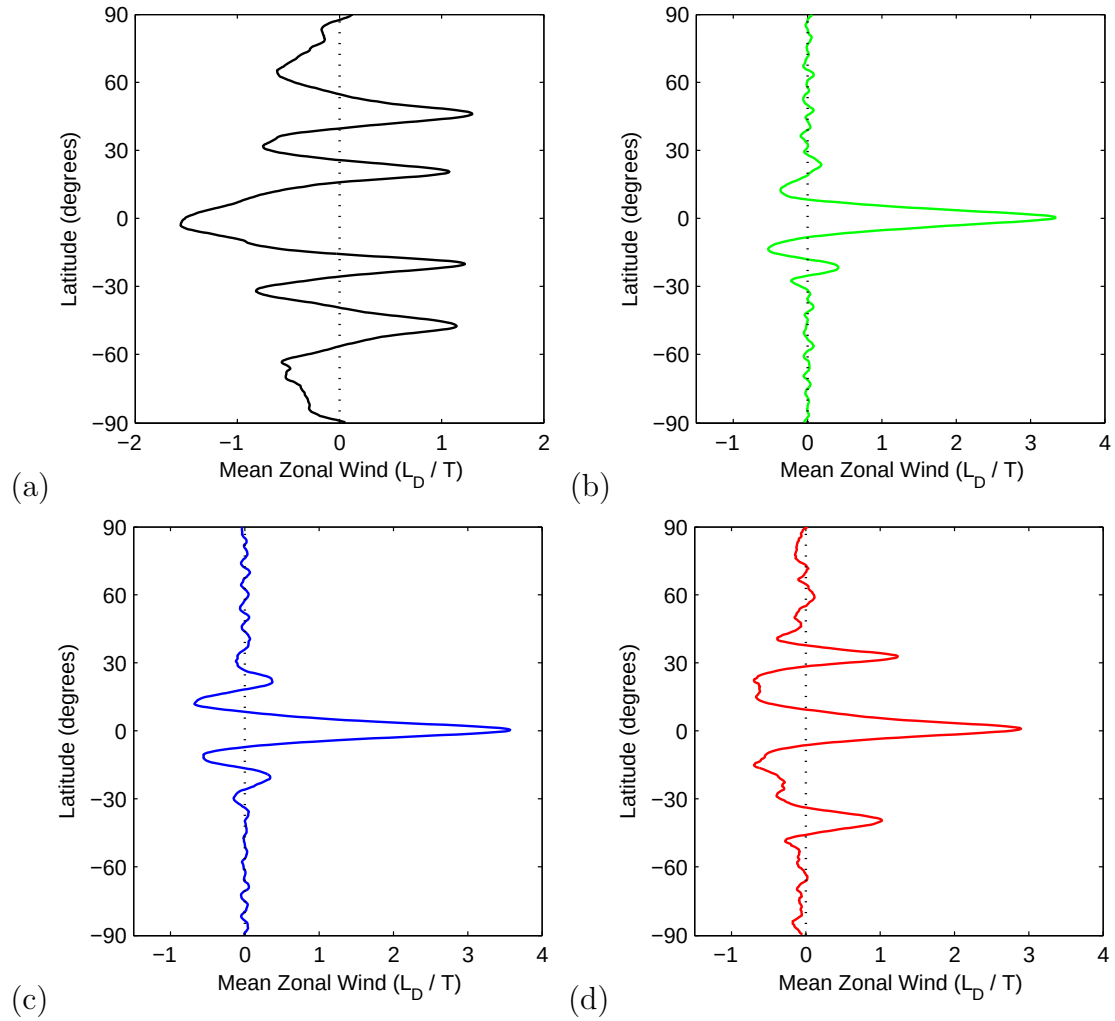


Figure 3.3: Mean zonal wind against latitude after 2×10^4 Jovian days for the four shallow water models: (a) standard shallow water, (b) height-damped, (c) momentum-conserving height-damped, (d) thermal.

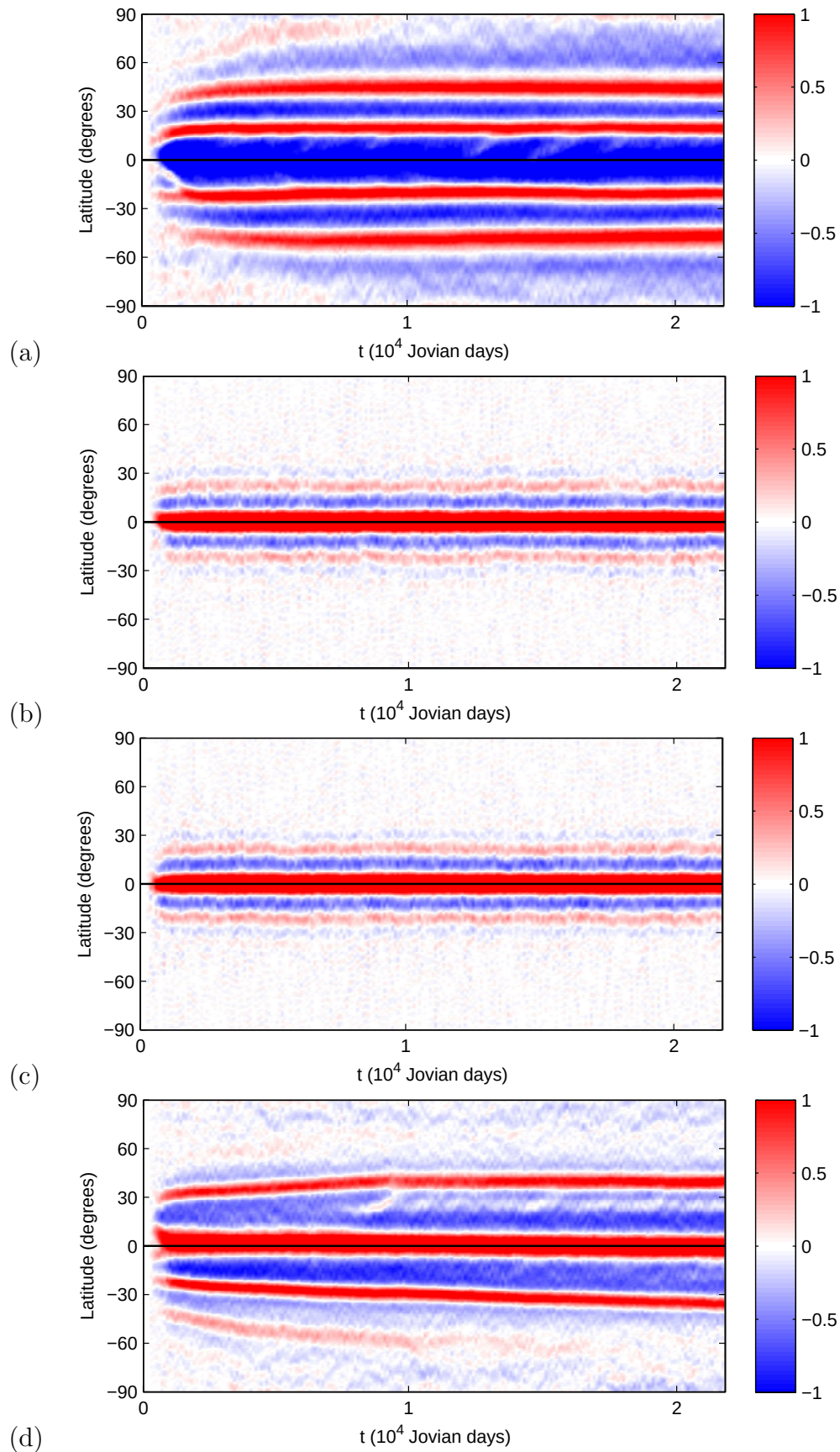


Figure 3.4: Evolution of the mean zonal wind for the four shallow water models: (a) standard shallow water, (b) height-damped, (c) momentum-conserving height-damped, (d) thermal.

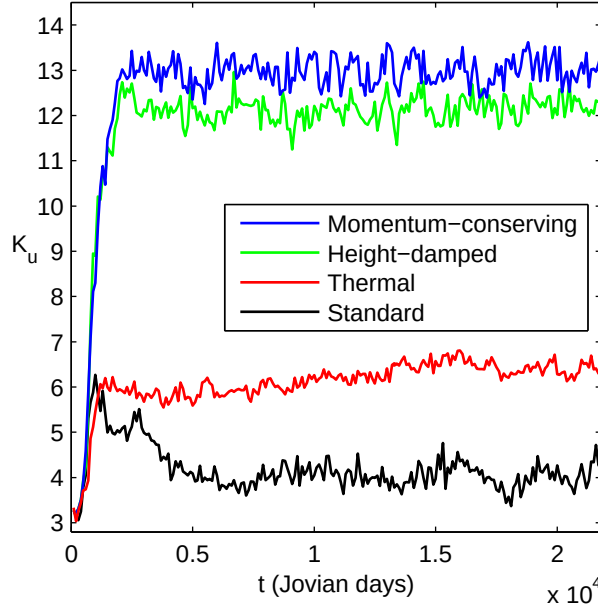


Figure 3.5: Evolution of the kurtosis of the zonal velocity.

Figure 3.3 shows the corresponding instantaneous zonally averaged zonal velocity, $\bar{u} = y_{\max}^{-1} \int u(x, y, t) dx$, in units of deformation radii per Jovian day. In figures 3.3, 3.5–3.10 the standard shallow water model is denoted by a black line, the height-damped model by a green line, the momentum-conserving height-damped model by a blue line, and the thermal model by a red line. The standard shallow water model shows a strong retrograde equatorial jet, as found in previous simulations with Jovian parameters (e.g. see Cho and Polvani, 1996*a,b*, Iacono, Struglia and Ronchi, 1999, Iacono, Struglia, Ronchi and Nicastro, 1999, Scott and Polvani, 2007, Showman, 2007, Vasavada and Showman, 2005) while the other three models show strong prograde equatorial jets. Figure 3.4 shows the evolution of the mean zonal wind as a function of latitude and time for the four shallow water models, and demonstrates the robustness of the retrograde equatorial jet for the standard shallow water simulation, and the robustness of the prograde equatorial jets in the two height-damped and thermal shallow water simulations. This figure also shows that the two height-damped models produce only very weak jets away from the equator. However, the thermal model produces stronger jets at around 40° either side of the equator, while the standard shallow water model shows two pairs of jets of roughly equal magnitudes at 20° and 45° either side of the equator.

These differences are quantified by calculating the kurtosis, or flatness, of the

Table 3.3: Average diagnostic quantities in the statistically steady state for two sets of the Jovian simulations in §3.3 with different initial random seeds. The error estimates quantify the differences between the diagnostics for each pair of simulations.

	$\langle\langle\epsilon_f\rangle\rangle$	$\langle\langle\xi\rangle\rangle$	$\langle\langle\langle\omega\rangle_N\rangle\rangle$
Standard	1300 ± 7	0.58 ± 0.001	-0.023 ± 0.001
Height-damped	2900 ± 3	0.51 ± 0.002	0.059 ± 0.0005
Momentum-conserving	2800 ± 1	0.54 ± 0.0007	0.062 ± 0.0004
Thermal	2200 ± 21	0.49 ± 0.006	0.046 ± 0.0007

Table 3.4: Average diagnostic quantities in the statistically steady state for two sets of the Jovian simulations in §3.3 with different initial random seeds. The error estimates quantify the differences between the diagnostics for each pair of simulations.

	$\langle\langle\mathcal{K}_u\rangle\rangle$	$\langle\langle\mathcal{K}_\omega\rangle\rangle$	$\langle\langle\mathcal{S}_\omega\rangle\rangle$
Standard	4.05 ± 0.03	3.53 ± 0.03	-0.033 ± 0.02
Height-damped	12.17 ± 0.09	3.64 ± 0.003	-0.0014 ± 0.004
Momentum-conserving	13.03 ± 0.07	3.73 ± 0.009	0.0007 ± 0.0009
Thermal	6.43 ± 0.08	4.59 ± 0.0008	-0.053 ± 0.02

pointwise zonal velocity field,

$$\mathcal{K}_u(t) = \langle u^4 \rangle / \langle u^2 \rangle^2, \quad (3.3.1)$$

where $\langle \cdot \rangle$ denotes an average over the entire simulation domain (planets 1 and 2). Figure 3.5 shows that the kurtosis begins near three, as for a Gaussian random variable, in each simulation. It increases due to the formation of coherent structures (such as zonal jets) and finally levels off as the flow reaches a statistical steady state. The two height-damped models produce zonal velocity fields with a kurtosis that is roughly twice as large as for the other two models. While the kurtosis also includes contributions from vortices and filaments, these much larger kurtosis values primarily reflect the dominance of the equatorial jets in the two height-damped models. Although subject to more statistical noise during simulations, the kurtosis of the zonal average $\bar{u}$ is also noticeably much larger in the two height-damped simulations. After 2×10^4 Jovian days, $\mathcal{K}_{\bar{u}} = 2.54$ for the standard shallow water model, $\mathcal{K}_{\bar{u}} = 19.69$ for the height-damped model, $\mathcal{K}_{\bar{u}} = 20.34$ for the momentum-conserving height-damped model, and $\mathcal{K}_{\bar{u}} = 10.54$ for the thermal model.

Tables 3.3 and 3.4 show a number of averaged diagnostic quantities computed from the simulations, with $\langle\langle \cdot \rangle\rangle$ denoting the time-average of a quantity over the period between 2×10^4 and 2.18×10^4 Jovian days. For example, $\langle\langle \mathcal{K}_u \rangle\rangle$ is the time-average of the kurtosis of the zonal velocity, the quantity whose time series is shown in figure

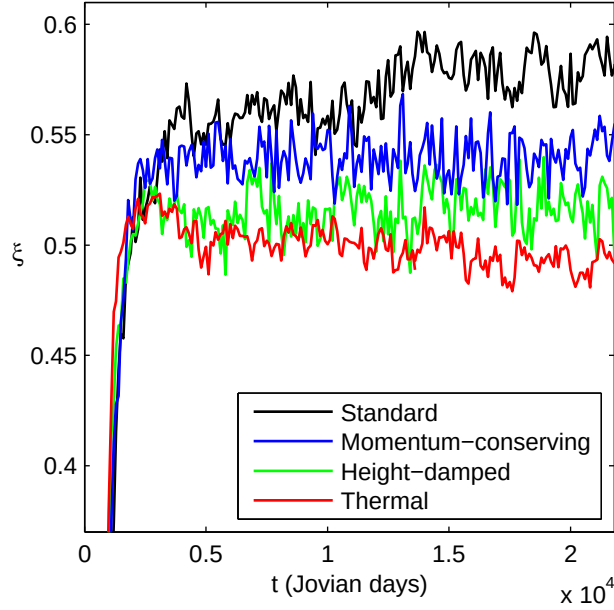


Figure 3.6: Evolution of the fraction of kinetic energy in the zonally averaged flow.

3.5. Two sets of simulations were run with different initial random seeds. The mean and half differences are presented to give an indication of the repeatability of the different diagnostics.

All four models produce jet speeds comparable to the Jovian observations shown in figure 1.1a. The forcing strengths for the four simulations were adjusted by the algorithm described in §3.2.4 to give the same statistically steady kinetic energy $0.37 (L_D/T)^2$ per unit area. Figure 3.6 shows the fraction of the kinetic energy in the zonally averaged flow, as defined by

$$\xi(t) = \langle \bar{h} \bar{u}^2 \rangle / \langle h (u^2 + v^2) \rangle, \quad (3.3.2)$$

where $\bar{h}$ and $\bar{u}$ are the zonal means of h and u . The corresponding averages $\langle \langle \xi \rangle \rangle$ are shown in table 3.3. The zonal mean flow accounts for roughly half the kinetic energy in the three radiatively damped models, and a slightly larger fraction $\langle \langle \xi \rangle \rangle = 0.58$ in the standard shallow water model. All four models produce zonal velocities in the Jovian parameter regime when forced with an amplitude determined by the same desired total kinetic energy. Another such diagnostic is the average forcing amplitude $\langle \langle \epsilon_f \rangle \rangle$, which shows that the three simulations with some form of radiative damping required a forcing amplitude roughly twice as large as that for the standard shallow water model.

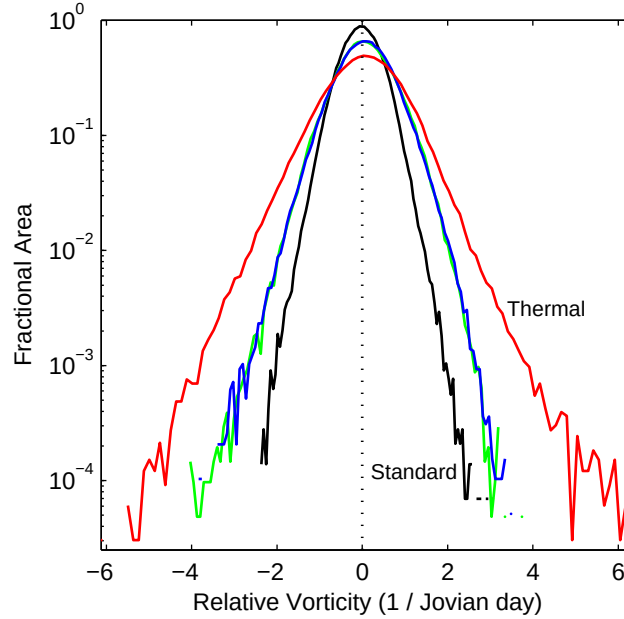


Figure 3.7: Histogram of the dimensionless relative vorticity at 2×10^4 Jovian days.

Figure 3.7 shows histograms of the relative vorticity in the northern hemisphere of planet 1 after 2×10^4 Jovian days for the four shallow water models. The black line denotes the standard shallow water model, the green line the height-damped model, the blue line the momentum-conserving height-damped model and the red line the thermal model. These histograms approximate the relative vorticity probability density functions that would arise from an ensemble of simulations with different random seeds. Any breaks in the histograms are due to particular bins having no entries. The tails of the histogram for the thermal model extend further out than for the other models, demonstrating that this model attains more extreme values for the relative vorticity. The peak of the histogram for the thermal model is correspondingly lower and broader than for the other three models.

This difference between the thermal model and the other three models was quantified by computing the kurtosis of the relative vorticity field (McWilliams, 1984, Weiss and McWilliams, 1993)

$$\mathcal{K}_\omega(t) = \langle \omega^4 \rangle / \langle \omega^2 \rangle^2, \quad (3.3.3)$$

where $\langle \cdot \rangle$ again denotes an average over the entire simulation domain. The kurtosis of the relative vorticity field is a measure of intermittency, and hence of the formation of coherent structures. A spatially uniform field has a kurtosis of one, whereas a Gaussian random variable has a kurtosis of three. Figure 3.8 shows the evolution of the kurtosis of relative vorticity for all four simulations. The kurtosis for the three

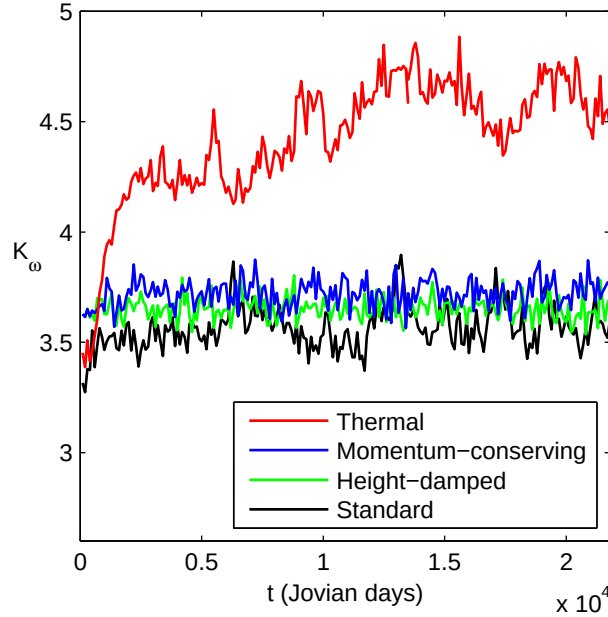


Figure 3.8: Evolution of the kurtosis of the dimensionless relative vorticity.

non-thermal simulations reaches a statistically steady state by around 1000 Jovian days, and then fluctuates between 3.4 and 3.8. By contrast, the kurtosis for the thermal model continues to grow for over 10^4 Jovian days, and show subsequent slow oscillations over thousands of days around a mean of ~ 4.6 . This difference from the other models quantifies the broader tails of the approximate probability distribution function for the relative vorticity field in the thermal simulation, as shown in figure 3.7.

Shallow water theory demonstrates an asymmetry between cyclonic and anticyclonic relative vorticity (Polvani et al., 1994, Scott and Polvani, 2007). This asymmetry is evaluated for the four different models by plotting the evolution of the mean relative vorticity $\langle \omega \rangle_N$ over the northern hemisphere of planet 1 in figure 3.9. The time-average $\langle \langle \omega \rangle_N \rangle$ is shown in table 3.3. The standard shallow water equations produce a net anticyclonic vorticity, while the other three models all produce a net cyclonic vorticity.

The direction of the equatorial jet is linked to the sign of the mean vorticity in the northern hemisphere (see Iacono, Struglia and Ronchi, 1999, Vallis, 2006). Applying Stokes' theorem to the northern hemisphere of planet 1 gives

$$\iint \omega \, dx \, dy = \int_E u \, dx + \int_N u \, dx, \quad (3.3.4)$$

where the integral on the left is an area integral over the northern hemisphere of planet 1, and the integrals on the right are line integrals around the equator and northern

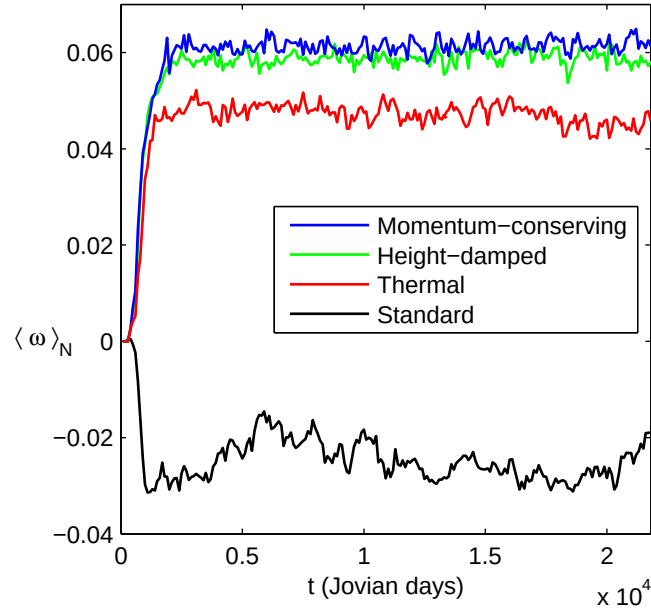


Figure 3.9: Evolution of the mean of the dimensionless relative vorticity.

boundary respectively. The periodicity of the domain has been used to cancel the left and right boundary integrals. The last integral would vanish on a sphere, for which the north pole is a point. The last integral is much smaller than the integral around the equator in these simulations since $\bar{u}$ at the northernmost points of the domain is much smaller than $\bar{u}$ at the equator (see figure 3.3). A northern hemisphere with net anticyclonic vorticity must thus exhibit a negative $\bar{u}$ at the equator and hence a retrograde equatorial jet, while net cyclonic vorticity implies a prograde equatorial jet. This consistency relation holds for all four shallow water models.

The cyclone/anticyclone asymmetry also applies to individual coherent structures (Polvani et al., 1994). Voyager and Cassini images show that 90% of the stable, long-lived, compact oval-shaped spots in Jupiter's atmosphere are anticyclonic (Li et al., 2004, Mac Low and Ingersoll, 1986). Cyclonic vorticity tends to occur in filamentary regions, rather than compact oval-shaped spots (Li et al., 2004, Mac Low and Ingersoll, 1986, Vasavada and Showman, 2005). This asymmetry is quantified using the skewness of the relative vorticity field (Iacono, Struglia and Ronchi, 1999, Scott and Polvani, 2007),

$$\mathcal{S}_\omega(t) = \langle (\omega - \langle \omega \rangle_N)^3 \rangle_N / \langle (\omega - \langle \omega \rangle_N)^2 \rangle_N^{3/2}, \quad (3.3.5)$$

where $\langle \cdot \rangle_N$ denotes the average of the quantity inside the angle brackets over the northern hemisphere of planet 1. Figure 3.10 shows the evolution of the skewness of the relative vorticity for the four different models. The skewness in all four models is

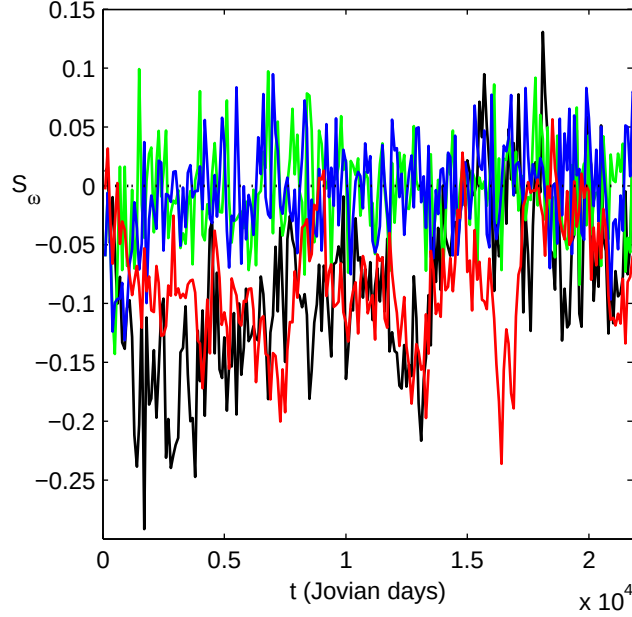


Figure 3.10: Evolution of the skew of the dimensionless relative vorticity. $\mathcal{S}_\omega$ for the two height-damped simulations (blue and green lines) oscillates around zero, while $\mathcal{S}_\omega$ for the standard shallow water (black line) and thermal (red line) simulations is distinctly negative.

close to zero. The skewnesses of the two height-damped models oscillate around zero, while the skewnesses of the standard shallow water and thermal models are more robustly negative. The latter two models thus exhibit more intense anticyclones than cyclones, in line with observations. However, the skewness time series show large fractional changes between simulations with different initial random seeds, as quantified in table 3.4.

3.3.1 Explanation for the close similarity between simulations of the two height-damped shallow water models

Numerical simulations of the two height-damped shallow water models produce remarkably similar results in the relative vorticity fields shown in figure 3.1b–c, the mean zonal velocities shown in figure 3.3b–c and figure 3.4b–c, and in the statistics shown in table 3.3 and table 3.4. To explain this similarity, both equation sets are rescaled for a local quasi-geostrophic regime (see chapter 4 for details of the local quasi-geostrophic limit). An f -plane is considered, where the Coriolis parameter is approximated by a constant value f_0 , and the scalings

$$\mathbf{x} = L\tilde{\mathbf{x}}, \quad \mathbf{u} = U\tilde{\mathbf{u}}, \quad t = (L/U)\tilde{t}, \quad h = h_0 \left(1 + \frac{f_0 UL}{g'h_0} \tilde{\eta} \right), \quad (3.3.6)$$

are introduced, where L and U are horizontal length and velocity scales respectively, h_0 is the mean layer depth, and g' is the reduced gravity. A superscript tilde denotes a dimensionless variable or operator. The momentum-conserving height-damped shallow water equations, corresponding to (1.3.11) with the Heaviside function omitted as in (3.2.2), become

$$Ro \left(\frac{\partial \tilde{\eta}}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\tilde{\eta} \tilde{\mathbf{u}}) \right) + Bu \tilde{\nabla} \cdot \tilde{\mathbf{u}} = -\kappa Ro \tilde{\eta}, \quad (3.3.7a)$$

$$Ro \left(\frac{\partial \tilde{\mathbf{u}}}{\partial \tilde{t}} + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) \tilde{\mathbf{u}} \right) + \hat{\mathbf{z}} \times \tilde{\mathbf{u}} = -\tilde{\nabla} \tilde{\eta} + Ro^2 \frac{\kappa}{Bu + Ro \tilde{\eta}} \tilde{\eta} \tilde{\mathbf{u}}, \quad (3.3.7b)$$

with parameters

$$Ro = \frac{U}{f_0 L}, \quad Bu = \frac{g' h_0}{f_0^2 L^2}, \quad \text{and} \quad \kappa = \frac{L}{U \tau_{\text{rad}}}. \quad (3.3.8)$$

The radiative damping term on the right-hand side of (3.3.7a) has been scaled to balance the time derivative on the left-hand side (cf. chapter 4 for the local quasi-geostrophic scaling of the radiatively damped thermal shallow water equations). This scaling preserves $\tilde{\nabla} \cdot \tilde{\mathbf{u}} = O(Ro)$, which enables the introduction of a geostrophic streamfunction, so $\tilde{\mathbf{u}} = \hat{\mathbf{z}} \times \tilde{\nabla} \tilde{\psi}$. Having chosen this scaling, the only difference between this momentum-conserving model and the rescaled version of the non-momentum-conserving Scott and Polvani (2007, 2008) model is the $O(Ro^2)$ last term that appears in (3.3.7b). The two models thus become identical in the local quasi-geostrophic approximation that retains only $O(Ro)$ corrections to geostrophic balance, thus explaining the close similarity between simulations of the two height-damped shallow water models.

The local quasi-geostrophic potential vorticity equation is

$$\frac{\partial \tilde{q}}{\partial \tilde{t}} + \hat{\mathbf{z}} \cdot (\tilde{\nabla} \tilde{\psi} \times \tilde{\nabla} \tilde{q}) = \frac{\kappa}{Bu} \tilde{\psi}, \quad \text{where} \quad \tilde{q} = \tilde{\nabla}^2 \tilde{\psi} - \frac{\tilde{\psi}}{Bu} \quad (3.3.9)$$

for both models. The radiative height-damping thus appears as a hypoviscous dissipation proportional to $\tilde{\psi}$ with coefficient $\kappa/Bu = f_0^2 L^3 / (g' h_0 U \tau_{\text{rad}})$ proportional to the square of the Coriolis parameter f_0 (see Scott and Polvani, 2007). Thus this dissipation coefficient is proportional to f^2 , so it becomes ineffective at low latitudes. This may explain why both these models produce a strong equatorial jet, with comparatively little activity at mid and high latitudes compared with the standard shallow water or thermal models. Radiative damping of the height equation thus reproduces a feature of the Rayleigh friction confined to latitudes outside 16.38° from the equator in the three-dimensional simulations of Schneider and Liu (2009). This friction

models the magnetohydrodynamic drag experienced by Taylor columns that extend into the electrically conducting inner region at depths below 96% of Jupiter’s radius, corresponding to pressures above 10^5 bars. Taylor columns near the equator cross into the other hemisphere without penetrating the deep interior, and so experience no drag in this model (see §1.3.6 for further details).

3.3.2 Energy spectra for Jovian simulations

The energy spectra for simulations of the four different shallow water models in the Jovian parameter regime are now investigated. Following Scott and Polvani (2008), narrow-band forcing was previously centred around dimensionless wavenumber 42. This led to realistic plots for the field variables, but the resulting inertial range was too small for anything meaningful to be said about the slope of the energy spectra. Thus further simulations with the forcing centred around dimensionless wavenumber 100 were run, where all the other parameters were left unchanged.

In the absence of rotation, numerical simulations of forced two-dimensional turbulence typically produce energy spectra that follow the classic Kolmogorov scaling $E(k) \propto k^{-5/3}$ in the inertial range, though in two dimensions the inertial range arises from an inverse cascade of energy from the injection scale to larger scales. Analyses of power spectra reconstructed from Cassini observations of Jupiter are consistent with an inverse cascade within Jupiter’s atmosphere (Choi and Showman, 2011). Coupling to Rossby waves produces a much steeper slope $E(k_x, k_y) \propto k_y^{-5}$ when the zonal wavenumber k_x is small, and the usual scaling $E(k) \propto k^{-5/3}$ elsewhere (see §1.3 and Huang et al., 2001). The k_y^{-5} scaling arises from equating the fluid velocity scale $(2E(k_x, k_y)k_y)^{1/2}$ at wavenumber k_y with the Rossby wave phase speed $c_p = -\beta/k_y^2$. This steeper spectrum appears most readily when large-scale friction is weak (Huang et al., 2001).

The kinetic energy density $\frac{1}{2}h|\mathbf{u}|^2$ for the shallow water equations is a cubic expression in the field variables h and $\mathbf{u}$. Replacing h by its mean h_0 gives a quadratic approximation whose spectrum may be computed using standard Fourier techniques based on Parseval’s theorem (Farge and Sadourny, 1989, Warn, 1986). This approximation differs from the true kinetic energy by just over 4% for the thermal simulations at 2×10^4 Jovian days, and by less than 1% for the other three simulations. The higher error in the thermal simulations could be due to the presence of an additional force-compensated wave mode in the thermal shallow water equations (see §4.3 and Ripa, 1996a). Compensating fluctuations in temperature and height in this mode leave the pressure unchanged, so near geostrophic balance imposes a weaker constraint on the

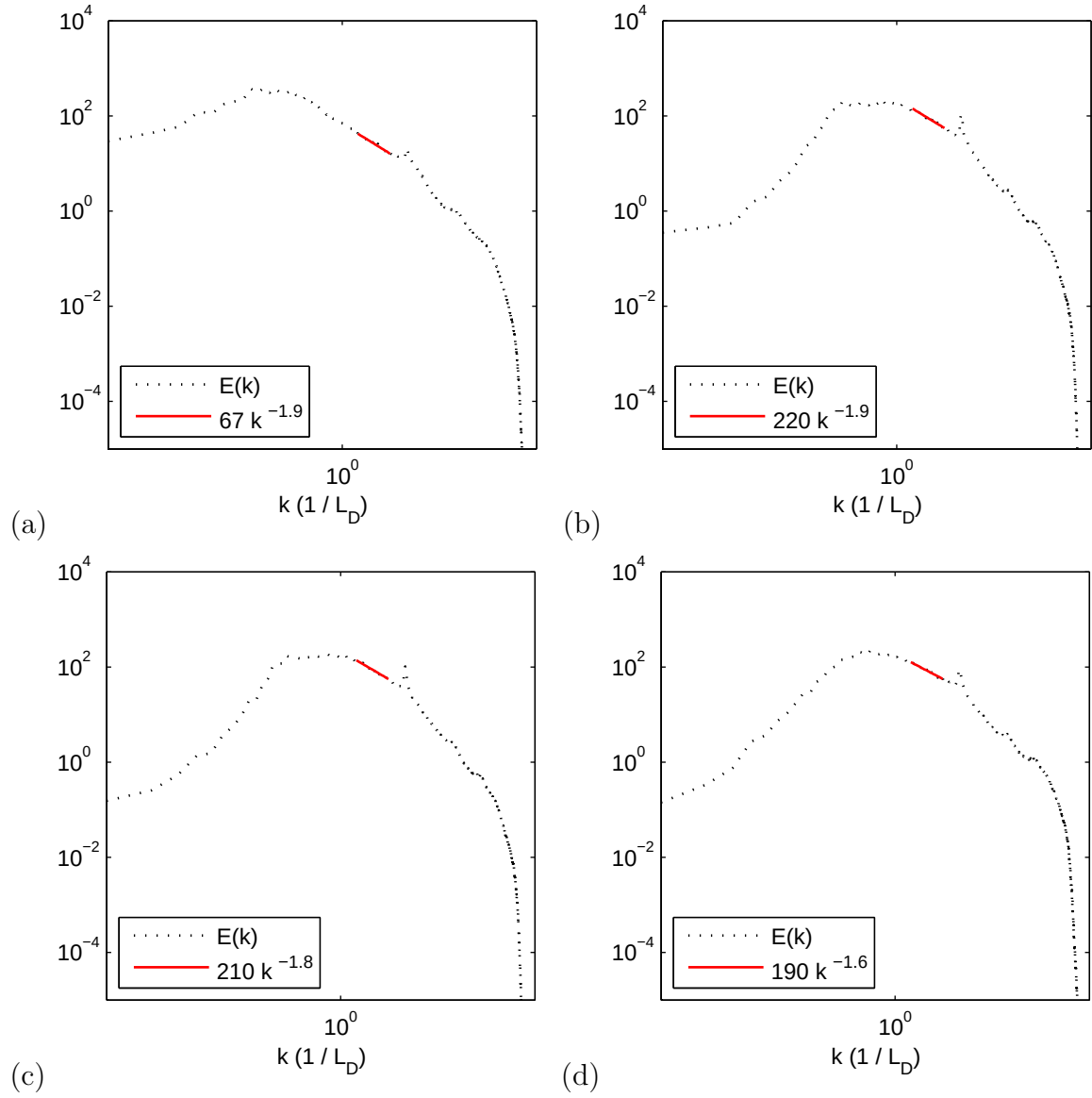


Figure 3.11: Non-zonal energy spectra for (a) the standard shallow water model, (b) the height-damped model, (c) the momentum-conserving height-damped model, and (d) the thermal model. Best fit lines to the inertial range are shown by red lines.

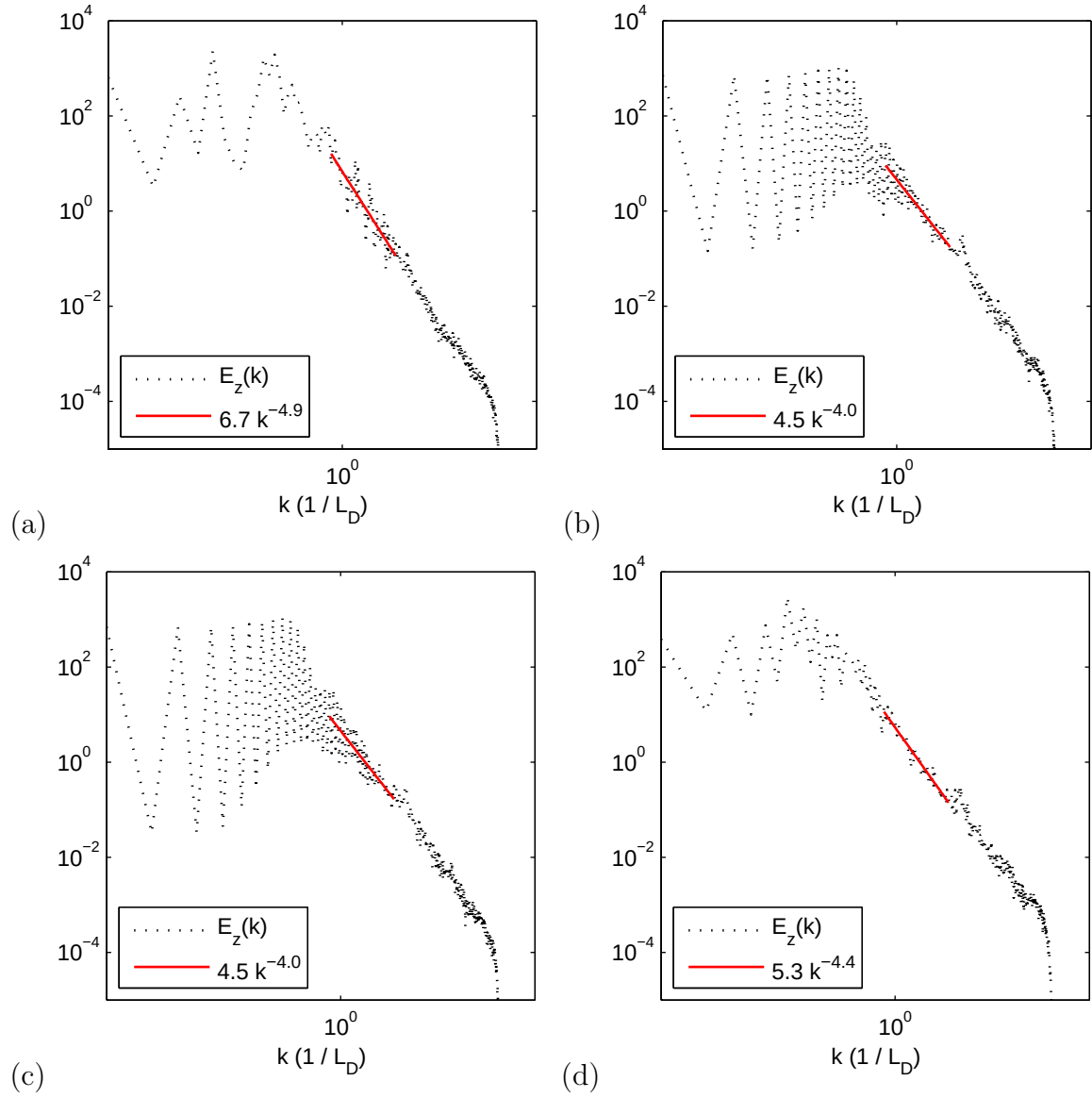


Figure 3.12: Zonal energy spectra for (a) the standard shallow water model, (b) the height-damped model, (c) the momentum-conserving height-damped model, and (d) the thermal model. Best fit lines to the inertial range are shown by red lines.

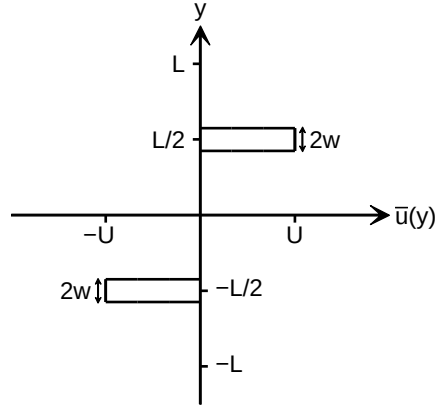


Figure 3.13: A step function $\bar{u}(y)$ of period $2L$ which approximates a zonal flow with two equatorial jets in an otherwise quiescent fluid for the square planet domain sketched in figure 1.7.

height fluctuations in these equations than in the three non-thermal models. An average forcing strength $\langle\langle\epsilon_f\rangle\rangle$ was calculated over the period from 2×10^4 to 2.18×10^4 Jovian days. The saved simulation output at 2×10^4 days was then used to initialise a new simulation. This simulation was then run with a constant forcing amplitude equal to $\langle\langle\epsilon_f\rangle\rangle$, and finally an averaged energy spectrum was calculated using data from each time step between 2×10^4 and 2.1×10^4 Jovian days.

Figure 3.11 shows the non-zonal energy spectra $E(k)$ for the four different shallow water models, where “non-zonal” excludes contributions from $k_x = 0$, calculated over the entire simulation domain. All four plots show a peak around the injection wavenumber $L_D k = 200\pi/232 \approx 2.71$. Least-squares best fit lines for the inertial ranges are also shown, each of which has a slope roughly consistent with the Kolmogorov scaling $E(k) \propto k^{-5/3}$.

Figure 3.12 shows the corresponding zonal energy spectra $E_z(k)$ comprising contributions from $k_x = 0$ (so $k = k_y$) and least-square fits to their inertial ranges. These spectra are much steeper than the Kolmogorov scaling, with slopes ranging between -4.9 and -4.0 . However, they are less steep than the theoretical -5 power law described above. This is perhaps due to the presence of significant friction in these models, which damps the large-scale energy. Galperin et al. (2001) fitted slopes of -4.7 ± 1.3 over a short range of wavenumbers to zonal energy spectra obtained from Voyager observations of Jupiter. The slopes of the zonal energy spectra from all four shallow water models lie within these observational error bars.

The zonal energy spectra shown in figure 3.12 exhibit pronounced zig-zag patterns between odd and even wavenumbers, most noticeably for the two height-damped

shallow water models. To elucidate this feature consider a zonal flow $\bar{u}(y)$ with two top-hat equatorial jets, each of width $2w$, in otherwise quiescent fluid as sketched in figure 3.13. This crudely approximates the zonal flows shown in figure 3.3. The cosine modes in the Fourier series of $\bar{u}(y)$ all have vanishing coefficients, because $\bar{u}(y)$ is an odd function of y . This explains the zig-zag patterns in figure 3.12, which are due to even modes having significantly smaller coefficients than odd modes.

3.4 Neptunian simulations

Cho and Polvani (1996*b*) and Scott and Polvani (2007) performed simulations of the standard rotating shallow water equations in parameter regimes relevant for Uranus and Neptune. They found velocities that were comparable to those of Uranus and Neptune, and a broad, strong retrograde equatorial jet, as observed on Neptune (see figure 1.1*b*). In this section, simulations of the radiatively damped thermal shallow water model are presented for Neptunian parameter values, as defined in tables 3.1 and 3.2. Neptune is a smaller and colder planet than Jupiter, with only 53 deformation radii around its circumference, compared with 232 for Jupiter. The radiative time scale is 396 Neptunian days, compared with 43 Jovian days for Jupiter at the same 25 millibar pressure level. Given the smaller ratio between the domain size and the deformation radius, the forcing was centred at wavenumber $k_c = 10$ (rather than wavenumber 42, as used in the Jovian simulations). $\mathcal{E}_\infty$ was set as 2000, or about 0.71 per unit area, giving a velocity scale $u_{\text{RMS}} \approx 1.19 L_D/T$, so the actual Rossby number is $u_{\text{RMS}}/(2\Omega L_D) \approx 0.095$. A larger $\mathcal{E}_\infty$ would have given larger zonal velocities closer to the observations shown in figure 1.1*b*, but the corresponding larger Rossby number implies large free-surface deformations that led to unstable simulations. The force-compensated mode allows free-surface deformations in a thermal shallow water simulation to be larger than those in an equivalent standard shallow water simulation with the same Rossby number (see §3.3.2). As before, t_{ramp} was set as 1000 planetary days, while t_{adj} was set as 50 planetary days.

These simulations revealed another difficulty that was not encountered in the Jovian simulations. The zonal momentum equation for the radiatively damped thermal shallow water equations, without forcing or friction, may be written in the conservation form

$$\frac{\partial}{\partial t} \left(h(u + R(y)) \right) + \frac{\partial}{\partial x} \left(hu(u + R(y)) + \frac{1}{2Ro^2} \Theta h^2 \right) + \frac{\partial}{\partial y} \left(hv(u + R(y)) \right) = 0, \quad (3.4.1)$$

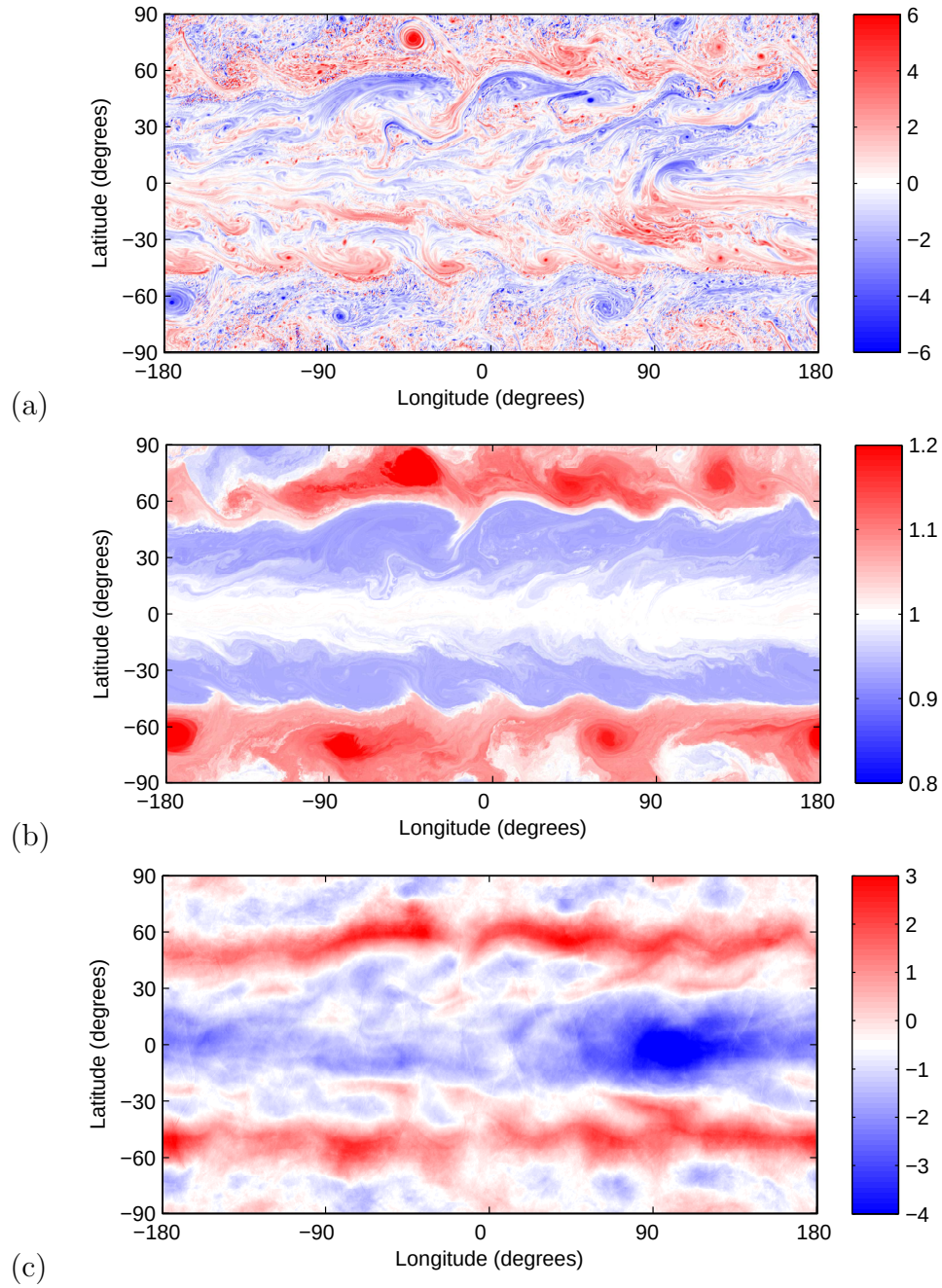


Figure 3.14: Dimensionless (a) relative vorticity, (b) temperature, and (c) zonal velocity from a Neptunian thermal simulation after 10^4 Neptunian days.

where $R(y)$ is the zonal component of the scaled vector potential for the latitude-dependent Coriolis parameter,

$$R(y) = -\frac{1}{Ro} \int_0^y f(y') dy'. \quad (3.4.2)$$

The total zonal momentum over the entire simulation domain thus changes only through forcing and friction, but since this domain contains two planets, the integrated zonal momentum over each individual planet may drift. However, the total momentum over planet 1 had changed by less than 2% after 2×10^4 days in the Jovian simulations. The drifts are larger in the Neptunian simulations, presumably because they contain much larger coherent structures (compare figure 3.14 to figures 3.1 and 3.2) that transfer significant zonal momentum when they cross the polar lines dividing the two planets. These drifts were counteracted by adding a drag term

$$-\frac{1}{\tau_u h} \left\{ \langle h(u + R(y)) \rangle_{1,2} - \langle R(y) \rangle_{1,2} \right\} \quad (3.4.3)$$

to the right-hand side of the zonal momentum equation (3.2.3c). The notation $\langle \cdot \rangle_{1,2}$ indicates an average over planet 1 when evaluated at points in planet 1, and an average over planet 2 when evaluated at points in planet 2. This term thus causes the total zonal momentum over each of planets 1 and 2 to relax towards its initial value (recall that $h = 1$ and $u = 0$ are the initial conditions). The relaxation time τ_u was set to be 200 Neptunian days.

Figure 3.14 shows the relative vorticity, temperature, and zonal velocity fields after 10^4 Neptunian days for planet 1 in the simulation domain. The zonal velocity field is far less zonally coherent than the equivalent field for Jupiter shown in figure 3.2b. The correlation between temperature fluctuations and relative vorticity fluctuations is 0.36 for the northern hemisphere at 10^4 Neptunian days for this simulation. However, two otherwise identical simulations initialised with different seeds for the random number generator gave correlations of 0.11 and -0.15 . Thus there appears to be no robust correlation between temperature fluctuations and relative vorticity fluctuations of the kind that exists in the Jovian thermal simulations (see page 73).

