

Robust multigrid methods for nearly incompressible elasticity using macro elements

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We present a mesh-independent and parameter-robust multigrid solver for the Scott–Vogelius discretisation of the nearly incompressible linear elasticity equations on meshes with a macro element structure. The discretisation achieves exact representation of the limiting divergence constraint at moderate polynomial degree. Both the relaxation and multigrid transfer operators exploit the macro structure for robustness and efficiency. For the relaxation, we use the existence of local Fortin operators on each macro cell to construct a local space decomposition with parameter-robust convergence. For the transfer, we construct a robust prolongation operator by performing small local solves over each coarse macro cell. The necessity of both components of the algorithm is confirmed by numerical experiments.

Keywords: linear elasticity; multigrid; preconditioning; macro elements; parameter-robustness

1. Introduction

We consider the linear elasticity equations on a simply connected domain $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$, given by

$$\begin{aligned} -\nabla \cdot (\mathbf{E}\mathbf{u} + \gamma(\nabla \cdot \mathbf{u})\mathbf{I}_d) &= \mathbf{f} && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_D, \\ (\mathbf{E}\mathbf{u} + \gamma(\nabla \cdot \mathbf{u})\mathbf{I}_d)\mathbf{n} &= \mathbf{h} && \text{on } \Gamma_N, \end{aligned} \tag{1.1}$$

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where $\mathbf{E}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^\top)$, $\gamma \geq 0$, and \mathbf{f} and \mathbf{h} are given data. Here $\gamma = \lambda/2\mu$ where μ and λ are the Lamé parameters describing an isotropic, homogeneous material. As $\gamma \rightarrow \infty$ this corresponds to the nearly incompressible case and the equations become difficult to solve (Schöberl, 1999b; Lee *et al.*, 2009; Dohrmann & Widlund, 2009).

In weak form the elasticity equations can be expressed as: given $\mathbf{f} \in H^{-1}(\Omega; \mathbb{R}^d)$ and $\mathbf{h} \in H^{-1/2}(\Gamma_N; \mathbb{R}^d)$, find $\mathbf{u} \in V := \{\mathbf{v} \in H^1(\Omega; \mathbb{R}^d) : \mathbf{v}|_{\Gamma_D} = \mathbf{0}\}$ such that

$$(\mathbf{E}\mathbf{u}, \mathbf{E}\mathbf{v}) + \gamma(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle + \langle \mathbf{h}, \mathbf{v} \rangle \quad (1.2)$$

for all $\mathbf{v} \in V$. Here (\cdot, \cdot) denotes the standard L^2 inner product and $\langle \cdot, \cdot \rangle$ refers to the dual pairing. The numerical solution of these equations in the nearly incompressible limit using the finite element method has attracted significant attention. Choosing finite dimensional subspaces $V_h \subset V$ and $Q_h \subseteq \text{div}(V_h) \subset L^2(\Omega)$, we consider the problem: find $\mathbf{u} \in V_h$ such that

$$a_{h,\gamma}(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle + \langle \mathbf{h}, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V_h. \quad (1.3)$$

Here $a_{h,\gamma}$ is defined as

$$a_{h,\gamma}(\mathbf{u}, \mathbf{v}) := a(\mathbf{u}, \mathbf{v}) + \gamma c_h(\mathbf{u}, \mathbf{v}), \quad (1.4)$$

and

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &:= (\mathbf{E}\mathbf{u}, \mathbf{E}\mathbf{v}), \\ c_h(\mathbf{u}, \mathbf{v}) &:= (\Pi_{Q_h}(\nabla \cdot \mathbf{u}), \Pi_{Q_h}(\nabla \cdot \mathbf{v})), \end{aligned} \quad (1.5)$$

where Π_{Q_h} is the projection onto Q_h . Here Q_h is the pressure space in an associated mixed formulation of the problem. We note that these equations also arise in iterated penalty and augmented Lagrangian formulations of the Stokes and Navier–Stokes equations. In this context it is highly desirable that the property $\text{div}(H^1) = L^2$ is preserved for the discretised problem, that is $\text{div}(V_h) = Q_h$ (John *et al.*, 2017). An example of a such a discretisation is the $[\mathbb{P}_k]^d - \mathbb{P}_{k-1}^{\text{disc}}$ Scott–Vogelius discretisation of degree k . In this work we restrict ourselves to discretisations preserving this exactness property.

As γ is increased, two separate issues arise: first, locking may occur (Babuška & Suri, 1992). This can be avoided by considering discretisations $V_h \times Q_h$ that satisfy a discrete inf-sup condition. The second issue is that the arising linear systems become poorly conditioned since the problem becomes nearly singular, due to the nullspace of the divergence operator. The focus of this work is the development of a multigrid method with robust performance as γ becomes large for the Scott–Vogelius discretisation.

Schöberl developed a robust multigrid scheme for nearly incompressible elasticity in his doctoral thesis (Schöberl, 1999a,b). He proved that the key ingredients are a robust prolongation and a robust relaxation scheme and showed how to construct these for the $[\mathbb{P}_2]^2 - \mathbb{P}_0$ element in two dimensions. His insight was that the prolongation needs to map divergence-free vector fields on the coarse grid to (nearly) divergence-free vector fields on the fine grid and that a robust relaxation can be built from a space decomposition that provides a stable local decomposition of the kernel of the divergence. While Schöberl considered additive relaxation, Lee *et al.* (2007, 2009) later developed a similar kernel decomposition condition for parameter-robust multiplicative relaxation.

The proof by Schöberl for the kernel decomposition is based on an explicit construction of a Fortin operator that is used in the proof of inf-sup stability for the $[\mathbb{P}_2]^2 - \mathbb{P}_0$ element. Lee *et al.* consider the Scott–Vogelius element, but rely on the existence of a local basis for C^1 piecewise polynomials to construct this decomposition. As the existence of such a space is only known for high-order polynomials, they have to consider at least $k \geq 4$ and are limited to two dimensions (Morgan & Scott, 1975). Also relying on a local basis for C^1 piecewise polynomials, Wu & Zheng (2014) extend the results of Lee *et al.* to the additive case. Lastly we mention that it is also possible to develop domain decomposition preconditioners for this problem, as shown by Dohrmann & Widlund (2009).

1.1 Contributions

In this work we consider meshes with a macro element structure with the main requirement being that the inf-sup condition is satisfied on each macro element. Our main contribution is the construction of a particular localised Fortin operator obtained by “gluing” together Fortin operators on each macro element, the existence of which is guaranteed by inf-sup stability on the macro elements. Using this operator we are then able to construct robust relaxation and prolongation schemes for lower polynomial degrees k than previously possible. As examples we consider the Scott–Vogelius element on Alfeld and Powell–Sabin splits. We emphasize that in contrast to previous work considering the Scott–Vogelius element in this context, our approach does not require an explicit construction of a local basis of a C^1 space. Finally, we note that the strategy developed here can be applied to find robust multigrid preconditioners for other elements. These include the popular $[\mathbb{Q}_k]^d - \mathbb{P}_{k-1}^{\text{disc}}$ or the $[\mathbb{Q}_k]^d - \mathbb{Q}_{k-2}^{\text{disc}}$ elements (Bernardi & Maday, 1997; Schwab & Suri, 1999; Matthies & Tobiska, 2002; Heuveline & Schieweck, 2007) on quadrilateral and hexahedral meshes, since for these elements inf-sup stability is usually proven using the macro element approach.

1.2 Structure

The rest of this work is structured as follows. In Section 2 we develop robust smoothers in the context of subspace correction methods. In Section 3 we motivate the need for a special prolongation operator and give a different proof of robustness for a modification of the prolongation introduced previously by Schöberl. In Section 4 we state a convergence theorem for multigrid W-cycles using these two ingredients. Finally, in Section 5 we report numerical examples in two and in three dimensions that clearly demonstrate that standard geometric and algebraic multigrid methods fail for this problem, and that both the robust smoothing and robust prolongation are necessary to obtain an effective solver in the high γ limit.

2. Smoothing

We define the operator $A_{h,\gamma} : V_h \rightarrow V_h^*$ by

$$\langle A_{h,\gamma} \mathbf{u}, \mathbf{v} \rangle := a_{h,\gamma}(\mathbf{u}, \mathbf{v}), \quad (2.1)$$

and drop the subscript γ to denote the case of $\gamma = 0$, i.e.

$$\langle A_h \mathbf{u}, \mathbf{v} \rangle := a(\mathbf{u}, \mathbf{v}). \quad (2.2)$$

Many smoothers commonly used in multigrid can be expressed as subspace correction methods (Xu, 1992). We consider a decomposition

$$V_h = \sum_i V_i \quad (2.3)$$

where the sum is not necessarily direct. For each subspace i we denote the natural inclusion by $I_i : V_i \rightarrow V_h$ and we define the restriction A_i of $A_{h,\gamma}$ onto V_i as

$$\langle A_i \mathbf{u}_i, \mathbf{v}_i \rangle := \langle A_{h,\gamma} I_i \mathbf{u}_i, I_i \mathbf{v}_i \rangle \quad \text{for all } \mathbf{u}_i, \mathbf{v}_i \in V_i. \quad (2.4)$$

The additive Schwarz preconditioner associated with the space decomposition $\{V_i\}$ is then defined by the action of its inverse:

$$D_{h,\gamma}^{-1} := \sum_i I_i A_i^{-1} I_i^*. \quad (2.5)$$

In this work we focus on the case when exact solves are used on each subspace, but we note that the work of Xu (1992) also considers inexact solves, i.e. $D_{h,\gamma}^{-1} := \sum_i I_i R_i I_i^*$ with $R_i \approx A_i^{-1}$.

The method is also known as the parallel subspace correction method (Xu, 1992). We denote the norms induced by the operator and the preconditioner by

$$\begin{aligned}\|\mathbf{u}\|_{A_{h,\gamma}}^2 &= \langle A_{h,\gamma} \mathbf{u}, \mathbf{u} \rangle \\ \|\mathbf{u}\|_{D_{h,\gamma}}^2 &= \langle D_{h,\gamma} \mathbf{u}, \mathbf{u} \rangle\end{aligned}\tag{2.6}$$

and recall a key result in the theory of subspace correction methods (Widlund (1992, Lemma 1), see Schöberl (1999b, Theorem 4.1) or Xu (2001, Eqn. (4.11)) for a proof):

$$\|\mathbf{u}_h\|_{D_{h,\gamma}}^2 = \inf_{\substack{\mathbf{u}_i \in V_i \\ \sum_i \mathbf{u}_i = \mathbf{u}_h}} \sum_i \|\mathbf{u}_i\|_{A_i}^2.\tag{2.7}$$

To prove spectral equivalence of $D_{h,\gamma}$ and $A_{h,\gamma}$, we want to obtain a bound of the form

$$c_1 D_{h,\gamma} \leq A_{h,\gamma} \leq c_2 D_{h,\gamma},\tag{2.8}$$

where $M \leq N$ means that $\|\mathbf{u}\|_M \leq \|\mathbf{u}\|_N$ for all \mathbf{u} . A bound for the number of iterations required by the conjugate gradient method for $A_{h,\gamma}$ preconditioned by $D_{h,\gamma}$ then behaves like $\sqrt{c_2/c_1}$ (Elman *et al.*, 2014, eqn. (2.18)).

The second inequality in (2.8) measures the interaction between subspaces and can be bounded in a parameter-independent way by the maximum overlap of the subspaces N_O (Xu (1992, Lemma 4.6), Schöberl (1999b, Lemma 3.2)), which is bounded on shape regular meshes.

The first inequality in (2.8) is harder to obtain and usually depends not only on the smoother but also on the PDE and the mesh size. We demonstrate this for the case of Jacobi relaxation, i.e. when $V_i = \{\alpha \boldsymbol{\varphi}_i : \alpha \in \mathbb{R}\}$ where $\{\boldsymbol{\varphi}_i\}$ is the set of basis functions used for V_h . We note that the decomposition $\mathbf{u}_h = \sum \mathbf{u}_i$, $\mathbf{u}_i \in V_i$ is unique, and hence,

$$\begin{aligned}\|\mathbf{u}_h\|_{D_{h,\gamma}}^2 &= \sum_i \|\mathbf{u}_i\|_{A_{h,\gamma}}^2 \preceq (1+\gamma) \sum_i \|\mathbf{u}_i\|_1^2 \preceq \frac{1+\gamma}{h^2} \sum_i \|\mathbf{u}_i\|_0^2 \\ &\preceq (1+\gamma) h^{-2} \|\mathbf{u}_h\|_0^2 \preceq (1+\gamma) h^{-2} \|\mathbf{u}_h\|_{A_{h,\gamma}}^2,\end{aligned}\tag{2.9}$$

where $a \preceq b$ means that there exists a constant C independent of h and γ such that $a \leq Cb$. This bound is parameter dependent and degrades for large γ , i.e. $D_{h,\gamma}$ becomes a poor preconditioner for $A_{h,\gamma}$ as $\gamma \rightarrow \infty$.

To obtain a bound independent of γ , one requires a space decomposition that respects the nullspace of the divergence operator, which we denote by

$$\mathcal{N}_h = \{\mathbf{v}_h \in V_h : \Pi_{Q_h}(\nabla \cdot \mathbf{v}_h) = 0\}.\tag{2.10}$$

To build intuition, we consider a $\mathbf{u}_0 \in \mathcal{N}_h$. If the space decomposition satisfies

$$\mathcal{N}_h = \sum_i V_i \cap \mathcal{N}_h,\tag{2.11}$$

then \mathbf{u}_0 can be written as

$$\mathbf{u}_0 = \sum_i \mathbf{u}_{0,i}, \quad \mathbf{u}_{0,i} \in V_i \cap \mathcal{N}_h.\tag{2.12}$$

Redoing the first steps of the calculation in (2.9), and using that each of the $\mathbf{u}_{0,i}$ are divergence-free, the second term in (1.4) vanishes and we obtain

$$\|\mathbf{u}_0\|_{D_{h,\gamma}}^2 \leq \sum_i \|\mathbf{u}_{0,i}\|_{A_{h,\gamma}}^2 \preceq \sum_i \|\mathbf{u}_{0,i}\|_1^2. \quad (2.13)$$

We now make this idea rigorous and prove γ -independent spectral equivalence of $D_{h,\gamma}$ and $A_{h,\gamma}$. A key assumption, which we will need to check for each element and space decomposition individually, is that the splitting in (2.12) is stable, so that the last term in (2.13) can be bounded. This statement was proven for general parameter-dependent problems by Schöberl but for completeness we include a proof for the special case of elasticity.

PROPOSITION 2.1 (Schöberl (1999b, Theorem 4.1)) Let $\{V_i\}$ be a space decomposition of V_h with overlap N_O and assume that the pair $V_h \times \mathcal{Q}_h$ is inf-sup stable for the mixed problem

$$B((\mathbf{u}, p), (\mathbf{v}, q)) := a(\mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) - (\nabla \cdot \mathbf{u}, q). \quad (2.14)$$

Assume that any $\mathbf{u}_h \in V_h$ and $\mathbf{u}_0 \in \mathcal{N}_h$ satisfy

$$\begin{aligned} \inf_{\substack{\mathbf{u}_h = \sum \mathbf{u}_i \\ \mathbf{u}_i \in V_i}} \sum_i \|\mathbf{u}_i\|_1^2 &\leq c_1(h) \|\mathbf{u}_h\|_0^2, \\ \inf_{\substack{\mathbf{u}_0 = \sum \mathbf{u}_{0,i} \\ \mathbf{u}_{0,i} \in \mathcal{N}_h \cap V_i}} \sum_i \|\mathbf{u}_{0,i}\|_1^2 &\leq c_2(h) \|\mathbf{u}_0\|_0^2, \end{aligned} \quad (2.15)$$

where $\|\cdot\|_0$ denotes the standard L^2 norm, and $\|\cdot\|_1$ denotes the standard H^1 norm. Then it holds that

$$(c_1(h) + c_2(h))^{-1} D_{h,\gamma} \preceq A_{h,\gamma} \leq N_O D_{h,\gamma}, \quad (2.16)$$

with constants independent of γ .

Proof. Let $\mathbf{u}_h \in V_h$, and consider a decomposition $\mathbf{u}_h = \mathbf{u}_0 + \mathbf{u}_1$ obtained by solving

$$B((\mathbf{u}_1, p_1), (\mathbf{v}_h, q_h)) = (\nabla \cdot \mathbf{u}_h, q_h) \quad \text{for all } (\mathbf{v}_h, q_h) \in V_h \times \mathcal{Q}_h. \quad (2.17)$$

Testing with $\mathbf{v}_h = 0$ we obtain that $\Pi_{\mathcal{Q}_h}(\nabla \cdot \mathbf{u}_1) = \Pi_{\mathcal{Q}_h}(\nabla \cdot \mathbf{u}_h)$ and hence $\Pi_{\mathcal{Q}_h}(\nabla \cdot \mathbf{u}_0) = 0$. Furthermore, denoting $\|(\mathbf{v}_h, q_h)\| = \|\mathbf{v}_h\|_1 + \|q_h\|_0$, by stability we have

$$\begin{aligned} \|\mathbf{u}_1\|_1 &\preceq \sup_{\substack{\mathbf{v}_h \in V_h \\ q_h \in \mathcal{Q}_h}} \frac{B((\mathbf{u}_1, p_1), (\mathbf{v}_h, q_h))}{\|(\mathbf{v}_h, q_h)\|} \\ &\stackrel{(2.17)}{\leq} \sup_{\substack{\mathbf{v}_h \in V_h \\ q_h \in \mathcal{Q}_h}} \frac{\|\Pi_{\mathcal{Q}_h}(\nabla \cdot \mathbf{u}_h)\|_0 \|q_h\|_0}{\|(\mathbf{v}_h, q_h)\|} \\ &\leq \|\Pi_{\mathcal{Q}_h}(\nabla \cdot \mathbf{u}_h)\|_0 \end{aligned} \quad (2.18)$$

and hence $\|\mathbf{u}_1\|_1 \preceq \|\mathbf{u}_h\|_1$ and $\|\mathbf{u}_1\|_1 \preceq \gamma^{-1/2} \|\mathbf{u}_h\|_{A_{h,\gamma}}$. Using $\mathbf{u}_0 = \mathbf{u}_h - \mathbf{u}_1$ we obtain in addition that

$\|\mathbf{u}_0\|_1 \preceq \|\mathbf{u}_h\|_1$ and conclude

$$\begin{aligned}
 \|\mathbf{u}_h\|_{D_h}^2 &\leq \inf_{\substack{\mathbf{u}_1 = \sum \mathbf{u}_{1,i} \\ \mathbf{u}_{1,i} \in V_i}} \sum_i \underbrace{\|\mathbf{u}_{1,i}\|_{A_{h,\gamma}}^2}_{\leq (1+\gamma)\|\mathbf{u}_{1,i}\|_1^2} + \inf_{\substack{\mathbf{u}_0 = \sum \mathbf{u}_{0,i} \\ \mathbf{u}_{0,i} \in \mathcal{N}_h \cap V_i}} \sum_i \underbrace{\|\mathbf{u}_{0,i}\|_{A_{h,\gamma}}^2}_{=\|\mathbf{u}_{0,i}\|_1^2} \\
 &\stackrel{(2.15)}{\preceq} (1+\gamma)c_1(h)\|\mathbf{u}_1\|_0^2 + c_2(h)\|\mathbf{u}_0\|_0^2 \\
 &\preceq (1+\gamma)c_1(h)\|\mathbf{u}_1\|_1^2 + c_2(h)\|\mathbf{u}_0\|_1^2 \\
 &\preceq (c_1(h) + c_2(h))\|\mathbf{u}_h\|_{A_{h,\gamma}}^2.
 \end{aligned} \tag{2.19}$$

□

We now discuss two approaches to finding space decompositions $\{V_i\}$ that satisfy the kernel decomposition property in (2.11).