Figure 3.15 shows the instantaneous mean zonal wind $\bar{u}$ in units of Neptunian deformation radii per Neptunian day, with blue dashed lines showing one standard deviation either side of the mean. The strong retrograde equatorial jet, and two mid-latitude prograde jets, are consistent with observations of Neptune. The large deviations from the zonal mean are consistent with the large error bars and substantial deviations between Hubble Space Telescope and Voyager 2 observations shown in

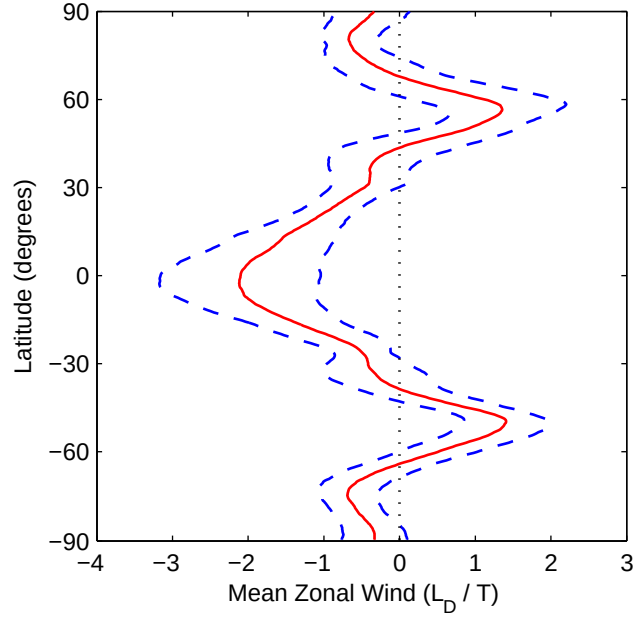


Figure 3.15: Mean zonal wind against latitude after 10^4 Neptunian days from a Neptunian thermal simulation. The blue dashed lines show one standard deviation either side of the zonal mean.

figure 1.1b, and with the large oscillations in zonal velocities found in more recent Keck Telescope observations (see Martin et al., 2012). However, the maximum zonal wind speed at the equator is significantly weaker than the observed value. Speeds of around two deformation radii per day are obtained, while the observations plotted in figure 1.1b are closer to eight deformation radii per day. The same discrepancy has arisen in previous three-dimensional general circulation model simulations of Neptune (e.g. see Liu and Schneider, 2010), although previous shallow water simulations were able to obtain realistic velocities (e.g. Cho and Polvani, 1996*b*, Scott and Polvani, 2007). Figure 3.16 shows the evolution of the mean zonal wind as a function of latitude and time for the Neptunian thermal shallow water simulation. This figure demonstrates the robustness of the retrograde equatorial jet throughout the simulation, while the two prograde mid-latitude jets drift towards higher latitudes before reforming at $\sim \pm 40^\circ$ latitude.

Table 3.5: Average diagnostic quantities in the statistically steady state for two sets of the Neptunian simulations in §3.4 with different initial random seeds. The error estimates quantify the differences between the diagnostics for each pair of simulations.

$\langle\langle\epsilon_f\rangle\rangle$	$\langle\langle\xi\rangle\rangle$	$\langle\langle\omega_N\rangle\rangle$	$\langle\langle\mathcal{K}_u\rangle\rangle$	$\langle\langle\mathcal{K}_\omega\rangle\rangle$	$\langle\langle\mathcal{S}_\omega\rangle\rangle$
4600 ± 1	0.65 ± 0.03	-0.16 ± 0.05	4.75 ± 0.03	5.45 ± 0.5	0.19 ± 0.03

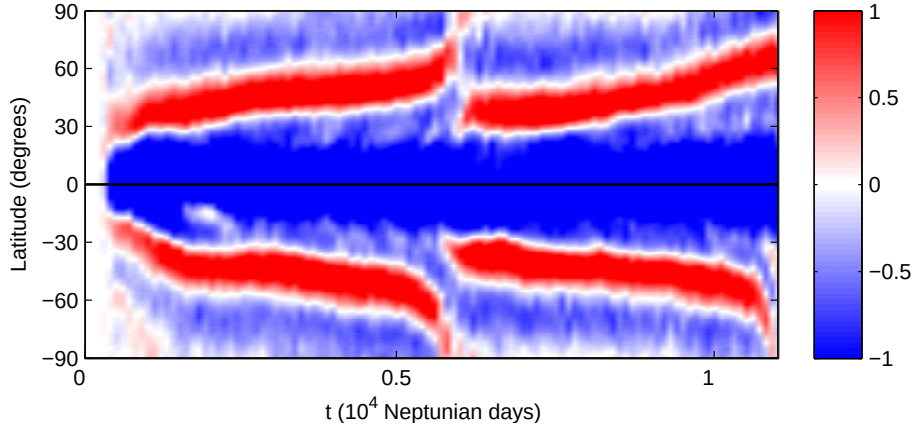


Figure 3.16: Evolution of the mean zonal wind from a Neptunian thermal simulation.

Table 3.5 shows a number of averaged diagnostic quantities computed from the Neptunian simulations, with $\langle\langle\cdot\rangle\rangle$ denoting the time-average of a quantity over the period between 10^4 and 1.1×10^4 Neptunian days. The two sets of simulations were identical except for being initialised with different random seeds. The mean and half differences are presented to give an indication of the repeatability of the different diagnostics.

Figure 3.17a shows the evolution of the kurtosis of the pointwise zonal velocity field, $\mathcal{K}_u$, as defined in (3.3.1). The kurtosis begins near three, as for a Gaussian random variable, increases due to the formation of coherent structures, and levels off to an average value of 4.75 as the flow reaches a statistical steady state (this average is lower than that for the Jovian thermal simulation in §3.3, see table 3.4). Figure 3.17b shows the fraction of kinetic energy in the zonally averaged flow, ξ , as defined by (3.3.2). The fraction ξ is larger for the Neptunian thermal simulation than the Jovian thermal simulation in §3.3, if the dip in ξ around 6000 Neptunian days is excluded, demonstrating that the zonal mean flow is generally more dominant in the Neptunian thermal simulation compared to the Jovian. The large dip in ξ near 6000 Neptunian days corresponds with the two prograde mid-latitude jets reforming at $\sim \pm 40^\circ$ after having previously drifted to higher latitudes, as illustrated in figure 3.16. The average forcing amplitude, $\langle\langle\epsilon_f\rangle\rangle$, is over twice as large for the Neptunian thermal simulation compared to the Jovian (compare table 3.3 with table 3.5).

Figure 3.17c shows a histogram of the relative vorticity in the northern hemisphere of planet 1 after 10^4 Neptunian days for the Neptunian thermal simulation. The tails of the histogram extend further out than for the Jovian thermal simulation in figure 3.7, demonstrating that the Neptunian simulation attains more extreme values for the relative vorticity. The differences between the Neptunian and Jovian histograms

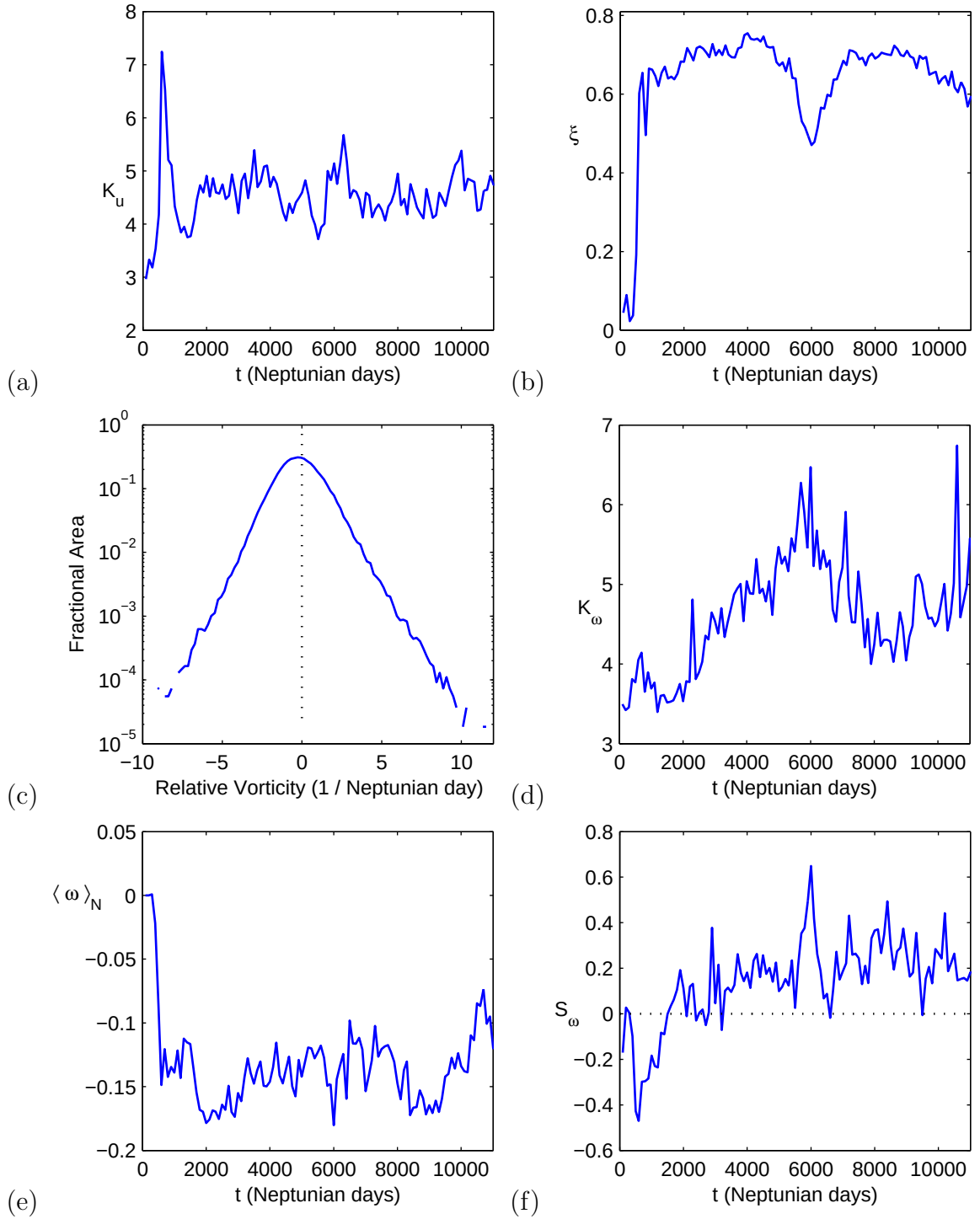


Figure 3.17: Evolution of (a) the kurtosis of the zonal velocity, and (b) the fraction of kinetic energy in the zonally averaged flow; (c) a histogram of the relative vorticity at 10^4 Neptunian days; evolution of (d) the kurtosis of the relative vorticity, (e) the mean of the relative vorticity, and (f) the skew of the relative vorticity.

of relative vorticity can be further quantified by computing the kurtosis of the relative vorticity field, $\mathcal{K}_\omega$, as defined in (3.3.3). The evolution of $\mathcal{K}_\omega$ is shown in figure 3.17d for the Neptunian thermal simulation, and although subject to greater variability than its Jovian thermal counterpart in figure 3.8, $\mathcal{K}_\omega$ is generally larger for the Neptunian thermal simulation. The larger average value for $\langle\langle\mathcal{K}_\omega\rangle\rangle$ in table 3.5 compared to table 3.4 quantifies the broader tails of the approximate probability distribution function for the Neptunian relative vorticity field.

Figure 3.17e shows the evolution of the mean relative vorticity, $\langle\omega\rangle_N$, over the northern hemisphere of planet 1 for the Neptunian thermal simulation, while the time average $\langle\langle\omega\rangle_N\rangle$ is given in table 3.5. The Neptunian thermal simulation produces a net anticyclonic vorticity, with average $\langle\langle\omega\rangle_N\rangle = -0.16$, while the Jovian thermal simulation in §3.3 produced a net cyclonic vorticity with average $\langle\langle\omega\rangle_N\rangle = +0.046$. Figure 3.17f depicts the evolution of the skewness of the relative vorticity field, $\mathcal{S}_\omega$, as defined in (3.3.5). The Neptunian thermal simulation produced a positive average skewness of $\langle\langle\mathcal{S}_\omega\rangle\rangle = 0.19$, while the Jovian thermal simulation in §3.3 produced a negative average skewness of $\langle\langle\mathcal{S}_\omega\rangle\rangle = -0.053$. Thus while the Jovian thermal simulation demonstrated more intense anticyclones than cyclones, the reverse is true for the Neptunian thermal simulation.

3.5 Sensitivity of the results to the choice of the radiative time scale

This section investigates the sensitivity of the results in §3.3 and §3.4 to the value used for the radiative time scale τ_{rad} .

3.5.1 Jovian simulations

Consider a layer that extends downwards from the 100 millibar pressure level, as opposed to the 25 millibar level considered in §3.3. From Seiff et al. (1998), $\mathcal{T} \approx 100$ K at the 100 millibar level for Jupiter. From §3.1, $c_p = 1.26 \times 10^4 \text{ J kg}^{-1} \text{ K}^{-1}$ is the specific heat for a hydrogen atmosphere in the Jovian temperature range, $g = 26 \text{ m s}^{-2}$ for Jupiter, while $\sigma = 5.670 \times 10^{-8} \text{ J m}^{-2} \text{ s}^{-1} \text{ K}^{-4}$. Thus the radiative time scale for Jupiter calculated at the 100 millibar pressure level is 2968 hours, or 300 Jovian days, as calculated from (3.1.1). This section considers simulations of the height-damped shallow water equations (3.2.1), the momentum-conserving height-damped shallow water equations (3.2.2), and the radiatively damped thermal shallow water equations (3.2.3). Simulations are performed exactly as detailed in §3.2, the only difference

Table 3.6: Average diagnostic quantities in the statistically steady state for two sets of Jovian height-damped shallow water simulations with different initial random seeds, for three different values of the radiative time scale τ_{rad} . The error estimates quantify the differences between the diagnostics for each pair of simulations, where $\langle\langle\cdot\rangle\rangle$ denotes the time-average of a quantity between 2×10^4 and 2.18×10^4 Jovian days.

	$\langle\langle\epsilon_f\rangle\rangle$	$\langle\langle\xi\rangle\rangle$	$\langle\langle\omega_N\rangle\rangle$	$\langle\langle\mathcal{K}_u\rangle\rangle$	$\langle\langle\mathcal{K}_\omega\rangle\rangle$	$\langle\langle\mathcal{S}_\omega\rangle\rangle$
$\tau_{\text{rad}} = 62 \text{ h}$	5000 ± 9	0.47 ± 0.003	0.056 ± 0.0004	14.47 ± 0.08	5.76 ± 0.03	0.098 ± 0.0009
$\tau_{\text{rad}} = 429 \text{ h}$	2900 ± 3	0.51 ± 0.002	0.059 ± 0.005	12.17 ± 0.09	3.64 ± 0.003	-0.0014 ± 0.004
$\tau_{\text{rad}} = 2968 \text{ h}$	2100 ± 32	0.47 ± 0.03	-0.036 ± 0.001	6.84 ± 0.2	3.46 ± 0.005	-0.13 ± 0.006

being that three different values of the radiative time scale are considered ($\tau_{\text{rad}} = 2968$ hours, as calculated at the 100 millibar level; $\tau_{\text{rad}} = 429$ hours, as calculated at the 25 millibar level; and an even shorter radiative time scale of $\tau_{\text{rad}} = 62$ hours, which corresponds to the 4 millibar pressure level if (3.1.1) is used with c_p , g and $\mathcal{T}$ taken to be the Jovian values used in §3.1 for the 25 millibar pressure level).

Figures 3.18–3.20 show the evolution of the mean zonal wind as a function of latitude and time for a range of values for τ_{rad} for the three different shallow water models. All three shallow water models demonstrate a switch from a super-rotating equatorial jet for $\tau_{\text{rad}} = 62$ hours and $\tau_{\text{rad}} = 429$ hours, to a sub-rotating equatorial jet for $\tau_{\text{rad}} = 2968$ hours. The two height-damped shallow water simulations produce very similar plots for the mean zonal wind (compare figure 3.18 to figure 3.19, and see the explanation in §3.3.1). The two height-damped models demonstrate stronger mid-latitude jets as τ_{rad} increases. This is caused by the radiative height-damping appearing as a hypoviscous dissipation (with coefficient proportional to $1/\tau_{\text{rad}}$) in the local quasi-geostrophic potential vorticity equation (3.3.9) associated with these height-damped shallow water models. Thus as τ_{rad} increases, the strength of the dissipation decreases.

The correlation between temperature fluctuations and relative vorticity fluctuations is positive at 2×10^4 Jovian days for the thermal simulations with $\tau_{\text{rad}} = 62$ hours (three identical simulations initialised with different seeds for the random number generator gave values of 0.40, 0.41 and 0.42), $\tau_{\text{rad}} = 429$ hours (three different seeds gave values of 0.29, 0.32 and 0.33) and $\tau_{\text{rad}} = 2968$ hours (three different seeds all gave a value of 0.29). This suggests that cyclonic regions are typically warmer than anticyclonic regions for all these Jovian thermal simulations.

In tables 3.6–3.8, average diagnostic quantities are recorded for the three radiatively damped shallow water models for three different values of τ_{rad} , where $\langle\langle\cdot\rangle\rangle$ denotes

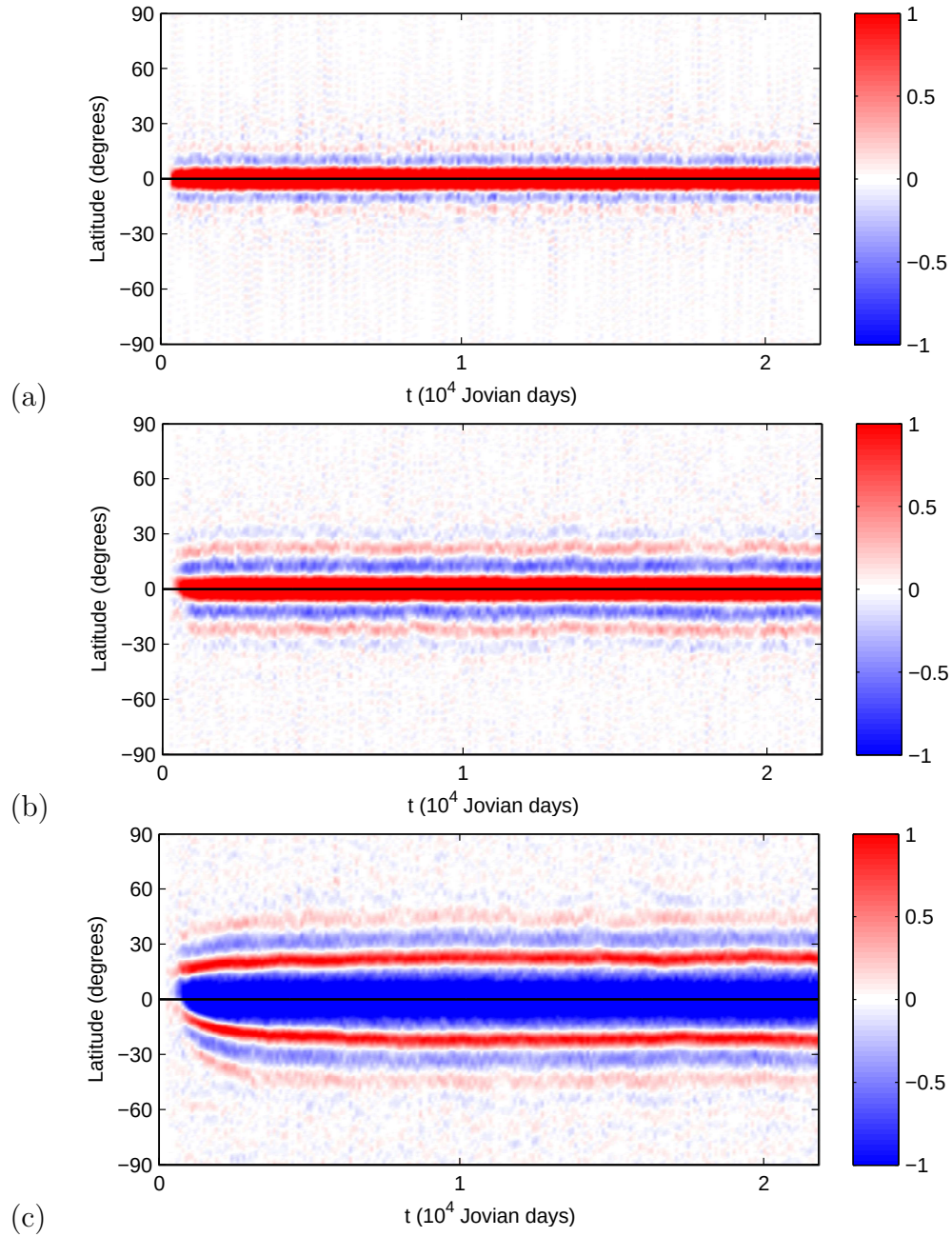


Figure 3.18: Evolution of the mean zonal wind for Jovian height-damped shallow water simulations with (a) $\tau_{\text{rad}} = 62$ hours, (b) $\tau_{\text{rad}} = 429$ hours and (c) $\tau_{\text{rad}} = 2968$ hours.

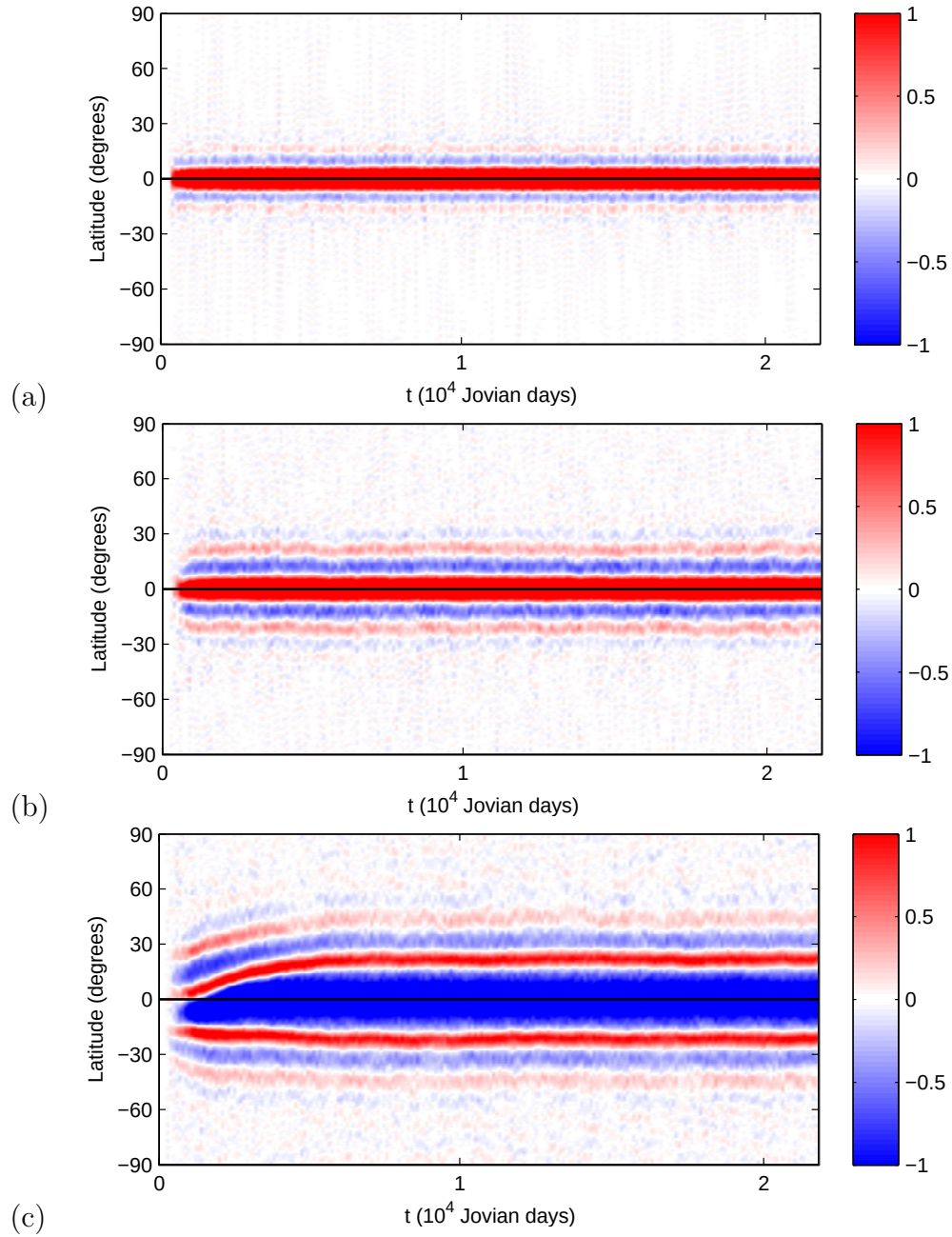


Figure 3.19: Evolution of the mean zonal wind for Jovian momentum-conserving height-damped shallow water simulations with (a) $\tau_{\text{rad}} = 62$ hours, (b) $\tau_{\text{rad}} = 429$ hours and (c) $\tau_{\text{rad}} = 2968$ hours.

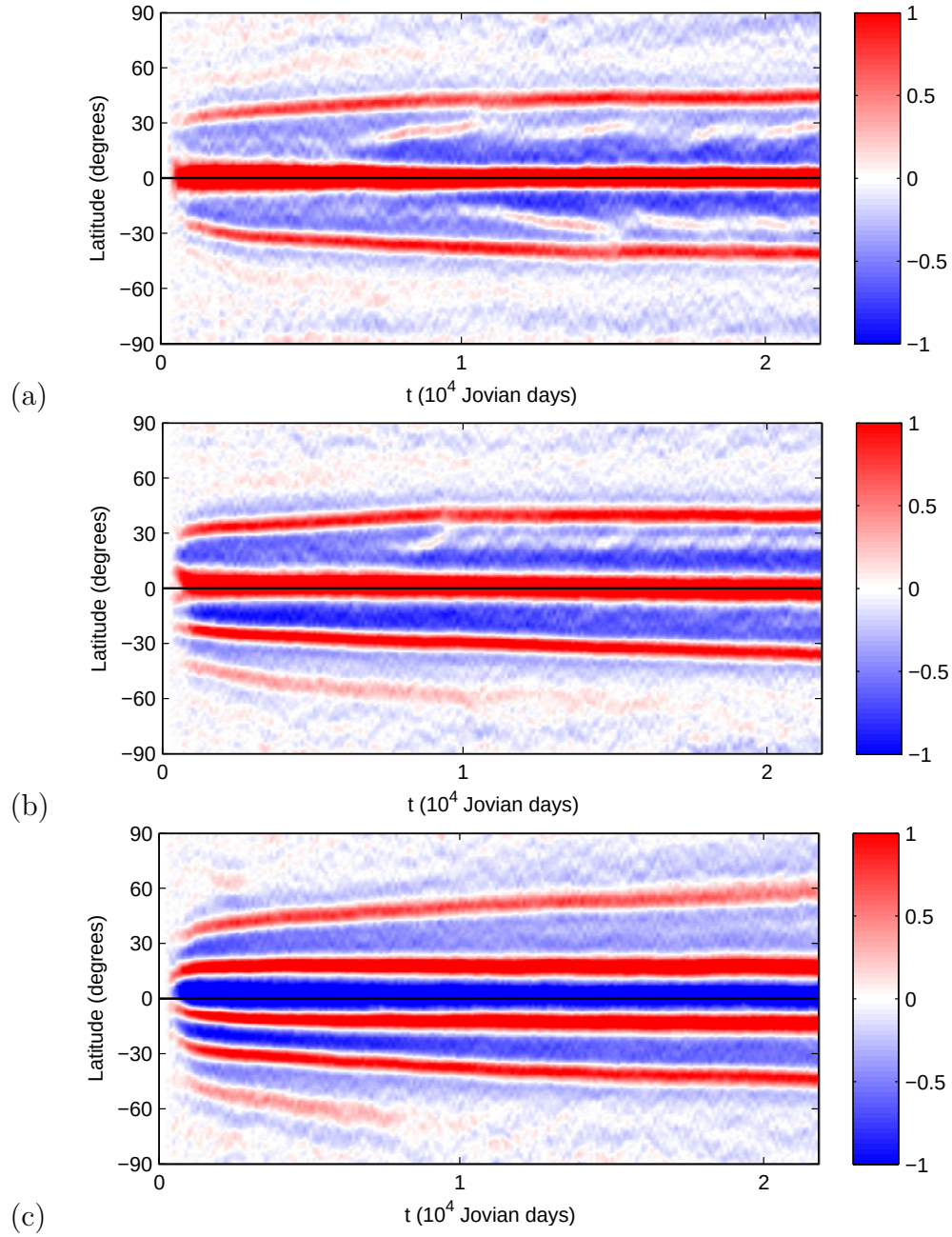


Figure 3.20: Evolution of the mean zonal wind for Jovian thermal shallow water simulations with (a) $\tau_{\text{rad}} = 62$ hours, (b) $\tau_{\text{rad}} = 429$ hours and (c) $\tau_{\text{rad}} = 2968$ hours.

Table 3.7: Average diagnostic quantities in the statistically steady state for two sets of Jovian momentum-conserving height-damped shallow water simulations with different initial random seeds, for three different values of the radiative time scale τ_{rad} . The error estimates quantify the differences between the diagnostics for each pair of simulations, where $\langle\langle\cdot\rangle\rangle$ denotes the time-average of a quantity between 2×10^4 and 2.18×10^4 Jovian days.

	$\langle\langle\epsilon_f\rangle\rangle$	$\langle\langle\xi\rangle\rangle$	$\langle\langle\omega_N\rangle\rangle$	$\langle\langle\mathcal{K}_u\rangle\rangle$	$\langle\langle\mathcal{K}_\omega\rangle\rangle$	$\langle\langle\mathcal{S}_\omega\rangle\rangle$
$\tau_{\text{rad}} = 62 \text{ h}$	3800 ± 24	0.54 ± 0.004	0.059 ± 0.0002	18.91 ± 0.1	7.01 ± 0.04	0.074 ± 0.008
$\tau_{\text{rad}} = 429 \text{ h}$	2800 ± 1	0.54 ± 0.0007	0.062 ± 0.0004	13.03 ± 0.07	3.73 ± 0.009	0.0007 ± 0.0009
$\tau_{\text{rad}} = 2968 \text{ h}$	2100 ± 60	0.47 ± 0.04	-0.036 ± 0.00008	6.69 ± 0.2	3.50 ± 0.01	-0.16 ± 0.001

Table 3.8: Average diagnostic quantities in the statistically steady state for two sets of Jovian thermal shallow water simulations with different initial random seeds, for three different values of the radiative time scale τ_{rad} . The error estimates quantify the differences between the diagnostics for each pair of simulations, where $\langle\langle\cdot\rangle\rangle$ denotes the time-average of a quantity between 2×10^4 and 2.18×10^4 Jovian days.

	$\langle\langle\epsilon_f\rangle\rangle$	$\langle\langle\xi\rangle\rangle$	$\langle\langle\omega_N\rangle\rangle$	$\langle\langle\mathcal{K}_u\rangle\rangle$	$\langle\langle\mathcal{K}_\omega\rangle\rangle$	$\langle\langle\mathcal{S}_\omega\rangle\rangle$
$\tau_{\text{rad}} = 62 \text{ h}$	2800 ± 35	0.49 ± 0.005	0.056 ± 0.002	8.55 ± 0.002	5.88 ± 0.1	0.069 ± 0.003
$\tau_{\text{rad}} = 429 \text{ h}$	2200 ± 21	0.49 ± 0.006	0.046 ± 0.0007	6.43 ± 0.08	4.59 ± 0.0008	-0.053 ± 0.02
$\tau_{\text{rad}} = 2968 \text{ h}$	1800 ± 30	0.52 ± 0.01	-0.022 ± 0.002	4.58 ± 0.02	4.47 ± 0.02	0.081 ± 0.02

the time average of a quantity between 2×10^4 and 2.18×10^4 Jovian days. Each simulation is run twice, with two different initial random seeds for the random number generator, and the error estimates quantify the differences between the diagnostics for each pair of simulations. The first diagnostic, the average forcing amplitude $\langle\langle\epsilon_f\rangle\rangle$, is found to decrease as τ_{rad} increases for the three radiatively damped shallow water models. This demonstrates that as the dissipation (proportional to $1/\tau_{\text{rad}}$) decreases, the simulation requires a smaller forcing strength to reach the same desired total kinetic energy. The second diagnostic is the average fraction of kinetic energy in the zonally averaged flow $\langle\langle\xi\rangle\rangle$, as defined by (3.3.2), which is not found to significantly alter as τ_{rad} varies for all three shallow water models. The third diagnostic is the mean of the relative vorticity over the northern hemisphere of the top planet $\langle\langle\omega_N\rangle\rangle$. The average is found to switch from positive (for $\tau_{\text{rad}} = 62$ hours and $\tau_{\text{rad}} = 429$ hours) to negative (for $\tau_{\text{rad}} = 2968$ hours) for all three shallow water models, representing the switch in equatorial jet direction as τ_{rad} increases (see the explanation for this on page 82). The fourth and fifth diagnostics are the average kurtosis of u ($\langle\langle\mathcal{K}_u\rangle\rangle$) and ω ($\langle\langle\mathcal{K}_\omega\rangle\rangle$), as defined by (3.3.1) and (3.3.3) respectively. Both these average kurtoses decrease as τ_{rad} increases for all three shallow water models. Finally the

sixth diagnostic is the average skew of ω over the northern hemisphere of planet 1 $\langle\langle\mathcal{S}_\omega\rangle\rangle$, as defined by (3.3.5). The average is found to switch from positive to negative as τ_{rad} increases for the two height-damped models, however there does not appear to be any discernible pattern for the thermal model as τ_{rad} varies.

Jovian simulations using the value of τ_{rad} calculated at the 100 millibar pressure level demonstrate a sub-rotating equatorial jet. However, simulations using the value of τ_{rad} calculated at the 25 millibar pressure level (and simulations using an even smaller radiative time scale) demonstrate a super-rotating equatorial jet. This same switch from a super- to a sub-rotating equatorial jet as τ_{rad} is increased is detailed for further simulation runs in figure 6.1 for the height-damped shallow water equations and the radiatively damped thermal shallow water equations. In conclusion, these simulation runs demonstrate that all three radiatively damped shallow water models only produce the correct equatorial jet direction for Jupiter (i.e. a super-rotating equatorial jet) if the radiative time scale is sufficiently (and perhaps unrealistically) short. However, the theory in chapter 6 suggests that lengthening the frictional time scale beyond 1000 Jovian days (or 9900 hours) would increase the tendency to form a super-rotating equatorial jet for longer radiative time scales, but running simulations until they reach statistical steady states then becomes prohibitively expensive.

3.5.2 Neptunian simulations

Again consider a layer that extends downwards from the 100 millibar pressure level, as opposed to the 25 millibar pressure level considered in §3.4. From Hammel and Lockwood (2007), $\mathcal{T} \approx 50$ K at the 100 millibar level. Using the Neptunian values for c_p and g given in §3.1 gives a radiative time scale of 48991 hours at the 100 millibar pressure level using (3.1.1), or 2737 Neptunian days. This section compares the Neptunian simulations from §3.4, where $\tau_{\text{rad}} = 7088$ hours, to Neptunian radiatively damped thermal shallow water simulations with $\tau_{\text{rad}} = 48991$ hours.

Figure 3.21 shows the evolution of the mean zonal wind as a function of latitude and time for Neptunian simulations of the radiatively damped thermal shallow water equations (3.2.3) with (a) $\tau_{\text{rad}} = 7088$ hours and (b) $\tau_{\text{rad}} = 48991$ hours. Both plots demonstrate a robust sub-rotating equatorial jet throughout the simulations, and two prograde mid-latitude jets which drift towards higher latitudes before reforming at $\sim \pm 40^\circ$ latitude.

If the correlation between temperature fluctuations and relative vorticity fluctuations is considered for the northern hemisphere at 10^4 Neptunian days for $\tau_{\text{rad}} = 48991$ hours, then three identical simulations initialised with different seeds for the random

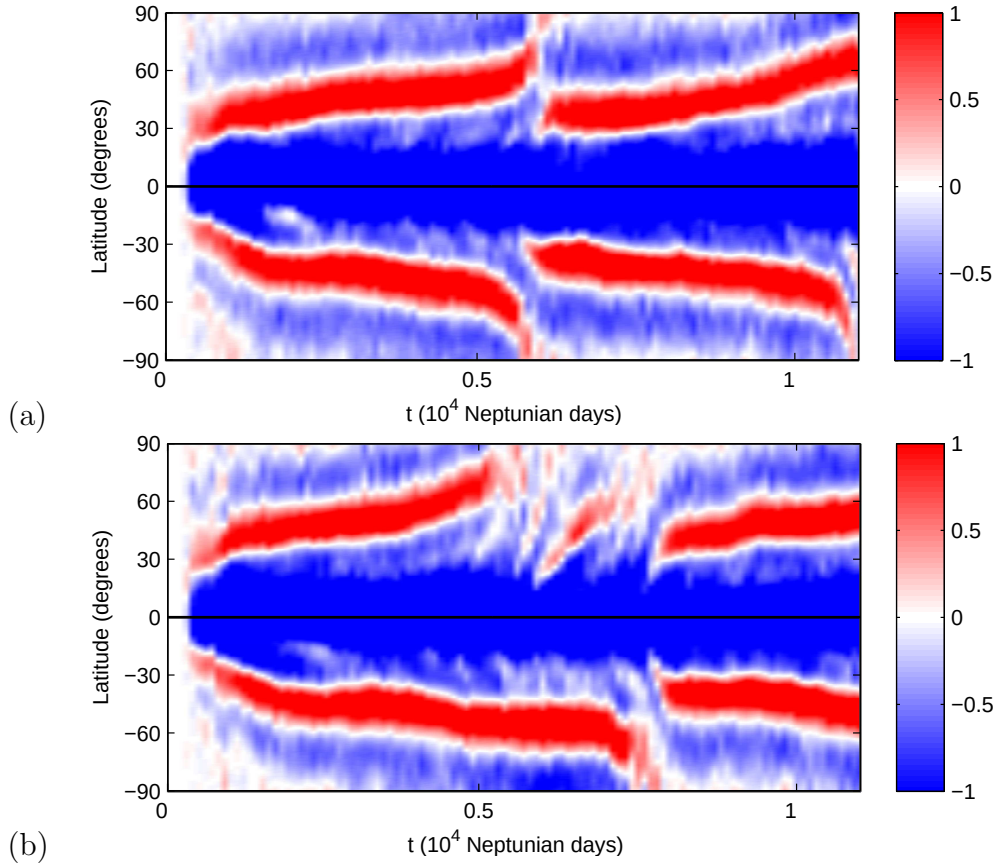


Figure 3.21: Evolution of the mean zonal wind for Neptunian thermal shallow water simulations with (a) $\tau_{\text{rad}} = 7088$ hours and (b) $\tau_{\text{rad}} = 48991$ hours.

number generator gave values of 0.03, 0.17 and 0.37. The correlation is again not robust for these Neptunian simulations, as was found for the Neptunian simulations with $\tau_{\text{rad}} = 7088$ hours in §3.4. The cause for the Neptunian thermal simulations giving positive, almost zero and negative values for the correlation, while the Jovian thermal simulations robustly produce a positive correlation, may be due to the much larger features present in the Neptunian simulations compared to the Jovian simulations. For example, the main contribution to the positive correlation of 0.36 for the Neptunian thermal simulation in figure 3.14 is the large vortex near 80° latitude and -40° longitude. Such large features greatly influence the correlation as they evolve. They are also able to move from one section of the square planet domain to another during the simulation, thus greatly altering the value of the correlation as they move into and out of the northern hemisphere of the top planet. However, the Jovian simulations in §3.3 and §3.5.1 display much smaller and more evenly spaced features, which individually do not make as large a contribution to the correlation. This leads to the correlation being much more robust for the Jovian thermal simulations as an

Table 3.9: Average diagnostic quantities in the statistically steady state for two sets of Neptunian thermal shallow water simulations with different initial random seeds, for two different values of the radiative time scale τ_{rad} . The error estimates quantify the differences between the diagnostics for each pair of simulations, where $\langle\langle\cdot\rangle\rangle$ denotes the time-average of a quantity between 10^4 and 1.1×10^4 Neptunian days.

	$\langle\langle\epsilon_f\rangle\rangle$	$\langle\langle\xi\rangle\rangle$	$\langle\langle\omega_N\rangle\rangle$	$\langle\langle\mathcal{K}_u\rangle\rangle$	$\langle\langle\mathcal{K}_\omega\rangle\rangle$	$\langle\langle\mathcal{S}_\omega\rangle\rangle$
$\tau_{\text{rad}} = 7088 \text{ h}$	4600 ± 1	0.65 ± 0.03	-0.16 ± 0.05	4.75 ± 0.03	5.45 ± 0.5	0.19 ± 0.03
$\tau_{\text{rad}} = 48991 \text{ h}$	4000 ± 50	0.67 ± 0.003	-0.17 ± 0.05	5.00 ± 0.03	3.97 ± 0.1	0.044 ± 0.02

individual feature evolving, or moving into or out of the northern hemisphere of the top planet, does not greatly alter the value for the correlation.

Table 3.9 records average diagnostic quantities for the two values of τ_{rad} , where $\langle\langle\cdot\rangle\rangle$ denotes the time average of a quantity between 10^4 and 1.1×10^4 Neptunian days. Each simulation is run twice, with two different initial seeds for the random number generator, and the error estimates quantify the differences between the diagnostics for each pair of simulations. As in §3.5.1, $\langle\langle\epsilon_f\rangle\rangle$ decreases as τ_{rad} increases, while $\langle\langle\xi\rangle\rangle$ does not change significantly as τ_{rad} varies. However, $\langle\langle\omega_N\rangle\rangle$ is negative for both values of τ_{rad} since both simulations demonstrate a strong sub-rotating equatorial jet (see page 82). The average $\langle\langle\mathcal{K}_u\rangle\rangle$ increases slightly as τ_{rad} increases, in contrast to the Jovian case, while the average $\langle\langle\mathcal{K}_\omega\rangle\rangle$ decreases as τ_{rad} increases in agreement with §3.5.1. Finally $\langle\langle\mathcal{S}_\omega\rangle\rangle$ is positive for both values of τ_{rad} for these Neptunian simulations.

In conclusion, Neptunian thermal simulations using the value of τ_{rad} calculated at the 100 millibar pressure level look very similar to the Neptunian thermal simulations using the value of τ_{rad} calculated at the 25 millibar pressure level.

3.6 Summary

The standard shallow water theory for a single active layer in the equivalent barotropic approximation captures many features of Jovian planetary atmospheres (e.g. see Cho and Polvani, 1996*a,b*, Dowling and Ingersoll, 1989, Iacono, Struglia and Ronchi, 1999, Iacono, Struglia, Ronchi and Nicastro, 1999, Marcus, 1988, Vasavada and Showman, 2005). Forced, dissipative simulations with isotropic random forcing organise themselves into a mixture of alternating zonal jets and long-lived coherent vortices (e.g. see Scott and Polvani, 2007). The approach to a statistically steady state takes many

thousands of rotation periods. Some dissipation at large-scales, such as Rayleigh friction, appears necessary to dissipate the slow leakage of energy past the Rhines scale that would otherwise cause the jets to break up into large coherent vortices (Vasavada and Showman, 2005). These shallow water simulations all produce retrograde equatorial jets when run for Jovian parameter values, in contrast to the long-lived prograde equatorial jets observed on Jupiter and Saturn.

Scott and Polvani (2007, 2008) introduced a model for radiative relaxation to equilibrium into the shallow water mass equation, following an earlier model for the terrestrial stratosphere (Jukes, 1989, Polvani et al., 1995). Their simulations produced prograde equatorial jets for Jovian parameters.

This chapter considers the radiatively damped thermal shallow water equations as an alternative model for gas giant planets. Numerical simulations were performed for four models: the thermal model, the Scott and Polvani (2007, 2008) model, its momentum-conserving variant, and the standard shallow water model, in a doubly-periodic Cartesian geometry that exploits an efficient fast Fourier transform library for GPUs. A narrow-band isotropic spectral forcing was imposed, like that employed in many previous simulations (e.g. Scott and Polvani, 2007, 2008, Smith, 2004, Smith et al., 2002, Srinivasan and Young, 2012). The amplitude of the forcing was controlled to reach a specified kinetic energy in all four simulations.

Simulations with Jovian parameter values, where the radiative cooling rate was calculated at the 25 millibar pressure level, produced a mixture of vortices and robust zonal jets, whose amplitudes decayed with distance from the equator for all four models. The three radiatively damped models each produced a zonal flow that accounted for roughly half the total kinetic energy. They each produced a super-rotating equatorial jet, with the associated net cyclonic vorticity in the northern hemisphere. The standard shallow water model produced a sub-rotating equatorial jet and net anticyclonic vorticity. The standard shallow water and thermal simulations robustly demonstrated more intense anticyclones than cyclones. Cyclonic regions in the thermal simulations were typically warmer than anticyclonic regions. The standard shallow water and thermal simulations produced substantial jets at mid-latitudes, while the two height-damped shallow water simulations produced a strong equatorial jet with very little activity at other latitudes. This difference was reflected in the kurtosis of the zonal velocity, which was roughly twice as large in the two height-damped models as in the thermal model. Conversely, the thermal model is distinguished by having a larger kurtosis in the relative vorticity than the other three models. All four

models produced non-zonal energy spectra slopes that were consistent with the classic Kolmogorov $E(k) \propto k^{-5/3}$ scaling in the inertial range. The zonal energy spectra were all much steeper, though less steep than the $E_z(k) \propto k^{-5}$ predicted by theory. The spectral slopes for all four models lie within the error bars for the slope of the observed zonal energy spectrum for Jupiter (Galperin et al., 2001).

A radiatively damped thermal shallow water simulation was also performed for Neptunian parameter values, where the radiative time scale was recalculated at the same 25 millibar pressure level for this much colder planet. Simulations then produced a retrograde equatorial jet, and two prograde mid-latitude jets, as seen on Neptune, but I was unable to run a stable simulation with sufficient forcing to match the jet speeds seen on Neptune. This may be because the observed maximum jet speed of eight deformation radii per Neptunian day implies a Rossby number close to unity, and hence large deformations of the free surface that destabilise the thermal shallow water model. However, the Neptunian simulation produced velocities closer to those observed on Uranus (see Liu and Schneider, 2010, for observed $\bar{u}$ values for Uranus), and previous papers have used a single simulation to represent both Uranus and Neptune (including Lian and Showman, 2010, Scott and Polvani, 2007).

The sensitivity of the results to the value used for the radiative time scale was considered. The Neptunian thermal simulations using the value for τ_{rad} calculated at the 100 millibar level looked very similar to the Neptunian thermal simulations using the value for τ_{rad} calculated at the 25 millibar pressure level. However, Jovian simulations using the value of τ_{rad} calculated at the 100 millibar level demonstrated a sub-rotating equatorial jet for all three radiatively damped models, as opposed to the super-rotating equatorial jet demonstrated in simulations using the value of τ_{rad} at the 25 millibar pressure level. All three radiatively damped shallow water models only produce the correct equatorial jet direction for Jupiter if the radiative time scale is sufficiently (and perhaps unrealistically) short. However, the theory in chapter 6 suggests that lengthening the frictional time scale beyond 1000 Jovian days (or 9900 hours) would increase the tendency to form a super-rotating equatorial jet for longer radiative time scales, but running simulations until they reach statistical steady states then becomes prohibitively expensive.

Adapting simulations to run on GPUs gave a substantial (factors of three to five) performance increase over using two conventional multi-core CPUs with little additional programming effort. However simulations still each took over a week to run for 2×10^4 Jovian days at 1024^2 resolution in the Jovian regime, and over two weeks to run in the Neptunian regime. The radiatively damped thermal shallow water

equations are studied in future chapters by considering their local quasi-geostrophic limit (chapter 4) and their global quasi-geostrophic limit (chapter 5). Chapter 6 provides a possible explanation for the difference in equatorial jet direction in Jovian simulations of the standard shallow water equations (which produce a sub-rotating equatorial jet), and the height-damped and radiatively damped thermal shallow water equations (which both produce a super-rotating equatorial jet for τ_{rad} sufficiently short).

Chapter 4

The local quasi-geostrophic theory of the radiatively damped thermal shallow water equations on a mid-latitude beta-plane

This chapter is based upon Warneford and Dellar (2013).

4.1 Introduction

Vortical motions typically evolve on time scales much longer than those characteristic of inertia-gravity waves. Quasi-geostrophic (QG) theory, first introduced by Charney (1948), is the simplest example of an intermediate or balanced model that exploits this separation of time scales to produce a simplified model, while still retaining the time evolution omitted from purely geostrophic flow. Quasi-geostrophic theory is a key framework in dynamic meteorology and oceanography, and may be applied to both shallow water and three-dimensional continuously stratified equations. It filters out inertia-gravity waves, replacing them with an elliptic relation between the potential vorticity and the geostrophic streamfunction. In this respect it closely resembles the derivation of incompressible fluid equations from compressible fluid equations, where sound waves are filtered out in favour of an elliptic equation for the pressure.

Quasi-geostrophic equations may be derived by posing expansions of the velocity, layer depth and other variables in a small Rossby number, the dimensionless parameter that characterises the significance of inertia relative to the Coriolis force. It is also necessary to assume small (order Rossby number) displacements of the layer depth, in shallow water theory, or of the isopycnal surfaces, in the three-dimensional

Boussinesq equations, from their uniform equilibrium positions for a state of rest (Majda, 2003, Pedlosky, 1987, Salmon, 1998, White, 2002). An alternative derivation of quasi-geostrophic theory projects the underlying equations onto a basis of the linear wave modes for this unperturbed rest state, and retains nonlinear interactions only between the vortical modes (Leith, 1980, Majda, 2003, Remmel and Smith, 2009, Salmon, 1998). This gives an explicit origin to the filtering out of inertia-gravity waves. The usage of the linear wave modes of the unperturbed state as coordinates for the projection is the origin of the restriction of quasi-geostrophic theory to small perturbations of the isopycnals, which is logically distinct from the small Rossby number approximation (Leith, 1980, Salmon, 1988*b*). Charney and Flierl (1981), Hoskins et al. (1985) and White (2002) give broader surveys and reviews of quasi-geostrophic theory, while Majda (2003) gives an introduction to the rigorous mathematical theory of the quasi-geostrophic limit.

Quasi-geostrophic theory is attractive for numerical simulation, because the time step is limited only by the maximum velocity of the fluid, rather than by the typically much larger velocity of inertia-gravity waves. The geostrophic streamfunction may be reconstructed from the instantaneous potential vorticity distribution via the inversion of a modified Helmholtz operator, which is diagonalised by a Fourier or spherical harmonic transform for the local quasi-geostrophic limit considered in this chapter. Numerical simulations of multi-layer quasi-geostrophic equations have evolved from pioneering experiments in numerical weather prediction (e.g. Charney and Phillips, 1953) to large-scale simulations of rotating stratified turbulence (e.g. Reinaud et al., 2003). The presence of only quadratic nonlinearities in quasi-geostrophic theory also makes them amenable to the techniques of statistical mechanics (e.g. Majda and Wang, 2006, Salmon, 1998), notably the use of cumulant expansions to study zonally symmetric mean flows in geostrophic turbulence on spheres and β -planes (Farrell and Ioannou, 1993, 2003, 2007, Marston et al., 2008, Srinivasan and Young, 2012, Tobias et al., 2011).

This chapter considers the local quasi-geostrophic limit of the radiatively damped thermal shallow water equations on a mid-latitude β -plane (see §1.4.1). A global quasi-geostrophic limit is considered in chapter 5, i.e. a quasi-geostrophic limit that is valid for the entire planet. Ripa (1996*b*) derived a local quasi-geostrophic form of the thermal shallow water equations, which resemble Young's (1994) subinertial mixed layer model in the limit where vertical momentum mixing is much faster than the inertial time scale (see §2.3.1). A formal derivation of these local thermal quasi-geostrophic equations with radiative cooling is provided in this chapter. The first

derivation involves an asymptotic expansion for small Rossby number. The second derivation is presented via a nonlinear balanced model that corresponds to a projection of the dynamics onto the slow normal modes for perturbations around a uniform rest state in the radiatively damped thermal shallow water equations. The local thermal quasi-geostrophic equations are then the small Rossby number limit of this balanced model. The effect of radiative cooling on the Rossby waves in the local thermal quasi-geostrophic equations is considered, and a comparison is made between waves in the thermal shallow water equations and those in their local quasi-geostrophic limit. The group velocity is shown to be the velocity with which the wave energy is transported for the local thermal quasi-geostrophic equations. The energy density and flux for the local thermal quasi-geostrophic equations are derived from the local quasi-geostrophic limit of a pseudo-energy density and flux for the thermal shallow water equations. This derivation involves the small ageostrophic correction to the leading order geostrophic velocity, the correction that is eliminated in the derivation of the closed quasi-geostrophic evolution equations. Finally, the non-canonical Hamiltonian structure of the local thermal quasi-geostrophic equations (as stated by Ripa, 1996*b*) is derived from a small Rossby number decomposition of a pseudo-energy and Poisson bracket for the thermal shallow water equations.

4.2 Derivation by a direct asymptotic expansion

This section formalises the derivation of the local quasi-geostrophic limit of the thermal shallow water equations given in Ripa (1996*b*), by performing a direct asymptotic expansion in a small Rossby number. First, the radiatively damped thermal shallow water equations (1.3.12) are considered on a mid-latitude β -plane (see §1.4.1). Thus x , y , z are treated as Cartesian coordinates, while the latitude-dependence of the Coriolis parameter in spherical geometry is still captured. The β -plane approximation makes analysis of the equations more tractable, and approximates $f = 2\Omega \sin \phi$ by a truncated Taylor series expansion in the variable $y = a(\phi - \phi_0)$, where ϕ_0 is a reference latitude and a is the planetary radius. This gives $f \approx f_0 + \beta^* y$, which varies linearly in y . The derivation relies upon the leading-order velocity satisfying a leading-order geostrophic balance condition with f approximated by f_0 . This approximation is only valid outside the equatorial region.