2.1 Characterisation of the kernel using exact de Rham complexes

We begin by recalling some fundamental de Rham complexes.

The smooth de Rham complex in two dimensions is given by

$$\mathbb{R} \xrightarrow{\text{id}} C^\infty(\Omega) \xrightarrow{\text{curl}} [C^\infty(\Omega)]^2 \xrightarrow{\text{div}} C^\infty(\Omega) \xrightarrow{\text{null}} 0, \tag{2.20}$$

and in three dimensions

$$\mathbb{R} \xrightarrow{\text{id}} C^\infty(\Omega) \xrightarrow{\text{grad}} [C^\infty(\Omega)]^3 \xrightarrow{\text{curl}} [C^\infty(\Omega)]^3 \xrightarrow{\text{div}} C^\infty(\Omega) \xrightarrow{\text{null}} 0. \tag{2.21}$$

Such a complex is called exact if the kernel of an operator is given by the range of the preceding operator in the sequence, e.g. when $\text{range curl} = \ker \text{div}$. It is well known that these complexes are exact precisely when the domain is simply connected (Arnold *et al.*, 2006, p. 18). Such an exactness property is of interest here because it allows us to characterise divergence-free vector fields as the curls of potentials.

Several lower regularity variants of these complexes exist. Likely the best-known ones are the complexes

$$\mathbb{R} \xrightarrow{\text{id}} H^1(\Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{\text{null}} 0, \tag{2D} (2.22)$$

$$\mathbb{R} \xrightarrow{\text{id}} H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{\text{null}} 0. \tag{3D} (2.23)$$

In the last decades, a significant effort has been made to find finite element spaces that form exact sub-complexes of (2.22) and (2.23) (Arnold *et al.*, 2006). For this work, we are interested in characterising the kernel of the divergence of vector fields with H^1 regularity. Hence, we study the so-called Stokes complexes where the function spaces enjoy higher regularity, given by

$$\mathbb{R} \xrightarrow{\text{id}} H^2(\Omega) \xrightarrow{\text{curl}} [H^1(\Omega)]^2 \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{\text{null}} 0, \tag{2.24}$$

and in three dimensions

$$\mathbb{R} \xrightarrow{\text{id}} H^2(\Omega) \xrightarrow{\text{grad}} H^1(\text{curl}, \Omega) \xrightarrow{\text{curl}} [H^1(\Omega)]^3 \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{\text{null}} 0, \tag{2.25}$$

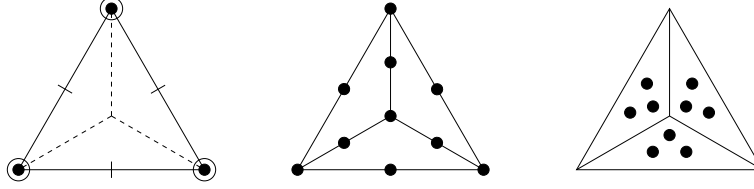


FIG. 1: A 2D exact Stokes complex on barycentrically refined meshes.

where $H^1(\text{curl}, \Omega) = \{\mathbf{u} \in [H^1(\Omega)]^3 : \text{curl } \mathbf{u} \in [H^1(\Omega)]^3\}$. Discrete subcomplexes of these Stokes complexes are much harder to construct and often result in high order polynomials due to the high regularity requirements.

Assume now that we have been given a discrete exact subsequence of (2.24) or (2.25) of the form

$$\cdots \rightarrow \Sigma_h \xrightarrow{\text{curl}} V_h \xrightarrow{\text{div}} Q_h \xrightarrow{\text{null}} 0. \quad (2.26)$$

Then for a divergence-free discrete vector field $\mathbf{u}_h \in V_h$, we can write it as the curl of a potential $\Phi_h \in \Sigma_h$. We note that Φ_h is a vector field in three dimensions but a scalar field in two dimensions.

Assume Σ_h has a basis given by $\{\Phi_j\}$, so that Φ_h can be written as $\Phi_h = \sum_j c_j \Phi_j$ for some coefficients c_j . Now we can define a divergence-free decomposition of \mathbf{u}_h as $\mathbf{u}_h = \sum_j \mathbf{u}_j$ where $\mathbf{u}_j = c_j \nabla \times \Phi_j$. Hence, a space decomposition $\{V_i\}$ such that $\nabla \times \Phi_j \in V_i$ for some i for all basis functions Φ_j decomposes the kernel. To understand how to choose a decomposition $\{V_i\}$ that satisfies this property, we have to examine the support of the basis functions $\{\Phi_j\}$.

In two dimensions and on barycentrically refined meshes, choosing Σ_h to be the Hsieh-Clough-Tocher (HCT) finite element space together with continuous \mathbb{P}_2^2 finite element functions for V_h and discontinuous \mathbb{P}_1 finite element functions for Q_h yields an exact discrete complex (John *et al.*, 2017, p. 514). The three elements are displayed in Figure 1. We call the mesh prior to barycentric refinement the *macro mesh* and its cells *macro elements*. For a given vertex v_i in the macro mesh, we define the *macrostar*(v_i) of the vertex as the union of all macro elements touching the vertex¹. We then see that for every HCT basis function Φ_j there exists a vertex v_i such that $\text{supp}(\Phi_j) \subset \text{macrostar}(v_i)$. Hence, also $\text{supp}(\nabla \times \Phi_j) \subset \text{macrostar}(v_i)$ and if we define

$$V_i = \{\mathbf{v} \in V_h : \text{supp}(\mathbf{v}) \subset \text{macrostar}(v_i)\} \quad (2.27)$$

then these subspaces decompose the kernel. More recently in Fu *et al.* (2020) an $H^1(\text{curl}, \Omega)$ -conforming element on barycentrically refined tetrahedral meshes was introduced that forms an exact sequence with piecewise cubic continuous velocities and piecewise quadratic discontinuous pressures. Hence, by the same argument we obtain that the macro star around vertices provides a decomposition of the kernel of the divergence in three dimensions.

We note that in two dimensions a quintic basis for Σ_h is known even without macro element structure (Morgan & Scott, 1975), which was used by Lee *et al.* (2009) to construct a robust relaxation method for sufficiently high polynomial degrees.

¹The *star* operation is a standard concept in algebraic topology (Munkres, 1984, §2); given a simplicial complex, the star of a simplex p is the union of the interiors of all simplices that contain p . The macro star operation is merely the star applied to vertices on the macro mesh.

2.2 Decomposing the kernel by a localised Fortin operator

In the previous section we introduced discrete exact sequences as a tool to construct a space decomposition that also decomposes the kernel of the divergence. When such an exact sequence exists, the approach is clearly attractive since the space decomposition can be found by simply studying the support of the basis functions in Σ_h . However, an exact sequence only guarantees the existence of some $\Phi_h \in \Sigma_h$ so that $\nabla \times \Phi_h = \mathbf{u}_h$, but does not make statements about its norm. The proof of exactness in Fu *et al.* (2020) for example is based on a counting argument and hence does not provide any stability bounds.

In two dimensions, this is not an issue as it is straightforward to obtain an element in Σ_h with bounded norm, as we will now argue. For a divergence-free vector field \mathbf{u}_h we know (Girault & Raviart, 1986, Theorem 3.3) that there exists a $\Phi \in H_0^2(\Omega)$ such that $\nabla \times \Phi = \mathbf{u}_h$ and $\|\Phi\|_2 \preceq \|\mathbf{u}_h\|_1$. Since $\nabla \times$ in two dimensions simply corresponds to the rotated gradient, we see that any two Φ that satisfy $\nabla \times \Phi = \mathbf{u}_h$ are equal up to a constant, and hence we have in fact $\Phi \in \Sigma_h$.

In three dimensions, the second step in this argument fails. It was proven in Costabel & McIntosh (2010, Proposition 4.1) that the *regularised Poincaré* operator provides a bounded linear map

$$R : [H^s(\Omega)]^3 \rightarrow [H^{s+1}(\Omega)]^3 \text{ s.t. } \nabla \times R(\mathbf{u}) = \mathbf{u} \text{ for all } \mathbf{u} \in H^s \text{ with } \nabla \cdot \mathbf{u} = 0, \quad (2.28)$$

for any $s \in \mathbb{R}$ and domain Ω that is star-like with respect to some ball. However, although the norm of the potential obtained from this map is bounded by the norm of \mathbf{u} , we cannot directly infer this property for the discrete potential as the uniqueness property that we exploited in two dimensions does not hold. In fact, we can add any gradient to the potential Φ and still preserve $\nabla \times \Phi = \mathbf{u}$. This motivates the development of a different strategy.

We briefly recall the approach of Schöberl for constructing the space decomposition. We start with a $\Phi \in H^2$ and consider a smooth partition of unity $\{\rho_i\}$. Considering $\mathbf{u}_i = \nabla \times (\rho_i \Phi)$, we immediately obtain that $\nabla \cdot \mathbf{u}_i = 0$ with $\sum_i \mathbf{u}_i = \mathbf{u}$. However, this construction may not yield an element of the finite element space. We thus consider $\mathbf{u}_i = I_h(\nabla \times (\rho_i \Phi))$ for an appropriate interpolation operator I_h that preserves the divergence in a suitable sense.