4.2.1 The quasi-geostrophic scaling

Following Salmon (1998), the variables in (1.3.12) are rescaled using

$$\mathbf{x} = L\tilde{\mathbf{x}}, \quad \mathbf{u} = U\tilde{\mathbf{u}}, \quad t = (L/U)\tilde{t}, \quad (4.2.1)$$

where L and U are horizontal length and velocity scales, respectively. The superscript tilde denotes a dimensionless variable. The advective time scale L/U is used, so that $\partial\mathbf{u}/\partial t$ is comparable to $(\mathbf{u} \cdot \nabla)\mathbf{u}$ in the dimensionless momentum equation. The layer depth is written as

$$h = h_0 \left(1 + \frac{f_0 U L}{\Theta_0 h_0} \tilde{\eta} \right), \quad (4.2.2)$$

for small displacements from a constant mean depth h_0 . This scaling for $\tilde{\eta}$ is set by a leading-order geostrophic balance between the Coriolis force and the pressure gradient in (1.3.12c). The temperature is written as

$$\Theta = \Theta_0 \left(1 + 2 \frac{f_0 U L}{\Theta_0 h_0} \tilde{\theta} \right), \quad (4.2.3)$$

with a factor of two for future convenience in the definition of the geostrophic streamfunction.

Substituting these scalings into (1.3.12) gives the dimensionless equations

$$Ro \left(\frac{\partial \tilde{\eta}}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\tilde{\eta} \tilde{\mathbf{u}}) \right) + Bu \left(\tilde{\nabla} \cdot \tilde{\mathbf{u}} \right) = 0, \quad (4.2.4a)$$

$$\frac{\partial \tilde{\theta}}{\partial \tilde{t}} + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) \tilde{\theta} = -2\lambda \left(\tilde{\theta} + \frac{1}{2} \tilde{\eta} + \frac{Ro}{Bu} \tilde{\eta} \tilde{\theta} \right), \quad (4.2.4b)$$

$$Ro \left(\frac{\partial \tilde{u}}{\partial \tilde{t}} + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) \tilde{u} \right) - (1 + \tilde{\beta} \tilde{y}) \tilde{v} = -\frac{\partial \tilde{\eta}}{\partial \tilde{x}} - \frac{\partial \tilde{\theta}}{\partial \tilde{x}} - \frac{Ro}{Bu} \left(2\tilde{\theta} \frac{\partial \tilde{\eta}}{\partial \tilde{x}} + \tilde{\eta} \frac{\partial \tilde{\theta}}{\partial \tilde{x}} \right), \quad (4.2.4c)$$

$$Ro \left(\frac{\partial \tilde{v}}{\partial \tilde{t}} + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) \tilde{v} \right) + (1 + \tilde{\beta} \tilde{y}) \tilde{u} = -\frac{\partial \tilde{\eta}}{\partial \tilde{y}} - \frac{\partial \tilde{\theta}}{\partial \tilde{y}} - \frac{Ro}{Bu} \left(2\tilde{\theta} \frac{\partial \tilde{\eta}}{\partial \tilde{y}} + \tilde{\eta} \frac{\partial \tilde{\theta}}{\partial \tilde{y}} \right), \quad (4.2.4d)$$

with the dimensionless parameters

$$Ro = \frac{U}{f_0 L}, \quad Bu = \left(\frac{L_D}{L} \right)^2, \quad \tilde{\beta} = \frac{\beta^* L}{f_0}, \quad \lambda = \frac{L}{2U\tau_{\text{rad}}}, \quad (4.2.5)$$

where Ro is the Rossby number, Bu is the Burger number and $L_D = \sqrt{\Theta_0 h_0}/f_0$ is the deformation radius, the horizontal length scale on which the Coriolis force becomes comparable to the pressure gradient. The scalings in (4.2.2) and (4.2.3) are now recognised as

$$h = h_0 \left(1 + \frac{Ro}{Bu} \tilde{\eta} \right), \quad \Theta = \Theta_0 \left(1 + 2 \frac{Ro}{Bu} \tilde{\theta} \right). \quad (4.2.6)$$

4.2.2 Asymptotic expansion for small Rossby number

The quasi-geostrophic limit describes rapidly rotating flows, meaning flows with small Rossby number which are strongly affected by Coriolis forces. Standard shallow water quasi-geostrophic theory also requires small deviations of the layer depth from a uniform value, as expressed in (4.2.6). Thermal shallow water quasi-geostrophic theory restricts the temperature to be a small deviation (of the same relative magnitude) from a uniform value Θ_0 (as in Young, 1994). The Rossby number is set as $Ro = \epsilon$, where ϵ is a small parameter, while $\tilde{\beta}$ is set as $\tilde{\beta} = \epsilon\beta$, where β remains $O(1)$ as $\epsilon \rightarrow 0$ (recognising that fractional changes in the Coriolis parameter must be small on the horizontal length scale L of the flow Salmon, 1998). The parameters Bu and λ are assumed to remain $O(1)$ as $\epsilon \rightarrow 0$. The tilde notation is dropped, with the understanding that all variables and operators are now dimensionless.

The velocity $\mathbf{u}$ is expanded in powers of ϵ ,

$$\mathbf{u} = \mathbf{u}_0 + \epsilon \mathbf{u}_1 + \cdots, \quad (4.2.7)$$

while the other variables η and θ are left unexpanded (as in Muraki et al., 1999, Warn et al., 1995, White, 2002). Following Van Kampen's (1985) general theory, closed evolution equations for the unexpanded slow variables η and θ are obtained that remain valid on long time scales with $t = O(1/\epsilon)$. Pedlosky (1987) and Majda (2003) avoid this step and expand η and θ as well, but they first supplement the continuity, momentum and temperature equations, with the exact evolution equation for the shallow water potential vorticity $\Pi = h^{-1}(f + v_x - u_y)$. This supplementary equation identifies the potential vorticity as a slow variable whose evolution should be captured by the asymptotic description.

Substituting (4.2.7) into (4.2.4) and truncating at leading order gives

$$\nabla \cdot \mathbf{u}_0 = 0, \quad v_0 = \partial_x(\eta + \theta), \quad u_0 = -\partial_y(\eta + \theta), \quad (4.2.8a)$$

$$\partial_t \theta + (\mathbf{u}_0 \cdot \nabla) \theta = -\lambda(2\theta + \eta). \quad (4.2.8b)$$

We thus have a geostrophic flow with streamfunction $\psi = \eta + \theta$. This simple expression justifies the factor of two in (4.2.3), and in the second expression of (4.2.6). At next order we obtain the $O(\epsilon)$ continuity and velocity equations,

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (\eta \mathbf{u}_0) + Bu(\nabla \cdot \mathbf{u}_1) = 0, \quad (4.2.9a)$$

$$\frac{\partial u_0}{\partial t} + (\mathbf{u}_0 \cdot \nabla) u_0 - v_1 - \beta y v_0 = -\frac{1}{Bu} \left(2\theta \frac{\partial \eta}{\partial x} + \eta \frac{\partial \theta}{\partial x} \right), \quad (4.2.9b)$$

$$\frac{\partial v_0}{\partial t} + (\mathbf{u}_0 \cdot \nabla) v_0 + u_1 + \beta y u_0 = -\frac{1}{Bu} \left(2\theta \frac{\partial \eta}{\partial y} + \eta \frac{\partial \theta}{\partial y} \right). \quad (4.2.9c)$$

Substituting (4.2.8a) into (4.2.9) and eliminating $\nabla \cdot \mathbf{u}_1$ between (4.2.9a) and (4.2.9b,c) gives

$$\frac{\partial q}{\partial t} + [\psi, q] = \frac{1}{Bu} [\eta, \theta], \quad (4.2.10)$$

where the local quasi-geostrophic potential vorticity

$$q = \nabla^2 \psi - \frac{\eta}{Bu} + \beta y \quad (4.2.11)$$

is a consistent linear approximation to the shallow water potential vorticity Π and $[\cdot, \cdot]$ is the two-dimensional Jacobian defined by $[\psi, q] = \hat{\mathbf{z}} \cdot (\nabla \psi \times \nabla q)$. In (4.2.11), the first term is the relative vorticity and the second term denotes contributions to potential vorticity due to variations in the free surface height, with the Burger number measuring this term's relative importance.

Eliminating $\eta = \psi - \theta$ gives the closed local thermal quasi-geostrophic system

$$\frac{\partial q}{\partial t} + [\psi, q] = \frac{1}{Bu} [\psi, \theta], \quad (4.2.12a)$$

$$\frac{\partial \theta}{\partial t} + [\psi, \theta] = -\lambda (\theta + \psi), \quad (4.2.12b)$$

$$q = \nabla^2 \psi - \frac{(\psi - \theta)}{Bu} + \beta y. \quad (4.2.12c)$$

The usual local quasi-geostrophic shallow water potential vorticity is modified by the additional θ/Bu term in (4.2.12c), which acts as an additional source in the modified Helmholtz equation $\nabla^2 \psi - \psi/Bu = q - \beta y - \theta/Bu$. Potential vorticity is not materially conserved, due to the additional term on the right-hand side of (4.2.12a). However, the total potential vorticity inside a closed isotherm (line of constant θ) is conserved (4.7.10), as for the radiatively damped thermal shallow water equations (cf. §2.3.2).

4.3 Derivation by a projection onto vortical modes

An alternative derivation of the local thermal quasi-geostrophic equations (4.2.12) is now presented that explicitly projects out inertia-gravity waves. The thermal shallow water equations (1.3.12) support interactions between four different wave modes: the usual two inertia-gravity waves, the vortical or Rossby wave of the standard shallow water equations, and an additional thermal wave due to the extra temperature equation. A balanced model is sought first that describes only the low-frequency dynamics represented by the Rossby and thermal waves by projecting out the high-frequency dynamics associated with the inertia-gravity waves (see Leith, 1980, Majda, 2003, Remmel and Smith, 2009, Salmon, 1998, Theiss and Mohebalhojeh, 2009, White,

2002). Ripa (1996*a*) studied linear waves in the thermal shallow water equations, but this section instead uses a different set of variables that give particularly simple forms for the different wave modes.

Following Viúdez and Dritschel's (2004) treatment of the standard shallow water equations, the radiatively damped thermal shallow water equations (1.3.12) are reformulated in terms of new variables: the potential vorticity Π , the velocity divergence D , and the acceleration divergence γ . Writing $h = h_0 + \eta$ and $\Theta = \Theta_0 + \theta$, these new variables are given by

$$\Pi = \frac{1}{h_0 + \eta} \left(f_0 + \beta^* y + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right), \quad (4.3.1a)$$

$$D = \nabla \cdot \mathbf{u}, \quad (4.3.1b)$$

$$\gamma = \nabla \cdot \left(- (f_0 + \beta^* y) \hat{\mathbf{z}} \times \mathbf{u} - (\Theta_0 + \theta) \nabla \eta - \frac{1}{2} (h_0 + \eta) \nabla \theta \right). \quad (4.3.1c)$$

Again a mid-latitude β -plane is considered on which $f = f_0 + \beta^* y$. In these new variables the radiatively damped thermal shallow water equations become

$$\frac{\partial \Pi}{\partial t} = \left\{ - (\mathbf{u} \cdot \nabla) \Pi + \frac{1}{2} \frac{[\eta, \theta]}{(h_0 + \eta)} \right\}, \quad (4.3.2a)$$

$$\frac{\partial \theta}{\partial t} = \left\{ - \frac{1}{h_0 \tau_{\text{rad}}} (h_0 \theta + \Theta_0 \eta + \theta \eta) - (\mathbf{u} \cdot \nabla) \theta \right\}, \quad (4.3.2b)$$

$$\frac{\partial D}{\partial t} = \gamma + \left\{ - (\mathbf{u} \cdot \nabla) D - D^2 + 2[u, v] \right\}, \quad (4.3.2c)$$

$$\begin{aligned} \frac{\partial \gamma}{\partial t} = & -f_0^2 D + \Theta_0 h_0 \nabla^2 D + \frac{1}{\tau_{\text{rad}}} \left\{ \frac{1}{2} h_0 \nabla^2 \theta + \frac{1}{2} \Theta_0 \nabla^2 \eta \right\} \\ & + \left\{ -2f_0 \beta^* y D - \beta^{*2} y^2 D - 2\beta^* (f_0 + \beta^* y) v + \beta^* \Theta_0 \frac{\partial \eta}{\partial x} + \frac{1}{2} \beta^* h_0 \frac{\partial \theta}{\partial x} \right\} \\ & + \{\text{nonlinear terms}\}, \end{aligned} \quad (4.3.2d)$$

where $[\cdot, \cdot]$ is the two-dimensional Jacobian as above. The equation set (4.3.2) may be rewritten in matrix form as

$$\frac{\partial \boldsymbol{\eta}}{\partial t} = -i\mathcal{L}\boldsymbol{\eta} + \mathcal{N}(\boldsymbol{\eta}), \quad (4.3.3)$$

where

$$\boldsymbol{\eta} = \begin{pmatrix} \Pi \\ \theta \\ D \\ \gamma \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i(\Theta_0 h_0 \nabla^2 - f_0^2) & 0 \end{pmatrix}, \quad (4.3.4)$$

and $\mathcal{N}(\boldsymbol{\eta})$ is a column vector containing the terms from (4.3.2) in curly brackets. The linear operator $\mathcal{L}$ characterises the wave modes of the undamped f -plane thermal

shallow water equations. All coupling between these modes is expressed through $\mathcal{N}(\boldsymbol{\eta})$, both the nonlinear coupling and the linear coupling through terms involving β^* and τ_{rad} . This choice of decomposition follows the approach of Leith (1980) and Salmon (1998).

The eigenmodes of $\mathcal{L}$ for an unbounded or periodic domain are the Fourier modes

$$\boldsymbol{\eta} = \hat{\boldsymbol{\eta}} \exp(i(kx + ly - \omega t)), \quad (4.3.5)$$

with $\hat{\boldsymbol{\eta}}$ a constant vector. Substituting (4.3.5) into (4.3.3) and setting $\mathcal{N} = \mathbf{0}$ gives

$$\omega_{R_1} = 0, \quad \omega_{R_2} = 0, \quad \omega_{G_{\pm}} = \pm \sqrt{f_0^2 + \Theta_0 h_0 (k^2 + l^2)}, \quad (4.3.6)$$

with the corresponding eigenvectors

$$\mathbf{e}_{R_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_{R_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_{G_{\pm}} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \mp i \sqrt{f_0^2 + \Theta_0 h_0 (k^2 + l^2)} \end{pmatrix}. \quad (4.3.7)$$

The thermal mode with eigenvector $\mathbf{e}_{R_2}$ was called the force-compensated mode by Ripa (1996a). Young and Chen (1995) named the equivalent mode in their subinertial mixed layer equations the ‘buoyancy-compensated mode’. The fluctuation in θ is balanced by a compensating fluctuation in η that leaves the pressure, and hence the velocity field, unchanged. This mode has a particularly simple eigenvector $\mathbf{e}_{R_2}$ in the variables Π, θ, D, γ because the height η has been eliminated in favour of the acceleration divergence γ . The combined contribution to γ from $\nabla \eta$ and $\nabla \theta$ vanishes precisely because this mode is force-compensated.

The pair $\{\omega_{R_1}, \mathbf{e}_{R_1}\}$ represents the low-frequency Rossby mode, the pair $\{\omega_{R_2}, \mathbf{e}_{R_2}\}$ represents the low-frequency thermal mode and the pairs $\{\omega_{G_+}, \mathbf{e}_{G_+}\}$ and $\{\omega_{G_-}, \mathbf{e}_{G_-}\}$ represent the high-frequency inertia-gravity modes. The space $\mathcal{R}$ spanned by the Rossby and thermal modes is called the slow or Rossby manifold, while the $\mathcal{G}$ space spanned by the inertia-gravity modes is called the fast or inertia-gravity manifold. The manifolds $\mathcal{R}$ and $\mathcal{G}$ are orthogonal and span the whole space, so any vector $\boldsymbol{\eta}$ may be decomposed uniquely into two components, $\boldsymbol{\eta} = \boldsymbol{\eta}_R + \boldsymbol{\eta}_G$, with $\boldsymbol{\eta}_R$ in $\mathcal{R}$ and $\boldsymbol{\eta}_G$ in $\mathcal{G}$. This clean separation into fast and slow manifolds motivates the introduction of the Π, θ, D, γ variables in (4.3.1).

Nonlinear interactions are now restricted to include only the Rossby and thermal modes. A vector $\hat{\boldsymbol{\eta}}_R$ on the Rossby manifold may be expressed as

$$\hat{\boldsymbol{\eta}}_R = \begin{pmatrix} \hat{\Pi}_R \\ \hat{\theta}_R \\ \hat{D}_R \\ \hat{\gamma}_R \end{pmatrix} = \hat{y}_{R_1} \mathbf{e}_{R_1} + \hat{y}_{R_2} \mathbf{e}_{R_2} = \begin{pmatrix} \hat{y}_{R_1} \\ \hat{y}_{R_2} \\ 0 \\ 0 \end{pmatrix}, \quad (4.3.8)$$

where $\hat{y}_{R_1}(k, l, t)$ and $\hat{y}_{R_2}(k, l, t)$ are the coordinates of $\hat{\boldsymbol{\eta}}_R$ on the Rossby manifold, and hats denote a Fourier transform in x and y . Transforming the lower two components of (4.3.8) back to physical space gives the balance relations

$$D_R = 0 \quad \text{and} \quad \gamma_R = 0. \quad (4.3.9)$$

Evolution equations for $\hat{\Pi}_R$ and $\hat{\theta}_R$ are found by Fourier transforming (4.3.3) in x and y , and substituting $\hat{\boldsymbol{\eta}} = \hat{\boldsymbol{\eta}}_R$ as given by (4.3.8), to obtain

$$\partial_t \hat{\Pi}_R \mathbf{e}_{R_1} + \partial_t \hat{\theta}_R \mathbf{e}_{R_2} = -i \left(\hat{\Pi}_R \hat{\mathcal{L}} \mathbf{e}_{R_1} + \hat{\theta}_R \hat{\mathcal{L}} \mathbf{e}_{R_2} \right) + \hat{\mathcal{N}}, \quad (4.3.10)$$

where $\hat{\mathcal{N}}$ and $\hat{\mathcal{L}}$ are the Fourier transforms of $\mathcal{N}$ and $\mathcal{L}$. From (4.3.6) it is known that $\mathbf{e}_{R_1}$ and $\mathbf{e}_{R_2}$ are eigenvectors of $\hat{\mathcal{L}}$ with zero eigenvalues, so (4.3.10) reduces to

$$\partial_t \hat{\Pi}_R \mathbf{e}_{R_1} + \partial_t \hat{\theta}_R \mathbf{e}_{R_2} = \hat{\mathcal{N}}. \quad (4.3.11)$$

Projecting (4.3.11) onto $\mathbf{e}_{R_1}$ and inverting the Fourier transform gives

$$\partial_t \Pi_R + (\mathbf{u}_R \cdot \nabla) \Pi_R = \frac{1}{2} \frac{[\eta_R, \theta_R]}{(h_0 + \eta_R)}, \quad (4.3.12)$$

while projecting (4.3.11) onto $\mathbf{e}_{R_2}$ and inverting the Fourier transform gives

$$\partial_t \theta_R + (\mathbf{u}_R \cdot \nabla) \theta_R = -\frac{1}{h_0 \tau_{\text{rad}}} (h_0 \theta_R + \Theta_0 \eta_R + \eta_R \theta_R). \quad (4.3.13)$$

A balanced model is thus obtained consisting of the two evolution equations (4.3.12) and (4.3.13), and the two balance relations (4.3.9). Dropping the R subscripts for convenience, the complete balanced model is

$$\frac{\partial \Pi}{\partial t} + (\mathbf{u} \cdot \nabla) \Pi = \frac{1}{2} \frac{[\eta, \theta]}{(h_0 + \eta)}, \quad (4.3.14a)$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla) \theta = -\frac{1}{\tau_{\text{rad}}} \left(\theta + \frac{\eta \Theta_0}{h_0} + \frac{\eta \theta}{h_0} \right), \quad (4.3.14b)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (4.3.14c)$$

$$(f_0 + \beta^* y) \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) - \beta^* u = \frac{3}{2} \nabla \eta \cdot \nabla \theta + (\Theta_0 + \theta) \nabla^2 \eta + \frac{1}{2} (h_0 + \eta) \nabla^2 \theta, \quad (4.3.14d)$$

where Π is defined by (4.3.1a). This generalises the δ - γ balanced model of Mohebalhojeh and Dritschel (2001) to the radiatively damped thermal shallow water equations. The derivation is not restricted to mid-latitude β -planes, and provides a route towards developing a spherical version of thermal quasi-geostrophic theory (see

chapter 5 and Charney and Stern, 1962, Kuo, 1959, Schubert et al., 2009, Verkley, 2009).

The earlier local thermal quasi-geostrophic equations now emerge as the leading-order approximation to (4.3.14) for small Rossby number. Non-dimensionalising (4.3.1a) and (4.3.14) using the variables

$$\mathbf{x} = L\tilde{\mathbf{x}}, \quad \mathbf{u} = U\tilde{\mathbf{u}}, \quad t = (L/U)\tilde{t}, \quad \eta = (f_0 UL/\Theta_0)\tilde{\eta}, \quad \theta = (2f_0 UL/h_0)\tilde{\theta}, \quad (4.3.15)$$

as in (4.2.1)–(4.2.3), the dimensionless equations

$$\left(\frac{\partial}{\partial \tilde{t}} + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) \right) \left\{ \frac{Ro(\tilde{v}_{\tilde{x}} - \tilde{u}_{\tilde{y}}) + 1 + \tilde{\beta}\tilde{y}}{Ro + (Ro^2/Bu)\tilde{\eta}} \right\} = \frac{1}{Bu} \left(\frac{[\tilde{\eta}, \tilde{\theta}]}{1 + (Ro/Bu)\tilde{\eta}} \right), \quad (4.3.16a)$$

$$\frac{\partial \tilde{\theta}}{\partial \tilde{t}} + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) \tilde{\theta} = -\lambda \left(2\tilde{\theta} + \tilde{\eta} + 2\frac{Ro}{Bu}\tilde{\eta}\tilde{\theta} \right), \quad (4.3.16b)$$

$$\tilde{\nabla} \cdot \tilde{\mathbf{u}} = 0, \quad (4.3.16c)$$

$$\begin{aligned} (1 + \tilde{\beta}\tilde{y}) \left(\frac{\partial \tilde{v}}{\partial \tilde{x}} - \frac{\partial \tilde{u}}{\partial \tilde{y}} \right) - \tilde{\beta}\tilde{u} &= \tilde{\nabla}^2 \tilde{\eta} + \tilde{\nabla}^2 \tilde{\theta} \\ &+ \frac{Ro}{Bu} \left(3\tilde{\nabla} \tilde{\eta} \cdot \tilde{\nabla} \tilde{\theta} + 2\tilde{\theta} \tilde{\nabla}^2 \tilde{\eta} + \tilde{\eta} \tilde{\nabla}^2 \tilde{\theta} \right), \end{aligned} \quad (4.3.16d)$$

are obtained, with parameters Ro , Bu , $\tilde{\beta}$, λ as defined in (4.2.5). The tilde notation is dropped, with the understanding that all variables and operators are now dimensionless in this section. As in §4.2.2, the Rossby number is set as $Ro = \epsilon$, while $\tilde{\beta} = \epsilon\beta$, where ϵ is a small parameter, and the other dimensionless parameters β , Bu and λ are all assumed to remain $O(1)$ as $\epsilon \rightarrow 0$. Expanding $\mathbf{u}$ in powers of ϵ as in (4.2.7) gives the equation set

$$\left(\frac{\partial}{\partial t} + (\mathbf{u}_0 \cdot \nabla) \right) \left\{ \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} - \frac{\eta}{Bu} + \beta y \right\} = \frac{1}{Bu} [\eta, \theta], \quad (4.3.17a)$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u}_0 \cdot \nabla) \theta = -\lambda (2\theta + \eta), \quad (4.3.17b)$$

$$\nabla \cdot \mathbf{u}_0 = 0, \quad (4.3.17c)$$

$$\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} = \nabla^2 \eta + \nabla^2 \theta, \quad (4.3.17d)$$

at leading order. Equation (4.3.17c) allows the introduction of a geostrophic streamfunction ψ such that

$$u_0 = -\partial_y \psi, \quad v_0 = \partial_x \psi. \quad (4.3.18)$$

Equation (4.3.17d) then implies that $\nabla^2 \psi = \nabla^2 \eta + \nabla^2 \theta$, thus we take $\psi = \eta + \theta$. Substituting (4.3.18) into (4.3.17a,b), and eliminating η using $\eta = \psi - \theta$, leads to the previous local thermal quasi-geostrophic equations (4.2.12).

4.4 Linear plane waves for the local thermal quasi-geostrophic equations

To investigate the properties of the local thermal quasi-geostrophic equations (4.2.12), the effect of the inclusion of radiative cooling on the Rossby waves is considered. Linear plane waves about a rest state are considered, as in §2.7.1 for the radiatively damped thermal shallow water equations. Substituting (4.2.12c) into (4.2.12a), we obtain the system

$$\frac{\partial}{\partial t} \left(\nabla^2 \psi - \frac{(\psi - \theta)}{Bu} \right) + [\psi, \nabla^2 \psi] + \beta \frac{\partial \psi}{\partial x} = 0, \quad (4.4.1)$$

$$\frac{\partial \theta}{\partial t} + [\psi, \theta] = -\lambda (\theta + \psi). \quad (4.4.2)$$

Linearising the equations, assuming that $\psi \ll 1$ and $\theta \ll 1$, gives

$$\frac{\partial}{\partial t} \left(\nabla^2 \psi - \frac{(\psi - \theta)}{Bu} \right) + \beta \frac{\partial \psi}{\partial x} = 0, \quad (4.4.3)$$

$$\frac{\partial \theta}{\partial t} = -\lambda (\theta + \psi). \quad (4.4.4)$$

Waves are sought of the form $\psi = \text{Re}\{\hat{\psi} \exp(i(kx + ly - \omega t))\}$ and $\theta = \text{Re}\{\hat{\theta} \exp(i(kx + ly - \omega t))\}$, where ω is the frequency and k and l are the wavenumbers in the x and y directions respectively:

$$\begin{pmatrix} i\omega(k^2 + l^2 + 1/Bu) + i\beta k & -i\omega/Bu \\ \lambda & \lambda - i\omega \end{pmatrix} \begin{pmatrix} \hat{\psi} \\ \hat{\theta} \end{pmatrix} = \mathbf{0}. \quad (4.4.5)$$

For a non-trivial solution, the determinant of the above matrix must be zero, which gives the dispersion relation

$$\left(k^2 + l^2 + \frac{1}{Bu}\right) \omega^2 + \left(\beta k + i\lambda \left(k^2 + l^2 + \frac{2}{Bu}\right)\right) \omega + i\beta \lambda k = 0. \quad (4.4.6)$$

Considering the case where λ is small, and setting $\lambda = \varepsilon \tau$ in the dispersion relation (4.4.6), where $0 < \varepsilon \ll 1$, gives

$$\left(k^2 + l^2 + \frac{1}{Bu}\right) \omega_S^2 + \left(\beta k + i\varepsilon \tau \left(k^2 + l^2 + \frac{2}{Bu}\right)\right) \omega_S + i\varepsilon \tau \beta k = 0. \quad (4.4.7)$$

Solving and expanding for small ε then gives the two roots

$$\omega_{S+} = 0 - i\varepsilon \tau + O(\varepsilon^2), \quad (4.4.8)$$

$$\omega_{S-} = -\frac{\beta k}{k^2 + l^2 + 1/Bu} - i\varepsilon \left(\frac{\tau}{Bu(k^2 + l^2 + 1/Bu)} \right) + O(\varepsilon^2). \quad (4.4.9)$$

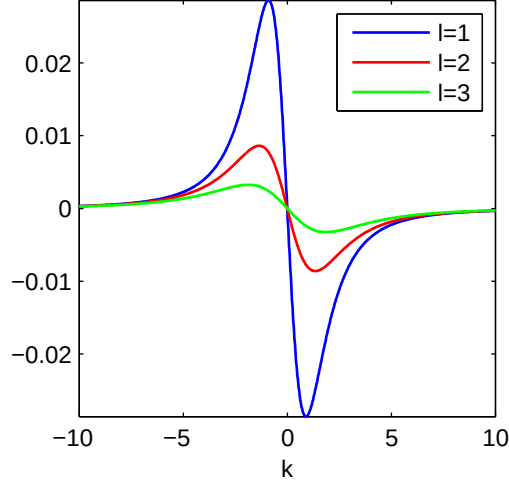


Figure 4.1: The real part of the dispersion relation ω_{L-} (4.4.12) is illustrated for $l = 1, 2, 3$ (where $Bu = 1$ and $\beta = 0.34$ in this figure).

The real part of ω_{S-} is the dispersion relation for a Rossby wave, and since τ , ε and Bu are all positive, the Rossby waves decay with time. The waves characterised by the ω_{S+} dispersion relation also decay with time.

Considering the case where λ is large, and setting $\lambda = \tau/\varepsilon$ in the dispersion relation (4.4.6), where $0 < \varepsilon \ll 1$, gives

$$\varepsilon \left(k^2 + l^2 + \frac{1}{Bu} \right) \omega_L^2 + \left(\varepsilon \beta k + i\tau \left(k^2 + l^2 + \frac{2}{Bu} \right) \right) \omega_L + i\beta\tau k = 0. \quad (4.4.10)$$

Solving and expanding for small ε gives the two roots

$$\omega_{L+} = - \frac{\beta k}{k^2 + l^2 + 2/Bu} + O(\varepsilon), \quad (4.4.11)$$

$$\omega_{L-} = - \frac{i}{\varepsilon} \left(\frac{\tau (k^2 + l^2 + 2/Bu)}{k^2 + l^2 + 1/Bu} \right) - \frac{\beta k}{Bu (k^2 + l^2 + 1/Bu) (k^2 + l^2 + 2/Bu)} + O(\varepsilon). \quad (4.4.12)$$

The first of these roots, ω_{L+} , is the dispersion relation for a Rossby wave with a factor of two in the effective Burger number. Figure 4.1 displays the real part (the $O(1)$ term) of ω_{L-} , and shows that ω_{L-} is again the dispersion relation for a Rossby wave. The imaginary part of ω_{L-} (the $O(1/\varepsilon)$ term) is always negative since τ , ε and Bu are all positive, hence these Rossby waves also decay with time due to the inclusion of the radiative term.

4.4.1 Comparison between linear plane waves for the thermal shallow water equations, and the local thermal quasi-geostrophic equations

This section compares waves in the thermal shallow water equations (1.3.12) to those in their local quasi-geostrophic limit (4.2.12). For ease of comparison between the two models, both equation sets are considered on a mid-latitude β -plane, and dissipation due to radiative cooling is neglected.

To find the dispersion relation for the thermal shallow water equations on a mid-latitude β -plane, we set $f = f_0 + \beta^* y$, $h = h_0 + \eta'$, $\Theta = \Theta_0 + \theta'$ and $\mathbf{u} = \mathbf{0} + \mathbf{u}'$ in (1.3.12), with infinite τ_{rad} . Linearising the subsequent equations, under the assumption that the dashed variables denote small perturbations, gives

$$\frac{\partial \eta'}{\partial t} + h_0 \nabla \cdot \mathbf{u}' = 0, \quad (4.4.13a)$$

$$\frac{\partial \theta'}{\partial t} = 0, \quad (4.4.13b)$$

$$\frac{\partial u'}{\partial t} - (f_0 + \beta^* y) v' + \Theta_0 \frac{\partial \eta'}{\partial x} + \frac{h_0}{2} \frac{\partial \theta'}{\partial x} = 0, \quad (4.4.13c)$$

$$\frac{\partial v'}{\partial t} + (f_0 + \beta^* y) u' + \Theta_0 \frac{\partial \eta'}{\partial y} + \frac{h_0}{2} \frac{\partial \theta'}{\partial y} = 0. \quad (4.4.13d)$$

If we seek waves that are harmonic in longitude and time of the form

$\eta' = \text{Re}(\tilde{\eta}(y) \exp(i(kx - \omega t)))$, $u' = \text{Re}(\tilde{u}(y) \exp(i(kx - \omega t)))$, $v' = \text{Re}(\tilde{v}(y) \exp(i(kx - \omega t)))$ and $\theta' = \text{Re}(\tilde{\theta}(y) \exp(i(kx - \omega t)))$, then equations (4.4.13a)–(4.4.13d) may be combined into a single ordinary differential equation for $\tilde{v}(y)$:

$$\frac{d^2 \tilde{v}}{dy^2} = \left(k^2 + \frac{(f_0 + \beta^* y)^2}{\Theta_0 h_0} - \frac{\omega^2}{\Theta_0 h_0} + \frac{k \beta^*}{\omega} \right) \tilde{v}. \quad (4.4.14)$$

Away from the equator $\beta^* y \ll f_0$, and following Zaqqarashvili et al. (2007), equation (4.4.14) reduces to

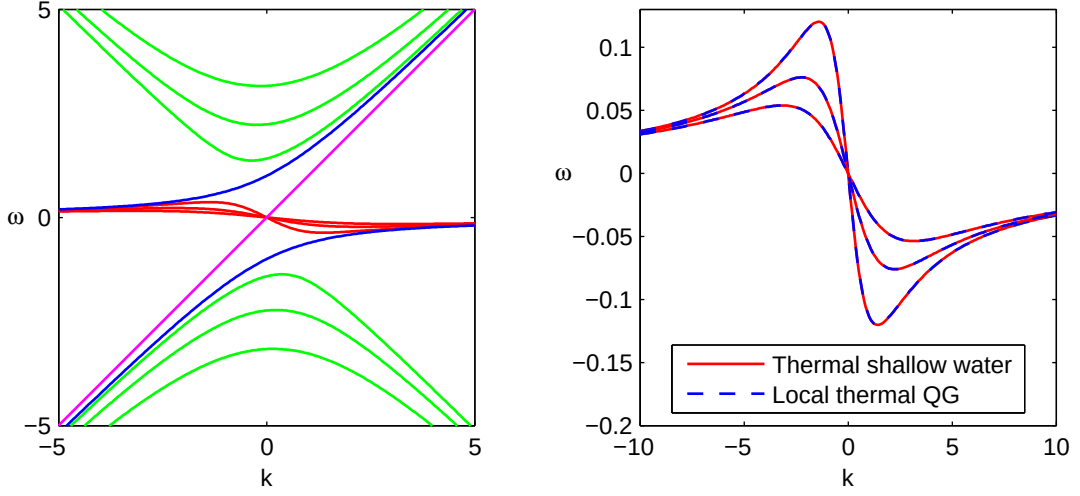
$$\frac{d^2 \tilde{v}}{dy^2} = \left(k^2 + \frac{f_0^2}{\Theta_0 h_0} - \frac{\omega^2}{\Theta_0 h_0} + \frac{k \beta^*}{\omega} \right) \tilde{v}. \quad (4.4.15)$$

Performing a Fourier analysis of the form $\tilde{v} = \hat{v} \exp(i l y)$ gives the dispersion relation

$$\omega^3 - (f_0^2 + \Theta_0 h_0 (k^2 + l^2)) \omega - \beta^* \Theta_0 h_0 k = 0. \quad (4.4.16)$$

Non-dimensionalising by scaling ω with U/L and $\mathbf{k}$ with $1/L$ gives the dimensionless dispersion relation

$$\frac{Ro^2}{Bu} \omega^3 - \left(\frac{1}{Bu} + k^2 + l^2 \right) \omega - \frac{\tilde{\beta}}{Ro} k = 0, \quad (4.4.17)$$



(a) Dispersion relation for the thermal shallow water equations, showing the inertia-gravity waves (green lines), Rossby waves (red lines), Yanai waves (blue lines) and Kelvin wave (pink line).

(b) Dispersion relation for the local thermal quasi-geostrophic equations (blue dashed lines), and the Rossby wave branch of the dispersion relation for the thermal shallow water equations (red solid lines).

Figure 4.2: Dispersion relations for the thermal shallow water equations and the local thermal quasi-geostrophic equations.

where Ro , Bu and $\tilde{\beta}$ are defined in (4.2.5).

To find the equivalent dispersion relation for the local thermal quasi-geostrophic equations, set $\lambda = 0$ in (4.4.6), which gives $\omega = 0$ or

$$\omega = -\frac{\beta k}{k^2 + l^2 + 1/Bu}, \quad (4.4.18)$$

which is the dispersion relation for a Rossby wave.

Figure 4.2a displays the dimensionless dispersion relation for the thermal shallow water equations (4.4.17), where $Ro = Bu = \tilde{\beta} = 1$, and the first few Rossby and inertia-gravity waves ($l = 1, 2, 3$) have been plotted. The Yanai wave has also been plotted, which corresponds to $l = 0$ in the dispersion relation (4.4.17), and joins the Rossby and inertia-gravity wave branches. The top branch resembles a Rossby wave for $k < 0$, and an inertia-gravity wave for $k > 0$. Finally the Kelvin wave has been plotted, which is distinguished by having no meridional velocity ($v = 0$). Its dimensionless frequency is given by $\omega = (\sqrt{Bu}/Ro)k$, thus the Kelvin wave remains nondispersive.

The blue dashed lines in figure 4.2b show the dispersion relation for the local thermal quasi-geostrophic equations (4.4.18). The red solid lines on the same figure

show the Rossby wave part of the dispersion relation for the thermal shallow water equations (4.4.17). The first few Rossby waves ($l = 1, 2, 3$) have been plotted for $Bu = 1$, $Ro = 1/(4\pi)$ and $\tilde{\beta}/Ro = \beta = 0.34$. Thus the quasi-geostrophic limit removes the fast inertia-gravity waves, and figure 4.2b shows that the Rossby waves of the thermal shallow water equations are well approximated by the local quasi-geostrophic dispersion relation.

4.5 Energy and the group velocity property

The local thermal quasi-geostrophic equations (4.2.12) imply the energy conservation law

$$\partial_t E_{\text{QG}} + \nabla \cdot \mathbf{F}_{\text{QG}} = S_{\text{QG}}, \quad (4.5.1)$$

for the energy density and energy flux

$$E_{\text{QG}} = \frac{1}{2} (|\nabla\psi|^2 + \psi^2/Bu), \quad \mathbf{F}_{\text{QG}} = -\psi\nabla\psi_t + \psi^2\hat{\mathbf{z}} \times \nabla(q/2 - \theta/Bu), \quad (4.5.2)$$

and a source term due to radiative cooling,

$$S_{\text{QG}} = -(\lambda/Bu)\psi(\theta + \psi). \quad (4.5.3)$$

This energy equation may be derived by considering $\psi\partial_t(q - \theta/Bu)$. The expression E_{QG} is identical to the energy density for the standard local quasi-geostrophic equations (e.g. Weinstein, 1983), but the relation between ψ and the prognostic variables has changed. For standard local quasi-geostrophic theory, McIntyre and Shepherd (1987) gave the equivalent formulae in the infinite deformation radius ($Bu \rightarrow \infty$) limit, while Pedlosky (1987) and Longuet-Higgins (1964) gave related formulae for linear waves in fluids with a finite deformation radius.

The aim of this section is to show that the local quasi-geostrophic energy (4.5.2) propagates with the group velocity, the same property that was demonstrated for the thermal shallow water equations in §2.6. The approach used in §2.6 is not used here since it is straightforward to find explicit expressions for the frequency and the group velocity for the local quasi-geostrophic equations. Instead the simpler method given in Pedlosky (1987, §3.21) is followed. In this section we set $\lambda = 0$ and let $q = \beta y + q'$, $\psi = 0 + \psi'$ and $\theta = 0 + \theta'$, so (4.2.12) become

$$\frac{\partial q'}{\partial t} + [\psi', q'] + \beta \frac{\partial \psi'}{\partial x} = \frac{1}{Bu} [\psi', \theta'], \quad (4.5.4a)$$

$$\frac{\partial \theta'}{\partial t} + [\psi', \theta'] = 0, \quad (4.5.4b)$$

$$q' = \nabla^2 \psi' - \frac{(\psi' - \theta')}{Bu}. \quad (4.5.4c)$$

Linearising these equations, using the assumption that the dashed variables denote small perturbations, gives

$$\frac{\partial q'}{\partial t} + \beta \frac{\partial \psi'}{\partial x} = 0, \quad (4.5.5a)$$

$$\frac{\partial \theta'}{\partial t} = 0, \quad (4.5.5b)$$

$$q' = \nabla^2 \psi' - \frac{(\psi' - \theta')}{Bu}. \quad (4.5.5c)$$

Consider a solution of the form

$$\begin{aligned} q' &= \text{Re}(\hat{q} \exp(i(kx + ly - \omega t))), & \psi' &= \text{Re}(\hat{\psi} \exp(i(kx + ly - \omega t))), \\ \theta' &= \text{Re}(\hat{\theta} \exp(i(kx + ly - \omega t))), \end{aligned} \quad (4.5.6)$$

where k is the wavenumber in the x direction, l is the wavenumber in the y direction, and $\omega(k, l)$ is the frequency. This gives $\hat{\theta} = 0$ and the dispersion relation for the Rossby waves

$$\omega(k, l) = -\frac{k\beta}{k^2 + l^2 + 1/Bu}. \quad (4.5.7)$$

The components of the group velocity $\mathbf{C}_g = (C_g^x, C_g^y)$ are given by

$$C_g^x = \frac{\partial \omega}{\partial k} = \frac{\beta(k^2 - l^2 - 1/Bu)}{(k^2 + l^2 + 1/Bu)^2}, \quad C_g^y = \frac{\partial \omega}{\partial l} = \frac{2\beta kl}{(k^2 + l^2 + 1/Bu)^2}. \quad (4.5.8)$$

Defining $\langle a \rangle$ to be the average of a over one period of a wave means that oscillating terms vanish on integration. Substituting (4.5.6) into (4.5.2) gives

$$\langle E_{QG} \rangle = \frac{1}{4} \left(k^2 + l^2 + \frac{1}{Bu} \right) |\hat{\psi}|^2, \quad (4.5.9)$$

$$\langle F_{QG}^x \rangle = -\frac{1}{2}(\beta/2 + k\omega) |\hat{\psi}|^2, \quad \langle F_{QG}^y \rangle = -\frac{1}{2}l\omega |\hat{\psi}|^2, \quad (4.5.10)$$

where F_{QG}^x and F_{QG}^y are the x and y components of $\mathbf{F}_{QG}$. Using (4.5.7), we obtain

$$\langle F_{QG}^x \rangle = \frac{\beta(k^2 - l^2 - 1/Bu)}{4(k^2 + l^2 + 1/Bu)} |\hat{\psi}|^2, \quad (4.5.11)$$

$$\langle F_{QG}^y \rangle = \frac{\beta kl}{2(k^2 + l^2 + 1/Bu)} |\hat{\psi}|^2, \quad (4.5.12)$$

which gives the desired relation

$$\mathbf{C}_g \langle E_{QG} \rangle = \langle \mathbf{F}_{QG} \rangle. \quad (4.5.13)$$

Again, the group velocity is the velocity with which the wave energy is transported for the local thermal quasi-geostrophic equations.

4.6 The local quasi-geostrophic limit of the radiatively damped thermal shallow water energy equation

However, equation (4.5.1) is not the local quasi-geostrophic limit of the corresponding energy conservation law

$$\partial_t E + \nabla \cdot \mathbf{F} = S, \quad (4.6.1)$$

for the radiatively damped thermal shallow water equations (1.3.12). The thermal shallow water energy density and flux are

$$E = \frac{1}{2} (h |\mathbf{u}|^2 + \Theta h^2), \quad \mathbf{F} = \mathbf{u} \left(\frac{1}{2} h |\mathbf{u}|^2 + \Theta h^2 \right), \quad (4.6.2)$$

and the source term due to radiative cooling is

$$S = -\frac{1}{2} \frac{h^2}{\tau_{\text{rad}}} \left(\frac{\Theta h}{h_0} - \Theta_0 \right). \quad (4.6.3)$$

Applying the quasi-geostrophic scalings (4.2.1)–(4.2.3) to E , and expanding as in §4.2.2, leads to the expression

$$E(\mathbf{u}, h, \Theta) = E(\mathbf{0}, h_0, \Theta_0) + \frac{h_0 f_0^2 L^2}{2} \left[\epsilon^2 |\tilde{\nabla} \tilde{\psi}|^2 + 2\epsilon(\tilde{\theta} + \tilde{\eta}) + \epsilon^2 \frac{(\tilde{\eta}^2 + 4\tilde{\theta}\tilde{\eta})}{Bu} + O(\epsilon^3) \right], \quad (4.6.4)$$

where $\tilde{\psi}$, $\tilde{\eta}$ and $\tilde{\theta}$ are the dimensionless variables introduced in §4.2. The kinetic energy, proportional to $|\tilde{\nabla} \tilde{\psi}|^2$, is $O(\epsilon^2)$, while the potential energy contains an $O(\epsilon)$ term. The kinetic energy is thus asymptotically smaller than the potential energy, and the potential energy is itself not sign-definite. The energy difference $E(\mathbf{u}, h, \Theta) - E(\mathbf{0}, h_0, \Theta_0)$ is thus not a satisfactory measure of the magnitude of deviations from an underlying basic state with $\mathbf{u} = \mathbf{0}$, $h = h_0$ and $\Theta = \Theta_0$.

4.6.1 Available potential energy and pseudo-energy

This difficulty was first encountered by Lorenz (1955) in defining the potential energy of a disturbance to an ideal gas in hydrostatic balance, and also arises when deriving an acoustic energy equation from the underlying energy equation for a compressible gas (e.g. Lighthill, 1978). Lorenz (1955) combined the mass conservation law with the energy conservation law to yield a new conserved quantity, the kinetic plus available potential energy (APE) that is quadratic to leading order in the disturbance quantities (for reviews see Tailleux, 2013, Winters et al., 1995).

Shepherd (1993) gave a general discussion of the construction of an energy equation for disturbances around an equilibrium $\boldsymbol{\eta}_0$ of a Hamiltonian system, which must satisfy

$$\mathbf{J} \frac{\delta \mathcal{H}}{\delta \boldsymbol{\eta}} = 0 \quad \text{at} \quad \boldsymbol{\eta} = \boldsymbol{\eta}_0. \quad (4.6.5)$$

This condition implies $\delta \mathcal{H} / \delta \boldsymbol{\eta} = 0$ for a canonical Hamiltonian system with an invertible $\mathbf{J}$. The Hamiltonian thus has the expected Taylor expansion $\mathcal{H}(\boldsymbol{\eta}_0 + \delta \boldsymbol{\eta}) = \mathcal{H}(\boldsymbol{\eta}_0) + \delta \boldsymbol{\eta} \cdot \mathcal{H}'(\boldsymbol{\eta}_0) + O(|\delta \boldsymbol{\eta}|^3)$. However, all we may conclude from (4.6.5) for a non-canonical system is

$$\frac{\delta \mathcal{H}}{\delta \boldsymbol{\eta}} = \frac{\delta \mathcal{C}}{\delta \boldsymbol{\eta}} \quad \text{at} \quad \boldsymbol{\eta} = \boldsymbol{\eta}_0, \quad (4.6.6)$$

for some Casimir functional $\mathcal{C}$. Lorenz's (1955) combination of energy conservation with mass conservation thus generalises to finding an additional conservation law, and its associated Casimir $\mathcal{C}$, for which the first variation of the combined functional $\mathcal{H} - \mathcal{C}$, known as the pseudo-energy, vanishes at equilibrium. The same observation underlies the energy-Casimir method for establishing the stability of non-canonical Hamiltonian systems (Arnold, 1965*a,b*, Arnold and Khesin, 1998, Holm et al., 1985, McIntyre and Shepherd, 1987, Shepherd, 1990).

4.6.2 Pseudo-energy for the thermal shallow water equations

Following Shepherd's (1993) algorithm, a pseudo-energy for the thermal shallow water equations is constructed from the solution of

$$\frac{\delta \mathcal{C}}{\delta \mathbf{u}} = \frac{\delta \mathcal{H}}{\delta \mathbf{u}} = \mathbf{0}, \quad \frac{\delta \mathcal{C}}{\delta h} = \frac{\delta \mathcal{H}}{\delta h} = h_0 \Theta_0, \quad \frac{\delta \mathcal{C}}{\delta \Theta} = \frac{\delta \mathcal{H}}{\delta \Theta} = \frac{1}{2} h_0^2, \quad (4.6.7)$$

evaluated at the basic state $\mathbf{u} = \mathbf{0}$, $h = h_0$ and $\Theta = \Theta_0$. A solution is given by

$$\mathcal{C}(h, \Theta) = \int \frac{3}{4} h_0 \Theta h + \frac{3}{8} \Theta_0 h_0 h - \frac{1}{8} \frac{h_0}{\Theta_0} \Theta^2 h \, dx \, dy. \quad (4.6.8)$$

The Π -containing term in (2.5.42) may be omitted because the variational derivatives are evaluated at a rest state with $\mathbf{u} = \mathbf{0}$.

The corresponding disturbance pseudo-energy is

$$\mathcal{A} = \mathcal{H}(\mathbf{u}, h, \Theta) - \mathcal{H}(\mathbf{0}, h_0, \Theta_0) - \mathcal{C}(h, \Theta) + \mathcal{C}(h_0, \Theta_0). \quad (4.6.9)$$

Expanding $\mathcal{A}$ for small disturbances to the background state, and transforming to the dimensionless variables of (4.3.15), gives the local quasi-geostrophic energy at leading order,

$$\mathcal{A} = h_0 f_0^2 L^4 \epsilon^2 \left[\frac{1}{2} \int |\tilde{\nabla} \tilde{\psi}|^2 + \frac{\tilde{\psi}^2}{B u} \, d\tilde{x} \, d\tilde{y} + O(\epsilon) \right]. \quad (4.6.10)$$

The kinetic energy and potential energy are now of the same order in ϵ , and the whole expression is positive definite. Moreover, $\mathcal{A}$ is an exact invariant of the undamped (infinite τ_{rad}) thermal shallow water equations. Initial disturbances to the background state that are small, in the sense that $\mathcal{A}$ is small, thus remain small disturbances in the sense measured by $\mathcal{A}$. However, $\mathcal{A}$ does not control the amplitude of the force-compensated mode, in which the disturbances $\tilde{\theta}$ and $\tilde{\eta}$ to Θ and h respectively occur in antiphase, and so do not alter $\tilde{\psi} = \tilde{\eta} + \tilde{\theta}$.

A second Casimir satisfying the conditions (4.6.7) is

$$\mathcal{C}_R = \int \sqrt{\Theta\Theta_0} h h_0 \, dx \, dy. \quad (4.6.11)$$

The general pseudo-energy definition (4.6.9) then takes the form proposed by Ripa (1995) and Røed (1997),

$$\mathcal{A}_R = \frac{1}{2} \int h |\mathbf{u}|^2 + \left(\sqrt{\Theta} h - \sqrt{\Theta_0} h_0 \right)^2 \, dx \, dy, \quad (4.6.12)$$

which combines the kinetic energy with a measure of the disturbance potential energy. Røed (1997) named the latter “available gravitational energy” following the work of Holliday and McIntyre (1981) and Andrews (1981) for continuously stratified fluids. The available gravitational energy may be computed locally for a fluid column, unlike Lorenz’s (1955) available potential energy that is defined relative to the potential energy minimising rearrangement of the fluid in the whole domain. Expanding $\mathcal{A}_R$ for small deviations in h and Θ , and transforming to dimensionless variables, gives an expression that coincides with the pseudo-energy (4.6.10) at leading order, but this approach has not previously been used in the derivation of a quasi-geostrophic theory. The pseudo-energy (4.6.12) is a sum of squares, while the pseudo-energy defined using the Casimir (4.6.8) only has this property for small disturbances to the background state. In the following $\mathcal{A}_R$ is used instead of the $\mathcal{A}$ constructed from the Casimir in (4.6.8), because $\mathcal{A}_R$ gives simpler intermediate expressions. The final results, which depend only upon expansions truncated at $O(\epsilon^2)$ for small disturbances, are identical whether one uses the Casimir from (4.6.8) or (4.6.11) to construct a pseudo-energy.