PROPOSITION 2.2 Assume that Ω is star-like with respect to some ball and let $I_h : V \rightarrow V_h$ be a Fortin operator, i.e. it satisfies

- I_h is linear and continuous,
- $(q_h, \nabla \cdot (I_h(\mathbf{v}))) = (q_h, \nabla \cdot \mathbf{v})$ for all $q_h \in Q_h$ and $\mathbf{v} \in V$,
- $I_h(\mathbf{v}_h) = \mathbf{v}_h$ for $\mathbf{v}_h \in V_h$.

Furthermore, let $\{\Omega_i\}$ be a covering of Ω with an associated smooth partition of unity $\{\rho_i\}$ satisfying

$$\begin{aligned} \|\rho_i\|_{L^\infty} &\leq 1, \\ \|\rho_i\|_{W^{1,\infty}} &\preceq h^{-1}, \\ \|\rho_i\|_{W^{2,\infty}} &\preceq h^{-2}, \\ \text{supp}(\rho_i) &\subset \Omega_i, \end{aligned} \quad (2.29)$$

then the space decomposition $\{V_i\}$ with

$$V_i := \{I_h(v) : v \in V, \text{supp}(v) \subset \Omega_i\} \quad (2.30)$$

satisfies

$$\inf_{\substack{\mathbf{u}_h = \sum \mathbf{u}_i \\ \mathbf{u}_i \in V_i}} \sum_i \|\mathbf{u}_i\|_1^2 \preceq h^{-2} \|\mathbf{u}_h\|_0^2, \quad (2.31)$$

and

$$\inf_{\substack{\mathbf{u}_0 = \sum \mathbf{u}_{0,i} \\ \mathbf{u}_{0,i} \in \mathcal{N}_h \cap V_i}} \sum_i \|\mathbf{u}_{0,i}\|_1^2 \preceq h^{-4} \|\mathbf{u}_0\|_0^2. \quad (2.32)$$

REMARK 2.1 This is an abstraction of Schöberl's approach. He used this strategy for a particular covering and a particular Fortin operator.

REMARK 2.2 In the multigrid context the goal is to choose the covering and the Fortin operator so that the spaces $\{V_i\}$ are small, as we use direct methods to solve the problems on the spaces V_i .

Proof. To prove the first statement, for $\mathbf{u}_h \in V_h$ we define

$$\mathbf{u}_i = I_h(\rho_i \mathbf{u}_h) \in V_i, \quad (2.33)$$

and observe that

$$\sum_i \mathbf{u}_i = I_h \left(\sum_i \rho_i \mathbf{u}_h \right) = I_h(\mathbf{u}_h) = \mathbf{u}_h \quad (2.34)$$

and

$$\begin{aligned} \|\mathbf{u}_i\|_{H^1(\Omega)}^2 &\preceq \|\rho_i \mathbf{u}_h\|_{H^1(\Omega_i)}^2 \\ &\leq \|\mathbf{u}_h\|_{L^2(\Omega_i)}^2 \|\nabla \rho_i\|_{L^\infty(\Omega_i)}^2 + \|\mathbf{u}_h\|_{H^1(\Omega_i)}^2 \|\rho_i\|_{L^\infty(\Omega_i)}^2 \\ &\preceq h^{-2} \|\mathbf{u}_h\|_{L^2(\Omega_i)}^2. \end{aligned} \quad (2.35)$$

Summing over i yields (2.31).

For the second splitting (2.32) we first note that given a $\mathbf{u}_0 \in \mathcal{N}_h$ by Girault & Raviart (1986, Theorem 3.3) in 2D and by Costabel & McIntosh (2010, Proposition 4.1) in 3D there exists a $\Phi \in H^2$ such that $\nabla \times \Phi = \mathbf{u}_0$ and $\|\Phi\|_2 \preceq \|\mathbf{u}_0\|_1$ and $\|\Phi\|_1 \preceq \|\mathbf{u}_0\|_0$.

At this point we have used the fact that discretely divergence-free vector fields are exactly divergence-free. This is not always necessary: for example in the case of the $[\mathbb{P}_2]^2 - \mathbb{P}_0$ element, one can modify \mathbf{u}_0 in the interior of each cell to obtain an exactly divergence-free field.

The estimates are obtained in a manner similar to the previous case: we define

$$\mathbf{u}_{0,i} = I_h(\nabla \times (\rho_i \Phi)) \quad (2.36)$$

and then calculate

$$\begin{aligned} \|\mathbf{u}_{0,i}\|_{H^1(\Omega)}^2 &\preceq \|\nabla \times (\rho_i \Phi)\|_{H^1(\Omega_i)}^2 \\ &\preceq \|\rho_i \Phi\|_{H^2(\Omega_i)}^2 \\ &\leq \|\Phi\|_{L^2(\Omega_i)}^2 \|\nabla^2 \rho_i\|_{L^\infty(\Omega_i)}^2 + \|\Phi\|_{H^1(\Omega_i)}^2 \|\nabla \rho_i\|_{L^\infty(\Omega_i)}^2 + \|\Phi\|_{H^2(\Omega_i)}^2 \|\rho_i\|_{L^\infty(\Omega_i)}^2 \\ &\leq h^{-4} \|\Phi\|_{L^2(\Omega_i)}^2 + h^{-2} \|\Phi\|_{H^1(\Omega_i)}^2 + \|\Phi\|_{H^2(\Omega_i)}^2. \end{aligned} \quad (2.37)$$

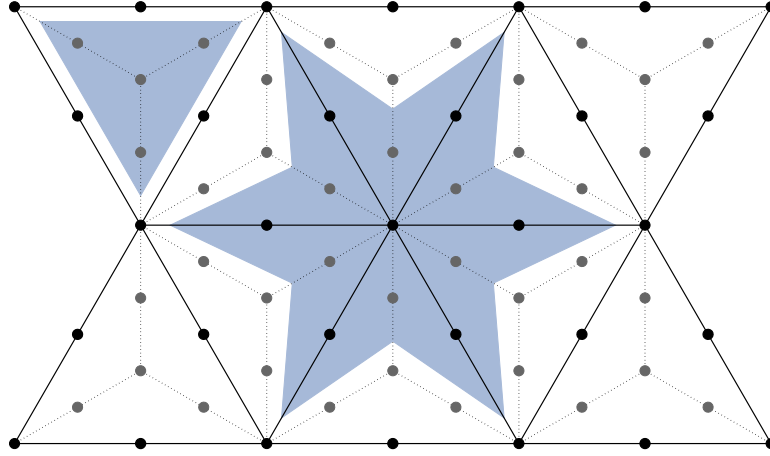


FIG. 2: The two types of patches obtained by taking the star around vertices in a barycentrically refined mesh.

Summing over i and denoting the overlap by N_O we obtain

$$\begin{aligned}
 \sum_i \|\mathbf{u}_{0,i}\|_{H^1(\Omega)}^2 &\leq N_O (h^{-4} \|\Phi\|_{L^2(\Omega)}^2 + h^{-2} \|\Phi\|_{H^1(\Omega)}^2 + \|\Phi\|_{H^2(\Omega)}^2) \\
 &\leq N_O (h^{-4} \|\mathbf{u}_0\|_{L^2(\Omega)}^2 + h^{-2} \|\mathbf{u}_0\|_{L^2(\Omega)}^2 + \|\mathbf{u}_0\|_{H^1(\Omega)}^2) \\
 &\leq N_O h^{-4} \|\mathbf{u}_0\|_{L^2(\Omega)}^2,
 \end{aligned} \tag{2.38}$$

where we used an inverse inequality in the last step. \square

Schöberl uses an operator I_h that is essentially the same one as in the classical proof for the inf-sup stability of the $[\mathbb{P}_2]^2 - \mathbb{P}_0$ element, with minor modifications so that it uses only values on element boundaries. This leads to small subspaces V_i when Ω_i is defined as a domain within the star around each vertex, see Figure 2. However, as of writing the authors are not aware of a similar construction of a Fortin operator for the Scott–Vogelius element in either two or three dimensions that could be modified.² To obtain a Fortin operator with the needed locality, we instead use that the **finite element pair** is known to be inf-sup stable on a single macro element: we begin by considering a Fortin operator \tilde{I}_h that preserves the divergence with respect to pressures that are piecewise constant on macro elements and then enrich this operator with local Fortin operators on each macro element.

Given a domain Ω , we consider a simplicial mesh $\mathcal{T}_h = \{K^h\}$ with $\bigcup_{K^h \in \mathcal{T}_h} K^h = \overline{\Omega}$ and $(K_1^h)^\circ \cap (K_2^h)^\circ = \emptyset$ for distinct $K_1^h, K_2^h \in \mathcal{T}_h$. The elements $K^h \in \mathcal{T}_h$ will be referred to as the *macro cells*.

LEMMA 2.1 Let \tilde{Q}_h be defined by

$$\tilde{Q}_h := \{\tilde{q}_h \in L^2(\Omega) : \tilde{q}_h|_K \equiv \text{const for all } K \in \mathcal{T}_h\} \tag{2.39}$$

and assume that there exists a Fortin operator $\tilde{I}_h : V \rightarrow V_h$ for the pair $V_h \times \tilde{Q}_h$, that is

²During the review process we were made aware of recent work (Boffi *et al.*, 2021, Theorem 3.6) in which a Fortin operator is constructed in two dimensions for a similar problem using a rotated gradient and different boundary conditions.