4.6.3 The local quasi-geostrophic limit of the pseudo-energy flux for the thermal shallow water equations

The pseudo-energy (4.6.12) may be written as $\mathcal{A}_R = \int E_P(\mathbf{x}, t) \, d\mathbf{x}$. The pseudo-energy density

$$E_P = \frac{1}{2} h |\mathbf{u}|^2 + \frac{1}{2} \left(\sqrt{\Theta} h - \sqrt{\Theta_0} h_0 \right)^2, \quad (4.6.13)$$

satisfies the conservation equation

$$\partial_t E_P + \nabla \cdot \mathbf{F}_P = S_P, \quad (4.6.14)$$

with pseudo-energy flux

$$\mathbf{F}_P = \mathbf{u} \left(\frac{1}{2} h |\mathbf{u}|^2 + \Theta h^2 - \sqrt{\Theta \Theta_0} h h_0 \right), \quad (4.6.15)$$

and a source term due to radiative cooling,

$$S_P = \frac{h}{2\tau_{\text{rad}}} \left(\frac{\Theta h}{h_0} - \Theta_0 \right) \left(\sqrt{\frac{\Theta_0}{\Theta}} h_0 - h \right). \quad (4.6.16)$$

The local quasi-geostrophic scaling is now applied to E_P , $\mathbf{F}_P$, and S_P . The equations (4.2.1)–(4.2.3) are used to non-dimensionalise, and then the tilde notation is dropped with the understanding that all variables and operators in this section are now dimensionless. Expanding as in §4.2.2 with $u_0 = -\partial\psi/\partial y$, $v_0 = \partial\psi/\partial x$ and $\eta = \psi - \theta$, and dividing by U/L gives

$$\frac{\partial E_P}{\partial t} + \nabla \cdot \mathbf{F}_P = S_P, \quad (4.6.17)$$

where

$$E_P = \frac{1}{2} h_0 f_0^2 L^2 \epsilon^2 \left(|\nabla \psi|^2 + \frac{\psi^2}{Bu} + O(\epsilon) \right), \quad (4.6.18a)$$

$$S_P = -h_0 f_0^2 L^2 \epsilon^2 \left(\frac{\lambda}{Bu} \psi (\theta + \psi) + O(\epsilon) \right), \quad (4.6.18b)$$

and

$$\mathbf{F}_P = h_0 f_0^2 L^2 \epsilon^2 \left(\mathbf{u}_0 \left(\frac{1}{2} |\nabla \psi|^2 + \frac{\psi}{\epsilon} + \frac{1}{Bu} (\psi^2 + \psi\theta - \frac{3}{2}\theta^2) \right) + \mathbf{u}_1 \psi + O(\epsilon) \right). \quad (4.6.18c)$$

The term proportional to $\psi \mathbf{u}_0 = -\nabla \times (\hat{\mathbf{z}} \psi^2/2)$ has zero divergence, so may be omitted from $\mathbf{F}_P$. The vanishing of this otherwise asymptotically large term is essential for the rigorous mathematical theory of the quasi-geostrophic limit (Majda, 2003). Eliminating $\mathbf{u}_1$ using the thermal shallow water momentum equations (4.2.9b,c) gives

$$\mathbf{F}_P = h_0 f_0^2 L^2 \epsilon^2 \left(-\psi \nabla \psi_t + \psi^2 \hat{\mathbf{z}} \times \nabla \left(\frac{q}{2} - \frac{\theta}{Bu} \right) + O(\epsilon) \right), \quad (4.6.19)$$

which explains the origin of the time derivative term $\psi \nabla \psi_t$. The local thermal quasi-geostrophic energy conservation equation is thus recovered

$$\frac{\partial}{\partial t} \left(\frac{|\nabla \psi|^2}{2} + \frac{\psi^2}{2Bu} \right) + \nabla \cdot \left(-\psi \nabla \psi_t + \psi^2 \hat{\mathbf{z}} \times \nabla \left(\frac{q}{2} - \frac{\theta}{Bu} \right) \right) = -\frac{\lambda}{Bu} \psi (\theta + \psi) \quad (4.6.20)$$

at leading order in ϵ by applying the local quasi-geostrophic scalings to the thermal shallow water pseudo-energy conservation equation (4.6.14). Although the derivation of the local thermal quasi-geostrophic equations (4.2.12) eliminates the ageostrophic velocity $\mathbf{u}_1$, $\mathbf{u}_1$ must be calculated to obtain the local thermal quasi-geostrophic energy equation as a limit of the corresponding thermal shallow water pseudo-energy equation. Setting $\theta = 0$ and $\lambda = 0$ in (4.6.20) gives the energy conservation law for the local quasi-geostrophic form of the standard shallow water equations (e.g. Pedlosky, 1987). This derivation has some similarities with §2 of McIntyre and Shepherd (1987) if their basic state is set as $\Psi = Q = U = 0$.

4.7 Non-canonical Hamiltonian structure of the local thermal quasi-geostrophic equations

The local thermal quasi-geostrophic equations (4.2.12) with $\lambda = 0$ have a non-canonical Hamiltonian structure, as given by Ripa (1996*b*). The field variables are $\boldsymbol{\eta} = (q, \theta)$, and the Hamiltonian for an infinite domain is

$$\mathcal{H}_{\text{QG}} = \frac{1}{2} \int |\nabla \psi|^2 + \frac{\psi^2}{Bu} \, dx \, dy. \quad (4.7.1)$$

The Hamiltonian for a bounded domain contains additional terms (e.g. Holm, 1986, Ripa, 1993, 1996*b*) that enforce the constraints derived by McWilliams (1977) for the circulations around each connected component of the boundary. Taking variations of $\mathcal{H}_{\text{QG}}$ gives

$$\delta \mathcal{H}_{\text{QG}} = \int \nabla \psi \cdot (\nabla \delta \psi) + \frac{1}{Bu} \psi \delta \psi \, dx \, dy, \quad (4.7.2)$$

$$= \int \left(-\nabla^2 \delta \psi + \frac{1}{Bu} \delta \psi \right) \psi \, dx \, dy, \quad (4.7.3)$$

$$= \int \left(\frac{\delta \theta}{Bu} - \delta q \right) \psi \, dx \, dy, \quad (4.7.4)$$

$$= \int \frac{\delta \mathcal{H}_{\text{QG}}}{\delta \theta} \delta \theta + \frac{\delta \mathcal{H}_{\text{QG}}}{\delta q} \delta q \, dx \, dy, \quad (4.7.5)$$

after an integration by parts, where the integral over the boundary is assumed to vanish. Hence the variational derivatives of $\mathcal{H}_{\text{QG}}$ are

$$\frac{\delta \mathcal{H}_{\text{QG}}}{\delta q} = -\psi, \quad \frac{\delta \mathcal{H}_{\text{QG}}}{\delta \theta} = \frac{\psi}{Bu}. \quad (4.7.6)$$

The evolution equations for q and θ may thus be written in matrix notation as

$$\frac{\partial \boldsymbol{\eta}}{\partial t} = \mathbf{J}_{\text{QG}} \frac{\delta \mathcal{H}_{\text{QG}}}{\delta \boldsymbol{\eta}}, \quad \text{with} \quad \mathbf{J}_{\text{QG}} = - \begin{pmatrix} [q, \cdot] & [\theta, \cdot] \\ [\theta, \cdot] & 0 \end{pmatrix}. \quad (4.7.7)$$

The corresponding Poisson bracket is

$$\{\mathcal{F}, \mathcal{G}\}_{\text{QG}} = \int q \left[\frac{\delta \mathcal{F}}{\delta q}, \frac{\delta \mathcal{G}}{\delta q} \right] + \theta \left(\left[\frac{\delta \mathcal{F}}{\delta q}, \frac{\delta \mathcal{G}}{\delta \theta} \right] + \left[\frac{\delta \mathcal{F}}{\delta \theta}, \frac{\delta \mathcal{G}}{\delta q} \right] \right) dx dy. \quad (4.7.8)$$

Morrison and Hazeltine (1984) derived this Poisson bracket for two-dimensional reduced magnetohydrodynamics, in which θ is the magnetic flux function, verified that it satisfies the Jacobi identity, and found its Casimir functionals

$$\mathcal{C}_{\text{QG}} = \int q F(\theta) + G(\theta) dx dy, \quad (4.7.9)$$

for arbitrary functions $F(\theta)$ and $G(\theta)$. Taking $G(\theta) = 0$, and $F(\theta) = \mathcal{H}(\theta - \theta_*)$ to be the step function that is unity for $\theta \geq \theta_*$ and zero otherwise, (4.7.9) implies that the total potential vorticity inside any contour of constant θ is conserved, in the sense that

$$\frac{d}{dt} \int_{S(\theta_*)} q d\mathbf{x} = 0, \quad (4.7.10)$$

where $S(\theta_*)$ is the material surface bounded by the $\theta = \theta_*$ contour.

4.7.1 Derivation of the local thermal quasi-geostrophic Hamiltonian and Poisson bracket

This Hamiltonian structure is now derived from the pseudo-energy (4.6.12) and Poisson bracket (2.5.40) of the thermal shallow water equations. Bokhove (2002) gave such a derivation for the standard shallow water equations using variables similar to those in §4.3. However, the derivation detailed in this section instead uses the simpler change of variables used by Tassi et al. (2009) for an analogous derivation of the Charney–Hasegawa–Mima equation in plasma physics.

The change of variables $(\mathbf{u}, h, \Theta) \mapsto (\Pi, D, \check{h}, \check{\Theta})$ is made, which is defined by

$$\Pi = \frac{1}{h} \left(f_0 + \beta^* y + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right), \quad D = \nabla \cdot \mathbf{u}, \quad \check{h} = h, \quad \check{\Theta} = \Theta, \quad (4.7.11)$$

which corresponds to a Helmholtz decomposition of $\mathbf{u}$ with potentials $h\Pi$ and D . This decomposition has the unique inverse

$$\mathbf{u} = \hat{\mathbf{z}} \times \nabla \Delta^{-1} (\Pi \check{h} - f_0 - \beta^* y) + \nabla \Delta^{-1} D, \quad h = \check{h}, \quad \Theta = \check{\Theta}, \quad (4.7.12)$$

where the Green's function Δ^{-1} for the two-dimensional Laplacian is defined by

$$\Delta^{-1}F = -\frac{1}{2\pi} \int \ln |\mathbf{x} - \mathbf{x}'| F(\mathbf{x}') d\mathbf{x}'. \quad (4.7.13)$$

The hat notation is introduced for $\check{h}$ and $\check{\Theta}$ so that $\mathcal{F}_{\check{h}} = \delta\mathcal{F}/\delta\check{h}$ denotes the variational derivative of $\mathcal{F}$ with respect to $\check{h}$ keeping Π , D and $\check{\Theta}$ fixed, while $\mathcal{F}_h$ denotes the variational derivative keeping $\mathbf{u}$ and Θ fixed.

In terms of these new variables, the pseudo-energy (4.6.12) becomes

$$\begin{aligned} \mathcal{A}_R = & \int \frac{1}{2} \check{h} (|\nabla \Delta^{-1}(\Pi \check{h} - f_0 - \beta^* y)|^2 + 2 [\Delta^{-1}(\Pi \check{h} - f_0 - \beta^* y), \Delta^{-1} D] + |\nabla \Delta^{-1} D|^2) \\ & + \frac{1}{2} \check{\Theta} \check{h}^2 - \sqrt{\check{\Theta} \Theta_0} \check{h} h_0 + \frac{1}{2} \Theta_0 h_0^2 dx dy, \end{aligned} \quad (4.7.14)$$

using the result $|\hat{\mathbf{z}} \times \nabla a + \nabla b|^2 = |\nabla a|^2 + 2[a, b] + |\nabla b|^2$ for all scalar fields a and b . The Poisson bracket becomes

$$\begin{aligned} \{\mathcal{F}, \mathcal{G}\} = & \int \Pi \{[\mathcal{F}_D, \mathcal{G}_D] + [\mathcal{F}_{\Pi/\check{h}}, \mathcal{G}_{\Pi/\check{h}}] + \nabla \mathcal{F}_D \cdot \nabla (\mathcal{G}_{\Pi/\check{h}}) - \nabla \mathcal{G}_D \cdot \nabla (\mathcal{F}_{\Pi/\check{h}})\} \\ & + (\mathcal{F}_{\check{h}} - \Pi/\check{h} \mathcal{F}_{\Pi}) \nabla^2 \mathcal{G}_D - (\mathcal{G}_{\check{h}} - \Pi/\check{h} \mathcal{G}_{\Pi}) \nabla^2 \mathcal{F}_D \\ & + \frac{1}{\check{h}} \{ \mathcal{G}_{\check{\Theta}} [\check{\Theta}, \mathcal{F}_{\Pi/\check{h}}] - \mathcal{F}_{\check{\Theta}} [\check{\Theta}, \mathcal{G}_{\Pi/\check{h}}] + \mathcal{F}_{\check{\Theta}} \nabla \check{\Theta} \cdot \nabla \mathcal{G}_D - \mathcal{G}_{\check{\Theta}} \nabla \check{\Theta} \cdot \nabla \mathcal{F}_D \} dx dy, \end{aligned} \quad (4.7.15)$$

where $\mathcal{F}_D = \delta\mathcal{F}/\delta D$ etc. The local quasi-geostrophic scalings are now applied to (4.7.14) and (4.7.15) using (4.2.1), scaling h with h_0 and Θ with Θ_0 . This equates to scaling Π with f_0/h_0 , D with U/L , $\check{h}$ with h_0 and $\check{\Theta}$ with Θ_0 . The tilde notation is dropped, with the understanding that all variables and operators are now dimensionless, and the dimensionless variables are decomposed as

$$\mathbf{u} = \mathbf{u}_0 + \epsilon \mathbf{u}_1, \quad h = 1 + \frac{\epsilon}{Bu} \eta, \quad \Theta = 1 + 2 \frac{\epsilon}{Bu} \theta, \quad (4.7.16)$$

where $\epsilon = Ro$, $u_0 = -\partial\psi/\partial y$, $v_0 = \partial\psi/\partial x$, $\eta = \psi - \theta$, and Ro and Bu are defined in (4.2.5). This corresponds to the following decomposition in the dimensionless new variables

$$\Pi = 1 + \epsilon q, \quad D = \epsilon D_1, \quad \check{h} = 1 + \frac{\epsilon}{Bu} \eta, \quad \check{\Theta} = 1 + 2 \frac{\epsilon}{Bu} \theta, \quad (4.7.17)$$

where $q = \nabla^2 \psi - \eta/Bu + \beta y$, $D_1 = \nabla \cdot \mathbf{u}_1$ and $\beta = \tilde{\beta}/\epsilon = \beta^* L^2/U$. Then the pseudo-energy (4.7.14) becomes

$$\mathcal{A}_R = h_0 f_0^2 L^4 \epsilon^2 \left[\frac{1}{2} \int |\nabla \Delta^{-1}(q + \eta/Bu - \beta y)|^2 + \frac{\psi^2}{Bu} dx dy + O(\epsilon) \right], \quad (4.7.18)$$

since $\int [a, b] dx dy = 0$ for all scalar fields a and b for which the integral over the boundary vanishes. Similarly the thermal shallow water Poisson bracket (4.7.15) becomes

$$\begin{aligned} \{\mathcal{F}, \mathcal{G}\} = & \frac{1}{h_0 f_0 L^4} \left[\frac{1}{\epsilon^3} \int q [\mathcal{F}_{D_1}, \mathcal{G}_{D_1}] + \nabla \mathcal{F}_{D_1} \cdot \nabla \mathcal{G}_q - \nabla \mathcal{G}_{D_1} \cdot \nabla \mathcal{F}_q \right. \\ & + (\mathcal{G}_q - Bu \mathcal{G}_\eta) \nabla^2 \mathcal{F}_{D_1} - (\mathcal{F}_q - Bu \mathcal{F}_\eta) \nabla^2 \mathcal{G}_{D_1} dx dy \\ & + \frac{1}{\epsilon^2} \int q \nabla \cdot (\mathcal{G}_q \nabla \mathcal{F}_{D_1} - \mathcal{F}_q \nabla \mathcal{G}_{D_1}) \\ & + \theta \nabla \cdot (\mathcal{G}_\theta \nabla \mathcal{F}_{D_1} - \mathcal{F}_\theta \nabla \mathcal{G}_{D_1}) dx dy \\ & + \frac{1}{\epsilon} \int q [\mathcal{F}_q, \mathcal{G}_q] + \theta ([\mathcal{F}_q, \mathcal{G}_\theta] + [\mathcal{F}_\theta, \mathcal{G}_q]) \\ & + q \nabla \cdot \left(\frac{\eta}{Bu} (\mathcal{F}_q \nabla \mathcal{G}_{D_1} - \mathcal{G}_q \nabla \mathcal{F}_{D_1}) \right) \\ & \left. + \theta \nabla \cdot \left(\frac{\eta}{Bu} (\mathcal{F}_\theta \nabla \mathcal{G}_{D_1} - \mathcal{G}_\theta \nabla \mathcal{F}_{D_1}) \right) dx dy + O(1) \right], \quad (4.7.19) \end{aligned}$$

using the transformations

$$\begin{aligned} \frac{\delta \mathcal{F}}{\delta \Pi} &= \frac{h_0}{\epsilon f_0 L^2} \frac{\delta \mathcal{F}}{\delta q}, \quad \frac{\delta \mathcal{F}}{\delta D} = \frac{1}{\epsilon U L} \frac{\delta \mathcal{F}}{\delta D_1}, \quad \frac{\delta \mathcal{F}}{\delta \tilde{h}} = \frac{Bu}{\epsilon h_0 L^2} \frac{\delta \mathcal{F}}{\delta \eta}, \\ \frac{\delta \mathcal{F}}{\delta \tilde{\Theta}} &= \frac{Bu}{2\epsilon \Theta_0 L^2} \frac{\delta \mathcal{F}}{\delta \theta}. \end{aligned} \quad (4.7.20)$$

The evolution of any functional $\mathcal{F}$ is given by $d\mathcal{F}/dt = \{\mathcal{F}, \mathcal{H}\} = \{\mathcal{F}, \mathcal{A}_R\}$, since $\mathcal{A}_R$ only differs from the Hamiltonian $\mathcal{H}$ by the addition of a Casimir. Non-dimensionalising as above, and dropping the tildes, we obtain the dimensionless equation

$$\frac{d\mathcal{F}_{\text{nd}}}{dt_{\text{nd}}} = \{\mathcal{F}_{\text{nd}}, \mathcal{A}_{\text{nd}}\}_{\text{nd}}, \quad (4.7.21)$$

where $\mathcal{A}_{\text{nd}} = \mathcal{A}_R / (h_0 f_0^2 L^4 \epsilon^2)$ is the dimensionless pseudo-energy, $\{\cdot, \cdot\}_{\text{nd}} = \epsilon h_0 f_0 L^4 \{\cdot, \cdot\}$ is the dimensionless Poisson bracket, $t_{\text{nd}} = (U/L)t$ is the dimensionless time, and $\mathcal{F}_{\text{nd}}$ is a dimensionless functional. Retaining only the leading-order term in $\mathcal{A}_{\text{nd}}$ gives the dimensionless local quasi-geostrophic Hamiltonian,

$$\mathcal{A}_{\text{nd}} = \frac{1}{2} \int |\nabla \Delta^{-1} (q + \eta/Bu - \beta y)|^2 + \frac{\psi^2}{Bu} dx dy + O(\epsilon) = \mathcal{H}_{\text{QG}} + O(\epsilon), \quad (4.7.22)$$

since $q + \eta/Bu - \beta y = \nabla^2 \psi$. Since $\delta \mathcal{A}_{\text{nd}} / \delta D_1 = 0$, all terms in $\{\cdot, \cdot\}_{\text{nd}}$ that contain variational derivatives with respect to D_1 may be neglected. This removes many otherwise formally large terms at $O(\epsilon^{-2})$ and $O(\epsilon^{-1})$ from the dimensionless Poisson bracket $\{\cdot, \cdot\}_{\text{nd}}$. Retaining only the largest surviving term gives

$$\{\mathcal{F}, \mathcal{G}\}_{\text{QG}} = \int q [\mathcal{F}_q, \mathcal{G}_q] + \theta ([\mathcal{F}_q, \mathcal{G}_\theta] + [\mathcal{F}_\theta, \mathcal{G}_q]) dx dy, \quad (4.7.23)$$

which is synonymous with the Poisson bracket (4.7.8) for the local thermal quasi-geostrophic equations.

4.8 Summary

In this chapter I have developed the theory of the local quasi-geostrophic limit of the radiatively damped thermal shallow water equations on a mid-latitude β -plane, as originally derived by Ripa (1996*b*). This chapter provides a formal derivation of the local thermal quasi-geostrophic equations as an expansion in a small Rossby number, an expansion that retains additional terms needed for the interpretation of the local thermal quasi-geostrophic energy equation. A more accurate nonlinear balanced model is also derived by projecting onto the slow normal modes of the thermal shallow water equations. This balanced model was shown to reduce to the local thermal quasi-geostrophic theory for small disturbances. The global quasi-geostrophic form of the radiatively damped thermal shallow water equations is developed in chapter 5, i.e. a quasi-geostrophic limit that is valid for the entire planet.

Radiative cooling was shown to damp the Rossby waves in the local thermal quasi-geostrophic equations, as was also demonstrated for linear plane waves in the radiatively damped thermal shallow water equations in §2.7.1. Waves in the thermal shallow water equations were then compared to those in their local quasi-geostrophic limit. The quasi-geostrophic limit was shown to remove the fast inertia-gravity waves, and the Rossby waves of the thermal shallow water equations were shown to be well approximated by the corresponding local thermal quasi-geostrophic dispersion relation.

The group velocity was shown to be the velocity with which the wave energy is transported for the local thermal quasi-geostrophic equations. The local thermal quasi-geostrophic energy equation was then derived from the local quasi-geostrophic limit of the conservation of pseudo-energy equation for the radiatively damped thermal shallow water equations. The local thermal quasi-geostrophic evolution equations form a closed system that does not involve the ageostrophic velocity $\mathbf{u}_1$. However, $\mathbf{u}_1$ needed to be calculated to interpret the local quasi-geostrophic energy flux as the limit of a pseudo-energy flux for the thermal shallow water equations. Finally, a Hamiltonian and Poisson bracket for the local thermal quasi-geostrophic equations was derived using a small Rossby number decomposition of a pseudo-energy and Poisson bracket for the thermal shallow water equations. The local thermal quasi-geostrophic theory shares the non-canonical Hamiltonian structure of two-dimensional reduced magnetohydrodynamics (Morrison and Hazeltine, 1984) and Gilman's (1967) quasi-geostrophic magnetohydrodynamics, further cementing the parallels between

the thermal and magnetohydrodynamic extensions of the shallow water equations (Dellar, 2003).

Chapter 5

The global quasi-geostrophic theory of the standard shallow water equations, the height-damped shallow water equations, and the radiatively damped thermal shallow water equations

5.1 Introduction

Inertia-gravity waves evolve on time scales much shorter than those of vortical motions. The idea behind quasi-geostrophic theory is to exploit this separation of time scales to produce a simplified balanced model, whilst retaining the time evolution omitted from purely geostrophic flow. It is attractive for numerical simulation because the time step is no longer limited by the large velocity of inertia-gravity waves. The presence of only quadratic nonlinearities makes quasi-geostrophic equations amenable to the use of cumulant expansions to study zonally symmetric mean flows in geostrophic turbulence on spheres and β -planes (Farrell and Ioannou, 1993, 2003, 2007, Marston et al., 2008, Srinivasan and Young, 2012, Tobias et al., 2011).

Chapter 4 developed the local quasi-geostrophic theory of the radiatively damped thermal shallow water equations on a mid-latitude β -plane, where x , y and z were treated as Cartesian coordinates, while f was approximated by $f_0 + \beta^*y$ to capture the latitude-dependence of the Coriolis parameter in spherical geometry. A formal derivation was presented involving an asymptotic expansion for small Rossby number.

A second derivation was then offered, which involved projecting the dynamics onto the slow normal modes of the thermal shallow water equations. In this interpretation, the local thermal quasi-geostrophic equations may be considered to be a balanced model (Allen and Newberger, 1993, Bokhove, 2002, McWilliams and Gent, 1980, Mohebalhojeh and McIntyre, 2007, White, 2002). These two derivations filtered out the inertia-gravity waves, replacing them with an elliptic relation for the geostrophic streamfunction ψ in terms of the potential vorticity q and temperature θ :

$$q = \nabla^2 \psi - \frac{(\psi - \theta)}{Bu} + \beta y. \quad (5.1.1)$$

However, a global quasi-geostrophic theory for the radiatively damped thermal shallow water equations is desirable in order to provide a simple framework in which to study zonal jets and the direction of the equatorial jet. The first derivation of the local thermal quasi-geostrophic equations in chapter 4, which involved performing an asymptotic expansion for small Ro number, does not make sense near the equator where f_0 becomes close to zero and so the Rossby number becomes formally large. However the second derivation, where a balanced model is developed, offers a route towards developing a quasi-geostrophic theory that is valid for the global domain. A global quasi-geostrophic theory has already been derived for a fully stratified atmosphere by Kuo (1959) and Charney and Stern (1962), and more recently from the standard shallow water equations by Theiss (2004), Schubert et al. (2009) and Verkley (2009). Williams (1985) derived a general geostrophic equation from the standard shallow water equations on a sphere, where the quasi-geostrophic regime was shown to be a subset of this general geostrophic equation (also see McCalpin, 1995). Theiss and Mohebalhojeh (2009) derived a balanced model on an equatorial β -plane from the standard shallow water equations, using a method similar to that used in §4.3. This equatorial balanced model represents an equatorial counterpart of the standard local quasi-geostrophic equations on a mid-latitude β -plane. The Verkley (2009) derivation assumes that the height field may be written as a mean value plus a small perturbation. Conservation of mass then implies that the divergence of the flow is small, thus $\mathbf{u}$ is approximated by a divergence-free velocity and a streamfunction ψ is introduced. Verkley then obtains a geostrophic balance relation $g'\eta = f\psi$ on linearising the divergence equation and using the fact that $\nabla \cdot \mathbf{u}$ is small and f is slowly varying. Given ψ , this balance equation is solved for η (it does not make sense to solve for ψ given η since $f = 0$ at the equator). Finally the global quasi-geostrophic potential vorticity equation is determined from the standard shallow water potential vorticity equation $\partial \Pi / \partial t + (\mathbf{u} \cdot \nabla) \Pi = 0$, where $\Pi = h^{-1}(f + \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u})$, by linearising

the definition of Π and dictating that the advection be given by the non-divergent velocity as defined by ψ . The global standard quasi-geostrophic equations are then given by

$$\frac{\partial q}{\partial t} + [\psi, q] = 0, \quad q = \left(\nabla^2 - \frac{f^2}{Bu} \right) \psi + \frac{f}{Ro}, \quad (5.1.2)$$

using the same notation as the remainder of this chapter, where $[\psi, q] = \hat{\mathbf{z}} \cdot \nabla \psi \times \nabla q$.

In this chapter I derive a global quasi-geostrophic version of the radiatively damped thermal shallow water equations, which are referred to as the global thermal quasi-geostrophic equations. A derivation of a global quasi-geostrophic version of the Scott and Polvani (2007, 2008) equations is also presented, and the resulting equation set is referred to as the global height-damped quasi-geostrophic equations. The global standard quasi-geostrophic equations are the infinite radiative time scale limit of the global height-damped quasi-geostrophic equations. These global quasi-geostrophic equation sets are valid for the whole sphere and for the square planet domain described in §1.4.2. An expansion for small Rossby number is performed, where the Rossby number is defined as $U/2\Omega L$ (this is equivalent to the mid-latitude β -plane definition of the Rossby number at the poles). The effect of radiative cooling on equatorially trapped waves in the global thermal and height-damped quasi-geostrophic equations is examined, where both equation sets are considered on an equatorial β -plane. The group velocity is demonstrated to be the velocity with which the wave energy is transported for the global thermal and height-damped quasi-geostrophic equations in the absence of radiative cooling. The Hamiltonian and Poisson bracket for the global thermal quasi-geostrophic equations are derived from a small Rossby number decomposition of a pseudo-energy and Poisson bracket for the thermal shallow water equations.

Jovian global quasi-geostrophic simulations are performed, for both the thermal and height-damped cases, for a range of radiative time scales. The equatorial jet is shown to switch from super- to sub-rotating as the radiative time scale is increased. The models are solved in the doubly-periodic Cartesian geometry described in §1.4.2, and compared to corresponding shallow water simulations in the Jovian regime in order to determine similarities and differences between shallow water and global quasi-geostrophic models. For these global quasi-geostrophic simulations, the potential vorticity equation must be solved at each time step for the geostrophic streamfunction. However, Fourier transforming the algebraic expression for the streamfunction (5.1.2) leads to a suite of pentadiagonal linear systems to be solved due to the $f(y)$ dependence (the equivalent expression for the local thermal quasi-geostrophic limit (5.1.1) is diagonalised by Fourier transforming). Considering the odd and even Fourier

modes in the y direction separately leads to two tridiagonal systems to be solved for each Fourier mode in the x direction (this is similar to the method used in Spatz, 2000, cf. §1.4). Efficient simulations of these global quasi-geostrophic models thus require the development of a Thomas algorithm. The Thomas algorithm is sequential, but efficient GPU code must be parallel, so I have developed a Thomas algorithm that works in parallel across the different Fourier modes in the x direction on GPUs. Another possibility would have been to use cyclic reduction, however this method is not efficient for the size of problems solved in this chapter (for a review of the cyclic reduction method see Gander and Golub, 1997).

5.2 Derivation of the global thermal quasi-geostrophic equations

5.2.1 Potential vorticity and horizontal divergence

This section presents the global quasi-geostrophic form of the radiatively damped thermal shallow water equations:

$$\frac{\partial h}{\partial t} + \nabla \cdot (h \mathbf{u}) = 0, \quad (5.2.1a)$$

$$\frac{\partial \Theta}{\partial t} + (\mathbf{u} \cdot \nabla) \Theta = -(\Theta h / h_0 - \Theta_0) / \tau_{\text{rad}}, \quad (5.2.1b)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + f \hat{\mathbf{z}} \times \mathbf{u} = -\nabla (\Theta h) + \frac{1}{2} h \nabla \Theta. \quad (5.2.1c)$$

Following Verkley (2009), the momentum equations (5.2.1c) are rewritten as

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left(\frac{1}{2} |\mathbf{u}|^2 \right) + (\nabla \times \mathbf{u}) \times \mathbf{u} + f \hat{\mathbf{z}} \times \mathbf{u} = -\nabla (\Theta h) + \frac{1}{2} h \nabla \Theta. \quad (5.2.2)$$

Taking the curl and horizontal divergence of (5.2.2) gives

$$\frac{\partial \omega}{\partial t} + \nabla \cdot ((\omega + f) \mathbf{u}) = \frac{1}{2} \hat{\mathbf{z}} \cdot (\nabla h \times \nabla \Theta), \quad (5.2.3a)$$

$$\frac{\partial D}{\partial t} + \nabla \cdot ((\omega + f) \hat{\mathbf{z}} \times \mathbf{u} + \Theta \nabla h + \frac{1}{2} h \nabla \Theta + \nabla \left(\frac{1}{2} |\mathbf{u}|^2 \right)) = 0, \quad (5.2.3b)$$

where $\omega = \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u}$ is the relative vorticity, and $D = \nabla \cdot \mathbf{u}$ is the horizontal divergence. The horizontal divergence of $\mathbf{u}$ is eliminated from (5.2.3a) using (5.2.1a), to give the thermal shallow water potential vorticity equation

$$\frac{\partial \Pi}{\partial t} + (\mathbf{u} \cdot \nabla) \Pi = \frac{1}{2h} \hat{\mathbf{z}} \cdot (\nabla h \times \nabla \Theta), \quad \text{where} \quad \Pi = \frac{f + \omega}{h}. \quad (5.2.4)$$

The source term on the right-hand side may be derived from the depth average of the baroclinic torque in a continuously stratified fluid (see §2.3.2).

5.2.2 The quasi-geostrophic scaling

Consider the equation set given by (5.2.1a), (5.2.1b), (5.2.3b) and (5.2.4). A non-dimensionalisation similar to that in §4.2.1 is introduced

$$\mathbf{x} = L\tilde{\mathbf{x}}, \quad \mathbf{u} = U\tilde{\mathbf{u}}, \quad t = (L/U)\tilde{t}, \quad f = 2\Omega\tilde{f}, \quad (5.2.5)$$

where L and U are horizontal length and velocity scales respectively, and Ω is the angular velocity. The depth and temperature of the active layer are written as

$$h = h_0 \left(1 + \frac{2\Omega UL}{\Theta_0 h_0} \tilde{\eta} \right), \quad \Theta = \Theta_0 \left(1 + 2 \frac{2\Omega UL}{\Theta_0 h_0} \tilde{\theta} \right). \quad (5.2.6)$$

Finally, the horizontal divergence and potential vorticity are non-dimensionalised by

$$D = \frac{U}{L} \tilde{D}, \quad \Pi = \frac{U}{L h_0} \tilde{\Pi}. \quad (5.2.7)$$

Substituting these scalings into (5.2.1a), (5.2.1b), (5.2.3b) and (5.2.4), and dropping the tilde notation with the understanding that all variables and operators are dimensionless, we obtain

$$Ro \left(\frac{\partial \eta}{\partial t} + \nabla \cdot (\eta \mathbf{u}) \right) + Bu \nabla \cdot \mathbf{u} = 0, \quad (5.2.8a)$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla) \theta = -2\lambda \left(\theta + \frac{1}{2} \eta + \frac{Ro}{Bu} \eta \theta \right), \quad (5.2.8b)$$

$$\frac{\partial \Pi}{\partial t} + (\mathbf{u} \cdot \nabla) \Pi = \frac{1}{Bu} \left(\hat{\mathbf{z}} \cdot (\nabla \eta \times \nabla \theta) \right), \quad (5.2.8c)$$

$$Ro \frac{\partial D}{\partial t} + \nabla \cdot \left((Ro \omega + f) \hat{\mathbf{z}} \times \mathbf{u} + \nabla (\eta + \theta) + \frac{Ro}{Bu} (2\theta \nabla \eta + \eta \nabla \theta) + Ro \nabla \left(\frac{1}{2} |\mathbf{u}|^2 \right) \right) = 0. \quad (5.2.8d)$$

The dimensionless shallow water potential vorticity is now,

$$\Pi = \frac{f/Ro + \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u}}{1 + (Ro/Bu)\eta}, \quad (5.2.9)$$

and the dimensionless parameters are

$$Ro = \frac{U}{2\Omega L}, \quad Bu = \frac{\Theta_0 h_0}{(2\Omega)^2 L^2} \quad \text{and} \quad \lambda = \frac{L}{2U\tau_{\text{rad}}}. \quad (5.2.10)$$

The scaling in (5.2.6) for η is now recognised as $h_0 Ro/Bu$, the scaling for θ is now recognised as $2\Theta_0 Ro/Bu$, while the scaling for Π in (5.2.7) can be rewritten as $(2\Omega/h_0)Ro$ (see page 140 for an explanation of this scaling).

5.2.3 Quasi-geostrophic balance

The Rossby number is set as $Ro = \epsilon$, where ϵ is a small parameter, and Bu and λ are assumed to remain $O(1)$ as $\epsilon \rightarrow 0$. The variables $\mathbf{u}$, η , θ and D are assumed to be $O(1)$ field variables as $\epsilon \rightarrow 0$. Expanding the definition of Π for small ϵ gives

$$\Pi = \frac{f}{\epsilon} + \left(\hat{\mathbf{z}} \cdot \nabla \times \mathbf{u} - \frac{f\eta}{Bu} \right) + O(\epsilon) = q + O(\epsilon), \quad (5.2.11)$$

where q is defined to be the shallow water potential vorticity truncated at $O(1)$,

$$q = \frac{f}{\epsilon} + \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u} - \frac{f\eta}{Bu}. \quad (5.2.12)$$

Truncating the equation set (5.2.8) at $O(1)$ gives

$$\nabla \cdot \mathbf{u} = 0, \quad (5.2.13a)$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla) \theta = -2\lambda \left(\theta + \frac{1}{2}\eta \right), \quad (5.2.13b)$$

$$\frac{\partial q}{\partial t} + (\mathbf{u} \cdot \nabla) q = \frac{1}{Bu} \hat{\mathbf{z}} \cdot (\nabla \eta \times \nabla \theta), \quad (5.2.13c)$$

$$\nabla \cdot (f\hat{\mathbf{z}} \times \mathbf{u} + \nabla(\eta + \theta)) = 0. \quad (5.2.13d)$$

Since from (5.2.13a) the horizontal velocity divergence is zero, a streamfunction ψ is introduced such that

$$u = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \psi}{\partial x}. \quad (5.2.14)$$

Substituting (5.2.14) into (5.2.13d) gives the balance condition

$$\nabla^2 (-f\psi + \eta + \theta) + \nabla \cdot (\psi \nabla f) = 0. \quad (5.2.15)$$

The variables ψ , η and θ vary on some length scale L which is taken to be the deformation radius $L_D = \sqrt{\Theta_0 h_0}/2\Omega$, while f varies on the much larger planetary radius, thus f can be assumed to be slowly varying, and so the last term in (5.2.15) can be neglected. Integrating the resulting equation and setting the arbitrary constant of integration to zero, gives the geostrophic relation

$$f\psi = \eta + \theta. \quad (5.2.16)$$

This relation generalises the f -plane geostrophic relation to the global domain. Given ψ and θ this relation can be used to find η .

Using (5.2.14) to eliminate $\mathbf{u}$ in favour of the streamfunction ψ from (5.2.13b,c), and using (5.2.16) to eliminate η , gives the global thermal quasi-geostrophic equations

$$\frac{\partial q}{\partial t} + [\psi, q] = \frac{1}{Bu} [f\psi, \theta], \quad (5.2.17a)$$

$$\frac{\partial \theta}{\partial t} + [\psi, \theta] = -\lambda (\theta + f\psi), \quad (5.2.17b)$$

where $[a, b] = \hat{\mathbf{z}} \cdot \nabla a \times \nabla b$, and the global thermal quasi-geostrophic potential vorticity is

$$q = \left(\nabla^2 - \frac{f^2}{Bu} \right) \psi + \frac{f}{Bu} \theta + \frac{f}{Ro}. \quad (5.2.17c)$$

These equations reduce to the thermal quasi-geostrophic equations on an f -plane when f is set to unity (to see this set $\beta = 0$ in (4.2.12)). The streamfunction ψ can be obtained from q and θ by inverting (5.2.17c). Note that the definition of the global thermal quasi-geostrophic potential vorticity contains a term of the form f/ϵ . In the global thermal quasi-geostrophic equations, the inclusion of this term only affects the left-hand side of (5.2.17a) by adding a term of the form $\psi_x f_y / \epsilon$. In the standard derivation to obtain local quasi-geostrophic equations on a mid-latitude β -plane, one is required to assume that fractional changes in the Coriolis parameter are small on the horizontal length scale L of the flow (Salmon, 1998). In the global domain this is equivalent to assuming that $df/dy = O(\epsilon)$, so the term $\psi_x f_y / \epsilon$ in (5.2.17a) may be considered to be an $O(1)$ term. This also justifies neglecting the last term in (5.2.15). More systematically, equations can be derived where the leading order dynamics occur on the synoptic scale $L = L_D$, while f is only considered to vary on the global scale a , so $f = f(\epsilon y)$, where $\epsilon = L_D/a \ll 1$ (as done in Theiss, 2004). Assuming that $\epsilon \approx \epsilon$ gives $df/dy = O(\epsilon)$. This same scaling gives $\beta = O(1)$ when deriving local quasi-geostrophic equations on mid-latitude β -planes, as in §4.2.2.

5.3 Derivation of the global height-damped quasi-geostrophic equations

In this section the corresponding global quasi-geostrophic form of the Scott and Polvani (2007, 2008) height-damped shallow water model:

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = -(h - h_0) / \tau_{\text{rad}}, \quad (5.3.1a)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + f \hat{\mathbf{z}} \times \mathbf{u} = -g' \nabla h, \quad (5.3.1b)$$

is derived. Rewriting (5.3.1b) in terms of the horizontal velocity divergence D and the shallow water potential vorticity Π , and non-dimensionalising as in §5.2.2 with Θ_0 replaced by the reduced gravity g' , gives the dimensionless equation set

$$Ro \left(\frac{\partial \eta}{\partial t} + \nabla \cdot (\eta \mathbf{u}) \right) + Bu \nabla \cdot \mathbf{u} = -2\lambda Ro \eta, \quad (5.3.2a)$$

$$\frac{\partial \Pi}{\partial t} + (\mathbf{u} \cdot \nabla) \Pi = \frac{2\lambda(Ro/Bu)\Pi\eta}{1 + (Ro/Bu)\eta}, \quad (5.3.2b)$$

$$Ro \frac{\partial D}{\partial t} + \nabla \cdot ((Ro \omega + f) \hat{\mathbf{z}} \times \mathbf{u} + \nabla \eta + Ro \nabla \left(\frac{1}{2} |\mathbf{u}|^2 \right)) = 0, \quad (5.3.2c)$$

where the dimensionless shallow water potential vorticity is defined in (5.2.9).

The global quasi-geostrophic form of (5.3.2) is derived following the same procedure as §5.2.3. However, the balance relation is now given by $f\psi = \eta$. The right-hand side of (5.3.2b) must also be expanded, bearing in mind the expansion of the shallow water potential vorticity (5.2.11), to give $2\lambda f\eta/Bu$ at $O(1)$. On eliminating $\mathbf{u}$ and η in favour of the streamfunction ψ , we obtain the global height-damped quasi-geostrophic equations

$$\frac{\partial q}{\partial t} + [\psi, q] = \frac{2\lambda}{Bu} f^2 \psi, \quad (5.3.3a)$$

$$q = \left(\nabla^2 - \frac{f^2}{Bu} \right) \psi + \frac{f}{Ro}, \quad (5.3.3b)$$

where the global height-damped quasi-geostrophic potential vorticity is the same as (5.2.17c) on setting $\theta = 0$. The radiative damping thus appears as a hypoviscous dissipation proportional to ψ with coefficient proportional to f^2 , so it becomes ineffective at low latitudes (cf. §3.3.1). The global standard quasi-geostrophic equations are then given by setting $\lambda = 0$ in (5.3.3). The global quasi-geostrophic form of the momentum-conserving height-damped shallow water equations (1.3.11, with $\mathcal{H} \equiv 1$) is the same as for the height-damped shallow water equations (cf. §3.3.1 where the local quasi-geostrophic limit of both equation sets were considered on an f -plane).

5.4 Equatorially trapped waves

To investigate the properties of the global thermal quasi-geostrophic equations (5.2.17), the effect of radiative cooling on the equatorially trapped waves is considered. The effect of radiative cooling on linear plane waves about a rest state was previously considered for the radiatively damped thermal shallow water equations with constant f in §2.7.1, and for the local thermal quasi-geostrophic equations on a mid-latitude β -plane in §4.4. If the global thermal quasi-geostrophic equations are considered on a

mid-latitude β -plane, then after assuming $\beta^*y \ll f_0$ away from the equator, a dispersion relation is obtained which is identical to (4.4.6) on redefining the dimensionless constants.

This section instead considers equatorially trapped waves on an equatorial β -plane (see §1.4.1). The dimensionless traditional Coriolis parameter is expanded around the equator to give $f(y) \approx \beta y$. The variables are expanded as $q = f(y)/Ro + q'$, $\theta = 0 + \theta'$ and $\psi = 0 + \psi'$, where the dashed variables are assumed to denote small perturbations. The linearised equations are then given by

$$\begin{aligned} \frac{\partial q'}{\partial t} + \frac{\beta}{Ro} \frac{\partial \psi'}{\partial x} &= 0, \quad \frac{\partial \theta'}{\partial t} = -\lambda (\theta' + \beta y \psi'), \\ q' &= \nabla^2 \psi' - \frac{(\beta y)^2}{Bu} \psi' + \frac{\beta y}{Bu} \theta'. \end{aligned} \quad (5.4.1)$$

Waves are sought which are harmonic in longitude and time of the form $q'(x, y, t) = \text{Re}\{\hat{q}(y) \exp(i(kx - \omega t))\}$, $\theta'(x, y, t) = \text{Re}\{\hat{\theta}(y) \exp(i(kx - \omega t))\}$ and $\psi'(x, y, t) = \text{Re}\{\hat{\psi}(y) \exp(i(kx - \omega t))\}$. The three resulting equations may then be combined into a single ordinary differential equations for $\hat{\psi}(y)$ of the form

$$\frac{d^2 \hat{\psi}}{dy^2} = (Ay^2 - B)\hat{\psi}, \quad A = \frac{\beta^2}{Bu} \left(\frac{\omega + 2i\lambda}{\omega + i\lambda} \right), \quad B = -k^2 - \frac{\beta}{Ro} \frac{k}{\omega}, \quad (5.4.2)$$

the same equation that governs a quantum harmonic oscillator. The solutions that decay as $y \rightarrow \pm\infty$ may be written using the Hermite polynomials $H_n(\xi)$ as $\hat{\psi} = H_n(\xi) \exp(-\xi^2/2)$, where $\xi = yA^{1/4}$. The dispersion relation $A^{-1/2}B = 2n + 1$ for $n = 1, 2, \dots$ implies

$$-\frac{\sqrt{Bu}}{\beta} \left(\frac{\omega + i\lambda}{\omega + 2i\lambda} \right)^{1/2} \left(k^2 + \frac{\beta}{Ro} \frac{k}{\omega} \right) = 2n + 1. \quad (5.4.3)$$

Considering the case where λ is small, λ is set to be $\varepsilon\tau$, where $0 < \varepsilon \ll 1$, in (5.4.3), which gives

$$\sqrt{Bu} (\omega + i\tau\varepsilon)^{1/2} \left(\omega k^2 + \frac{\beta}{Ro} k \right) + \beta (2n + 1) \omega (\omega + 2i\tau\varepsilon)^{1/2} = 0. \quad (5.4.4)$$

When $\varepsilon = 0$, the two roots

$$\omega_R = -\frac{\beta k}{Ro \left(k^2 + \beta(2n + 1)/\sqrt{Bu} \right)} \quad \text{and} \quad \omega_0 = 0 \quad (5.4.5)$$

are obtained. The first root represents an equatorially trapped Rossby mode and the second root represents an equatorially trapped thermal, or force-compensated,

mode (see §4.3). This thermal mode does not exist for the global height-damped quasi-geostrophic equations. Calculating the perturbed Rossby mode at $O(\varepsilon)$ gives

$$\omega_R = -\frac{\beta k}{Ro \left(k^2 + \beta(2n+1)/\sqrt{Bu} \right)} - i\varepsilon\tau \frac{\beta(2n+1)}{2\sqrt{Bu} \left(k^2 + \beta(2n+1)/\sqrt{Bu} \right)} + O(\varepsilon^2), \quad (5.4.6)$$

so the equatorially trapped Rossby wave is damped by the radiative cooling term. Perturbing the thermal mode to $O(\varepsilon)$ gives,

$$\omega_0 = 0 - i\varepsilon\tau + O(\varepsilon^2), \quad (5.4.7)$$

so the additional equatorially trapped thermal wave is also damped by the radiative cooling term.

Performing the same analysis for the global height-damped quasi-geostrophic equations from §5.3 gives just the one root

$$\omega_R = -\frac{\beta k}{Ro \left(k^2 + \beta(2n+1)/\sqrt{Bu} \right)} - i\varepsilon\tau \frac{\beta(2n+1)}{\sqrt{Bu} \left(k^2 + \beta(2n+1)/\sqrt{Bu} \right)} + O(\varepsilon^2) \quad (5.4.8)$$

which represents an equatorially trapped Rossby wave, again damped by the inclusion of radiative cooling. In chapter 6 the effect of the radiative cooling term on equatorially trapped waves is considered in the radiatively damped thermal shallow water equations, and in the height-damped shallow water equations, on an equatorial β -plane.

5.5 Energy and the group velocity property

The global thermal quasi-geostrophic equations (5.2.17) imply the energy conservation law

$$\partial_t E_{QG} + \nabla \cdot \mathbf{F}_{QG} = S_{QG}, \quad (5.5.1)$$

for the energy density and flux

$$E_{QG} = \frac{1}{2} (|\nabla\psi|^2 + f^2\psi^2/Bu), \quad (5.5.2a)$$

$$\mathbf{F}_{QG} = -\psi\nabla\psi_t + \psi^2 (\hat{\mathbf{z}} \times \nabla(q/2) - (f/Bu)\hat{\mathbf{z}} \times \nabla\theta), \quad (5.5.2b)$$

and a source term due to radiative cooling,

$$S_{QG} = -(\lambda f/Bu)\psi(\theta + f\psi). \quad (5.5.3)$$

This reduces to the local thermal quasi-geostrophic energy conservation law (4.5.1)–(4.5.3) on setting $f = 1$ in (5.5.1)–(5.5.3). The local thermal quasi-geostrophic energy conservation law was previously derived from the pseudo-energy conservation law (4.6.14) for the radiatively damped thermal shallow water equations. However, this derivation required calculating the $O(\epsilon)$ correction ($\mathbf{u}_1$) to geostrophic flow, which was not required to derive the closed set of quasi-geostrophic evolution equations. A similar derivation is not provided here because the global quasi-geostrophic theory is thought of as a balance model instead of an expansion in Rossby number.

The global height-damped quasi-geostrophic equations (5.3.3) also imply an energy conservation law of the form (5.5.1), where E_{QG} and $\mathbf{F}_{\text{QG}}$ are defined by setting $\theta = 0$ in (5.5.2), and

$$S_{\text{QG}} = -2(\lambda f^2 / Bu) \psi^2, \quad (5.5.4)$$

which is twice (5.5.3) on setting $\theta = 0$. Setting $\lambda = 0$ in the global height-damped quasi-geostrophic energy conservation law gives the energy conservation law for the global standard quasi-geostrophic equations.

The global thermal quasi-geostrophic energy density (5.5.2a) is now shown to propagate with the group velocity for Rossby waves. This property was demonstrated for the thermal shallow water equations in §2.6, and for the local thermal quasi-geostrophic equations in §4.5. The approach used in §2.6 and Masuda (1978) is followed. The traditional Coriolis parameter f is assumed to be an arbitrary function of y , subject to the condition that it behaves sufficiently well for normal modes to exist, and the domain is considered to be either periodic or extend to infinity in the y direction. The boundary conditions that ψ and $\partial\psi/\partial y$ are periodic in y , or that ψ and $\partial\psi/\partial y$ tend to zero as $y \rightarrow \pm\infty$ must be imposed, depending on the domain under consideration. Dissipation due to radiative cooling is neglected for the remainder of this section.