- \tilde{I}_h is linear and continuous,
- $(\tilde{q}_h, \nabla \cdot (\tilde{I}_h(\mathbf{v}))) = (\tilde{q}_h, \nabla \cdot \mathbf{v})$ for all $\tilde{q}_h \in \tilde{Q}_h$ and $\mathbf{v} \in V$,
- $\tilde{I}_h(\mathbf{v}_h) = \mathbf{v}_h$ for $\mathbf{v}_h \in V_h$,
- there exists a covering $\{\Omega_i\}$ of Ω such that for all vertices v_i in the macro mesh it holds that $\Omega_i \subset \text{macro star}(v_i)$ and

$$\tilde{I}_h(\mathbf{v}) \in V_i \text{ for all } \mathbf{v} \in V \text{ such that } \text{supp}(\mathbf{v}) \subset \Omega_i \quad (2.40)$$

where $\{V_i\}$ are the subspaces defined in (2.27), that is

$$V_i = \{\mathbf{v} \in V_h : \text{supp}(\mathbf{v}) \subset \text{macro star}(v_i)\}. \quad (2.41)$$

Furthermore assume that the **finite element pair** is stable on each macro cell. Then there exists a linear map $I_h : V \rightarrow V_h$ such that

- I_h is linear and continuous,
- $(q_h, \nabla \cdot (I_h(\mathbf{v}))) = (q_h, \nabla \cdot \mathbf{v})$ for all $q_h \in Q_h$ and $\mathbf{v} \in V$,
- $I_h(\mathbf{v}_h) = \mathbf{v}_h$ for $\mathbf{v}_h \in V_h$,
- the covering $\{\Omega_i\}$ has the property that for all vertices v_i in the macro mesh it holds

$$I_h(\mathbf{v}) \in V_i \text{ for all } \mathbf{v} \in V \text{ such that } \text{supp}(\mathbf{v}) \subset \Omega_i. \quad (2.42)$$

REMARK 2.3 The key difference between \tilde{I}_h and I_h is that the former only has to preserve the divergence with respect to test functions in the smaller space \tilde{Q}_h , but I_h preserves the divergence with respect to test functions in the full space Q_h .

REMARK 2.4 The advantageous consequence of this result is that the macro star gives a kernel-capturing space decomposition with small support, suitable for use as a multigrid relaxation method.

Before proving this statement, we give two examples.

For the first example, consider the Scott–Vogelius element on Alfeld splits as shown in Figure 3. By Qin (1994, Section 4.6) and Zhang (2004) the **finite element pair** is inf-sup stable for $k = d$ in both two and three dimensions.

We begin by considering the two dimensional case. The covering $\{\Omega_i\}$ is shown as the blue shaded region in the figure and the operator \tilde{I}_h can be chosen to be the standard Fortin operator for the $[\mathbb{P}_2]^2 - \mathbb{P}_0$ finite element pair, constructed as follows. Let I_1 be a Scott–Zhang interpolant based on the integration domains shown in red in Figure 3. First, note that $I_1(\mathbf{v}_h) = \mathbf{v}_h$ for all $\mathbf{v}_h \in V_h$, and that by construction $I_1(\mathbf{v}) \in V_i$ for all \mathbf{v} with support in Ω_i . Let $I_2 : V \rightarrow V_h$ be defined to be zero on all degrees of freedom except the degrees of freedom on the macro edges, which instead are chosen so that

$$\int_E I_2(\mathbf{v}) \, ds = \int_E \mathbf{v} \, ds, \quad (2.43)$$

for all edges E in the macro mesh. Then let $\tilde{I}_h(\mathbf{v}) = I_1(\mathbf{v}) + I_2(\mathbf{v} - I_1(\mathbf{v}))$. By construction,

$$(\nabla \cdot I_2(\mathbf{v}), \tilde{q}_h) = (\nabla \cdot \mathbf{v}, \tilde{q}_h) \text{ for all } \tilde{q}_h \in \tilde{Q}_h, \mathbf{v} \in V, \quad (2.44)$$

and hence the same property holds for \tilde{I}_h . The existence of a global Fortin operator with the required locality properties then follows from Lemma 2.1. Comparing Figures 2 and 3 we note that the subspaces obtained here are larger but that fewer are obtained, since we only obtain one subspace per vertex in the mesh prior to barycentric refinement.

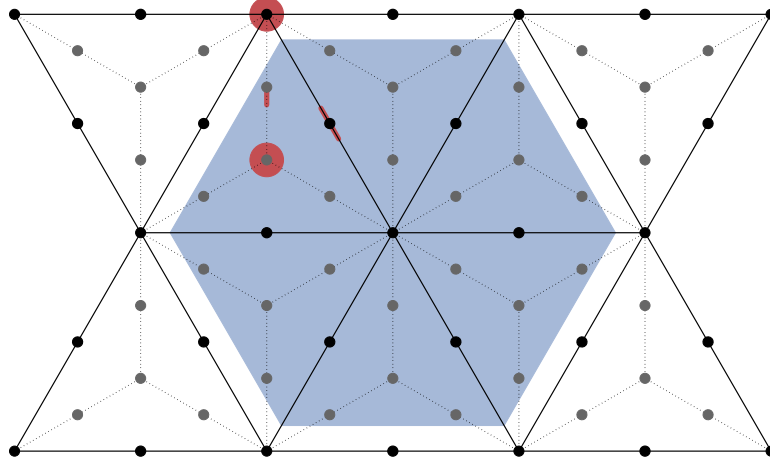


FIG. 3: Domain Ω_i around a vertex v_i (blue), covering all degrees of freedom inside $\text{macrostar}(v_i)$. The figure also shows integration regions (in red) for vertex and edge degrees of freedom used in the Scott–Zhang operator I_1 . Note that the integration regions are chosen so that only those associated with $\text{macrostar}(v_i)$ intersect with Ω_i .

Essentially the same construction can be used in three dimensions. I_1 is again defined as a Scott–Zhang interpolant, I_2 is set to be zero on all vertex and edge degrees of freedom, and the value on each facet F is chosen so that

$$\int_F I_2(\mathbf{v}) \, dx = \int_F \mathbf{v} \, dx \quad \text{for all } \mathbf{v} \in V. \quad (2.45)$$

Finally we note that a similar statement holds for the second example of the Powell–Sabin split, shown in Figure 4. Due to the presence of singular vertices/edges, one has to consider a slightly smaller pressure space $Q_h = \{\nabla \cdot \mathbf{v}_h : \mathbf{v}_h \in V_h\}$, but then the element pair $V_h \times Q_h$ is inf-sup stable for $k = d - 1$ (Zhang, 2008, 2011; Guzmán *et al.*, 2020). We highlight that the Powell–Sabin split introduces vertices on the macro edges in 2D and on the macro facets in 3D. The degrees of freedom on these vertices are used in the construction of I_2 so that (2.43) holds in 2D and (2.45) holds in 3D respectively. Finally, we note that to implement the smoother one does not need an explicit description of Q_h , but only of V_h , so the more complicated nature of Q_h for these splits does not raise any practical issues.

REMARK 2.5 When compared using elements of the same degree, Alfeld splits are more efficient, as the macro star(v_i) for that split contains fewer degrees of freedom and hence the local solves are cheaper. However, Powell–Sabin splits enable the use of lower order discretisations. The lower degree offered by Powell–Sabin splits could make them more attractive when additional singular terms are present in the equations that are captured by the support of C^1 elements, such as when interior penalty stabilisation is used for the Navier–Stokes equations; see Farrell *et al.* (2021b, Remark 4.1) for an example.

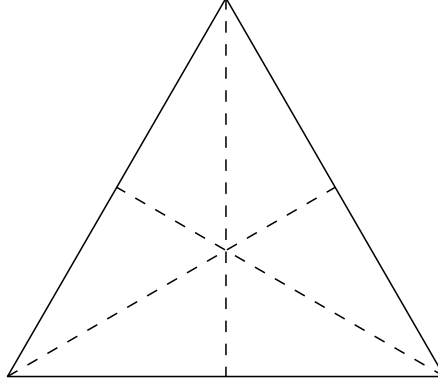


FIG. 4: Powell–Sabin split in two dimensions.

Proof of Lemma 2.1. The idea is to combine the global Fortin operator \tilde{I}_h that preserves the discrete divergence with respect to pressures that are constant on macro cells with suitable local Fortin operators.

To treat the divergence with respect to the remaining pressures in $Q_h \setminus \tilde{Q}_h$ we now consider the macro elements separately. For each such macro triangle K we define the spaces $V_{h,0}(K) = \{\mathbf{v}_h|_K : \mathbf{v}_h \in V_h, \text{supp}(\mathbf{v}_h) \subset K\}$ and $Q_h(K) = \{q_h|_K : q_h \in \tilde{Q}_h^\perp\}$. The space $V_{h,0}(K)$ consists of velocity fields supported in a macro cell and the space $Q_h(K)$ consists of pressures on a macro cell that integrate to zero. We note that $Q_h = \sum_{K \in \mathcal{T}_h} Q_h(K) \oplus \tilde{Q}_h$. Since we assume that this pair is inf-sup stable for each macro element, we know that there exist Fortin operators $I_h^K : V(K) \rightarrow V_{h,0}(K)$, where $V(K) = \{\mathbf{v}|_K : \mathbf{v} \in V\}$, such that

- $I_h^K(\mathbf{v}_h) = \mathbf{v}_h$ for $\mathbf{v}_h \in V_{h,0}(K)$
- I_h^K is bounded as a map $V(K) \mapsto V_{h,0}(K)$
- $(q_h, \nabla \cdot \mathbf{v}) = (q_h, \nabla \cdot (I_h^K(\mathbf{v})))$ for all $q_h \in Q_h(K)$

for all K (Fortin (1977), Ern & Guermond (2004, Lemma 4.19)).

Now we define

$$I_h(\mathbf{v}) = \tilde{I}_h(\mathbf{v}) + \sum_K I_h^K((\mathbf{v} - \tilde{I}_h(\mathbf{v}))|_K). \quad (2.46)$$

Clearly, I_h is linear and $I_h(\mathbf{v}_h) = \mathbf{v}_h$ for all $\mathbf{v}_h \in V_h$. In addition, I_h is continuous with continuity constant only dependent on the continuity constant of \tilde{I}_h and the local Fortin operators, and I_h satisfies the locality property in (2.42).

Furthermore, we note that the discrete divergence of vector fields in $V_{h,0}(K)$ with respect to \tilde{Q}_h is zero. It follows that

$$\begin{aligned} & (q_h, \nabla \cdot (I_h(\mathbf{v}))) \\ &= \underbrace{(q_h, \nabla \cdot (\tilde{I}_h(\mathbf{v})))}_{=(q_h, \nabla \cdot \mathbf{v})} + \sum_K \underbrace{(q_h, \nabla \cdot \overbrace{I_h^K((\mathbf{v} - \tilde{I}_h(\mathbf{v}))|_K))}^{\in V_{h,0}(K)}}_{=0} \\ &= (q_h, \nabla \cdot \mathbf{v}) \end{aligned} \quad (2.47)$$

for all $q_h \in \tilde{Q}_h$ and $\mathbf{v} \in V$.

Lastly, we show that I_h preserves the discrete divergence with respect to the local pressures in $Q_h(K)$. For $\mathbf{v} \in V$, $K \in \mathcal{T}_h$, and $q_h \in Q_h(K)$, we have

$$\begin{aligned} & (q_h, \nabla \cdot (I_h(\mathbf{v}))) \\ &= (q_h, \nabla \cdot (\tilde{I}_h(\mathbf{v}))) + (q_h, \nabla \cdot (I_h^K((\mathbf{v} - \tilde{I}_h \mathbf{v})|_K))) \\ &= (q_h, \nabla \cdot (\tilde{I}_h(\mathbf{v}))) + (q_h, \nabla \cdot (\mathbf{v} - \tilde{I}_h(\mathbf{v}))) \\ &= (q_h, \nabla \cdot \mathbf{v}), \end{aligned} \tag{2.48}$$

as desired. \square

3. Prolongation

The second key ingredient for a robust multigrid scheme is a robust prolongation operator \tilde{P}_H that maps coarse grid functions to fine grid functions with a continuity constant independent of γ . To build intuition for this requirement, let P_H be a given prolongation operator and calculate

$$\begin{aligned} \|\mathbf{u}_H\|_{A_{H,\gamma}}^2 &= \|\mathbf{u}_H\|_{A_H}^2 + \gamma \|\nabla \cdot \mathbf{u}_H\|_0^2 \\ \|P_H \mathbf{u}_H\|_{A_{h,\gamma}}^2 &= \|P_H \mathbf{u}_H\|_{A_h}^2 + \gamma \|\nabla \cdot (P_H \mathbf{u}_H)\|_0^2. \end{aligned} \tag{3.1}$$

The key difficulty lies in the second term of this norm. If $V_H \not\subset V_h$, interpolation is not exact. Hence for a divergence-free vector field $\mathbf{u}_H \in \mathcal{N}_H$ the second term in $\|\mathbf{u}_H\|_{A_{H,\gamma}}^2$ vanishes, but it may not hold that $P_H \mathbf{u}_H$ is divergence-free, and so the corresponding term in $\|P_H \mathbf{u}_H\|_{A_{h,\gamma}}^2$ might be large. Hence the continuity constant of the prolongation operator P_H in the energy norm is not independent of γ .

Some macro structures cause non-nestedness, and some do not. For example, Alfeld splits induce a non-nested mesh structure, as seen in Figure 5. On the other hand, in two dimensions and on regular meshes, the additional nodes induced by regular refinement are a subset of those induced by the Powell–Sabin split and the resulting hierarchy is nested, again demonstrated in Figure 5. The work of Lee *et al.* (2009) considered uniform refinements, which are nested, and hence they do not require any modification of the prolongation operator. For the remainder of this section we consider macro meshes that induce non-nested hierarchies, such as in the case of Alfeld splits.

To remedy the γ -dependence of the standard prolongation operator, we must modify it to map fields that are divergence-free on the coarse grid to fields that are (nearly) divergence-free on the fine grid, i.e. that $\|\nabla \cdot (\tilde{P}_H \mathbf{u}_H)\|_0^2$ is of order $\mathcal{O}(\gamma^{-1})$. We now describe a modification of the standard prolongation operator that satisfies this condition. This type of modification goes back to Schöberl's work, although we give a different derivation and proof.

Let $\mathbf{u}_H \in \mathcal{N}_H$ be a divergence-free function on the coarse-grid and denote the standard prolongation induced by the interpolation operator on the finite element space by $P_H \mathbf{u}_H$. We are interested in finding a small perturbation $\tilde{\mathbf{u}}_h$ such that $\tilde{P}_H \mathbf{u}_H = P_H \mathbf{u}_H - \tilde{\mathbf{u}}_h \in \mathcal{N}_h$. This could be achieved by solving

$$\begin{aligned} & \min_{\tilde{\mathbf{u}}_h \in V_h} a(\tilde{\mathbf{u}}_h, \tilde{\mathbf{u}}_h) \\ & \text{s.t.} \quad \Pi_{Q_h}(\nabla \cdot \tilde{\mathbf{u}}_h) = \Pi_{Q_h}(\nabla \cdot P_H \mathbf{u}_H). \end{aligned} \tag{3.2}$$

This corresponds to solving a Stokes-like problem in $V_h \times Q_h$. We now relax this problem in two ways. First, we do not need to enforce that $P_H \mathbf{u}_H - \tilde{\mathbf{u}}_h$ has zero divergence, as it is enough if it is suitably

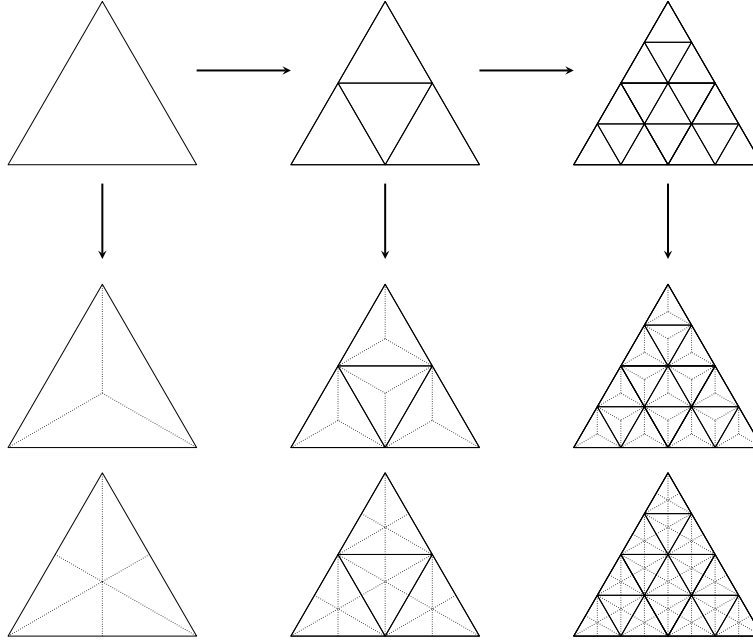


FIG. 5: A three level multigrid hierarchy using either Alfeld splits or Powell–Sabin splits at each level.

small, i.e. we can instead find $\tilde{\mathbf{u}}_h \in V_h$ that minimises

$$\min_{\tilde{\mathbf{u}}_h \in V_h} a(\tilde{\mathbf{u}}_h, \tilde{\mathbf{u}}_h) + \gamma \|\Pi_{Q_h}(\nabla \cdot (P_H \mathbf{u}_H - \tilde{\mathbf{u}}_h))\|_0^2. \quad (3.3)$$

This corresponds to: find $\tilde{\mathbf{u}}_h \in V_h$ such that

$$a_{h,\gamma}(\tilde{\mathbf{u}}_h, \mathbf{v}_h) = \gamma (\Pi_{Q_h}(\nabla \cdot P_H \mathbf{u}_H), \Pi_{Q_h}(\nabla \cdot \mathbf{v}_h)) \quad \text{for all } \mathbf{v}_h \in V_h. \quad (3.4)$$

Clearly at this stage we have not gained much, since we now need to solve a global problem involving the nearly singular bilinear form $a_{h,\gamma}$. (Recall that the projection onto Q_h of the divergence is the identity, as $\nabla \cdot V_h = Q_h$.) However, it turns out that under certain assumptions to be stated in the following proposition, one can instead solve the same problem on smaller spaces $\hat{V}_h \subset V_h$ and $\hat{Q}_h \subset Q_h$:

$$\min_{\tilde{\mathbf{u}}_h \in \hat{V}_h} a(\tilde{\mathbf{u}}_h, \tilde{\mathbf{u}}_h) + \gamma \|\Pi_{\hat{Q}_h}(\nabla \cdot (P_H \mathbf{u}_H - \tilde{\mathbf{u}}_h))\|_0^2, \quad (3.5)$$

or equivalently: find $\tilde{\mathbf{u}}_h \in \hat{V}_h$ such that

$$a_{h,\gamma}(\tilde{\mathbf{u}}_h, \hat{\mathbf{v}}_h) = \gamma (\Pi_{\hat{Q}_h}(\nabla \cdot (P_H \mathbf{u}_H)), \Pi_{\hat{Q}_h}(\nabla \cdot \hat{\mathbf{v}}_h)) \quad \text{for all } \hat{\mathbf{v}}_h \in \hat{V}_h. \quad (3.6)$$

How exactly one chooses these subspaces will depend on the discretisation under consideration and we will again illustrate this for the Scott–Vogelius element on Alfeld splits.