First consider $\lambda = 0$, $q = f(y)/Ro + q'$, $\psi = 0 + \psi'$ and $\theta = 0 + \theta'$ in (5.2.17), and assume that dashed variables denote small perturbations, to obtain the analogue of (5.4.1) for general $f(y)$,

$$\begin{aligned} \frac{\partial q'}{\partial t} + \frac{1}{Ro} \frac{df}{dy} \frac{\partial \psi'}{\partial x} &= 0, \quad \frac{\partial \theta'}{\partial t} = 0, \\ q' &= \nabla^2 \psi' - \frac{f(y)^2}{Bu} \psi' + \frac{f(y)}{Bu} \theta'. \end{aligned} \quad (5.5.5)$$

A normal mode solution of the form

$$\psi' = \tilde{\psi}(y) \cos(kx - \omega t) \quad \text{and} \quad \theta' = \tilde{\theta}(y) \cos(kx - \omega t) \quad (5.5.6)$$

is considered, and from (5.5.5) it is possible to deduce that

$$\tilde{\theta} = 0, \quad q' = \left(\frac{d^2 \tilde{\psi}}{dy^2} - \left(k^2 + \frac{f(y)^2}{Bu} \right) \tilde{\psi} \right) \cos(kx - \omega t), \quad (5.5.7)$$

and thus obtain the normal mode equation

$$\omega \left(\frac{d^2 \tilde{\psi}}{dy^2} - \left(k^2 + \frac{f(y)^2}{Bu} \right) \tilde{\psi} \right) - \frac{k}{Ro} \frac{df}{dy} \tilde{\psi} = 0, \quad (5.5.8)$$

which is in Sturm–Liouville form. Since $\tilde{\theta} = 0$ and $\lambda = 0$, this derivation also holds for the global standard quasi-geostrophic equations, i.e. for the global height-damped quasi-geostrophic equations when radiative cooling is neglected. Following an approach introduced by Chandrasekhar (1961) for hydrodynamic stability calculations, (5.5.8) is multiplied by $\tilde{\psi}$ and integrated over y to obtain the integral formula

$$\int \omega \left(\frac{d\tilde{\psi}}{dy} \right)^2 + \omega \left(k^2 + \frac{f(y)^2}{Bu} \right) \tilde{\psi}^2 + \frac{k}{Ro} \frac{df}{dy} \tilde{\psi}^2 dy = \left[\omega \tilde{\psi} \frac{d\tilde{\psi}}{dy} \right]. \quad (5.5.9)$$

The term in square brackets vanishes due to the boundary conditions. To find an expression for the group velocity, (5.5.9) is perturbed by considering a variation δk of k , and a corresponding variation $\delta \omega$ of ω ,

$$\begin{aligned} & \delta \omega \int \left(\frac{d\tilde{\psi}}{dy} \right)^2 + \left(k^2 + \frac{f(y)^2}{Bu} \right) \tilde{\psi}^2 dy + \delta k \int 2k\omega \tilde{\psi}^2 + \frac{1}{Ro} \frac{df}{dy} \tilde{\psi}^2 dy \\ & + \delta \tilde{\psi} \int 2\omega \left(k^2 + \frac{f(y)^2}{Bu} \right) \tilde{\psi} + \frac{2k}{Ro} \frac{df}{dy} \tilde{\psi} - 2\omega \frac{d^2 \tilde{\psi}}{dy^2} dy + \left[2\omega \frac{d\tilde{\psi}}{dy} \delta \tilde{\psi} \right] = 0. \end{aligned} \quad (5.5.10)$$

The term in square brackets vanishes again due to the boundary conditions. The integral multiplied by $\delta \tilde{\psi}$ vanishes due to (5.5.8), leaving

$$\frac{\partial \omega}{\partial k} \left\{ \frac{1}{4} \int \left(\frac{d\tilde{\psi}}{dy} \right)^2 + \left(k^2 + \frac{f(y)^2}{Bu} \right) \tilde{\psi}^2 dy \right\} = - \int \frac{k\omega}{2} \tilde{\psi}^2 + \frac{1}{4Ro} \frac{df}{dy} \tilde{\psi}^2 dy. \quad (5.5.11)$$

The energy equation for the linearised global quasi-geostrophic equations (5.5.5) with $\theta' = 0$ and $\lambda = 0$ is

$$\begin{aligned} & \frac{\partial E_L}{\partial t} + \nabla \cdot \mathbf{F}_L = 0, \quad \text{where} \quad E_L = \frac{1}{2} \left(|\nabla \psi'|^2 + \frac{f(y)^2}{Bu} \psi'^2 \right), \\ & \mathbf{F}_L = (F_L^x, F_L^y) = \left(-\psi' \frac{\partial^2 \psi'}{\partial x \partial t} - \frac{1}{Ro} \frac{df}{dy} \frac{\psi'^2}{2}, -\psi' \frac{\partial^2 \psi'}{\partial y \partial t} \right). \end{aligned} \quad (5.5.12)$$

This equation holds for the linearised global standard, and thermal, quasi-geostrophic equations since $\theta = 0$ and $\lambda = 0$. Then using the definition $\psi' = \tilde{\psi}(y) \cos(kx - \omega t)$, gives

$$\begin{aligned}\langle E_L \rangle &= \int \overline{E_L} dy = \frac{1}{4} \int \left(\frac{d\tilde{\psi}}{dy} \right)^2 + \left(k^2 + \frac{f(y)^2}{Bu} \right) \tilde{\psi}^2 dy, \\ \langle F_L^x \rangle &= \int \overline{F_L^x} dy = - \int \frac{k\omega}{2} \tilde{\psi}^2 + \frac{1}{4Ro} \frac{df}{dy} \tilde{\psi}^2 dy,\end{aligned}\tag{5.5.13}$$

where $\bar{a}$ denotes the average of a quantity over one period of a wave. Comparing (5.5.11) with (5.5.13) establishes

$$C_g^x \langle E_L \rangle = \langle F_L^x \rangle,\tag{5.5.14}$$

where $C_g^x = \partial\omega/\partial k$ is the group velocity in the x direction. The group velocity in the y direction, C_g^y , is zero since these waves are trapped in the y direction (cf. chapter 6 where equatorially trapped waves in the height-damped shallow water equations and the radiatively damped thermal shallow water equations are considered). Hence the group velocity may be regarded as the velocity with which the wave energy is transported for the global standard, height-damped and thermal quasi-geostrophic equations, when dissipation due to radiative cooling is neglected.

5.6 Non-canonical Hamiltonian structure of the global thermal quasi-geostrophic equations

The non-canonical Hamiltonian structure of the global thermal quasi-geostrophic equations (5.2.17) is outlined, where dissipation by radiation is neglected throughout this section. The Hamiltonian and Poisson bracket of the global thermal quasi-geostrophic equations are shown to be derivable from a pseudo-energy and Poisson bracket of the thermal shallow water equations, as was shown for the local thermal quasi-geostrophic equations in chapter 4. Since Noether's theorem enables conservation laws to be related to symmetries, by making approximations to the thermal shallow water equations that preserve its symmetries, the global thermal quasi-geostrophic equations automatically possess equivalent conservation laws.

For the global thermal quasi-geostrophic equations (5.2.17) with $\lambda = 0$, the field variables are $\boldsymbol{\eta} = (q, \theta)$, and the Hamiltonian for an infinite domain is given by

$$\mathcal{H}_{QG} = \frac{1}{2} \int |\nabla \psi|^2 + \frac{f^2 \psi^2}{Bu} dS,\tag{5.6.1}$$

where dS is an area element which is equivalent to $dx dy$ in Cartesian coordinates, however the following results also hold in spherical geometry. The Hamiltonian (5.6.1) differs from its local quasi-geostrophic form, as defined in (4.7.1), due to the inclusion of the latitudinally varying f^2 term. The variational derivatives of $\mathcal{H}_{\text{QG}}$ are

$$\frac{\delta \mathcal{H}_{\text{QG}}}{\delta q} = -\psi, \quad \frac{\delta \mathcal{H}_{\text{QG}}}{\delta \theta} = \frac{f\psi}{Bu}, \quad (5.6.2)$$

and the global thermal quasi-geostrophic evolution equations may be written in matrix notation as

$$\frac{\partial \boldsymbol{\eta}}{\partial t} = \mathbf{J}_{\text{QG}} \frac{\delta \mathcal{H}_{\text{QG}}}{\delta \boldsymbol{\eta}}, \quad \text{with} \quad \mathbf{J}_{\text{QG}} = - \begin{pmatrix} [q, \cdot] & [\theta, \cdot] \\ [\theta, \cdot] & 0 \end{pmatrix}, \quad (5.6.3)$$

where $[a, b] = \hat{\mathbf{z}} \cdot \nabla a \times \nabla b$. The Poisson bracket is defined by

$$\{\mathcal{F}, \mathcal{G}\}_{\text{QG}} = \int q \left[\frac{\delta \mathcal{F}}{\delta q}, \frac{\delta \mathcal{G}}{\delta q} \right] + \theta \left(\left[\frac{\delta \mathcal{F}}{\delta q}, \frac{\delta \mathcal{G}}{\delta \theta} \right] + \left[\frac{\delta \mathcal{F}}{\delta \theta}, \frac{\delta \mathcal{G}}{\delta q} \right] \right) dS. \quad (5.6.4)$$

Both the Poisson tensor $\mathbf{J}_{\text{QG}}$ and the Poisson bracket $\{\cdot, \cdot\}_{\text{QG}}$ are identical to those given in §4.7 for the local quasi-geostrophic limit of the thermal shallow water equations, although q and $\mathcal{H}_{\text{QG}}$ are defined differently for the global quasi-geostrophic form.

The Casimir functionals of this Poisson bracket are

$$\mathcal{C}_{\text{QG}} = \int q F(\theta) + G(\theta) dS, \quad (5.6.5)$$

for arbitrary functions F and G (Morrison and Hazeltine, 1984). Taking $G(\theta) = 0$ and $F(\theta) = \mathcal{H}(\theta - \theta_*)$ in (5.6.5), shows that the total potential vorticity inside any contour of constant θ is also conserved for the global thermal quasi-geostrophic equations, in the sense that

$$\frac{d}{dt} \int_{S(\theta_*)} q dS = 0, \quad (5.6.6)$$

where $S(\theta_*)$ is the material surface bounded by the $\theta = \theta_*$ contour (cf. (4.7.10) where the total potential vorticity inside any contour of constant θ is also shown to be conserved for the local thermal quasi-geostrophic equations).

5.6.1 Derivation of the global thermal quasi-geostrophic Hamiltonian and Poisson bracket

The Hamiltonian structure of the global thermal quasi-geostrophic equations is derived from that of the thermal shallow water equations, following the same procedure

as that used in chapter 4 where an equivalent derivation for the local thermal quasi-geostrophic equations was presented. As was shown in §4.6, a pseudo-energy of the thermal shallow water equations must be considered, as opposed to the Hamiltonian, when forming such a quasi-geostrophic limit. An additional conservation law is considered, and its associated Casimir $\mathcal{C}$, for which the first variation of $\mathcal{H} - \mathcal{C}$, known as the pseudo-energy, vanishes at the basic state $\mathbf{u} = \mathbf{0}$, $h = h_0$ and $\Theta = \Theta_0$. As in §4.7.1, the form of the pseudo-energy proposed by Ripa (1995) and Røed (1997) is used

$$\mathcal{A}_R = \frac{1}{2} \int h |\mathbf{u}|^2 + \left(\sqrt{\Theta} h - \sqrt{\Theta_0} h_0 \right)^2 dS, \quad (5.6.7)$$

which combines the kinetic energy with a measure of the disturbance potential energy. Then $\mathcal{A}_R$ only differs from the Hamiltonian $\mathcal{H}$ of the thermal shallow water equations by the Casimir $\mathcal{C} = \int \sqrt{\Theta \Theta_0} h h_0 - \Theta_0 h_0^2 / 2 dS$.

As in §4.7.1 a change of variables is made $(\mathbf{u}, h, \Theta) \mapsto (\Pi, D, \check{h}, \check{\Theta})$ defined by

$$\Pi = \frac{f + \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u}}{h}, \quad D = \nabla \cdot \mathbf{u}, \quad \check{h} = h, \quad \check{\Theta} = \Theta. \quad (5.6.8)$$

This decomposition has the unique inverse

$$\mathbf{u} = \hat{\mathbf{z}} \times \nabla \Delta^{-1} (\Pi \check{h} - f) + \nabla \Delta^{-1} D, \quad h = \check{h}, \quad \Theta = \check{\Theta}, \quad (5.6.9)$$

where Δ^{-1} is the inverse of the Laplace–Beltrami operator on a sphere. Substituting (5.6.9) into the pseudo-energy (5.6.7) gives

$$\begin{aligned} \mathcal{A}_R = & \int \frac{1}{2} \check{h} (|\nabla \Delta^{-1} (\Pi \check{h} - f)|^2 + 2 [\Delta^{-1} (\Pi \check{h} - f), \Delta^{-1} D] + |\nabla \Delta^{-1} D|^2) \\ & + \frac{1}{2} \check{\Theta} \check{h}^2 - \sqrt{\check{\Theta} \Theta_0} \check{h} h_0 + \frac{1}{2} \Theta_0 h_0^2 dS, \end{aligned} \quad (5.6.10)$$

and substituting the above change of variables into the thermal shallow water Poisson bracket again gives (4.7.15), where $\mathcal{F}_{\check{h}} = \delta \mathcal{F} / \delta \check{h}$ denotes the variational derivative of $\mathcal{F}$ with respect to $\check{h}$ keeping Π , D and $\check{\Theta}$ fixed, while $\mathcal{F}_h$ denotes the variational derivative keeping $\mathbf{u}$ and Θ fixed. To see where (4.7.15) arises from requires the use of

$$\delta \Pi = \frac{\hat{\mathbf{z}} \cdot \nabla \times \delta \mathbf{u} - \Pi \delta h}{h}, \quad \delta D = \nabla \cdot \delta \mathbf{u}, \quad \delta \check{h} = \delta h, \quad \text{and} \quad \delta \check{\Theta} = \delta \Theta. \quad (5.6.11)$$

Substituting these expressions into the defining property of variational derivatives,

$$\begin{aligned} \delta \mathcal{F} &= \int \mathcal{F}_u \delta u + \mathcal{F}_v \delta v + \mathcal{F}_h \delta h + \mathcal{F}_\Theta \delta \Theta dS, \\ &= \int \mathcal{F}_\Pi \delta \Pi + \mathcal{F}_D \delta D + \mathcal{F}_{\check{h}} \delta \check{h} + \mathcal{F}_{\check{\Theta}} \delta \check{\Theta} dS, \end{aligned} \quad (5.6.12)$$

gives the variational derivatives of a functional $\mathcal{F}$ with respect to $\mathbf{u}$, h and Θ in terms of the variational derivatives of $\mathcal{F}$ with respect to Π , D , $\check{h}$ and $\check{\Theta}$:

$$\mathcal{F}_{\mathbf{u}} = -\nabla \mathcal{F}_D + \nabla \times (\mathcal{F}_{\Pi}/\check{h} \hat{\mathbf{z}}), \quad \mathcal{F}_h = \mathcal{F}_{\check{h}} - (\Pi/\check{h}) \mathcal{F}_{\Pi}, \quad \mathcal{F}_{\Theta} = \mathcal{F}_{\check{\Theta}}. \quad (5.6.13)$$

Equations (5.6.10) and (4.7.15) are non-dimensionalised using the scalings in (5.2.5), and scaling h with its mean value h_0 and Θ with its mean value Θ_0 . This equates to scaling Π with $2\Omega/h_0$, D with U/L , $\check{h}$ with h_0 , and $\check{\Theta}$ with Θ_0 . The tilde notation is neglected with the assumption that all variables and operators are dimensionless. The quasi-geostrophic form is then obtained by assuming that $u = -\partial\psi/\partial y$ and $v = \partial\psi/\partial x$ so that $D = 0$, and decomposing the other dimensionless variables as

$$\Pi = \epsilon q = \epsilon \left(\frac{f}{\epsilon} + \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u} - \frac{f\eta}{Bu} \right), \quad \check{h} = 1 + \frac{\epsilon}{Bu} \eta, \quad \check{\Theta} = 1 + 2 \frac{\epsilon}{Bu} \theta, \quad (5.6.14)$$

where $f\psi = \eta + \theta$ and $\epsilon = Ro$, with Ro and Bu defined in (5.2.10). On substituting this decomposition into the dimensionless pseudo-energy, $\mathcal{A}_{\text{nd}}$, we obtain

$$\mathcal{A}_{\text{nd}} = \frac{\mathcal{A}_{\text{R}}}{h_0(2\Omega)^2 L^4 \epsilon^2} = \frac{1}{2} \int |\nabla \psi|^2 + \frac{f^2 \psi^2}{Bu} + O(\epsilon) \, dS, \quad (5.6.15)$$

where $\mathcal{A}_{\text{R}}$ is defined in (5.6.10). Since $D = 0$, all variational derivatives with respect to D are neglected in the dimensionless Poisson bracket, then introducing the decomposition gives

$$\begin{aligned} \{\mathcal{F}, \mathcal{G}\}_{\text{nd}} &= h_0(2\Omega)L^4\epsilon \{\mathcal{F}, \mathcal{G}\} \\ &= \int q [\mathcal{F}_q, \mathcal{G}_q] + \theta ([\mathcal{F}_q, \mathcal{G}_{\theta}] + [\mathcal{F}_{\theta}, \mathcal{G}_q]) + O(\epsilon) \, dS. \end{aligned} \quad (5.6.16)$$

The evolution of a functional $\mathcal{F}$ is determined by $d\mathcal{F}/dt = \{\mathcal{F}, \mathcal{H}\} = \{\mathcal{F}, \mathcal{A}_{\text{R}}\}$, since $\mathcal{A}_{\text{R}}$ only differs from $\mathcal{H}$ by a Casimir. Non-dimensionalising this equation gives $d\mathcal{F}_{\text{nd}}/dt_{\text{nd}} = \{\mathcal{F}_{\text{nd}}, \mathcal{A}_{\text{nd}}\}_{\text{nd}}$, where $t_{\text{nd}} = (U/L)t$ is the dimensionless time, and $\mathcal{F}_{\text{nd}}$ is a dimensionless functional. Retaining only the leading order term in $\mathcal{A}_{\text{nd}}$ gives the global thermal quasi-geostrophic Hamiltonian (5.6.1), and retaining only the leading order term in $\{\cdot, \cdot\}_{\text{nd}}$ gives the global thermal quasi-geostrophic Poisson bracket (5.6.4).

5.7 Numerical formulation for simulations of the global quasi-geostrophic models

Numerical solutions of the global thermal quasi-geostrophic model (5.2.17) and the global height-damped quasi-geostrophic model (5.3.3) are computed for Jovian parameter values in §5.8, where a range of values for the dimensionless radiative cooling

rate λ are considered. The simulation results are compared to corresponding simulations from chapter 3 for the standard shallow water model, the height-damped shallow water model and the radiatively damped thermal shallow water model.

The simulations of the two global quasi-geostrophic models are made as similar to the corresponding Jovian shallow water model simulations in §3.3 as possible. The deformation radius is taken as the horizontal length scale $L = L_D$, where $L_D = \sqrt{\Theta_0 h_0} / (2\Omega)$, and $g' = \Theta_0$ for the global height-damped quasi-geostrophic model. Comparing the radius of Jupiter (as defined by a in table 3.1) with the deformation radius, it is found that there are approximately 232 deformation radii around the circumference of the planet. The planetary rotation period $T = 2\pi/\Omega$ is taken as the time scale, so the horizontal velocity scale is $U = L_D/T = L_D\Omega/(2\pi)$. Given these assumptions, the dimensionless parameters are set as $Bu = 1$ and $Ro = 1/(4\pi)$. The Jovian simulations in §3.3 set $\tau_{\text{rad}} = 429$ hours (or 43 Jovian days), which corresponds to a dimensionless radiative cooling rate of $\lambda = 0.01153$. Simulation results for this value of λ are considered for both the global thermal and height-damped quasi-geostrophic models. Simulation results for $\lambda = 0.001$ and $\lambda = 0.1$ are also considered, which correspond to radiative cooling time scales of $\tau_{\text{rad}} = 4950$ hours (or 500 Jovian days) and $\tau_{\text{rad}} = 50$ hours (or 5 Jovian days) respectively. Rayleigh friction is included to dissipate energy at large-scales with a time scale τ_{fric} of 9900 hours (or 1000 Jovian days).

The equations sets are solved in the same doubly-periodic Cartesian geometry used in chapter 3, where further details of this square planet domain are given in §1.4.2. This domain enables the superior memory bandwidth and floating point performance of GPUs to be exploited, using a spectral spatial representation based on fast Fourier transforms. The dimensionless Coriolis parameter is still fully varying in latitude, and is defined in the simulations by $f(y) = \cos(2\pi y/y_{\text{max}})$, where y_{max} is the dimensionless length of the side of the domain.

The models are discretised using a standard pseudo-spectral technique on a grid of 1024×1024 Fourier collocation points. The large system of ordinary differential equations are then integrated using the standard fourth-order Runge–Kutta scheme. The Hou and Li (2007) spectral filter is again used to prevent the build-up of enstrophy at the highest resolved wavenumbers. The flow is driven by applying an isotropic random forcing F to $q - f(y)/Ro$. The forcing is chosen to be comparable to that used in the shallow water simulations in §3.3, and is localised to a narrow annulus of wavenumbers $|\mathbf{k}| \in [k_c - 2, k_c + 2]$ centred around wavenumber $k_c = 42$. Each mode is forced with amplitude ϵ_f using random phases that are δ -correlated (white)

in time, where the MT19937 variant of the Mersenne Twister algorithm (Matsumoto and Nishimura, 1998) is used to generate these uniformly distributed random phases. The forcing amplitude ϵ_f is adjusted at each time step so that the kinetic energy $\int \int (\psi_x^2 + \psi_y^2)/2 \, dx \, dy$ roughly follows the profile $\mathcal{E}(t) = \mathcal{E}_\infty \tanh(t/t_{\text{ramp}})$ to mimic the forcing used in §3.2.4, where $\mathcal{E}_\infty = 2 \times 10^4$, $t_{\text{ramp}} = 1000$ Jovian days and $t_{\text{adj}} = 50$ Jovian days in the quasi-geostrophic simulations (see §3.2.4 for further details of this forcing). Detailed simulation results for one particular value of the seed for the random number generator are presented. In tables 5.1 and 5.2 the repeatability of the results is considered by comparing various statistics of the flow for two runs with different initial random seeds.

5.7.1 Potential vorticity inversion

Solving quasi-geostrophic equations numerically requires the geostrophic streamfunction ψ to be computed from the potential vorticity equation. For the local thermal quasi-geostrophic equations the defining equation for potential vorticity (5.1.1) may be rewritten as

$$\tilde{q} = \nabla^2 \psi - \frac{(\psi - \theta)}{Bu}, \quad (5.7.1)$$

where $\tilde{q} = q - \beta y$. Then Fourier transforming in x and y gives

$$\left(k^2 + l^2 + \frac{1}{Bu}\right) \hat{\psi}(k, l) = -\hat{q}(k, l) + \frac{1}{Bu} \hat{\theta}(k, l) \quad (5.7.2)$$

which is in diagonal form, where $\hat{q}$ denotes the Fourier transform of $\tilde{q}$, $\hat{\theta}$ denotes the Fourier transform of θ and $\hat{\psi}$ denotes the Fourier transform of ψ . Thus the geostrophic streamfunction may be easily reconstructed from the instantaneous potential vorticity and temperature fields.

For the global thermal quasi-geostrophic equations the situation is not so simple. The definition of potential vorticity (5.2.17c) may be rewritten as

$$\tilde{q} = \nabla^2 \psi - \frac{f(y)^2}{Bu} \psi + \frac{f(y)}{Bu} \theta, \quad (5.7.3)$$

where $\tilde{q} = q - f(y)/Ro$. Theiss (2004) solved the equivalent equation in his global quasi-geostrophic model by Fourier transforming in the x direction and solving the resulting ordinary differential equation in the y direction with a finite difference scheme. If (5.7.3) is Fourier transformed in both the x and y directions instead, we obtain

$$\begin{aligned} \frac{1}{4Bu} \hat{\psi}(k, l-2) + \left(k^2 + l^2 + \frac{1}{2Bu}\right) \hat{\psi}(k, l) + \frac{1}{4Bu} \hat{\psi}(k, l+2) \\ = -\hat{q}(k, l) + \frac{1}{2Bu} \hat{\theta}(k, l-1) + \frac{1}{2Bu} \hat{\theta}(k, l+1), \end{aligned} \quad (5.7.4)$$

since $f(y) = \cos(2\pi y/y_{\max})$ and $f(y)^2 = 1/2 + (1/2)\cos(4\pi y/y_{\max})$. Given instantaneous values for the potential vorticity and temperature, this equation must be solved at each time step for the geostrophic streamfunction ψ . Fourier transforming the potential vorticity equation has led to a suite of pentadiagonal linear systems to be solved for ψ . To solve this system efficiently the odd and even Fourier modes in the y direction are considered separately (this is similar to the method used in Spatz, 2000, cf. §1.4). There are now two tridiagonal linear systems to solve in order to find ψ for each Fourier mode in the x direction, as opposed to one pentadiagonal linear system. I have developed a Thomas algorithm for GPUs to solve these tridiagonal systems in parallel across the different Fourier modes in the x direction.

The Thomas algorithm (e.g. see Morton and Mayers, 1994) is used to solve tridiagonal systems of the form

$$\begin{pmatrix} b_0 & c_0 & & & 0 \\ a_1 & b_1 & c_1 & & \\ & a_2 & b_2 & \ddots & \\ & & \ddots & \ddots & c_{N-2} \\ 0 & & & a_{N-1} & b_{N-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-1} \end{pmatrix}, \quad (5.7.5)$$

where the coefficients a_i , b_i and c_i , and the right-hand sides d_i , are all given. The Thomas algorithm finds the solution $\mathbf{x} = (x_0, x_1, \dots, x_{N-1})$ in $O(N)$ operations, as opposed to the $O(N^3)$ operations required by a general Gaussian elimination or LU factorisation. The first step in the algorithm removes the a_i coefficients, and forms new coefficients c'_i and d'_i , where

$$c'_i = \begin{cases} \frac{c_i}{b_i} & \text{for } i = 0 \\ \frac{c_i}{b_i - c'_{i-1}a_i} & \text{for } i = 1, \dots, N-2 \end{cases} \quad (5.7.6)$$

and

$$d'_i = \begin{cases} \frac{d_i}{b_i} & \text{for } i = 0 \\ \frac{d_i - d'_{i-1}a_i}{b_i - c'_{i-1}a_i} & \text{for } i = 1, \dots, N-1 \end{cases} \quad (5.7.7)$$

where $b_0 \neq 0$ and $b_i - c'_{i-1}a_i \neq 0$ for $i = 1, \dots, N-1$. To elucidate this consider the first equation in (5.7.5) and divide by b_0 to obtain $x_0 + (c_0/b_0)x_1 = (d_0/b_0)$, i.e. $x_0 + c'_0x_1 = d'_0$. If the first j equations are assumed to have been reduced to

$$x_{i-1} + c'_{i-1}x_i = d'_{i-1} \quad \text{for } i = 1, \dots, j, \quad (5.7.8)$$

and the next equation is still in its original form

$$a_j x_{j-1} + b_j x_j + c_j x_{j+1} = d_j, \quad (5.7.9)$$

then eliminating x_{j-1} by substituting (5.7.8) for $i = j$ into (5.7.9), and dividing by $b_j - c'_{j-1}a_j$, gives an equation for just x_j and x_{j+1}

$$x_j + \frac{c_j}{b_j - c'_{j-1}a_j} x_{j+1} = \frac{d_j - d'_{j-1}a_j}{b_j - c'_{j-1}a_j}. \quad (5.7.10)$$

Comparing with $x_j + c'_j x_{j+1} = d'_j$ gives (5.7.6) and (5.7.7). The tridiagonal system is now of the form

$$\begin{pmatrix} 1 & c'_0 & & & 0 \\ & 1 & c'_1 & & \\ & & 1 & \ddots & \\ & & & \ddots & c'_{N-2} \\ 0 & & & & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{pmatrix} = \begin{pmatrix} d'_0 \\ d'_1 \\ d'_2 \\ \vdots \\ d'_{N-1} \end{pmatrix}, \quad (5.7.11)$$

and $\mathbf{x} = (x_0, x_1, \dots, x_{N-1})$ is solved for by back substitution

$$x_{N-1} = d'_{N-1}, \quad (5.7.12a)$$

$$x_i = d'_i - c'_i x_{i+1} \quad \text{for } i = N-2, \dots, 0. \quad (5.7.12b)$$

These steps are all sequential, but efficient GPU code must be parallel, so I have developed a Thomas algorithm to solve these tridiagonal systems in parallel across the different Fourier modes in the x direction on GPUs. The code is initialised by inputting the elements of the matrices on the host for each Fourier mode k in the x direction, where $\mathbf{M}_{k,\text{even}}$ denotes the matrices for the even Fourier modes in the y direction, and $\mathbf{M}_{k,\text{odd}}$ denotes the matrices for the odd Fourier modes in the y direction. Throughout this section, a subscript k denotes the particular x Fourier mode, and the subscripts “even” and “odd” distinguish between the tridiagonal systems for even and odd Fourier modes in the y direction respectively. The a_{k,even_i} , b_{k,even_i} and c_{k,even_i} from (5.7.5) are given by

$$a_{k,\text{even}_i} = 1/(4Bu), \quad \text{for } i = 1, \dots, N-1, \quad (5.7.13a)$$

$$b_{k,\text{even}_i} = 1/(2Bu) + k^2 + l_{\text{even}_i}^2, \quad \text{for } i = 0, \dots, N-1, \quad (5.7.13b)$$

$$c_{k,\text{even}_i} = 1/(4Bu), \quad \text{for } i = 0, \dots, N-2, \quad (5.7.13c)$$

where $N = 512$ for the simulation results in §5.8, and l_{even_i} are the even multiples of $(2\pi/y_{\text{max}})$ from $-512(2\pi/y_{\text{max}})$ to $510(2\pi/y_{\text{max}})$ for $i = 0, \dots, N-1$. The a_{k,odd_i} and c_{k,odd_i} are the same as the even case, while

$$b_{k,\text{odd}_i} = 1/(2Bu) + k^2 + l_{\text{odd}_i}^2, \quad \text{for } i = 0, \dots, N-1, \quad (5.7.14)$$

where l_{odd_i} are the odd multiples of $(2\pi/y_{\text{max}})$ from $-511(2\pi/y_{\text{max}})$ to $511(2\pi/y_{\text{max}})$ for $i = 0, \dots, N-1$. These expressions for the matrix coefficients are then copied to the GPU. These coefficients satisfy the diagonally dominant condition $|b_i| \geq |a_i| + |c_i|$ for $i = 0, \dots, N-1$, where $a_0 = 0 = c_{N-1}$, for each Fourier mode in the x direction, and for the tridiagonal systems for both even and odd Fourier modes in the y direction. This ensures that the Thomas algorithm never encounters a division by zero.

Each time the Helmholtz solver is called, it inputs the right-hand sides (the d_{k,even_i} and d_{k,odd_i} from (5.7.5)), where this part of the code is parallelised for the Fourier modes in both the x and y directions. The c'_{k,even_i} and d'_{k,even_i} are then calculated using the formulae (5.7.6) and (5.7.7), where $c'_i = c'_{k,\text{even}_i}$, $d'_i = d'_{k,\text{even}_i}$, $a_i = a_{k,\text{even}_i}$, $b_i = b_{k,\text{even}_i}$ and $c_i = c_{k,\text{even}_i}$, for each Fourier mode k in the x direction. The c'_{k,odd_i} and d'_{k,odd_i} are calculated similarly. The x_{k,even_i} are then calculated using (5.7.12) with $x_i = x_{k,\text{even}_i}$, $c'_i = c'_{k,\text{even}_i}$ and $d'_i = d'_{k,\text{even}_i}$ for each Fourier mode k in the x direction. The x_{k,odd_i} are calculated similarly. These sections of the code can only be parallelised across the Fourier modes in the x direction, since these routines are sequential in the y direction. The odd and even sets of tridiagonal solvers were run in parallel, by creating two so-called CUDA streams. This same method was used for the global standard and height-damped quasi-geostrophic equations, the only difference being that the $\hat{\theta}$ terms were set equal to zero in (5.7.4). This method solves the modified Helmholtz equation (5.7.4) efficiently in spectral space. One time step of a global quasi-geostrophic simulation with the Thomas algorithm incorporated only takes 14% longer than one time step of an equivalent shallow water simulation from §3.3.

5.7.2 Time step control

The global thermal quasi-geostrophic evolution equations (5.2.17a,b) may be rewritten as

$$\frac{\partial \tilde{q}}{\partial t} + (\mathbf{u} \cdot \nabla) \tilde{q} = \frac{f}{Bu} (\mathbf{u} \cdot \nabla) \theta - \frac{\psi}{Bu} \frac{df}{dy} \frac{\partial \theta}{\partial x} - \frac{1}{Ro} \frac{df}{dy} v, \quad (5.7.15a)$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla) \theta = -\lambda(\theta + f\psi), \quad (5.7.15b)$$

where

$$\tilde{q} = q - \frac{f}{Ro} = \left(\nabla^2 - \frac{f^2}{Bu} \right) \psi + \frac{f}{Bu} \theta, \quad (5.7.15c)$$

and $\mathbf{u} = \hat{\mathbf{z}} \times \nabla \psi$. In matrix form these become

$$\frac{\partial}{\partial t} \begin{pmatrix} \tilde{q} \\ \theta \end{pmatrix} + \mathbf{X} \frac{\partial}{\partial x} \begin{pmatrix} \tilde{q} \\ \theta \end{pmatrix} + \mathbf{Y} \frac{\partial}{\partial y} \begin{pmatrix} \tilde{q} \\ \theta \end{pmatrix} = \mathbf{b}, \quad (5.7.16)$$

where the two matrices are defined by

$$\mathbf{X} = \begin{pmatrix} u & (\psi \, df/dy - fu)/Bu \\ 0 & u \end{pmatrix} \quad \text{and} \quad \mathbf{Y} = \begin{pmatrix} v & -fv/Bu \\ 0 & v \end{pmatrix}, \quad (5.7.17)$$

and the term on the right-hand side is defined by

$$\mathbf{b} = - \begin{pmatrix} v \, (df/dy) / Ro \\ \lambda (\theta + f\psi) \end{pmatrix}, \quad (5.7.18)$$

with $u = -\partial_y \psi$ and $v = \partial_x \psi$.

The usual approach to stability treats u and v , and here the two matrices $\mathbf{X}$ and $\mathbf{Y}$, as constants. The Fourier coefficients $\hat{q}$ and $\hat{\theta}$ (of $\tilde{q}$ and θ) with wavenumber $\mathbf{k} = (k, l)$ thus evolve according to

$$\frac{\partial}{\partial t} \begin{pmatrix} \hat{q} \\ \hat{\theta} \end{pmatrix} + ik\mathbf{X} \begin{pmatrix} \hat{q} \\ \hat{\theta} \end{pmatrix} + il\mathbf{Y} \begin{pmatrix} \hat{q} \\ \hat{\theta} \end{pmatrix} = \hat{\mathbf{b}}. \quad (5.7.19)$$

The limitation on the stable time step typically comes from the differentiated terms proportional to k and l , so the algebraic term $\hat{\mathbf{b}}$ on the right-hand side may be neglected.

In the usual case these matrices are diagonal,

$$\mathbf{X}_D = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_D = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}, \quad (5.7.20)$$

so the system decouples into two scalar equations for $\hat{q}$ and $\hat{\theta}$. The first of these is

$$\frac{\partial \hat{q}}{\partial t} = -i(ku + lv)\hat{q}. \quad (5.7.21)$$

The standard fourth-order Runge–Kutta scheme for $\partial_t \hat{q} = -i\omega \hat{q}$ is stable when $\omega \Delta t < 2\sqrt{2}$, which implies the usual Courant–Friedrichs–Lewy stability condition $k_{\max} u_{\max} \Delta t < 2\sqrt{2}$, where $k_{\max}$ is the maximum of $|\mathbf{k}|$ and $u_{\max}$ is the maximum of $|\mathbf{u}|$.

However, the two matrices in (5.7.17) are each proportional to Jordan blocks, and so cannot be diagonalised. The system (5.7.19) for prescribed values of u and

v is thus only degenerately hyperbolic. This suggests replacing the eigenvalues u and v that appear in the usual Courant–Friedrichs–Lewy stability condition with some other property of the matrices $\mathbf{X}$ and $\mathbf{Y}$. The representation of $\tilde{q}$ and θ as Fourier series, with its associated Parseval theorem, motivates the use of the ℓ_2 norm $\|(\hat{q}, \hat{\theta})\|_2 = (|\hat{q}|^2 + |\hat{\theta}|^2)^{1/2}$. The ℓ_2 operator norm of a general matrix $\mathbf{A}$ is then defined by

$$\|\mathbf{A}\|_2 = \max\{\|\mathbf{A}\boldsymbol{\eta}\|_2 : \|\boldsymbol{\eta}\|_2 = 1\}, \quad (5.7.22)$$

which implies the property $\|\mathbf{A}\boldsymbol{\eta}\|_2 \leq \|\mathbf{A}\|_2 \|\boldsymbol{\eta}\|_2$ for all vectors $\boldsymbol{\eta}$. Moreover, $\|\mathbf{A}\|_2$ coincides with the largest singular value of the matrix $\mathbf{A}$, which may be readily calculated from the singular value decomposition.

The norms of the $\mathbf{X}$ and $\mathbf{Y}$ matrices are

$$\|\mathbf{X}\|_2 = \left(u^2 + \frac{1}{2}\Delta^2 + |\Delta|\sqrt{u^2 + \Delta^2/4}\right)^{1/2}, \quad (5.7.23a)$$

$$\|\mathbf{Y}\|_2 = |v| \left(1 + \frac{1}{2}\frac{f^2}{Bu^2} + \frac{|f|}{Bu}\sqrt{1 + f^2/(4Bu)}\right)^{1/2}, \quad (5.7.23b)$$

respectively, where $\Delta = (\psi \, df/dy - fu)/Bu$. One would thus expect a stability condition of the form

$$k_{\max} \max(\|\mathbf{X}\|_2 + \|\mathbf{Y}\|_2) \Delta t < 2\sqrt{2}, \quad (5.7.24)$$

for the global thermal quasi-geostrophic equations, which reduces to the standard Courant–Friedrichs–Lewy condition if $\mathbf{X}$ and $\mathbf{Y}$ are taken to be diagonal.

The global height-damped and standard quasi-geostrophic simulations are determined by the usual diagonal case (5.7.21). Thus the usual Courant–Friedrichs–Lewy stability condition $k_{\max} u_{\max} \Delta t < 2\sqrt{2}$ applies for these models, where $k_{\max}$ is the maximum of $|\mathbf{k}|$ and $u_{\max}$ is the maximum of $|\mathbf{u}|$.

In the numerical simulations in §5.8 it was necessary to replace $2\sqrt{2}$ by a smaller constant to ensure stable long-time integrations when using the standard fourth-order Runge–Kutta integrator. This leads to the time step Δt in the global quasi-geostrophic simulations being roughly twice that for the equivalent shallow water simulations from chapter 3 with the same radiative time scale.

5.8 Jovian simulations for varying radiative cooling rate

Figure 5.1 shows the instantaneous relative vorticity $\omega = \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u} = \nabla^2 \psi$, where $u = -\partial\psi/\partial y$ and $v = \partial\psi/\partial x$, after 2×10^4 Jovian days, for Jovian global height-

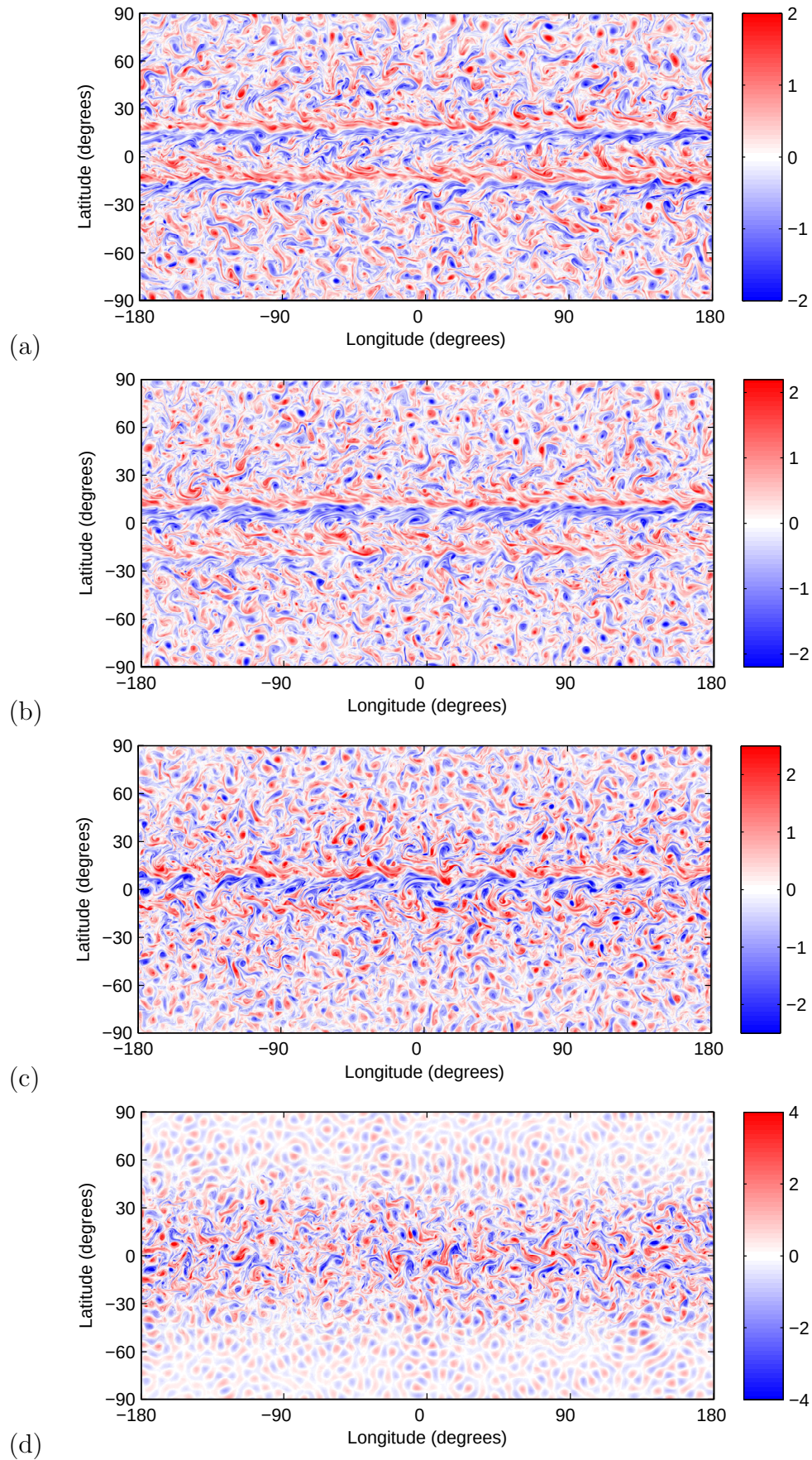


Figure 5.1: Dimensionless relative vorticity after 2×10^4 Jovian days for Jovian global height-damped quasi-geostrophic simulations with (a) $\lambda = 0$, (b) $\lambda = 0.001$, (c) $\lambda = 0.01153$ and (d) $\lambda = 0.1$.

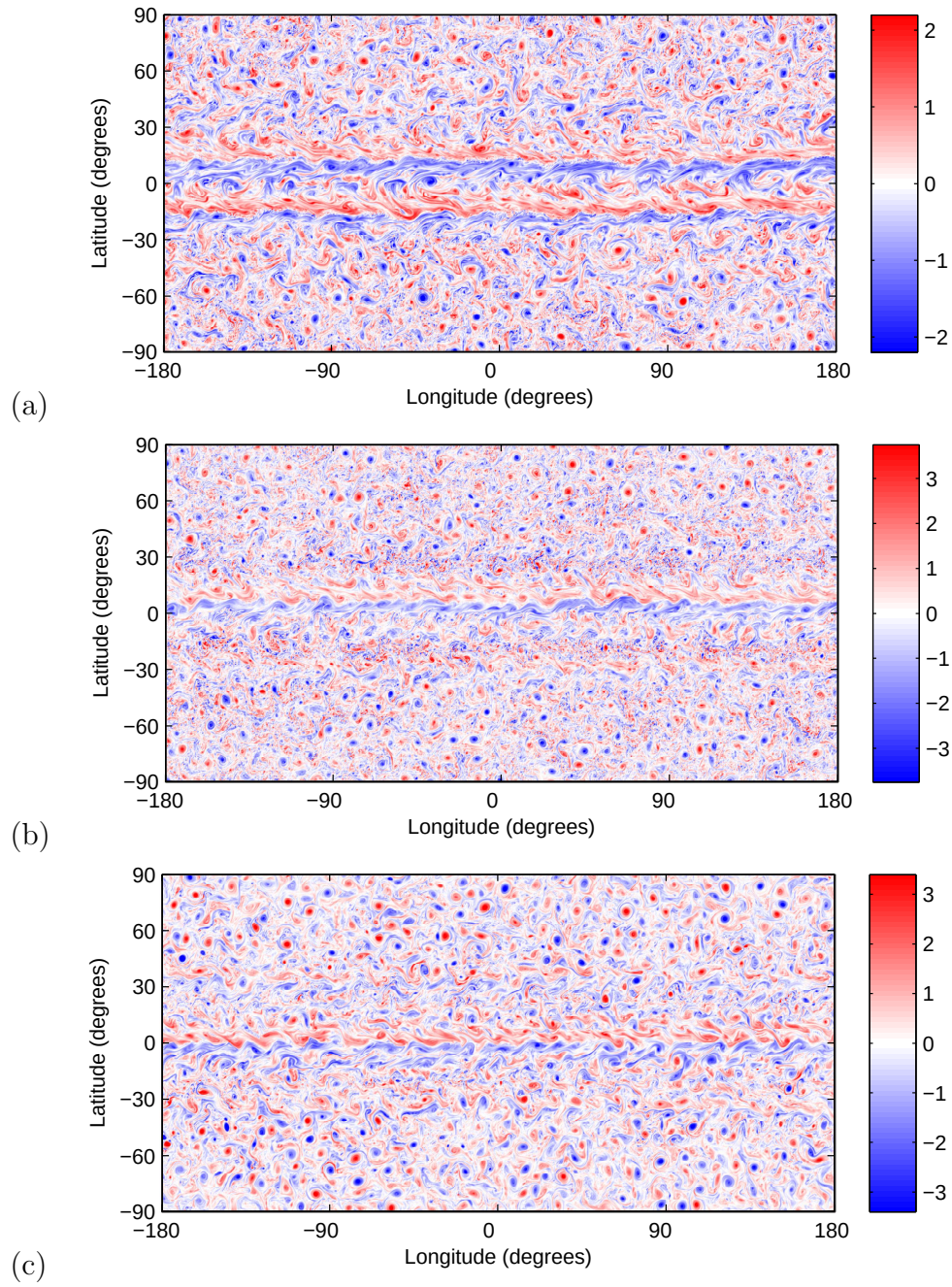


Figure 5.2: Dimensionless relative vorticity after 2×10^4 Jovian days for Jovian global thermal quasi-geostrophic simulations with (a) $\lambda = 0.001$, (b) $\lambda = 0.01153$ and (c) $\lambda = 0.1$.

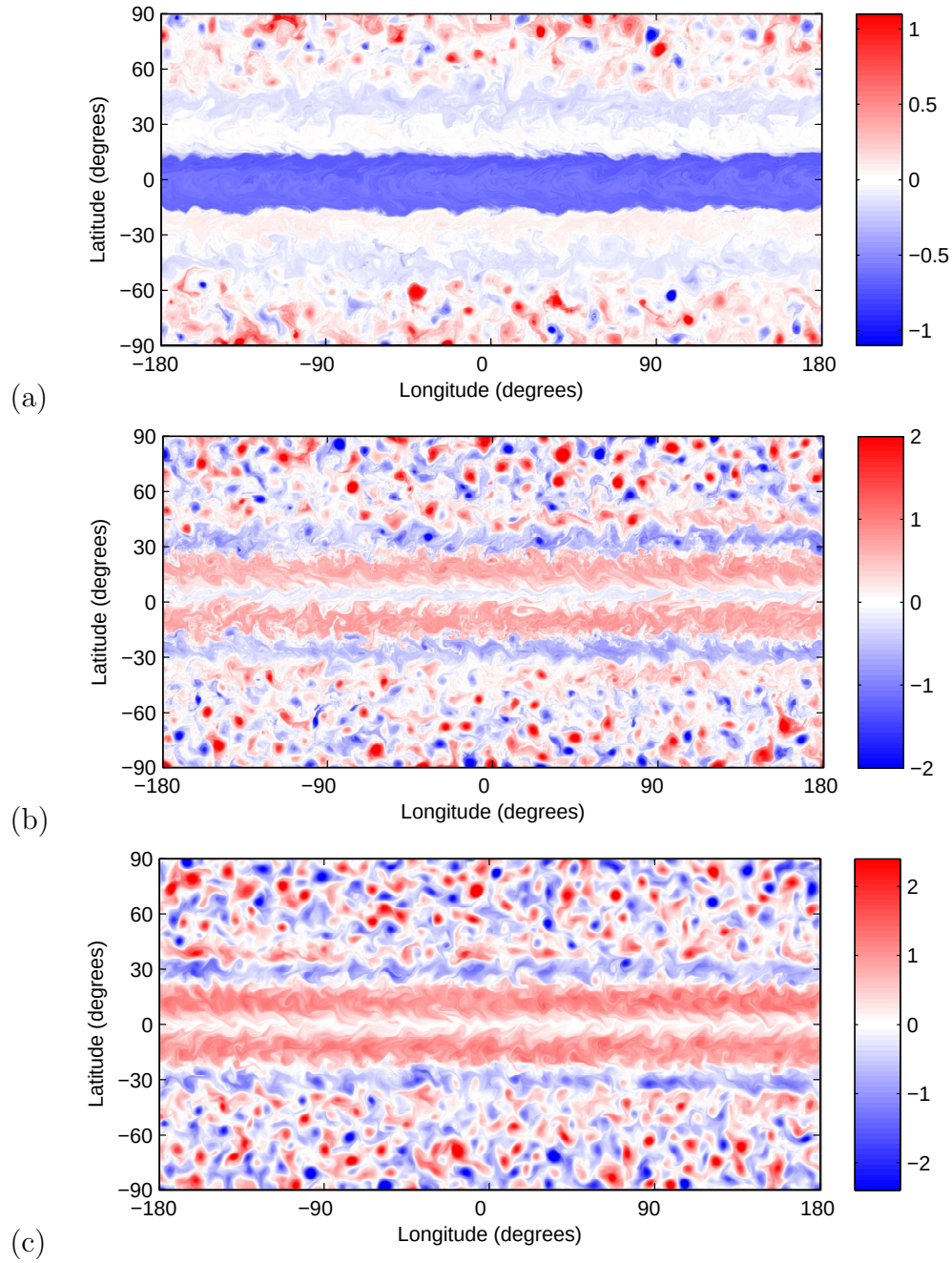


Figure 5.3: θ after 2×10^4 Jovian days for Jovian global thermal quasi-geostrophic simulations with (a) $\lambda = 0.001$, (b) $\lambda = 0.01153$ and (c) $\lambda = 0.1$.

damped quasi-geostrophic simulations with (a) $\lambda = 0$, (b) $\lambda = 0.001$, (c) $\lambda = 0.01153$ and (d) $\lambda = 0.1$. As usual only the top half of the square planet domain is shown, i.e. the top planet in figure 1.7, for ease of interpretation. Simulations of the global height-damped quasi-geostrophic model demonstrate a mixture of vortices and zonally elongated structures for each value of λ , as observed on Jupiter. Comparing the standard shallow water simulation in figure 3.1a to the global standard quasi-geostrophic simulation in figure 5.1a, it is clear that the global standard quasi-geostrophic simulation has lost the two distinct jets at $\pm 45^\circ$ latitude that are evident in the standard shallow water simulation. If the height-damped shallow water simulation in figure 3.1b is compared to the global height-damped quasi-geostrophic simulation with the same radiative cooling time scale in figure 5.1c, it is clear that the main jet has moved slightly north in the global height-damped quasi-geostrophic simulation.

Figure 5.2 shows an equivalent plot for the global thermal quasi-geostrophic simulations with (a) $\lambda = 0.001$, (b) $\lambda = 0.01153$ and (c) $\lambda = 0.1$. Again these simulations demonstrate a mixture of vortices and zonally elongated structures for each value of λ , as observed on Jupiter. Comparing the Jovian thermal shallow water simulation in figure 3.1d to the global thermal quasi-geostrophic simulation with the same radiative cooling time scale in figure 5.2b, it is clear that the jet-like structures at mid-latitudes are less evident in the global thermal quasi-geostrophic model. Figure 5.3 shows θ after 2×10^4 Jovian days for the global thermal quasi-geostrophic simulations for different values of λ . When $\lambda = 0.001$, θ is largely negative in the equatorial region; θ is largely positive in the equatorial region for $\lambda = 0.1$. The temperature fluctuations in the northern hemisphere of figure 5.3 all have a positive correlation coefficient (between 0.36 and 0.47) with respect to the relative vorticity fluctuations shown in the northern hemisphere of figure 5.2, as was also found for the Jovian thermal shallow water simulations in §3.3 and §3.5.1. Cyclonic regions are thus typically warmer than anticyclonic regions for these global thermal quasi-geostrophic simulations.

Figure 5.4 compares the instantaneous mean zonal wind, $\bar{u} = y_{\max}^{-1} \int u(x, y, t) dx$, at 2×10^4 Jovian days, for Jovian simulations of the standard shallow water and global standard quasi-geostrophic models. Both models produce a broad sub-rotating equatorial jet, in contrast to the broad super-rotating equatorial jet observed on Jupiter, although zonal jet speeds are comparable to those on Jupiter (see figure 1.1a). Some activity is lost at higher latitudes in the global standard quasi-geostrophic simulation, however the two prograde mid-latitude jets at roughly $\pm 20^\circ$ are much stronger. The increased dominance of the equatorial jet in the global standard quasi-geostrophic simulation compared to the standard shallow water simulation is reflected

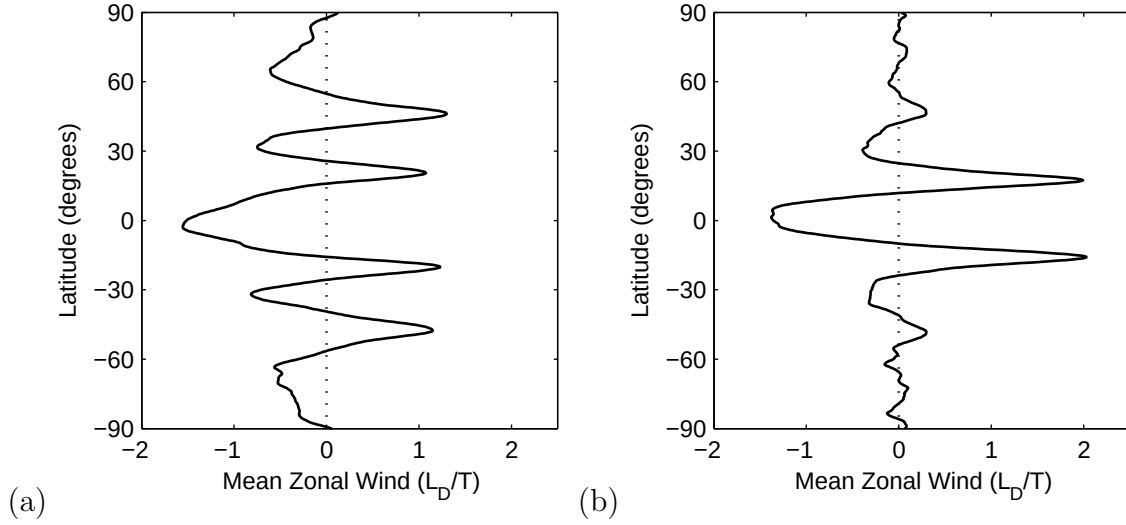


Figure 5.4: Mean zonal wind against latitude after 2×10^4 Jovian days for Jovian simulations of (a) the standard shallow water model and (b) the global standard quasi-geostrophic model.

in the larger mean time-average value for the kurtosis $\mathcal{K}_u$, which is defined in (3.3.1), and takes the value 4.05 for the standard shallow water simulations (see table 3.4) and 5.25 for the global standard quasi-geostrophic simulations (see table 5.1).

Figure 5.5 compares the instantaneous mean zonal wind at 2×10^4 Jovian days for the Jovian height-damped shallow water simulation from §3.3, and global height-damped quasi-geostrophic simulations for different values of λ . While the height-damped shallow water simulation produces a strong super-rotating equatorial jet, the equivalent global height-damped quasi-geostrophic simulation ($\lambda = 0.01153$) produces a big zig-zag with a zero crossing close to the equator. However as λ increases, i.e. as the radiative time scale decreases, the equatorial jet switches from sub- to super-rotating for the Jovian global height-damped quasi-geostrophic simulations (as was found for the Jovian height-damped shallow water simulations in §3.5.1). The radiative time scale must be made shorter than the value used in the height-damped shallow water simulation in §3.3, to create a prograde equatorial jet in Jovian global height-damped quasi-geostrophic simulations. The super-rotating equatorial jet for the global height-damped quasi-geostrophic simulation with $\lambda = 0.1$ is much weaker than that for the Jovian height-damped shallow water simulation from §3.3, and weaker than the equatorial jet speed observed on Jupiter (see figure 1.1a). There also appears to be less activity at higher latitudes as λ increases for the global height-damped quasi-geostrophic simulations. This is caused by the radiative height-damping appearing as a hypoviscous dissipation proportional to ψ with a coefficient

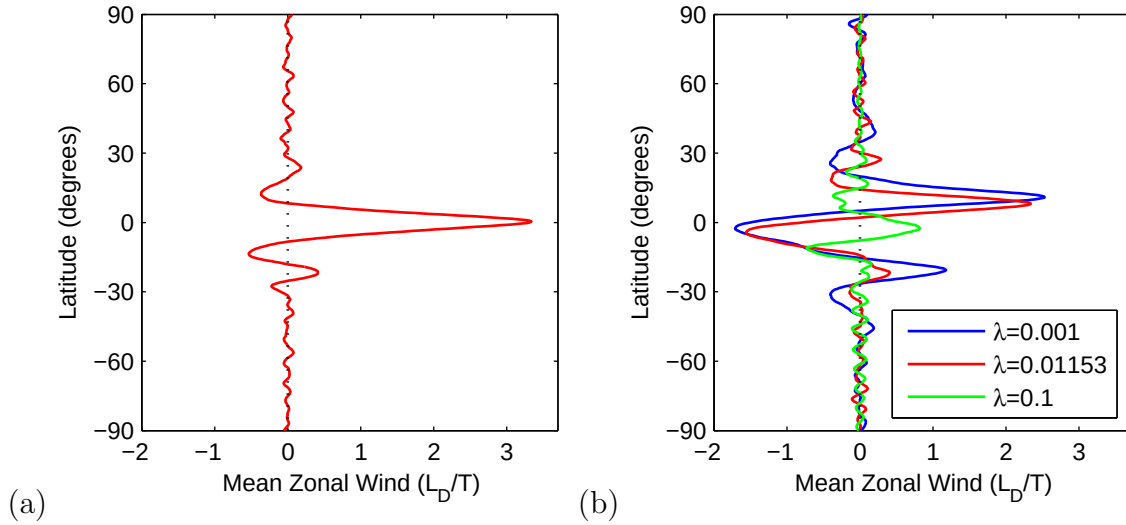


Figure 5.5: Mean zonal wind against latitude after 2×10^4 Jovian days for Jovian simulations of (a) the height-damped shallow water model with $\lambda = 0.01153$ and (b) the global height-damped quasi-geostrophic model. The right-hand figure contains plots of $\bar{u}$ for different values of the dimensionless radiative cooling rate λ .

proportional to $\lambda f(y)^2$ in (5.3.3a). This dissipation is thus ineffective at low latitudes, while at higher latitudes the dissipation strength increases as λ increases (cf. page 85).

Figure 5.6 compares the instantaneous mean zonal wind at 2×10^4 Jovian days for the Jovian thermal shallow water simulation from §3.3, and global thermal quasi-geostrophic simulations for different values of λ . While the Jovian thermal shallow water simulation produces a super-rotating equatorial jet, the equivalent global thermal quasi-geostrophic simulation ($\lambda = 0.01153$) appears to be on the border between sub- and super-rotating. As for the global height-damped quasi-geostrophic simulations, as λ increases (or the radiative time scale decreases) the equatorial jet switches from sub- to super-rotating for the global thermal quasi-geostrophic simulations (as was found for Jovian simulations of the thermal shallow water equations in §3.5.1). A smaller radiative time scale is required for Jovian global thermal quasi-geostrophic simulations, compared to the value used for the Jovian thermal shallow water simulation in §3.3, to create a super-rotating equatorial jet. Unlike the global height-damped quasi-geostrophic simulation, however, the super-rotating equatorial jet for the global thermal quasi-geostrophic simulation with $\lambda = 0.1$ is roughly the same magnitude as that for the Jovian thermal shallow water simulation from §3.3, and is roughly the same magnitude as the equatorial jet speed observed on Jupiter (see figure 1.1a). The mid-latitude jets at approximately $\pm 40^\circ$ for $\lambda = 0.01153, 0.1$ are also roughly the

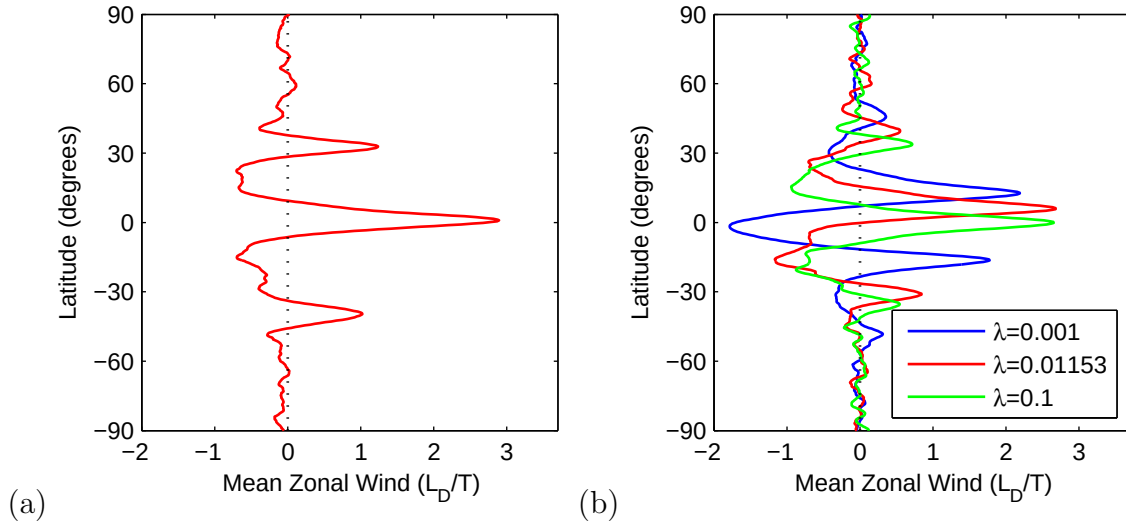


Figure 5.6: Mean zonal wind against latitude after 2×10^4 Jovian days for Jovian simulations of (a) the radiatively damped thermal shallow water model with $\lambda = 0.01153$ and (b) the global thermal quasi-geostrophic model. The right-hand figure contains plots of $\bar{u}$ for different values of the dimensionless radiative cooling rate λ .

same magnitude as the mid-latitude jet speeds observed on Jupiter.

Figures 5.7 and 5.8 show the evolution of $\bar{u}$ as a function of latitude and time for Jovian global height-damped and thermal quasi-geostrophic simulations for a range of λ values. For the global height-damped quasi-geostrophic simulations, the solution remains incredibly steady throughout the simulation for each value of λ , with $\lambda = 0.1$ demonstrating the most variability. This is also true for the global thermal quasi-geostrophic simulation with $\lambda = 0.1$, and to a lesser extent for $\lambda = 0.001$. Comparing the global standard quasi-geostrophic simulation in figure 5.7a to the corresponding standard shallow water simulation in figure 3.4a, some activity has been lost at higher latitudes and the two prograde mid-latitude jets at $\pm 20^\circ$ are broader in the global standard quasi-geostrophic simulation. Comparing the global height-damped quasi-geostrophic simulation in figure 5.7c to the corresponding Jovian height-damped shallow water simulation in figure 3.4b, the height-damped shallow water simulation produced a robust super-rotating equatorial jet, but the equivalent global height-damped quasi-geostrophic simulation with $\lambda = 0.01153$ gives a zonal velocity close to zero at the equator. For the global height-damped quasi-geostrophic simulations, the value of $\bar{u}$ at the equator switches from negative, to close to zero, to positive, as λ increases. The mid-latitude jets also become weaker as λ increases, as also shown in figure 5.5. Comparing the global thermal quasi-geostrophic simulation in figure 5.8b to the corresponding Jovian thermal shallow water simulation in figure 3.4d, the

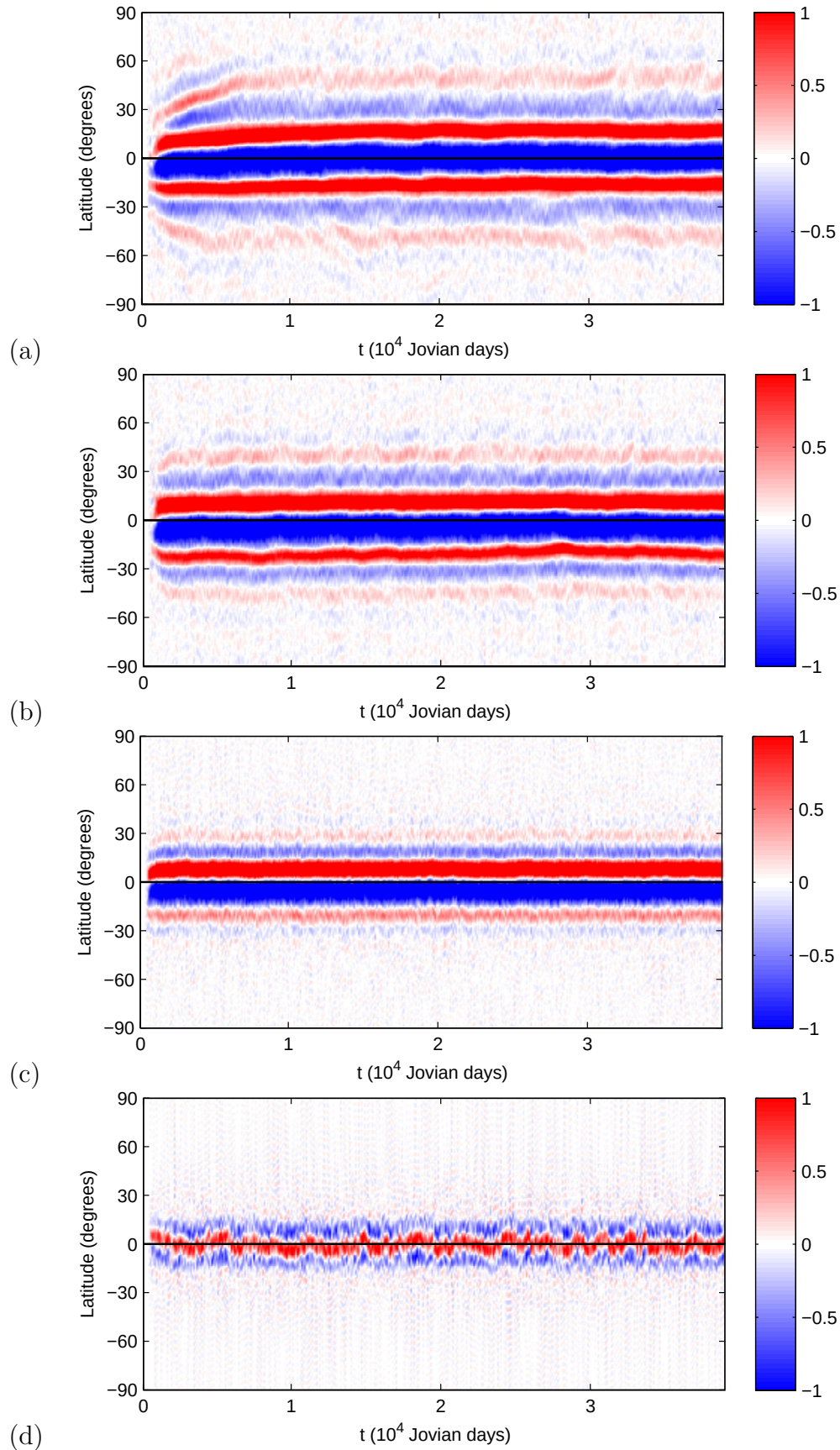


Figure 5.7: Evolution of $\bar{u}$ for Jovian global height-damped quasi-geostrophic simulations with (a) $\lambda = 0$, (b) $\lambda = 0.001$, (c) $\lambda = 0.01153$ and (d) $\lambda = 0.1$.

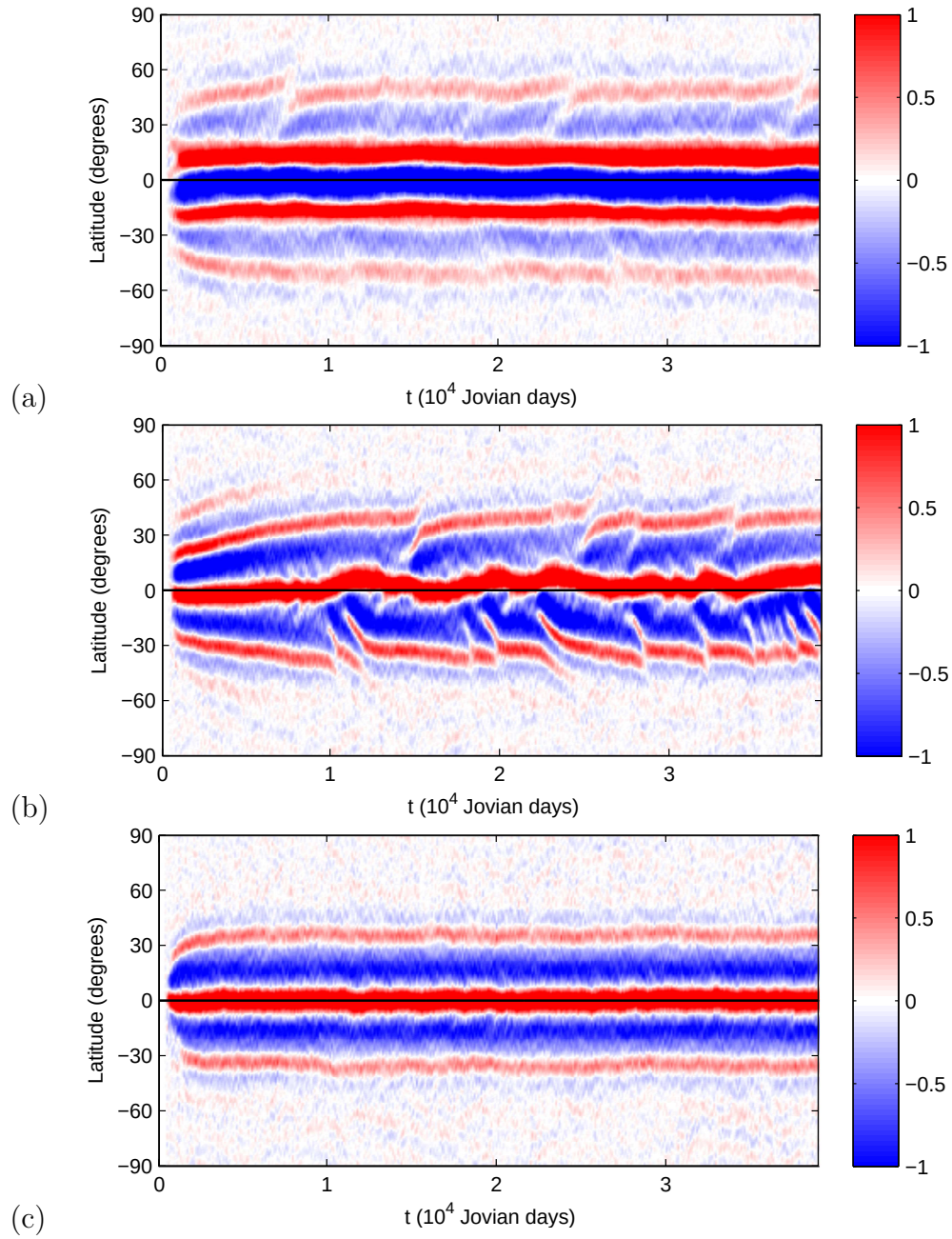


Figure 5.8: Evolution of $\bar{u}$ for Jovian global thermal quasi-geostrophic simulations with (a) $\lambda = 0.001$, (b) $\lambda = 0.01153$ and (c) $\lambda = 0.1$.

Table 5.1: Average diagnostic quantities in the statistically steady state for two sets of global height-damped quasi-geostrophic simulations with different initial random seeds. The error estimates quantify the differences between the diagnostics for each pair of simulations.

	$\langle\langle\epsilon_f\rangle\rangle$	$\langle\langle\xi\rangle\rangle$	$\langle\langle\langle\omega\rangle_N\rangle\rangle$	$\langle\langle\mathcal{K}_u\rangle\rangle$	$\langle\langle\mathcal{K}_\omega\rangle\rangle$	$\langle\langle\mathcal{S}_\omega\rangle\rangle$
$\lambda = 0$	1800 ± 6	0.53 ± 0.003	-0.027 ± 0.0008	5.25 ± 0.02	3.43 ± 0.004	$+0.027 \pm 0.009$
$\lambda = 0.001$	2200 ± 8	0.53 ± 0.0007	-0.026 ± 0.0007	6.16 ± 0.02	3.48 ± 0.005	-0.022 ± 0.0005
$\lambda = 0.01153$	3800 ± 8	0.39 ± 0.002	-0.013 ± 0.001	7.17 ± 0.06	3.75 ± 0.02	-0.011 ± 0.01
$\lambda = 0.1$	7900 ± 6	0.06 ± 0.0009	$+0.012 \pm 0.0006$	6.74 ± 0.06	5.81 ± 0.002	$+0.047 \pm 0.002$

Table 5.2: Average diagnostic quantities in the statistically steady state for two sets of global thermal quasi-geostrophic simulations with different initial random seeds. The error estimates quantify the differences between the diagnostics for each pair of simulations.

	$\langle\langle\epsilon_f\rangle\rangle$	$\langle\langle\xi\rangle\rangle$	$\langle\langle\langle\omega\rangle_N\rangle\rangle$	$\langle\langle\mathcal{K}_u\rangle\rangle$	$\langle\langle\mathcal{K}_\omega\rangle\rangle$	$\langle\langle\mathcal{S}_\omega\rangle\rangle$
$\lambda = 0.001$	2100 ± 7	0.54 ± 0.005	-0.027 ± 0.0002	5.27 ± 0.08	3.64 ± 0.03	$+0.058 \pm 0.01$
$\lambda = 0.01153$	3100 ± 11	0.44 ± 0.002	$+0.012 \pm 0.001$	5.70 ± 0.1	4.37 ± 0.02	$+0.031 \pm 0.004$
$\lambda = 0.1$	3800 ± 3	0.44 ± 0.003	$+0.046 \pm 0.0002$	6.59 ± 0.02	4.93 ± 0.03	$+0.15 \pm 0.04$

thermal shallow water simulation produced a robust super-rotating equatorial jet, but the corresponding global thermal quasi-geostrophic simulation with $\lambda = 0.01153$ meanders between producing a super- and sub-rotating equatorial jet. Also the zonal jets in the global thermal quasi-geostrophic simulation with $\lambda = 0.01153$ frequently merge and reform around $\pm 40^\circ$ latitude, while they are much more robust in the corresponding thermal shallow water simulation. The switch from a sub- to super-rotating equatorial jet is evident in the global thermal quasi-geostrophic simulations as λ increases, while the mid-latitude jets persist.

5.8.1 Average diagnostic quantities

To quantify the differences between the global height-damped and thermal quasi-geostrophic simulations for different values of λ , some average diagnostic quantities are computed from the simulations. Tables 5.1 and 5.2 show the time-averages of these diagnostic quantities in the statistically steady state, with $\langle\langle\cdot\rangle\rangle$ denoting the time-average of a quantity between 2×10^4 and 2.18×10^4 Jovian days (for consistency with §3.3). Each global quasi-geostrophic simulation was run twice, with different initial seeds for the random number generator, and the mean and half differences are presented in tables 5.1 and 5.2 to give an indication of the repeatability of the

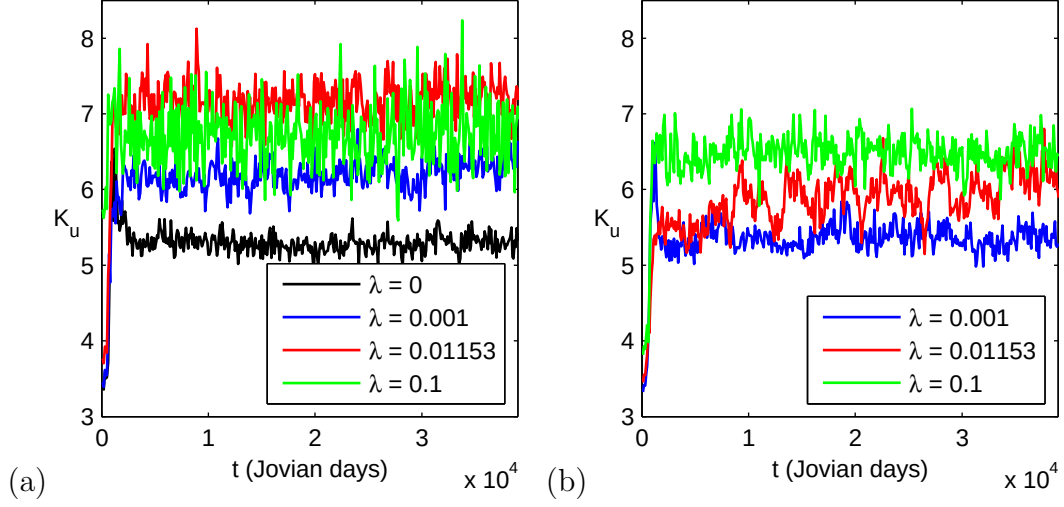


Figure 5.9: Evolution of the kurtosis of the zonal velocity for Jovian simulations of the global (a) height-damped and (b) thermal quasi-geostrophic models for different values of the dimensionless radiative cooling rate λ .

different diagnostics.

Figure 5.9 shows the evolution of the kurtosis of $u = -\partial\psi/\partial y$, as defined in (3.3.1). The kurtosis begins near three for each simulation, as for a Gaussian random variable, increases due to the formation of coherent structures, and levels off as the simulation reaches a statistically steady state. The kurtosis K_u is larger for the global height-damped quasi-geostrophic simulations compared to their thermal counterparts with the same value of λ , quantifying the greater dominance of the equatorial jet in the height-damped simulations. Increasing λ increases $\langle\langle K_u \rangle\rangle$ in general, although the global height-damped quasi-geostrophic simulation with $\lambda = 0.1$ does not follow this pattern.

Figure 5.10 shows the evolution of the fraction of kinetic energy in the zonally averaged flow for the global quasi-geostrophic simulations, defined by

$$\xi(t) = \langle \bar{u}^2 \rangle / \langle u^2 + v^2 \rangle, \quad (5.8.1)$$

where $u = -\partial\psi/\partial y$, $v = \partial\psi/\partial x$, $\bar{u}$ is the zonal mean of u , and $\langle \cdot \rangle$ denotes an average over the entire simulation domain (planets 1 and 2 from figure 1.7). The corresponding averages are shown in tables 5.1 and 5.2. Increasing λ decreases $\langle\langle \xi \rangle\rangle$, where this decrease in the fraction of kinetic energy in the zonally averaged flow as λ increases is evident from figures 5.1 and 5.2. The global height-damped quasi-geostrophic simulation with $\lambda = 0.1$ has almost no kinetic energy in the zonally averaged flow, with the average of $\langle\langle \xi \rangle\rangle$ for the two simulation runs with different

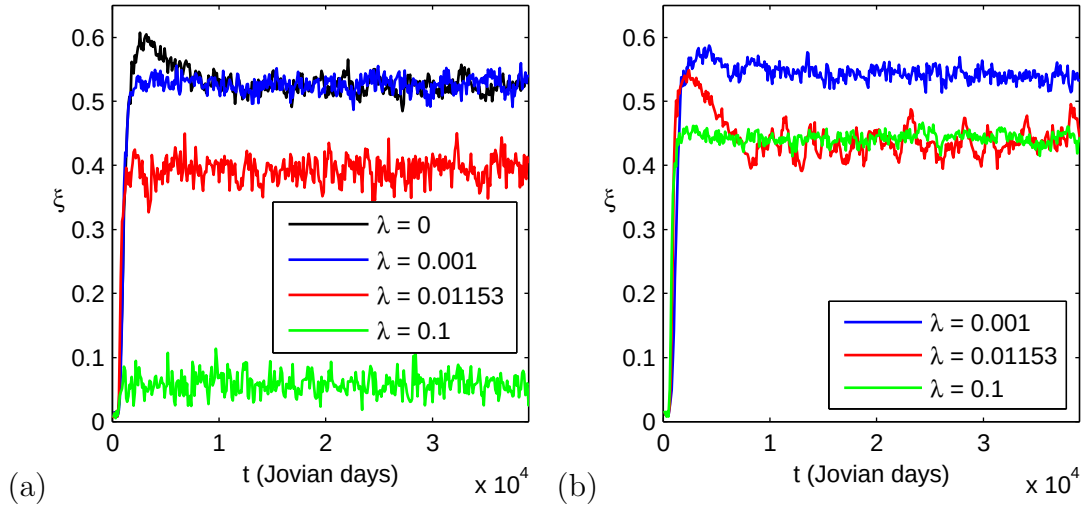


Figure 5.10: Evolution of the fraction of kinetic energy in the zonally averaged flow for Jovian simulations of the global (a) height-damped and (b) thermal quasi-geostrophic models for different values of the dimensionless radiative cooling rate λ .

initial random seeds being 0.06. The evolution of ξ for the global thermal quasi-geostrophic simulations shows the most variability for $\lambda = 0.01153$, which is probably caused by the jets continually merging and reforming at $\pm 40^\circ$ latitude (see figure 5.8b).

As λ increases, $\langle\langle\epsilon_f\rangle\rangle$ increases, for both the global height-damped and thermal quasi-geostrophic simulations (see tables 5.1 and 5.2). The forcing amplitude for all the simulations was adjusted to give the same statistically steady total kinetic energy. Thus as the dissipation due to radiative cooling increases, the simulation requires larger forcing to reach the same desired total kinetic energy.

In figure 5.11 the evolution of the kurtosis of the relative vorticity $\omega = \nabla^2\psi$ is plotted, as defined in (3.3.3). The kurtosis $\mathcal{K}_\omega$ converges much faster for the global height-damped quasi-geostrophic simulations in general, where the global thermal quasi-geostrophic simulations for $\lambda = 0.01153$ and $\lambda = 0.1$ take close to 10^4 Jovian days to reach a statistical steady state. For both the height-damped and thermal models, increasing λ increases $\langle\langle\mathcal{K}_\omega\rangle\rangle$.

Figure 5.12 shows the evolution of the mean of the relative vorticity over the northern hemisphere of the top planet in figure 1.7 for simulations of the two global quasi-geostrophic models. The time-average $\langle\langle\omega_N\rangle\rangle$ is shown in tables 5.1 and 5.2. As λ increases, $\langle\langle\omega_N\rangle\rangle$ increases, and switches from demonstrating a net anticyclonic vorticity to a net cyclonic vorticity. The global thermal quasi-geostrophic simulation with $\lambda = 0.01153$, and to a much lesser extent the height-damped simulation with

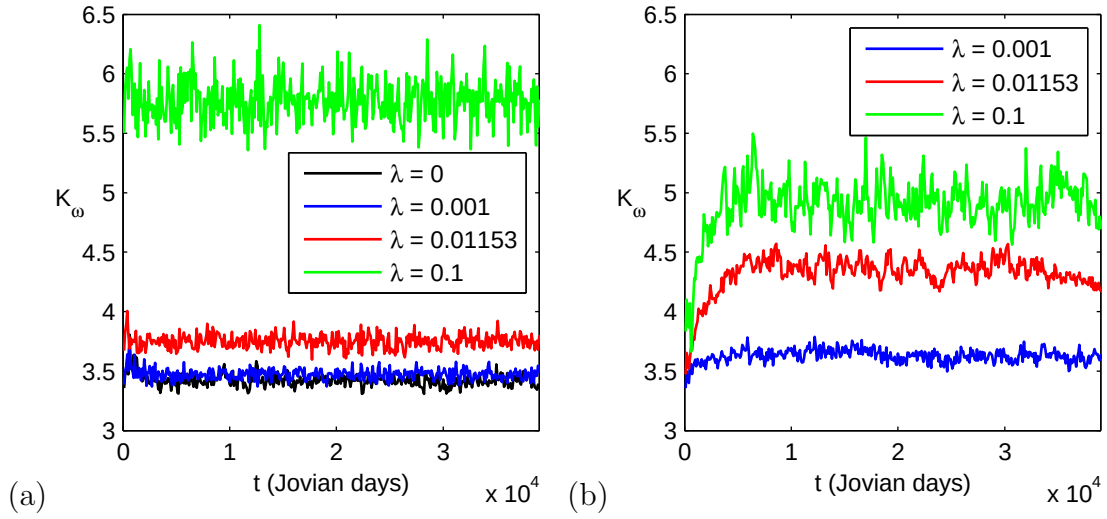


Figure 5.11: Evolution of the kurtosis of the relative vorticity for Jovian simulations of the global (a) height-damped and (b) thermal quasi-geostrophic models for different values of the dimensionless radiative cooling rate λ .

$\lambda = 0.1$, are in a region where the net relative vorticity repeatedly switches between cyclonic and anticyclonic during the simulation. The result (3.3.4) also holds for these global quasi-geostrophic simulations. However, only when $\bar{u}$ at the equator is much larger in magnitude than $\bar{u}$ at the north pole can we conclude that a northern hemisphere with net anticyclonic vorticity must exhibit a retrograde equatorial jet and vice versa. From figures 5.4–5.6, it is clear that this holds for all the simulations at 2×10^4 Jovian days, excluding the global thermal quasi-geostrophic simulation with $\lambda = 0.01153$.

Figure 5.13 shows the evolution of the skew of the relative vorticity for the northern hemisphere of the top planet in figure 1.7 for the two global quasi-geostrophic models, where $\mathcal{S}_\omega$ is defined by (3.3.5), and tables 5.1 and 5.2 show the time-average $\langle\langle\mathcal{S}_\omega\rangle\rangle$. There does not appear to be a clear pattern for how $\mathcal{S}_\omega$ varies with λ for the global quasi-geostrophic simulations. All the global thermal quasi-geostrophic simulations, and two of the height-damped simulations, demonstrate a positive average value for $\langle\langle\mathcal{S}_\omega\rangle\rangle$, which is consistent with cyclones being more intense than anticyclones. However the skew oscillates between being positive and negative for all the global quasi-geostrophic simulations.

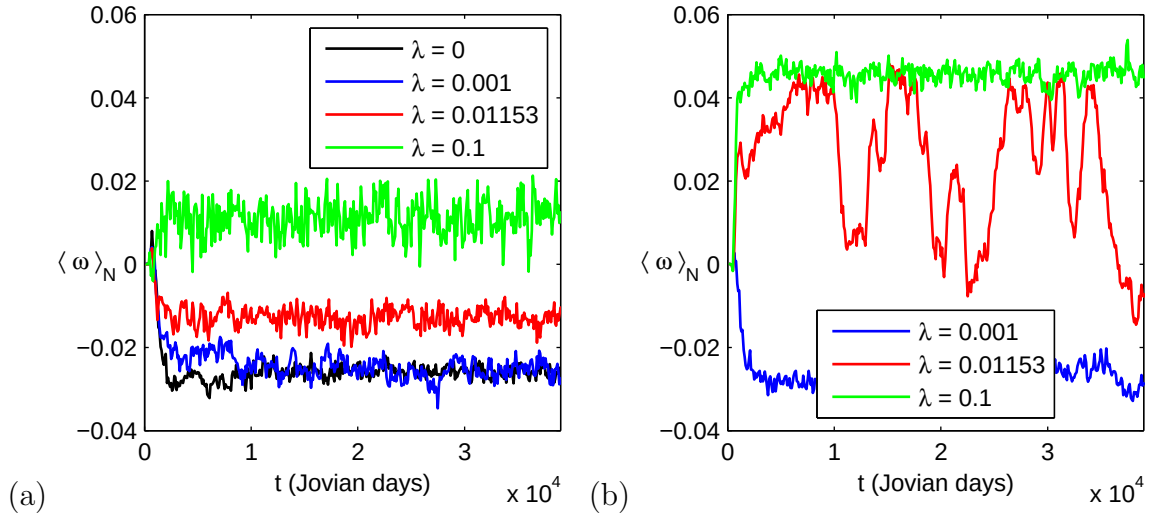


Figure 5.12: Evolution of the mean of the relative vorticity for Jovian simulations of the global (a) height-damped and (b) thermal quasi-geostrophic models for different values of the dimensionless radiative cooling rate λ .

5.9 Summary

In this chapter I have derived the global quasi-geostrophic form of the radiatively damped thermal shallow water equations (5.2.1), and the height-damped shallow water equations (5.3.1). The global quasi-geostrophic form of the standard shallow water equations (1.3.6) is then the infinite radiative time scale limit of the global height-damped quasi-geostrophic equations. To derive these equation sets the momentum equations were manipulated to obtain evolution equations for the shallow water potential vorticity $\Pi = (f + \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u})/h$ and the horizontal divergence $D = \nabla \cdot \mathbf{u}$. A quasi-geostrophic scaling was applied, then an expansion was performed for small Ro number (where $Ro = U/2\Omega L$). The geostrophic relation for the global quasi-geostrophic equation sets was shown to be $f\psi = \eta + \theta$, where $\theta = 0$ for the non-thermal models. Given ψ and θ this relation is used to find η . The resulting global quasi-geostrophic equation sets reduce to their local quasi-geostrophic counterparts on an f -plane on setting f to be unity.

Radiative cooling was demonstrated to damp the equatorially trapped waves for both the global thermal and height-damped quasi-geostrophic equation sets on equatorial β -planes. In chapter 6, the effects of Newtonian cooling and Rayleigh friction on equatorially trapped waves in the radiatively damped thermal shallow water equations, and the height-damped shallow water equations, are considered. The energy conservation law was stated for the global quasi-geostrophic equation sets. The global

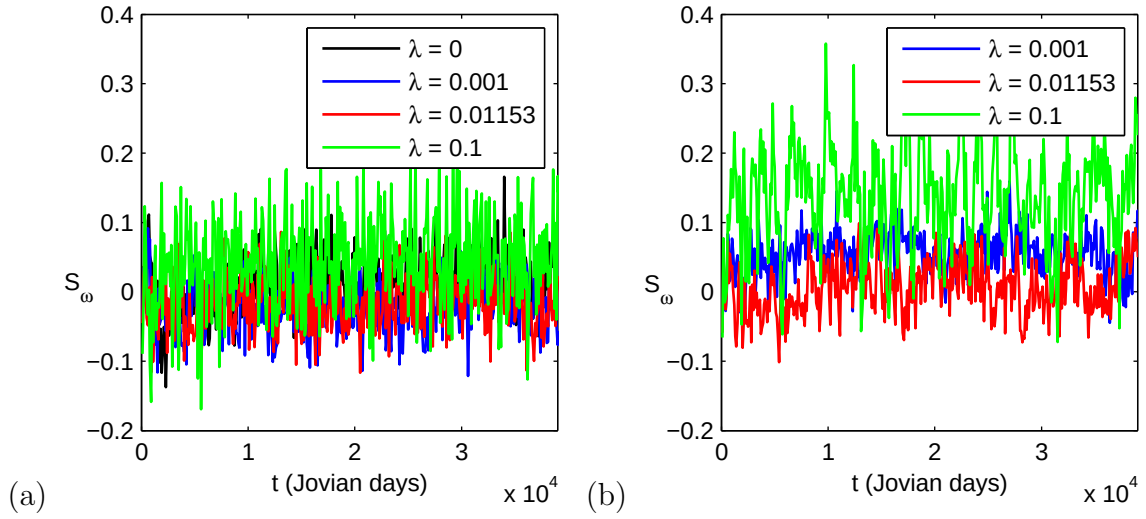


Figure 5.13: Evolution of the skew of the relative vorticity for Jovian simulations of the global (a) height-damped and (b) thermal quasi-geostrophic models for different values of the dimensionless radiative cooling rate λ .

quasi-geostrophic energy density (5.5.2a) was demonstrated to propagate with the group velocity for the global standard, height-damped and thermal quasi-geostrophic equations, when radiative cooling was neglected. The non-canonical Hamiltonian structure of the global thermal quasi-geostrophic equations was derived from a pseudo-energy and Poisson bracket for the thermal shallow water equations.

Jovian numerical simulations of the global thermal and height-damped quasi-geostrophic models were performed. The global quasi-geostrophic simulations were made as similar to the corresponding Jovian shallow water simulations in §3.3 as possible. Fourier transforming the definition of potential vorticity for these global quasi-geostrophic models leads to a suite of pentadiagonal linear systems to be solved for ψ given an instantaneous value for the potential vorticity q (and also an instantaneous value for the temperature θ for the global thermal quasi-geostrophic model). To solve these systems efficiently the odd and even Fourier modes in the y direction were considered separately, thus two tridiagonal linear systems were considered for each Fourier mode in the x direction as opposed to one pentadiagonal linear system. Thus the main addition needed by these global quasi-geostrophic codes was the development of a Thomas algorithm to solve these tridiagonal systems which works in parallel on GPUs for Fourier modes in the x direction.

Global quasi-geostrophic simulations were performed for a range of values of the dimensionless radiative cooling rate ($\lambda = 0, 0.001, 0.01153, 0.1$). All the global quasi-geostrophic simulations demonstrated a mixture of vortices and zonally elongated

structures, as observed on Jupiter. The global thermal quasi-geostrophic simulations all demonstrated cyclonic regions which were typically warmer than anticyclonic regions. The mean zonal winds $\bar{u}$ for the global quasi-geostrophic simulations were compared to $\bar{u}$ from Jovian simulations of the corresponding shallow water models from §3.3. Both the standard shallow water and global standard quasi-geostrophic models produced a broad sub-rotating equatorial jet, in contrast to the broad super-rotating equatorial jet observed on Jupiter. Some activity was lost at higher latitudes in the global standard quasi-geostrophic simulation, however the two prograde mid-latitude jets at roughly $\pm 20^\circ$ were much stronger than those in the standard shallow water simulation. The global height-damped and thermal quasi-geostrophic simulations did not produce a super-rotating equatorial jet for the radiative time scale used in the Jovian shallow water simulations in §3.3. However, both models demonstrated a switch from a sub- to a super-rotating equatorial jet as λ was increased, i.e. as the radiative time scale was decreased. Thus a smaller radiative time scale (than that used in the Jovian shallow water simulations in §3.3) is required for the Jovian global quasi-geostrophic simulations to produce a super-rotating equatorial jet. This same switch in equatorial jet direction is also found in Jovian simulations of the height-damped shallow water equations, and the radiatively damped thermal shallow water equations, as the radiative time scale is decreased (see §3.5.1 and figure 6.1). The super-rotating equatorial jet in the global height-damped quasi-geostrophic simulation with $\lambda = 0.1$ was shown to be much weaker than the equatorial jet speed observed on Jupiter. Less activity was observed at higher latitudes as λ was increased in global height-damped quasi-geostrophic simulations. The super-rotating equatorial jet for the global thermal quasi-geostrophic simulation with $\lambda = 0.1$ was roughly the same speed as that observed on Jupiter, as were the mid-latitude jets at $\pm 40^\circ$. The global thermal quasi-geostrophic simulation with $\lambda = 0.01153$ demonstrated the most variability through the simulation, where jets merged and reformed around $\pm 40^\circ$ latitude, while the equatorial jet meandered between super- and sub-rotating.

The differences between the global height-damped and thermal quasi-geostrophic simulations were quantified for different values of λ by considering the evolution of various diagnostic quantities computed from the simulation results. The kurtosis of u was larger for the global height-damped quasi-geostrophic simulations compared to their thermal counterparts with the same value for λ , quantifying the greater dominance of the equatorial jet in height-damped simulations. It was found that increasing λ decreased the average fraction of kinetic energy in the zonally averaged flow, while it increased the average forcing amplitude and the kurtosis of the relative vorticity.

The mean of the relative vorticity over the northern hemisphere of planet 1 from figure 1.7 was found to increase as λ increases, and switch from demonstrating a net anticyclonic vorticity to a net cyclonic vorticity. This reflects the switch from a sub- to super-rotating equatorial jet as λ increases in global quasi-geostrophic simulations. The skew of the relative vorticity for the northern hemisphere of planet 1 from figure 1.7 was found to oscillate between positive and negative values for all the global quasi-geostrophic simulations. However the time-average of this skew for all the global thermal quasi-geostrophic simulations, and two of the height-damped simulations, was positive when averaged between 2×10^4 and 2.18×10^4 Jovian days, which is consistent with cyclones being more intense than anticyclones.

The next chapter investigates the role of equatorially trapped waves in establishing the direction of the equatorial jet in Jovian simulations of the standard and height-damped shallow water equations, and the radiatively damped thermal shallow water equations. The fact that simulations of the global thermal and height-damped quasi-geostrophic models, along with their shallow water counterparts, switch from producing a sub- to a super-rotating equatorial jet as the radiative time scale is decreased, shows that the mechanism responsible for the switch in equatorial jet direction must lie within quasi-geostrophic theory. Hence equatorially trapped Rossby waves are the main focus in the explanation for the equatorial jet direction in chapter 6.

Chapter 6

Super- and sub-rotating equatorial jets: Newtonian cooling versus Rayleigh friction

This chapter is based upon Warneford and Dellar (2014a).

6.1 Introduction

Forced-dissipative simulations often include a Rayleigh friction term to damp the large-scales, with time scale τ_{fric} , and absorb the slow leakage of energy past the Rhines scale that would eventually cause the zonal jets to break up into large coherent vortices. In chapter 3, numerical simulations of the standard shallow water equations (i.e. (6.1.1) with infinite τ_{rad}), the height-damped shallow water equations

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = -(h - h_0) / \tau_{\text{rad}}, \quad (6.1.1a)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + f \hat{\mathbf{z}} \times \mathbf{u} = -g' \nabla h - \mathbf{u} / \tau_{\text{fric}}, \quad (6.1.1b)$$

and the radiatively damped thermal shallow water equations

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0, \quad (6.1.2a)$$

$$\frac{\partial \Theta}{\partial t} + (\mathbf{u} \cdot \nabla) \Theta = -(\Theta h / h_0 - \Theta_0) / \tau_{\text{rad}}, \quad (6.1.2b)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + f \hat{\mathbf{z}} \times \mathbf{u} = -\nabla (\Theta h) + \frac{1}{2} h \nabla \Theta - \mathbf{u} / \tau_{\text{fric}}, \quad (6.1.2c)$$

were performed. To drive the flow, a divergence-free isotropic random forcing $\mathbf{F}$ was included in the momentum equations of these models, as described in §3.2.4. Thus the standard shallow water simulation contained dissipation solely by Rayleigh friction,

while the height-damped and radiatively damped thermal shallow water simulations contained dissipation by a Rayleigh friction term and a Newtonian cooling term, with radiative time scale τ_{rad} . The simulation of the standard shallow water model for Jovian parameter values in chapter 3 produced a sub-rotating (retrograde) equatorial jet. This is invariably the case for both freely decaying and forced-dissipative simulations of the standard shallow water equations in the Jovian regime (Cho and Polvani, 1996*a,b*, Iacono, Struglia and Ronchi, 1999, Iacono, Struglia, Ronchi and Nicastro, 1999, Scott and Polvani, 2007, Showman, 2007). However, Jovian simulations of the height-damped shallow water model and the radiatively damped thermal shallow water model in chapter 3 produced strong super-rotating (prograde) equatorial jets, in line with Jovian observations (see figure 1.1a), for τ_{rad} sufficiently short. The Jovian simulations in §3.3 set $\tau_{\text{fric}} = 9900$ hours (or 1000 Jovian days), while the simulations which contained Newtonian cooling set $\tau_{\text{rad}} = 429$ hours (or 43 Jovian days). Henceforth the Newtonian cooling terms are referred to as cooling terms, while the Rayleigh friction term is referred to as friction, for brevity.

Equatorially trapped waves play an important role in near-equatorial dynamics (e.g. Gill, 1982, Khouider et al., 2013, Majda and Klein, 2003, McCreary, 1985). This chapter investigates the role of these waves in establishing the direction of the equatorial jet in Jovian simulations of the standard and height-damped shallow water equations, and the radiatively damped thermal shallow water equations. They are conveniently studied using the equatorial β -plane approximation, expressed in Cartesian geometry with a latitude-dependent Coriolis parameter, that is valid within about 30° of the equator. Expanding $f = 2\Omega \sin \phi$ around the equator gives $f(y) \approx \beta^* y$, where $\beta^* = 2\Omega/a$, $y = a\phi$, and a is the radius of the planet (see §1.4.1). The standard shallow water equations on the equatorial β -plane, with no dissipation or forcing, support a spectrum of equatorially trapped westward-propagating Rossby waves, originally discovered by Matsuno (1966), and described in §6.2. As shown in §6.3.1, the terms on the right-hand side of the shallow water zonal mean zonal momentum equation (6.3.3) are identically zero for these trapped Rossby waves (cf. evanescent waves in a spatially uniform system).

The focus on equatorially trapped Rossby waves is due to the global quasi-geostrophic simulations in chapter 5. Numerical simulations of the global thermal quasi-geostrophic equations (5.2.17) produce a super-rotating equatorial jet, as do simulations of the global height-damped quasi-geostrophic equations (5.3.3), when the radiative time scale is sufficiently short. However, numerical simulations of both the global thermal and global height-damped quasi-geostrophic equations produce

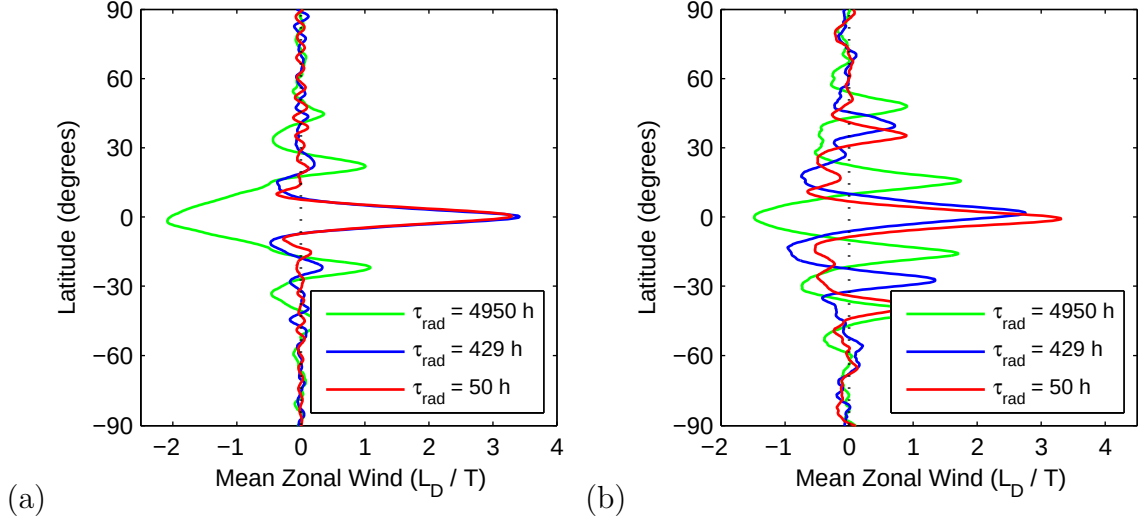


Figure 6.1: Mean zonal wind against latitude after 10^4 Jovian days for simulations of (a) the height-damped shallow water model and (b) the radiatively damped thermal shallow water model. All parameters are the same as for the simulations in §3.3, except the radiative time scale which is allowed to vary and takes the values 4950 hours (green line), 429 hours (blue line) and 50 hours (red line).

a sub-rotating equatorial jet when τ_{rad} is sufficiently long. Simulations of the global standard quasi-geostrophic equations ((5.3.3) with $\lambda = 0$) also produce a sub-rotating equatorial jet. This same switch in equatorial jet direction is also found in Jovian simulations of the height-damped shallow water equations, and the radiatively damped thermal shallow water equations (see figure 6.1 and §3.5.1). Although the precise value of τ_{rad} for which the switch in equatorial direction occurs is different for the shallow water models and their corresponding global quasi-geostrophic limits, the global quasi-geostrophic forms of the three different shallow water models produce equatorial jets in the same direction as their parent models away from this switching region. The mechanism responsible for this switch in equatorial jet direction for the shallow water simulations must lie within quasi-geostrophic theory. This justifies the subsequent neglect of the effects of equatorially trapped inertia-gravity, Kelvin and Yanai waves in the explanation for the equatorial jet direction.

Sardeshmukh and Hoskins (1988) introduced a diagnostic for sources of Rossby wave activity using a Helmholtz decomposition of the velocity $\mathbf{u} = \mathbf{u}_\chi + \mathbf{u}_\psi$ into divergent ($\mathbf{u}_\chi$) and rotational ($\mathbf{u}_\psi$) parts. The dimensionless absolute vorticity $\omega_a = f(y)/Ro + v_x - u_y$ in the forced height-damped shallow water equations (3.2.1) evolves

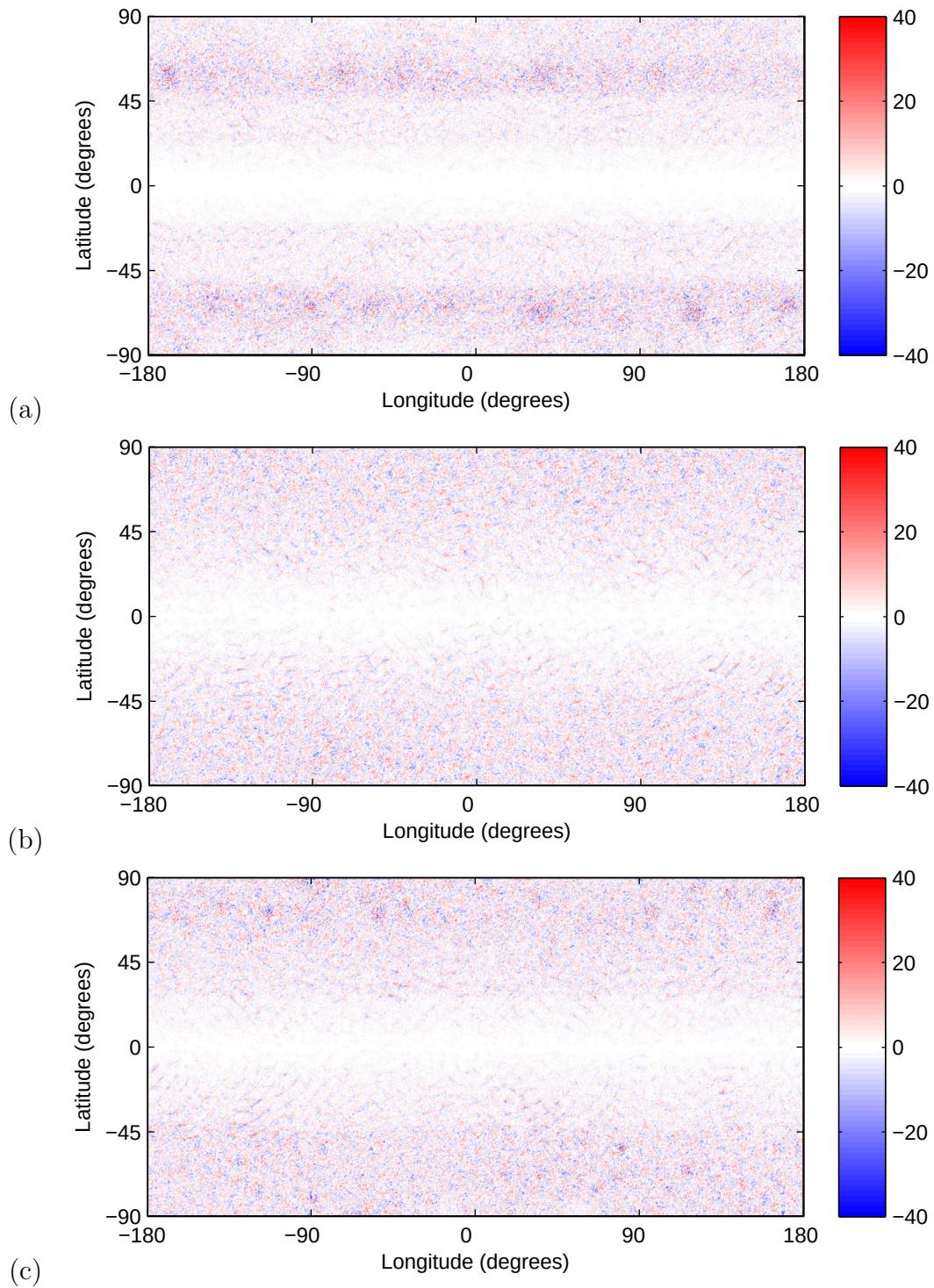


Figure 6.2: Rossby wave source R' in units of $(\text{Jovian day})^{-2}$ for (a) a standard shallow water simulation with dissipation by friction alone, (b) a height-damped shallow water simulation with dissipation by cooling and friction, and (c) a radiatively damped thermal shallow water simulation with dissipation by cooling and friction (see §3.3 for details of the simulations and further simulation outputs).