PROPOSITION 3.1 (Robust prolongation) Assume we can split $Q_h = \tilde{Q}_h \oplus \hat{Q}_h$ and that $\tilde{Q}_h \subseteq Q_h$. Let $P_H : V_H \rightarrow V_h$ be a prolongation operator that is continuous in the $\|\cdot\|_1$ norm and preserves the divergence with respect to \tilde{Q}_h , i.e.

$$(\nabla \cdot (P_H \mathbf{v}_H), \tilde{q}_H) = (\nabla \cdot \mathbf{v}_H, \tilde{q}_H) \quad \text{for all } \tilde{q}_H \in \tilde{Q}_h, \mathbf{v}_H \in V_H. \quad (3.7)$$

Assume in addition that there exists a $\hat{V}_h \subset V_h$ such that

$$(\nabla \cdot \hat{\mathbf{v}}_h, \tilde{q}_H) = 0 \quad \text{for all } \tilde{q}_H \in \tilde{Q}_h, \hat{\mathbf{v}}_h \in \hat{V}_h, \quad (3.8)$$

and such that the pairing $\hat{V}_h \times \hat{Q}_h$ is inf-sup stable, i.e.

$$\inf_{\hat{q}_h \in \hat{Q}_h} \sup_{\hat{\mathbf{v}}_h \in \hat{V}_h} \frac{(\hat{q}_h, \nabla \cdot \hat{\mathbf{v}}_h)}{\|\hat{\mathbf{v}}_h\|_1 \|\hat{q}_h\|_0} \geq c \quad (3.9)$$

for some mesh independent $c > 0$. For $\mathbf{u}_H \in V_H$, define $\tilde{\mathbf{u}}_h$ as the solution to

$$a_{h,\gamma}(\tilde{\mathbf{u}}_h, \hat{\mathbf{v}}_h) = \gamma(\Pi_{Q_h}(\nabla \cdot (P_H \mathbf{u}_H)), (\Pi_{Q_h}(\nabla \cdot (\hat{\mathbf{v}}_h)))) \quad \text{for all } \hat{\mathbf{v}}_h \in \hat{V}_h. \quad (3.10)$$

Then the prolongation $\tilde{P}_H : V_H \rightarrow V_h$ defined by

$$\tilde{P}_H \mathbf{u}_H = P_H \mathbf{u}_H - \tilde{\mathbf{u}}_h \quad (3.11)$$

is continuous in the energy norm with continuity constant independent of γ .

REMARK 3.1 The problems in (3.6) and (3.10) are equivalent by the assumption in (3.8).

REMARK 3.2 This prolongation operator is very similar to the one used by Schöberl (1999b, Theorem 4.2) and Benzi & Olshanskii (2006, Lemma 5.1). The difference is that in Schöberl's work the problem in (3.10) is replaced with

$$a_{h,\gamma}(\tilde{\mathbf{u}}_h, \hat{\mathbf{v}}_h) = a_{h,\gamma}(P_H \mathbf{u}_H, \hat{\mathbf{v}}_h) \quad \text{for all } \hat{\mathbf{v}}_h \in \hat{V}_h. \quad (3.12)$$

We note that this problem only differs in the right-hand side, and hence the two methods have essentially the same cost. We use the version in (3.10) for two reasons: first, it reduces to the standard prolongation for $\gamma = 0$, and second, it can be viewed as a local version of the global problem in (3.4), whereas solving a global version of (3.12) would lead to $\tilde{\mathbf{u}}_h = P_H \mathbf{u}_H$ and hence $\tilde{P}_H \mathbf{u}_H = 0$.

The proofs of Schöberl and Benzi & Olshanskii are based on an equivalent mixed problem. We will give a different proof motivated by the formulation as an optimisation problem and use the existence of a Fortin operator arising from inf-sup stability.

Before giving the proof, we again consider the case of the Scott–Vogelius element on Alföld splits. In this case, the hierarchy is constructed as shown in Figure 5. Due to the barycentric refinement at each level, we do not have nested function spaces and hence the prolongation is not exact and a divergence-free function on the coarse grid may be prolonged to a function on the fine grid with nonzero divergence. However, we observe that interpolation is exact on the boundaries of coarse grid macro cells. This means that flux across these boundaries is preserved and hence the divergence with respect to functions in

$$\tilde{Q}_H := \{q \in L^2 : q \equiv \text{const on coarse grid macro cells } K \in \mathcal{T}_H\}, \quad (3.13)$$

is preserved, i.e. we have for the standard interpolation operator $P_H : V_H \rightarrow V_h$ that

$$(\nabla \cdot \mathbf{u}_H, \tilde{q}_H) = (\nabla \cdot (P_H \mathbf{u}_H), \tilde{q}_H) \quad \text{for all } \mathbf{u}_H \in V_H, \tilde{q}_H \in \tilde{Q}_H. \quad (3.14)$$

Hence requirement (3.7) of Proposition 3.1 is satisfied. To correct for any extra divergence gained due to interpolation inside a coarse grid macro cell we will solve local problems. To this end, we define the spaces

$$\begin{aligned} \hat{Q}_h &:= \{q_h \in Q_h : \Pi_{\tilde{Q}_H} q_h = 0\} \\ \hat{V}_h &:= \{\mathbf{v}_h \in V_h : \text{supp}(\mathbf{v}_h) \subset K \text{ for some } K \in \mathcal{T}_H\}. \end{aligned} \quad (3.15)$$

The space \hat{V}_h consists of local patches of velocity degrees of freedom contained in coarse grid macro cells, as shown in Figure 6. We highlight that these patches decouple and hence solves involving \hat{V}_h can be performed independently, leading to good performance. The pair $\hat{V}_h \times \hat{Q}_h$ is inf-sup stable in both

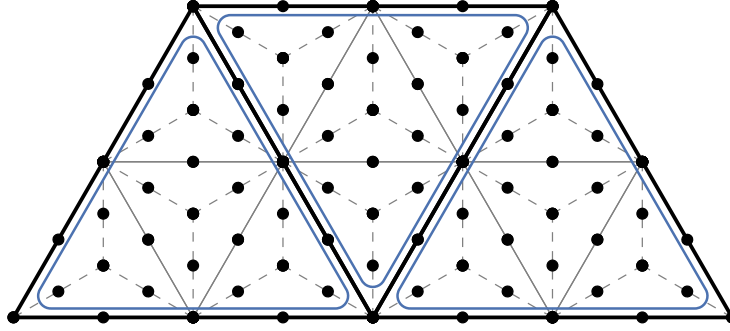


FIG. 6: Degrees of freedom of the \hat{V}_h on which we perform local solves to obtain a robust prolongation in two dimensions.

two and three dimensions, so it remains to check

$$(\nabla \cdot \hat{\mathbf{v}}_h, \tilde{q}_H) = 0 \quad \text{for all } \tilde{q}_H \in \tilde{Q}_H, \hat{\mathbf{v}}_h \in \hat{V}_h, \quad (3.16)$$

which follows from the requirement that the support of vector fields $\hat{\mathbf{v}}_h \in \hat{V}_h$ is contained in coarse grid macro cells and the definition of \tilde{Q}_H . A robust prolongation operator can now be constructed as in Proposition 3.1.

Proof of Proposition 3.1. We denote

$$J(\tilde{\mathbf{v}}_h) := a(\tilde{\mathbf{v}}_h, \tilde{\mathbf{v}}_h) + \gamma \|\Pi_{\hat{Q}_h}(\nabla \cdot (P_H \mathbf{u}_H - \tilde{\mathbf{v}}_h))\|_0^2, \quad (3.17)$$

and observe that $\tilde{\mathbf{u}}_h$ is the unique minimiser of J in \hat{V}_h . By inf-sup stability of the pairing $\hat{V}_h \times \hat{Q}_h$ there exists a continuous Fortin operator $I : V \rightarrow \hat{V}_h$ that satisfies

$$\Pi_{\hat{Q}_h}(\nabla \cdot (I\mathbf{v})) = \Pi_{\hat{Q}_h}(\nabla \cdot \mathbf{v}) \quad \text{for all } \mathbf{v} \in V. \quad (3.18)$$

Let $\tilde{\mathbf{u}}_h := I(P_H \mathbf{u}_H) \in \hat{V}_h$ and observe that

$$J(\tilde{\mathbf{u}}_h) \leq J(\tilde{\mathbf{u}}_h) = a(\tilde{\mathbf{u}}_h, \tilde{\mathbf{u}}_h) \leq \|P_H \mathbf{u}_H\|_1^2 \leq \|\mathbf{u}_H\|_1^2. \quad (3.19)$$