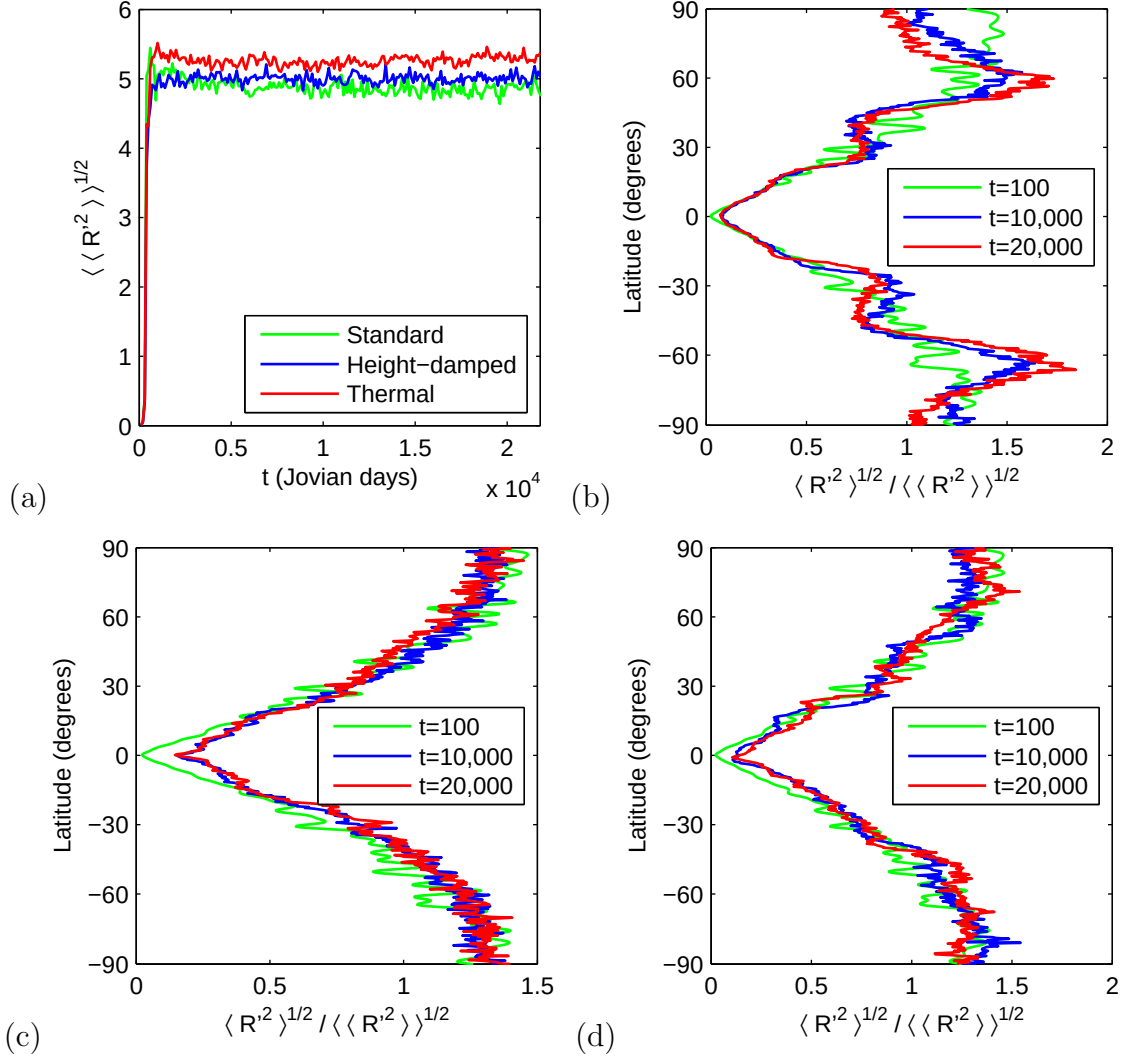


Figure 6.3: (a) Evolution of the total root-mean-square average of the Rossby wave source field, $\langle \langle R'^2 \rangle \rangle^{1/2}$, for simulations of the three different shallow water models. Root-mean-square zonal averages of the Rossby wave source field $\langle R'^2 \rangle^{1/2}$ normalised by $\langle \langle R'^2 \rangle \rangle^{1/2}$ are plotted at 100, 10^4 and 2×10^4 Jovian days for (b) a standard shallow water simulation, (c) a height-damped shallow water simulation, and (d) a radiatively damped thermal shallow water simulation (see §3.3 for details of the simulations and further simulation outputs).

according to

$$\partial_t \omega_a + (\mathbf{u}_\psi \cdot \nabla) \omega_a = R + \hat{\mathbf{z}} \cdot \nabla \times \mathbf{F} - \gamma (\omega_a - f(y)/Ro), \quad (6.1.3)$$

where $R = -\nabla \cdot (\omega_a \mathbf{u}_\chi)$, and the non-dimensionalisation from §3.2.1 has been applied with $Ro = 1/(4\pi)$ fixed. The corresponding equation for the forced radiatively damped thermal shallow water model (3.2.3) adds an additional term $B = \hat{\mathbf{z}} \cdot (\nabla h \times \nabla \Theta)/(2Ro^2)$ to the right-hand side of (6.1.3). The source term R is identified as the dimensionless source of Rossby wave activity and can be expanded as $R = -\omega_a(\nabla \cdot \mathbf{u}_\chi) - (\mathbf{u}_\chi \cdot \nabla) \omega_a$, where the first term represents stretching of the absolute vorticity and the second term represents advection of the absolute vorticity by the divergent flow. Following Schneider and Liu (2009), the non-zonal part $R' = R - \langle R \rangle$ is used in this section to diagnose the source of absolute vorticity fluctuations and thus the source of Rossby wave activity.

The other terms on the right-hand sides of the absolute vorticity evolution equations are neglected as sources of Rossby waves. The $-\gamma \omega_a$ term is neglected since it is a damping term and thus does not excite Rossby waves. The $\gamma f(y)/Ro$ term is independent of x , and thus does not have a horizontal wave number associated with it and cannot excite Rossby waves. The forcing term, $\hat{\mathbf{z}} \cdot \nabla \times \mathbf{F}$, is neglected in (6.1.3) since the forcing at any time step is uncorrelated to the forcing at the previous time step and so cannot excite motions that require coherence such as Rossby waves. The Rossby wave time scale is estimated for the radiatively damped thermal shallow water equations (3.2.3) with $\kappa = \gamma = 0$ and the forcing $\mathbf{F}$ neglected. If a mid-latitude β -plane is considered so $f = f_0 + \beta y$, then following the same procedure as that used in §4.4.1 gives the dispersion relation

$$Ro^2 \omega^3 - (k^2 + l^2 + f_0^2) \omega - \beta k / Ro = 0. \quad (6.1.4)$$

Neglecting the ω^3 term on the basis that we are considering low frequency Rossby waves gives the approximate Rossby wave dispersion relation

$$\omega = -\frac{(\beta/Ro)k}{k^2 + l^2 + f_0^2}. \quad (6.1.5)$$

Calculating the Rossby wave time scale (given by $1/|\omega|$) for a mid-latitude β -plane centred around a latitude of 45° and setting $l = 0, k = 40$ gives a time scale of 166 Jovian days (for comparison with the forcing we only consider $|\mathbf{k}| \in [40, 44]$). This is much longer than the forcing time scale meaning that the forcing cannot excite Rossby waves, thus this term is neglected as a source of Rossby waves. The total root-mean-square average of the non-zonal part of B (the extra term on the right-hand

side of the radiatively damped thermal shallow water absolute vorticity equation) is $\langle\langle B'^2 \rangle\rangle^{1/2} = 0.36$ at 2×10^4 Jovian days, while the root-mean-square zonal average at the equator is $\langle B'^2 \rangle^{1/2} = 0.09$ for the radiatively damped thermal shallow water simulation from §3.3. For comparison, the total root-mean-square average of R' for the radiatively damped thermal shallow water simulation from §3.3 is $\langle\langle R'^2 \rangle\rangle^{1/2} = 5.33$ at 2×10^4 Jovian days, while the root-mean-square zonal average is $\langle R'^2 \rangle^{1/2} = 0.65$ at the equator. Thus the non-zonal part of B is an order of magnitude smaller than the non-zonal part of R , thus this term is also neglected as a source of Rossby waves.

Figure 6.2 shows the instantaneous R' field at 2×10^4 Jovian days for the Jovian simulations from §3.3 of the standard and height-damped shallow water models, and the radiatively damped thermal shallow water model. Figure 6.3a shows the evolution of the total root-mean-square average of the Rossby wave source field, $\langle\langle R'^2 \rangle\rangle^{1/2}$, for the three shallow water simulations. Figures 6.3b–d show the root-mean-square zonal averages $\langle R'^2 \rangle^{1/2}$ at 100, 10^4 and 2×10^4 Jovian days for the three shallow water simulations, where each line is normalised by $\langle\langle R'^2 \rangle\rangle^{1/2}$ calculated at the corresponding time. These plots demonstrate that Rossby waves are preferentially generated away from the equator throughout the simulations for the three shallow water models. Figure 6.3a shows that $\langle\langle R'^2 \rangle\rangle^{1/2}$ has converged to a statistically steady state by 1000 Jovian days for the three shallow water models, and figures 6.3b–d explicitly demonstrate that there is little change in the shape of $\langle R'^2 \rangle^{1/2} / \langle\langle R'^2 \rangle\rangle^{1/2}$ as time evolves for the three simulations. If the analogue of figure 6.2 is created for B' , then this figure again shows that any Rossby waves created by B' are also preferentially generated away from the equator. Thus in the theory that follows, any Rossby waves present in the near-equatorial region are assumed to have been excited by sources outside that region.

This chapter provides a possible explanation for the directions of the equatorial jets in Jovian simulations of the standard and height-damped shallow water models, and also the radiatively damped thermal shallow water model. The inclusion of either form of dissipation, cooling or friction, is shown to damp the equatorially trapped Rossby waves. This alters the phase relation between the various terms in the shallow water zonal mean zonal momentum equation (defined by (6.3.3) and (6.3.7) for the height-damped shallow water equations and radiatively damped thermal shallow water equations respectively). This creates non-zero contributions to the zonal mean zonal acceleration. Dissipation by cooling leads to a positive zonal mean zonal acceleration, consistent with the formation of super-rotating equatorial jets, while

dissipation by friction leads to a negative zonal mean zonal acceleration, consistent with the formation of sub-rotating equatorial jets.

In calculating the transport of zonal momentum by Rossby waves, this approach represents a shallow water version of Andrews and McIntyre's (1976) study of the competing effects of friction and cooling on equatorial Rossby waves in a three-dimensional stratified atmosphere. A key difference is that the mean zonal accelerations at the equator calculated by Andrews and McIntyre (1976) were always westward ($\langle u \rangle_t \leq 0$ in the notation of §6.3) and so could not explain the formation of super-rotating equatorial jets. Imamura et al. (2004) performed a similar analysis for a modified equatorial Kelvin wave solution in the presence of a zonal mean meridional circulation. This solution possessed a non-zero meridional wave component which induced a meridional flux of zonal momentum. Showman and Polvani (2011) recently looked at a variant of the Matsuno–Gill problem for the response due to a steady longitudinally-varying forcing as a model for solar forcing on a tidally locked exoplanet. They found that the interaction between the resulting standing Rossby and Kelvin waves generated a super-rotating equatorial jet, but this model relies on a tidally-locked solar forcing, and so does not apply to Jupiter with its 9.9 hour rotation period. Arnold et al. (2012) found super-rotation in an idealised multi-level GCM forced with a zonally varying equatorial heating, and identified a Rossby–Kelvin wave resonance as being responsible for eddy momentum flux convergence at the equator. However, their theory requires an equatorial source of Rossby waves, which is not visible for the Jovian simulations from §3.3 (see figures 6.2 and 6.3).

6.2 Rossby waves on an equatorial beta-plane

This section investigates the properties of equatorially trapped waves in the three different shallow water models in the presence of dissipation.

6.2.1 Height-damped shallow water equations

The linearised form of the height-damped shallow water equations (6.1.1) on an equatorial β -plane may be written as (Gill, 1982, Matsuno, 1966)

$$h'_t + u'_x + v'_y = -\kappa h', \quad (6.2.1a)$$

$$u'_t - \frac{1}{2}yv' + h'_x = -\gamma u', \quad (6.2.1b)$$

$$v'_t + \frac{1}{2}yu' + h'_y = -\gamma v'. \quad (6.2.1c)$$

The dashes denote small perturbations about a rest state with uniform depth h_0 . A non-dimensionalisation has been applied using the internal wave speed $c = \sqrt{g'h_0}$ as the velocity scale, the equatorial deformation radius $L_{\text{eq}} = \sqrt{c/2\beta^*}$ as the horizontal length scale ($\beta^* = 2\Omega/a$), h_0 as the depth scale, and the advective time scale $T = L_{\text{eq}}/c$ for time. Table 3.1 gives Jovian values of all these parameters (where $\Theta_0 = g'$). The dimensionless cooling and friction rates are $\kappa = T/\tau_{\text{rad}}$ and $\gamma = T/\tau_{\text{fric}}$. The standard shallow water simulation in §3.3 took $\kappa = 0$ and $\gamma = 0.000342$, while the height-damped shallow water simulation took $\kappa = 0.00788$ and $\gamma = 0.000342$. In these variables, the latitude is $\phi = (L_{\text{eq}}/a)y = \sqrt{Ro}y$ where $Ro = c/(4\Omega a)$ is the Rossby number based on the internal wave speed and the planetary diameter $2a$. The Jovian parameters in table 3.1 give $Ro \approx 0.014$, so $y = 5$ corresponds to a latitude of 33° in the subsequent plots.

Following Matsuno (1966) and Gill (1982), waves are sought that are harmonic in longitude and time of the form $h'(x, y, t) = \text{Re}\{\hat{h}(y) \exp(i(kx - \omega t))\}$ etc. With no dissipation ($\kappa = \gamma = 0$) the three equations (6.2.1) may be combined into a single ordinary differential equation for the meridional velocity $\hat{v}(y)$,

$$\frac{d^2 \hat{v}}{dy^2} = (Ay^2 - B)\hat{v}, \quad A = \frac{1}{4}, \quad B = \omega^2 - k^2 - \frac{k}{2\omega}, \quad (6.2.2)$$

the same equation that governs a quantum harmonic oscillator. The solutions that decay as $y \rightarrow \pm\infty$ may be written using the Hermite polynomials $H_n(\xi)$ as

$$\hat{v} = H_n(\xi) \exp(-\xi^2/2), \quad \xi = yA^{1/4}. \quad (6.2.3)$$

The dispersion relation $A^{-1/2}B = 2n + 1$ for $n = -1, 0, 1, 2, \dots$ gives a cubic equation for ω ,

$$\omega^2 - k^2 - \frac{k}{2\omega} = n + \frac{1}{2}. \quad (6.2.4)$$

The Rossby wave branch of solutions is characterised by $n \geq 1$ and $0 < -\omega/k \ll 1$. The other two roots give inertia-gravity waves with $|\omega| \geq |k|$. The cases $n = 0$ and $n = -1$ give the Yanai and Kelvin waves respectively. Figure 6.4 shows these different branches of the dispersion relation, all of which represent trapped waves localised within a few deformation radii of the equator at $y = 0$. The inertia-gravity, Yanai and Kelvin wave branches all enter the super-inertial frequency regime $|\omega| \geq |f|$, and so do not exist in quasi-geostrophic theory. Since the mechanism responsible for the equatorial jet direction in the shallow water models is known to lie within quasi-geostrophic theory, the effects of inertia-gravity, Kelvin and Yanai waves are not

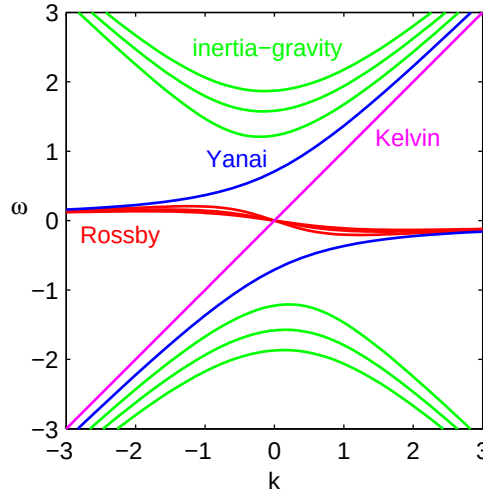


Figure 6.4: Dispersion relation for equatorially trapped waves in the standard shallow water equations with no dissipation (so ω is real) on the equatorial β -plane. The dispersion relation is shown for both positive and negative k and ω . By convention, however, it would only be plotted for $\omega > 0$ since the dispersion relation is invariant under the transformation $(k, \omega) \rightarrow (-k, -\omega)$. However, only $k > 0$ and negative Rossby wave frequencies $\text{Re}(\omega) < 0$ are considered in the work that follows.

considered in the explanation for the direction of the equatorial jet which is detailed in the remainder of this chapter.

Restoring the friction and cooling terms in (6.2.1) changes the A and B coefficients in the ordinary differential equation (6.2.2) to

$$A = \frac{1}{4} \left(\frac{\omega + i\kappa}{\omega + i\gamma} \right), \quad B = \omega^2 - k^2 - \frac{k}{2(\omega + i\gamma)} + i\omega(\kappa + \gamma) - \kappa\gamma, \quad (6.2.5)$$

so the dispersion relation for equatorially trapped Rossby waves becomes

$$\omega^2 - k^2 - \frac{k}{2(\omega + i\gamma)} + i\omega(\kappa + \gamma) - \kappa\gamma = \left(n + \frac{1}{2} \right) \left(\frac{\omega + i\kappa}{\omega + i\gamma} \right)^{1/2} \quad \text{for } n = 1, 2, \dots, \quad (6.2.6)$$

for the height-damped shallow water model.

Figure 6.5 shows the real and imaginary parts of the complex frequency ω for the first few equatorial Rossby waves ($n = 1, 2, 3$) for different values of γ and κ . The real part of ω is virtually unchanged for weak damping ($\kappa = 0.001$ or $\gamma = 0.001$), while $\text{Im}(\omega)$ becomes negative as expected. The damping due to cooling ($\kappa = 0.001$ and $\gamma = 0$) is largest for small k , and tends to zero as $k \rightarrow \infty$. By contrast, the damping due to friction ($\gamma = 0.001$ and $\kappa = 0$) is monotonically increasing with k , being weakest for small k , and tending to $-\gamma$ as $k \rightarrow \infty$.

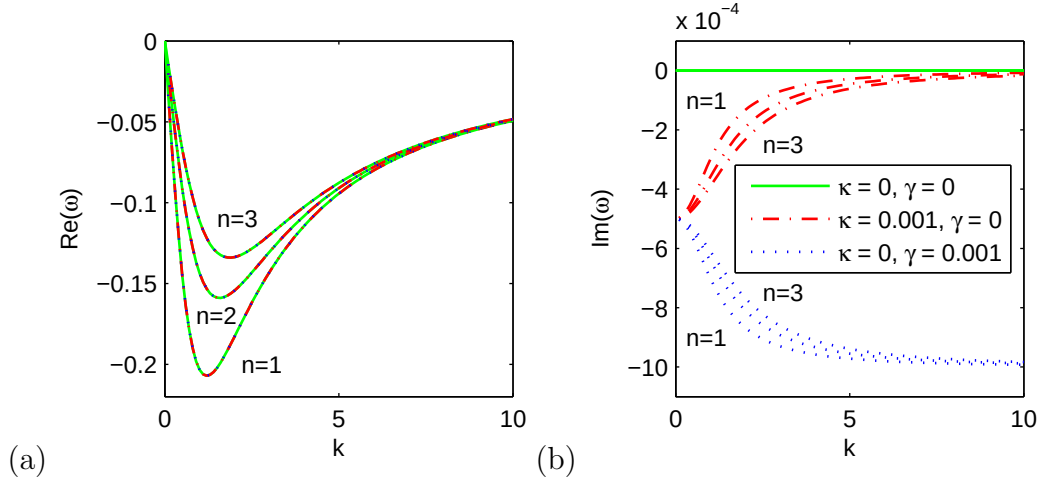


Figure 6.5: The (a) real and (b) imaginary parts of the dimensionless equatorial Rossby wave frequency ω for the height-damped shallow water model with no dissipation (green solid lines), dissipation by cooling alone (red dash-dot lines), and dissipation by friction alone (blue dotted lines).

Figure 6.5b suggests a symmetry about the line $\text{Im}(\omega) = -\varepsilon/2$ between small cooling ($\kappa = \varepsilon$ and $\gamma = 0$) and small friction ($\kappa = 0$ and $\gamma = \varepsilon$). This symmetry can be investigated by expanding the roots of (6.2.6) as $\omega = \omega_0 + \varepsilon \omega_{1,\text{rad}} + \varepsilon^2 \omega_{2,\text{rad}} + \dots$ for cooling and $\omega = \omega_0 + \varepsilon \omega_{1,\text{fric}} + \varepsilon^2 \omega_{2,\text{fric}} + \dots$ for friction. In both cases the real frequency ω_0 is given by (6.2.4) at leading order. The $O(\varepsilon)$ corrections are both purely dissipative:

$$\omega_{1,\text{rad}} = \frac{1}{2}i\omega_0(-4\omega_0^2 + 2n + 1)/(k + 4\omega_0^3), \quad (6.2.7a)$$

$$\omega_{1,\text{fric}} = \frac{1}{2}i\omega_0(-4\omega_0^2 - 2n - 1 - 2k/\omega_0)/(k + 4\omega_0^3). \quad (6.2.7b)$$

Observing that

$$\omega_{1,\text{rad}} + i/2 = \frac{i}{2} \left(\frac{k + (2n + 1)\omega_0}{k + 4\omega_0^3} \right) = -(\omega_{1,\text{fric}} + i/2), \quad (6.2.8)$$

establishes the symmetry about $\text{Im}(\omega) = -\varepsilon/2$ for the dispersion relation truncated at $O(\varepsilon)$.

Figure 6.5b shows that $\text{Im}(\omega)$ is a function of both n and k away from $k = 0$ for dissipation by cooling alone (red dash-dot lines) and by friction alone (blue dotted lines). However, close to $k = 0$ the imaginary part of the dimensionless equatorial Rossby wave frequency ω appears to be independent of n for dissipation by cooling alone and by friction alone. To establish this property, first consider the case of small cooling ($\kappa = \varepsilon$ and $\gamma = 0$), where the roots of (6.2.6) are expanded as $\omega =$

$\omega_0 + \varepsilon \omega_{1,\text{rad}} + \varepsilon^2 \omega_{2,\text{rad}} + \dots$. As detailed above, the leading order frequency ω_0 is real and given by (6.2.4). Expanding ω_0 for small k as $\omega_0 = \omega_{0,k} + k \omega_{1,k} + k^2 \omega_{2,k} + \dots$ in

$$\omega_0^2 - k^2 - \frac{k}{2\omega_0} = n + \frac{1}{2}, \quad (6.2.9)$$

gives $\omega_0 = -k/(2n+1) + O(k^3)$ for small k . Substituting $\omega_0 = -k/(2n+1) + O(k^3)$ in (6.2.8) gives

$$\omega_{1,\text{rad}} + i/2 = O(k^2). \quad (6.2.10)$$

Performing the same analysis for the small friction case ($\kappa = 0$ and $\gamma = \varepsilon$) gives $\omega_{1,\text{fric}} + i/2 = O(k^2)$. Thus for small k and small cooling $\text{Im}(\omega) = -i\kappa/2$ at leading order in k and κ , while for small k and small friction $\text{Im}(\omega) = -i\gamma/2$ at leading order in k and γ . This establishes the independence of $\text{Im}(\omega)$ on n close to $k = 0$ in figure 6.5b.

6.2.2 Radiatively damped thermal shallow water equations

The same procedure is now followed for the radiatively damped thermal shallow water equations (6.1.2). The equatorial β -plane is again considered, and linearising for small perturbations about a state of rest with uniform depth h_0 and uniform temperature Θ_0 gives

$$h'_t + u'_x + v'_y = 0, \quad (6.2.11a)$$

$$\Theta'_t = -\kappa(h' + \Theta'), \quad (6.2.11b)$$

$$u'_t - \frac{1}{2}yv' + h'_x + \frac{1}{2}\Theta'_x = -\gamma u', \quad (6.2.11c)$$

$$v'_t + \frac{1}{2}yu' + h'_y + \frac{1}{2}\Theta'_y = -\gamma v'. \quad (6.2.11d)$$

The non-dimensionalisation is as before with Θ_0 introduced as the temperature scale, and $c = \sqrt{h_0\Theta_0}$ taken as the velocity scale. The dimensionless cooling and friction rates are again defined by $\kappa = T/\tau_{\text{rad}}$ and $\gamma = T/\tau_{\text{fric}}$ respectively, where the radiatively damped thermal shallow water simulation in §3.3 took $\kappa = 0.00788$ and $\gamma = 0.000342$. After seeking waves that are harmonic in longitude and time, the four equations (6.2.11) may again be combined into the single ordinary differential equation (6.2.2) for $\hat{v}(y)$, with coefficients

$$A = \frac{1}{4} \frac{\omega(\omega + i\kappa)}{(\omega + i\gamma)(\omega + i\kappa/2)}, \quad B = \frac{\omega(\omega + i\gamma)(\omega + i\kappa)}{\omega + i\kappa/2} - k^2 - \frac{k}{2(\omega + i\gamma)}, \quad (6.2.12)$$

so the dispersion relation for equatorially trapped Rossby waves becomes

$$\frac{\omega(\omega + i\gamma)(\omega + i\kappa)}{\omega + i\kappa/2} - k^2 - \frac{k}{2(\omega + i\gamma)} = \left(n + \frac{1}{2}\right) \left(\frac{\omega(\omega + i\kappa)}{(\omega + i\gamma)(\omega + i\kappa/2)}\right)^{1/2}, \quad (6.2.13)$$

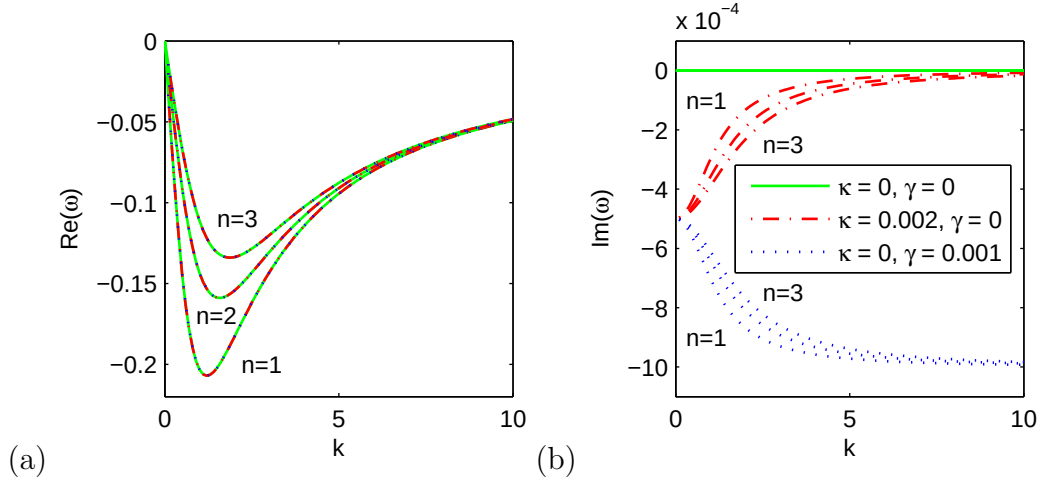


Figure 6.6: The real (a) and imaginary (b) parts of the dimensionless equatorial Rossby wave frequency ω for the radiatively damped thermal shallow water model with no dissipation (green solid lines), dissipation by cooling alone (red dash-dot lines), and dissipation by friction alone (blue dotted lines).

for the radiatively damped thermal shallow water model.

Figure 6.6 shows the real and imaginary parts of the complex frequency ω for the first few equatorial Rossby waves ($n = 1, 2, 3$) and three different sets of κ and γ values. Again, the real part of ω is virtually unchanged by weak dissipation ($\kappa = 0.002$ or $\gamma = 0.001$), while $\text{Im}(\omega)$ becomes negative. Figure 6.6 closely resembles figure 6.5, but κ has been doubled to 0.002. A perturbation analysis similar to before shows that $\text{Im}(\omega)$ is symmetric about $-\varepsilon/2$ between the cases of small cooling ($\kappa = 2\varepsilon$ and $\gamma = 0$) and small friction ($\kappa = 0$ and $\gamma = \varepsilon$). While figure 6.6b shows that $\text{Im}(\omega)$ is a function of both n and k away from $k = 0$ for dissipation by cooling alone (red dash-dot lines) and friction alone (blue dotted lines), close to $k = 0$ the imaginary part of ω appears to be independent of n . The independence of $\text{Im}(\omega)$ on n close to $k = 0$ can be established by a perturbation analysis similar to that detailed on page 185.

6.3 Acceleration of the zonal mean zonal flow

The acceleration of the zonal mean zonal flow caused by these solutions for decaying equatorially trapped Rossby waves is now calculated.

6.3.1 Height-damped shallow water equations

Consider the full version of the height-damped shallow water model (6.1.1) on an equatorial β -plane. Combining the mass equation with the zonal momentum equation gives the flux form of the zonal momentum equation,

$$(hu)_t + \left(hu^2 + \frac{1}{2}h^2\right)_x + (huv)_y - \frac{1}{2}y hv = -\gamma hu - \kappa u(h - 1), \quad (6.3.1)$$

using the non-dimensionalisation of §6.2.1. Following standard practice (Andrews and McIntyre, 1976, Eliassen and Palm, 1961, Vallis, 2006), the velocity and other fields are now decomposed as $\mathbf{u} = \langle \mathbf{u} \rangle + \mathbf{u}'$ into a zonal average $\langle \mathbf{u} \rangle$ and a deviation $\mathbf{u}'$. Combining the zonal average of (6.3.1) with the zonal average of the dimensionless mass conservation equation,

$$\langle h \rangle_t + \langle hv \rangle_y = -\kappa (\langle h \rangle - 1), \quad (6.3.2)$$

gives the zonal mean zonal momentum equation

$$\langle u \rangle_t = \underbrace{\langle v \rangle^* \left(\frac{1}{2}y - \langle u \rangle_y\right)}_{\text{Term 1}} - \underbrace{\frac{1}{\langle h \rangle} \partial_y \langle (hv)' u' \rangle}_{\text{Term 2}} - \underbrace{\frac{\kappa}{\langle h \rangle} \langle h' u' \rangle}_{\text{Term 3}} - \underbrace{\gamma \langle u \rangle^*}_{\text{Term 4}} - \underbrace{\frac{1}{\langle h \rangle} \langle h' u' \rangle_t}_{\text{Term 5}}, \quad (6.3.3)$$

where $\langle a \rangle^* = \langle ha \rangle / \langle h \rangle$ is defined to be the mass-weighted zonal mean of a quantity a (Showman and Polvani, 2011, Thuburn and Lagneau, 1999). Term 1 is the zonal acceleration due to the advection of absolute zonal momentum by the mean meridional circulation. Term 2 is the zonal acceleration due to the convergence of the shallow water Eliassen–Palm flux $-\langle (hv)' u' \rangle$ (Thuburn and Lagneau, 1999). Terms 3 and 4 are the zonal accelerations due directly to cooling and friction, respectively. Term 5 is the zonal acceleration due to the rate of change of zonal eddy momentum.

These general expressions are now evaluated for the equatorially trapped Rossby wave solutions from §6.2.1. These represent small perturbations about a state of rest with unit depth in dimensionless variables. The meridional velocity perturbation is $v' = \text{Re}\{\hat{v}(y) \exp(i(kx - \omega t))\}$, with $\hat{v}(y)$ defined by (6.2.3) and (6.2.5). The corresponding zonal velocity and height perturbations are $u' = \text{Re}\{\hat{u}(y) \exp(i(kx - \omega t))\}$ and $h' = \text{Re}\{\hat{h}(y) \exp(i(kx - \omega t))\}$, where

$$\hat{u}(y) = i \left(\frac{k(d\hat{v}/dy) - \frac{1}{2}y(\omega + i\kappa)\hat{v}}{k^2 - (\omega + i\gamma)(\omega + i\kappa)} \right), \quad \hat{h}(y) = -i \left(\frac{(i\omega - \gamma)\hat{u} + \frac{1}{2}y\hat{v}}{k} \right), \quad (6.3.4)$$

for $k \neq 0$. The vanishing denominator at $k^2 = (\omega + i\gamma)(\omega + i\kappa)$ in $\hat{u}(y)$ gives the dispersion relation for equatorially trapped Kelvin waves, which are distinguished by having zero meridional velocity ($\hat{v} = 0$).

Using standard formulae for the averages of products, the five terms on the right-hand side of (6.3.3) are

$$\langle v \rangle^* \left(\frac{1}{2}y - \langle u \rangle_y \right) = \frac{1}{4}y \operatorname{Re}\{\hat{h}(y)\hat{v}(y)^*\}, \quad (6.3.5a)$$

$$-\frac{1}{\langle h \rangle} \partial_y \langle (hv)' u' \rangle = -\partial_y \langle u' v' \rangle = -\partial_y \left(\frac{1}{2} \operatorname{Re}\{\hat{u}(y)\hat{v}(y)^*\} \right), \quad (6.3.5b)$$

$$-\frac{\kappa}{\langle h \rangle} \langle h' u' \rangle = -\frac{1}{2}\kappa \operatorname{Re}\{\hat{h}(y)\hat{u}(y)^*\}, \quad (6.3.5c)$$

$$-\gamma \langle u \rangle^* = -\frac{1}{2}\gamma \operatorname{Re}\{\hat{h}(y)\hat{u}(y)^*\}, \quad (6.3.5d)$$

$$-\frac{1}{\langle h \rangle} \langle h' u' \rangle_t = -\frac{1}{2} \operatorname{Re}\{-i\omega \hat{h}(y)\hat{u}(y)^*\} - \frac{1}{2} \operatorname{Re}\{\hat{h}(y)(-i\omega \hat{u}(y))^*\}, \quad (6.3.5e)$$

where a superscript $*$ denotes a complex conjugate. The cubic contribution $-\partial_y \langle h' u' v' \rangle$ has been omitted from (6.3.5b) since perturbations are assumed to be small. The total zonal acceleration $\langle u \rangle_t$ is the sum of these five terms, each of which is quadratic in the perturbation quantities. A normalisation is applied that allows a fair comparison between the mean zonal accelerations generated by waves with different n and k values. Since the quantities (6.3.5) are all proportional to the square of the wave amplitude, it is appropriate to normalise with a quantity that is also quadratic in the wave amplitude. Thus u' , v' and h' are normalised by making the quadratic approximation to the dimensionless disturbance pseudo-energy equal to unity,

$$\frac{1}{2} \int u'^2 + v'^2 + h'^2 \, dx \, dy = 1. \quad (6.3.6)$$

The pseudo-energy for the height-damped shallow water equations is defined by (4.6.12) on setting $\Theta = \Theta_0 = g'$, and takes the above form when expanded for small amplitude disturbances from a rest state of constant depth (Shepherd, 1993). Verkley (2009), following Longuet-Higgins (1968b), used a different normalisation since the velocity was non-dimensionalised with $a\Omega$, leading to a factor of $4/\varepsilon$ in front of h'^2 , where $\varepsilon = 4\Omega^2 a^2 / (g' h_0)$ is Lamb's parameter.

All five terms are identically zero for undamped ($\kappa = \gamma = 0$) equatorially trapped Rossby waves, since ω , A , and $\hat{v}$ are all real, while $\hat{u}$ and $\hat{h}$ are purely imaginary. Figure 6.7 shows $\langle u \rangle_t$ as a function of y for equatorial Rossby waves with $n = 1, 2, 3$ as damped by friction alone ($\kappa = 0$ and $\gamma = 0.001$) and by cooling alone ($\kappa = 0.001$ and $\gamma = 0$). The y -dependence of $\langle u \rangle_t$ for waves with $n = 1, 2, 3$ dissipated by cooling ($\kappa > 0$) implies $\langle u \rangle_t > 0$ in a region near the equator. This is consistent with the formation of a super-rotating equatorial jet. For $n = 1$ the tips of the two near-equatorial peaks are located at $\pm 7^\circ$, while for $n = 3$ the tips of the two near-equatorial peaks are located at $\pm 6^\circ$ (where the conversion between y and latitude is

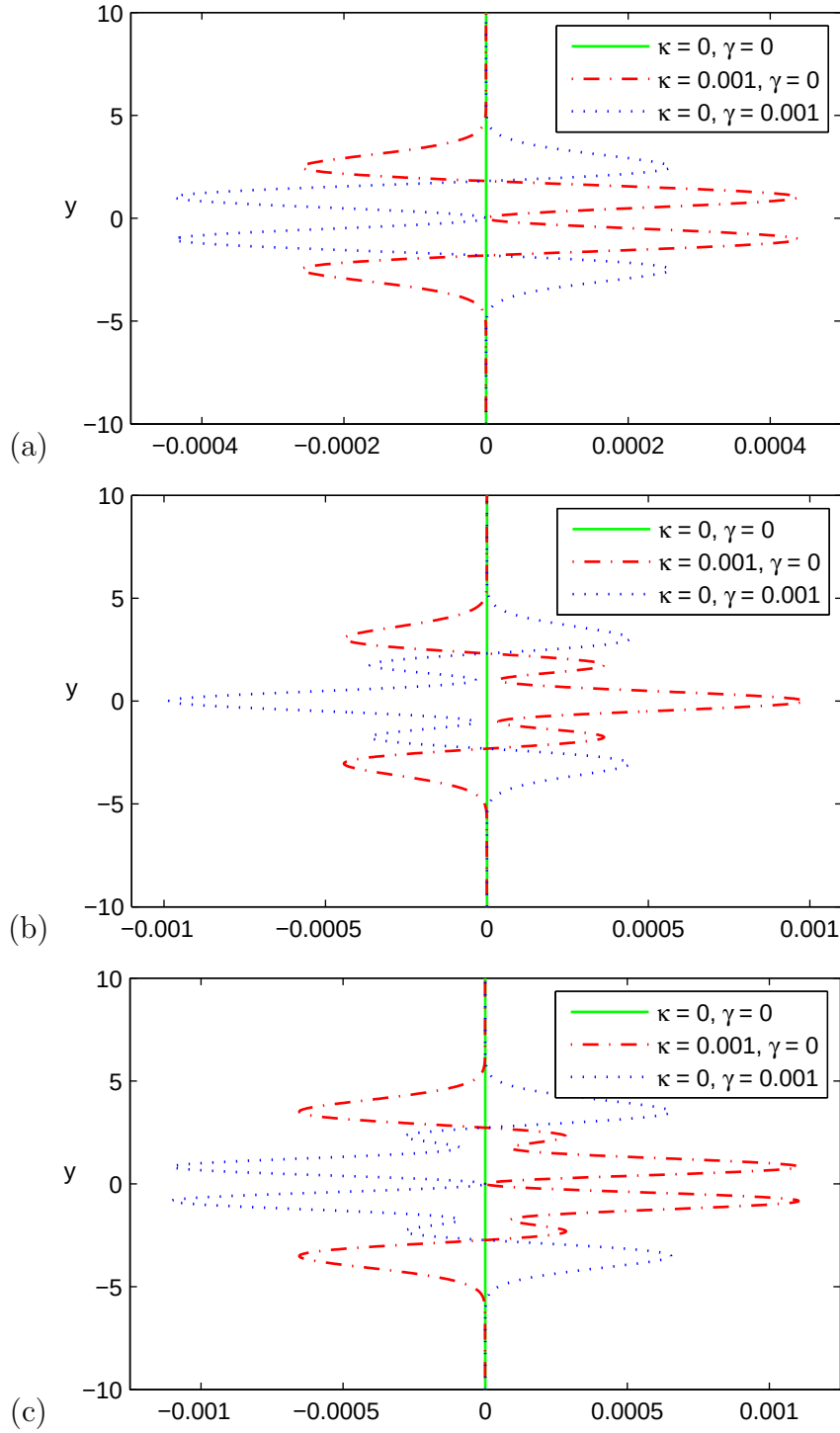


Figure 6.7: The sum of the five zonal acceleration terms on the right-hand side of (6.3.3) for the three cases of no dissipation (green solid line), dissipation by cooling ($\kappa \neq 0$, red dash-dot line) and dissipation by friction ($\gamma \neq 0$, blue dotted line). All terms are plotted for $k = 1$. Plots (a), (b), (c) correspond to $n = 1, 2, 3$ respectively.

defined by $\phi = (L_{\text{eq}}/a)y = \sqrt{Ro} y$ with $Ro = c/(4\Omega a)$, and table 3.1 gives $Ro \approx 0.014$ for Jupiter). For $n = 2$ the large eastward peak extends from roughly -7° to 7° latitude. The latitudinal structure of $\langle u \rangle_t$ is thus consistent with the locations of the two “horns”, or local maxima of the eastward velocity, that are observed in the broad Jovian equatorial jet shown in figure 1.1a.

Conversely, the $\langle u \rangle_t$ for frictionally dissipated waves shown in figure 6.7 is negative near the equator, consistent with the formation of a sub-rotating equatorial jet. A perturbation analysis similar to that in §6.2.1 shows that $\langle u \rangle_t$ due to weak cooling ($\kappa = \varepsilon$ and $\gamma = 0$) is equal to minus the $\langle u \rangle_t$ due to weak friction ($\gamma = \varepsilon$ and $\kappa = 0$). Thus as figure 6.7 suggests, there is a line of symmetry about zero between the cases of small cooling and small friction. Moreover, $\langle u \rangle_t$ is very small at the equator for both friction and cooling when n is odd. Increasing k while keeping n, κ and γ fixed increases the magnitude of $\langle u \rangle_t$ without changing its general shape. Increasing one of κ or γ while keeping k and n fixed increases the magnitude of $\langle u \rangle_t$ without changing its general shape.

Figure 6.8 shows the five separate terms in (6.3.3) alongside their sum $\langle u \rangle_t$ (black line) for $n = k = 1$ and dissipation due to either (a) cooling or (b) friction. The Eliassen–Palm flux convergence (Term 2, red line) makes the largest contribution for both types of dissipation. The other terms have a smaller effect. In figure 6.8a, the zonal acceleration due directly to friction is identically zero (green line). Accelerations due to the advection of absolute zonal momentum by the mean meridional circulation (pink line) and accelerations due directly to cooling (blue line) are eastward near the equator, while accelerations due to the rate of change of zonal eddy momentum (cyan line) are westward near the equator. For figure 6.8b, accelerations due directly to cooling are now identically zero (blue line), while accelerations due to the advection of absolute zonal momentum by the mean meridional circulation (pink line) have changed direction. Accelerations due directly to friction are now eastward (green line) near the equator. The main contribution for $n = 2$, shown in figure 6.9, is again zonal accelerations due to the convergence of the Eliassen–Palm flux (Term 2, red line). However, the magnitude of Terms 3, 4 and 5 now decrease towards zero at the equator. For $n = 3$ (not shown) the Eliassen–Palm flux convergence is again the main contribution to $\langle u \rangle_t$ near the equator.

6.3.2 Radiatively damped thermal shallow water equations

The zonal acceleration due to equatorially trapped Rossby waves is now computed for the radiatively damped thermal shallow water model (6.1.2) on an equatorial β -plane,

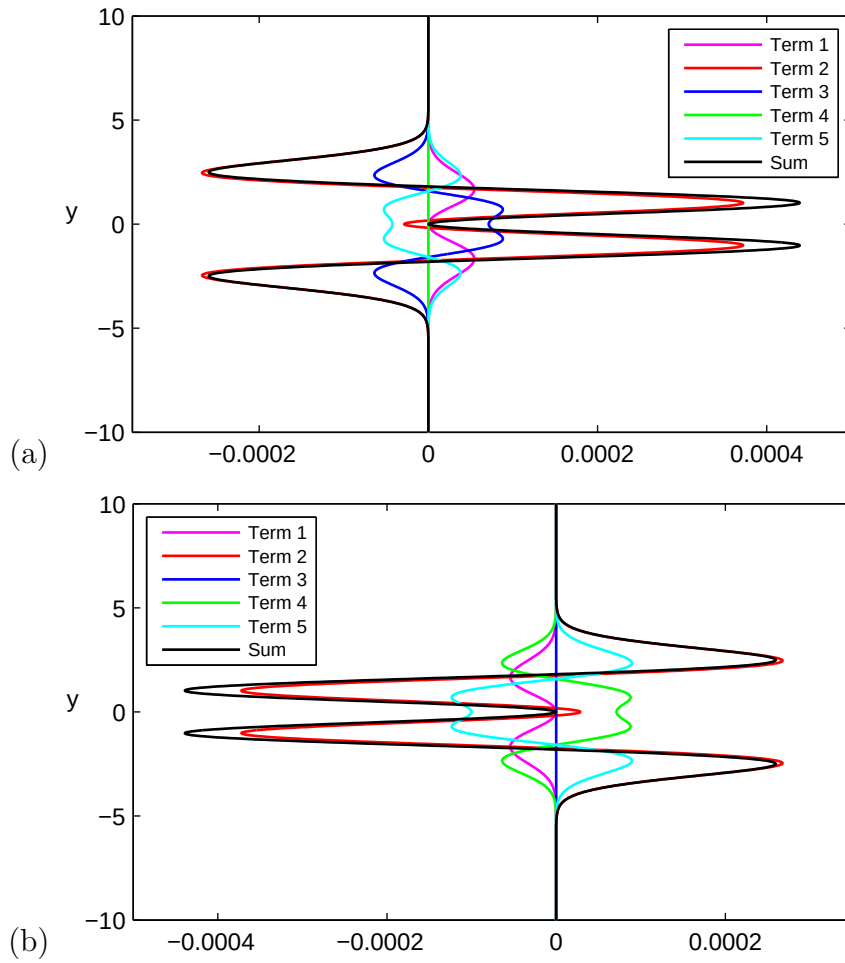


Figure 6.8: The five zonal acceleration terms from the right-hand side of (6.3.3), and their sum, for (a) dissipation solely by cooling ($\kappa = 0.001, \gamma = 0$) and (b) dissipation solely by friction ($\kappa = 0, \gamma = 0.001$). All terms are calculated for $n = k = 1$.

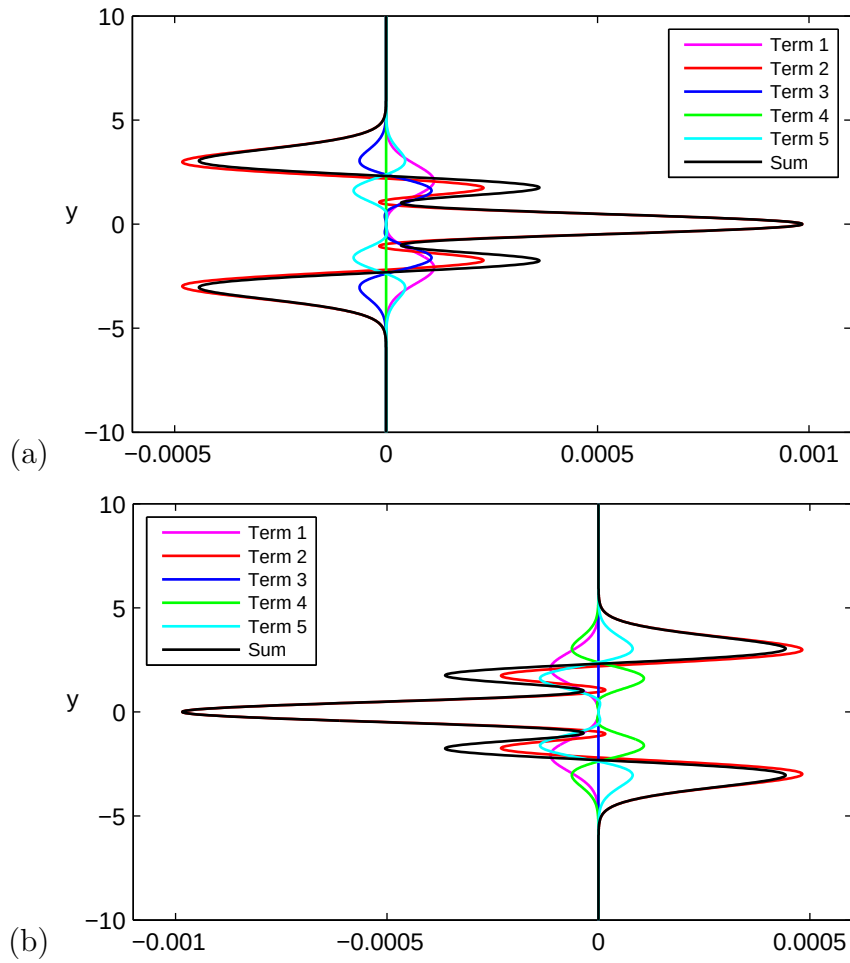


Figure 6.9: The five zonal acceleration terms from the right-hand side of (6.3.3), and their sum, for (a) dissipation solely by cooling ($\kappa = 0.001, \gamma = 0$) and (b) dissipation solely by friction ($\kappa = 0, \gamma = 0.001$). All terms are calculated for $n = 2$ and $k = 1$.

where the same non-dimensionalisation used in §6.2.2 is applied. The corresponding zonal mean zonal momentum equation is

$$\langle u \rangle_t = \underbrace{\langle v \rangle^* \left(\frac{1}{2}y - \langle u \rangle_y \right)}_{\text{Term 1}} - \underbrace{\frac{1}{\langle h \rangle} \partial_y \langle (hv)' u' \rangle}_{\text{Term 2}} - \underbrace{\gamma \langle u \rangle^*}_{\text{Term 4}} - \underbrace{\frac{1}{\langle h \rangle} \langle h' u' \rangle_t}_{\text{Term 5}}. \quad (6.3.7)$$

There is no explicit cooling term proportional to κ , because cooling appears only in the temperature equation. However, the four terms are still labelled as Term 1, Term 2, Term 4, and Term 5 to aid comparison with §6.3.1.

Again, these general expressions are evaluated for the equatorially trapped Rossby wave solutions from §6.2.2. These represent small perturbations about a state of rest with unit depth and temperature in dimensionless variables. The meridional velocity perturbation for an equatorial Rossby wave is again $v' = \text{Re}\{\hat{v}(y) \exp(i(kx - \omega t))\}$, with $\hat{v}(y)$ defined by (6.2.3) and (6.2.12). The corresponding perturbations to the height, temperature, and zonal velocity are $h' = \text{Re}\{\hat{h}(y) \exp(i(kx - \omega t))\}$, $\Theta' = \text{Re}\{\hat{\Theta}(y) \exp(i(kx - \omega t))\}$ and $u' = \text{Re}\{\hat{u}(y) \exp(i(kx - \omega t))\}$ respectively, where

$$\begin{aligned} \hat{h}(y) &= -i \left(\frac{ik\hat{u} + d\hat{v}/dy}{\omega} \right), & \hat{\Theta}(y) &= -i \left(\frac{\kappa\hat{h}}{\omega + i\kappa} \right), \\ \hat{u}(y) &= i \left(\frac{k(1 + i\kappa/(2\omega)) d\hat{v}/dy - \frac{1}{2}y(\omega + i\kappa)\hat{v}}{k^2(1 + i\kappa/(2\omega)) - (\omega + i\gamma)(\omega + i\kappa)} \right), \end{aligned} \quad (6.3.8)$$

for $\omega \neq 0$ and $\omega \neq -i\kappa$. The remaining vanishing denominator for $\hat{u}$ at $k^2 = (\omega + i\gamma)(\omega + i\kappa) / (1 + i\kappa/(2\omega))$ gives the dispersion relation for the equatorially trapped Kelvin waves as modified by friction and cooling. As before, the disturbances u' , v' , h' and Θ' are normalised such that

$$\frac{1}{2} \int u'^2 + v'^2 + (h' + \Theta'/2)^2 dx dy = 1. \quad (6.3.9)$$

The left-hand side contains the dimensionless quadratic approximation to the disturbance pseudo-energy for the thermal shallow water equations (4.6.12), when expanded for small amplitude disturbances from a rest state of constant depth and temperature. This normalisation allows a fair comparison between the mean zonal accelerations generated by waves with different k and n values.