We conclude

$$\begin{aligned}
& \|\tilde{P}_H \mathbf{u}_H\|_{A_h, \gamma}^2 \\
& \leq \|P_H \mathbf{u}_H - \tilde{\mathbf{u}}_h\|_{A_h}^2 + \gamma \|\Pi_{Q_h}(\nabla \cdot (P_H \mathbf{u}_H - \tilde{\mathbf{u}}_h))\|_0^2 \\
& \leq \underbrace{\|P_H \mathbf{u}_H\|_{A_h}^2}_{\leq \|\mathbf{u}_H\|_{A_h}^2} + \underbrace{\|\tilde{\mathbf{u}}_h\|_{A_h}^2 + \gamma \|\Pi_{\tilde{Q}_h}(\nabla \cdot (P_H \mathbf{u}_H - \tilde{\mathbf{u}}_h))\|_0^2}_{=J(\tilde{\mathbf{u}}_h) \stackrel{(3.19)}{\leq} \|\mathbf{u}_H\|_1^2} + \gamma \|\Pi_{\tilde{Q}_H}(\nabla \cdot (P_H \mathbf{u}_H - \tilde{\mathbf{u}}_h))\|_0^2 \\
& \leq \|\mathbf{u}_H\|_1^2 + \gamma \underbrace{\|\Pi_{\tilde{Q}_H}(\nabla \cdot (P_H \mathbf{u}_H))\|_0^2}_{\stackrel{(3.7)}{=} \|\Pi_{\tilde{Q}_H}(\nabla \cdot \mathbf{u}_H)\|_0^2} + \gamma \underbrace{\|\Pi_{\tilde{Q}_H}(\nabla \cdot \tilde{\mathbf{u}}_h)\|_0^2}_{\stackrel{(3.8)}{=} 0} \\
& \leq \|\mathbf{u}_H\|_1^2 + \gamma \|\Pi_{\tilde{Q}_H}(\nabla \cdot \mathbf{u}_H)\|_0^2 \\
& \leq \|\mathbf{u}_H\|_{A_h, \gamma}^2.
\end{aligned} \tag{3.20}$$

□

4. Multigrid convergence

We have seen that the bounds in the estimates for the relaxation behave like $c_1 \sim h^{-2}$ and $c_2 \sim h^{-4}$. The reason for the fast growth of c_2 is that the splitting into divergence-free functions is constructed by expressing $\mathbf{u}_0 = \nabla \times \Phi$ for a potential $\Phi \in H^2$, introducing second derivatives into the estimates. This quartic growth of c_2 complicates a standard multigrid analysis, as one needs to show that a coarse-grid solve reduces the error in such a way that the multigrid scheme yields mesh-independent convergence. Following the structure of the proof by Schöberl for the $[\mathbb{P}_2]^2 - \mathbb{P}_0$ element (Schöberl, 1999b, §4.4, §4.6.1), with suitable modifications, one can obtain a parameter-robust multigrid convergence result.

THEOREM 4.1 Define a multigrid method using the relaxation defined in Section 2, the prolongation operator defined in Section 3, and the adjoint of the prolongation operator as restriction operator. Then a W-cycle scheme with sufficiently many smoothing steps leads to a solver with convergence independent of the number of levels in the multigrid hierarchy and the parameter γ .

5. Numerical example

We conclude with a numerical example that corroborates the multigrid convergence theorem and that demonstrates the necessity of both components of the multigrid scheme. The example is implemented using the Firedrake (Rathgeber *et al.*, 2016) finite element package, with the local solves performed using the PCPATCH (Farrell *et al.*, 2021a) preconditioner recently included in PETSc (Balay *et al.*, 2019).

We consider the problem (1.1) on a domain $\Omega = [0, 1]^d$ for a range of parameter values γ . We pick a zero right-hand-side $\mathbf{f} = 0$, homogeneous Dirichlet conditions on the boundary $\{x_1 = 0\}$, a constant traction \mathbf{h} pulling down in x_2 direction with magnitude $1/2$ on the boundary $\{x_1 = 1\}$, and homogeneous Neumann conditions on the remaining boundaries. The problem is discretised with Scott–Vogelius elements on Alfed splits of a regular simplicial mesh. We apply conjugate gradients preconditioned by multigrid W-cycles with two Chebyshev smoothing iterations per level to solve the problem. We consider four variants of the algorithm, considering all combinations of robust and standard components. In addition, we also compare to the HYPRE BoomerAMG algorithm (Henson & Yang, 2002) using

symmetric SOR smoothing on each process and additive updates between processes (Baker *et al.*, 2011). Each solver was terminated when the Euclidean norm of the residual was reduced by eight orders of magnitude, up to a maximum of 200 iterations.

Results for the two-dimensional problem with $[\mathbb{P}_2]^2$ are given in Table 1, and results for the three-dimensional problem with $[\mathbb{P}_3]^3$ are given in Table 2. The algebraic multigrid algorithm exhibits growth of iteration counts as the mesh is refined. For all four geometric multigrid variants, mesh independence is observed, but parameter-robustness is only achieved when both robust relaxation and robust transfer are employed.

Refinements	DoF	0	1	10	10^2	γ 10^3	10^4	10^6	10^8
Robust relaxation & robust transfer									
1	1 602	9	9	10	14	15	15	15	15
2	6 274	9	9	11	15	15	15	15	15
3	24 834	9	9	11	15	15	15	15	15
4	98 818	8	9	10	15	15	15	15	15
5	394 242	8	8	10	14	15	15	15	15
Robust relaxation & standard transfer									
1	1 602	9	9	11	20	79	180	>200	>200
2	6 274	9	9	11	20	>200	>200	>200	>200
3	24 834	9	9	11	20	>200	>200	>200	>200
4	98 818	8	9	11	20	>200	>200	>200	>200
5	394 242	8	9	11	20	>200	>200	>200	>200
Jacobi relaxation & robust transfer									
1	1 602	21	23	51	173	>200	>200	>200	>200
2	6 274	21	24	53	180	>200	>200	>200	>200
3	24 834	21	23	53	181	>200	>200	>200	>200
4	98 818	21	23	52	179	>200	>200	>200	>200
5	394 242	20	23	52	179	>200	>200	>200	>200
Jacobi relaxation & standard transfer									
1	1 602	21	24	48	>200	>200	>200	>200	>200
2	6 274	21	25	46	>200	>200	>200	>200	>200
3	24 834	21	24	46	>200	>200	>200	>200	>200
4	98 818	21	24	46	>200	>200	>200	>200	>200
5	394 242	20	24	46	>200	>200	>200	>200	>200
Algebraic multigrid									
1	1 602	17	19	34	93	180	>200	>200	>200
2	6 274	19	21	37	108	>200	>200	>200	>200
3	24 834	20	23	40	120	>200	>200	>200	>200
4	98 818	22	25	46	132	>200	>200	>200	>200
5	394 242	23	26	49	142	>200	>200	>200	>200

Table 1: Iteration counts in two dimensions for the $[\mathbb{P}_2]^2$ element for five different geometric and algebraic multigrid variants. The geometric multigrid results are obtained with a 4×4 coarse grid.

To give an idea of relative computational costs, we report runtimes of the different solver components in two and three dimensions in Table 3. To cleanly separate setup and application times, the solver was adjusted to use Richardson instead of Chebyshev iteration for the relaxation; when configuring the Chebyshev relaxation, PETSc automatically estimates the eigenvalues of the preconditioned operator,

Refinements	DoF	0	1	10	10^2	γ 10^3	10^4	10^6	10^8
Robust relaxation & robust transfer									
1	23 871	17	15	14	18	23	25	26	25
2	185 115	18	16	16	21	26	29	31	30
3	1 458 867	18	16	16	22	27	31	32	31
Robust relaxation & standard transfer									
1	23 871	17	20	38	124	>200	>200	>200	>200
2	185 115	18	23	>200	>200	>200	>200	>200	>200
3	1 458 867	18	22	>200	>200	>200	>200	>200	>200
Jacobi relaxation & robust transfer									
1	23 871	53	57	114	>200	>200	>200	>200	>200
2	185 115	60	64	114	>200	>200	>200	>200	>200
3	1 458 867	58	64	114	>200	>200	>200	>200	>200
Jacobi relaxation & standard transfer									
1	23 871	53	66	183	>200	>200	>200	>200	>200
2	185 115	60	73	>200	>200	>200	>200	>200	>200
3	1 458 867	58	73	>200	>200	>200	>200	>200	>200
Algebraic multigrid									
1	23 871	42	45	80	>200	>200	>200	>200	>200
2	185 115	46	52	90	>200	>200	>200	>200	>200
3	1 458 867	51	56	97	>200	>200	>200	>200	>200

Table 2: Iteration counts in three dimensions for the $[\mathbb{P}_3]^3$ element for five different geometric and algebraic multigrid variants. The geometric multigrid results are obtained with a $2 \times 2 \times 2$ coarse grid.

which requires applications of the subspace correction in the overall preconditioner setup. Both relaxation and prolongation are of $\mathcal{O}(1)$ complexity in γ ; the cost of relaxation is linear in the number of vertices of the macro mesh on shape regular meshes, while the cost of prolongation is linear in the number of macro cells.

Dimension	Relaxation		Transfer		Other
	Setup	Application	Setup	Application	
2	19.0	15.5	6.8	7.5	4.1
3	83.6	77.0	20.7	17.0	12.8

Table 3: Runtime (in seconds) of the different solver components for simple two level examples in two and three dimensions. To simplify timing the code is run in serial and we employ Richardson iteration on each level. In 2D the fine grid has 221954 dofs, and in 3D the fine grid has 185115 dofs.

6. Conclusion

We have demonstrated that it is possible to construct multigrid methods in both two and three dimensions that are robust with respect to the parameter γ in the nearly incompressible elasticity equations (1.1). This includes situations where the meshes are nonnested, for which there is a need to impose incompressibility constraints as part of the prolongation operator. Our main contribution is to guide the choice of subspace decomposition used by the smoother without the need for any explicit description of the divergence-free space, by utilizing local Fortin operators guaranteed by local inf-sup conditions. This approach can likely be applied more generally when a local inf-sup condition is known.

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Code availability

For reproducibility, we cite archives of the exact software versions used to produce the results in this paper. All major Firedrake components as well as the code used to obtain the shown iteration counts have been archived on Zenodo (zenodo/Firedrake-20210702.0, 2021). An installation of Firedrake with components matching those used to produce the results in this paper can be obtained following the instructions at <https://www.firedrakeproject.org/download.html>.

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