Figure 6.10 shows the net zonal acceleration $\langle u \rangle_t$ generated by cooling alone and by friction alone, for the first three equatorial Rossby waves $n = 1, 2, 3$ with $k = 1$. Again, all terms are identically zero in the absence of dissipation. For $n = 2$ and dissipation by cooling, $\langle u \rangle_t > 0$ in a region near the equator. For $n = 1$ and $n = 3$,

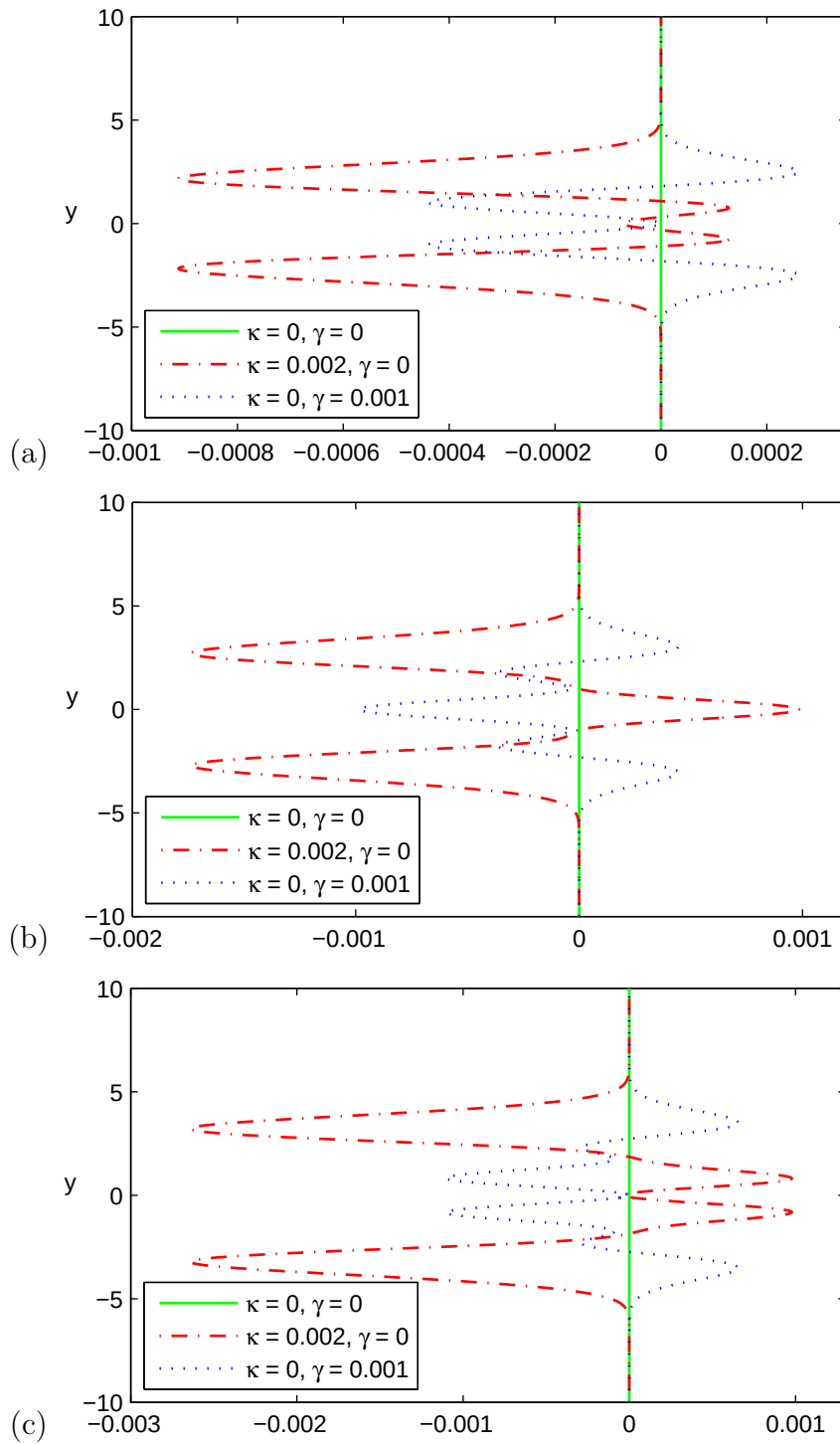


Figure 6.10: The sum of the four zonal acceleration terms on the right-hand side of (6.3.7) for the three cases of no dissipation (green solid line), dissipation by cooling alone (red dash-dot line), and dissipation by friction alone (blue dotted line). All terms are plotted for $k = 1$. Plots (a),(b),(c) correspond to $n = 1, 2, 3$ respectively.

$\langle u \rangle_t < 0$ in a very small region close to the equator. However, $\langle u \rangle_t > 0$ from -7° to -2° and from 2° to 7° for $n = 1$. For $n = 3$, $\langle u \rangle_t > 0$ from -12° to -1° and from 1° to 12° (where the conversion between y and latitude is defined by $\phi = (L_{\text{eq}}/a)y = \sqrt{Ro} y$ with $Ro = c/(4\Omega a)$, and table 3.1 gives $Ro \approx 0.014$ for Jupiter). Moreover, the positive $\langle u \rangle_t$ at the equator for $n = 2$ is far larger in magnitude than the negative accelerations created at the equator by decaying waves with comparable amplitude and $n = 1$ or $n = 3$ (recall that the quadratic approximation to the dimensionless disturbance pseudo-energy has been set to unity for each wave). The tips of the two equatorial peaks are located at $\pm 5^\circ$ for $n = 1$ and $n = 3$, while for $n = 2$ the large eastward peak extends from roughly -7° to 7° latitude. The latitudinal structure of $\langle u \rangle_t$ is thus again consistent with the formation of a super-rotating equatorial jet through dissipation by cooling, and with the locations of the two peaks of maximal eastward velocity in the broad Jovian equatorial jet at $\pm 7^\circ$ shown in figure 1.1a.

The corresponding data for frictionally dissipated waves in figure 6.10 show that $\langle u \rangle_t < 0$ in regions near the equator for $n = 1, 2, 3$, again consistent with the formation of sub-rotating equatorial jets. There is no longer a line of symmetry between the accelerations due to small cooling and small friction. However, increasing k while keeping n, κ and γ fixed increases the magnitude of $\langle u \rangle_t$ in both cases. Increasing κ or γ for fixed k and n increases the magnitude of $\langle u \rangle_t$.

Figures 6.11 and 6.12 show the four separate terms from (6.3.7) for the cases of dissipation solely by either cooling or friction for waves with $k = 1$, and $n = 1$ and $n = 2$ respectively. Term 2, the Eliassen–Palm flux convergence, is primarily responsible for the positive zonal acceleration in the near-equatorial region due to cooling for $n = 1$ and 2 (red lines). However, the net zonal acceleration at higher latitudes created by cooling is now primarily due to Term 1, the advection of absolute zonal momentum by the mean meridional circulation (pink lines). This term creates large, negative values of $\langle u \rangle_t$ at higher latitudes. By contrast, the Eliassen–Palm flux convergence was the dominant contribution at all latitudes in the height-damped shallow water model. For $n = 1$ and dissipation by cooling, Term 1 encourages westward flow from roughly -30° to 30° (pink line), Term 4 is identically zero (green line), while Term 5 (cyan line) is partially responsible (along with Term 2) for $\langle u \rangle_t$ being negative in a very small region about the equator for $n = k = 1$. For $n = 2$ and dissipation by cooling, the Eliassen–Palm flux convergence is responsible for the positive zonal acceleration in the near-equatorial region (red line), while the magnitudes of Terms 1 and 5 decrease towards zero at the equator. For $n = 3$ (not shown) the Eliassen–Palm flux convergence is again the main contribution to $\langle u \rangle_t$ near the equator.

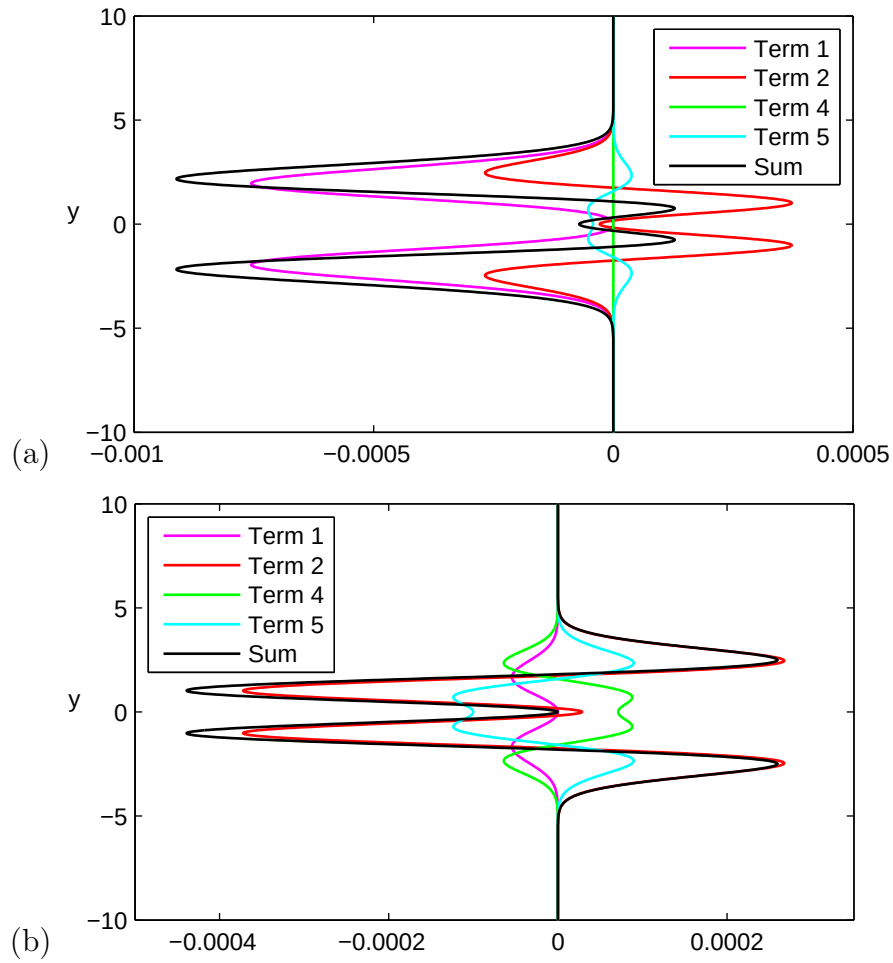


Figure 6.11: The zonal acceleration terms from the right-hand side of (6.3.7), and their sum, for (a) dissipation by cooling alone ($\kappa = 0.002, \gamma = 0$) and (b) dissipation by friction alone ($\kappa = 0, \gamma = 0.001$). All terms are plotted for $n = k = 1$.

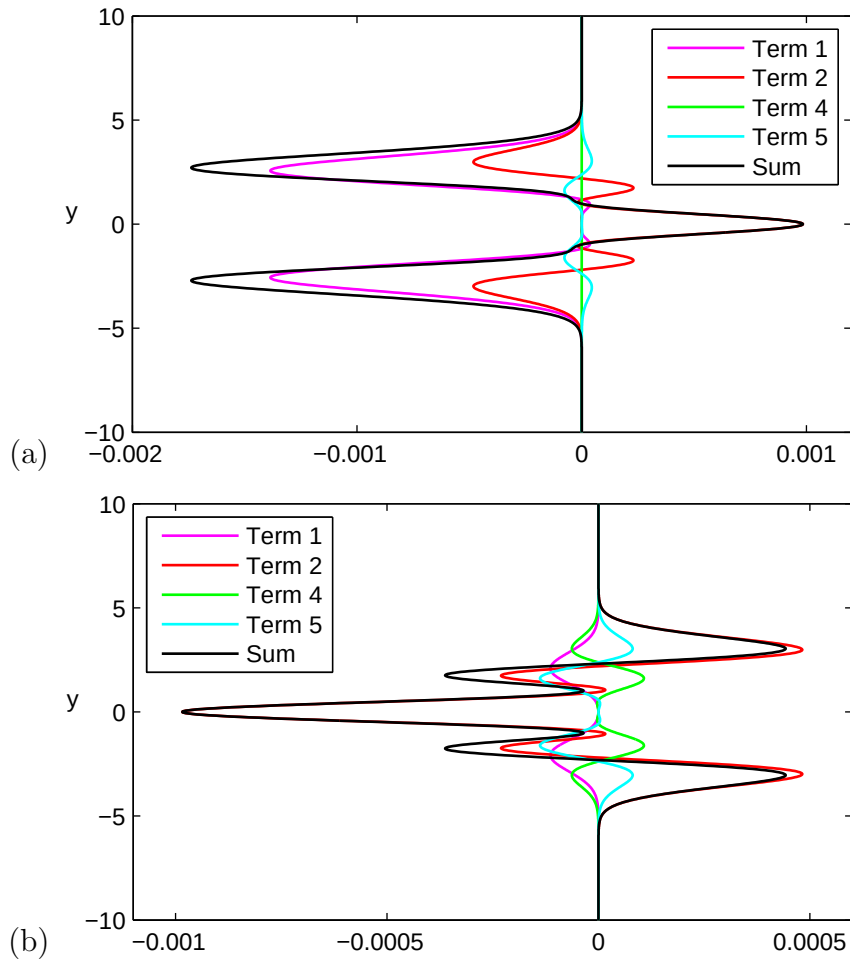


Figure 6.12: The zonal acceleration terms from the right-hand side of (6.3.7), and their sum, for (a) dissipation by cooling alone ($\kappa = 0.002, \gamma = 0$) and (b) dissipation by friction alone ($\kappa = 0, \gamma = 0.001$). All terms are plotted for $n = 2$ and $k = 1$.

For dissipation by friction and $n = 1$, the Eliassen–Palm flux convergence is primarily responsible for $\langle u \rangle_t$ being negative in the near-equatorial region (Term 2, red line). However, Term 2 contributes a positive zonal acceleration close to the equator, but this is outweighed by a large, negative contribution from Term 5 due to the rate of change of zonal eddy momentum (cyan line). At other latitudes, the Eliassen–Palm flux convergence is the dominant contribution. Terms 1 and 5 encourage westward flow near the equator (pink and cyan lines respectively), while Term 4 encourages eastward flow (green line). For $n = 2$ and dissipation by friction, the Eliassen–Palm flux convergence is the dominant contribution to $\langle u \rangle_t$ at all latitudes (red line), except for two intermediate regions where Term 5 is both dominant and negative (cyan line). The magnitudes of Terms 1, 4 and 5 decrease towards zero near the equator. For $n = 3$ (not shown) the Eliassen–Palm flux convergence is again the main contribution to $\langle u \rangle_t$ near the equator, and the other three terms are much smaller in magnitude in the equatorial region.

6.3.3 Discussion

I have also performed a similar analysis for decaying equatorially trapped Kelvin, Yanai, and inertia-gravity waves in the height-damped shallow water model and the radiatively damped thermal shallow water model. The zonal accelerations due to these waves are all an order of magnitude smaller than those due to decaying Rossby waves of comparable amplitude. Thus, as expected from the global quasi-geostrophic simulations in chapter 5, these other types of wave make no significant contribution to setting the direction of the equatorial jet.

The Jovian height-damped shallow water simulation in §3.3 took the dimensionless cooling rate to be $\kappa = 0.00788$, which is roughly 23 times larger than the dimensionless friction rate $\gamma = 0.000342$ in the simulation. In §6.3.1 an antisymmetry was established in $\langle u \rangle_t$ between the cases of small cooling ($\kappa = \varepsilon$ and $\gamma = 0$) and small friction ($\kappa = 0$ and $\gamma = \varepsilon$) for the height-damped shallow water equations. For $\kappa \gg \gamma$, this theory thus suggests that $\langle u \rangle_t > 0$ in a region near the equator, consistent with the super-rotating equatorial jet observed in the Jovian simulation in §3.3. Conversely, for the purely frictional case ($\kappa = 0$) the zonal mean zonal acceleration is negative throughout the near-equatorial region (see figure 6.7, blue dotted lines), consistent with the sub-rotating equatorial jet in the standard shallow water simulation in §3.3.

No corresponding symmetry between κ and γ exists in the radiatively damped thermal shallow water model. Figure 6.13 re-plots figure 6.10 for $\kappa = 0.00788$ and $\gamma = 0.000342$. The positive magnitude of $\langle u \rangle_t$ due to dissipation by cooling is observed

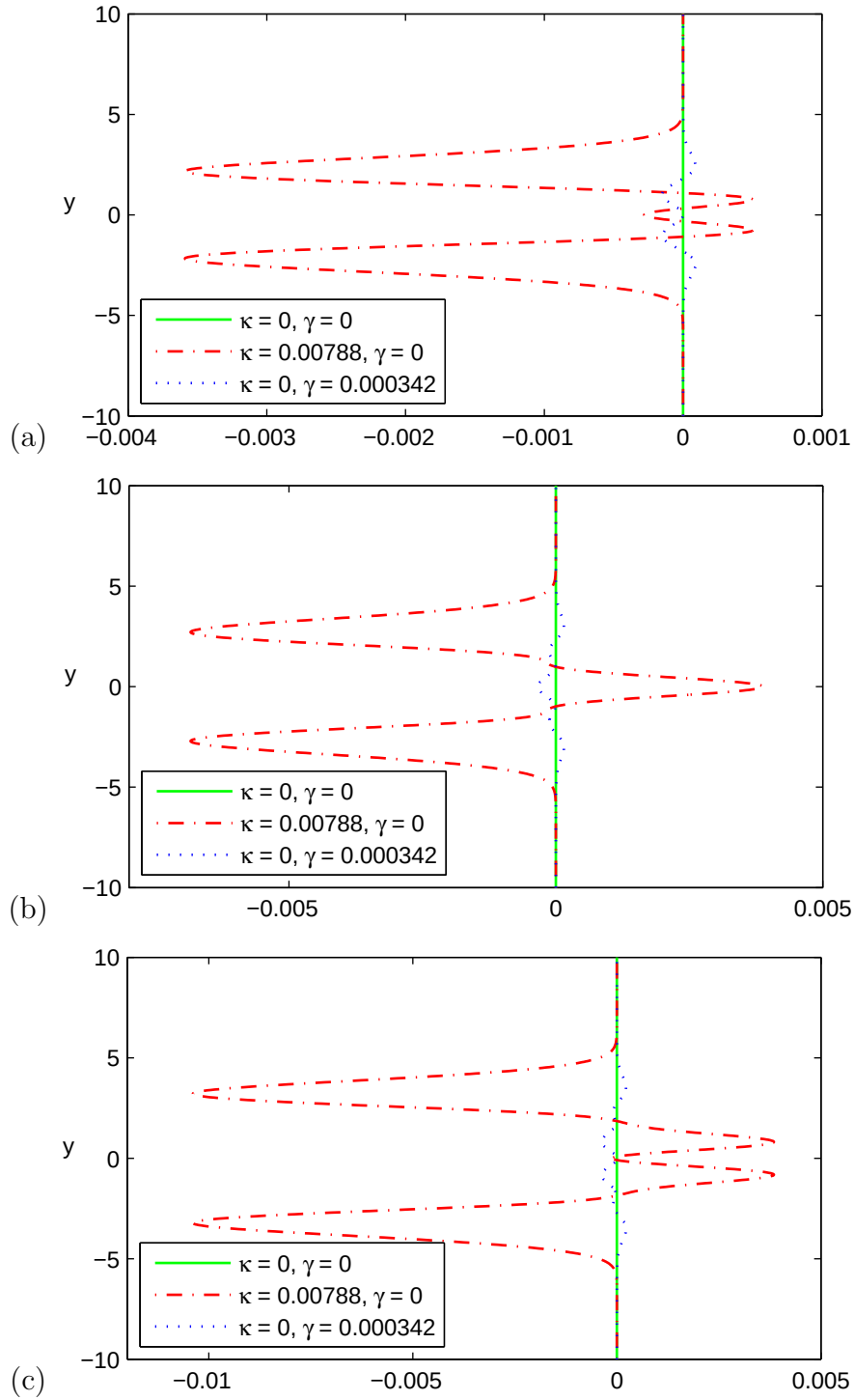


Figure 6.13: The sum of the four zonal acceleration terms on the right-hand side of (6.3.7) for the three cases of no dissipation (green solid line), dissipation by cooling alone ($\kappa = 0.00788$, $\gamma = 0$), and dissipation by friction alone (blue dotted line, $\gamma = 0.000342$). When either κ or γ is non-zero, they are set to the values in the Jovian radiatively damped thermal shallow water simulation detailed in §3.3. All terms are plotted for $k = 1$. Plots (a),(b),(c) correspond to $n = 1, 2, 3$ respectively.

to be much greater than the negative magnitude of $\langle u \rangle_t$ due to dissipation by friction in the near-equatorial region in figure 6.13. This theory thus suggests that $\langle u \rangle_t > 0$ in the near-equatorial region, again consistent with the formation of a super-rotating equatorial jet (as seen in the Jovian simulation in §3.3). For $n = 1$ and $n = 3$ there is a very small region at the equator where decaying Rossby waves produce $\langle u \rangle_t < 0$, but there is a much larger positive contribution from $n = 2$. Thus this theory still suggests a super-rotating equatorial jet, if waves with even and odd n are assumed to be excited with comparable amplitudes.

In §5.8, simulations of the global height-damped quasi-geostrophic equations (5.3.3) and the global thermal quasi-geostrophic equations (5.2.17) were run for different values of the radiative time scale τ_{rad} , where τ_{fric} was fixed at 9900 hours. As the radiative time scale was increased (i.e. as λ was decreased in the notation of §5.8), the equatorial jet was observed to switch from super- to sub-rotating for both models. This same switch in equatorial jet direction is also observed as τ_{rad} is increased in Jovian height-damped shallow water simulations, and Jovian radiatively damped thermal shallow water simulations (see figure 6.1 and §3.5.1). Increasing τ_{rad} decreases κ . Decreasing κ , while keeping all other parameters fixed, decreases the magnitude of $\langle u \rangle_t$ due to cooling for both shallow water models. Thus, this switch from a super-rotating to a sub-rotating equatorial jet as τ_{rad} is increased (and thus κ is decreased) is perhaps due to the positive contribution to $\langle u \rangle_t$ in the near-equatorial region due to dissipation by cooling decreasing in magnitude as κ decreases.

In the Neptunian radiatively damped thermal shallow water simulation in §3.4, $\tau_{\text{rad}} = 7,088$ hours and $\tau_{\text{fric}} = 17,900$ hours. These time scales correspond to $\kappa = 0.000414$ and $\gamma = 0.000164$, using the values in table 3.1. Figure 6.14 re-plots figure 6.10 for $\kappa = 0.000414$ and $\gamma = 0.000164$. For $n = 1$ in figure 6.14, the negative magnitude of $\langle u \rangle_t$ due to friction is greater than the positive magnitude of $\langle u \rangle_t$ due to cooling in the near-equatorial region, while the reverse is true for $n = 2, 3$. The absolute difference between the maximum positive amplitude of $\langle u \rangle_t$ due to cooling, and the maximum absolute amplitude of $\langle u \rangle_t$ due to friction in the equatorial region is 0.0000455 for $n = 1$, 0.0000423 for $n = 2$ and 0.0000214 for $n = 3$. This scenario is on the border between predicting either a super- or sub-rotating equatorial jet for this theory, highlighting that this theory is less useful for predicting the direction of the equatorial jet when the values of κ and γ are close.

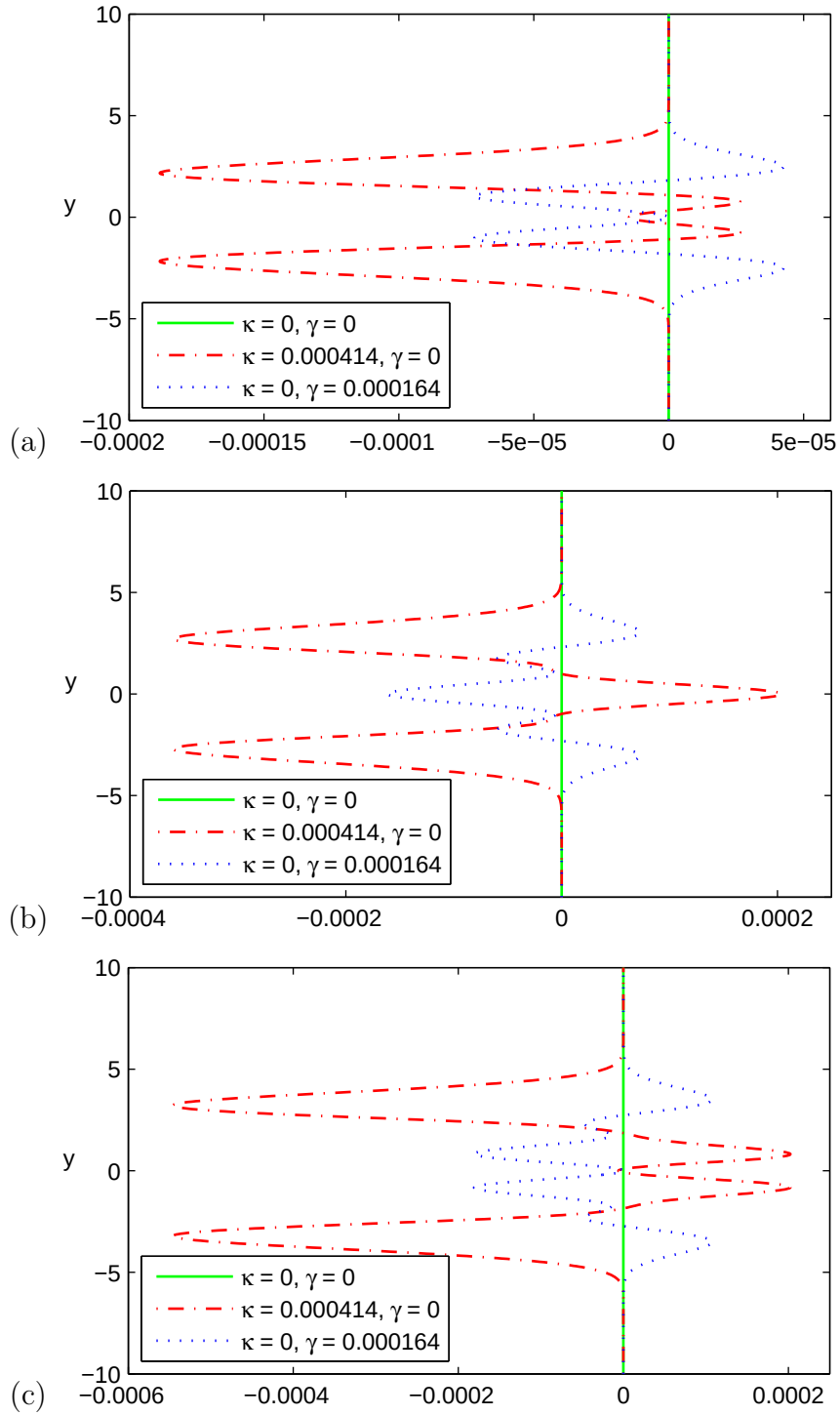


Figure 6.14: The sum of the four zonal acceleration terms on the right-hand side of (6.3.7) for the three cases of no dissipation (green solid line), dissipation by cooling alone (red dash-dot line, $\kappa = 0.000414$), and dissipation by friction alone (blue dotted line, $\gamma = 0.000164$). When either κ or γ is non-zero, they are set to the values in the Neptunian radiatively damped thermal shallow water simulation detailed in §3.4. All terms are plotted for $k = 1$. Plots (a),(b),(c) correspond to $n = 1, 2, 3$ respectively.

6.4 Summary

Numerical simulations of shallow water equations with isotropic random forcing reliably produce mixtures of coherent vortices and alternating zonal jets. These simulations typically include Rayleigh friction to absorb the gradual inverse cascade of energy past the Rhines scale that otherwise causes the zonal jets to break up into domain-sized coherent vortices after very long times. Such simulations invariably produce sub-rotating equatorial jets when run with Jovian parameters. However, in chapter 3 it was demonstrated that simulations which model radiative effects by relaxing the height field produce a super-rotating equatorial jet for Jovian parameters, as do simulations of the radiatively damped thermal shallow water equations with a Newtonian cooling term in their separate temperature equation, for τ_{rad} sufficiently short.

This chapter provides a possible explanation for the different directions of the equatorial jets in three different shallow water models in the Jovian regime. The different contributions to the equation for $\langle u \rangle_t$, the zonal mean zonal acceleration, were identified for small perturbations to a uniform rest state, and evaluated for equatorially trapped Rossby waves decaying through a combination of Newtonian cooling and Rayleigh friction. For the standard shallow water equations, with or without height-damping, the resulting zonal acceleration was shown to arise predominantly from the convergence of the Eliassen–Palm flux $\langle u'v' \rangle$ due to small-amplitude disturbances in shallow water. This flux vanishes for undamped Rossby waves, due to the phase relation between u' and v' . Dissipation alters this phase relation to create a net flux of zonal momentum in the meridional direction. The varying spatial structure of this flux for the height-damped shallow water equations suggests that super-rotating equatorial jets will form in simulations damped primarily by Newtonian cooling, and sub-rotating equatorial jets will form in simulations damped primarily by Rayleigh friction. This supports the findings of Schneider and Liu (2009) and Liu and Schneider (2010), who had to make the friction in their model very weak in the equatorial region in order to obtain a large-scale prograde jet at the equator. Moreover, the spatial structure of $\langle u \rangle_t$ due to dissipation by Newtonian cooling was shown to be consistent with the presence and locations of two “horns”, or peaks of maximal eastward velocity, located in the broad Jovian equatorial jet, and a slightly smaller velocity precisely on the equator (see figure 1.1a).

An equivalent analysis was also performed for the radiatively damped thermal shallow water model. The overall zonal mean zonal acceleration $\langle u \rangle_t$ due to equatori-

ally trapped Rossby waves decaying due to Newtonian cooling and Rayleigh friction was again shown to be consistent with the formation of super-rotating equatorial jets in simulations for which cooling was the primary source of dissipation. However, the Eliassen–Palm flux convergence was only the largest contribution to $\langle u \rangle_t$ very close to the equator. Instead, the zonal acceleration due to the advection of absolute zonal momentum by the mean meridional circulation dominated at higher latitudes. Moreover, it was shown that a net positive $\langle u \rangle_t$ at the equator only arises from Newtonian cooling if the positive contributions from even mode numbers ($n = 2$ is shown) outweigh the smaller negative contributions from odd mode numbers ($n = 1$ and $n = 3$ are shown) under the assumption that both odd and even wave modes are excited with roughly equal amplitudes.

This work relies upon two major simplifying assumptions. Firstly, trapped equatorial Rossby waves on a background rest state were considered, which enabled the use of known analytical expressions for these waves in terms of Hermite polynomials. A more complete theory would calculate the Rossby wave spectrum supported by a background mean zonal flow $\bar{u}(y)$. Secondly, the mean zonal accelerations were calculated for freely decaying Rossby waves. The waves were thus assumed to be excited by sources outside the equatorial region, and the effect of this excitation on the zonal mean flow was neglected. Some support for this second assumption is offered by the profiles shown in figures 6.2 and 6.3 for the quantity R' used by Sardeshmukh and Hoskins (1988) to diagnose sources of Rossby waves. These sources were shown to be predominantly located away from the equatorial region throughout the simulations from §3.3. Although this work falls short of a full wave-mean flow interaction theory in these two respects, I believe that it offers at least a step towards a theoretical explanation of the origins of equatorial super- or sub-rotation in numerical simulations.

Chapter 7

Conclusions

The main aims of this thesis have been to derive and analyse the radiatively damped thermal shallow water equations, to consider their local and global quasi-geostrophic limits, whilst simultaneously considering their utility as a model for Jovian atmospheres. The summaries at the end of chapters 2, 3, 4, 5 and 6 provide detailed reviews of the key results obtained in the corresponding chapter. Here a summary of the conclusions is presented, and possible areas for future work are suggested.

7.1 Summary of results

Shallow water theory provides a widely used conceptual model in terms of a single homogeneous fluid layer, and is an approximation which is valid when the ratio of vertical to horizontal length scales is small. The horizontal velocity is assumed to be depth-independent in the active layer, so the fluid moves in columns. Thermal shallow water theory extends standard shallow water theory by permitting horizontal variations in the thermodynamic properties of the fluid within the active layer. The main focus of chapter 2 was to derive and analyse the key properties of thermal shallow water theory. Three derivations were presented: the first two involved averaging vertically over the fluid layer, while the third projected the three-dimensional Lagrangian for a Boussinesq fluid onto columnar fluid motions and then applied Hamilton's principle of least action. The thermal shallow water potential vorticity equation was derived by considering the particle relabelling symmetry of the thermal shallow water Lagrangian. The loss of the material conservation of potential vorticity was discussed, where the source term on the right-hand side of the thermal shallow water potential vorticity equation was derived from the depth average of the baroclinic torque in a continuously stratified fluid. The thermal shallow water equations were instead shown to possess an integral conservation law for the total potential vorticity

inside a closed isotherm. The non-canonical Hamiltonian structure of the thermal shallow water equations was also discussed.

To form a model for gas giant planets, heating or cooling by radiation was incorporated by adding a Newtonian cooling term to the thermal shallow water temperature equation. This was motivated by previous models for the terrestrial stratosphere (Jukes, 1989, Polvani et al., 1995, Shell and Held, 2004) and gas giant planets (Scott and Polvani, 2007, 2008), which added a Newtonian cooling term to the shallow water mass conservation equation instead. These models thus broke the key conservation properties for mass and momentum which are preserved for the radiatively damped thermal shallow water model. Chapter 3 compared simulations of these radiatively damped thermal shallow water equations to simulations of the standard shallow water equations, the Scott and Polvani (2007, 2008) height-damped shallow water equations, and a momentum-conserving variant of the Scott and Polvani (2007, 2008) model. Turbulent self-organisation into zonal jets takes many eddy turn-over times. Thus simulations were performed on GPUs in order to exploit their high memory bandwidth and floating point performance. The four models were solved in a doubly-periodic Cartesian geometry using Fourier spectral methods based on an efficient fast Fourier transform library for GPUs. Rayleigh friction was included to dissipate energy at large-scales, while a narrow-band isotropic spectral forcing was included to inject turbulence. For Jovian parameter values, all four models reproduced vortices and robust zonal jets of roughly the correct speeds. The standard and thermal simulations robustly demonstrated more intense anticyclones than cyclones. Only the three models that incorporated some form of Newtonian cooling reproduced the Jovian super-rotating equatorial jet when the radiative time scale τ_{rad} was sufficiently short. Cyclonic regions were typically warmer than anticyclonic regions in the Jovian thermal simulations. When parameter values were changed to fit Neptune, turbulent simulations of the radiatively damped thermal shallow water equations switched to demonstrating a sub-rotating equatorial jet and two prograde mid-latitude jets (in agreement with Neptunian observations). The sensitivity of the results to the value used for the radiative time scale was considered. Neptunian thermal simulations using the values for τ_{rad} calculated at the 25 and 100 millibar pressure levels looked very similar. However, Jovian simulations using the value of τ_{rad} at the 100 millibar level demonstrated a sub-rotating equatorial jet for all three radiatively damped models, as opposed to the super-rotating equatorial jet demonstrated in simulations using the value of τ_{rad} at the 25 millibar pressure level. Thus all three radiatively damped shallow water models only produce the correct equatorial jet direction for Jupiter if the

radiative time scale is sufficiently short. However, the theory in chapter 6 suggests that lengthening the frictional time scale beyond 1000 Jovian days (or 9900 hours) would increase the tendency to form a super-rotating equatorial jet for longer radiative time scales, but running simulations until they reach statistical steady states then becomes prohibitively expensive. The radiatively damped thermal shallow water model is the first shallow water model to reproduce equatorial super-rotation in a Jovian regime and equatorial sub-rotation in a Neptunian regime, while also conserving mass and momentum. However, I was unable to run a stable simulation with sufficient forcing to match the jet speeds seen on Neptune. In fact the Neptunian simulation detailed in §3.4 exhibits velocities closer to those observed on Uranus.

Quasi-geostrophic theory exploits the separation of time scales between fast inertia-gravity waves and slow vortical motions to produce a simplified model, while still retaining the time evolution omitted from purely geostrophic flow. It filters out the rapidly moving inertia-gravity waves, replacing them with an elliptic relation between the potential vorticity and the geostrophic streamfunction. The local quasi-geostrophic limit of the radiatively damped thermal shallow water equations on a mid-latitude β -plane was derived in chapter 4. Two derivations were provided, both of which formalised the derivation given in Ripa (1996*b*). The first derivation involved an asymptotic expansion for small Rossby number. The second derivation was presented via a nonlinear balanced model that corresponds to a projection of the dynamics onto the slow normal modes of the thermal shallow water equations. The local thermal quasi-geostrophic equations were then shown to emerge as the small Rossby number limit of this balanced model. They form a closed system that does not involve the small ageostrophic correction $\mathbf{u}_1$ to the leading order geostrophic velocity $\mathbf{u}_0$. However, it was shown that $\mathbf{u}_1$ must be calculated in order to interpret the local thermal quasi-geostrophic energy flux as the local quasi-geostrophic limit of a pseudo-energy flux for the thermal shallow water equations. The non-canonical Hamiltonian structure of the local thermal quasi-geostrophic equations (as stated by Ripa, 1996*b*) was derived from a small Rossby number decomposition of a pseudo-energy and Poisson bracket for the thermal shallow water equations. Since Noether's theorem enables conservation laws to be related to symmetries, by making approximations to the thermal shallow water equations that preserve its symmetries, the local thermal quasi-geostrophic equations automatically inherit equivalent conservation laws.

Chapter 5 extends this analysis to the global domain, and considers a global quasi-geostrophic form of the radiatively damped thermal shallow water equations that is

valid for the whole sphere, and for the square planet domain described in §1.4.2. The derivation involves rewriting the thermal shallow water momentum equations as evolution equations for the potential vorticity and horizontal divergence, applying the quasi-geostrophic scaling, and performing an expansion for small Rossby number (where the Rossby number was defined as $Ro = U/2\Omega L$). The extension of the geostrophic relation to the global domain was then defined by $f\psi = \eta + \theta$. The resulting global thermal quasi-geostrophic equations are valid for the entire planet, and reduce to their local quasi-geostrophic form on an f -plane by setting f to be unity. As for the local quasi-geostrophic limit, the Hamiltonian and Poisson bracket for the global thermal quasi-geostrophic equations were shown to arise as limiting forms of the analogous properties in the thermal shallow water equations. Jovian global quasi-geostrophic simulations were performed for a range of radiative cooling rates for this global thermal quasi-geostrophic model, and for the global quasi-geostrophic form of the Scott and Polvani (2007, 2008) height-damped shallow water model. Simulations were made to be as similar to the corresponding Jovian shallow water simulations from §3.3 as possible, to enable a fair comparison between simulations of the shallow water models and their global quasi-geostrophic counterparts. Solving quasi-geostrophic equations numerically requires the solution of an elliptic relation for the geostrophic streamfunction at each time step. The elliptic relation is diagonalised by a Fourier transform for the local quasi-geostrophic limit. However, this is not the case for the global quasi-geostrophic limit where Fourier transforming leads to a suite of pentadiagonal linear systems to be solved. Considering the odd and even Fourier modes in the y direction separately instead gives two tridiagonal systems to solve for each Fourier mode in the x direction. Thus an efficient GPU implementation for the global quasi-geostrophic simulations required the development of a Thomas algorithm that solved the tridiagonal systems in parallel across the different x Fourier modes. All the global quasi-geostrophic simulations demonstrated a mixture of vortices and zonally elongated structures, while the global thermal quasi-geostrophic simulations all demonstrated cyclonic regions that were typically warmer than anticyclonic regions. The equatorial jet was shown to switch from sub- to super-rotating as the radiative cooling rate was increased for both these global quasi-geostrophic models.

Chapter 6 investigates the role of equatorially trapped waves in establishing the direction of the equatorial jet for Jovian simulations of the shallow water models detailed in chapter 3. Since simulations of the Scott and Polvani (2007, 2008) height-damped shallow water model and the radiatively damped thermal shallow water model (along with their global quasi-geostrophic counterparts) switch from producing a sub- to a

super-rotating equatorial jet as the radiative time scale is decreased, the mechanism responsible for this switch must lie within quasi-geostrophic theory. Hence the explanation for the equatorial jet direction in this chapter focused on equatorially trapped Rossby waves. Rossby waves were shown to be preferentially generated away from the equator throughout the Jovian standard, height-damped and thermal shallow water simulations from §3.3. Thus in this chapter, any Rossby waves present in the near-equatorial region were assumed to be excited by sources outside that region, and the effect of this excitation on the zonal mean flow was neglected. The two key forms of dissipation (Newtonian cooling and Rayleigh friction) from the shallow water simulations in chapter 3, were shown to damp these equatorially trapped Rossby waves. The different contributions to the zonal mean zonal acceleration were identified for small perturbations to a uniform rest state, and were shown to be zero for undamped Rossby waves. However, dissipation created a net flux of zonal momentum in the meridional direction. The resulting zonal acceleration in the near-equatorial region was shown to rise predominantly from the convergence of the shallow water Eliassen–Palm flux. Dissipation by Newtonian cooling led to a positive zonal mean zonal acceleration close to the equator, consistent with the formation of super-rotating equatorial jets, while dissipation by Rayleigh friction led to a negative zonal mean zonal acceleration close to the equator, consistent with the formation of sub-rotating equatorial jets. While this is not a full wave-mean flow interaction theory, it suggests that a super-rotating equatorial jet forms in height-damped and thermal shallow water simulations if the positive magnitude of $\langle u \rangle_t$ due to dissipation by Newtonian cooling outweighs the negative magnitude of $\langle u \rangle_t$ due to dissipation by Rayleigh friction in the near-equatorial region (which typically occurs when $\tau_{\text{fric}} \gg \tau_{\text{rad}}$). This provides a possible explanation for why standard shallow water simulations for Jovian parameters invariably produce sub-rotating equatorial jets, since they typically set τ_{fric} to be finite and τ_{rad} to be infinite.

7.2 Future work

As discussed in §2.3.1, the thermal shallow water equations offer a single layer model that captures some aspects of baroclinic instability in continuously stratified fluids. Capturing this instability required two or more layers in previous models. I have already performed simulations of the local thermal quasi-geostrophic equations which

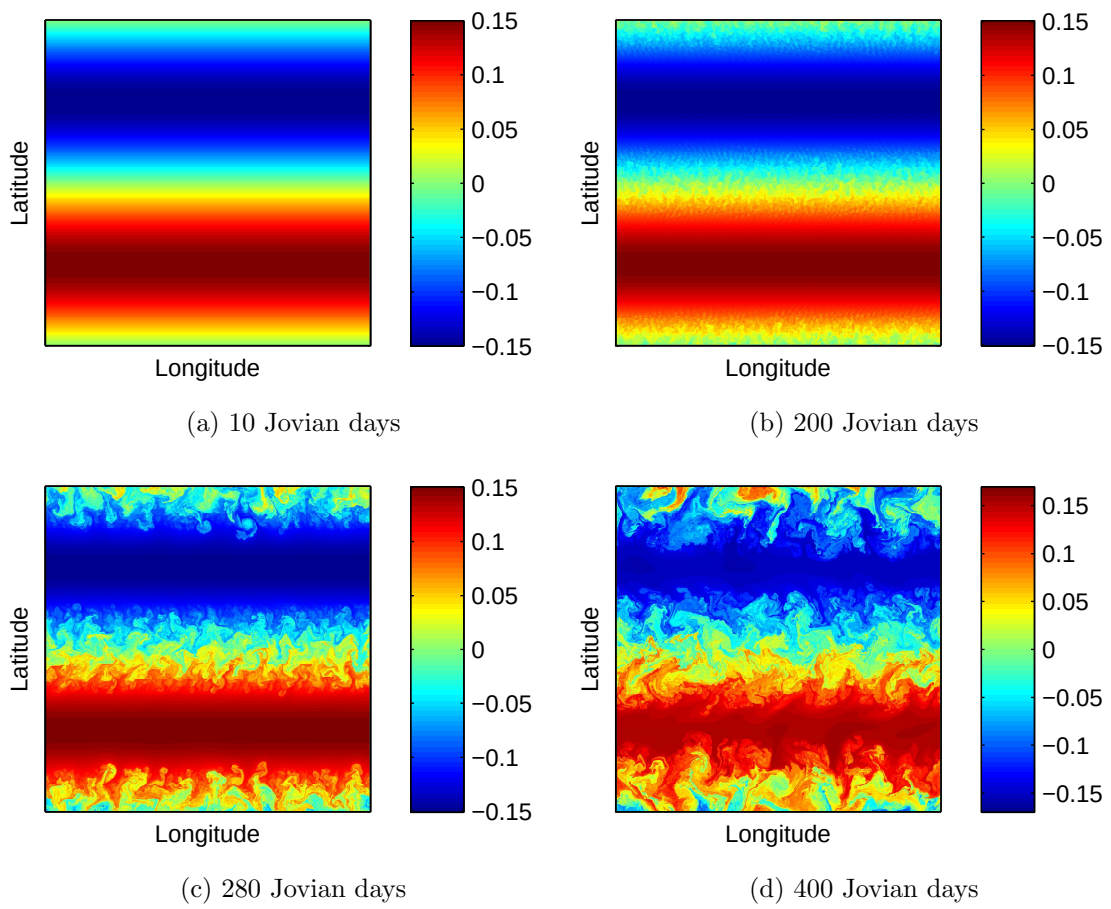


Figure 7.1: Initial evolution of θ for a Jovian simulation of the local thermal quasi-geostrophic equations (7.2.1) on a doubly-periodic mid-latitude β -plane, where an instability (as opposed to any additional random forcing) drives the flow.

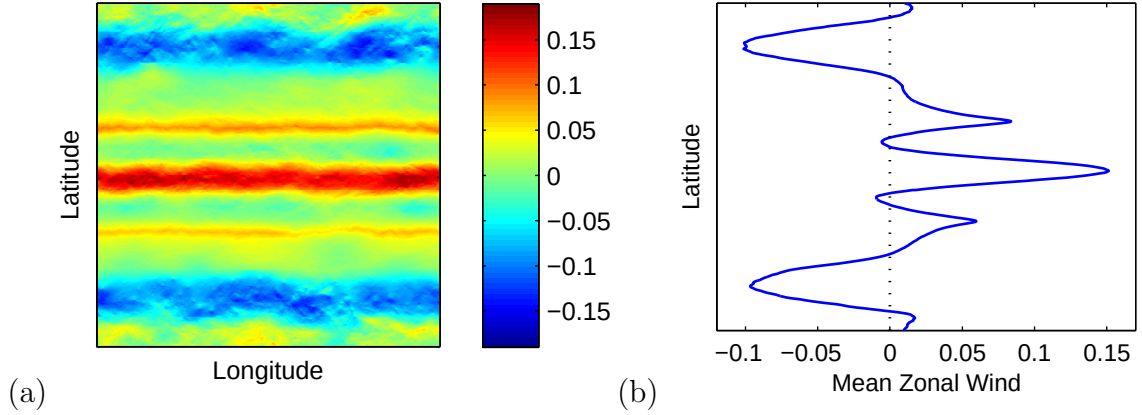


Figure 7.2: Dimensionless (a) u and (b) $\bar{u}$ after 4×10^4 Jovian days for a Jovian simulation of the local thermal quasi-geostrophic equations (7.2.1) on a doubly-periodic mid-latitude β -plane, where an instability (as opposed to any additional random forcing) drives the flow.

use this instability to drive the flow:

$$\frac{\partial q}{\partial t} + [\psi, q] = \frac{1}{Bu} [\psi, \theta], \quad (7.2.1a)$$

$$\frac{\partial \theta}{\partial t} + [\psi, \theta] = -\lambda (\theta + \psi - 2\theta_0(y)), \quad (7.2.1b)$$

$$q = \nabla^2 \psi - \frac{(\psi - \theta)}{Bu} + \beta y. \quad (7.2.1c)$$

Simulations were performed on a doubly-periodic mid-latitude β -plane centred around a latitude of 45° . The length of the side of the doubly-periodic β -plane for figures 7.1 and 7.2 was taken to be $y_{\max} = 15$ deformation radii. Jovian parameter values were used, and perturbed initial conditions of the form $\theta_0(y) = \psi_0(y) = 0.15 \sin(2\pi y/y_{\max})$ and $q_0(y) = -0.15 (2\pi/y_{\max})^2 \sin(2\pi y/y_{\max}) + \beta y$ were taken. I did not add any additional random forcing to these simulations, instead they were driven by the temperature gradient which underwent an instability (see figure 7.1) that provided sufficient driving for a small-scale turbulence that self-organised into zonal jets (see figure 7.2). The instability inherent in the thermal shallow water equations, and its local quasi-geostrophic limit, provides a more natural route for driving the flow and introducing turbulence into planetary atmospheres than the random forcing described in §3.2.4. An extension to the above work would be to determine how simulations of the instability in (7.2.1) compare to simulations of a baroclinic instability in a standard two-layer local quasi-geostrophic model (cf. §1.3.5).

The typical approach to investigate a new fluid dynamical model is to establish qualitative properties, e.g. conservation laws combined with direct numerical simulations, as I have done in this thesis for the radiatively damped thermal shallow water equations and its local and global quasi-geostrophic limits. However, for chaotic or turbulent flows, especially those involving random forcing to model omitted small-scale processes, it is statistical properties that are of most interest. These are the properties that are robust under the inevitable small imperfections of the model, and between different realisations of the random forcing. Modern techniques based on cumulant expansions offer a direct approach to computing these statistical properties, and more insight at much less computational cost than running large ensembles of direct numerical simulations (Srinivasan and Young, 2012, Tobias et al., 2011). An extension of this thesis would be to investigate the cumulant equations for the global thermal quasi-geostrophic model and seek a numerical solution like that in Tobias et al. (2011). A key aim of this work would be to extend the theory in chapter 6 to include effects of the random forcing.

There has been discussion in recent years as to whether climate change could lead to equatorial super-rotation in the Earth's atmosphere, which is currently sub-rotating (Mitchell and Vallis, 2010, Shell and Held, 2004, Showman and Polvani, 2010). This idea is intriguing given the reversal in equatorial jet direction from sub- to super-rotating in simulations of the radiatively damped thermal shallow water model (and its global quasi-geostrophic form) as the radiative time scale decreases. An area for future work would be to apply the radiatively damped thermal shallow water model to the Earth's atmosphere with the aim of gaining new insights into this possible transition, since any changes in greenhouse gases and cloud cover will affect the Earth's radiative time scale.

The role of zonal flows in the toroidal fusion devices known as tokamaks has been recognised in recent years (Diamond et al., 2005). A recent discovery has shown reversals in direction of intrinsic rotation with increased dissipation from collisions between particles in the plasma (Rice et al., 2011). This again sounds intriguingly similar to the switch in equatorial jet direction observed in radiatively damped thermal shallow water (and global thermal quasi-geostrophic) simulations with increased dissipation from radiative cooling. The thermal quasi-geostrophic equations also resemble the Hasegawa–Wakatani equations, an established approximate fluid model for turbulence in plasmas. Applying the techniques used in this thesis, particularly those in chapter 6, to the Hasegawa–Wakatani equations could provide useful insights

into this reversal mechanism, and also into behaviours likely to be observed in future experiments such as the International Thermonuclear Experimental Reactor (ITER).

A further possible application of the thermal shallow water equations is as a model for exoplanets. Exoplanetary modelling is a rapidly expanding and exciting area (Rauscher and Menou, 2010, Showman and Polvani, 2011, Thrastarson and Cho, 2011), especially since the launch of the Kepler mission in 2009. Tidally-locked Jovian exoplanets which are subjected to very strong and longitudinally varying solar heating could be modelled by incorporating a longitudinal variation in the background temperature of the radiatively damped thermal shallow water model:

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0, \quad (7.2.2a)$$

$$\frac{\partial \Theta}{\partial t} + (\mathbf{u} \cdot \nabla) \Theta = -(\Theta h/h_0 - \Theta_0(x)) / \tau_{\text{rad}}, \quad (7.2.2b)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla(\Theta h) + \frac{1}{2}h\nabla\Theta - \mathbf{u}/\tau_{\text{fric}}, \quad (7.2.2c)$$

where e.g. $\Theta_0(x) = A \cos(2\pi x/x_{\text{max}})$ with x_{max} the length of the side of the square planet domain in figure 1.7. Such idealised theoretical models would help to make sense of the rapidly growing body of observational data, and identify the possible range of atmospheric flows that could exist on exoplanets.

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